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COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES OF A HALF PLANE

SAM ELLIOTT AND ANDREW WYNN

Abstract. We use induction and interpolation techniques to prove that a composition operator induced by a map ϕ is bounded on the weighted Bergman space $\mathcal{A}^2_\alpha(\mathbb{H})$ of the right half-plane if and only if ϕ fixes ∞ non-tangentially, and has a finite angular derivative λ there. We further prove that in this case the norm, essential norm, and spectral radius of the operator are all equal, and given by $\lambda^{(2+\alpha)/2}$.

1. Introduction

Analytic composition operators have been studied in a number of contexts, primarily on spaces of functions in the unit disc of the the complex plane. It has long been known as a consequence of the Littlewood subordination principle that all such operators are bounded on all the Hardy spaces, as well as a large class of other spaces of functions.

On the half-plane, however, things are somewhat more complicated. It is well know that there are unbounded composition operators on the half-plane. Indeed, in [9], Valentin Matache proved that a composition operator is bounded on the Hardy space H^2 of the half plane if and only if the inducing map fixes the point at infinity, and has a finite angular derivative λ there. Later, in [5] the first named author and Michael Jury sharpened this result, and showed that in the case when such a composition operator is bounded, the norm, essential norm and spectral radius of the operator are all equal to $\sqrt{\lambda}$. This in particular strengthened a result on non-compactness of composition operators produced by Matache in [8].

Noting that the Hardy space is effectively the Bergman space with weight $\alpha = -1$, we will take the known situation as a base case, and use induction and interpolation techniques to extend the results to all weighted Bergman spaces. In particular, we provide a formula for the norm which agrees with the known results for the Hardy space case. For a thorough discussion of Bergman spaces and their composition operators, see [3] or [7].

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2. Preliminaries

Let $\mathbb{H}$ denote the right half-plane $\{\Re z > 0\}$. For $\alpha > -1$, the weighted Bergman space $\mathcal{A}^2_\alpha(\mathbb{H})$ contains those analytic functions $F : \mathbb{H} \to \mathbb{C}$ for which

$$\|F\|^2_{\mathcal{A}^2_\alpha(\mathbb{H})} := \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{\infty} x^\alpha |F(x+iy)|^2 dxdy < \infty.$$

For each $\alpha > -1$, the functions $\{k^\alpha_\omega ; \omega \in \mathbb{H}\}$ defined by

$$k^\alpha_\omega(z) := \frac{2^\alpha(1+\alpha)}{(\overline{\omega} + z)^{2+\alpha}}, \qquad z \in \mathbb{H}, \tag{2.1}$$

are the reproducing kernels for $\mathcal{A}^2_\alpha(\mathbb{H})$. As such, they have the property that

$$\langle f, k_\omega \rangle_{\mathcal{A}^2_\alpha(\mathbb{H})} = f(\omega), \quad f \in \mathcal{A}^2_\alpha(\mathbb{H}), \omega \in \mathbb{H}. \tag{2.2}$$

In order to prove our result, we will show that a certain kernel is positive. We say a kernel $K(z,w)$ on $\mathbb{H} \times \mathbb{H}$ is positive if

$$\sum_{i,j=1}^{n} c_i \overline{c_j} K(x_i, x_j) \geq 0$$

for all $n \geq 1$, and all scalars $c_1, \dots c_n \in \mathbb{C}$ and points $x_1, \dots x_n \in \mathbb{H}$.

Proposition 2.1 (Nevanlinna). *A holomorphic function ψ in $\mathbb{H}$ has positive real part if and only if the kernel*

$$\frac{\psi(z) + \overline{\psi(w)}}{z + \overline{w}}$$

is positive.

A sequence of points $z_n = x_n + iy_n$ in $\mathbb{H}$ is said to tend non-tangentially to ∞ if

(1) $x_n \to \infty$,
(2) the ratio $|y_n|/x_n$ is uniformly bounded.

We then say that a map $\phi : \mathbb{H} \to \mathbb{H}$ fixes infinity non-tangentially, and write $\phi(\infty) = \infty$, if $\phi(z_n) \to \infty$ whenever $z_n \to \infty$ non-tangentially. If $\phi(\infty) = \infty$, we say that ϕ has a finite angular derivative at ∞ if the non-tangential limit

$$\lim_{z \to \infty} \frac{z}{\phi(z)} \tag{2.3}$$

exists and is finite, under these circumstances, we write $\phi'(\infty)$ for this quantity.

If we let ψ be the self-map of $\mathbb{D}$ equivalent to ϕ via the standard Möbius identification of the disc with the half-plane given by $\tau(\zeta) = \frac{1+\zeta}{1-\zeta}$, that is $\psi = \tau^{-1} \circ \phi \circ \tau$, then (2.3) is equal, by the Julia-Carathéodory theorem, the non-tangential limit of $\psi'(\zeta)$ as $\zeta \to 1$, which is where the terminology comes from. Indeed, we have the following half-plane version of the Julia-Carathéodory theorem, proved in [5].

Lemma 2.2 (Half plane Julia-Carathéodory theorem)**.** *Let $\phi : \mathbb{H} \to \mathbb{H}$ be holomorphic. The following are equivalent:*

(1) $\phi(\infty) = \infty$ *and* $\phi'(\infty)$ *exists;*

(2) $\sup_{z\in\mathbb{H}} \frac{\Re z}{\Re\phi(z)} < \infty$*;*

(3) $\limsup_{z\to\infty} \frac{\Re z}{\Re\phi(z)} < \infty$*.*

Moreover the quantities in (2) and (3) are both equal to the angular derivative $\phi'(\infty)$*.*

Lemma 2.3. *Suppose that $K(\omega, z)$ is a positive kernel on $\mathbb{H}\times\mathbb{H}$ and let $c \geq 0$ be a positive constant. Then $\tilde{K}(\omega, z) := K(\omega, z) + c$ is a positive kernel on $\mathbb{H}\times\mathbb{H}$.*

Proof. Since the analytic function $\psi(z) = z$ on $\mathbb{H}$ has positive real part, Proposition 2.1 implies that $L(\omega, z) \equiv 1$ is positive. Since $\tilde{K} = K + cL$, it follows that $\tilde{K}$ is positive. □

3. Main Results

For a natural number $n \geq 1$ and a holomorphic function $\phi : \mathbb{H} \to \mathbb{H}$ with finite angular derivative λ at infinity, we define the kernel $K^n(\omega, z)$ on $\mathbb{H}\times\mathbb{H}$ by

$$K^n(\omega, z) := \frac{(\phi(z) + \overline{\phi(\omega)})^n - \lambda^{-n}(z + \bar{\omega})^n}{(z + \bar{\omega})^n}, \qquad \omega, z \in \mathbb{H}.$$

Lemma 3.1. *Suppose that $\phi : \mathbb{H} \to \mathbb{H}$ has finite angular derivative $0 < \lambda < \infty$ at infinity. Then for every natural number $n \geq 0$, the kernel K^{2^n} is positive.*

Proof. It is shown in [5] that K^1 is positive. Now suppose that K^{2^n} is positive for some natural number $n \geq 1$. Then, using the fact that the numerator of $K^{2^{n+1}}$ is the difference of two squares,

$$\begin{aligned}
K^{2^{n+1}}(\omega, z) &= \frac{\left((\phi(z) + \overline{\phi(\omega)})^{2^n}\right)^2 - \left(\lambda^{-2^n}(z + \bar{\omega})^{2^n}\right)^2}{(z + \bar{\omega})^{2^{n+1}}} \\
&= \frac{(\phi(z) + \overline{\phi(\omega)})^{2^n} - \lambda^{-2^n}(z + \bar{\omega})^{2^n}}{(z + \bar{\omega})^{2^n}} \cdot \frac{(\phi(z) + \overline{\phi(\omega)})^{2^n} + \lambda^{-2^n}(z + \bar{\omega})^{2^n}}{(z + \bar{\omega})^{2^n}} \\
&= K^{2^n}(\omega, z) \cdot \left(\frac{(\phi(z) + \overline{\phi(\omega)})^{2^n}}{(z + \bar{\omega})^{2^n}} + \lambda^{-2^n} \right) \\
&= K^{2^n}(\omega, z) \left(K^{2^n}(\omega, z) + 2 \cdot \lambda^{-2^n} \right).
\end{aligned}$$

By assumption that K^{2^n} is positive and Lemma 2.3, this is the product of two positive kernels, and hence, $K^{2^{n+1}}$ is positive by the Schur product theorem [1]. The result now follows by induction. □

As a result of Lemma 3.1, it is possible to provide conditions for boundedness of composition operators on weighted Bergman spaces, for certain integer weights.

Proposition 3.2. *Let $\phi : \mathbb{H} \to \mathbb{H}$ be holomorphic and let $n \geq 1$ be a natural number. The composition operator $C_\phi : \mathcal{A}^2_{2^n-2}(\mathbb{H}) \to \mathcal{A}^2_{2^n-2}(\mathbb{H})$ is bounded if and only if ϕ has finite angular derivative $0 < \lambda < \infty$ at infinity, in which case $\|C_\phi\| = \lambda^{2^{n-1}}$.*

Proof. Let $n \geq 1$ be a natural number and define $\alpha := 2^n - 2$. Following [5], if it can be shown that

$$\lambda^{2^n} \langle k^\alpha_\omega, k^\alpha_z \rangle_{\mathcal{A}^2_\alpha(\mathbb{H})} - \langle C^*_\phi k^\alpha_\omega, C^*_\phi k^\alpha_z \rangle_{\mathcal{A}^2_\alpha(\mathbb{H})} \tag{3.1}$$

is a positive kernel, then $C_\phi : \mathcal{A}^2_\alpha(\mathbb{H}) \to \mathcal{A}^2_\alpha(\mathbb{H})$ is bounded with $\|C_\phi\| \leq \lambda^{2^{n-1}}$. Using the fact that $C^*_\phi k^\alpha_\omega = k^\alpha_{\phi(\omega)}$ and (2.2), it follows that (3.1) is equal to

$$2^\alpha(1+\alpha)\left(\frac{\lambda^{2^n}}{(z+\bar{\omega})^{2^n}} - \frac{1}{(\phi(z)+\overline{\phi(\omega)})^{2^n}}\right).$$

This can easily be seen to factorise as

$$\lambda^{2^n}\frac{2^\alpha(1+\alpha)}{(\phi(z)+\overline{\phi(\omega)})^{2^n}} \cdot \frac{(\phi(z)+\overline{\phi(\omega)})^{2^n} - \lambda^{-2^n}(z+\bar{\omega})^{2^n}}{(z+\bar{\omega})^{2^n}},$$

which is just

$$\lambda^{2^n}\langle k^\alpha_{\phi(\omega)}, k^\alpha_{\phi(z)} \rangle_{\mathcal{A}^2_\alpha(\mathbb{H})} \cdot K^{2^n}(\omega, z).$$

This is positive, being the product of positive kernels and positive scalars.

For the converse, the calculation is similar to the Hardy space case. If the composition operator $C_\phi : \mathcal{A}^2_\alpha(\mathbb{H}) \to \mathcal{A}^2_\alpha(\mathbb{H})$ is bounded and $\|C_\phi\| \leq M$ then,

$$\begin{aligned}
\frac{2^\alpha(1+\alpha)}{2^{2+\alpha}(\Re\phi(z))^{2+\alpha}} = \|k^\alpha_{\phi(z)}\|^2_{\mathcal{A}^2_\alpha(\mathbb{H})} &= \|C^*_\phi k^\alpha_z\|^2_{\mathcal{A}^2_\alpha(\mathbb{H})} \\
&\leq M^2 \|k^\alpha_z\|^2_{\mathcal{A}^2_\alpha(\mathbb{H})} \\
&= M^2 \frac{2^\alpha(1+\alpha)}{2^{2+\alpha}(\Re z)^{2+\alpha}}.
\end{aligned}$$

As such,

$$\frac{\Re(z)}{\Re(\phi(z))} \leq M^{2/(2+\alpha)},$$

hence by Lemma 2.2, ϕ has finite angular derivative

$$\phi'(\infty) = \lambda \leq \|C_\phi\|^{2/(2+\alpha)} = \|C_\phi\|^{2^{-(n-1)}}.$$

By the first part of the proof, the norm of C_ϕ must be at most $\lambda^{2^{n-1}}$, and by the second part it must be at least that large. It follows that indeed

$$\|C_\phi\| = \lambda^{2^{n-1}}.$$

□

Proposition 3.2 tells us that the result holds for particular integral values of α of arbitrarily large size. We proceed by interpolating for the spaces $\mathcal{A}^2_\alpha(\mathbb{H})$, where $2^n < \alpha < 2^{n+1}$. The following weighted version of the Paley-Wiener Theorem (see [4] or [6]) will be useful.

Lemma 3.3. *The Bergman space $\mathcal{A}^2_\alpha(\mathbb{H})$ is isometrically isomorphic, via the Laplace transform $\mathcal{L}$, to the space $L^2(\mathbb{R}_+, d\mu_\alpha)$. Here,*

$$d\mu_\alpha = \frac{\Gamma(1+\alpha)}{2^\alpha t^{\alpha+1}} dt,$$

and dt is Lebesgue measure on $\mathbb{R}_+ := (0, \infty)$.

Theorem 3.4. *Let $\phi : \mathbb{H} \to \mathbb{H}$ be holomorphic and $\alpha > -1$. The composition operator $C_\phi : \mathcal{A}^2_\alpha(\mathbb{H}) \to \mathcal{A}^2_\alpha(\mathbb{H})$ is bounded if and only if ϕ has finite angular derivative $0 < \lambda < \infty$ at infinity, in which case $\|C_\phi\| = \lambda^{(2+\alpha)/2}$.*

Proof. Let $\alpha > -1$. By Proposition 3.2, the result holds if α is of the form $\alpha = 2^n - 2$. Hence, it may be assumed without loss of generality that there exists a natural number $n \geq 0$ such that $\alpha \in (2^n - 2, 2^{n+1} - 2)$. Write $A := 2^n - 2$, $B := 2^{n+1} - 2$. In the following, for simplicity, write $L^2(d\mu)$ for $L^2(\mathbb{R}_+, d\mu)$. Define a linear operator

$$T : L^2(d\mu_A) \to L^2(d\mu_A); \qquad T : L^2(d\mu_B) \to L^2(d\mu_B)$$

by $T := \mathcal{L}^{-1} \circ C_\phi \circ \mathcal{L}$. Since $\mathcal{L}$ is an isometric isomorphism between the respective spaces (Lemma 3.3), Proposition 3.2 implies that

$$\|T\|_{L^2(d\mu_A) \to L^2(d\mu_A)} = \|C_\phi\|_{\mathcal{A}^2_A(\mathbb{H}) \to \mathcal{A}^2_A(\mathbb{H})} = \lambda^{2^{n-1}} = \lambda^{(2+A)/2};$$
$$\|T\|_{L^2(d\mu_B) \to L^2(d\mu_B)} = \|C_\phi\|_{\mathcal{A}^2_B(\mathbb{H}) \to \mathcal{A}^2_B(\mathbb{H})} = \lambda^{2^n} = \lambda^{(2+B)/2}.$$

(Note that in the case $n = 0$, $\mathcal{A}^2_A(\mathbb{H})$ should be replaced by the Hardy space $H^2(\mathbb{H})$). Since $\alpha \in (A, B)$, there exists $\theta \in (0, 1)$ such that $\alpha = A(1-\theta) + B\theta$. By the Stein-Weiss interpolation theorem [2, Corollary 5.5.4],

$$\|T\|_{L^2(dw) \to L^2(dw)} \leq \lambda^{(2+A)(1-\theta)/2} \lambda^{(2+B)\theta/2} = \lambda^{(2+\alpha)/2}, \tag{3.2}$$

where

$$dw = \frac{\Gamma(1+A)^{1-\theta}\Gamma(1+B)^\theta}{2^{A(1-\theta)+B\theta} t^{A(1-\theta)+B\theta+1}} dt = \frac{\Gamma(1+A)^{1-\theta}\Gamma(1+B)^\theta}{2^\alpha t^{1+\alpha}} dt$$

By Lemma 3.3, for any $g \in \mathcal{A}^2_\alpha(\mathbb{H})$ there exists $f \in L^2(d\mu_\alpha)$ such that $\mathcal{L}f = g$ and $\|g\|_{\mathcal{A}^2_\alpha(\mathbb{H})} = \|f\|_{L^2(d\mu_\alpha)}$. Thus,

$$\begin{aligned}
\|C_\phi g\|_{\mathcal{A}^2_\alpha(\mathbb{H})} = \|C_\phi(\mathcal{L}f)\|_{\mathcal{A}^2_\alpha(\mathbb{H})} &= \|\mathcal{L}(Tf)\|_{\mathcal{A}^2_\alpha(\mathbb{H})} \\
&= \|Tf\|_{L^2(d\mu_\alpha)} \\
&= \frac{\Gamma(1+\alpha)^{1/2}}{\Gamma(1+A)^{(1-\theta)/2}\Gamma(1+B)^{\theta/2}} \|Tf\|_{L^2(dw)} \\
\text{(by (3.2))} &\leq \frac{\lambda^{(2+\alpha)/2}\Gamma(1+\alpha)^{1/2}}{\Gamma(1+A)^{(1-\theta)/2}\Gamma(1+B)^{\theta/2}} \|f\|_{L^2(dw)} \\
&= \lambda^{(2+\alpha)/2}\|f\|_{L^2(d\mu_\alpha)} \\
&= \lambda^{(2+\alpha)/2}\|g\|_{\mathcal{A}^2_\alpha(\mathbb{H})}.
\end{aligned}$$

As such, C_ϕ is bounded with $\|C_\phi\| \leq \lambda^{(2+\alpha)/2}$.

For the converse assume that C_ϕ is bounded. Then by exactly the same proof as the second half of Proposition 3.2, it follows that ϕ has finite angular derivative λ and that $\|C_\phi\| \geq \lambda^{(2+\alpha)/2}$. □

The following results, concerning the spectral radius and essential norm of C_ϕ, can be deduced from Theorem 3.4 by the methods used in [5] for the Hardy space $H^2(\mathbb{H})$.

Theorem 3.5. *If C_ϕ is bounded on $\mathcal{A}^2_\alpha(\mathbb{H})$, then its spectral radius and norm are equal.*

Theorem 3.6. *Every bounded composition operator on $\mathcal{A}^2_\alpha(\mathbb{H})$ has essential norm equal to its operator norm. In particular, since the zero operator is not a composition operator, there are no compact composition operators on any of the spaces $\mathcal{A}^2_\alpha(\mathbb{H})$.*

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Department of Pure Mathematics, University of Leeds, Leeds, LS2 9JT, UK
E-mail address: samuel@maths.leeds.ac.uk

Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK
E-mail address: andrew.wynn@ucl.ac.uk