On attractors, spectra and bifurcations of random dynamical systems

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by

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Declaration

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

In this thesis a number of related topics in random dynamical systems theory are studied: local attractors and attractor-repeller pairs, the exponential dichotomy spectrum and bifurcation theory.

We review two existing theories in the literature on local attractors for random dynamical systems on compact metric spaces and associated attractor-repeller pairs and Morse decompositions, namely, local weak attractors and local pullback attractors. We extend the theory of past and future attractor-repeller pairs for nonautonomous systems to the setting of random dynamical systems, and define local strong attractors, which both pullback and forward attract a random neighbourhood. Some examples are given to illustrate the nature of these different attractor concepts. For linear systems considered on the projective space, it is shown that a local strong attractor that attracts a uniform neighbourhood is an object with sufficient properties to prove an analogue of Selgrade's Theorem on the existence of a unique finest Morse decomposition.

We develop the dichotomy spectrum for random dynamical systems and investigate its relationship to the Lyapunov spectrum. We demonstrate the utility of the dichotomy spectrum for random bifurcation theory in the following example. Crauel and Flandoli [CF98] studied the stochastic differential equation formed from the deterministic pitchfork normal form with additive noise. It was shown that for all parameter values this system possesses a unique invariant measure given by a globally attracting random fixed point with negative Lyapunov exponent, and hence the deterministic bifurcation scenario is destroyed by additive noise. Here, however, we show that one may still observe qualitative changes in the dynamics at the underlying deterministic bifurcation point, in terms of: a loss of hyperbolicity of the dichotomy spectrum; a loss of uniform attractivity; a qualitative change in the

distribution of finite-time Lyapunov exponents; and that whilst for small parameter values the systems are topologically equivalent, there is a loss of uniform topological equivalence.

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Publications

The material in Sections 3.1 and 3.2, and in Chapter 4 of this thesis is joint work with Doan Thai Son, Jeroen Lamb and Martin Rasmussen, and has been submitted for publication in the paper [CDLR13].

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Chapter 1

Introduction

The theory of random dynamical systems seeks to model physical systems which are under the influence of random perturbations. The mathematical framework incorporates ideas from the well developed fields of dynamical systems, probability and stochastic analysis. The qualitative theory of random dynamical systems is still in its relatively early stages, and much of current research is involved in translating concepts from classical dynamical systems theory into the context of random dynamical systems. There are fundamental questions as to what the 'correct' corresponding concepts and definitions are in the setting of random dynamical systems, which requires exploration of definitions and direct exploration of case studies. In this thesis we explore three important and related concepts in dynamical systems theory: local attractors and attractor-repeller pairs, dynamical spectral theory and bifurcation theory. Our focus throughout this thesis is on linear theory. One of the inherent properties of random and nonautonomous systems is the nonuniformity of the dynamics, and many of the results in the thesis are a qualification of this aspect. We shall now give an overview of the thesis topics.

The fundamental theory of local attractors, attractor-repeller pairs and Morse decompositions for autonomous dynamical systems was established by Conley in his monograph [Con78]. There are many nonequivalent definitions of local attractors for autonomous dynamical systems, and for random dynamical systems there are additional degrees of freedom on the choice of the type of convergence to the attractor, given by the facts that they are nonautonomous systems and are equipped with a probability measure. A comparison of a number of different attraction concepts for random dynamical systems has been carried out by Ashwin and Ochs [AO03]. To date, two theories of attractor-repeller pairs for random dynamical systems can be found in the literature: Ochs [Och99] demonstrated the existence of attractor-repeller pairs corresponding to so-called local weak attractors, which use the notion of convergence in probability; Liu [Liu06, Liu07a, Liu07b] demonstrated the existence of attractor-repeller pairs corresponding to so-called local pullback attractors, which use the notion of almost sure pullback convergence. For general nonautonomous systems, in particular not necessarily possessing a compact base, Rasmussen [Ras06] has demonstrated that by restricting one's attention to only the past or the future time domain of the system and using suitable attraction and repulsion concepts, one can obtain so-called past and future attractor-repeller pairs. In this thesis we shall review the results on weak and pullback attractor-repeller pairs, and extend the concept of past and future attractor-repeller pairs to the setting of random dynamical systems (Theorem 2.7.5). We shall also define strong attraction, in which both forward and pullback convergence are required. Our main aim with this survey of different types of local attractors is to find a type of attractor-repeller pair with sufficient properties to be able to prove an analogue of Selgrade's Theorem (see below).

Linear theory is important in the study of nonlinear dynamical systems, since the stability of solutions of nonlinear systems can often be derived from the stability of their linearization. There are a number of dynamical spectral concepts available to study linear dynamical systems. For random dynamical systems, the Multiplicative Ergodic Theorem of Oseledets [Ose68] guarantees the existence of Lyapunov exponents for almost all initial conditions, and if the system is ergodic there are finitely many — the Lyapunov spectrum. This strong result has formed the basis of much important research in random dynamical systems thus far [Arn98]. An important concept in linear nonautonomous dynamical systems is that of an exponential dichotomy (see [Cop78]), and the associated dichotomy, or Sacker-Sell, spectrum [SS78]. We shall define the concept of an exponential dichotomy for linear random dynamical systems and demonstrate the associated spectral theorem, namely that the dichotomy spectrum consists of a finite union of closed intervals (Theorem 3.2.4), and that each of these spectral intervals is associated with an invariant linear subspace (Theorem 3.2.5). We note that the concept of the dichotomy spectrum for random dynamical systems has also recently been investigated by Wang and Cao [WC14] with a different set-up to ours (which we discuss in Section 3.4).

An alternative approach to studying linear systems is to consider the induced system on the projective space. For a linear flow on a vector bundle with compact base for which the base flow is chain transitive, Selgrade's Theorem [Sel75] demonstrates the existence of a unique so-called finest Morse decomposition of the projective bundle. Analogous results have been shown for nonautonomous systems with a noncompact base using the theory of past and future attractor-repeller pairs [Ras07, Ras08], and we shall use similar techniques to demonstrate an analogue of Selgrade's Theorem for linear random dynamical systems using local strong attractors (Theorem 3.3.6).

The theory of bifurcations in random dynamical systems is still in its developmental stages (see [Arn98, Chapter 9] for a detailed description and historical account). A bifurcation in a parameterized family of random dynamical systems is considered to be a qualitative change in the set of invariant measures of the system. There are two notions of 'invariant measure' available for random dynamical systems: if the system admits a Markov semi-group, one may consider the so-called stationary measures of the semi-group; the notion of an invariant measure, on the other hand, is a random measure that is invariant (in a nonautonomous sense) under the random flow. There are then two associated notions of bifurcation: the concept of a phenomenological bifurcation relates to a qualitative change in the density of a stationary measure, whilst the concept of a dynamical bifurcation relates to a qualitative change in the set of invariant measures of the system and is associated to a loss of hyperbolicity of the Lyapunov spectrum [Arn98, Theorem 9.2.3]. The two concepts are independent, with one type of bifurcation able to occur without the other [Arn98, p. 473, 476]. The phenomenological bifurcation concept has a number of significant disadvantages: it is restricted to the Markovian case; stationary measures provide only the time averaged position of each trajectory, and qualitative changes are not related, in general, to a change in stability (i.e. a change in the Lyapunov spectrum); the concept is not coordinate independent; and stationary measures are in a one-to-one correspondence with invariant measures that only depend on the past of the system, but additional invariant measures are often present. For these reasons, the more recent dynamical bifurcation concept is now favoured in the literature.

Initial studies have sought to find analogues of elementary deterministic bifurcation scenarios — the saddle-node, transcritical, pitchfork and Hopf bifurcations [AB92, CF98, ASNSH96, SH96, ABSH99]. In particular, in [AB92] the authors consider stochastic differential equations formed from the transcritical, pitchfork and saddle-node normal forms, with noise added to the parameter. For the transcritical and pitchfork cases one obtains dynamical bifurcation patterns similar to the deterministic cases: the trivial equilibria of the corresponding deterministic systems persist, and the other equilibria are replaced by invariant random Dirac measures (also known as random fixed points), with the stability of these invariant measures being the same as the corresponding deterministic equilibria. In [CIS99] the authors find necessary and sufficient conditions for a certain class of parameterized one-dimensional stochastic differential equations, with zero as a fixed point, to undergo similar transcritical and pitchfork dynamical bifurcations of random fixed points. Crauel and Flandoli in [CF98] consider the normal form of the deterministic pitchfork bifurcation with additive noise, that is, the one-dimensional stochastic differential equation driven by a Wiener process $\{W_t\}_{t\in\mathbb{R}}$ given by

$$\mathrm{d}x_t = \big(\alpha x_t - x_t^3\big)\mathrm{d}t + \sigma \mathrm{d}W_t\,, \tag{1.0.1}$$

with real parameters α (the bifurcation parameter) and σ (the noise intensity). The associated random dynamical system undergoes a phenomenological bifurcation at the parameter value $\alpha = 0$, where the stationary distribution transitions from a one-peak to a two-peak form. However, the authors demonstrate that for arbitrary nonzero noise intensity there is no dynamical bifurcation: for all values of the parameter α there exists a unique ergodic invariant measure given by a random fixed point, for which the associated Lyapunov exponent is negative, and the random fixed point is globally attractive. Hence one may say that the deterministic pitchfork bifurcation at $\alpha = 0$ is destroyed by additive noise. Here we take a different approach to the bifurcation theory of random dynamical systems and we make further investigations to the random dynamical system generated by (1.0.1). We argue that in some sense the bifurcation at $\alpha = 0$ is not destroyed, as one can still observe qualitative changes in the dynamics:

(i) The random fixed point is uniformly attractive only if $\alpha < 0$ (Theorem 4.2.3).

(ii) There is a change in the practical observability of the Lyapunov exponent (by finite-time Lyapunov exponents, Theorem 4.2.5).

(iii) There is a qualitative change in the dichotomy spectrum associated to the random fixed point (Theorem 4.3.1).

(iv) Whilst for $|\alpha|$ sufficiently small, the resulting dynamics are topologically equivalent (Theorem 4.4.1), there does not exist a uniformly continuous topological conjugacy between the dynamics of cases with positive and negative parameter α (Theorem 4.4.4).

It is our opinion that additive noise is a natural scenario to consider, and the bifurcation theory of such systems is of interest for applications.

We now give an overview of the thesis contents and state its main contributions. In the remainder of Chapter 1 (Section 1.1) we give formal definitions and introduce some basic concepts for random dynamical systems, and establish notation and basic assumptions held throughout the thesis.

In Chapter 2 we make a survey of different types of attractor-repeller pairs for random dynamical systems defined with respect to three different concepts of attraction. First we review some ideas on attractors and attractor-repeller pairs for deterministic systems in Section 2.1, and discuss the three different attraction concepts for random dynamical systems that we will consider in Section 2.2. Then in Section 2.3 we recall results on local weak set attractors from [Och99, CDS04]. We prepare for the theories of local pullback, past and future attractors with some results on pullback limit sets, which are well studied in the literature in terms of global attractors. The theory of pullback attractor-repeller pairs was first demonstrated in the papers [Liu06, Liu07a, Liu07b, LSZ08]; here we aim to give a clear and concise exposition of the construction of the pullback attractor-repeller pair in Section 2.5, the defining result being Theorem 2.5.11. In Section 2.6 we give some (linear) examples to demonstrate that the three local attraction concepts are not equivalent, and demonstrate that pullback attractor-repeller pairs may have some undesirable properties (see Remark 2.5.13). We then proceed to extend the theory of past and future attractor-repeller pairs to the setting of random dynamical systems in Section 2.7, the main result being Theorem 2.7.5. We define the notion of a local strong attractor in Section 2.8, and finish the chapter with a discussion in Section 2.9, principally on the nature of pullback attractor-repeller pairs.

Chapter 3 is on linear theory. In Section 3.1 the notion of an exponential dichotomy for a linear random dynamical system is defined and some basic properties are given. Then in Section 3.2 the dichotomy spectrum is defined and the spectral theorem (Theorem 3.2.4) and existence of the spectral manifolds (Theorem 3.2.5) are demonstrated. We also demonstrate a necessary and sufficient condition to obtain a bounded spectrum (Proposition 3.2.7), and investigate the relationship of the dichotomy spectrum to the Lyapunov spectrum (Remark 3.2.9). In Section 3.3 we study past attractor-repeller pairs and strong attractor-repeller pairs in the projective space of linear systems, and prove an analogue of Selgrade's Theorem (Theorem 3.3.6) using strong attractor-repeller pairs. Our reasons for using local strong attractors are discussed in the introduction to Section 3.3 and Remark 3.3.4. In Section 3.4 we discuss our results on the dichotomy spectrum in relation to those obtained in [WC14], the possibility of obtaining a version of Selgrade's Theorem for weak attractor-repeller pairs, and also introduce the concept of the Morse spectrum for random dynamical systems.

In Chapter 4 we contribute to the bifurcation theory of random dynamical systems by a case study of the random dynamical system generated by (1.0.1). In Section 4.1 we review the key results obtained by Crauel and Flandoli [CF98]. Then in Section 4.2 we show the transition to nonuniform attractivity of the random fixed point (Theorem 4.2.3), and the existence of a positive measure set of positive finite-time Lyapunov exponents for $\alpha > 0$ (Theorem 4.2.5). The loss of hyperbolicity of the dichotomy spectrum associated to the random fixed point for $\alpha > 0$ (Theorem 4.3.1) is demonstrated in Section 4.3, and we also show that limits of the set of finite-time Lyapunov exponents are contained in the dichotomy spectrum (Theorem 4.3.2). The results on topological equivalence (Theorem 4.4.1) and uniform topological equivalence (Theorem 4.4.4) are given in Section 4.4, and we finish with a discussion on future work in Section 4.5.

1.1 Mathematical set-up, notation and assumptions

Here we give formal definitions and basic facts regarding the theory of random dynamical systems, and establish the notation used in the thesis. Also note that we will hold Assumption 1.1.4 throughout the thesis. For a comprehensive introduction to the theory of random dynamical systems, we refer the reader to the monograph of Ludwig Arnold [Arn98].

Time

The *time set* $\mathbb{T}$ is equal to one of the following: $\mathbb{R}$ or $\mathbb{Z}$, in which case we speak of *continuous* or *discrete*, respectively, *two-sided time*; $\mathbb{R}_0^+ := \{t \in \mathbb{R} \colon t \geq 0\}$ or $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, in which case we speak of *continuous* or *discrete*, respectively, *one-sided time*. We also take the time set to be equipped with its Borel σ-algebra, $\mathcal{B}(\mathbb{T})$. Given a one-sided time set $\mathbb{T}$, we denote the corresponding two-sided time set by $\bar{\mathbb{T}}$, i.e. if $\mathbb{T} = \mathbb{R}_0^+$ then $\bar{\mathbb{T}} = \mathbb{R}$, and if $\mathbb{T} = \mathbb{N}_0$ then $\bar{\mathbb{T}} = \mathbb{Z}$. We also define $\mathbb{T}_0^+ := \mathbb{T} \cap \mathbb{R}_0^+$ and $\mathbb{T}_0^- := \bar{\mathbb{T}} \cap \mathbb{R}_0^-$ where $\mathbb{R}_0^- := \{t \in \mathbb{R} \colon t \leq 0\}$.

Dynamical system, metric dynamical system, random dynamical system

Definition 1.1.1. A *dynamical system* (DS) on a *state space* X with time set $\mathbb{T}$ is a mapping

$$\phi \colon \mathbb{T} \times X \to X, \qquad (t, x) \mapsto \phi(t, x),$$

where the mappings $\phi_t := \phi(t, \cdot) \colon X \to X$ satisfy

(i) $\phi_0 = \mathrm{id}_X$

(ii) $\phi_{t+s} = \phi_t \circ \phi_s$ for all $t, s \in \mathbb{T}$

The noise in a random dynamical system is modelled by a *metric dynamical system* (see e.g. [Arn98, Appendix A]).

Definition 1.1.2. A *metric dynamical system* (metric DS) consists of a probability space $(\Omega, \mathcal{F}, \mu)$, and a dynamical system $\theta \colon \mathbb{T} \times \Omega \to \Omega$ which is a $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F})$-measurable mapping and is measure preserving, i.e. μ is a *θ-invariant measure*: $\theta_t\mu(\cdot) := \mu(\theta_t^{-1} \cdot) = \mu(\cdot)$ for all $t \in \mathbb{T}$.

A metric DS shall often be written as the quadruple $(\Omega, \mathcal{F}, \mu, \theta)$, where the time set $\mathbb{T}$ for θ should be clear from the context. The measure preserving property essentially means that the metric DS is modelling a stationary stochastic process. Given a metric DS $(\Omega, \mathcal{F}, \mu, \theta)$ a measurable set $F \in \mathcal{F}$ is called *θ-invariant* if $\theta_t^{-1} F = F$ for all $t \in \mathbb{T}$, and a measurable function $f\colon \Omega \to \mathbb{R}$ is called θ-invariant if $f(\theta_t \omega) = f(\omega)$ for all $t \in \mathbb{T}$ and μ almost all $\omega \in \Omega$; the metric DS is called *ergodic* if every θ-invariant set has either measure one or measure zero.

The formal definition of a random dynamical system is as follows.

Definition 1.1.3. A *random dynamical system* (RDS) on the measurable *state space* $(X, \mathcal{B})$ over a metric DS $(\Omega, \mathcal{F}, \mu, \theta)$ with time set $\mathbb{T}$ is a mapping

$$\varphi\colon \mathbb{T} \times \Omega \times X \to X, \qquad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B})$-measurable, and which forms a *cocycle* over θ, that is, the mappings $\varphi(t, \omega) := \varphi(t, \omega, \cdot)\colon X \to X$ satisfy

(i) $\varphi(0, \omega) = \mathrm{id}_X$ for all $\omega \in \Omega$, and

(ii) $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$ for all $\omega \in \Omega$ and $t, s \in \mathbb{T}$.

For simplicity we will normally just speak of an RDS φ without mentioning the underlying metric DS. An RDS φ may equivalently be described as a *skew product* of the metric DS θ and cocycle φ, that is, the mapping

$$\Theta\colon \mathbb{T} \times \Omega \times X \to \Omega \times X, \qquad (t, \omega, x) \mapsto \Theta_t(\omega, x) := (\theta_t \omega, \varphi(t, \omega) x).$$

The skew product Θ is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{F} \otimes \mathcal{B})$-measurable and is a dynamical system with state space $\Omega \times X$.

Assumption 1.1.4. We shall always hold the following assumptions on an RDS φ throughout the thesis.

(i) The metric DS θ associated to φ is invertible, i.e. even if φ is only defined with a one-sided time set $\mathbb{T}$, the underlying metric DS is defined for the corresponding two-sided time set $\bar{\mathbb{T}}$ (this is justified by the procedure of 'natural extension', see [Arn98, Appendix A.1]).

(ii) The metric DS θ associated to φ is ergodic.

(iii) The probability space $(\Omega, \mathcal{F}, \mu)$ is complete.

(iv) The state space X is a complete separable metric space (hence a Polish space) equipped with its Borel σ-algebra $\mathcal{B}(X)$.

(v) The RDS φ is continuous, i.e. for each fixed $\omega \in \Omega$ the mapping

$$\varphi\colon \mathbb{T} \times X \to X, \qquad (t,x) \mapsto \varphi(t,\omega)x$$

is continuous.

If φ is an RDS defined for two-sided time then we have that (see [Arn98, Theorem 1.1.6]) for all $(t,\omega) \in \mathbb{T} \times \Omega$

$$\varphi(t,\omega)^{-1} = \varphi(-t, \theta_t\omega)$$

and with Assumption 1.1.4 (v), $\varphi(t,\omega)\colon X \to X$ is a homeomorphism.

Products of random mappings

In discrete time an RDS is generated by a so-called *product of random mappings*. Let $(\Omega, \mathcal{F}, \mu, \theta)$ be a metric dynamical system with discrete time $\mathbb{T}$, and $f\colon \Omega \times X \to X$ be measurable. Then the mappings $\varphi(n,\omega)\colon X \to X$ given by

$$\varphi(n,\omega) := \begin{cases} f(\theta^{n-1}\omega) \circ \cdots \circ f(\omega), & n \geq 1, \\ \mathrm{id}_X, & n = 0 \end{cases}$$

define a one-sided time random dynamical system. The RDS is two-sided iff the mappings $f(\omega)\colon X \to X$ are measurably invertible for all $\omega \in \Omega$, and then one additionally has

$$\varphi(n,\omega) := f(\theta^n\omega)^{-1} \circ \cdots \circ f(\theta^{-1}\omega)^{-1}, \qquad n \leq -1.$$

Stochastic differential equations

When dealing with RDS generated by (one-dimensional) stochastic differential equations (SDE) driven by the Wiener process, the following metric dynamical system is used. Let $\Omega := C_0(\mathbb{R},\mathbb{R}) := \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, and let Ω be equipped with the compact-open topology and the Borel σ-algebra $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$. Let μ denote the Wiener probability measure on $(\Omega,\mathcal{F})$. The metric dynamical system is then given by the Wiener shift $\theta : \mathbb{R}\times\Omega \to \Omega$, defined by $\theta(t,\omega(\cdot)) := \omega(\cdot + t) - \omega(t)$, and it is well-known that θ is ergodic [Arn98, Appendix A.3]. On $(\Omega,\mathcal{F})$, we have the natural filtration

$$\mathcal{F}_s^t := \sigma\{\omega(u) - \omega(v) : s \le u, v \le t\} \quad \text{for all } s \le t\,,$$

with $\theta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}$. The independent increments property of the Wiener process means that the filtrations $\mathcal{F}_s^t$ and $\mathcal{F}_u^v$ for $s < t \le u < v$ are independent.

Set-valued functions, invariant sets

Since RDS are nonautonomous systems one would not, in general, expect there to exist a set $D \subset X$ which is invariant under the cocyle φ for all $t \in \mathbb{T}$ and all $\omega \in \Omega$, i.e. such that $\varphi(t,\omega)^{-1}D = D$. Instead one considers set valued functions $D\colon \Omega \to 2^X$, $\omega \mapsto D(\omega)$, that transform in a 'stationary' way under the cocycle, which means that $\varphi(t,\omega)^{-1}D(\theta_t\omega) = D(\omega)$, and call this property invariance. One may identify a set valued mapping with its *graph*,

$$\operatorname{graph}(D) := \{(\omega, x) \in \Omega \times X \colon x \in D(\omega)\}.$$

In the other direction, for a subset $D \subset \Omega \times X$ we call the set valued mapping $D(\omega) := \{x \in X \colon (\omega, x) \in D\}$ the *ω-sections* or *ω-fibers* of D. In general we shall say that D has some topological property if $D(\omega)$ has that property for all $\omega \in \Omega$. The formal definitions of invariant sets for an RDS are as follows.

Definition 1.1.5. Let φ be an RDS and consider a set $D \subset \Omega \times X$.

(i) D is called *forward invariant* if for all $\omega \in \Omega$ and $t \geq 0$

$$\varphi(t,\omega)D(\omega) \subset D(\theta_t\omega).$$

(ii) D is called *backward invariant* if for all $\omega \in \Omega$ and $t \geq 0$

$$\varphi(t,\omega)^{-1}D(\theta_t\omega) \subset D(\omega),$$

or equivalently for two sided time, if for all $\omega \in \Omega$ and $t \leq 0$

$$\varphi(t,\omega)D(\omega) \subset D(\theta_t\omega).$$

(iii) D is called *invariant* if for all $\omega \in \Omega$ and $t \in \mathbb{T}$

$$D(\omega) = \varphi(t,\omega)^{-1}D(\theta_t\omega),$$

or equivalently for two-sided time,

$$\varphi(t,\omega)D(\omega) = D(\theta_t\omega).$$

The above notions of invariance of $D \subset \Omega \times X$ for an RDS φ equate to the usual notions of invariance of the set D under the skew product Θ, for example D is invariant under φ iff $\Theta_t^{-1}D = D$ for all $t \in \mathbb{T}$. The following lemma gives an elementary property of forward invariant sets.

Lemma 1.1.6 ([CDS04] Lemma 4.1). *Let $D \in \Omega \times X$ be a forward invariant set for an RDS φ, then for every $s \leq t$ we have*

$$\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subset \varphi(s,\theta_{-s}\omega)D(\theta_{-s}\omega).$$

Proof. Using the cocycle property and forward invariance we have

$$\begin{aligned}\varphi(s+t-s,\theta_{-t}\omega)D(\theta_{-t}\omega) &= \varphi(s,\theta_{-s}\omega)\varphi(t-s,\theta_{-t}\omega)D(\theta_{-t}\omega)\\ &\subset \varphi(s,\theta_{-s}\omega)D(\theta_{t-s}\theta_{-t}\omega)\\ &= \varphi(s,\theta_{-s}\omega)D(\theta_{-s}\omega).\end{aligned}$$

□

Random sets

Let (Y,d) be an arbitrary metric space. The *Hausdorff semi-distance* between two subsets of Y, $\operatorname{dist}\colon 2^Y \times 2^Y \to \mathbb{R}_0^+ \cup \{\infty\}$, is defined as follows: for two nonempty subsets $A, B \subset Y$,

$$\operatorname{dist}(A,B) := \sup_{a\in A}\inf_{b\in B} d(a,b),$$

$\operatorname{dist}(\emptyset,\emptyset) := 0$, $\operatorname{dist}(\emptyset, A) := 0$ and $\operatorname{dist}(A,\emptyset) := \infty$. For a singleton set $A = \{a\}$ we write $\operatorname{dist}(a,B) := \operatorname{dist}(A,B)$ etc. We also define the function $\tilde{d}\colon 2^Y \times 2^Y \to \mathbb{R}_0^+$ by

$$\tilde{d}(A,B) := \inf_{a\in A, b\in B} d(a,b),$$

with $\tilde{d}(A,\emptyset) = \tilde{d}(\emptyset, A) := 0$ and $\tilde{d}(\emptyset,\emptyset) := 0$.

In order to talk about convergence to invariant sets under the flow of a random dynamical system in a well defined probabilistic way, we require the notion of a *random set*. Such a notion is also known as a *measurable multifunction*, and the classic reference for the analytical theory of such set-valued functions is Castaing and Valadier [CV77]; here and in Appendix A we mainly follow the exposition by Crauel [Cra02b, Chapters 1 & 2].

Definition 1.1.7. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. A set valued map $D : \Omega \to 2^X$ taking values in the subsets of a Polish space X is said to be a *random set* if the mapping $\omega \mapsto \operatorname{dist}(x, D(\omega))$ is measurable for each $x \in X$, where dist is defined with respect to some complete metric on X. If C is a random set such that $C(\omega)$ is a closed set for each $\omega \in \Omega$, then C is called a *closed random set*. A random set U is called an *open random set* if its complement, i.e. the mapping $\omega \mapsto U^c(\omega) := (U(\omega))^c$, is a closed random set. A random set K for which $K(\omega)$ is compact for each $\omega \in \Omega$ will be called a *compact random set*.

Remark 1.1.8. An open random set U is indeed also a random set, i.e. the distance mapping $\omega \mapsto \operatorname{dist}(x, U(\omega))$ is measurable for each $x \in X$ (Proposition A.0.7 (xi)). However, a set valued map V for which $V(\omega)$ is open for all $\omega \in \Omega$ and the mapping $\omega \mapsto \operatorname{dist}(x, V(\omega))$ is measurable for each $x \in X$ is not necessarily an open random set; see [Cra02b, Remark 2.11 (i)] for an example.

The notion of measurability in the definition of a closed random set is slightly stronger than that of a set D being a product measurable set, i.e. $D \in \mathcal{F} \otimes \mathcal{B}$, such that the ω-sections are closed for all $\omega \in \Omega$; in that case $\omega \mapsto \operatorname{dist}(x, D(\omega))$ is generally not $\mathcal{F}$-measurable, but is measurable with respect to a larger σ-algebra, the *universal completion* of $\mathcal{F}$, which we label $\mathcal{F}^u$ (Definition A.0.1). In the case that $(\Omega, \mathcal{F}, \mu)$ is a complete probability space then $\mathcal{F}^u$ coincides with $\mathcal{F}$ (see Remark A.0.2). We list some basic theorems and facts relating to random sets in Appendix A, which are largely from [Cra02b, Chapters 1 & 2] and [Chu02, Section 1.3]. A useful theorem to note is the Projection Theorem A.0.6 which will be used in proving measurability of certain objects; this theorem only guarantees $\mathcal{F}^u$-measurability. Here we will assume that the probability space $(\Omega, \mathcal{F}, \mu)$ is complete (Assumption 1.1.4), but will make the distinction of $\mathcal{F}^u$-measurability clear in proofs. In the non-complete case Lemmas A.0.8 and A.0.9 may be used to replace $\mathcal{F}^u$-measurable objects with almost equal $\mathcal{F}$-measurable ones. Another useful theorem to note is the Representation Theorem A.0.4, which gives a convenient way of representing a closed random set.

We will use the following definitions of closed and open balls with a random radius, which are closed and open, respectively, random sets that are measurable with respect to $\mathcal{F}^u$ (see Remark A.0.3):

Definition 1.1.9. Given a nonnegative random variable $\eta\colon \Omega \to \mathbb{R}_0^+$ we define the *closed random ball* around a random set D of radius η, using the Hausdorff semi-distance, by

$$\bar{B}_{\eta(\omega)}(D(\omega)) := \{x \in X\colon \operatorname{dist}(x, D(\omega)) \leq \eta(\omega)\},$$

and the *open random ball* by

$$B_{\eta(\omega)}(D(\omega)) := \{x \in X : \operatorname{dist}(x, D(\omega)) < \eta(\omega)\}.$$

Stationary measures, invariant measures, disintegration, random fixed point

For an RDS φ one would not, in general, expect there to exist a probability measure ρ on the state space $(X, \mathcal{B})$ which is invariant under φ in the sense that $\varphi(t, \omega)\rho(B) := \rho(\varphi(t, \omega)^{-1}B) = \rho(B)$ for all $(t, \omega, B) \in \mathbb{T} \times \Omega \times \mathcal{B}$. There are two notions of an 'invariant' measure for RDS. If the RDS induces a Markov semigroup (as is the case for RDS generated by products of independent and identically distributed random maps or time-homogeneous stochastic differential equations driven by a Wiener process, see [Arn98, Chapter 2]) with transition probabilities $\mathcal{P}\colon X \times \mathcal{B} \to [0, 1]$, then a probability measure ρ on the state space $(X, \mathcal{B})$ is called *stationary* if

$$\rho(B) = \int_X \mathcal{P}(x, B)\mathrm{d}\rho(x) \quad \text{for all } B \in \mathcal{B}.$$

On the other hand, the following definition is a natural extension of the notion of an invariant measure to random dynamical systems.

Definition 1.1.10. Given an RDS φ with state space $(X, \mathcal{B})$, time set $\mathbb{T}$, over a metric DS $(\Omega, \mathcal{F}, \mu, \theta)$, and with associated skew product Θ, a probability measure ν on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ is called *invariant* if

(i) $\Theta_t \nu = \nu$ for all $t \in \mathbb{T}$

(ii) $\Pi_\Omega \nu = \mu$

where Π_Ω denotes the projection onto Ω.

Condition (ii) in the above definition is imposed because the marginal measure $\Pi_\Omega \nu$ on $(\Omega, \mathcal{F})$ is necessarily a θ-invariant measure, and the noise system is considered to be an external influence on the dynamics, which is already equipped with the invariant measure μ.

Given a probability measure ν on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ there exists (by [Arn98, Proposition 1.4.3], and in consideration of Assumption 1.1.4) a μ-almost surely unique *disintegration* of ν with respect to μ, that is, a function $\nu_{\cdot}(\cdot)\colon \Omega \times \mathcal{B} \to [0,1]$ such that

(i) $\omega \mapsto \nu_\omega(B)$ is $\mathcal{F}$-measurable,

(ii) ν_ω is μ-almost surely a probability measure on $(X, \mathcal{B})$,

(iii) for all $A \in \mathcal{F} \otimes \mathcal{B}$ one has

$$\nu(A) = \int_\Omega \nu_\omega(A(\omega))\mathrm{d}\mu(\omega).$$

Then Definition 1.1.10 (i) may be equivalently characterised by the relation (see [Arn98, Theorem 1.4.5])

$$\varphi(t,\omega)\nu_\omega = \nu_{\theta_t\omega} \qquad \text{for all } t \in \mathbb{T}, \mu\text{-a.s.} \tag{1.1.1}$$

If an invariant measure ν is supported by a random variable $a\colon \Omega \to X$ then ν is called a *random Dirac measure* and we write $\nu_\omega = \delta_{a(\omega)}$. By (1.1.1), one has $\varphi(t,\omega)a(\omega) = a(\theta_t\omega)$, and hence the graph of a is invariant under φ, and one also refers to $\{a(\omega)\}_{\omega\in\Omega}$ as a *random fixed point*.

Topological equivalence

The following notion of topological equivalence for RDS extends that for classical dynamical systems; here one considers a conjugacy that depends on the noise.

Definition 1.1.11. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\theta : \mathbb{T} \times \Omega \to \Omega$ a metric dynamical system and (X_1, d_1), (X_2, d_2) be metric spaces. Then two random dynamical systems $\varphi_1 : \mathbb{T} \times \Omega \times X_1 \to X_1$ and $\varphi_2 : \mathbb{T} \times \Omega \times X_2 \to X_2$, both over θ, are called *topologically equivalent* if there exists a *random conjugacy* (or *random coordinate transformation*) $h : \Omega \times X_1 \to X_2$ fulfilling the following properties:

(i) For almost all $\omega \in \Omega$, the mapping $x \mapsto h(\omega, x)$ is a homeomorphism from X_1 to X_2.

(ii) The mappings $(\omega, x_1) \mapsto h(\omega, x_1)$ and $(\omega, x_2) \mapsto h^{-1}(\omega, x_2)$ are measurable.

(iii) The random dynamical systems φ_1 and φ_2 are *cohomologous*, i.e.

$$\varphi_2(t,\omega,h(\omega,x)) = h(\theta_t\omega,\varphi_1(t,\omega,x)) \quad \text{for all } x \in X_1 \text{ and almost all } \omega \in \Omega\,.$$

Linear RDS, projected linear RDS

Definition 1.1.12. An RDS $\varphi\colon \mathbb{T}\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$ is called *linear* if for all vectors $x, y \in \mathbb{R}^d$ and all scalars $\alpha, \beta \in \mathbb{R}$, one has $\varphi(t,\omega)(\alpha x + \beta y) = \alpha\varphi(t,\omega)x + \beta\varphi(t,\omega)y$ for all $(t,\omega) \in \mathbb{T}\times\Omega$. For such a linear RDS φ there exists a corresponding matrix valued function $\Phi\colon \mathbb{T}\times\Omega \to \mathbb{R}^{d\times d}$ such that $\Phi(t,\omega)x = \varphi(t,\omega)x$ for all $(t,\omega,x) \in \mathbb{T}\times\Omega\times\mathbb{R}^d$, and we identify the RDS φ with Φ.

We always take $\mathbb{R}^d$ to be equipped with the Euclidean norm $\|\cdot\|$, and the associated metric $d(\cdot,\cdot)$ and Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. The real projective space of $\mathbb{R}^d$ is the quotient space $\mathbb{P}^{d-1} := (\mathbb{R}^d\setminus\{0\})/\sim$, where $x \sim y$ if $x = \alpha y$ for some $\alpha \in \mathbb{R}\setminus\{0\}$. We denote by $\mathbb{P}(x)$ the equivalence class of $x \in \mathbb{R}^d\setminus\{0\}$, and for $A \subset \mathbb{P}^{d-1}$, denote $\mathbb{P}^{-1}A := \{x \in \mathbb{R}^d\colon \mathbb{P}(x) \in A\}$. We equip $\mathbb{P}^{d-1}$ with the metric $d_{\mathbb{P}}\colon \mathbb{P}^{d-1}\times\mathbb{P}^{d-1} \to [0,\sqrt{2}]$, defined by

$$d_{\mathbb{P}}(\hat{x},\hat{y}) := \min\left\{\left|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right|, \left|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right|\right\}$$

for $\hat{x},\hat{y} \in \mathbb{P}^{d-1}$, and where $x \in \mathbb{P}^{-1}\{\hat{x}\}$ and $y \in \mathbb{P}^{-1}\{\hat{y}\}$ are arbitrary nonzero vectors. Then $(\mathbb{P}^{d-1}, d_{\mathbb{P}})$ is a compact metric space. The notation $\operatorname{dist}_{\mathbb{P}}$ denotes the Hausdorff semi-distance on $\mathbb{P}^{d-1}$ with respect to the metric $d_{\mathbb{P}}$.

A linear RDS $\Phi\colon \mathbb{T}\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$ induces an RDS $\mathbb{P}\Phi\colon \mathbb{T}\times\Omega\times\mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ on the real projective space called the *projected linear RDS*, given by

$$\mathbb{P}\Phi(t,\omega)\mathbb{P}(x) := \mathbb{P}(\Phi(t,\omega)x) \tag{1.1.2}$$

for every $(t,\omega,x) \in \mathbb{T}\times\Omega\times\mathbb{R}^d\setminus\{0\}$ (for further details see [Arn98, Section 6.2]).

Chapter 2

Random attractor-repeller pairs

This chapter is concerned with attractor-repeller pairs of random dynamical systems on compact metric spaces. We survey different types of attractor-repeller pairs from the literature that arise from three different concepts of attraction for random dynamical systems. The main goal of this chapter is to find an attractor-repeller pair with suitable properties in order to prove an analogue of Selgrade's Theorem, which will be achieved in the next chapter.

We begin in Section 2.1 with a brief overview of attractors and attractor-repeller pairs in autonomous and nonautonomous systems. Then in Section 2.2 we define the three concepts of attraction for RDS that we will consider here: *forward*, *pullback* and *weak*. In Section 2.3 we recall definitions and results on so-called weak attractor-repeller pairs from Ochs [Och99] and Crauel et al. [CDS04]; this is the most basic type of attractor-repeller pair in the sense that the other types considered here are also weak attractor-repeller pairs. The other types of attractor-repeller pairs involve local pullback attractors, and we prepare for them in Section 2.4 with some results on pullback Ω-limit sets, which are the nonautonomous analogues of ω-limit sets and may be used to characterize local pullback attractors. In Section 2.5 we present the construction of so-called pullback attractor-repeller pairs, which was first demonstrated by Liu in the collection of papers [Liu06, Liu07a, Liu07b]. We then give a series of examples in Section 2.6 to illustrate the nature of the different types of attractor-repeller pairs. The first of these examples is a counter-example to [LJS08, Lemma 4.3] on the pullback dynamics of a local pullback attractor; but a weaker statement still holds [LSZ08, Remark 3.3 (ii)] (see Remark 2.5.13 (i)). The other examples demonstrate the nonequivalence of the three local attractor concepts, and show that pullback attractor-repeller pairs may have some undesirable properties (see Remark 2.5.13 (ii)). In Section 2.7 we extend the idea of so-called past and future attractor-repeller pairs for nonautonomous systems [Ras06] to the setting of random dynamical systems. We then define *strong attraction* which combines pullback and forward attraction; in Subsection 3.3.2 it will be shown that the properties of a local strong attractor are sufficient to obtain an analogue of Selgrade's Theorem using similar techniques used for so-called all-time attractor repeller pairs in nonautonomous systems [Ras08]. Finally, in Section 2.9 we discuss the results of this chapter and future work.

2.1 Attractors for autonomous and nonautonomous dynamical systems

Here we give an informal overview of the concept of an attractor. We begin with the case of autonomous dynamical systems and discuss fundamental results on attractor-repeller pairs. Broadly speaking, an attractor is an invariant compact subset of the state space of a dynamical system, which is in some sense stable, so that it 'attracts' other points in the state space. An attractor may be thought of as a generalization of a stable fixed point or limit cycle. There have been many different definitions of attractors proposed in the literature, and a brief historical account may be found in [Mil85]. One of the main differences noted there is the notion of stability associated to the attractor, for example asymptotic stability versus Lyapunov stability. Another distinction is whether there is attraction of individual points from some subset of the state space, which gives the notion of a *point attractor*, or attraction of a family of bounded (or compact) sets, which gives the notion of a *set attractor* (of course, a set attractor is also a point attractor). In [Mil85], the author considers a point attractor for which the set of points that converge to the attractor are required to have positive Lebesgue measure, and this object is known as a *Milnor* or *measure attractor*. If the attractor attracts all points or all compact sets in the entire state space then one speaks of a *global point attractor* or *global set attractor*, respectively, whereas if attraction only holds within some open neighbourhood of the attractor, one speaks of a *local point attractor* or *local set attractor*, respectively.

Charles Conley, in the monograph [Con78], proved a number of fundamental results relating to local attractors of dynamical systems on compact metric spaces. In what follows let $\phi\colon \mathbb{R} \times X \to X$ denote a continuous dynamical system on a compact metric space (X, d), and let $\omega(D)$ and $\alpha(D)$ denote the ω-limit set and α-limit set of $D \subset X$, respectively. Conley's definition of a *local attractor* of ϕ is a compact invariant set A such that there is an open neighbourhood $U \supset A$ with $\omega(U) = A$ (such an attractor is both Lyapunov and asymptotically stable). The dual of ϕ is the DS $\phi^-(t, x) := \phi(-t, x)$, for all $t \in \mathbb{R}$ and $x \in X$, called the *reverse time system.* A *local repeller* is the dual concept of a local attractor, that is, it is a local attractor of the DS ϕ^-. The existence of a local attractor A implies the existence of a corresponding local repeller, given by $R := \{x \in X\colon \omega(x) \cap A = \emptyset\}$; also $A = \{x \in X\colon \alpha(x) \cap R = \emptyset\}$, and hence A and R are dual.

The pair (A, R), called an *attractor-repeller pair*, gives a coarse but simple description of the dynamics: the ω-limit set of any point in $X\backslash R$ is contained in A, and the α-limit set of any point in $X\backslash A$ is contained in R. Conley also showed that there exists a Lyapunov function on X that takes the value 1 on R, 0 on A, and is decreasing along trajectories of points in $X\backslash(A \cup R)$. In particular, if a dynamical system on a noncompact space has a global attractor, then the asymptotic dynamics can be further analysed by attractor-repeller pair decomposition on the global attractor. Furthermore, Conley demonstrated that by taking the intersection $\cap_i(A_i \cup R_i)$ over all the attractor-repeller pairs (A_i, R_i) for ϕ gives the chain recurrent set of ϕ, and there exists a so-called complete Lyapunov function, which is constant on each of the connected components of the chain recurrent set, and decreases along trajectories of all other points. Hence, the state space is decomposed into points which are chain recurrent and points for which trajectories are gradient-like; this decomposition result has been called *the fundamental theorem of dynamical systems* [Nor95]. Another important concept is that of a *Morse decomposition*: for a finite nested sequence of attractors, $\emptyset =: A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n := X$ (which implies that the corresponding local repellers satisfy $X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset$), the (nonempty) sets $M_i := A_i \cap R_{i-1}$, $i \in \{1, \ldots, n\}$ are called *Morse sets* and the collection $\{M_1, \ldots, M_n\}$ a *Morse decomposition*. Precisely when the chain recurrent set has a finite number of connected components, these components are given by the Morse sets of a so-called finest Morse decomposition.

An important result in the theory of Morse decompositions is Selgrade's Theorem ([Sel75], see also [CK00] Chapter 5), which establishes the existence of a unique finest Morse decomposition for the projected flow of a linear flow on a vector bundle for which the base flow is chain transitive.

There is also the important notion of the Conley index, for which we refer the reader to the introductory article [Mis99]. This article also discusses the notion of continuation of attractor-repeller pairs, which is a result on the stability of attractor-repeller pairs under perturbations of the dynamical system. Conley's results were adapted to discrete time dynamical systems by Franks ([Fra88]) and extended to the case of noncompact metric spaces and semi-dynamical systems by Rybakowski [Ryb87] and Hurley [Hur91, Hur92, Hur95, Hur98]. For articles giving a more detailed introduction to the topics mentioned here, we refer the reader to [ACMC$^+$06] and [Mis99].

Conley's definition of a local attractor given above is equivalent to the one below, which serves as the basis for the definitions of local attractors for RDS that we will consider here.

Definition 2.1.1. Let $\phi\colon \mathbb{R} \times X \to X$ be a dynamical system on a compact metric space (X, d). A closed invariant set A is called a *local attractor* of ϕ if there exists an open neighbourhood $U \supset A$ such that

$$\lim_{t\to\infty} \operatorname{dist}(\phi_t U, A) = 0.$$

The concept of an attractor-repeller pair has also been extended to nonautonomous dynamical systems. In the case of a skew product flow with a compact base space and compact state space, the methods for the autonomous case may be applied on the product space. In the general nonautonomous case the base space is the time axis and hence noncompact, and so one cannot directly apply the methods used for autonomous systems. Rasmussen [Ras06] has demonstrated that by restricting to only the past or the future time domain of a nonautonomous system with suitable notions of local attraction and repulsion, one can obtain so-called past attractor-repeller pairs or future attractor-repeller pairs. Furthermore, in the case of linear nonautonomous dynamical systems, if there exists a so-called uniform attractor then there exists also a corresponding uniform repeller, and the pair is called an all-time attractor-repeller pair. It is then possible to prove analogues of

Selgrade's theorem using past, future and all-time attractor-repeller pairs [Ras07, Ras08]. (For a comprehensive exposition on past, future and all-time attractor-repeller pairs and Morse decompositions see [KR11, Chapters 3,4].)

2.2 Attraction concepts for random dynamical systems

An RDS is a nonautonomous system, which gives an additional degree of freedom and leads to separate attraction concepts for different time domains of the system. Moreover, the RDS comes equipped with a probability measure and so we may consider convergence with respect to the different notions of probabilistic convergence (here we only consider almost sure convergence and convergence in probability).

The most natural definition of set attraction for an RDS φ is the following:

Definition 2.2.1. A random set A is said to *forward attract* a random set D if

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\omega)D(\omega), A(\theta_t\omega)) = 0 \qquad \mu\text{-a.s.} \tag{2.2.1}$$

The notion of *pullback attraction* is a more abstract concept.

Definition 2.2.2. For given $(t,\omega) \in \mathbb{T}\times\Omega$, the *pullback mapping* corresponding to an RDS φ is the mapping

$$\varphi(t,\theta_{-t}\omega)\colon X \to X.$$

The pullback mapping describes the dynamics starting from a past value of the noise, $\theta_{-t}\omega$, up to the present time (recall that if $\mathbb{T}$ is one-sided, we still assume that θ is defined on the corresponding two-sided time set $\bar{\mathbb{T}}$).

Definition 2.2.3. A random set A is said to *pullback attract* a random set D if

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)) = 0 \qquad \mu\text{-a.s.} \tag{2.2.2}$$

The mathematical advantage of starting from earlier times and evolving up to the present time is that one is converging to a fixed set. The disadvantage is that pullback attraction does not generally imply forward attraction (and vice versa), and hence may not truly describe the asymptotic dynamics of the RDS, although it does imply forward attraction in the weaker sense described below.

Weak attraction uses the weaker notion of convergence in probability as opposed to almost sure convergence.

Definition 2.2.4. A random set A is said to *weakly attract* a random set D if

$$\lim_{t\to\infty} \mu\{\omega\colon \operatorname{dist}(\varphi(t,\omega)D(\omega), A(\theta_t\omega)) > \varepsilon\} = 0 \qquad \text{for every } \varepsilon > 0. \tag{2.2.3}$$

Remark 2.2.5. (i) Let A and D be random sets. Since θ preserves μ, one has

$$\mu\{\operatorname{dist}(\varphi(t,\omega)D(\omega), A(\theta_t\omega)) > \varepsilon\} = \mu\{\operatorname{dist}(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)) > \varepsilon\}, \tag{2.2.4}$$

and so if A weakly attracts D, then

$$\lim_{t\to\infty} \mu\{\omega\colon \operatorname{dist}(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)) > \varepsilon\} = 0 \qquad \text{for every } \varepsilon > 0, \tag{2.2.5}$$

hence, one may say that A both 'forward attracts D in probability' and 'pullback attracts D in probability'.

(ii) Note that since almost sure convergence implies convergence in probability, forward and pullback attraction both imply weak attraction, and in particular pullback attraction implies weak forward attraction (recall also that convergence in probability implies almost sure convergence of a subsequence).

(iii) Forward, pullback, and weak *repulsion* are all defined similarly with $\lim_{t\to-\infty}$; forward repulsion requires $\mathbb{T}$ to be two-sided, whilst in the case of weak repulsion for one-sided time, one may consider weak repulsion in the pullback sense as described in (i).

We make the following distinctions relating to the time set $\mathbb{T}$ of an RDS φ:

(i) The *past* of φ corresponds to the time domain $\mathbb{T}_0^-$,

(ii) the *future* of φ corresponds to the time domain $\mathbb{T}_0^+$,

(iii) the *entire time* of φ corresponds to the time domain $\mathbb{T}$.

Pullback attraction and forward repulsion are then considered to be attraction concepts for the past of the RDS, as they only involve the dynamics of φ on $\mathbb{T}_0^-$. Similarly, forward attraction and pullback repulsion are considered to be attraction concepts for the future of the RDS. Weak attraction and repulsion are considered attraction concepts for the entire time, since by Remark 2.2.5 (i) we may take a forward or a pullback limit.

Examples using these three attraction concepts are compared in the papers [Sch02] (for SDEs) and [AO03] (for discrete time RDS), where it is shown that only the relationships of forward attraction and pullback attraction implying weak attraction hold in general (see also [Cra02a]). In Section 2.6 we also demonstrate this fact with some examples of projected linear discrete time random dynamical systems. We also note that a wider range of attraction concepts for RDS are studied and compared in the paper [AO03], including Milnor-type random attractors.

Global attractors for RDS have mainly been studied in the sense of pullback attraction, and were first considered in the works [Sch92] and [CF94]. Global attractors in the sense of weak attraction were first studied in [Och99]. In [AS01a] the authors show that there exists a certain weakly attracting global attractor if and only if there exists a random Lyapunov function whose zero set is equal to the attractor. The notions of chain recurrence, Lyapunov functions and the Conley index have been studied in the context of pullback attraction in [Liu06, Liu07a, Liu07b, LJS08, Liu08]. Here we study constructions and properties of attractor-repeller pairs on compact metric spaces corresponding to local random attractors defined with respect to the three attraction concepts given above.

2.3 Weak attractor-repeller pairs

Weak attractors were first considered in [Och99], with many fundamental results proved for global, local, and so-called relative weak attractors in terms of both set and point attraction, and in particular Morse decompositions by local weak set attractors are demonstrated. Further analysis of Morse decomposition by local weak set attractors is carried out in the paper [CDS04]. In this section we recall basic definitions and results on local weak set attractors from these two papers, which here shall be referred to as just local weak attractors. All of the types of attractor-repeller pairs we shall consider in this thesis are also weak attractor-repeller pairs, and hence all of the results in this section are applicable to these other attractor-repeller pair types as well. We hold the following assumptions in this section.

Assumption 2.3.1. Let (X, d) be a compact metric space (hence a Polish space), and φ a continuous RDS with two-sided time ($\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$).

Definition 2.3.2. An invariant compact random set A is called a *local weak attractor* if there exists a forward invariant open random set U with $U(\omega) \supset A(\omega)$ μ-a.s. such that each closed random set $C \subset U$ is weakly attracted to A, i.e.

$$\lim_{t\to\infty} \mu\{\text{dist}(\varphi(t,\omega)C(\omega), A(\theta_t\omega)) > \varepsilon\} = 0 \qquad \text{for every } \varepsilon > 0. \tag{2.3.1}$$

The neighbourhood U is said to be a *weak attracting neighbourhood of* A. The set

$$B(A)(\omega) := \{x \in X \colon \varphi(t,\omega)x \in U(\theta_t\omega) \text{ for some } t \geq 0\}$$

is called the *basin of attraction of* A.

Definition 2.3.3. An invariant compact random set R is called a *local weak repeller* if there exists a backward invariant open random set U with $U(\omega) \supset R(\omega)$ μ-a.s. such that each closed random set $C \subset U$ is weakly repelled to R, i.e.

$$\lim_{t\to-\infty} \mu\{\text{dist}(\varphi(t,\omega)C(\omega), R(\theta_t\omega)) > \varepsilon\} = 0 \qquad \text{for every } \varepsilon > 0.$$

The neighbourhood U is said to be a *weak repelling neighbourhood of* R. The set

$$B(R)(\omega) := \{x \in X \colon \varphi(t,\omega)x \in U(\theta_t\omega) \text{ for some } t \leq 0\}$$

is called the *basin of repulsion of* R.

We generally only state results pertaining to attractors, with analogous results holding for repellers. The first results here describe important properties of the basin of attraction.

Lemma 2.3.4 ([CDS04, Lemma 4.2])**.** *The basin of attraction of a local weak attractor A is an invariant open random set and A weakly attracts all closed random sets C such that $C(\omega) \subset B(A)(\omega)$ μ-a.s., i.e. (2.3.1) holds.*

Theorem 2.3.5 ([Och99, Theorem 1])**.** *Let A be a local weak attractor, then any invariant closed random set $C \subset B(A)$ is contained in A μ-a.s. In particular, A is (up to a set of measure zero) the maximal local weak attractor in $B(A)$.*

The following important result demonstrates that the existence of a weak attractor implies the existence of a weak repeller.

Proposition 2.3.6 ([CDS04, Proposition 5.1])**.** *Let A be a local weak attractor. Then*

$$R := X \backslash B(A)$$

is a local weak repeller with basin of repulsion $B(R) = X \backslash A$.

Note that by applying the repeller version of Theorem 2.3.5 to R in the above proposition, R is the maximal local weak repeller in $X \backslash A$. Since $R = X \backslash B(A)$, this implies the following corollary.

Corollary 2.3.7 ([CDS04, Corollary 5.1])**.** *Let A be a local weak attractor with weak attracting neighbourhood U. Then the basin of attraction $B(A)$ is independent of U up to a set of zero measure.*

To summarize the above results, given a local weak attractor A, $R := X \backslash B(A)$ is the almost surely unique, maximal local weak repeller in $X \backslash A$, and similarly, $A = X \backslash B(R)$ and is the almost surely unique, maximal local weak attractor in $X \backslash R$.

Definition 2.3.8. Let A be a local weak attractor, then the local weak repeller $R = X \backslash B(A)$ is called the *repeller corresponding to A*, and A the *attractor corresponding to R*, and (A, R) is called a *weak attractor-repeller pair*.

The following theorem shows that local weak attractors (and repellers) are invariant under random coordinate transformations.

Theorem 2.3.9 ([Och99, Theorem 4]). *Let φ be an RDS on X, and let Y be another compact metric space. Let $h\colon \Omega \times X \to Y$ be a random conjugacy from X to Y. Then if A is a local weak attractor for φ in X with basin of attraction $B(A)$, $h(A)$ is a local weak attractor for $\psi(t,\omega) := h_{\theta_t\omega}\varphi(t,\omega)h_\omega$ in Y with basin of attraction $h(B(A))$.*

The following theorem is important in obtaining Morse decompositions for RDS, and demonstrates that a nested sequence of local weak attractors leads to an oppositely nested sequence of corresponding weak repellers.

Theorem 2.3.10 ([CDS04, Theorem 5.1]). *Suppose that A_1 and A_2 are local weak attractors such that $A_1(\omega) \subsetneq A_2(\omega)$ μ-a.s., and with corresponding repellers R_1 and R_2, then $R_1(\omega) \supsetneq R_2(\omega)$ μ-a.s.*

The following definition of a *Morse decomposition* holds for all types of attractor-repeller pairs we will consider in this thesis, that is, weak, pullback, past, future or strong attractor-repeller pairs.

Definition 2.3.11. Suppose that (A_i, R_i), $i \in \{0, \dots, n\}$, are weak, pullback, past, future, or strong attractor-repeller pairs almost surely satisfying

$$\emptyset = A_0(\omega) \subsetneq A_1(\omega) \subsetneq \cdots \subsetneq A_n(\omega) = X,$$

(equivalently, $X = R_0(\omega) \supsetneq R_1(\omega) \supsetneq \cdots \supsetneq R_n(\omega) = \emptyset$). Then the family of compact random sets $\mathcal{M} := \{M_1, \dots, M_n\}$ defined by

$$M_i := A_i \cap R_{i-1}, \qquad i \in \{1, \dots, n\}$$

is called a *weak, pullback, past, future*, or *strong*, respectively, *Morse decomposition* of X, and the sets M_i are called *Morse sets*.

One can introduce a partial ordering on the equivalence class of Morse decompositions for an RDS that differ only on a set of measure zero.

Definition 2.3.12. The Morse decomposition $\mathcal{M} = \{M_1, \ldots, M_n\}$ is said to be *finer* than the Morse decomposition $\tilde{\mathcal{M}} = \{\tilde{M}_1, \ldots, \tilde{M}_m\}$, if for each $i \in \{1, \ldots, m\}$ there exists a $j \in \{1, \ldots, n\}$ such that $M_j(\omega) \subset \tilde{M}_i(\omega)$ μ-a.s. A minimal element of this partial ordering is called a *finest* Morse decomposition.

The next set of results give some basic properties of weak Morse decompositions and show that the weak Morse sets describe the asymptotic dynamics of an RDS, in terms of weak convergence, and in that they support all invariant measures.

Definition 2.3.13 ([CDS04, Definition 3.3])**.** An invariant random set M is called *isolated* if there exists an open random set U with $U(\omega) \supset M(\omega)$ μ-a.s. such that for each random variable x satisfying

$$\varphi(t, \omega)x(\omega) \in U(\theta_t\omega) \qquad \text{for all } t \in \mathbb{T}, \mu\text{-a.s.},$$

then $x(\omega) \in M(\omega)$ μ-a.s.

Lemma 2.3.14 ([CDS04, Lemma 5.1])**.** *Morse sets are invariant compact random sets, which are almost surely nonempty and pairwise disjoint, and are isolated.*

Theorem 2.3.15 ([CDS04, Theorem 5.2])**.** *Suppose that $\{M_1, \ldots, M_n\}$ is a weak Morse decomposition given by weak attractor-repeller pairs (A_i, R_i), $i \in \{1, \ldots, n\}$, and let $M := \cup_{i=1}^n M_i$. Then,*

(i) Every X-valued random variable x is both weakly attracted and repelled to the set M.

(ii) If a random variable x is weakly attracted by M_i and weakly repelled by M_j, then $i \leq j$, and $i = j$ if and only if $x(\omega) \in M_i(\omega)$ μ-a.s.

(iii) If $x_1, \ldots, x_k$ are X-valued random variables such that for some $j_0, \ldots j_k \in \{1, \ldots, n\}$, x_i is weakly repelled by $M_{j_{i-1}}$ and weakly attracted by M_{j_i}, then $j_0 \leq j_k$. Furthermore $j_0 < j_k$ if and only if $\mu(x_i \notin M) > 0$ for some i, otherwise $j_0 = \cdots = j_k$.

Theorem 2.3.16 ([Och99, Theorem 2 & Remark 11])**.** *Every φ-invariant measure is supported by the Morse sets of a weak Morse decomposition, i.e. let $\{M_1, \ldots, M_n\}$ be a weak*

Morse decomposition, $M := \cup_{i=1}^n M_i$*, and* ν *be a* φ*-invariant measure, then* $\nu_\omega(M(\omega)) = 1$ μ*-a.s.*

2.4 Pullback limit sets

In autonomous systems local attractors and repellers may be characterized in terms of ω-limit sets and α-limit sets, respectively. For nonautonomous dynamical systems there exist analogous notions of limit sets which are invariant sets in the sense of Definition 1.1.5, and in the next section they will be used as an alternative characterization of *local pullback attractors* and *local pullback repellers*. These are called *pullback* Ω*-limit sets* and *pullback* α*-limit sets* (it should be clear when the use of a capital omega refers to a pullback limit set, and when it refers to the probability space), and in this section we give some fundamental results; these results may be found in the literature in some close form, but we include them here with their proofs both for completeness and since similar techniques used in the proofs are called for later.

The material presented in this section is in a more general setting than is required for the remainder of the chapter.

Assumption 2.4.1. Let (X, d) be a complete separable metric space, and φ a continuous RDS with one- or two-sided time $\mathbb{T}$.

Definition 2.4.2. Given a set $D \subset \Omega \times X$, the *(pullback)* Ω*-limit set* of D is defined as

$$\Omega_D(\omega) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega) D(\theta_{-t}\omega)},$$

and if the time set $\mathbb{T}$ is two-sided, the *(pullback)* α*-limit set* of D is defined as

$$\alpha_D(\omega) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(-t, \theta_t\omega) D(\theta_t\omega)}.$$

(Equivalently,

$$\Omega_D(\omega) = \{x \in X : \text{ there exists a sequence of times } \{t_i\}_{i \in \mathbb{N}} \text{ with } t_i \to \infty \text{ for } i \to \infty,$$
$$\text{and a sequence of points } x_i \in D(\theta_{-t_i}\omega) \text{ such that } \lim_{i \to \infty} \varphi(t_i, \theta_{-t_i}\omega)x_i = x\},$$

and similarly for α_D.)

The following lemma gives a measurability result for pullback images of random sets (the proof follows the proof of Proposition 1.5.1 in [Chu02], which in turn takes the idea from [CF94]).

Lemma 2.4.3. *Let D be a random set. Then for each $x \in X$ the mapping*

$$(t, \omega) \mapsto \operatorname{dist}(x, \varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega))$$

is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{B}(\bar{\mathbb{R}}_0^+))$-measurable, and hence for each fixed $t \in \mathbb{T}$, $\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega)$ is also a random set.

Proof. From Proposition A.0.7 (ii), $\overline{D}$ is a closed random set. Then by the Representation Theorem A.0.4, we have that $\overline{D(\omega)} = g(\omega, Y)$ for $\omega \in \Omega\backslash E$ where $E := \{\omega\colon D(\omega) = \emptyset\}$, Y is a Polish space, and $g\colon \Omega \times Y \to X$ is a mapping such that for all $\omega \in \Omega$, $g(\omega, \cdot)$ is continuous and for all $y \in Y$, $g(\cdot, y)$ is measurable. Then for each $x \in X$ and $n \in \mathbb{N}$ we define

$$\rho_n(t, \omega) := d(x, \varphi(t, \theta_{-t}\omega)g(\theta_{-t}\omega, y_n))$$

where $\{y_n\}_{n\in\mathbb{N}}$ is a countable dense set of points in Y, and let

$$\rho(t, \omega) := \inf_{n\in\mathbb{N}} \rho_n(t, \omega).$$

By definition, the mapping $(t, \omega) \mapsto \theta_t\omega$ is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F})$-measurable, and then so is the mapping $(t, \omega) \mapsto \theta_{-t}(\omega) =: \theta_-(t, \omega)$. Then we have $\theta_-^{-1}E \in \mathcal{B}(\mathbb{T}) \otimes \mathcal{F}$ and

$$\begin{aligned}\operatorname{dist}(x, \varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega)) =& \operatorname{dist}(x, \varphi(t, \theta_{-t}\omega)\overline{D(\theta_{-t}\omega)}) \\ =& \begin{cases} \rho(t, \omega), & (t, \omega) \in (\mathbb{T} \times \Omega)\backslash\theta_-^{-1}E, \\ \infty, & (t, \omega) \in \theta_-^{-1}E. \end{cases}\end{aligned}$$

We now demonstrate that ρ_n is a $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}_0^+)$-measurable function, which then also implies that ρ is measurable, and this proves the result. It is straightforward to show that the mapping $\psi(t, \omega) := (t, \theta_-(t, \omega))$ is $(\mathcal{B}(\mathbb{T})\otimes\mathcal{F}, \mathcal{B}(\mathbb{T})\otimes\mathcal{F})$-measurable and that the mapping

$G_n(t,\omega) := (t,\omega,g(\omega,y_n))$ is $(\mathcal{B}(\mathbb{T})\otimes\mathcal{F},\mathcal{B}(\mathbb{T})\otimes\mathcal{F}\otimes\mathcal{B})$-measurable, and by definition $(t,\omega,x)\mapsto\varphi(t,\omega)x$ is $(\mathcal{B}(\mathbb{T})\otimes\mathcal{F}\otimes\mathcal{B},\mathcal{B})$-measurable. Therefore the composition $\varphi\circ G_n\circ\psi$ is $(\mathcal{B}(\mathbb{T})\otimes\mathcal{F},\mathcal{B})$-measurable, and it follows that ρ_n is a $(\mathcal{B}(\mathbb{T})\otimes\mathcal{F},\mathcal{B}(\mathbb{R}_0^+))$-measurable function. □

The following two results are based on the results [CF94, Lemma 3.2, Proposition 3.6 & Theorem 3.11], [Cra99, Lemma 3.5] and [Liu06, Theorem 3.1]. The following lemma asserts the invariance of pullback limit sets, and measurability of pullback limit sets of random sets.

Lemma 2.4.4. *Let $D\subset\Omega\times X$, then Ω_D is a closed, forward invariant set, and is also invariant if $\mathbb{T}$ is two-sided. If D is a random set then Ω_D is a closed random set.*

Proof. The fact that $\Omega_D(\omega)$ is closed for each $\omega\in\Omega$ follows from its definition as an intersection of closed sets. To show forward invariance, let $x\in\Omega_D(\omega)$, then there exist sequences $t_i\to\infty$ and $x_i\in D(\theta_{-t_i}\omega)$ such that $\lim_{i\to\infty}\varphi(t_i,\theta_{-t_i}\omega)x_i = x$. Then for $t\geq 0$, using the continuity of φ,

$$\begin{aligned}\varphi(t,\omega)x &= \varphi(t,\omega)\lim_{i\to\infty}\varphi(t_i,\theta_{-t_i}\omega)x_i\\ &= \lim_{i\to\infty}\varphi(t+t_i,\theta_{-t_i}\omega)x_i\\ &= \lim_{i\to\infty}\varphi(\tilde{t}_i,\theta_{-\tilde{t}_i}\theta_t\omega)x_i\end{aligned}$$

with $\tilde{t}_i = t+t_i$, and since $\tilde{t}_i\to\infty$ with $x_i\in D(\theta_{-\tilde{t}_i}\theta_t\omega)$ we have $\varphi(t,\omega)x\in\Omega_D(\theta_t\omega)$. Then the invariance of Ω_D for two-sided time follows from the fact that $\varphi(t,\omega)$ is then a homeomorphism on X for all $(t,\omega)\in\mathbb{T}\times\Omega$.

We now show the measurability of $\omega\mapsto\operatorname{dist}(x,\Omega_D(\omega))$ for any $x\in X$. Since,

$$\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\supset\bigcup_{t\geq u}\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\qquad\text{for } u\geq s\geq 0,$$

the Ω-limit set may be formed by taking a countable intersection, i.e.

$$\Omega_D(\omega) = \bigcap_{n\in\mathbb{N}}\overline{\bigcup_{t\geq n}\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)}. \tag{2.4.1}$$

We also have the following:

$$\operatorname{dist}\left(x, \bigcup_{t\geq\tau} \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\right) = \inf_{t\geq\tau} \operatorname{dist}(x, \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)) \tag{2.4.2}$$

for $\tau \geq 0$. Lemma 2.4.3 gives that $(t,\omega) \mapsto \operatorname{dist}(x, \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega))$ is a $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{B}(\bar{\mathbb{R}}_0^+))$-measurable function. Then in the discrete time case the infimum in (2.4.2) is over a countable set, and hence $\cup_{t\geq\tau}\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)$ is an $\mathcal{F}$-measurable random set. Then taking a countable intersection as in (2.4.1) and using Proposition A.0.7 (ii), Ω_D is an $\mathcal{F}$-measurable closed random set. We now deal with the continuous time case. For arbitrary $a \in \bar{\mathbb{R}}_0^+$ we have

$$\begin{aligned}\{\omega\colon \inf_{t\geq\tau} \operatorname{dist}(x, \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)) < a\} \\ = \Pi_\Omega\{(t,\omega)\colon \operatorname{dist}(x, \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)) < a, t \geq \tau\},\end{aligned}$$

and so by the Projection Theorem A.0.6, we have that (2.4.2) is $\mathcal{F}^u$-measurable. Then once again taking a countable intersection as in (2.4.1), gives us that $\Omega_D(\omega)$ is an $\mathcal{F}^u$-measurable closed random set. □

The following theorem gives sufficient conditions for pullback limit sets to be random sets that are nonempty and compact μ-a.s.

Theorem 2.4.5. *Let D be a random set that is nonempty μ-a.s., and such that*

$$\bigcup_{t\geq T(\omega)} \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)$$

is pre-compact μ-a.s. for some random variable $T\colon \Omega \to \mathbb{R}_0^+$. Then the following statements hold:

(i) Ω_D is a closed random set that is both nonempty and compact μ-a.s.

(ii) If $\mathbb{T}$ is one-sided, $\Omega_D(\omega)$ is forward invariant and

$$\varphi(t,\omega)\Omega_D(\omega) = \Omega_D(\theta_t\omega) \qquad \textit{for all } t \in \mathbb{T},\ \mu\textit{-a.s.},$$

whilst if $\mathbb{T}$ is two-sided then Ω_D is invariant (cf. Definition 1.1.5).

(iii) *Ω_D pullback attracts D.*

Proof. (i) The fact that Ω_D is a closed random set is given by Lemma 2.4.4. By the nonempty condition and ergodicity of θ, for μ almost all $\omega \in \Omega$ there exists a sequence of times $t_i \to \infty$ such that $D(\theta_{-t_i}\omega) \neq \emptyset$, and so combining this with the pre-compactness condition, $\Omega_D(\omega)$ is almost surely the intersection of nonempty centred compact sets, and hence is itself nonempty and compact μ-a.s.

(ii) We show that $\Omega_D(\theta_t\omega) \subset \varphi(t,\omega)\Omega_D(\omega)$ for all $t \in \mathbb{T}$, μ-a.s., which combined with the forward invariance given by Lemma 2.4.4, gives that $\varphi(t,\omega)\Omega_D(\omega) = \Omega_D(\theta_t\omega)$ μ-a.s. Let $y \in \Omega_D(\theta_t\omega)$, then $y = \lim_{i\to\infty} \varphi(t_i, \theta_{-t_i+t}\omega)y_i$ for some sequences $t_i \to \infty$ and $y_i \in D(\theta_{-t_i+t}\omega)$, and we have $y = \lim_{i\to\infty} \varphi(t,\omega)\varphi(t_i - t, \theta_{-t_i+t}\omega)y_i$. Define $\tilde{y}_i := \varphi(\tilde{t}_i, \theta_{-\tilde{t}_i}\omega)y_i$, with $\tilde{t}_i := t_i - t$, so that $\tilde{y}_i \in \varphi(\tilde{t}_i, \theta_{-\tilde{t}_i}\omega)D(\theta_{-\tilde{t}_i}\omega)$. For μ-a.e. $\omega \in \Omega$, $\overline{\cup_{s\geq T(\omega)}\varphi(s,\theta_{-s}\omega)D(\theta_{-s}\omega)}$ is compact, and so there exists a convergent subsequence $\tilde{y}_{i_j} \to \tilde{y} \in \Omega_D(\omega)$. By the continuity of φ we have that $y = \varphi(t,\omega)\tilde{y}$, and hence the assertion is proved. The invariance of Ω_D for the case of two-sided time is given by Lemma 2.4.4.

(iii) To show that $\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega), \Omega_D(\omega)) = 0$ μ-a.s., let $F \in \mathcal{F}$ be the full measure set where the pre-compactness condition holds, and assume there is an $\omega \in F$ such that the pullback attraction does not hold. Then there is a sequence of times $t_i \to \infty$ and points $x_i \in D(\theta_{-t_i}\omega)$ with $\operatorname{dist}(\varphi(t_i,\theta_{-t_i}\omega)x_i, \Omega_D(\omega)) \geq \varepsilon$ for all $i \in \mathbb{N}$ and some $\varepsilon > 0$. Since $\overline{\cup_{t\geq T(\omega)}\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)}$ is compact there is then a convergent subsequence of the points $\varphi(t_i,\theta_{-t_i}\omega)x_i$, converging to a point $x \in \overline{\cup_{t\geq T(\omega)}\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)} \backslash B_\varepsilon(\Omega_D(\omega))$, which is a contradiction. □

2.5 Pullback attractor-repeller pairs

The material in this section is an exposition on the construction of pullback attractor-repeller pairs due to Liu [Liu06, Liu07a, Liu07b]. Here we review the key results from those papers, and try to give a clear and concise account of the construction of pullback attractor-repeller pairs. We first give some basic definitions, then show how local pullback attractors may be alternatively described in terms of pullback Ω-limit sets. We then establish the independence of the basin of attraction of a local pullback attractor with respect to different pullback attracting neighbourhoods. The main result is Theorem 2.5.11 which shows that the existence of a local pullback attractor implies the existence of a corresponding local pullback repeller. Finally, we discuss some undesirable properties of these objects, which will be demonstrated by examples in the next section.

Assumption 2.5.1. Let (X, d) be a compact metric space (hence a Polish space), and φ a continuous RDS with two-sided time ($\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$).

Definition 2.5.2. An invariant compact random set A is called a *local pullback attractor* if there is an open random set U with $U(\omega) \supset A(\omega)$ μ-a.s. such that

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega), A(\omega)) = 0 \quad \mu\text{-a.s.}$$

The random set U is called a *pullback attracting neighbourhood of* A. The set

$$B(A)(\omega) := \{x \in X : \varphi(t, \omega)x \in U(\theta_t\omega) \text{ for some } t \geq 0\}$$

is called the *basin of attraction of* A. A local pullback attractor is called *trivial* if $\mu(A(\omega) \in \{\emptyset, X\}) = 1$.

Definition 2.5.3. An invariant compact random set R is called a *local pullback repeller* if there is an open random set U with $U(\omega) \supset R(\omega)$ μ-a.s. such that

$$\lim_{t\to-\infty} \operatorname{dist}(\varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega), R(\omega)) = 0 \quad \mu\text{-a.s.}$$

The random set U is called a *pullback repelling neighbourhood of* R. The set

$$B(R)(\omega) := \{x \in X : \varphi(t, \omega)x \in U(\theta_t\omega) \text{ for some } t \leq 0\}$$

is called the *basin of repulsion of* R. A local pullback repeller is called *trivial* if $\mu(R(\omega) \in \{\emptyset, X\}) = 1$.

Definition 2.5.4. Let U be an open random set. If there exists a random variable $T : \Omega \to$

$\mathbb{R}_0^+$ such that

$$\overline{\bigcup_{t \geq T(\omega)} \varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega)} \subset U(\omega) \quad \mu\text{-a.s.},$$

then U is called a *pullback absorbing set*, and T is called the *absorbtion time*.

Definition 2.5.5. Let U be an open random set. If there exists a random variable $T\colon \Omega \to \mathbb{R}_0^+$ such that

$$\overline{\bigcup_{t \geq T(\omega)} \varphi(-t, \theta_{t}\omega)U(\theta_{t}\omega)} \subset U(\omega) \quad \mu\text{-a.s.},$$

then U is called a *pullback expelling set*, and T is called the *expulsion time*.

All of the results in this section are stated for attractors only; analogous results also hold for repellers. Under Assumption 2.5.1 we may make the following modification of Theorem 2.4.5.

Theorem 2.5.6. *Let D be a random set, then $\Omega_D(\omega)$ is an invariant compact random set which pullback attracts D.*

The existence of a local pullback attractor may be guaranteed by the existence of a pullback absorbing set.

Proposition 2.5.7. *The Ω-limit set of a pullback absorbing set U is a local pullback attractor with pullback attracting neighbourhood U.*

Proof. Since U is a pullback absorbing set $\Omega_U(\omega) \subset U(\omega)$ almost surely, and by Theorem 2.5.6 Ω_U pullback attracts U. □

This demonstrates a mathematical advantage of pullback attractors over forward attractors, that one can show the existence of a pullback attractor when one has a pullback absorbing set and construct it in a fiber-wise fashion (by taking the pullback Ω-limit set), whilst there is no analogous method for forward attractors of nonautonomous systems. The following result shows that the pullback attracting neighbourhood of a local pullback attractor is a pullback absorbing set, and hence local pullback attractors may also be characterized by Ω-limit sets of pullback absorbing sets.

Proposition 2.5.8. *Let A be a local pullback attractor with pullback attracting neighbourhood U, then*

(i) $\Omega_U(\omega) = A(\omega)$ *μ-a.s., and*

(ii) U *is a pullback absorbing set.*

Proof. (i). Let $F \in \mathcal{F}$ be the set of full measure where A pullback attracts U and $U(\omega) \supset A(\omega)$, and assume that $\Omega_U(\omega) \neq A(\omega)$ for some $\omega \in F$. Then since $A(\omega) \subset \Omega_U(\omega)$ (because of the invariance of $A(\omega)$), there exists an $x \in \Omega_U(\omega) \backslash A(\omega)$, and there exist sequences $t_i \to \infty$ and $x_i \in U(\theta_{-t_i}\omega)$ such that $x = \lim_{i\to\infty} \varphi(t_i, \theta_{-t_i}\omega)x_i$. Since $A(\omega)$ is compact there is an $\varepsilon > 0$ such that $x \notin B_\varepsilon(A(\omega))$, and hence for some $N \in \mathbb{N}$, $\varphi(t_i, \theta_{-t_i}\omega)x_i \notin B_\varepsilon(A(\omega))$ for all $i \geq N$. This contradicts pullback attraction of U to A for $\omega \in F$.

(ii). Define $\eta(\omega) := \tilde{d}(A(\omega), U^c(\omega))$ which is measurable by similar arguments to the proof of Lemma A.0.10. For μ almost all $\omega \in \Omega$ there exists a time T such that $\mathrm{dist}(\varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega), A(\omega)) \leq \frac{\eta(\omega)}{2}$ for all $t \geq T$, and hence $\overline{\cup_{t\geq T}\varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega)} \subset U(\omega)$. It remains to show that there exists a measurable such T. Define

$$\begin{aligned} T(\omega) &:= \inf\left\{\tau \in \mathbb{R}_0^+ : \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega)} \subset \bar{B}_{\eta(\omega)/2}(A(\omega))\right\} \\ &= \inf\left\{\tau \in \mathbb{R}_0^+ : \mathrm{dist}\left(\overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega)}, \bar{B}_{\eta(\omega)/2}(A(\omega))\right) = 0\right\}. \end{aligned}$$

One can show that

$$\varrho(t, \omega) := \mathrm{dist}\left(\varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega), \bar{B}_{\eta(\omega)/2}(A(\omega))\right)$$

is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{B}(\bar{\mathbb{R}}_0^+))$-measurable by using similar arguments to the proofs of Lemmas A.0.10 and 2.4.3. We have

$$\mathrm{dist}\left(\overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega)}, \bar{B}_{\eta(\omega)/2}(A(\omega))\right) = \sup_{t\geq\tau} \varrho(t, \omega) =: \varrho_\tau(\omega),$$

and then for $a \in \mathbb{R}_0^+ \cup \{\infty\}$,

$$\{\omega\colon \varrho_\tau(\omega) > a\} = \Pi_\Omega\{(t,\omega)\colon \varrho(t,\omega) > a, t \geq \tau\},$$

which is an $\mathcal{F}^u$ measurable set by the Projection Theorem A.0.6. Let $F_\tau := \{\omega\colon \varrho_\tau(\omega) = 0\}$. Note that $F_{\tau_1} \subset F_{\tau_2}$ for $\tau_1 \leq \tau_2$, and hence

$$\{\omega\colon T(\omega) < a\} = \bigcup_{\tau \in \mathbb{Q}_0^+, \tau < a} F_\tau$$

so that T is $\mathcal{F}^u$ measurable. □

The following lemma, which shows that there exists a forward invariant pullback attracting neighbourhood for any given local pullback attractor, is needed to demonstrate that a local pullback attractor is also a local weak attractor (note that the definition of a local weak attractor, Definition 2.3.2, requires a forward invariant neighbourhood), and is also required for the proof of Theorem 2.5.11.

Lemma 2.5.9 ([Liu07b, Lemma 3.1]). *Let A be a local pullback attractor with pullback attracting neighbourhood U. Then there exists a forward invariant pullback attracting neighbourhood of A.*

Proof. Define

$$\tilde{U}(\omega) := \operatorname{int}\left(\overline{\bigcup_{t \geq 0} \varphi(t, \theta_{-t}\omega) U(\theta_{-t}\omega)}\right). \tag{2.5.1}$$

By Proposition 1.5.1 of [Chu02] and Proposition A.0.7 (xii), $\tilde{U}$ is an $\mathcal{F}^u$-measurable forward invariant open random set. We will show that $\Omega_{\tilde{U}}(\omega) = \Omega_U(\omega)$ for all $\omega \in \Omega$, and then by Proposition 2.5.8 and Theorem 2.5.6, we obtain that A pullback attracts $\tilde{U}$. We first note the fact that for a continuous function f and an arbitrary collection of sets $\{B_i\}$ one has

$$\overline{\cup_i f(B_i)} = \overline{\cup_i f(\overline{B_i})}. \tag{2.5.2}$$

Then using (2.5.2), we obtain

$$\Omega_{\tilde{U}}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} \varphi(s,\theta_{-s}\omega)\tilde{U}(\theta_{-s}\omega)}$$
$$= \bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} \varphi(s,\theta_{-s}\omega)\mathrm{int}\left(\overline{\overline{\bigcup_{t\geq 0} \varphi(t,\theta_{-t}\theta_{-s}\omega)U(\theta_{-t}\theta_{-s}\omega)}}\right)}.$$

Since $U(\omega)$ is open and $\varphi(t,\omega)$ is a homeomorphism it follows that the latter expression is equal to

$$\bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} \varphi(s,\theta_{-s}\omega)\overline{\overline{\bigcup_{t\geq 0} \varphi(t,\theta_{-t}\theta_{-s}\omega)U(\theta_{-t}\theta_{-s}\omega)}}}$$

and using (2.5.2) again, this is equal to

$$\bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} \varphi(s,\theta_{-s}\omega)\bigcup_{t\geq 0} \varphi(t,\theta_{-t}\theta_{-s}\omega)U(\theta_{-t}\theta_{-s}\omega)}.$$

Finally, the last expression is easily simplified to give

$$\bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau}\bigcup_{t\geq 0} \varphi(s+t,\theta_{-s-t}\omega)U(\theta_{-s-t}\omega)} = \bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} \varphi(s,\theta_{-s}\omega)U(\theta_{-s}\omega)}$$
$$= \Omega_U(\omega).$$

□

Lemma 2.5.10. *The basin of attraction $B(A)$ of a local pullback attractor A is independent, up to a zero measure set, of the pullback attracting neighbourhood. If one chooses a forward invariant pullback attracting neighbourhood of A (the existence of which is guaranteed by Lemma 2.5.9) then $B(A)$ is an invariant open random set. Furthermore, A is a local weak attractor with basin of attraction $B(A)$.*

Proof. Let $B(A,U)$ denote the basin of attraction of the local pullback attractor A defined with respect to the pullback attracting neighbourhood U of A. Given a forward invariant pullback attracting neighbourhood $\tilde{U}$ of A, since A pullback attracts $\tilde{U}$ it also weakly attracts all closed random sets in $\tilde{U}$, and so is a local weak attractor with (forward invariant) weak attracting neighbourhood $\tilde{U}$ and basin of attraction $B(A, \tilde{U})$. Lemma 2.3.4 asserts that $B(A, \tilde{U})$ is an invariant open random set.

Now consider two arbitrary pullback attracting neighbourhoods U_1 and U_2 of A. We now show that $B(A, U_1) \subset B(A, U_2)$ almost surely. Since U_1 is pullback attracted to A, by similar arguments to the proof of Proposition 2.5.8 (ii), there exists a measurable $T\colon \Omega \to \mathbb{R}_0^+$ such that

$$\overline{\bigcup_{t \geq T(\omega)} \varphi(t, \theta_{-t}\omega)U_1(\theta_{-t}\omega)} \subset U_2(\omega) \quad \mu\text{-a.s.} \tag{2.5.3}$$

Let $E \subset \Omega$ be the exceptional set in (2.5.3), and choose $N \in \mathbb{N}$ such that the set $F_N := \{\omega \in \Omega\colon T(\omega) \leq N\} \backslash E$ has $\mu(F_N) > 0$. Then by Birkhoff's ergodic theorem, there exists a full measure set $G \subset \Omega$ such that for all $\omega \in G$ there exists a sequence of times $\{t_i\}_{i \in \mathbb{N}}$ with $t_i \to \infty$ and such that $\theta_{t_i}\omega \in F_N$ for each i. By (2.5.3), for each $i \in \mathbb{N}$

$$\overline{\bigcup_{t \geq N} \varphi(t, \theta_{-t}\theta_{t_i}\omega)U_1(\theta_{-t}\theta_{t_i}\omega)} \subset U_2(\theta_{t_i}\omega).$$

Let $x \in B(A, U_1)(\omega)$, which means that there exists an $s \in \mathbb{T}_0^+$ such that $\varphi(s, \omega)x \in U_1(\theta_s\omega)$, and choose $i \in \mathbb{N}$ such that $t_i \geq N + s$. Then $\varphi(t_i - s, \theta_s\omega)U_1(\theta_s\omega) \subset U_2(\theta_{t_i}\omega)$, and hence $x \in B(A, U_2)(\omega)$, and therefore $B(A, U_1)(\omega) \subset B(A, U_2)(\omega)$ μ-a.s. Reversing the roles of U_1 and U_2 then gives the almost sure opposite inclusion, and so $B(A, U_1) = B(A, U_2)$ almost surely, and this completes the proof. □

The following result from [Liu07a] says that the existence of a local pullback attractor implies the existence of a μ-a.s. uniquely determined local pullback repeller (in the proof given here we also explicitly prove the fact that the complement of a forward invariant pullback absorbing set is a pullback expelling set).

Theorem 2.5.11 ([Liu07a] Lemma 5.1)**.** *Let A be a local pullback attractor, then $R := X \backslash B(A)$ is a local pullback repeller with basin of repulsion $X \backslash A$. Furthermore, (A, R) is also a weak attractor-repeller pair.*

Proof. Let U be the forward invariant pullback attracting neighbourhood of A given by

Lemma 2.5.9. By Proposition 2.5.8 U is a pullback absorbing set, i.e. there exists a non-negative random variable T such that

$$\overline{\bigcup_{t\geq T(\omega)} \varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega)} \subset U(\omega) \qquad \mu\text{-a.s.} \tag{2.5.4}$$

Since U is a forward invariant open random set, $V := \overline{U}^c$ is a backward invariant open random set (see [Arn98, Exercise 1.6.10] and Proposition A.0.7 (xi)). The proof is divided into four parts.

Part 1. We show that V is a pullback expelling set, and hence α_V is a local pullback repeller. Lemma 1.1.6 implies that

$$\bigcup_{t\geq T(\omega)} \varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega) = \varphi(T(\omega),\theta_{-T(\omega)}\omega)U(\theta_{-T(\omega)}\omega). \tag{2.5.5}$$

Since $\varphi(t,\omega)$ is a homeomorphism for all $(t,\omega) \in \mathbb{T}\times\Omega$, we have

$$\overline{\varphi(T(\omega),\theta_{-T(\omega)}\omega)U(\theta_{-T(\omega)}\omega)} = \varphi(T(\omega),\theta_{-T(\omega)}\omega)\overline{U(\theta_{-T(\omega)}\omega)},$$

and hence using (2.5.5) and (2.5.4),

$$\overline{U(\theta_{-T(\omega)}\omega)} \subset \varphi(-T(\omega),\omega)U(\omega).$$

Then,

$$\overline{U(\theta_{-T(\omega)}\omega)}^c \supset \varphi(-T(\omega),\omega)U(\omega)^c \supset \overline{\varphi(-T(\omega),\omega)\overline{U(\omega)}^c}$$

and so,

$$V(\theta_{-T(\omega)}\omega) \supset \overline{\varphi(-T(\omega),\omega)V(\omega)}.$$

Since V is backward invariant one can apply a backward invariance version of Lemma 1.1.6 to obtain

$$\overline{\bigcup_{t\geq T(\omega)} \varphi(-t,\theta_{t-T(\omega)}\omega)V(\theta_{t-T(\omega)}\omega)} = \overline{\varphi(-T(\omega),\omega)V(\omega)} \subset V(\theta_{-T(\omega)}\omega), \tag{2.5.6}$$

that is, V is expelling for $\theta_{-T(\omega)}\omega$. Now the idea is to show that the ergodicity of θ and measurability of T imply there is also a time in the future when V is expelling, and combining this with the backward invariance of V then means that it is truly an expelling set. Let $E \in \mathcal{F}$ be the null set in (2.5.4), and choose $N \in \mathbb{N}$ such that the set $F_N := \{\omega \in \Omega \colon T(\omega) \leq N\} \backslash E$ has $\mu(F_N) > 0$. Then by Birkhoff's ergodic theorem, there exists a full measure set $G \in \mathcal{F}$ such that for each $\omega \in G$ there exists a sequence of times $\{t_i\}_{i\in\mathbb{N}}$ with $t_i \to \infty$ such that $\theta_{t_i}\omega \in F_N$ for each i. Choose $i \in \mathbb{N}$ such that $t_i \geq N$, then using (2.5.6) we have

$$\begin{aligned} \overline{\bigcup_{t\geq T(\theta_{t_i}\omega)} \varphi(-t, \theta_{t-T(\theta_{t_i}\omega)+t_i}\omega)V(\theta_{t-T(\theta_{t_i}\omega)+t_i}\omega)} &= \overline{\varphi(-T(\theta_{t_i}\omega), \theta_{t_i}\omega)V(\theta_{t_i}\omega)} \\ &\subset V(\theta_{-T(\theta_{t_i}\omega)+t_i}\omega), \end{aligned} \tag{2.5.7}$$

Now we have,

$$\begin{aligned} \overline{\varphi(-t_i, \theta_{t_i}\omega)V(\theta_{t_i}\omega)} &= \varphi(T(\theta_{t_i}\omega) - t_i, \theta_{-T(\theta_{t_i}\omega)+t_i}\omega)\overline{\varphi(-T(\theta_{t_i}\omega), \theta_{t_i}\omega)V(\theta_{t_i}\omega)} \\ &\subset \varphi(T(\theta_{t_i}\omega) - t_i, \theta_{-T(\theta_{t_i}\omega)+t_i}\omega)V(\theta_{-T(\theta_{t_i}\omega)+t_i}\omega) \\ &\subset V(\omega) \end{aligned}$$

with the first line above using the cocycle property and that $\varphi(t, \omega)$ is a homeomorphism, the second line using (2.5.7), and the last inclusion using the backward invariance of V. Hence we have, again using the backward invariance of V, that

$$\overline{\bigcup_{t\geq t_i} \varphi(-t, \theta_t\omega)V(\theta_t\omega)} \subset V(\omega),$$

that is, almost surely there exists a time t_i such that $V(\omega)$ is expelling. Now the t_i must be replaced by a measurable function $\tilde{T} \colon \Omega \to \mathbb{R}_0^+$. We do not need a measurable $\tilde{T}$ to take the pullback α-limit set, and so first we may define $R(\omega) := \alpha_V(\omega)$ which by the expelling property and the reverse time version of Theorem 2.5.6 is a local pullback repeller with repelling neighbourhood $V(\omega)$. Then the reverse time version of Proposition 2.5.8 (ii) demonstrates that there exists a measurable $\tilde{T}$, and V is a pullback expelling set.

Part 2. In this step we show that $R = X\backslash B(A)$ μ-a.s. Since $R(\omega) \subset V(\omega)$ μ-a.s., we must have that $R(\omega) \subset X\backslash B(A)(\omega)$ μ-a.s., since otherwise there is an $x \in R(\omega) \cap B(A)(\omega)$, and by the definition of the basin of attraction there exists a $t \geq 0$ such that $\varphi(t,\omega)x \in U(\theta_t\omega)$, but by the invariance of R one also has $\varphi(t,\omega)x \in R(\theta_t\omega)$, which contradicts that $V(\theta_t\omega) \cap U(\theta_t\omega) = \emptyset$. Now we demonstrate the opposite inclusion, $R(\omega) \supset X\backslash B(A)(\omega)$ μ-a.s. We first show that $\overline{U(\omega)} \subset B(A)(\omega)$ almost surely. Let F_N be as defined in Part 1, then for almost all $\omega \in \Omega$ there exists an $s \geq N$ such that $\theta_s\omega \in F_N$ and then $\overline{\cup_{t\geq N}\varphi(t,\theta_{-t+s}\omega)U(\theta_{-t+s}\omega)} \subset U(\theta_s\omega)$, which implies that $\varphi(s,\omega)\overline{U(\omega)} \subset U(\theta_s\omega)$. Then $X\backslash B(A)(\omega) \subset V(\omega)$, and since $X\backslash B(A)(\omega)$ is invariant by Lemma 2.5.10, and $R(\omega) = \alpha_V(\omega)$, the inclusion follows.

Part 3. We show that $B(R) = X\backslash A$ μ-a.s. It is clear that $B(R)(\omega) \subset X\backslash A(\omega)$ almost surely. Now let $x \in X\backslash B(R)(\omega)$, then $\varphi(-t,\omega)x \in \overline{U(\theta_{-t}\omega)}$ for all $t \geq 0$, or equivalently,

$$x \in \varphi(t,\theta_{-t}\omega)\overline{U(\theta_{-t}\omega)}$$

Then since for almost all $\omega \in \Omega$,

$$\lim_{t\to\infty} \operatorname{dist}(A(\omega), \varphi(t,\theta_{-t}\omega)\overline{U(\theta_{-t}\omega)}) = 0$$

it follows that $x \in A(\omega)$.

Part 4. Since Lemma 2.5.10 gives that a local pullback attractor A is also a local weak attractor with the same basin of attraction, and the corresponding local pullback repeller R is the complement of the basin of attraction of A, Proposition 2.3.6 shows that (A,R) is also a weak attractor-repeller pair. □

Definition 2.5.12. The repeller R in Theorem 2.5.11 is called the *repeller corresponding to* A, and (A,R) is called a *pullback attractor-repeller pair*.

Remark 2.5.13. (i) Lemma 4.3 in [LJS08] states: Let A be a local pullback attractor and $B(A)$ its basin of attraction, then for any random closed set $C \subset B(A)$, A pullback attracts C. Example 2.6.4 in the next section is a counter-example to this statement. In [LSZ08, Remark 3.3 (ii)] the authors also make a weakened version of this statement, and this still holds: Let U be a forward invariant pullback attracting neighbourhood of A, then for any random set $D \subset B(A)$ such that there exists a deterministic time $T \geq 0$ with $\varphi(T,\omega)D(\omega) \subset U(\theta_T\omega)$ for all $\omega \in \Omega$, A pullback attracts D.

(ii) Example 2.6.9 in the next section is an RDS with a deterministic (i.e. independent of $\omega \in \Omega$) pullback attractor-repeller pair (A, R). A pullback attracts all deterministic closed sets in its (deterministic) basin of attraction, but deterministic closed sets in the basin of repulsion are not pullback repelled to R. Let V denote the pullback repelling neighbourhood of R. Since any deterministic point x arbitrarily close to R is not pullback repelled, there must be a sequence of times $t_i \to \infty$ such that $x \notin V(\theta_{t_i}\omega)$, and for this example this implies

$$\liminf_{t\to\infty} \operatorname{dist}(V(\theta_t\omega), R(\theta_t\omega)) = 0 \qquad \mu\text{-a.s.}$$

It can be seen directly in this example that with V constructed according to the proof of Theorem 2.5.11, V may be arbitrarily small with positive probability.

2.6 Examples of attractor-repeller pairs

In this section we give some examples which are designed to explore the definitions of local weak, pullback and forward (Definition 2.7.2) attractors. The examples are of projected linear RDS (as defined by (1.1.2)). They are based around a *non-tempered random variable* (see [Arn98, Section 4.1]) and were inspired by Example 1 in [AO03].

Definition 2.6.1. Let $(\Omega, \mathcal{F}, \mu, \theta)$ be an ergodic metric DS. A random variable $\chi\colon \Omega \to (0, \infty)$ is called *tempered from above* if

$$\limsup_{t\to\infty} \frac{1}{t} \ln \chi(\theta_{\pm t}\omega) = 0 \qquad \mu\text{-a.s.}$$

Remark 2.6.2. If a random variable χ as given in the above definition is not tempered from above, then it follows (see [Arn98, Proposition 4.1.3]) that

$$\limsup_{t\to\infty} \frac{1}{t} \ln \chi(\theta_{\pm t}\omega) = \infty \qquad \mu\text{-a.s.}$$

Remark 2.6.3. We follow [AO03, Theorem 2] to construct a random variable that is not tempered from above. Let $(\Omega, \mathcal{F}, \mu, \theta)$ be an ergodic metric DS with time set $\mathbb{T} = \mathbb{Z}$ and such that the probability space is nonatomic. Let $\{U_k\}_{k\in\mathbb{N}}$ be a partition of Ω with $\mu(U_k) > 0$ for all $k \in \mathbb{N}$. Then there exists a sequence $\{N_k\}_{k\in\mathbb{N}}$, $N_k \in \mathbb{N}_0$, with $\lim_{k\to\infty} N_k = \infty$, and for almost all $\omega \in \Omega$ there exists an $m \in \mathbb{N}_0$ and a sequence $\{n_k\}_{k\in\mathbb{N}}$, $n_k \in \mathbb{N}_0$ (both depending on ω), with $\lim_{k\to\infty} n_k = \infty$ such that $n_k \leq N_k$, and $\theta^{n_k+m}\omega \in U_k$. Let $\alpha > 1$ and define the random variable $\beta\colon \Omega \to [1, \infty)$ by

$$\beta(\omega) := \alpha^{N_k^2} \qquad \text{for } \omega \in U_k.$$

Then for almost all $\omega \in \Omega$ we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \ln \beta(\theta^n\omega) &= \limsup_{n\to\infty} \frac{1}{n+m} \ln \beta(\theta^{n+m}\omega) \\
&\geq \limsup_{k\to\infty} \frac{1}{n_k+m} \ln \beta(\theta^{n_k+m}\omega) \\
&= \limsup_{k\to\infty} \frac{1}{n_k+m} N_k^2 \ln \alpha \\
&= \infty,
\end{aligned}$$

and so β is not tempered from above.

We will also need a similar random variable that is not tempered from above along the negative orbit of θ: Similarly to the above, there exists a sequence $\{\tilde{N}_k\}_{k\in\mathbb{N}}$, $\tilde{N}_k \in \mathbb{N}_0$, with $\lim_{k\to\infty} \tilde{N}_k = \infty$, and for almost all $\omega \in \Omega$ there exist an $\tilde{m} \in \mathbb{N}_0$ and a sequence $\{\tilde{n}_k\}_{k\in\mathbb{N}}$, $\tilde{n}_k \in \mathbb{N}_0$ (both depending on ω), with $\lim_{k\to\infty} \tilde{n}_k = \infty$ such that $\tilde{n}_k \leq \tilde{N}_k$, and $\theta^{-\tilde{n}_k-\tilde{m}}\omega \in U_k$. Let $\alpha > 1$ and define the random variable $\tilde{\beta}\colon \Omega \to [1, \infty)$ by

$$\tilde{\beta}(\omega) := \alpha^{\tilde{N}_k^2} \qquad \text{for } \omega \in U_k.$$

First we state the counter-example mentioned in Remark 2.5.13 (ii). The idea is to take a deterministic attractor-repeller pair, and a closed random set whose values along a subsequence of the negative orbit in the base space converge to the repeller faster than the rate of attraction.

Example 2.6.4. Let $\alpha > 1$, and take the metric DS and non-tempered random variable $\tilde{\beta}$ given in Remark 2.6.3. Define the linear RDS $\Phi\colon \mathbb{Z} \times \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ by the deterministic system

$$\Phi(n, \omega) := \begin{pmatrix} \alpha^n & 0 \\ 0 & 1 \end{pmatrix} \tag{2.6.1}$$

and consider the induced RDS $\mathbb{P}\Phi$ in the projective space $\mathbb{P}^1$. Since the RDS is deterministic, the pullback mapping equals the forward mapping, i.e. $\Phi(n, \theta^{-n}\omega) = \Phi(n, \omega)$. Take $\eta \in (0, \sqrt{2})$ and define $U(\omega) := B_\eta(\mathbb{P}((1, 0)))$. Then $A(\omega) := \mathbb{P}((1, 0))$ is a local pullback (and local forward, Definition 2.7.2) attractor for $\mathbb{P}\Phi$ with basin of attraction $B(A)(\omega) = \mathbb{P}^1 \backslash \mathbb{P}((0, 1))$ and the corresponding local pullback repeller obtained by Theorem 2.5.11 is $R(\omega) = \mathbb{P}((0, 1))$. Now consider the closed random set $C(\omega) := \mathbb{P}((1, \tilde{\beta}(\omega)))$. Taking the pullback images of C along the subsequence of times $\{\tilde{n}_k + \tilde{m}\}_{k\in\mathbb{N}}$ one obtains

$$\begin{aligned} \lim_{k\to\infty} \mathbb{P}\Phi(\tilde{n}_k + \tilde{m}, \theta^{-\tilde{n}_k - \tilde{m}}\omega)C(\theta^{-\tilde{n}_k - \tilde{m}}\omega) &= \lim_{k\to\infty} \mathbb{P}((\alpha^{\tilde{n}_k + \tilde{m}}, \beta(\theta^{-\tilde{n}_k - \tilde{m}}\omega))) \\ &= \lim_{k\to\infty} \mathbb{P}((\alpha^{\tilde{n}_k + \tilde{m}}, \alpha^{\tilde{N}_k^2})) \\ &= \mathbb{P}((0, 1)), \end{aligned}$$

that is, along this subsequence the closed random set C pullback converges to the repeller, and hence C is not pullback attracted to A.

The next two examples demonstrate the nonequivalence of local forward attraction and local pullback attraction.

Example 2.6.5. Let $\alpha > 1$ and take the metric DS and nontempered random variable β described in Remark 2.6.3. Define

$$H(\omega) := \begin{pmatrix} 1 & 0 \\ 0 & \beta(\omega) \end{pmatrix},$$

Let $\Psi\colon \mathbb{Z} \times \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ be the discrete time linear RDS obtained by applying the random coordinate transformation H to Φ given by (2.6.1), that is,

$$\Psi(n,\omega) = H(\theta^n\omega)\Phi(n,\omega)H^{-1}(\omega)$$
$$= \begin{pmatrix} \alpha^n & 0 \\ 0 & \frac{\beta(\theta^n\omega)}{\beta(\omega)} \end{pmatrix} \tag{2.6.2}$$

The pullback mapping for Ψ is given by

$$\Psi(n,\theta^{-n}\omega) = \begin{pmatrix} \alpha^n & 0 \\ 0 & \frac{\beta(\omega)}{\beta(\theta^{-n}\omega)} \end{pmatrix}. \tag{2.6.3}$$

Since the second term on the diagonal of (2.6.3) is bounded by $\beta(\omega)$ it is easy to see that for the induced RDS $\mathbb{P}\Psi$ on the projective space $\mathbb{P}^1$, $A(\omega) := \mathbb{P}((1,0))$ pullback attracts the neighbourhood $U(\omega) := B_\eta(\mathbb{P}((1,0)))$, for any $\eta \in (0,\sqrt{2})$. However, A is not a local forward attractor: for the subsequence of times $\{n_k+m\}_{k\in\mathbb{N}}$, (2.6.2) becomes

$$\Psi(n_k+m,\omega) = \begin{pmatrix} \alpha^{n_k+m} & 0 \\ 0 & \frac{\alpha^{N_k^2}}{\beta(\omega)} \end{pmatrix}$$

and the images of the neighbourhood $U(\omega)$ under $\mathbb{P}\Psi$ along this subsequence grow arbitrarily large.

Example 2.6.6. Let $\alpha > 1$, and take the metric DS and non-tempered random variable $\tilde{\beta}$ given in Remark 2.6.3. Now define the linear RDS $\tilde{\Psi}\colon \mathbb{Z}\times\Omega\times\mathbb{R}^2 \to \mathbb{R}^2$ by

$$\tilde{\Psi}(n,\omega) := \begin{pmatrix} \alpha^n & 0 \\ 0 & \frac{\tilde{\beta}(\omega)}{\tilde{\beta}(\theta^n\omega)} \end{pmatrix}.$$

Since the second term on the diagonal is bounded by $\tilde{\beta}(\omega)$, one can see that A is a forward attractor of $\mathbb{P}\Psi$, with A and U defined as in Example 2.6.5. The pullback mapping along the sequence $\{\tilde{n}_k + \tilde{m}\}_{k\in\mathbb{N}}$ is

$$\tilde{\Psi}(\tilde{n}_k + \tilde{m}, \theta^{-\tilde{n}_k-\tilde{m}}\omega) = \begin{pmatrix} \alpha^{\tilde{n}_k+\tilde{m}} & 0 \\ 0 & \frac{\tilde{\beta}(\theta^{-\tilde{n}_k-\tilde{m}}\omega)}{\tilde{\beta}(\omega)} \end{pmatrix}$$

which, similarly to the lack of forward attraction in Example 2.6.5, shows that A does not pullback U under $\mathbb{P}\tilde{\Psi}$.

Remark 2.6.7. The attractor A for the (autonomous, deterministic) RDS $\mathbb{P}\Phi$ in Example 2.6.4 both forward and pullback attracts deterministic closed sets in its basin of attraction. The RDS in Examples 2.6.5 and 2.6.6 may be obtained by random coordinate transformations of $\mathbb{P}\Phi$, but do not forward, respectively pullback, attract deterministic sets in their basins of attraction. It is well known that the class of sets which are attracted by a pullback attractor (the so-called *attraction universe*, see Section 2.9) is not invariant under arbitrary random coordinate transformations [Och99, Remark 8], unlike in the case of local weak attractors (Theorem 2.3.9).

The following example shows that there exist local weak attractors that are not local forward or local pullback attractors.

Example 2.6.8. Let $\alpha > 1$, β, $\tilde{\beta}$ and the metric DS θ be as given in Remark 2.6.3 above. Define the linear RDS $\hat{\Psi}\colon \mathbb{Z} \times \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$, by

$$\hat{\Psi}(n, \omega) := \begin{pmatrix} \alpha^n & 0 & 0 \\ 0 & \frac{\beta(\theta^n\omega)}{\beta(\omega)} & 0 \\ 0 & 0 & \frac{\tilde{\beta}(\omega)}{\tilde{\beta}(\theta^n\omega)} \end{pmatrix}. \tag{2.6.4}$$

Then the set $\hat{A}(\omega) := \mathbb{P}((1, 0, 0))$ is not a local pullback or forward attractor for $\mathbb{P}\hat{\Psi}$ with the attracting neighbourhood $\hat{U}(\omega) := B_\eta(\hat{A})$, $\eta \in (0, \sqrt{2})$, since as in Examples 2.6.5 and 2.6.6, the second element on the diagonal hinders forward attraction, whilst the third element on the diagonal hinders pullback attraction. However, $\hat{A}$ is a local weak attractor with forward invariant basin of attraction $B(\hat{A}) = \mathbb{P}^2\backslash\mathbb{P}(\text{span}\{(0, 1, 0), (0, 0, 1)\})$. To see this, first note that for $n \to \infty$, α^n is arbitrarily larger than the other two terms on the diagonal of (2.6.4) with arbitrarily large probability (since θ preserves μ), and then for any deterministic closed set $C \subset B(\hat{A})$,

$$\lim_{n\to\infty} \mu(\mathrm{dist}_{\mathbb{P}}(\mathbb{P}\hat{\Psi}(n,\omega)C, \hat{A}(\omega)) > \varepsilon) = 0 \qquad \text{for all } \varepsilon > 0.$$

Define the family of deterministic closed sets

$$C_\zeta := \mathbb{P}^2 \backslash B_\zeta(\mathbb{P}(\mathrm{span}\{(0,1,0),(0,0,1)\}))$$

for $\zeta \in (0, \sqrt{2})$. Since for any closed random set $K(\omega) \subset B(\hat{A})$ and any $\delta > 0$ there exists a ζ such that $\mu(K(\omega) \subset C_\zeta) > 1 - \delta$, it follows from [Och99, Theorem 3] that $\hat{A}$ is a local weak attractor.

Finally, we revisit Example 2.6.5 in order to demonstrate the comments on pullback attractor-repeller pairs in Remark 2.5.13 (ii), and we also show that the pullback attractor-repeller pair is also a weak and a past attractor-repeller pair (which will be introduced in Section 2.7).

Example 2.6.9. Consider the RDS in Example 2.6.5.

Pullback attractor-repeller pair. Since Ψ possesses a local pullback attractor A (with pullback attracting neighbourhood U), by Theorem 2.5.11 there must also exist a corresponding pullback repeller R, given by the complement of the basin of attraction $B(A)$. All points in $\mathbb{P}^1$ except $\mathbb{P}((0,1))$ enter U in forward time, since by ergodicity there exist times $n \in \mathbb{N}$ when α^n is arbitrarily larger than $\beta(\theta^n\omega)/\beta(\omega)$, and then any point may be mapped arbitrarily close to A under $\mathbb{P}\Psi(n,\omega)$. Hence the corresponding local pullback repeller is $R = \mathbb{P}((0,1))$. The pullback mapping of Ψ in backward time is given by

$$\Psi(-n, \theta^n\omega) = \begin{pmatrix} \alpha^{-n} & 0 \\ 0 & \frac{\beta(\omega)}{\beta(\theta^n\omega)} \end{pmatrix},$$

for $n \geq 0$. For the sequence $\{n_k + m\}_{k\in\mathbb{N}}$, this becomes

$$\Psi(-n_k - m, \theta^{n_k+m}\omega) = \begin{pmatrix} \alpha^{-n_k-m} & 0 \\ 0 & \alpha^{-N_k^2}\beta(\omega) \end{pmatrix},$$

from which one can see that for any deterministic point $x \in B(R)\backslash R$, one has

$$\lim_{k\to\infty} \mathrm{dist}_{\mathbb{P}}(\mathbb{P}\Psi(-n_k - m, \theta^{n_k+m}\omega)x, A) = 0.$$

Note that one starts with a local pullback attractor that pullback attracts an open ball of constant radius, but the corresponding local pullback repeller pullback repels no open ball of constant radius. Since β is arbitrarily large with positive probability, one can see from (2.6.3) that the forward invariant pullback attracting neighbourhood $\tilde{U}$ given by (2.5.1) may be arbitrarily large with positive probability. Then the pullback repelling neighbourhood $V := \overline{\tilde{U}}^c$ used in the proof of Theorem 2.5.11 is arbitrarily small with positive probability.

Weak attractor-repeller pair. Since

$$\lim_{n\to\infty} \mu\left(\frac{\beta(\theta^n\omega)/\beta(\omega)}{\alpha^n} > \varepsilon\right) = 0 \qquad \text{for all } \varepsilon > 0$$

and

$$\lim_{n\to\infty} \mu\left(\frac{\alpha^{-n}}{\beta(\theta^{-n}\omega)/\beta(\omega)} > \varepsilon\right) = 0 \qquad \text{for all } \varepsilon > 0,$$

one can see that all deterministic closed sets in the respective basins of A and R are weakly attracted and repelled, respectively. Then similarly to Example 2.6.8 above, [Och99, Theorem 3] demonstrates that A is a local weak attractor, and R is a local weak repeller, and by Proposition 2.3.6, (A, R) is a weak attractor-repeller pair of $\mathbb{P}\Psi$.

Past attractor-repeller pair. Now consider the backward time mapping, given by

$$\Psi(-n, \omega) = \begin{pmatrix} \alpha^{-n} & 0 \\ 0 & \frac{\beta(\theta^{-n}\omega)}{\beta(\omega)} \end{pmatrix}$$

for $n \geq 0$, from which it is easy to see that R is a local forward repeller of $\mathbb{P}\Psi$ (Definition 2.7.3), with, for example, forward repelling neighbourhood $\tilde{V}(\omega) = B_\eta(R)$ for arbitrary $\eta \in (0, \sqrt{2})$. Then (A, R) is a past attractor-repeller pair of $\mathbb{P}\Psi$ (see Theorem 2.7.5).

2.7 Past and future attractor-repeller pairs

The examples in the previous section show that almost sure local attraction may not hold in both the pullback and forward sense. The ergodicity of the metric DS implies that attraction in the past and future are equivalent in terms of weak convergence, i.e. weak pullback and weak forward attraction (Remark 2.2.5), but this is not a strong enough property to ensure equivalence of attraction in the past and future in terms of almost sure convergence, i.e. equivalence of pullback and forward attraction. Furthermore, the local pullback attractor A in Example 2.6.5 pullback attracts an open ball of constant radius, i.e. A has a *uniform pullback attracting neighbourhood* (Definition 2.7.4), whilst the corresponding local pullback repeller given by Theorem 2.5.11 does not pullback repel any open ball of constant radius. These facts demonstrate a disparity between the past and future dynamics of the RDS. This leads us to consider the past and future time domains of the system separately, by considering so-called *past attractor-repeller pairs* and *future attractor-repeller pairs*. Past attractor-repeller pairs consist of a *local forward repeller* (Definition 2.7.3) with a uniform forward repelling neighbourhood paired with a corresponding local pullback attractor, and future attractor-repeller pairs are the reverse time versions, that is, a *local forward attractor* (Definition 2.7.2) with a uniform forward attracting neighbourhood paired with a corresponding local pullback repeller. The construction of past attractor-repeller pairs works only in one direction: one constructs the local pullback attractor fiber-wise from the local forward repeller using an Ω-limit set, but there is no analogous method to construct a local forward repeller from a local pullback attractor.

For a past attractor-repeller pair, the corresponding local pullback attractor has a uniform pullback attracting neighbourhood (Theorem 2.7.5) (and similarly for future attractor-repeller pairs). We will need a pullback attractor possessing a uniform attracting neighbourhood in the proof of Proposition 3.3.3, which will subsequently be needed to obtain our analogue of Selgrade's Theorem (Theorem 3.3.6).

Here we will extend the fundamental theory of past and future attractor-repeller pairs for nonautonomous systems to the setting of random dynamical systems, following the presentation given in [KR11, Chapter 4]. We will only make statements and proofs for past attractor-repeller pairs, as corresponding statements for future attractor-repeller pairs are obtained by a reversal of time.

Assumption 2.7.1. Let (X, d) be a compact metric space (hence a Polish space), and φ a continuous RDS with two-sided time ($\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$).

Definition 2.7.2. An invariant compact random set A is called a *local forward attractor* if there exists an open random set U with $U(\omega) \supset A(\omega)$ μ-a.s. such that

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\omega)U(\omega), A(\theta_t\omega)) = 0 \qquad \mu\text{-a.s.} \tag{2.7.1}$$

The random set U is called a *forward attracting neighbourhood of A.*

Definition 2.7.3. An invariant compact random set R is called a *local forward repeller* if there exists an open random set U with $U(\omega) \supset R(\omega)$ μ-a.s. such that

$$\lim_{t\to-\infty} \operatorname{dist}(\varphi(t,\omega)U(\omega), R(\theta_t\omega)) = 0 \qquad \mu\text{-a.s.} \tag{2.7.2}$$

The random set U is called a *forward repelling neighbourhood of R.*

Definition 2.7.4. If A is a local weak, forward, pullback or strong attractor with attracting neighbourhood U given by an open ball of constant radius, i.e. $U(\omega) = B_\eta(A(\omega))$ for some $\eta > 0$, then U is called a *uniform weak, forward, pullback* or *strong*, respectively *attracting neighbourhood* (and similarly for the definition of a *uniform repelling neighbourhood* in the case of a local weak, forward, pullback or strong repeller).

The theorem below establishes the existence of past attractor-repeller pairs in the setting of random dynamical systems, the proof of which is an adaptation of the version for general nonautonomous systems, given in [KR11, Theorem 4.1].

Theorem 2.7.5. *Let R be a local forward repeller with uniform forward repelling neighbourhood $B_\eta(R)$ for some $\eta > 0$. Then there exists an almost surely uniquely determined local pullback attractor A given by*

$$A := \Omega_{X\setminus B_\eta(R)}$$

with $A(\omega) \subset X\backslash B_\eta(R(\omega))$ almost surely, and such that $B_\varepsilon(A)$ is a uniform pullback attracting neighbourhood for A for any $0 < \varepsilon < \eta$. The attractor-repeller pair (A, R) is called a past attractor-repeller pair. *Furthermore, (A, R) is a weak attractor-repeller pair and a pullback attractor-repeller pair (the pullback repelling neighbourhood for R may be nonuniform).*

Proof. Since R is a forward repeller, for any $\xi \in (0, \eta)$ and for almost all $\omega \in \Omega$ there exists a $T(\omega)$ such that for all $t \geq T(\omega)$

$$\operatorname{dist}(\varphi(-t,\omega)B_\eta(R(\omega)), R(\theta_{-t}\omega)) < \xi,$$

or equivalently, $\varphi(-t,\omega)B_\eta(R(\omega)) \subset B_\xi(R(\theta_{-t}\omega))$. Then

$$B_\eta(R(\omega)) \subset \varphi(t,\theta_{-t}\omega)B_\xi(R(\theta_{-t}\omega)) \quad \text{for all } t \geq T(\omega),$$

which implies that

$$B_\eta(R(\omega)) \subset \bigcap_{t\geq T(\omega)} \varphi(t,\theta_{-t}\omega)\overline{B_\xi(R(\theta_{-t}\omega))}$$

and so

$$X\backslash B_\eta(R(\omega)) \supset \bigcup_{t\geq T(\omega)} \varphi(t,\theta_{-t}\omega)X\backslash\overline{B_\xi(R(\theta_{-t}\omega))}.$$

Let $D_\xi(\omega) := X\backslash\overline{B_\xi(R(\omega))}$, then we have

$$D_\xi(\omega) \supset X\backslash B_\eta(R(\omega)) \supset \overline{\bigcup_{t\geq T(\omega)} \varphi(t,\theta_{-t}\omega)D_\xi(\theta_{-t}\omega)}. \tag{2.7.3}$$

Hence the family of open random sets D_ξ, $\xi \in (0, \eta)$, are pullback absorbing sets (there exists a measurable absorbtion time by the same argument in the proof of Proposition 2.5.8 (ii)), and by Proposition 2.5.7 their Ω-limit sets give pullback attractors Ω_{D_ξ} that pullback attract the neighbourhoods D_ξ. Moreover, by (2.7.3) the Ω_{D_ξ} are all almost surely contained within $X\backslash B_\eta(R(\omega))$ and pullback attract this set: hence they pullback attract one another, which due to the invariance of the Ω_{D_ξ} (Theorem 2.5.6) means that for arbitrary

$\xi_1, \xi_2 \in (0,\eta)$, $\Omega_{D_{\xi_1}}(\omega) = \Omega_{D_{\xi_2}}(\omega)$ for almost all $\omega \in \Omega$. Then there exists an almost surely unique local pullback attractor A with $A(\omega) \subset X \backslash B_\eta(R(\omega))$ almost surely, that pullback attracts the D_ξ. Since for any $\varepsilon \in (0,\eta)$ there exists a $\xi \in (0,\eta)$ such that $B_\varepsilon(A(\omega)) \subset D_\xi(\omega)$, A pullback attracts $B_\varepsilon(A)$, which may serve as a uniform pullback attracting neighbourhood for A.

We now demonstrate the fact that (A,R) is a weak attractor-repeller pair. It follows from Theorem 2.7.8 (i), that R is also a local weak repeller with basin of repulsion $B(R) = X \backslash A$, and then by the reverse time version of Proposition 2.3.6, A is the corresponding local weak attractor.

Finally, (A,R) is a pullback attractor-repeller pair: since A is a local weak attractor with corresponding local weak repeller R, and A is a local pullback attractor, it follows from Theorem 2.5.11 that R is the corresponding pullback repeller.

□

Remark 2.7.6.

(i) In the case of a general nonautomonous dynamical system different forward repellers may give rise to the same pullback attractor, although these nonunique past repellers converge in backward time (see [KR11, Proposition 4.5]). For an RDS, since Theorem 2.7.5 shows that a past attractor-repeller pair is also a weak attractor-repeller pair, Proposition 2.3.6 demonstrates that the local pullback attractor corresponds to an almost surely unique local forward repeller.

(ii) Example 2.6.9 demonstrates a past attractor-repeller pair.

Definition 2.7.7. For the future time domain version of Theorem 2.7.5, i.e. beginning with a local forward attractor A and obtaining a local pullback repeller R, the pair (A,R) is called a *future attractor-repeller pair*.

The following theorem describes the dynamics of a past attractor-repeller pair. The proof of this result is an adaptation of the version for nonautonomous systems, given in [KR11, Theorem 4.4].

Theorem 2.7.8. *Let (A,R) be a past attractor-repeller pair. Then the following statements hold:*

(i) Convergence in backward time: let $K \subset X \backslash A(\omega)$ be a compact set. Then

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(-t,\omega)K, R(\theta_{-t}\omega)) = 0 \qquad \mu\text{-a.s.}$$

(ii) Pullback convergence: let C be a compact random set such that

$$\liminf_{t\to\infty} \operatorname{dist}(C(\theta_{-t}\omega), R(\theta_{-t}\omega)) > 0 \qquad \mu\text{-a.s.}, \tag{2.7.4}$$

then,

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)C(\theta_{-t}\omega), A(\omega)) = 0 \qquad \mu\text{-a.s.}$$

Proof. Let η and D_ξ be as defined in the proof of Theorem 2.7.5; there it was shown that A pullback attracts the sets D_ξ.

(i). Since A pullback attracts the D_ξ, for almost all $\omega \in \Omega$ there exists a $T(\omega) \geq 0$ such that for all $t \geq T(\omega)$

$$K \cap \varphi(t,\theta_{-t}\omega)D_\xi(\theta_{-t}\omega) = \emptyset.$$

Then $\varphi(-t,\omega)K \cap D_\xi(\theta_{-t}\omega) = \emptyset$ for all $t \geq T(\omega)$, that is, $\operatorname{dist}(\varphi(-t,\omega)K, R(\theta_{-t}\omega)) \leq \xi$.

(ii). By (2.7.4), for almost all $\omega \in \Omega$ there exists $T(\omega) \geq 0$ and $\xi \in (0,\eta)$ such that for all $t \geq T(\omega)$, $C(\theta_{-t}\omega) \subset D_\xi(\theta_{-t}\omega)$, and since A pullback attracts D_ξ, A also pullback attracts C. □

Remark 2.7.9. Suppose that $\{R_0, \ldots, R_n\}$ is a finite set of local forward repellers with

$$X = R_0(\omega) \supsetneq R_1(\omega) \supsetneq \cdots \supsetneq R_n(\omega) = \emptyset \tag{2.7.5}$$

holding almost surely, and let (A_i, R_i), $i \in \{0, \ldots, n\}$, be the corresponding past attractor-repeller pairs. Since by Theorem 2.7.5 the past attractor-repeller pairs are also weak attractor-repeller pairs, Theorem 2.3.10 implies that the corresponding pullback attractors are also almost surely nested:

$$\emptyset = A_n(\omega) \subsetneq A_{n-1}(\omega) \subsetneq \cdots \subsetneq A_0(\omega) = X \qquad \mu\text{-a.s.} \tag{2.7.6}$$

The definitions of *past* and *future Morse decompositions* are given by Definition 2.3.11. The following result gives some basic properties of past Morse sets, and is the analogue of [KR11, Proposition 4.7] for nonautonomous systems.

Proposition 2.7.10. *The Morse sets of a past Morse decomposition $\{M_1, \ldots, M_n\}$ are invariant compact random sets that are almost surely nonempty and pairwise disjoint, and are* uniformly isolated*, i.e. there exists a $\beta > 0$ such that for $i \neq j$*

$$B_\beta(M_i(\omega)) \cap B_\beta(M_j(\omega)) = \emptyset \qquad \mu\text{-a.s.}$$

Proof. Theorem 2.7.5 states that past attractor-repeller pairs are also weak attractor-repellers pairs, and then most of the statements here follow from Lemma 2.3.14. Theorem 2.7.5 states that the past attractor-repeller pair (A, R) corresponding to a local forward repeller R with uniform forward repelling neighbourhood $B_\eta(R)$ for some $\eta > 0$, satisfies $A(\omega) \cap B_\eta(R(\omega)) = \emptyset$ almost surely; the fact that the Morse sets are uniformly isolated is then a simple consequence of this uniform isolation of each past attractor-repeller pair. □

The following theorem shows that random variables which are bounded away from the repeller boundaries are pullback attracted to the Morse sets; the proof is an adaptation of the version for nonautonomous systems, given in [KR11, Theorem 4.9].

Theorem 2.7.11. *Let $\{M_1, \ldots, M_n\}$ be a past Morse decomposition formed from the sequence of past attractor-repeller pairs (A_i, R_i), $i \in \{0, \ldots, n\}$. Then for any random variable x satisfying*

$$\liminf_{t\to\infty} \operatorname{dist}(x(\theta_{-t}\omega), \cup_{i=1}^{n-1} \partial R_i(\theta_{-t}\omega)) > 0 \qquad \mu\text{-a.s.} \tag{2.7.7}$$

one has

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t, \theta_{-t}\omega)x(\theta_{-t}\omega), \cup_{i=1}^{n} M_i(\omega)) = 0 \qquad \mu\text{-a.s.}$$

Proof. First note that the invariance of the forward repellers and pullback attractors implies that (2.7.5) and (2.7.6) hold on a θ-invariant set of full measure, $\tilde{\Omega} \subset \Omega$. Define $\tilde{\Omega}_i := \{\omega \in \tilde{\Omega} \colon x(\omega) \in R_{i-1}(\omega) \backslash R_i(\omega)\}$, $i \in \{1, \ldots, n\}$, then the $\tilde{\Omega}_i$ form a partition of $\tilde{\Omega}$. Also define

$\mathbb{T}_i(\omega) := \{t \in \mathbb{T}\colon \theta_t\omega \in \tilde{\Omega}_i\}$, then for all $\omega \in \tilde{\Omega}$ the $\mathbb{T}_i(\omega)$ form a partition of $\mathbb{T}$. The condition (2.7.7) implies that for almost all $\omega \in \tilde{\Omega}$

$$\liminf_{t\to-\infty, t\in\mathbb{T}_i(\omega)} \operatorname{dist}(x(\theta_t\omega), R_i(\theta_t\omega)) > 0$$

and hence by Theorem 2.7.8, for almost all $\omega \in \tilde{\Omega}$

$$\lim_{t\to\infty, t\in\mathbb{T}_i(\omega)} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)x(\theta_{-t}\omega), A_i(\omega)) = 0. \tag{2.7.8}$$

Now assume there exists $\varepsilon > 0$, and a sequence $t_n \to \infty$ with $t_n \in \mathbb{T}_i(\omega)$ such that

$$\operatorname{dist}(\varphi(t_n,\theta_{-t_n}\omega)x(\theta_{-t_n}\omega), M_i(\omega)) > \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{2.7.9}$$

By compactness one may assume that for $n \to \infty$, $\varphi(t_n,\theta_{-t_n}\omega)x(\theta_{-t_n}\omega) \to y \in R_{i-1}(\omega)$. Also, by (2.7.8), $y \in A_i(\omega)$, and so $y \in M_i(\omega)$, contradicting (2.7.9). It follows that for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)x(\theta_{-t}\omega), \cup_{i=1}^n M_i(\omega)) = 0.$$

□

Remark 2.7.12. Comparing Theorems 2.7.11 and 2.7.8, note that we have not shown forward repulsion to the Morse sets in analogy to Theorem 2.7.8 (i). In the case of nonautonomous systems there exists a counter-example to such convergence [KR11, p. 80], but we have not as yet proved or disproved this for RDS.

2.8 Local strong attractors

We make the following definition of a *local strong attractor*, which is a compact random set that is both a local forward and local pullback attractor.

Definition 2.8.1. An invariant compact random set A is called a *local strong attractor* if there is an open random set U with $U(\omega) \supset A(\omega)$ μ-a.s. such that both

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\omega)U(\omega), A(\theta_t\omega)) = 0 \quad \mu\text{-a.s.},$$

and

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega), A(\omega)) = 0 \quad \mu\text{-a.s.}$$

The random set U is called a *strong attracting neighbourhood of* A. The set

$$B(A)(\omega) := \{x \in X \colon \varphi(t,\omega)x \in U(\theta_t\omega) \text{ for some } t \geq 0\}$$

is called the *basin of attraction of* A. A local strong attractor is called *trivial* if $\mu(A(\omega) \in \{\emptyset, X\}) = 1$.

Definition 2.8.2. An invariant compact random set R is called a *local strong repeller* if there is an open random set U with $U(\omega) \supset R(\omega)$ μ-a.s. such that both

$$\lim_{t\to-\infty} \operatorname{dist}(\varphi(t,\omega)U(\omega), R(\theta_t\omega)) = 0 \quad \mu\text{-a.s.},$$

and

$$\lim_{t\to-\infty} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega), R(\omega)) = 0 \quad \mu\text{-a.s.}$$

The random set U is called a *strong repelling neighbourhood of* A. The set

$$B(R)(\omega) := \{x \in X \colon \varphi(t,\omega)x \in U(\theta_t\omega) \text{ for some } t \leq 0\}$$

is called the *basin of repulsion of* R. A local strong repeller is called *trivial* if $\mu(A(\omega) \in \{\emptyset, X\}) = 1$.

In the next chapter it will be shown that for a projected linear RDS with a local strong attractor with a uniform strong attracting neighbourhood there exists a corresponding local strong repeller with a uniform strong repelling neighbourhood (Theorem 3.3.5). Example 2.6.9 shows that strong attractor-repeller pairs need not exist in general, since there the repeller is both a local pullback and forward repeller, hence is a strong repeller (with a nonuniform neighbourhood), but the corresponding attractor is not a local forward attractor.

2.9 Discussion

We shall mainly discuss pullback attractor-repeller pairs. The examples in Section 2.6 make use of non-tempered random variables, and this may be considered pathological. In the theory of global pullback attractors it is usual to consider pullback attraction in the *universe of tempered random sets* [IS01], which we shall now explain. A *universe* is a family of random sets $\mathcal{D}$ such that if D, D' are random sets such that $D' \subset D$ and $D \in \mathcal{D}$, then also $D' \in \mathcal{D}$. An invariant compact random set $A \in \mathcal{D}$ is called a $\mathcal{D}$-pullback attractor if A pullback attracts all random sets $D \in \mathcal{D}$ (it is easy to see that such a pullback attractor is unique). A random set D is called a *tempered random set* if there exists an $x \in X$ and a tempered real random variable r such that $D(\theta_t\omega) \subset B_{r(\theta_t\omega)}(x)$ for all $t \in \mathbb{T}$, and clearly the family of tempered random sets forms a universe. A number of examples of global pullback attractors for tempered random sets have been studied (see [IS01], [Arn98, Chapter 9]). Considering attraction in the universe of tempered random sets is natural since one generally deals with an exponential rate of attraction, and since otherwise one is considering different sets in each fiber (with measurability, as a random set, being the only restriction) and then random sets may 'fly away' too quickly along the θ-orbit. This is the point in Example 2.6.4, where the random set converges to the repeller with a super-exponential rate. The attraction universe of tempered random sets is invariant only under random coordinate changes satisfying a sufficiently strong growth condition (see [IS01]). It is then clear that in the context of local pullback attractors one should also restrict the class of coordinate changes to obtain attraction of the same universe of sets in the basin under that coordinate change. From this point of view the differences between the deterministic Example 2.6.4 and Example 2.6.9 obtained by (the projection of) a non-tempered random coordinate change are not so surprising. However, the other aspect which we wish to emphasize is the disparity between the properties of the local pullback attractor and corresponding local pullback repeller in Example 2.6.9; the attraction universe of the local pullback attractor is very different to the attraction universe of the corresponding local pullback repeller. An interesting question is whether or not there exists a sufficiently well behaved, yet interesting for applications, class of RDS for which pullback attractor-repeller pairs have attraction and repulsion universes that are similar (e.g. random sets which are bounded away from the repeller or attractor, respectively).

Since Example 2.6.9 is based on a linear RDS, the fact that A is not a forward attractor is directly linked to the fact that R is not a pullback repeller for a uniform neighbourhood: they are both associated to the same time domain, and linearity means that convergence to A (or lack of) is in correspondence with convergence to R. Indeed, in the linear case a local strong attractor with uniform strong attracting neighbourhood has a corresponding local strong repeller with uniform strong repelling neighbourhood (Theorem 3.3.5). We have not investigated the existence of strong attractor repeller pairs with uniform neighbourhoods in the nonlinear case. Also, in relation to past attractor-repeller pairs, we have not investigated whether there exist systems with local pullback attractors with uniform neighbourhoods but with no corresponding local forward repellers. An interesting question from [Och99, p. 3] is whether every weak attractor is a pullback attractor for a suitable universe, which in our context translates to whether every local weak attractor also pullback attracts some random neighbourhood.

The theory of weak attractor-repeller pairs is much cleaner, although of course at the expense of weaker convergence results. On the other hand, it seems that weak attraction is a sufficiently strong property to obtain many important dynamical results, such as the facts that weak Morse decompositions support all invariant measures (Theorem 2.3.16) and the existence of weak global attractors are in correspondence with the existence of random Lyapunov functions [AS01a].

Chapter 3

Linear random dynamical systems

In this chapter we focus on linear random dynamical systems, in terms of spectral concepts and decompositions into invariant subspaces. Spectral studies of random dynamical systems have so far focused on Lyapunov exponents [Arn98, Con97]. Here we develop an alternative spectral theory based on exponential dichotomies that is related to the Sacker–Sell (or dichotomy) spectrum for nonautonomous differential equations. The original construction due to Sacker and Sell [SS78] requires a compact base space (which can be obtained, for instance, from an almost periodic differential equation). Alternative approaches to the dichotomy spectrum [AS01b, BAG93, Ras09, Ras10, Sie02] hold in the general noncompact case, and we use similar techniques for the construction of the dichotomy spectrum by combining them with ergodic properties of the base flow. We note that the relationship between the dichotomy spectrum and Lyapunov spectrum has also been explored in [JPS87] in the special case that the base space of a random dynamical system is a compact Hausdorff space, but our set-up does not require a topological structure on the base. The dichotomy spectrum for RDS has also been investigated in [WC14] with a different set-up to ours, and we shall discuss this in Section 3.4. We shall demonstrate the utility of the dichotomy spectrum in the bifurcation theory of random dynamical systems in Chapter 4 (Theorem 4.3.1).

Another approach to spectral theory is provided by Selgrade's Theorem [Sel75] (alternatively see [CK00, Theorem 5.2.5]). This states that for a linear flow on a vector bundle with compact base space, and such that the flow on the base space is chain transitive, there exists a unique finest Morse decomposition of the induced flow on the projective bundle. This allows one to define the Morse spectrum: for each Morse set in the finest Morse decomposition one takes the set of limit points as time tends to infinity of finite time exponential growth rates for trajectories starting in the Morse set, and the Morse spectrum is then the union over all of the Morse sets of these sets of limit points (see [Grü00] for this formulation, and [CK00, Chapter 5] for the original formulation using (ε, T)-chains). The relationship between the Lyapunov, dichotomy, and Morse spectral concepts, among others, are discussed in [CK96, Grü00]. These results have also been extended to the noncompact setting [Ras07, Ras08, CKR08, Ras10], and here we use similar techniques to demonstrate the existence of a (almost surely) unique finest Morse decomposition for projectivized linear RDS.

This chapter is organized as follows. We first introduce the notion of an exponential dichotomy for a linear RDS in Section 3.1, and in Section 3.2 the dichotomy spectrum is defined and fundamental properties of the spectrum are shown, including its relation to the Lyapunov spectrum. In Section 3.3 we demonstrate an analogue of Selgrade's Theorem for RDS using strong attractor-repeller pairs, and we finish with a discussion on the material in this chapter in Section 3.4.

Many of the results in this chapter are extensions to the setting of random dynamical systems of results for linear nonautonomous systems given in [KR11, Chapter 5].

We will use the following notation and assumption in this chapter:

Assumption 3.0.1. Let $\Phi\colon \mathbb{T} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ denote a linear RDS (Definition 1.1.12), and we shall always assume the time set $\mathbb{T}$ is two-sided.

Below we state the Multiplicative Ergodic Theorem of Oseledets [Ose68] for the case of a linear RDS on $\mathbb{R}^d$ with two-sided time (for statements of the theorem in other settings and for proofs see, for example, [Arn98, Via14]).

Theorem 3.0.2 (Oseledets' Multiplicative Ergodic Theorem)**.** *Let Φ be a real linear RDS such that the following integrability condition holds:*

$$\sup_{t\in[0,1]} \ln^+(\|\Phi(t,\cdot)^{\pm 1}\|) \in L^1(\mu), \tag{3.0.1}$$

where $\ln^+(x) := \max\{0, \ln(x)\}$. *Then there exists a* θ*-invariant set* $\tilde{\Omega} \in \mathcal{F}$ *of full measure such that for every* $\omega \in \tilde{\Omega}$ *the following statements hold:*

(i) there exists a direct sum decomposition of $\mathbb{R}^d$*:*

$$O_1(\omega) \oplus \cdots \oplus O_p(\omega) = \mathbb{R}^d,$$

(ii) the O_i, $i \in \{1, \ldots, p\}$, *are invariant, i.e.* $\Phi(t,\omega)O_i(\omega) = O_i(\theta_t\omega)$ *for all* $t \in \mathbb{T}$, *and are closed random sets such that* $\dim(O_i) := \dim(O_i(\omega))$ *is constant on* $\tilde{\Omega}$,

(iii) there exist real numbers, $\lambda_1 < \lambda_2 \cdots < \lambda_p$, *such that*

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \ln\|\Phi(t,\omega)v\| = \lambda_i \quad \Leftrightarrow \quad v \in O_i(\omega)\backslash\{0\}.$$

The numbers λ_i *are called the* Lyapunov exponents *of* Φ, *the sets* O_i *are called the* Oseledets subspaces, *and* $\dim(O_i)$ *is called the* multiplicity *of the corresponding Lyapunov exponent* λ_i. *The set of all Lyapunov exponents* $\Lambda := \{\lambda_1, \ldots, \lambda_p\}$ *is called the* Lyapunov spectrum *of* Φ.

3.1 Exponential dichotomies*

An exponential dichotomy is the nonautonomous analogue of hyperbolicity, and describes a splitting of the state space of a linear system into complementary subspaces of initial conditions, for which the growths of the corresponding trajectories are exponentially bounded, in forwards time for one subspace and in backwards time for the other subspace, by distinct exponential growth rates. The subspaces are described using so-called *invariant projectors*.

Definition 3.1.1. An *invariant projector* of a linear RDS Φ is a measurable function $P : \Omega \to \mathbb{R}^{d\times d}$ with

$$P(\omega) = P(\omega)^2 \quad \text{and} \quad P(\theta_t\omega)\Phi(t,\omega) = \Phi(t,\omega)P(\omega) \quad \text{for all } t \in \mathbb{T} \text{ and } \omega \in \Omega\,.$$

*The material in this section follows that of [CDLR13, Section 2].

Definition 3.1.2. A random set M (Definition 1.1.7) is called a *linear random set* if $M(\omega)$ is a linear subspace of $\mathbb{R}^d$ for each $\omega \in \Omega$.

Given linear random sets M_1, M_2,

$$\omega \mapsto M_1(\omega) \cap M_2(\omega) \quad \text{and} \quad \omega \mapsto M_1(\omega) + M_2(\omega)$$

are also linear random sets, denoted by $M_1 \cap M_2$ and $M_1 + M_2$, respectively.

The *range*

$$\mathcal{R}(P) := \big\{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in \mathcal{R}P(\omega)\big\}$$

and the *null space*

$$\mathcal{N}(P) := \big\{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in \mathcal{N}P(\omega)\big\}$$

of an invariant projector P are invariant linear random sets such that $\mathcal{R}(P) \oplus \mathcal{N}(P) = \Omega \times \mathbb{R}^d$.

The following proposition uses the ergodicity of θ to show that the dimensions of the range and the null space of an invariant projector are almost surely constant.

Proposition 3.1.3. *Let $P : \Omega \to \mathbb{R}^{d\times d}$ be an invariant projector of a linear RDS Φ. Then*

(i) The mapping $\omega \mapsto \operatorname{rk} P(\omega)$ is measurable, and

(ii) $\operatorname{rk} P(\omega)$ is almost surely constant.

Proof. (i) We first show that the mapping $A \mapsto \operatorname{rk} A$ on $\mathbb{R}^{d\times d}$ is lower semi-continuous. For this purpose, let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of matrices in $\mathbb{R}^{d\times d}$ which converges to $A \in \mathbb{R}^{d\times d}$, and define $r := \operatorname{rk} A$. Then there exist nonzero vectors $x_1, \dots, x_r$ such that $Ax_1, \dots, Ax_r$ are linearly independent, which implies that $\det[Ax_1, \dots, Ax_r, x_{r+1}, \dots, x_d] \neq 0$ for some vectors $x_{r+1}, \dots, x_d \in \mathbb{R}^d$. Since $\lim_{k\to\infty} A_k = A$, one gets

$$\lim_{k\to\infty} \det[A_k x_1, \dots, A_k x_r, x_{r+1}, \dots, x_d] = \det[Ax_1, \dots, Ax_r, x_{r+1}, \dots, x_d]\,.$$

Hence, there exists a $k_0 \in \mathbb{N}$ such that vectors $A_k x_1, \dots, A_k x_r$ are linearly independent for $k \geq k_0$, and thus, $\operatorname{rk} A_k \geq r$ for all $k \geq k_0$. Consequently, the lower semi-continuity of the mapping $A \mapsto \operatorname{rk} A$ is proved. Therefore, the map $\mathbb{R}^{d\times d} \to \mathbb{N}_0, A \mapsto \operatorname{rk} A$ is the limit of a monotonically increasing sequence of continuous functions by [Ton52], and thus is measurable. The proof of this part is complete. (ii) By invariance of P, we get that

$$P(\theta_t\omega) = \Phi(t,\omega)P(\omega)\Phi(t,\omega)^{-1},$$

which implies that $\operatorname{rk} P(\theta_t\omega) = \operatorname{rk} P(\omega)$. This together with ergodicity of θ and measurability of the map $\omega \mapsto \operatorname{rk} P(\omega)$ as shown in (i) gives that $\operatorname{rk} P(\omega)$ is almost surely constant. □

According to Proposition 3.1.3, the rank of an invariant projector P can be defined via

$$\operatorname{rk} P := \dim \mathcal{R}(P) := \dim \mathcal{R}P(\omega) \quad \text{for almost all } \omega \in \Omega\,,$$

and one sets

$$\dim \mathcal{N}(P) := \dim \mathcal{N}P(\omega) \quad \text{for almost all } \omega \in \Omega.$$

The following definition of an exponential dichotomy describes uniform exponential splitting of a linear RDS.

Definition 3.1.4. Let Φ be a linear RDS, and let $\gamma \in \mathbb{R}$ and $P_\gamma : \Omega \to \mathbb{R}^{d\times d}$ be an invariant projector of Φ. Then Φ is said to admit an *exponential dichotomy* with growth rate γ, constants $\alpha > 0$, $K \geq 1$ and projector P_γ if for almost all $\omega \in \Omega$, one has

$$\begin{aligned} \|\Phi(t,\omega)P_\gamma(\omega)\| &\leq Ke^{(\gamma-\alpha)t} \quad \text{for all } t \geq 0\,,\\ \|\Phi(t,\omega)(\mathbb{1} - P_\gamma(\omega))\| &\leq Ke^{(\gamma+\alpha)t} \quad \text{for all } t \leq 0\,. \end{aligned}$$

The following proposition shows that if a linear RDS satisfies the integrability condition of Oseledets' Theorem, then the ranges and null spaces of invariant projectors are given by sums of Oseledets subspaces.

Proposition 3.1.5. *Let Φ be a linear RDS which satisfies the integrability condition of Oseledets' Multiplicative Ergodic Theorem 3.0.2. Let $\lambda_1 > \cdots > \lambda_p$ and $O_1(\omega), \dots, O_p(\omega)$ denote the Lyapunov exponents and the associated Oseledets subspaces of Φ, respectively,* *and suppose that Φ admits an exponential dichotomy with growth rate $\gamma \in \mathbb{R}$ and projector P_γ. Then the following statements hold:*

(i) $\gamma \notin \{\lambda_1, \dots, \lambda_p\}$.

(ii) Define $k := \max\big\{i \in \{0, \dots, p\} : \lambda_i > \gamma\big\}$ *with the convention that* $\lambda_0 = \infty$. *Then for almost all* $\omega \in \Omega$, *one has*

$$\mathcal{N}P(\omega) = \bigoplus_{i=1}^{k} O_i(\omega) \quad \textit{and} \quad \mathcal{R}P(\omega) = \bigoplus_{i=k+1}^{p} O_i(\omega)\,.$$

Proof. (i) Suppose to the contrary that $\gamma = \lambda_k$ for some $k \in \{1, \dots, p\}$. Because of the Multiplicative Ergodic Theorem 3.0.2, we have that for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,\omega)v\| = \lambda_k = \gamma \qquad \text{for all } v \in O_k(\omega) \setminus \{0\}\,. \tag{3.1.1}$$

On the other hand, for almost all $\omega \in \Omega$ and for all $v \in \mathcal{R}P_\gamma(\omega)$ we get $\|\Phi(t,\omega)v\| \leq Ke^{(\gamma-\alpha)t}\|v\|$ for all $t \geq 0$. Thus,

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,\omega)v\| \leq \gamma - \alpha \quad \text{for all } v \in \mathcal{R}P_\gamma(\omega)\,,$$

which together with (3.1.1) implies that $O_k(\omega) \cap \mathcal{R}P_\gamma(\omega) = \{0\}$. Similarly, using the fact that

$$\lim_{t\to-\infty} \frac{1}{t} \ln \|\Phi(t,\omega)v\| = \lambda_k = \gamma \quad \text{for all } v \in O_k(\omega) \setminus \{0\}$$

and Definition 3.1.4, we obtain that $O_k(\omega) \cap \mathcal{N}P_\gamma(\omega) = \{0\}$. Consequently, $O_k(\omega) = \{0\}$ almost surely, which leads to a contradiction.

(ii) Let $v \in \mathcal{R}P_\gamma(\omega) \setminus \{0\}$ be arbitrary. Then, according to Definition 3.1.4 and the definition of k we obtain almost surely that

$$\lim_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,\omega)v\| \leq \gamma - \alpha < \lambda_k. \tag{3.1.2}$$

Now we write v in the form $v = v_i + v_{i+1} + \dots + v_p$, where $i \in \{1, \dots, p\}$ with $v_i \neq 0$ and

$v_j \in O_j(\omega)$ for all $j = i, \dots, p$. Using the fact that for $j \in \{i, \dots, p\}$ with $v_j \neq 0$

$$\lim_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,\omega)v_j\| = \lambda_j \le \lambda_i,$$

we obtain that

$$\lim_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,\omega)v\| = \lambda_i,$$

which together with (3.1.2) implies that $i \ge k+1$ and therefore $\mathcal{R}P_\gamma(\omega) \subset \bigoplus_{i=k+1}^p O_i(\omega)$ almost surely. Similarly, we also get that $\mathcal{N}P_\gamma(\omega) \subset \bigoplus_{i=1}^k O_i(\omega)$ almost surely. On the other hand,

$$\mathbb{R}^d = \mathcal{N}P_\gamma(\omega) \oplus \mathcal{R}P_\gamma(\omega) = \bigoplus_{i=1}^k O_i(\omega) \oplus \bigoplus_{i=k+1}^p O_i(\omega).$$

Consequently, we have that $\mathcal{R}P_\gamma(\omega) = \bigoplus_{i=k+1}^p O_i(\omega)$ and $\mathcal{N}P_\gamma(\omega) = \bigoplus_{i=1}^k O_i(\omega)$ hold for almost all $\omega \in \Omega$, and the proof is complete. □

The monotonicity of the exponential function implies the following basic criteria for the existence of exponential dichotomies.

Lemma 3.1.6. *Suppose that the linear RDS Φ admits an exponential dichotomy with growth rate γ and projector P_γ. Then the following statements are fulfilled:*

(i) If $P_\gamma \equiv \mathbb{1}$ almost surely, then Φ admits an exponential dichotomy with growth rate ζ and invariant projector $P_\zeta \equiv \mathbb{1}$ for all $\zeta > \gamma$.

(ii) If $P_\gamma \equiv 0$ almost surely, then Φ admits an exponential dichotomy with growth rate ζ and invariant projector $P_\zeta \equiv 0$ for all $\zeta < \gamma$.

Definition 3.1.7. Given $\gamma \in \mathbb{R}$, a function $g : \mathbb{R} \to \mathbb{R}^d$ is called γ^+*-exponentially bounded* if $\sup_{t\in\mathbb{R}_0^+} \|g(t)\|e^{-\gamma t} < \infty$. Accordingly, one says that a function $g : \mathbb{R} \to \mathbb{R}^d$ is γ^-*-exponentially bounded* if $\sup_{t\in\mathbb{R}_0^-} \|g(t)\|e^{-\gamma t} < \infty$.

We define for all $\gamma \in \mathbb{R}$

$$\mathcal{S}^\gamma := \left\{(\omega, x) \in \Omega \times \mathbb{R}^d : \Phi(\cdot,\omega)x \text{ is } \gamma^+\text{-exponentially bounded}\right\},$$

and

$$\mathcal{U}^\gamma := \left\{(\omega, x) \in \Omega \times \mathbb{R}^d : \Phi(\cdot, \omega)x \text{ is } \gamma^-\text{-exponentially bounded}\right\}.$$

It is obvious that $\mathcal{S}^\gamma$ and $\mathcal{U}^\gamma$ are invariant linear random sets of Φ, and given $\gamma \leq \zeta$, the relations $\mathcal{S}^\gamma \subset \mathcal{S}^\zeta$ and $\mathcal{U}^\gamma \supset \mathcal{U}^\zeta$ are fulfilled.

The relationship between the projectors of exponential dichotomies with growth rate γ and the sets $\mathcal{S}^\gamma$ and $\mathcal{U}^\gamma$ will now be discussed. The proof of the following result is adapted from the corresponding version for nonautonomous systems given in [KR11, Proposition 5.5].

Proposition 3.1.8. *If the linear RDS Φ admits an exponential dichotomy with growth rate γ and projector P_γ, then $\mathcal{N}(P_\gamma) = \mathcal{U}^\gamma$ and $\mathcal{R}(P_\gamma) = \mathcal{S}^\gamma$ almost surely.*

Proof. Suppose that Φ admits an exponential dichotomy with growth rate γ, constants α, K and projector P_γ. This means that for almost all $\omega \in \Omega$, one has

$$\begin{aligned} \|\Phi(t, \omega)P_\gamma(\omega)\| &\leq Ke^{(\gamma-\alpha)t} \quad \text{for all } t \geq 0\,, \\ \|\Phi(t, \omega)(\mathbb{1} - P_\gamma(\omega))\| &\leq Ke^{(\gamma+\alpha)t} \quad \text{for all } t \leq 0\,. \end{aligned} \tag{3.1.3}$$

We now prove the relation $\mathcal{N}(P_\gamma) = \mathcal{U}^\gamma$ almost surely. ($\supset$) Choose $(\omega, x) \in \mathcal{U}^\gamma$ with ω in the full measure set $F \in \mathcal{F}$ where both (3.1.3) and Birkhoff's Ergodic Theorem hold, and with x arbitrary. We have that $\|\Phi(t, \omega)x\| \leq Ce^{\gamma t}$ for all $t \leq 0$ and some real constant $C > 0$. Write $x = x_1 + x_2$ with $x_1 \in \mathcal{R}P_\gamma(\omega)$ and $x_2 \in \mathcal{N}P_\gamma(\omega)$. By Birkhoff's Ergodic Theorem there exists a sequence $t_i \to -\infty$ such that for all $i \in \mathbb{N}$ one has $\theta_{t_i}\omega \in F$, and hence

$$\begin{aligned} \|x_1\| &= \|\Phi(-t_i, \theta_{t_i}\omega)\Phi(t_i, \omega)P_\gamma(\omega)x\| = \|\Phi(-t_i, \theta_{t_i}\omega)P_\gamma(\theta_{t_i}\omega)\Phi(t_i, \omega)x\| \\ &\leq Ke^{-(\gamma-\alpha)t_i}\|\Phi(t_i, \omega)x\| \leq CKe^{-(\gamma-\alpha)t_i}e^{\gamma t_i} = CKe^{\alpha t_i}\,. \end{aligned}$$

The right hand side of this inequality converges to zero in the limit $i \to \infty$. This implies $x_1 = 0$, and thus, $(\omega, x) \in \mathcal{N}(P_\gamma)$. ($\subset$) Choose $(\omega, x) \in \mathcal{N}(P_\gamma)$. Thus, for all $t \leq 0$ and almost all $\omega \in \Omega$, the relation $\|\Phi(t, \omega)x\| \leq Ke^{(\gamma+\alpha)t}\|x\|$ is fulfilled. This means that $\Phi(\cdot, \omega)x$ is γ^--exponentially bounded. The proof of the statement concerning the range of the projector is treated analogously. □

3.2 The dichotomy spectrum*

In this section we introduce the dichotomy spectrum for random dynamical systems. The dichotomy spectrum is defined as the set of values γ for which a linear RDS Φ does not admit an exponential dichotomy with growth rate γ. The growth rates $\gamma = \pm\infty$ are not excluded from our considerations; one says that Φ admits an exponential dichotomy with growth rate ∞ if there exists a $\gamma \in \mathbb{R}$ such that Φ admits an exponential dichotomy with growth rate γ and projector $P_\gamma \equiv \mathbb{1}$. Accordingly, one says that Φ admits an exponential dichotomy with growth rate $-\infty$ if there exists a $\gamma \in \mathbb{R}$ such that Φ admits an exponential dichotomy with growth rate γ and projector $P_\gamma \equiv 0$. In what follows we use the notation $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ for the extended real line.

Definition 3.2.1. The *dichotomy spectrum* of a linear RDS Φ is defined by

$$\Sigma := \left\{\gamma \in \overline{\mathbb{R}} : \Phi \text{ does not admit an exponential dichotomy with growth rate } \gamma\right\}.$$

The corresponding *resolvent set* is defined by $\rho := \overline{\mathbb{R}} \setminus \Sigma$.

The aim of the following lemma is to analyse the topological structure of the resolvent set. The proof is an adaptation of the version for nonautonomous systems, given in [KR11, Lemma 5.10].

Lemma 3.2.2. *Consider the resolvent set ρ of a linear RDS Φ. Then $\rho \cap \mathbb{R}$ is open. More precisely, for all $\gamma \in \rho \cap \mathbb{R}$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(\gamma) \subset \rho$. Furthermore, the relation $\operatorname{rk} P_\zeta = \operatorname{rk} P_\gamma$ is (almost surely) fulfilled for all $\zeta \in B_\varepsilon(\gamma)$ and every invariant projector P_γ and P_ζ of the exponential dichotomies of Φ with growth rates γ and ζ, respectively.*

Proof. Choose $\gamma \in \rho$ arbitrarily. Since Φ admits an exponential dichotomy with growth rate γ, there exists an invariant projector P_γ and constants $\alpha > 0$, $K \geq 1$ such that for

*The material in this section follows that of [CDLR13, Section 3].

almost all $\omega \in \Omega$, one has

$$\|\Phi(t,\omega)P_\gamma(\omega)\| \le Ke^{(\gamma-\alpha)t} \quad \text{for all } t \ge 0\,,$$
$$\|\Phi(t,\omega)(\mathbb{1} - P_\gamma(\omega))\| \le Ke^{(\gamma+\alpha)t} \quad \text{for all } t \le 0\,.$$

Set $\varepsilon := \frac{1}{2}\alpha$, and choose $\zeta \in B_\varepsilon(\gamma)$. It follows that for almost all $\omega \in \Omega$,

$$\|\Phi(t,\omega)P_\gamma(\omega)\| \le Ke^{(\zeta-\frac{\alpha}{2})t} \quad \text{for all } t \ge 0\,,$$
$$\|\Phi(t,\omega)(\mathbb{1} - P_\gamma(\omega))\| \le Ke^{(\zeta+\frac{\alpha}{2})t} \quad \text{for all } t \le 0\,.$$

This yields $\zeta \in \rho$, and it follows that $\operatorname{rk} P_\zeta = \operatorname{rk} P_\gamma$ for any projector P_ζ of the exponential dichotomy with growth rate ζ. This finishes the proof of this lemma. □

The next lemma relates the topological structure of the resolvent set to the invariant projectors. The proof is an adaptation of the version for nonautonomous systems given in [KR11, Lemma 5.11].

Lemma 3.2.3. *Consider the resolvent set ρ of a linear RDS Φ, and let $\gamma_1, \gamma_2 \in \rho \cap \mathbb{R}$ such that $\gamma_1 < \gamma_2$. Moreover, choose invariant projectors P_{γ_1} and P_{γ_2} for the corresponding exponential dichotomies with growth rates γ_1 and γ_2. Then the relation $\operatorname{rk} P_{\gamma_1} \le \operatorname{rk} P_{\gamma_2}$ holds. In addition, $[\gamma_1, \gamma_2] \subset \rho$ is fulfilled if and only if $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$, and in this case one has that $P_\gamma = P_\zeta$ almost surely for all $\gamma, \zeta \in [\gamma_1, \gamma_2]$.*

Proof. The relation $\operatorname{rk} P_{\gamma_1} \le \operatorname{rk} P_{\gamma_2}$ is a direct consequence of Proposition 3.1.8, since $\mathcal{S}^{\gamma_1} \subset \mathcal{S}^{\gamma_2}$ and $\mathcal{U}^{\gamma_1} \supset \mathcal{U}^{\gamma_2}$. Now assume that $[\gamma_1, \gamma_2] \subset \rho$. Arguing contrapositively, suppose that $\operatorname{rk} P_{\gamma_1} \ne \operatorname{rk} P_{\gamma_2}$, and choose invariant projectors P_ζ, $\zeta \in (\gamma_1, \gamma_2)$, for the exponential dichotomies of Φ with growth rate ζ. Define

$$\zeta_0 := \sup \left\{ \zeta \in [\gamma_1, \gamma_2] : \operatorname{rk} P_\zeta \ne \operatorname{rk} P_{\gamma_2} \right\}.$$

Due to Lemma 3.2.2, there exists an $\varepsilon > 0$ such that $\operatorname{rk} P_{\zeta_0} = \operatorname{rk} P_\zeta$ for all $\zeta \in B_\varepsilon(\zeta_0)$. This is a contradiction to the definition of ζ_0. Conversely, let $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$, then Proposition 3.1.8 together with the fact that $\mathcal{S}^{\gamma_1} \subset \mathcal{S}^{\gamma_2}$ and $\mathcal{U}^{\gamma_1} \supset \mathcal{U}^{\gamma_2}$ yields that $\mathcal{R}(P_{\gamma_1}) =$

$\mathcal{R}(P_{\gamma_2})$ and $\mathcal{N}(P_{\gamma_1}) = \mathcal{N}(P_{\gamma_2})$ almost surely, hence $P_{\gamma_1} = P_{\gamma_2}$ almost surely and P_{γ_2} is also an invariant projector of the exponential dichotomy with growth rate γ_1. Thus, one obtains for almost all $\omega \in \Omega$,

$$\|\Phi(t,\omega)P_{\gamma_2}(\omega)\| \le K_1 e^{(\gamma_1-\alpha_1)t} \quad \text{for all } t \ge 0$$

for some $K_1 \ge 1$ and $\alpha_1 > 0$, and

$$\left\|\Phi(t,\omega)(\mathbb{1} - P_{\gamma_2}(\omega))\right\| \le K_2 e^{(\gamma_2+\alpha_2)t} \quad \text{for all } t \le 0$$

with some $K_2 \ge 1$ and $\alpha_2 > 0$. For all $\gamma \in [\gamma_1, \gamma_2]$ these two inequalities imply, by setting $K := \max\{K_1, K_2\}$ and $\alpha := \min\{\alpha_1, \alpha_2\}$, that for almost all $\omega \in \Omega$

$$\begin{aligned} \|\Phi(t,\omega)P_{\gamma_2}(\omega)\| &\le K e^{(\gamma-\alpha)t} \quad \text{for all } t \ge 0\,, \\ \|\Phi(t,\omega)(\mathbb{1} - P_{\gamma_2}(\omega))\| &\le K e^{(\gamma+\alpha)t} \quad \text{for all } t \le 0\,. \end{aligned}$$

This means that $\gamma \in \rho$, and thus, $[\gamma_1, \gamma_2] \subset \rho$. Now for arbitrary $\gamma, \zeta \in [\gamma_1, \gamma_2]$ with $\gamma \le \zeta$ one has $\operatorname{rk} P_\gamma \le \operatorname{rk} P_\zeta$, and since the relation $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$ also holds, one must have that $\operatorname{rk} P_\gamma = \operatorname{rk} P_\zeta$. Then Proposition 3.1.8 together with the fact that $\mathcal{S}^\gamma \subset \mathcal{S}^\zeta$ and $\mathcal{U}^\gamma \supset \mathcal{U}^\zeta$ yields that $\mathcal{R}(P_\gamma) = \mathcal{R}(P_\zeta)$ and $\mathcal{N}(P_\gamma) = \mathcal{N}(P_\zeta)$ almost surely, and hence $P_\gamma = P_\zeta$ almost surely. □

For an arbitrarily chosen $a \in \mathbb{R}$, define

$$[-\infty, a] := (-\infty, a] \cup \{-\infty\}\,, \qquad [a, \infty] := [a, \infty) \cup \{\infty\}$$

and

$$[-\infty, -\infty] := \{-\infty\}, \qquad [\infty, \infty] := \{\infty\}, \qquad [-\infty, \infty] := \overline{\mathbb{R}}\,.$$

The following *Spectral Theorem* shows that the dichotomy spectrum consists of at least one and at most d closed intervals; it is an extension of the corresponding result for nonautonomous systems given in [KR11, Theorem 5.12] to the setting of random dynamical systems.

Theorem 3.2.4. *Let Φ be a linear RDS with dichotomy spectrum Σ. Then there exists an $n \in \{1, \dots, d\}$ such that*

$$\Sigma = [a_1, b_1] \cup \dots \cup [a_n, b_n]$$

with $-\infty \le a_1 \le b_1 < a_2 \le b_2 < \dots < a_n \le b_n \le \infty$.

Proof. Due to Lemma 3.2.2, the resolvent set $\rho \cap \mathbb{R}$ is open. Thus, $\Sigma \cap \mathbb{R}$ is the disjoint union of closed intervals. The relation $(-\infty, b_1] \subset \Sigma$ implies $[-\infty, b_1] \subset \Sigma$, because the assumption of the existence of a $\gamma \in \mathbb{R}$ such that Φ admits an exponential dichotomy with growth rate γ and projector $P_\gamma \equiv 0$ leads to $(-\infty, \gamma] \subset \rho$ using Lemma 3.1.6, and this is a contradiction. Analogously, it follows from $[a_n, \infty) \subset \Sigma$ that $[a_n, \infty] \subset \Sigma$. To show the relation $n \le d$, assume to the contrary that $n \ge d + 1$. Thus, there exist

$$\zeta_1 < \zeta_2 < \dots < \zeta_d \in \rho$$

such that the $d + 1$ intervals $(-\infty, \zeta_1)$, (ζ_1, ζ_2), $\dots$, (ζ_d, ∞) have nonempty intersection with the spectrum Σ. It follows from Lemma 3.2.3 that

$$0 \le \operatorname{rk} P_{\zeta_1} < \operatorname{rk} P_{\zeta_2} < \dots < \operatorname{rk} P_{\zeta_d} \le d$$

is fulfilled for invariant projectors P_{ζ_i} of the exponential dichotomy with growth rate ζ_i, $i \in \{1, \dots, n\}$. This implies either $\operatorname{rk} P_{\zeta_1} = 0$ or $\operatorname{rk} P_{\zeta_d} = d$. Thus, either

$$[-\infty, \zeta_1] \cap \Sigma = \emptyset \quad \text{or} \quad [\zeta_d, \infty] \cap \Sigma = \emptyset$$

is fulfilled, and this is a contradiction. To show $n \ge 1$, assume that $\Sigma = \emptyset$. This implies $\{-\infty, \infty\} \subset \rho$. Thus, there exist $\zeta_1, \zeta_2 \in \mathbb{R}$ such that Φ admits an exponential dichotomy with growth rate ζ_1 and projector $P_{\zeta_1} \equiv 0$ and an exponential dichotomy with growth rate ζ_2 and projector $P_{\zeta_2} \equiv \mathbb{1}$. Applying Lemma 3.2.3, one gets $(\zeta_1, \zeta_2) \cap \Sigma \neq \emptyset$. This contradiction yields $n \ge 1$ and finishes the proof of the theorem. □

Each spectral interval is associated to a so-called spectral manifold, which are refinements of the stable and unstable manifolds obtained by the ranges and null spaces of invariant projectors of exponential dichotomies. These are described in the following theorem, the proof of which is an adaptation of the version for nonautonomous systems given in [KR11, Theorem 5.14].

Theorem 3.2.5. *Consider the dichotomy spectrum*

$$\Sigma = [a_1, b_1] \cup \cdots \cup [a_n, b_n]$$

of the linear RDS Φ *and define the invariant projectors* $P_{\gamma_0} := 0$, $P_{\gamma_n} := \mathbb{1}$, *and for* $i \in \{1, \ldots, n-1\}$, *choose* $\gamma_i \in (b_i, a_{i+1})$ *and projectors* P_{γ_i} *of the exponential dichotomy of* Φ *with growth rate* γ_i. *Then the sets*

$$\mathcal{W}_i(\omega) := \mathcal{R}(P_{\gamma_i}(\omega)) \cap \mathcal{N}(P_{\gamma_{i-1}}(\omega)) \quad \textit{for all } i \in \{1, \ldots, n\}$$

are linear subsets of $\mathbb{R}^d$ *for all* $\omega \in \Omega$, *the so-called* spectral manifolds, *such that for almost all* $\omega \in \Omega$ *they form a direct sum decomposition*

$$\mathcal{W}_1(\omega) \oplus \cdots \oplus \mathcal{W}_n(\omega) = \mathbb{R}^d$$

with $\mathcal{W}_i(\omega) \neq \{0\}$ *for* $i \in \{1, \ldots, n\}$.

Proof. The sets $\mathcal{W}_1, \ldots, \mathcal{W}_n$ obviously have linear fibers. We first show that $\mathcal{W}_i(\omega) \neq \{0\}$ almost surely for all $i \in \{1, \ldots, n\}$. If $\mathcal{W}_1(\omega) \neq \{0\}$ does not hold almost surely, then Proposition 3.1.3 implies that $P_{\gamma_1}(\omega) = 0$ almost surely, and Lemma 3.1.6 implies $[-\infty, \gamma_1] \cap \Sigma = \emptyset$, which is a contradiction. A similar argument may be used for $\mathcal{W}_n$. In the case $1 < i < n$, using Lemma 3.2.3, one obtains

$$\dim \mathcal{W}_i = \dim \big(\mathcal{R}(P_{\gamma_i}) \cap \mathcal{N}(P_{\gamma_{i-1}})\big) = \operatorname{rk} P_{\gamma_i} + d - \operatorname{rk} P_{\gamma_{i-1}} - \dim \big(\mathcal{R}(P_{\gamma_i}) + \mathcal{N}(P_{\gamma_{i-1}})\big) \geq 1.$$

Now the relation $\mathcal{W}_1(\omega) \oplus \cdots \oplus \mathcal{W}_n(\omega) = \mathbb{R}^d$ μ-a.s. will be proved. For $1 \leq i < j \leq n$, due to Proposition 3.1.8, the relations $\mathcal{W}_i \subset \mathcal{R}(P_{\gamma_i})$ and $\mathcal{W}_j \subset \mathcal{N}(P_{\gamma_{j-1}}) \subset \mathcal{N}(P_{\gamma_i})$ are almost surely fulfilled. This yields that, almost surely,

$$\mathcal{W}_i(\omega) \cap \mathcal{W}_j(\omega) \subset \mathcal{R}(P_{\gamma_i}(\omega)) \cap \mathcal{N}(P_{\gamma_i}(\omega)) = \{0\}\,.$$

One also obtains

$$\begin{aligned}\mathbb{R}^d &= \mathcal{W}_1(\omega) + \mathcal{N}(P_{\gamma_1}(\omega)) = \mathcal{W}_1(\omega) + \mathcal{N}(P_{\gamma_1}(\omega)) \cap \big(\mathcal{R}(P_{\gamma_2}(\omega)) + \mathcal{N}(P_{\gamma_2}(\omega))\big) \\ &= \mathcal{W}_1(\omega) + \mathcal{N}(P_{\gamma_1}(\omega)) \cap \mathcal{R}(P_{\gamma_2}(\omega)) + \mathcal{N}(P_{\gamma_2}(\omega)) = \mathcal{W}_1(\omega) + \mathcal{W}_2(\omega) + \mathcal{N}(P_{\gamma_2}(\omega)) \,.\end{aligned}$$

using the fact that for linear subspaces $E, F, G \subset \mathbb{R}^d$ with $E \supset G$ fulfill $E \cap (F + G) = (E \cap F) + G$. It follows inductively that

$$\mathbb{R}^d = \mathcal{W}_1(\omega) + \cdots + \mathcal{W}_n(\omega) + \mathcal{N}(P_{\gamma_n}(\omega)) = \mathcal{W}_1(\omega) + \cdots + \mathcal{W}_n(\omega)$$

for almost all $\omega \in \Omega$. □

Remark 3.2.6. If the linear RDS Φ under consideration fulfils the conditions of the Multiplicative Ergodic Theorem, then Proposition 3.1.5 implies that the spectral manifolds $\mathcal{W}_i$ of the above theorem are given by direct sums of Oseledets subspaces.

The remaining part of this section on the dichotomy spectrum will be devoted to the study of boundedness properties of the spectrum. Firstly, a criterion for boundedness from above and below is provided by the following proposition.

Proposition 3.2.7. *Consider a linear RDS Φ, let Σ denote the dichotomy spectrum of Φ, and define*

$$\alpha^{\pm}(\omega) := \begin{cases} \ln^+\big(\|\Phi(1,\omega)^{\pm 1}\|\big) & : \mathbb{T} = \mathbb{Z}\,, \\ \ln^+\big(\sup_{t\in[0,1]}\|\Phi(t,\omega)^{\pm 1}\|\big) & : \mathbb{T} = \mathbb{R}\,. \end{cases}$$

Then Σ is bounded from above if and only if

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \alpha^+(\omega) < \infty\,,$$

and Σ is bounded from below if and only if

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \alpha^-(\omega) < \infty\,.$$

Consequently, if the dichotomy spectrum Σ is bounded, then Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem.

Proof. Suppose that Σ is bounded from above. Then there exist $K \geq 1$ and $\Gamma \in \mathbb{R}$ such that for almost all $\omega \in \Omega$

$$\|\Phi(t,\omega)\| \leq K e^{\Gamma t} \quad \text{for all } t \geq 0\,,$$

which implies that $\operatorname{ess\,sup}_{\omega\in\Omega} \alpha^+(\omega) \leq \ln(K) + |\Gamma|$. On the other hand, suppose that $\operatorname{ess\,sup}_{\omega\in\Omega} \alpha^+(\omega) < \infty$. Then there exists a full measure set $F \in \mathcal{F}$ such that for all $\omega \in F$ we have $\alpha^+(\omega) \leq \beta$ for some positive number β. Define

$$\widetilde{\Omega} := \bigcap_{n\in\mathbb{Z}} \theta_n F\,.$$

Due to the measure preserving property of θ, we get that $\mu\big(\widetilde{\Omega}\big) = 1$. Then for all $\omega \in \widetilde{\Omega}$, we have

$$\|\Phi(t,\omega)\| \leq \|\Phi(t-\lfloor t\rfloor, \theta_{\lfloor t\rfloor}\omega)\|\|\Phi(1,\theta_{\lfloor t\rfloor-1}\omega)\|\cdots\|\Phi(1,\omega)\| \leq e^{\beta(t+1)} \quad \text{for all } t \geq 0.$$

Let $\gamma > \beta$ be arbitrary and $\varepsilon < \gamma - \beta$. Then

$$\|\Phi(t,\omega)\| \leq e^{\beta} e^{(\gamma-\varepsilon)t} \quad \text{for all } t \geq 0\,,$$

which implies that Φ admits an exponential dichotomy with growth rate γ and projector $P_\gamma \equiv \mathbb{1}$, and hence $\Sigma \subset [-\infty, \gamma)$. Similarly, we get that Σ is bounded from below if and only if $\operatorname{ess\,sup}_{\omega\in\Omega} \alpha^-(\omega) < \infty$. This finishes the proof of this proposition. $\square$

The following example shows that there exist linear random dynamical systems which satisfy the integrability condition of the Multiplicative Ergodic Theorem, but which have no bounded dichotomy spectrum.

Example 3.2.8. Let $(\Omega, \mathcal{F}, \mu)$ be a nonatomic probability space and $\theta\colon \mathbb{Z}\times\Omega \to \Omega$ be a metric dynamical system which is ergodic. Then there exists, by using [Hal56, Lemma 2,

p. 71], a measurable set U of the form

$$U = \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{k} \theta_j U_k, \tag{3.2.1}$$

where U_i, $i \in \mathbb{N}$, are measurable sets such that

(i) for all $k, \ell \in \mathbb{N}$, $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, \ell\}$, we have

$$\theta_j U_k \cap \theta_i U_\ell = \emptyset \quad \text{whenever } k \neq \ell \text{ or } i \neq j\,,$$

(ii) $0 < \mu(U_k) \leq \frac{1}{k^3}$ for all $k \in \mathbb{N}$.

We now define a random variable $a : \Omega \to \mathbb{R}$ by

$$a(\omega) := \begin{cases} 1 : \omega \in \Omega \setminus U\,, \\ k : k \text{ is even and } \omega \in \theta_j U_k\,, \\ \frac{1}{k} : k \text{ is odd and } \omega \in \theta_j U_k\,, \end{cases}$$

with $j \in \{0, \dots, k\}$. Using the random variable a, we define a discrete-time scalar linear RDS $\Phi : \mathbb{Z} \times \Omega \to \mathbb{R}$ by

$$\Phi(n, \omega) = \begin{cases} a(\theta_{n-1}\omega) \cdots a(\omega) & : n \geq 1\,, \\ 1 & : n = 0\,, \\ a(\theta_{-1}\omega)^{-1} \cdots a(\theta_n \omega)^{-1} & : n \leq -1\,. \end{cases}$$

A direct computation yields that

$$\mathbb{E} \ln^+(\|\Phi(1,\omega)\|) = \sum_{k=1}^{\infty} (2k+1)\mu(U_{2k}) \ln(2k) \leq \sum_{k=1}^{\infty} (2k+1) \frac{\ln(2k)}{8k^3} < \infty\,,$$

and

$$\mathbb{E}\ln^+(\|\Phi(1,\omega)^{-1}\|) = \sum_{k=1}^{\infty}(2k+2)\mu(U_{2k+1})\ln(2k+1)$$
$$\leq \sum_{k=1}^{\infty}(2k+2)\frac{\ln(2k+1)}{(2k+1)^3} < \infty\,.$$

Then the linear system Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem. The fact that the dichotomy spectrum of Φ is unbounded from above follows from

$$\|\Phi(n,\omega)\| = k^n \quad \text{for all } \omega \in U_k \text{ with } k \text{ even and } 0 \leq n \leq k+1\,.$$

Similarly, one can prove that the spectrum is unbounded from below.

Remark 3.2.9. [JPS87, Theorem 2.3] contains the following statement. Let M be a compact connected Hausdorff space and $\Psi\colon M \times \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}^d$ a continuous linear cocycle with two-sided time, over a continuous dynamical system $\phi\colon \mathbb{T} \times M \to M$ equipped with a unique ergodic measure μ, then

$$\partial\Sigma \subset \Lambda \subset \Sigma.$$

In particular, the endpoints of the intervals in the dichotomy spectrum are given by Lyapunov exponents. Example 3.2.8 shows that the first inclusion does not hold in general for a linear RDS. The metric DS in this example may be chosen to satisfy the above set-up (e.g. an irrational circle rotation equipped with the Lebesgue measure); the difference here arises from the lack of continuity of the cocycle with respect to the base, which allows the time-one mapping of the cocycle to be arbitrarily large with positive probability, whilst still satisfying the integrability condition of Oseledets Theorem. The second inclusion was shown for linear RDS in Proposition 3.1.5.

3.3 Morse decompositions for linear systems

A linear RDS $\Phi\colon \mathbb{T} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ induces an RDS $\mathbb{P}\Phi\colon \mathbb{T} \times \Omega \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ on the real projective space, given by

$$\mathbb{P}\Phi(t,\omega)\mathbb{P}(x) := \mathbb{P}(\Phi(t,\omega)x)$$

for every $(t,\omega,x) \in \mathbb{T}\times\Omega\times\mathbb{R}^d$ (see Section 1.1: *Linear RDS, projected linear RDS*, p. 25). Since the real projective space is a compact metric space, one can analyse the dynamics of $\mathbb{P}\Phi$ using attractor-repeller pairs, as discussed in Chapter 2. The main result in this section is Theorem 3.3.6, which is an analogue of Selgrade's Theorem on the existence of a unique finest Morse decomposition of $\mathbb{P}\Phi$. This is achieved using strong attractor-repeller pairs with uniform attracting and repelling neighbourhoods, the existence of which are demonstrated in Theorem 3.3.5. The main ingredient in the proof of Theorem 3.3.6 is the fact that a strong attractor-repeller pair corresponds to a direct sum decomposition of the state space, and then since the state space is finite dimensional, there can only exist a finite number of nested attractor-repeller pairs. The fact that the attractor and corresponding repeller are linear subspaces relies on Proposition 3.3.3, which itself relies on a local pullback attractor having a uniform pullback attracting neighbourhood. For this reason we cannot proceed with the techniques employed here using pullback attractor-repeller pairs, since Remark 2.5.13 (ii) asserts that a local pullback attractor with uniform pullback attracting neighbourhood may have a corresponding local pullback repeller with a nonuniform pullback repelling neighbourhood. We are also unable to prove an analogue of Selgrade's Theorem using past or future attractor-repeller pairs — see Remark 3.3.4.

The following two lemmas will be needed in this section.

Lemma 3.3.1 ([KR11, Lemma A1]). *For all $\eta > 0$ there exists a $\delta \in (0,1)$ such that for all nonzero vectors $x, y \in \mathbb{R}^d$ satisfying*

$$\frac{\langle x,y\rangle^2}{\|x\|^2\|y\|^2} \geq 1-\delta,$$

one has

$$d_{\mathbb{P}}(\mathbb{P}x, \mathbb{P}y) \leq \eta.$$

Lemma 3.3.2. *Let* $V, W \subset \mathbb{R}^d$ *be linear subspaces with* $\dim(W) > \dim(V)$. *Then,*

$$\operatorname{dist}_{\mathbb{P}}(\mathbb{P}W, \mathbb{P}V) = \sqrt{2}.$$

Proof. Define the linear subspace $V^{\perp} := \{x \in \mathbb{R}^d \colon \langle x, v\rangle = 0 \text{ for all } v \in V\}$, then the linear subspace $V^{\perp} \cap W$ is nontrivial. Let $w \in (V^{\perp} \cap W)\setminus\{0\}$, then for all $v \in V$ we have

$$\begin{aligned}
d_{\mathbb{P}}(\mathbb{P}w, \mathbb{P}v) &= \min\left\{\left\| \frac{v}{\|v\|} \pm \frac{w}{\|w\|}\right\|\right\} \\
&= \min\left\{\sqrt{\left\langle \frac{v}{\|v\|}, \frac{v}{\|v\|}\right\rangle + \left\langle \frac{w}{\|w\|}, \frac{w}{\|w\|}\right\rangle \pm 2\left\langle \frac{v}{\|v\|}, \frac{w}{\|w\|}\right\rangle}\right\} \\
&= \sqrt{2}
\end{aligned}$$

Then since $d_{\mathbb{P}}(x, y) \leq \sqrt{2}$ for all $x, y \in \mathbb{P}^{d-1}$, the result is implied. □

3.3.1 Past attractor-repeller pairs for linear systems

The following result shows that for a projected linear RDS, a pullback attractor with a uniform pullback attracting neighbourhood almost surely corresponds to a linear subspace. The proof is an adaptation of the nonautonomous case, given in [KR11, Proposition 5.18].

Proposition 3.3.3. *Let* A *be a nontrivial local pullback attractor of* $\mathbb{P}\Phi$ *with pullback attracting neighbourhood given by the random open ball* $B_\eta(A)$, *for some constant* $\eta > 0$. *Then for almost all* $\omega \in \Omega$ *one has that for all compact sets* $C \subset \mathbb{S}^{d-1}\setminus\mathbb{P}^{-1}A(\omega)$,

$$\lim_{t\to-\infty} \frac{\sup_{u\in\mathbb{S}^{d-1}\cap\mathbb{P}^{-1}A(\omega)}\|\Phi(t,\omega)u\|}{\inf_{v\in C}\|\Phi(t,\omega)v\|} = 0.$$

Moreover, the set $\mathbb{P}^{-1}A(\omega)$ *is almost surely a linear subspace of* $\mathbb{R}^d$.

Proof. The proof of this proposition is divided into five parts. For fixed $\omega \in \Omega$ let C denote an arbitrary compact set such that $C \subset \mathbb{S}^{d-1}\setminus\mathbb{P}^{-1}A(\omega)$. For any two vectors $u, v \in \mathbb{R}^d$, define the linear subspace $L_{u,v} := \text{span}(\{u, v\})$.

Part 1. It will be shown that for almost all $\omega \in \Omega$, nonzero $u \in \mathbb{P}^{-1}A(\omega)$ and $v \in C$ such that $\mathbb{P}u$ is a boundary point of $A(\omega) \cap \mathbb{P}L_{u,v}$ relative to $\mathbb{P}L_{u,v}$, one has

$$\lim_{t\to-\infty} \frac{\|\Phi(t,\omega)u\|}{\|\Phi(t,\omega)v\|} = 0. \tag{3.3.1}$$

By the definition of A,

$$\lim_{t\to\infty} \mathrm{dist}_{\mathbb{P}}(\mathbb{P}\Phi(t,\theta_{-t}\omega)B_\eta(A(\theta_{-t}\omega)), A(\omega)) = 0 \qquad \mu\text{-a.s.} \tag{3.3.2}$$

By Lemma 3.3.1 there exists a $\delta \in (0,1)$ such that $d_{\mathbb{P}}(\mathbb{P}x, \mathbb{P}y) \le \frac{\eta}{2}$ holds for all $x, y \in \mathbb{R}^d \backslash \{0\}$ with

$$\frac{\langle x,y\rangle^2}{\|x\|^2\|y\|^2} \ge 1-\delta. \tag{3.3.3}$$

Let $F \in \mathcal{F}$ be the set of full measure where the limit in (3.3.2) holds and A is nontrivial, and assume there exists an $\omega \in F$ such that (3.3.1) does not hold, then there exists a $\gamma > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = -\infty$ such that

$$\frac{\|\Phi(t_n,\omega)v\|}{\|\Phi(t_n,\omega)u\|} \le \gamma, \qquad \text{for all } n \in \mathbb{N}.$$

For nonzero $c \in \mathbb{R}$ with $|c|$ sufficiently small, this implies (using the Cauchy-Schwarz inequality) that for all $n \in \mathbb{N}$ the following holds:

$$\frac{\langle \Phi(t_n,\omega)(cv+u), \Phi(t_n,\omega)u\rangle^2}{\|\Phi(t_n,\omega)(cv+u)\|^2\|\Phi(t_n,\omega)u\|^2} \ge 1-\delta.$$

Hence, for $|c| > 0$ sufficiently small, by (3.3.3) one has that

$$\mathrm{dist}_{\mathbb{P}}(\mathbb{P}\Phi(t_n,\omega)\mathbb{P}(cv+u), A(\theta_{t_n}\omega)) \le \frac{\eta}{2} \qquad \text{for all } n \in \mathbb{N}.$$

Then,

$$\begin{aligned}\operatorname{dist}_{\mathbb{P}}(\mathbb{P}(cv+u),A(\omega)) &= \lim_{n\to\infty}\operatorname{dist}_{\mathbb{P}}(\mathbb{P}(cv+u),A(\omega))\\ &= \lim_{n\to\infty}\operatorname{dist}_{\mathbb{P}}(\mathbb{P}\Phi(-t_n,\theta_{t_n}\omega)\underbrace{\mathbb{P}\Phi(t_n,\omega)\mathbb{P}(cv+u)}_{\in B_\eta(A(\theta_{t_n}\omega))},A(\omega))\\ &=0,\end{aligned}$$

using the pullback attraction to A given in (3.3.2). This is a contradiction since $\mathbb{P}u$ was assumed to be a boundary point of $A(\omega)\cap\mathbb{P}L_{u,v}$ in $\mathbb{P}L_{u,v}$, and hence (3.3.1) holds for all $\omega\in F$.

Part 2. It will be shown that for each $\omega\in F$ (with F defined as in Part 1), that for arbitrary nonzero $u\in\mathbb{P}^{-1}A(\omega)$ and $v\in C$, the intersection $A(\omega)\cap\mathbb{P}L_{u,v}$ is a singleton. First note that any point in $\mathbb{P}L_{u,v}\backslash\{\mathbb{P}u\}$ is given by $\mathbb{P}(v+cu)$ for some $c\in\mathbb{R}$. It follows from Part 1 that for all $\omega\in F$

$$\lim_{t\to-\infty}\frac{\langle\Phi(t,\omega)(v+cu),\Phi(t,\omega)v\rangle^2}{\|\Phi(t,\omega)(v+cu)\|^2\|\Phi(t,\omega)v\|^2}=1$$

if $\mathbb{P}u$ is a boundary point of $A(\omega)\cap\mathbb{P}L_{u,v}$ relative to $\mathbb{P}L_{u,v}$. This then implies together with Lemma 3.3.1 that

$$\lim_{t\to-\infty}d_{\mathbb{P}}(\mathbb{P}\Phi(t,\omega)\mathbb{P}(v+cu),\mathbb{P}\Phi(t,\omega)\mathbb{P}v)=0 \qquad \mu\text{-a.s.}$$

If $\mathbb{P}(u+cv)\in A(\omega)$ then this implies that there is a $T\leq 0$ such that $\mathbb{P}\Phi(t,\omega)\mathbb{P}v\in B_\eta(A(\theta_t\omega))$ for all $t\leq T$, and then

$$\begin{aligned}\operatorname{dist}_{\mathbb{P}}(\mathbb{P}v,A(\omega)) &= \lim_{t\to-\infty}\operatorname{dist}_{\mathbb{P}}(\mathbb{P}\Phi(-t,\theta_t\omega)\mathbb{P}\Phi(t,\omega)\mathbb{P}v,A(\omega))\\ &=0\end{aligned}$$

which is a contradiction since $\mathbb{P}v$ is in the compact set $\mathbb{P}C$. Therefore, $A(\omega)\cap\mathbb{P}L_{u,v}$ consists of a single point.

Part 3. It now follows directly from Parts 1 and 2 that for almost all $\omega\in\Omega$, and all nonzero $u \in \mathbb{P}^{-1}A(\omega)$ and $v \in C$ one has

$$\lim_{t \to -\infty} \frac{\|\Phi(t,\omega)u\|}{\|\Phi(t,\omega)v\|} = 0.$$

Part 4. It is now shown that $\mathbb{P}^{-1}A(\omega)$ is a linear subspace of $\mathbb{R}^d$ μ almost surely. Part 2 asserted that for all $\omega \in F$ and for $u \in \mathbb{P}^{-1}A(\omega)$ and $v \in C$, $A(\omega) \cap \mathbb{P}L_{u,v}$ consists of a single point. Now this implies that for any two nonzero vectors $x, y \in \mathbb{R}^d$, we have μ-a.s. that $A(\omega) \cap \mathbb{P}L_{x,y}$ is either a single point, the empty set, or equals the set $\mathbb{P}L_{x,y}$. This implies that $\mathbb{P}^{-1}A(\omega)$ is a linear subspace.

Part 5. We finally show that

$$\lim_{t \to -\infty} \frac{\sup_{u \in \mathbb{S}^{d-1} \cap \mathbb{P}^{-1}A(\omega)} \|\Phi(t,\omega)u\|}{\inf_{v \in C} \|\Phi(t,\omega)v\|} = 0 \qquad \mu\text{-a.s.}$$

Let F be as defined in Part 1, and assume there exists an $\omega \in F$ such that the above does not hold, so that there exists a sequence of times $\{t_n\}_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} t_n = -\infty$, and sequences $\{u_n\}_{n \in \mathbb{N}}$ in $\mathbb{S}^{d-1} \cap \mathbb{P}^{-1}A(\omega)$ and $\{v_n\}_{n \in \mathbb{N}}$ in C, which by compactness one may assume converge with $\lim_{n \to \infty} u_n = u$ for some $u \in \mathbb{S}^{d-1} \cap \mathbb{P}^{-1}A(\omega)$ and $\lim_{n \to \infty} v_n = v$ for some $v \in C$, and there exists a $\gamma > 0$ such that the following holds:

$$\frac{\|\Phi(t_n,\omega)v_n\|}{\|\Phi(t_n,\omega)u_n\|} \le \gamma \qquad \text{for all } n \in \mathbb{N}.$$

Then, similarly to Part 1, for all nonzero $c \in \mathbb{R}$ with $|c|$ sufficiently small, this implies that for all $n \in \mathbb{N}$

$$\frac{\langle \Phi(t_n,\omega)(cv_n + u_n), \Phi(t_n,\omega)u_n \rangle^2}{\|\Phi(t_n,\omega)(cv_n + u_n)\|^2 \|\Phi(t_n,\omega)u_n\|^2} \ge 1 - \delta$$

holds, with $\delta \in (0,1)$ chosen as in Part 1. Hence by Lemma 3.3.1, for $|c| > 0$ sufficiently small one obtains

$$\operatorname{dist}_{\mathbb{P}}(\mathbb{P}\Phi(t_n,\omega)\mathbb{P}(cv_n + u_n), A(\theta_{t_n}\omega)) \le \eta \qquad \text{for all } n \in \mathbb{N}.$$

Part 2 gave that $A(\omega) \cap \mathbb{P}L_{u,v}$ is a singleton, and so $\mathbb{P}(cv + u) \notin A(\omega)$. Hence there exists an $n_0 \in \mathbb{N}$ and a $\beta > 0$ such that $\mathbb{P}(cv_n + u_n) \notin B_\beta(A(\omega))$ for all $n \ge n_0$. Then using the pullback attraction to A in a similar argument to the end of Part 1, we get a contradiction. □

Remark 3.3.4. By the above proposition and Theorem 2.7.5, given a local forward repeller R of $\mathbb{P}\Phi$ with a uniform forward repelling neighbourhood, one obtains a corresponding local pullback attractor A that almost surely corresponds to a linear subspace. In the nonautonomous theory the local forward repeller is nonunique, and may be chosen to be the projection of any invariant linear subspace complementary to the pullback attractor, so that one obtains a direct sum decomposition by the local forward repeller and local pullback attractor [KR11, Proposition 5.19]. For RDS we have seen (Remark 2.7.6 (i)) that R is almost surely unique, and we are unable to determine whether R must also necessarily correspond to a linear subspace, which we shall need to show the existence of a unique finest Morse decomposition in Theorem 3.3.6. For this reason we proceed by considering local strong attractors. Given a local strong attractor of $\mathbb{P}\Phi$ with a uniform strong attracting neighbourhood, since it is also a local forward attractor, by Theorem 2.7.5 there exists a corresponding local pullback repeller with a uniform pullback repelling neighbourhood. Then, since they are both pullback objects we can apply Proposition 3.3.3 to obtain that they both correspond to linear subspaces. Furthermore, it is shown in Theorem 3.3.5 that the local pullback repeller is in fact a local strong repeller, and that the attractor-repeller pair forms a direct sum decomposition of $\mathbb{R}^d$.

3.3.2 Strong attractor-repeller pairs for linear systems

The following theorem demonstrates the existence of strong attractor-repeller pairs with uniform neighbourhoods for projected linear RDS. The proof uses techniques from the proof of [KR11, Theorem 5.26] on the existence of so-called all-time attractor-repeller pairs for projected linear nonautonomous systems.

Theorem 3.3.5. *Given a local strong attractor A of $\mathbb{P}\Phi$ with attracting neighbourhood $B_\eta(A)$ for some $\eta > 0$, then the compact random set R given by the pullback α-limit set*

$$R(\omega) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \mathbb{P}\Phi(-t, \theta_t \omega)(\mathbb{P}^{d-1} \backslash B_\eta(A(\theta_t \omega)))}$$

is a local strong repeller with a repelling neighbourhood given by $B_\varepsilon(R)$ for some $\varepsilon > 0$. Furthermore, the following properties hold almost surely:

(i) A and R are uniformly isolated:

$$B_\eta(A(\omega)) \cap R(\omega) = \emptyset \tag{3.3.4}$$

(ii) $\mathbb{P}^{-1}A(\omega)$ and $\mathbb{P}^{-1}R(\omega)$ are linear subspaces and form a direct sum decomposition

$$\mathbb{P}^{-1}A(\omega) \oplus \mathbb{P}^{-1}R(\omega) = \mathbb{R}^d, \tag{3.3.5}$$

such that the dimensions of $\mathbb{P}^{-1}A(\omega)$ and $\mathbb{P}^{-1}R(\omega)$ are almost surely constant.

The pair (A, R) is called a strong attractor-repeller pair*. Moreover, (A, R) is also a past, future, pullback and weak attractor-repeller pair.*

Proof. By the reverse time version of Theorem 2.7.5 the compact random set R is a local pullback repeller such that (i) holds, and (A, R) is also a future, pullback and weak attractor-repeller pair. Next we demonstrate (ii). Since A is a local pullback attractor and R a local pullback repeller, Proposition 3.3.3 asserts that almost surely both $\mathbb{P}^{-1}A(\omega)$ and $\mathbb{P}^{-1}R(\omega)$ are linear subspaces of $\mathbb{R}^d$. Now define the closed random set $K\colon \Omega \to 2^{\mathbb{P}^{d-1}}$ by

$$K(\omega) := \begin{cases} \{x \in \mathbb{P}^{d-1}\colon \mathrm{dist}_{\mathbb{P}}(x, R(\omega)) \geq \sqrt{2}\}, & \omega \in F, \\ A(\omega), & \omega \in \Omega\backslash F, \end{cases}$$

where $F \in \mathcal{F}$ is the set of full measure where $\mathbb{P}^{-1}A(\omega)$ and $\mathbb{P}^{-1}R(\omega)$ are linear subspaces. Note that since

$$\mathrm{dist}_{\mathbb{P}}(x, R(\omega)) \geq \sqrt{2} \Leftrightarrow \langle \hat{x}, \hat{\xi} \rangle = 0$$

for any $\hat{x} \in \mathbb{P}^{-1}\{x\}$ and for all $\hat{\xi} \in \mathbb{P}^{-1}R(\omega)$, it means that for $\omega \in F$, $\mathbb{P}^{-1}K(\omega)$ is the orthogonal complement of $\mathbb{P}^{-1}R(\omega)$; hence $\mathbb{P}^{-1}R(\omega) \oplus \mathbb{P}^{-1}K(\omega) = \mathbb{R}^d$, and also $\dim(\mathbb{P}^{-1}K(\omega)) = d - \dim(\mathbb{P}^{-1}R(\omega))$. Note that $\omega \mapsto \dim(\mathbb{P}^{-1}A(\omega))$ is a θ-invariant function, since by the invertibility of Φ and invariance of A,

$$\begin{aligned}\dim(\mathbb{P}^{-1}A(\omega)) &= \dim(\mathbb{P}^{-1}(\mathbb{P}\Phi(t,\omega)A(\omega))) \\ &= \dim(\mathbb{P}^{-1}A(\theta_t\omega)).\end{aligned}$$

Also, since Φ is invertible, for all $\omega \in F$ and for all $t \in \mathbb{T}$, $\mathbb{P}^{-1}(\mathbb{P}\Phi(t,\omega)K(\omega))$ is also a linear subspace of dimension $d - \dim(\mathbb{P}^{-1}R(\omega))$. Since K is a compact set valued function such that $K(\omega) \cap R(\omega) = \emptyset$ μ-a.s., we have by the reversed time version of Theorem 2.7.8 (i) that

$$\lim_{t\to\infty} \mathrm{dist}_{\mathbb{P}}(\mathbb{P}\Phi(t,\omega)K(\omega), A(\theta_t\omega)) = 0 \qquad \mu\text{-a.s.}$$

Hence by the contrapositive of Lemma 3.3.2, there exists a $t > 0$ such that

$$\dim(\mathbb{P}^{-1}A(\theta_t\omega)) \geq d - \dim(\mathbb{P}^{-1}R(\omega))$$

and by θ-invariance we have $\dim(\mathbb{P}^{-1}A(\omega)) \geq d - \dim(\mathbb{P}^{-1}R(\omega))$ almost surely. Since $A(\omega) \cap R(\omega) = \emptyset$, we must have $\dim(\mathbb{P}^{-1}A(\omega)) = d - \dim(\mathbb{P}^{-1}R(\omega))$ and hence

$$\mathbb{P}^{-1}A(\omega) \oplus \mathbb{P}^{-1}R(\omega) = \mathbb{R}^d \quad \mu\text{-a.s.}$$

To show that $\dim(\mathbb{P}^{-1}A(\omega))$ is constant μ-a.s., we already have that it is a θ-invariant function and hence just need to show that it is measurable. For a linear random set M in $\mathbb{R}^d$ on a complete probability space, the proof of Corollary 4.7 in [Via14] demonstrates that the measurability of $\mathrm{dist}(\hat{x}, M(\omega))$ for all $\hat{x} \in \mathbb{R}^d$ is equivalent to the measurability of $\mathrm{dist}_{\mathbb{P}}(x, \mathbb{P}M(\omega))$ for all $x \in \mathbb{P}^{d-1}$; hence $\mathbb{P}^{-1}A(\omega)$ is a linear random set in $\mathbb{R}^d$ on F. Then by Lemma 5.2.1 in [Arn98], $\omega \mapsto \dim(\mathbb{P}^{-1}A(\omega))$ is measurable (restricted to F) and so by the ergodicity of θ, $\dim(\mathbb{P}^{-1}A(\omega))$ is almost surely constant.

Next we show that R is a local forward repeller as well as a local pullback repeller. The proof of this is divided into two parts.

Part 1. It will be shown that for almost all $\omega \in \Omega$ and compact sets C such that

$C \cap A(\omega) = \emptyset$, one has

$$\lim_{t\to-\infty} \inf_{0\neq v\in\mathbb{P}^{-1}C} \frac{\|\Phi(t,\omega)v_r\|}{\|\Phi(t,\omega)v\|} = \lim_{t\to-\infty} \sup_{0\neq v\in\mathbb{P}^{-1}C} \frac{\|\Phi(t,\omega)v_r\|}{\|\Phi(t,\omega)v\|} \\ = 1$$

where $v = v_a + v_r$ with $v_a \in \mathbb{P}^{-1}A(\omega)$ and $v_r \in \mathbb{P}^{-1}R(\omega)$. The infimum relation follows from

$$\begin{aligned}\lim_{t\to-\infty} \inf_{0\neq v\in\mathbb{P}^{-1}C} \frac{\|\Phi(t,\omega)v_r\|}{\|\Phi(t,\omega)v\|} &\geq \left(\lim_{t\to\infty} \sup_{0\neq v\in\mathbb{P}^{-1}C} \frac{\|\Phi(t,\omega)v_a\|}{\|\Phi(t,\omega)v_r\|} + 1\right)^{-1} \\ &= \left(\lim_{t\to-\infty} \sup_{v\in\mathbb{P}^{-1}C, v_a\neq 0} \frac{\|v_a\|\|\Phi(t,\omega)\frac{v_a}{\|v_a\|}\|}{\|v_r\|\|\Phi(t,\omega)\frac{v_r}{\|v_r\|}\|} + 1\right)^{-1} \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to-\infty} \inf_{0\neq v\in\mathbb{P}^{-1}C} \frac{\|\Phi(t,\omega)v_r\|}{\|\Phi(t,\omega)v\|} &\leq \left(\lim_{t\to-\infty} \sup_{0\neq v\in\mathbb{P}^{-1}C} \left|1 - \frac{\|\Phi(t,\omega)v_a\|}{\|\Phi(t,\omega)v_r\|}\right|\right)^{-1} \\ &= \left(\lim_{t\to-\infty} \sup_{v\in\mathbb{P}^{-1}C, v_a\neq 0} \left|1 - \frac{\|v_a\|\|\Phi(t,\omega)\frac{v_a}{\|v_a\|}\|}{\|v_r\|\|\Phi(t,\omega)\frac{v_r}{\|v_r\|}\|}\right|\right)^{-1} \\ &= 1.\end{aligned}$$

with the last line in each relation above following from Proposition 3.3.3, and the fact that $\|v_a\|/\|v_r\|$ is bounded. This fact can be seen by considering the projection $P \in \mathbb{R}^{d\times d}$ with range $\mathbb{P}^{-1}A(\omega)$ and null space $\mathbb{P}^{-1}R(\omega)$, which demonstrates that the following set is compact

$$\{v_a \colon v \in \mathbb{P}^{-1}C \cap \mathbb{S}^{d-1}\} = P(\mathbb{P}^{-1}C \cap \mathbb{S}^{d-1}),$$

and that the following set is compact and hence bounded away from zero (since $C \cap A(\omega) = \emptyset$):

$$\{v_r \colon v \in \mathbb{P}^{-1}C \cap \mathbb{S}^{d-1}\} = (\mathbb{1} - P)(\mathbb{P}^{-1}C \cap \mathbb{S}^{d-1}).$$

The supremum relation follows analogously.

Part 2. It will now be shown that for almost all $\omega \in \Omega$ and all compact sets $C \subset \mathbb{P}^{d-1}$ with $C \cap A(\omega) = \emptyset$, one has

$$\lim_{t \to -\infty} \operatorname{dist}_{\mathbb{P}}(\mathbb{P}\Phi(t,\omega)C, R(\theta_t\omega)) = 0. \tag{3.3.6}$$

By the Cauchy-Schwarz inequality and Proposition 3.3.3 one obtains

$$0 \leq \lim_{t\to-\infty} \sup_{v \in \mathbb{S}^{d-1}\cap\mathbb{P}^{-1}C} \frac{\langle \Phi(t,\omega)v_a, \Phi(t,\omega)v_r\rangle^2}{\|\Phi(t,\omega)v\|^2 \|\Phi(t,\omega)v_r\|^2} \leq \lim_{t\to-\infty} \sup_{v \in \mathbb{S}^{d-1}\cap\mathbb{P}^{-1}C} \frac{\|\Phi(t,\omega)v_a\|^2}{\|\Phi(t,\omega)v\|^2} = 0$$

and

$$\begin{aligned}
0 &\leq \lim_{t\to-\infty} \sup_{v \in \mathbb{S}^{d-1}\cap\mathbb{P}^{-1}C} \frac{2|\langle \Phi(t,\omega)v_a, \Phi(t,\omega)v_r\rangle|}{\|\Phi(t,\omega)v\|^2} \\
&\leq \lim_{t\to-\infty} \sup_{v \in \mathbb{S}^{d-1}\cap\mathbb{P}^{-1}C} 2\frac{\|\Phi(t,\omega)v_a\|}{\|\Phi(t,\omega)v\|}\frac{\|\Phi(t,\omega)v_r\|}{\|\Phi(t,\omega)v\|} \\
&= \lim_{t\to-\infty} \sup_{v \in \mathbb{S}^{d-1}\cap\mathbb{P}^{-1}C} 2\frac{\|\Phi(t,\omega)v_a\|}{\|\Phi(t,\omega)v\|} \\
&= 0.
\end{aligned}$$

Then since

$$\begin{aligned}
\frac{\langle \Phi(t,\omega)v, \Phi(t,\omega)v_r\rangle^2}{\|\Phi(t,\omega)v\|^2\|\Phi(t,\omega)v_r\|^2} &= \frac{(\langle \Phi(t,\omega)v_a, \Phi(t,\omega)v_r\rangle + \langle \Phi(t,\omega)v_r, \Phi(t,\omega)v_r\rangle)^2}{\|\Phi(t,\omega)v\|^2\|\Phi(t,\omega)v_r\|^2} \\
&= \frac{\langle \Phi(t,\omega)v_a, \Phi(t,\omega)v_r\rangle^2}{\|\Phi(t,\omega)v\|^2\|\Phi(t,\omega)v_r\|^2} + \frac{\|\Phi(t,\omega)v_r\|^2}{\|\Phi(t,\omega)v\|^2} \\
&\qquad + \frac{2\langle \Phi(t,\omega)v_a, \Phi(t,\omega)v_r\rangle}{\|\Phi(t,\omega)v\|^2},
\end{aligned}$$

it follows by the above and Part 1 that

$$\lim_{t\to-\infty} \sup_{v \in \mathbb{S}^{d-1}\cap\mathbb{P}^{-1}C} \frac{\langle \Phi(t,\omega)v, \Phi(t,\omega)v_r\rangle^2}{\|\Phi(t,\omega)v\|^2\|\Phi(t,\omega)v_r\|^2} = 1.$$

Using Lemma 3.3.1, this implies the assertion. It follows from (3.3.6) and (3.3.4) that R forward repels a neighbourhood, say $B_{\eta/2}(R)$, and hence it is a local strong repeller, and

(A, R) is also a past attractor-repeller pair. □

Below we give the analogue of Selgrade's Theorem for strong Morse decompositions. The proof is an adaptation of that for [KR11, Theorem 5.23] on the existence of a finest past Morse decomposition for a projected linear nonautonomous system.

Theorem 3.3.6. *There exists a unique finest strong Morse decomposition* $\{M_1, \ldots, M_n\}$ *of* $\mathbb{P}\Phi$*. Moreover,* $n \leq d$*, and for almost all* $\omega \in \Omega$ *the Morse sets correspond to a direct sum decomposition:*

$$\mathbb{P}^{-1}M_1(\omega) \oplus \cdots \oplus \mathbb{P}^{-1}M_n(\omega) = \mathbb{R}^d.$$

Proof. It is first shown that for any two local pullback attractors A and $\tilde{A}$ that differ on a positive measure set, we have that either $A(\omega) \subset \tilde{A}(\omega)$ or $\tilde{A}(\omega) \subset A(\omega)$, almost surely. In the case that one or both of A and $\tilde{A}$ are trivial this is simple, and so assume that both are nontrivial. Let $F \in \mathcal{F}$ be the set of full measure where Proposition 3.3.3 holds for both A and $\tilde{A}$, and suppose there is an $\omega \in F$ such that there exist vectors

$$x \in \mathbb{S}^{d-1} \cap (\mathbb{P}^{-1}A(\omega) \backslash \mathbb{P}^{-1}\tilde{A}(\omega)) \quad \text{and} \quad \tilde{x} \in \mathbb{S}^{d-1} \cap (\mathbb{P}^{-1}\tilde{A}(\omega) \backslash \mathbb{P}^{-1}A(\omega)).$$

By Proposition 3.3.3 one obtains that both

$$\lim_{t \to -\infty} \frac{\|\Phi(t, \omega)x\|}{\|\Phi(t, \omega)\tilde{x}\|} = 0 \quad \text{and} \quad \lim_{t \to -\infty} \frac{\|\Phi(t, \omega)\tilde{x}\|}{\|\Phi(t, \omega)x\|} = 0,$$

which are contradictory statements. Proposition 3.3.3 also establishes that the ω-sections of local pullback attractors almost surely correspond to linear subspaces. Therefore there are at most $d + 1$ local strong attractors of $\mathbb{P}\Phi$, namely $\{A_0, \ldots, A_n\}$, $n \leq d$, such that

$$\emptyset = A_0(\omega) \subsetneq A_1(\omega) \subsetneq \cdots \subsetneq A_n(\omega) = \mathbb{P}^{d-1} \quad \mu\text{-a.s.}$$

Let $\{M_1, \ldots, M_n\}$ denote the corresponding strong Morse decomposition, and let $\{\tilde{M}_1, \ldots, \tilde{M}_m\}$ be another strong Morse decomposition obtained from a set of local strong attractors $\{\tilde{A}_0, \ldots, \tilde{A}_m\}$, $m \leq d$ with $\emptyset = \tilde{A}_0(\omega) \subsetneq \tilde{A}_1(\omega) \subsetneq \cdots \subsetneq \tilde{A}_m(\omega) = \mathbb{P}^{d-1}$ μ-a.s. Now for each $i \in \{0, \ldots, m\}$, there exists an $n_i \in \{0, \ldots, n\}$ such that $\tilde{A}_i(\omega) = A_{n_i}(\omega)$ almost surely. Hence we have that $M_{n_i}(\omega) \subset \tilde{M}_i(\omega)$ almost surely, and $\mathcal{M} := \{M_1, \dots M_n\}$ is the unique (up to a null set) finest strong Morse decomposition.

Now the decomposition,

$$\mathbb{P}^{-1}M_1(\omega) \oplus \cdots \oplus \mathbb{P}^{-1}M_n(\omega) = \mathbb{R}^d \quad \mu\text{-a.s.}$$

will be shown. Using (3.3.5) we have that, μ-a.s.,

$$\begin{aligned}
\mathbb{R}^d &= \mathbb{P}^{-1}A_1(\omega) \oplus \mathbb{P}^{-1}R_1(\omega) \\
&= \mathbb{P}^{-1}M_1(\omega) \oplus (\mathbb{P}^{-1}R_1(\omega) \cap (\mathbb{P}^{-1}A_2(\omega) \oplus \mathbb{P}^{-1}R_2(\omega))) \\
&= \mathbb{P}^{-1}M_1(\omega) \oplus (\mathbb{P}^{-1}R_1(\omega) \cap \mathbb{P}^{-1}A_2(\omega)) \oplus \mathbb{P}^{-1}R_2(\omega) \\
&= \mathbb{P}^{-1}M_1(\omega) \oplus \mathbb{P}^{-1}M_2(\omega) \oplus \mathbb{P}^{-1}R_2(\omega),
\end{aligned}$$

where the third line follows from the fact that for linear subspaces $E, F, G \subset \mathbb{R}^d$ with $G \subset E$, the following holds:

$$E \cap (F + G) = (E \cap F) + G.$$

The result then follows by induction. □

3.4 Discussion

As noted earlier, the dichotomy spectrum for RDS has also been investigated by Wang and Cao [WC14] with a different set-up to ours, and we shall now describe the key differences in the set-up and results. There the authors consider a random linear skew product, that is, they take an invertible, bounded and continuous linear skew product with a compact metric base space and Banach state space, and then randomize it by placing an ergodic metric DS as the base to this system. Their definition of an exponential dichotomy is weaker than ours, in particular they allow the constants K and ε to be certain random variables, and hence have a nonuniform notion of an exponential dichotomy. Neither one of our set-ups includes the other. They obtain a similar spectral theory and spectral manifolds as we do: under the assumptions that the spectrum is hyperbolic (i.e. zero is in the spectrum), the compact base is connected and that an integrability condition holds, they obtain a nonempty and bounded spectrum and associated invariant linear spectral manifolds. This contrasts to our theory where we allow the spectrum to be unbounded and nonhyperbolic. It would be interesting to see if it is possible to allow for nonuniformity of the constants within our set-up. Given the importance of the dichotomy spectrum in nonautonomous systems (see the introduction in [WC14] for an overview and the references therein), it will be important to investigate the use of the dichotomy spectrum in analogous situations in random dynamics.

We now discuss the results of Section 3.3. We have established an analogue of Selgrade's Theorem in Theorem 3.3.6 using strong Morse decompositions. As noted in Remark 3.3.4, we required the definition of a local strong attractor in order to obtain that the components of the attractor-repeller pair correspond to linear subspaces. This relied on Propostion 3.3.3, which explicitly uses pullback arguments, i.e. using the fact that pullback convergence is convergence to a fixed set at time zero, and we would not expect to be able to prove directly that a local forward attractor corresponds to a linear subspace. We believe it should be possible to establish that the components of a weak attractor-repeller pair for a projected linear RDS correspond to linear subspaces, by using weak attraction in the pullback sense. Establishing this result would be desirable, as it would then apply to all the other attractor-repeller pair types considered in Chapter 2. However, the analysis proves more difficult as linearity of the set needs to be established using random variables, as opposed to the case of almost sure pullback attraction where one works pointwise in ω from a full measure set. In [CDS04, Theorem 6.1] it is shown that the projection of the Oseledets subspaces form a weak Morse decomposition in the projective space. Since in general the angle between the Oseledets subspaces may not be bounded away from zero, and the Morse sets of a strong Morse decomposition are uniformly isolated (this follows from Theorem 3.3.5 (i)), it follows that the decomposition by Oseledets subspaces will not be a strong Morse decomposition in general. On the other hand, we believe it should be straightforward to prove an analogue of Theorem 6.9 in [Ras09], that is, the projection of the spectral manifolds of the dichotomy spectrum represents a strong Morse decomposition.

The next step from obtaining the existence of a finest Morse decomposition is to define the *Morse spectrum* and investigate its basic properties; we have yet to do this for RDS and it forms future research. Following the deterministic definition given in [Grü00], a natural definition for the Morse spectrum for a linear RDS Φ is the following. Let $\{M_1, \dots, M_n\}$ be the finest Morse decomposition of $\mathbb{P}\Phi$, then for each $\omega \in \Omega$, $i \in \{1, \dots, n\}$ define

$$\begin{aligned}\Xi(M_i)(\omega) := \{\xi \in \overline{\mathbb{R}} \colon \text{ there exists a sequence } \{(T_k, t_k, x_k)\}_{k \in \mathbb{N}} \text{ with } T_k, t_k \in \mathbb{T},\\ x_k \in \mathbb{P}^{-1} M(\theta_{t_k}\omega) \backslash \{0\} \text{ such that } \lim_{k \to \infty} T_k = \infty \text{ and } \lim_{k \to \infty} \lambda^{T_k}(\theta_{t_k}\omega, x_k) = \xi\},\end{aligned}$$

where

$$\lambda^T(\omega, x) := \frac{1}{T} \ln \frac{\|\Phi(T, \omega)x\|}{\|x\|}$$

is the finite-time Lyapunov exponent. Then the Morse spectrum is given by

$$\Xi(\omega) := \bigcup_{i=1}^{n} \Xi(M_i)(\omega).$$

Given the ergodicity of θ this should be almost surely constant. For nonautonomous systems it has been shown that the Morse spectrum coincides with the dichotomy spectrum [Ras10], and it would be interesting to see if an analogous result can be obtained for random dynamical systems.

Chapter 4

Pitchfork bifurcation with additive noise*

Despite its importance for applications, relatively little progress has been made towards the development of a bifurcation theory for random dynamical systems. Main contributions have been made by Ludwig Arnold and co-workers [Arn98], distinguishing between *phenomenological* (P-) and *dynamical* (D-) bifurcations. P-bifurcations refer to qualitative changes in the profile of densities of stationary probability measures [SN90, HL84]. This concept carries substantial drawbacks such as providing reference only to static properties and not being independent of the choice of coordinates. D-bifurcations refer to the bifurcation of a new invariant measure from a given reference invariant measure, in the sense of weak convergence, and are associated with a qualitative change in the Lyapunov spectrum [Arn98, Theorem 9.2.3]. They have been studied mainly in the case of multiplicative noise [Bax94, CIS99, Wan14], and numerically [ABSH99, KO99].

Here we contribute to the bifurcation theory of random dynamical systems by shedding new light on the influential paper *Additive noise destroys a pitchfork bifurcation* by Crauel and Flandoli [CF98], in which the one-dimensional stochastic differential equation

$$\mathrm{d}x_t = \big(\alpha x_t - x_t^3\big)\mathrm{d}t + \sigma \mathrm{d}W_t\,, \tag{4.0.1}$$

*The material in this chapter follows that of [CDLR13, Sections 4 & 5].

with two-sided Wiener process $\{W_t\}_{t\in\mathbb{R}}$ and parameters $\alpha, \sigma \in \mathbb{R}$, was studied. In the deterministic (noise-free) case, $\sigma = 0$, this system has a pitchfork bifurcation of equilibria: if $\alpha < 0$ there is one equilibrium ($x = 0$) which is globally attractive, and if $\alpha > 0$, the trivial equilibrium is repulsive and there are two additional attractive equilibria $\pm\sqrt{\alpha}$. [CF98] establish the following facts in the presence of noise, i.e. when $|\sigma| > 0$:

(i) For all $\alpha \in \mathbb{R}$, there is a unique invariant measure given by a globally attracting random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$.

(ii) The Lyapunov exponent associated to $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is negative for all $\alpha \in \mathbb{R}$.

As a result, [CF98] concludes that the pitchfork bifurcation is destroyed by the additive noise. (This refers to the absence of D-bifurcation, as (4.0.1) admits a qualitative change P-bifurcation from a unimodal distribution to a bimodal distribution, see [Arn98, p. 473].) However, we are inclined to argue that the pitchfork bifurcation is not destroyed by additive noise, on the basis of the following additional facts concerning the dynamics near the bifurcation point that we obtain here:

(i) The attracting random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is uniformly attractive only if $\alpha < 0$ (Theorem 4.2.3).

(ii) At the bifurcation point there is a change in the practical observability of the Lyapunov exponent: when $\alpha < 0$ all finite-time Lyapunov exponents are negative, but when $\alpha > 0$ there is a positive probability to observe positive finite-time Lyapunov exponents, irrespective of the length of time interval under consideration (Theorem 4.2.5).

(iii) The bifurcation point $\alpha = 0$ is characterized by a qualitative change in the dichotomy spectrum associated to $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ (Theorem 4.3.1). In addition, we show that the dichotomy spectrum is directly related to the observability range of the finite-time Lyapunov spectrum (Theorem 4.3.2).

In light of these findings, we thus argue for the recognition of qualitative properties of the dichotomy spectrum as an additional indicator for bifurcations of random dynamical systems.

In analogy to the corresponding bifurcation theory for one-dimensional deterministic dynamical systems, we finally study whether the pitchfork bifurcation with additive noise can be characterized in terms of a breakdown of topological equivalence. We recall that two random dynamical systems (θ, φ_1) and (θ, φ_2) are said to be topologically equivalent if there are families $\{h_\omega\}_{\omega\in\Omega}$ of homeomorphisms of the state space such that $\varphi_2(t, \omega, h_\omega(x)) = h_{\theta_t\omega}(\varphi_1(t, \omega, x))$, almost surely (Definition 1.1.11). We establish the following results for the stochastic differential equation (4.0.1):

(i) Throughout the bifurcation, i.e. for $|\alpha|$ sufficiently small, the resulting dynamics are topologically equivalent (Theorem 4.4.1).

(ii) There does not exist a uniformly continuous topological conjugacy between the dynamics of cases with positive and negative parameter α (Theorem 4.4.4).

These results lead us to propose the association of bifurcations of random dynamical systems with a breakdown of *uniform* topological equivalence, rather than the weaker form of general topological equivalence with no requirement on uniform continuity of the involved conjugacy. Note that uniformity of equivalence transformations plays an important role in the notion of equivalence for nonautonomous linear systems (i.e. in contrast to random systems, the base set of nonautonomous systems is not a probability but a topological space), see [Pal79].

This chapter is organized as follows. We first review in Section 4.1 the main results of Crauel and Flandoli [CF98]. We then show in Section 4.2 the change from uniform to nonuniform attractivity of the random fixed points $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ at the bifurcation point $\alpha = 0$, and also that there is a positive probability to observe positive finite-time Lyapunov exponents for $\alpha > 0$. In Section 4.3 we demonstrate that at the bifurcation point, one losses hyperbolicity of the dichotomy spectrum obtained from the linearization along the random fixed point. In Section 4.4 we demonstrate the topological equivalence of the systems corresponding to α values sufficiently close to zero, and the lack of uniform topological equivalence between systems with negative and positive α values. Finally, we discuss our results and future work in Section 4.5.

4.1 Existence of a unique random attracting fixed point

We first look at the case where $\sigma = 0$ in (4.0.1) i.e. the deterministic initial value problem

$$\dot{x} = \alpha x - x^3, \quad x(0) = x_0\,, \tag{4.1.1}$$

and we denote the solution by $\phi\colon \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$. For $\alpha < 0$, the ordinary differential equation (4.1.1) has one equilibrium ($x = 0$) which is globally attractive. For positive α, the trivial equilibrium becomes repulsive, and there are two additional equilibria, given by $\pm\sqrt{\alpha}$, which are attractive. This also means that the global attractor K_α of the deterministic equation undergoes a bifurcation from a trivial to a nontrivial object. It is given by

$$K_\alpha := \begin{cases} \{0\} & : \alpha \le 0\,, \\ \big[-\sqrt{\alpha}, \sqrt{\alpha}\big] & : \alpha > 0\,. \end{cases} \tag{4.1.2}$$

It was shown in [CF98] that such an attractor bifurcation does not persist to the randomly perturbed system, i.e. for $|\sigma| > 0$, and we will now explain the details.

Since the solutions of (4.0.1) explode in backward time, we need the notion of a local RDS, see [Arn98, Section 1.2] for further details.

Definition 4.1.1. Suppose that $\mathbb{T} = \mathbb{R}$, and $(\Omega, \mathcal{F}, \mu, \theta)$ is a metric DS. A *local continuous random dynamical system* over θ on a topological space X is a measurable mapping

$$\varphi\colon D \to X, \qquad (t, \omega, x) \mapsto \varphi(t, \omega, x)$$

where $D \subset \mathbb{R} \times \Omega \times X$ is a measurable set, with the following properties: For all $\omega \in \Omega$

(i) The random *domain*

$$D(\omega) := \{(t, x) \in \mathbb{R} \times X \colon (t, \omega, x) \in D\} \subset \mathbb{R} \times X$$

is nonvoid and open, and

$$\varphi(\omega)\colon D(\omega) \to X, \qquad (t, x) \mapsto \varphi(t, \omega, x)$$

is continuous.

(ii) For each $x \in X$

$$D(\omega, x) := \{t \in \mathbb{R} \colon (t, \omega, x) \in D\} \subset \mathbb{R}$$

is an open interval containing 0, hence can be written as

$$D(\omega, x) =: (\kappa^-(\omega, x), \kappa^+(\omega, x)).$$

(iii) $\varphi(\omega)$ satisfies the *local cocycle property*:

$$\varphi(0, \omega) = \mathrm{id}_X$$

and for all $x \in X$ and all $s \in D(\omega, x)$ we have the following property: $t \in D(\theta_s\omega, \varphi(s, \omega, x))$ if and only if $t + s \in D(\omega, x)$. In this case we have,

$$\varphi(t + s, \omega)x = \varphi(t, \theta_s\omega)\varphi(s, \omega)x.$$

Some basic properties of local continuous RDS are given in [Arn98, Theorem 1.2.3]. By [Arn98, Theorem 2.3.36] the stochastic differential equation (4.0.1) generates a local continuous RDS $\varphi : D \to \mathbb{R}$ over the standard metric DS $(\Omega, \mathcal{F}, \mu, \theta)$ representing the Wiener process. Solutions may explode only in backward time, and for each $(\omega, x) \in \Omega \times \mathbb{R}$ we have $D(\omega, x) = (\kappa(\omega, x), \infty)$, where the measurable function $\kappa \colon \Omega \times X \to \mathbb{R}^-$ is the *explosion time* of the trajectory $\varphi(\cdot, \omega)x$ starting at x at time $t = 0$.

It can be shown [Arn98, p. 474] that for any $\alpha \in \mathbb{R}$, $|\sigma| > 0$, the Markov semigroup associated with (4.0.1) admits a unique stationary measure $\rho_{\alpha,\sigma}$ which is equivalent to the Lebesgue measure, with density

$$p_{\alpha,\sigma}(x) = N_{\alpha,\sigma} \exp\left(\tfrac{1}{\sigma^2}(\alpha x^2 - \tfrac{1}{2}x^4)\right), \tag{4.1.3}$$

where $N_{\alpha,\sigma}$ is a normalization constant. This stationary measure ρ corresponds to an invariant measure ν of the RDS φ, and ν has the disintegration given by

$$\nu_\omega = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho \quad \text{for almost all } \omega \in \Omega\,.$$

It was shown in [CF98] that ν_ω is a Dirac measure concentrated on a random variable $a_\alpha(\omega)$, and is the unique invariant measure for the random dynamical system. The random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is the only solution of (4.0.1) which does not explode in backward time (that is, $D(\omega, a_\alpha(\omega)) = \mathbb{R}$). Linearizing along this random fixed point yields a negative Lyapunov exponent, given by

$$\lambda_\alpha = -\frac{2}{\sigma^2}\int_{\mathbb{R}} (\alpha x - x^3)^2 p_{\alpha,\sigma}(x)\,\mathrm{d}x\,.$$

Moreover, the random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is the unique global pullback attractor (see Definition 4.1.2 below) of φ, which implies that the attractor bifurcation associated with the deterministic pitchfork bifurcation (that is, K_α bifurcates from a singleton to a nontrivial object) is destroyed by noise.

In the previous chapters we have considered notions of local attraction for RDS. Here we deal with a global random attractor that attracts in the pullback sense (see in particular [CF94, FS96, CDF97, Cra99]).

Definition 4.1.2. Let φ be an RDS with complete separable metric state space (X, d). A closed random set A is called a *global pullback attractor* if

(i) $A(\omega)$ is compact μ-a.s.

(ii) A is invariant.

(iii) For all bounded deterministic sets $B \subset X$ one has

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t, \theta_{-t}\omega)B, A(\omega)) = 0 \qquad \mu\text{-a.s.}$$

A global pullback attractor, when it exists, is almost surely unique, and the existence of a global pullback attractor may be guaranteed by the existence of a compact *global pullback absorbing set* [Cra99, Theorem 4.3].

Definition 4.1.3. Let φ be an RDS with complete separable metric state space (X, d). A bounded random set G is called a *global pullback absorbing set* if for any bounded deterministic set $B \subset X$ there exists a (possibly random) time $T_B(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega)B \subset G(\omega) \qquad \text{for all } t \geq T_B(\omega) \text{ and almost all } \omega \in \Omega.$$

4.2 Qualitative changes in uniform attractivity

In order to demonstrate a qualitative change in the attractivity of the unique random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ of (4.0.1) we first establish that for all parameter values $\alpha \in \mathbb{R}$, for an arbitrarily large time interval this random fixed point may remain arbitrarily close to the origin with positive probability.

Proposition 4.2.1. *Consider the RDS φ generated by* (4.0.1)*, and let $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ be its unique random fixed point corresponding to the parameter value $\alpha \in \mathbb{R}$. Then for any $\varepsilon > 0$ and $T \geq 0$, there exists a measurable set $\mathcal{A} \in \mathcal{F}_{-\infty}^{T}$ (see Section 1.1:* Stochastic differential equations*, p. 19) of positive measure such that*

$$a_\alpha(\theta_s\omega) \in (-\varepsilon, \varepsilon) \quad \textit{for all } s \in [0, T] \textit{ and } \omega \in \mathcal{A}\,.$$

Proof. The unique stationary measure $\rho_{\alpha,\sigma}$ for the Markov semigroup associated to (4.0.1) with $|\sigma| > 0$ is equivalent to the Lebesgue measure with the density function given by (4.1.3). The invariant measure $\delta_{a(\omega)}$ and stationary measure ρ are in correspondence by the following relations: the invariant measure is obtained as the limit of the pullback images of the stationary measure, i.e.

$$\delta_{a(\omega)} = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho \quad \text{for almost all } \omega \in \Omega\,,$$

and the stationary measure is obtained as the expectation of the invariant measure, i.e.

$$\rho(\cdot) = \int_\Omega \delta_{a(\omega)}(\cdot)\mathrm{d}\mu(\omega) \tag{4.2.1}$$

(see [CF98]). Now define

$$\eta := \frac{\varepsilon e^{-|\alpha|T}}{2(1+|\sigma|)}.$$

Since the support of ρ is the entire real line, it follows from (4.2.1) that the set

$$A_1 := \{\omega \in \Omega \colon a_\alpha(\omega) \in (-\eta, \eta)\} \tag{4.2.2}$$

has positive probability for any $\alpha \in \mathbb{R}$. The global pullback attractor $\{a_\alpha(\omega)\}$ is measurable with respect to the past of the noise $\mathcal{F}_{-\infty}^0$ (again see [CF98]), and hence $A_1 \in \mathcal{F}_{-\infty}^0$. Define

$$A_2 := \left\{\omega \in \Omega \colon \sup_{t\in[0,T]} |\omega(t)| \le \tfrac{\eta}{2}\right\} \in \mathcal{F}_0^T$$

which, by [IW81, Section 6.8], has positive probability. Since the sets A_1 and A_2 are independent, the set $\mathcal{A} := A_1 \cap A_2 \in \mathcal{F}_{-\infty}^T$ also has positive probability. Choose and fix an arbitrary $\omega \in \mathcal{A}$. By the definition of A_1 we have that $|a_\alpha(\omega)| < \eta$. Since $a_\alpha(\omega)$ is a random fixed point of φ it follows, using the integral from of (4.0.1), that

$$a_\alpha(\theta_t\omega) = a_\alpha(\theta_s\omega) + \int_s^t \left(\alpha a_\alpha(\theta_r\omega) - a_\alpha(\theta_r\omega)^3\right) \mathrm{d}r + \sigma(\omega(t) - \omega(s)). \tag{4.2.3}$$

Choose and fix an arbitrary $t \in [0,T]$. Define $\mathcal{I} := \{s \in [0,t] \colon a_\alpha(\theta_s\omega) = 0\}$; by continuity the set $\mathcal{I}$ is closed (but possibly empty). We consider the following three cases:

Case 1. If $t \in \mathcal{I}$, then $|a_\alpha(\theta_t\omega)| = 0$.

Case 2. If $\mathcal{I}$ is not empty and $t \notin \mathcal{I}$, then $s := \sup \mathcal{I} < t$ and $a_\alpha(\theta_s\omega) = 0$. By the definition of $\mathcal{I}$ and continuity, we have either $a_\alpha(\theta_r\omega) > 0$ for all $r \in (s,t]$ or $a_\alpha(\theta_r\omega) < 0$ for all $r \in (s,t]$. Using this observation and (4.2.3), we obtain that

$$|a_\alpha(\theta_t\omega)| \le |\sigma|\eta + \int_s^t |\alpha||a_\alpha(\theta_r\omega)| \,\mathrm{d}r.$$

Case 3. If $\mathcal{I}$ is empty, then either $a_\alpha(\theta_s\omega) > 0$ for all $s \in [0,t]$ or $a_\alpha(\theta_s\omega) < 0$ for all $s \in [0,t]$. Using (4.2.3) and noting that $|a_\alpha(\omega)| < \eta$, we arrive at the following inequality:

$$|a_\alpha(\theta_t\omega)| \le (1+|\sigma|)\eta + \int_0^t |\alpha||a_\alpha(\theta_s\omega)| \,\mathrm{d}s.$$

In view of the three cases above, we have that

$$|a_\alpha(\theta_t\omega)| \le (1+|\sigma|)\eta + \int_0^t |\alpha||a_\alpha(\theta_s\omega)|\,\mathrm{d}s \quad \text{for all } t\in[0,T]\,.$$

Then, using Gronwall's inequality, we obtain that

$$|a_\alpha(\theta_t\omega)| \le (1+|\sigma|)\eta e^{|\alpha|t} < \varepsilon \quad \text{for all } t\in[0,T]\,.$$

Thus we have that for all $\omega\in\mathcal{A}$, $a_\alpha(\theta_t\omega)\in(-\varepsilon,\varepsilon)$ for all $t\in[0,T]$, which completes the proof. □

We now give a detailed description of the random bifurcation scenario for the stochastic differential equation (4.0.1) by means of both *asymptotic* and *finite-time* dynamical behaviour. The change in asymptotic behaviour at the bifurcation point $\alpha=0$ is apparent as a qualitative change in the uniformity of attraction of the unique random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$. Then in terms of finite-time dynamics, after the bifurcation point, the (asymptotic) Lyapunov exponent is not observable by a finite-time Lyapunov exponent (over an arbitrarily large time interval) with nonzero probability; however, before the bifurcation, the (asymptotic) Lyapunov exponent can be approximated by the finite-time Lyapunov exponent almost surely. Finite-time Lyapunov exponents for random dynamical systems have not been considered in the literature so far, but play an important role in the description of Lagrangian coherent structures in fluid dynamics [HY00].

Definition 4.2.2. Let $\varphi\colon \mathbb{T}\times\Omega\times X\to X$ be an RDS with a random fixed point $\{a(\omega)\}_{\omega\in\Omega}$. Then $\{a(\omega)\}_{\omega\in\Omega}$ is called *locally uniformly attractive* if there exists a $\delta>0$ such that

$$\lim_{t\to\infty}\sup_{x\in B_\delta(a(\omega))}\operatorname*{ess\,sup}_{\omega\in\Omega} d(\varphi(t,\omega)x, a(\theta_t\omega)) = 0.$$

Theorem 4.2.3. *Consider the stochastic differential equation* (4.0.1) *with the unique attracting random fixed point* $\{a_\alpha(\omega)\}_{\omega\in\Omega}$*. Then the following statements hold:*

(i) For $\alpha<0$*, the random fixed point* $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ *is locally uniformly attractive; in*

fact, it is even globally uniformly exponentially attractive, i.e.

$$|\varphi(t,\omega,x)-\varphi(t,\omega,a_\alpha(\omega))|\leq e^{\alpha t}|x-a_\alpha(\omega)| \quad \textit{for all } x\in\mathbb{R}\,. \tag{4.2.4}$$

(ii) For $\alpha>0$, the random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is not locally uniformly attractive.

Proof. (i) Let $x\in\mathbb{R}$ be arbitrary such that $x\neq a_\alpha(\omega)$. Since φ is monotone with respect to initial conditions [Arn98, Theorem 1.8.4 (i)], we may assume that $\varphi(t,\omega,x)>\varphi(t,\omega,a_\alpha(\omega))$ for all $t\geq 0$. The integral form of (4.0.1),

$$\varphi(t,\omega)x = x+\int_0^t \big(\alpha\varphi(s,\omega)x-(\varphi(s,\omega)x)^3\big)\,\mathrm{d}s+\sigma\omega(t),$$

yields that

$$\varphi(t,\omega)x-\varphi(t,\omega)a_\alpha(\omega)\leq x-a_\alpha(\omega)+\alpha\int_0^t\big(\varphi(s,\omega)x-\varphi(s,\omega)a_\alpha(\omega)\big)\,\mathrm{d}s\,.$$

Using Gronwall's inequality implies (4.2.4), which finishes this part of the proof.

(ii) Suppose to the contrary that there exists $\delta>0$ such that

$$\lim_{t\to\infty}\sup_{x\in(-\delta,\delta)}\operatorname*{ess\,sup}_{\omega\in\Omega}|\varphi(t,\omega,a_\alpha(\omega)+x)-a_\alpha(\theta_t\omega)|=0\,,$$

which implies that there exists $N\in\mathbb{N}$ such that

$$\sup_{x\in(-\delta,\delta)}\operatorname*{ess\,sup}_{\omega\in\Omega}|\varphi(t,\omega,a_\alpha(\omega)+x)-a_\alpha(\theta_t\omega)|<\frac{\sqrt{\alpha}}{4}\quad\text{for all } t\geq N\,. \tag{4.2.5}$$

According to Proposition 4.2.1, there exists $\mathcal{A}\in\mathcal{F}_{-\infty}^0$ of positive probability such that $a_\alpha(\omega)\in(-\frac{\delta}{2},\frac{\delta}{2})$. Consider the corresponding deterministic ($\sigma=0$) system to (4.0.1), that is the differential equation (4.1.1) which has the two attractive fixed points $-\sqrt{\alpha}$ and $\sqrt{\alpha}$, and let $\phi(\cdot,x_0)$ denote the solution which satisfies $x(0)=x_0$. Then there exists $T>N$ such that

$$\phi(T,\delta/2)>\frac{\sqrt{\alpha}}{2}\quad\text{and}\quad\phi(T,-\delta/2)<-\frac{\sqrt{\alpha}}{2}\,. \tag{4.2.6}$$

For any $\varepsilon > 0$, we define

$$\mathcal{A}_\varepsilon^+ := \left\{\omega \in \Omega : \sup_{t\in[0,T]} |\omega(t)| < \varepsilon\right\}.$$

By [IW81, Section 6.8], $\mathcal{A}_\varepsilon^+ \in \mathcal{F}_0^T$ has positive probability, and thus, $\mu(\mathcal{A} \cap \mathcal{A}_\varepsilon^+) = \mu(\mathcal{A})\mu(\mathcal{A}_\varepsilon^+)$ is positive. Due to the compactness of $[0,T]$, there exists $\varepsilon > 0$ such that for all $\omega \in \mathcal{A}_\varepsilon^+$, we have

$$|\varphi(T,\omega,\delta/2) - \phi(T,\delta/2)| < \frac{\sqrt{\alpha}}{4} \quad \text{and} \quad |\varphi(T,\omega,-\delta/2) - \phi(T,-\delta/2)| < \frac{\sqrt{\alpha}}{4},$$

which implies together with (4.2.6) that

$$\varphi(T,\omega,\delta/2) > \frac{\sqrt{\alpha}}{4} \quad \text{and} \quad \varphi(T,\omega,-\delta/2) < -\frac{\sqrt{\alpha}}{4}.$$

Since $|a_\alpha(\omega)| < \frac{\delta}{2}$ for all $\omega \in \mathcal{A} \cap \mathcal{A}_\varepsilon^+$, we obtain that for all $\omega \in \mathcal{A} \cap \mathcal{A}_\varepsilon^+$

$$\begin{aligned}\sup_{x\in(-\delta,\delta)} &|\varphi(T,\omega,a_\alpha(\omega)+x) - a_\alpha(\theta_T\omega)| \\ &\geq \max\left\{\varphi(T,\omega,\delta/2) - a_\alpha(\theta_T\omega)|, |\varphi(T,\omega,-\delta/2) - a_\alpha(\theta_T\omega)|\right\}.\end{aligned}$$

Consequently,

$$\sup_{x\in(-\delta,\delta)} \operatorname*{ess\,sup}_{\omega\in\Omega} |\varphi(t,\omega,a_\alpha(\omega)+x) - a_\alpha(\theta_t\omega)| > \frac{\sqrt{\alpha}}{4},$$

which contradicts (4.2.5) and the proof is complete. □

For the description of the bifurcation via finite-time dynamics, we consider finite-time Lyapunov exponents.

Definition 4.2.4. Given a linear RDS $\Phi\colon \mathbb{T} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, for $T > 0$, $\omega \in \Omega$ and $x \in \mathbb{R}^d \backslash \{0\}$ the *finite-time Lyapunov exponent* is defined as

$$\lambda^T(\omega,x) := \frac{1}{T} \ln \frac{\|\Phi(T,\omega)x\|}{\|x\|}.$$

Clearly the (classical) Lyapunov exponent λ (see Theorem 3.0.2) associated with the vector $x \in \mathbb{R}^d \backslash \{0\}$ is given by

$$\lambda(\omega, x) = \lim_{T\to\infty} \lambda^T(\omega, x)\,.$$

The finite-time Lyapunov exponent associated with the linearization along the random fixed point $a_\alpha(\omega)$ is given by

$$\lambda_\alpha^T(\omega) := \frac{1}{T} \ln \left| \frac{\partial \varphi_\alpha}{\partial x}(T, \omega, a_\alpha(\omega)) \right|\,.$$

In contrast to the classical Lyapunov exponent, the finite-time Lyapunov exponent is, in general, a non-constant random variable.

Theorem 4.2.5. *Consider the stochastic differential equation* (4.0.1) *with the unique attracting random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$. For any finite time $T > 0$, let $\lambda_\alpha^T(\omega)$ denote the finite-time Lyapunov exponent associated with the linearization along $\{a_\alpha(\omega)\}_{\omega\in\Omega}$. Then the following statements hold:*

(i) For $\alpha < 0$, the random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is finite-time attractive, i.e.

$$\lambda_\alpha^T(\omega) \le \alpha < 0 \quad \textit{for all } \omega \in \Omega\,.$$

(ii) For $\alpha > 0$, the random fixed point $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is not finite-time attractive, i.e.

$$\mu\big\{\omega \in \Omega : \lambda_\alpha^T(\omega) > 0\big\} > 0.$$

Proof. (i) This follows directly from Theorem 4.2.3 (i).

(ii) Let $\varepsilon := \frac{\sqrt{\alpha}}{2} > 0$. According to Proposition 4.2.1, there exists a measurable set $\mathcal{A} \in \mathcal{F}_{-\infty}^T$ of positive probability such that for all $\omega \in \mathcal{A}$

$$a_\alpha(\theta_s\omega) \in (-\varepsilon, \varepsilon) \quad \text{for all } s \in [0, T]\,.$$

We will estimate $\lambda_\alpha^T(\omega)$ for arbitrary $\omega \in \mathcal{A}$. Let $\Phi_\alpha(t, \omega) := \frac{\partial\varphi_\alpha}{\partial x}(t, \omega, a_\alpha(\omega))$ denote the linearized RDS along the random fixed point $a_\alpha(\omega)$. Note that the linearized equation along the random fixed point $a_\alpha(\omega)$ is given by

$$\dot{\xi}(t) = (\alpha - 3a_\alpha(\theta_t\omega)^2)\xi(t)\,,$$

from which we derive that

$$\Phi_\alpha(t,\omega) = \exp\left(\int_0^t \big(\alpha - 3a_\alpha(\theta_s\omega)^2\big)\mathrm{d}s\right).$$

We thus get

$$\lambda_\alpha^T(\omega) = \alpha - \frac{1}{T}\int_0^T 3a_\alpha(\theta_t\omega)^2\mathrm{d}t \geq \frac{\alpha}{4}\,,$$

which completes the proof. □

This theorem implies that the change in the sign of finite-time Lyapunov exponents indicates a qualitative change in the dynamics. This means that the bifurcation is observable in practice, since finite-time Lyapunov exponents are numerically computable quantities. Note that the numerical approximation of classical Lyapunov exponents is difficult in general. In the special case of random matrix products with positive matrices, however, [Pol10] recently established explicit bounds for the numerical approximation of (classical) Lyapunov exponents.

4.3 The dichotomy spectrum at the bifurcation point

We will compute the dichotomy spectrum of the linearization around the unique attracting random fixed point $\{a_\alpha(\omega)\}$ of the system (4.0.1). As a direct consequence, we observe that hyperbolicity is lost at the bifurcation point $\alpha = 0$.

Theorem 4.3.1. *Let $\Phi_\alpha(t,\omega) := \frac{\partial\varphi_\alpha}{\partial x}(t,\omega,a_\alpha(\omega))$ denote the linearized RDS along the random fixed point $a_\alpha(\omega)$. Then the dichotomy spectrum Σ_α of Φ_α is given by*

$$\Sigma_\alpha = [-\infty, \alpha] \quad \textit{for all } \alpha \in \mathbb{R}\,.$$

Proof. From the linearized equation along $a_\alpha(\omega)$

$$\dot{\xi}(t) = (\alpha - 3a_\alpha(\theta_t\omega)^2)\xi(t)\,,$$

we derive that

$$\Phi_\alpha(t,\omega) = \exp\left(\int_0^t \big(\alpha - 3a_\alpha(\theta_s\omega)^2\big)\mathrm{d}s\right)\,. \tag{4.3.1}$$

Consequently,

$$|\Phi_\alpha(t,\omega)| \le e^{\alpha|t|} \quad \text{for all } t \in \mathbb{R}\,,$$

which implies that $\Sigma_\alpha \subset [-\infty,\alpha]$. Thus, it is sufficient to show that $[-\infty,\alpha] \subset \Sigma_\alpha$. For this purpose, let $\gamma \in (-\infty,\alpha]$ be arbitrary. Suppose the opposite, that Φ_α admits an exponential dichotomy with growth rate γ, invariant projector P_γ and positive constants K,ε. We now consider the two possible cases: (i) $P_\gamma = \mathbb{1}$ and (ii) $P_\gamma = 0$:

Case (i). $P_\gamma = \mathbb{1}$, i.e. we have

$$\Phi_\alpha(t,\omega) \le Ke^{(\gamma-\varepsilon)t} \quad \text{for all } t \ge 0\,. \tag{4.3.2}$$

Choose and fix $T > 0$ such that $e^{\frac{\varepsilon}{4}T} > K$. According to Proposition 4.2.1, there exists a measurable set $\mathcal{A} \in \mathcal{F}_{-\infty}^T$ of positive measure such that

$$a_\alpha(\theta_s\omega) \in \big(-\sqrt{\varepsilon}/2, \sqrt{\varepsilon}/2\big) \quad \text{for all } \omega \in \mathcal{A} \text{ and } s \in [0,T]\,.$$

From (4.3.1) we derive that

$$|\Phi_\alpha(T,\omega)| \ge e^{T\left(\alpha - \frac{3\varepsilon}{4}\right)} > Ke^{(\gamma-\varepsilon)T}\,,$$

which leads to a contradiction to (4.3.2).

Case (ii): $P_\gamma = 0$, i.e. we have for almost all $\omega \in \Omega$

$$\Phi_\alpha(t,\theta_{-t}\omega) \ge \frac{1}{K}e^{(\gamma+\varepsilon)t} \quad \text{for all } t \ge 0\,,$$

which together with (4.3.1) implies that

$$\frac{\ln K + (\alpha - \gamma)t}{3} \geq \int_0^t a_\alpha(\theta_s \theta_{-t}\omega)^2 \, \mathrm{d}s \,. \tag{4.3.3}$$

Choose and fix $T > 0$ such that

$$\frac{(T-1)^3}{3} > \frac{\ln K + (\alpha - \gamma)T}{3} \,.$$

Consider the following integral equation

$$x(t) = \int_0^t \big(\alpha x(s) - x(s)^3\big)\mathrm{d}s + \frac{t^4}{4} - \alpha \frac{t^2}{2} + t \,.$$

Clearly, the explicit solution of the above equation is $x(t) = t$. Due to the compactness of $[0, T]$, there exists an $\varepsilon > 0$ such that for any $x(0) \in (-\varepsilon, \varepsilon)$ and $\omega(t)$ with $\sup_{t\in[0,T]} |\omega(t) - \frac{t^4}{4} + \alpha \frac{t^2}{2} - t| \leq \varepsilon$ then the solution $x(t)$ of the following equation

$$x(t) = x(0) + \int_0^t (\alpha x(s) - x(s)^3) \, \mathrm{d}s + \omega(t)$$

satisfies that $\sup_{t\in[0,T]} |x(t) - t| \leq 1$. According to Proposition 4.2.1, there exists a measurable set $\mathcal{A}_\varepsilon^- \in \mathcal{F}_{-\infty}^0$ of positive measure such that $a_\alpha(\omega) \in (-\varepsilon, \varepsilon)$ for all $\omega \in \mathcal{A}_\varepsilon^-$. Define $\mathcal{A}_\varepsilon^+ \in \mathcal{F}_0^T$ by

$$\mathcal{A}_\varepsilon^+ := \left\{\omega \in \Omega : \sup_{t\in[0,T]} |\omega(t) - t^4/4 + \alpha t^2/2 - t| \leq \varepsilon\right\}.$$

Therefore, for all $\omega \in \mathcal{A}_\varepsilon^- \cap \mathcal{A}_\varepsilon^+$, we get

$$\sup_{t\in[0,T]} |a_\alpha(\theta_t\omega) - t| \leq 1 \,,$$

which implies that

$$\int_0^T a_\alpha(\theta_s\omega)^2 \, \mathrm{d}s > \frac{(T-1)^3}{3} > \frac{\ln K + (\alpha - \gamma)T}{3} \,.$$

Note that $\mu(\mathcal{A}_\varepsilon^- \cap \mathcal{A}_\varepsilon^+) = \mu(\mathcal{A}_\varepsilon^-)\mu(\mathcal{A}_\varepsilon^+) > 0$. Then for $\omega \in \theta_T(\mathcal{A}_\varepsilon^- \cap \mathcal{A}_\varepsilon^+)$, the above leads to a contradiction to (4.3.3), and the proof is complete. □

We have seen in Theorem 4.2.5 that the bifurcation of (4.0.1) manifests itself also via finite-time Lyapunov exponents: before the bifurcation, all finite-time Lyapunov exponents are negative, and after the bifurcation, one observes positive finite-time Lyapunov exponents with positive probability for arbitrarily large times. This implies in particular that for positive α the set of all finite-time Lyapunov exponents observed on a set of full measure does not converge to the (classical) Lyapunov exponent when time tends to infinity. The following theorem makes precise the fact that, in contrast to the Lyapunov spectrum, the dichotomy spectrum includes limits of the set of finite-time Lyapunov exponents.

Theorem 4.3.2. *Let Φ be a linear RDS on $\mathbb{R}^d$ with dichotomy spectrum Σ. Let $\lambda^T(\omega, x)$ denote the finite-time Lyapunov exponent (Definition 4.2.4) for $T > 0$, $\omega \in \Omega$ and $x \in \mathbb{R}^d \setminus \{0\}$. Then*

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) = \sup \Sigma$$

provided that $\sup \Sigma < \infty$ *and*

$$\lim_{T\to\infty} \operatorname*{ess\,inf}_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) = \inf \Sigma$$

provided that $\inf \Sigma > -\infty$.

Proof. By definition of $\lambda^T(\omega, x)$, we get that for all $T, S \geq 0$

$$(T + S) \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^{T+S}(\omega, x) \leq T \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) + S \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^S(\omega, x).$$

This implies that the sequence $\{T \operatorname{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x)\}_{T\geq 0}$ is subadditive. We thus obtain that the limit $T \to \infty$ exists and so

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) = \limsup_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x).$$

We first prove that provided $\sup \Sigma < \infty$, we have

$$\gamma := \limsup_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) = \sup \Sigma.$$

Since $\sup \Sigma < \infty$ it follows that there exists $K > 0$ such that

$$\|\Phi(t,\omega)\| \le K e^{t \sup \Sigma} \quad \text{for all } t \ge 0\,. \tag{4.3.4}$$

Assume first that $\gamma < \sup \Sigma$. This means that there exists a $t_0 > 0$ such that for all $t \ge t_0$ and for almost all $\omega \in \Omega$, we have $\|\Phi(t,\omega)\| \le e^{t(\gamma+\sup\Sigma)/2}$. Thus, together with (4.3.4), we obtain for all $t \ge 0$ and for almost all $\omega \in \Omega$ that

$$\|\Phi(t,\omega)\| \le \widehat{K} e^{t(\gamma+\sup\Sigma)/2}, \qquad \widehat{K} := \max\{1, K e^{t_0(\sup\Sigma-\gamma)/2}\}.$$

Hence, $\sup \Sigma \le (\gamma + \sup \Sigma)/2$, which is a contradiction. Assume now that $\gamma > \sup \Sigma$. This means in particular that $\sup \Sigma < \infty$. Hence, there exists a $K > 0$ such that for almost all $\omega \in \Omega$, we have

$$\|\Phi(t,\omega)x\| \le K e^{t(\gamma+\sup\Sigma)/2} \|x\| \quad \text{for all } x \in \mathbb{R}^d\,.$$

This leads to $\lambda^t(\omega, x) \le (\gamma + \sup \Sigma)/2$ for all $x \in \mathbb{R}^d \setminus \{0\}$, and thus,

$$\gamma = \limsup_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) \le (\gamma + \sup\Sigma)/2,$$

which is a contradiction. This proves the first equality. Similar arguments may be used to show that

$$\lim_{T\to\infty} \operatorname*{ess\,inf}_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) = \inf \Sigma$$

provided that $\inf \Sigma > -\infty$. □

In the following example, we explicitly construct a linear RDS with $\sup \Sigma = \infty$ but with

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) < \infty.$$

An example of a linear RDS with $\inf \Sigma = -\infty$ but with

$$\lim_{T\to\infty} \operatorname*{ess\,inf}_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d\setminus\{0\}} \lambda^T(\omega, x) > -\infty$$

can be constructed analogously. This example shows the importance of the assumption $\sup \Sigma < \infty$ or $\inf \Sigma > -\infty$ in the above theorem.

Example 4.3.3. Following the construction in Example 3.2.8, there exist infinitely many measurable sets $\{U_n\}_{n\in\mathbb{N}}$ of positive measure such that for $n \geq 2$, $U_n, \theta U_n, \theta^2 U_n$ are pairwise disjoint. We define a random mapping $A : \Omega \to \mathbb{R}$ as follows:

$$A(\omega) = \begin{cases} \frac{1}{n} : \omega \in U_n \cup \theta^2 U_n\,, n \geq 2\,, \\ n : \omega \in \theta U_n\,, n \geq 2\,, \\ 1 : \text{otherwise.} \end{cases}$$

Let Φ denote the discrete-time RDS generated by A. Since $\ln \|A(\cdot)\|$ is neither bounded from above nor from below, we get that $\Sigma(\Phi) = [-\infty, \infty]$. On the other hand, it is easy to see that for all $T \geq 2$ we get that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} |\Phi(T, \omega)| = 1,$$

which implies that

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \frac{1}{T} \ln |\Phi(T, \omega)| = 0.$$

4.4 Topological equivalence

This section deals with topological equivalence of random dynamical systems ([IS01, IL02, LL05, Arn98]). This concept has not been used so far to study bifurcations of random dynamical systems, and the main aim of this section is to discuss topological equivalence for the stochastic differential equation (4.0.1). The concept of topological equivalence for random dynamical systems [Arn98, Definition 9.2.1] differs from the corresponding deterministic notion of topological equivalence in the sense that instead of one homeomorphism (mapping orbits to orbits), the random version is given by a family of homeomorphisms $\{h_\omega\}_{\omega\in\Omega}$. The precise definition is given by Definition 1.1.11. A bifurcation is then described by means of a lack of topological equivalence at the bifurcation point.

The following theorem says that near the bifurcation point $\alpha = 0$, all systems of (4.0.1) are equivalent; since we have a local RDS φ, the definition of topological equivalence used is a modification of that given in Definition 1.1.11.

Theorem 4.4.1. *Let $\varphi_\alpha : D_\alpha \to \mathbb{R}$ denote the local continuous RDS generated by the SDE* (4.0.1)*. Then there exists an $\varepsilon > 0$ such that for all $\alpha \in (-\varepsilon, \varepsilon)$ the random dynamical systems φ_α are topologically equivalent to the dynamical system $(e^{-t}x)_{t,x\in\mathbb{R}}$, in the following sense: there exists a conjugacy $h : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for almost all $\omega \in \Omega$, we have*

$$\varphi_\alpha(t, \omega, h(\omega, x)) = h(\theta_t\omega, e^{-t}x) \quad \textit{for all } \; t \in (\kappa_\alpha(\omega, h(\omega, x)), \infty), x \in \mathbb{R}\,.$$

Proof. Let $a_\alpha(\omega)$ denote the unique random fixed point of (4.0.1). According to the results in [CF98], we obtain that

$$\mathbb{E}a_\alpha(\omega)^2 = \frac{\int_{-\infty}^{\infty} u^2 \exp\left(\frac{1}{\sigma^2}\left(\alpha u^2 - \frac{1}{2}u^4\right)\right)\, \mathrm{d}u}{\int_{-\infty}^{\infty} \exp\left(\frac{1}{\sigma^2}\left(\alpha u^2 - \frac{1}{2}u^4\right)\right)\, \mathrm{d}u}\,.$$

and therefore,

$$\lim_{\alpha\to 0} \mathbb{E}a_\alpha(\omega)^2 = \frac{\int_{-\infty}^{\infty} u^2 \exp\left(-\frac{u^4}{2\sigma^2}\right) \mathrm{d}u}{\int_{-\infty}^{\infty} \exp\left(-\frac{u^4}{2\sigma^2}\right) \mathrm{d}u} > 0.$$

Then there exists an $\varepsilon > 0$ such that for all $\alpha \in (-\varepsilon, \varepsilon)$, we have

$$\delta := \frac{3}{4}\mathbb{E}a_\alpha(\omega)^2 - \alpha > 0\,.$$

We define the local continuous RDS $\psi\colon \widetilde{D} \to \mathbb{R}$ by

$$\psi(t, \omega, x) := \varphi_\alpha(t, \omega, x + a_\alpha(\omega)) - a_\alpha(\theta_t\omega)\,. \tag{4.4.1}$$

where $\widetilde{D} := \{(t, \omega, x) \in \mathbb{R} \times \Omega \times \mathbb{R}\colon t > \kappa_\alpha(\omega, x + a_\alpha(\omega))\}$. By using the transformation function $f(\omega, x) := x - a_\alpha(\omega)$, the random dynamical systems φ_α and ψ are topologically equivalent. Hence, it is sufficient to show that ψ is topologically equivalent to the dynamical system $(e^{-t}x)_{t,x\in\mathbb{R}}$; the proof of this is divided into four parts.

Part 1. We first summarise some properties of ψ:

(i) Since $a_\alpha(\omega)$ is a random fixed point of φ_α, it follows that

$$\psi(t,\omega,0) = 0 \quad \text{for all } t\in\mathbb{R} \text{ and } \omega\in\Omega\,. \tag{4.4.2}$$

(ii) Due to the monotonicity of φ_α, for $x_1 > x_2$, we have

$$\psi(t,\omega,x_1) > \psi(t,\omega,x_2) \quad \text{for all } \omega\in\Omega \text{ and } t\in\widetilde{D}(\omega,x_1)\cap\widetilde{D}(\omega,x_2)\,. \tag{4.4.3}$$

(iii) From (4.0.1), we derive that

$$\begin{aligned}\psi(t,\omega,x) = x + \int_0^t \psi(s,\omega,x)\big(\alpha - a_\alpha(\theta_s\omega)^2 - a_\alpha(\theta_s\omega)\varphi_\alpha(s,\omega,a_\alpha(\omega)+x)\\ - \varphi_\alpha(s,\omega,a_\alpha(\omega)+x)^2\big)\,\mathrm{d}s\,,\end{aligned}$$

consequently,

$$\begin{aligned}\psi(t,\omega,x) = x\exp\Bigg(\int_0^t \alpha - a_\alpha(\theta_s\omega)^2 - a_\alpha(\theta_s\omega)\varphi_\alpha(s,\omega,a_\alpha(\omega)+x)\\ - \varphi_\alpha(s,\omega,a_\alpha(\omega)+x)^2\,\mathrm{d}s\Bigg)\,.\end{aligned} \tag{4.4.4}$$

Part 2. We shall now demonstrate some estimates on ψ. According to Birkhoff's ergodic theorem, there exists an invariant set $\widetilde{\Omega}$ of full measure such that

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t a_\alpha(\theta_s\omega)^2\,\mathrm{d}s = \mathbb{E}a_\alpha(\omega)^2\,. \tag{4.4.5}$$

Choose and fix $\omega\in\widetilde{\Omega}$. From (4.4.5), there exists $T>0$ such that for all $|t|>T$ we have

$$\left|\frac{1}{t}\int_0^t a_\alpha(\theta_s\omega)^2\,\mathrm{d}s - \mathbb{E}a_\alpha(\omega)^2\right| \le \delta\,. \tag{4.4.6}$$

The elementary inequality $u^2 + uv + v^2 \geq \frac{3}{4}u^2$ for $u, v \in \mathbb{R}$ implies with (4.4.4) that for $x > 0$

$$\psi(t,\omega,x) \leq x \exp\left(\int_0^t \alpha - \frac{3}{4}a_\alpha(\theta_s\omega)^2 \,\mathrm{d}s\right),$$

then for $t \geq T$, (4.4.6) implies the following estimate

$$\psi(t,\omega,x) \leq xe^{-\frac{\delta}{4}t}, \quad \text{for all } x > 0. \tag{4.4.7}$$

For negative time ψ explodes, and we have

$$\lim_{t\to\tilde{\kappa}(\omega,x)+} \psi(t,\omega,x) = \infty \quad \text{for } x > 0 \quad \text{and} \quad \lim_{t\to\tilde{\kappa}(\omega,x)+} \psi(t,\omega,x) = -\infty \quad \text{for } x < 0. \tag{4.4.8}$$

Part 3. We now show the required conjugacy. By (4.4.2) and (4.4.3), for $x > 0$ we have $\psi(s,\omega,x) > 0$ for all $s \in \widetilde{D}(\omega,x)$, and consequently by (4.4.7) and (4.4.8) we obtain that

$$\lim_{r\to\infty} \int_r^\infty \psi(s,\omega,x)\,\mathrm{d}s = 0 \quad \text{and} \quad \lim_{r\to\tilde{\kappa}(\omega,x)+} \int_r^\infty \psi(s,\omega,x)\,\mathrm{d}s = \infty.$$

Hence there exists a unique $r(\omega,x)$ such that

$$\int_{r(\omega,x)}^\infty \psi(s,\omega,x)\,\mathrm{d}s = 1. \tag{4.4.9}$$

Similarly, $r(\omega,x)$ for $x < 0$ is defined to satisfy

$$\int_{r(\omega,x)}^\infty \psi(s,\omega,x)\,\mathrm{d}s = -1, \tag{4.4.10}$$

and we define $r(\omega,0) := -\infty$. Using the local cocycle property of ψ, we obtain that

$$r(\omega,x) = r(\theta_s\omega, \psi(s,\omega,x)) + s. \tag{4.4.11}$$

Define the function

$$g(\omega,x) := \begin{cases} e^{r(\omega,x)}, & x>0\,,\\ 0, & x=0\,,\\ -e^{r(\omega,x)}, & x<0\,. \end{cases}$$

We will now show that g transforms the random dynamical system ψ to the dynamical system $(e^{-t}x)_{t,x\in\mathbb{R}}$. For any $x > 0$, we have $\psi(s,\omega,x) > 0$ and thus from the definition of the function g it follows that

$$g(\theta_s\omega,\psi(s,\omega,x)) = e^{r(\theta_s\omega,\psi(s,\omega,x))},$$

which implies together with (4.4.11) that

$$g(\theta_s\omega,\psi(s,\omega,x)) = e^{r(\omega,x)-s} = e^{-s}g(\omega,x)\,.$$

Similarly, for $x < 0$ we also have $g(\theta_s\omega,\psi(s,\omega,x)) = e^{-s}g(\omega,x)$ for all $s\in(\tilde{\kappa}(\omega,x),\infty)$, $\omega\in\Omega$.

Part 4. We will show that $g_\omega:\mathbb{R}\to\mathbb{R}$, $x\mapsto g(\omega,x)$ is a homeomorphism, and that g is jointly measurable. Choose and fix $\omega\in\widetilde{\Omega}$.

Injectivity: From the definition of g, it is easily seen that for $x_1 > 0 > x_2$ we have

$$g_\omega(x_1) > 0 > g_\omega(x_2).$$

On the other hand, based on strict monotonicity of ψ we get that for $x_1 > x_2 > 0$

$$\int_{r(\omega,x_2)}^{\infty}\psi(s,\omega,x_1)\,\mathrm{d}s > \int_{r(\omega,x_2)}^{\infty}\psi(s,\omega,x_2)\,\mathrm{d}s = 1.$$

Consequently, $r(\omega,x_1) > r(\omega,x_2)$ and thus $g_\omega(x_1) > g_\omega(x_2)$. Similarly, for $0 > x_1 > x_2$ we also have $g_\omega(x_1) > g_\omega(x_2)$. Therefore, g_ω is strictly increasing and thus injective.

Continuity: We first show that $\lim_{x\to 0+} g_\omega(x) = 0$. Let $\varepsilon > 0$ be arbitrary. Choose

$\tilde{T} > T$ such that $\frac{4}{\delta}e^{-\frac{\delta}{4}\tilde{T}} < \frac{1}{3}$ and $e^{-\tilde{T}} < \varepsilon$. By (4.4.7), for all $t \geq \tilde{T}$ we have

$$\psi(t, \omega, x) \leq e^{-\frac{\delta}{4}t}x\,.$$

As a consequence, for all $x \in (0, 1)$ we get

$$\int_{\tilde{T}}^{\infty} \psi(s, \omega, x)\,\mathrm{d}s \leq \int_{\tilde{T}}^{\infty} e^{-\frac{\delta}{4}s}\,\mathrm{d}s < \frac{1}{3}\,. \tag{4.4.12}$$

Since $\lim_{x\to 0}\psi(s, \omega, x) = 0$, $\left[-\tilde{T}, \tilde{T}\right]$ is a compact interval and $\lim_{x\to 0}\tilde{\kappa}(\omega, x) = -\infty$, there exists $\delta^* > 0$ such that

$$\int_{-\tilde{T}}^{\tilde{T}} \psi(s, \omega, \delta^*)\,\mathrm{d}s < \frac{1}{3},$$

which together with (4.4.12) implies that

$$\int_{-\tilde{T}}^{\infty} \psi(s, \omega, x)\,\mathrm{d}s < \frac{2}{3} \quad \text{for all } x \in \big(0, \min\{1, \delta^*\}\big)\,.$$

Therefore, $r(\omega, x) < -\tilde{T}$ and thus $g_\omega(x) < \varepsilon$ for all $x \in \big(0, \min\{1, \delta^*\}\big)$. Hence, $\lim_{x\to 0+} g_\omega(x) = 0$. One can similarly show that $\lim_{x\to 0-} g_\omega(x) = 0$, and thus g_ω is continuous at 0. The continuity of g on the whole real line can be proved in a similar way.

Surjectivity: It is easy to prove surjectivity from

$$\lim_{x\to\infty} g_\omega(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} g_\omega(x) = -\infty\,.$$

Measurability: By the definition of g, in order to prove the joint measurability of g it is enough to show the joint measurability of the mapping $(\omega, x) \mapsto r(\omega, x)$. Since the map $x \mapsto r(\omega, x)$ is continuous for each fixed $\omega \in \Omega$, it follows from e.g. [Cra02b, Lemma 1.1] that it is sufficient to show that the map $\omega \mapsto r(\omega, x)$ is measurable for each fixed $x \in \mathbb{R}$. Choose and fix an arbitrary $x > 0$, and let $\beta \in \mathbb{R}$ be arbitrary. Then, by the definition of

$r(\omega, x)$ we have

$$\Big\{\omega : r(\omega, x) \le \beta\Big\} = \Big\{\omega : \int_\beta^\infty \psi(t, \omega, x)\,\mathrm{d}t \le 1\Big\}$$
$$= \bigcap_{n\in\mathbb{N}, n\ge\beta} \Big\{\omega : \int_\beta^n \psi(t, \omega, x)\,\mathrm{d}t < 1\Big\}.$$

It should be clear that for each $n \in \mathbb{N}$, the map $\omega \mapsto \int_\beta^n \psi(t, \omega, x)\,\mathrm{d}t$ is measurable, and consequently the map $\omega \mapsto r(\omega, x)$ is measurable. The case $x < 0$ is similar, and we have defined $r(\omega, 0) = -\infty$ for all $\omega \in \Omega$. Thus we obtain the measurability of the map $\omega \mapsto r(\omega, x)$ for all $x \in \mathbb{R}$.

This completes the proof of this theorem. □

The above theorem implies that the stochastic differential equation (4.0.1) does not admit a bifurcation at $\alpha = 0$ which is induced by the concept of topological equivalence. In addition, because of the observations in Theorem 4.3.1, this concept of equivalence is not in correspondence with the dichotomy spectrum (linear systems which are hyperbolic and non-hyperbolic can be equivalent). We will now show that the concept of *uniform topological equivalence* is the right tool to obtain the bifurcations studied here.

Definition 4.4.2. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\theta : \mathbb{T} \times \Omega \to \Omega$ an ergodic metric dynamical system and (X_1, d_1), (X_2, d_2) be metric spaces. Then two random dynamical systems $\varphi_1 : \mathbb{T} \times \Omega \times X_1 \to X_1$ and $\varphi_2 : \mathbb{T} \times \Omega \times X_2 \to X_2$, both over θ, are called *uniformly topologically equivalent* with respect to a random fixed point $\{a(\omega)\}_{\omega\in\Omega}$ of φ_1 if there exists a conjugacy $h : \Omega \times X_1 \to X_2$ fulfilling the following properties:

(i) For almost all $\omega \in \Omega$, the mapping $x \mapsto h(\omega, x)$ is a homeomorphism from X_1 to X_2.

(ii) The mappings $(\omega, x_1) \mapsto h(\omega, x_1)$ and $(\omega, x_2) \mapsto h^{-1}(\omega, x_2)$ are measurable.

(iii) The random dynamical systems φ_1 and φ_2 are *cohomologous*, i.e.

$$\varphi_2(t, \omega, h(\omega, x)) = h(\theta_t\omega, \varphi_1(t, \omega, x)) \quad \text{for all } x \in X_1 \text{ and almost all } \omega \in \Omega\,.$$

(iv) We have

$$\lim_{\delta\to 0} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_2(h(\omega,x),h(\omega,a(\omega))) = 0$$

and

$$\lim_{\delta\to 0} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(h(\omega,a(\omega)))} d_1(h^{-1}(\omega,x),a(\omega)) = 0\,.$$

Note that, in comparison to the concept of topological equivalence (Definition 1.1.11), we added (iv) to take uniformity into account. We now show that uniform topological equivalence preserves local uniform attractivity.

Proposition 4.4.3. *Let $(\Omega,\mathcal{F},\mu)$ be a probability space, $\theta:\mathbb{T}\times\Omega\to\Omega$ a metric dynamical system and (X_1,d_1), (X_2,d_2) be metric spaces. Let $\varphi_1:\mathbb{T}\times\Omega\times X_1\to X_1$ and $\varphi_2:\mathbb{T}\times\Omega\times X_2\to X_2$ be two random dynamical systems, both over θ, which are uniformly topologically equivalent with respect to a random fixed point $\{a(\omega)\}_{\omega\in\Omega}$ of φ_1. Let $h:\Omega\times X_1\to X_2$ denote the conjugacy. Then $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1 if and only if $\{h(\omega,a(\omega))\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_2.*

Proof. Suppose that $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1. Let $\eta>0$, then there exists a $\gamma>0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\gamma(a(\omega))} d_2(h(\omega,x),h(\omega,a(\omega))) \le \eta\,.$$

Since $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1, there exists a $\delta>0$ and a $T>0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_1(\varphi_1(t,\omega,x),a(\theta_t\omega)) \le \frac{\gamma}{2} \quad \text{for all } t\ge T\,.$$

Hence, for all $t\ge T$, we have

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_2(h(\theta_t\omega,\varphi_1(t,\omega,x)),h(\theta_t\omega,a(\theta_t\omega))) \le \eta\,,$$

which means that for all $t\ge T$,

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_2(\varphi_2(t,\omega,h(\omega,x)),h(\theta_t\omega,a(\theta_t\omega))) \le \eta\,.$$

There exists a $\beta > 0$ such that

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{x \in B_\beta(h(\omega, a(\omega)))} d_1(h^{-1}(\omega, x), a(\omega)) \leq \frac{\delta}{2},$$

and finally, this means that for all $t \geq T$ we have

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{x \in B_\beta(h(\omega, a(\omega)))} d_2(\varphi_2(t, \omega, x), h(\theta_t \omega, a(\theta_t \omega))) \leq \eta,$$

which finishes the proof that $\{h(\omega, a(\omega))\}_{\omega \in \Omega}$ is locally uniformly attractive for φ_2. The converse is proved similarly. □

As a corollary to this proposition, it follows that (4.0.1) admits a bifurcation.

Theorem 4.4.4. *The stochastic differential equation* (4.0.1) *admits a random bifurcation at* $\alpha = 0$ *which is induced by the concept of uniform topological equivalence.*

Proof. This is a direct consequence of Theorem 4.2.3 and Proposition 4.4.3. □

4.5 Discussion

We have shown how the pitchfork normal form with additive noise admits a bifurcation in terms of a break-down of the uniformity of the dynamics, which manifests via nonuniform attractivity of the global attractor, the existence of positive finite-time Lyapunov exponents, a loss of hyperbolicity of the exponential dichotomy spectrum and a loss of uniform topological equivalence. The bifurcation is observable since finite-time Lyapunov exponents are numerically computable. It would be interesting to see if one could calculate the distribution of the finite-time Lyapunov exponents for this example.

In future work we would like to investigate the use of these concepts in other elementary examples, and ultimately generalize these ideas. In particular, one could investigate the use of a lack of (uniform or nonuniform) topological equivalence as a defining concept for bifurcations in general one-dimensional random dynamical systems, and investigate the use of the dichotomy spectrum in higher dimensional examples.

Appendix A

Random Sets

Here we give some fundamental theorems and facts on *random sets* (Definition 1.1.7). The classic reference on random sets, or *measurable multifunctions*, is [CV77]. The material presented here has been adapted from [Cra02b, Chapters 1 & 2] and [Chu02, Section 1.3], to which we refer the reader for further details, and also to the references given therein.

In what follows let X be a Polish space and let $\mathcal{B}$ denote its Borel σ-algebra. Let d be a complete metric on X and $\mathrm{dist}\colon 2^X \times 2^X \to \mathbb{R}_0^+$ denote the Hausdorff semi-distance on the subsets of X with respect to d, as defined in Section 1.1 (p. 21). Also, let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

Definition A.0.1. For a given measurable space $(\Omega, \mathcal{F})$, the *universal completion of* $\mathcal{F}$ is defined to be the σ-algebra given by $\mathcal{F}^u := \cap_\nu \bar{\mathcal{F}}^\nu$, where $\bar{\mathcal{F}}^\nu$ denotes the completion of $\mathcal{F}$ with respect to the probability measure ν and the intersection is taken over all probability measures on $\mathcal{F}$.

Remark A.0.2. Note that for any given probability measure μ, $(\bar{\mathcal{F}}^\mu)^u = \bar{\mathcal{F}}^\mu$; that is, if a σ-algebra is already complete with respect to some probability measure, it coincides with its universal completion. Also note that $\mathcal{F}^u \subset \bar{\mathcal{F}}^\mu$, i.e. $\mathcal{F}^u$-measurability implies $\bar{\mathcal{F}}^\mu$-measurability.

Remark A.0.3. We are only able to demonstrate that the closed and open random balls (Definition 1.1.9) are closed, respectively open, random sets with respect to $\mathcal{F}^u$ (including the case of a constant radius). It is straightforward to show that their graphs are measurable subsets of $\mathcal{F} \otimes \mathcal{B}$ (this is similar to the proof of (i)⇒(iii) in [Cra02b, Proposition 2.4]), but this only implies that they are $\mathcal{F}^u$-measurable using Proposition A.0.7 (ix). See also [Cra02b, Remark 2.11 (ii)].

The next three results are useful in establishing measurability results.

Theorem A.0.4 (Representation Theorem, [Iof79])**.** *Let $C\colon \Omega \to X$ be a closed random set taking values in a Polish space X. Then there exist a Polish space Y and a mapping $g\colon \Omega \times Y \to X$ such that*

(i) $g(\omega, \cdot)$ is continuous for all $\omega \in \Omega$, and $g(\cdot, y)$ is measurable for all $y \in Y$,

(ii) let d_X and d_Y be metrics on X and Y, respectively, then for all $\omega \in \Omega$ and $y_1, y_2 \in Y$ one has:

$$d_X(g(\omega, y_1), g(\omega, y_2)) \leq d_Y(y_1, y_2)(1 + d_X(g(\omega, y_1), g(\omega, y_2))),$$

(iii) for all $\omega \in \Omega$ such that $C(\omega) \neq \emptyset$, one has $C(\omega) = g(\omega, Y)$.

The preceding theorem leads to the following one.

Theorem A.0.5 (Selection Theorem, [Cra02b, Theorem 2.6])**.** *A set valued map $C\colon \Omega \to 2^X$ taking values in the nonvoid closed subsets of a Polish space X is a closed random set if and only if there exists a countable sequence $\{c_n\}_{n\in\mathbb{N}}$ of measurable maps $c_n\colon \Omega \to X$ such that $C(\omega) = \overline{\{c_n(\omega)\colon n \in \mathbb{N}\}}$ for all $\omega \in \Omega$. In particular, if C is a closed random set, then there exists a measurable selection, i.e., a measurable map $c\colon \Omega \to X$ such that $c(\omega) \in C(\omega)$ for all $\omega \in \Omega$.*

Theorem A.0.6 (Projection Theorem, [Cra02b, Theorem 2.12])**.** *Let $(\Omega, \mathcal{F})$ be a measurable space and X a Polish space. Then the projection onto Ω of any set $A \in \mathcal{F} \otimes \mathcal{B}$, given by*

$$\Pi_\Omega(A) := \{\omega \in \Omega\colon (\omega, x) \in A \text{ for some } x \in X\},$$

is universally measurable, i.e. $\Pi_\Omega(A) \in \mathcal{F}^u$.

The following proposition lists some key facts about random sets. They have been collated from [Chu02, Proposition 1.3.1] and [Cra02b, Proposition 2.4, Proposition 2.9 & Corollary 2.10]. We first note the elementary fact that the continuity of $a \mapsto d(x,a)$ implies that for any $x \in X$ and any set $A \subset X$,

$$\operatorname{dist}(x, A) = \operatorname{dist}(x, \overline{A}). \tag{A.0.1}$$

Proposition A.0.7. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a probability space,* X *be a Polish space and* $D\colon \Omega \to 2^X$ *a set valued mapping, then the following hold:*

(i) *D is a random set if and only if the set $\{\omega\colon D(\omega) \cap U \neq \emptyset\}$ is measurable for any open set $U \subset X$,*

(ii) *D is a random set if and only if $\omega \mapsto \overline{D(\omega)}$ is a closed random set (by* (A.0.1)*),*

(iii) *D is a compact random set if and only if $D(\omega)$ is compact for every $\omega \in \Omega$ and the set $\{\omega\colon D(\omega) \cap C \neq \emptyset\}$ is measurable for any closed set $C \subset X$,*

(iv) *if $\{D_n\}_{n\in\mathbb{N}}$ is a sequence of closed random sets with nonvoid intersection and there exists an $n_0 \in \mathbb{N}$ such that D_{n_0} is a compact random set, then $\cap_{n\in\mathbb{N}} D_n$ is a compact random set,*

(v) *if $\{D_n\}_{n\in\mathbb{N}}$ is a sequence of random sets, then $D = \cup_{n\in\mathbb{N}} D_n$ is also a random set,*

(vi) *if $f\colon \Omega \times X \to X$ is a mapping such that $f(\omega, \cdot)$ is continuous for all ω and $f(\cdot, x)$ is measurable for all x, then $\omega \mapsto f(\omega, D(\omega))$ is a random set if D is a random set (and is a compact random set if D is a compact random set),*

(vii) *D is a random set if and only if for every $\delta > 0$,* $\operatorname{graph}(B_\delta(D)) \in \mathcal{F} \otimes \mathcal{B}$

(viii) *if D is a closed random set, then* $\operatorname{graph}(D) \in \mathcal{F} \otimes \mathcal{B}$ *(and hence if D is an open random set,* $\operatorname{graph}(D) = \operatorname{graph}(D^c)^c \in \mathcal{F} \otimes \mathcal{B}$*),*

(ix) *if $\mathcal{F}$ is universally complete and D is a closed or an open set valued map such that* $\operatorname{graph}(D) \in \mathcal{F} \otimes \mathcal{B}$*, then D is a closed or an open, respectively, random set,*

(x) if D is a closed random set, then $\omega \mapsto \overline{D^c(\omega)}$ is also a closed random set,

(xi) if D is an open random set, then $\overline{D}$ is a closed random set (and it follows by (A.0.1) *that $\omega \mapsto \operatorname{dist}(x, D(\omega))$ is $\mathcal{F}$-measurable for each $x \in X$),*

(xii) If D is a closed random set then $\omega \mapsto \operatorname{int}(D(\omega))$ is an open random set.

The following two results allow one to replace $\mathcal{F}^u$-measurable objects with almost equal $\mathcal{F}$-measurable versions.

Lemma A.0.8 ([Cra02b, Lemma 2.7])**.** *Suppose that $C\colon \Omega \to 2^X$ is a set valued map taking values in the closed subsets of a Polish space X, such that* $\operatorname{graph}(C) \in \bar{\mathcal{F}}^\mu \otimes \mathcal{B}$. *Then there exists a closed random set $\tilde{C}\colon \Omega \to 2^X$ (which is $\mathcal{F}$-measurable) and $\tilde{C}(\omega) = C(\omega)$ μ-almost surely.*

As a particular application of the above lemma, if $C \subset \mathcal{F} \otimes \mathcal{B}$ is a product measurable set with closed ω-sections, then since $\mathcal{F} \otimes \mathcal{B} \subset \bar{\mathcal{F}}^\mu \otimes \mathcal{B}$, there exists an $\mathcal{F}$-measurable random set $\tilde{C}$ which is almost surely equal to C.

Lemma A.0.9 ([Cra02b, Lemma 1.2])**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, Y a separable metric space. Then for any $\bar{\mathcal{F}}^\mu$-measurable map $f\colon \Omega \to Y$ there exists an $\mathcal{F}$-measurable map $\tilde{f}\colon \Omega \to Y$ with $\tilde{f}(\omega) = f(\omega)$ μ-almost surely.*

The following lemma demonstrates that the distance between two random sets is a measurable function.

Lemma A.0.10. *Let D_1 and D_2 be random sets, then $\omega \mapsto \operatorname{dist}(D_1(\omega), D_2(\omega))$ is measurable.*

Proof. By Proposition A.0.7 (ii), $\overline{D_{1,2}}$ are closed random sets and we define $E_{1,2} := \{\omega\colon D_{1,2}(\omega) = \emptyset\}$. Then let $g\colon \Omega \times Y \to X$ be a representation of $\overline{D_1}$ and $h\colon \Omega \times Z \to X$ be a representation of $\overline{D_2}$ as described in Theorem A.0.4, and let $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ be dense subsets in Y and Z, respectively. Then using (A.0.1),

$$\operatorname{dist}(D_1(\omega), D_2(\omega)) = \operatorname{dist}(\overline{D_1(\omega)}, \overline{D_2(\omega)})$$
$$= \begin{cases} \sup_{n\in\mathbb{N}} \inf_{m\in\mathbb{N}} d(g(\omega, y_n), h(\omega, z_m)), & \omega \in \Omega \backslash (E_1 \cup E_2), \\ 0, & \omega \in E_1, \\ \infty, & \omega \in E_2 \backslash E_1, \end{cases}$$

and measurability follows since $d(g(\omega, y_n), h(\omega, z_m))$ is measurable for each n and m. $\square$

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