



Synchronization by small time delays

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ABSTRACT

Synchronization is a phenomenon observed in all of the living and in much of the non-living world, for example in the heart beat, Huygens' clocks, the flashing of fireflies and the clapping of audiences. Depending on the number of degrees of freedom involved, different mathematical approaches have been used to describe it, most prominently integrate-and-fire oscillators and the Kuramoto model of coupled oscillators. In the present work, we study a very simple and general system of *smoothly evolving oscillators, which continue to interact even in the synchronized state*. We find that under very general circumstances, *synchronization generically occurs in the presence of a (small) time delay*. Strikingly, the synchronization time is *inversely proportional* to the time delay.

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1. Introduction

The emergence of coherent structures in time and space through synchronization occurs across the entire breadth of science: vibrating atoms, firing neurons, flashing fireflies, clapping audiences, etc. and has therefore been studied intensively from a mathematical viewpoint [1–3].

Synchronization is often analyzed in models which explicitly favor phase synchronization, e.g. in the seminal Kuramoto model [4,2] and in diffusively coupled models (see e.g. Refs. [5,6]). In these schemes the net-interaction between oscillators indeed vanishes in the synchronized state.

However, in many cases, such as fireflies [7–9], cardiac cells, neuronal system and applauding audiences [3,10] the interaction between oscillators consists in the exchange of brief pulses, which persist even when the system fully synchronizes. Since Mirollo and Strogatz's influential 1990 paper [11] such systems are often described by a set of non-analytically evolving integrate-and-fire oscillators. Each oscillator is described by a load variable, which is taken to have a concave dependence on a monotonously increasing phase. When the load reaches a certain threshold, relaxation occurs instantaneously and a pulse is sent to all connected oscillators. Receiving oscillators jump discontinuously forward by a given amount. For such systems Mirollo and Strogatz showed [11] that full synchronization always occurs. Later Ernst, Pawelzik and Geisel [12] demonstrated that for excitatory-only couplings, synchronization depends on phase lag, whereas the presence of inhibitory couplings leads to full in-phase synchronization.

The treatment of pulse oscillators in terms of non-analytic integrate-and-fire oscillators is more a tradition than a necessity. In the present paper we assume that each oscillator is represented by a phase $\theta_i(t)$ whose time, t , derivative is always equal to a constant rate plus a sum of *smooth* but narrow pulses emitted by surrounding oscillators coupled with strength J . Synchronization (asymptotically vanishing phase difference) always occurs for this system if pulses arrive with a non-zero time lag δt for a very wide class of adjacencies, including the mean-field setting often considered in the literature.

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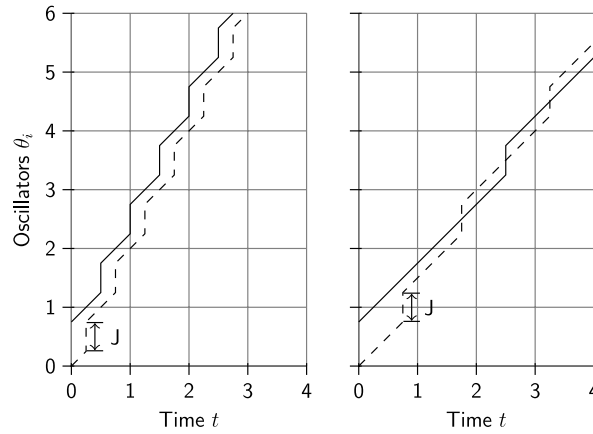


Fig. 1. The time evolution of two oscillators (solid and dashed lines) exchanging pulses according to Eq. (1). Left panel: $\delta t = 0$, right panel: $\delta t = 0.5$.

Given the previous results for pulse oscillators [11,12] and to illuminate the more detailed discussion below it is natural to begin our analysis of two pulse exchanging phases by considering Dirac's delta pulses for the interaction.

$$\begin{aligned}\dot{\theta}_1(t) &= \omega + J \sum_{n \in \mathbb{Z}} \delta(\theta_2(t - \delta t) - n) \\ \dot{\theta}_2(t) &= \omega + J \sum_{n \in \mathbb{Z}} \delta(\theta_1(t - \delta t) - n).\end{aligned}\quad (1)$$

Integrating the time derivatives tells us that θ_1 “jumps” each time θ_2 passes through an integer value, $\theta_1(t) \mapsto \theta_1(t) + J$, and vice versa for θ_2 . Let $\theta_1(0) > \theta_2(0)$, it is straight forward to see, Fig. 1, that the two phases are unable to synchronize though in the case of a finite time delay they may leapfrog each other, as the jump of one oscillator can make the other skip a jump forward.

Obviously Dirac delta pulses are unrealistic. Pulses emitted by real systems will have a finite width and a smooth time dependence (Eq. (2)). The introduction of smooth pulses changes the behavior in an essential way. As will be explained below synchronization now takes place whenever a time lag is present, $\delta t > 0$, and in this case complete synchronization occurs for all smooth pulses.

General model—We now consider n coupled oscillators, each described by a single degree of freedom θ_i , with $i = 1, 2, \dots, n$, and each with the same eigenfrequency ω :

$$\dot{\theta}_i(t) = \omega + \sum_j J_{ij} \sigma(\theta_j(t - \delta t)). \quad (2)$$

Oscillators are coupled through an adjacency matrix J_{ij} and a feedback function $\sigma(\theta)$ which has period ξ . It is only through $\sigma(\theta)$ that periodicity is implemented: $\sigma(\theta_i)$ describes the effect that the state of θ_i (say, the flashing of a firefly) has on any other oscillator. As opposed to other models often studied in synchronization, such as the Kuramoto model [2], the effect of σ does not disappear in the synchronized state.

We chose $\dot{\theta}_i > 0$ at all times such that $\theta_i(t)$ are monotonically increasing functions in time. This is achieved by choosing $\sigma(\theta) > 0$. In our numerical study below we use a comb of normalized Gaussians with period ξ and width w , $\sigma(\theta) = \sum_{n=-\infty}^{\infty} \exp(-(x + n\xi)^2 / (4w^2)) (4\pi w^2)^{-1/2} = \xi^{-1} \vartheta_3(\pi x / \xi, \exp(-4w^2 \pi^2 / \xi^2))$, the Jacobi theta function.

In natural systems time delay is inevitable. We show that $\delta t > 0$ is crucial for synchronization. We use this term in a strong sense: for any pair i, j of oscillators $\lim_{t \rightarrow \infty} \theta_i(t) - \theta_j(t) = \mu_{ij} \xi$ with $\mu_{ij} \in \mathbb{Z}$, i.e. the phase difference between any two oscillators converges to an integer multiple of the period of σ , which implies $\lim_{t \rightarrow \infty} \dot{\theta}_i - \dot{\theta}_j = 0$. By inspection it is clear that for the synchronized state to exist indefinitely, $\sum_j J_{ij} = \tilde{J}$ needs to be independent of i , which means that if synchronization takes place, the difference between any $\theta_i(t)$ and the solution $\tilde{\theta}(t)$ of

$$\dot{\tilde{\theta}}(t) = \omega + \tilde{J} \sigma(\tilde{\theta}(t - \delta t)) \quad (3)$$

with appropriate initial conditions vanishes asymptotically. Provided $\tilde{J} \neq 0$, the eigenfrequency ω can be absorbed into σ , using $\sigma(\theta) \rightarrow \sigma(\theta) + \omega/\tilde{J}$.

Simple two oscillator case—We now demonstrate that under very general conditions the system in Eq. (2) will synchronize in the long time limit. First we consider the simple case of two oscillators, i.e. $n = 2$ and $J_{ij} = 1 - \delta_{ij}$. By considering $\dot{\theta}_1/\dot{\theta}_2$, it is easy to show that $\theta_1(t) - \theta_2(t)$ is periodic if $\delta t = 0$, i.e. synchronization in the strong sense above does not occur without time delay, rather, entrainment is inevitable. However, integrating the equation of motion Eq. (2) numerically on the basis of a simple Euler method suggests differently. Better numerical schemes, such as the Runge-Kutta [13] method, remove

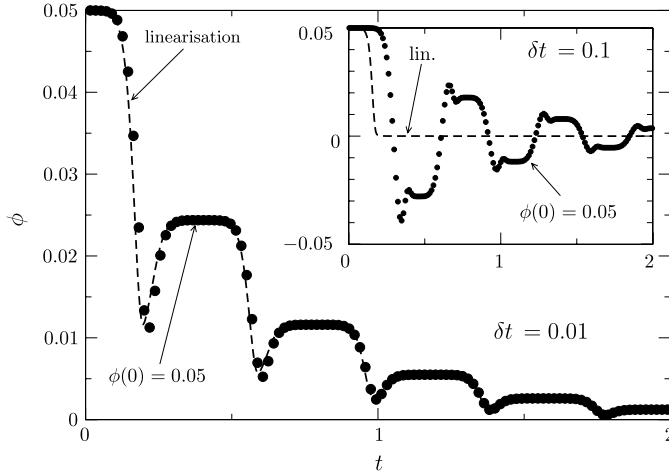


Fig. 2. Comparison of $\phi(t) = (\theta_1(t) - \theta_2(t))/2$ from a numerical integration of Eq. (2) (filled circles) and the linear approximation (dashed line) Eq. (6), with $\phi(t) = 0.05$ and $\delta t = 0.01$. Parameters are $n = 2$, $J_{ij} = 1 - \delta_{ij}$, $\omega = 2$, $\xi = 1.01$ and $w = 0.1$, so that $\psi \approx 0.3934$. The inset compares full solution and linear approximation for $\phi(0) = 0.05$ and $\delta t = 0.1$.

the spurious synchronization, which depends on the integration time step and therefore hints at the rôle of the time delay effectively implemented by the forward derivative used in the most naïve Euler scheme.

We now analyze in detail the effect of a time delay by considering Eq. (2) with $\delta t > 0$. A linear stability analysis for small δt and small deviations $\theta_i(t) - \bar{\theta}(t)$ reveals that any positive δt leads to a synchronized state. We present the calculation briefly in the following for $n = 2$ and $J_{ij} = 1 - \delta_{ij}$.

The equations of motion of $\phi(t) = (1/2)(\theta_1(t) - \theta_2(t))$ and $\bar{\theta} = (1/2)(\theta_1(t) + \theta_2(t))$ are

$$\dot{\phi}(t) = \frac{1}{2}(\sigma(\theta_2(t - \delta t)) - \sigma(\theta_1(t - \delta t))) \quad (4)$$

$$\dot{\bar{\theta}}(t) = \frac{1}{2}(\sigma(\theta_2(t - \delta t)) + \sigma(\theta_1(t - \delta t))) + \omega \quad (5)$$

which we study to first order in ϕ and δt and find

$$\phi(t) = \phi(0) \frac{\dot{\bar{\theta}}(0)}{\dot{\bar{\theta}}(t)} \exp\left(-2\delta t \bar{J}^2 \int_{\bar{\theta}(0)}^{\bar{\theta}(t)} d\theta \frac{\sigma'^2(\theta)}{\omega + \bar{J}\sigma(\theta)}\right) \quad (6)$$

with $t(\bar{\theta}) = \int_{\bar{\theta}(0)}^{\bar{\theta}} d\theta (\omega + \sigma(\theta))^{-1}$. As the integrand is strictly positive, synchronization takes place in this approximation for all $\delta t > 0$. Fig. 2 shows that the linearized solution is a very good approximation of the full system Eq. (2).

The characteristic time to synchronization is estimated in the following way. Define

$$\psi = \int_0^\xi d\bar{\theta} \frac{1}{\omega + \bar{J}\sigma(\bar{\theta})} \approx \int_0^\xi \frac{d\bar{\theta}}{\dot{\bar{\theta}}} \quad (7)$$

$$S = \int_0^\xi d\bar{\theta} \frac{\sigma'^2(\bar{\theta})}{\omega + \bar{J}\sigma(\bar{\theta})} \quad (8)$$

where ψ , to leading order, is the time for $\sigma(\bar{\theta}(t))$ to go through one period, i.e. $\bar{\theta}(t + \psi) \approx \bar{\theta}(t) + \xi$. S corresponds to the integral in the exponent of Eq. (6) for $\bar{\theta}(t) = \bar{\theta}(0) + \xi$. As the integrand is periodic we estimate

$$\int_{\bar{\theta}(0)}^{\bar{\theta}(t)} d\theta \frac{\sigma'^2(\theta)}{\omega + \bar{J}\sigma(\theta)} \approx \frac{t}{\psi} S \quad (9)$$

and rewrite Eq. (6) $\phi(t) = \phi(0) \frac{\dot{\bar{\theta}}(0)}{\dot{\bar{\theta}}(t)} e^{-t/\tau}$, with the characteristic synchronization time

$$\tau \simeq \frac{\psi}{2\bar{J}^2 \delta t S}, \quad (10)$$

inversely proportional to the time delay δt .

Network of oscillators—The above picture can be extended to arbitrary coupling matrices J_{ij} , or a (weighted) network adjacency matrix. The only constraint is $\sum_j J_{ij} = \bar{J}$ independent from i , similar to a stochastic matrix. Motivated by the observation that the two parameters used above, $\bar{\theta}$ and ϕ , are based on the eigenvectors of the matrix $J_{ij} = 1 - \delta_{ij}$ studied for

$n = 2$, we consider the time evolution of $\langle i|\phi(t)\rangle$, i.e. of “normal modes”, where $\langle i|$ is the i th left eigenvector of J , which, we assume for simplicity, has n linearly independent eigenvectors. For simplicity, we normalize $\langle i|j\rangle = \delta_{ij}$. The matrix J is not necessarily symmetric so generally $(\langle i|)^\dagger \neq |i\rangle$. Due to the stochastic property, there is a pair of left and right eigenvectors with eigenvalue \tilde{J} , which in the following is denoted by $\langle 1|$ and $|1\rangle = \sum_i^n |e_i\rangle$ respectively, where $|e_i\rangle$ denotes the canonical basis of the \mathbb{R}^n .

The state of the entire system is written in vector form as $|\theta(t)\rangle = \sum_i^n |e_i\rangle \theta_i(t)$. The column vector $|\phi(t)\rangle$ is the deviation $|\phi(t)\rangle = |\theta(t)\rangle - \bar{\theta}(t)|1\rangle$ of $|\theta(t)\rangle$ from $\bar{\theta}(t) = \langle 1|\theta(t)\rangle$ anticipating that $\bar{\theta}(t)$ represents the asymptotically synchronized state. Following the procedure above, one finds

$$\langle i|\phi(t)\rangle = A_i \left(\frac{T(\bar{\theta}(t))}{T_0} \right)^{\frac{\lambda_i}{\tilde{J}}} \times \exp \left(\lambda_i (\tilde{J} - \lambda_i) \delta t \int_{\bar{\theta}(0)}^{\bar{\theta}(t)} \frac{\sigma'^2(\theta')}{T(\theta')} d\theta' \right) \quad (11)$$

where $T_0 = T(\bar{\theta}(0))$ and $T(\bar{\theta}) = \omega + \tilde{J} \left(\sigma(\bar{\theta}) - \delta t \dot{\bar{\theta}}(\bar{\theta}) \sigma'(\bar{\theta}) \right) = \dot{\bar{\theta}}(\bar{\theta}) + \mathcal{O}(\delta t^2)$.

The amplitudes A_i are determined by the initial projections $\langle i|\phi(0)\rangle = A_i$. Eq. (11) also applies to $i = 1$, yet $\langle 1|\phi(t)\rangle = 0$ by construction so that $A_1 = 0$. The special case $\tilde{J} = 0$ (so that $\dot{\bar{\theta}} = \omega$ to linear order) coincides with the limit $\tilde{J} \rightarrow 0$, where

$$\lim_{\tilde{J} \rightarrow 0} \left(\frac{T(\bar{\theta}(t))}{T_0} \right)^{\frac{\lambda_i}{\tilde{J}}} = e^{\lambda_i \sigma(\bar{\theta}) / \omega - \delta t \lambda_i \sigma'(\bar{\theta})} \quad (12)$$

to leading order, assuming $T_0 = \omega$ for simplicity.

Since $T(\bar{\theta})$ is periodic, the long-term behavior of $\langle i|\phi(t)\rangle$ depends crucially on the sign of the real part of $\lambda_i(\tilde{J} - \lambda_i)$. If it is negative, the projection has an approximate synchronization time

$$\tau_i = \frac{\int_0^\xi d\bar{\theta} \frac{1}{\omega + \tilde{J} \sigma(\bar{\theta})}}{-\Re \left(\lambda_i (\tilde{J} - \lambda_i) \right) \delta t \int_0^\xi d\bar{\theta} \frac{\sigma'^2(\bar{\theta})}{\omega + \tilde{J} \sigma(\bar{\theta})}}, \quad (13)$$

corresponding to Eq. (10). Here $\Re(\cdot)$ denotes the real part. The usual mean-field setup $J_{ij} = a(1 - \delta_{ij})$ has one eigenvalue $\tilde{J} = (n-1)a$ and $n-1$ eigenvalues $\lambda_i = -a$, so that $\lambda_i(\tilde{J} - \lambda_i) = -na^2$ has a negative real part provided a^2 has a positive one, i.e. in particular for all real a . The mean-field theory thus always synchronizes. The same applies generally to the lattice Laplacian, which has $\tilde{J} = 0$ so that $\lambda_i(\tilde{J} - \lambda_i) = -\lambda_i^2$. For example, the Laplacian of the complete graph (all-to-all), $J_{ij} = 1/(N-1)$ for $i \neq j$ and $J_{ii} = -1$ has one eigenvalue 0 and $N-1$ eigenvalues $\lambda_i = -N/(N-1)$, so that $\lambda_i(\tilde{J} - \lambda_i) = -(N/(N-1))^2$ is real and negative. Large classes of adjacency matrices have been analyzed for their spectrum [14,15] and many of them, in particular many binary, symmetric ones, display synchronization in all modes.

As an example consider a random Gilbert graph [16] with N nodes and probability p for an edge connecting any two nodes. We now consider what happens as we change the number of edges in the graph (simultaneously adjusting the weights of the edges so the required condition $\sum_{ij} J_{ij} = \tilde{J}$ remains fulfilled, see discussion just before Eq. (3)). The entire graph will synchronize in the fully connected regime for values of p above the percolation threshold $p_c = 1/N$. When p is lowered below p_c the graph starts to fall apart. All the nodes belonging to one connected sub-cluster will still synchronize but the different clusters will not synchronize to the same frequency. This is because the synchronization mechanism we consider involves that interaction between oscillators persists in the synchronized state. And the synchronized state of a sub-cluster will depend on the size of the cluster through the spectral properties (i.e. \tilde{J} and λ_i in Eq. (13) of the cluster). So as the graph disintegrates below the percolation transition synchronization remains only within each sub-cluster. If one starts with a non-synchronized set of nodes with a connectivity $p < p_c$, synchronization will happen (asymptotically in time) within subgraphs as they are formed when p approaches p_c from below. The different sub-graphs will again synchronize to different states and only as p passes through the percolation threshold p_c will the entire graph become synchronized in the long time limit.

The perturbative result Eq. (11) can be compared to the numerical integration of the system. We used a fourth order Runge–Kutta integration scheme [13] and show in Fig. 3 that the derived synchronization time compares very well (for time delays up to 5%–10% of the synchronized period) to that of the linearized result, Eq. (6) and to the estimate equation (10).

Mechanism—How is synchronization achieved? Fig. 4 shows $\sigma(\theta_i(t))$ and $\dot{\theta}_i(t)$ as a function of t for $n = 2$. Synchronization occurs because θ_2 experiences a greater increase in speed by $\sigma(\theta_1(t - \delta t))$ than θ_1 does by $\sigma(\theta_2(t - \delta t))$. This asymmetry comes about because θ_2 is relatively fast itself when $\sigma(\theta_2(t))$ goes through its maximum and θ_1 is relatively slow when $\sigma(\theta_1(t))$ goes through its maximum. As a result the maximum $\sigma(\theta_1(t))$ is broadened as a function of time, and $\sigma(\theta_2(t))$ is narrowed (this effect is minute and thus not visible in Fig. 4). Therefore, the maximum of $\sigma(\theta_1(t - \delta))$ enters into θ_2 for a longer time period than $\sigma(\theta_2(t - \delta))$ enters into θ_1 , leading to a speedup of θ_2 relative to θ_1 . In summary, synchronization is a result of oscillator i being slow or fast when going through the maximum of the function $\sigma(\theta_i)$. What rôle has the time delay in this? The time delay ensures that the trailing oscillator θ_2 receives a boost at a time when $\sigma(\theta_2(t))$ goes through a maximum, while the leading θ_1 receives its boost at a time when $\sigma(\theta_1(t))$ goes through a local minimum. Without the time delay, the effect of the speed-up and the slowdown would indeed be perfectly symmetric.

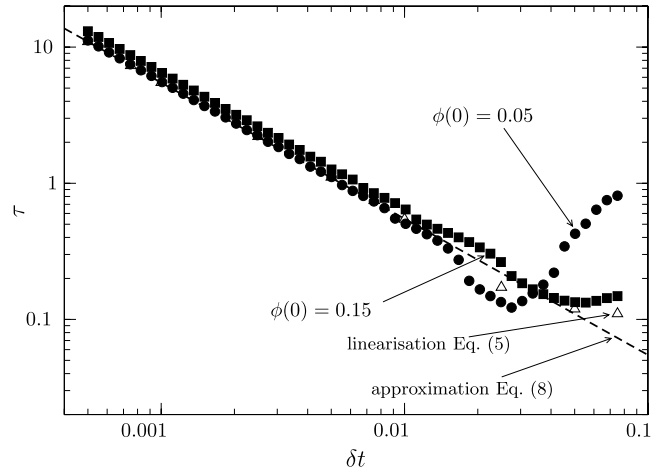


Fig. 3. Plot of the synchronization time estimated from the window averaged phase difference $\theta_1(t) - \theta_2(t)$ (average taken over a time period t^* so that $\hat{\theta}(t) = \hat{\theta}(t - t^*) - \xi$). The filled symbols refer to results based on Eq. (2), the empty triangles to Eq. (6) and the line to Eq. (10). Parameters as in Fig. 2.

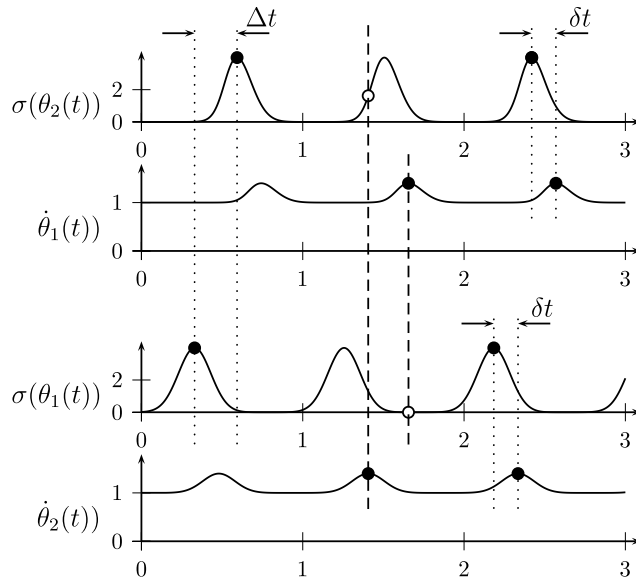


Fig. 4. We see that θ_1 is ahead of θ_2 , with the maximum of $\sigma(\theta_2(t))$ displaced by Δt (as indicated) to the right relative to that of $\sigma(\theta_1(t))$, as initialized. At a given time, θ_2 still has to pass through the maximum of $\sigma(\theta_2(t))$ when θ_1 already has. The phase speed $\dot{\theta}_{1,2}(t)$ is essentially $\sigma(\theta_{2,1}(t))$ shifted by δt to the right, as indicated by the dotted lines. As a result, the maximum of θ_2 nearly aligns with the maximum of $\sigma(\theta_2(t))$, i.e. θ_2 is fast when $\sigma(\theta_2(t))$ goes through the maximum (dashed line), thereby narrowing it. In turn, θ_1 passes very quickly through a low value of $\sigma(\theta_1(t))$, relatively broadening in turn the maximum of $\sigma(\theta_1(t))$.

We notice that the mechanism underlying the synchronization supported by Eq. (2) is a kind of Doppler effect that makes the received pulse change its duration when the sending oscillator changes its speed.

Eq. (2) provides a remarkably simple mechanism for synchronization. Because oscillators lagging behind by a certain amount catch up in every period of $\sigma(\bar{\theta})$ by an amount of phase difference proportional to the phase difference at the beginning of the period, the model can immediately be extended to one with different eigenfrequencies ω_i of oscillators or some variation in $\sum_j J_{ij}$ with i or of σ and even δt .

An analysis of the effect of inhomogeneity in the eigenfrequencies and the time delays follows the derivation given above. Along the lines of Eq. (11), each mode is attracted to the origin (the synchronized state) by a spring force $-1/\tau_i$ and driven away from it by the various “mismatches”, such as a deviation of ω_i from the mean value or $\sum_j J_{ij}$ from $\sum_{ij} J_{ij}/N$. Many of these setups lead to a balancing of the forces, i.e. entrainment, whose amplitude is inversely related to the time delay.

We illustrate the robustness of the synchronization mechanism by considering pairs of oscillators and study the effect of random time delays. In Fig. 5 we show the behavior for a specific choice of different delay times. We notice that the behavior is very similar to the one shown in Fig. 2. The inset shows the asymptotic behavior averaged over a time widow given by the periodicity of the σ function in Eq. (2).

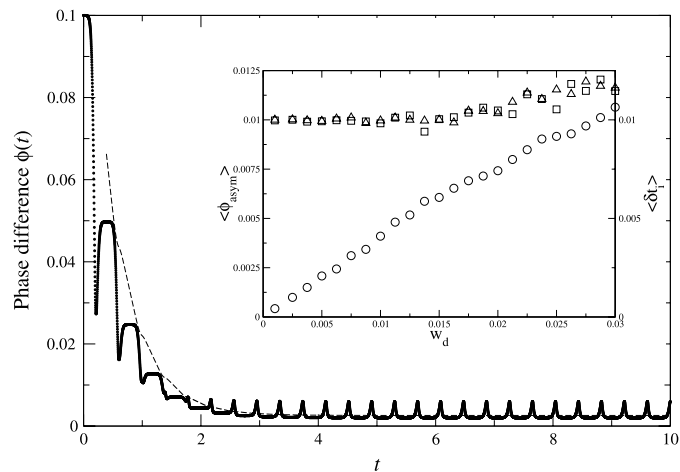


Fig. 5. Filled dark circles show the phase difference between $\theta_1(t)$ and $\theta_2(t)$ (with $\delta t_1 = 0.011$ and with $\delta t_2 = 0.009$). Parameters used: are $\xi = 1.01$, $w = 0.10$, $\omega = 2.00$ and $\phi(0) = 0.10$. The dashed line shows the period averaged phase difference. Inset: Left y-axis: Circles show $\langle \phi_{\text{asym}} \rangle$, which is the average of the asymptotic phase difference (averaged over one time period) of a sample of 500 realizations as a function of the standard deviation w_d for the Gaussian distribution of the time delays. Right y-axis: Since only positive time delays were considered, the distribution of the delay times is not strictly Gaussian and the average value of the realized time delays, δt_i , is accordingly weakly dependent on w_d as seen from the open squares (refer to δt_1) and triangles (δt_2).

Synchronization by time delay is a viable explanation for natural synchronization phenomena whenever oscillators respond to the duration of the received pulse. Fireflies are known to be able to change the pulse duration and female fireflies are sensitive to that Ref. [8]. The exact way a clapping audience reaches synchrony [3,10] can be analyzed sufficiently accurately to establish whether people change the duration of the individual clap [17] in the process of reaching synchrony. If this is the case, synchronization of clapping may occur due to a mechanism similar to the one described in the present paper.

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