Smooth densities for solutions to stochastic differential equations with jumps

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Abstract

We consider a solution x_t to a generic Markovian jump diffusion and show that for any $t_0 > 0$ the law of x_{t_0} has a C^∞ density with respect to the Lebesgue measure under a uniform version of the Hörmander conditions. Unlike previous results in the area the result covers a class of infinite activity jump processes. The result is accomplished using carefully crafted refinements to the classical arguments used in proving the smoothness of density via Malliavin calculus. In particular, we provide a proof that the semimartingale inequality of J. Norris persists for discontinuous semimartingales when the jumps are small.

1. Introduction

This paper focuses on the study of the stochastic differential equation

$$x_t = x + \int_0^t Z(x_{s-}) \mathrm{d}s + \int_0^t V(x_{s-}) \mathrm{d}W_s + \int_0^t \int_E Y(x_{s-}, y)(\mu - \nu)(\mathrm{d}y, \mathrm{d}s), \qquad (1.1)$$

and addresses the fundamental problem of finding a sufficient condition for the existence of a smooth (C^∞) density for the solution at positive times. For diffusion processes the pioneering work of Bismut [5] and Stroock [16,17] provides a probabilistic framework for establishing such a result under the Hörmander conditions on the vector fields. As is pointed out in [17] it is, given the existence of alternative methods based on partial differential equations, difficult to justify the

effort involved in the probabilistic proof of this result purely for the sake of diffusion processes. From the outset it was always understood that this approach should be used as a template for investigating the smoothness properties for different probabilistic objects, not amenable to analysis by the PDE theory. We now switch our focus to the question: *when does a solution to the* SDE (1.1) *admit a smooth density?*

We point out that we are by no means the first to consider this problem and several prominent landmarks are worthy of comment. The first comprehensive account of these ideas was presented in [4], where a smoothness result is proved under a uniform ellipticity on the diffusion vector fields (in fact [4] also explores how a smooth density can be acquired through the jump component). Further progress was made in [13] and [11] where existence of the density was shown under a version of the Hörmander conditions which are local in the starting point. Both these works were successful in establishing a criterion for a smooth density namely that the inverse of the (reduced) Malliavin covariance matrix has finite L^p norms for $p \geq 2$.

Verification of this criterion usually occurs by way of subtle estimates on the reduced covariance matrix which are in general difficult to establish. In the diffusion case a streamlined approach to obtaining these estimates has been achieved by a semimartingale inequality known as Norris's lemma (see [12] or [14]). This result, interesting in its own right, provides an estimate for the probability that a continuous semimartingale is small on a set where its quadratic variation is comparatively large. Traditionally, this result has been presented as a quantitative form of the uniqueness of the Doob–Meyer decomposition for continuous semimartingales, however the appearance of similar estimates in the context of fractional Brownian motion with $H > 1/2$ (not a semimartingale, see [2]) has made it seem as though Norris's lemma expresses something fundamental rather than anything tied to the particular structure of continuous semimartingales.

Some recent work in the case of jump diffusions has been undertaken in [6,9,10,15]. The article [9] proves a smoothness result under uniform Hörmander conditions and under the assumption that the underlying jump process is of finite activity. This is achieved by fixing some $T > 0$, conditioning on $N_T = n$, the number of jumps until time T, and noticing that this gives rise to some (random) interval $[S_1(\omega), S_2(\omega))$ with $0 \leq S_1 < S_2 < T$ such that $S_2(\omega) - S_1(\omega) \geq T(n+1)^{-1}$ and

$$\left\{ x_t^x : S_1 \leq t < S_2 \right\} \overset{\mathcal{D}}{=} \left\{ \tilde{x}_t^{x_{S_1}^x} : 0 \leq t < S_2 - S_1 \right\}$$

where $\tilde{x}_t^x$ is the diffusion process

$$\tilde{x}_t^x = x + \int_0^t Z(\tilde{x}_s^x) \mathrm{d}s + \int_0^t V(\tilde{x}_s^x) \mathrm{d}W_s.$$

The usual diffusion Norris lemma may be applied to give estimates for the Malliavin covariance matrix arising from $\tilde{x}_t$ on this interval which can then easily be related to the covariance matrix for x_t. In this paper we pursue this idea further by proving that the quality of the estimate which features in Norris's lemma is preserved when jumps are introduced provided that these jumps are small enough so that they do not interfere too much. We then develop the conditioning argument outlined above by splitting up the sample path into disjoint intervals on which the jumps are small, and then estimating the Malliavin covariance matrix on the largest of these intervals. The outcome of this reasoning will be the conclusion that a solution to (1.1) has a smooth density under uniform Hörmander conditions (indeed, the same conditions as in [9]) and subject to some restrictions on the rate at which the jump measure accumulates small jumps. These conditions are sufficiently flexible to admit some jump diffusions based on infinite activity jump processes.

This paper is arranged as follows. We first present some preliminary results and notation on Malliavin calculus. Subsequently, we state and prove our new version of Norris's lemma and then illustrate how it may be utilized in concert with classical arguments to verify the C^∞ density criterion for the solution to (1.1).

2. Preliminaries

Let x_t denote the solution to the SDE

$$x_t^x = x + \int_0^t Z(x_{s-}^x)\mathrm{d}s + \int_0^t V(x_{s-}^x)\mathrm{d}W_s + \int_0^t \int_E Y(x_{s-}^x, y)(\mu - v)(\mathrm{d}y, \mathrm{d}s), \qquad (2.1)$$

where $W_t = \left(W_t^1, \ldots, W_t^d\right)$ is an $\mathbb{R}^d$-valued Brownian motion on some probability space $(\Omega, \mathcal{F}_t, P)$ and μ is a $(\Omega, \mathcal{F}_t, P)$-Poisson random measure on $E \times [0, \infty)$ for some topological[1] space E such that v, the compensator of μ, is of the form $G(\mathrm{d}y)\mathrm{d}t$ for some σ-finite measure G. The vector fields $Z : \mathbb{R}^e \to \mathbb{R}^e$, $Y(\cdot, y) : \mathbb{R}^e \to \mathbb{R}^e$ and $V = (V_1, \ldots, V_d)$, where $V_i : \mathbb{R}^e \to \mathbb{R}^e$ for $i \in \{1, \ldots, d\}$, and where we will understand throughout that

$$\int_0^t U(s)\mathrm{d}W_s = \sum_{i=1}^d \int_0^t U_i(s)\,\mathrm{d}W_s^i,$$

for any $U(t) = (U_1(t), \ldots, U_d(t))$, with $U(t) \in \mathbb{R}^e$ and such that the stochastic integral makes sense. At times we will write simply x_t, dropping the emphasis on the starting point.

We introduce some notation, firstly for $p \in \mathbb{R}$ let

$$L_+^p(G) = \left\{ f : E \to \mathbb{R}^+ : \int_E f(y)^p G(\mathrm{d}y) < \infty \right\},$$

and define

$$L_+^{p,\infty}(G) = \bigcap_{q \geq p} L_+^q(G).$$

We will always assume that at least the following conditions are in force.

Condition 1. $Z, V_1, \ldots, V_d \in C_b^\infty(\mathbb{R}^e)$.

Condition 2. *For some $\rho_2 \in L_+^{2,\infty}(G)$ and every $n \in \mathbb{N}$*

$$\sup_{y \in E, x \in \mathbb{R}^e} \frac{1}{\rho_2(y)}|\nabla_1^n Y(x, y)| < \infty.$$

Condition 3. $\sup_{y \in E, x \in \mathbb{R}^e} |(I + \nabla_1 Y(x, y))^{-1}| < \infty$.

We now define the processes $J_{t \leftarrow 0}^{x,I}$ and $J_{0 \leftarrow t}^{x,I}$ considered as linear maps from $\mathbb{R}^e$ to $\mathbb{R}^e$ as the solutions to the following SDEs

$$J_{t \leftarrow 0}^{x,I} = I + \int_0^t \nabla Z(x_{s-}^x) J_{s- \leftarrow 0}^{x,I}\mathrm{d}s + \int_0^t \nabla V(x_{s-}^x) J_{s- \leftarrow 0}^{x,I}\mathrm{d}W_s$$

$$+ \int_0^t \int_E \nabla_1 Y(x_{s-}^x, y) J_{s- \leftarrow 0}^{x,I}(\mu - v)(\mathrm{d}y, \mathrm{d}s) \qquad (2.2)$$

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[1] We will later need some vector space structure on E and will principally be concerned with the case $E = \mathbb{R}^n$.

and

$$J_{0\leftarrow t}^{x,I} = I - \int_0^t J_{0\leftarrow s-}^{x,I} \left(\nabla Z(x_{s-}^x) - \sum_{i=1}^d \nabla V_i(x_{s-}^x)^2 \right.$$

$$\left. - \int_E (I + \nabla_1 Y(x_{s-}, y))^{-1} \nabla_1 Y(x_{s-}^x, y) G(dy) \right) ds - \int_0^t J_{0\leftarrow s-}^{x,I} \nabla V(x_{s-}^x) dW_s$$

$$- \int_0^t \int_E J_{0\leftarrow s-}^{x,I} (I + \nabla_1 Y(x_{s-}^x, y))^{-1} \nabla_1 Y(x_{s-}^x, y)(\mu - \nu)(dy, ds). \qquad (2.3)$$

The following result may then be verified (see for instance [11]).

Theorem 1. *Under Conditions 1–3 the systems of* SDEs *(2.1), (2.2) and (2.1), (2.3) have unique solutions with*

$$\sup_{0 \le s \le t} |J_{s\leftarrow 0}^{x,I}| \quad and \quad \sup_{0 \le s \le t} |J_{0\leftarrow s}^{x,I}| \in L^p$$

for all $t \ge 0$ *and* $p < \infty$. *Moreover,*

$$J_{0\leftarrow t}^{x,I} = \left(J_{t\leftarrow 0}^{x,I} \right)^{-1} \quad for\ all\ t \ge 0\ almost\ surely.$$

We define the reduced Malliavin covariance matrix

$$C_{0,t}^{x,I} = C_t^{x,I} = \int_0^t \sum_{i=1}^d J_{0\leftarrow s-}^{x,I} V_i(x_{s-}^x) \otimes J_{0\leftarrow s-}^{x,I} V_i(x_{s-}^x) ds$$

which we will sometimes refer to simply as C_t suppressing the dependence on the initial conditions. The following well-known result provides a sufficient condition for the process x_t to have a C^∞ density in terms of the moments of the inverse of C_t (see [11]).

Theorem 2. *Fix* $t_0 > 0$ *and* $x \in \mathbb{R}^e$ *and suppose that for every* $p \ge 2$ $\left| \left(C_{t_0}^{x,I} \right)^{-1} \right| \in L^p$, *then* $x_{t_0}^x$ *has a* C^∞ *density with respect to the Lebesgue measure.*

3. Norris's lemma

From now on we set $E = \mathbb{R}^n$. The following result provides an exponential martingale type inequality for a class of local martingales based on stochastic integrals with respect to a Poisson random measure when the jumps of the local martingale are bounded. Interesting discussions on results of this type can be found in [1,7].

Lemma 1. *Let* μ *be a Poisson random measure on* $E \times [0, \infty)$ *with compensator* ν *of the form* $\nu(dy, dt) = G(dy)dt$. *Let* $f(t, y)$ *be a real-valued previsible process having the property that*

$$\sup_{y \in E} \sup_{0 \le s \le t} |f(s, y)| < A \quad a.s.$$

and

$$\int_0^t \int_E f(s, y)^2 G(dy)ds < \infty \quad a.s.$$

for every $0 < t < \infty$ *and some* $A < \infty$. *Then, if* $M_t = \int_0^t \int_E f(s, y)(\mu - \nu)(\mathrm{d}y, \mathrm{d}s)$ *the following inequality holds*

$$P\left(\sup_{0 \leq s \leq t} |M_s| \geq \delta, \langle M \rangle_t < \rho\right) \leq 2 \exp\left(-\frac{\delta^2}{2(A\delta + \rho)}\right).$$

Proof. Consider $Z_t = \exp(\theta M_t - \alpha \langle M \rangle_t)$ with $0 < \theta < A^{-1}$ and $\alpha = 2^{-1}\theta^2(1 - \theta A)^{-1}$. For any $x \in \mathbb{R}$ we have

$$g_\theta(x) := e^{\theta x} - 1 - \theta x = \sum_{k=2}^{\infty} \frac{\theta^k x^k}{k!} \leq \frac{\theta^2 x^2}{2} \sum_{k=0}^{\infty} (\theta A)^k = \frac{\theta^2 x^2}{2(1 - \theta A)} = \alpha x^2. \tag{3.1}$$

We may deduce that Z is a supermartingale by writing

$$Z_t = \exp\left(\theta M_t - \int_0^t \int_E g_\theta(f(s, y))G(\mathrm{d}y)\mathrm{d}s\right)$$
$$\exp\left(\int_0^t \int_E \left(g_\theta(f(s, y)) - \alpha f(s, y)^2\right) G(\mathrm{d}t)\mathrm{d}s\right)$$

and, using Itô's formula the first term of the product is a non-negative local martingale (and hence a supermartingale) and the second term decreases in t by (3.1). Define the stopping time $T = \inf\{s \geq 0 : \langle M \rangle_s > \rho\}$ then, since $Z_0 = 1$, taking $\theta = \delta(\rho + A\delta)^{-1}$ and applying Doob's supermartingale inequality give

$$P\left(\sup_{0 \leq s \leq t} |M_s| \geq \delta, \langle M \rangle_t < \rho\right) \leq P\left(\sup_{0 \leq s \leq T} Z_s \geq e^{\delta\theta - \alpha\rho}\right) \leq \exp\left(-\frac{\delta^2}{2(A\delta + \rho)}\right).$$

Finally, we complete the proof by applying the same argument to $-M$. $\qquad\square$

From now on we will assume that the following technical conditions on the jump measure G and the jump vector field Y are in force:

Condition 4. $\sup_{x \in \mathbb{R}^e} \int_E |Y(x, y)|G(\mathrm{d}y) < \infty.$

Condition 5. *For some* $\kappa \geq n$ *we have*

$$\limsup_{\epsilon \downarrow 0} \frac{1}{f(\epsilon)} \int_{|y| > \epsilon} G(\mathrm{d}y) < \infty, \tag{3.2}$$

where $f : (0, \infty) \to \mathbb{R}$ *is defined by*

$$f(x) = \begin{cases} -\log x^{-1} & \text{if } \kappa = n \\ x^{-\kappa + n} & \text{if } \kappa > n. \end{cases} \tag{3.3}$$

Moreover, for any $\beta > 0$ *we have*

$$\int_E |y|^{\kappa - n + \beta} G(\mathrm{d}y) < \infty,$$

and

$$\limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon^\beta} \int_{|y| < \epsilon} |y|^{\kappa - n + \beta} G(\mathrm{d}y) < \infty. \tag{3.4}$$

Condition 6. *There exists a function $\phi \in L^1_+(G)$ which has the properties that for some $\alpha > 0$*

$$\limsup_{y \to 0} \frac{\phi(y)}{|y|^{\kappa - n + \alpha}} < \infty,$$

and, for some positive constant $C < \infty$ and every $k \in \mathbb{N} \cup \{0\}$

$$\sup_{x \in \mathbb{R}^e} |\nabla_1^k Y(x, y)| \le C\phi(y).$$

Conditions 4–6 may at first sight appear somewhat opaque, however they will be a crucial ingredient in our subsequent arguments, in particular they enable us to quantify the rate at which the total mass of the jump measure increases near zero. To develop intuition for their implications consider the following straightforward example: take $n = 1$ and $Y(x, y) = \widetilde{Y}(x)y$ for some C^∞-bounded $\widetilde{Y} : \mathbb{R}^e \to \mathbb{R}^e$ (this puts us in the setup of [9]). Also, define the measure G on $\mathbb{R}$ by taking $G(\mathrm{d}y) = |y|^{-\kappa} 1_{\{|y| \le 1\}} \mathrm{d}y$. We then see what is needed to verify each of the conditions in turn, firstly, Condition 4 will be satisfied provided

$$\sup_{x \in \mathbb{R}^e} \int_E |\widetilde{Y}(x)y| G(\mathrm{d}y) = \sup_{x \in \mathbb{R}^e} |\widetilde{Y}(x)| \int_E |y| G(\mathrm{d}y) = 2 \sup_{x \in \mathbb{R}^e} |\widetilde{Y}(x)| \int_0^1 y^{1-\kappa} \mathrm{d}y < \infty,$$

which will hold so long as $\kappa < 2$. The constraint that $\kappa \ge 1$ in Condition 5 ensures that the jump measure is of infinite activity and (3.2) and (3.3) are trivially verified by integration. Since we are in the setting $1 \le \kappa < 2$, we may find $\alpha \in (0, 1)$ such that $\kappa + \alpha < 2$ to ensure that $\phi(y) := |y|$ is $O(|y|^{\kappa - n + \alpha})$ as $y \to 0$ and hence Condition 6 is also satisfied.

Suppose now that $\Upsilon : [0, t_0] \times E \to \mathbb{R}$ is some given, real-valued, previsible process. It will at times be important for us to impose the following condition on Υ.

Condition 7. *Let G satisfy Condition 5. Then there exists some previsible process D_t taking values in $[0, \infty)$ with $\sup_{0 \le t \le t_0} D_t \in L^p$ for all $p \ge 1$, and a function $\phi \in L^1_+(G)$ such that*

$$|\Upsilon(t, y)| \le D_t \phi(y) \quad \text{for all } t \in [0, t_0] \text{ and } y \in E, \tag{3.5}$$

and for some $\alpha = \alpha_\Upsilon > 0$

$$K_\phi := \limsup_{y \to 0} \frac{\phi(y)}{|y|^{\kappa - n + \alpha}} < \infty. \tag{3.6}$$

Equipped with these remarks we are now in a position to state and prove the following lemma which will be fundamental to providing the estimates on the reduced covariance matrix we need later.

Lemma 2 (*Norris-Type Lemma*). *Fix $t_0 > 0$ and for every $\epsilon > 0$ suppose $\beta^\epsilon(t), \gamma^\epsilon(t) = (\gamma_1^\epsilon(t), \ldots, \gamma_d^\epsilon(t)), u^\epsilon(t) = (u_1^\epsilon(t), \ldots, u_d^\epsilon(t))$ are previsible processes taking values in $\mathbb{R}, \mathbb{R}^d$ and $\mathbb{R}^d$ respectively. Suppose further that $\zeta^\epsilon(t, y)$ and $f^\epsilon(t, y)$ are real-valued previsible processes satisfying Condition 7 such that the functions ϕ^ζ and ϕ^f do not depend on ϵ and moreover for every $q \ge 1$*

$$\sup_{\epsilon > 0} E \left[\sup_{0 \le t \le t_0} \left(D_t^{\zeta, \epsilon} \right)^q + \sup_{0 \le t \le t_0} \left(D_t^{f, \epsilon} \right)^q \right] < \infty. \tag{3.7}$$

Let $\alpha = \min(\alpha_\zeta, \alpha_f)$, $\delta > 0$, $z = 3\delta(\kappa - n + \alpha)^{-1}$ and define the processes a^ϵ and Y^ϵ as the solutions to the SDEs

$$a^\epsilon(t) = \alpha + \int_0^t \beta^\epsilon(s)\mathrm{d}s + \sum_{i=1}^d \int_0^t \gamma_i^\epsilon(s)\mathrm{d}W_s^i + \int_0^t \int_{|y|<\epsilon^z} \zeta^\epsilon(s, y)(\mu - \nu)(\mathrm{d}s, \mathrm{d}y)$$

$$Y^\epsilon(t) = y + \int_0^t a^\epsilon(s)\mathrm{d}s + \sum_{i=1}^d \int_0^t u_i^\epsilon(s)\mathrm{d}W_s^i + \int_0^t \int_{|y|<\epsilon^z} f^\epsilon(s, y)(\mu - \nu)(\mathrm{d}s, \mathrm{d}y).$$

Assume that for some $p \geq 2$ the quantity

$$\sup_{\epsilon>0} E\left[\sup_{0\leq t\leq t_0} \left(|\beta^\epsilon(t)| + |\gamma^\epsilon(t)| + |a^\epsilon(t)| + |u^\epsilon(t)| \right. \right.$$

$$\left. \left. + \int_E (|\zeta^\epsilon(t, y)|^2 + |f^\epsilon(t, y)|^2) G(\mathrm{d}y) \right)^p \right] \tag{3.8}$$

is finite, and for some $\rho_1, \rho_2 \in L_+^{2,\infty}(G)$ we have

$$\sup_{\epsilon>0} \left(E\left[\left(\sup_{0\leq t\leq t_0} \sup_{y\in E} \frac{|\zeta^\epsilon(t, y)|}{\rho_1(y)} \right)^p \right] + E\left[\left(\sup_{0\leq t\leq t_0} \sup_{y\in E} \frac{|f^\epsilon(t, y)|}{\rho_2(y)} \right)^p \right] \right) < \infty.$$

Then we can find finite constants c_1, c_2 and c_3 which do not depend on ϵ, such that for any $q > 8$ and any $l, r, v, w > 0$ with $18r + 9v < q - 8$, there exists $\epsilon_0 = \epsilon_0(t_0, q, r, v, l)$ such that if $\epsilon \leq \epsilon_0 < 1$ and $\delta w^{-1} > \max(q/2 - r + v/2, (\kappa - n + \alpha)/4\alpha)$ we have

$$P\left(\int_0^{t_0} \left(Y^\epsilon(t) \right)^2 \mathrm{d}t < \epsilon^{qw}, \right.$$

$$\left. \times \int_0^{t_0} \left(\left| a^\epsilon(t) - \int_{|y|<\epsilon^z} f^\epsilon(t, y) G(\mathrm{d}y) \right|^2 + |u^\epsilon(t)|^2 \right) \mathrm{d}t \geq l\epsilon^w \right)$$

$$\leq c_1 \epsilon^{rwp} + c_2 \epsilon^{wp/4} + c_3 \exp\left(-\epsilon^{-vw/2} \right).$$

Moreover, we have $\epsilon_0(t_0, q, r, v, l) = t_0^{-k} \epsilon_0(q, r, v, l)$ for some $k > 0$.

Proof. Let $0 < C < \infty$ denote a generic constant which varies from line to line and which does not depend on ϵ. We begin with some preliminary remarks. Firstly, the hypotheses of the theorem are sufficient to imply (by Theorem A6 of [3]) that

$$\sup_{\epsilon>0} \left(\max \left(E\left[\sup_{0\leq t\leq t_0} |Y^\epsilon(t)|^p \right], E\left[\sup_{0\leq t\leq t_0} |a^\epsilon(t)|^p \right] \right) \right) < \infty.$$

Secondly, by hypothesis we can find previsible processes $D_t^{\zeta,\epsilon}$ and $D_t^{f,\epsilon}$ and functions ϕ^ζ and ϕ^f not depending on ϵ such that

$$|\zeta^\epsilon(t, y)| \leq D_t^{\zeta,\epsilon} \phi^\zeta(y) \quad \text{and} \quad |f^\epsilon(t, y)| \leq D_t^{f,\epsilon} \phi^f(y). \tag{3.9}$$

Let $D_t^\epsilon = \max(D_t^{\zeta,\epsilon}, D_t^{f,\epsilon})$ and $\phi(y) = \max(\phi^\zeta(y), \phi^f(y))$ and (using the notation of (3.6)) $K = \max(K_\zeta, K_f, 1)$, then for some $\epsilon^* > 0$ we have

$$\phi(y) \leq K|y|^{\kappa-n+\alpha} \tag{3.10}$$

for $|y| \leq \epsilon^*$. Consequently taking $\epsilon \leq \min(\epsilon^*, 1)$ and using the definition of z we see that for $|y| < \epsilon^z$

$$\phi(y) \leq K\epsilon^{z(\kappa - n + \alpha)} = K\epsilon^{3\delta}. \tag{3.11}$$

Now, we define

$$A = \left\{ \int_0^{t_0} \left(Y^\epsilon(t) \right)^2 dt < \epsilon^{qw}, \right.$$

$$\left. \int_0^{t_0} \left(\left| a^\epsilon(t) - \int_{|y| < \epsilon^z} f^\epsilon(t, y) G(dy) \right|^2 + |u^\epsilon(t)|^2 \right) dt \geq l\epsilon^w \right\}$$

and let

$$\theta_t = |\beta^\epsilon(t)| + |\gamma^\epsilon(t)| + |a^\epsilon(t)| + |u^\epsilon(t)| + \int_{|y| < \epsilon^z} (|\zeta^\epsilon(t, y)|^2 + |f^\epsilon(t, y)|^2) G(dy).$$

Taking $\psi = \alpha(\kappa - n + \alpha)^{-1} \leq 1$ we see using (3.9) and (3.11) that on the set $\left\{ \sup_{0 \leq t \leq t_0} |D_t^\epsilon| \leq K^{-1}\epsilon^{-\psi\delta} \right\}$ we have

$$\sup_{0 \leq t \leq t_0} \max(|\zeta^\epsilon(t, y)|, |f^\epsilon(t, y)|) \leq \epsilon^{-\psi\delta}\epsilon^{3\delta} \leq \epsilon^{2\delta}. \tag{3.12}$$

Define the stopping time $T = \min(\inf\{s \geq 0 : \sup_{0 \leq u \leq s} \theta_s > \epsilon^{-rw}\}, t_0)$, let $A_1 = \{T < t_0\}$, $A_2 = \{\sup_{0 \leq t \leq t_0} |D_t^\epsilon| > K^{-1}\epsilon^{-\psi\delta}\}$, $A_3 = A \cap A_1^c \cap A_2^c$ and observe that

$$P(A) \leq P(A_1) + P(A_2) + P(A_3).$$

Using (3.7), the finiteness of (3.8) and Chebyshev's inequality gives

$$P(A_1) \leq \epsilon^{rwp} E\left[\sup_{0 \leq t \leq t_0} \theta_s^p \right] \leq C\epsilon^{rwp} \quad \text{and} \quad P(A_2) \leq \epsilon^{\delta\psi p} E\left[\sup_{0 \leq t \leq t_0} D_t^p \right] \leq C\epsilon^{\delta\psi p},$$

while on the set A_3 the processes a^ϵ and Y^ϵ satisfy, by virtue of (3.12), the SDEs

$$da^\epsilon(t) = \beta^\epsilon(t)dt + \sum_{i=1}^d \gamma_i^\epsilon(t)dW_t^i + \int_{|y| < \epsilon^z} \zeta^\epsilon(t, y) 1_{\{|\zeta^\epsilon(t,y)| < \epsilon^{2\delta}\}}(\mu - \nu)(dt, dy),$$

$$dY^\epsilon(t) = a^\epsilon(t)dt + \sum_{i=1}^d u_i^\epsilon(t)dW_t^i + \int_{|y| < \epsilon^z} f^\epsilon(t, y) 1_{\{|f^\epsilon(t,y)| < \epsilon^{2\delta}\}}(\mu - \nu)(dt, dy),$$

with $a^\epsilon(0) = \alpha$, $Y^\epsilon(0) = y$. We now define the following processes

$$A_t = \int_0^t a^\epsilon(s)ds, \quad M_t = \sum_{i=1}^d \int_0^t u_i^\epsilon(s)dW_s^i, \qquad Q_t = \sum_{i=1}^d \int_0^t A(s)\gamma_i^\epsilon(s)dW_s^i,$$

$$N_t = \sum_{i=1}^d \int_0^t Y^\epsilon(s-)u_i^\epsilon(s)dW_s^i,$$

$$P_t = \int_0^t \int_{|y| < \epsilon^z} f^\epsilon(s, y) 1_{\{|f^\epsilon(s,y)| < \epsilon^{2\delta}\}}(\mu - \nu)(ds, dy),$$

$$L_t = \int_0^t \int_{|y| < \epsilon^z} Y^\epsilon(s-) f^\epsilon(s, y) 1_{\{|f^\epsilon(s,y)| < \epsilon^{2\delta}\}} (\mu - \nu)(ds, dy),$$

$$H_t = \int_0^t \int_{|y| < \epsilon^z} A(s) \zeta^\epsilon(s, y) 1_{\{|\zeta^\epsilon(s,y)| < \epsilon^{2\delta}\}} (\mu - \nu)(ds, dy),$$

$$J_t = \int_0^t \int_{|y| < \epsilon^z} f^\epsilon(s, y)^2 1_{\{|f^\epsilon(s,y)| < \epsilon^{2\delta}\}} (\mu - \nu)(ds, dy),$$

and for $\delta_j > 0$, $\rho_j > 0$, $j \in \{1, \ldots, 7\}$ define the sets

$$B_1 = \left\{ \langle N \rangle_T < \rho_1, \ \sup_{0 \le t \le T} |N_t| \ge \delta_1 \right\}, \qquad B_2 = \left\{ \langle M \rangle_T < \rho_2, \ \sup_{0 \le t \le T} |M_t| \ge \delta_2 \right\},$$

$$B_3 = \left\{ \langle Q \rangle_T < \rho_3, \ \sup_{0 \le t \le T} |Q_t| \ge \delta_3 \right\}, \qquad C_1 = \left\{ \langle P \rangle_T < \rho_4, \ \sup_{0 \le t \le T} |P_t| \ge \delta_4 \right\},$$

$$C_2 = \left\{ \langle L \rangle_T < \rho_5, \ \sup_{0 \le t \le T} |L_t| \ge \delta_5 \right\}, \qquad C_3 = \left\{ \langle H \rangle_T < \rho_6, \ \sup_{0 \le t \le T} |H_t| \ge \delta_6 \right\},$$

$$C_4 = \left\{ \langle J \rangle_T < \rho_7, \ \sup_{0 \le t \le T} |J_t| \ge \delta_7 \right\}.$$

The exponential martingale inequality for continuous semimartingales gives $P(B_j) \le 2e^{-\delta_j^2/2\rho_j}$ for $j = 1, 2, 3$. Since the jumps in P and J are bounded by $\epsilon^{2\delta}$ and $\epsilon^{4\delta}$ respectively, an application of Lemma 1 gives

$$P(C_1) \le 2 \exp\left(\frac{-\delta_4^2}{2(\epsilon^{2\delta}\delta_4 + \rho_4)} \right) \quad \text{and} \quad P(C_4) \le 2 \exp\left(\frac{-\delta_7^2}{2(\epsilon^{4\delta}\delta_7 + \rho_7)} \right).$$

For C_2 and C_3 we use the fact that $\sup_{0 \le t \le T} |a^\epsilon(t)| \in L^p$ and $\sup_{0 \le t \le T} |Y^\epsilon(t)| \in L^p$ uniformly in ϵ to see

$$P(C_2) \le P\left(\langle L \rangle_T < \rho_5, \ \sup_{0 \le t \le T} |L_t| \ge \delta_5, \ \sup_{0 \le t \le T} |Y^\epsilon(t)| \le \epsilon^{-\delta} \right)$$

$$+ P\left(\sup_{0 \le t \le T} |Y^\epsilon(t)| > \epsilon^{-\delta} \right)$$

$$\le 2 \exp\left(\frac{-\delta_5^2}{2(\epsilon^\delta \delta_5 + \rho_5)} \right) + C\epsilon^{\delta p},$$

where the second term comes from Chebyshev's inequality and the first follows from Lemma 1 in concert with the observation that, on the set $\{\sup_{0 \le t \le T} |Y^\epsilon(t)| \le \epsilon^{-\delta}\}$, we have

$$L_t = \int_0^t \int_{|y| < \epsilon^z} Y^\epsilon(s-) f^\epsilon(s, y) 1_{\{|f^\epsilon(s,y)| < \epsilon^{2\delta}, |Y^\epsilon(s-)| \le \epsilon^{-\delta}\}} (\mu - \nu)(ds, dy)$$

for $0 \leq t \leq T$. Hence, the jumps in L are bounded by ϵ^δ on this set. The same argument may also be applied to C_3 to give

$$P(C_3) \leq 2 \exp \left(\frac{-\delta_6^2}{2(\epsilon^\delta \delta_6 + \rho_6)} \right) + C\epsilon^{\delta p}.$$

We now show that $A_3 \subset \left(\cup_{j=1}^3 B_j \right) \cup \left(\cup_{j=1}^4 C_j \right)$ whence on choosing appropriate values for δ_j and ρ_j the proof shall be complete. To do this suppose that $\omega \notin \left(\cup_{j=1}^3 B_j \right) \cup \left(\cup_{j=1}^4 C_j \right)$, $T(\omega) = t_0$, $\int_0^T Y_t^\epsilon(\omega)^2 dt < \epsilon^{qw}$ and $\sup_{0 \leq t \leq T} |D_t^\epsilon(\omega)| < K^{-1}\epsilon^{-\psi\delta}$. Then

$$\langle N \rangle_T = \int_0^T (Y^\epsilon(t-))^2 |u^\epsilon(t)|^2 dt < \epsilon^{(-2r+q)w} =: \rho_1,$$

and since $\omega \notin B_1$, $\sup_{0 \leq t \leq T} \left| \sum_{i=1}^d \int_0^t Y^\epsilon(s-)u_i^\epsilon(s)dW_s^i \right| < \delta_1 := \epsilon^{q_1}$, where $q_1 = (q/2 - r - v/2)w$. By the same reasoning we have

$$\langle L \rangle_T = \int_0^T \int_{|y| < \epsilon^z} Y^\epsilon(t-)^2 f^\epsilon(s, y)^2 1_{\{|f^\epsilon(t,y)| < \epsilon^{2\delta}\}} G(dy)dt < \epsilon^{(-2r+q)w} =: \rho_5,$$

since $\omega \notin C_2$ we may let $\delta_5 = \epsilon^{q_1}$ to give $\sup_{0 \leq t \leq T} |L_t| < \delta_5$. Since we also have

$$\sup_{0 \leq t \leq T} \left| \int_0^t Y^\epsilon(s-)a^\epsilon(s)ds \right| \leq \left(t_0 \int_0^T Y^\epsilon(s-)^2 a^\epsilon(s)^2 ds \right)^{1/2} < t_0^{1/2} \epsilon^{(-r+q/2)w},$$

it follows that

$$\sup_{0 \leq t \leq T} \left| \int_0^t Y^\epsilon(s-)dY^\epsilon(s) \right| < t_0^{1/2} \epsilon^{(-r+q/2)w} + 2\epsilon^{q_1}.$$

Itô's formula now gives $Y^\epsilon(t)^2 = y^2 + 2\int_0^t Y^\epsilon(s-)dY^\epsilon(s) + \langle M \rangle_t + [P]_t$, and we notice that because

$$\langle J \rangle_T = \int_0^T \int_{|y| < \epsilon^z} f^\epsilon(s, y)^4 1_{\{|f^\epsilon(s,y)| < \epsilon^{2\delta}\}} G(dy)dt$$

$$\leq \epsilon^{4\delta} \int_0^T \int_{|y| < \epsilon^z} f^\epsilon(s, y)^2 1_{\{|f^\epsilon(s,y)| < \epsilon^{2\delta}\}} G(dy)dt \leq \epsilon^{4\delta - rw} =: \rho_7,$$

and since $\omega \notin C_4$ we must have $\sup_{0 \leq t \leq T} |J_t| = \sup_{0 \leq t \leq T} |[P]_t - \langle P \rangle_t| \leq \delta_7 := \epsilon^{2\delta - (r+v)w}$. Consequently,

$$\langle M \rangle_t + \langle P \rangle_t \leq Y^\epsilon(t)^2 - y^2 - 2\int_0^t Y^\epsilon(s-)dY^\epsilon(s) + \sup_{0 \leq t \leq T} |[P]_t - \langle P \rangle_t|$$

and hence,

$$\int_0^T \langle M \rangle_t \, dt + \int_0^T \langle P \rangle_t \, dt < \epsilon^{qw} + t_0^{3/2} \epsilon^{(-r+q/2)w} + 2t_0 \epsilon^{q_1} + t_0 \epsilon^{2\delta - (r+v)w}.$$

We notice that $2\delta - (r+v)w > (q-3r)w > q_1$, $qw > q_1$ and $(q/2-r)w > q_1$ and so provided

$$\epsilon < \min \left(1, t_0^{-1/(2\delta - (r+v)w - q_1)}, t_0^{-3/2((-r+q/2)w - q_1)} \right)$$

we get

$$\int_0^T \langle M \rangle_t \, dt + \int_0^T \langle P \rangle_t \, dt < (2t_0 + 3)\epsilon^{q_1}.$$

$\langle M \rangle_t$ and $\langle P \rangle_t$ are increasing processes, so for any $0 < \gamma < T$

$$\gamma \langle M \rangle_{T-\gamma} < (2t_0 + 3)\epsilon^{q_1} \quad \text{and} \quad \gamma \langle P \rangle_{T-\gamma} < (2t_0 + 3)\epsilon^{q_1}.$$

Since these processes are also continuous we get $\langle M \rangle_T \leq \gamma^{-1}(2t_0 + 3)\epsilon^{q_1} + \gamma\epsilon^{-2rw}$ and $\langle P \rangle_T \leq \gamma^{-1}(2t_0 + 3)\epsilon^{q_1} + \gamma\epsilon^{-2rw}$. By defining $\rho_2 = \rho_4 := 2(2t_0 + 3)^{1/2}\epsilon^{-2rw+q_1/2}$ and $\gamma = (2t_0 + 3)^{1/2}\epsilon^{q_1/2}$, we get $\langle M \rangle_T < \rho_2$ and $\langle P \rangle_T < \rho_4$, and since $\omega \notin B_2 \cup C_1$ we have

$$\sup_{0 \leq t \leq T} |M_t| < \delta_2 := \epsilon^{(q/8 - 5r/4 - 5v/8)w} =: \epsilon^{q_2}, \qquad \sup_{0 \leq t \leq T} |P_t| < \delta_4 = \epsilon^{q_2}.$$

Since $\int_0^T Y^\epsilon(t)^2 dt < \epsilon^{qw}$, Chebyshev's inequality gives

$$\text{Leb}\left\{t \in [0, T] : |Y_t^\epsilon(\omega)| \geq \epsilon^{qw/3}\right\} \leq \epsilon^{qw/3}$$

so that

$$\text{Leb}\left\{t \in [0, T] : |y + A_t(\omega)| \geq \epsilon^{qw/3} + 2\epsilon^{q_2}\right\} \leq \epsilon^{qw/3}.$$

Then, for each $t \in [0, T]$, there exists some $s \in [0, T]$ such that $|s - t| \leq \epsilon^{qw/3}$ and $|y + A_s(\omega)| < \epsilon^{qw/3} + 2\epsilon^{q_2}$, which yields

$$|y + A_t| \leq |y + A_s| + \left|\int_s^t a^\epsilon(\tau) d\tau\right| < (1 + \epsilon^{-rw})\epsilon^{qw/3} + 2\epsilon^{q_2}.$$

In particular we have $|y| < (1 + \epsilon^{-rw})\epsilon^{qw/3} + 2\epsilon^{q_2}$ and, for all $t \in [0, T]$, since $q_2 < (q/3 - r)w$, we have

$$|A_t| < 2\left((1 + \epsilon^{-rw})\epsilon^{qw/3} + 2\epsilon^{q_2}\right) \leq 8\epsilon^{q_2}.$$

This implies that

$$\langle Q \rangle_T = \int_0^T A(t)^2 |\gamma^\epsilon(t)|^2 dt < 64 t_0 \epsilon^{2q_2 - 2rw} =: \rho_3$$

$$\langle H \rangle_T = \int_0^T \int_{|y| < \epsilon^z} A(t)^2 \zeta^\epsilon(t, y)^2 1_{\{|\zeta^\epsilon(t,y)| < \epsilon^{2\delta}\}} G(dy) dt \leq \rho_3 =: \rho_6,$$

and since $\omega \notin B_3 \cup C_3$ we must have

$$\sup_{0 \leq t \leq T} |Q_t| = \sup_{0 \leq t \leq T} \left|\sum_{i=1}^d \int_0^t A(s)\gamma_i^\epsilon(s) dW_i(s)\right| < \delta_3 := \epsilon^{(q/8 - 9r/4 - 9v/8)w} =: \epsilon^{q_3}$$

$$\sup_{0 \leq t \leq T} |H_t| = \sup_{0 \leq t \leq T} \left|\int_0^t \int_{|y| < \epsilon^z} A(s)\zeta^\epsilon(s, y) 1_{\{|\zeta^\epsilon(s,y)| < \epsilon^{2\delta}\}}(\mu - \nu)(ds, dy)\right| < \delta_6 := \epsilon^{q_3}.$$

Now we observe using (3.9), (3.10), Condition 5, $\sup_{0 \leq t \leq T} |D_t^\epsilon(\omega)| < K^{-1}\epsilon^{-\psi\delta}$, the definition of ψ, and the fact that ϕ^f does not depend on ϵ

$$\int_0^{t_0} \left| \int_{|y|<\epsilon^z} f^\epsilon(t, y) G(\mathrm{d}y) \right|^2 \mathrm{d}t \le t_0 \left(\epsilon^{-\delta\psi} \int_{|y|<\epsilon^z} |y|^{\kappa-n+\alpha} G(\mathrm{d}y) \right)^2$$

$$\le C t_0 \epsilon^{-2\delta\psi+2z\alpha} = C t_0 \epsilon^{4\delta\alpha/(\kappa-n+\alpha)}.$$

An application of Itô's formula then gives

$$\int_0^T \left(\left| a^\epsilon(t) - \int_{|y|<\epsilon^z} f^\epsilon(t, y) G(\mathrm{d}y) \right|^2 + |u^\epsilon(t)|^2 \right) \mathrm{d}t$$

$$\le 2 \int_0^T a^\epsilon(t)^2 \mathrm{d}t + \int_0^T |u^\epsilon(t)|^2 \mathrm{d}t + 2 \int_0^T \left| \int_{|y|<\epsilon^z} f^\epsilon(t, y) G(\mathrm{d}y) \right|^2 \mathrm{d}t$$

$$\le 2 \int_0^T a^\epsilon(t) \mathrm{d}A(t) + \langle M \rangle_T + 2 C t_0 \epsilon^{4\delta\alpha/(\kappa-n+\alpha)}$$

$$= 2 \left(a^\epsilon(T) A(T) - \int_0^T A(t) \beta^\epsilon(t) \mathrm{d}t - \sum_{i=1}^d \int_0^T A(t) \gamma_i^\epsilon(t) \mathrm{d}W_t^i \right.$$

$$\left. - \int_0^T \int_{|y|<\epsilon^z} A(s) \zeta^\epsilon(s, y) 1_{\{|\zeta^\epsilon(s,y)|<\epsilon^{2\delta}\}} (\mu - \nu)(\mathrm{d}s, \mathrm{d}y) \right) + \langle M \rangle_T$$

$$+ 2 C t_0 \epsilon^{4\delta\alpha/(\kappa-n+\alpha)}$$

$$\le 16(1 + t_0) \epsilon^{q_2-rw} + 4\epsilon^{q_3} + 4(2t_0 + 3)^{1/2} \epsilon^{-2rw+q_1/2} + 2 C t_0 \epsilon^{4\delta\alpha/(\kappa-n+\alpha)}$$

$$\le l \epsilon^w$$

provided

$$\epsilon < \min \left(\left(\frac{l}{64(1 + t_0)} \right)^{(q_2-rw-w)^{-1}}, \left(\frac{l}{16} \right)^{(q_3-w)^{-1}}, \right.$$

$$\left. \left(\frac{l}{16(2t_0 + 3)^{1/2}} \right)^{(-2rw+q_1/2-w)^{-1}}, \left(\frac{l}{8 C t_0} \right)^{\left(\frac{4\delta\alpha}{\kappa-n+\alpha}-w \right)^{-1}} \right),$$

where the last inequality follows from $q_2 - rw > w, q_3 > w, q_1/2 - 2rw > w$ and $\delta > w(\kappa - n + \alpha)/4\alpha$. Finally, by the choice of δ_j and ρ_j and the assumption that $\delta > (-r + q/2 + v/2)w$ (which also implies that $\delta > -rw + q_1/2 - q_2/4$ and $\delta > 2q_2 - 2rw - q_3$) we see that $\epsilon^\delta \delta_5 < \rho_5$, $\epsilon^{2\delta} \delta_4 < \rho_4$, and $\epsilon^{4\delta} \delta_7 < \rho_7$ if

$$\epsilon < \left(2(2t_0 + 3)^{1/2} \right)^{(2\delta+q_2-q_1/2+2rw)^{-1}}.$$

We also note that $64 t_0 \epsilon^\delta \delta_6 < \rho_6$, giving

$$P(C_3) \le 2 \exp \left(-\frac{\delta_6^2}{2 \left(\frac{1}{64 t_0} + 1 \right) \rho_6} \right) + C \epsilon^{\delta p} \le 2 \exp \left(-\frac{\epsilon^{-vw}}{2(1 + 64 t_0)} \right) + C \epsilon^{\delta p}.$$

Putting all of this together, these choices for δ_j and ρ_j enable us to deduce that

$$P \left(\bigcup_{j=1}^3 B_j \right) \le 2 \left(\exp \left(-\frac{1}{2} \epsilon^{-vw} \right) + \exp \left(-\frac{1}{4(2t_0 + 3)^{1/2}} \epsilon^{-vw} \right) \right.$$

$$\left. + \exp \left(-\frac{1}{128 t_0} \epsilon^{-vw} \right) \right),$$

and

$$P\left(\bigcup_{j=1}^{4} C_j\right) \le 2\left(2\exp\left(-\frac{1}{4}\epsilon^{-vw}\right) + \exp\left(-\frac{1}{8(2t_0+3)^{1/2}}\epsilon^{-vw}\right)\right.$$
$$\left. + \exp\left(-\frac{\epsilon^{-vw}}{2(1+64t_0)}\right) + C\epsilon^{\delta p}\right).$$

The proof is finished on noting that $\delta p > w/4$, and the dependence of ϵ_0 on t_0 follows immediately from the proof. $\square$

4. Uniform Hörmander condition

We now present our uniform Hörmander condition.

Condition 8 (*UH*). *Let $V_0 = Z - \frac{1}{2}\sum_{i=1}^{d}\nabla V_i V_i$ and assume that Conditions 2 and 4 hold. Recursively define the following families of vector fields*

$$\mathcal{L}_0 = \{V_1, \ldots, V_d\}$$
$$\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{[V_i, K], i = 1, \ldots, d : K \in \mathcal{L}_n\}$$
$$\cup \left\{[V_0, K] - \int_E [Y, K](\cdot, y)G(\mathrm{d}y) : K \in \mathcal{L}_n\right\}.$$

Then there exists some smallest integer $j_0 \ge 1$ and a constant $c > 0$ such that for any $u \in \mathbb{R}^e$ with $|u| = 1$ we have

$$\inf_{x \in \mathbb{R}^e} \sum_{j=0}^{j_0} \sum_{K \in \mathcal{L}_j} \left(u^T K(x)\right)^2 \ge c.$$

The next important result is a development of an idea presented in [9], it enables us to estimate the Malliavin covariance matrix on a time interval where the Poisson random measure records no jumps of size greater than some truncation parameter. As in [9] the key idea is to make explicit the dependence of the estimate on the length of the time interval under consideration.

Theorem 3. *Let $t > 0$ and let x_t satisfy the SDE*

$$x_t = x + \int_0^t Z(x_{s-})\mathrm{d}s + \int_0^t V(x_{s-})\mathrm{d}W_s + \int_0^t \int_E Y(x_{s-}, y)(\mu - v)(\mathrm{d}y, \mathrm{d}s)$$

and assume that the following conditions are satisfied:

$$Z, V_1, \ldots, V_d \in C_b^{\infty}(\mathbb{R}^e), \tag{4.1}$$

for every $y \in E$ $Y(\cdot, y) \in C_b^{\infty}(\mathbb{R}^e)$ and, for some $\rho_2 \in L_+^{2;\infty}(G)$ and every $n \in \mathbb{N} \cup \{0\}$

$$\sup_{y \in E, x \in \mathbb{R}^e} \frac{1}{\rho_2(y)}|\nabla_1^n Y(x, y)| < \infty, \tag{4.2}$$

$$\sup_{y \in E, x \in \mathbb{R}^e} |(I + \nabla_1 Y(x, y))^{-1}| < \infty \quad and \quad \sup_{x \in \mathbb{R}^e} |(I + \nabla_1 Y(x, \cdot))^{-1}| \in L_+^{2;\infty}(G).$$

Further assume Conditions 4–6 and condition (UH) *hold. For some* $0 < t < t_0, \delta, \alpha > 0$ *and* $z = 3\delta(\kappa - n + \alpha)^{-1}$ *define the set* $A_t = A_t(\epsilon)$ *by*

$$A_t = \left\{ \omega : (\text{supp}\,\mu(\cdot, \cdot)) \cap [0, t) \times E \subseteq [0, t) \times \{|y| \leq \epsilon^z\} \right\}.$$

Then, $P\left(\left\{\sup_{0 \leq s \leq t} |x_s - x_s(\epsilon)| > 0\right\} \cap A_t\right) = 0$, *where* $x_t(\epsilon)$ *is the solution to the* SDE

$$dx_t(\epsilon) = \left(Z(x_{t-}(\epsilon)) - \int_{|y| \geq \epsilon^z} Y(x_{t-}(\epsilon), y)G(dy) \right) dt + V(x_{t-}(\epsilon))dW_t$$

$$+ \int_{|y| < \epsilon^z} Y(x_{t-}(\epsilon), y)(\mu - \nu)(dy, dt). \tag{4.3}$$

Moreover if we let the reduced Malliavin covariance matrix associated with $x_t(\epsilon)$ *be denoted by* $C_t(\epsilon)$ *then we have for any* $p \geq 1$ *and some* $\epsilon_0(p) > 0, K(p) \geq 1$, *that*

$$\sup_{|u|=1} P\left(\left\{u^T C_t u \leq \epsilon\right\} \cap A_t\right) = \sup_{|u|=1} P(u^T C_t(\epsilon)u \leq \epsilon) \leq \epsilon^p$$

for $0 \leq \epsilon \leq t^{K(p)}\epsilon_0(p)$, *provided that*

$$16\delta > \max\left(8 - r + \frac{v}{2}, \frac{\kappa - n + \alpha}{4\alpha} \right),$$

where $r, v > 0$ *are such that* $18r + 9v < 8$.

Proof. The indistinguishability of the processes x and $x(\epsilon)$ on A_t is trivial. For the remainder of the proof we first note that condition (UH) enables us to identify a smallest integer j_0 and a constant $c > 0$ such that, for any $u \in \mathbb{R}^e$ with $|u| = 1$

$$\inf_{x \in \mathbb{R}^e} \sum_{j=0}^{j_0} \sum_{K \in \mathcal{L}_j} \left(u^T K(x)\right)^2 \geq c.$$

For $j = 0, 1, \ldots, j_0$ set $m(j) = 2^{-4j}$ and define

$$E_j = \left\{ \sum_{K \in \mathcal{L}_j} \int_0^t \left(u^T(\epsilon)J_{0 \leftarrow s}(\epsilon)K(x_s(\epsilon))\right)^2 ds \leq \epsilon^{m(j)} \right\},$$

where $J_{t \leftarrow 0}(\epsilon)$ denotes the Jacobian of the flow associated with $x_t(\epsilon)$ and $J_{0 \leftarrow t}(\epsilon)$ denotes its inverse (which exists by the assumptions on the vector fields as in Theorem 1). It is straightforward to note, using (4.2), L^p inequalities for stochastic integrals based on Poisson random measures (see [3], Lemma A.14) and Gronwall's inequality that for any $p < \infty$

$$\sup_{\epsilon \geq 0} \mathbb{E}\left[\sup_{0 \leq s \leq t} |J_{t \leftarrow 0}(\epsilon)|^p \right] < \infty. \tag{4.4}$$

Let C denote a constant which varies from line to line and does not depend on ϵ. Then, as usual we have

$$\left\{u^T C_t(\epsilon)u \leq \epsilon\right\} = E_0 \subset \left(E_0 \cap E_1^c\right) \cup (E_1 \cap E_2^c) \cup \cdots \cup (E_{j_0-1} \cap E_{j_0}^c) \cup F$$

where $F = E_0 \cap E_1 \cap \cdots \cap E_{j_0}$. Define the stopping time

$$S = \min \left(\inf \left\{ s \geq 0 : \sup_{0 \leq z \leq s} |J_{0 \leftarrow z}(\epsilon) - I| \geq \frac{1}{2} \right\}, t \right),$$

and notice that by choosing $0 < \beta < m(j_0)$ we discover that $P(F) \leq P(S < \epsilon^\beta) \leq C \epsilon^{q\beta/2}$ for $\epsilon \leq \epsilon_1$ and any $q \geq 2$ (see [14,9] for details), where as in [9], ϵ_1 satisfies

$$\epsilon_1 < \min \left(t^{1/\beta}, \left(\frac{c}{4(j_0 + 1)} \right)^{1/(m(j_0) - \beta)} \right).$$

We notice that for any $K \in C_b^\infty(\mathbb{R}^e)$ we have

$$
\begin{aligned}
\mathrm{d}u^T J_{0 \leftarrow t}(\epsilon) K(x_t(\epsilon)) &= u^T J_{0 \leftarrow t-}(\epsilon) \Bigg([V_0, K](x_{t-}(\epsilon)) - \int_E [Y, K](x_{t-}(\epsilon), y) G(\mathrm{d}y) \\
&\quad + \frac{1}{2} \sum_{i=1}^d [V_i, [V_i, K]](x_{t-}(\epsilon)) + \int_{|y| < \epsilon^z} ((I + \nabla_1 Y(x_{t-}(\epsilon), y)^{-1}) K(x_{t-}(\epsilon)) \\
&\quad + Y(x_{t-}(\epsilon), y)) - K(x_{t-}(\epsilon)) G(\mathrm{d}y)) \Bigg) \mathrm{d}t + u^T J_{0 \leftarrow t-}(\epsilon) \sum_{i=1}^d [V_i, K](x_{t-}(\epsilon)) \mathrm{d}W_t^i \\
&\quad + u^T J_{0 \leftarrow t-}(\epsilon) \int_{|y| < \epsilon^z} (I + \nabla_1 Y(x_{t-}(\epsilon), y)^{-1}) K(x_{t-}(\epsilon)) \\
&\quad + Y(x_{t-}(\epsilon), y)) - K(x_{t-}(\epsilon))(\mu - \nu)(\mathrm{d}y, \mathrm{d}t).
\end{aligned}
$$

We now verify the conditions of Lemma 2 in the case where

$$
\begin{aligned}
Y^\epsilon(t) &= u^T J_{0 \leftarrow t}(\epsilon) K(x_t(\epsilon)) \\
a^\epsilon(t) &= u^T J_{0 \leftarrow t}(\epsilon) \Bigg([V_0, K](x_t(\epsilon)) - \int_E [Y, K](x_t(\epsilon), y) G(\mathrm{d}y) \\
&\quad + \frac{1}{2} \sum_{i=1}^d [V_i, [V_i, K]](x_t(\epsilon)) \\
&\quad + \int_{|y| < \epsilon^z} ((I + \nabla_1 Y(x_t(\epsilon), y)^{-1}) K(x_t(\epsilon) + Y(x_t(\epsilon), y)) - K(x_t(\epsilon)) G(\mathrm{d}y)) \Bigg).
\end{aligned}
$$

$$=: u^T J_{0 \leftarrow t} \tilde{K}(x_t(\epsilon)),$$

where $\tilde{K} \in C_b^\infty(\mathbb{R}^e)$. To do this we observe, using the notation of Lemma 2 that

$$f^\epsilon(t, y) = u^T J_{0 \leftarrow t-}(\epsilon)(I + \nabla_1 Y(x_{t-}(\epsilon), y)^{-1}) K(x_{t-}(\epsilon) + Y(x_{t-}(\epsilon), y)) - K(x_{t-}(\epsilon))$$

and hence for some $0 < C < \infty$

$$|f^\epsilon(t, y)| \leq C \left| u^T J_{0 \leftarrow t-}(\epsilon) \right| \max \left(\sup_{x \in \mathbb{R}^e} |K(x)|, \sup_{x \in \mathbb{R}^e} |\nabla K(x)| \right)$$

$$\left(\sup_{x \in \mathbb{R}^e, y \in E} |(I + \nabla_1 Y(x, y))^{-1}| \, |\nabla_1 Y(x_{t-}(\epsilon), y)| + |Y(x_{t-}(\epsilon), y)| \right).$$

Condition 6 then gives that $|f^\epsilon(t, y)| \leq C \left| u^T J_{0 \leftarrow t-}(\epsilon) \right| \phi(y)$ where $\phi \in L_+^1(G)$ does not depend on ϵ, $C = C(K) < \infty$ and where and for some $\alpha > 0$ (which does not depend on ϵ

or K!) we have

$$\limsup_{y \to 0} \frac{\phi(y)}{|y|^{\kappa - n + \alpha}} < \infty.$$

Finally, using the notation of (3.7), we notice that Cauchy–Schwarz gives

$$|u^T J_{0 \leftarrow t-}(\epsilon)|^2 \leq \sum_{i=1}^{e} |J_{0 \leftarrow t-}(\epsilon) e_i|^4 =: D_t^{f, \epsilon}, \tag{4.5}$$

where e_i is the standard basis in $\mathbb{R}^e$. Hence by (4.4) we have for any $p < \infty$

$$\sup_{\epsilon \geq 0} E \left[\sup_{0 \leq s \leq t} \left(D_s^{f, \epsilon} \right)^p \right] < \infty.$$

We have therefore verified the conditions of Lemma 2 for the process $f^{\epsilon}(t, y)$ and they may be also checked for the process $\zeta^{\epsilon}(t, y)$ in the same manner. Now let us note that for $j \in \{0, 1, \ldots, j_0 - 1\}$

$$P(E_j \cap E_{j+1}^c) = P \left(\sum_{K \in \mathcal{L}_j} \int_0^t \left(u^T J_{0 \leftarrow s}(\epsilon) K(x_s(\epsilon)) \right)^2 ds \leq \epsilon^{m(j)}, \right.$$

$$\sum_{K \in \mathcal{L}_{j+1}} \int_0^t \left(u^T J_{0 \leftarrow s}(\epsilon) K(x_s(\epsilon)) \right)^2 ds > \epsilon^{m(j+1)} \Bigg)$$

$$\leq \sum_{K \in \mathcal{L}_j} P \left(\int_0^t (v^T J_{0 \leftarrow s}(\epsilon)) K(x_s(\epsilon))^2 ds \leq \epsilon^{m(j)}, \right.$$

$$\sum_{k=1}^{d} \int_0^t (u^T J_{0 \leftarrow s}(\epsilon) V_k(x_s(\epsilon))^2) ds + \int_0^t u^T J_{0 \leftarrow s-}(\epsilon) \left([V_0, K](x_{s-}(\epsilon)) \right.$$

$$\left. - \int_E [Y, K](x_{s-}(\epsilon), y) G(dy) + \frac{1}{2} \sum_{i=1}^{d} [V_i, [V_i, V_k]](x_{s-}(\epsilon)) \right) ds > \frac{\epsilon^{m(j+1)}}{n(j)} \Bigg). \tag{4.6}$$

Since the other hypotheses of Lemma 2 are trivial to verify we estimate the terms in the sum on the right-hand side of (4.6) by j_0 applications of this lemma, with $z = 3\delta(\kappa - n + \alpha)^{-1}$ and the choices

$$q = 16, r, v > 0 \quad \text{such that } 18r + 9v < 8 \quad \text{and} \quad w_j = 2^{-4(j+1)}.$$

Since $w_j \leq w_0 = 16^{-1}$ we see that the right-hand side of (4.6) will be $o(\epsilon^p)$ for all $j \in \{0, 1, \ldots, j_0 - 1\}$ if

$$16\delta > \max \left(8 - r + v/2, \frac{\kappa - n + \alpha}{4\alpha} \right),$$

and $\epsilon \leq \epsilon_2(p)$ where ϵ_2 can be chosen as $\epsilon_3 t^{-k^*}$ for some $k^* > 0$ with ϵ_3 independent of t. Setting $\epsilon_0 = \min(\epsilon_1, \epsilon_2)$ and noticing by (4.5) that all the estimates are uniform over $|u| = 1$ gives the result. $\square$

5. C^∞ density under the Hörmander condition

We now state and prove our main result.

Theorem 4. *Suppose that x_t is the solution to the* SDE

$$x_t = x + \int_0^t Z(x_{s-})\mathrm{d}s + \int_0^t V(x_{s-})\mathrm{d}W_s + \int_0^t \int_E Y(x_{s-}, y)(\mu - v)(\mathrm{d}y, \mathrm{d}s)$$

and that the conditions of Theorem 3 are in force. Then, for any $t_0 > 0$ the law of x_{t_0} has a C^∞ density with respect to the Lebesgue measure under the uniform Hörmander Condition 8 provided, in the notation of Theorem 3, we have

$$16m(j_0) > 3(\kappa - n)\max\left(\frac{8 - r + v/2}{\kappa - n + \alpha}, \frac{1}{4\alpha}\right), \tag{5.1}$$

where $m(j) = 2^{-4j}$ and j_0 is the integer described in (UH).

Remark 1. Note that (5.1) is always true when $\kappa = n$.

Proof. By Theorem 2 it suffices to check that $\left|C_{t_0}^{-1}\right| \in L^p$ for all $p \geq 2$. Let $\Lambda = \inf_{|u|=1} u^T C_{t_0} u$ be the smallest eigenvalue of C_{t_0}. Then it is sufficient to show that $\Lambda^{-1} \in L^p$ for all $p \geq 2$. However, we may write

$$E[\Lambda^{-p}] = C_1 \int_0^\infty \epsilon^{-k} P(\Lambda \leq \epsilon^2)\mathrm{d}\epsilon \leq C_2 + C_3 \int_0^1 \epsilon^{-k} P(\Lambda \leq \epsilon^2)\mathrm{d}\epsilon,$$

for some $k > 1$. By a routine compactness argument we may show (see [12]) that

$$P(\Lambda \leq \epsilon) \leq C_2 \epsilon^{-e} \sup_{|u|=1} P(u^T C_{t_0} u \leq \epsilon),$$

so that for some $k' > 1$

$$E[\Lambda^{-p}] \leq C_3 + C_4 \int_0^1 \epsilon^{-k'} \sup_{|u|=1} P(u^T C_{t_0} u \leq \epsilon^2)\mathrm{d}\epsilon. \tag{5.2}$$

Now we define a Poisson process N_ϵ on $\mathbb{R}^+$ for $\epsilon > 0$ by

$$N_\epsilon(t) = \int_0^t \int_{|y|>\epsilon^z} \mu(\mathrm{d}y, \mathrm{d}s),$$

whose rate is given as

$$\lambda(\epsilon) = \int_{|y|>\epsilon^z} G(\mathrm{d}y).$$

By (3.2) we know that

$$\limsup_{\epsilon \to 0} \frac{\lambda(\epsilon)}{f(\epsilon^z)} < \infty. \tag{5.3}$$

We may find a (random) subinterval $[t_1, t_2) \subseteq [0, t_0)$ such that $t_2 - t_1 \geq t_0(N_\epsilon(t_0) + 1)^{-1}$ on which the Poisson random measure μ records no jumps of absolute value greater than ϵ^z and, as such, the underlying process x_t solves the SDE (4.3) started at x_{t_1} on this interval. We emphasize

the dependence of C_{t_0} on the starting point (x, I) of the process $(x_t, J_{0 \leftarrow t})$. Then, using the fact that $J_{0 \leftarrow t}^{x,V} = V J_{0 \leftarrow t}^{x,I}$, $J_{0 \leftarrow t} = J_{t \leftarrow 0}^{-1}$, the (strong) Markov property, and the two observations that $t_2 - t_1 \geq t_0 (N_\epsilon(t_0) + 1)^{-1}$ and

$$\mathrm{span}\{u^T J_{0 \leftarrow t}^{x,I} : u \in \mathbb{R}^e, |u| = 1\} = \mathbb{R}^e \quad \text{a.s. for every } t > 0 \text{ and } x \in \mathbb{R}^e$$

we see that for any $q < \infty$

$$\sup_{|u|=1} P(u^T C_{t_0}^{x,I} u \leq \epsilon^2)$$

$$\leq \sup_{|u|=1} P\left(u^T C_{t_1,t_2}^{x_{t_1}^x, J_{0 \leftarrow t_1}^{x,I}} u \leq \epsilon^2 \right)$$

$$= \sup_{|u|=1} P\left(u^T J_{0 \leftarrow t_1}^{x,I} C_{t_1,t_2}^{x_{t_1}^x,I} \left(J_{0 \leftarrow t_1}^{x,I} \right)^T u \leq \epsilon^2 \right)$$

$$= \sup_{|u|=1} P\left(\frac{u^T J_{0 \leftarrow t_1}^{x,I} C_{t_1,t_2}^{x_{t_1}^x,I} \left(J_{0 \leftarrow t_1}^{x,I} \right)^T u}{|u^T J_{0 \leftarrow t_1}^{x,I}|^2} \leq \frac{\epsilon^2}{|u^T J_{0 \leftarrow t_1}^{x,I}|^2} \right)$$

$$\leq \sup_{|u|=1} P\left(u^T C_{t_1,t_2}^{x_{t_1}^x,I} u \leq \epsilon \right) + \sup_{|u|=1} P\left(|u^T J_{0 \leftarrow t_1}^{x,I}|^{-1} \geq \epsilon^{-1/2} \right)$$

$$= \sup_{|u|=1} P\left(u^T C_{t_2-t_1}^{x_{t_1}^x,I}(\epsilon) u \leq \epsilon \right) + \mathrm{O}(\epsilon^q)$$

$$\leq \sup_{|u|=1} P\left(u^T C_{t_0(N_\epsilon(t_0)+1)^{-1}}^{x_{t_1},I}(\epsilon) u \leq \epsilon \right) + \mathrm{O}(\epsilon^q). \tag{5.4}$$

An application of Theorem 3 yields

$$\sup_{|u|=1} P\left(u^T C_{t_0(N_\epsilon(t_0)+1)^{-1}}^{x_{t_1},I}(\epsilon) u \leq \epsilon \right) \quad \text{is O }(\epsilon^q)$$

for any $q \geq 2$ if $\epsilon \leq \epsilon_0 t_0^{1/K(q)} (N_\epsilon(t_0) + 1)^{-1/K(q)}$ provided that $\delta > \max(8 - r + v/2, (\kappa - n + \alpha)/4\alpha)$. From this, (5.2) and (5.4) we get that

$$E[\Lambda^{-p}] \leq C_5 + C_6 \int_0^1 \epsilon^{-k'} P\left(N_\epsilon(t_0) > \left\lfloor t_0 \left(\frac{\epsilon_0}{\epsilon} \right)^{1/K(q)} \right\rfloor \right) d\epsilon.$$

From the proof of Theorem 3 we see that $K(q) = K(q, \epsilon) = \beta^{-1}$ for ϵ small enough, where $\beta < m(j_0)$, and hence to see that $E[\Lambda^{-p}] < \infty$ it will suffice to show

$$P\left(N_\epsilon(t_0) > \left\lfloor t_0 \left(\frac{\epsilon_0}{\epsilon} \right)^\beta \right\rfloor \right) \quad \text{is o}(\epsilon^q) \quad \text{as } \epsilon \to 0 \text{ for any } q > 0.$$

Chebyshev's inequality and (5.3) yield

$$P\left(N_\epsilon(t_0) > \left\lfloor t_0 \left(\frac{\epsilon_0}{\epsilon} \right)^\beta \right\rfloor \right) \leq \exp\left(-t_0 \left(\frac{\epsilon_0}{\epsilon} \right)^\beta + (e-1) t_0 \lambda(\epsilon) \right)$$

$$\leq \exp\left(-t_0 \left(\frac{\epsilon_0}{\epsilon} \right)^\beta + C(e-1) t_0 f(\epsilon^z) \right) \quad \text{as } \epsilon \to 0,$$

which, by the definition of f is seen to be $o(\epsilon^q)$ for any $q > 0$ if

$$\beta > \frac{3\delta(\kappa - n)}{(\kappa - n + \alpha)}.$$

Since β and δ may take any values subject to the constraints $\beta < m(j_0)$ and $16\delta > \max(8 - r + v/2, (\kappa - n + \alpha)/4\alpha)$, this condition becomes

$$16m(j_0) > 3(\kappa - n) \max\left(\frac{8 - r + v/2}{\kappa - n + \alpha}, \frac{1}{4\alpha}\right). \quad \square$$

Condition (5.1) exposes the qualitative structure of the problem structure of the problem quite well in that it becomes easier to satisfy with smaller values of j_0 (so that $\mathbb{R}^e$ is spanned with brackets of smaller length), or with smaller values of κ (less intense jumps) or larger values of α (corresponding to better behaved vector fields). One might think that the use of the lower bound $t_0(m + 1)^{-1}$ on the size of the longest interval is somewhat crude. Indeed, conditional on $N_\epsilon(t_0) = m$, the distribution function of the longest interval is known (see Feller [8]):

$$F(x) = \sum_{i=1}^{m} (-1)^{-i} \binom{m}{i} \left(1 - \frac{ix}{t_0}\right)_+^{i-1}$$

and more explicit calculation may be performed using this, however they seem to lead to no improvement in the eventual criterion obtained. Clearly, the use of only part of the covariance matrix in forming the estimate is an area in which improvement would allow further insight to be gained.

Acknowledgements

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