On Memristor Ideality and Reciprocity

Panayiotis S. Georgiou[a,1,*], Mauricio Barahona[b], Sophia N. Yaliraki[c], Emmanuel M. Drakakis[d]

[a] *Department of Bioengineering, Imperial College London*
[b] *Department of Mathematics, Imperial College London*
[c] *Department of Chemistry, Imperial College London*
[d] *Department of Bioengineering, Imperial College London, South Kensington Campus, London SW7 2AZ, United Kingdom*

Abstract

Starting from the constitutive properties that underpin the ideal memristor as originally defined by Leon Chua, we identify the conditions under which two memristors comply with the reciprocity theorem. In particular, we explore the minimal set of requirements for an ideal memristor and the physical implications of not complying with these criteria. Then, we show that reciprocity is satisfied when two identical ideal memristors with the same initial memristance are connected such that the output of the first one is taken as the input of the second; the output of the second memristor then matches exactly the initial input of the first device. We also discuss under which conditions non-ideal memristors can be reciprocal and how this property may be exploited in applications.

1. Introduction

In 1971 Leon Chua postulated the existence of the *memristor*, a new fundamental passive two-terminal circuit element to be added to the classic trio of the resistor, the capacitor and the inductor [1]. As its name indicates, the memristor behaves similarly to a conventional resistor but it has memory in the sense that its instantaneous resistance depends on the history of its input. A few years later, Chua and Kang extended the conceptual framework to include systems that share common characteristics with memristors but whose behavior cannot be adequately captured by the stricter original definition of the memristor. To enable the modeling of such systems, they introduced the term *memristive systems* [2]. For several decades, experimental devices exhibiting memristive behavior remained unclassified as such [3, 4], with research interest mostly confined to theoretical studies [5, 6]. This state of affairs changed in 2008, when Hewlett Packard (HP) fabricated a nano-scale device whose operation was described using a memristor model, thus reigniting experimental and theoretical interest in this elusive element [7].

The fabrication of the HP memristor has attracted the attention of the research community, which has been motivated in great measure by the many potential applications of memristors. This activity has resulted in the fabrication of a wide range of experimental memristive devices demonstrating attractive properties such as nano-scale dimensions [4], low-power consumption [8], high-speed resistance switching [9], and intrinsic non-volatile memory [10]. These properties render memristors ideal candidates for improving the performance of existing applications (e.g. computer memories, re-configurable digital circuits [11, 12]) or enabling new ones (e.g. learning/adaptive circuits) [13].

The experimental devices presented so far exhibit qualitative characteristics that point to a memristor (e.g. nonlinearity, memory, hysteresis, zero-crossing), but they deviate significantly from the strict definition of a memristor as originally articulated by Chua [4, 14, 15, 16]. Thus, they can only be satisfactorily modeled under the broader definition of a memristive system. For this reason, the terms memristor and memristive system are frequently used interchangeably in the literature, although they have different mathematical properties.

In our view, both notions can be useful depending on the problem being tackled. When it comes to device modeling, utilizing memristive systems is necessary if an accurate model is to be developed. However, theoretical studies aimed at understanding the fundamental characteristics of memristors as isolated elements or as part of larger networks, or in combination with other circuit elements, are greatly simplified by the use of the stricter definition of the memristor. Henceforth, we will refer to the original element as introduced in [1] as the *ideal memristor*.

Here, we revisit the fundamental mathematical properties underpinning the ideal memristor and highlight the implications of omitting any of them. In doing so, one can understand the origin of the observable behavior of non-ideal devices, and identify the factors responsible for

*Corresponding ahuthor
Email address: `ps.georgiou@imperial.ac.uk` (Panayiotis S. Georgiou)

[1] *Acknowledgments:* The authors would like to thank EPSRC for supporting Dr. Georgiou through the Doctoral Prize Fellowship and Prof. Robert Spence (Imperial College) for his insightful comments.

their non-ideal operation. Moreover, many of the theoretical results developed for the ideal element can be applied to the non-ideal case, if its operation is appropriately restricted. We will refer to such memristive devices as *piecewise ideal*. Such piecewise ideal memristors can exhibit properties usually associated with strict ideality.

In recent work, we have shown that the dynamics of ideal memristors complies with the Bernoulli differential equation [17, 18, 19]. This reformulation allowed us to obtain a set of general solutions and to evaluate analytically the output response of HP's ideal memristor model for various input signals, both as a single element and as part of series or parallel network configurations. Additionally, using the analytic expressions it was possible to investigate various properties of ideal memristors such as hysteresis, harmonic distortion and the effect of parasitic resistance [18, 19].

In this paper, we build on these results to investigate another interesting property of ideal memristors, namely, under what conditions do ideal memristors behave as reciprocal elements. In particular, we identify the requirements on the input/output signals and the device itself which lead to compliance with the *reciprocity theorem* [20]. When these requirements are met, memristors exhibit a useful behavior that may be exploited in applications. The final part of this work discusses the applicability of reciprocity on non-ideal devices and its potential applications.

2. The Ideal Memristor

2.1. Analysis of the Ideal Memristor

This section introduces and analyzes the *ideal memristor* [18]. The theory presented provides us with the necessary tools to investigate the compliance of ideal memristors with the reciprocity theorem (Section 3).

The memristor is a 2-terminal passive circuit element that relates the charge q:

$$q(t) = \int_{-\infty}^{t} i(\tau)d\tau = q_0 + \int_{0}^{t} i(\tau)d\tau, \qquad (1)$$

with the flux-linkage φ:

$$\varphi(t) = \int_{-\infty}^{t} v(\tau)d\tau = \varphi_0 + \int_{0}^{t} v(\tau)d\tau, \qquad (2)$$

where v is the voltage across and i is the current through the memristor, and q_0 and φ_0 are respectively the initial charge and flux-linkage values at $t = 0$.

The memristor is characterized by the constitutive relation:

$$f_{\mathcal{M}}(q, \varphi) = 0, \qquad (3)$$

which relates the variables q and φ. More simply, the memristor is a circuit element whose input-output response is defined by a charge-flux ($q - \varphi$) curve [1, 21].

An *ideal memristor* is characterized by a unique and time-invariant $q - \varphi$ curve complying with the following criteria:

1.1 *nonlinear;*

1.2 *continuously differentiable*;

1.3 *strictly monotonically increasing.*

This three criteria for ideality were extracted from the original publication introducing the memristor [1]. The only exception is that we impose *strict* monotonicity rather than the lighter requirement imposed by Leon Chua of *non-strict* monotonicity.

Assuming an ideal memristor, it is always possible to express the constitutive relation (3) as an explicit function of both q *and* φ:

$$\varphi = \hat{\varphi}(q), \qquad (4)$$
$$q = \hat{q}(\varphi). \qquad (5)$$

The memristor is then referred to as charge-controlled, for the case (4), or flux-controlled, for the case (5). Under the same assumptions, it follows that:

$$\hat{q}^{-1}(q) = \hat{\varphi}(q) \quad \text{and} \quad \hat{\varphi}^{-1}(\varphi) = \hat{q}(\varphi). \qquad (6)$$

Using (1) and (2), and taking the time derivative of (4) and (5) results in the representation of the memristor on the current-voltage ($i - v$) plane. For the charge-controlled case it is given by:

$$v = \mathcal{M}(q)i(t), \qquad (7)$$

where $\mathcal{M}(q) = d\hat{\varphi}(q)/dq$ is the *incremental memristance*, measured in Ohms (Ω), which corresponds to the gradient of the $q - \varphi$ curve at an operating point (OP) $Q(q_a, \varphi_a)$ (Figure 1). Similarly, for the flux-controlled case, the $i - v$ representation is given by:

$$i = \mathcal{W}(\varphi)v(t), \qquad (8)$$

where $\mathcal{W}(\varphi) = d\hat{q}(\varphi)/d\varphi$ is the *incremental memductance*, measured in Siemens (S), which corresponds to the gradient of the $\varphi - q$ curve at an OP $Q(q_a, \varphi_a)$ [1, 22].

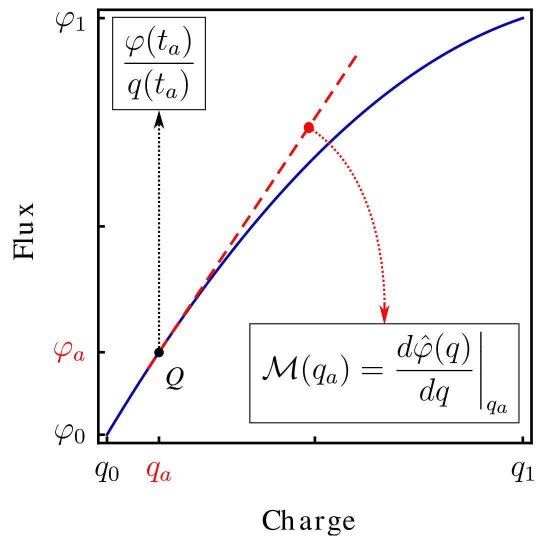
From (7) and (8), we see that at any OP $Q(q_a, \varphi_a)$ along the $q - \varphi$ curve the following holds [3]:

$$\mathcal{M}(q) = \frac{1}{\mathcal{W}(\varphi)} = \frac{v(t)}{i(t)}, \qquad (9)$$

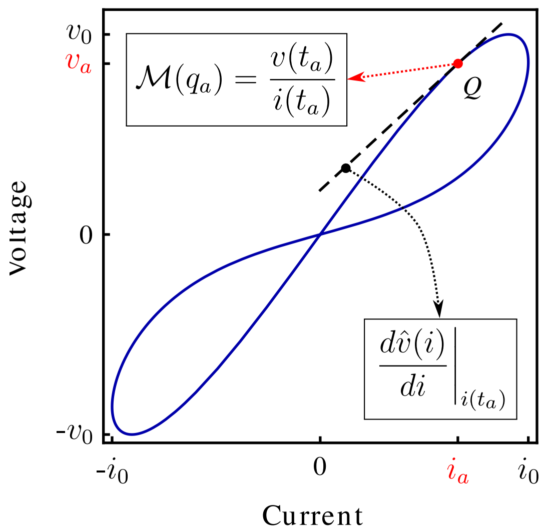
namely, the incremental memristance is equal to both the instantaneous resistance ($v(t)/i(t)$) and to the reciprocal of the incremental memductance. This result is of practical importance as it relates the memristance (or memductance) with the driving signal and its output: (9) indicates that the incremental memristance can be obtained from experimental data by sampling the input and output of the device and evaluating their ratio.

The definition of ideal memristors allows also the substitution of (6) in (9):

$$\mathcal{M}(q) = \frac{1}{\mathcal{W}(\hat{q}^{-1}(q))} \quad \Leftrightarrow \quad \mathcal{M}(\hat{\varphi}^{-1}(\varphi)) = \frac{1}{\mathcal{W}(\varphi)}. \qquad (10)$$



(a)



(b)

Figure 1: (a) An ideal $q-\varphi$ curve satisfying the criteria (1.1)-(1.3) and (b) its corresponding $i-v$ response when driven by a sinusoidal excitation. In both figures the same operating point (OP) $Q(q_a, \varphi_a)$ is shown at $t = t_a$. At Q in (a) the instantaneous memristance $\varphi(t)/q(t)$ and the incremental memristance $\mathcal{M}(q) = d\hat{\varphi}(q)/dq$ are shown. For the same OP (b) shows the instantaneous resistance $v(t)/i(t) = \mathcal{M}(q)$ and the incremental resistance $d\hat{v}(i)/di$. The incremental memristance is equal to the instantaneous resistance when evaluated at the same OP. However, all four quantities will be equal with each other only when the $q-\varphi$ curve becomes linear. In this case the memristor becomes indistinguishable from a linear resistor.

For an ideal memristor, the $q-\varphi$ curve is uniquely invertible, and its memristance (memductance) is uniquely defined at any point along the $q-\varphi$ curve irrespective of whether it is driven by a voltage or by a current input [23]. Therefore, the two representations, charge-controlled and flux-controlled, are equivalent to each other and can be used interchangeably to our convenience. On the other hand, for a non-ideal memristor each value of q or φ may correspond to multiple memristance values. Therefore, (10) does not hold in general. In this case, (10) may hold locally only if the $q-\varphi$ curve is appropriately restricted to be monotonic sub-functions, each one complying with the criteria of ideality [24].

Equations (7) and (8) are the representation of the memristor on the $i-v$ plane and can be viewed as a generalization of Ohm's law with a dynamic and nonlinear resistance that depends on the history of the input [22]. These expressions reveal key features of the response: the *multiplicative output function*; the *zero-crossing*; the *non-volatile memory*; and the relation between the incremental memristance and the instantaneous resistance [1, 22]. More specifically, the output of the component at any time is equal to the product between the input and a nonlinear function representing the memristance (multiplicative output function) and it is zero whenever the input is zero (zero-crossing property). Furthermore, replacing q in (7) with (1) and φ in (8) with (2) yields:

$$v(t) = \mathcal{M}\left(\int_{-\infty}^{t} i(\tau)d\tau\right) i(t), \tag{11}$$

$$i(t) = \mathcal{W}\left(\int_{-\infty}^{t} v(\tau)d\tau\right) v(t), \tag{12}$$

which clearly reveal the *non-volatile memory property* of the memristor [18]. The value of the memristance (memductance) is determined by the entire past history of the input signal. Therefore, the memristance (memductance) keeps changing as long as an input is applied on the device. Once the signal is removed the memristor will maintain its state indefinitely, or until the time instant at which an input is applied again [1, 5, 25].

Figure 1a shows a $q-\varphi$ curve satisfying the criteria listed above and therefore representing an ideal memristor. Its corresponding $i-v$ response is shown in Figure 1b when the device is driven by a sinewave input. The figure also illustrates that in the general nonlinear case:

$$\frac{\varphi(t)}{q(t)} \neq \frac{d\hat{\varphi}[q(t)]}{dq(t)} = \frac{v(t)}{i(t)} \neq \frac{d\hat{v}[i(t)]}{di(t)}. \tag{13}$$

Namely, the instantaneous ($\varphi(t)/q(t)$) is not equal to the incremental ($d\hat{\varphi}[q(t)]/dq(t)$) memristance; and the instantaneous resistance ($v(t)/i(t)$) is not equal to the incremental ($d\hat{v}[i(t)]/di(t)$) resistance. These four quantities only become equal when the memristor degenerates to the special case of the linear resistor. However, the incremental memristance is always equal to the instantaneous resistance and reduces to a constant value in the linear case. In this work we only refer to the incremental memristance and instantaneous resistance, so the terms *incremental* and *instantaneous* will be generally omitted.

2.2. Exploring the Criteria for Ideality

We now describe the effect of each of the conditions (1.1)-(1.3) on the $q-\varphi$ curve of an ideal memristor and discuss the consequences of not complying with these criteria through examples.

An ideal memristor is characterized by a $q-\varphi$ curve complying with the properties:

- *nonlinear*: The nonlinearity of the $q-\varphi$ curve distinguishes the memristor from a linear resistor. From (7) and (8), it was established that the memristance is equal to the slope of the $q-\varphi$ curve. Therefore, a linear $q-\varphi$ corresponds to a device with a constant memristance, which is indistinguishable from a constant linear resistor according to (9).
- *continuously differentiable*: This property, which ensures that the derivative $(d\hat{\varphi}(q)/dq)$ of (4) exists and is continuous, implies that the memristance (i.e., the gradient of $q-\varphi$) is a continuous function defined at every point along the $q-\varphi$ curve and is finite: $|\mathcal{M}(q)| < +\infty, \forall q$. Moreover, from (9), it follows that $\mathcal{W}(\varphi) \neq 0, \forall\varphi$. By applying the same arguments but reversing the roles of memristance and memductance, we deduce that the memductance is also finite and the memristance is non-zero. Putting everything together leads to: $0 \neq |\mathcal{M}(q)| < +\infty, \forall q$ and $0 \neq |\mathcal{W}(\varphi)| < +\infty, \forall\varphi$. This result can be further restricted to show that the memristance and the memductance are in fact positive by taking into account the increasing monotonicity requirement (1.3): $0 < \mathcal{M}(q) < +\infty, \forall q,$ and $0 < \mathcal{W}(\varphi) < +\infty, \forall\varphi$.
- *strictly monotonically increasing*: A strictly monotonic $q-\varphi$ curve has a unique inverse such that (6) holds with the implications already explained in the previous section. If the $q-\varphi$ curve is monotonically *increasing*, its gradient (i.e., the memristance) is positive and non-zero leading to a strictly passive device, i.e., a device which does not require an internal power source to operate. A direct consequence of these restrictions is that the $q-\varphi$ curve of an ideal memristor must be a one-to-one function. It should be stressed here that, although an ideal memristor complying with the three criteria detailed here is always passive, the converse is not always true (see Table 1).

An ideal memristor characterized by a $q-\varphi$ curve satisfying the above criteria has a unique response for any input waveform and initial state. More specifically, assuming

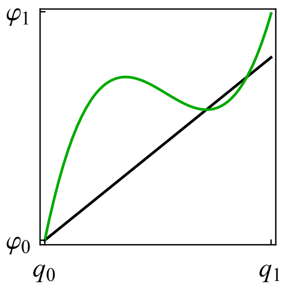
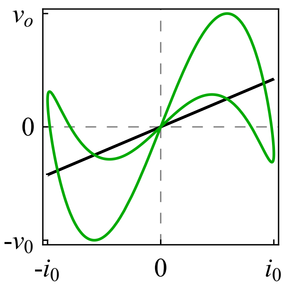
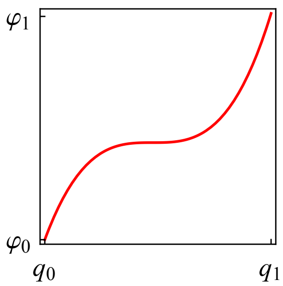
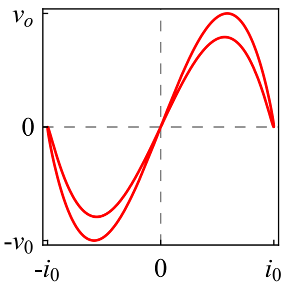
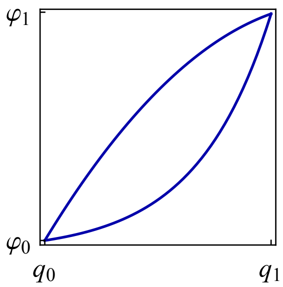
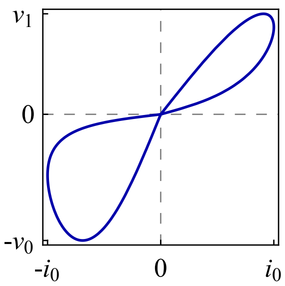
	Charge-Flux	**Current-Voltage**	**Comments and $\varphi(q)$**
Example 1			The $\varphi(q)$ function is not monotonic hence the $i-v$ response is not restricted in the 1st and 3rd quadrant and the memristor cannot be passive. The extrema of $\varphi(q)$ correspond to zero memristance and appear as four additional zero-crossings on the $i-v$. The linear $\varphi(q)$ shows that a linear resistor is the degenerative case of a linear memristor. $\varphi(q) = q^3 - 4.5q^2 + 6q$
Example 2			The $\varphi(q)$ function is not strictly increasing hence the memristance is zero at the point of inflection. In the $i-v$ plane this appears as two additional zero-crossings at which the voltage is zero for a non-zero input current. $\varphi(q) = q^3 - 3q^2 + 3q$
Example 3			The $\varphi(q)$ is a double-valued function hence it is not monotonic. Also, at the transition point between the two branches it is non-continuously differentiable. This causes discontinuities in the memristance and results to a non-symmetric $i-v$ response. $\varphi(q) = \begin{cases} q(5-q)/3, & i(t) \geq 0 \\ (\mathrm{e}^{1.55q} - 1)/10, & i(t) < 0 \end{cases}$

Table 1: Examples of non-ideal charge-flux curves with their respective current-voltage responses under a sinusoidal input. The first example is also compared with the linear case $\varphi(q) = \kappa q$ indicated by the black solid line.

the same initial memristance $\mathcal{M}_0$ (or memductance $\mathcal{W}_0$), a specific amount of charge Δq (or flux $\Delta\varphi$) flowing through the device induces the same change in the memristance (memductance) taking also into account the polarity of the driving signal. As we will see later through examples, this does not necessary hold for non-ideal devices.

Table 1 illustrates three examples of non-ideal $q-\varphi$ curves and their respective $i-v$ responses when the system is driven by a sinusoidal current input $i = i_0 \sin(t)$.

Example 1 in Table 1 violates the monotonicity requirement (1.3). In particular, the function $\varphi(q)$ is increasing ($\mathcal{M}(q) > 0$) up to a maximum ($\mathcal{M}(q) = 0$) and then it decreases ($\mathcal{M}(q) < 0$) down to a local minimum ($\mathcal{M}(q) > 0$) after which it increases ($\mathcal{M}(q) > 0$). When the device is in the region of operation where $\mathcal{M}(q) < 0$, the voltage and current have opposite polarities. This results in a partially active memristor with an $i-v$ response that is not restricted to the first and third quadrant of the plane. The two local extrema of $\varphi(q)$ correspond to the additional four zero-crossings of the $i-v$ function (other than the origin) at which the output voltage is zero although the input current is non-zero. Such a memristor can only exist as an active device.

Example 2 in Table 1 shows a $\varphi(q)$ curve that is not strictly increasing due to a point of inflection thus violating again (1.3). At the point of inflection, the memristance is zero. On the $i-v$ plane this appears as two points (in addition to the origin) at which the voltage output is zero although the current input is non-zero. At these two points the memristor is instantaneously acting as a resistanceless wire without a voltage drop across it.

The response ensuing from Example 2 is non-unique in the sense that starting from the same initial memristance the same amount of charge Δq can cause different changes in the memristance. More specifically, assume that the $\varphi(q)$ function consists of two branches: a first branch on the left of the point of inflection and the second on its right. For every point on the left branch there is a corresponding point on the right branch that shares the same memristance. However, driving the same amount of charge Δq through the device causes the memristance to decrease, if the OP lies on the left branch, but it causes the memristance to increase, if the OP lies on the right branch. Therefore, the same input can cause two different responses depending on which of the two branches the OP lies on. The same observations apply in Example 1 for the two increasing sub-functions of the $q-\varphi$ curve.

Example 3 in Table 1 shows a $q-\varphi$ curve which is a piecewise (double-valued) function violating (1.2) and (1.3). This could correspond to a device where each of its characteristic sub-functions is chosen depending on the polarity of the input. As a result, the $i-v$ response shown is not symmetric for a periodic zero-mean input. Additionally, when switching from one branch of $\varphi(q)$ to the other, discontinuities are introduced in the memristance as the gradient is not uniquely defined at the switching point. This example can still be classified as passive since its sub-functions are strictly increasing. However, it cannot be classified as an ideal memristor. This example highlights that passivity does not imply ideality. On the other hand, an ideal $q-\varphi$ curve guarantees passivity.

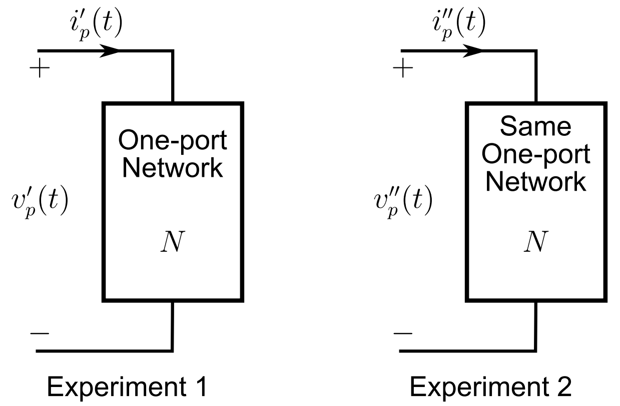


Figure 2: Two experiments, or states, of the *same* one-port network consisting of N branches. In each experiment i_p, v_p denote the port current and voltage drop respectively and i_j, v_j denote respectively the current and voltage drop of the j-th branch, where $j = 1, 2, 3, \ldots, N$. The signals of the first experiment are indicated by a single prime ($'$) and those of the second, by a double prime ($''$).

It is also instructive to note that the response of the memristor in Example 3 is unique although $\varphi(q)$ is double-valued. Because of the double-valued $q-\varphi$ curve, the device may have the same initial memristance on both branches. Thus applying the same charge Δq on each of these initial states will result to two different responses. However, once the polarity of the input is taken into account this ambiguity is immediately resolved.

Example 3 in Table 1 is the most interesting case of the non-ideal memristors. Although its $\varphi(q)$ function is not ideal, each of its constituent sub-functions satisfy all the criteria of ideality. These memristors will be referred to as *piecewise ideal*. This is an important characteristic since it may lead to properties of ideal memristors to apply separately on each of the sub-functions of the $q-\varphi$ curve. We expect many practical devices to exhibit such $q-\varphi$ curves [16, 15].

3. Memristors as Reciprocal Elements

In the previous section we have analyzed the ideal memristor and its fundamental constituent properties. In particular, we have shown how these properties restrict the form of the $q-\varphi$ curve, and thus the behavior of the ideal device. In this section, we identify the conditions under which ideal memristors comply with the *reciprocity theorem*. The configuration that satisfies this theorem leads to an interesting behavior: if the output of a memristor is used to drive another identical memristor, the output of the second reproduces exactly the waveform used to drive the first device. We also explore the applicability of reciprocity to non-ideal memristors. Finally, we discuss the

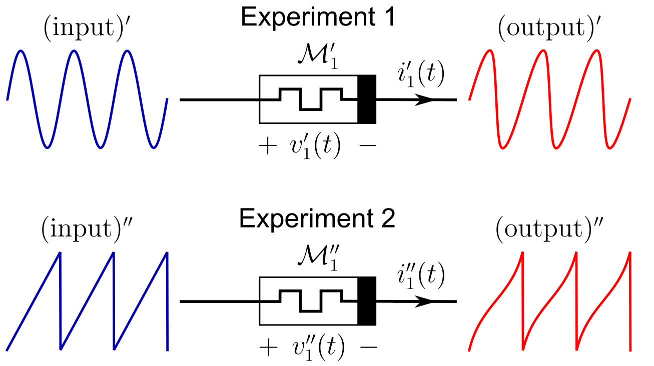


Figure 3: Two experiments on the *same*, or two identical memristors. The two memristors share an identical $q-\varphi$ curve. The branch voltage, current and the memristance are denoted by i_1, v_1 and $\mathcal{M}_1$ respectively, while the single ($'$) and double ($''$) prime indicate the first and second experiments respectively.

idea of exploiting reciprocity for signal encryption.

3.1. Definition of reciprocity

We first introduce some notation. As shown in Figure 2 the reciprocity theorem considers *two states of the same network*. Consider a circuit network consisting of N branches with j denoting the j-th branch. The single prime ($'$) is used to denote quantities of the first state, and the double prime ($''$) to denote those of the second state. Each state is a set of N pairs of branch voltage drops and currents resulting from two distinct excitations of the same frequency. The two states can also be viewed as two different *experiments* on the same network, or on two identical ones [26, 27]. Hence, the voltage and current of the j-th branch are denoted v_j', i_j' for the first experiment and v_j'', i_j'' for the second experiment.

Consider two states of the *same* circuit network resulting from two distinct input signals having the same frequency. The network is reciprocal if it satisfies the condition [27]:

$$\sum_{j=1}^{N}\left(i_j' v_j'' - i_j'' v_j'\right) = 0, \tag{14}$$

where the summation is taken over all the branches of the network. Although we will focus here on a one-port network, the definition extends to multi-port networks. Additionally, it can be shown that a network composed of reciprocal elements is itself reciprocal. Note that the condition (14) may also be stated in terms of port voltages and currents. However, the two are equivalent and there is no need to introduce this second representation [27].

3.2. Conditions under which ideal memristors are reciprocal

We now investigate the conditions under which an ideal memristor satisfies the reciprocity condition stated in (14). As shown in Figure 3, we will focus on a network consisting of only a single memristor, therefore, (14) reduces to the following condition for a single branch:

$$i_1' v_1'' - i_1'' v_1' = 0, \tag{15}$$

which by using (7) can be re-expressed in terms of the memristances:

$$\mathcal{M}_1' = \mathcal{M}_1'', \tag{16}$$

where $\mathcal{M}_1' = v_1'/i_1'$ and $\mathcal{M}_1'' = v_1''/i_1''$. The memristance depends on time t through its controlling variable, which is either the charge $q = q(t)$ or the flux $\varphi = \varphi(t)$. Depending on the choice of input signal in each of the two experiments, the reciprocity condition (16) becomes one of the following four conditions:

$$\mathcal{M}_1'(q) = \mathcal{M}_1''(q), \tag{17a}$$
$$\mathcal{M}_1'(\varphi) = \mathcal{M}_1''(\varphi), \tag{17b}$$
$$\mathcal{M}_1'(q) = \mathcal{M}_1''(\varphi), \tag{17c}$$
$$\mathcal{M}_1'(\varphi) = \mathcal{M}_1''(q). \tag{17d}$$

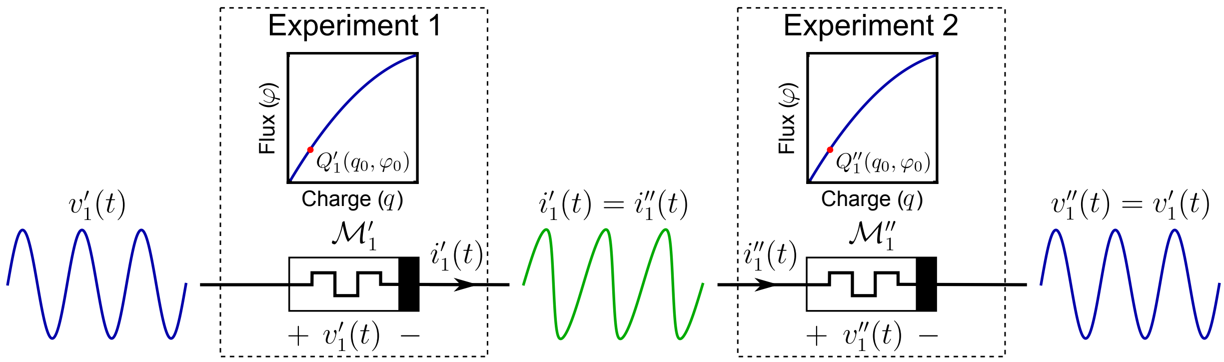
It is important to emphasize that the appropriate condition must hold for all $t \geq 0$.

Consider first the cases (17a) and (17b), in which both experiments have the *same* controlling quantity. The reciprocity theorem requires that the two experiments are performed on the same memristor (i.e. characterized by an identical $q-\varphi$ curve); hence the two memristances, $\mathcal{M}_1'$ and $\mathcal{M}_1''$ will also have an identical form. Consider also equal initial memristances $\mathcal{M}_1'(q_0) = \mathcal{M}_1''(q_0)$. The equality in (17a) holds if $q = q_1' = q_1''$, $\forall t > 0$, where $q_1'(t) = \int_0^t i_1'(\tau)d\tau$ and $q_1''(t) = \int_0^t i_1''(\tau)d\tau$. The new equality $q = q_1' = q_1''$ is only true *for all* t if $i_1' = i_1''$. Similarly, in the case of (17b) the requirement is satisfied if $v_1' = v_1''$. Hence, the first two conditions are straightforward and do not provide any valuable insights; they simply verify that, for two identical ideal memristors, applying the same input should always have exactly the same effect at all instants in time.

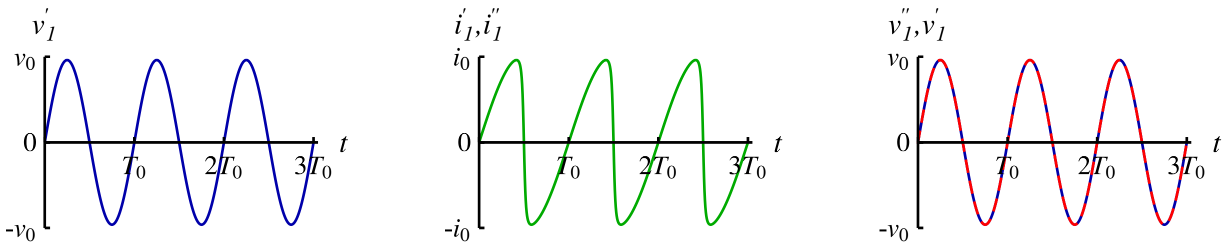
Consider now the other two cases (17c) and (17d), in which the controlling quantity is *different* in each experiment. We will discuss (17c) but the same arguments hold for (17d) as well. The input and output signals satisfying these two conditions reveal an interesting property of ideal memristors. The first experiment in (17c) is current-driven with input i_1' and $q_1'(t) = \int_0^t i_1'(\tau)d\tau$, while the second experiment is voltage-driven with input v_1'' and $\varphi_1''(t) = \int_0^t v_1''(\tau)d\tau$. The two memristors are identical and hence share the same $q-\varphi$ curve. We also assume that the initial memristance is identical in both cases: $\mathcal{M}_1'(q_0) = \mathcal{M}_1''(\varphi_0)$. It then follows that (17c) can be re-expressed as follows:

$$\mathcal{M}_1\left(q_1'(t)\right) = \mathcal{M}_1\left(\varphi_1''(t)\right) \qquad \forall\, t > 0, \tag{18}$$

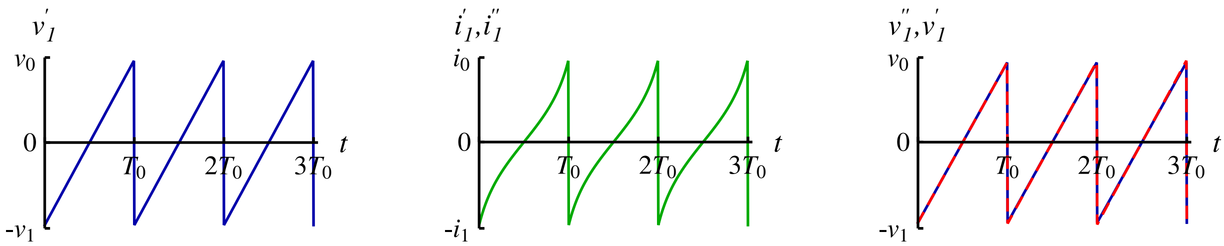
where it is important to note that the first memristance is viewed as a function of q and the second memristance is a



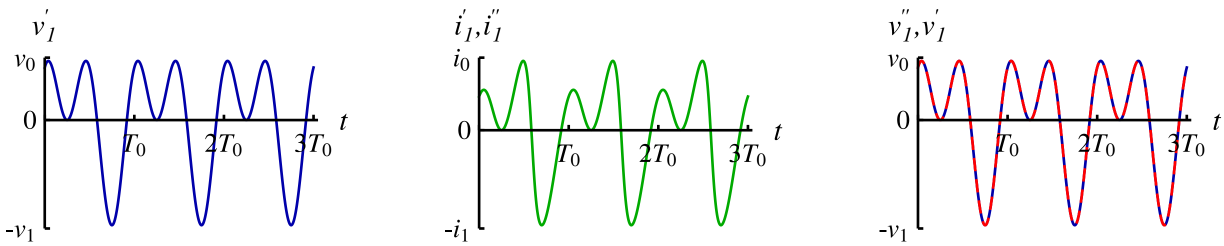
(a)



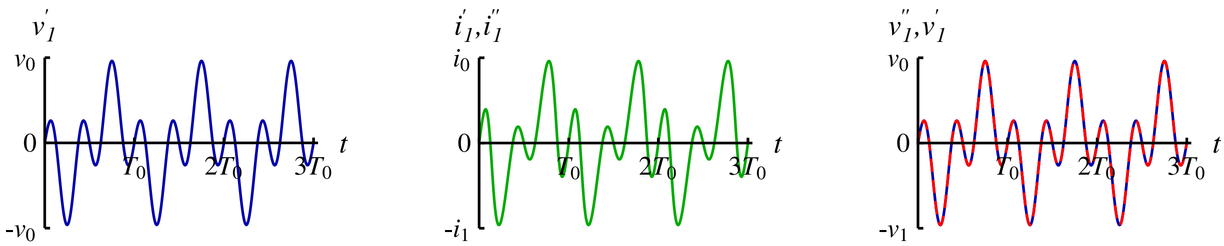
(b) sinusoidal input



(c) sawtooth input



(d) sum of two sinusoidals input



(e) product of two sinusoidals input

Figure 4: Demonstration of the effect of reciprocity. The two memristors are configured as in (4a) with the output of the first one used to drive the second ($i_1' = i_1''$) and assuming that the two devices are *ideal*, *identical* and *have the same initial memristance* such that (14) is satisfied. As a result of this configuration the output of the second memristor is identical to the input of the first memristor ($v_1' = v_1''$). This effect is demonstrated in (4b)-(4e) for various input waveforms. The first column of plots shows the input of the first memristor (v_1'), the second column the output of the first, which is also the input of the second ($i_1' = i_1''$), and the third column shows the output of the second memristor superimposed on the input of the first one to illustrate that the two waveforms match exactly. It is assumed that the memristors are characterized by HP's ideal memristor model [7] whose output response can be found in References [18].

function of φ. As we have seen in (4), for ideal memristors, the flux is uniquely defined by an explicit function of the charge, thus (18) can be re-expressed as:

$$\mathcal{M}_1\left(q_1'(t)\right) = \mathcal{M}_1\left(\hat{\varphi}_1''\left(q_1''(t)\right)\right) \qquad \forall\, t > 0. \tag{19}$$

Additionally, for ideal memristors each value of the charge corresponds to a unique memristance. Hence (19) is true only if $q_1' = q_1''$, or equivalently, if $i_1' = i_1''$, $\forall\, t$. Since $i_1' = i_1''$, then it follows from (19) that

$$v_1' = i_1'(t)\mathcal{M}_1\left(q_1'\right) = i_1''(t)\mathcal{M}_1\left(\hat{\varphi}_1''(q_1'')\right) = v_1''. \tag{20}$$

Therefore, the reciprocity property holds for two identical and ideal memristors configured such that the output of the first is equal to the input of the second and provided their initial memristances, $\mathcal{M}_1'(q_0)$ and $\mathcal{M}_1''(\varphi_0)$, are the same.

As part of the derivation of (20) we obtained the requirement that $i_1' = i_1''$, namely, the input of the first memristor is equal to the output of the second. If we consider the case (17d), in which the first experiment is flux-controlled and the second charge-controlled, it is possible to establish the same result but with the roles of the voltage and current reversed. This reveals an important consequence of reciprocity for ideal memristors connected such that the output of the first is the input of the second memristor: the second memristor cancels the effect of the first one and returns as its output the original input of the first memristor.

In conclusion, the reciprocity condition (15) holds for two *ideal* and *identical* memristors with *the same initial memristance* when:

$$(\text{output})' = (\text{input})''. \tag{21}$$

The effect of this configuration is then

$$(\text{input})' = (\text{output})'', \tag{22}$$

where the input and the output can be voltage or current waveforms. Figure 4a illustrates the case (17d) and shows how the second memristor cancels the effect of the first one and outputs the original input waveform used to drive the first memristor.

It is important to note that although the discussion above has focused on single memristors, the results on reciprocity extend to networks consisting of ideal memristors. This follows from the fact that *a network of reciprocal elements, is itself reciprocal* [27]. In the case of memristors, we may arrive at the same conclusion following an alternative path: Chua's *Closure* and *Existence and Uniqueness* theorems show that a network of ideal memristors is equivalent to a single unique memristor [1]. Therefore, if the equivalent memristors representing the two networks fulfill the conditions summarized above, they will behave as reciprocal.

Figures 4b-4e demonstrate the operation of two ideal, identical HP memristors [7] with the same initial memristance arranged such that the output of a voltage-driven memristor is fed as an input to a second current-driven, as in Figure 4a. We show the response to four different input waveforms and demonstrate how the second memristor cancels the effect of the first one and outputs the waveform originally fed as input to the first memristor.

This 'canceling' property of memristors depends on the set of requirements detailed above. If any of these conditions is not satisfied, then the reciprocity property is not guaranteed to hold. Figure 5 illustrates two such scenarios. Figure 5a shows the case where the two memristors are identical but their initial memristance is not. Clearly, the output of the second memristor is not the same as the first input. A similar lack of 'perfect cancellation' is observed in Figure 5b where the two memristors are not identical because they do not share the same $q-\varphi$ curves.

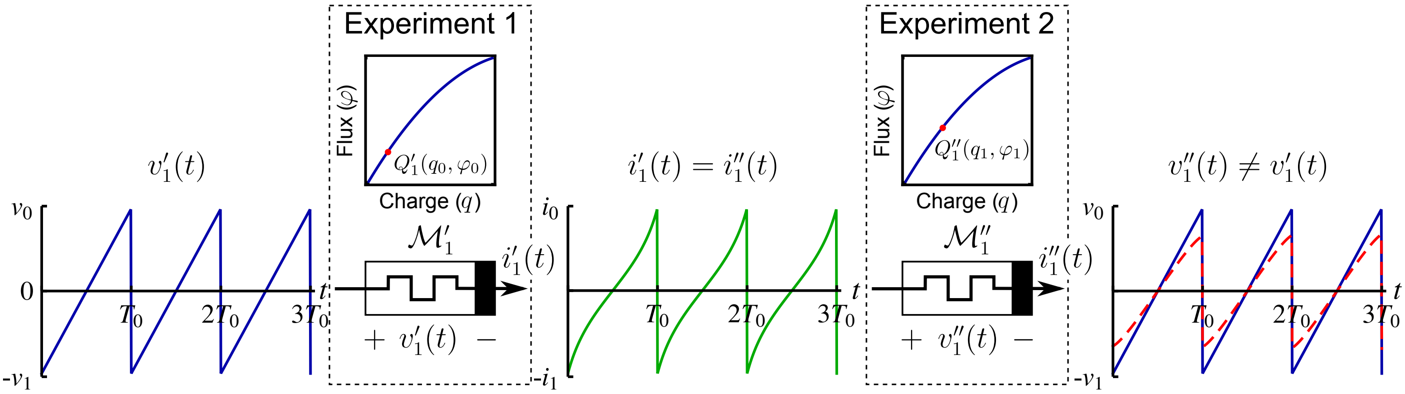
3.3. Reciprocity for non-ideal memristors

For a *non-ideal* memristor it is not always possible to uniquely express the flux as an explicit function of the charge (and vice versa), thus (19) and the steps that follow do not hold in general. However, it is still possible to operate non-ideal memristors, and specifically *piecewise ideal* memristors, as reciprocal elements locally, as we now show.

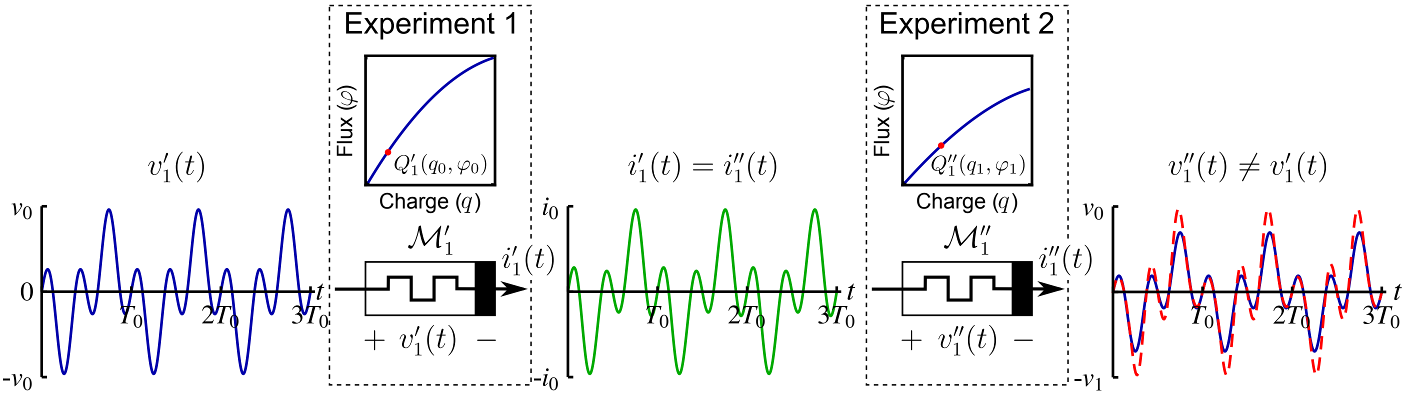
First, consider two identical, non-ideal memristors with the same initial memristance characterized by the $q-\varphi$ curve shown in Example 1 of Table 1. Irrespective of whether the controlling quantities of the two experiments are as in (17c) or (17d), if the input to the first memristor operates the device only within the increasing part of the $q-\varphi$ curve (i.e. the locally ideal sub-function) then the devices will appear as complying with the reciprocity theorem. On the other hand, if the memristors have their controlling quantities according to (17c) and the input to the first device is such that it is driven beyond the maximum of the $q-\varphi$ curve, then because the $q-\varphi$ curve does not have a unique inverse, at least in theory, it is possible for the second memristor to follow multiple paths. Depending on which path is followed, the configuration may, or may not comply with reciprocity.

Consider now two identical, non-ideal memristors with the same initial memristance characterized either by the $q-\varphi$ curve of Example 2 or Example 3 of Table 1. In both cases the configuration will satisfy the reciprocity theorem for any input signal irrespective of the region of operation. This is because for any value of the charge there is a unique value of the flux and vice versa. We would like to note here that the apparent ambiguity in the third example is resolved by taking into account the polarity of the input signal. Both of these examples, especially the third one, are of particular value since they indicate the potential applicability of reciprocity on real non-ideal devices. An example of the operation of such a configuration of interconnected *piecewise ideal memristors* is shown in Figure 6.

It may be possible to utilize the compliance of memristors with the reciprocity theorem in practical applications.



(a) sawtooth input



(b) product of sinusoidals input

Figure 5: Failure to satisfy the reciprocity theorem because the two memristors are not identical. It is assumed that the memristors are ideal and characterized by HP's model [7] as in Figure 4. The devices in the two experiments are connected such that the output of the first one drives the second. In (5a) the two memristors differ in their initial memristance. In (5b) the memristors are characterized by non-identical $q-\varphi$ curves due to different device widths. As a result in both examples the output of the second memristor does not match exactly the input of the first one ($v_1' \neq v_1''$).

For example, a proposed application is to use a block of memristors for non-linearly transforming, or encrypting, a signal that needs to be securely transmitted. According to the reciprocity theorem the signal can only be decrypted if the receiver has an identical copy of the block used to encrypt the signal at the transmitter. In other words, at both ends of this encrypted communication system the block of memristors and its initial state will be acting as the unique key for encrypting/decrypting the signal. If the receiver lacks an exact copy of the transmitter's block of memristors then the signal will not be retrieved in its original form. This could prevent unauthorized interception of the transmitted information. This proposed application will be the object of future research.

4. Discussion

In this work we have investigated and identified the conditions under which ideal memristors comply with the reciprocity theorem of classical circuit theory. Our investigation began first by clarifying the properties that give rise to the ideal memristor and how these properties shape the characteristic $q-\varphi$ curve of the ideal device. Devices that fail to comply with the ideality requirements may exhibit effects such as zero memristance (or infinite conductance); non-unique responses; or inexistence as passive devices. We then identified the requirements on the input/output signals and the device itself under which ideal memristors behave as reciprocal elements. More specifically, it was shown that two ideal and identical memristors with the same initial memristance comply with the reciprocity theorem in (14) when the output of the first memristor is used as the input of the second device. A direct consequence of this configuration is that the second memristor cancels the effect of the first one and outputs the original input waveform used to drive the first memristor. In the case of identical, non-ideal memristors with the same initial memristance, reciprocity may be satisfied depending on the specific $q-\varphi$ curve characterizing the device and the input signal applied. More specifically, if the device is operated within a locally ideal region of its $q-\varphi$ curve satisfying (1.1)-(1.3) then the reciprocity theorem will still be satisfied (e.g. Example 3 in Table 1). A proposed application discussed is to exploit reciprocity in an encrypted

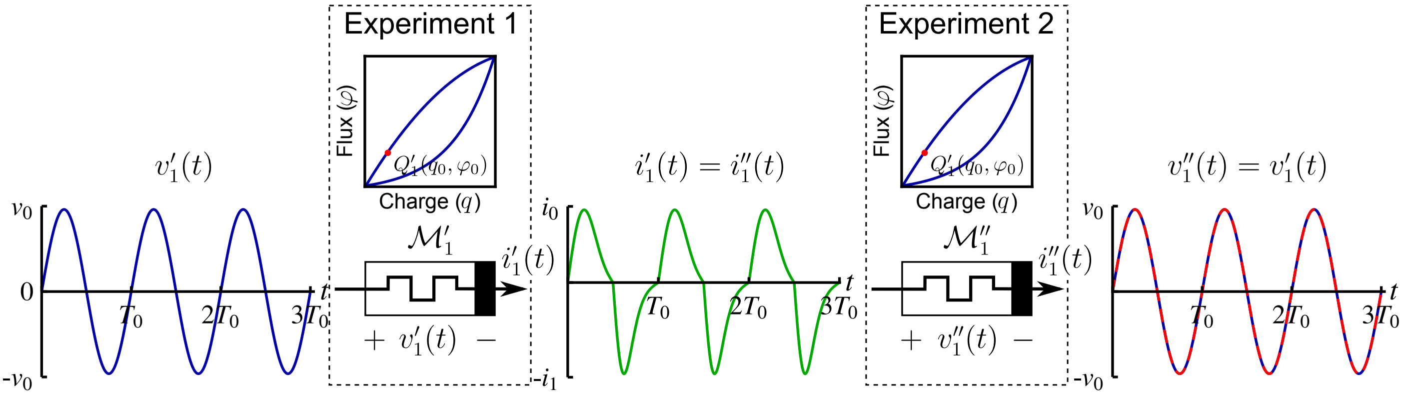


Figure 6: Reciprocity in non-ideal memristors: The memristors in the two experiments are characterized by the same non-ideal $q-\varphi$ curve of Example 3 in Table 1 and share the same initial memristance. As in Figure 4a the output of the first devices is fed as an input to the second. The configuration satisfies reciprocity because each of the monotonic sub-functions of the $q-\varphi$ is locally ideal (piecewise ideal) and the device has a unique response at each point along the curve. As a result, the final output matches the initial input ($v_1'' = v_1'$).

communication system. In such a system the network of memristors itself and its initial state will be the encryption/decryption key.

The study of reciprocity showcases the use of the ideal memristor as a fundamental tool for investigating the properties of these devices and for extending our understanding of non-ideal devices. For example, by viewing non-ideal devices as locally ideal may allow us to apply results developed for the ideal memristor to non-ideal components.

While significant research on practical non-ideal devices continues, through our work we would like to demonstrate that there is also a lot to be learned by studying the ideal element as originally defined by Leon Chua [1]. The results presented here and in our previous works [18, 19, 17] can be used as examples to motivate the use of this approach as an alternative means of comprehending the behavior of practical memristive devices [20].

References

[1] L. O. Chua, Memristor-the missing circuit element, IEEE Trans. Circuit Theory 18 (5) (1971) 507–519.
[2] L. O. Chua, S. M. Kang, Memristive devices and systems, Proc. IEEE 64 (2) (1976) 209–223.
[3] C. Baatar, W. Porod, T. Roska, Cellular Nanoscale Sensory Wave Computing, 1st Edition, Springer Publishing Company, Incorporated, 2009.
[4] R. Waser, R. Dittmann, G. Staikov, K. Szot, Redox-based resistive switching memories - nanoionic mechanisms, prospects, and challenges, Adv. Mater. 21 (25-26) (2009) 2632–2663.
[5] G. F. Oster, D. M. Auslander, The memristor: A new bond graph element, J. Dyn. Syst-t Asme 94 (3) (1972) 249–252.
[6] E. M. Drakakis, A. J. Payne, A bernoulli cell-based investigation of the non-linear dynamics in log-domain structures, Analog Integr. Circuits Signal Process. 22 (2000) 127–146.
[7] D. B. Strukov, G. S. Snider, D. R. Stewart, W. R. Stanley, The missing memristor found, Nat. 453 (7191) (2008) 80–83.
[8] C. Schindler, M. Weides, M. N. Kozicki, R. Waser, Low current resistive switching in Cu-SiO_2 cells, Appl. Phys. Lett. 92 (12) (2008) 122910.
[9] A. C. Torrezan, J. P. Strachan, G. Medeiros-Ribeiro, R. S. Williams, Sub-nanosecond switching of a tantalum oxide memristor, Nanotechnology 22 (48) (2011) 485203.
[10] S. H. Jo, K.-H. Kim, W. Lu, High-density crossbar arrays based on a Si memristive system, Nano Lett. 9 (2) (2009) 870–874.
[11] I. Vourkas, G. Sirakoulis, A novel design and modeling paradigm for memristor-based crossbar circuits, IEEE Trans. Nano. 11 (6) (2012) 1151–1159.
[12] I. Vourkas, G. C. Sirakoulis, Memristor-based combinational circuits: A design methodology for encoders/decoders, Microelectronics Journal 45 (1) (2014) 59–70.
[13] D. B. Strukov, D. R. Stewart, J. L. Borghetti, X. Li, M. D. Pickett, G. Ribeiro, W. Robinett, G. S. Snider, J. P. Strachan, W. Wu, Q. Xia, J. J. Yang, W. R. Stanley, Hybrid CMOS/memristor circuits, IEEE Int. Symp. Circ. S.
[14] J. J. Yang, M. D. Pickett, X. Li, D. A. A. Ohlberg, D. R. Stewart, W. R. Stanley, Memristive switching mechanism for metal/oxide/metal nanodevices, Nature Nanotechnology 3 (7) (2008) 429–433.
[15] F. Alibart, L. Gao, B. D. Hoskins, D. B. Strukov, High precision tuning of state for memristive devices by adaptable variation-tolerant algorithm, Nanotechnology 23 (7) (2012) 075201.
[16] S. H. Jo, K.-H. Kim, T. Chang, S. Gaba, W. Lu, Si memristive devices applied to memory and neuromorphic circuits, IEEE Int. Symp. Circ. S. (2010) 13–16.
[17] E. M. Drakakis, S. Yaliraki, M. Barahona, Memristors and bernoulli dynamics, in: IEEE CNNA, 2010, pp. 1–6.
[18] P. S. Georgiou, S. N. Yaliraki, E. M. Drakakis, M. Barahona, Quantitative measure of hysteresis for memristors through explicit dynamics, Proc. R. Soc. A 468 (2144) (2012) 2210–2229.
[19] P. Georgiou, M. Barahona, S. Yaliraki, E. Drakakis, Device properties of bernoulli memristors, Proc. IEEE 100 (6) (2012) 1938–1950.
[20] P. S. Georgiou, M. Barahona, S. N. Yaliraki, E. M. Drakakis, Ideal memristors as reciprocal elements, ICECS 20th Conference (2013) 301–304.
[21] L. O. Chua, Device modeling via basic nonlinear circuit elements, IEEE Trans. Circuits Syst. 27 (11) (1980) 1014–1044.
[22] L. O. Chua, Resistance switching memories are memristors, Appl. Phys. A: Mater. Sci. Process. 102 (2011) 765–783.
[23] Y. V. Pershin, M. Di Ventra, Memory effects in complex materials and nanoscale systems, Advances in Physics 60 (2) (2011) 145–227.
[24] T. M. Apostol, Mathematical analysis, 1st Edition, Addison-Wesley Series in Mathematics, Addison-Wesley Pub. Co., 1957.
[25] G. F. Oster, A note on memristors, IEEE Trans. Circuits Syst. 21 (1) (1974) 152–152.
[26] P. Penfield, R. Spence, S. Duinker, Tellegen's theorem and electrical networks, Research monographs, MIT Press, 1970.
[27] R. Spence, Linear Active Networks, Wiley-Interscience, 1970.