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Contractivity of transport distances for the kinetic Kuramoto equation

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Abstract We present synchronization and contractivity estimates for the kinetic Kuramoto model obtained from the Kuramoto phase model in the mean-field limit. For identical Kuramoto oscillators, we present an admissible class of initial data leading to time-asymptotic complete synchronization, that is, all measure valued solutions converge to the traveling Dirac measure concentrated on the initial averaged phase. In the case of non-identical oscillators, we show that the velocity field converges to the average natural frequency proving that the oscillators move asymptotically with the same frequency under suitable assumptions on the initial configuration. If two initial Radon measures have the same natural frequency density function and strength of coupling, we show that the Wasserstein p -

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distance between corresponding measure valued solutions is exponentially decreasing in time. This contraction principle is more general than previous L^1 -contraction properties of the Kuramoto phase model.

Keywords Kuramoto model · complete synchronization · Wasserstein distance · contraction

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1 Introduction

The objective of this paper is to present a contraction property of the kinetic Kuramoto equation in the transport distances. The synchronization phenomena exhibited by various biological systems are ubiquitous in nature, e.g., the flashing of fireflies, chorusing of crickets, synchronous firing of cardiac pacemakers, and metabolic synchrony in yeast cell suspensions (see for instance [1, 5]). Winfree and Kuramoto [20, 31] pioneered the mathematical treatment of these synchronized phenomena. They introduced phase models for large weakly coupled oscillator systems, and showed that the synchronized behavior of complex biological systems can emerge from the competing mechanisms of intrinsic randomness and sinusoidal couplings. The kinetic Kuramoto equation has been widely used in the literature [1] to analyze the phase transition from a completely disordered state to a partially ordered state as the coupling strength increases from zero. Suppose that $g = g(\Omega)$ is an integrable steady probability density function for natural frequencies with a compact support (see (2.1) for details). Let $f = f(\theta, \Omega, t)$ be the probability density function of Kuramoto oscillators in $\theta \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ with a natural frequency Ω at time t as in [21]. The kinetic Kuramoto equation (KKE) is given as follows:

$$\begin{aligned} \partial_t f + \partial_\theta(\omega[f]f) &= 0, & (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}, \quad t > 0, \\ \omega[f](\theta, \Omega, t) &= \Omega - K \int_{\mathbb{T}} \sin(\theta - \theta_*) \rho(\theta_*, t) d\theta_*, & \rho(\theta_*, t) := \int_{\mathbb{R}} f d\Omega_*, \end{aligned} \quad (1.1)$$

subject to the initial data:

$$f(\theta, \Omega, 0) = f_0(\theta, \Omega), \quad \int_{\mathbb{T}} f_0 d\theta = g(\Omega). \quad (1.2)$$

Note that KKE (1.1) can be regarded as a scalar conservation law with a nonlocal flux, and it has been derived from a previously proposed Kuramoto model [9, 21]. However, to the best of authors' knowledge, few studies have investigated the qualitative properties of the KKE, such as an asymptotic behavior and stability of some equilibria.

The main results of this paper can be summarized as follows. First, we present sufficient conditions for the emergence of completely synchronized states. More precisely, when many coupled limit-cycle oscillators have the same natural frequency (identical oscillators), and the support of the initial

Radon measure in phase is confined in a half circle, we show that complete phase synchronization occurs asymptotically. This means that the measure-valued solution approaches asymptotically to a multiple of a Dirac delta concentrated on the initial average phase value and the common natural frequency value. In the case of non-identical oscillators, we obtain that the support of the velocity field associated to the measure-valued solution decreases exponentially fast to one point, when the diameter of the support of the initial Radon measure in phase is less than π and in natural frequency is bounded, and the coupling strength K is large enough. In fact, the velocity field converges to zero showing that the oscillators move asymptotically with the same frequency. For this purpose, we lift the finite-dimensional result for the Kuramoto model (KM) to the infinite-dimensional KKE. Second, we present a contraction property of the KKE in the Wasserstein p -distance for measure valued solutions with the same natural frequency distribution by using a strategy similar to the one described in [8, 22]. We define a cumulative distribution function of a density function f for the KKE, say F , and we derive a new integro-differential equation using its pseudo-inverse function. Then, we use simple techniques for the optimal mass transport in one-dimension, i.e., the equivalence relation between the Wasserstein p -distance and the L^p -distance of the corresponding pseudo-inverse of F in order to obtain the exponential decay estimate of the Wasserstein p -distance between two measure-valued solutions.

The rest of this paper is organized as follows. In Section 2, we briefly review the Kuramoto model and its mean-field version (the KKE), and we provide several a priori estimates. In Section 3, we revisit an existence theory of measure valued solutions to the KKE, and we present several a priori estimates, in particular, we provide a finite-time stability estimate for measure-valued solutions in a bounded Lipschitz distance. In Section 4, we show the emergence of completely synchronized states by lifting the corresponding results for the KM to the KKE using the argument of the particle-in-cell method. This strategy has been employed in the Cucker-Smale flocking model in [7]. Section 5 is devoted to the contraction property of the KKE using the method of optimal mass transport as described in [8, 22].

2 Preliminaries

In this section, we briefly review the particle Kuramoto model and its kinetic mean-field model. Consider an ensemble of sinusoidally coupled nonlinear oscillators that can be visualized as active rotors on the circle \mathbb{S}^1 . Throughout the paper, we will identify a rotor with an oscillator. Let $x_i = e^{\sqrt{-1}\theta_i}$ be the position of the i -th rotor. Then, the dynamics of x_i is completely determined by that of phase θ_i . In the absence of coupling, the phase equation for θ_i is simply given by the decoupled ODE system:

$$\frac{d\theta_i}{dt} = \Omega_i, \quad \text{i.e., } \theta_i(t) = \theta_i(0) + \Omega_i t,$$

where Ω_i is the natural phase-velocity (frequency) and is assumed to be a random variable extracted from the density function $g = g(\Omega)$:

$$\begin{aligned} g(-\Omega) &= g(\Omega), \quad \text{supp}(g) \text{ is bounded,} \\ \int_{\mathbb{R}} \Omega g(\Omega) d\Omega &= 0, \quad \int_{\mathbb{R}} g(\Omega) d\Omega = 1. \end{aligned} \quad (2.1)$$

In the seminal work [20] of Kuramoto, he derived a coupled phase model heuristically from the complex Ginzburg-Landau system. The KM is given by

$$\frac{d\theta_i}{dt} = \Omega_i - \frac{K}{N} \sum_{j=1}^N \sin(\theta_i - \theta_j), \quad t > 0, \quad i = 1, \dots, N, \quad (2.2)$$

subject to initial data:

$$\theta_i(0) = \theta_{i0}. \quad (2.3)$$

Note that the first term on the R.H.S. of (2.2) represents the intrinsic randomness, whereas the second term denotes the nonlinear attractive coupling. Hence, synchronized states for system (2.2) will emerge, when the nonlinear coupling dominates the intrinsic randomness.

Remark 2.1 The assumptions (2.1) are very crucial for our analysis after this section. For example, in the hypothesis of Lemma 5.1, the coupling strength K should be at least bigger than the diameter of natural frequencies. For this, we need to assume that g has a compact support, which guarantee $K < \infty$.

The system (2.2) has been extensively studied over the last three decades, and it remains a popular subject in nonlinear dynamics and statistical physics (see review articles and a book [1, 3, 13, 18, 27, 29]). In [19, 20], Kuramoto first observed that in the mean-field limit ($N \rightarrow \infty$), the system (2.2) with a unimodal distribution function $g(\Omega)$ (which is assumed to be one-humped and symmetric with respect to mean frequency $\Omega_c^p := \frac{1}{N} \sum_{i=1}^N \Omega_i$) has a continuous dynamical phase transition at a critical value of the coupling strength K_{cr} :

$$K_{cr} = \frac{2}{\pi g(0)}, \quad \text{in the mean-field limit.}$$

Moreover, he introduced an asymptotic order parameter $r^\infty \in [0, 1]$ to measure the degree of the phase synchronization in mean-field limit:

$$r^\infty(K) := \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{i=1}^N e^{\sqrt{-1}\theta_i(t)} \right|,$$

and he observed that this quantity r^∞ changes from zero to a non-zero value, when the coupling strength K exceeds a critical value K_{cr} . Note that for an initial phase configuration that is uniformly distributed on the

interval $[0, 2\pi)$, the quantity r^∞ is exactly zero, whereas for a completely synchronized configuration $\theta_i = \theta_c$ for $i = 1, \dots, N$, r^∞ becomes the unity. Therefore we can regard r^∞ as the “order parameter” measuring the degree of synchronization. Before we conclude this subsection, we recall a complete phase synchronization result from [14]. For this purpose, we introduce the diameters of the phase and frequency configurations $\theta = (\theta_1, \dots, \theta_N)$, $w = (w_1, \dots, w_N)$, and the natural frequency set as follows.

$$D_\theta(t) := \max_{1 \leq i, j \leq N} |\theta_i(t) - \theta_j(t)|, \quad D_w(t) := \max_{1 \leq i, j \leq N} |w_i(t) - w_j(t)|,$$

and

$$D_\Omega := \max_{1 \leq i, j \leq N} |\Omega_i - \Omega_j|,$$

where $w_i(t) := \dot{\theta}_i(t)$.

Proposition 2.1 [14] *Suppose that the natural frequencies, the coupling strength and initial configuration satisfy*

$$D_\Omega = 0, \quad K > 0, \quad D_0 := D_\theta(0) < \pi,$$

and let $\theta = \theta(t)$ be the smooth solution to the system (2.2)-(2.3) with initial phase configuration θ_0 . Then we have

$$e^{-Kt} D_0 \leq D_\theta(t) \leq e^{-K\alpha t} D_0, \quad t \geq 0, \quad (2.4)$$

where α is the positive constant only depending on the diameter of the initial phase configuration given by

$$\alpha := \frac{\sin D_0}{D_0}.$$

We also recall the estimate of existence of a trapping region for non-identical oscillators from [10] as follows.

Proposition 2.2 [10] *Let $\theta = \theta(t)$ be the global smooth solution to (2.2)-(2.3) satisfying*

$$0 < D_0 < \pi, \quad D_\Omega > 0, \quad K > K_e := \frac{D_\Omega}{\sin D_0}. \quad (2.5)$$

Then we obtain

- (i) $\sup_{t \geq 0} D_\theta(t) \leq D_0 < \pi$.
- (ii) $\exists t_0 > 0$ such that $\sup_{t \geq t_0} D_\theta(t) \leq D^\infty$,

where D^∞ is defined by

$$D^\infty := \arcsin \left[\frac{D_\Omega}{K} \right] \in \left(0, \frac{\pi}{2} \right).$$

Furthermore we have

$$D_w(t_0) e^{-K(t-t_0)} \leq D_w(t) \leq D_w(t_0) e^{-K(\cos D^\infty)(t-t_0)}, \quad t \geq t_0.$$

Remark 2.2 1. Let $\Omega_{i_1}, \dots, \Omega_{i_\ell}$ be the distinct natural frequencies, and $\{\mathcal{P}_1, \dots, \mathcal{P}_\ell\}$ be the partition of the set $\{1, \dots, N\}$ such that $\mathcal{P}_k := \{j : \Omega_j = \Omega_{i_k}\}$. Then it follows from [10] that if the initial configurations satisfy (2.5), we have

$$\lim_{t \rightarrow \infty} \theta_i(t) = \lim_{t \rightarrow \infty} \theta_j(t) \quad \text{for all } i, j \in \mathcal{P}_k, \quad k \in \{1, \dots, \ell\}.$$

2. If we set the average phase and natural frequency of the particles as

$$\theta_c^p(t) := \frac{1}{N} \sum_{i=1}^N \theta_i(t), \quad \Omega_c^p := \frac{1}{N} \sum_{i=1}^N \Omega_i,$$

then from the particle KM (2.2), one can easily obtain

$$\theta_c^p(t) = \theta_c^p(0) + \Omega_c^p t, \quad \text{for all } t \geq 0.$$

Without loss of generality, we may assume that $\Omega_c^p = 0$ using the phase-shift framework. Then we notice that identical and non-identical oscillators satisfying the assumptions in Propositions 2.1 and 2.2 satisfy

$$\begin{cases} \theta_i(t) \rightarrow \theta_c(0) & \text{as } t \rightarrow \infty, \text{ for identical oscillators,} \\ \theta_i(t) \in (\theta_c^p(0) - D^\infty, \theta_c^p(0) + D^\infty) & \text{for } t \geq t_0, \text{ for non-identical oscillators.} \end{cases}$$

Note that the conditions and decay estimates (2.4) are independent of the particle-number N . For the related synchronization estimates for the KM, we refer to [10–12, 15, 17, 23, 24].

We rewrite the system (2.2) as a dynamical system on the extended phase space $\mathbb{T} \times \mathbb{R}$ for (θ_i, Ω_i) : For $i = 1, \dots, N$,

$$\frac{d\theta_i}{dt} = \Omega_i - \frac{K}{N} \sum_{j=1}^N \sin(\theta_i - \theta_j), \quad \frac{d\Omega_i}{dt} = 0, \quad t > 0. \quad (2.6)$$

Throughout this paper, we will use the interval $[0, 2\pi)$ to denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, i.e., $\theta \in [0, 2\pi)$ implies that θ satisfies $\theta + 2\pi\mathbb{Z} = \theta$.

3 Existence theory of measure valued solutions

In this section, we briefly review the existence of measure valued solutions to (1.1). For the KM, the rigorous mean-field limit was first done by Lancelotti [21] using Neunzert's general theory for the Vlasov equation [25, 28]. Optimal transport arguments allow to generalize these results in several ways for granular and flocking models [2, 6]. H. Chiba recently obtained the same mean-field limit based on functional tools [9]. For a later use and reader's convenience, we present several estimates for measure valued solutions to the KKE.

3.1 Measure-theoretic framework

In this subsection, we discuss a measure-theoretic formulation of the KKE.

Let $\mathcal{M}([0, 2\pi) \times \mathbb{R})$ be the set of nonnegative Radon measures on $[0, 2\pi) \times \mathbb{R}$, which can be regarded as nonnegative bounded linear functionals on $\mathcal{C}_0([0, 2\pi) \times \mathbb{R})$. For a Radon measure $\nu \in \mathcal{M}([0, 2\pi) \times \mathbb{R})$, we use the standard duality relation:

$$\langle \nu, h \rangle := \int_0^{2\pi} \int_{\mathbb{R}} h(\theta, \Omega) \nu(d\theta, d\Omega), \quad h \in \mathcal{C}_0([0, 2\pi) \times \mathbb{R}),$$

where \mathcal{C}_0 denotes the set of continuous functions vanishing at infinity. Note that since $\theta \in [0, 2\pi)$ is a 2π -periodic variable, $h(\theta, \Omega)$ is a 2π -periodic function with respect to θ on $[0, 2\pi) \times \mathbb{R}$. We set $\mathcal{C}_w([0, T]; \mathcal{M}([0, 2\pi) \times \mathbb{R}))$ a space of all weakly continuous time-dependent measures. Then definition of a measure-valued solution to equation (1.1) is given as follows.

Definition 1 For $T \in [0, \infty)$, let $\mu \in \mathcal{C}_w([0, T]; \mathcal{M}([0, 2\pi) \times \mathbb{R}))$ be a measure valued solution to (1.1) with an initial Radon measure $\mu_0 \in \mathcal{M}([0, 2\pi) \times \mathbb{R})$ if and only if μ satisfies the following conditions:

1. μ is weakly continuous:

$$\langle \mu_t, h \rangle \text{ is continuous as a function of } t, \quad \forall h \in \mathcal{C}_0([0, 2\pi) \times \mathbb{R}).$$

2. μ satisfies the integral equation: $\forall h \in \mathcal{C}_0^1([0, 2\pi) \times \mathbb{R} \times [0, T])$,

$$\langle \mu_t, h(\cdot, \cdot, t) \rangle - \langle \mu_0, h(\cdot, \cdot, 0) \rangle = \int_0^t \langle \mu_s, \partial_s h + \omega[\mu] \partial_\theta h \rangle ds, \quad (3.1)$$

where $\omega[\mu](\theta, \Omega, s)$ is defined by

$$\omega[\mu](\theta, \Omega, s) := \Omega - K(\mu_s * \sin)\theta. \quad (3.2)$$

Here $*$ denotes the standard convolution, i.e.,

$$(\mu_s * \sin)\theta = \int_0^{2\pi} \int_{\mathbb{R}} \sin(\theta - \theta_*) \mu_s(d\theta_*, d\Omega).$$

Remark 3.1 (i) Let $f = f(\theta, \Omega, t)$ be a classical solution to (1.1). Then the measure $\mu := f d\Omega d\theta$ is a measure valued solution to (1.1).

(ii) Note that the empirical measure

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{\theta_i(t)} \otimes \delta_{\Omega_i(t)}, \quad \text{where } (\theta_i(t), \Omega_i(t)) \text{ is a solution of (2.6),}$$

is a measure valued solution to (1.1) in the sense of Definition 3.1. Thus, the solutions to the KM (2.2) can be treated as measure valued solutions via

an empirical measure. Here, δ_{z_*} is the Dirac measure concentrated at $z = z_*$.

(iii) Since the density function $g(\Omega)$ has a compact support and the dynamics (1.1) only governs the θ -variable, we can see that the projected Ω -support of μ_t also has a compact support as well. Under the assumption of the compact support of g , we can expand the class of test functions to $\mathcal{C}([0, 2\pi) \times \mathbb{R})$.

(iv) By choosing $h = \Omega$ in (3.1) (see above comment in (iii)), we have

$$\langle \mu_t, \Omega \rangle = \langle \mu_0, \Omega \rangle, \quad t > 0.$$

(v) As we mentioned before, $\theta \in [0, 2\pi)$ is regarded as a 2π -periodic variable, i.e., $\theta + 2\pi\mathbb{Z} = \theta$. Thus we can consider $h = \theta$ a test function.

Lemma 3.1 *Suppose that the density function $g = g(\Omega)$ has a compact support and the initial measure satisfies*

$$\langle \mu_0, \Omega \rangle = 0,$$

and let $\mu \in \mathcal{C}_w([0, T]; \mathcal{M}([0, 2\pi) \times \mathbb{R}))$ be a measure valued solution to (1.1). Then for $t \geq 0$, we have

$$\langle \mu_t, 1 \rangle = \langle \mu_0, 1 \rangle = 1, \quad \langle \mu_t, \theta \rangle = \langle \mu_0, \theta \rangle, \quad t \geq 0.$$

Proof We first notice from Remark 3.1 that the class of test functions can be expanded to $\mathcal{C}([0, 2\pi) \times \mathbb{R})$. This yields that we can use $h = 1$ in (3.1). Then, the R.H.S. of (3.1) will be zero hence, we have conservation of total mass. For the time evolution of the first moment of θ , it follows from Remark 3.1 (iv) that

$$\langle \mu_t, \Omega \rangle = 0, \quad t > 0.$$

We now set $h(\theta) = \theta$ in (3.1) and use (3.2) to get

$$\begin{aligned} \langle \mu_t, \theta \rangle &= \langle \mu_0, \theta \rangle + \int_0^t \langle \mu_s, \omega[\mu_s] \rangle ds \\ &= \langle \mu_0, \theta \rangle + \int_0^t \left(\langle \mu_s, \Omega \rangle - K \langle \mu_s, (\mu_s * \sin)\theta \rangle \right) ds \\ &= \langle \mu_0, \theta \rangle - K \int_0^t \langle \mu_s, (\mu_s * \sin)\theta \rangle ds = \langle \mu_0, \theta \rangle, \end{aligned}$$

where we used the anti-symmetry of $\sin(\theta - \theta_*)$ to determine that $\langle \mu_s, (\mu_s * \sin)\theta \rangle = 0$.

3.2 A priori local stability estimate and approximate solutions

In this part, we recall the stability estimate for measure valued solutions to (1.1) in the bounded Lipschitz distance, and present that the measure valued solutions to the KKE can be approximated as a sum of Dirac measures.

The stability estimate is crucial to the global existence of a measure valued function for the KKE. First, we review the definition of the bounded Lipschitz distance presented in [16, 25, 28]. We define the admissible set \mathcal{S} of test functions as

$$\mathcal{S} := \left\{ h : [0, 2\pi) \times \mathbb{R} \rightarrow \mathbb{R} : \|h\|_{L^\infty} \leq 1, \quad \text{Lip}(h) \leq 1 \right\},$$

where

$$\text{Lip}(h) := \sup_{(\theta_1, \Omega_1) \neq (\theta_2, \Omega_2)} \frac{|h(\theta_1, \Omega_1) - h(\theta_2, \Omega_2)|}{|(\theta_1, \Omega_1) - (\theta_2, \Omega_2)|}.$$

Definition 2 [25, 28] Let $\mu, \nu \in \mathcal{M}([0, 2\pi) \times \mathbb{R})$ be two Radon measures. Then the bounded Lipschitz distance $d(\mu, \nu)$ between μ and ν is given by

$$d(\mu, \nu) := \sup_{h \in \mathcal{S}} |\langle \mu, h \rangle - \langle \nu, h \rangle|.$$

Remark 3.2 The space of Radon measures $\mathcal{M}([0, 2\pi) \times \mathbb{R})$ equipped with the metric $d(\cdot, \cdot)$ is a complete metric space.

Remark 3.3 The bounded Lipschitz distance d for compactly supported probability measures is equivalent to the Kantorovich-Rubinstein distance W_1 (see [28]):

$$W_1(\mu, \nu) := \inf_{\gamma \in \Pi(\mu, \nu)} \int_0^{2\pi} \int_{\mathbb{R}} \int_0^{2\pi} \int_{\mathbb{R}} |(\theta - \theta_*, \Omega - \Omega_*)| \gamma(d(\theta, \Omega), d(\theta_*, \Omega_*)),$$

where $\Pi(\mu, \nu)$ is the set of all product measures on $([0, 2\pi) \times \mathbb{R}) \times ([0, 2\pi) \times \mathbb{R})$ such that their marginals are μ and ν , respectively. Both equivalent distances endow the weak-* convergence of measures with metric structure in bounded sets.

Remark 3.4 For any $h \in \mathcal{C}([0, 2\pi) \times \mathbb{R})$ with $\|h\|_{L^\infty} \leq a$ and $\text{Lip}(h) \leq b$, we have

$$|\langle \mu, h \rangle - \langle \nu, h \rangle| \leq \max\{a, b\}d(\mu, \nu).$$

We next present that a measure valued solution to the KKE can be approximated as a sum of Dirac measures using a Lancellotti's argument [21].

Proposition 3.1 [21] 1. For any $\mu_0 \in \mathcal{M}([0, 2\pi) \times \mathbb{R})$, the equation (1.1) has a unique solution $\mu \in \mathcal{C}_w(0, T; \mathcal{M}([0, 2\pi) \times \mathbb{R}))$ with initial data μ_0 .
2. Let $\mu_t, \nu_t \in \mathcal{C}_w(0, T; \mathcal{M}([0, 2\pi) \times \mathbb{R}))$ be two solutions to the equation (1.1) with initial data $\mu_0, \nu_0 \in \mathcal{M}([0, 2\pi) \times \mathbb{R})$, respectively. Then there exists a constant $C > 0$ depending on T such that

$$d(\mu_t, \nu_t) \leq Cd(\mu_0, \nu_0).$$

We note that Proposition 3.1 can be extended using a simple approximation argument of measures by smooth positive densities as follows.

Theorem 3.1 *For any $\mu_0 \in \mathcal{M}([0, 2\pi) \times \mathbb{R})$, let μ_t be a unique measure valued solution to KKE (1.1) with initial data μ_0 . Then μ_t can be approximated as a sum of Dirac measures of the form:*

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{\theta_i(t)} \otimes \delta_{\Omega_i(t)}.$$

Furthermore, there holds

$$d(\mu_t, \mu_t^N) \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Proof Although this proof can be easily obtained from Proposition 3.1 by a density argument, we give the details in Appendix A for reader sake.

From now on, let us assume that the initial measure is a smooth absolutely continuous measure with respect to Lebesgue with connected support. These assumptions can be eliminated by standard mollifier approximation as above. Therefore, we will proceed by working on smooth solutions and obtaining estimates depending only on quantities that pass to the limit in the weak-* sense, and thus, stable estimates under this approximation. To avoid too much repetition, this procedure will not be specified in the proofs below and the statements of the results will be written directly for measure valued solutions.

4 Asymptotic complete synchronization estimate

In this section, we present an asymptotic synchronization estimate for the KKE (1.1) by lifting corresponding results for the KM (2.2) using the argument of the particle-in-cell method [26] discussed in the previous section.

Let $\mu \in \mathcal{C}_w([0, T]; \mathcal{M}([0, 2\pi) \times \mathbb{R}))$ be a measure valued solution to (1.1), and let $R(t)$ and $P(t)$ be the orthogonal θ and Ω -projections of $\text{supp}(\mu_t)$ respectively, i.e.,

$$\begin{aligned} R(t) &:= \mathbb{P}_\theta \text{supp}(\mu_t) = \{ \theta \in [0, 2\pi) : (\theta, \Omega) \in \text{supp}(\mu_t) \}, \\ P(t) &:= \mathbb{P}_\Omega \text{supp}(\mu_t) = \{ \Omega \in \mathbb{R} : (\theta, \Omega) \in \text{supp}(\mu_t) \}. \end{aligned}$$

Then it is easy to see that

$$P(t) = P(0), \quad t \geq 0.$$

We also set

$$\begin{aligned} D_\theta(\mu_t) &:= \text{diam}(R(t)), \quad D_\Omega(\mu_t) := \text{diam}(P(t)), \quad M(t) := \langle \mu_t, 1 \rangle, \\ \theta_c(t) &:= \frac{1}{M(t)} \langle \mu_t, \theta \rangle, \quad \Omega_c(t) := \frac{1}{M(t)} \langle \mu_t, \Omega \rangle, \end{aligned}$$

where $\text{diam}(A) := \sup_{x,y \in A} |x - y|$ for $A \subset \mathbb{R}$. We observe from Lemma 3.1 that

$$M(t) = \langle \mu_t, 1 \rangle = \langle \mu_0, 1 \rangle = M(0) = 1,$$

and since $\Omega_c(0) = 0$, we obtain

$$\Omega_c(t) = 0 \quad \text{and} \quad \theta_c(t) = \theta_c(0), \quad t \geq 0.$$

Lemma 4.1 *Suppose that the oscillators are identical, i.e., $g(\Omega) = \delta_{\Omega_c(0)}(\Omega)$, and let μ_0 be a given initial measure in $\mathcal{M}([0, 2\pi) \times \mathbb{R})$ satisfying*

$$\langle \mu_0, \theta \rangle = \pi, \quad D_\theta(\mu_0) < \pi, \quad K > 0.$$

Then the measure valued solution μ to (1.1) - (1.2) with an initial datum μ_0 satisfies

$$D_\theta(\mu_t) \leq D_\theta(\mu_0)e^{-K\bar{\alpha}t}, \quad t \geq 0,$$

where

$$\bar{\alpha} = \frac{\sin D_\theta(\mu_0)}{D_\theta(\mu_0)}.$$

Proof We use the approximation argument in Theorem 3.1 giving the initial particle approximation μ_0^N defined as in (A.1). Then, it follows from Proposition 2.1 that the approximate measure valued solution $\mu_t^N \in \mathcal{M}([0, 2\pi) \times \mathbb{R})$ satisfies

$$D_\theta(\mu_t^N) \leq D_\theta(\mu_0^N)e^{-K\bar{\alpha}_N t}, \quad t \geq 0,$$

where

$$\bar{\alpha}_N = \frac{\sin D_\theta(\mu_0^N)}{D_\theta(\mu_0^N)}.$$

Since Theorem 3.1 implies that $d(\mu_t, \mu_t^N) \rightarrow 0$ as $N \rightarrow \infty$. Hence, $D_\theta(\mu_t^N) \rightarrow D_\theta(\mu_t)$ as $N \rightarrow \infty$ and we obtain the desired result.

Remark 4.1 Throughout the paper, without loss of generality, we assume that $\langle \mu_0, \theta \rangle = \pi$ in order to avoid any possible confusion arising from the periodicity of θ . In fact, if the oscillators satisfy the assumption in Lemma 4.1 (or Lemma 5.1), the orthogonal θ -projection of $\text{supp}(\mu_t)$, $R(t)$ is confined to the interval $(0, 2\pi)$ for all $t \geq 0$ (see Remark 2.2). This property will also be significantly used in Section 5 (see Lemma 5.2).

We now show that the measure valued solution to the system (1.1) for identical oscillators will converge to a multiple of the Dirac measure concentrated on $(\theta_c(0), \Omega_c(0))$ in the phase space (θ, Ω) . We set

$$\mu_\infty(d\theta, d\Omega) := \delta_{\theta_c(0)}(\theta) \otimes \delta_{\Omega_c(0)}(\Omega).$$

Theorem 4.1 *Suppose that the oscillators are identical, i.e., $g(\Omega) = \delta_{\Omega_c(0)}(\Omega)$, and let $\mu_0 \in \mathcal{M}([0, 2\pi) \times \mathbb{R})$ be a given initial probability measure satisfying*

$$\theta_c(0) = \pi, \quad D_\theta(\mu_0) < \pi \quad \text{and} \quad K > 0.$$

Then the measure valued solution μ_t to (1.1) with initial datum μ_0 satisfies

$$\lim_{t \rightarrow \infty} d(\mu_t, \mu_\infty) = 0,$$

where $d = d(\cdot, \cdot)$ is the bounded Lipschitz distance defined in Section 3.

Proof Since the oscillators are identical ($g(\Omega) = \delta_{\Omega_c(0)}(\Omega)$), it is enough to consider a test function of one variable θ for the convergence estimate of the measure value solution. Let $h = h(\theta) \in \mathcal{C}([0, 2\pi])$ be any test function satisfying

$$\|h\|_{L^\infty} \leq 1, \quad \text{Lip}(h) \leq 1.$$

Note that h can also be regarded as a test function in $\mathcal{C}([0, 2\pi] \times \mathbb{R})$. Then we have

$$\begin{aligned} & \left| \int_{[0, 2\pi] \times \mathbb{R}} h(\theta) \mu_t(d\theta, d\Omega) - \int_{[0, 2\pi] \times \mathbb{R}} h(\theta) \mu_\infty(d\theta, d\Omega) \right| \\ &= \left| \int_0^{2\pi} h(\theta) \bar{\mu}_t(d\theta) - h(\pi) \right| \leq \int_0^{2\pi} |\theta - \pi| \bar{\mu}_t(d\theta) \leq D_\theta(\mu_0) e^{-K\bar{\alpha}t}. \end{aligned}$$

where we used Lemma 4.1, and $\bar{\mu}_t(d\theta)$ is a θ -marginal of the measure μ_t , i.e.,

$$\bar{\mu}_t(d\theta) := \int_{\mathbb{R}} \mu_t(d\theta, d\Omega).$$

This implies that

$$d(\mu_t, \mu_\infty) \leq D_\theta(\mu_0) e^{-K\bar{\alpha}t} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

5 Stability estimate of the KKE

In this section, we present the strict contractivity of measure valued solutions to the KKE by using the method of optimal mass transport [8, 22, 30]. The strict contractivity result generalizes the L^1 -contraction result for the KM in [10].

5.1 Alternative formulation of the KKE

In this part, we derive an alternative form of the KKE, which is more convenient for deriving estimates in terms of the Wasserstein distance. First, we study the existence of an invariant set for the KKE.

Lemma 5.1 *Suppose that the initial probability measure μ_0 and the coupling strength K satisfy*

$$0 < D_\theta(\mu_0) < \pi, \quad 0 < D_\Omega(\mu_0) < \infty, \quad K > \frac{D_\Omega(\mu_0)}{\sin D_\theta(\mu_0)}.$$

Then, there exist $t_0 > 0$ and $D^\infty \in (0, \frac{\pi}{2})$ such that the measure valued solution μ to (1.1) with initial datum μ_0 satisfies

$$D_\theta(\mu_t) \leq D^\infty, \quad t \geq t_0.$$

Proof We apply the argument similar to that in the proof of Lemma 4.1. Let $N > 0$ be given. Then we have the following approximation μ_0^N for μ_0 :

$$\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{\theta_{i0}} \otimes \delta_{\Omega_{i0}}.$$

We now solve the Cauchy problem for KM:

$$\begin{cases} \frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), & t > 0, \\ \frac{d\Omega_i}{dt} = 0. \end{cases}$$

subject to initial data $(\theta_i(0), \Omega_i(0)) = (\theta_{i0}, \Omega_{i0})$. Theorem 3.1 implies that

$$d(\mu_t, \mu_t^N) \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

and thus, $D_\theta(\mu_t^N) \rightarrow D_\theta(\mu_t)$ and $D_\Omega(\mu_t^N) \rightarrow D_\Omega(\mu_t)$ as $N \rightarrow \infty$. Hence we can take N large enough such that $D_\Omega(\mu_0^N)$ and $D_\theta(\mu_0^N)$ satisfies the conditions of Proposition 2.2. Thus, we find that there exist $t_0^N > 0$ and $D^{\infty, N}$ such that

$$D_\theta(\mu_t^N) \leq D^{\infty, N}, \quad t \geq t_0^N, \quad \text{for } N \text{ large enough,}$$

where

$$t_0^N := \frac{D_\theta(\mu_0^N) - D^{\infty, N}}{K \sin D_\theta(\mu_0^N) - D_\Omega(\mu_0^N)}, \quad D^{\infty, N} := \arcsin \left[\frac{D_\Omega(\mu_0^N)}{K} \right] \in \left(0, \frac{\pi}{2} \right).$$

We now let $N \rightarrow \infty$ to obtain the desired result.

In the remainder of this section, from Remark 4.1, we assume that

$$R(t) \subset (0, 2\pi) \quad \text{and} \quad t \geq 0. \quad (5.1)$$

Under this assumption the solution is given by a smooth particle density function $f(\theta, \Omega, t)$ in L^1 for all $t \geq 0$. For a given Ω , we consider a one-particle density function f as a function of θ . Then we define the pseudo cumulative distribution function of f :

$$F(\theta, \Omega, t) := \int_0^\theta f(\theta_*, \Omega, t) d\tilde{\theta}, \quad (\theta, \Omega, t) \in [0, 2\pi) \times \mathbb{R} \times \mathbb{R}_+,$$

and a pseudo-inverse ϕ of $F(\cdot, \Omega, t)$ as a function of θ :

$$\phi(\eta, \Omega, t) := \inf\{\theta : F(\theta, \Omega, t) > \eta\}, \quad \eta \in [0, g(\Omega)].$$

As long as there is no confusion, we use the notation $F^{-1}(\eta, \Omega, t) = \phi$ as the pseudo inverse of F as θ -function. Then it is easy to see that

$$F(\phi(\eta, \Omega, t), \Omega, t) = \eta. \quad (5.2)$$

Lemma 5.2 *Let μ be a measure-valued solution to (1.1)-(1.2), and let ϕ be the pseudo-inverse function of the cumulative distribution function F . Then we have*

$$\begin{aligned} (i) \quad & \max\{\theta \mid \theta \in R(t)\} = \max_{\Omega \in \text{supp}(g)} \phi(g(\Omega), \Omega, t). \\ (ii) \quad & \min\{\theta \mid \theta \in R(t)\} = \min_{\Omega \in \text{supp}(g)} \phi(0, \Omega, t). \\ (iii) \quad & \max_{\Omega \in \text{supp}(g)} \phi(g(\Omega), \Omega, t) - \min_{\Omega \in \text{supp}(g)} \phi(0, \Omega, t) \leq D^\infty, \quad t \geq t_0. \end{aligned}$$

Proof Since the estimate for (ii) is similar to that of (i) and the estimate for (iii) follows from the estimates (i) and (ii), we only provide the proof for the estimate (i). For notational simplicity, we set

$$\theta_M := \max\{\theta \mid \theta \in R(t)\}.$$

Then, by definition of μ_t , we have

$$\theta_M = \max\{\theta \mid \theta \in \text{supp}_\theta(f(\theta, \Omega, t)) \text{ and } \Omega \in \text{supp}(g)\},$$

where $\text{supp}_\theta(f(\theta, \Omega, t))$ is the θ -projection of $\text{supp}(f(\theta, \Omega, t))$. This yields

$$\theta_M = \max\{\phi(g(\Omega), \Omega, t) \text{ such that } \Omega \in \text{supp}(g)\},$$

by definition of the pseudo-inverse function. This completes the proof.

Next, we derive an integro-differential equation for the pseudo inverse ϕ . It follows from (5.1) that the smooth solution $f(\theta, \Omega, t)$ to (1.1)-(1.2) satisfies

$$f(0, \Omega, t) = 0, \quad \Omega \in \mathbb{R}, \quad t \geq 0.$$

We differentiate the relation (5.2) in t and use $\partial_\theta F = f$ to get

$$\partial_t F(\theta, \Omega, t) \Big|_{\theta=\phi(\eta, \Omega, t)} + f(\theta, \Omega, t) \Big|_{\theta=\phi(\eta, \Omega, t)} \partial_t \phi(\eta, \Omega, t) = 0.$$

This yields

$$\begin{aligned} & \partial_t \phi(\eta, \Omega, t) \\ &= -\frac{1}{f(\theta, \Omega, t)} \partial_t F(\theta, \Omega, t) \Big|_{\theta=\phi(\eta, \Omega, t)} \\ &= \frac{1}{f(\theta, \Omega, t)} \Big|_{\theta=\phi(\eta, \Omega, t)} \times (\omega[f]f)(\cdot, \Omega, t) \Big|_{\theta=0}^{\theta=\phi(\eta, \Omega, t)} \\ &= \Omega + K \int_{\mathbb{R}} \int_0^{2\pi} \sin(\theta_* - \phi(\eta, \Omega, t)) f(\theta_*, \Omega_*, t) d\theta_* d\Omega_* \quad \text{using (5.1)} \\ &= \Omega + K \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \sin(\phi(\eta_*, \Omega_*, t) - \phi(\eta, \Omega, t)) d\eta_* d\Omega_*, \end{aligned}$$

where we used $\theta_* = \phi(\eta_*, \Omega_*, t)$ and relation (5.2) to see $f(\theta_*, \Omega_*, t)d\theta_* = d\eta_*$. Hence, the pseudo-inverse ϕ satisfies the following integro-differential equation:

$$\partial_t \phi = \Omega + K \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \sin(\phi_* - \phi) d\eta_* d\Omega_*. \quad (5.3)$$

where we used abbreviated notations:

$$\phi_* := \phi(\eta_*, \Omega_*, t), \quad \phi := \phi(\eta, \Omega, t).$$

The following results is a simple consequence of the change of variables and Lemma 3.1.

Lemma 5.3 *Let μ_t be a measure valued solution to (1.1) - (1.2) with an associated pseudo-inverse function ϕ . Then, we have*

$$\int_{\mathbb{R}} \int_0^{g(\Omega)} \phi d\eta d\Omega = \int_{\mathbb{R}} \int_0^{2\pi} \theta \mu_t(d\theta, d\Omega), \quad \frac{d}{dt} \int_{\mathbb{R}} \int_0^{g(\Omega)} \phi d\eta d\Omega = 0.$$

5.2 Strict contractivity in the Wasserstein distance

In this part, we present the proof of the strict contraction property of the KKE.

For the one-dimensional case, it is well known [8, 30] that the Wasserstein p -distance $W_p(\mu_1, \mu_2)$ between two measures μ_1 and μ_2 is equivalent to the L^p -distance between the corresponding pseudo-inverse functions ϕ_1 and ϕ_2 respectively. Thus, we set

$$W_p(\mu_1, \mu_2)(\Omega, t) := \|\phi_1(\cdot, \Omega, t) - \phi_2(\cdot, \Omega, t)\|_{L^p(0, g(\Omega))}, \quad 1 \leq p \leq \infty.$$

Since $W_p(\mu_1, \mu_2)$ depends on Ω , we introduce a modified metric on the phase-space (θ, Ω) :

$$\widetilde{W}_p(\mu_1, \mu_2)(t) := \|W_p(\mu_1, \mu_2)(\cdot, t)\|_{L^p(\mathbb{R})}, \quad 1 \leq p \leq \infty.$$

Below, we assume that the density function $g(\Omega)$ has compact support. Then, it is easy to see that $\widetilde{W}_p(\mu_1, \mu_2)$ is a metric that satisfies

$$\lim_{p \rightarrow \infty} \widetilde{W}_p(\mu_1, \mu_2)(t) = \widetilde{W}_\infty(\mu_1, \mu_2)(t), \quad t \geq 0. \quad (5.4)$$

Recall that the sgn function is defined by

$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Lemma 5.4 *Let Φ be a measurable function defined on $[0, g(\Omega)] \times \mathbb{R}$ satisfying*

$$|\Phi(\eta, \Omega)| < \frac{\pi}{2} \quad \text{and} \quad \int_{\mathbb{R}} \int_0^{g(\Omega)} \Phi(\eta, \Omega) d\eta d\Omega = 0.$$

Then for $1 \leq p < \infty$, we have

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{g(\Omega)} \int_0^{g(\Omega_*)} \left[|\Phi(\eta, \Omega)|^{p-1} \operatorname{sgn}(\Phi(\eta, \Omega)) - |\Phi(\eta_*, \Omega_*)|^{p-1} \operatorname{sgn}(\Phi(\eta_*, \Omega_*)) \right] \\ & \quad \times \sin\left(\frac{\Phi(\eta_*, \Omega_*) - \Phi(\eta, \Omega)}{2}\right) d\eta_* d\eta d\Omega_* d\Omega \leq -\frac{2}{\pi} \int_{\mathbb{R}} \int_0^{g(\Omega)} |\Phi(\eta)|^p d\eta d\Omega. \end{aligned}$$

Proof For notational simplicity, we set

$$\begin{aligned} \Phi &:= \Phi(\eta, \Omega), \quad \Phi_* := \Phi(\eta_*, \Omega_*), \quad \text{and} \\ \Delta(\eta, \eta_*, \Omega, \Omega_*) &:= \left[|\Phi|^{p-1} \operatorname{sgn}(\Phi) - |\Phi_*|^{p-1} \operatorname{sgn}(\Phi_*) \right] \sin\left(\frac{\Phi_* - \Phi}{2}\right), \end{aligned}$$

and we decompose the domain $[0, g(\Omega)] \times \mathbb{R}$ as the disjoint union of three subsets:

$$\mathcal{P} := \{(\eta, \Omega) \mid \Phi(\eta, \Omega) > 0\}, \quad \mathcal{Z} := \{(\eta, \Omega) \mid \Phi(\eta, \Omega) = 0\},$$

and

$$\mathcal{N} := \{(\eta, \Omega) \mid \Phi(\eta, \Omega) < 0\}.$$

Then it follows from the condition $\int_{\mathbb{R}} \int_0^{g(\Omega)} \Phi d\eta d\Omega = 0$ that

$$\int_{\mathcal{P}} |\Phi| d\eta d\Omega = \int_{\mathcal{N}} |\Phi| d\eta d\Omega. \quad (5.5)$$

We use $[0, g(\Omega)] \times \mathbb{R} = \mathcal{P} \cup \mathcal{Z} \cup \mathcal{N}$ to obtain

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \int_0^{g(\Omega)} \Delta(\eta, \eta_*, \Omega, \Omega_*) d\eta d\eta_* d\Omega d\Omega_* \\ &= \underbrace{\left(\int_{\mathcal{P} \times \mathcal{Z}} + \cdots + \int_{\mathcal{N} \times \mathcal{P}} \right)}_{\text{distinct signs}} + \underbrace{\left(\int_{\mathcal{P} \times \mathcal{P}} + \int_{\mathcal{N} \times \mathcal{N}} + \int_{\mathcal{Z} \times \mathcal{Z}} \right)}_{\text{same signs}} \Delta(\eta, \eta_*, \Omega, \Omega_*) d\eta d\eta_* d\Omega d\Omega_* \end{aligned}$$

We now consider the following sub-integrals separately.

$$I(A, B) := \int_{A \times B} \Delta(\eta, \eta_*, \Omega, \Omega_*) d\eta d\eta_* d\Omega d\Omega_*, \quad A, B \in \{\mathcal{P}, \mathcal{Z}, \mathcal{N}\}.$$

We claim the following:

Case	A	B	$I(A, B) \leq$
I	\mathcal{P}	\mathcal{Z}	$-\frac{\mathcal{L}(\mathcal{Z})}{\pi} \int_{\mathcal{P}} \Phi_* ^p d\eta_* d\Omega_*$
II	\mathcal{N}	\mathcal{Z}	$-\frac{\mathcal{L}(\mathcal{Z})}{\pi} \int_{\mathcal{N}} \Phi_* ^p d\eta_* d\Omega_*$
III	\mathcal{Z}	\mathcal{P}	$-\frac{\mathcal{L}(\mathcal{Z})}{\pi} \int_{\mathcal{P}} \Phi ^p d\eta d\Omega$
IV	\mathcal{Z}	\mathcal{N}	$-\frac{\mathcal{L}(\mathcal{Z})}{\pi} \int_{\mathcal{N}} \Phi ^p d\eta d\Omega$
V	\mathcal{P}	\mathcal{N}	$-\frac{1}{\pi} \left[\mathcal{L}(\mathcal{P}) \int_{\mathcal{N}} \Phi ^p d\eta d\Omega + \mathcal{L}(\mathcal{N}) \int_{\mathcal{P}} \Phi_* ^p d\eta_* d\Omega_* \right. \\ \left. + \int_{\mathcal{N}} \Phi ^{p-1} d\eta d\Omega \int_{\mathcal{P}} \Phi_* d\eta_* d\Omega_* \right. \\ \left. + \int_{\mathcal{P}} \Phi_* ^{p-1} d\eta_* d\Omega_* \int_{\mathcal{N}} \Phi d\eta d\Omega \right]$
VI	\mathcal{N}	\mathcal{P}	$-\frac{1}{\pi} \left[\mathcal{L}(\mathcal{N}) \int_{\mathcal{P}} \Phi ^p d\eta d\Omega + \mathcal{L}(\mathcal{P}) \int_{\mathcal{N}} \Phi_* ^p d\eta_* d\Omega_* \right. \\ \left. + \int_{\mathcal{P}} \Phi ^{p-1} d\eta d\Omega \int_{\mathcal{N}} \Phi_* d\eta_* d\Omega_* \right. \\ \left. + \int_{\mathcal{N}} \Phi_* ^{p-1} d\eta_* d\Omega_* \int_{\mathcal{P}} \Phi d\eta d\Omega \right]$
VII	\mathcal{P}	\mathcal{P}	$-\frac{1}{\pi} \left[\mathcal{L}(\mathcal{P}) \int_{\mathcal{P}} \Phi ^p d\eta d\Omega + \mathcal{L}(\mathcal{P}) \int_{\mathcal{P}} \Phi_* ^p d\eta_* d\Omega_* \right. \\ \left. - \int_{\mathcal{P}} \Phi ^{p-1} d\eta d\Omega \int_{\mathcal{P}} \Phi_* d\eta_* d\Omega_* \right. \\ \left. - \int_{\mathcal{P}} \Phi_* ^{p-1} d\eta_* d\Omega_* \int_{\mathcal{P}} \Phi d\eta d\Omega \right]$
VIII	\mathcal{N}	\mathcal{N}	$-\frac{1}{\pi} \left[\mathcal{L}(\mathcal{N}) \int_{\mathcal{N}} \Phi ^p d\eta d\Omega + \mathcal{L}(\mathcal{N}) \int_{\mathcal{N}} \Phi_* ^p d\eta_* d\Omega_* \right. \\ \left. - \int_{\mathcal{N}} \Phi ^{p-1} d\eta d\Omega \int_{\mathcal{N}} \Phi_* d\eta_* d\Omega_* \right. \\ \left. - \int_{\mathcal{N}} \Phi_* ^{p-1} d\eta_* d\Omega_* \int_{\mathcal{N}} \Phi d\eta d\Omega \right]$
IX	\mathcal{Z}	\mathcal{Z}	0

where $\mathcal{L}(A)$ denotes the Lebesgue measure of the set A :

$$\mathcal{L}(A) := \int_A 1 d\eta d\Omega.$$

We also note that

$$\mathcal{L}(\mathcal{P}) + \mathcal{L}(\mathcal{Z}) + \mathcal{L}(\mathcal{N}) = \int_{\mathbb{R}} \int_0^{g(\Omega)} 1 d\eta d\Omega = \int_{\mathbb{R}} g(\Omega) d\Omega = 1.$$

Case I: In this case, we use the definition of $\Delta(\eta, \eta_*, \Omega, \Omega_*)$ and the inequality

$$\sin x \geq \frac{2}{\pi}x, \quad \text{for } x \in \left[0, \frac{\pi}{2}\right],$$

to determine that

$$\Delta(\eta, \eta_*, \Omega, \Omega_*) = -|\Phi_*|^{p-1} \sin \frac{|\Phi_*|}{2} \leq -\frac{1}{\pi} |\Phi_*|^p.$$

This yields

$$I(\mathcal{P}, \mathcal{Z}) \leq -\frac{1}{\pi} \int_{\mathcal{P} \times \mathcal{Z}} |\Phi_*|^p d\eta d\eta_* d\Omega d\Omega_* = -\frac{\mathcal{L}(\mathcal{Z})}{\pi} \int_{\mathcal{P}} |\Phi_*|^p d\eta_* d\Omega_*.$$

Case II - Case IV: The estimates are basically the same as in Case I. Hence, we omit their estimates.

Case V: In this case, we have

$$\begin{aligned}\Delta(\eta, \eta_*, \Omega, \Omega_*) &= -(|\Phi|^{p-1} + |\Phi_*|^{p-1}) \sin\left(\frac{|\Phi_*| + |\Phi|}{2}\right) \\ &\leq -\frac{1}{\pi} \left(|\Phi|^p + |\Phi_*|^p + |\Phi|^{p-1} |\Phi_*| + |\Phi_*|^{p-1} |\Phi| \right).\end{aligned}$$

This yields the desired result.

Case VI: Once we interchange $\mathcal{P} \longleftrightarrow \mathcal{N}$, the same estimate holds.

Case VII: In this case, we need to consider two subcases:

$$\text{Either } \Phi > \Phi_* > 0 \quad \text{or} \quad \Phi_* \geq \Phi > 0.$$

By considering each case, we have

$$\begin{aligned}\Delta(\eta, \eta_*, \Omega, \Omega_*) &= (|\Phi|^{p-1} - |\Phi_*|^{p-1}) \sin\left(\frac{\Phi_* - \Phi}{2}\right) \\ &\leq \frac{1}{\pi} (|\Phi|^{p-1} - |\Phi_*|^{p-1}) (|\Phi_*| - |\Phi|) \\ &= -\frac{1}{\pi} \left(|\Phi|^p + |\Phi_*|^p - |\Phi|^{p-1} |\Phi_*| - |\Phi_*|^{p-1} |\Phi| \right).\end{aligned}$$

This yields the desired result.

Case VIII: The estimate is exactly the same as in Case VII. Hence we omit its proof.

Case IX: The estimate is trivial.

We now add all cases and use (5.5) to find

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \int_0^{g(\Omega)} \Delta(\eta, \eta_*, \Omega, \Omega_*) d\eta d\eta_* d\Omega d\Omega_* \\ \leq -\frac{2}{\pi} \left(\mathcal{L}(\mathcal{P}) + \mathcal{L}(\mathcal{Z}) + \mathcal{L}(\mathcal{N}) \right) \int_{\mathbb{R}} \int_0^{g(\Omega)} |\Phi|^p d\eta d\Omega \\ = -\frac{2}{\pi} \int_{\mathbb{R}} \int_0^{g(\Omega)} |\Phi|^p d\eta d\Omega.\end{aligned}$$

Theorem 5.1 *Suppose that two initial measures $\mu_0, \nu_0 \in \mathcal{M}([0, 2\pi] \times \mathbb{R})$ and K satisfy*

- (i) $0 < D_\theta(\nu_0) \leq D_\theta(\mu_0) < \pi$, $\int_{[0, 2\pi] \times \mathbb{R}} \theta \mu_0(d\theta, d\Omega) = \int_{[0, 2\pi] \times \mathbb{R}} \theta \nu_0(d\theta, d\Omega) = \pi$.
- (ii) $K > D_\Omega(\mu_0) \max\left\{ \frac{1}{\sin D_\theta(\mu_0)}, \frac{1}{\sin D_\theta(\nu_0)} \right\}$,

and let μ_t and ν_t be two measure valued solutions to (1.1) - (1.2) corresponding to initial data μ_0 and ν_0 , respectively. Then, there exists $t_0 > 0$ such that

$$\widetilde{W}_p(\mu_t, \nu_t) \leq \exp \left[-\frac{2K \cos D^\infty}{\pi} (t - t_0) \right] \widetilde{W}_p(\mu_{t_0}, \nu_{t_0}), \quad t > t_0, \quad 1 \leq p \leq \infty.$$

Proof First, we consider the case where $p \in [1, \infty)$. Note that the Wasserstein distance in one-space dimension is equivalent to the L^p -distance of its corresponding pseudo inverse distribution function. Remember that we are assuming that the solutions are smooth, hence it is more convenient to obtain the L^p -estimate from equation (5.3). Denoting by ϕ_i , $i = 1, 2$ the pseudo inverse functions associated to μ_t and ν_t respectively, we get

$$\partial_t \phi_i = \Omega + K \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \sin(\phi_{i*} - \phi_i) d\eta_* d\Omega_*,$$

for $i = 1, 2$. Then the above equations imply that

$$\begin{aligned} \partial_t(\phi_1 - \phi_2) &= K \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \left(\sin(\phi_{1*} - \phi_1) - \sin(\phi_{2*} - \phi_2) \right) d\eta_* d\Omega_*, \\ &= 2K \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \cos \left(\frac{\phi_{1*} - \phi_1}{2} + \frac{\phi_{2*} - \phi_2}{2} \right) \sin \left(\frac{\phi_{1*} - \phi_1}{2} - \frac{\phi_{2*} - \phi_2}{2} \right) d\eta_* d\Omega_*. \end{aligned} \quad (5.6)$$

We multiply (5.6) by $p \operatorname{sgn}(\phi_1 - \phi_2) |\phi_1 - \phi_2|^{p-1}$ and integrate over $[0, g(\Omega)] \times \mathbb{R}$ using the symmetry $(\eta, \Omega) \iff (\eta_*, \Omega_*)$ to obtain

$$\begin{aligned} &\frac{d}{dt} \|\phi_1 - \phi_2\|_{L^p}^p \\ &= 2pK \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{g(\Omega)} \int_0^{g(\Omega_*)} \left[\cos \left(\frac{\phi_{1*} - \phi_1}{2} + \frac{\phi_{2*} - \phi_2}{2} \right) \sin \left(\frac{\phi_{1*} - \phi_1}{2} - \frac{\phi_{2*} - \phi_2}{2} \right) \right. \\ &\quad \left. \times \left[|\phi_1 - \phi_2|^{p-1} \operatorname{sgn}(\phi_1 - \phi_2) - |\phi_{1*} - \phi_{2*}|^{p-1} \operatorname{sgn}(\phi_{1*} - \phi_{2*}) \right] \right] d\eta_* d\eta d\Omega_* d\Omega. \end{aligned}$$

It follows from the proof of Lemma 5.4 that for all $a, b \in \mathbb{R}$,

$$\left(|a|^{p-1} \operatorname{sgn}(a) - |b|^{p-1} \operatorname{sgn}(b) \right) \sin \left(\frac{b-a}{2} \right) \leq 0.$$

On the other hand, Lemma 5.1 implies that there exists t_0 such that

$$D_\theta(\mu_t) \leq D^\infty, \quad D_\theta(\nu_t) \leq D^\infty, \quad t \geq t_0,$$

and we use Lemma 5.2 to obtain

$$\begin{aligned} &\max_{\Omega \in \operatorname{supp}(g)} \phi_1(g(\Omega), \Omega, t) - \min_{\Omega \in \operatorname{supp}(g)} \phi_1(0, \Omega, t) \leq D^\infty, \\ &\max_{\Omega \in \operatorname{supp}(g)} \phi_2(g(\Omega), \Omega, t) - \min_{\Omega \in \operatorname{supp}(g)} \phi_2(0, \Omega, t) \leq D^\infty, \quad t \geq t_0. \end{aligned}$$

Then, this yields

$$0 < \cos D^\infty \leq \cos \left(\frac{\phi_{1*} - \phi_1}{2} + \frac{\phi_{2*} - \phi_2}{2} \right).$$

Hence, we obtain

$$\frac{d}{dt} \|\phi_1 - \phi_2\|_{L^p}^p \leq 2pK \cos D^\infty \mathcal{J}$$

where

$$\begin{aligned} \mathcal{J} := & \int_{\mathbb{R} \times \mathbb{R}} \int_0^{g(\Omega)} \int_0^{g(\Omega_*)} \sin \left(\frac{\phi_{1*} - \phi_{2*}}{2} - \frac{\phi_1 - \phi_2}{2} \right) \\ & \times [|\phi_1 - \phi_2|^{p-1} \operatorname{sgn}(\phi_1 - \phi_2) - |\phi_{1*} - \phi_{2*}|^{p-1} \operatorname{sgn}(\phi_{1*} - \phi_{2*})] d\eta_* d\eta d\Omega_* d\Omega. \end{aligned}$$

If we set $\Phi := \phi_1 - \phi_2$, then

$$|\Phi_* - \Phi| \leq |\phi_{1*} - \phi_1| + |\phi_{2*} - \phi_2| \leq 2D^\infty < \pi, \quad t > t_0.$$

Since μ_0, ν_0 have the same center of mass, it follows from Lemma 5.3 that

$$\int_{\mathbb{R}} \int_0^{g(\Omega)} \Phi d\eta d\Omega = 0, \quad t > t_0.$$

Thus, we can apply Lemma 5.4 with $\Phi = \phi_1 - \phi_2$ to obtain

$$\frac{d}{dt} \widetilde{W}_p^p(\mu_t, \nu_t) \leq -\frac{2pK \cos D^\infty}{\pi} \widetilde{W}_p^p(\mu_t, \nu_t), \quad t \geq t_0.$$

This yields

$$\widetilde{W}_p(\mu_t, \nu_t) \leq \exp \left(-\frac{2K \cos D^\infty}{\pi} (t - t_0) \right) \widetilde{W}_p(\mu_{t_0}, \nu_{t_0}). \quad (5.7)$$

In the case of $p = \infty$, we use (5.4) and (5.7) to obtain

$$\widetilde{W}_\infty(\mu_t, \nu_t) \leq \exp \left(-\frac{2K \cos D^\infty}{\pi} (t - t_0) \right) \widetilde{W}_\infty(\mu_{t_0}, \nu_{t_0}).$$

This completes the proof for smooth solutions. As mentioned above a simple approximation argument as in Subsection 3.3 finishes the proof for measure valued solutions.

Corollary 5.1 *There exists a unique stationary state μ_∞ in the set of measures μ_0 satisfying the assumptions in Theorem 5.1. Moreover, given any measure μ_0 in that set, we have*

$$\widetilde{W}_p(\mu_t, \mu_\infty) \leq \exp \left[-\frac{2K \cos D^\infty}{\pi} (t - t_0) \right] \widetilde{W}_p(\mu_{t_0}, \mu_\infty), \quad t > t_0, \quad 1 \leq p \leq \infty.$$

Furthermore, this stationary state satisfies that $\omega[\mu_\infty] = 0$.

Proof This is an easy consequence of Theorem 5.1 and the continuity in time of the solutions, see [4] for more details. For the asymptotic behaviour of the measure valued solutions μ_t , we can show that the support of the velocity field associated to μ_t shrinks to a single point exponentially fast using a similar argument as in the proof of Lemma 5.1 together with Proposition 2.2. This means that $\omega[\mu_\infty] = 0$.

Remark 5.1 The assumption in Theorem 5.1 on the initial measures to have equal mean in θ is not restricted. Due to Lemma 5.3, the mean in θ is preserved in time. Thus, we can always restrict to the equal mean in θ case by translational invariance of (1.1).

A Proof of Theorem 3.1

In this part, we provide the details for the proof of Theorem 3.1.

For given $\mu_0 \in \mathcal{M}([0, 2\pi] \times \mathbb{R})$, we set for $0 < \varepsilon < 1$,

$$\tilde{\mu}_0^\varepsilon := ((1-\varepsilon)\mu_0 + \varepsilon\chi_{\mathcal{R}}) * \eta_\varepsilon = \int_{[0, 2\pi] \times \mathbb{R}} \eta_\varepsilon^1(\theta - \tilde{\theta}) \eta_\varepsilon^2(\Omega - \tilde{\Omega}) ((1-\varepsilon)\mu_0 + \varepsilon\chi_{\mathcal{R}})(d\tilde{\theta}, d\tilde{\Omega}),$$

where $\eta_\varepsilon = (\eta_\varepsilon^1, \eta_\varepsilon^2)$, η_ε^1 is a periodic compactly supported mollifier with period 2π and η_ε^2 is a standard compactly supported mollifier satisfying $\|\eta_\varepsilon^1\|_{L^1([0, 2\pi])} = 1$, $\|\eta_\varepsilon^2\|_{L^1(\mathbb{R})} = 1$, and $\chi_{\mathcal{R}}$ is the uniform probability measure on rectangle \mathcal{R} enclosing $\text{supp}(\mu_0)$ such that $\mathcal{R} \subset [0, 2\pi] \times [-C, C]$ with $C > 0$, where $\text{supp}(\mu_0)$ is the smallest closed set $D \subset [0, 2\pi] \times \mathbb{R}$ such that $\mu_t(D) = 1$, i.e., $\text{supp}(\mu_0) = \text{cl}\{z \in [0, 2\pi] \times \mathbb{R} : \mu_0(B(z, r)) > 0 \text{ for all } r > 0\}$. It is straightforward to check that

$$d(\tilde{\mu}_0^\varepsilon, \mu_0) \simeq W_1(\tilde{\mu}_0^\varepsilon, \mu_0) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0$$

and $\text{supp}(\tilde{\mu}_0^\varepsilon) \subset [0, 2\pi] \times [-C_\varepsilon, C_\varepsilon]$.

We remark that if $\text{supp}(\mu_0) \subset (0, 2\pi) \times [-C, C]$, then $\text{supp}(\tilde{\mu}_0^\varepsilon) \subset (0, 2\pi) \times [-C_\varepsilon, C_\varepsilon]$ using a standard compactly supported mollifier η_ε for sufficiently small ε . Since $\tilde{\mu}_0^\varepsilon$ is absolutely continuous with respect to Lebesgue measure $d\theta d\Omega$ and with connected support, then for all ε the initial approximated measure $\tilde{\mu}_0^\varepsilon$ can be approximated by a Dirac comb of uniform masses. This means that there exist point distributions $\{(\theta_{i0}^{\varepsilon, N}, \Omega_{i0}^{\varepsilon, N})\}_{i=1, \dots, N}$, whose dependence on N is elapsd for clarity, such that

$$\lim_{N \rightarrow \infty} d(\tilde{\mu}_0^\varepsilon, \tilde{\mu}_0^{\varepsilon, N}) = 0, \quad \tilde{\mu}_0^{\varepsilon, N} := \frac{1}{N} \sum_{i=1}^N \delta_{\theta_{i0}^{\varepsilon, N}} \otimes \delta_{\Omega_{i0}^{\varepsilon, N}}. \quad (\text{A.1})$$

Then we solve the KM with N -particles:

$$\frac{d\theta_i^{\varepsilon, N}}{dt} = \Omega_i^{\varepsilon, N} - \frac{K}{N} \sum_{j=1}^N \sin(\theta_i^{\varepsilon, N} - \theta_j^{\varepsilon, N}), \quad \frac{d\Omega_i^{\varepsilon, N}}{dt} = 0, \quad t > 0, \quad i = 1, \dots, N, \quad (\text{A.2})$$

with initial data:

$$(\theta_i^{\varepsilon, N}(0), \Omega_i^{\varepsilon, N}(0)) = (\theta_{i0}^{\varepsilon, N}, \Omega_{i0}^{\varepsilon, N}).$$

With solutions $(\theta_i^{\varepsilon, N}(t), \Omega_i^{\varepsilon, N}(t))$ of (A.2), the approximate solution $\tilde{\mu}_t^{\varepsilon, N}$ for the measure valued solution can be constructed as a sum of Dirac measures, i.e.,

$$\tilde{\mu}_t^{\varepsilon, N} := \frac{1}{N} \sum_{i=1}^N \delta_{\theta_i^{\varepsilon, N}(t)} \otimes \delta_{\Omega_i^{\varepsilon, N}(t)}.$$

Then, $\tilde{\mu}_t^{\varepsilon, N}$ is clearly a measure valued solution to KKE (1.1). As consequences of Proposition 3.1, the KKE (1.1) has a unique measure valued solutions μ_t and $\tilde{\mu}_t^\varepsilon$ with initial data μ_0 and $\tilde{\mu}_0^\varepsilon$, respectively. Furthermore, the measure valued solutions satisfy the stability estimates:

$$d(\tilde{\mu}_t^\varepsilon, \tilde{\mu}_t^{\varepsilon, N}) \leq C d(\tilde{\mu}_0^\varepsilon, \tilde{\mu}_0^{\varepsilon, N}) \quad \text{and} \quad d(\tilde{\mu}_t^\varepsilon, \mu_t) \leq C d(\tilde{\mu}_0^\varepsilon, \mu_0).$$

This obviously yields

$$\begin{aligned} d(\tilde{\mu}_t^{\varepsilon, N}, \mu_t) &\leq d(\tilde{\mu}_t^{\varepsilon, N}, \tilde{\mu}_t^\varepsilon) + d(\mu_t, \tilde{\mu}_t^\varepsilon) \\ &\leq C(d(\tilde{\mu}_0^{\varepsilon, N}, \tilde{\mu}_0^\varepsilon) + d(\mu_0, \tilde{\mu}_0^\varepsilon)). \end{aligned}$$

For any $\delta > 0$, we first choose a sufficiently small ε such that $d(\mu_0, \tilde{\mu}_0^\varepsilon) < \frac{\delta}{2C}$. Then we next choose $N = N(\varepsilon)$ large enough so that $d(\tilde{\mu}_0^{\varepsilon, N}, \tilde{\mu}_0^\varepsilon) < \frac{\delta}{2C}$. Hence we have

$$d(\tilde{\mu}_t^{\varepsilon, N}, \mu_t) < \delta,$$

and this completes the proof by selecting our desired approximations $\mu_t^N = \tilde{\mu}_t^{\varepsilon, N}$.

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