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EXPRESS LETTER

The Ricker wavelet and the Lambert *W* function

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SUMMARY

The Ricker wavelet has been widely used in the analysis of seismic data, as its asymmetrical amplitude spectrum can represent the attenuation feature of seismic wave propagation through viscoelastic homogeneous media. However, the frequency band of the Ricker wavelet is not analytically determined yet. The determination of the frequency band leads to an inverse exponential equation. To solve this equation analytically a special function, the Lambert *W* function, is needed. The latter provides a closed and elegant expression of the frequency band of the Ricker wavelet, which is a sample application of the Lambert *W* function in geophysics and there have been other applications in various scientific and engineering fields in the past decade. Moreover, the Lambert *W* function is a variation of the Ricker wavelet amplitude spectrum. Since the Ricker wavelet is the second derivative of a Gaussian function and its spectrum is a single-valued smooth curve, numerical evaluation of the Lambert *W* function can be implemented by a stable interpolation procedure, followed by a recursive computation for high precision.

Key words: Time-series analysis; Numerical solutions; Computational seismology; Wave propagation.

INTRODUCTION

The Ricker wavelet is the second derivative of a Gaussian function. Theoretically, it is a solution of the Stokes differential equation, including the effect of Newtonian viscosity (Ricker 1943, 1944). It is applicable to seismic wave propagation through viscoelastic homogeneous media, that is the Voigt model. In reality, this purely Newtonian viscosity might be too simple, as the experimental seismic signals are often closer to the first or one-and-a-half derivatives of a Gaussian that are not symmetric (Hosken 1988) than to the second derivative, which is symmetric in the time domain. Nevertheless, the Ricker wavelet remains in vogue in the seismic field, because the various derivatives of a Gaussian function have similar spectra. At the beginning of the seismic chain, it can be used to represent a symmetrical source wavelet generated, for example, by correlating vibrator sweeps. At the end of the chain, it can be a desired wavelet presented on a processed seismic profile, so that the wave peak can correspond well to the depth of a subsurface reflector or to the time of a target reflection for geological interpretation. Thus, the Ricker wavelet is often a processed wavelet.

The amplitude spectrum of a Gaussian function is also in a Gaussian distribution. Mathematically, the similar spectra of the first, one-and-a-half, and second derivatives can be understood as the Gaussian spectrum multiplied by frequency-related factors ω, $\omega^{3/2}$ and ω^2, where ω is the angular frequency. Such frequency-dependent multiplications alter the amplitude spectra from the Gaussian symmetric to be asymmetric. Physically, however, these asymmetric spectra can represent the frequency-dependent attenuation feature of seismic waves propagated through viscoelastic media.

The Ricker wavelet is defined in the time domain by a single parameter, the most energetic frequency, which is the peak frequency in the amplitude spectrum. Because the amplitude spectrum of the Ricker wavelet is asymmetric, the peak frequency is not the central frequency, the geometric centre of the frequency band. However, the frequency band and the central frequency of the Ricker wavelet are not analytically determined yet.

Frequency band and central frequency are key parameters characterizing seismic resolution, and their variations along the wave propagation may directly reflect the attenuation property of the subsurface media (Wang 2004, 2014). But identifying them from field seismic data is a difficult task, because the Fourier spectra always have strong fluctuations with notches and spikes. Only an average quantity such as the mean frequency can be calculated numerically and the standard deviation can be evaluated statistically. Once the frequency band and the central frequency are analytically specified in a closed form, the relationship between the mean frequency and the central frequency and the relationship between the standard deviation and the frequency band can be established. All of these quantities can be expressed in terms of the peak frequency. I will address these relationships in a separate paper.

The frequency band is defined as the spectral breadth at a half of the peak amplitude in the frequency spectrum. That is, the bandwidth is measured by approximately −3 dB cut-off of the amplitude spectrum and equivalently −6 dB cut-off of the power spectrum. This paper shows that the problem of the band determination leads to an inverse exponential equation, for which an analytical solution is given by the Lambert W function.

The inverse exponential equation has a form of

$$z \exp z = x, \tag{1}$$

and the solution z is given by the Lambert W function,

$$z = W(x). \tag{2}$$

According to Corless *et al.* (1996), the mathematical history of $W(x)$ begins with Lambert (1758) who solved the trinomial equation $x = q + x^m$ and, subsequently, Euler (1779) who transformed the equation into the form $\ln x = v x^\alpha$. The latter can be expressed as eq. (1), $z \exp z = -\alpha v$, if set $x^\alpha = \exp(-z)$.

The Lambert W function provides a convenient and elegant solution to various scientific and engineering problems. Banwell & Jayakumar (2000) used it to express an analytical solution for current flow through a diode. Valluri *et al.* (2000) used it to determine Wien's displacement constant based on the Planck spectral distribution law for black body radiations, and to solve the inverse conformal transformation for the fringing fields of a capacitor. Packel & Yuen (2004) used the Lambert W function in a symbolic computation of projectile motion equations. Shafee (2007) used it to define the entropy of an information system. Tamm (2014) applied the inverse exponential equation (1) to invert the function $f(x) = x \ln x$, by setting $z = \ln x$.

Using the Lambert W function, we are able to present, for the first time, an analytical expression for the frequency band of the Ricker wavelet. This is a sample application of the Lambert W function in geophysics, in addition to various applications in the recent decade, mentioned above. Moreover, we also show that the Lambert W function is a variation of the Ricker wavelet amplitude spectrum. Since the spectrum of the Ricker wavelet, the second derivative of a Gaussian function, is a smooth function, we propose an interpolation procedure to find the numeric value of $W(x)$ for any given variable x, followed by recursive refining for high precision.

THE RICKER WAVELET AND THE FREQUENCY BAND

The Ricker wavelet in the time domain (Ricker 1953) is defined as

$$r(\tau) = \left(1 - \frac{1}{2}\omega_p^2 \tau^2\right) \exp\left(-\frac{1}{4}\omega_p^2 \tau^2\right), \tag{3}$$

where τ is the time, and ω_p is the most energetic frequency (in radians per second). The Fourier transform of the Ricker wavelet can be expressed as

$$R(\omega) = \frac{2\omega^2}{\sqrt{\pi}\omega_p^3} \exp\left(-\frac{\omega^2}{\omega_p^2}\right). \tag{4}$$

This frequency spectrum is real and non-negative in value, $|R(\omega)| = R(\omega)$. Thus, it is just the module of the Fourier transform of the even Ricker wavelet. Any possible time delay in the time domain will affect only the phase, not the magnitude. Therefore, eq. (4) may be referred to as the amplitude spectrum of the Ricker wavelet.

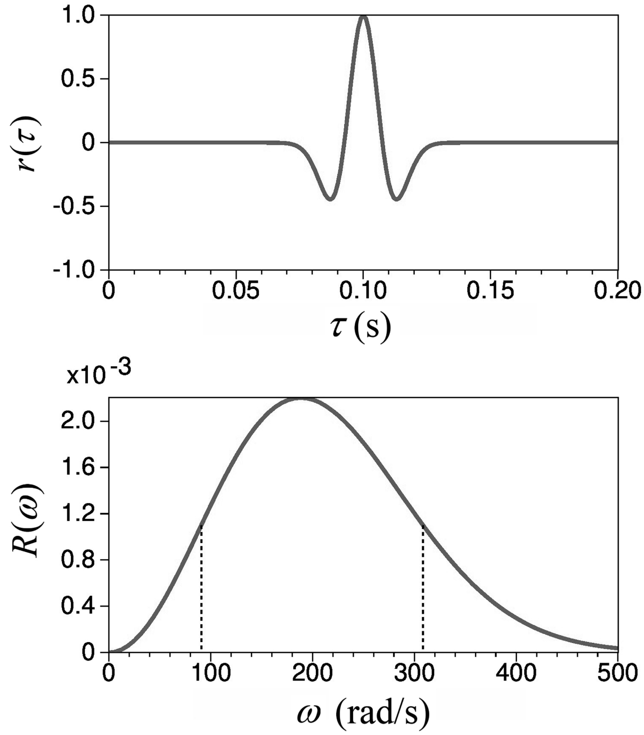

Figure 1. The Ricker wavelet $r(\tau)$ and the frequency spectrum $R(\omega)$, with the peak angular frequency of 60π rad s^{-1} (the ordinary frequency of 30 Hz). The frequency spectrum is just the amplitude spectrum of the Ricker wavelet, and two vertical dash lines indicate the frequency band, corresponding to a half of the peak amplitude.

It is also easy to verify that the most energetic frequency ω_p is the peak frequency with the greatest spectral content, if we set the derivative of the amplitude spectrum, with respect to the frequency, as zero, $\mathrm{d}R/\mathrm{d}\omega = 0$. A sample Ricker wavelet displayed in Fig. 1 demonstrates that the Ricker wavelet is symmetrical in the time domain and asymmetrical in the frequency domain. The sample wavelet has the peak frequency of 60π rid s^{-1} (equivalent to the ordinary frequency of 30 Hz).

When $\omega = \omega_p$, the peak amplitude is $R(\omega_p) = 2(e\sqrt{\pi}\omega_p)^{-1}$. Note that '$e$' is Euler's number, $e = \lim_{n\to\infty}(1 + n^{-1})^n \approx 2.71828$, and 'exp' represents an exponential function in this paper. The bandwidth of the amplitude spectrum is measured at a half of this peak, as

$$R(\omega) = \frac{1}{e\sqrt{\pi}\omega_p}, \tag{5}$$

which leads to the following equation:

$$\frac{\omega^2}{\omega_p^2} \exp\left(-\frac{\omega^2}{\omega_p^2}\right) = \frac{1}{2e}. \tag{6}$$

This is a special equation, and its specialty lies in the analytical solution:

$$\frac{\omega^2}{\omega_p^2} = -W\left(-\frac{1}{2e}\right). \tag{7}$$

where $W(x)$ is the Lambert function.

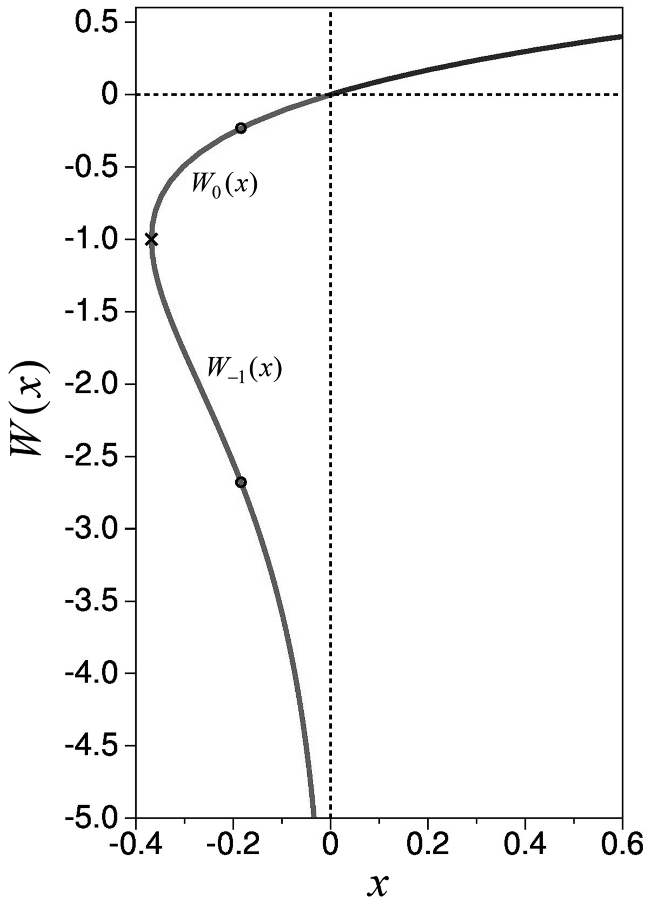

Figure 2. The Lambert W function has two branches, $W_{-1}(x)$ for $W(x) \leq -1$ and $W_0(x)$ for $W(x) \geq -1$. The cross mark indicates the branching point at $(-e^{-1}, -1)$. Two circle marks are points which define the frequency band of the Ricker wavelet.

Fig. 2 plots the Lambert function, $W(x)$, with respect to a real-valued variable x. The numeric evaluation will be discussed in the following sections. For $x = -(2e)^{-1} < 0$, $W(x)$ has two branches: $W_{-1}(x)$ for $W(x) \leq -1$, and $W_0(x)$ for $W(x) \geq -1$. Therefore, the frequency band, $[\omega_{\ell 1}, \omega_{\ell 2}]$, may be given analytically by

$$\omega_{\ell 1} = \omega_p \sqrt{-W_0\left(-\frac{1}{2e}\right)},$$

$$\omega_{\ell 2} = \omega_p \sqrt{-W_{-1}\left(-\frac{1}{2e}\right)}. \tag{8}$$

As marked with circles in Fig. 2, for $x = -(2e)^{-1}$, numeric approximations of $W_0(x)$ and $W_{-1}(x)$ are

$$W_0\left(-\frac{1}{2e}\right) \approx -0.231961,$$

$$W_{-1}\left(-\frac{1}{2e}\right) \approx -2.67835. \tag{9}$$

The frequency band is defined as

$$[\omega_{\ell 1}, \omega_{\ell 2}] = [0.481623, 1.63657]\, \omega_p, \tag{10}$$

and is centred at frequency

$$\omega_c = \frac{1}{2}(\omega_{\ell 1} + \omega_{\ell 2}) = 1.0590965\, \omega_p. \tag{11}$$

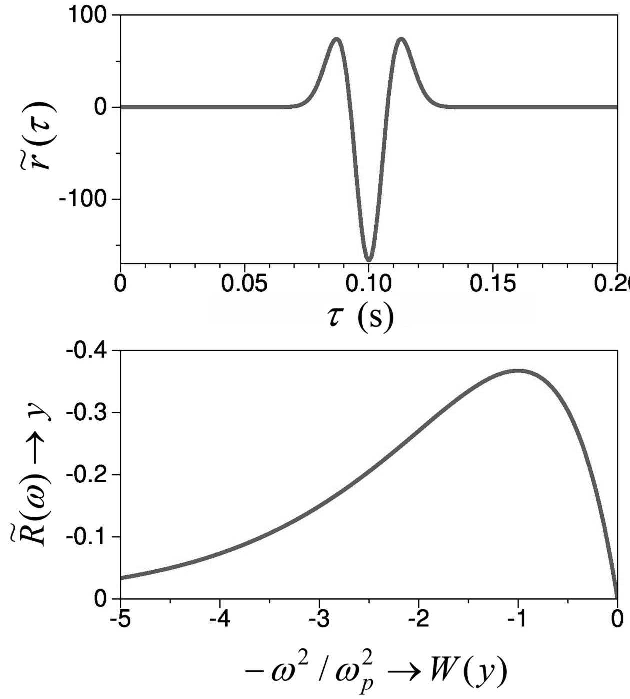

Figure 3. The Ricker wavelet $\tilde{r}(\tau)$ with a negative peak, and the frequency spectrum $\tilde{R}(\omega)$. The latter is the negative amplitude spectrum and is the Lambert $W(y)$ function for $y \leq 0$, if the frequency axis is transferred as $-\omega^2/\omega_p^2 \to W(y)$, and the amplitude axis is transferred as $\tilde{R}(\omega) \to y$.

This is, apparently, the first time for an analytical solution of the Ricker wavelet frequency band, thanks to the introduction of the special Lambert W function.

THE LAMBERT *W* FUNCTION VERSUS THE RICKER WAVELET

The Lambert W function is defined as the inverse function of

$$W(x) \exp W(x) = x. \tag{12}$$

For a real-valued variable x, the Lambert W function has two real branches. The bottom branch $W_{-1}(x) \leq -1$ monotonically decreases within the interval $x \in [-e^{-1}, 0)$. The upper branch for $W_0(x) \geq -1$ monotonically increases for $x \in [-e^{-1}, \infty)$. The branching point with a cross marked in Fig. 2 is located at $(-e^{-1}, -1)$.

In this section, we will show that the Lambert $W(x)$function (for $x < 0$) is, actually, a rotation of the Ricker wavelet amplitude spectrum.

Multiplying the Ricker wavelet in the time domain by a factor of $-\frac{1}{2}\sqrt{\pi}\omega_p$, one can have a scaled wavelet,

$$\tilde{r}(\tau) = -\frac{\sqrt{\pi}\omega_p}{2}\left(1 - \frac{1}{2}\omega_p^2\tau^2\right)\exp\left(-\frac{1}{4}\omega_p^2\tau^2\right). \tag{13}$$

This time-domain wavelet shown in Fig. 3 has an opposite polarity to the wavelet in Fig. 1. The frequency spectrum,

$$\tilde{R}(\omega) = -\frac{\omega^2}{\omega_p^2}\exp\left(-\frac{\omega^2}{\omega_p^2}\right), \tag{14}$$

is the negative amplitude spectrum and has a trough $\tilde{R}(\omega) = -e^{-1}$ at frequency $\omega = \omega_p$. A half of this negative peak is $-(2e)^{-1}$, at where the frequency band of this Ricker wavelet is defined.

This negative amplitude spectrum is similar to the Lambert $W(x)$ function for $x < 0$. The latter has a trough at $x = -e^{-1}$, and a half of this negative peak is at $x = -(2e)^{-1}$. Solutions $W_0[-(2e)^{-1}]$ and $W_{-1}[-(2e)^{-1}]$ are just the quantities we seek in order to define the frequency band.

In relabelling two axes $\tilde{R}(\omega) \to y$ and $-\omega^2/\omega_p^2 \to W(y)$, as shown in Fig. 3, we may obtain exactly eq. (12):

$$y = W(y) \exp W(y). \tag{15}$$

Thus, we prove that the Lambert W function is a rotated amplitude spectrum of the Ricker wavelet.

Since the spectrum (14) of the Ricker wavelet, the second derivative of a Gaussian, is a single-valued smooth curve, we can numerically evaluate the Lambert W function by interpolation. For any given $W(y)$ values, eq. (15) calculates y values correspondingly, and then interpolation can make y sufficiently close to a desired point x. By performing interpolation repeatedly, we can obtain the desired value $W(x)$.

Note that, for $-e^{-1} < x < 0$, two separate interpolations are needed for each x sample and are performed within the branches $W(y) < -1$ and $-1 < W(y) < 0$, respectively. As $W(y)$ is either increasing or decreasing monotonically in two branches, $W(y) < -1$ and $W(y) > -1$, the iterative interpolation is simple and numerically stable.

RECURSION FORMS OF THE LAMBERT *W* FUNCTION

To further improve the accuracy with high precision, the interpolated $W(x)$ can be used as an initial solution for the recursive algorithm presented in this section, to produce the final numeric value of $W(x)$.

For $W_{-1}(x) \le -1$, the bottom branch of the $W(x)$ function where $-e^{-1} \le x < 0$ (Fig. 2), we can multiply both sides of eq. (12) with -1, take the logarithm, and rewrite it as

$$W(x) = \ln \frac{-x}{-W(x)}. \tag{16}$$

It is clear that an analytical expression for $W(x)$ exhibits a degree of self-similarity. Hence, if we replace $W(x)$ in the denominator with the entire term on the right-hand side of eq. (16), we obtain a continued logarithm,

$$W_{-1}(x) = \ln \frac{-x}{-\ln \frac{-x}{-\ln \frac{-x}{\cdots}}}. \tag{17}$$

For $-1 \le W_0(x) < 1$, the first part of the upper branch of $W(x)$ function, we have $-e^{-1} \le x < e$, and the solution is

$$W(x) = \frac{x}{\exp W(x)}. \tag{18}$$

Replacing $W(x)$ in the denominator repeatedly leads to a continued exponential form,

$$W_0(x) = \frac{x}{\exp \frac{x}{\exp \frac{x}{\exp \cdots}}}. \tag{19}$$

For $W_0(x) \ge 1$ where $x \ge e$, we can re-arrange eq. (12) as $\exp W(x) = x/W(x)$, and take the natural logarithm,

$$W(x) = \ln \frac{x}{W(x)}. \tag{20}$$

This leads to again a continued logarithm,

$$W_0(x) = \ln \frac{x}{\ln \frac{x}{\ln \frac{x}{\cdots}}}. \tag{21}$$

The analytical forms (17), (19) and (21), clearly, can be used to make successive approximation, for which the final numeric value is given by the limit, when it exists.

As the previous interpolation step generates an initial solution sufficiently close to the final solution, the entire scheme of 'interpolation followed by recursive refining' is able to produce the numeric value with desired high precision, but is much more stable, if compared to conventional numeric schemes, such as Halley's root-finding method, based on the Taylor series of $f(x) = W(y) - x$, or an iteration method based on Padérational fraction approximation (Fritsch, Shafer & Crowley 1973).

CONCLUSIONS

The frequency band of the Ricker wavelet is expressed analytically in terms of the Lambert W function. The latter is a solution to an inverse exponential equation, and has been an attractive topic in various fields in the past decade. This paper shows a sample application in geophysics.

This paper also provides a relationship between the Lambert W function and the amplitude spectrum of the Ricker wavelet. Since the spectrum is a single-valued smooth curve, interpolation can be implemented to stably evaluate $W(x)$. Using this interpolated value as an initial solution, recursive refining then produces the final numeric value of the Lambert W function with high precision.

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