Far Field Perturbations Caused by a Roughness Element to the Three Dimensional Hypersonic Plate Flow Boundary Layer

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I certify that this thesis is a presentation of my own work, and all the published and unpublished work of others which are consulted in this thesis is always clearly attributed and fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

For the three dimensional hypersonic viscous compressible flat plate flow, when there is only small roughness on the wall, its effect can be considered as perturbation to two dimensional roughness-free plate flow. To study such a flow problem, we will assume there is only a single roughness element on the plate, of which the equation is in the self-similar form $\eta = \epsilon Y_0(\xi)$, where $\xi = zx^{-\frac{3}{4}}$ and $\epsilon \ll 1$, and thus the perturbed flow boundary layer equations will also have self-similar solutions. When solving the boundary layer equations, we use the Dorodnitsyn Transformation and write the solutions in coordinate asymptotic expansions. In these expansions, the leading order terms are the solutions to the two dimensional flat plate flow boundary layer equations, and the expression of these terms will be treated as already known since they can be obtained from the Blasius Equation.

The solutions for the perturbation terms show that the perturbations produced by the roughness are capable of propagating against the flow in the boundary layer. This is despite the fact that in the flow regime analysed in this thesis the longitudinal boundary-layer equation does not involve the pressure gradient, and this equation can be thought of as parabolic.

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Introduction

From V2/WAS Corporal, the first artificial object to reach hypersonic speed, to X-51 WaveRider, the latest scramjet demonstration aircraft for hypersonic flight within the atmosphere, considerable research has been carried out on hypersonic flow. Conventionally, the hypersonic flows refer to those with Mach Number $M > 5$, but some other flows with $3 < M < 5$ may also be considered as hypersonic[7]. Therefore, instead of Mach Number, a better way to distinguish hypersonic from supersonic is by looking at certain physics pheonomena which become progressively more important as the Mach number is increased to higher values [2]. In addition, for some problems, as the Mach number rises, its actual value is less important, provided only that it is sufficient large[4], so we are more interested in the limit situation $M \to \infty$, and take Mach number as another asymptotic large non-dimensional parameter besides the Reynolds number in the hypersonic problem.

A three dimensional viscous compressible flow is governed by the Navier-Stokes equations:

$$\begin{aligned} \hat{\rho}\left(\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} + \hat{w}\frac{\partial \hat{u}}{\partial \hat{z}}\right) &= -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\partial}{\partial \hat{x}}\left\{\hat{\mu}\left[\frac{4}{3}\frac{\partial \hat{u}}{\partial \hat{x}} - \frac{2}{3}\left(\frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}}\right)\right]\right\} \\ &\quad + \frac{\partial}{\partial \hat{y}}\left[\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}}\right)\right] + \frac{\partial}{\partial \hat{z}}\left[\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}}\right)\right], \end{aligned} \tag{1}$$

$$\begin{aligned} \hat{\rho}\left(\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} + \hat{w}\frac{\partial \hat{v}}{\partial \hat{z}}\right) &= -\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{\partial}{\partial \hat{y}}\left\{\hat{\mu}\left[\frac{4}{3}\frac{\partial \hat{v}}{\partial \hat{y}} - \frac{2}{3}\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{w}}{\partial \hat{z}}\right)\right]\right\} \\ &\quad + \frac{\partial}{\partial \hat{x}}\left[\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}}\right)\right] + \frac{\partial}{\partial \hat{z}}\left[\hat{\mu}\left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}}\right)\right], \end{aligned} \tag{2}$$

$$\hat{\rho}\left(\hat{u}\frac{\partial \hat{w}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{w}}{\partial \hat{y}}+\hat{w}\frac{\partial \hat{w}}{\partial \hat{z}}\right)=-\frac{\partial \hat{p}}{\partial \hat{z}}+\frac{\partial}{\partial \hat{z}}\left\{\hat{\mu}\left[\frac{4}{3}\frac{\partial \hat{w}}{\partial \hat{z}}-\frac{2}{3}\left(\frac{\partial \hat{u}}{\partial \hat{x}}+\frac{\partial \hat{v}}{\partial \hat{y}}\right)\right]\right\}$$
$$+\frac{\partial}{\partial \hat{x}}\left[\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{z}}+\frac{\partial \hat{w}}{\partial \hat{x}}\right)\right]+\frac{\partial}{\partial \hat{y}}\left[\hat{\mu}\left(\frac{\partial \hat{v}}{\partial \hat{z}}+\frac{\partial \hat{w}}{\partial \hat{y}}\right)\right], \tag{3}$$

$$\hat{\rho}\left(\hat{u}\frac{\partial \hat{h}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{h}}{\partial \hat{y}}+\hat{w}\frac{\partial \hat{h}}{\partial \hat{z}}\right)=\hat{u}\frac{\partial \hat{p}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{p}}{\partial \hat{y}}+\hat{w}\frac{\partial \hat{p}}{\partial \hat{z}}$$
$$+\frac{1}{Pr}\left[\frac{\partial}{\partial \hat{x}}\left(\hat{\mu}\frac{\partial \hat{h}}{\partial \hat{x}}\right)+\frac{\partial}{\partial \hat{y}}\left(\hat{\mu}\frac{\partial \hat{h}}{\partial \hat{y}}\right)+\frac{\partial}{\partial \hat{z}}\left(\hat{\mu}\frac{\partial \hat{h}}{\partial \hat{z}}\right)\right]$$
$$+\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{y}}+\frac{\partial \hat{v}}{\partial \hat{x}}\right)^2+\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{z}}+\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2+\hat{\mu}\left(\frac{\partial \hat{v}}{\partial \hat{z}}+\frac{\partial \hat{w}}{\partial \hat{y}}\right)^2$$
$$+\frac{4}{3}\hat{\mu}\left[\left(\frac{\partial \hat{u}}{\partial \hat{x}}-\frac{\partial \hat{v}}{\partial \hat{y}}\right)\frac{\partial \hat{u}}{\partial \hat{x}}+\left(\frac{\partial \hat{v}}{\partial \hat{y}}-\frac{\partial \hat{w}}{\partial \hat{z}}\right)\frac{\partial \hat{v}}{\partial \hat{y}}+\left(\frac{\partial \hat{w}}{\partial \hat{z}}-\frac{\partial \hat{u}}{\partial \hat{x}}\right)\frac{\partial \hat{w}}{\partial \hat{z}}\right], \tag{4}$$

$$\frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{x}}+\frac{\partial(\hat{\rho}\hat{v})}{\partial \hat{y}}+\frac{\partial(\hat{\rho}\hat{w})}{\partial \hat{z}}=0, \tag{5}$$

$$\hat{h}=\frac{\gamma}{\gamma-1}\frac{\hat{p}}{\hat{\rho}}, \tag{6}$$

in the Cartesian coordinate system ($O\hat{x}\hat{y}\hat{z}$). Usually, the $\hat{x}$-axis is chosen to be parallel to the free-stream velocity vector $(V_\infty, 0, 0)$, and the pressure, density and viscosity coefficient in the free-stream flow are denoted by p_∞, ρ_∞ and μ_∞, respectively.

Since the physics properties near the surface of the aircrafts are of most interest, the main subject of the hypersonic flow study is its boundary layer, which is quite different from the boundary layer in the supersonic or subsonic flow. In the boundary layer in a hypersonic flow, due to the viscosity effect, a large amount of kinetic energy of the fluid particles transforms into the heat near the wall, resulting in significant temperature increase in the boundary layer, so the magnitude of characteristic temperature in the boundary layer is much larger than that of the free-stream temperature T_∞ and is of the same order with that of the stagnation temperature T_0[8]. As a result, the viscosity coefficient, which is a function of the temperature, is much larger in the main part of the boundary layer than in the free-stream flow. We will denote it as μ_0, the viscosity coefficient at T_0. In addition, from the ideal gas equation

$$\hat{p}=\hat{\rho}RT,$$

the density $\hat{\rho}$ in the boundary layer is expected to be much smaller than ρ_∞ since $\hat{p} \approx p_\infty$.

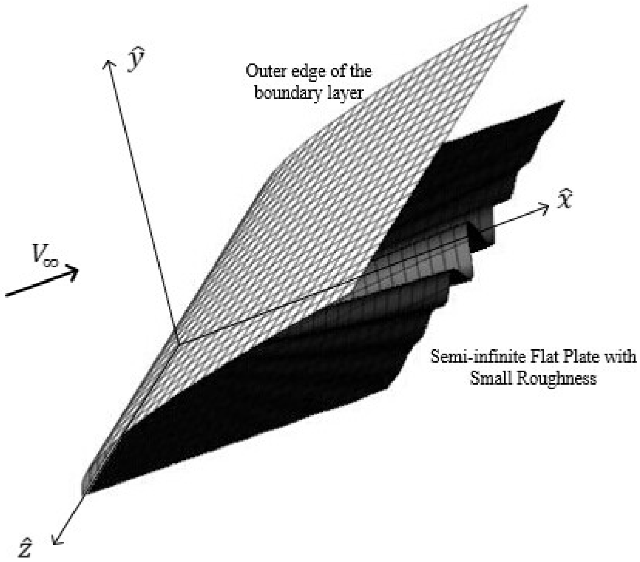

Figure 1: Three Dimensional Compressible Viscous Hypersonic Flow over Semi-infinite Flat Plate with Small Roughness

In this thesis, our goal is to investigate the process of the propagation of the perturbation in the boundary layer against the flow. Here the flow is considered in the three dimensional space as compressible and viscous, with only a weak interaction between the boundary layer and external inviscid flow, i.e. when the Mach wedge is assumed much thicker than the boundary layer. The plate, as shown in Figure 1, is infinite in the spanwise direction with infinitesimal thickness and longitudinal characteristic length L, and is parallel to the free-stream flow with the leading edge on the $\hat{z}$-axis. Accordingly, the Reynolds number is defined as $Re = \frac{V_\infty L \rho_\infty}{\mu_0}$, and the free-stream Mach Number is $M_\infty = \frac{V_\infty}{a_\infty}$, where a_∞ is the speed of sound in the free-stream flow. Besides, the heat capacity at constant pressure C_p and heat capacity ratio γ are taken to be constants as real gas effects, such that ionization and chemical reactions, will not be considered here. We shall also assume that the viscosity coefficient is linearly dependent on the temperature.

If the plate is ideally smooth without any roughness, all the fluid-dynamic variables will be independent of the spanwise coordinate $\hat{z}$, and also the spanwise velocity component $\hat{w}$ will be zero. Hence, it degenerates to, and will be referred to later as, a two dimensional flow problem. However, on the plate if there is some roughness, the two dimensional flow will be perturbed, and the spanwise velocity component and the derivatives of fluid-dynamic functions with respect to $\hat{z}$ will be no longer zero, and the flow must be considered as three dimensional. In fact, if the roughness is much thinner than the two dimensional flow boundary layer, such change caused by the roughness on the flow can be treated as small perturbation to the roughness-free case. In addition, we shall assume that the width of the roughness element is of $O(LRe^{-\frac{1}{4}}M_\infty)$. Ruban and Kravtsova have studied a similar perturbation problem, in which the roughness height is of $O(LhM_\infty Re^{-\frac{5}{8}})$ and h is a asymptotically small parameter[11], while in this thesis the thickness of the roughness considered is much larger and is of $O(L\epsilon M_\infty Re^{-\frac{1}{2}})$ with $\epsilon \ll 1$. In [9], it is presented that if the planform of a wing is has governing equation $z = cx^{\frac{3}{4}}$, the flow past the wing will be self-similar. As for our case, to make sure the perturbed flow admit self-similar solution, the governing equations of the roughness element should be in the form $y = \sqrt{x}f(zx^{-\frac{3}{4}})$, where $f(t)$ is an arbitrary function of t (details and reasons for these will be demonstrated in Chapter 1 and Chapter 2, respectively).

To deal with this flow problem, the thesis is structured as follows. Firstly, the boundary layer equations will be given in Chapter 1, as well as the pressure equation and the boundary conditions; then the boundary layer equations will be transformed into another form in the self-similar variables after Dorodnitsyn transformation (Chapter 2), and finally, by finding the solutions for the perturbation terms in the coordinate asymptotic expansions of the flow variables, we will figure out how the perturbation propagates to the far field, i.e. upstream to the leading edge of the plate ($\hat{x} \to 0$) or in the spanwise direction to the area where $\hat{z} \to \infty$.

Chapter 1

Boundary Layer Equations and Boundary Conditions

Since our three dimensional flow with surface roughness can be considered as a perturbed two dimensional flow, we will start with a brief analysis on the two dimensional flow governing equations in Section 1.1. We shall assume that the roughness is symmetric with respect to the $\hat{x}$-axis, and then without losing generality, the roughness element is set to sit on the $\hat{x}$-axis and we will only study the perturbation in the $\hat{z} > 0$ hemi-space.

1.1 Boundary Layer Equations

If the flat plate is infinite in the spanwise direction and is without roughness, the three dimensional flow problem reduces to two dimensional, and the governing Navier-Stokes Equations are

$$\begin{aligned}\hat{\rho}\left(\hat{u}\frac{\partial\hat{u}}{\partial\hat{x}}+\hat{v}\frac{\partial\hat{u}}{\partial\hat{y}}\right)=-\frac{\partial\hat{p}}{\partial\hat{x}}+\frac{\partial}{\partial\hat{x}}\left[\hat{\mu}\left(\frac{4}{3}\frac{\partial\hat{u}}{\partial\hat{x}}-\frac{2}{3}\frac{\partial\hat{v}}{\partial\hat{y}}\right)\right]\\+\frac{\partial}{\partial\hat{y}}\left[\hat{\mu}\left(\frac{\partial\hat{u}}{\partial\hat{y}}+\frac{\partial\hat{v}}{\partial\hat{x}}\right)\right],\end{aligned} \tag{1.1a}$$

$$\hat{\rho}\left(\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{v}}{\partial \hat{y}}\right) = -\frac{\partial \hat{p}}{\partial \hat{y}}+\frac{\partial}{\partial \hat{y}}\left[\hat{\mu}\left(\frac{4}{3}\frac{\partial \hat{v}}{\partial \hat{y}}-\frac{2}{3}\frac{\partial \hat{u}}{\partial \hat{x}}\right)\right] + \frac{\partial}{\partial \hat{x}}\left[\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{y}}+\frac{\partial \hat{v}}{\partial \hat{x}}\right)\right], \tag{1.1b}$$

$$\hat{\rho}\left(\hat{u}\frac{\partial \hat{h}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{h}}{\partial \hat{y}}\right) = \hat{u}\frac{\partial \hat{p}}{\partial \hat{x}}+\hat{v}\frac{\partial \hat{p}}{\partial \hat{y}}+\frac{1}{Pr}\left[\frac{\partial}{\partial \hat{x}}\left(\hat{\mu}\frac{\partial \hat{h}}{\partial \hat{x}}\right)+\frac{\partial}{\partial \hat{y}}\left(\hat{\mu}\frac{\partial \hat{h}}{\partial \hat{y}}\right)\right] + \hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{y}}+\frac{\partial \hat{v}}{\partial \hat{x}}\right)^2+\frac{4}{3}\hat{\mu}\left[\left(\frac{\partial \hat{u}}{\partial \hat{x}}-\frac{\partial \hat{v}}{\partial \hat{y}}\right)\frac{\partial \hat{u}}{\partial \hat{x}}+\left(\frac{\partial \hat{v}}{\partial \hat{y}}\right)^2\right], \tag{1.1c}$$

$$\frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{x}}+\frac{\partial(\hat{\rho}\hat{v})}{\partial \hat{y}} = 0, \tag{1.1d}$$

$$\hat{h} = \frac{\gamma}{\gamma-1}\frac{\hat{p}}{\hat{\rho}}. \tag{1.1e}$$

Referred to L, μ_0, V_∞, ρ_∞ and p_∞, the flow variables can be made dimensionless and scaled by

$$\hat{x} = Lx, \qquad \hat{y} = L\sigma_1 y, \qquad \hat{\mu} = \mu_0\mu,$$
$$\hat{u} = V_\infty u, \qquad \hat{v} = V_\infty\sigma_2 v, \qquad \hat{\rho} = \rho_\infty\chi\rho$$
$$\hat{p} = p_\infty + \rho_\infty V_\infty^2\sigma_3 p,$$

where the variables without 'hat' are of $O(1)$.

From the continuity equation (1.1d), we know $\sigma_1 = \sigma_2$. Then by balancing the largest viscosity term $\frac{\partial}{\partial \hat{y}}\left(\hat{\mu}\frac{\partial \hat{u}}{\partial \hat{y}}\right)$ in momentum equation (1.1a) with the convective term $\hat{\rho}\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}}$, it can be found that

$$\frac{\mu_0 V_\infty}{\sigma_1^2 L^2} = \frac{\rho_\infty \chi V_\infty^2}{L},$$

so $\sigma_1 = Re^{-\frac{1}{2}}\chi^{-\frac{1}{2}}$, where the Reynolds number is $Re = \frac{V_\infty L \rho_\infty}{\mu_0}$.

To describe the energy transformation process from fluid particle kinetic energy to the heat, in the energy equation (1.1c) the term $\hat{\rho}\hat{u}\frac{\partial \hat{h}}{\partial \hat{x}}$ should be balanced by the largest viscous term containing velocity variables, which is $\hat{\mu}\left(\frac{\partial \hat{u}}{\partial \hat{y}}\right)^2$, giving us $\hat{h} \sim V_\infty^2$. Considering the state equation (1.1e) along with this estimate of the magnitude of $\hat{h}$, we can write

$$V_\infty^2 \sim \frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty \chi} = \frac{a_\infty^2}{(\gamma-1)\,\chi},$$

$$\text{i.e.} \qquad \chi = M_\infty^{-2}.$$

Since the dimensional displacement thickness $\hat{\delta}$ is of the same order of magnitude with $\hat{y}$, the pressure distribution at the outer edge of the boundary layer, according to the Ackeret Formula, is

$$\begin{aligned} \hat{p} - p_\infty &= \rho_\infty V_\infty^2 \frac{d\hat{\delta}(\hat{x})/d\hat{x}}{\sqrt{M_\infty^2 - 1}} \sim \rho_\infty V_\infty^2 \frac{\hat{y}/\hat{x}}{\sqrt{M_\infty^2-1}} \\ &= \rho_\infty V_\infty^2 \frac{Re^{-\frac{1}{2}} M_\infty}{\sqrt{M_\infty^2-1}} = \rho_\infty V_\infty^2 Re^{-\frac{1}{2}}, \end{aligned} \tag{1.2}$$

and it follows that $\sigma_3 = Re^{-\frac{1}{2}}$. If the dimensionless form of the displacement thickness δ is given by $\hat{\delta} = LRe^{-\frac{1}{2}} M_\infty \delta$, then in the two dimensional flow it should be calculated as

$$\delta(x) = \int_{y_w}^{\delta_T} (1 - u\chi\rho)\, dy, \tag{1.3}$$

where δ_T denotes the dimensionless boundary layer thickness, or in the three dimensional flow[6],

$$\delta(x,z) = \int_{y_w}^{\delta_T} (1 - u\chi\rho)\, dy - \frac{\partial}{\partial z}\int_0^x \int_{y_w}^{\delta_T} (w\chi\rho)\, dy dx. \tag{1.4}$$

Remember that in hypersonic flows the boundary-layer thickness is a well defined quantity. Taking into consideration that $\chi = \frac{1}{M_\infty^2}$, in both two dimensional and three dimensional cases, we can write the relation between δ and δ_T as

$$\delta(x,z) = \delta_T(x,z)\left(1 - O\left(\frac{1}{M_\infty^2}\right)\right) + \cdots, \tag{1.5}$$

which means, to the leading order, the displacement thickness δ can be taken as equal to the boundary layer thickness δ_T.

Therefore, the scalings of the variables in the boundary layer of the two-dimensional flow are given as

$$\hat{x} = Lx, \tag{1.6a}$$

$$\hat{y} = LRe^{-\frac{1}{2}} M_\infty y, \tag{1.6b}$$

$$\hat{u} = V_\infty u(x, y), \tag{1.6c}$$

$$\hat{v} = V_\infty Re^{-\frac{1}{2}} M_\infty v(x, y), \tag{1.6d}$$

$$\hat{\rho} = \frac{\rho_\infty}{M_\infty^2} \rho(x, y), \tag{1.6e}$$

$$\hat{p} = p_\infty + \rho_\infty V_\infty^2 Re^{-\frac{1}{2}} p(x, y), \tag{1.6f}$$

$$\hat{h} = V_\infty^2 h(x, y), \tag{1.6g}$$

$$\hat{\mu} = \mu_0 \mu(x, y). \tag{1.6h}$$

Then in the flat plate flow problem with small surface roughness, the aforementioned two dimensional flow is perturbed so that there is a pressure difference along the span-wise direction. This leads to the fact that the velocity component in the z-direction is no longer zero, and perturbations of other variables will also arise. Nevertheless, since the roughness is much smaller compared with the thickness of the boundary layer, the order of magnitude of the flow variables will not be changed, so we can still use the scalings (1.6) for most of the the three dimensional flow variables, with the only extra work needed concerning the scalings of $\hat{w}$ and $\hat{z}$.

Since the non-zero spanwise velocity component $\hat{w}$ arises from the pressure difference in the $\hat{z}$-direction, the convective term should be balanced by the pressure term in Equation (3), which is

$$\hat{\rho}\hat{u}\frac{\partial \hat{w}}{\partial \hat{x}} \sim \frac{\partial \hat{p}}{\partial \hat{z}}. \tag{1.7}$$

Then if there is balance

$$\frac{\partial(\hat{\rho}\hat{u})}{\partial \hat{x}} \sim \frac{\partial(\hat{\rho}\hat{w})}{\partial \hat{z}} \tag{1.8}$$

in the three-dimensional continuity equation (5), we can combine it with Equation (1.7) to conclude that

$$\hat{z} \sim LM_\infty Re^{-\frac{1}{4}}, \tag{1.9}$$

$$\hat{w} \sim V_\infty M_\infty Re^{-\frac{1}{4}}, \tag{1.10}$$

and that the corresponding scalings are

$$\hat{z} = LRe^{-\frac{1}{4}} M_\infty z, \tag{1.11}$$

$$\hat{w} = V_\infty M_\infty Re^{-\frac{1}{4}} w(x, y, z), \tag{1.12}$$

where $z = O(1)$ and $w = O(1)$. In fact, if the span-wise width of the roughness element is much larger than $O(LM_\infty Re^{-\frac{1}{4}})$, then the range of the area affected by it will definitely also be much larger than $O(LM_\infty Re^{-\frac{1}{4}})$, so, according to (1.7), the span-wise velocity component $\hat{w}$ will be much smaller than $O(V_\infty M_\infty Re^{-\frac{1}{4}})$; as a result, the third term will be much smaller than the first and second term in the continuity equation (5), which means that only the next order term in the asymptotic solution, with respect to Re and M_∞, of $\hat{u}$ and $\hat{v}$ will 'feel' the perturbation caused by such a roughness element and which therefore is not what we mean to study. Hence, we will restrict the width of the single roughness element to $O(LM_\infty Re^{-\frac{1}{4}})$ to make sure the width of the region perturbed by it is also of $O(LM_\infty Re^{-\frac{1}{4}})$.

After the substitution of the scalings of all the variables into the three dimensional Navier-Stokes equations and working with the leading order terms, the boundary layer equations are obtained as

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{1.13a}$$

$$0 = \frac{\partial p}{\partial y}, \tag{1.13b}$$

$$\rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right), \tag{1.13c}$$

$$\rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w\frac{\partial h}{\partial z}\right) = \frac{1}{Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{1.13d}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{1.13e}$$

$$h = \frac{1}{(\gamma - 1)\rho}. \tag{1.13f}$$

Since the momentum equation (1.13b) tells us the pressure does not change across the boundary layer, we can also write $p(x, y, z)$ as $p(x, z)$ in the boundary layer. However, apart from providing this information, Equation (1.13b) does not seem to be helpful in finding the pressure distribution in the boundary layer, so we need to find another equation for p. Although the Ackeret Formula (1.2) is usually only for two dimensional flows, it is also applicable for our three dimensional flat plate flow, which will be proved in the Appendix A, so

$$p(x, z) = \frac{\partial \delta(x, z)}{\partial x} \tag{1.14}$$

is the pressure equation required and at the moment the displacement thickness δ is just taken as a known function. Now, we will need to formulate the boundary conditions for the equations (1.13).

1.2 Boundary Conditions

Since the boundary layer equations, except the state equation and continuity equation, are parabolic, we will need to find the boundary conditions for u, w, h at the leading edge of the plate ($x = 0$), the roughness surface ($y = y_w(x, z) = \epsilon y_0(x, z)$) and the outer edge of the boundary layer ($y = \delta$), respectively, where $\epsilon \ll 1$ and $y_0 = O(1)$. As for v, we only need to know its boundary condition at $y = y_w$.

Before looking at the boundary condition, we can adopt the Prandtl Transformation

$$x^* = x, \tag{1.15a}$$

$$y^* = y - \epsilon y_0, \tag{1.15b}$$

$$z^* = z, \tag{1.15c}$$

$$v = v^* + \epsilon \frac{\partial y_0}{\partial x} u^* + \epsilon \frac{\partial y_0}{\partial z} w^*, \tag{1.15d}$$

$$\delta = \delta^* + \epsilon y_0, \tag{1.15e}$$

to shift the boundary conditions from $y = y_w(x, z)$ to $y^* = 0$. In addition, after applying it to the boundary layer equation system and discarding the 'star' afterwards, all the equations, as well as the boundary conditions at $y = 0$ and $y = \delta$, are left unchanged, while the equation for p becomes

$$p = \frac{\partial \delta}{\partial x} + \epsilon \frac{\partial y_0}{\partial x}. \tag{1.16}$$

Now we will start to analyse the boundary conditions.
As the flow is undisturbed at the leading edge of the plate, the first boundary condition is

$$u = 1, \qquad w = 0 \qquad \text{at} \qquad x = 0. \tag{1.17}$$

At the roughness surface, because of the impermeability condition and the no-slip condition, we should have

$$u = v = w = 0 \qquad \text{at} \qquad y = 0. \tag{1.18}$$

Then at the outer edge of the boundary layer, the longitudinal velocity component $\hat{u}$ is almost V_∞ and the span-wise velocity, as shown in Appendix A, is of $O(V_\infty Re^{-\frac{1}{4}} M_\infty^{-1})$. Thus, to their respective leading orders,

$$u = 1, \qquad w = 0 \qquad \text{at} \qquad y = \delta(x). \tag{1.19}$$

As for the enthalpy term h, since $\hat{h} = h_\infty = \frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty}$ at both the leading edge of the plate and the outer edge of the boundary layer, we can get $h = \frac{1}{(\gamma-1)M_\infty{}^2}$. Therefore, if we only keep the leading order terms, these boundary conditions become

$$h = 0 \qquad \text{at} \qquad x = 0, \tag{1.20}$$

$$h = 0 \qquad \text{at} \qquad y = \delta(x). \tag{1.21}$$

In addition, the wall temperature is assumed to be constant, so the enthalpy h on the wall is also a constant, which is denoted by h_w,

$$\text{i.e.} \qquad h = h_w \qquad \text{at} \qquad y = 0. \tag{1.22}$$

We have now acquired all the equations and boundary conditions needed. However, in the pressure equation, the exact expression of δ is not known yet. Although (1.4) can give us the relation of the displacement thickness with the density and velocity, we can make use of the equivalence between the displacement thickness and boundary layer thickness in hypersonic flows to get a simpler formula for δ. To achieve this goal, we will introduce the Dorodnitsyn Transformation in the next chapter.

Chapter 2

Dorodnitsyn Self-Similar Solution

In this chapter, we will use Dorodnitsyn Transformation to get the relation between the pressure and density (or enthalpy) in the boundary layer. Meanwhile, adopting this transformation will also allow us to exclude the viscosity coefficient from the equations, which makes our future analysis process much simpler. After that, the self-similar form of the solution can be expected, and we will see how the boundary layer equations with respect to them look like.

2.1 Dorodnitsyn Transformation

The Dorodnitsyn variables are introduced as

$$\tilde{x} = x, \qquad \tilde{y} = \int_0^y \rho(x, y, z)\mathrm{d}y, \qquad \tilde{z} = z, \tag{2.1}$$

with the density and pressure being

$$\tilde{\rho}(\tilde{x}, \tilde{y}, \tilde{z}) = \rho(x, y, z), \tag{2.2}$$

$$\tilde{p}(\tilde{x}, \tilde{z}) = p(x, z). \tag{2.3}$$

Then, using the chain rule, we will have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial}{\partial \tilde{y}}, \tag{2.4a}$$

$$\frac{\partial}{\partial y} = \rho\frac{\partial}{\partial \tilde{y}}, \tag{2.4b}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \tilde{z}} + \frac{\partial \tilde{y}}{\partial z}\frac{\partial}{\partial \tilde{y}}. \tag{2.4c}$$

From the equation (2.4b), it is obviously that

$$\frac{\partial}{\partial \tilde{y}} = \frac{1}{\tilde{\rho}}\frac{\partial}{\partial y},$$

and therefore

$$\frac{\partial y}{\partial \tilde{y}} = \frac{1}{\tilde{\rho}}.$$

In addition, from (2.1) it follows that $\tilde{y} = 0$ at $y = 0$. The inverse transformation from $\tilde{y}$ to y hence is

$$y = \int_0^{\tilde{y}} \frac{1}{\tilde{\rho}} d\tilde{y}. \tag{2.5}$$

Since at the outer edge of the boundary layer $\rho = O(M_\infty^2)$, we can expect

$$\tilde{y}|_{y=\delta} = \int_0^{\delta} \rho(x, y, z)\mathrm{d}y = \infty \tag{2.6}$$

in a hypersonic flow. As a result, the dimensionless displacement thickness is given by

$$\delta = \int_0^{\infty} \frac{1}{\tilde{\rho}} d\tilde{y} \tag{2.7}$$

and the pressure distribution can be calculated as

$$\tilde{p}(\tilde{x}, \tilde{z}) = p(x, z) = \frac{\partial \delta}{\partial x} + \epsilon\frac{\partial y_0(x, z)}{\partial x} = \frac{\partial}{\partial \tilde{x}}\int_0^{\infty} \frac{1}{\tilde{\rho}} d\tilde{y} + \epsilon\frac{\partial y_0(\tilde{x}, \tilde{z})}{\partial \tilde{x}}. \tag{2.8}$$

By using the chain rule differentiation (2.4a)- (2.4c), the boundary layer equations (1.13) are transformed into

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} + \tilde{w}\frac{\partial \tilde{u}}{\partial \tilde{z}} = \frac{\partial}{\partial \tilde{y}}\left(\tilde{\mu}\tilde{\rho}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right), \tag{2.9a}$$

$$\tilde{u}\frac{\partial \tilde{w}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{w}}{\partial \tilde{y}} + \tilde{w}\frac{\partial \tilde{w}}{\partial \tilde{z}} = -\frac{1}{\tilde{\rho}}\frac{\partial \tilde{p}}{\partial \tilde{z}} + \frac{\partial}{\partial \tilde{y}}\left(\tilde{\mu}\tilde{\rho}\frac{\partial \tilde{w}}{\partial \tilde{y}}\right), \tag{2.9b}$$

$$\tilde{u}\frac{\partial \tilde{h}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{h}}{\partial \tilde{y}} + w\frac{\partial \tilde{h}}{\partial \tilde{z}} = \frac{1}{Pr}\frac{\partial}{\partial \tilde{y}}\left(\tilde{\mu}\tilde{\rho}\frac{\partial \tilde{h}}{\partial \tilde{y}}\right) + \tilde{\mu}\tilde{\rho}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}}\right)^2, \tag{2.9c}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \tag{2.9d}$$

$$\tilde{h} = \frac{1}{(\gamma - 1)\tilde{\rho}}. \tag{2.9e}$$

Here

$$\tilde{u}(\tilde{x}, \tilde{y}, \tilde{z}) = u(x, y, z), \tag{2.10a}$$

$$\tilde{v}(\tilde{x}, \tilde{y}, \tilde{z}) = u\frac{\partial \tilde{y}}{\partial x} + v\rho + w\frac{\partial \tilde{y}}{\partial z}, \tag{2.10b}$$

$$\tilde{w}(\tilde{x}, \tilde{y}, \tilde{z}) = w(x, y, z), \tag{2.10c}$$

$$\tilde{h}(\tilde{x}, \tilde{y}, \tilde{z}) = h(x, y, z), \tag{2.10d}$$

$$\tilde{\mu}(\tilde{x}, \tilde{y}, \tilde{z}) = \mu(x, y, z). \tag{2.10e}$$

From the state equation, it follows that

$$\tilde{\mu}\tilde{\rho} = \frac{\tilde{\mu}}{(\gamma - 1)\tilde{h}} = \frac{\hat{\mu}V_\infty^2}{\mu_0(\gamma - 1)\hat{h}}. \tag{2.11}$$

Because of $\hat{h} = C_pT$ and $\hat{\mu} \propto T$, the ratio of viscosity coefficient and the enthalpy should remain constant, say,

$$\frac{\hat{\mu}}{\hat{h}} = \frac{\mu_\infty}{h_\infty} = \frac{\mu_0}{h_0}. \tag{2.12}$$

Here h_0 denotes the enthalpy at the stagnation temperature T_0 and, from the Crocco's Integral,

$$h_0 = h_\infty + \frac{V_\infty^2}{2}. \tag{2.13}$$

Remembering

$$h_\infty = \frac{\gamma}{\gamma - 1}\frac{p_\infty}{\rho_\infty} = \frac{a_\infty^2}{\gamma - 1}, \tag{2.14}$$

we can combine it with Equation (2.11), (2.12) and (2.13), and then we will have

$$\tilde{\mu}\tilde{\rho} = \frac{2}{\gamma - 1}. \tag{2.15}$$

Therefore, the combination $\tilde{\mu}\tilde{\rho}$ in the boundary layer equations (2.9a)-(2.9e) only works as a constant. Furthermore, this constant can be eliminated by adopting the transformation

$$\bar{x} = \frac{\gamma - 1}{2}\tilde{x}, \qquad \bar{y} = \frac{\gamma - 1}{2}\tilde{y}, \qquad \bar{z} = \frac{\gamma - 1}{2}\tilde{z},$$

and the boundary layer equations and the pressure equation turn into

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{2.16a}$$

$$\bar{u}\frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{z}} + \frac{\partial^2 \bar{w}}{\partial \bar{y}^2}, \tag{2.16b}$$

$$\bar{u}\frac{\partial \bar{h}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{h}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{h}}{\partial \bar{z}} = \frac{1}{Pr}\frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2, \tag{2.16c}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \tag{2.16d}$$

$$\bar{h} = \frac{1}{(\gamma - 1)\bar{\rho}}, \tag{2.16e}$$

$$\bar{p} = \frac{\partial}{\partial \bar{x}}\int_0^\infty \frac{1}{\bar{\rho}}d\bar{y} + \epsilon\frac{\partial \bar{y}_0(\bar{x}, \bar{z})}{\partial \bar{x}}. \tag{2.16f}$$

Here $\bar{m}(\bar{x}, \bar{y}, \bar{z}) = \tilde{m}(\tilde{x}, \tilde{y}, \tilde{z})$ with m being u, v, w, h, μ or p, and $\bar{y}_0(\bar{x}, \bar{z}) = \frac{\gamma-1}{2}y_0(\tilde{x}, \tilde{z}) = \frac{\gamma-1}{2}y_0(\frac{2}{\gamma-1}\bar{x}, \frac{2}{\gamma-1}\bar{z})$.

If the state equation (2.16e) is substituted into (2.16b) and the pressure equation (2.16f), we can also get rid of $\bar{\rho}$, and this yields

$$\bar{u}\frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}} = -(\gamma - 1)\bar{h}\frac{\partial \bar{p}}{\partial \bar{z}} + \frac{\partial^2 \bar{w}}{\partial \bar{y}^2}, \tag{2.17a}$$

$$\bar{p} = (\gamma - 1)\frac{\partial}{\partial \bar{x}}\int_0^\infty \bar{h}d\bar{y} + \epsilon\frac{\partial \bar{y}_0(\bar{x}, \bar{z})}{\partial \bar{x}}. \tag{2.17b}$$

So far, we have reduced the number of dependent variables necessary for obtaining the pressure distribution and velocity profile of our flow. Then we can find out the self-similar form of the solution to our problem, in which there will be only two independent variables.

2.2 Self-similar Solution

The self-similar form of the solution can be found via adopting invariant affine transformations that leave all the boundary layer equations and boundary conditions unchanged, and these transformations are assumed to be

$$\begin{aligned}
&\bar{x} = ax_2, \qquad \bar{y} = by_2, \qquad \bar{z} = cz_2, \\
&\bar{u}(x,y,z) = du_2(x_2,y_2,z_2), \qquad \bar{v}(x,y,z) = ev_2(x_2,y_2,z_2), \\
&\bar{w}(x,y,z) = fw_2(x_2,y_2,z_2), \qquad \bar{h}(x,y,z) = gh_2(x_2,y_2,z_2). \\
&\bar{p}(x,z) = qp_2(x_2,z_2),
\end{aligned} \tag{2.18}$$

where a, b, c, d, e, f, g and q are constants.
Substitution of (2.18) into the boundary layer equations (2.16a),(2.17a),(2.16c),(2.16d) and the pressure equation (2.17b) results in

$$\frac{d^2}{a}u_2\frac{\partial u_2}{\partial x_2} + \frac{de}{b}v_2\frac{\partial u_2}{\partial y_2} + \frac{df}{c}w_2\frac{\partial u_2}{\partial z_2} = \frac{d}{b^2}\frac{\partial^2 u_2}{\partial {y_2}^2}, \tag{2.19a}$$

$$\frac{df}{a}u_2\frac{\partial w_2}{\partial x_2} + \frac{ef}{b}v_2\frac{\partial w_2}{\partial y_2} + \frac{f^2}{c}w_2\frac{\partial w_2}{\partial z_2} = -\frac{gq}{c}(\gamma-1)h_2\frac{\partial p_2}{\partial z_2} + \frac{f}{b^2}\frac{\partial^2 w_2}{\partial {y_2}^2}, \tag{2.19b}$$

$$\frac{dg}{a}u_2\frac{\partial h_2}{\partial x_2} + \frac{eg}{b}v_2\frac{\partial h_2}{\partial y_2} + \frac{fg}{c}w_2\frac{\partial h_2}{\partial z_2} = \frac{g}{b^2}\frac{1}{Pr}\frac{\partial^2 h_2}{\partial {y_2}^2} + \frac{d^2}{b^2}\left(\frac{\partial u_2}{\partial y_2}\right)^2, \tag{2.19c}$$

$$\frac{d}{a}\frac{\partial u_2}{\partial x_2} + \frac{e}{b}\frac{\partial v_2}{\partial y_2} + \frac{f}{c}\frac{\partial w_2}{\partial z_2} = 0, \tag{2.19d}$$

$$qp_2 = \frac{bg}{a}(\gamma-1)\frac{\partial}{\partial x_2}\int_0^\infty h_2 dy_2 + \epsilon\frac{\partial \bar{y}_0(ax_2, cz_2)}{a\partial x_2}. \tag{2.19e}$$

As for the boundary conditions, we have

$$\begin{aligned} &\bar{u} = \bar{v} = \bar{w} = 0,\ \bar{h} = h_w \quad \text{at} \quad \bar{y} = 0, \\ \text{i.e.} \quad &du_2 = ev_2 = fw_2 = 0,\ gh_2 = h_w \quad \text{at} \quad by_2 = 0, \end{aligned} \tag{2.20a}$$

$$\begin{aligned} &\bar{u} = 1,\ \bar{w} = 0,\ \bar{h} = 0 \quad \text{at} \quad \bar{y} = \infty, \\ \text{i.e.} \quad &du_2 = 1,\ fw_2 = 0,\ gh_2 = 0 \quad \text{at} \quad by_2 = \infty, \end{aligned} \tag{2.20b}$$

$$\begin{aligned} &\bar{u} = 1,\ \bar{w} = 0,\ \bar{h} = 0 \quad \text{at} \quad \bar{x} = 0, \\ \text{i.e.} \quad &du_2 = 1,\ fw_2 = 0,\ gh_2 = 0 \quad \text{at} \quad ax_2 = 0. \end{aligned} \tag{2.20c}$$

We shall assume that the affine transformations (2.18) leave the boundary layer equations, the pressure equation and boundary conditions unchanged, i.e.

$$u_2\frac{\partial u_2}{\partial x_2} + v_2\frac{\partial u_2}{\partial y_2} + w_2\frac{\partial u_2}{\partial z_2} = \frac{\partial^2 u_2}{\partial {y_2}^2}, \tag{2.21a}$$

$$u_2\frac{\partial w_2}{\partial x_2} + v_2\frac{\partial w_2}{\partial y_2} + w_2\frac{\partial w_2}{\partial z_2} = -(\gamma-1)h_2\frac{\partial p_2}{\partial z_2} + \frac{\partial^2 w_2}{\partial {y_2}^2}, \tag{2.21b}$$

$$u_2\frac{\partial h_2}{\partial x_2} + v_2\frac{\partial h_2}{\partial y_2} + w_2\frac{\partial h_2}{\partial z_2} = \frac{1}{Pr}\frac{\partial^2 h_2}{\partial {y_2}^2} + \left(\frac{\partial u_2}{\partial y_2}\right)^2, \tag{2.21c}$$

$$\frac{\partial u_2}{\partial x_2} + \frac{\partial v_2}{\partial y_2} + \frac{\partial w_2}{\partial z_2} = 0, \tag{2.21d}$$

$$p_2 = (\gamma-1)\frac{\partial}{\partial x_2}\int_0^\infty h_2 dy_2 + \epsilon\frac{\partial \bar{y}_0(x_2, z_2)}{\partial x_2}, \tag{2.21e}$$

and boundary conditions

$$u_2 = v_2 = w_2 = 0,\ h_2 = h_w \quad \text{at} \quad y_2 = 0, \tag{2.22a}$$

$$u_2 = 1,\ w_2 = 0,\ h_2 = 0 \quad \text{at} \quad y_2 = \infty, \tag{2.22b}$$

$$u_2 = 1,\ w_2 = 0,\ h_2 = 0 \quad \text{at} \quad x_2 = 0. \tag{2.22c}$$

By comparing the above two sets of equations and boundary conditions, we can find the relationship between the affine transformation coefficients,

$$\frac{d}{a} = \frac{e}{b} = \frac{f}{c}, \tag{2.23a}$$
$$\frac{d^2}{a} = \frac{d}{b^2}, \tag{2.23b}$$
$$\frac{df}{a} = \frac{gq}{c} = \frac{f}{b^2}, \tag{2.23c}$$
$$\frac{dg}{a} = \frac{g}{b^2} = \frac{d^2}{b^2}, \tag{2.23d}$$
$$q = \frac{bg}{a}, \tag{2.23e}$$
$$d = g = 1, \tag{2.23f}$$

and to make sure we can acquire a self-similar solution, a restriction should be put on the surface roughness governing equation such that

$$\frac{1}{aq}\bar{y}_0(ax_2, cz_2) = \bar{y}_0(x_2, z_2). \tag{2.24}$$

We see that a can be treated as a free parameter, while the other coefficients which satisfy (2.23) can be found as

$$\begin{aligned}
&b = a^{\frac{1}{2}}, \qquad c = a^{\frac{3}{4}}, \qquad d = 1,\\
&e = a^{-\frac{1}{2}}, \qquad f = a^{-\frac{1}{4}}, \qquad g = 1,\\
&q = a^{-\frac{1}{2}}.
\end{aligned}$$

It should also be noticed that the equations and boundary conditions for the variables with subscript $_2$ have the same form with those for the variables with 'bar'. This means that the solution to the boundary layer problem (2.21) and (2.22) should be

$$u_2 = \bar{u}(x_2, y_2, z_2), \tag{2.25a}$$
$$v_2 = \bar{v}(x_2, y_2, z_2), \tag{2.25b}$$

$$w_2 = \bar{w}(x_2, y_2, z_2), \tag{2.25c}$$

$$p_2 = \bar{p}(x_2, z_2), \tag{2.25d}$$

$$h_2 = \bar{h}(x_2, y_2, z_2). \tag{2.25e}$$

(2.25f)

Therefore, we can write

$$\bar{u}(\bar{x}, \bar{y}, \bar{z}) = u_2(x_2, y_2, z_2) = \bar{u}(a^{-1}\bar{x}, a^{-\frac{1}{2}}\bar{y}, a^{-\frac{3}{4}}\bar{z}), \tag{2.26a}$$

$$\bar{v}(\bar{x}, \bar{y}, \bar{z}) = a^{-\frac{1}{2}}v_2(x_2, y_2, z_2) = a^{-\frac{1}{2}}\bar{v}(a^{-1}\bar{x}, a^{-\frac{1}{2}}\bar{y}, a^{-\frac{3}{4}}\bar{z}), \tag{2.26b}$$

$$\bar{w}(\bar{x}, \bar{y}, \bar{z}) = a^{-\frac{1}{4}}w_2(x_2, y_2, z_2) = a^{-\frac{1}{4}}\bar{w}(a^{-1}\bar{x}, a^{-\frac{1}{2}}\bar{y}, a^{-\frac{3}{4}}\bar{z}), \tag{2.26c}$$

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}) = h_2(x_2, y_2, z_2) = \bar{h}(a^{-1}\bar{x}, a^{-\frac{1}{2}}\bar{y}, a^{-\frac{3}{4}}\bar{z}), \tag{2.26d}$$

$$\bar{p}(\bar{x}, \bar{z}) = a^{-\frac{1}{2}}p_2(x_2, z_2) = a^{-\frac{1}{2}}\bar{p}(a^{-1}\bar{x}, a^{-\frac{3}{4}}\bar{z}). \tag{2.26e}$$

Since a is an arbitrary constant and these functions does not really depend on it, to hide a the solution should have the form

$$\bar{u}(\bar{x}, \bar{y}, \bar{z}) = \bar{u}(1, \bar{x}^{-\frac{1}{2}}\bar{y}, \bar{x}^{-\frac{3}{4}}\bar{z}) =: U(\eta, \xi), \tag{2.27a}$$

$$\bar{v}(\bar{x}, \bar{y}, \bar{z}) = \bar{x}^{-\frac{1}{2}}\bar{v}(1, \bar{x}^{-\frac{1}{2}}\bar{y}, \bar{x}^{-\frac{3}{4}}\bar{z}) =: \bar{x}^{-\frac{1}{2}}V(\eta, \xi), \tag{2.27b}$$

$$\bar{w}(\bar{x}, \bar{y}, \bar{z}) = \bar{x}^{-\frac{1}{4}}\bar{w}(1, \bar{x}^{-\frac{1}{2}}\bar{y}, \bar{x}^{-\frac{3}{4}}\bar{z}) =: \bar{x}^{-\frac{1}{4}}W(\eta, \xi), \tag{2.27c}$$

$$\bar{h}(\bar{x}, \bar{y}, \bar{z}) = \bar{h}(1, \bar{x}^{-\frac{1}{2}}\bar{y}, \bar{x}^{-\frac{3}{4}}\bar{z}) =: H(\eta, \xi), \tag{2.27d}$$

$$\bar{p}(\bar{x}, \bar{z}) = \bar{x}^{-\frac{1}{2}}\bar{p}(1, \bar{x}^{-\frac{3}{4}}\bar{z}) =: \bar{x}^{-\frac{1}{2}}P(\xi), \tag{2.27e}$$

with η and ξ denoting $\bar{x}^{-\frac{1}{2}}\bar{y}$ and $\bar{x}^{-\frac{3}{4}}\bar{z}$, respectively. Similarly, we see that the condition (2.24) will be satisfied only if the roughness function is such that

$$\bar{y} = \epsilon\bar{y}_0(\bar{x}, \bar{z}) = \epsilon\sqrt{\bar{x}}Y_0\left(\frac{\bar{z}}{\bar{x}^{\frac{3}{4}}}\right), \qquad \text{i.e.} \qquad \eta = \epsilon Y_0(\xi), \tag{2.28}$$

where $Y_0(\xi)$ is an arbitrary function of ξ.

Now we need to substitute (2.27) into the boundary layer equations (2.21) and the boundary conditions (2.22). From the chain rule

$$\frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial \bar{x}} = -\frac{1}{2}\frac{\eta}{\bar{x}}\frac{\partial}{\partial \eta} - \frac{3}{4}\frac{\xi}{\bar{x}}\frac{\partial}{\partial \xi}, \tag{2.29a}$$

$$\frac{\partial}{\partial \bar{y}} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial \bar{y}} = \bar{x}^{-\frac{1}{2}}\frac{\partial}{\partial \eta}, \tag{2.29b}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial \bar{z}} = \bar{x}^{-\frac{3}{4}}\frac{\partial}{\partial \xi}, \tag{2.29c}$$

the boundary layer equations (2.21) become

$$\left(-\frac{1}{2}\eta U\frac{\partial}{\partial \eta} - \frac{3}{4}\xi U\frac{\partial}{\partial \xi}\right) U + V\frac{\partial U}{\partial \eta} + W\frac{\partial U}{\partial \xi} = \frac{\partial^2 U}{\partial \eta^2}, \tag{2.30a}$$

$$\begin{aligned}\left(-\frac{U}{4} - \frac{1}{2}\eta U\frac{\partial}{\partial \eta} - \frac{3}{4}\xi U\frac{\partial}{\partial \xi}\right) W + V\frac{\partial W}{\partial \eta} + W\frac{\partial W}{\partial \xi}& \\ = -(\gamma-1)H\frac{\partial P}{\partial \xi} + \frac{\partial^2 W}{\partial \eta^2}&,\end{aligned} \tag{2.30b}$$

$$\left(-\frac{1}{2}\eta U\frac{\partial}{\partial \eta} - \frac{3}{4}\xi U\frac{\partial}{\partial \xi}\right) H + V\frac{\partial H}{\partial \eta} + W\frac{\partial H}{\partial \xi} = \frac{1}{Pr}\frac{\partial^2 H}{\partial \eta^2} + \left(\frac{\partial U}{\partial \eta}\right)^2, \tag{2.30c}$$

$$\left(-\frac{1}{2}\eta\frac{\partial}{\partial \eta} - \frac{3}{4}\xi\frac{\partial}{\partial \xi}\right) U + \frac{\partial V}{\partial \eta} + \frac{\partial W}{\partial \xi} = 0, \tag{2.30d}$$

$$P = \left(\frac{1}{2} - \frac{3}{4}\xi\frac{\partial}{\partial \xi}\right)\left((\gamma-1)\int_0^\infty H d\eta + \epsilon Y_0(\xi)\right), \tag{2.30e}$$

and the corresponding boundary conditions are

$$\left.\begin{aligned} U &= 0,\\ V &= 0,\\ W &= 0,\\ H &= H_w \end{aligned}\right\} \quad \text{at} \quad \eta = 0, \tag{2.31}$$

and

$$\left.\begin{aligned} U &= 1,\\ W &= 0,\\ H &= 0 \end{aligned}\right\} \quad \text{at} \quad \eta = \infty. \tag{2.32}$$

Since both $\hat{x} \to 0$ and $\hat{z} \to \infty$ correspond to $\xi \to \infty$, to understand the behaviour of the perturbation induced by the surface roughness near the leading edge of the flat plate or considerably far from the $\hat{x}$-axis in the span-wise direction, we only need to know the solution for the perturbation terms as $\xi \to \infty$, which will be the task in the next chapter.

Chapter 3

Propagation of the Perturbations towards the Far Field

Since our flow considered is a perturbed two dimensional flow, we need to solve the two dimensional flow problem first. In this case, the boundary layer equations (2.30) can be written as

$$\left(V - \frac{1}{2}\eta U\right)\frac{\partial U}{\partial \eta} = \frac{\partial^2 U}{\partial \eta^2}, \tag{3.1a}$$

$$\left(V - \frac{1}{2}\eta U\right)\frac{\partial H}{\partial \eta} = \frac{1}{Pr}\frac{\partial^2 H}{\partial \eta^2} + \left(\frac{\partial U}{\partial \eta}\right)^2, \tag{3.1b}$$

$$-\frac{1}{2}\eta\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \eta} = 0, \tag{3.1c}$$

$$P = \frac{\gamma - 1}{2}\int_0^\infty H d\eta. \tag{3.1d}$$

They have to be solved with the boundary conditions

$$\begin{aligned} U &= 0, \\ V &= 0, \qquad \text{at} \qquad \eta = 0, \\ H &= h_w \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} U &\to 1, \\ H &\to 0 \end{aligned} \qquad \text{as} \qquad \eta \to \infty. \tag{3.3}$$

The solution to this two dimensional flow problem will be denoted by u_0, v_0 and h_0. To solve this equation system, we will introduce a new function $\phi' = U$ with $\phi(0) = 0$, and hence, $V = \frac{1}{2}\eta\phi' - \frac{1}{2}\phi$. From the longitudinal momentum equation (3.1a) and the continuity equation (3.1c), the function ϕ should satisfy the Blasius Equation

$$\phi''' + \frac{1}{2}\phi\phi'' = 0. \tag{3.4}$$

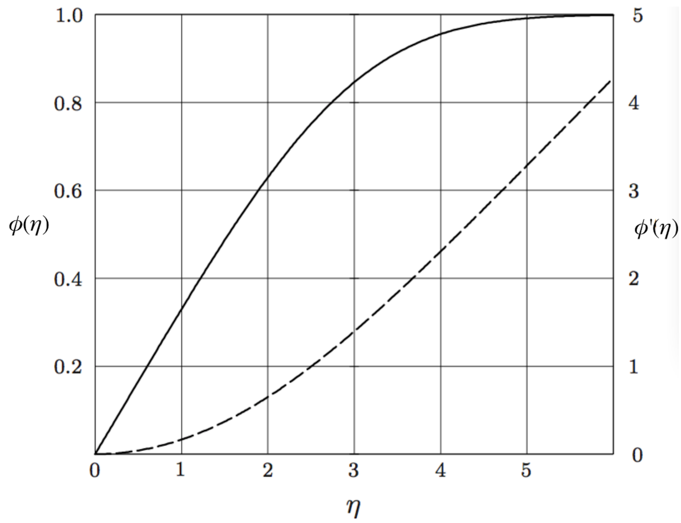

Figure 3.1: The solution to the Blasius Equaion. The solid line represents $\phi(\eta)$ and the dashed line represents $\phi'(\eta)$[10]

Apart form the boundary conditions in the definition of ϕ, another boundary condition for ϕ is $\phi'(\infty) = 0$, coming from the condition that $U \to 1$ as $\eta \to \infty$. Since the numerically solution to this equation, as shown in Figure 3.1, is well known, the solutions for u_0, v_0 and h_0 will be treated as known functions. Particularly, as $\eta \to 0$, the solution to the Blasius Equation is

$$\phi = \frac{1}{2}\lambda\eta^2 + O(\eta^5), \tag{3.5}$$

where $\lambda = \phi''(0)$ and its numerical value of λ is 0.3321 approximately[1]. It further follows from (3.5) that

$$U = \lambda\eta + O(\eta^4), \tag{3.6a}$$

$$V = \frac{1}{4}\lambda\eta^2 + O(\eta^5). \tag{3.6b}$$

Substituting (3.6) into the energy equation (3.1b) and using the fact that $\frac{dH}{d\eta} = 0$ at $\eta = 0$ gives us

$$H = h_w - \frac{Pr}{2}\lambda^2\eta^2 + O(\eta^5). \tag{3.7}$$

Besides, the pressure equation (3.1d) tells us that P is independent of η in the boundary layer for the two dimensional flow.

Now turn back to the three dimensional problem. Approaching the leading edge of the plate or the region far from the roughness, we will have $x \to 0$ or $z \to \infty$, and both of these two situations are equivalent to $\xi \to \infty$. In the $z > 0$ half space, this process can be visulized as going from the dark orange line $\xi = \xi_1$ to the light blue line $\xi = \xi_3$ in Figure 3.2. Therefore, to study the far field perturbation arising from the roughness, we need to

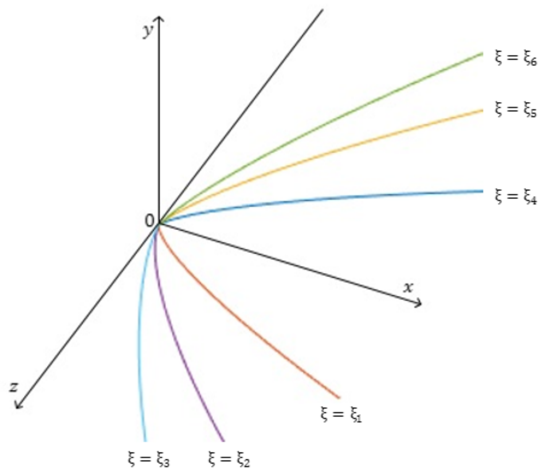

Figure 3.2: The contour lines of ξ

solve the perturbation term as $\xi \to \infty$. Since the span-wise width of the area perturbed by the roughness is expected to be larger than the width of the roughness itself, there must exist $\xi_0 > 0$ such that when $\xi > \xi_0$ the roughness height is zero but the flow variables can still 'feel' the perturbation. Therefore, as $\xi \to \infty$, the three dimensional flow pressure equation will become

$$P = \left(\frac{1}{2} - \frac{3}{4}\xi\frac{\partial}{\partial\xi}\right)\int_0^\infty (\gamma - 1)H d\eta. \tag{3.8}$$

As in the triple deck theory, we shall assume that the boundary layer splits into viscous sublayer and inviscid main part of the boundary layer, which is shown in Figure 3.3, and

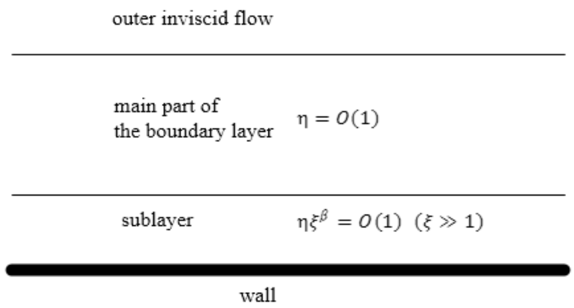

Figure 3.3: The Triple Deck in the 'Far Field'

that the thickness of the viscous sublayer is defined by $s = \eta\xi^\beta = O(1)$ where $\xi \to \infty$. Since the perturbation to P tends to zero as $\xi \to \infty$, in this sublayer the pressure is assumed to be

$$P = P_0 + \cdots + \epsilon\xi^{\gamma_p}e^{-A\xi^\alpha}p_{10} + \cdots . \tag{3.9}$$

Here α and A are positive constants and their value will be found later, and γ_p is an arbitrary constant parameter (Due to the fact that $|\gamma_p \ln\xi|$ is much smaller than $A\xi^\alpha$ when $\xi \to \infty$, the choice of this parameter γ_p is of no importance for further analysis). Accordingly, to keep the usual balance between the terms in the boundary layer equations, we shall seek the solution in the form

$$U = \xi^{-\beta}\lambda s + \cdots \qquad +\epsilon\xi^{\gamma_1}e^{-A\xi^\alpha}U_1(s) + \cdots , \tag{3.10a}$$

$$V = \xi^{-2\beta}\frac{\lambda}{4}s^2 + \cdots \qquad +\epsilon\xi^{\gamma_2}e^{-A\xi^\alpha}V_1(s) + \cdots , \tag{3.10b}$$

$$W = \qquad \epsilon\xi^{\gamma_3}e^{-A\xi^\alpha}W_1(s) + \cdots , \tag{3.10c}$$

$$H = h_w - \xi^{-2\beta}\frac{Pr}{2}\lambda^2 s^2 + \cdots \qquad +\epsilon\xi^{\gamma_h}e^{-A\xi^\alpha}H_1(s) + \cdots , \tag{3.10d}$$

where U_1, V_1, W_1 and H_1 are all of $O(1)$. Similar to the situation in the triple deck theory, we shall assume that U_1 tends to a constant, say, a_0 as $s \to \infty$. Then the solution for U in the main part of the boundary layer should be sought in the form

$$U = u_0(\eta) + \cdots + \epsilon \xi^{\gamma_1} e^{-A\xi^\alpha} U_2(\eta). \tag{3.11}$$

According to the lateral momentum equation, there should be balance between the convective term, the viscous term and the pressure term, say,

$$(-\frac{1}{2}\eta U \frac{\partial}{\partial \eta} - \frac{3}{4}\xi U \frac{\partial}{\partial \xi} - \frac{U}{4})W \sim H\frac{\partial P}{\partial \xi} \sim \frac{\partial^2 W}{\partial \eta^2}. \tag{3.12}$$

By only looking at the exponential perturbation terms in this balance, it can be concluded that

$$\xi^{\gamma_3+\alpha-\beta}\epsilon e^{-A\xi^\alpha} \sim \xi^{\gamma_p+\alpha-1}\epsilon e^{-A\xi^\alpha} \sim \xi^{\gamma_3+2\beta}\epsilon e^{-A\xi^\alpha}$$

$$\text{i.e.} \qquad \gamma_3 + \alpha - \beta = \gamma_p + \alpha - 1 = \gamma_3 + 2\beta. \tag{3.13}$$

If $Pr = 1$, it is known from the Crocco's Integral that

$$H = \frac{1 - U^2}{2}, \tag{3.14}$$

so even when $Pr \neq 1$, we can still use the estimation $H \sim \frac{1-U^2}{2}$ and this gives us

$$\gamma_h = \gamma_1 - \beta \tag{3.15}$$

and, from the pressure equation (3.8),

$$\begin{aligned} P &= \left(\frac{1}{2} - \frac{3}{4}\xi\frac{\partial}{\partial \xi}\right)\int_0^\infty (\gamma - 1)H d\eta \\ &\sim \xi \frac{\partial}{\partial \xi}\int_0^\infty H d\eta \\ &= \xi\frac{\partial}{\partial \xi}\int_0^\infty \frac{1-U^2}{2} d\eta. \end{aligned} \tag{3.16}$$

Using (3.11) in (3.16) and taking into account that the perturbation term of P is of $O(\epsilon\xi^{\gamma_p}e^{-\lambda\xi^{\alpha}})$, we find

$$\gamma_p = \gamma_1 + \alpha. \tag{3.17}$$

Finally, balancing the three terms in the continuity equation, say,

$$\left(-\frac{1}{2}\eta\frac{\partial}{\partial\eta} - \frac{3}{4}\xi\frac{\partial}{\partial\xi}\right)U \sim \frac{\partial W}{\partial\xi} \sim \frac{\partial V}{\partial\eta}, \tag{3.18}$$

we can see that

$$\gamma_1 + \alpha = \gamma_3 + \alpha - 1 = \gamma_2 + \beta. \tag{3.19}$$

Solving the equations (3.13),(3.15),(3.17) and (3.19), we have

$$\begin{aligned} &\alpha = \frac{3}{2}, \qquad && \beta = \frac{1}{2}, \\ &\gamma_1 = \gamma_p - \frac{3}{2}, \qquad && \gamma_2 = \gamma_3 = \gamma_p - \frac{1}{2}, \qquad \gamma_h = \gamma_p - 2. \end{aligned}$$

Therefore, in the sublayer, the coordinate asymptotic expansions of the variables are

$$U = \xi^{-\frac{1}{2}}\lambda s + \cdots \qquad +\epsilon\xi^{\gamma_p-\frac{3}{2}}e^{-A\xi^{\frac{3}{2}}}U_1(s) + \cdots, \tag{3.20a}$$

$$V = \xi^{-1}\frac{\lambda}{4}s^2 + \cdots \qquad +\epsilon\xi^{\gamma_p-\frac{1}{2}}e^{-A\xi^{\frac{3}{2}}}V_1(s) + \cdots, \tag{3.20b}$$

$$W = \qquad \epsilon\xi^{\gamma_p-\frac{1}{2}}e^{-A\xi^{\frac{3}{2}}}W_1(s) + \cdots, \tag{3.20c}$$

$$H = h_w - \xi^{-1}\frac{Pr}{2}\lambda^2 s^2 + \cdots \quad +\epsilon\xi^{\gamma_p-2}e^{-A\xi^{\frac{3}{2}}}H_1(s) + \cdots, \tag{3.20d}$$

$$P = P_0 + \cdots \qquad +\epsilon\xi^{\gamma_p}e^{-A\xi^{\frac{3}{2}}}p_{10} + \cdots. \tag{3.20e}$$

Substituting (3.20) into the boundary layer equations (2.30) and considering the perturbation terms only, we can arrive at

$$\lambda\left(\frac{9}{8}AsU_1(s) + V_1(s)\right) = U_1''(s), \tag{3.21a}$$

$$\frac{9}{8}A\lambda sW_1(s) = \frac{3(\gamma-1)}{2}h_w A p_{10} + W_1''(s), \tag{3.21b}$$

$$\frac{9}{8}A\lambda sH_1(s) - Pr\lambda^2 sV_1(s) = \frac{1}{Pr}H_1''(s) + 2\lambda U_1'(s), \tag{3.21c}$$

$$\frac{9}{8}AU_1(s) + V_1^{'}(s) - \frac{3}{2}AW_1(s) = 0. \tag{3.21d}$$

The no-slip and impermeability conditions for these equations are

$$U_1(0) = 0, \tag{3.22a}$$

$$V_1(0) = 0, \tag{3.22b}$$

$$W_1(0) = 0, \tag{3.22c}$$

$$H_1(0) = 0. \tag{3.22d}$$

To solve the set of equations (3.21), we introduce a new function $f = U_1 - \frac{4}{3}W_1$, and then the equations for f and V_1 can be deduced from Equations (3.21a) (3.21b) and (3.21d) as

$$\frac{9}{8}A\lambda s f(s) + \lambda V_1(s) = -2(\gamma - 1)h_w A p_{10} + f^{''}(s), \tag{3.23a}$$

$$\frac{9}{8}Af(s) + V_1^{'}(s) = 0. \tag{3.23b}$$

Here V_1 can be eliminated by differentiating (3.23a) with respect to s and using (3.23b). This leaves us with

$$\frac{9}{8}A\lambda s f^{'}(s) = f^{'''}(s). \tag{3.24}$$

The general solution to this Airy equation is

$$f^{'}(s) = C_1 Ai\left(\left(\frac{9}{8}A\lambda\right)^{\frac{1}{3}} s\right) + C_2 Bi\left(\left(\frac{9}{8}A\lambda\right)^{\frac{1}{3}} s\right), \tag{3.25}$$

where $Ai(s)$ is the Airy function and $Bi(s)$ the Bairy function. Since Bairy function $Bi(\left(\frac{9}{8}A\lambda\right)^{\frac{1}{3}} s)$ is exponentially growing when $s \to \infty$ while the derivative of neither U_1 nor W_1 is expected to be exponentially large as $s \to \infty$, the coefficient C_2 must be set to zero, and we will have

$$f(s) = C_1 \int_0^s Ai\left(\theta s\right) ds, \tag{3.26}$$

where θ denotes $\left(\frac{9}{8}A\lambda\right)^{\frac{1}{3}}$. It is well known that $\int_0^\infty A(z)dz = \frac{1}{3}$. Hence, it follows from (3.26) that

$$\lim_{s\to\infty} f(s) = \frac{C_1}{3\theta}. \tag{3.27}$$

Then according to Equation(3.23b),

$$V_1 = Bs + \cdots \qquad \text{as} \qquad s \to \infty \tag{3.28}$$

with $B = -\frac{3AC_1}{8\theta}$.

To find U_1, we need to know the behaviour of W_1 when $s \to \infty$ first. For this purpose, we will consider Equation (3.21b) with the following possibilities at the outer edge of the sublayer.

1. If $\frac{9}{8}A\lambda sW_1(s)$ and $\frac{3(\gamma-1)}{2}h_w Ap_{10}$ are the leading order terms here and balanced by each other, we could have $W_1 \to \frac{E}{s}$ as $s \to \infty$, with $E = \frac{4(\gamma-1)}{3}\frac{h_w p_{10}}{\lambda}$ being a constant, and hence, $W_1''(s) = O(s^{-3})$, which is much smaller than the other terms as $s \to \infty$;

2. If W_1'' and $\frac{3(\gamma-1)}{2}h_w Ap_{10}$ are assumed to be the leading order terms and balanced, then $W_1 = O(s^2)$, which makes $\frac{9}{8}A\lambda sW_1$ be of $O(s^3)$ and become the leading order term. Obviously it contradicts our assumption;

3. If $\frac{9}{8}A\lambda sW_1$ and W_1'' are the balanced leading order terms, the solution would be either of the same order with Bairy function, which makes the solution exponentially large near the outer edge of the sublayer, or Airy function, which results in the fact that these two terms are too much smaller than the constant term to be leading order terms, so conflict also arises here.

Therefore, only the 1st case will not lead to a conflict, and the solutions for W_1 and U_1 are

$$\begin{aligned} W_1 &= Es^{-1} + \cdots, \\ U_1 &= \frac{C_1}{3\theta} + \cdots \end{aligned} \qquad \text{as} \qquad s \to \infty. \tag{3.29}$$

This result verifies our earlier assumption about U_1 being a constant as $s \to \infty$ and shows us that $C_1 = 3\theta a_0$. Similarly, we can deduce from energy equation (3.21c) that

$$H_1 = \frac{8Pr\lambda B}{9A}s + \cdots \qquad \text{as} \qquad s \to \infty. \tag{3.30}$$

Now, using the principle of matching of the asymptotic expansions, we can see that in the main part of the boundary layer the solution has to be written as

$$U = u_0(\eta) + \cdots \quad + \epsilon\xi^{\gamma_p - \frac{3}{2}} e^{-A\xi^{\frac{3}{2}}} U_2(\eta) + \cdots, \tag{3.31a}$$

$$V = v_0(\eta) + \cdots \quad + \epsilon\xi^{\gamma_p} e^{-A\xi^{\frac{3}{2}}} V_2(\eta) + \cdots, \tag{3.31b}$$

$$W = \qquad \epsilon\xi^{\gamma_p - 1} e^{-A\xi^{\frac{3}{2}}} W_2(\eta) + \cdots, \tag{3.31c}$$

$$H = h_0(\eta) + \cdots \quad + \epsilon\xi^{\gamma_p - \frac{3}{2}} e^{-A\xi^{\frac{3}{2}}} H_2(\eta) + \cdots, \tag{3.31d}$$

$$P = P_0 + \cdots \quad + \epsilon\xi^{\gamma_p} e^{-A\xi^{\frac{3}{2}}} p_{10} + \cdots. \tag{3.31e}$$

Substitution of (3.31) into the boundary layer equations (2.30) results in

$$\frac{9A}{8}u_0 U_2 + u_0' V_2 = 0, \tag{3.32a}$$

$$\frac{3}{4}W_2 u_0 = (\gamma - 1)p_{10}h_0, \tag{3.32b}$$

$$\frac{9A}{8}u_0 H_2 + V_2 h_0' = 0, \tag{3.32c}$$

$$\frac{9A}{8}U_2 + V_2' = 0. \tag{3.32d}$$

It directly follows from (3.32b) that the solution for W_2 is

$$W_2 = \frac{4(\gamma - 1)p_{10}h_0}{3u_0}. \tag{3.33}$$

If we combine (3.32a) and (3.32d), it will give us

$$u_0 V_2' - u_0' V_2 = 0, \tag{3.34}$$

which means that V_2/u_0 is a constant in the main part of the boundary layer. To find this constant, we need to perform the matching of V_2 and u_0 with the solution in the viscous sublayer. We know that near the wall (as $\eta \to 0$)

$$u_0|_{\eta\to 0} = \lambda\eta, \tag{3.35}$$

and, from the matching

$$\xi^{\gamma_p-\frac{1}{2}}e^{-A\xi^{\frac{3}{2}}}V_1(s)|_{s\to\infty} = \xi^{\gamma_p}e^{-A\xi^{\frac{3}{2}}}V_2(\eta)|_{\eta\to 0}, \tag{3.36}$$

we see that the asymptotic behaviour of V_2 as $\eta \to 0$ is

$$V_2|_{\eta\to 0} = \xi^{-\frac{1}{2}}V_1|_{s\to\infty} = \xi^{-\frac{1}{2}}\frac{9}{8}Aa_0\eta\xi^{\frac{1}{2}} = -\frac{9}{8}Aa_0\eta. \tag{3.37}$$

This shows that the sought constant is $-(9Aa_0)/(8\lambda)$,

$$\text{i.e.} \qquad V_2 = -\frac{9Aa_0}{8\lambda}u_0. \tag{3.38}$$

The solutions for U_2 and H_2 can be obtained through substituting (3.38) into the momentum equation (3.32a) and the energy equation (3.32c). We have

$$U_2 = \frac{a_0}{\lambda}u_0^{'}, \tag{3.39}$$

$$H_2 = \frac{a_0}{\lambda}h_0^{'}. \tag{3.40}$$

Substituting the coordinate asymptotic expansions of H (3.31d) and P (3.31e) into the pressure equation (3.8), the value of p_{10} is obtained as

$$p_{10} = \frac{9A}{8}(\gamma-1)\int_0^\infty H_2 d\eta = -\frac{9Aa_0}{8\lambda}(\gamma-1)h_w, \tag{3.41}$$

and thus

$$W_2 = -\frac{3Aa_0(\gamma-1)^2h_w}{2\lambda}\frac{h_0}{u_0}. \tag{3.42}$$

Furthermore, the solution for f (3.26) tells us that

$$f^{''}(0) = C_1 \theta Ai^{'}(0) \tag{3.43}$$

while it is shown from the first equation regarding f, namely (3.23a), that

$$f^{''}(0) = 2(\gamma - 1) h_w A p_{10}, \tag{3.44}$$

so we can collect the above two equations and use the value of p_{10} given in (3.41) to know

$$A = 3^{\frac{1}{4}} \lambda^{\frac{5}{4}} \left| Ai^{'}(0) \right|^{\frac{3}{4}} (\gamma - 1)^{-\frac{3}{2}} h_w^{-\frac{3}{2}}. \tag{3.45}$$

Therefore, by substitution of the solution (3.38), (3.39), (3.40) and (3.42) into the coordinate asymptotic expansions (3.31), these expansions can describe the far field flow behaviour perturbed by the small roughness on the wall.

Conclusion and Discussion

For the three dimensional hypersonic viscous flow over semi-infinite flat plate with small roughness, near the leading edge of the plate or far from the roughness in the span-wise direction, the perturbed flow is described by

$$
\begin{aligned}
U &= u_0(\eta) + \cdots & &+ \left(\epsilon \xi^{\gamma_p} e^{-A\xi^{\frac{3}{2}}} a_0\right) \xi^{-\frac{3}{2}} \frac{1}{\lambda} u_0'(\eta) + \cdots , \\
V &= v_0(\eta) + \cdots & &- \left(\epsilon \xi^{\gamma_p} e^{-A\xi^{\frac{3}{2}}} a_0\right) \frac{9A}{8\lambda} u_0 + \cdots , \\
W &= & &- \left(\epsilon \xi^{\gamma_p} e^{-A\xi^{\frac{3}{2}}} a_0\right) \xi^{-1} \frac{3A(\gamma-1)^2 h_w}{2} \frac{h_0}{u_0} + \cdots , \\
H &= h_0(\eta) + \cdots & &+ \left(\epsilon \xi^{\gamma_p} e^{-A\xi^{\frac{3}{2}}} a_0\right) \xi^{-\frac{3}{2}} \frac{1}{\lambda} h_0' + \cdots , \\
P &= P_0 + \cdots & &- \left(\epsilon \xi^{\gamma_p} e^{-A\xi^{\frac{3}{2}}} a_0\right) \frac{9A}{8\lambda} (\gamma-1) h_w + \cdots
\end{aligned}
$$

in the main part of the boundary layer, or

$$
\begin{aligned}
U &= \xi^{-\frac{1}{2}} \lambda s + \cdots & &+ \left(\epsilon \xi^{\gamma_p - \frac{3}{2}} e^{-A\xi^{\frac{3}{2}}} a_0\right) + \cdots , \\
V &= \xi^{-1} \frac{\lambda}{4} s^2 + \cdots & &- \left(\epsilon \xi^{\gamma_p - \frac{1}{2}} e^{-A\xi^{\frac{3}{2}}} a_0\right) \frac{9A}{8} s + \cdots , \\
W &= & &- \left(\epsilon \xi^{\gamma_p - \frac{1}{2}} e^{-A\xi^{\frac{3}{2}}} a_0\right) \frac{3}{2} \frac{(\gamma-1)^2 h_w^2 A}{\lambda^2} s^{-1} + \cdots , \\
H &= h_w - \xi^{-1} \frac{Pr}{2} \lambda^2 s^2 + \cdots & &- \left(\epsilon \xi^{\gamma_p - 2} e^{-A\xi^{\frac{3}{2}}} a_0\right) Pr \lambda s + \cdots , \\
P &= P_0 + \cdots & &- \left(\epsilon \xi^{\gamma_p} e^{-A\xi^{\frac{3}{2}}} a_0\right) \frac{9A(\gamma-1) h_w}{8\lambda} + \cdots
\end{aligned}
$$

at the outer edge of the sublayer.

Here u_0, v_0, h_0 and P_0 are the solutions to the unperturbed two dimensional flow problem and

$$A = 3^{\frac{1}{4}}\lambda^{\frac{5}{4}}\left|Ai'(0)\right|^{\frac{3}{4}}(\gamma-1)^{-\frac{3}{2}}h_w^{-\frac{3}{2}}.$$

We can recall that in our case the perturbation propagates in the ξ-increasing direction, corresponding to travelling from the line $\xi = \xi_1$ to the line $\xi = \xi_3$ in Figure 3.2. From the definition of ξ, which is $\xi = \frac{\bar{z}}{\bar{x}^{\frac{3}{4}}}$, we know that $\xi \to \infty$ is equivalent to $\bar{x} \to 0$ for a fixed $\bar{z}$, or $\bar{z} \to \infty$ for a fixed $\bar{x}$. Therefore, one conclusion we can draw from our analysis is that in the boundary layer the perturbation can propagate not only in the spanwise direction, but also upstream towards the leading edge.

Furthermore, it can be shown that for a given a_0 and γ_p, the far field ($\xi \to \infty$) perturbation only depends on the enthalpy near the wall (or the wall temperature), which mainly affects the decay speed of the perturbation, and the undisturbed plate flow behaviour, provided that the heat capacity remains constant everywhere. In addition, it can be seen from the power of ξ that, as $\xi \to \infty$, the perturbation to U and H decays fastest, W has the second largest perturbation decay speed, and the roughness effect on P and V reduces slowest.

However, in fact only the relative value of the perturbation with respect to $\xi^{\gamma_p}a_0$ is explicit instead of the absolute value, as the exact values of a_0 and γ_p are not really known yet or proved to be the same for different types of roughness. It should be also noticed that no detail about the roughness is included in our solution, so it is probably that the value of these two parameters, especially a_0, which is the value of U_2 as $\eta \to 0$ and which indicates the strength of the perturbation, might be dependent on the detailed structure of the roughness. To find the values of a_0 and γ_p, we may need to understand the perturbation behaviour in the near field ($\xi = O(1)$), and this task can only be accomplished after obtaining the numerical solution of the entire boundary layer problem through the finite difference method[12].

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Appendix A

Ackeret Formula for Three Dimensional Hypersonic Flow

In the inviscid flow above the boundary layer, with the non-dimensionalization transformation

$$\hat{x} = Lx, \qquad \hat{y} = Ly, \qquad \hat{z} = Lz$$
$$\hat{u} = V_\infty u, \qquad \hat{v} = V_\infty v, \qquad \hat{w} = V_\infty w,$$
$$\hat{\rho} = \rho_\infty \rho, \qquad \hat{p} = p_\infty + \rho_\infty V_\infty^2 p, \qquad \hat{h} = V_\infty^2 h,$$

we have to solve the three dimensional Euler Equations*

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial u} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x}, \tag{A.1a}$$
$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial u} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y}, \tag{A.1b}$$
$$\rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial u} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}, \tag{A.1c}$$
$$\rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial u} + w\frac{\partial h}{\partial z}\right) = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}, \tag{A.1d}$$

*Here the symbols without 'hat' only denote dimensionless variables and some of them may not be $O(1)$, which is different from how they are defined in Section1.1.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{A.1e}$$

$$h = \frac{\gamma}{\gamma-1}\frac{1}{\rho}\left(\frac{1}{\gamma M_\infty^2} + p\right). \tag{A.1f}$$

We know that in the perturbation region

$$\hat{x} \sim L, \qquad \hat{z} \sim LRe^{-\frac{1}{4}}M_\infty, \qquad \hat{p} - p_\infty \sim \rho_\infty V_\infty^2 Re^{-\frac{1}{2}}, \tag{A.2}$$

so after the non-dimensionlization transformation,

$$x \sim 1, \qquad z \sim Re^{-\frac{1}{4}}M_\infty, \qquad p \sim Re^{-\frac{1}{2}}. \tag{A.3}$$

Since in the overlapping region of the boundary layer and the outer inviscid flow there is balance

$$\hat{\rho}\hat{u}\frac{\partial \hat{w}}{\partial \hat{x}} \sim \frac{\partial \hat{p}}{\partial \hat{z}},$$

considering $\hat{u} \sim V_\infty$ and $\hat{\rho} \sim \rho_\infty$ to the leading order, the order of magnitude of $\hat{w}$ can be found as

$$\hat{w} = O(V_\infty Re^{-\frac{1}{4}}M_\infty^{-1}). \tag{A.4}$$

In the inviscid region of the two dimensional flow, the impermeability condition is satisfied on the surface of the thin effective body, which is acquired by augmenting the real body with the displacement thickness of the boundary layer, so the deflection angle is quite small, or more precisely of $O(Re^{-\frac{1}{2}}M_\infty)$ as the dimensionless thickness of the boundary layer is of $O(Re^{-\frac{1}{2}}M_\infty)$. This tells us that the shock wave is supposed to be so weak that it degenerates to the Mach line $y = \frac{1}{M_\infty}x$ in the two dimensional plate flow[10]. Therefore, even in the three dimensional case, the dimensionless thickness of the shock layer is still

$$y = O(\frac{1}{M_\infty}), \tag{A.5}$$

because the roughness is too smaller than the boundary layer to change the oder of magnitude of the shock layer thickness.

As for the other dimensionless variables, we will seek their asymptotic expansions in the form

$$u = 1 + \epsilon_1 u_1 + \cdots, \qquad v = \epsilon_2 v_1 + \cdots, \tag{A.6}$$

$$\rho = 1 + \epsilon_3 \rho_1 + \cdots, \qquad h = \frac{1}{(\gamma - 1) M_\infty^2} + \epsilon_4 h_1 + \cdots, \tag{A.7}$$

where $u_1, v_1, p_1, \rho_1.h_1 \sim O(1)$ and the coefficients $\epsilon_i \ll 1$. To find out these coefficients, we will do the following analysis.

1. At $\hat{y} = \hat{\delta}(\hat{x}, \hat{z})$, the impermeability condition is

$$\hat{v} = \hat{u}\hat{\delta}_{\hat{x}} + \hat{w}\hat{\delta}_{\hat{z}}. \tag{A.8}$$

As the first term on the right-hand side is of $O(V_\infty Re^{-\frac{1}{2}} M_\infty)$ while the second term is of $O(V_\infty Re^{-\frac{1}{2}} M_\infty^{-1})$ being much smaller, the order of magnitude of $\hat{v}$ is supposed to be $O(V_\infty Re^{-\frac{1}{2}} M_\infty)$ and hence, $\epsilon_2 = Re^{-\frac{1}{2}} M_\infty$.

2. Considering the velocity perturbation across the front shock of the two-dimensional flow, it should be perpendicular to the shock wave, so

$$\frac{\epsilon_1 u_1}{\epsilon_2 v_1} = \frac{1}{M_\infty} \qquad \text{i.e.} \qquad \epsilon_1 = Re^{-\frac{1}{2}}.$$

As the order of magnitude of the variables is unchanged from the two dimensional flow to the three-dimensional, this result of ϵ_1 is still valid for the three-dimensional case.

3. In the boundary layer, the density is much smaller than that in the free-stream flow, and the flow in front of the shock wave is undisturbed, so this density loss in the boundary layer will be compensated for by the density increase in shock layer, which means that

$$\frac{L}{M_\infty} \epsilon_3 \rho_\infty \sim L Re^{-\frac{1}{2}} M_\infty \rho_\infty \qquad \text{i.e.} \qquad \epsilon_3 = Re^{-\frac{1}{2}} M_\infty^2. \tag{A.9}$$

4. Substitution of the asymptotic expansions of p and ρ into the state equation (A.1f),

we can obtain

$$\epsilon_4 h_1 = -\frac{Re^{-\frac{1}{2}}}{\gamma - 1}\rho_1 + \frac{\gamma}{\gamma - 1}Re^{-\frac{1}{2}}p_1, \qquad \text{i.e.} \qquad \epsilon_4 = Re^{-\frac{1}{2}}. \tag{A.10}$$

Collecting (A.3), (A.5), (A.4) and the above results, the dimensionless variables, or their variation from the free-stream value, in the shock layer can be scaled to $O(1)$ via

$$y = M_\infty^{-1} Y, \tag{A.11a}$$
$$z = Re^{-\frac{1}{4}} M_\infty Z, \tag{A.11b}$$
$$u(x, y, z) = 1 + Re^{-\frac{1}{2}} u_1(x, Y, Z) + \cdots, \tag{A.11c}$$
$$v(x, y, z) = Re^{-\frac{1}{2}} M_\infty v_1(x, Y, Z) + \cdots, \tag{A.11d}$$
$$w(x, y, z) = Re^{-\frac{1}{4}} M_\infty^{-1} w_1(x, Y, Z) + \cdots, \tag{A.11e}$$
$$\rho(x, y, z) = 1 + Re^{-\frac{1}{2}} M_\infty^2 \rho_1(x, Y, Z) + \cdots, \tag{A.11f}$$
$$p(x, y, z) = Re^{-\frac{1}{2}} p_1(x, Y, Z) + \cdots, \tag{A.11g}$$
$$h(x, y, z) = \frac{1}{(\gamma - 1) M_\infty^2} + Re^{-\frac{1}{2}} h_1(x, Y, Z) + \cdots, \tag{A.11h}$$

where Y, Z are the $O(1)$ independent variables. Then if we substitute them into the Euler Equations and keep the leading order terms, the equations become

$$\frac{\partial u_1}{\partial x} = -\frac{\partial p_1}{\partial x}, \tag{A.12a}$$
$$\frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial Y}, \tag{A.12b}$$
$$\frac{\partial w_1}{\partial x} = -\frac{\partial p_1}{\partial Z}, \tag{A.12c}$$
$$\frac{\partial \rho_1}{\partial x} + \frac{\partial v_1}{\partial Y} = 0, \tag{A.12d}$$
$$\frac{\partial h_1}{\partial x} = \frac{\partial p_1}{\partial x}, \tag{A.12e}$$
$$h_1 = \frac{\gamma}{\gamma - 1} p_1 - \frac{\rho_1}{\gamma - 1}. \tag{A.12f}$$

Corresponding to the scalings in this appendix, we will use

$$\hat{\delta}(\hat{x}, \hat{z}) = LRe^{-\frac{1}{2}} M_\infty \delta(x, Z) \tag{A.13}$$

for the displacement thickness and have $\delta(x, Z) = O(1)$.

To find the relation between the variables with subscript $_1$, we need the help of the non-dimensional jump conditions on the shock wave, which are

$$\rho(un_x + vn_y + wn_z)^2 + p = n_x^2, \tag{A.14a}$$

$$u\tau_x + v\tau_y + w\tau_z = \tau_x, \tag{A.14b}$$

$$h + \frac{1}{2}(u^2 + v^2 + w^2) = \frac{1}{(\gamma - 1)M_\infty^2} + \frac{1}{2}, \tag{A.14c}$$

$$\rho(un_x + vn_y + wn_z) = n_x, \tag{A.14d}$$

where $\vec{n} = (n_x, n_y, n_z)$ and $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ are the unit vectors normal and tangential to the shock wave in the coordinate system $Oxyz$, respectively. Compared with the shock layer in supersonic, transonic and subsonic flows, an important difference about the hypersonic flow shock layer is that some of the components in the normal and tangential unit vectors may not be of $O(1)$ for they might be asymptotically small.

For any given point M_0 on the shock wave, we can always find θ_0 and α_0 such that $\vec{n} = (\cos\alpha_0 \sin\theta_0, \sin\alpha_0 \sin\theta_0, \cos\theta_0)$ are the normal vector originating from M_0, with two unit vectors $\vec{\tau}_1 = (\cos\alpha_0 \cos\theta_0, \sin\alpha_0 \cos\theta_0, -\sin\theta_0)$ and $\vec{\tau}_2 = (\sin\alpha_0, -\cos\alpha_0, 0)$ perpendicular to it. Then for any unit tangential vector $\vec{\tau}$ of the shock wave at M_0, there must exist two constants n_1 and n_2 such that

$$\vec{\tau} = \frac{n_1\vec{\tau}_1 + n_2\vec{\tau}_2}{\| n_1\vec{\tau}_1 + n_2\vec{\tau}_2 \|},$$

so the satisfaction of the conditions

$$\cos\alpha_0 \cos\theta_0 u + \sin\alpha_0 \cos\theta_0 v - \sin\theta_0 w = \cos\alpha_0 \cos\theta_0,$$

$$\sin\alpha_0 u - \cos\alpha_0 v = \sin\alpha_0,$$

$$\text{i.e. } Re^{-\frac{1}{2}}\cos\alpha_0\cos\theta_0 u_1 + Re^{-\frac{1}{2}}M_\infty\sin\alpha_0\cos\theta_0 v_1 - Re^{-\frac{1}{4}}M_\infty^{-1}\sin\theta_0 w_1 = 0, \quad \text{(A.15)}$$

$$Re^{-\frac{1}{2}}\sin\alpha_0 u_1 - Re^{-\frac{1}{2}}M_\infty\cos\alpha_0 v_1 = 0 \quad \text{(A.16)}$$

will guarantee the second jump condition (A.14b) being satisfied for any $\vec{\tau}$ originating from M_0. Then it can be known from the above two equations that

$$v_1 = M_\infty^{-1}\frac{\sin\alpha_0}{\cos\alpha_0}u_1, \quad \text{(A.17)}$$

$$w_1 = Re^{-\frac{1}{4}}M_\infty\frac{\cos\theta_0}{\sin\theta_0\cos\alpha_0}u_1. \quad \text{(A.18)}$$

Substitution of these results into the jump condition (A.14a) gives us

$$\left(1 + Re^{-\frac{1}{2}}M_\infty^2\rho_1\right)\left(n_x + \frac{Re^{-\frac{1}{2}}u_1}{n_x}\right)^2 + Re^{-\frac{1}{2}}p_1 = n_x^2.$$

If we neglect the terms which are apparently much smaller than the others in the expansion of this equation, it becomes

$$Re^{-\frac{1}{2}}\left(2u_1 + M_\infty^2\rho_1 n_x^2 + p_1\right) + Re^{-1}\frac{u_1^2}{n_x^2} = 0. \quad \text{(A.19)}$$

To find another relationship between p_1 and ρ_1, we can multiply both hand-sides of the first jump condition (A.14a) by ρ and substitute (A.14d) into it; with considering the asymptotic expansions of p and ρ, this finally results in

$$p_1 = M_\infty^2 n_x^2\rho_1. \quad \text{(A.20)}$$

Combining the above two equations leads to the conclusion that

$$u_1 + p_1 = 0 \quad \text{(A.21)}$$

on the shock wave.
In addition, from Equation (A.12a), in the whole shock layer we should have

$$u_1 + p_1 = l_1(Y, Z). \tag{A.22}$$

Taking into consideration the result (A.21) and the fact that the projection of the shock wave upon the (Y, Z)-plane covers every pair of (Y, Z) with $Y > 0$, the function $l_1(Y, Z)$ should be constantly equal to 0 in the entire shock layer,

$$\text{i.e.} \qquad u_1 + p_1 = 0. \tag{A.23}$$

Similarly, we can find the relation between h_1 and p_1 in the shock layer: If we adopt the scalings (A.11c)-(A.11e) and (A.11h) into the third jump condition (A.14c), it turns into

$$(h_1 + u_1)|_{\text{shock}} = 0 \qquad \text{i.e.} \qquad h_1|_{\text{shock}} = p_1|_{\text{shock}},$$

and from the energy equation (A.12e) we can get

$$h_1 = p_1 + l_2(Y, Z);$$

It follows directly from the above results that in the whole shock layer

$$h_1 = p_1. \tag{A.24}$$

Furthermore, substituting it into the state equation (A.12f) tells us that

$$p_1 = \rho_1, \tag{A.25}$$

so everywhere between the shock wave and the outer edge of the boundary layer

$$p_1 = \rho_1 = h_1 = -u_1. \tag{A.26}$$

Besides, recalling $p_1 = M_\infty^2 n_x^2 \rho_1$ on the shock wave, we can know

$$n_x = \frac{1}{M_\infty},$$

so the shock wave equation is supposed to be

$$y = \frac{1}{M_\infty} x + \psi(z), \qquad \text{or} \qquad Y = x + \Psi(Z).$$

Now turning to the lateral momentum equation (A.12b) and the continuity equation (A.12d), they can be rewritten as

$$\frac{\partial v_1}{\partial x} = \frac{\partial u_1}{\partial Y} \qquad \text{i.e.} \qquad \frac{\partial v_1}{\partial x} = \frac{1}{M_\infty}\frac{\partial u_1}{\partial y}, \tag{A.27a}$$

$$\frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial Y} = 0 \qquad \text{i.e.} \qquad \frac{\partial u_1}{\partial x} - \frac{1}{M_\infty}\frac{\partial v_1}{\partial y} = 0. \tag{A.27b}$$

From (A.27a), we know that a potential function $\phi(x, Y, Z)$ can always be found such that

$$u_1 = \frac{\partial \phi}{\partial x},$$
$$v_1 = \frac{\partial \phi}{\partial Y},$$

and hence, (A.27b) becomes

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial Y^2} = 0, \tag{A.28}$$

to which the solution is

$$\phi(x, Y, Z) = f(x - Y, Z) + g(x + Y, Z). \tag{A.29}$$

For any given z_0, the unit tangential vectors of the intersection line of the shock wave and the plane $z = z_0$ are also tangential to the shock wave, and the third components of these vectors are zero, so if we denote such a vector by $(\tau_{0x}, \tau_{0y}, 0)$ in the coordinate system $Oxyz$, the corresponding jump conditions are

$$u_1 \tau_{0x} + M_\infty v_1 \tau_{0y} = 0, \qquad \text{i.e.} \qquad \frac{\partial \phi}{\partial x}\tau_{0x} + \frac{\partial \phi}{\partial y}\tau_{0y} = 0.$$

This means that ϕ is a constant C along the intersection ling of the shock wave and plane $z = z_0$, namely the line $\{y = \frac{1}{M_\infty}x + \psi(z), z = z_0\}$ or $\{Y = x + \Psi(Z), Z = Z_0\}$, i.e.

$$f(\Psi(Z_0), Z_0) + g(2x + 2\Psi(Z_0), Z_0) = C,$$

where $Z_0 = z_0 Re^{\frac{1}{4}} M_\infty^{-1}$. Since $f(\Psi(Z_0), Z_0)$ and C are constants for a given Z_0, the function $g(2x + 2\Psi(Z_0), Z_0)$ should also be only dependent on the parameter Z_0. Thus $g(x + Y, Z)$ is a function of Z only. Therefore, we can say

$$\phi(x, Y, Z) = f(x - Y, Z).$$

Now if we can obtain the expression of $f(x - Y, Z)$, we will be able to know u_1, thereby p_1.

According to the impermeability condition (A.8) and the scalings of u and v,

$$v(x, Re^{-\frac{1}{2}} M_\infty \delta, z) = Re^{-\frac{1}{2}} M_\infty \frac{\partial \delta(x, Z)}{\partial x} + \cdots, \tag{A.30}$$

while if we start from $v = Re^{-\frac{1}{2}} M_\infty v_1$ and the Taylor expansion of v_1 at $y = Re^{-\frac{1}{2}} M_\infty \delta$, which is

$$\begin{aligned} v_1(x, Re^{-\frac{1}{2}} M_\infty \delta, Z) &= v_1(x, 0, Z) + Re^{-\frac{1}{2}} M_\infty \delta \frac{\partial v_1}{\partial y}(x, 0, Z) + O(Re^{-1} M_\infty^2 \frac{\partial^2 v_1}{\partial y^2}) \\ &= v_1(x, 0, Z) + Re^{-\frac{1}{2}} M_\infty^2 \delta \frac{\partial v_1}{\partial Y}(x, 0, Z) + O(Re^{-1} M_\infty^4), \end{aligned} \tag{A.31}$$

the asymptotic expansion of $v(x, Re^{-\frac{1}{2}} M_\infty \delta, z)$ also can be written as

$$v(x, Re^{-\frac{1}{2}} M_\infty \delta, z) = Re^{-\frac{1}{2}} M_\infty v_1(x, 0, Z) + O(Re^{-1} M_\infty^3). \tag{A.32}$$

It can be known from the comparison of (A.30) and (A.32) that

$$v_1(x, 0, Z) = \frac{\partial \delta(x, Z)}{\partial x},$$

$$\text{i.e.} \qquad \frac{\partial \delta(x,Z)}{\partial x} = \frac{\partial \phi}{\partial Y}|_{Y=0} = -\frac{\partial f(x,Z)}{\partial x}, \tag{A.33}$$

so

$$f(x,Z) = -\delta(x,Z) + l_3(Z),$$

and hence,

$$\phi(x,Y,Z) = f(x-Y,Z) = -\delta(x-Y,Z) + l_3(Z).$$

Since the choice of $l_3(Z)$ contributes nothing to $\frac{\partial \phi}{\partial x}$ or $\frac{\partial \phi}{\partial Y}$, it can be set to zero to leave us

$$\phi(x,0,Z) = -\delta(x,Z). \tag{A.34}$$

Therefore, remembering $p_1 = -u_1$, we can conclude that

$$p_1(x,0,Z) = -u_1(x,0,Z) = \frac{\partial \delta(x,Z)}{\partial x}.$$

This result shows us that in our case of flat plate with small roughness, the three dimensional Ackeret Formula has the same form with the two dimensional one.