Stability of Schwarzschild-AdS for the spherically symmetric Einstein-Klein-Gordon system

Gustav Holzegel* and Jacques Smulevici†

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Abstract

In this paper, we study the global behavior of solutions to the spherically symmetric coupled Einstein-Klein-Gordon (EKG) system in the presence of a negative cosmological constant. For the Klein-Gordon mass-squared satisfying $a \geq -1$ (the Breitenlohner-Freedman bound being $a > -\frac{9}{8}$), we prove that the Schwarzschild-AdS spacetimes are asymptotically stable: Small perturbations of Schwarzschild-AdS initial data again lead to regular black holes, with the metric on the black hole exterior approaching, at an exponential rate, a Schwarzschild-AdS spacetime. The main difficulties in the proof arise from the lack of monotonicity for the Hawking mass and the asymptotically AdS boundary conditions, which render even (part of) the orbital stability intricate. These issues are resolved in a bootstrap argument on the black hole exterior, with the red-shift effect and weighted Hardy inequalities playing the fundamental role in the analysis. Both integrated decay and pointwise decay estimates are obtained. As a corollary of our estimates on the Klein-Gordon field, one obtains in particular exponential decay in time of spherically-symmetric solutions to the linear Klein-Gordon equation on Schwarzschild-AdS.

Contents

*Princeton University, Department of Mathematics, Fine Hall, Washington Road, Princeton, NJ 08544, United States.
†Max-Planck-Institut für Gravitationsphysik, Albert-Einstein Institut, Am Mühlenberg 1, 14476 Golm, Germany.

1 Introduction

The subject of black hole stability has undergone rapid development in the past ten years. There is now an extensive literature addressing the behavior of linear wave equations on black hole backgrounds [18, 2, 39, 17, 21, 44, 1, 19], the current state of the art being a decay result for ϕ satisfying $\Box_g \phi = 0$ with g being the metric of a subextremal ($|a| < M$) Kerr spacetime [20]. In addition, there has been progress in establishing improved decay rates [37, 36, 12, 22, 41, 43], in developing techniques to address non-linear problems on fixed backgrounds [38, 45] and, most recently, preliminary attempts to bridge the gap between these linear and the prospective full non-linear stability problem [29].

1.1 Scalar waves and the stability of black holes

While the problem of Kerr stability will, presumably, require the study of the Bianchi equations as in [9], rather than the scalar wave equation, there is nonetheless a coupled non-linear gravitational system, whose metric evolution is governed entirely[1] by a scalar field ϕ satisfying

$$\Box_g \phi = 0. \tag{1}$$

This is the well-known spherically-symmetric coupled Einstein-scalar field system

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi \mathbb{T}_{\mu\nu} = 8\pi \left[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 \right] \tag{2}$$

with Λ being the cosmological constant and $g_{\mu\nu}$ a spherically symmetric metric. The study of this system was initiated in the 1960ies. Over the years, a complete and satisfactory picture of the dynamics has emerged for the asymptotically flat case ($\Lambda = 0$). In a sequence of papers [5, 7, 6, 8], Christodoulou proved that generic initial data either evolve into regular black holes or into spacetimes which are geodesically complete and whose Bondi mass approaches zero along null-infinity. For the black hole case, quantitative rates of approach to the Schwarzschild metric on the domain of outer communication have been established in [15].[2]

The result of [15] in fact holds for any data containing a trapped surface and hence applies to a much larger class of data than perturbations of Schwarzschild. Its proof, however, exploits extensively the spherical symmetry: (1)-(2) reduces to a system of 1+1 dimensional PDEs, for which many additional analytical tools are available. On the other hand, it was shown in [13, 30] that the vectorfield techniques developed for the linear wave equation on a *fixed* Schwarzschild background are sufficiently robust to understand the dynamics of the coupled system (1)-(2) in a neighborhood of Schwarzschild, that is to say to prove asymptotic stability of Schwarzschild within this model.[3] This avoids the use of techniques which are special to 1+1 dimensional PDEs and connects, in a satisfactory manner, the numerous works on the linear scalar wave equation with a non-linear model of gravitational collapse, illustrating at the same time how appropriate the vectorfield method is for these types of non-linear applications.

[1] By this we mean that if $\phi = 0$, the spacetime is stationary. The metric g depends nonetheless non-linearly on ϕ via the Einstein equations.

[2] These polynomial rates are commonly known as "Price's law". Note that in [15] the system studied is actually the spherically symmetric Einstein-Maxwell-scalar field equations, which reduce to (1)-(2) if the Maxwell field vanishes.

[3] We remark that the paper [30] studies a five-dimensional version of the spherically-symmetric Einstein scalar field system, more precisely, a class of vacuum, $SU(2)$-symmetric spacetimes, also known as biaxial Bianchi IX. The method, however, easily specializes to the spherically-symmetric coupled scalar field system in four dimensions.

1.2 Linear wave equations in asymptotically de-Sitter and Anti-de-Sitter spacetimes

It is natural to ask how these results change when $\Lambda \neq 0$ in (2). The physical motivation to consider $\Lambda > 0$ originates from cosmology and the observed accelerated expansion of the universe. The case $\Lambda < 0$, on the other hand, appears naturally in string theory and in the context of the AdS-CFT correspondence, see Section 1.5.

For a positive cosmological constant, $\Lambda > 0$, the system (2) has been studied (without symmetry assumptions and for more general matter models) to prove stability of the trivial solution, de Sitter space [25, 42]. In the black hole context, there has also been work on the linear wave equation $\Box_g \phi = 0$ for g a fixed Schwarzschild-de Sitter metric [16, 3, 40] and more recently, Kerr-de Sitter metric [24, 47, 23].

For a negative cosmological constant, $\Lambda = -\frac{3}{l^2} < 0$, there are only few results available, the linear problem having recently been addressed in [28]. In the latter paper, the massive wave equation,

$$\Box_g \phi - \frac{2a}{l^2}\phi = 0\,, \tag{3}$$

with the mass-squared of the Klein-Gordon field a satisfying the Breitenlohner-Freedman (BF) bound $-a < \frac{9}{8}$, is studied for a class of stationary spacetimes $(\mathcal{M}, g)$, which are sufficiently close to a slowly rotating Kerr-AdS spacetime. A boundedness result is then proven for a certain class of solutions to (3). The existence and uniqueness (after imposing suitable boundary conditions) of this class of solutions to (3) on any asymptotically AdS spacetime was only assumed in [28], with a proof now available in [31]. (See also [46], which likewise proves well-posedness of (3) with Dirichlet conditions for asymptotically AdS spacetimes admitting a conformal compactification.) Note that even this local well-posedness statement is non-trivial in view of the non globally-hyperbolic nature of these spacetimes. We refer to the introduction of [28], as well as the original work [4] for more motivation and an explanation of the BF-bound.

1.3 The spherically symmetric Einstein-Klein-Gordon system

In the present paper, we study the corresponding non-linear coupled gravitational system, the so-called Einstein-Klein-Gordon (EKG) system. As in the asymptotically flat case, the metric evolution is governed by (3) in the sense that if $\phi = 0$, then the only spherically symmetric solutions are the Schwarzschild-Anti-de-Sitter spacetimes or Anti-de-Sitter, by a simple generalization of Birkhoff's theorem.

Hence, we are interested in triples $(\mathcal{M}, g, \phi)$ such that $(\mathcal{M}, g)$ is a spherically-symmetric spacetime satisfying

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi \mathbb{T}_{\mu\nu}, \tag{4}$$

$$\mathbb{T}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - \frac{a}{l^2}\phi^2 g_{\mu\nu}, \tag{5}$$

and such that ϕ satisfies the Klein-Gordon equation (3) associated to $(\mathcal{M}, g)$ with mass $a \in \mathbb{R}$ and where we recall that $\Lambda = -\frac{3}{l^2}$, with $l \in \mathbb{R}$. As for the linear case, we shall need a bound on a. In this paper, we will assume

$$a \geq -1\,, \tag{6}$$

which includes the important conformally invariant case, $a = -1$. In fact, several results of this paper will hold under the weaker BF-bound $a > -\frac{9}{8}$. The only argument which exploits $a \geq -1$ is contained in the proof of the integrated decay estimate (Proposition 3.3).

The study of the EKG system with $\Lambda < 0$ was initiated in [33], where we prove that this system is well-posed under appropriate regularity and boundary conditions, with the time of existence of solutions depending only on an invariant H^2-type norm for the Klein-Gordon-field. As applications, we formulated two extension principles and obtained the existence and uniqueness of a maximal development. These results form the basis for the analysis conducted in this paper.

1.4 The Stability of Schwarzschild-AdS spacetimes

The main result of this paper may be summarized as follows:

Theorem 1.1. *The Schwarzschild-Anti de Sitter spacetime is both orbitally and asymptotically stable within EKG in spherical symmetry provided the Klein-Gordon mass satisfies (6). Moreover, the convergence to Schwarzschild-AdS is exponential in time on the black hole exterior.*

In other words, small spherically symmetric perturbations of Schwarzschild-AdS spacetimes again evolve into black hole spacetimes with a regular event horizon and a complete null-infinity (orbital stability). Moreover, the scalar field decays exponentially towards the future (asymptotic stability). A more precise version of Theorem 1.1, which includes in particular the specific decay rates and the coordinate systems they are measured in is given in Section 3, Theorem 3.8.

Remark 1.2. *We emphasize that the stability statement of Theorem 1.1 is expected to be very special to spherical symmetry, as suggested by our recent analysis [32] of the Klein-Gordon equation on a fixed Kerr-AdS background (with no symmetry assumptions on the Klein-Gordon field). The underlying reason is that trapping is entirely absent in spherical symmetry, while in the general case the combination of the familiar trapping and the boundary conditions near infinity leads to a version of "stable trapping" which may reasonably be expected to trigger an instability in the non-linear problem. See [32] for further details.*

We spend the remainder of this introduction to discuss the techniques entering the proof of Theorem 1.1. As in the asymptotically-flat case, the system

(3)-(4)-(5) comes with a conservation law, which is manifest in the properties of a generalized Hawking mass (renormalized by a cosmological term). Contrary to the former case case, however, this Hawking mass does not enjoy any monotonicity properties, in view of the possibly negative zeroth order mass term in (5). This can also be understood from the vectorfield point of view: There is a certain geometric vectorfield, T, which is not Killing but nevertheless gives rise to a conservation law via the energy momentum tensor of ϕ. This is the well-known Kodama vectorfield in spherical symmetry [34]. From this perspective, the failure of monotonicity is simply the energy momentum tensor (5) not satisfying the dominant energy condition. In any case, this behavior implies that the extension principle developed for the asymptotically-flat case (see [14, 11, 10]) is not available, since it relied on the monotonicity properties of the Hawking mass. While the general structure of the Penrose diagram can in fact still be inferred from a generalized extension principle (originally developed in [35] and adapted to our problem in [33]), the lack of monotonicity turns proving the completeness of null-infinity into a difficult problem.[4] This is in stark contrast to (2), where the completeness followed (for any data containing a trapped surface) from the extension principle and the monotonicity alone [11].

Studying the massive wave equation on Schwarzschild-AdS in [28], we observed that while the energy density is not pointwise positive definite, one can still prove a global doubly-weighted Hardy inequality on spacelike slices, allowing one to absorb the zeroth order term by a derivative term and hence to establish positivity in an integrated sense. It turns out that for the non-linear system (3)-(4)-(5), this integrated monotonicity survives in a region away from the future event horizon (whose location is bootstrapped in the non-linear problem under consideration). Close to the horizon, on the other hand, we have the redshift available, which will allow us to control the wrong-signed zeroth order term. We will see two manifestations of the redshift here: One in the framework of vectorfields (and hence L^2-type estimates), the other in the context of pointwise estimates along characteristics (originally developed in [15]), the latter being a typical feature of 1+1 dimensional systems.

It is instructive to compare this situation with the case of the linear wave equation on slowly rotating asymptotically-flat Kerr spacetimes. For these spacetimes, there is a conserved energy associated with the Killing field ∂_t, which is negative in a region close to the horizon, as the latter vectorfield becomes spacelike there. This is the well-known ergo-sphere and the phenomenon of superradiance that it triggers. One of the main insights of [21] was that this problem can be resolved by exploiting the redshift effect as a stability mechanism near the horizon. It is very much in this fashion that we are able to remedy the problems near the horizon for (3)-(4)-(5). Our setting is easier in that we are dealing with a highly symmetric problem (in particular, there is no trapping!), but at the same time it introduces new difficulties since we are addressing a fully coupled problem and since for us the non-positivity is actually

[4] Note that the completeness is a crucial ingredient of the *orbital* stability statement of Schwarzschild-AdS within (3)-(4)-(5).

a global feature.

Following this strategy, which is unfolded in a bootstrap argument on the location of the event horizon, we will prove that the scalar field ϕ remains small outside the future event horizon, provided the initial data is chosen sufficiently small. In a second step, we prove an integrated decay estimate, which implies that ϕ has to decay towards the future. While the existence of such an estimate may already be suggested from the non-existence of stationary solutions in the linearized setting (cf. Appendix A), we emphasize that it is the spherical symmetry which allows such an *unrestricted* integrated decay estimate here: In the non-symmetric (linearized) setting this estimate is expected to lose exponentially in the angular momentum modes. See Remark 1.2 and our [32].

This almost completes the proof of the theorem except for an issue which has to do with the radial decay towards null-infinity. We recall from [33] that the well-posedness statement is formulated in terms of a weighted H^2-norm for ϕ. To establish global existence in this space we need to also commute with the vectorfield T mentioned above. This introduces a fair amount of error-terms (as T is not Killing) which are, however, easily controlled from previous bounds and by adding an ϵ of the integrated decay estimate that we can prove simultaneously for $T\phi$. It then follows that $T\phi$ satisfies the same boundedness and decay estimates as ϕ does, which in particular establishes improved decay for $T\phi$. From this, the radial decay for *all* first derivatives can be improved, depending on how close a is to the Breitenlohner-Freedman bound (the closer, the smaller the improvement). This last step uses a version of the redshift effect which is present for asymptotically AdS spacetimes near null-infinity.[5] Collecting the improvements one recovers global uniform boundedness in the H^2-spaces of [33].

Finally, one proves that the integrated decay estimate implies exponential decay of the energy. This relies on the characteristic r-weights in the energy of asymptotically AdS spacetimes: Unlike in the asymptotically-flat context, the integrated decay estimate here controls the energy integrated in time without any loss of r-weights. From the statement that the energy integrated in time is controlled by the energy itself, exponential decay follows.[6]

In summary, the paper settles (excluding questions about black hole interiors) the issue of global dynamics for (EKG) for the mass range (6) and with Dirichlet boundary conditions for ϕ near the Schwarzschild-AdS solution.

1.5 Hairy black holes and motivations from high-energy physics

We conclude this introduction by providing some background information about the system (3)-(4)-(5). An important motivation for the study of this system derives from high energy physics, more precisely, the AdS-CFT correspondence

[5] This improvement arising after commutation has been exploited in [31] in the context of weighted elliptic estimates on spatial slices and in [33] in the context of pointwise estimates.

[6] Note that the method of proof used in this paper to derive the decay statements of Section 3 may naturally be applied to the linear case, cf. Corollary 3.9.

and its potential applications to condensed matter physics [48, 27, 26]. In this field, systems such as (3)-(4)-(5) (coupled often also to electromagnetism or complex scalar fields) are considered as models describing phase transitions in superconductors. From the gravitational point of view, such phase transitions correspond to non-trivial (i.e. with non-identically vanishing scalar field) stationary black hole spacetimes, also known as black holes with "scalar hair". Such solutions are possible in principle because the "no-hair" theorems valid in the asymptotically flat case do not (typically) generalize to the case of a negative cosmological constant. The main result of this paper excludes the existence of such hairy black holes in a neighborhood of the Schwarzschild-AdS solution within the class of boundary conditions on ϕ considered.

As suggested from asymptotic expansions of (3), there is an important alternative class of boundary conditions, Neumann-conditions, for ϕ.[7] For this class, one could attempt to carry out a similar program as in [28, 33]: Prove a well-posedness statement for (3) and thereafter for the non-linear (3)-(4)-(5), now imposing Neumann conditions at the boundary. The global dynamics under these circumstances may be more complicated, even in a neighborhood of Schwarzschild. In particular, the aforementioned hairy black hole solutions may enter the picture. We postpone the analysis of this system to future work.

1.6 Outline

The outline of this paper is as follows. Section 2 contains all necessary background material to perform the analysis: We construct the Schwarzschild-AdS spacetime in Section 2.2 and introduce the functional framework adapted to our problem (Section 2.4.1). The existence of a maximal development is also recalled (Section 2.4). After these preliminaries, we present a detailed version of our main results in Section 3. The basic boundedness estimates are proven in Section 4. In Section 5 we derive integrated decay estimates via vectorfield methods. Higher-order estimates as well as improved decay estimates are derived in Section 6. Appendix A contains an independent result establishing the non-existence of stationary solutions for the linear wave equation on Schwarzschild-AdS satisfying the boundary conditions of [28, 31].

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[7] at least in the range $\frac{5}{8} \leq -a < \frac{9}{8}$. For $-a < \frac{5}{8}$, there is only one solution and no freedom to impose boundary conditions.

2 Preliminaries

2.1 The spherically-symmetric Einstein-Klein-Gordon system in double null coordinates

We start by recalling a standard result (see [10, 11, 15]) regarding the form of the equations in double null coordinates for spherically symmetric solutions. It may be paraphrased by saying that the analysis of the spherically symmetric EKG system reduces to the study of the equations (8)–(12) on a $1+1$ dimensional Lorentzian manifold.

Lemma 2.1. *Let $\mathcal{Q}$ be a C^3 two-dimensional manifold (possibly with boundary) and $\mathcal{M} = \mathcal{Q} \times \mathbb{S}^2$. Let g be a C^2 Lorentzian metric on $\mathcal{M}$ and ϕ a C^2-function on $\mathcal{M}$. Assume that $(\mathcal{M}, g, \phi)$ is a solution to the system (3)-(4)-(5) and that $(\mathcal{M}, g, \phi)$ is invariant under the natural action of $SO(3)$, i.e. the latter acting transitively, with spacelike orbits, and by isometry on the $\mathbb{S}^2$-part of $\mathcal{M}$.[8] The manifold $\mathcal{Q}$ then inherits a natural Lorentzian metric from $(\mathcal{M}, g)$ as follows. Let π denote the canonical projection $\pi : \mathcal{M} \to \mathcal{Q}$. For any $X_{\mathcal{Q}}, Y_{\mathcal{Q}} \in T\mathcal{Q}$, there exist unique vectors X and Y, which are orthogonal to the $\mathbb{S}^2$-orbits and project to $X_{\mathcal{Q}}, Y_{\mathcal{Q}}$. Therefore $g_{\mathcal{Q}}(X_{\mathcal{Q}}, Y_{\mathcal{Q}}) = g(X, Y)$ defines a natural metric on $\mathcal{Q}$. Denote by r the area-radius of the spheres of symmetry. Then, locally around any point of $\mathcal{M}$, there exist double null coordinates u, v such that the metric takes the form:*

$$g = -\Omega^2 du dv + r^2 d\sigma_{\mathbb{S}^2}, \tag{7}$$

such that $-\Omega^2 du dv = \pi^* g_{\mathcal{Q}}$, $\Omega > 0$ and $r > 0$ may be identified with C^2-functions depending only on (u, v) and where $d\sigma_{\mathbb{S}^2}$ denotes the standard metric on $\mathbb{S}^2$. Moreover, the Einstein-Klein-Gordon equations[9] reduce to the following set of equations on $\mathcal{Q}$:

$$\partial_u \left(\frac{r_u}{\Omega^2} \right) = -4\pi r \frac{(\partial_u \phi)^2}{\Omega^2}, \tag{8}$$

$$\partial_v \left(\frac{r_v}{\Omega^2} \right) = -4\pi r \frac{(\partial_v \phi)^2}{\Omega^2}, \tag{9}$$

$$r_{uv} = -\frac{\Omega^2}{4r} - \frac{r_u r_v}{r} + 4\pi r \left(\frac{a\Omega^2 \phi^2}{2l^2} \right) - \frac{3}{4} \frac{r}{l^2} \Omega^2, \tag{10}$$

$$(\log \Omega)_{uv} = \frac{\Omega^2}{4r^2} + \frac{r_u r_v}{r^2} - 4\pi \partial_u \phi \partial_v \phi, \tag{11}$$

$$\partial_u \partial_v \phi = -\frac{r_u}{r} \phi_v - \frac{r_v}{r} \phi_u - \frac{\Omega^2 a}{2l^2} \phi. \tag{12}$$

Note that (8) and (9) are the Raychaudhuri equations governing the evolution of area of the spheres of symmetry. Note also that the last equation

[8] For simplicity, we will exclude here any possible axis of symmetry, since this is sufficient for the purpose of this paper.

[9] By a small abuse of notation, we denote functions on $\mathcal{M}$ and their projections to $\mathcal{Q}$ by the same symbols.

is simply the wave operator associated with g acting on spherically symmetric scalar fields, which may be written shorthand as

$$0 = \Box_g \phi - \frac{2a\phi}{l^2} = -\frac{4}{\Omega^2}\left(\partial_u\partial_v\phi + \frac{r_u}{r}\phi_v + \frac{r_v}{r}\phi_u\right) - \frac{2a\phi}{l^2}. \tag{13}$$

We shall use the following first order notation:

$$r_u = \nu \quad ; \quad r_v = \lambda \quad ; \quad r\phi_u = \zeta \quad ; \quad r\phi_v = \theta \quad ; \quad \kappa = -\frac{\Omega^2}{4r_u} \quad ; \quad \gamma = \frac{\Omega^2}{4r_v}. \tag{14}$$

We can then rewrite the Raychaudhuri equations as

$$\partial_u\kappa = -\frac{\Omega^2}{\nu^2}r\pi(\partial_u\phi)^2 = -\frac{16}{\Omega^2}\kappa^2 r\pi(\partial_u\phi)^2 < 0 \quad ; \quad \partial_u \log\kappa = \frac{4\pi r}{\nu}(\partial_u\phi)^2 \tag{15}$$

and

$$\partial_v \log\gamma = \frac{4\pi r}{\lambda}(\partial_v\phi)^2. \tag{16}$$

We define the (renormalized) Hawking mass,

$$\varpi = \frac{r}{2}\left(1 + \frac{4r_u r_v}{\Omega^2}\right) - \frac{\Lambda}{6}r^3 = \frac{r}{2}\left(1 + \frac{4r_u r_v}{\Omega^2}\right) + \frac{r^3}{2l^2}, \tag{17}$$

which is seen to satisfy

$$\partial_u\varpi = -8\pi r^2\frac{r_v}{\Omega^2}(\partial_u\phi)^2 + \frac{4\pi r^2 a}{l^2}r_u\phi^2, \tag{18}$$

$$\partial_v\varpi = -8\pi r^2\frac{r_u}{\Omega^2}(\partial_v\phi)^2 + \frac{4\pi r^2 a}{l^2}r_v\phi^2. \tag{19}$$

The quantity

$$1 - \mu := 1 - \frac{2\varpi}{r} + \frac{r^2}{l^2} = -\frac{4r_u r_v}{\Omega^2} \tag{20}$$

will be used frequently. We finally collect some further identities, which will be useful to refer to in later computations. The volume element is $\sqrt{-g} = \frac{\Omega^2 r^2}{2}\sqrt{\sigma_{\mathbb{S}^2}}$ and hence

$$\sqrt{-g}g^{uv} = \frac{\Omega^2 r^2}{2}\left(\frac{-2}{\Omega^2}\right)\sqrt{\sigma_{\mathbb{S}^2}} = -r^2\sqrt{\sigma_{\mathbb{S}^2}}. \tag{21}$$

The following identities hold for the Christoffel symbols: ${\Gamma^\mu}_{uv} = 0$ for $\mu = \{u, v, \theta, \phi\}$ and

$${\Gamma^u}_{uu} = g^{uv}\left(g_{uv}\right)_u = \frac{2\Omega_u}{\Omega} \qquad , \qquad {\Gamma^v}_{vv} = g^{uv}\left(g_{uv}\right)_v = \frac{2\Omega_v}{\Omega}, \tag{22}$$

$$g^{\mu\nu}\Gamma^{u}{}_{\mu\nu} = \frac{4r_v}{r\Omega^2} = \frac{1}{r\gamma} \qquad , \qquad g^{\mu\nu}\Gamma^{v}{}_{\mu\nu} = \frac{4r_u}{r\Omega^2} = -\frac{1}{r\kappa}\,. \tag{23}$$

The square of the gradient of ϕ is

$$g(\nabla\phi, \nabla\phi) = \frac{-4}{\Omega^2}\phi_u\phi_v. \tag{24}$$

Finally, the wave equation for r may be rewritten using the Hawking mass ϖ:

$$r_{uv} = -\frac{\Omega^2\varpi}{2r^2} - \frac{\Omega^2 r}{2l^2} + \frac{2\pi r a\Omega^2\phi^2}{l^2}\,. \tag{25}$$

2.2 Schwarzschild-AdS

We now construct the Schwarzschild-AdS family from the point of view of Lemma 2.1. Let $M > 0$, $l > 0$ be given real parameters. Define r_{ASch} to be the unique positive real root of $F(X) = 1 - \frac{2M}{X} + \frac{X^2}{l^2}$. Let us also define the following dimensionless parameters:

$$c_1 = \frac{r_{ASch}}{l} =: R_h \quad , \quad c_2 = R_h^2 + 1 \quad , \quad c_3 = \frac{R_h}{R_h^2+1} \quad , \quad c_4 = \frac{R_h^2+1}{3R_h^2+1}\,,$$

as well as the function $h : (0,\infty) \to \mathbb{R}$,

$$h(r) = \log(3M - r_{ASch}) + \frac{1}{lc_3}\int_r^{3M} d\rho \frac{-c_3 \cdot \frac{\rho}{l} + c_4}{\frac{\rho^2}{l^2} + c_1\frac{\rho}{l} + c_2} + \frac{C}{lc_3}\,,$$

where $C = \int_{3M}^{\infty} \frac{dx}{F(x)}$. Note $h(r) \sim \log r$ as $r \to \infty$. Finally, let $j : (0,\infty) \to \mathbb{R}$,

$$j(r) = -4l^2 (c_3)^2 (r - r_{ASch})e^{-h(r)}\,.$$

One checks that j is monotonically decreasing, hence injective, and computes the constants

$$J_{max} = \lim_{r\to 0} j(r) = 4l^2(c_3)^2 r_{ASch}e^{-h(0)} > 0 \qquad J_{min} = \lim_{r\to\infty} j(r) = -4l^2(c_3)^2\,.$$

Define the manifold $\mathcal{Q} \subset \mathbb{R}^2$ by

$$\mathcal{Q} = \left\{(U,V) \in (-1,1)\times(-1,1) \;\middle|\; J_{min} < \tan\left(\frac{\pi}{2}U\right)\tan\left(\frac{\pi}{2}V\right) < J_{max}\right\}.$$

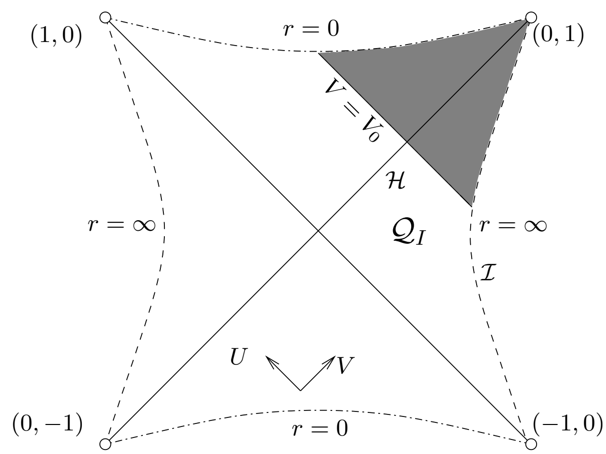
Finally, equip the manifold $\mathcal{M} = \mathcal{Q} \times \mathbb{S}^2$ with the metric

$$g = -\frac{e^{h(r)}}{4r}\left[\frac{r^2}{l^2} + \frac{r_{ASch}}{l^2}r + \frac{r_{ASch}^2}{l^2} + 1\right]\frac{\pi\, dU}{\cos^2\left(\frac{\pi}{2}U\right)}\frac{\pi\, dV}{\cos^2\left(\frac{\pi}{2}V\right)} + r^2 d\sigma_{\mathbb{S}^2}^2 \tag{26}$$

where the area radius $r = r(U,V)$ is defined implicitly via

$$\tan\left(\frac{\pi}{2}U\right)\tan\left(\frac{\pi}{2}V\right) = j(r) = -4l^2(c_3)^2(r - r_{ASch})e^{-h(r)}. \tag{27}$$

The manifold $(\mathcal{M}, g)$ with g given by (26) is, by definition, the maximally extended Schwarzschild AdS spacetime. Note that the set of (U, V) for which $\tan\left(\frac{\pi}{2}U\right)\tan\left(\frac{\pi}{2}V\right) = J_{max}$ corresponds to two disconnected boundary components of $\mathcal{Q} \subset \mathbb{R}^2$ with the area radius going to zero as this set is approached. One may easily check that the curvature blows up at this part of the boundary. The set of (U, V) for which $\tan\left(\frac{\pi}{2}U\right)\tan\left(\frac{\pi}{2}V\right) = J_{min}$ constitutes another two (also disconnected) boundary components of $\mathcal{Q}$ with the area radius approaching infinity as these points are approached from $\mathcal{Q}$. We will write $\mathcal{I}$ for the boundary component with $U < 0$. The remaining boundary components of $\mathcal{Q} \subset \mathbb{R}^2$ are simply the points $(0, -1)$, $(0, 1)$, $(1, 0)$, $(-1, 0)$. The set of points $(U, V) \in \mathcal{Q}$ with $U = 0$ and $V \geq 0$ is called the future event horizon of the black hole, which we have denoted by $\mathcal{H}$. The Penrose diagram of the spacetime is depicted below (see Appendix C of [15] for a formal introduction to Penrose diagrams). The spacetime is maximally extended in the sense that all its geodesics are either affine complete or they terminate at the curvature singularity at $r = 0$ in finite affine time.



Unfortunately, the coordinates of (26) are not very well suited to identify the metric as being asymptotically AdS. To remedy this, we now consider the submanifold $\mathcal{Q}_I = \mathcal{Q} \cap \{U < 0\} \cap \{V > 0\}$ and express the metric on $\mathcal{Q}_I$ in the more familiar Eddington Finkelstein (EF)-coordinates. Perform the following coordinate transformation to (u, v)-coordinates on $\mathcal{Q}_I$:

$$-2lc_3 e^{-\frac{u}{2lc_3}} = \tan\left(\frac{\pi}{2}U\right) \quad , \quad 2lc_3 e^{\frac{v}{2lc_3}} = \tan\left(\frac{\pi}{2}V\right) . \tag{28}$$

Note that the (u, v) coordinate system covers $\mathcal{Q}_I$ with range $(-\infty, \infty) \times (-\infty, \infty)$ and that the boundary component $U = 0$ of $\mathcal{Q}_I$ in $\mathcal{Q}$ corresponds to $u \to \infty$ and similarly $V = 0$ to $v \to -\infty$. Note also that from (27) we have $r > r_{ASch}$ on $\mathcal{Q}_I$ and $r = r_{ASch}$ on the boundary of $\mathcal{Q}_I$ in $\mathcal{Q}$. The metric g on $\mathcal{Q}_I \times \mathbb{S}^2$ can then be written as

$$g = -\frac{e^{h(r)}}{r}\left[\frac{r^2}{l^2} + \frac{r_{ASch}}{l^2}r + \frac{r_{ASch}^2}{l^2} + 1\right] 4l^2 (c_3)^2\, e^{\frac{v-u}{2lc_3}}\, du dv + r^2 d\sigma_{\mathbb{S}^2}^2 . \tag{29}$$

Using (27) and (28) one easily checks that

$$\frac{e^{h(r)}}{r}\left[\frac{r^2}{l^2}+\frac{r_{ASch}}{l^2}r+\frac{r_{ASch}^2}{l^2}+1\right]4l^2\left(c_3\right)^2 e^{\frac{v-u}{2lc_3}}=1-\frac{2M}{r}+\frac{r^2}{l^2}\,,$$

which takes (29) into the familiar EF-form on $\mathcal{Q}_I$. Note that in the EF-coordinates we have the familiar relations $2r_v = -2r_u = \left(1-\frac{2M}{r}+\frac{r^2}{l^2}\right)$. In particular, the set $\mathcal{I}$ gets "straightened out" in these coordinates corresponding now to a line along which $u+v$ is constant.

Consider finally a ray $V = V_0 > 0$ in $\mathcal{Q}$ as indicated in the figure above. As this ray may equivalently (and conveniently) be described by its EF-v-coordinate, v_0, we shall denote it by $N\left(v_0\right)$, and the past limit point where it intersects the set $\mathcal{I}$ by (u_0, v_0). We can introduce a regular coordinate system $(\tilde{u}, v)$ covering the shaded region by keeping the EF-coordinate v and defining $\tilde{u}$ along $N\left(v_0\right)$ such that $\tilde{u}\left(u_0\right) = u_0$ and $-2r_{\tilde{u}}\left(\tilde{u}, v_0\right) = 1+\frac{r^2(\tilde{u},v_0)}{l^2}$ holds along $N\left(v_0\right)$. In these coordinates we have $N\left(v_0\right) = \{q \in \mathcal{Q} \mid (\tilde{u}_q, v_q) \in (u_0, u_0+\pi l)\times\{v_0\}\}$. It is convenient to drop the tilde and (by a small abuse of notation) identify the set $N\left(v_0\right)$ with its image in regular coordinates, i.e. to write $N\left(v_0\right) = (u_0, u_0+\pi l)\times\{v_0\}$.

2.3 Perturbed Schwarzschild-AdS data

With $N\left(v_0\right)$ fixed, we will now prescribe perturbations of Schwarzschild data on $N\left(v_0\right)$ and study their maximal development as in [33]. Instead of merely referring to the results of [33], where characteristic initial data are constructed in appropriate function spaces, we shall give a self-contained construction of initial data on $N\left(v_0\right)$ in this section. This will enable us to introduce the smallness conditions on the matter fields. The notation follows [33] and the reader is invited to consult the latter paper for more details.

On $N\left(v_0\right) = (u_0, u_0+\pi l)\times\{v_0\}$, we define $\bar{r}\left(u\right) = l\tan\left(\frac{u_0-u+l\pi}{2l}\right)$. This fixes our u-coordinate, cf. [33].

The free-data

The free data then consists in a C^2-function $\bar{\phi} : N\left(v_0\right)\to\mathbb{R}$ satisfying the smallness bound

$$\bar{r}^{\frac{3}{2}+\frac{1}{2}s}\left(|\bar{\phi}|+|\bar{r}\frac{\bar{\phi}_u}{\bar{r}_u}|\right)+\left|\bar{r}^{\frac{7}{2}}\frac{\partial_u\frac{\bar{\phi}_u}{\bar{r}_u}}{\bar{r}_u}\right|\leq\epsilon \quad \text{everywhere on } N\left(v_0\right), \tag{30}$$

where $s = \min\left(\sqrt{9+8a}, 2\right)$, and in addition, being such that the combination

$$\bar{\Phi} = \bar{r}^2\left[\bar{r}\,\partial_u\left(\frac{\bar{\phi}_u}{\bar{r}_u}\right)-4\bar{\phi}_u-\frac{2a\bar{r}_u}{\bar{r}}\bar{\phi}\right] \tag{31}$$

is integrable, $\Pi(u) = \int_{u_0}^{u} \bar{\Phi}(\bar{u})\, d\bar{u} < \epsilon$ for any $u \in N(v_0)$, and moreover, the bound

$$\int_{N(v_0)} \left(\bar{\Phi}^2 \frac{\bar{r}^2}{\bar{r}_u} + \Pi^2 \bar{r}_u \right) du < \epsilon^2 \tag{32}$$

holds. Note that both (30) and (32) are independent of the choice of the u-coordinate.

Remark 2.2. *The radial decay imposed on ϕ above is discussed extensively in both [31] and [33]. Note that in [33], we choose the constant s in* (30) *to be* $\min(\sqrt{9+8a}, 1)$*, compared to* $\min(\sqrt{9+8a}, 2)$ *here. While this additional radial decay is not needed to establish well-posedness for the non-linear problem (cf. [33]), it nonetheless can be shown to propagate. This was already remarked in [33] and will be shown again explicitly later, Section 6.4.*

Deduced quantities

From $\bar{\phi}$ we define the quantity $\bar{\varpi}$ as the unique C^1-solution of

$$\partial_u \bar{\varpi} = 8\pi \bar{r}^2 \frac{1 - \frac{2\bar{\varpi}}{\bar{r}} + \frac{\bar{r}^2}{l^2}}{4\bar{r}_u} \left(\partial_u \bar{\phi}\right)^2 + \frac{4\pi \bar{r}^2 a}{l^2} \bar{r}_u \bar{\phi}^2 \quad , \quad \lim_{u \to u_0} \bar{\varpi}(u) = M \tag{33}$$

and the C^1 quantity $\overline{r_v}$ as

$$\overline{r_v} = \frac{1}{2} \left(1 - \frac{2\bar{\varpi}}{\bar{r}} + \frac{\bar{r}^2}{l^2}\right) \exp\left(\int_{u_0}^{u} \frac{4\pi \bar{r}}{\bar{r}_u} \left(\partial_u \bar{\phi}\right)^2 du \right) . \tag{34}$$

Note that $\overline{r_v}$ is independent on the choice of u-coordinate on the data. We also define the C^1 quantity

$$\bar{\Omega}^2 = -\frac{4\bar{r}_u \overline{r_v}}{1 - \frac{2\bar{\varpi}}{\bar{r}} + \frac{\bar{r}^2}{l^2}} , \tag{35}$$

and the shorthand $\bar{\kappa} = \frac{\overline{r_v}}{1-\mu}$, which both depend on the choice of coordinates. Finally, we define the C^1 quantity $\overline{T(\phi)}$ (see [33] as well as the discussion of the vectorfield T in Section 5.2) as the unique solution of the ODE

$$\partial_u \left(\bar{r}\bar{\kappa}\overline{T(\phi)} \right) = -\bar{r}\bar{r}_v \, \partial_u \frac{\bar{\phi}_{,u}}{\bar{r}_u} + \bar{\phi}_u \left[-2\bar{r}_v - 2\frac{\bar{\kappa}\bar{r}^2}{l^2} - \frac{2\bar{\kappa}\bar{\varpi}}{\bar{r}} + \frac{8\pi \bar{r}^2 a \bar{\kappa} \bar{\phi}^2}{l^2} \right] - \frac{a\bar{\Omega}^2 \bar{r}}{2l^2} \bar{\phi}$$

with the boundary condition $\bar{r}\bar{\kappa}\overline{T(\phi)} = 0$. It follows from the conditions (32) and (30) that

$$\int_{u_0}^{u_1} \bar{r}^2 \left(\frac{\bar{r}^2}{|\bar{r}_u|} \left[\partial_u \overline{T(\phi)} \right]^2 + \overline{T(\phi)}^2 |\bar{r}_u| \right) du < C\epsilon^2 \tag{36}$$

and also that $|\overline{r}^{\frac{3}{2}}\overline{T(\phi)}| < C\epsilon$, where $C > 0$ is a constant depending only on a, l and M. Moreover, defining the C^1 quantity $\overline{\phi_v} = \bar{\kappa}\overline{T(\phi)} + \frac{\overline{r_v}}{\bar{r}_u}\partial_u\bar{\phi}$, we see that it satisfies

$$\left(\overline{\phi_v}\right)_u = -\frac{\bar{r}_u}{\bar{r}}\overline{\phi_v} - \frac{\overline{r_v}}{\bar{r}}\bar{\phi}_u - \frac{\bar{\Omega}^2 a}{2l^2}\bar{\phi}. \tag{37}$$

Note that (36) does not depend on the choice of u-coordinate.

Definition 2.3. *An* <u>*ϵ-perturbed Schwarzschild-AdS data set*</u> *on $N(v_0)$ consists in a free function $\bar{\phi} : N(v_0) \to \mathbb{R}$ satisfying (30) and (32), together with the C^1 deduced quantities $(\bar{\varpi}, \bar{\Omega}, \overline{r_v})$ as defined above. In particular, (36) holds for any ϵ-perturbed Schwarzschild-AdS data set.*

Remark 2.4. *In [33], we constructed initial data with $-2\bar{r}_u = 1 - \frac{2M}{\bar{r}} + \frac{\bar{r}^2}{l^2} + o(\bar{r}^{-1})$. Using the coordinate transformation*

$$\frac{du^\star}{du} = \frac{1 + \frac{\bar{r}^2}{l^2}}{1 - \frac{2M}{\bar{r}} + \frac{\bar{r}^2}{l^2}} = 1 + \mathcal{O}\left(\frac{1}{\bar{r}^3}\right) \tag{38}$$

near infinity, the ϵ-perturbed data set becomes manifestly a $\mathcal{C}^{1+k}_{a,M}$ asymptotically AdS data set in the sense of [33].

2.4 Maximum development and set-up

From [33], it follows that any ϵ-perturbed Schwarzschild-AdS data set admits a unique (up to diffeomorphism) maximal development. We refer to [33] for the precise statement of this result. Note also that by the uniqueness, specifying $\phi = 0$ identically will yield (a subset of) the Schwarzschild spacetime as its maximum development.

Let us denote the quotient (by the symmetry group) of the maximal development of some ϵ-perturbed Schwarzschild-AdS data set by $\mathcal{Q} \subset \mathbb{R}^2$. With λ and ν defined as in (14), we let $\mathcal{R} \subset \mathcal{Q}$ denote the *regular* region, i.e. the set of points such that $\lambda > 0$, $\nu < 0$. From [33], we have in particular:

Proposition 2.5. *Let $(\mathcal{M}, g)$ be the maximal development of some ϵ-perturbed Schwarzschild-AdS data set defined on $N(v_0)$ and $\mathcal{Q} = \mathcal{M}/SO(3)$ its quotient. Let u_0 be the infimum of u on $N(v_0)$. For $u > u_0$ we denote by $N(u) \subset \mathcal{Q}$ the outgoing characteristic null-line $u = constant$ emanating from the initial data. Then the following is true:*

1. *The set*
$$\{u > u_0 \mid N(u) \in \mathcal{R} \text{ and } r \to \infty \text{ along } N(u)\}$$
is non-empty.

2. *Defining*
$$u_{\mathcal{H}} := \sup_{u > u_0}\{u \mid N(u) \in \mathcal{R} \text{ and } r \to \infty \text{ along } N(u)\}, \tag{39}$$

as well as the subregions

$$\mathcal{R}_{\mathcal{H}} := \mathcal{R} \cap \{u_0 < u < u_{\mathcal{H}}\} \qquad \text{and} \qquad \overline{\mathcal{R}}_{\mathcal{H}} := \mathcal{R}_{\mathcal{H}} \cup N(u_{\mathcal{H}}), \tag{40}$$

we have that

$$\sup_{\mathcal{R}_{\mathcal{H}}} v = \sup_{N(u_{\mathcal{H}})} v\,.$$

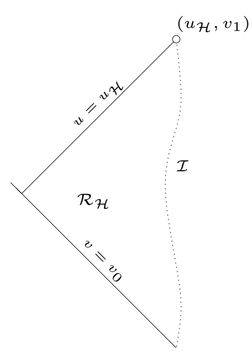
This says that first singularities[10] *cannot arise along* $u = u_{\mathcal{H}}$*.*

3. *Define the set "null infinity"* $\mathcal{I} = \{(u, v_\infty(u))\,|u_0 < u < u_{\mathcal{H}}\}$*, where* $v_\infty(u)$ *is the value of* v *such that we have* $\lim_{v \to v_\infty(u)} r(u,v) = \infty$*. There exists a double-null coordinate system* (u,v) *covering* $\mathcal{R}_{\mathcal{H}}$ *such that:*

$$\kappa = \frac{1}{2} \quad \text{on } \mathcal{I}, \qquad\qquad \frac{-r_u}{1+\frac{r^2}{l^2}} = \frac{1}{2} \quad \text{on } v = v_0. \tag{41}$$

Proof. By continuity, the data set contains a trapped surface, i.e. a point with $\lambda < 0$. Hence, Corollary A.2 of [33] applies, which yields 1. and 2. as well as the fact that the set $\mathcal{I}$ is well-defined. The existence of the coordinate system follows from a simple coordinate transformation.[11] □

We have depicted (a subset of) the Penrose diagram below. Note that $\mathcal{I}$ is not a straight line in the regular coordinates (41).



We emphasize that, for convenience, we have defined null-infinity here so as not to include the future limit point of the ray $u = u_{\mathcal{H}}$ even if $r \to \infty$ along $u = u_{\mathcal{H}}$, which is a priori possible. It will be an important part of the orbital stability to establish that r remains bounded along the horizon. In the asymptotically flat case, the monotonicity of the Hawking mass and the Raychaudhuri equation alone allow one to immediately conclude that r remains bounded along the horizon. Here we will have to work quite hard for this statement.

[10] See for instance [10] for an introduction to the study of singularities in spherically symmetric spacetimes.

[11] Note that under a change of null-coordinates defined by $\hat{u} = f(u)$, $\hat{v} = g(v)$, the quantities Ω^2, κ and γ transform as:

$$\widehat{\Omega}^2 = \frac{\Omega^2}{f'g'} \quad , \quad \hat{\kappa} = \frac{\kappa}{g'} \quad , \quad \hat{\gamma} = \frac{\gamma}{f'}\,.$$

2.4.1 Norms and Constants

We define the following norms on $\mathcal{R}_{\mathcal{H}}$. For any point $(u,v) \in \mathcal{R}_{\mathcal{H}}$ let $u_{\mathcal{I}}(v)$ denote the u-coordinate of the past limit point where the $v = const$-ray intersects $\mathcal{I}$. (If confusion is unlikely, we will sometimes abbreviate $u_{\mathcal{I}} = u_{\mathcal{I}}(v)$.)

$$\|\psi\|^2_{H^1_{AdS}(u,v)} = \int_{u_{\mathcal{I}}(v)}^{u} r^2 \left[\frac{r^2}{-r_u}(\partial_u \psi)^2 - r_u \psi^2\right](\bar{u},v)\, d\bar{u} + \int_{v_0}^{v} r^2 \left[\frac{1-\mu}{r_v}(\partial_v\psi)^2 + r_v\psi^2\right](u,\bar{v})\, d\bar{v}\,, \tag{42}$$

$$\|\psi\|^2_{H^1_{AdS,deg}(u,v)} = \int_{u_{\mathcal{I}}(v)}^{u} r^2 \left[\frac{1-\mu}{-r_u}(\partial_u \psi)^2(\bar{u},v) - r_u \psi^2\right](\bar{u},v)\, d\bar{u} + \int_{v_0}^{v} r^2 \left[\frac{1-\mu}{r_v}(\partial_v\psi)^2 + r_v\psi^2\right](u,\bar{v})\, d\bar{v}\,. \tag{43}$$

Note that both of these norms are independent of the choice of double null-coordinates. From [33], it follows in particular that they are continuous in (u,v). We also define spacetime energies capturing integrated decay:

$$\mathbb{I}_{deg}[\psi](\mathcal{D}) = \int_{\mathcal{D}} \frac{1}{r^2}\left[\frac{(\partial_u\psi)^2}{\gamma^2} + \frac{(\partial_v\psi)^2}{\kappa^2} + r^2\psi^2\right]\Omega^2 r^2(\bar{u},\bar{v})\, d\bar{u}d\bar{v}\,, \tag{44}$$

and also the non-degenerate integrated decay norm

$$\mathbb{I}[\psi](\mathcal{D}) = \int_{\mathcal{D}} \frac{1}{r^2}\left[\frac{r^4}{l^4}\frac{(\partial_u\psi)^2}{r_u^2} + \frac{(\partial_v\psi)^2}{\kappa^2} + r^2\psi^2\right]\Omega^2 r^2(\bar{u},\bar{v})\, d\bar{u}d\bar{v}\,. \tag{45}$$

In applications, the region $\mathcal{D}$ is often going to be

$$D(u,v) = \overline{\mathcal{R}_{\mathcal{H}}} \cap \{(u_0,u] \times [v_0,v]\}, \tag{46}$$

with $(u,v) \in \overline{\mathcal{R}_{\mathcal{H}}}$.

Finally, we denote by $B_{M,l}$ a constant which only depends on the fixed cosmological constant and the mass at infinity and by $B_{M,l,a}$ a constant which also depends on the fixed parameter a.

Remark 2.6. *The degenerate norm (43) originates from the conservation law associated with the Hawking mass, cf. (18) and (19). As this norm degenerates near the horizon (where $1-\mu$ is very small or zero) we need to control also (42), which will be achieved using the redshift near the horizon. Similarly, we have introduced two different energies measuring integrated decay. We will first control (44) and then, again using the redshift, (45). As mentioned in the introduction, a key property of the integrated decay energies is that they admit the same asymptotic r-weights as the energy itself (something that is not possible in the asymptotically-flat context). This will produce the exponential decay later and is a characteristic feature of the AdS asymptotic end.*

2.4.2 The constant r-curves r_X and r_Y

Since $\mathcal{R}_{\mathcal{H}}$ is part of the regular region, $r \geq r_{min} := r(u_{\mathcal{H}}, v_0)$ holds in $\mathcal{R}_{\mathcal{H}}$. By the Raychaudhuri equation, a point on the initial data-ray $v = v_0$ at which $r_v < 0$ cannot be part of $\mathcal{R}_{\mathcal{H}}$. Since moreover in Schwarzschild $r_v < 0$ holds for $r < r_{ASch}$ (recall r_{ASch} is the value of r on the horizon in Schwarzschild-AdS), we have by the smallness assumption on the data the lower bound $r_{min} \geq r_{ASch}(1 - C(\epsilon))$, with $C(\epsilon) \to 0$ as $\epsilon \to 0$. Let $c > 0$ be a small uniform constant (in particular, $c^{\frac{1}{3}}$ should still be much smaller than $a + \frac{9}{8}$) and define r_Y as the unique real solution of

$$1 - \frac{2M}{r_Y} + \frac{r_Y^2}{l^2} = c^{\frac{1}{3}}\,, \tag{47}$$

Note that for c small we have the estimate

$$0 < r_Y - r_{ASch} \leq c^{\frac{1}{3}} \frac{2r_{ASch}}{1 + \frac{3r_{ASch}^2}{l^2}}\,. \tag{48}$$

We choose c so small that in particular

$$\frac{2Ml^2\,(1 - \sqrt{c})}{r_Y^3} > \frac{1}{2} \tag{49}$$

holds. Since this estimate is true for $c = 0$ from $\frac{Ml^2}{r_{ASch}^3} = \frac{Ml^2}{2Ml^2 - r_{ASch}l^2} > \frac{1}{2}$, this is possible by continuity. Note that a-priori the curve $r = r_Y$ could lie outside of $\mathcal{R}_{\mathcal{H}}$, namely, if r_{min} happens to be much larger than r_{ASch}. In the same manner, we define a curve $r = r_X$ by solving

$$1 - \frac{2M}{r_X} + \frac{r_X^2}{l^2} = d^{\frac{1}{3}}\,. \tag{50}$$

We assume $d > c$ (hence $r_X > r_Y$). As for r_Y, we have

$$0 \leq r_X - r_{ASch} \leq B_{M,l} d^{1/3},$$

By continuity, we can choose d so that the following estimate holds:

$$\log \frac{r_X}{r_{min}} < \frac{1}{2|a|}\,. \tag{51}$$

3 The main results

The main theorem can be found at the end of this section. We use this section to outline the sequence of propositions leading to the theorem. Some propositions are proven right away, while the proof of the three key propositions containing the crucial estimates is postponed to Sections 4 to 6.

For the results below, recall the mass bound (6), the definition of an ϵ-perturbed Schwarzschild-AdS data set (Definition 2.3) and that of the region $\mathcal{R}_{\mathcal{H}}$ associated with it, (40).

Step 1: We first establish *uniform* bounds in the region $\mathcal{R}_{\mathcal{H}}$.

Proposition 3.1 (Basic estimates)**.** *There is an $\epsilon > 0$ such that the solution arising from an ϵ-perturbed Schwarzschild-AdS data set satisfies the following estimate for $(u,v) \in \mathcal{R}_{\mathcal{H}}$:*

$$\begin{aligned} &|\varpi - M|^{\frac{1}{2}} + |2\kappa - 1|^{\frac{1}{2}} + |r^{\frac{3}{2}}\phi| + \left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right| + \|\phi\|_{H^1_{AdS}(u,v)} \\ &\qquad \leq B_{M,l,a}\left[\|\phi\|_{H^1_{AdS}(u_{\mathcal{H}},v_0)} + \sup_{v=v_0, u\leq u_{\mathcal{H}}}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right|\right]. \end{aligned} \tag{52}$$

We remark that Proposition 3.1 actually holds for the entire Breitenlohner-Freedman range $a > -\frac{9}{8}$ and not only the range (6). The proof of this proposition is contained in Section 4.

Proposition 3.2 (Improved and higher order bounds)**.** *Let*

$$\begin{aligned} \mathbb{N}[\phi](v_0) = \Big[&\|\phi\|_{H^1_{AdS}(u_{\mathcal{H}},v_0)} + \|T\phi\|_{H^1_{AdS}(u_{\mathcal{H}},v_0)} \\ &+ \sup_{I(v_0)}\left|r^{\frac{3}{2}+\frac{s}{2}}\frac{\zeta}{\nu}\right| + \sup_{I(v_0)}\left|r^{\frac{5}{2}}\frac{\partial_u(T\phi)}{\nu}\right| + \sup_{I(v_0)}\left|r^{\frac{7}{2}}\frac{\partial_u\frac{\partial_u\phi}{r_u}}{r_u}\right|\Big] < \infty \end{aligned} \tag{53}$$

with $I(v_0) = \{(u,v_0) \mid u \leq u_{\mathcal{H}}\} \subset N(v_0)$ be a second order norm on the initial data. There is an $\epsilon > 0$ such that for any ϵ-perturbed Schwarzschild-AdS data set we have the following estimates for $(u,v) \in \mathcal{R}_{\mathcal{H}}$:

$$\left|r^{\frac{7}{2}}\frac{\partial_u\frac{\partial_u\phi}{r_u}}{r_u}\right| + \left|r^{\frac{5}{2}}\frac{\partial_u(T\phi)}{\nu}\right| + \|T\phi\|_{H^1_{AdS}(u,v)} \leq B_{M,l,a}\cdot\mathbb{N}[\phi](v_0), \tag{54}$$

and, for any $\delta > 0$ and $s = \min\left(\sqrt{9+8a}, 2-2\delta\right)$,

$$|r^{\frac{3}{2}+\frac{s}{2}}\phi| + \left|r^{\frac{3}{2}+\frac{s}{2}}\frac{\zeta}{\nu}\right| + |r^{\frac{1}{2}+\frac{s}{2}}\phi_v| \leq C_\delta\cdot B_{M,l,a}\cdot\mathbb{N}[\phi](v_0), \tag{55}$$

where $T\phi = \frac{1}{4\kappa}\partial_v\phi + \frac{1}{4\gamma}\partial_u\phi$.

Note that all these bounds are independent on the choice of u-coordinate. For a discussion of the role of the vectorfield T and its relation to the Hawking mass see Section 5.2 and the discussion in the introduction.

Proposition 3.3. *(Integrated decay) We have the integrated decay estimates*

$$\|\phi\|^2_{H^1_{AdS}(u,v)} + \mathbb{I}[\phi](D(u,v)) \leq B_{M,l,a}\|\phi\|^2_{H^1_{AdS}(u,v_0)}, \tag{56}$$

$$\mathbb{I}[T\phi](D(u,v)) \leq B_{M,l,a}\cdot\mathbb{N}^2[\phi](v_0). \tag{57}$$

Remark 3.4. *The proof of Proposition 3.1 will not require the construction of an integrated decay estimate. It exploits the redshift in terms of pointwise estimates, a characteristic feature of spherical symmetry [15]. On the other hand, the proof of both Propositions 3.2 and 3.3 will require integrated decay for both ϕ and $T\phi$. It is precisely in our construction of the integrated decay estimate that the restriction (6) enters.*

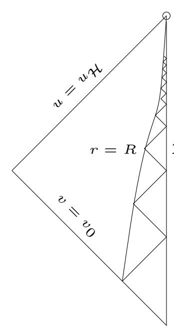
The estimate (55) can be improved further by another commutation with T in case that $\sqrt{9+8a} \geq 2$, cf. Remark 6.9.

The proofs of Propositions 3.2 and 3.3, which in view of the remark are intertwined, will be carried out in Sections 5 and 6.

Step 2: The above estimates allow one to prove what is essentially the completeness of null-infinity.

Proposition 3.5. *Let* $v_m = \sup_{v \geq v_0}\{v \mid (u_{\mathcal{H}}, v) \in \mathcal{Q}\}$. *We must have* $v_m = \infty$.

Proof. Consider the family of curves of constant area radius r. In view of $r_u < 0$ and $r_v > 0$ holding in $\mathcal{R}_{\mathcal{H}}$, these curves are seen to be timelike in $\mathcal{R}_{\mathcal{H}}$ and to foliate $\mathcal{R}_{\mathcal{H}}$. Now, either none of these constant r-curves has future limit point $(u_{\mathcal{H}}, v_m)$, meaning that all of them intersect the horizon, or one of them, say $r = R$, has (and hence all later ones, $r > R$, as constant r curves cannot intersect). In the latter case, we consider the infinite "zig-zag"-curve as in the diagram below



and observe that the v-length of each constant u-piece is uniformly bounded below. Namely, in view of the bound on κ and the fact that $\frac{1-\mu}{r^2} \leq \frac{2}{l^2}$ to the right of the curve $r = R$ (for R sufficiently large depending only on M and l) we have for each constant u-piece $\mathcal{U}_i$

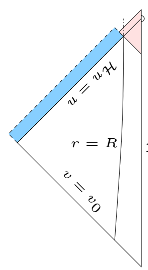
$$\int_{\mathcal{U}_i} dv \geq \frac{l^2}{2}\int_{\mathcal{U}_i} \frac{\kappa\left(1-\mu\right)}{r^2} dv = \frac{l^2}{2}\int_{\mathcal{U}_i} \frac{r_v}{r^2} dv \geq \frac{l^2}{2R}\,.$$

Since there are infinitely many $\mathcal{U}_i$ in the zig-zag curve ($r = R$ is timelike!), $v_m = \infty$ follows.

We turn to the first case (all constant r-curves intersecting the horizon and hence $\lim_{v \rightarrow v_m} r\left(u_{\mathcal{H}}, v\right) = \infty$). Assuming $v_m = V < \infty$ (otherwise, we are done) we will show that this contradicts the fact that $u = u_{\mathcal{H}}$ is the last u-ray along which $r = \infty$ can be reached. Pick $r = R$ very large, the corresponding curve intersecting $u_{\mathcal{H}}$ at $q = (u_{\mathcal{H}}, v_q)$, say. In view of the assumptions and the uniform bounds on κ and ϖ of Propositions 3.1 and 3.2, we have $\frac{1-\mu}{r^2} > c > 0$ in $\overline{\mathcal{R}_{\mathcal{H}}}$ for a constant c. Indeed, this is obvious in $\mathcal{R}_{\mathcal{H}} \cap \{r \geq R\}$ by computation, and immediate by compactness in $\overline{\mathcal{R}_{\mathcal{H}}} \cap \{r \leq R\}$, since $r_v = 0$ cannot hold anywhere in $\overline{\mathcal{R}_{\mathcal{H}}} \cap \{r \leq R\}$ (this would contradict that $r \rightarrow \infty$ along any $u = const$ ray in $\overline{\mathcal{R}_{\mathcal{H}}}$). It follows that $\gamma = -\frac{r_u}{1-\mu}$ is bounded on the data $[u_0, u_{\mathcal{H}}] \times \{v_0\}$. Using the bound on $1-\mu$, κ and ϕ one easily obtains (integrating (16) in v) that γ is uniformly bounded in $\overline{\mathcal{R}_{\mathcal{H}}}$. By a change of u coordinate (cf. the proof of Lemma 6.2 in [33]), one achieves that $\gamma = \frac{1}{2}$ holds on $\mathcal{I}$. In this new coordinate system, one has that $[\partial_u + \partial_v] \frac{1}{r} = 0$ holds on $\mathcal{I}$ and hence that $u = v$ on $\mathcal{I}$. Moreover, $\gamma_u = 0$ on $\mathcal{I}$ and, integrating (16) from $\mathcal{I}$, the uniform bounds

$$\left| r^3 \left(\gamma - \frac{1}{2} \right) \right| + |r^2 \gamma_u| < C \,. \tag{58}$$

With this established, all assumptions of the extension principle of Proposition 8.2 of [33] hold and we can extend the solution to a larger triangle, as depicted below.



This contradicts the assumption that $u_{\mathcal{H}}$ is the last ray along which $r \to \infty$ can be reached. □

Step 3: The estimate (56) implies exponential decay in v.

Corollary 3.6. *Define the flux*

$$\mathbb{F}(v) = \int_{u_{\mathcal{I}}}^{u_{\mathcal{H}}} \left[\frac{(\partial_u \phi)^2}{-r_u} r^4 + \phi^2 \; r^2 \left(-r_u\right) \right] (\bar{u}, v) \, d\bar{u} \tag{59}$$

and the region $D^\star(v_1, v_2) = D(u_{\mathcal{H}}, v_2) \cap \{v \geq v_1\}$. *Then*

$$\mathbb{F}(v) \leq \mathbb{F}(v_0) \, exp\left(-B_{M,l,a} \cdot v\right) \,. \tag{60}$$

Proof. Note that

$$\mathbb{I}[\phi]\left(D^\star(v_1, v_2)\right) \geq b_{M,l} \int_{v_1}^{v_2} d\bar{v} \; \mathbb{F}(\bar{v}) \tag{61}$$

where $b_{M,l}$ is a small positive constant depending only on M and l. Applying the estimate (56) in the region $D^\star(v_1, v_2)$ (i.e. not starting from $v = v_0$ but from $v = v_1$) yields

$$\mathbb{F}(v_2) + b_{M,l} \int_{v_1}^{v_2} d\bar{v} \; \mathbb{F}(\bar{v}) \leq B_{M,l,a} \cdot \mathbb{F}(v_1) \,, \tag{62}$$

which implies (60). □

From the estimate

$$|\phi\,(u,v)\,| \leq B_{M,l,a} \cdot \frac{1}{r^{\frac{3}{2}}} \cdot \sqrt{\mathbb{F}\,(v)}\,, \tag{63}$$

which follows from Cauchy Schwarz, we conclude that $|\phi|$ decays pointwise exponentially in v. Note that the exponential decay of the flux $\mathbb{F}\,(v)$ implies that $\varpi \to M$ and $\kappa \to \frac{1}{2}$ exponentially in v, uniformly along any $u = const$ ray including the horizon. In this sense the metric converges exponentially to a Schwarzschild-AdS metric of mass M. In particular, we have

Proposition 3.7. *The Lorentzian Penrose inequality* $\sup_{\mathcal{H}} r \leq r_{ASch}$ *holds. In fact, the area radius* r *converges exponentially in* v *to* r_{ASch} *along the horizon.*

Proof. We will show that the assumption that $r \geq r_{ASch} + M\delta$ along all of $\mathcal{H}$ leads to a contradiction for any $\delta > 0$. Let hence $\delta > 0$ be given and $r \geq r_{ASch} + M\delta$ hold along $\mathcal{H}$. Then, by the exponential decay of the Hawking mass along the event horizon, we can pick a $v_i > v_0$ such that

$$\frac{1-\mu}{r^2} \geq \left(\frac{1}{r^2} - \frac{2M}{r^3} + \frac{1}{l^2}\right) - \frac{2|M-\varpi|}{r^3} \geq b_{M,l}\delta$$

holds on $\mathcal{H} \cap \{v \geq v_i\}$. We then have, on the one hand,

$$\int_{v_i}^{v} \frac{r_v}{r^2} dv = -\frac{1}{r\,(v)} + \frac{1}{r(v_i)}\,, \tag{64}$$

which is uniformly bounded for all $v_i < v < \infty$. On the other hand,

$$\int_{v_i}^{v} \frac{r_v}{r^2} dv = \int_{v_i}^{v} \frac{\kappa\,(1-\mu)}{r^2} dv \geq b_{M,l}\delta \cdot (v - v_i)\ , \tag{65}$$

which can become arbitrarily large as $v \to \infty$ along $\mathcal{H}$, cf. Proposition 3.5.

With $r \leq r_{ASch}$ established, we prove exponential decay $r \to r_{ASch}$ along the horizon. From the exponential decay of ϖ and the finiteness of r along $\mathcal{H}$:

$$\int_{v}^{\infty} (r_{ASch} - r)\, d\bar{v} \leq B_{M,l} \int_{v}^{\infty} (1-\mu)\, d\bar{v} + B_{M,l,a} e^{-B_{M,l,a} v}$$

$$\leq B_{M,l} \int_{v}^{\infty} \lambda\ d\bar{v} + B_{M,l,a} e^{-B_{M,l,a} v} \leq B_{M,l}\,(r_{ASch} - r) + B_{M,l,a} e^{-B_{M,l,a} v}\,,$$

holds along $u = u_{\mathcal{H}}$, from which exponential decay follows. □

We summarize the statements of Propositions 3.1, 3.2,3.3, 3.5 and 3.7 as

Theorem 3.8. *Given an* ϵ*-perturbed Schwarzschild-AdS data set on* $N\,(v_0)$ *in the sense of Definition 2.3, its associated maximum development is a black hole spacetime with a regular future event horizon* $\mathcal{H}$*, and a complete null-infinity* $\mathcal{I}$*. Moreover, the estimates of Propositions 3.1, 3.2 and 3.3 hold for any* (u,v) *in* $\mathcal{R}_{\mathcal{H}}$*. This implies in particular that* ϕ *decays exponentially in* v *on the latter set: The non-degenerate energy flux through any null-hypersurface of constant* v*,* $\mathbb{F}\,(v)$ *defined in (59), satisfies (60).*

We finally remark that the techniques of this paper (in particular, the integrated decay estimate of Section 5) are naturally applied to the study of spherically symmetric solutions of the wave equation (3) on a fixed Schwarzschild-AdS background. In this uncoupled case, we have $\kappa = \frac{1}{2}$, $\gamma = \frac{1}{2}$ identically in Eddington-Finkelstein coordinates, and $T = \partial_t$ becomes the familiar timelike Killing field. See [15] for further discussion of the relation between the coupled and uncoupled problem in the (completely analogous) asymptotically-flat context. We have, in particular, the following result:

Corollary 3.9. *Let $(\mathcal{M}, g)$ be a fixed Schwarzschild-AdS spacetime with Eddington Finkelstein coordinate system (u, v). Let $a \in \mathbb{R}$ be a given mass-(squared) satisfiying (6). Then, spherically-symmetric solutions of the Klein-Gordon equation (3) decay exponentially in the Eddington Finkelstein coordinate v on the black hole exterior. In particular, the estimate (60) holds for the non-degenerate energy flux of the Klein-Gordon field.*

4 Proof of Proposition 3.1: Basic Estimates

Proposition 3.1 will be proven by a bootstrap.

4.1 The bootstrap regions and the bootstrap assumptions

We define, for $\tilde{u} \in [u_0, u_{\mathcal{H}}]$,

$$\widehat{\mathcal{B}}(\tilde{u}) = \mathcal{R}_{\mathcal{H}} \cap \{u_0 \leq u < \tilde{u}\} . \tag{66}$$

Note that $\mathcal{R}_{\mathcal{H}} = \widehat{\mathcal{B}}(u_{\mathcal{H}})$. Let

$$u_{max} = \sup_u \Big(\text{ conditions (68)-(72) hold in } \widehat{\mathcal{B}}(u) \Big) \tag{67}$$

1. **Auxiliary bound:**

$$|r^3 \phi^2| < \frac{Ml^2}{8\pi |a|} . \tag{68}$$

2. **Smallness of matter fields:**

$$\frac{4\pi(-a)}{l^2} \int_{v_0}^{v} d\bar{v}\, \mathbf{1}_{\{r \leq r_Y\}} r^2\, r_v\, \phi^2(u, \bar{v}) < M \cdot c \,, \tag{69}$$

$$2\pi \int_{v_0}^{v} \mathbf{1}_{\{r \geq r_Y\}} \frac{\phi_v^2}{\kappa} r^2 (u, \bar{v})\, d\bar{v} < M\sqrt{c}, \tag{70}$$

$$\frac{4\pi(-a)}{l^2} \int_{u_{\mathcal{I}}}^{u} d\bar{u}\, \mathbf{1}_{\{r \leq r_Y\}} r^2 |r_u|\, \phi^2(\bar{u}, v) < M \cdot c \,, \tag{71}$$

$$2\pi\int_{u_{\mathcal{I}}}^{u} d\bar{u}\mathbf{1}_{\{r\geq r_Y\}}\frac{\phi_u^2}{\gamma}r^2\left(\bar{u},v\right) < M\sqrt{c}, \tag{72}$$

for any (u,v) in $\widehat{\mathcal{B}}\left(u\right)$ where $\mathbf{1}_{\{\ldots\}}$ is the indicator function.

Finally, define the bootstrap region $\mathcal{B}=\widehat{\mathcal{B}}\left(u_{\max}\right)\subset\mathcal{R}_{\mathcal{H}}$.

We would like to prove that in fact $\mathcal{B}=\mathcal{R}_{\mathcal{H}}$. Now $\mathcal{B}$ is open in $\mathcal{R}_{\mathcal{H}}$ by continuity (recall from [33] that all the norms entering the bootstrap are continuous in both u and v (uniformly as $\mathcal{I}$ is approached) and that u and v are finite for any $(u,v)\in\mathcal{R}_{\mathcal{H}}$) and also non-empty by Cauchy stability. Hence we are done if we could show that $\mathcal{B}$ is also closed in $\mathcal{R}_{\mathcal{H}}$. To do this, we assume $u_{max}<u_{\mathcal{H}}$ fixed (otherwise there is nothing to show) and prove that in $\overline{\mathcal{B}}$ the bounds (68)-(72) can be improved.

4.2 Overview of the argument

We will show that the bootstrap assumptions imply that $M-c\leq\varpi\leq M$. This is done by exploiting Hardy inequalities both in the u- and the $v-$ direction but restricted to the region $r\geq r_Y$, as well as bootstrap assumption (69) for the bad term in the region $r\leq r_Y$. Importantly, it will turn out that the Hardy inequalities do not require the entire good-signed derivative term (provided a satisfies the Breitenlohner-Freedman bound). This enables us in turn to control ϕ-flux trough characteristic lines by the mass difference and hence to improve (70) and (72) from $\sqrt{c}$ to c. With the improvement for the mass-flux, we invoke the redshift estimate (Section 4.6) and estimates from infinity (Section 4.5) to prove the pointwise bounds $|r^3\phi^2|+|r^{\frac{3}{2}}\left(\frac{\phi_u}{r_u}\right)^2|<c$ everywhere, improving (68). Finally, we improve (69) and (71) using that the r difference in the region $r\leq r_Y$ is $c^{\frac{1}{3}}$-small (Section 4.7).

4.3 Integrated positivity for ϖ in $r\geq r_Y$

An immediate consequence of the bootstrap assumptions is

Lemma 4.1. *In the region $\mathcal{B}$ we have*

$$|\varpi-M|\leq 2M\sqrt{c} \tag{73}$$

which follows from integrating (18) using the bootstrap assumptions, and

Lemma 4.2. *In the region $\mathcal{B}\cap\{r\geq r_Y\}$ we have*

$$\frac{1-\mu}{r^2}\geq\frac{1}{8r_Y^2}c^{\frac{1}{3}}\,. \tag{74}$$

Proof. We write

$$1-\frac{2\varpi}{r}+\frac{r^2}{l^2}=\left(1-\frac{2M}{r}+\frac{r^2}{l^2}\right)+\frac{2M-2\varpi}{r}\,. \tag{75}$$

Since $-\frac{2M}{r}$ is increasing in r we can estimate it from below by $-\frac{2M}{r_Y}$. The mass difference is estimated by Lemma 4.1. Hence, for $r \geq r_Y$, we have by (47),

$$1-\frac{2\varpi}{r}+\frac{r^2}{l^2}\geq c^{\frac{1}{3}}-\frac{4M}{r_Y}\sqrt{c}+\frac{r^2-r_Y^2}{l^2}\geq\frac{1}{2}c^{\frac{1}{3}}+\frac{r^2-r_Y^2}{l^2}\,. \tag{76}$$

Dividing by r^2 we discard the the second term in the region $r \leq 2r_Y$, while in $r \geq 2r_Y$ the second term is larger than $\frac{3}{4l^2} > \frac{1}{8r_Y^2}c^{\frac{1}{3}}$. □

Lemma 4.3. *For any* $\aleph < \frac{9}{8}$, *the following inequality holds in* $\overline{\mathcal{B}} \cap \{r \geq r_Y\}$*:*

$$\frac{2}{9}\frac{\aleph}{l^2}h^2\left(r-r_Y\right)^2\leq\frac{\left(-r_u\right)r_v}{\Omega^2}\quad,\quad where\quad h=1+\frac{r_Y}{r}+\left(\frac{r_Y}{r}\right)^2\,. \tag{77}$$

Proof. Define

$$\tilde{g}=+r_v\frac{\left(-r_u\right)}{\Omega^2}-\frac{2}{9}\aleph\hat{h}^2\frac{\left(-a\right)}{l^2}\left(r-r_Y\right)^2\,. \tag{78}$$

Note that $\tilde{g}\left(r_Y\right)\geq\frac{1}{32}c^{\frac{1}{3}}$ by Lemma 4.2.

We first show that the same bound is valid on the initial data for $r \geq r_Y$. Again, this holds trivially where $r = r_Y$ intersects the data (call the u-coordinate of that point u_Y). The derivative in the u-direction satisfies

$$\begin{aligned}\tilde{g}_u=r_u\Bigg[&4\pi r\left(\frac{\phi_u}{r_u}\right)^2\frac{r_u r_v}{\Omega^2}+\frac{\varpi}{2r^2}-\frac{2\pi a r\phi^2}{l^2}\\&+\frac{r}{2l^2}\left(1-\frac{8}{9}\aleph-\frac{8}{9}\aleph\left(\frac{r_Y}{r}\right)^3\left(1-2\left(\frac{r_Y}{r}\right)^3\right)\right)\Bigg]\,.\end{aligned} \tag{79}$$

We have

$$\tilde{g}\left(u,v_0\right)=g\left(u_Y,v_0\right)+\int_u^{u_Y}\left(-\tilde{g}_u\right)\,, \tag{80}$$

and we want to show $\tilde{g}\left(u,v_0\right)\geq 0$. Note that the third term in the square bracket has a good sign for $a<0$ while for $a>0$ we can use (68). We estimate

$$\begin{aligned}\tilde{g}\left(u,v_0\right)\geq\frac{1}{32}c^{\frac{1}{3}}-\frac{\pi}{r_Y}\int_u^{u_Y}\frac{\phi_u^2}{\gamma}r^2d\bar{u}\\+\int_u^{u_Y}\frac{-r_u r}{2l^2}\left(1-\frac{8}{9}\aleph-\frac{8}{9}\aleph\left(\frac{r_Y}{r}\right)^3\left(1-2\left(\frac{r_Y}{r}\right)^3\right)+\frac{\left(M-\epsilon\right)l^2}{r^3}\right)du\end{aligned} \tag{81}$$

and observe that the second term can be estimated by the $H^1_{AdS,deg}$ norm on the data and is hence ϵ-small. We conclude that the first line is already positive for sufficiently small data. Thus, $\tilde{g} > 0$ holds on the data if we can show that the second line is positive. Letting $A = \frac{8}{9}\aleph$, we have

$$\begin{aligned}
&\int_u^{u_Y} \frac{-r_u r}{2l^2}\left(1 - \frac{8}{9}\aleph - \frac{8}{9}\aleph\left(\frac{r_Y}{r}\right)^3\left(1 - 2\left(\frac{r_Y}{r}\right)^3\right) + \frac{(M-\epsilon)\, l^2}{r^3}\right)du \\
&= \frac{r_Y^2}{2l^2}\left[-\frac{x^2}{2}(1-A) - \frac{A}{x} + \frac{A}{2x^4} + \frac{(M-\epsilon)\, l^2}{r_Y^3}\frac{1}{x}\right]_{x=r/r_Y}^{x=1} \\
&> \frac{r_Y^2}{2l^2}\left(-\frac{1}{2} + \left(\frac{r}{r_Y}\right)^2\frac{1-A}{2} + A\frac{r_Y}{r} - \frac{A}{2}\left(\frac{r_Y}{r}\right)^4 + \frac{1}{2}\left(1 - \frac{r_Y}{r}\right)\right) \\
&= \frac{r_Y^2}{2l^2} z\left(\frac{r}{r_Y}, A\right), \qquad (82)
\end{aligned}$$

where we used that $\frac{(M-\epsilon)l^2}{r_Y^3} > \frac{1}{2}$ holds by (49) to estimate the last term in the penultimate step. On the other-hand, we easily have:

Lemma 4.4. *For any $0 \leq A < 1$, the function*

$$z\,(x, A) = \frac{1-A}{2}x^2 + \frac{A}{x} - \frac{A}{2x^4} - \frac{1}{2x} = \frac{1}{x^4}\left(\frac{1-A}{2}x^6 + (A - \frac{1}{2})x^3 - \frac{A}{2}\right)$$

is non-negative in $[1, \infty)$.

Proof. Note that $z(x, A)$ is linear in A. For $A = 0$, we have $z(x, 0)x^4 = \frac{x^6}{2} - \frac{x^3}{2}$ which is non-negative on $[1, \infty)$. For $A = 1$, we have $z(x, 1)x^4 = \frac{x^3}{2} - \frac{1}{2}$, which is also non-negative on $[1, \infty)$. □

To establish $\tilde{g} \geq 0$ in the entire region $\overline{\mathcal{B}} \cap \{r \geq r_Y\}$, it suffices to show that the bound (78) is propagated in the v-direction. We compute

$$\begin{aligned}
\tilde{g}_v &= -\pi r\frac{\phi_v^2}{\kappa} + r_{,v}\frac{\varpi}{2r^2} - r_v\frac{2\pi a r\phi^2}{l^2} \\
&\quad + r_v\frac{r}{2l^2}\left(1 - \frac{8}{9}\aleph - \frac{8}{9}\aleph\left(\frac{r_Y}{r}\right)^3\left(1 - 2\left(\frac{r_Y}{r}\right)^3\right)\right). \qquad (83)
\end{aligned}$$

In analogy to the previous case, we would like to show that $g\,(v) \geq \frac{1}{32}c^{\frac{1}{3}} + \int_{v_0}^v \mathbf{1}_{r\geq r_Y}\tilde{g}_v$ is positive. We note that the bad first term can now be estimated from the bootstrap assumption (70):

$$\int_{v(r_Y)}^v \pi r\frac{\phi_v^2}{\kappa}d\bar{v} \leq \pi\frac{1}{r_Y}\int_{v(r_Y)}^v \pi r^2\frac{\phi_v^2}{\kappa} \leq \pi\frac{M}{r_Y}\sqrt{c}\,. \qquad (84)$$

In view of $\frac{1}{32}c^{\frac{1}{3}} - \pi\frac{M}{r_Y}c^{\frac{1}{2}} > 0$ for sufficiently small c, we conclude that this term cannot drive $\tilde{g}$ to zero. To establish positivity of the integral for the other terms, we simply repeat the argument we followed in the u-direction reducing the problem to Lemma 4.4. □

For the next Lemma, recall that $u_{\mathcal{I}}$ denotes the u-value where the $v = const$ curve intersects $\mathcal{I}$ and similarly $v_{\mathcal{I}}$ denotes the v-value where the $u = const$ curve intersects $\mathcal{I}$.

Lemma 4.5. *For any $\aleph < \frac{9}{8}$ fixed, we have for any $(u,v) \in \mathcal{B} \cap \{r \geq r_Y\}$,*

$$\int_{u_{\mathcal{I}}}^{u} \frac{4\pi r^2 \aleph}{l^2} \left(-r_u\right) \phi^2 \left(\bar{u}, v\right) d\bar{u} \leq \int_{u_{\mathcal{I}}}^{u} 8\pi r^2 \frac{r_v}{\Omega^2} \left(\partial_u \phi\right)^2 \left(\bar{u}, v\right) d\bar{u} \tag{85}$$

and for any fixed $u = const$ curve in $\mathcal{B}$

$$\int_{v}^{v_{\mathcal{I}}} \frac{4\pi r^2 \aleph}{l^2} \left(r_v\right) \phi^2 \left(u, \bar{v}\right) d\bar{v} \leq \int_{v}^{v_{\mathcal{I}}} 8\pi r^2 \frac{-r_u}{\Omega^2} \left(\partial_v \phi\right)^2 \left(u, \bar{v}\right) d\bar{v} \, . \tag{86}$$

Proof. We have by integration by parts:

$$\int_{u_{\mathcal{I}}}^{u} \frac{4\pi r^2 \aleph}{l^2} \left(-r_u\right) \phi^2 du = -\frac{4\pi\aleph}{l^2} \frac{r^2 h \left(r - r_Y\right)}{3} \phi^2 \Big|_{u_{\mathcal{I}}}^{u} - \int_{u_{\mathcal{I}}}^{u} \frac{8\pi\aleph r^2 h \left(r - r_Y\right)}{3l^2} \phi \phi_u du \, ,$$

where we recall $h = 1 + \frac{r_Y}{r} + \left(\frac{r_Y}{r}\right)^2$. Of the boundary terms on the right-hand side, one has a good (negative) sign, while the other vanishes by the decay of ϕ as $r \to \infty$. For the remaining term, we apply Cauchy-Schwarz:

$$\begin{aligned} &\int_{u_{\mathcal{I}}}^{u} \frac{8\pi\aleph r^2 h \left(r - r_Y\right)}{3l^2} \phi \phi_u du \\ \leq \frac{8\pi\aleph}{3l^2} &\left(\int_{u_{\mathcal{I}}}^{u} r^2 (-r_u) \phi^2\right)^{1/2} \left(\int_{u_{\mathcal{I}}}^{u} \frac{r^2 h^2 \left(r - r_Y\right)^2}{-r_u} \phi_u^2\right)^{1/2} , \end{aligned}$$

from which we deduce that:

$$\frac{4\pi\aleph}{l^2} \int_{u_{\mathcal{I}}}^{u} r^2 (-r_u) \phi^2 du \leq \frac{16\pi}{9} \frac{\aleph}{l^2} \int_{u_{\mathcal{I}}}^{u} \frac{r^2 h^2 \left(r - r_Y\right)^2}{-r_u} \phi_u^2 du.$$

An application of Lemma 4.3 now yields the result. The inequality in the v-direction is similar. □

Corollary 4.6. *In the region $\mathcal{B} \cap \{r \geq r_Y\}$ the estimate $\varpi \leq M$ holds.*

Proof. We have

$$\varpi - M = \int_{u_{\mathcal{I}}}^{u} \partial_u \varpi du = \int_{u_{\mathcal{I}}}^{u} du \left[-8\pi r^2 \frac{r_v}{\Omega^2} \left(\partial_u \phi\right)^2 + \frac{4\pi r^2 \left(-a\right)}{l^2} \left(-r_u\right) \phi^2\right]$$

and by Lemma 4.5 the right-hand side is negative. □

4.4 Improving bootstrap assumptions (70) and (72)

From the conservation of Hawking mass we estimate for $(u,v) \in \mathcal{B}$,

$$
\begin{aligned}
B_{M,l,a} \, \|\phi\|^2_{H^1_{AdS,deg}(u_{\mathcal{H}},v_0)} \geq \int_{u_0}^{u} \left(2\pi \frac{\phi_u^2}{\gamma} - \frac{4\pi a}{l^2}\phi^2 r_u\right) r^2 (\bar{u}, v_0)\, d\bar{u} = \\
\int_{u_{\mathcal{I}}}^{u} \left[2\pi \frac{\phi_u^2}{\gamma} - \frac{4\pi a}{l^2}\phi^2 r_u\right] r^2(\bar{u},v)\, d\bar{u} + \int_{v_0}^{v}\left[2\pi\frac{\phi_v^2}{\kappa} + \frac{4\pi a}{l^2}\phi^2 r_v\right] r^2(u,\bar{v})\, d\bar{v} \\
\geq \frac{1}{2}\left(a + \frac{9}{8}\right) \|\phi\|^2_{H^1_{AdS,deg}(u,v)} + \frac{9}{8}\frac{4\pi}{l^2}\int_{u_{\mathcal{I}}}^{u} \mathbf{1}_{r \leq r_Y} \phi^2 r_u r^2 (\bar{u}, v)\, d\bar{u} \\
-\frac{9}{8}\frac{4\pi}{l^2}\int_{v_0}^{v} \mathbf{1}_{r\leq r_Y}\phi^2 r_v r^2(u,\bar{v})\, d\bar{v}\,, \quad (87)
\end{aligned}
$$

where we used the Hardy inequalities established in Lemma 4.5. Using the bootstrap assumptions (69) and (71) for the terms in $r \leq r_Y$, we establish

Corollary 4.7. *For any $(u,v) \in \mathcal{B}$ we have*

$$\|\phi\|_{H^1_{AdS,deg}(u,v)} \leq C_a \cdot M \cdot \sqrt{c} \tag{88}$$

with C_a a uniform constant, depending only on how close a is to the BF-bound.

This improves in particular bootstrap assumptions (70) and (72) and also shows that the overall mass-variation is c-small.

4.5 Estimating ϕ in $r \geq r_X$

Next we derive a pointwise smallness bound for ϕ in $r \geq r_X$ (not r_Y!), by integrating in u from infinity:

Lemma 4.8. *For all $(u,v) \in \mathcal{B} \cap \{r \geq r_X\}$, we have*

$$|r^{\frac{3}{2}}\phi(u,v)| \leq B_{M,l} \cdot d^{-1/6} \|\phi\|_{H^1_{AdS,deg}(u,v)} \leq B_{M,l,a} \cdot \sqrt{c}\,. \tag{89}$$

Proof. Integrating out from infinity (where ϕ vanishes) we find

$$
\begin{aligned}
|\phi\left(u_{r\geq r_X}, v\right)| \leq 0 + \left|\int du \phi_u\right| \leq \sqrt{\int du \zeta^2 \frac{\lambda}{\Omega^2}}\sqrt{\int du \frac{4}{r^2(1-\mu)}(-r_u)} \\
\leq \sqrt{\frac{32}{3}\frac{r_X}{r^{\frac{3}{2}}}} d^{-\frac{1}{6}} \|\phi\|_{H^1_{AdS,deg}(u,v)}\,,
\end{aligned}
$$

where we used the upper bound on the v-flux as well as the estimate $8r_X^2 \frac{1-\mu}{r^2} \geq d^{1/3}$ (cf. Lemma 4.2), which holds in the region where $r \geq r_X$. □

Note that on $r = r_Y$ we would only obtain $c^{\frac{1}{3}}$-smallness, as the bad $(1-\mu)^{-1}$-weight would bring in an inverse c-smallness.

4.6 The red-shift effect: $\frac{\zeta}{\nu}$ estimate in $r \leq r_X$

Recall $\zeta = r\phi_u$. Inspired by [15] we write the wave equation as

$$\partial_v \left(\frac{\zeta}{\nu}\right) = -\phi_v + \frac{2r\kappa a\phi}{l^2} - \frac{\zeta}{\nu}\left[2\kappa\frac{\varpi}{r^2} + \frac{2\kappa r}{l^2} - 8\pi r\frac{a}{l^2}\kappa\phi^2\right]. \tag{90}$$

The following estimate is a refinement of the red-shift estimates in [15] necessitated by the fact that $a \neq 0$. Recall that $I(v_0) = \{(u, v_0) \mid u \leq u_{\mathcal{H}}\} \subset N(v_0)$.

Lemma 4.9. *For any $(u,v) \in \mathcal{B} \cap \{r \leq r_X\}$ we have*

$$\left|\frac{\zeta}{\nu}\right| + |\phi| \leq B_{M,l}\left(\sup_{D(u,v)} \|\phi\|_{H^1_{AdS,deg}(u,v)} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right|\right) \leq B_{M,l,a}\cdot\sqrt{c}\,. \tag{91}$$

Proof. Let us denote the redshift weight

$$\rho = 2\kappa\left[\frac{\varpi}{r^2} + \frac{r}{l^2} - 4\pi r\frac{a}{l^2}\phi^2\right]. \tag{92}$$

Note that $\frac{\rho}{\kappa} > \left(\frac{2r_{min}}{l^2} + \frac{M}{r_X^2}\right)$ using (68).[12] Integrating (90) we find

$$\frac{\zeta}{\nu}(u,v) = \left(\frac{\zeta}{\nu}(u,v_0)\right)\cdot\exp\left(\int_{v_0}^{v} -\rho(u,\bar{v})\,d\bar{v}\right)$$
$$+\int_{v_0}^{v} d\bar{v}\left[\exp\left(-\int_{\bar{v}}^{v}\rho(u,\hat{v})\,d\hat{v}\right)\left(-\phi_v + \frac{2r\kappa a\phi}{l^2}\right)(u,\bar{v})\right]. \tag{93}$$

Let us study the inhomogeneous term. For the ϕ_v-term we need to estimate

$$\left|\int_{v_0}^{v} d\bar{v}\left[\exp\left(-\int_{\bar{v}}^{v}\rho(u,\hat{v})\,d\hat{v}\right)\phi_v\right]\right|$$
$$\leq \sqrt{\int_{v_0}^{v} d\bar{v}\frac{1}{r^2}\kappa\cdot\exp\left(-2\int_{\bar{v}}^{v}\rho(u,\hat{v})\,d\hat{v}\right)}\sqrt{\int_{v_0}^{v}\frac{\phi_v^2}{\kappa}r^2(u,\bar{v})\,d\bar{v}}\,. \tag{94}$$

The second square root can be controlled from the energy, while the first can be estimated by a constant:

$$\int_{v_0}^{v} d\bar{v}\frac{1}{r^2}\kappa\cdot\exp\left(-2\int_{\bar{v}}^{v}\rho(u,\hat{v})\,d\hat{v}\right) = \int_{v_0}^{v} d\bar{v}\frac{\kappa}{2r^2\rho}\partial_{\bar{v}}\exp\left(-2\int_{\bar{v}}^{v}\rho(u,\hat{v})\,d\hat{v}\right),$$

and we can take out the supremum of $\frac{\kappa}{2r^2\rho}$ because the derivative of the exponential has a positive sign, i.e. the integrand is positive everywhere. This finally

[12] Due to the cosmological term, we actually have a global redshift at work. In the asymptotically-flat case, the strength of the redshift degenerates at infinity, in view of the absence of that term. We will exploit this good term which grows in r ("the redshift at infinity") later in the estimates near infinity.

yields

$$\begin{aligned}
&\int_{v_0}^{v} d\bar{v} \frac{1}{r^2} \kappa \cdot \exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})\, d\hat{v}\right) \\
&\leq \sup\left(\frac{\kappa}{2r^2\rho}\right) \cdot \left[1 - \exp\left(-2\int_{v_0}^{v} \rho(u,\bar{v})\, d\bar{v}\right)\right] \\
&\leq \frac{1}{4} \cdot \sup\left[\left(\varpi + \frac{r^3}{l^2} - 4\pi r^3 \frac{a}{l^2}\phi^2\right)^{-1}\right]. \qquad (95)
\end{aligned}$$

The ϕ-term in (93) is more delicate because a smallness bound on ϕ is not available close to the horizon. The only thing we have at our disposal is that $\int \phi^2 r_v dv$ is controlled by the energy (which is not immediately useful because r_v may be very small in the region under consideration). The idea is to integrate the inhomogeneity by parts, since a v-derivative falling on r will generate the required factor of r_v. We write

$$\begin{aligned}
&\int_{v_0}^{v} d\bar{v} \left[\exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})\, d\hat{v}\right) \left(\frac{2r\kappa a\phi}{l^2}\right)(u,\bar{v})\right] \\
&= \int_{v_0}^{v} d\bar{v} \frac{2r\kappa a\phi}{l^2\rho} \partial_{\bar{v}} \left[\exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})\, d\hat{v}\right)(u,\bar{v})\right]. \qquad (96)
\end{aligned}$$

When we integrate by parts, the boundary terms that arise are one on data (which is ϵ-small by assumption) and one at (u,v), which is

$$\left| \left(\frac{ar}{l^2} \frac{1}{\left[\frac{\varpi}{r^2} + \frac{r}{l^2} - 4\pi r \frac{a}{l^2}\phi^2\right]}\right) \phi(u,v) \right| \leq |a||\phi|. \qquad (97)$$

To analyze the main term we compute

$$\begin{aligned}
&\partial_v \left(\phi \frac{2r\kappa}{\rho}\right) = \frac{2r\kappa}{\rho}\left(1 + \frac{2r\kappa}{\rho}\frac{8\pi a}{l^2}\phi^2\right)\phi_v \\
&+ \left(\frac{2r\kappa}{\rho}\right)^2 \left(\frac{3\varpi}{r^4} - \frac{4\pi a}{rl^2}\phi^2\right)\phi\ r_v - \left(\frac{2r\kappa}{\rho}\right)^2 2\pi \frac{\phi}{r}\frac{\phi_v^2}{\kappa}. \qquad (98)
\end{aligned}$$

Note again that the factor $\frac{2r\kappa}{\rho}$ is both bounded above and below. Hence the term proportional to ϕ_v can (after using (68)) be estimated as before (cf. (94)). For the term proportional to r_v we use Cauchy-Schwarz

$$\begin{aligned}
&\left|\int_{v_0}^{v} d\bar{v} \left[\exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})\, d\hat{v}\right)\phi r_v\right]\right| \\
&\leq \sqrt{\int_{v_0}^{v} d\bar{v} \frac{1}{r^2}\kappa(1-\mu) \cdot \exp\left(-2\int_{\bar{v}}^{v} \rho(u,\hat{v})\, d\hat{v}\right)} \sqrt{\int_{v_0}^{v} \phi^2 r^2\ r_v(u,\bar{v})\, d\bar{v}}, \quad (99)
\end{aligned}$$

recovering the $H^1_{AdS,deg}$-norm. Finally, for the cubic term in (98) we apply the pointwise auxiliary bootstrap assumption (68) to ϕ and estimate the remainder by the *square* of the $H^1_{AdS,deg}$-norm. Since this norm itself is $\sqrt{c}$ small by Corollary 4.7, we have $\|\phi\|^2_{H^1_{AdS,deg}} \leq M\sqrt{c}\|\phi\|_{H^1_{AdS,deg}}$.

We summarize that (93) finally turns into the estimate

$$\left|\frac{\zeta}{\nu}(u,v)\right| \leq B_{M,l}\left[\sup_{D(u,v)}\|\phi\|_{H^1_{AdS,deg}(u,v)} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right|\right] + |a|\ |\phi|(u,v)\ , \quad (100)$$

valid for $(u,v)\in\mathcal{B}\cap\{r\leq r_X\}$. Note that the pointwise norm on $r^{\frac{3}{2}}\frac{\zeta}{\nu}$ controls in particular the ϕ-term picked up on the data in the integration by parts. From this we derive an estimate for ϕ by integrating from the fixed $r = r_X$-curve towards the horizon:

$$|\phi(u,v)| \leq |\phi(u_{r_X},v)| + \int_{u_{r_X}}^{u}\left|\frac{\zeta}{\nu}\right|\frac{(-r_u)}{r}d\bar{u} \quad (101)$$

leads, after applying Lemma 4.8 and (100), to

$$\sup_{D(u,v)\cap\{r\leq r_X\}}|\phi(u,v)| \leq B_{M,l}\sup_{D(u,v)}\|\phi\|_{H^1_{AdS,deg}(u,v)} + \log\frac{r_X}{r_{min}}\Bigg[B_{M,l}\left(\sup_{D(u,v)}\|\phi\|_{H^1_{AdS,deg}(u,v)} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right|\right) + |a|\sup_{D(u,v)\cap\{r\leq r_X\}}|\phi(u,v)|\Bigg],$$

from which the estimate (91) follows for ϕ recalling our choice (51). Revisiting the estimate (100), we obtain the same bound for $r^{\frac{3}{2}}\frac{\zeta}{\nu}$. □

4.7 Improving assumptions (68), (69) and (71)

Note that assumption (68) has already been improved in view of Lemma 4.9 and Lemma 4.8.

Using the global pointwise smallness bound for ϕ in the region $r \leq r_X$ established in Lemma 4.9, we can improve both (69) and (71), using that the r-difference in the region $r\leq r_Y$ is $c^{\frac{1}{3}}$ small. For (69):

$$\frac{4\pi|a|}{l^2}\int_{v_0}^{v} d\bar{v}\,\mathbf{1}_{\{r\leq r_Y\}}r^2\,r_v\,\phi^2(u,\bar{v})$$
$$\leq \frac{4\pi|a|}{l^2}\sup_{r\leq r_Y}|r^2\phi^2|(r_Y - r_{min}) \leq B_{M,l,a}c^{\frac{4}{3}} < \frac{1}{2}M\cdot c\,. \quad (102)$$

Assumption (71) is improved completely analogously. This improves the last of the bootstrap assumptions and we conclude that $\mathcal{B} = \mathcal{R}_{\mathcal{H}}$. In the final subsection we explain how this implies the estimates of Proposition 3.1.

4.8 Conclusions

Note first that inserting the estimate of Lemma 4.9 into (87) actually yields

$$\|\phi\|^2_{H^1_{AdS,deg}(u,v)} \leq B_{M,l,a}\left[\|\phi\|^2_{H^1_{AdS,deg}(u_{\mathcal{H}},v_0)} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right|^2\right], \tag{103}$$

after exploiting the $c^{\frac{1}{3}}$-smallness. From the general estimate

$$\sup_{D(u,v)}\|\phi\|_{H^1_{AdS}(u,v)} \leq B_{M,l}\left[\sup_{D(u,v)}\|\phi\|_{H^1_{AdS,deg}(u,v)} + \sup_{D(u,v)\cap\{r\leq r_X\}}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right|\right],$$

and Lemma 4.9, we conclude that (103) also holds for the non-degenerate norm on the left-hand side.

To finally conclude Proposition 3.1 we need the pointwise bound on κ and a bound for $r^{\frac{3}{2}}\frac{\zeta}{\nu}$ in the region $r \geq r_X$. For κ, we integrate (15) to obtain

$$\kappa(u,v) = \frac{1}{2}\exp\left(\int_{u_{\mathcal{I}}}^{u}\frac{4\pi r}{\nu}(\partial_u\phi)^2 du\right).$$

Clearly, $\kappa(u,v) \leq \frac{1}{2}$ globally. To derive a lower bound on κ in the region $r \geq r_X$ we estimate

$$\kappa(u,v) \geq \frac{1}{2}\exp\left(-\sup_{r\geq r_X}\frac{1}{r(1-\mu)}\int_{u_{\mathcal{I}}}^{u} r^2\frac{\lambda}{\Omega^2}(\partial_u\phi)^2 du\right). \tag{104}$$

Now since in $r \geq r_X$ we have $8r_X^2\frac{(1-\mu)}{r^2} \geq d^{\frac{1}{3}}$, we can conclude that

$$\kappa(u,v) \geq \frac{1}{2}\exp\left(-\frac{8}{r_X}d^{-\frac{1}{3}}\|\phi\|^2_{H^1_{AdS,deg}(u,v)}\right) \quad \text{in } \mathcal{R}_{\mathcal{H}}\cap\{r\geq r_X\}. \tag{105}$$

From $r \leq r_X$ we continue to integrate up to the boundary of $\mathcal{R}_{\mathcal{H}}$, now using the bound on $\frac{\zeta}{\nu}$ established in Lemma 4.9:

$$\begin{aligned}\kappa(u,v) &\geq \frac{1}{2}\exp\left(-B_{M,l}\|\phi\|^2_{H^1_{AdS,deg}(u,v)}\right)\exp\left(\int_{u_{r_X}}^{u}\frac{4\pi r}{\nu}(\partial_u\phi)^2 du\right)\\ &\geq \frac{1}{2}\exp\left(-B_{M,l}\sup_{\mathcal{R}_{\mathcal{H}}}\|\phi\|^2_{H^1_{AdS,deg}(u,v)}\right)\exp\left(-\sup_{\mathcal{R}_{\mathcal{H}}\cap\{r\leq r_X\}}\left|\frac{\zeta}{\nu}r^{\frac{3}{2}}\right|^2\int_{u_{r_X}}^{u}\frac{-\nu}{r^4}du\right)\\ &\geq \frac{1}{2} - B_{M,l}\left(\sup_{\mathcal{R}_{\mathcal{H}}}\|\phi\|^2_{H^1_{AdS,deg}(u,v)} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right|^2\right),\end{aligned}$$

where we used Lemma 4.9 in the last step.

For the r-weighted estimated for $\frac{\zeta}{\nu}$ we prove

Lemma 4.10. *In the entire region $\mathcal{B}$ we have*

$$\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right| + |\phi| \leq B_{M,l,a}\left(\sup_{\mathcal{B}}\|\phi\|_{H^1_{AdS,deg}(u,v)} + \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right|\right). \tag{106}$$

Proof. In view of Lemma 4.8 and Lemma 4.9, we only need to derive the bound for $\frac{\zeta}{\nu}r^{\frac{3}{2}}$ in the region $r \geq r_X$. We compute

$$\partial_v \left(r^n \frac{\zeta}{\nu}\right) = -r^n \phi_v + \frac{2\kappa a \phi}{l^2} r^{n+1} - r^n \frac{\zeta}{\nu}\left[-\frac{n\lambda}{r} + \frac{2\kappa\varpi}{r^2} + \frac{2\kappa r}{l^2} - \frac{8\pi r a}{l^2}\kappa\phi^2\right], \tag{107}$$

and observe that

$$-n\frac{\lambda}{r} + 2\kappa\frac{\varpi}{r^2} + \frac{2\kappa r}{l^2} - 8\pi r \frac{a}{l^2}\kappa\phi^2 = -n\frac{\kappa(1-\mu)}{r} + \frac{2\kappa\varpi}{r^2} + \frac{2\kappa r}{l^2} - 8\pi r\frac{a}{l^2}\kappa\phi^2$$
$$= \kappa\left[2(n+1)\frac{\varpi}{r^2} + \frac{r}{l^2}(2-n) - \frac{n}{r} - 8\pi r \frac{a}{l^2}\phi^2\right].$$

Choosing $n = \frac{3}{2}$ we see that we gain an exponential decay factor for large r. We integrate (107) in v from $r = r_X$ (where we already established the bound, Lemma 4.9) or from the initial data to any point in $\mathcal{R}_{\mathcal{H}}$, which leads to the estimate

$$\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}(u,v)\right| \leq \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right| + B_{M,l,a}\cdot\frac{1}{\sqrt{r}}\int_{v_0}^{v}\mathbf{1}_{\{r\geq r_X\}}\sqrt{r}\left[r^{\frac{3}{2}}|\phi_v| + \frac{2\kappa}{l^2}|\phi| r^{\frac{5}{2}}\right]$$
$$\leq \sup_{I(v_0)}\left|r^{\frac{3}{2}}\frac{\zeta}{\nu}\right| + B_{M,l,a}\cdot\frac{1}{\sqrt{r}}\|\phi\|_{H^1_{AdS,deg}(u,v)}\sqrt{\int_{v_0}^{v}\mathbf{1}_{\{r\geq r_X\}} r_v dv}$$

where we used both Cauchy-Schwarz and that $r_v \geq \frac{1}{B_{M,l,a}}r^2$ holds in $r \geq r_X$. The desired estimate follows. – We remark that later we will improve this estimate considerably using commutation. $\square$

5 Vectorfields and an integrated decay estimate

5.1 Vectorfield identities

Let $X = X^u(u,v)\partial_u + X^v(u,v)\partial_v$ be a vectorfield. We have the following formula for its deformation tensor $2^{(X)}\pi_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha$:

$$2^{(X)}\pi^{\alpha\beta} = g^{\alpha\gamma}\partial_\gamma X^\beta + g^{\beta\delta}\partial_\delta X^\alpha + g^{\alpha\gamma}g^{\beta\delta}g_{\gamma\delta,\mu}X^\mu, \tag{108}$$

and hence the following non-vanishing components:

$$^{(X)}\pi^{uu} = -\frac{2}{\Omega^2}\partial_v X^u \quad , \quad ^{(X)}\pi^{vv} = -\frac{2}{\Omega^2}\partial_u X^v, \tag{109}$$

$$^{(X)}\pi^{uv} = -\frac{1}{\Omega^2}(\partial_v X^v + \partial_u X^u) - \frac{2}{\Omega^2}\left(\frac{\Omega_u}{\Omega}X^u + \frac{\Omega_v}{\Omega}X^v\right), \tag{110}$$

$$^{(X)}\pi^{AB} = \frac{1}{r}g^{AB}(r_u X^u + r_v X^v). \tag{111}$$

Let ψ be a spherically symmetric solution to the equation[13]

$$\Box_g \psi - \frac{2a}{l^2}\psi = \mathfrak{q}\left[\psi\right] . \tag{112}$$

Then the energy momentum tensor

$$\mathbb{T}_{\mu\nu}\left[\psi\right] = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} (\partial \psi)^2 - \frac{a}{l^2}\psi^2 g_{\mu\nu} \tag{113}$$

satisfies

$$\nabla^\mu \mathbb{T}_{\mu\nu}\left[\psi\right] = \left(\nabla_\nu \psi\right) \mathfrak{q}\left[\psi\right] . \tag{114}$$

For future use we collect its components

$$\mathbb{T}_{uu}\left[\psi\right] = \left(\partial_u \psi\right)^2 \quad , \quad \mathbb{T}_{vv}\left[\psi\right] = \left(\partial_v \psi\right)^2 \quad , \quad \mathbb{T}_{uv}\left[\psi\right] = \frac{a\Omega^2}{2l^2}\psi^2 , \tag{115}$$

$$g^{AB}\mathbb{T}_{AB}\left[\psi\right] = \frac{4}{\Omega^2}\partial_u \psi \partial_v \psi - 2\frac{a}{l^2}\psi^2 . \tag{116}$$

We want to make use of the following multiplier identity

$$\nabla^\mu J^{X,\mathfrak{f}}_\mu \left[\psi\right] = K^{X,\mathfrak{f}}\left[\psi\right] , \tag{117}$$

where $\mathfrak{f}\left(u,v\right)$ is a C^2-function and

$$\begin{aligned} J^{X,\mathfrak{f}}_\mu \left[\psi\right] &= \mathbb{T}_{\mu\nu}\left[\psi\right] X^\nu + \mathfrak{f}\psi \nabla_\mu \psi - \frac{1}{2}\psi^2 \nabla_\mu \mathfrak{f} \\ K^{X,\mathfrak{f}}\left[\psi\right] &= \mathbb{T}_{\mu\nu}\left[\psi\right] {}^{(X)}\pi^{\mu\nu} + \left(X\psi\right)\mathfrak{q}\left[\psi\right] \\ &\quad + \mathfrak{f}\left[g^{\mu\nu}\partial_\mu \psi \partial_\nu \psi\right] + \left(-\frac{1}{2}\Box \mathfrak{f} + \frac{2a}{l^2}\mathfrak{f}\right)\psi^2 + \mathfrak{f}\psi \mathfrak{q}\left[\psi\right] . \end{aligned}$$

We compute

$$\begin{aligned} K^{X,\mathfrak{f}}\left[\psi\right] &= -\frac{2}{\Omega^2}\left(\partial_v X^u\right)\left(\partial_u \psi\right)^2 - \frac{2}{\Omega^2}\left(\partial_u X^v\right)\left(\partial_v \psi\right)^2 \\ &+ \left(\partial_u \psi\right)\left(\partial_v \psi\right)\left[\frac{4r_u}{\Omega^2 r}X^u + \frac{4r_v}{\Omega^2 r}X^v - \frac{4}{\Omega^2}\mathfrak{f}\right] + \left(X\left[\psi\right] + \mathfrak{f}\psi\right)\mathfrak{q}\left[\psi\right] \\ &- \frac{a}{l^2}\psi^2\left[-2\mathfrak{f} + \frac{l^2}{2a}\Box \mathfrak{f} + \partial_u X^u + \left(2\frac{r_u}{r} + 2\frac{\Omega_u}{\Omega}\right)X^u + \partial_v X^v + \left(2\frac{r_v}{r} + 2\frac{\Omega_v}{\Omega}\right)X^v\right] \end{aligned} \tag{118}$$

We finally remark that the identity (117) will typically be integrated over the spacetime region $D\left(u,v\right) \times \mathbb{S}^2$ with the diamond-shaped $D\left(u,v\right) \subset \mathcal{Q}$ defined in section 2.4.

[13] In applications, ψ will be $T\phi$ and hence $\mathfrak{q}\left[T\phi\right]$ equal to the error arising from commutation with the vectorfield T.

5.2 The vectorfield $T = \frac{1}{4\kappa}\partial_v + \frac{1}{4\gamma}\partial_u$

The non-vanishing components of the deformation tensor of T are

$$ {}^{(T)}\pi^{uu} = \frac{1}{2\Omega^2}\frac{\gamma_v}{\gamma^2} = 8\pi \ r \left(\frac{\partial_v \phi}{\Omega^2}\right)^2 \quad , \quad {}^{(T)}\pi^{vv} = \frac{1}{2\Omega^2}\frac{\kappa_u}{\kappa^2} = -8\pi \ r \left(\frac{\partial_u \phi}{\Omega^2}\right)^2 . $$

This is because

$$ 2^{(T)}\pi^{uv} = \frac{1}{2\Omega^2}\frac{\kappa_v}{\kappa^2} + \frac{1}{2\Omega^2}\frac{\gamma_u}{\gamma^2} - \frac{1}{2}\left(\frac{2}{\Omega^2}\right)^2 T\left(\Omega^2\right) = 0 \, , \tag{119} $$

$$ {}^{(T)}\pi^{AB} = \frac{2}{r^3} T\left(r\right) g^{AB} = 0 \, , \tag{120} $$

with the last two identities following from

$$ \frac{2}{\Omega^2}\frac{\kappa_v}{\kappa^2} = -\frac{2}{\Omega^2}\partial_v\left(\frac{1}{\kappa}\right) = \frac{2}{\Omega^2}\partial_v\left(\frac{4r_u}{\Omega^2}\right) = \frac{8r_{uv}}{\Omega^4} + \frac{2}{\Omega^4}\frac{1}{\kappa}\partial_v\Omega^2 \, , \tag{121} $$

$$ \frac{2}{\Omega^2}\frac{\gamma_u}{\gamma^2} = -\frac{2}{\Omega^2}\partial_u\left(\frac{1}{\gamma}\right) = -\frac{2}{\Omega^2}\partial_u\left(\frac{4r_v}{\Omega^2}\right) = -\frac{8r_{uv}}{\Omega^4} + \frac{2}{\Omega^4}\frac{1}{\gamma}\partial_u\Omega^2 \, , \tag{122} $$

on the one hand, and $T\left(r\right) = 0$ on the other. It follows that ${}^{(T)}\pi^{ab}\mathbb{T}_{ab}\left[\phi\right] = 0$, which means that for our non-linear system T is not Killing, but nevertheless leads to a conservation law in view of the identity (117) becoming

$$ \nabla^a\left(\mathbb{T}_{ab}\left[\phi\right] T^b\right) = 0 \, . \tag{123} $$

Inspecting the boundary-terms generated by T it becomes apparent that the Hawking mass is a potential for the energy fluxes of the vectorfield T through a hypersurface.

5.3 An integrated decay estimate for ϕ

Recall the norms defined in section 2.4. We define the fluxes

$$ \begin{aligned} \mathbb{F}\left(u,v\right) &= \|\phi\|^2_{H^1_{AdS}(u,v)} + \|\phi\|^2_{H^1_{AdS}(u,v_0)} \, , \\ \mathbb{F}_{deg}\left(u,v\right) &= \|\phi\|^2_{H^1_{AdS,deg}(u,v)} + \|\phi\|^2_{H^1_{AdS,deg}(u,v_0)} \, . \end{aligned} \tag{124} $$

Proposition 5.1. *For any $(u,v) \in \mathcal{R}_{\mathcal{H}}$ we have*

$$ \begin{aligned} \int_{u_{\mathcal{I}}}^{u} \frac{\left(\partial_u\phi\right)^2}{-\nu} r^2\left(\bar{u},v\right) d\bar{u} + \int_{v_0}^{v} \kappa\phi^2 r^2\left(u,\bar{v}\right) d\bar{v} + \mathbb{I}\left[\phi\right]\left(D\left(u,v\right)\right) \\ \leq B_{M,l,a}\left[\|\phi\|^2_{H^1_{AdS,deg}(u,v)} + \|\phi\|^2_{H^1_{AdS}(u,v_0)}\right] . \end{aligned} \tag{125} $$

Note that the boundary terms on the left are almost equal to the non-degenerate H^1_{AdS}-norm, *except* that their r-weight at infinity is weaker.

Proposition 5.1 will follow from the sequence of propositions proven in the remainder of this subsection. We apply (117) with a vectorfield X for which

$$X^u = -\frac{r_v}{\Omega^2}\mathfrak{F}(r) \quad \text{and} \quad X^v = -\frac{r_u}{\Omega^2}\mathfrak{F}(r) \ , \tag{126}$$

with $\mathfrak{F}$ a bounded C^3-function. We compute

$$\partial_v X^u = 4\pi r \frac{(\partial_v \phi)^2}{\Omega^2}\mathfrak{F}(r) - \frac{r_v}{\Omega^2}\mathfrak{F}'(r)\, r_v \, , \tag{127}$$

$$\partial_u X^v = 4\pi r \frac{(\partial_u \phi)^2}{\Omega^2}\mathfrak{F}(r) - \frac{r_u}{\Omega^2}\mathfrak{F}'(r)\, r_u \, , \tag{128}$$

$$\partial_u X^u + 2\frac{\Omega_u}{\Omega}X^u + \partial_v X^v + 2\frac{\Omega_v}{\Omega}X^v = -2\frac{r_v r_u}{\Omega^2}\mathfrak{F}'(r) - 2\frac{r_{vu}}{\Omega^2}\mathfrak{F}(r) \ . \tag{129}$$

We split

$$K^{X,\mathfrak{f}}[\phi] = K^{X,\mathfrak{f}}_{main}[\phi] + K^{X,\mathfrak{f}}_{error}[\phi] \ , \tag{130}$$

where

$$\begin{aligned} K^{X,\mathfrak{f}}_{main}[\phi] =& 2\mathfrak{F}'(r)\left[\frac{r_v}{\Omega^2}\partial_u\phi + \frac{r_u}{\Omega^2}\partial_v\phi\right]^2 \\ &+ (\partial_u\phi)(\partial_v\phi)\left[-\frac{4 r_u r_v}{\Omega^2\Omega^2}\left(\mathfrak{F}' + \frac{2}{r}\mathfrak{F}\right) - \frac{4}{\Omega^2}\mathfrak{f}\right] \\ &- \frac{a}{l^2}\phi^2\left[-2\mathfrak{f} - 2\frac{r_v r_u}{\Omega^2}\left(\mathfrak{F}' + \frac{2}{r}\mathfrak{F}\right) - 2\frac{r_{vu}}{\Omega^2}\mathfrak{F}(r) + \frac{l^2}{2a}\Box_g \mathfrak{f}\right] \end{aligned} \tag{131}$$

and

$$K^{X,\mathfrak{f}}_{error}[\phi] = -\frac{16}{\Omega^4}\pi r (\partial_u\phi)^2(\partial_v\phi)^2\mathfrak{F}(r) \ . \tag{132}$$

We may choose

$$\mathfrak{f} = -\frac{r_u r_v}{\Omega^2}\left(\mathfrak{F}' + \frac{2}{r}\mathfrak{F}\right) = \frac{1}{4}(1-\mu)\left(\mathfrak{F}' + \frac{2}{r}\mathfrak{F}\right) \ , \tag{133}$$

so that

$$\begin{aligned} K^{X,\mathfrak{f}}_{main}[\phi] =& 2\mathfrak{F}'(r)\left(\frac{r_v}{\Omega^2}\partial_u\phi + \frac{r_u}{\Omega^2}\partial_v\phi\right)^2 \\ &- \frac{a}{l^2}\phi^2\left[-2\frac{r_{vu}}{\Omega^2}\mathfrak{F}(r) + \frac{l^2}{2a}\Box_g\left(-\frac{r_u r_v}{\Omega^2}\left(\mathfrak{F}' + \frac{2}{r}\mathfrak{F}\right)\right)\right] \ . \end{aligned} \tag{134}$$

We would like to find a bounded, monotonically increasing $\mathfrak{F}$, since this will make the boundary term in the multiplier identity controllable by the energy and will, in addition, give the derivative term in (134) a sign. Exploiting the remaining freedom in $\mathfrak{F}$ to make the square bracket in (134) globally positive is difficult (if not impossible). However, the next proposition shows that the zeroth order term can be absorbed by the derivative term for a simple choice of $\mathfrak{F}$.

Proposition 5.2. *We have for any $a \geq -1$ the estimate*

$$\int_{D(u,v)} \frac{1}{r^6}\left(\frac{1}{4\gamma}\phi_u - \frac{1}{4\kappa}\phi_v\right)^2 \frac{\Omega^2}{2} r^2 \; du \; dv \leq B_{M,l,a} \cdot \mathbb{F}_{deg}\left(u,v\right) .$$

Proof. Apply the identity (117) with $\mathfrak{F}\left(r\right) = -\frac{1}{r^2}$ and $\mathfrak{f} = 0$. We first look at the boundary terms. We have

$$(4\pi)^{-1} \int_{D(u,v)\times\mathbb{S}^2} \nabla^\mu J^{X,\mathfrak{f}}_\mu\left[\phi\right] = \int_{v_0}^{v} \left(\mathbb{T}_{vv}X^v + \mathbb{T}_{uv}X^u\right) r^2\left(u,\bar{v}\right) d\bar{v}$$
$$\int_{u_{\mathcal{I}}}^{u} \left(\mathbb{T}_{uu}X^u + \mathbb{T}_{uv}X^v\right) r^2\left(\bar{u},v\right) d\bar{u} - \int_{u_0}^{u} \left(\mathbb{T}_{uu}X^u + \mathbb{T}_{uv}X^v\right) r^2\left(\bar{u},v_0\right) d\bar{u}$$

because the boundary term on $\mathcal{I}$ vanishes. It is not hard to see that

$$\Big| \int_{D(u,v)\times\mathbb{S}^2} \nabla^\mu J^{X,\mathfrak{f}}_\mu\left[\phi\right] \Big| \leq B_{M,l,a} \cdot \mathbb{F}_{deg}\left(u,v\right) .$$

We turn to the spacetime term. Observe that

$$\frac{2a}{l^2}\phi^2 \frac{r_{vu}}{\Omega^2}\mathfrak{F}\left(r\right) = \frac{a}{l^2 r^2}\phi^2\left(\frac{r}{l^2} + \frac{\varpi}{r^2} - \frac{4\pi r a \phi^2}{l^2}\right) . \tag{135}$$

This term has the same sign as a since the bracket is positive (cf. 68). Hence for $a > 0$ we are done immediately, gaining in addition control over the spacetime integral $\int \phi^2 r \Omega^2 \; dudvd\sigma_{\mathbb{S}^2}$. Returning to $a < 0$, we can write both

$$\frac{a}{l^2 r^2}\phi^2\left(\frac{r}{l^2} + \frac{\varpi}{r^2} - \frac{4\pi r a \phi^2}{l^2}\right) = \frac{a}{l^2 r^2}\left(\frac{1}{2r_u}\partial_u\left(1-\mu\right) - \frac{8\pi r \; r_v}{r_u \Omega^2}\left(\partial_u \phi\right)^2\right)\phi^2$$

and

$$\frac{a}{l^2 r^2}\phi^2\left(\frac{r}{l^2} + \frac{\varpi}{r^2} - \frac{4\pi r a \phi^2}{l^2}\right) = \frac{a}{l^2 r^2}\left(\frac{1}{2r_v}\partial_v\left(1-\mu\right) - \frac{8\pi r \; r_u}{r_v \Omega^2}\left(\partial_v \phi\right)^2\right)\phi^2$$

Integrating this zeroth order terms yields (in view of $\sqrt{g} = \frac{\Omega^2}{2}r^2$)

$$\int_{D(u,v)} \frac{a}{l^2 r^2}\phi^2\left(\frac{r}{l^2} + \frac{\varpi}{r^2} - \frac{4\pi r a \phi^2}{l^2}\right)\frac{\Omega^2}{2} r^2 dudv =$$
$$\frac{1}{2}\int_{D(u,v)} \frac{a}{l^2}\phi^2\left(-\kappa \cdot \partial_u\left(1-\mu\right) - \frac{4\pi r \; r_v}{r_u}\left(\partial_u\phi\right)^2\right) du \; dv$$
$$+\frac{1}{2}\int_{D(u,v)} \frac{a}{l^2}\phi^2\left(\gamma \cdot \partial_v\left(1-\mu\right) - \frac{4\pi r \; r_u}{r_v}\left(\partial_v\phi\right)^2\right) du \; dv \, . \tag{136}$$

Integrating the first term in the second line by parts, we see that if the derivative hits the κ it will cancel with the second term in that line. Similarly for the third line and the derivative falling on γ. This means that

$$\int_{D(u,v)} \frac{a}{l^2 r^2}\phi^2 \left(\frac{r}{l^2} + \frac{\varpi}{r^2} - \frac{4\pi r a \phi^2}{l^2}\right)\frac{\Omega^2}{2} r^2 \, du \, dv$$
$$= \int_{D(u,v)} \frac{4a}{l^2}\gamma\kappa\left(1-\mu\right)\phi\left(\frac{1}{4\gamma}\phi_u - \frac{1}{4\kappa}\phi_v\right) du \, dv - \int_{u_0}^{u} \frac{a}{4l^2}\phi^2\left(-r_u\right)\left(\bar{u}, v_0\right) d\bar{u}$$
$$+ \int_{u_{\mathcal{I}}}^{u} \frac{a}{4l^2}\phi^2\left(-r_u\right)\left(\bar{u}, v\right) d\bar{u} - \int_{v_0}^{v} \frac{a}{4l^2}\phi^2\left(r_v\right)\left(u, \bar{v}\right) d\bar{v}$$

since again the boundary term on $\mathcal{I}$ vanishes. Clearly, the boundary terms are manifestly controlled by $\mathbb{F}_{deg}\left(u,v\right)$. For the remaining spacetime term we apply Cauchy's estimate $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$:

$$\int_{D(u,v)} \frac{4|a|}{l^2}\gamma\kappa\left(1-\mu\right)\phi\left(\frac{1}{4\gamma}\phi_u - \frac{1}{4\kappa}\phi_v\right) du \, dv$$
$$\leq \int_{D(u,v)} \frac{|a|}{2l^2}\phi^2\left(\frac{r}{l^2} + \frac{\varpi}{r^2} - \frac{4\pi r a\phi^2}{l^2}\right)\frac{\Omega^2}{2} \, du \, dv +$$
$$\int_{D(u,v)} \frac{32|a|}{l^2}\frac{\gamma^2\kappa^2\left(1-\mu\right)^2}{\Omega^4}\left(\frac{r}{l^2} + \frac{\varpi}{r^2} - \frac{4\pi r a\phi^2}{l^2}\right)^{-1}\left(\frac{\phi_u}{4\gamma} - \frac{\phi_v}{4\kappa}\right)^2\frac{\Omega^2}{2} du \, dv$$

and using that $\frac{\gamma^2\kappa^2(1-\mu)^2}{\Omega^4} = \frac{1}{16}$ we obtain

$$\frac{1}{2}\int_{D(u,v)} \frac{|a|}{l^2 r^2}\phi^2\left(\frac{r}{l^2} + \frac{\varpi}{r^2} - \frac{4\pi r a\phi^2}{l^2}\right)\frac{\Omega^2}{2} r^2 \, du \, dv \leq B_{M,l,a}\cdot\mathbb{F}_{deg}\left(u,v\right)$$
$$+ \int_{D(u,v)} \frac{2|a|}{r}\left(1 + \frac{\varpi l^2}{r^3} - 4\pi a\phi^2\right)^{-1}\left(\frac{1}{4\gamma}\phi_u - \frac{1}{4\kappa}\phi_v\right)^2\frac{\Omega^2}{2} du \, dv \, .$$

Finally, in view of the estimate (recall the bounds (49) and (68))

$$1 - |a|\left(1 + \frac{\varpi l^2}{r^3} - 4\pi a\phi^2\right)^{-1} \geq \frac{Ml^2}{4r^3} \, , \tag{137}$$

which holds for all $0 \geq a \geq -1$, we conclude

$$\int_{D(u,v)} K^{X,f}\left[\phi\right] \geq \int_{D(u,v)} \frac{2Ml^2}{r^4}\left(\frac{1}{4\gamma}\phi_u - \frac{1}{4\kappa}\phi_v\right)^2\frac{\Omega^2}{2} dudv - B_{M,l,a}\cdot\mathbb{F}_{deg}\left(u,v\right),$$

as $K^{X,0}_{error}\left[\phi\right]$ has a good sign. The proposition follows. □

Repeating the proof above with a slightly different r-weight, we can derive the following more general Hardy inequality, which will be important later to optimize the radial weights near infinity.

Corollary 5.3. *Set* $y = r^n\left(\frac{2r}{l^2}+\frac{2\varpi}{r^2}-\frac{8\pi r a\phi^2}{l^2}\right)+nr^{n-1}\left(1-\mu\right)$*. We have, for* $0 \leq n \leq 1$*, the estimate*

$$\int_{D(u,v)} y\ \phi^2\Omega^2 dudv \leq \int_{D(u,v)} \Omega^2 \frac{r^{2n}}{y}\left(\frac{\phi_u}{\gamma}-\frac{\phi_v}{\kappa}\right)^2 dudv + B_{M,l,a}\cdot \mathbb{F}_{deg}\left(u,v\right)$$

Proof. Write $y = \frac{1}{r_u}\partial_u\left(r^n\left(1-\mu\right)\right) - r^n\frac{16\pi r r_v}{r_u\Omega^2}\left(\partial_u\phi\right)^2$ and analogously for the v-derivative. Then repeat the proof above ($n=0$ being the previous case). □

Remark 5.4. *Note that the proof of Proposition 5.2 shows that if we have the strict inequality* $a > -1$*, we obtain in addition control over the spacetime integral of* ϕ^2 *(with the control degenerating as* $a \rightarrow -1$*, cf. (137)). Moreover, the weight* $\frac{1}{r^6}$ *for the derivative-term improves to* $\frac{1}{r^3}$ *in case that* $a > -1$*. The next Proposition shows that the control over the zeroth order term actually does not degenerate as* $a \rightarrow -1$*. It also retrieves control over the missing derivative and optimizes the r-weights near infinity.*

Proposition 5.5. *We have for any* $a \geq -1$ *the estimate*

$$\int_{D(u,v)}\left(\phi^2+\frac{1}{r^2}\frac{1}{\gamma^2}\left(\partial_u\phi\right)^2+\frac{1}{r^2}\frac{1}{\kappa^2}\left(\partial_v\phi\right)^2\right)\frac{\Omega^2}{2}r^2\ du\ dv \leq B_{M,l,a}\cdot\mathbb{F}_{deg}\left(u,v\right)$$

Proof. As mentioned we first wish to prove uniform control over the spacetime integral of ϕ^2 as $a \rightarrow -1$. In view of Remark 5.4 we can assume $a \leq -\frac{1}{2}$ for this "Step 1", since for $a > -1/2$ we already control the spacetime integral of ϕ^2.

Step 1: We choose $\mathfrak{F} = \frac{1}{r^5}$ in (134). With this choice the derivative term is controlled from Proposition 5.2. The zeroth order term with $r_{uv}\mathfrak{F}$ has a globally (positive) sign. The zeroth order term with the $\Box_g$ in it will be integrated by parts via the identity

$$-\frac{1}{2}\int\phi^2\Box_g\mathcal{G} = -\frac{1}{2}\int\nabla^\alpha\left(\phi^2\nabla_\alpha\mathcal{G}\right)+\int\phi g^{\alpha\beta}\nabla_\alpha\mathcal{G}\nabla_\beta\phi \tag{138}$$

producing boundary terms controlled by the energy as well as a cross-term of the form $\phi\left(\phi_u/\gamma+\phi_v/\kappa\right)$. The latter appears because $\mathcal{G} = -\frac{r_u r_v}{\Omega^2}\left(\mathfrak{F}'+\frac{2}{r}\mathfrak{F}'\right) = \frac{1}{4}\left(1-\mu\right)\left(\mathfrak{F}'+\frac{2}{r}\mathfrak{F}'\right)$ depends only on r except for a harmless error arising from the $\varpi\left(u,v\right)$ in $\left(1-\mu\right)$. Since the derivative appearing in the cross term is already controlled by Proposition 5.2, it suffices to borrow a δ from the good $r_{uv}\mathfrak{F}$ term, leading to overall control of the zeroth order spacetime term:

$$\int_{D(u,v)}\left(\frac{1}{r^4}\phi^2\right)\frac{\Omega^2}{2}r^2\ du\ dv\ \leq B_{M,l,a}\cdot\mathbb{F}_{deg}\left(u,v\right)+\epsilon\cdot\mathbb{I}_{deg}\left[\phi\right]\left(D\left(u,v\right)\right). \tag{139}$$

Here the last term appears to estimate the error-terms mentioned and to control $\int_{D(u,v)} K^{X,\mathfrak{f}}_{error} \leq \epsilon\cdot\mathbb{I}_{deg}\left[\phi\right]\left(D\left(u,v\right)\right)$, the ϵ coming from the pointwise bound on $\frac{\phi_u}{r_u}$ of Proposition 3.1.

Step 2: To retrieve the missing derivative, we revisit (131). It is apparent that choosing $\mathfrak{F} = \frac{1}{r^5}$ and $\mathfrak{f} = 0$, we can dominate the mixed derivative term by the good-signed quadratic terms, while the zeroth order term is controlled by (139). The boundary terms are again controlled by the energy. Hence

$$\int_{D(u,v)} \left(\frac{1}{r^4}\phi^2 + \frac{1}{r^6}\frac{1}{\gamma^2}\left(\partial_u \phi\right)^2 + \frac{1}{r^6}\frac{1}{\kappa^2}\left(\partial_v \phi\right)^2 \right) \frac{\Omega^2}{2} r^2 \, du \, dv$$
$$\leq B_{M,l,a} \cdot \mathbb{F}_{deg}\left(u,v\right) + \epsilon \cdot \mathbb{I}_{deg}\left[\phi\right]\left(D\left(u,v\right)\right) .$$

Step 3: Finally, we optimize the weights near infinity by choosing $\mathfrak{F} = -\frac{1}{r}$ in (134). Note that the dominant term near infinity of the zeroth order term becomes

$$-\frac{1}{8}\Box_g \left(\left(1-\mu\right) \left(\mathfrak{F}' + \frac{2\mathfrak{F}}{r} \right) \right) - \frac{a}{l^2}\mathfrak{F}\left(\frac{r}{l^2} + \frac{M}{r^2} \right) = \frac{a}{l^4} + \text{l.o. terms in } r \quad (140)$$

in this case. On the other hand, the derivative term in (134) becomes $\frac{1}{8r^2}\left(\frac{1}{\gamma}\phi_u - \frac{1}{\kappa}\phi_v\right)^2$. By Corollary 5.3, applied with $n = 1$ we can absorb the zeroth order term by the derivative term near infinity provided that $-a < \frac{9}{8}$. This finally yields the proposition, since an integrated decay estimate away from infinity was proved in Step 2. Note once more that for any bounded $\mathfrak{F}$ we have $\int_{D(u,v)} K^{X,\mathfrak{f}}_{error} \leq \epsilon \cdot \mathbb{I}_{deg}\left[\phi\right]\left(D\left(u,v\right)\right)$ and that this error is finally absorbed by the main term. $\Box$

Up until now we proved (Proposition 5.5)

$$\mathbb{I}_{deg}\left[\phi\right]\left(D\left(u,v\right)\right) \leq B_{M,l,a} \cdot \mathbb{F}_{deg}\left(u,v\right) . \quad (141)$$

The missing ingredient to reach Proposition 5.1 as stated is to go from the degenerate to the non-degenerate spacetime term, and to obtain the missing boundary term. This is achieved with the (future-directed, null) redshift vectorfield

$$Y = \left(-r_u\right)^{-1} \partial_u \, . \quad (142)$$

From (117) we obtain

$$K^{Y,0}\left[\psi\right] = \frac{\left(\partial_u \psi\right)^2}{2r_u^2}\left(\frac{2\varpi}{r^2} + \frac{2r}{l^2} \right) + \frac{\partial_u \psi}{r_u}\frac{1}{r\kappa}\partial_v \psi + \frac{a}{l^2}\psi^2 \left[\frac{2}{r}\right] , \quad (143)$$

$$J^{Y,0}\left[\psi\right]\left(Y,\partial_u\right) = \frac{\left(\partial_u \psi\right)^2}{r_u^2}\left(-r_u\right) \quad , \quad J^{Y,0}\left[\psi\right]\left(Y,\partial_v\right) = \frac{2a\kappa}{2l^2}\psi^2 \, , \quad (144)$$

which in turn leads to the estimate

$$\int_{u_{\mathcal{I}}}^{u} \frac{(\partial_u \phi)^2}{r_u^2} r^2 (-r_u)(\bar{u}, v)\, d\bar{u} + \int_{D(u,v)} \frac{r (\partial_u \phi)^2}{r_u^2} \Omega^2 r^2 du dv$$
$$\leq B_{M,l,a} \int_{D(u,v)} \left[\frac{(\partial_v \phi)^2}{r^3 \kappa^2} + \frac{\phi^2}{r} \right] \Omega^2 r^2 d\bar{u} d\bar{v} + B_{M,l,a} \int_{v_0}^{v} \frac{1}{l^2} \kappa \phi^2 r^2 (u, \bar{v})\, d\bar{v}$$
$$+ \int_{u_{\mathcal{I}}}^{u} \frac{(\partial_u \phi)^2}{r_u^2} r^2 (-r_u)(\bar{u}, v_0)\, d\bar{u}\,. \tag{145}$$

The first term on the right hand side is controlled by the degenerate term $\mathbb{I}_{deg}[\phi](D(u,v))$ that we already control by (141). The second term, a boundary term, can be converted into a spacetime-term, which is partly absorbed by the left and partly adds a term of the first type to the right hand side of (145). Indeed,

$$\int_{v_0}^{v} \frac{1}{l^2} \kappa \phi^2 r^2 d\bar{v} = \int_{u_0}^{u} d\bar{u}\ \partial_u \int_{v_0}^{v^\star(u)} \frac{1}{l^2} \kappa \phi^2 r^2 d\bar{v}\,, \tag{146}$$

where $v^\star(u)$ is the v-value where the ray of constant u intersects either $\mathcal{I}$ or the constant v-ray (whatever happens first). The right hand side of the previous equation is equal to

$$= \frac{1}{l^2} \int_{D(u,v)} \left(-r\pi \frac{(\partial_u \phi)^2}{r_u^2} \phi^2 - \frac{1}{2} \phi \frac{\partial_u \phi}{r_u} - \frac{1}{2r} \phi^2 \right) \Omega^2 r^2 d\bar{u} d\bar{v}\,, \tag{147}$$

observing that the boundary term on $\mathcal{I}$ vanishes in view of the decay of ϕ and that $v^\star(u)$ is constant in u after the point where u intersects the point where $v = const$ meets $\mathcal{I}$. A simple application of Cauchy's inequality then shows that (145) also holds without the v-boundary term. Proposition 5.1 then follows since the last term in (145) is on data (requiring, however, the non-degenerate norm).

5.4 Proof of the estimate (56) of Proposition 3.3

In view of the general estimate

$$|\phi r^{\frac{3}{2}}(u,v)| \leq B_{M,l} \sup_{D(u,v)} \|\phi\|_{H^1_{AdS}(\bar{u},\bar{v})}\,, \tag{148}$$

we obtain from the estimate (87),

$$\|\phi\|^2_{H^1_{AdS,deg}(u,v)} \leq B_{M,l,a} \left[\|\phi\|^2_{H^1_{AdS,deg}(u,v_0)} + c^{\frac{1}{3}} \sup_{(\bar{u},\bar{v}) \in D(u,v)} \|\phi\|^2_{H^1_{AdS}(\bar{u},\bar{v})} \right].$$

We can insert this estimate on the right hand side of the estimate of Proposition 5.1 and also add it to the resulting equation. This yields

$$\|\phi\|^2_{H^1_{AdS}(u,v)} + \mathbb{I}[\phi](D(u,v))$$
$$\leq B_{M,l,a} \left[\|\phi\|^2_{H^1_{AdS}(u,v_0)} + c^{\frac{1}{3}} \sup_{(\bar{u},\bar{v}) \in D(u,v)} \|\phi\|^2_{H^1_{AdS}(\bar{u},\bar{v})} \right].$$

Taking the sup in $D(u,v)$ and absorbing the $c^{\frac{1}{3}}$-term on the right yields (56).

6 Proof of Proposition 3.2: Improved and higher order bounds

For the estimates in this section, recall the initial data norm (53).

6.1 Further consequences of the bootstrap assumptions

In this section we will derive estimates for higher derivatives. These estimates are not sharp but sufficient to control the error in the commuted estimates later.

Lemma 6.1. *We have the pointwise estimate*

$$\left| r^{\frac{7}{2}} \frac{\partial_u \frac{\phi_u}{r_u}}{r_u} \right| \leq B_{M,l,a} \cdot \mathbb{N}[\phi](v_0) \tag{149}$$

Proof. We derive the following evolution equation:

$$\begin{aligned} \partial_v \left(\frac{\partial_u \frac{\phi_u}{r_u}}{r_u} \right) &= \left[-4\kappa \left(\frac{\varpi}{r^2} + \frac{r}{l^2} - \frac{4\pi a \phi^2}{l^2} + \frac{1-\mu}{4r} \right) \right] \left(\frac{\partial_u \frac{\phi_u}{r_u}}{r_u} \right) \\ + \frac{2}{r^2}\phi_v + 8\pi r \frac{\kappa a}{l^2} \phi \left(\frac{\phi_u}{r_u} \right)^2 &- \frac{2\kappa a \phi}{l^2 r} + \frac{\phi_u}{r_u} \left(\frac{2\lambda}{r^2} - \frac{1}{r r_u} \partial_u \left(r \frac{r_{uv}}{r_u} \right) + \frac{2a\kappa}{l^2} \right) \end{aligned} \tag{150}$$

Note the exponential decay factor (redshift) in the square bracket in the first line. Integrating and estimating the errors as in Lemma 4.9 yields the result. See also Section 7 of our [33], where the same computation is carried out. $\square$

Interestingly enough, we were able to derive this pointwise bound for a second u-derivative without second v-derivatives appearing anywhere in the estimate. Note also that at this point we do not yet have a pointwise bound for ϕ_v available as we can not integrate from infinity, this being part of the difficulty of the AdS-end. The way we are going to establish such a bound is via commutation: We consider the wave equation for $T\phi$ and prove its H^1-energy estimate, which in turn implies a pointwise bound on $T\phi$ which finally yields a pointwise bound on ϕ_v via the pointwise bound on ϕ_u already proven. In fact, our argument requires bootstraping the pointwise bound on ϕ_v (cf. Lemma 6.7) as the error in the H^1-estimate for $T\phi$ seems to require pointwise control on ϕ_v.

6.2 The wave equation for $T\phi$

We turn to the commutation of the wave equation by the vectorfield $T = \frac{1}{4\kappa}\partial_v \phi + \frac{1}{4\gamma}\partial_u \phi$.

Lemma 6.2. *Let ψ be a solution of the equation $\Box_g \psi = 0$ and X be a vectorfield. Then*

$$\Box_g (X\psi) = \mathfrak{q}[X\psi] \qquad \textit{with} \tag{151}$$

$$\mathfrak{q}[X\psi] = 2^{(X)}\pi^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi + \left[2\nabla^{\alpha\,(X)}\pi_{\alpha\mu} - \nabla_\mu\left(tr^{(X)}\pi\right)\right]\nabla^\mu\psi\,. \tag{152}$$

Proof. Note that on functions f

$$\mathcal{L}_X\nabla_\alpha f = \nabla_\alpha\mathcal{L}_X f = X^\beta\nabla_\beta\nabla_\alpha f + \left(\nabla_\alpha X^\beta\right)\nabla_\beta f\,, \tag{153}$$

while on (co)-vectors V_β (writing $\pi_{\alpha\beta}$ for ${}^{(X)}\pi_{\alpha\beta}$ for the moment)

$$\begin{aligned}\mathcal{L}_X\nabla_\alpha V_\beta - \nabla_\alpha\mathcal{L}_X V_\beta = X^\gamma\left[\nabla_\gamma,\nabla_\alpha\right]V_\beta - V^\gamma\left[\nabla_\gamma,\nabla_\alpha\right]X_\beta \\ + \left(\nabla_\gamma\nabla_\alpha X_\beta - 2\nabla_\alpha\pi_{\gamma\beta}\right)V^\gamma\,.\end{aligned}$$

Contracting with $g^{\alpha\beta}$ we obtain the formula

$$g^{\alpha\beta}\mathcal{L}_X\nabla_\alpha V_\beta = g^{\alpha\beta}\nabla_\alpha\mathcal{L}_X V_\beta + V^\gamma\nabla_\gamma\left(tr\ \pi\right) - 2V^\gamma\nabla^\alpha\pi_{\gamma\alpha}\,. \tag{154}$$

With $V_\beta = \nabla_\beta\psi$ combining (153) and (154) yields

$$g^{\alpha\beta}\mathcal{L}_X\nabla_\alpha\nabla_\beta\psi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\mathcal{L}_X\psi + \nabla_\gamma\left(tr\ \pi\right)\nabla^\gamma\psi - 2\nabla^\alpha\pi_{\gamma\alpha}\nabla^\gamma\psi\,. \tag{155}$$

Finally, the desired formula follows from

$$X\left(\Box_g\psi\right) = \mathcal{L}_X\left(g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi\right) = -2\pi^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi + g^{\alpha\beta}\mathcal{L}_X\nabla_\alpha\nabla_\beta\psi \tag{156}$$

after applying (155) to the second term on the right hand side. □

From the computations

$$\begin{aligned}2g^{\alpha\beta}\left(\nabla_\alpha{}^{(T)}\pi_{\beta\gamma}\right)\nabla^\gamma\psi &= 2\left(\frac{2}{\Omega^2}\right)^2\left[\partial_u\psi\ \partial_u\left(2\pi r\phi_v^2\right) - \partial_v\psi\ \partial_v\left(2\pi r\phi_u^2\right)\right] \\ &\quad - \frac{8}{\gamma r\Omega^2}\partial_v\psi\left(\pi r\phi_u^2\right) - \frac{8}{\kappa r\Omega^2}\partial_u\psi\left(\pi r\phi_v^2\right) \\ &= \frac{-4}{\gamma r\Omega^2}\partial_v\psi\left(\pi r\phi_u^2\right) - \frac{4}{\kappa r\Omega^2}\partial_u\psi\left(\pi r\phi_v^2\right) - \pi\frac{8}{\Omega^2\gamma}\psi_u\phi_u\phi_v \\ &\quad - \frac{8\pi}{\Omega^2\kappa}\psi_v\phi_u\phi_v - \frac{16}{\Omega^2}r\frac{a}{l^2}\left(\partial_u\psi\right)\phi\phi_v + \frac{16}{\Omega^2}r\frac{a}{l^2}\left(\partial_v\psi\right)\phi\phi_u\end{aligned}$$

and

$$\begin{aligned}&2^{(T)}\pi^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi = 2^{(T)}\pi^{\alpha\beta}\left(\partial_\alpha\partial_\beta\psi - \Gamma^\delta_{\alpha\beta}\partial_\delta\psi\right) \\ &\quad = 16\frac{\pi r}{\Omega^2}\left[\left(\partial_v\phi\right)^2\partial_u\left(\frac{\partial_u\psi}{\Omega^2}\right) - \left(\partial_u\phi\right)^2\partial_v\left(\frac{\partial_v\psi}{\Omega^2}\right)\right] \\ &\quad = \frac{\pi r\phi_v^2}{\kappa^2}\left[\frac{\partial_u\left(\frac{\partial_u\psi}{r_u}\right)}{r_u}\right] - \frac{4\pi^2r^2\phi_v^2}{\kappa^2}\left(\frac{\phi_u}{r_u}\right)^2\frac{\psi_u}{r_u} \\ &\qquad - 16\frac{\pi r}{\Omega^2}\left(\partial_u\phi\right)^2\left[-\frac{1}{r_u}\partial_v\left(T\left[\psi\right]\right) + \frac{r_{uv}}{r_u}\frac{1}{r_u}\frac{1}{\kappa}\partial_v\psi - \frac{4}{\Omega^2}r\pi\phi_v^2\frac{\phi_u}{r_u} + \frac{\phi_{uv}}{4\gamma r_u}\right],\end{aligned}$$

we find using the Lemma applied with $\psi = \phi$,

$$\mathfrak{q}\left[T\phi\right] = -4\pi r \left(\frac{\phi_u}{r_u}\right)^2 \frac{1}{\kappa}\partial_v\left(T\phi\right) + \frac{\pi r \phi_v^2}{\kappa^2}\left[\frac{\partial_u\left(\frac{\partial_u \phi}{r_u}\right)}{r_u}\right]$$
$$+ \left(-16\pi r\frac{r_{uv}}{\Omega^2} - 2\pi\left(1-\mu\right)\right)\frac{\phi_v}{\kappa}\left(\frac{\phi_u}{r_u}\right)^2 + 3\pi\frac{\phi_v^2}{\kappa^2}\left(\frac{\phi_u}{r_u}\right)$$
$$- \pi\left(\frac{\phi_u}{r_u}\right)^3\left(1-\mu\right)^2 - \frac{2\pi a r}{l^2}\left(1-\mu\right)\phi\left(\frac{\phi_u}{r_u}\right)^2 . \tag{157}$$

6.3 Estimates for $T\phi$

For any point (u, v) in $\mathcal{R}_{\mathcal{H}}$, we define the higher order energy

$$E\left[T\phi\right](u,v) = \int_{u_{\mathcal{I}}}^{u}\left[2\pi r^2\frac{1}{\gamma}\left(\partial_u(T\phi)\right)^2 - 4a\pi\frac{r^2}{l^2}r_u\left(T\phi\right)^2\right](\bar{u},v)\,d\bar{u}$$
$$+ \int_{v_0}^{v}\left[2\pi r^2\frac{1}{\kappa}\left(\partial_v(T\phi)\right)^2 + 4a\pi\frac{r^2}{l^2}r_v\left(T\phi\right)^2\right](u,\bar{v})\,d\bar{v}\,. \tag{158}$$

Lemma 6.3. *The energy $E\left[T\phi\right](u,v)$ is almost conserved in that*

$$E\left[T\phi\right](u,v) = E\left[T\phi\right](u,v_0) + \int_{D(u,v)}\mathfrak{Q}\;\;\Omega^2 r^2 du dv$$

$$\textit{where}\qquad \mathfrak{Q} = \frac{1}{\Omega^2 r^2}\Bigg(- 32 r^3\pi^2\frac{1}{\Omega^2}\left(\partial_v\phi\right)^2\left(\partial_u(T\phi)\right)^2$$
$$+32r^3\pi^2\frac{1}{\Omega^2}\left(\partial_u\phi\right)^2\left(\partial_v(T\phi)\right)^2 + \pi r^2\Omega^2\cdot\left(TT\phi\right)\cdot\mathfrak{q}\left[T\phi\right]\Bigg) \tag{159}$$

holds.

Proof. The quantity $T\phi$ satisfies the wave equation $\Box_g(T\phi) = \mathfrak{q}\left[T\phi\right]$. Integrating the energy identity (117) for the energy momentum tensor associated with $T\phi$ with the vectorfield $T = \frac{1}{4\kappa}\partial_v + \frac{1}{4\gamma}\partial_u$ yields the above identity using that the energy-flux through the boundary at infinity is zero by the local well-posedness result [33]. $\Box$

Lemma 6.4. *For any $(u, v) \in \mathcal{R}_{\mathcal{H}}$ we have the estimate*

$$\|T\phi\|_{H^1_{AdS}(u,v)} \leq B_{M,l,a}\cdot\mathbb{N}\left[\phi\right](v_0) + B_{M,l}\left(\int_{D(u,v)}|\mathfrak{Q}|\;\;\Omega^2 r^2 du dv\right)^{\frac{1}{2}} \tag{160}$$

Proof. We can prove this estimate in the same way that we proved it for ϕ itself in the context of the bootstrap of Section 4 (in which case there was no commutation error $\mathfrak{Q}$, of course). In fact, a bootstrap is no-longer necessary, since the important Hardy inequalities have already been established in $\mathcal{R}_{\mathcal{H}}$.

Integrating $T\phi$ from infinity one derives the pointwise bound

$$|T\phi| \leq B_{M,l} \cdot r^{-\frac{3}{2}} \cdot \|T\phi\|_{H^1_{AdS,deg}(u,v)} \qquad \text{in } \mathcal{R}_{\mathcal{H}} \cap \{r \geq r_X\} \tag{161}$$

using Cauchy-Schwarz as in Lemma 4.8. Repeating the redshift estimate of Lemma 4.9, now with $\partial_v \left(\frac{\partial_u (T\phi)}{r_u}\right)$, we derive for any $(u,v) \in \mathcal{R}_{\mathcal{H}} \cap \{r \leq r_X\}$

$$\left|\frac{\partial_u (T\phi)}{\nu}\right| + |T\phi| \leq B_{M,l,a} \left[\sup_{(\bar{u},\bar{v}) \in D(u,v)} \|T\phi\|_{H^1_{AdS,deg}(\bar{u},\bar{v})} + \mathbb{N}[\phi](v_0)\right]. \tag{162}$$

Indeed, the only difference to Lemma 4.9 is that there is now an error-term $r\kappa\mathfrak{q}[T\psi]$ on the right hand side of the equation due to the commutation term in the wave equation. For this error-term we note that (using definition (92))

$$\begin{aligned} &\left|\int_{v_0}^{v} d\bar{v} \left[\exp\left(-\int_{\bar{v}}^{v} \rho(u,\hat{v})\, d\hat{v}\right) r\kappa\mathfrak{q}[T\phi]\right]\right| \\ &\leq \epsilon \left(\sup_{(\bar{u},\bar{v}) \in D(u,v)} \|T\phi\|_{H^1_{AdS,deg}(\bar{u},\bar{v})} + B_{M,l,a} \cdot \mathbb{N}[\phi](v_0)\right), \end{aligned} \tag{163}$$

which follows by inspecting the terms in $\mathfrak{q}[T\phi]$ (cf. (157)) individually:

- the first term is estimated as in Lemma 4.9, cf. the estimate (94). The same holds for the first term in the second line. Note the smallness factor arising from the pointwise bound on $\frac{\phi_u}{r_u}$, Proposition 3.1.
- terms which have ϕ_v^2 can be estimated using the pointwise bound on $\frac{\phi_u}{r_u}$ and (149) and the $H^1_{AdS,deg}$-norm for ϕ. Note that $\|\phi\|^2_{H^1_{AdS,deg}(u,v)} \leq B_{M,l} \cdot \epsilon \cdot \|\phi\|_{H^1_{AdS,deg}(u,v)}$ from Proposition 3.1.
- the terms which have $(1-\mu)$ have $\lambda = \kappa(1-\mu)$ in them and are easily integrated using the pointwise bound on $\frac{\phi_u}{r_u}$

With the pointwise bound on $|T\phi|$ we repeat the conservation of energy argument (87), now modified to the almost conservation (159): For any $(u,v) \in \mathcal{R}_{\mathcal{H}}$,

$$\begin{aligned} \|T\phi\|^2_{H^1_{AdS,deg}(u_{\mathcal{H}},v_0)} &\geq \int_{u_0}^{u} \left(2\pi \frac{(\partial_u (T\phi))^2}{\gamma} - \frac{4\pi a}{l^2} (T\phi)^2 r_u\right) r^2 (\bar{u}, v_0)\, d\bar{u} \\ = \int_{u_{\mathcal{I}}}^{u} \left(\mathbf{1}_{\{r \geq r_Y\}} + \mathbf{1}_{\{r \leq r_Y\}}\right) &\left(2\pi \frac{(\partial_u (T\phi))^2}{\gamma} - \frac{4\pi a}{l^2} (T\phi)^2 r_u\right) r^2 (\bar{u}, v)\, d\bar{u} + \\ \int_{v_0}^{v} \left(\mathbf{1}_{r \geq r_Y} + \mathbf{1}_{r \leq r_Y}\right) &\left(2\pi \frac{(\partial_v (T\phi))^2}{\kappa} + \frac{4\pi a}{l^2} (T\phi)^2 r_v\right) r^2 (u, \bar{v})\, d\bar{v} + \int_{D(u,v)} \mathfrak{Q} \end{aligned}$$

Using the Hardy inequalities of Lemma 4.5 (which clearly hold for ϕ replaced by any ψ satisfying the same boundary conditions at infinity, hence in particular for $T\phi$), we continue to estimate

$$\|T\phi\|^2_{H^1_{AdS,deg}(u_{\mathcal{H}},v_0)} \geq \int_{D(u,v)} \mathfrak{Q} + \frac{9}{8}\frac{4\pi}{l^2}\int_{u_{\mathcal{I}}}^{u} \mathbf{1}_{r\leq r_Y}|T\phi|^2 r_u r^2\left(\bar{u},v\right)d\bar{u}$$

$$-\frac{9}{8}\frac{4\pi}{l^2}\int_{v_0}^{v} \mathbf{1}_{r\leq r_Y}|T\phi|^2 r_v r^2\left(u,\bar{v}\right)d\bar{v} + \frac{1}{2}\left(a+\frac{9}{8}\right)\|T\phi\|^2_{H^1_{AdS,deg}(u,v)}$$

$$\geq \left(a+\frac{9}{8}\right)\|T\phi\|^2_{H^1_{AdS,deg}(u,v)} + \int_{D(u,v)} \mathfrak{Q}$$

$$-B_{M,l}c^{\frac{1}{3}}\left(\sup_{\mathcal{R}_{\mathcal{H}}\cap\{\bar{v}\leq v\}\cap\{\bar{u}\leq u\}}\|T\phi\|^2_{H^1_{AdS,deg}(\bar{u},\bar{v})} + B_{M,l,a}\cdot\mathbb{N}^2\left[\phi\right](v_0)\right),$$

where we inserted the pointwise estimate (162) in the last step and exploited the $c^{\frac{1}{3}}$ smallness of the r-difference is the region $r \leq r_Y$. Taking the sup over all $(\bar{u},\bar{v}) \in D(u,v)$ of this estimate we arrive at the estimate of Lemma 6.4 for the $H^1_{AdS,deg}$-norm on the left hand side. However, in view of (162) and the remarks of Section 4.8, the estimate also holds for the H^1_{AdS}-norm. □

Corollary 6.5. *For any $(u,v) \in \mathcal{R}_{\mathcal{H}}$ we have the estimate*

$$\left|r^{\frac{3}{2}}\frac{r\partial_u\left(T\phi\right)}{r_u}\right| \leq B_{M,l,a}\cdot\mathbb{N}\left[\phi\right](v_0) + B_{M,l}\left(\int_{D(u,v)}|\mathfrak{Q}|\ \ \Omega^2 r^2 du dv\right)^{\frac{1}{2}}. \quad (164)$$

Proof. This is immediate in $r \leq r_X$ from (162) and the Lemma 6.4. For $r \geq r_X$ one repeats the proof of Lemma 4.10. □

Since $T\phi$ satisfies the wave equation with an inhomogeneous error-term on the right hand side, we can prove the same integrated decay estimate for $T\phi$ that we proved for ϕ, corrected only by the error-term arising from commutation:

Lemma 6.6. *For any $(u,v) \in \mathcal{R}_{\mathcal{H}}$ we have the estimate*

$$\|T\phi\|^2_{H^1_{AdS}(u,v)} + \mathbb{I}\left[T\phi\right](D(u,v)) \leq B_{M,l,a}\cdot\mathbb{N}^2\left[\phi\right](v_0) + B_{M,l}\int_{D(u,v)}\mathfrak{P}\ \Omega^2 r^2 du dv \quad (165)$$

with

$$\mathfrak{P} = \left(\left|\frac{r}{r_u}\partial_u\left(T\phi\right)\right| + \frac{1}{r\kappa}\left|\partial_v\left(T\phi\right)\right| + \frac{1-\mu}{r^2}\left|T\phi\right|\right)\left|\mathfrak{q}\left[T\phi\right]\right| + |\mathfrak{Q}|. \quad (166)$$

Proof. We are proving an estimate for the same wave equation as in Proposition 5.1, except that there is the inhomogeneity $q\left[T\phi\right]$ on the right hand side. Looking at formula (118), we see that this inhomogeneous term enters the vectorfield estimates as the spacetime error-term

$$\int_{D(u,v)}\left(X\left[T\phi\right] + \mathfrak{f}\ T\phi\right)\mathfrak{q}\left[T\phi\right]. \quad (167)$$

Checking carefully which pairs of multipliers $(X, \mathfrak{f})$ were used to derive the integrated decay estimate, we see that the first term in (166) accounts for these terms. The boundary terms in the integrated decay estimate are again controlled by the $\|T\phi\|_{H^1_{AdS}(u,v)}$-norm. Inserting the estimate (160) for these will produce the last term in (166). □

Lemma 6.7. *Let* $|\frac{\phi_v}{\kappa}| < 1$ *hold in* $\tilde{D}(u,v)$*. Then*

$$\int_{\tilde{D}(u,v)} \mathfrak{P} \, \Omega^2 r^2 du dv \leq B_{M,l,a} \cdot \epsilon \cdot \left(\mathbb{I}[T\phi] \left(\tilde{D}(u,v) \right) + \mathbb{I}[\phi] \left(\tilde{D}(u,v) \right) \right), \quad (168)$$

with the ϵ*-factor arising from the smallness of the* $\mathbb{N}[\phi](v_0)$ *norm.*

Proof. Inspecting the terms in (157), this is an easy application of Cauchy's inequality after using the pointwise bounds for ϕ_v from the assumption and the smallness bounds we already established for $\frac{\phi_u}{r_u}$ and (149). □

We can finally derive the second estimate (57) of Proposition 3.3 by a bootstrap on the size of $|\frac{\phi_v}{\kappa}| < 1$. Recall the region $\widehat{\mathcal{B}}(\tilde{u})$ from (66). Let

$$u_{max} = \sup_u \left(\left| \frac{\phi_v}{\kappa} \right| < 1 \text{ holds in } \widehat{\mathcal{B}}(u) \right). \quad (169)$$

and $\mathcal{B} = \widehat{\mathcal{B}}(u_{\max})$. By the local well-posedness, $\mathcal{B}$ is non-empty and by continuity of the pointwise norm, the region $\mathcal{B}$ is open. We show that $\mathcal{B}$ is also closed, which implies $\mathcal{B} = \mathcal{R}_{\mathcal{H}}$. Combining Lemma 6.6 and Lemma 6.7 with Proposition 5.1, we obtain that for $(u,v) \in \mathcal{B}$ we have

$$\left| r^{\frac{5}{2}} \frac{\partial_u (T\phi)}{r_u} \right|^2 + \|T\phi\|^2_{H^1_{AdS}(u,v)} + \mathbb{I}[T\phi](D(u,v)) \leq B_{M,l,a} \cdot \mathbb{N}^2[\phi](v_0). \quad (170)$$

Integrating $T\phi = 0 + \int du \, \partial_u (T\phi) \, du$ from infinity (where $T\phi$ vanishes) yields in view of the previous estimate,

$$|T\phi| \leq B_{M,l,a} \cdot \mathbb{N}[\phi](v_0) \cdot r^{-\frac{3}{2}}. \quad (171)$$

Finally, from the relation $\phi_v = \kappa(T\phi) + \kappa(1-\mu) \frac{\phi_u}{r_u}$ and the pointwise bounds already established for the right hand side we obtain in particular $\frac{\phi_v}{\kappa} < \frac{1}{2}$ in $\mathcal{B}$. The bootstrap closes, hence (170) holds in all of $\mathcal{R}_{\mathcal{H}}$, which implies both the estimate (57) of Proposition 3.3 and (54) of Proposition 3.2.

6.4 Improved r-weighted bounds for ϕ and first derivatives

It remains to establish (55) of Proposition 3.2:

Lemma 6.8. *We have, in* $\mathcal{R}_{\mathcal{H}} \cap \{r \geq r_X\}$ *the bounds*

$$|\phi(u,v)| \leq C_\delta \cdot B_{M,l,a} \cdot \mathbb{N}[\phi](v_0) \cdot r^{max\left(2p-3, -\frac{5}{2}+\delta\right)} \quad (172)$$

$$\left|r^2 \frac{\phi_u}{r_u}(u,v)\right| + |\phi_v(u,v)| \leq C_\delta \cdot B_{M,l,a} \cdot \mathbb{N}[\phi](v_0) \cdot r^{max\left(2p-4,-\frac{3}{2}+\delta\right)} \quad (173)$$

for any $\delta > 0$.

Proof. On the initial data and on $r = r_X$ these bounds hold by assumption and Proposition 3.1 respectively. Let now

$$p = \frac{3}{4} - \sqrt{\frac{9}{16} - \frac{-a}{2}}\,. \quad (174)$$

One derives the following evolution equation for

$$A = r^n \frac{\zeta}{\nu} + 2pr^n\phi\ : \quad (175)$$

$$\partial_v A = A\left[\frac{\lambda}{r}(n+2p-1) - \rho\right] + f$$
$$f = (2p-1)\, r^n \kappa T(\phi) + 2\phi\kappa r^{n+1} p\left[\frac{1}{r^2}(1-2p) + \frac{4\varpi p}{r^3} - 8\pi\frac{a}{l^2}\phi^2\right] \quad (176)$$

where we recall the redshift weight ρ from (92). This computation exploits an important cancellation: The zeroth order term in f decays better (in r) than naively expected while we have already shown improved decay for $T\phi$ by our commutation argument. Note in this context that the conformally coupled case, $p = \frac{1}{2}$ is special.[14] Noting that

$$\frac{\lambda}{r}(n+2p-1) - \rho = \frac{\kappa}{r}\left(\frac{n-3+2p}{l^2}r^2 + (n+2p-1) + \text{terms decaying in } r\right)$$

we choose $n = \min\left(3-2p, \frac{5}{2}-\delta\right)$. Note that for $n = 3-2p$ we have (using that $\kappa \leq 8d^{-\frac{1}{3}}\frac{l^2\lambda}{r^2}$ in $r \geq r_X$)

$$\int_{v_0}^{v} \mathbf{1}_{r\geq r_X}\left(\frac{\lambda}{r}(n+2p-1) - \rho\right)(u,\bar{v})\, d\bar{v} \leq B_{M,l} \quad (177)$$

uniformly, while for $n = \frac{5}{2} - \delta$ we obtain an exponential decay factor in (176). Either way, integrating (176), one easily obtains the following estimate for A in all of $r \geq r_X$:

$$|A(u,v)| \leq B_{M,l} \cdot \int_{v_0}^{v} \mathbf{1}_{r\geq r_X}|f|(u,\bar{v})\, d\bar{v}\,. \quad (178)$$

To estimate this, we exploit the pointwise bounds available for both ϕ and $T\phi$. For instance, from (171),

$$\int_{v_0}^{v} \mathbf{1}_{r\geq r_X} r^n |\kappa T\phi|(u,\bar{v})\, d\bar{v} \leq B_{M,l,a} \cdot \mathbb{N}[\phi](v_0)\int_{v_0}^{v} \mathbf{1}_{r\geq r_X} r^{n-\frac{3}{2}}\frac{r_v}{r^2}(u,\bar{v})\, d\bar{v}$$
$$\leq B_{M,l,a} \cdot \mathbb{N}[\phi](v_0)\, C_\delta\,.$$

[14] In particular, commutation by T is not necessary to obtain the improved estimates of the Proposition since the $T\phi$-term drops out of (176).

The ϕ-term can be estimated in the same way. What we have shown so far is

$$\left| r^{\min\left(3-2p,\frac{5}{2}-\delta\right)}\left(\frac{\zeta}{\nu}+2p\phi\right)\right| \leq B_{M,l,a}\cdot\mathbb{N}\left[\phi\right]\left(v_0\right)C_\delta\,, \tag{179}$$

which in view of $2p < \frac{3}{2}$ is an improvement over previous estimates of the two summands individually.

With the improved decay for the quantity above one can re-estimate ϕ from infinity by integrating $r^{2p}\phi$. The latter quantity vanishes at infinity, since $2p < \frac{3}{2}$, which is the decay we already established for ϕ.

$$\begin{aligned} r^{2p}\phi\left(u,v\right) &= \int_{u_{\mathcal{I}(v)}}^{u}\frac{\partial_u\left(r^{2p}\phi\right)}{\nu}\nu\left(\bar{u},v\right)d\bar{u} \\ &\leq B_{M,l,a}\cdot\mathbb{N}\left[\phi\right]\left(v_0\right)\int_{u_{\mathcal{I}(v)}}^{u} r^{\max\left(4p-4,2p-\frac{7}{2}+\delta\right)}\left(-\nu\right)\left(\bar{u},v\right)d\bar{u} \\ &\leq C_\delta\cdot B_{M,l,a}\cdot\mathbb{N}\left[\phi\right]\left(v_0\right)r^{\max\left(4p-3,2p-\frac{5}{2}+\delta\right)} \end{aligned} \tag{180}$$

and hence

$$\left|\phi\right|\left(u,v\right)\leq C_\delta\cdot B_{M,l,a}\cdot\mathbb{N}\left[\phi\right]\left(v_0\right)\cdot r^{max\left(2p-3,-\frac{5}{2}+\delta\right)}\,, \tag{181}$$

which is the first estimate of the Lemma. The second immediately follows by combining the first with (179). For the bound on ϕ_v note that $\phi_v = \frac{\Omega^2}{-r_u}\left(T\phi\right) - r_v\frac{\phi_u}{r_u}$ and use the previous bounds. □

Remark 6.9. *For $3-2p > \frac{5}{2}$, one can actually improve the decay further by another commutation with T (which will improve the pointwise decay for $T\phi$ to what we have just shown for ϕ) to establish the heuristically expected r^{3-2p} decay for ϕ. Since the gain is not needed, we do not concern ourselves with optimizing the result in that direction.*

A Absence of stationary solutions in the linear case

We present here an elementary argument to establish the non-existence of stationary solutions for the wave equation on Schwarzschild-AdS backgrounds satisfying the boundary conditions of [28, 31]. For spherically-symmetric solutions this is of course implied by Corollary 3.9. The simple computation is given here because it motivated the analysis of the present paper and the subsequent [32].

Assume that there was a stationary solution ψ of (3) on a fixed AdS-Schwarzschild background,

$$g = -F\left(r\right)dt^2 + F\left(r\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2 d\varphi^2\right) \quad , \quad F\left(r\right) = 1-\frac{2M}{r}+\frac{r^2}{l^2}\,.$$

In view of $\partial_t \psi = 0$, ψ must satisfy

$$\frac{1}{r^2}\partial_m \left(r^2 g^{mn} \partial_n \psi\right) + \frac{2a}{l^2}\psi = 0 \qquad \text{with } m,n = \{r,\theta,\varphi\} \,. \tag{182}$$

Multiplying this equation by $r^2\psi$ and integrating over a constant t-slice with $drd\theta d\varphi$ we obtain after integrating by parts,

$$\int dr\ d\theta\ d\varphi\ r^2 \left[F\left(r\right)\left(\partial_r \psi\right)^2 + r^2 g^{AB}\partial_A\psi\partial_B\psi - \frac{2a}{l^2}\psi^2\right] = 0\,. \tag{183}$$

Note that the boundary-terms vanish both at infinity (in view of the boundary conditions of [28]) and at the horizon (since $g^{rr} = F\left(r\right) = 0$ there) in this computation. By the Hardy inequalities proven in [28] this implies that $\psi = 0$, as the zeroth order term can be absorbed by the derivative term for $-a > \frac{9}{8}$. Hence there are no non-trivial stationary solutions for the wave equation on Schwarzschild satisfying the boundary conditions.

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