APPROXIMATE SOLUTIONS OF CERTAIN SHALLOW SHELL PROBLEMS

by

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Volume 1

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ABSTRACT

The numerical solutions for the shallow doubly curved shells presented in this thesis are based on the shallow curved plate theory.

Thin shells of constant thickness and rectangular plan-form and subject to uniformly distributed normal loading are considered.

The solution methods include the Rayleigh-Ritz, Galerkin and Lagrangian multiplier methods. These are referred to as indirect methods in this thesis. The method of lines, in which the derivatives in one direction are replaced by finite difference expressions, is also discussed.

Various approximating functions are considered in conjunction with the indirect methods.

Initially the indirect methods and approximating functions are applied to translational shell problems with Levy-type solutions. In this way the indirect solutions are compared with available exact solutions.

The indirect methods are then extended to translational shells with clamped, hinged or normal slide (1) conditions on any two opposite boundaries. Several numerical examples are given and the convergence

of the solutions discussed. An overall equilibrium check is presented.

In a similar manner, the indirect methods are applied to ruled surface hyperbolic paraboloids with clamped, hinged, normal slide (1), normal gable or normal slide (2) conditions on any two opposite boundaries. Several numerical examples are again given and the solutions discussed. An overall equilibrium check is presented.

The behaviour of the shell is then studied by varying certain non-dimensional parameters.

The method of lines is applied to translational and ruled surface shells with two opposite boundaries clamped and the remaining two either clamped or free. A system of linear first order ordinary differential equations with constant coefficients is obtained and is solved using the matrix progression method. The numerical difficulties encountered are discussed. In order to reduce the accumulation of roundoff errors the shell is segmented. A stiffness method is then used to restore equilibrium at the segment junctions. Numerical examples are presented.

Wherever possible comparisons are made with other available solutions.

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NOTATION

In the following the subscripts i and j range over the values 1 and 2.

General

(x_1, x_2, z)	Right handed orthogonal cartesian system of axes
$(\bar{i}_1, \bar{i}_2, \bar{j})$	Unit vectors in the x_1, x_2 and z directions respectively
$\bar{P} \equiv \bar{P}(\alpha_1, \alpha_2)$	The position vector of a point P measured in the (x_1, x_2, z) reference frame
(α_1, α_2)	Curvilinear co-ordinates of the shell middle surface
γ	Co-ordinate measured normal to the (α_1, α_2) set
ds^2	The metric of the surface
A_{11}, A_{12}, A_{22}	The coefficients of the first fundamental quadratic form
K_{ij}	The undeformed curvatures of the shell middle surface.
K_1, K_2	Principal curvatures

$K_G = K_1 K_2$	Gaussian curvature
$c = \frac{K_1}{K_2}$	
l_1, l_2	Plan lengths of the shell in the α_1 and α_2 directions respectively
f_1, f_2	Defined for translational shells in figure 2.4
$f, \bar{F}$	Defined for ruled surface shells in figure 2.5
$\beta_1 = \frac{\alpha_1}{T_1}$	
$\beta_2 = \frac{\alpha_2}{T_2}$	
X_i, Z	Loading pressures in the α_i and γ directions respectively
u_i, w	Components of the middle surface displacement (referred to as "displacements")
h	Thickness of the shell
ν	Poisson's ratio
E	Young's modulus of elasticity
$(\varepsilon_{ij}),(\varepsilon_{ij})_\gamma$	Components of strain on the middle and γ surfaces respectively.

$(\sigma_{ij}),(\sigma_{ij})_\gamma$	Components of stress on the middle and γ surfaces respectively
$\delta_{ij} \begin{cases} = 1, & i=j \\ = 0, & i \neq j \end{cases}$	Kronecker delta
$e_{ij} = \varepsilon_{ij}$, $k_{ij} = -w_{,ij}$	Strain resultants
n_{ij}	Membrane stress-resultants
q_i	Shear stress-resultants
r_i	Kirchhoff shear stress-resultants
m_{ij}	Stress-couples

$$K = \frac{Eh}{(1-\nu^2)}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

ϕ Pücher stress-resultant function

$$\nabla^2 \equiv \frac{\partial^2}{\partial \alpha_1^2} + \frac{\partial^2}{\partial \alpha_2^2}$$

$$\nabla^4 \equiv \nabla^2 \nabla^2$$

$$\nabla_R^2 \equiv K_{22} \frac{\partial^2}{\partial \alpha_1^2} - 2K_{12} \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} + K_{11} \frac{\partial^2}{\partial \alpha_2^2}$$

col $\{ \quad \}$ Denotes a column matrix

$\underline{I}$ The unit matrix

Indirect Methods

V_o	Potential energy of elastic deformation (strain energy)
V_1	Potential energy of the surface loads
V_2	Potential energy of the applied boundary loads
$V = V_o + V_1 + V_2$	Total potential energy of the deformed shell
N_{ij}, Q_i, M_{ij}	Applied boundary loads

$$R_1 = Q_1 + M_{12,2}$$

$$R_2 = Q_2 + M_{12,1}$$

δu_1, δu_2, δw	Displacement variations (or the "virtual displacements")
u_1^m, u_2^m, w_m; U_1^n, U_2^n, W_n	Independent sets of kinematically admissible functions (unless otherwise stated)
a_m, b_m, c_m; a_{mn}, b_{mn}, c_{mn}	Arbitrary constants
δa_m, δb_m, δc_m	Arbitrary variations in the constants a_m, b_m and c_m respectively
δa_{mn}, δb_{mn}, δc_{mn}	Arbitrary variations in the constants a_{mn}, b_{mn} and c_{mn} respectively.

S — The number of approximating functions

$T_1 = \dfrac{l_1}{i}$

i — Non-zero positive integer

$\bar{r} = \dfrac{T_1}{T_2}$

$r = \dfrac{l_1}{l_2}$

$\bar{u}_i$, $\bar{w}$ — Non-dimensional forms of u_i and w respectively

$\bar{n}_{ij}, \bar{q}_i, \bar{r}_i, \bar{m}_{ij}$ — Non-dimensional forms of n_{ij}, q_i, r_i and m_{ij} respectively

$$\bar{\rho}_T = \frac{h}{T_1} \cdot \frac{1}{K_2 T_1} = -\frac{1}{8\bar{r}} \cdot \frac{h}{T_1} \cdot \frac{l_2}{f_2}$$

$$\rho_T = \frac{h}{l_1} \cdot \frac{1}{K_2 l_1} = -\frac{1}{8r} \cdot \frac{h}{l_1} \cdot \frac{l_2}{f_2}$$

Non-dimensional shell parameters for translastional shells

$$\rho_R = \frac{h}{l_1} \cdot \frac{1}{K_{12} l_1} = \frac{h}{l_1} \cdot \frac{l_2}{f} = -\frac{1}{2} \cdot \frac{h}{l_1} \cdot \frac{l_2}{\bar{f}}$$

Non-dimensional shell parameter for ruled surface shells

$\overline{Z}_o = \dfrac{Z_o(1-\nu^2)}{Eh\,K_2}$ where Z_o is a constant

$\overline{Z} = \dfrac{Z(1-\nu^2)}{Eh\,K_2}$ or $\dfrac{Z(1-\nu^2)}{Eh\,K_{12}}$

$\lambda_m, \overline{\lambda}_m$ Lagrangian multipliers

$\lambda_m^e, \overline{\lambda}_m^e$ Constants associated with the Lagrangian multipliers

L_1^e, L_2^e Sets of independent functions

Q Normal reactive force at a corner of the shell and is positive when acting in the $(-y)$ direction

E_1, E_2, E_3 Errors in equilibrium for one quarter of the shell

$\overline{E}_1, \overline{E}_2, \overline{E}_3$ Non-dimensional forms of E_1, E_2 and E_3 respectively

m, n, e Positive integers

$\overline{Q}$ Non-dimensional form of Q

<u>Method of Lines</u>

$\Theta \quad = \quad w_{,2}$

$\underline{Y} \quad = \quad \text{col} \left\{ n_{22} \; n_{12} \; m_{22} \; r_2 \; u_2 \; u_1 \; \Theta \; w \right\}$ (8 x 1) Matrix

$\underline{L} \quad = \quad \text{col} \left\{ . \quad . \quad . \; Zl_2 \; . \quad . \quad . \quad . \right\}$ (8 x 1) Matrix

N, $\beta_1^{\,k}$ — The region bounded by $\beta_1 = 0$ and $\beta_1 = 1$ is divided into 2N equal divisions by the lines $\beta_1^{\,k}$ $(k = 0, 1, 2, \ldots, 2N)$

$$a = \frac{1}{2N}$$

y^k — Denotes the value of a displacement, stress-resultant or stress-couple along the line $\beta_1^{\,k}$

$\underline{y} \quad = \quad \text{col} \left\{ y^1 \;\; y^2 \;\; \ldots\ldots \; y^k \; \ldots\ldots \; y^N \right\}$ (N x 1) Matrix

Z^k — Uniformly distributed normal loading along the line $\beta_1^{\,k}$

$\underline{Z} \; = \quad \text{col} \left\{ Z^1 \;\; Z^2 \;\; \ldots\ldots \; Z^k \; \ldots\ldots \; Z^N \right\}$ (N x 1) Matrix

$\underline{X} \; = \quad X(\beta_2) \quad = \quad \text{col} \left\{ \underline{n}_{22}, \;\; \underline{n}_{12}, \;\; \underline{m}_{22}, \;\; \underline{r}_2 \right\}$ Actions (4N x 1) Matrix

$\underline{U} \equiv \underline{U}(\beta_2) = \text{col}\left\{ \underline{u}_2, \underline{u}_1, \underline{\theta}, \underline{w} \right\}$ Displacements (4N x 1) Matrix

$\underline{F} \equiv \underline{F}(\beta_2) = \text{col}\left\{ \underline{X}, \underline{U} \right\}$ (8N x 1) Matrix

$\overline{\underline{Z}} = \text{col}\left\{ . \quad . \quad . \quad {}_{1_2}\underline{Z} \quad . \quad . \quad . \quad . \right\}$ (8N x 1) Matrix

$\underline{A}$ Matrix defined by equations (7.33)

$\underline{F}^{(p)} = - \underline{A}^{-1} \overline{\underline{Z}} = \text{col}\left\{ \underline{X}^{(p)}, \underline{U}^{(p)} \right\}$ The particular solution

$\underline{X}^{(p)} = \text{col}\left\{ \underline{n}_{22}^{(p)}, \underline{n}_{12}^{(p)}, \underline{m}_{22}^{(p)}, \underline{r}_2^{(p)} \right\}$ (4N x 1) Matrix

$\underline{U}^{(p)} = \text{col}\left\{ \underline{u}_2^{(p)}, \underline{u}_1^{(p)}, \underline{\theta}^{(p)}, \underline{w}^{(p)} \right\}$ (4N x 1) Matrix

$\underline{G}(\beta_2) = e^{-\underline{A}\beta_2}$ The distribution or transfer matrix (8N x 8N)

$\underline{G}_{ij}(\beta_2)$ Submatrices of $\underline{G}(\beta_2)$ (4N x 4N)

$\underline{F}_q \equiv \underline{F}\Big|_{\beta_2 = q}$

$\underline{C} \equiv \underline{C}(\beta_2) = \underline{F} - \underline{F}^{(p)}$

$\overline{\underline{X}}_1^m, \overline{\underline{X}}_2^m$ Actions at edges 1 and 2 of segment m

$\overline{\underline{U}}_1^m, \overline{\underline{U}}_2^m$ Displacements at edges 1 and 2 of segment m

$\underline{\bar{F}}_1^m = \text{col}\left\{ \underline{\bar{X}}_1^m, \underline{\bar{U}}_1^m \right\}$ (8N x 1) Matrix

$\underline{\bar{F}}_2^m = \text{col}\left\{ \underline{\bar{X}}_2^m, \underline{\bar{U}}_2^m \right\}$ (8N x 1) Matrix

M	Number of segments into which the shell is divided
$\underline{\bar{G}} = \underline{G}(b)$	
b	Length of the segment in the β_2 direction
$\underline{S}^m$	Stiffness matrix of the segment m
$\underline{\bar{X}}_1^{om}$, $\underline{\bar{X}}_2^{om}$	Clamped edge solution at edges 1 and 2 of segment m respectively
$\underline{S}$	Assembled stiffness matrix of the shell.

CHAPTER 1

INTRODUCTION

This thesis is concerned with the numerical solution of the shallow curved plate equations. In the following a brief review of some of the references noted will precede a discussion on the scope of the research.

1.1 A Brief Review

The shallow cylindrical shell equations have been derived by Donnell[(1),(2)]* for shell buckling problems. Jenkins[(3)] presented the stiffness (displacement) matrix method for transversely continuous shell and edge beam problems using a Levy-type solution of the Donnell equations. Extended Levy methods of solution have been used by Newman[(4)], Lu[(5)] and Gunasekera[(6)]. Chuang and Veletsos[(7)] have

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* These numbers correspond to references given at the end of this thesis.

used the Rayleigh-Ritz and Lagrangian multiplier methods and also a finite difference technique. Several types of approximating functions are considered by Chuang and Veletsos but only with Levy-type solutions.

For the case of translational shells the equations for the shallow curved plate theory are obtainable from the work of Marguerre[8] and of Vlasov[9]. Navier-type solutions have been discussed by Ambartsumyan[10] and by Flugge and Conrad[11] and Levy-type solutions by Bouma[12] and by Apeland[13]. The extended Levy method has been discussed by Ansah[14] and by Gunasekera[6]. Noor and Veletsos[15] extended the work of Chuang and Veletsos[7] to translational shells. Further suggested solution procedures have used a variational method[16],[17], a finite difference[18] technique and a discrete element technique[19].

For the case of ruled surface hyperbolic paraboloid shells, it has been shown that the Navier and Levy-type solutions correspond to unrealistic boundary conditions[20]. However such solutions are of interest and have been discussed by Apeland and Popov[21],[22]. Variational methods have been suggested by Tottenham[23],[24] and by Chetty[25],[26]. Further suggested solution procedures have used a finite difference technique[27],[28], a discrete element technique[19], an extended Levy method[29],[6] and a finite element method[30],[31]. Various approximate methods have been suggested for this problem[32],[33],[34],[35].

In this thesis the Rayleigh-Ritz, Galerkin and Lagrangian multiplier methods (referred to as "indirect methods" in this thesis) will be used. A useful review of these and other indirect methods is given by Finlayson and Scriven[36]. This reference includes an extensive bibliography.

Use will also be made of Rayleigh functions[37], which have been applied to shell problems by Vlasov[9], Oniashvilli[38], Morice[39] and by Munro[35].

Consideration will also be given to the method of lines in which, for two dimensional problems, the derivatives in one direction are replaced by finite difference expressions. Smirnov[40] attributes the method to Rothe[41]. The method was later applied by Hartree[42], Slobodyansky[43] and by Faddeyeva[44]. The latter two references are discussed by Mikhlin[45]. A further description of the method is given by Berezin and Zhidkov[46]. Jenkins and Tottenham[47] applied the method of lines to doubly curved shells, but did not present any numerical results. Chetty[25] applied this method to ruled surface hyperbolic paraboloids and presented solutions for two sets of conditions on all boundaries (viz., clamped and normal gable conditions).

However, Chetty[25] made no study of the convergence of the solution as the number of lines varied.

1.2 Scope of Research

The scope of the research will be to:

(a) apply indirect methods (Rayleigh-Ritz, Galerkin and Lagrangian multiplier methods) in conjunction with various approximating functions to translational shells for which an exact solution is possible,*

(b) apply the indirect methods and approximating functions to translational shells with various boundary conditions for which an exact solution is not possible,

(c) apply the indirect methods and approximating functions to ruled surface shells with various boundary conditions,

(d) study the behaviour of translational and ruled surface shells as certain non-dimensional parameters are varied,

(e) apply the method of lines to translational and ruled surface shells.

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*This exact solution is obtainable from a Levy-type solution.

CHAPTER 2

SHALLOW CURVED PLATE THEORY

In this chapter assumptions in addition to those made in the classical theory of thin shells will be first discussed. The required shell equations will be then derived vectorially and variationally. In the vectorial treatment, the fundamental variables are directed quantities (displacements and forces) and in the variational treatment, the fundamental quantities are scalars (potential energy).

2.1 Assumptions and Geometry of the Shell Middle Surface

Let (x_1,x_2,z) be a right handed orthogonal cartesian system of axes and let $\bar{i}_1$, $\bar{i}_2$ and $\bar{j}$ be unit vectors in x_1, x_2 and z directions respectively (figure 2.1).

Let $\bar{P}$ be the position vector of a point P measured in the (x_1,x_2,z) reference frame and let it be a function of two parameters α_1 and α_2. As α_1 and α_2 vary a surface is described. Let this represent the middle surface of the shell.

The curve described when one parameter is varied while the other is kept constant is a parametric curve.

In parametric form:

$$x_1 \equiv x_1(a_1, a_2) \tag{2.1}$$

$$x_2 \equiv x_2(a_1, a_2) \tag{2.2}$$

$$z \equiv z(a_1, a_2) \tag{2.3}$$

Then the position vector $\overline{P}$ is given by:

$$\overline{P} = x_1 \,\overline{i}_1 + x_2 \,\overline{i}_2 + z \,\overline{j} \tag{2.4}$$

Using comma notation to represent partial differentiation with respect to a_1 or a_2, the partial derivatives of $\overline{P}$ are given by:

$$\overline{P}_{,1} = x_{1,1}\overline{i}_1 + x_{2,1}\overline{i}_2 + z_{,1}\overline{j} \tag{2.5}$$

$$\overline{P}_{,2} = x_{1,2}\overline{i}_1 + x_{2,2}\overline{i}_2 + z_{,2}\overline{j} \tag{2.6}$$

Let the magnitude of the vectors $\overline{P}_{,1}$ and $\overline{P}_{,2}$ be A_{11} and A_{22} respectively.

The first fundamental quadratic form of the surface is given by (figure 2.2):

$$ds^2 = d\overline{P}\cdot d\overline{P} = A_{11}^2 da_1^2 + 2A_{12}da_1 da_2 + A_{22}^2 da_2^2 \tag{2.7}$$

where A_{11}, A_{22} and A_{12} are termed the coefficients of the first fundamental quadratic form and are defined by:

$$A_{11}^2 = \overline{P}_{,1} \cdot \overline{P}_{,1} = (x_{1,1})^2 + (x_{2,1})^2 + (z_{,1})^2 \qquad (2.8)$$

$$A_{22}^2 = \overline{P}_{,2} \cdot \overline{P}_{,2} = (x_{1,2})^2 + (x_{2,2})^2 + (z_{,2})^2 \qquad (2.9)$$

$$A_{12} = \overline{P}_{,1} \cdot \overline{P}_{,2} = A_{11}A_{22} \cos \chi =$$

$$= (x_{1,1})(x_{1,2}) + (x_{2,1})(x_{2,2}) + (z_{,1})(z_{,2}) \qquad (2.10)$$

The quantity ds^2 is termed the metric of the surface.

Consider, for example, the middle surface of the circular cylindrical shell of radius R given in figure 2.3.

Then

$$x_1 = \alpha_1 \qquad (2.11)$$

$$x_2 = R \sin \left(\frac{l_2}{2R}\right) - R \sin \left(\frac{l_2 - 2\alpha_2}{2R}\right) \qquad (2.12)$$

$$z = R \cos \left(\frac{l_2}{2R}\right) - R \cos \left(\frac{l_2 - 2\alpha_2}{2R}\right) \qquad (2.13)$$

Substituting equations (2.11), (2.12) and (2.13) into equations (2.8), (2.9) and (2.10) yields:

$$A_{11}^2 = 1 \tag{2.14}$$

$$A_{22}^2 = 1 \tag{2.15}$$

$$A_{12} = 0 \tag{2.16}$$

If the parameters chosen were α_1 and ϕ (figure 2.3) then

$$A_{11}^2 = 1 \tag{2.17}$$

$$A_{22}^2 = R^2 \tag{2.18}$$

$$A_{12} = 0 \tag{2.19}$$

Surfaces which have zero Gaussian curvature, such as the cylindrical surface in the above example, are developable surfaces and are isometric to a plane.[48] For such surfaces an (α_1, α_2) set exists such that A_{11} and A_{22} are constants and A_{12} is zero. For other surfaces this is not the case.[49]

For surfaces which are not isometric to a plane the curved plate approximation consists of taking $A_{12} = 0$ and A_{11} and A_{22} as constants, which, in particular, may be taken as unity.

Then equation (2.7) becomes:

$$ds^2 = da_1^{\,2} + da_2^{\,2} \qquad (2.20)$$

The shallow shell static approximation can be stated as:

$$K_{ij}\, m_{rs} << n_{pq} \qquad (2.21a)$$

where K_{ij} is the undeformed curvature of the middle surface, m_{rs} is a stress-couple, n_{pq} is a stress-resultant, and i,j,p,q,r,s range over the values 1 and 2. The quantities m_{rs} and n_{pq} will be defined in section (2.2).

The shallow shell kinematic approximation can be stated as:(50)

$$\gamma K_{ij} << 1 \qquad (2.21b)$$

where γ is measured in the direction normal to the (a_1, a_2) set (refer to sections 2.1.1 and 2.2.).

When the shallow shell approximations are made in conjunction with the geometric simplifications of the curved plate approximations the shallow curved plate theory results.(35),(50)

This thesis will be restricted to shallow curved plates which are thin, of constant thickness and rectangular plan-form.

The loading will be static and all problems will be linearised. Linearisation will be achieved by assuming:

(a) linearly elastic constitutive relations

(b) small (infinitesimal) displacements

(c) linearised strain-displacement relations.

2.1.1 Simplified Geometry of the Shell Middle Surface

In this section the shell types used in this thesis will be discussed in conjunction with the curved plate approximation.

Let the middle surface of the shell, in terms of the reference frame (x_1, x_2, z) be defined by:

$$z = \tfrac{1}{2}a_1 x_1^2 + \tfrac{1}{2}a_2 x_2^2 + a_3 x_1 x_2 + a_4 x_1 + a_5 x_2 + a_6 \qquad (2.22)$$

where a_i $(i = 1,2,\ldots,6)$ are constants.

Let the (α_1, α_2) set be defined by the intersection of the x_1 = constant and x_2 = constant planes with the middle surface of the shell and let γ be mutually orthogonal to the (α_1, α_2) set.

Within the limits of the curved plate approximations:

(a) the (α_1, α_2) set is sensibly orthogonal

and (b) the products of the slopes, $\frac{\partial z}{\partial x_1}$ and $\frac{\partial z}{\partial x_2}$, of the undeformed middle surface of the shell are negligible compared with unity.

From (b) and equation (2.22) the undeformed curvatures of the shell middle surface are constant and given by:

$$K_{11} \simeq \frac{\partial^2 z}{\partial x_1^{\;2}} = a_1 \qquad (2.23)$$

$$K_{12} \simeq \frac{\partial^2 z}{\partial x_1 \partial x_2} = a_3 \qquad (2.24)$$

$$K_{22} \simeq \frac{\partial^2 z}{\partial x_2^{\;2}} = a_2 \qquad (2.25)$$

K_{11} is the undeformed curvature of the a_1 line, K_{22} is the undeformed curvature of the a_2 line and K_{12} is the undeformed twist of the middle surface.

The shells considered are classified under translational and ruled surface shells.

I Translational Shells (figure 2.4):

When $a_3 = 0$, equation (2.22) defines a translational shell.

Let K_1 and K_2 be the principal curvatures and let K_G denote the Gaussian curvature.

Then

$$K_G = K_1 K_2$$

Within the curved plate approximations $K_1 \simeq K_{11}$ and $K_2 \simeq K_{22}$.

Then the equation of the middle surface of a translational shell is given by:

$$z = \frac{K_2}{2}\left[c(x_1^2 - l_1x_1) + (x_2^2 - l_2x_2)\right] \qquad (2.26)$$

where
$$c = \frac{K_1}{K_2} \qquad (2.27)$$

$$K_1 = -\frac{8f_1}{l_1^2} \qquad (2.28)$$

$$K_2 = -\frac{8f_2}{l_2^2} \qquad (2.29)$$

where f_1, f_2, l_1 and l_2 are defined in figure 2.4.

Translational shells may be further classified according to their Gaussian curvature into:

(a) elliptic paraboloids for which $K_G > 0$

(b) hyperbolic paraboloids for which $K_G < 0$

and (c) parabolic cylinders for which $K_G = 0$.

Note that with a circular cylinder it is not necessary to resort to the geometric simplifications of the curved plate approximations (refer to the example considered in section 2.1).

II Ruled Surface Shells (figure 2.5):

This classification follows when $K_{11} = 0 = K_{22}$ and $K_{12} \neq 0$.

Such shells will be referred to as "ruled surface hyperbolic paraboloid shells" or simply "ruled surface shells".

In figure 2.5 two alternative definitions for the equation of the middle surface are given.

In figure 2.5a the equation of the middle surface is given by:

$$z = K_{12} x_1 x_2 \qquad (2.30)$$

where
$$K_{12} = \frac{f}{l_1 l_2} \qquad (2.31)$$

and where f, l_1 and l_2 are defined in figure 2.5a. This form is only symmetric about one diagonal.

In figure 2.5b the equation of the middle surface is given by:

$$z = K_{12}\left(-\frac{l_2 x_1}{2} - \frac{l_1 x_2}{2} + x_1 x_2\right) \qquad (2.32)$$

where $K_{12} = -\frac{2\bar{f}}{l_1 l_2}$ (2.33)

where $\bar{f}(= -\frac{f}{2})$, l_1 and l_2 are defined in figure 2.5b. This form is symmetric about either diagonal, and will be used in conjunction with the overall equilibrium check in section 6.4.

Vlasov[9] considers that the simplifications made are such that the theory is sufficiently accurate if the maximum (rise/length) ratio does not exceed $\frac{1}{5}$.

2.2 Vectorial Treatment of Shell Equations

Consider a differential element of the shell (figure 2.6a and 2.6b).

Using the Einstein summation convention, the equations of equilibrium for a shallow curved plate are:

$$n_{ij,j} + X_i = 0 \quad (2.34)$$

$$K_{ij}n_{ij} + q_{i,i} + Z = 0 \quad (2.35)$$

$$m_{ij,j} - q_i = 0 \quad (2.36)$$

where n_{ij} and q_i will be termed stress-resultants, m_{ij} will be termed stress-couples, K_{ij} are the undeformed curvatures of the shell middle surface, X_i and Z are the loading pressures in the α_i and γ directions respectively and i and j range over the values 1 and 2. Comma notation is used to represent partial differentiation.

The components of strain on the middle surface are given by:

$$\mathcal{E}_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} - 2K_{ij}w) \qquad (2.37)$$

where u_i and w are the middle surface displacement components (hereafter referred to as "displacements"). After neglecting γK_{11} and γK_{22} compared with unity (Love's first approximation), the components of strain on the γ surface (figure 2.7) of the shell are given by:

$$(\mathcal{E}_{ij})_{\gamma} = \mathcal{E}_{ij} - \gamma w,_{ij} \qquad (2.38)$$

Again, after neglecting γK_{11} and γK_{22} compared with unity and assuming the middle and centroidal surfaces to coincide (figure 2.8),[3],[20] the stress-resultants n_{ij} and the stress-couples m_{ij} are defined by:

$$n_{ij} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} (\sigma_{ij})_{\gamma} d\gamma \qquad (2.39)$$

$$m_{ij} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \gamma(\sigma_{ij})_{\gamma} d\gamma \qquad (2.40)$$

where h is the shell thickness and $(\sigma_{ij})_\gamma$ are the components of stress on the γ surface.

The constitutive relations are:

$$\sigma_{ij} = \frac{E}{(1-\nu^2)} \left[(1-\nu)\,\varepsilon_{ij} + \nu\,\delta_{ij}\,\varepsilon_{pp} \right] \qquad (2.41)$$

where E is Young's modulus of elasticity of the shell material, is Poisson's ratio and δ_{ij} is the Kronecker delta.

Introducing the strain-resultants e_{ij} and k_{ij} defined by:

$$e_{ij} = \varepsilon_{ij} \qquad (2.42)$$

$$k_{ij} = -w_{,ij} \qquad (2.43)$$

then, from equations (2.37) to (2.43) inclusive, the stress-resultants, n_{ij}, and the stress couples, m_{ij}, can be expressed in the following form:

$$n_{ij} = K\left[(1-\nu)e_{ij} + \nu\,\delta_{ij}e_{pp}\right] \qquad (2.44)$$

$$m_{ij} = D\left[(1-\nu)k_{ij} + \nu\,\delta_{ij}k_{pp}\right] \qquad (2.45)$$

where

$$K = \frac{Eh}{(1-\nu^2)} \qquad (2.46)$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad (2.47)$$

In the following the forms of the shell equations referred to in this thesis will be derived.

2.2.1. Shell Equations in Terms of the Displacement w and the Pücher Stress-Resultant Function ϕ

The Pücher stress-resultant function is defined by:

$$n_{11} = \phi_{,22} - \int X_1 da_1 \qquad (2.48)$$

$$n_{12} = -\phi_{,12} \qquad (2.49)$$

$$n_{22} = \phi_{,11} - \int X_2 da_2 \qquad (2.50)$$

The equations of equilibrium (2.34), (2.35) and (2.36), after eliminating q_i and substituting for m_{ij} and n_{ij} by equations (2.43), (2.45), (2.48), (2.49) and (2.50), reduce to the single equation:

$$D\nabla^4 w - \nabla_R^4 \phi = Z - K_{11}\int X_1 da_1 - K_{22}\int X_2 da_2 \qquad (2.51)$$

where

$$\nabla^4 \equiv \nabla^2 \nabla^2$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial \alpha_1^{\;2}} + \frac{\partial^2}{\partial \alpha_2^{\;2}}$$

$$\nabla_R^2 \equiv K_{22} \frac{\partial^2}{\partial \alpha_1^{\;2}} - 2K_{12} \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} + K_{11} \frac{\partial^2}{\partial \alpha_2^{\;2}}$$

The second equation linking w and ϕ is obtained from the compatibility equation:

$$\varepsilon_{11,22} - 2\varepsilon_{12,12} + \varepsilon_{22,11} = - Eh \nabla_R^2 w \qquad (2.52)$$

which, from equations (2.42), (2.44), (2.48), (2.49) and (2.50), yields

$$\nabla^4 \phi + Eh \nabla_R^2 w = \frac{\partial^2}{\partial \alpha_2^{\;2}} \int X_1 d\alpha_1 + \frac{\partial^2}{\partial \alpha_1^{\;2}} \int X_2 d\alpha_2 - $$

$$- \nu \left(\frac{\partial X_1}{\partial \alpha_1} + \frac{\partial X_2}{\partial \alpha_2}\right) \qquad (2.53)$$

2.2.2. Shell Equations in Terms of the Displacements u_1, u_2 and w

Equations (2.34) and (2.35), after substituting for q_i from equation (2.36) expressing n_{ij} and m_{ij} in terms of the displacements u_1, u_2 and w from equations (2.37), (2.42), (2.43), (2.44) and (2.45), yield the required equations, which in matrix form are:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ -w \end{bmatrix} + \frac{(1-\nu^2)}{Eh} \begin{bmatrix} X_1 \\ X_2 \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2.54)$$

where

$$L_{11} = \left[\frac{\partial^2}{\partial \alpha_1^2} + \frac{(1-\nu)}{2} \frac{\partial^2}{\partial \alpha_2^2}\right]$$

$$L_{12} = \left[\frac{(1+\nu)}{2} \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2}\right] = L_{21}$$

$$L_{13} = \left[(1-\nu)K_{12}\frac{\partial}{\partial \alpha_2} + (K_{11} + \nu K_{22})\frac{\partial}{\partial \alpha_1}\right] = L_{31}$$

$$L_{22} = \left[\frac{(1-\nu)}{2} \frac{\partial^2}{\partial \alpha_1^2} + \frac{\partial^2}{\partial \alpha_2^2}\right]$$

$$L_{23} = \left[(1-\nu)K_{12}\frac{\partial}{\partial \alpha_1} + (K_{22} + \nu K_{11})\frac{\partial}{\partial \alpha_2}\right] = L_{32}$$

$$L_{33} = \left[\frac{h^2}{12} \nabla^4 + K_{11}^2 + 2(1-\nu)K_{12}^2 + 2\nu K_{11}K_{22} + K_{22}^2 \right]$$

2.2.3 Shell Equations in Terms of Four Actions $(n_{22}, n_{12}, r_2, m_{22})$ and Four Displacements (u_2, u_1, w, Θ)

The form of the equations derived in this section will be used in conjunction with the method of lines (Chapter 7), as discussed by Jenkins and Tottenham.(47)

The equations of equilibrium (2.34), (2.35) and (2.36) when written out in full are:

$$n_{11,1} + n_{12,2} + X_1 = 0 \qquad (2.55)$$

$$n_{12,1} + n_{22,2} + X_2 = 0 \qquad (2.56)$$

$$K_{11}n_{11} + 2K_{12}n_{12} + n_{22}K_{22} + q_{1,1} + q_{2,2} + Z = 0 \qquad (2.57)$$

$$m_{11,1} + m_{12,2} - q_1 = 0 \qquad (2.58)$$

$$m_{12,1} + m_{22,2} - q_2 = 0 \qquad (2.59)$$

The Kirchhoff shears are given by:

$$r_1 = q_1 + m_{12,2} \qquad (2.60)$$

$$r_2 = q_2 + m_{12,1} \qquad (2.61)$$

Equations (2.44) and (2.45) when written out in full are:

$$n_{11} = \frac{Eh}{(1-\nu^2)}\left[u_{1,1} + \nu u_{2,2} - (K_{11} + \nu K_{22})w\right] \qquad (2.62)$$

$$n_{22} = \frac{Eh}{(1-\nu^2)}\left[u_{2,2} + \nu u_{1,1} - (K_{22} + \nu K_{11})w\right] \qquad (2.63)$$

$$n_{12} = \frac{Eh}{2(1+\nu)}\left[u_{1,2} + u_{2,1} - 2K_{12}w\right] \qquad (2.64)$$

$$m_{11} = \frac{-Eh^3}{12(1-\nu^2)}\left[w,_{11} + \nu w,_{22}\right] \qquad (2.65)$$

$$m_{22} = \frac{-Eh^3}{12(1-\nu^2)}\left[w,_{22} + \nu w,_{11}\right] \qquad (2.66)$$

$$m_{12} = -\frac{Eh^3}{12(1+\nu)}\left[w,_{12}\right] \qquad (2.67)$$

From equations (2.62), (2.63), (2.65) and (2.66) the following are obtained:

$$n_{11} - \nu n_{22} = Eh(u_{1,1} - K_{11}w) \qquad (2.68)$$

$$n_{22} - \nu n_{11} = Eh(u_{2,2} - K_{22}w) \qquad (2.69)$$

$$m_{11} - \nu m_{22} = - \frac{Eh^3}{12}(w,_{11}) \qquad (2.70)$$

$$m_{22} - \nu m_{11} = - \frac{Eh^3}{12}(w,_{22}) \qquad (2.71)$$

From equations (2.55) and (2.68) the following is obtained

$$n_{12,2} + \nu n_{22,1} + Eh(u_{1,11} - K_{11}w,_1) + X_1 = 0 \qquad (2.72)$$

Defining

$$\Theta = w,_2 \qquad (2.73)$$

then equations (2.59), (2.61) and (2.67) yield the following:

$$m_{22,2} - r_2 - \frac{Eh^3}{6(1+\nu)}\Theta,_{11} = 0 \qquad (2.74)$$

From equations (2.57), (2.58), (2.61), (2.68) and (2.70), the following equation is obtained:

$$r_{2,2} + 2K_{12}n_{12} + (K_{22} + \nu K_{11})n_{22} + \nu m_{22,11} +$$

$$+ EhK_{11}u_{1,1} - EhK_{11}{}^2w - \frac{Eh^3}{12} w,_{1111} + Z = 0 \qquad (2.75)$$

The required equations are given by (2.56), (2.63), (2.64), (2.66), (2.72), (2.73), (2.74) and (2.75), and can be arranged in the following matrix form:

$$\frac{\partial}{\partial \alpha_2}\begin{bmatrix} n_{22} \\ n_{12} \\ m_{22} \\ r_2 \\ u_2 \\ u_1 \\ \theta \\ w \end{bmatrix} + \begin{bmatrix} \cdot & A_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{21} & \cdot & \cdot & \cdot & \cdot & A_{26} & \cdot & A_{28} \\ \cdot & \cdot & \cdot & A_{34} & \cdot & \cdot & A_{37} & \cdot \\ A_{41} & A_{42} & A_{43} & \cdot & \cdot & A_{46} & \cdot & A_{48} \\ A_{51} & \cdot & \cdot & \cdot & \cdot & A_{56} & \cdot & A_{58} \\ \cdot & A_{62} & \cdot & \cdot & A_{65} & \cdot & \cdot & A_{68} \\ \cdot & \cdot & A_{73} & \cdot & \cdot & \cdot & \cdot & A_{78} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A_{87} & \cdot \end{bmatrix}\begin{bmatrix} n_{22} \\ n_{12} \\ m_{22} \\ r_2 \\ u_2 \\ u_1 \\ \theta \\ w \end{bmatrix} + \begin{bmatrix} X_2 \\ X_1 \\ \cdot \\ Z \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \underline{O} \qquad (2.76)$$

where

$$A_{12} = \frac{\partial}{\partial \alpha_1} = A_{65}$$

$$A_{21} = \nu \frac{\partial}{\partial \alpha_1} = A_{56}$$

$$A_{26} = Eh \frac{\partial^2}{\partial \alpha_1{}^2}$$

$$A_{28} = -EhK_{11} \frac{\partial}{\partial \alpha_1} = -A_{46}$$

$$A_{34} = -1 = A_{37}$$

$$A_{37} = -\frac{Eh^3}{6(1+\nu)} \frac{\partial^2}{\partial \alpha_1^2}$$

$$A_{41} = (K_{22} + \nu K_{11}) = -A_{58}$$

$$A_{42} = 2K_{12} = -A_{68}$$

$$A_{43} = \nu \frac{\partial^2}{\partial \alpha_1^2} = A_{78}$$

$$A_{48} = \left[-EhK_{11}^2 - \frac{Eh^3}{12} \frac{\partial^4}{\partial \alpha_1^4}\right]^*$$

$$A_{51} = -\frac{(1-\nu^2)}{Eh}$$

$$A_{62} = -\frac{2(1+\nu)}{Eh}$$

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*Jenkins and Tottenham[47] neglected the term EhK_{11}^2 as being small. This term has been retained in equations (2,76), which is consistent with the other shell equations derived.

$$A_{73} = \frac{12(1-\nu^2)}{Eh^3}$$

or, more compactly:

$$\underline{Y}_{,2} + \underline{A}\,\underline{Y} + \underline{L} = O \qquad (2.77)$$

where $\underline{Y} = \text{col}\left\{n_{22} n_{12} m_{22} r_2 u_2 u_1 \Theta w\right\}$

$\underline{L} = \text{col}\left\{X_2 X_1 \,.\, Z \,.\,.\,.\,.\right\}$

2.3 Variational Treatment of Shell Equations

In the following, the term "kinematically admissible displacements" means displacements which satisfy the internal compatibility conditions and the kinematic conditions on that part of the surface where displacements are prescribed; the term "statically admissible stresses" means stresses which satisfy the internal equilibrium conditions and the equilibrium conditions on that part of the surface where external forces are prescribed.

In studying the equilibrium of an elastic system, two principles* may be applied:

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*These are, respectively, particular cases of two general principles applicable to any mechanical system in equilibrium, viz.:

(a) the principle of virtual displacements

(b) the principle of virtual changes in the stressed state.

(a) the principle of minimum total potential energy, in which variations with respect to kinematically admissible displacements are considered

(b) the principle of minimum complementary energy, in which variations with respect to statically admissible stresses are considered.

Generally, if kinematically admissible displacements are assumed, the equilibrium conditions are violated, and if statically admissible stresses are assumed, the compatibility conditions are violated. However, it follows from (a) above that stable equilibrium corresponds to those kinematically admissible displacements for which the total potential energy is a minimum, and from (b) that the satisfaction of the compatibility conditions corresponds to those statically admissible stresses for which the complementary energy is a minimum.[51]

For the application of these two principles, suitable kinematically admissible displacements and statically admissible stresses must be found. For (a), internal compatibility is satisfied by selecting displacements which are continuous. However, for (b), internal equilibrium must be established by selecting stresses which satisfy the equations of equilibrium. Since these equations are differential equations such stresses are not always easy to find. Further, the displacement approach offers a more direct formulation for the boundary conditions of the problem.

In view of the foregoing, only principle (a) will be considered. Further the assumptions made in section (2.1) relating to thin elastic shallow curved plates will still apply. The displacements considered will be small and kinematically admissible.* Similarly, the variations in the displacements (or the "virtual displacements") will be small and kinematically admissible and will vanish wherever the displacements are prescribed.

For the problem under consideration, the total potential energy, V, of the deformed shell (rectangular plan-form) is the sum of:

(i) the potential energy of elastic deformation (strain energy), V_o, given by:

$$V_o = \frac{1}{2} \int_o^{l_1} \int_o^{l_2} \int_{-\frac{h}{2}}^{+\frac{h}{2}} (\sigma_{ij})_\gamma (\varepsilon_{ij})_\gamma \, d\alpha_1 d\alpha_2 d\gamma \qquad (2.78)$$

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* It is sometimes useful to relax the prescribed kinematic conditions. This will be discussed further in conjunction with the Lagrangian multiplier method in section (3.1.3).

where i and j range over the values 1 and 2, $(\sigma_{ij})_\gamma$ and $(\varepsilon_{ij})_\gamma$ are respectively the stress and strain components on the γ surface of the shell (figure 2.7), and γK_{11} and γK_{22} are considered small compared with unity,

(ii) the potential energy of the surface loads, V_1, given by:

$$V_1 = - \int_0^{l_1} \int_0^{l_2} (X_i u_i + Zw) \, da_1 da_2 \qquad (2.79)$$

where X_i and Z are respectively the surface loads corresponding to, but independent of, the displacements u_i and w,

and (iii) the potential energy of the applied boundary loads, V_2, given by:

$$V_2 = \sum_n V_2^{\,n} \qquad (2.80)$$

where

$$V_2^{\,n} = - \int_0^{l_i} F_m v_m \, da_i \qquad (2.81)$$

and where, for each boundary n, F_m is the applied boundary load corresponding to, but independent of, the boundary displacement v_m.*

*When the boundary is flexible, F_m is dependent on v_m and in the subsequent integrations the relationship between F_m and v_m must be considered.

Then it follows that

$$V = V_o + V_1 + V_2 \qquad (2.82)$$

From equations (2.38), (2.39), (2.40), (2.42), (2.43) and (2.78) the following is obtained:

$$V_o = \tfrac{1}{2}\int_o^{l_1}\int_o^{l_2}(n_{ij}e_{ij} + m_{ij}k_{ij})d\alpha_1 d\alpha_2 \qquad (2.83)$$

Substituting for n_{ij}, m_{ij} and k_{ij} by equations (2.44), (2.45) and (2.43) respectively, equation (2.83) becomes:

$$V_o = \tfrac{1}{2}\int_o^{l_1}\int_o^{l_2}\left\{\frac{Eh}{(1-\nu^2)}\left[e_{11}^2 + 2(1-\nu)e_{12}^2 + e_{22}^2 + 2\nu e_{11}e_{22}\right]\right.$$

$$\left. + \frac{Eh^3}{12(1-\nu^2)}\left[w,_{11}^2 + 2(1-\nu)w,_{12}^2 + w,_{22}^2 + 2\nu w,_{11}w,_{22}\right]\right\}d\alpha_1 d\alpha_2 \quad (2.84)$$

Equation (2.80) can be written in the form (figure 2.6):

$$V_2 = -\left[\int_o^{l_2}(N_{11}u_1 + N_{12}u_2 + Q_1 w - M_{11}w,_1 - M_{12}w,_2)d\alpha_2\right]_{\alpha_1=o}^{\alpha_1=l_1}$$

$$-\left[\int_o^{l_1}(N_{12}u_1 + N_{22}u_2 + Q_2 w - M_{12}w,_1 - M_{22}w,_2)d\alpha_1\right]_{\alpha_2=o}^{\alpha_2=l_2} = O \quad (2.85)$$

where N_{11}, N_{12}, N_{22}, Q_1, Q_2, M_{11}, M_{12} and M_{22} are the applied boundary loads and u_1, u_1 or u_2, u_2, w, w, $(w_{,1})$, $(w_{,1})$ or $(w_{,2})$, and $(w_{,2})$ are the corresponding displacements respectively. The minus signs in the terms containing M_{11}, M_{12} and M_{22} in equation (2.85) are due to the sign conventions adopted for M_{11} and the corresponding slopes.

From equations (2.79), (2.82), (2.84) and (2.85), the total potential energy of the deformed shell becomes:

$$V = \frac{1}{2}\int_0^{l_1}\int_0^{l_2}\left\{\frac{Eh}{(1-\nu^2)}\left[e_{11}^2 + 2(1-\nu)e_{12}^2 + e_{22}^2 + 2\nu e_{11}e_{12}\right] + \right.$$

$$\left. + \frac{Eh^3}{12(1-\nu^2)}\left[w_{,11}^2 + 2(1-\nu)w_{,12}^2 + w_{,22}^2 + 2\nu w_{,11}w_{,22}\right]\right\} da_1 da_2 -$$

$$- \int_0^{l_1}\int_0^{l_2}(X_1u_1 + X_2u_2 + Zw)da_1da_2 -$$

$$- \left[\int_0^{l_1}(N_{12}u_1 + N_{22}u_2 + Q_2w - M_{12}w_{,1} - M_{22}w_{,2})da_1\right]_{a_2=0}^{a_2=l_2}$$

$$- \left[\int_0^{l_2}(N_{11}u_1 + N_{12}u_2 + Q_1w - M_{11}w_{,1} - M_{12}w_{,2})da_2\right]_{a_1=0}^{a_1=l_1} \qquad (2.86)$$

For stable equilibrium, the total potential energy of the deformed shell is a minimum and therefore assumes a stationary value,

$$\text{i.e.} \quad \delta V = 0 \qquad (2.87)$$

which, with equation (2.86), yields:

$$\int_0^{l_1}\int_0^{l_2}\Bigg\{ \frac{Eh}{(1-\nu^2)}\Big[e_{11}\delta e_{11} + 2(1-\nu)e_{12}\delta e_{12} + e_{22}\delta e_{22} + \nu e_{11}\delta e_{22} + \nu e_{22}\delta e_{11}\Big] + \frac{Eh^3}{12(1-\nu^2)}\Big[w,_{11}\delta w,_{11} + 2(1-\nu)w,_{12}\delta w,_{12} + w,_{22}\delta w,_{22} + \nu w,_{11}\delta w,_{22} + \nu w,_{22}\delta w,_{11}\Big]\Bigg\} da_1 da_2 -$$

$$- \int_0^{l_1}\int_0^{l_2}(X_1\delta u_1 + X_2\delta u_2 + Z\delta w)da_1 da_2 -$$

$$- \Big[\int_0^{l_2}(N_{11}\delta u_1 + N_{12}\delta u_2 + Q_1\delta w - M_{11}\delta w,_1 - M_{12}\delta w,_2)da_2\Big]_{a_1=0}^{a_1=l_1}$$

$$- \Big[\int_0^{l_1}(N_{12}\delta u_1 + N_{22}\delta u_2 + Q_2\delta w - M_{12}\delta w,_1 - M_{22}\delta w,_2)da_1\Big]_{a_2=0}^{a_2=l_2} = 0 \quad (2.88)$$

Using equations (2.44) and (2.45), equation (2.88) becomes:

$$\int_0^{l_1}\int_0^{l_2}\Big[(n_{11}\delta e_{11} + 2n_{12}\delta e_{12} + n_{22}\delta e_{22}) - (m_{11}\delta w_{,11} +$$

$$+ 2m_{12}\delta w_{,12} + m_{22}\delta w_{,22})\Big]\, d\alpha_1 d\alpha_2 \; -$$

$$\int_0^{l_1}\int_0^{l_2}(X_1\delta u_1 + X_2\delta u_2 + Z\delta w)d\alpha_1 d\alpha_2 \; -$$

$$- \Big[\int_0^{l_2}(N_{11}\delta u_1 + N_{12}\delta u_2 + Q_1\delta w - M_{11}\delta w_{,1} - M_{12}\delta w_{,2})d\alpha_2\Big]_{\alpha_1=o}^{\alpha_1=l_1} \; -$$

$$- \Big[\int_0^{l_1}(N_{12}\delta u_1 + N_{22}\delta u_2 + Q_2\delta w - M_{12}\delta w_{,1} - M_{22}\delta w_{,2})d\alpha_1\Big]_{\alpha_2=o}^{\alpha_1=l_2} = O \quad (2.89)$$

Using equations (2.37), (2.42), (2.43), (2.44) and (2.45), and the relations given in table 2.1, equation (2.89) becomes:

$$\frac{-Eh}{(1-\nu^2)}\int_o^{l_1}\int_o^{l_2}\left\{\left[L_{11}u_1 + L_{12}u_2 - L_{13}w + \frac{X_1(1-\nu^2)}{Eh}\right]\delta u_1 + \right.$$

$$+\left[L_{21}u_1 + L_{22}u_2 - L_{23}w + \frac{X_2(1-\nu^2)}{Eh}\right]\delta u_2 +$$

$$\left. + \left[L_{21}u_1 + L_{22}u_2 - L_{23}w + \frac{Z(1-\nu^2)}{Eh}\right]\delta w\right\} d\alpha_1 d\alpha_2 +$$

$$+\left\{\int_o^{l_2}\left[(n_{11}-N_{11})\delta u_1 + (n_{12}-N_{12})\delta u_2 - (m_{11}-M_{11})\delta w_{,1} + (r_1-R_1)\delta w\right] d\alpha_2\right\}_{\alpha_1=o}^{\alpha_1=l_1} +$$

$$+\left\{\int_o^{l_1}\left[(n_{22}-N_{22})\delta u_2 + (n_{12}-N_{12})\delta u_1 - (m_{22}-M_{22})\delta w_{,2} + (r_2-R_2)\delta w\right] d\alpha_1\right\}_{\alpha_2=o}^{\alpha_2=l_2} -$$

$$-\left\{\left[2(m_{12}-M_{12})\delta w\right]_{\alpha_1=o}^{\alpha_1=l_1}\right\}_{\alpha_2=o}^{\alpha_2=l_2} = 0 \qquad (2.90)$$

where

$$R_1 = Q_1 + M_{12,2} \qquad (2.91)$$

$$R_2 = Q_2 + M_{12,1} \qquad (2.92)$$

and where the partial differential operators L_{ij} $(i,j = 1,2,3)$ are defined by equations (2.54).

Equation (2.90) yields directly the three equations of equilibrium together with the four boundary conditions (static or kinematic) which need to be specified along each boundary.

For either of the principles discussed in this section, the boundary conditions are subdivided into:

(i) Those which are essential for the application of the principle (the "essential"[52],[53] or "imposed"[54] boundary conditions)

and (ii) those which are realised by virtue of the principle itself (the "additional",[52] "natural",[54] or "suppressible"[53] boundary conditions).

Only the principle of minimum total potential energy is considered in this thesis. For this problem the kinematic boundary conditions are termed the "imposed" boundary conditions and the static boundary conditions are termed the "natural" boundary conditions.

CHAPTER 3

SOLUTION METHODS

This thesis is mainly concerned with the application of indirect methods (Rayleigh-Ritz, Galerkin and Lagrangian multiplier methods) in conjunction with various types of approximating functions. However, consideration is also given to the method of lines in which the derivatives in one direction are replaced by finite difference expressions.

3.1 Indirect Methods and Approximating Functions

In this thesis, solution methods will be referred to as "indirect methods" when the functions in the series representation for the dependent variables do not satisfy the partial differential equations and all boundary conditions term-by-term. Solution methods will be referred to as "direct methods" when the functions do satisfy the partial differential equations and all boundary conditions term-by-term (e.g. Navier and Levy-type solutions).

In the following, the term "kinematically admissible functions" means functions which are continuous and differentiable and which satisfy the imposed boundary conditions where prescribed.

3.1.1 Rayleigh-Ritz Method

The variational equation (2.90) forms the basis of the Rayleigh-Ritz method used in this thesis for the solution of thin shallow curved plates.

In this approach, the displacements are considered in the following series form:

$$u_1 = \sum_m \sum_n a_{mn} u_1^m(\alpha_1) U_1^n(\alpha_2) \tag{3.1}$$

$$u_2 = \sum_m \sum_n b_{mn} u_2^m(\alpha_1) U_2^n(\alpha_2) \tag{3.2}$$

$$w = \sum_m \sum_n c_{mn} w_m(\alpha_1) W_n(\alpha_2) \tag{3.3}$$

where u_1^m, U_1^n, u_2^m, U_2^n, w_m and W_n represent independent sets of kinematically admissible functions, a_{mn}, b_{mn} and c_{mn} are arbitrary constants to be determined, and m and n are positive integers.

The displacement variations may be selected in the following forms:

$$\delta u_1 = \sum_m \sum_n u_1^m(\alpha_1) U_1^n(\alpha_2) \delta a_{mn} \tag{3.4}$$

$$\delta u_2 = \sum_m \sum_n u_2^m(\alpha_1) U_2^n(\alpha_2) \delta b_{mn} \tag{3.5}$$

$$\delta w = \sum_m \sum_n w_m(\alpha_1) W_n(\alpha_2) \delta c_{mn} \tag{3.6}$$

where δa_{mn}, δb_{mn} and δc_{mn} are arbitrary variations in the constants a_{mn}, b_{mn} and c_{mn} respectively.

Substituting equations (3.1) to (3.6) inclusive into equations (2.90), integrating the resulting expressions and noting that δa_{mn}, δb_{mn} and δc_{mn} are arbitrary, yields a set of simultaneous linear equations in terms of the constants a_{mn}, b_{mn} and c_{mn}. By using truncated series, the problem is reduced from one with infinite degrees of freedom to one with finite degrees of freedom.

Equations (3.1), (3.2) and (3.3) represent a family of kinematically admissible displacements and the Rayleigh-Ritz method attempts to find those constants (a_{mn}, b_{mn}, c_{mn}) for which the equilibrium conditions within the shell and on its boundaries are satisfied.

From equation (2.90) it follows that, when a kinematic boundary condition is prescribed, the corresponding boundary integral vanishes. When a static boundary condition is prescribed, and is not satisfied by the chosen functions, the corresponding boundary integral remains. The Rayleigh-Ritz method will seek out this "natural" boundary condition.

3.1.2 Galerkin Method

If the functions given in equations (3.1), (3.2) and (3.3) are chosen such that all the boundary conditions, static and kinematic, are satisfied, then all the boundary integrals in equation (2.90) vanish and the Galerkin equations are obtained.

The Galerkin method has a wider application than the Rayleigh-Ritz method, since it is not restricted to variational problems.[51),(55),(56] However, the Galerkin and Rayleigh-Ritz methods become equivalent when:

(a) applied to variational problems associated with quadratic functionals (as in this thesis)

and (b) the kinematically admissible functions given in equations (3.1), (3.2) and (3.3) satisfy, in addition, the static boundary conditions where they are prescribed.

3.1.3 Lagrangian Multiplier Method

It is sometimes useful to relax the kinematic boundary conditions by selecting functions which are not kinematically admissible. Use can then be made of the Lagrangian multiplier method,[54] in which the kinematic boundary conditions violated are applied as constraint conditions.

Suppose the kinematic boundary condition

$$u_1(\alpha_1, o) = 0 \qquad (3.7)$$

is prescribed and that the corresponding functions given in equation (3.1) are chosen such that this condition is not satisfied.

The Lagrangian multiplier method introduces another variable $\lambda_1(\alpha_1)$, the Lagrangian multiplier, such that

$$\delta V + \int_o^{l_1} \lambda_1(\alpha_1)\delta u_1(\alpha_1, o)d\alpha_1 = O \qquad (3.8)$$

where V is the total potential energy of the deformed shell given by equation (2.86). The corresponding constraint condition is given by equation (3.7).

Substituting the series given by equation (3.1) for u_1 in equations (3.8) and (3.7) yields respectively:

$$\delta V + \int_o^{l_1} \lambda_1(\alpha_1)u_1^m(\alpha_1)U_1^n(o)d\alpha_1\delta a_{mn} = O \qquad (3.9)$$

$$a_{mn}u_1^m(\alpha_1)U_1^n(o) = O \qquad (3.10)$$

where the Einstein summation convention is used.

The Lagrangian multiplier method conveniently reduces the constrained variational problem to one of free variation. Note that the series for u_1 no longer vanishes term-by-term on the boundary (α_2=o), but is replaced by the condition that the series as a whole vanishes (equation

3.10). The Lagrangian multiplier has a physical meaning - it is the generalised reactive force associated with the corresponding constraint condition. The Rayleigh-Ritz method may be considered as a particular case of the Lagrangian multiplier method with all multipliers set to zero.

The multiplier $\lambda_1(a_1)$ is a general function of a_1 and cannot readily be determined in this form. However, $\lambda_1(a_1)$ can be expressed as the following series:

$$\lambda_1(a_1) = \sum_k \lambda_1^k L_1^k(a_1) \tag{3.11}$$

where $L_1^k(a_1)$ represents a set of independent functions, λ_1^k are constants and k is a positive integer.

Substitution of equation (3.11) into (3.9) yields:

$$\delta V + \int_0^{l_1} \lambda_1^k L_1^k(a_1) u_1^m(a_1) U_1^n(o) da_1 \delta a_{mn} = 0 \tag{3.12}$$

The constraint condition (3.10) can be rearranged in the form:

$$\left[a_{mn} U_1^n(o) \right] u_1^m(a_1) = 0 \tag{3.13}$$

Since each $u_1^m(a_1)$ is independent, then for all a_1 the following condition holds:

$$a_{mn} U_1^n(o) = O \qquad (3.14)$$

Assuming that the same number, S (say), of functions are chosen for $u_1^m(a_1)$ and $L_1^k(a_1)$, then equations (3.12) and (3.14) introduce an additional S unknowns, λ_1^k, together with an additional S equations given by equation (3.14). The problem can now be conveniently solved.

Similar remarks apply to any other prescribed kinematic condition which may be violated.

In particular, functions $L_1^k(a_1)$ and $u_1^m(a_1)$ may represent the same set of orthogonal functions. Then equations (3.12) and (3.14) become:

$$\delta V + g\lambda_1^m U_1^n(o)\delta a_{mn} = O \qquad (3.15)$$

$$a_{mn} U_1^n(o) = O \qquad (3.16)$$

since:

$$\int_o^{l_1} u_1^k u_1^m da_1 = g \text{ (say)}, \quad \text{if } m = k$$

$$= O, \quad \text{if } m \neq k.$$

Application of the Lagrangian multiplier method in this way will provide, in general, two values for the generalised reactive force associated with the prescribed constraint condition. These are given by:

(a) the displacement derivatives

and (b) the Lagrangian multiplier.

Ideally, they should be the same, but generally they will be different. In particular, the displacement functions could be chosen such that (a) was zero, e.g. as for a cosine or sine series.

It will be demonstrated in subsequent chapters, that the Lagrangian multiplier gives a better estimate of the generalised reactive force than the corresponding displacement derivative.

3.1.4 Approximating Functions

The selection of suitable approximating functions is the essential feature of the indirect methods discussed in this chapter. Such functions may be simple or complicated and need not be orthogonal, although this latter property is very useful and convenient. The derivatives of the functions should be well defined since the stress-resultants and stress-couples are dependent on them. A physical insight into the problem at hand greatly assists the choice of suitable functions, which may possibly lead to a rapid convergence of the solution.

The functions to be studied in this thesis are classified in table 3.1. The origin is located at one corner of the shell (figures 2.4 and 2.5). Of the functions tabulated only IA, IB, IIA and IIB are orthogonal.

Class I Functions

Functions IC were used by Chuang and Veletsos[7] in the variational solution of a shallow cylindrical shell. These functions were later applied to doubly curved shallow shells by Noor and Veletsos.[15] Both these references included the function $(1-2\beta_i)$ in this set. The reason for omitting this function will be discussed in the next chapter in conjunction with the numerical results.

Functions ID were also considered by Chuang and Veletsos.[7]

Functions IE were originally proposed by Filonenko-Boroditch,[57] who referred to them as "almost orthogonal" functions. These functions have been used to represent the displacement w by Buziarova,[58] for the bending solution of a clamped plate, and by Noor and Veletsos,[15] for the bending solution of a clamped shell. Although these functions satisfy the clamped boundary conditions on w, they satisfy the additional conditions that the normal and Kirchhoff shears vanish on the boundary. This will undoubtedly affect the boundary value of the moment.

Functions IF have been obtained by modifying functions IE such that the normal and Kirchhoff shears no longer vanish. Note that the shape of the corresponding cosine and sine functions of IF are similar and numerical difficulties could be introduced as more terms are taken in the series.

Class II Functions

Rayleigh functions are functions of the type:

$$F_m \equiv F_m(\beta_i) = A_m \sin a_m \beta_i + B_m \sinh a_m \beta_i + C_m \cos a_m \beta_i + \cosh a_m \beta_i \qquad (3.17)$$

and have been tabulated in detail in references (9) and (60) up to m = 4 and m = 5 respectively.

The Rayleigh functions used in this thesis were provided by Kuo,[59] who has calculated them out to m = 27.

Further details of Rayleigh functions are given in Appendix I.

Functions IIB will be used only to represent displacement u_1. They were also used by Chuang and Veletsos[7] to represent displacements u_1, u_2 and w for a cylindrical shell with free boundaries at $a_2 = 0, l_2$. The Rayleigh-Ritz method was used. Deep thin inextensible gables were assumed at $\alpha_1 = O, l_1$ such that a Levy-type solution was possible (refer to chapter 4 for a description of these terms). Their results showed poor convergence. However, these functions have been incorrectly used with w. It can be shown that the natural boundary condition $r_2(\alpha_1, o) = O$ for a free boundary becomes, on using functions IIB and the series form for w given by equation (3.3), $W_{n,2}(o) = O$. Similarly the natural boundary condition $m_{22}(\alpha_1, o) = O$ for $\nu = O$ reduces to $W_{n,22}(o) = O$, which is identically satisfied by functions IIB. The coupling of these two conditions is valid only for the constant 1 of functions IIB. However, in general, the coupling of these conditions seems to invite difficulties.

3.2 Method of Lines

Equations (2.76) form the basis of the method of lines used in this thesis for the solution of shallow curved plates.

In this method, the derivatives in one direction (α_1 in this thesis) are replaced by finite difference expressions. In this way equations (2.76) are reduced to a system of linear first order ordinary differential equations with constant coefficients.

Thus the boundary value problem may be considered as an equivalent initial value problem in which four of the dependent variables are specified by the initial boundary conditions. The initial values of the remaining four dependent variables must be determined such that the final four boundary conditions are satisfied.

Integration of this system of first order ordinary differential equations is the immediate problem. The matrix progression method[(61),(62)] offers a convenient and systematic approach for the numerical solution of these differential equations. The application of this numerical procedure in conjunction with the method of lines has been discussed in detail by Jenkins and Tottenham,[(47)] who give several illustrative examples. However, no numerical results are presented.

The matrix progression method is similar in principle to the transfer matrix method.[(63)]

Due to the limited number of significant figures used in practice, the integration of such problems may introduce serious roundoff errors. This problem may be overcome by segmenting the path of integration.[(64)] The influence coefficients for each segment are then determined by integration and the solution obtained by restoring equilibrium and/or compatability.

A further way to overcome this numerical problem is to "bring up the initial boundary".[(65),(66)] This idea is used with the matrix progression method. In this approach the integration path is divided into steps. The boundary conditions are then brought up for each step, in such a form that they may be used as the initial boundary for the next step. This procedure continues until the final boundary is reached, where the known boundary conditions are applied. The solution at this final boundary is then obtained. The solution at each step follows by back substitution.

In this thesis (Chapter 7) the matrix progression method will be used. Whenever necessary the integration path will be segmented. A stiffness approach will be proposed, in which the stiffness matrix for each segment can be obtained from the transfer (or distribution) matrix (refer to Chapter 7). The assembled stiffness matrix for the shell will be in tri-diagonal form, which is readily solved by partitioning.

CHAPTER 4

APPLICATION OF THE INDIRECT METHODS TO TRANSLATIONAL SHELLS WITH LEVY-TYPE SOLUTIONS

In this chapter, the proposed indirect methods will be applied to shell problems whose exact solutions are known. In this way the convergence of various types of approximating functions may be studied.

4.1 Non-Dimensional Form of Equations

Levy-type solutions are available for shallow translational curved plates of rectangular plan-form (figure 2.4) supported on two opposite edges by normally-located deep thin inextensible gables (defined in Table 4.3).

For convenience a loading function will be selected such that a one-term Levy expansion provides the required exact solution. Normal gables will be assumed at $\alpha_1 = O, l_1$.

The selected loading is

$$X_1 = O = X_2 \tag{4.1}$$

$$Z = Z_o \sin j\pi\beta_1 \tag{4.2}$$

where:

Z_o is a constant

j is a non-zero positive integer

$$\beta_1 = \frac{\alpha_1}{l_1} \tag{4.3}$$

The applied boundary loads will be assumed to be zero.

The Levy solution procedure implies displacement distributions of the type:*

$$u_1 = l_1 U_1(\beta_2) \cos j\pi\beta_1 \tag{4.4}$$

$$u_2 = l_2 U_2(\beta_2) \sin j\pi\beta_1 \tag{4.5}$$

$$w = \frac{1}{K_2} W(\beta_2) \sin j\pi\beta_1 \tag{4.6}$$

where the origin is located in one corner of the shell as shown in figure 2.4, and

---

*The functions $U_1(\beta_2)$, $U_2(\beta_2)$ and $W(\beta_2)$ should be more correctly written as $U_1{}^j(\beta_2)$, $U_2{}^j(\beta_2)$ and $W^j(\beta_2)$. However, to avoid confusion with other functions, the j superscript is dropped from this notation. This does not affect the subsequent deviations in any way.

$$\bar{I}_1 = \frac{I_1}{I} \tag{4.7}$$

$$\beta_2 = \frac{\alpha_2}{I_2} \tag{4.8}$$

For convenience, the functions U_1, U_2 and W in equations (4.4), (4.5) and (4.6) have been non-dimensionalised.

In the indirect procedure, the functions U_1, U_2 and W will be approximated by the following truncated series:

$$U_1 = \sum_m a_m U_1^m \tag{4.9}$$

$$U_2 = \sum_m b_m U_2^m \tag{4.10}$$

$$W = \sum_m c_m W_m \tag{4.11}$$

where a_m, b_m and c_m are constants to be determined, U_1^m, U_2^m and W_m represent sets of independent kinematically admissible functions and m is a positive integer.

The corresponding displacement variations may be selected in the following forms:

$$\delta u_1 = l_1 \sum_m U_1^m \cos j\pi\beta_1 \; \delta a_m \qquad (4.12)$$

$$\delta u_2 = l_2 \sum_m U_2^m \sin j\pi\beta_1 \; \delta b_m \qquad (4.13)$$

$$\delta w = \frac{1}{K_2} \sum_m W_m \sin j\pi\beta_1 \; \delta c_m \qquad (4.14)$$

where δa_m, δb_m and δc_m are arbitrary variations in the constants a_m, b_m and c_m respectively.

For the special case being considered the variational equation (2.90) after:*

(a) setting the applied boundary loads to zero,

(b) non-dimensionalising the co-ordinates to the (β_1, β_2) set defined by equations (4.3) and (4.8),

(c) setting K_{12} to zero and replacing K_{11} and K_{22} by K_1 and K_2 respectively,

(d) substitution of equations (4.1), (4.2), (4.4), (4.5), (4.6), (4.7), (4.9), (4.10), (4.11), (4.12), (4.13) and (4.14), and

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*The boundary integrals at $a_1 = 0$, l_1 automatically vanish since all boundary conditions are satisfied there (Table 4.3).

(e) integrating the equations with respect to β_1,

reduces to the following three independent equations, since δa_m, δb_m and δc_m are arbitrary:

$$\int_o^1 \Big[\pi^2 a_i U_1^i - \frac{(1-\nu)}{2} \bar{r}^{-2} a_i U^i_{1,22} - \frac{(1+\nu)}{2} \pi b_k U^k_{2,2} + $$

$$+ (c+\nu)\pi c_p W_p \Big] U_1^m d\beta_2 - 2\bar{r}\, \bar{n}_{12}(o) U_1^m(o) = 0 \qquad (4.15)$$

$$\int_o^1 \Big[\frac{(1+\nu)}{2} \pi a_i U^i_{1,2} + \frac{(1-\nu)}{2\bar{r}^{-2}} \pi^2 b_k U_2^k - b_k U^k_{2,22} + $$

$$+ (1+\nu c) c_p W_{p,2} \Big] U_2^m d\beta_2 - 2\bar{n}_{22}(o) U_2^m(o) = 0 \qquad (4.16)$$

$$\int_o^1 \Big[(c+\nu)\pi a_i U_1^i - (1+\nu c) b_k U^k_{2,2} + \frac{\bar{\rho}_T^{\,2}}{12} (\pi^4 c_p W_p - $$

$$- 2\bar{r}^{-2} \pi^2 c_p W_{p,22} + \bar{r}^{-4} c_p W_{p,2222}) + (1+2\nu c + c^2) c_p W_p - $$

$$- \bar{Z}_o \Big] W_m d\beta_2 + 2\bar{r}^{-2} \bar{m}_{22}(o) W_{m,2}(o) - 2\bar{r}\, \bar{r}_2(o) W_m(o) = 0 \qquad (4.17)$$

where

$$\bar{r} = \frac{\bar{l}_1}{l_2} \qquad (4.18)$$

$$\bar{\rho}_T = \frac{h}{T_1} \cdot \frac{1}{K_2 \bar{T}_1} = -\frac{1}{8\bar{r}} \left(\frac{h}{T_1}\right)\left(\frac{l_2}{r_2}\right) \qquad (4.19)$$

$$c = \frac{K_1}{K_2} \qquad (4.20$$

$$\bar{Z}_o = \frac{Z_o(1-\nu^2)}{EhK_2} \qquad (4.21)$$

i, k, m, p are positive integers and $\bar{n}_{12}$, $\bar{n}_{22}$, $\bar{m}_{22}$ and $\bar{r}_2$ are, in this case, functions of β_2 only and are the non-dimensional forms (given in Table 4.1) of n_{12}, n_{22}, m_{22} and r_2 respectively. In equations (4.15), (4.16), (4.17) and Table 4.1 the Einstein summation convention is adopted and comma notation is used to represent differentiation with respect to β_2.

In deriving expressions for the boundary integrals, the boundary conditions were assumed symmetric about $\beta_2 = 0.5$. If this were not the case equations (4.15), (4.16) and (4.17) would be modified in the following way:

replace $\left[-2\bar{r}\;\bar{n}_{12}(o)U_1^m(o)\right]$ by $\left[\bar{r}\;\bar{n}_{12}(\beta_2)U_1^m(\beta_2)\right]_{\beta_2=0}^{\beta_2=1}$

replace $\left[-2\bar{n}_{22}(o)U_2^m(o)\right]$ by $\left[\bar{n}_{22}(\beta_2)U_2^m(\beta_2)\right]_{\beta_2=0}^{\beta_2=1}$

replace $\left[-2\bar{r}\;\bar{r}_2(o)W_m(o)\right]$ by $\left[\bar{r}\,\bar{r}_2(\beta_2)W_m(\beta_2)\right]_{\beta_2=0}^{\beta_2=1}$

replace $\left[+2\bar{r}^{2}\,m_{22}(o)W_{m,2}(o)\right]$ by $\left[-\bar{r}^{2}\,m_{22}(\beta_2)W_{m,2}(\beta_2)\right]_{\beta_2=0}^{\beta_2=1}$

It is evident from the foregoing that the problem is specified through the non-dimensional parameters $\bar{\rho}_T$, c, $\bar{r}$ and ν.*

With $\bar{Z}_o = 1$ equations (4.15), (4.16) and (4.17) are the equations used for the solutions presented in this chapter.

The actual values of the displacements, stress-resultants and stress-couples, for any loading of the type given by equation (4.2), are obtained from the non-dimensional forms given in Table 4.1 by the factors given in Table 4.2.

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*The single parameter $\bar{\rho}_T$ could have been replaced by the separate parameters $(\frac{h}{\bar{T}_1})$ and $(\frac{l_2}{f_2})$. However, the use of $\bar{\rho}_T$ covers a wider range of shells.

Equations (4.15), (4.16) and (4.17) are the Galerkin equations modified by expressions corresponding to the relevant boundary integrals in equation (2.90).

4.1.1 Modification for the Lagrangian Multiplier Method

In this section only homogeneous kinematic boundary conditions will be considered.

For the problem considered here, a maximum of four homogeneous kinematic conditions may be prescribed along $\alpha_2 = 0$ and $\alpha_2 = l_2$, viz.:

$$u_1 = 0 \tag{4.22}$$

$$u_2 = 0 \tag{4.23}$$

$$w = 0 \tag{4.24}$$

$$w_{,2} = 0 \tag{4.25}$$

Assume that the conditions given by equations (4.22) to (4.25) inclusive are now applied as constraint conditions.

Then following the procedure described in section (3.1.3) and <u>assuming that the boundary conditions are symmetric about the axis</u> $\alpha_2 = \frac{l_2}{2}$, the variational equation (2.90) is modified to:

Left hand side of equation (2.90) +

$$+ 2\int_{o}^{l_1} \lambda_1(\alpha_1)\delta u_1(\alpha_1,o)d\alpha_1 \quad +$$

$$+ 2\int_{o}^{l_1} \lambda_2(\alpha_1)\delta u_2(\alpha_1,o)d\alpha_1 \quad +$$

$$+ 2\int_{o}^{l_1} \lambda_3(\alpha_1)\delta w(\alpha_1,o)d\alpha_1 \quad +$$

$$+ 2\int_{o}^{l_1} \lambda_4(\alpha_1)\delta w,_2(\alpha_1,o)d\alpha_1 \quad = \quad 0 \tag{4.26}$$

where $\lambda_1(\alpha_1)$, $\lambda_2(\alpha_1)$, $\lambda_3(\alpha_1)$ and $\lambda_4(\alpha_1)$ are the Lagrangian multipliers corresponding to the displacements u_1, u_2, w and $(w,_2)$, respectively.

The constraint conditions are:

$$u_1(\alpha_1,o) \quad = \quad 0 \tag{4.27}$$

$$u_2(\alpha_1,o) \quad = \quad 0 \tag{4.28}$$

$$w(\alpha_1,o) \quad = \quad 0 \tag{4.29}$$

$$w,_2(\alpha_1,o) \quad = \quad 0 \tag{4.30}$$

Equations (4.26) to (4.30) inclusive completely define the problem.

Expressing, for this special problem, :

$$\lambda_1(\alpha_1) = \sum_i \lambda_1^i \cos i\pi\alpha_1 \qquad (4.31)$$

$$\lambda_2(\alpha_1) = \sum_i \lambda_2^i \sin i\pi\alpha_1 \qquad (4.32)$$

$$\lambda_3(\alpha_1) = \sum_i \lambda_3^i \sin i\pi\alpha_1 \qquad (4.33)$$

$$\lambda_4(\alpha_1) = \sum_i \lambda_4^i \sin i\pi\alpha_1 \qquad (4.34)$$

where λ_1^i, λ_2^i, λ_3^i and λ_4^i are constants and i is a non-zero positive integer, and proceeding as in sections (4.1) and (3.1.3), equations (4.26) to (4.30) inclusive reduce to the following :

$$\text{Left hand side of equation (4.15)} + 2\bar{r}\,\bar{\lambda}_1 U_1^m(o) = 0 \qquad (4.35)$$

$$\text{Left hand side of equation (4.16)} + 2\bar{\lambda}_2 U_2^m(o) = 0 \qquad (4.36)$$

$$\text{Left hand side of equation (4.17)} + 2\bar{r}\,\bar{\lambda}_3 W_m(o) + {}$$

$$+ 2\bar{r}^{-2}\bar{\lambda}_4 W_{m,2}(o) = 0 \qquad (4.37)$$

$$a_i U_1^i(o) = 0 \qquad (4.38)$$

$$b_k U_2^k(o) \quad = \quad 0 \tag{4.39}$$

$$c_p W_p(o) \quad = \quad 0 \tag{4.40}$$

$$c_p W_{p,2}(o) \quad = \quad 0 \tag{4.41}$$

where

$$\bar{\lambda}_1 \quad = \quad \frac{(1-\nu^2)}{Eh}\lambda_1 \tag{4.42}$$

$$\bar{\lambda}_2 \quad = \quad \frac{(1-\nu^2)}{Eh}\lambda_2 \tag{4.43}$$

$$\bar{\lambda}_3 \quad = \quad \frac{(1-\nu^2)}{Eh\,K_2\bar{T}_1}\lambda_3 \tag{4.44}$$

$$\bar{\lambda}_4 \quad = \quad \frac{(1-\nu^2)}{Eh\,K_2 T_1^{\,2}}\lambda_4 \tag{4.45}$$

and the j superscript is dropped from the notation.

<u>With</u> $\bar{Z}_o = 1$ <u>equations (4.35) to (4.41) inclusive are the equations used in conjunction with the Lagrangian multiplier method.</u>

As before the non-dimensional and actual values of the displacement, stress-resultants and stress-couples are obtained from Tables (4.1) and (4.2) respectively.

Interpretation of the Lagrangian multipliers

The Lagrangian multipliers provide the generalised reactive force associated with the corresponding constraint condition.

Then for the symmetric case considered:

$$n_{12}(\alpha_1, o) = \lambda_1(\alpha_1) \quad (4.46)$$

$$n_{22}(\alpha_1, o) = \lambda_2(\alpha_1) \quad (4.47)$$

$$r_2(\alpha_1, o) = \lambda_3(\alpha_1) \quad (4.48)$$

$$m_{22}(\alpha_1, o) = -\lambda_4(\alpha_1) \quad (4.49)$$

where the minus sign in equation (4.49) is due to the sign conventions adopted for m_{22} and $(w_{,2})$.

In non-dimensional form, equations (4.46) to (4.49) reduce to:

$$\bar{n}_{12}(o) = \bar{\lambda}_1 \quad (4.50)$$

$$\bar{n}_{22}(o) = \bar{\lambda}_2 \quad (4.51)$$

$$\bar{r}_2(o) = \bar{\lambda}_3 \quad (4.52)$$

$$\bar{m}_{22}(o) = -\bar{\lambda}_4 \quad (4.53)$$

Alternative expressions for $\bar{n}_{11}(o)$, $\bar{m}_{11}(o)$, $\bar{q}_1(o)$, $\bar{q}_2(o)$ and $\bar{r}_1(o)$ will now be derived.

The following expressions are obtained from Table 4.1:

$$\bar{n}_{11} - \nu \bar{n}_{22} = -\pi(1-\nu^2)a_m U_1^m - c(1-\nu^2)c_p W_p \tag{4.54}$$

$$\bar{m}_{11} - \nu \bar{m}_{22} = \frac{\bar{\rho}_T^2}{12} \pi^2(1-\nu^2)c_p W_p \tag{4.55}$$

$$\bar{q}_1 = \frac{-\bar{\rho}_T^2}{12} \left[-\pi^3 c_p W_p + \bar{r}^{-2} \pi c_p W_{p,22} \right] \tag{4.56}$$

$$\bar{q}_2 = \bar{r}_2 - \frac{\bar{\rho}_T^2}{12} \bar{r}\pi^2(1-\nu)c_p W_{p,2} \tag{4.57}$$

$$\bar{r}_1 = \frac{-\bar{\rho}_T^2}{12} \left[-\pi^3 c_p W_p + \bar{r}^{-2} \pi(2-\nu)c_p W_{p,22} \right] \tag{4.58}$$

$$\bar{r}^{-2} c_p W_{p,22} = \frac{-12}{\bar{\rho}_T^2} \bar{m}_{22} + \nu \pi^2 c_p W_p \tag{4.59}$$

Substituting equation (4.59) into equations (4.56) and (4.58) and rearranging equations (4.54) and (4.55) yields:

$$\bar{n}_{11} = -\pi(1-\nu^2)a_m U_1^m - c(1-\nu^2)c_p W_p + \nu \bar{n}_{22} \tag{4.60}$$

$$\bar{m}_{11} = \frac{\bar{\rho}_T^2}{12} \pi^2(1-\nu^2)c_p W_p + \nu \bar{m}_{22} \tag{4.61}$$

$$\bar{q}_1 = \frac{\bar{\rho}_T^2}{12} \pi^3(1-\nu)c_p W_p + \pi \bar{m}_{22} \qquad (4.62)$$

$$\bar{r}_1 = \frac{\bar{\rho}_T^2}{12} \pi^3(1-\nu)^2 c_p W_p + \pi(2-\nu)\bar{m}_{22} \qquad (4.63)$$

At the boundary $\beta_2 = O$, equations (4.60), (4.61), (4.62), (4.63) and (4.57), after substitution of equations (4.50), (4.51), (4.52) and (4.53), become:

$$\bar{n}_{11}(o) = -\pi(1-\nu^2)a_m U_1^m(o) - c(1-\nu^2)c_p W_p(o) + \nu \bar{\lambda}_2 \qquad (4.64)$$

$$\bar{m}_{11}(o) = \frac{\bar{\rho}_T^2}{12} \pi^2(1-\nu^2)c_p W_p(o) - \nu \bar{\lambda}_4 \qquad (4.65)$$

$$\bar{q}_1(o) = \frac{\bar{\rho}_T^2}{12} \pi^3(1-\nu)c_p W_p(o) - \pi \bar{\lambda}_4 \qquad (4.66)$$

$$\bar{r}_1(o) = \frac{\bar{\rho}_T^2}{12} \pi^3(1-\nu)^2 c_p W_p(o) - \pi(2-\nu)\bar{\lambda}_4 \qquad (4.67)$$

$$\bar{q}_2(o) = \frac{-\bar{\rho}_T^2}{12} \bar{r}\, \pi^2(1-\nu)c_p W_{p,2}(o) + \bar{\lambda}_3 \qquad (4.68)$$

<u>Equations (4.50) to (4.53) inclusive and (4.64) to (4.68) inclusive provide alternative* boundary values to those based on the displacement</u>

---

*As discussed in section (3.1.3), the values based on the Lagrangian multipliers are generally different from the corresponding values based on the displacement derivatives.

derivatives.

The actual values are obtained as before from Table 4.2.

This matter will be discussed further in sections (4.4) in conjunction with numerical examples.

4.2 Boundary Conditions

The boundary conditions to be considered in this chapter are given in Table 4.3.

Only boundary conditions which are symmetric about $\beta_2 = 0.5$ are considered.

Normal slide (1) is so numbered to distinguish it from normal slide (2), a boundary condition which will be introduced in Chapter 6.

The approximating functions chosen to specify a particular boundary condition are discussed in section (4.4).

4.3 Reduction to a System of Linear Algebraic Equations

For a particular set of approximating functions, equations (4.15), (4.16) and (4.17), with $\overline{Z}_o = 1$, reduce, on integration, to a system of linear algebraic equations which in matrix form are:

$$\begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} \\ \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} \\ \underline{A}_{31} & \underline{A}_{32} & \underline{A}_{33} \end{bmatrix} \begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} + \begin{bmatrix} \underline{O} \\ \underline{O} \\ \underline{g} \end{bmatrix} = \begin{bmatrix} \underline{O} \\ \underline{O} \\ \underline{O} \end{bmatrix} \qquad (4.69)$$

or, more compactly:

$$\underline{A} \quad \underline{\bar{a}} \quad + \quad \underline{\bar{g}} \quad = \quad \underline{O} \qquad (4.70)$$

where $\underline{\bar{a}} = \text{col}\left\{ \underline{a} \quad \underline{b} \quad \underline{c} \right\}$

$\underline{\bar{g}} = \text{col}\left\{ \underline{O} \quad \underline{O} \quad \underline{g} \right\}$

Typical elements of the submatrices in equation (4.69) are given in Table 4.4. The relevant integration formulae are given in Appendix 2.

4.3.1 Modification for the Lagrangian Multiplier Method

When the Lagrangian multiplier method is applied, the modified form given by equations (4.35) to (4.41) inclusive is used. These equations may also be reduced to a system of linear algebraic equations which in matrix form are:

$$
\begin{bmatrix}
\underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} & 2\bar{r}\underline{d}_1 & \cdot & \cdot & \cdot \\
\underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} & \cdot & 2\underline{d}_2 & \cdot & \cdot \\
\underline{A}_{31} & \underline{A}_{32} & \underline{A}_{33} & \cdot & \cdot & 2\bar{r}\underline{d}_3 & 2\bar{r}\underline{d}_4 \\
\underline{d}_1^T & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \underline{d}_2^T & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \underline{d}_3^T & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \underline{d}_4^T & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
\underline{a} \\ \underline{b} \\ \underline{c} \\ \bar{\lambda}_1 \\ \bar{\lambda}_2 \\ \bar{\lambda}_3 \\ \bar{\lambda}_4
\end{bmatrix}
+
\begin{bmatrix}
\cdot \\ \cdot \\ \underline{g} \\ \\ \cdot \\ \cdot \\ \cdot
\end{bmatrix}
= O \qquad (4.71)
$$

where:

typical elements of the submatrices $\underline{A}_{ij}(i,j = 1,2,3)$, $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{g}$ are, as before, given in Table 4.4,

$\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$ and $\bar{\lambda}_4$ are constants,

and typical elements of the column matrices $\underline{d}_1$, $\underline{d}_2$, $\underline{d}_3$ and $\underline{d}_4$ are respectively given by :

$$d_m^1 = U_1^m(o) \qquad (4.72)$$

$$d_m^2 = U_2^m(o) \qquad (4.73)$$

$$d_m^3 = W_m(o) \qquad (4.74)$$

$$d_m^4 = W_{m,2}(o) \qquad (4.75)$$

If there are less than four imposed boundary conditions, equations (4.71) are adjusted accordingly.

If S is the number of functions chosen to repesent each of U_1, U_2 and W,* then each submatrix $\underline{A}_{ij}$ is of order (S x S) and each column matrix $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}_1$, $\underline{d}_2$, $\underline{d}_3$ and $\underline{d}_4$ is of order (S x 1). Then there are 3S constants to be determined by equations (4.69) and (3S +4) constants to be determined by equations (4.71).

The solution of equations (4.71) form the basis of the numerical results presented in this chapter. When no Lagrangian multipliers are used these equations reduce to equations (4.69).

---

*It is not essential to adopt the same value of S for each of U_1, U_2 and W.

4.4 Convergence Study of the Approximating Functions

In this section the approximating functions given in table 3.1 will be applied to particular numerical examples. In the computer programs developed, provision is made for the symmetry of the problem by choosing the functions given in table 3.1 such that:

(a) U_1^m, W_m are symmetric functions about $\beta_2 = 0.5$

(b) U_2^m is an antimetric function about $\beta_2 = 0.5$

The Levy-type solutions given in this chapter were obtained from computer programs developed at Imperial College by Gunasekera(6) and by Samartin(49).

4.4.1 Numerical Examples

The examples and the corresponding approximating functions to be studied are given in table 4.5. The boundary conditions are defined in table 4.3. Details of the approximating functions are given in table 3.1.

For all examples, the shell parameters $\bar{\rho}_T$, $\bar{r}$ and ν will be set at the following values:

$$\bar{\rho}_T = 0.0152928$$

$$\bar{r} = 1.0$$

$$\nu = 0.25$$

The value for c is either -0.5 or +0.5 (refer to Table 4.5).

With $c = -0.5$ the above parameters correspond to an example discussed by Noor and Veletsos(15) using a $w - \phi$ formulation.

The results are presented in Tables 4.7 to 4.17 inclusive and figures 4.1 to 4.6 inclusive. The tabulated results* have been reproduced from the computer program output and, to avoid confusion, the notation used in the program and the corresponding notation used in this thesis are given in Table 4.6.

The exact values are given in Tables 4.18 to 4.21 inclusive.

Displacements, stress-resultants and stress couples are presented in non-dimensional form (Table 4.1). The actual values are obtainable from the expressions given in Table 4.2.

Whenever the Lagrangian multiplier method is used, the boundary value based on the multiplier is quoted separately from the corresponding value based on the displacement derivative. These latter values are tabulated under the caption "Values of....."

S is the number of functions used to represent each of U_1, U_2 and W, due allowance being made for the symmetry of the problem in the selection of these functions.

---

*The results in the tables are presented in floating point notation, e.g. 3.481, -5 means 3.481×10^{-5}.

4.4.2 Discussion

(a) Example A: Clamped at $\beta_2 = 0.1$

The exact values are given in Tables 4.18a and 4.18b.

(i) Examples A1, A2 and A3: Refer to Tables 4.7, 4.8 and 4.9a and figures 4.1 and 4.2.

These three examples differ by the functions chosen for W_m (refer to Table 4.5). The most rapid convergence of moments (stress-couples), normal shears (stress-resultants) and displacement w was obtained in example A2. Good convergence was also obtained in example A3 while the convergence in example A1 was somewhat slower.

It has been previously noted (section 3.1.4) that the functions IE, which are used to represent W in example A1 impose the additional boundary conditions of zero normal shear and zero Kirchhoff shear and this undoubtedly contributed to the slower convergence observed for this case.

The convergence of $\bar{n}_{11}$ and $\bar{n}_{22}$ was good, whereas $\bar{n}_{12}$ converged very slowly on the boundary.

(ii) Examples A3, A4, A5 and A6: Refer to Tables 4.9 to 4.12 inclusive and figures 4.3, 4.4 and 4.5.

These examples differ by the functions chosen for U_1^m (refer to Table 4.5). The Lagrangian multiplier method has been used in examples A4, A5 and A6 in an attempt to improve the convergence of $\bar{n}_{12}$ on the boundary. In figures 4.3 and 4.4 (examples A4, A5 and A6), the value of $\bar{n}_{12}$ on the boundary is based on the Lagrangian multiplier.

<u>c = - 0.5:</u> The convergence of $\bar{n}_{12}$ on the boundary was greatly improved in each of examples A4, A5 and A6, with A4 showing the most rapid convergence. The solution for $\bar{n}_{12}$ within the shell converged rapidly in example A4 but more slowly in examples A5 and A6.

Figure 4.5 illustrates the good convergence of $\bar{n}_{11}$ and $\bar{n}_{22}$.

<u>c = + 0.5:</u> A complete set of results for A3 is given in Table 4.9b. Only results for $\bar{u}_1$, $\bar{n}_{11}$, and $\bar{n}_{12}$ are given for A4, A5 and A6. The remaining results are similar to example A3.

In this case the magnitude of $\bar{n}_{12}$ is greater than for c = - 0.5. The convergence of $\bar{n}_{12}$ on the boundary was again slow in example A3, but better within the shell. Use of the Lagrangian multiplier method again improved the boundary convergence of $\bar{n}_{12}$.

In each example the Lagrangian multiplier provided a better estimate of the boundary value of $\bar{n}_{12}$ than the corresponding displacement derivative.

(iii) Example A7: Refer to Table 4.13 and figure 4.6.

The boundary value of $\bar{q}_2$ (= $\bar{r}_2$ for the clamped boundary conditions considered) based on the Lagrangian multiplier is very close to the exact solution after 6 functions. However, the corresponding value based on the displacement derivative is zero. This extreme difference is reflected in the slow convergence of the moments and normal shears. The solution is generally comparable with example A1.

(iv) Example A8: Refer to Table 4.14.

The solution generally converged rapidly. The boundary values based on the Lagrangian multipliers were very nearly exact after 6 functions. The corresponding values based on the displacement derivatives also compared closely with the exact values.

(b) Example B: Free at β_2 = O,1

The results and corresponding exact values are given in Tables 4.15 and 4.19 respectively.

The solution generally converged rapidly, with the (natural) boundary conditions for a free edge being approximately fulfilled.

(c) Example C: Hinged at β_2 = O,1

The results and corresponding exact values are given in Tables 4.16 and 4.20 respectively.

The boundary values of $\bar{n}_{11}$, $\bar{n}_{12}$ and $\bar{n}_{22}$ converged slowly. These values could be improved by choosing functions which violate the boundary conditions on u_1 and u_2 and applying the Lagrangian multiplier method (as in examples A4, A5 and A6). The values within the shell show better convergence.

The displacements, moments and normal shear $\bar{q}_1$ show good convergence, but $\bar{q}_2$ on the boundary is slowly convergent.

(d) Example D: Normal Slide (1) at β_2 = O,1

The results and corresponding exact values are given in Tables 4.17 and 4.21 respectively.

The convergence of the solution is generally good. Again $\bar{n}_{12}$ on the boundary is slow to converge.

4.4.3 Some Notes on Functions IC, ID and IF

(a) Functions IC

As S becomes large these functions may introduce numerical difficulties in the solution.

The set IC contains the constant unity and also a half-range Fourier series. However, unity itself can be represented by this Fourier series. Such a representation becomes better as S increases. Then it was not unexpected that some difficulty may be realised with these functions.

In order to investigate this problem two solutions were obtained for any problem associated with these functions (examples A4, A8 and B) using:

(i) an unscaled matrix $\underline{A}$

(ii) a scaled matrix $\underline{A}$, such that the diagonal elements are made unity.

Matrix $\underline{A}$ is defined by equations (4.70).

If the equations are well-conditioned scaling should not affect the solution.*

A maximum number of 16 functions was considered.

To four significant figures, the values of the displacements, stress-resultants and stress-couples were the same in (i) and (ii). However, the solution constants associated with functions IC were completely different for values of $S \geq 10$, although the solution for displacements, etc., was virtually the same.

Whenever functions IC were used, the solution converged rapidly. Due to this rapid convergence, the difficulties discussed above and

*To effectively study the stability of the solution of a set of equations, the effect on the solution of small perturbations of some of the matrix element values should be considered.

associated with a large number of terms of this series, were minimised. This investigation indicates, however, that some caution should be exercised in using these functions.

References (7) and (15) both included $(1-2\beta_2)$ in set IC. However, because of the similarity of this function with $\cos \pi\beta_2$, and to avoid possible further difficulties, this function was excluded from this set.

(b) Functions ID

Arguments similar to those used in the above discussion of functions IC apply to this set also. However, these functions were used only with displacement u_1 (example A5), in conjunction with the Lagrangian multiplier method. Accordingly, the corresponding constraint condition (equation 4.38) reduces to the condition that the constant (say a_o) associated with the constant unity* in set ID is zero. In example A5, this constant is set to zero before solving the system of linear algebraic equations. When used in this way, no difficulty was observed with these functions.

---

*This applies when the loading and boundary conditions are symmetric about $\beta_2 = 0.5$. If this is not the case, and if $u_1 = 0$ at $\beta_2 = 0,1$, it follows that the constants associated with unity and $(1-2\beta_2)$ in set ID are both zero.

The condition that a_o is zero may appear trivial, but the Lagrangian multiplier maintains this condition, and effectively gives a good estimate of the boundary value of the corresponding action (refer to example A5).

Functions ID, in conjunction with the Lagrangian multiplier method, may be effectively used to improve a particular stress-resultant, which is slowly convergent on the boundary but satisfactory elsewhere. (Refer to example A5 for c = + 0.5).

(c) Functions IF

In section (3.1.4) it was noted that difficulties with these functions could arise, due to the similarity of the corresponding forms of the cosine and sine sets.

Operations on the matrix $\underline{A}$ described above in (a) were again carried out. Only example A2 is affected.

For values of S up to 10, the values of displacement w, moments and normal shears were, to four significant figures, the same in cases (i) and (ii) described in (a) above. For values of S greater than 10, some of the values, particularly the normal shears, differed in the third, and sometimes the second, significant figure.

However, due to the rapid convergence observed with set IF (example A2), it was not necessary to consider a large number of terms and the difficulties were minimised. It is apparent that these functions should be used with caution.

4.5 Discussion of the Computer Programs

A separate computer program was developed for each of the examples given in table 4.5.

The approximating functions were selected in accordance with the symmetry of the problem (section 4.4). The same value of S for each of U_1, U_2 and W was considered.

Input, and therefore output, was in non-dimensional form. The output was arranged in tabular form and has been reproduced in tables 4.7 to 4.21 inclusive.

Further details of the computer programs are available at Imperial College.[69]

The computer programs were written in EXCHLF Autocode for the University of London Atlas computer.[70],[71]

CHAPTER 5

FURTHER APPLICATION OF THE INDIRECT METHODS TO TRANSLATIONAL SHELLS

In this chapter the proposed indirect methods will be applied to translational shells (figure 2.4) which are unsuitable for Levy-type solutions.

Only uniformly distributed normal loading (Z) will be considered.

5.1 Non-Dimensional Form of Equations

Let the displacement distributions assume the following forms:

$$u_1 = l_1 \sum_m \sum_n a_{mn} u_1^m(\beta_1) U_1^n(\beta_2) \tag{5.1}$$

$$u_2 = l_2 \sum_m \sum_n b_{mn} u_2^m(\beta_1) U_2^n(\beta_2) \tag{5.2}$$

$$w = \frac{1}{K_2} \sum_m \sum_n c_{mn} w_m(\beta_1) W_n(\beta_2) \tag{5.3}$$

where

$$\beta_1 = \frac{\alpha_1}{T_1} \tag{5.4}$$

$$\beta_2 = \frac{\alpha_2}{T_2} \tag{5.5}$$

a_{mn}, b_{mn} and c_{mn} are constants to be determined,

u_1^m, u_2^m, w_m, U_1^n, U_2^n and W_n represent sets of independent kinematically admissible functions,

and m and n are positive integers.

The corresponding displacement variations may be selected in the following forms:

$$\delta u_1 = l_1 \sum_m \sum_n u_1^m(\beta_1) U_1^n(\beta_2) \delta a_{mn} \tag{5.6}$$

$$\delta u_2 = l_2 \sum_m \sum_n u_2^m(\beta_1) U_2^n(\beta_2) \delta b_{mn} \tag{5.7}$$

$$\delta w = \frac{1}{K_2} \sum_m \sum_n w_m(\beta_1) W_n(\beta_2) \delta c_{mn} \tag{5.8}$$

where δa_{mn}, δb_{mn} and δc_{mn} are arbitrary variations in the constants a_{mn}, b_{mn} and c_{mn} respectively.

In the following derivation only the boundary integrals corresponding to n_{11} and n_{22} will be retained. In all other cases the boundary integrals will be assumed to vanish by virtue of the chosen functions.

Then the variational equation (2.90) after:

(a) setting X_1, X_2 and the applied boundary loads to zero,

(b) non-dimensionalising the co-ordinates to the (β_1, β_2) set defined by equations (5.4) and (5.5),

(c) setting K_{12} to zero and replacing K_{11} and K_{22} by K_1 and K_2 respectively, and

(d) substitution of equations (5.1), (5.2), (5.3), (5.6), (5.7) and (5.8),

reduces to the following three independent equations, since δa_{mn}, δb_{mn} and δc_{mn} are arbitrary:

$$\int_o^1 \int_o^1 \Big[-a_{ij} u^i_{1,11} U^j_1 - \frac{(1-\nu)}{2} r^2 a_{ij} u^i_1 U^j_{1,22} - \frac{(1+\nu)}{2} b_{kl} u^k_{2,1} U^l_{2,2} +$$

$$+ (c+\nu) c_{pq} w_{p,1} W_q \Big] u^m_1 U^n_1 d\beta_1 d\beta_2 - 2 \int_o^1 \bar{n}_{11}(o,\beta_2) u^m_1(o) U^n_1 d\beta_2 = 0 \quad (5.9)$$

$$\int_o^1 \int_o^1 \Big[- \frac{(1+\nu)}{2} a_{ij} u^i_{1,1} U^j_{1,2} - b_{kl} u^k_2 U^l_{2,22} - \frac{(1-\nu)}{2r^2} b_{kl} u^k_{2,11} U^l_2 +$$

$$+ (1+\nu c) c_{pq} w_p W_{q,2} \Big] u^m_2 U^n_2 d\beta_1 d\beta_2 - 2 \int_o^1 \bar{n}_{22}(\beta_1,o) u^m_2 U^n_2(o) d\beta_1 = 0 \quad (5.10)$$

$$\int_0^1 \int_0^1 \Big[-(c+\nu) a_{ij} u^i_{1,1} U^j_1 - (1+\nu c) b_{kl} u^k_2 U^l_{2,2} + \frac{\rho_T^{\,2}}{12} (c_{pq} w_{p,1111} W_q +$$

$$+ 2r^2 c_{pq} w_{p,11} W_{q,22} + r^4 c_{pq} w_p W_{q,2222}) + (1+2\nu c + c^2) c_{pq} w_p W_q -$$

$$- \bar{Z} \Big] w_m W_n d\beta_1 d\beta_2 = 0 \qquad (5.11)$$

where

$$r = \frac{l_1}{l_2} \qquad (5.12)$$

$$\rho_T = \frac{h}{l_1} \cdot \frac{1}{K_2 l_1} = -\frac{1}{8r} \frac{h}{l_1} \cdot \frac{l_2}{f_2} \qquad (5.13)$$

$$c = \frac{K_1}{K_2} \qquad (5.14)$$

$$\bar{Z} = \frac{Z(1-\nu^2)}{EhK_2} \qquad (5.15)$$

i, j, k, l, m, n, p, q are positive integers and $\bar{n}_{11}$ and $\bar{n}_{22}$ are functions of β_1 and β_2 and are the non-dimensional forms (given in Table 5.1) of n_{11} and n_{22} respectively. In equations (5.9), (5.10), (5.11) and Table 5.1, the Einstein summation convention is adopted and comma notation is used to represent differentiation with respect to β_1 and β_2.

In deriving expressions for the boundary integrals, the boundary conditions were assumed symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$. If this were not the case equations (5.9), (5.10) and (5.11) would be modified in the following way:

replace $\left[-2\int_0^1 \bar{n}_{11}(o,\beta_2)u_1^m(o)U_1^n d\beta_2\right]$ by

$$\left[+\int_0^1 \bar{n}_{11}(\beta_1,\beta_2)u_1^m(\beta_1)U_1^n d\beta_2\right]_{\beta_1=0}^{\beta_1=1}$$

replace $\left[-2\int_0^1 \bar{n}_{22}(\beta_1,o)u_2^m U_2^n(o)d\beta_1\right]$ by

$$\left[+\int_0^1 \bar{n}_{22}(\beta_1,\beta_2)u_2^m U_2^n(\beta_2)d\beta_1\right]_{\beta_2=0}^{\beta_2=1}$$

It is evident from the foregoing that the problem is specified through the non-dimensional parameters ρ_T, c, r and ν .*

---

*As noted in section (4.1) the separate parameters $(\frac{h}{T_1})$ and $(\frac{T_2}{f_2})$ could have been considered in place of the single parameter ρ_T.

With $\bar{Z} = 1$ equations (5.9), (5.10) and (5.11) are the equations used for the solutions presented in this chapter.

The actual values of the displacements, stress-resultants and stress-couples for any uniformly distributed normal loading Z are obtained from the non-dimensional forms given in Table 5.1 by the factors given in Table 5.2.

Equations (5.9), (5.10) and (5.11) are the Galerkin equations modified by expressions corresponding to the relevant boundary integrals in equation (2.90).

5.1.1 Modification for the Lagrangian Multiplier Method

In this section only the following homogeneous kinematic conditions will be considered:

$$u_1 = 0 \quad \text{at} \quad \alpha_1 = 0,\ l_1 \tag{5.16}$$

$$u_2 = 0 \quad \text{at} \quad \alpha_2 = 0,\ l_2 \tag{5.17}$$

$$w = 0 \text{ at } (\alpha_1, \alpha_2) = (o,o),(l_1,o),(o,l_2),(l_1,l_2) \tag{5.18}$$

Assume that the conditions given by equations (5.16), (5.17) and (5.18) are now applied as constraint conditions.

Then following the procedure described in section (3.1.3) and assuming that the boundary conditions are symmetric about $a_1 = 0.5\ l_1$ and $a_2 = 0.5\ l_2$, the variational equation (2.90) is modified to:

$$\text{Left hand side of equation (2.90)} + 2\int_0^{l_2} \lambda_1(a_2)\delta u_1(o,a_2)da_2 + 2\int_0^{l_1} \lambda_2(a_1)\delta u_2(a_1,o)da_1 + 4\ \lambda_3\delta w(o,o) = 0 \qquad (5.19)$$

where $\lambda_1(a_2)$, $\lambda_2(a_1)$ and λ_3 (a constant) are the Lagrangian multipliers corresponding to the displacements u_1, u_2 and w respectively.

The constraint conditions are:

$$u_1(o,a_2) = 0 \qquad (5.20)$$

$$u_2(a_1,o) = 0 \qquad (5.21)$$

$$w(o,o) = 0 \qquad (5.22)$$

Equations (5.19) to (5.22) inclusive completely define the problem.

Expressing

$$\lambda_1(\alpha_2) = \sum_e \lambda_1^e L_1^e(\alpha_2) \tag{5.23}$$

$$\lambda_2(\alpha_1) = \sum_e \lambda_2^e L_2^e(\alpha_1) \tag{5.24}$$

where λ_1^e and λ_2^e are constants,

$L_1^e(\alpha_2)$ and $L_2^e(\alpha_1)$ represent sets of independent functions,

and e is a positive integer,

and proceeding as in sections (5.1) and (3.1.3), equations (5.19) to (5.22) inclusive reduce to the following:

$$\text{Left hand side of equation (5.9)} + 2\bar{\lambda}_1^e u_1^m(o) \int_o^1 L_1^e U_1^n d\beta_2 = 0 \tag{5.25}$$

$$\text{Left hand side of equation (5.10)} + 2\bar{\lambda}_2^e U_2^n(o) \int_o^1 L_2^e u_2^m d\beta_1 = 0 \tag{5.26}$$

$$\text{Left hand side of equations (5.11)} + 4\bar{\lambda}_3 w_m(o) W_n(o) = 0 \tag{5.27}$$

$$a_{ij} u_1^i(o) = 0 \quad (5.28)$$

$$b_{kl} U_2^l(o) = 0 \quad (5.29)$$

$$c_{pq} w_p(o) W_q(o) = 0 \quad (5.30)$$

where $$\bar{\lambda}_1^e = \frac{(1-\nu^2)}{Eh} \lambda_1^e \quad (5.31)$$

$$\bar{\lambda}_2^e = \frac{(1-\nu^2)}{Eh} \lambda_2^e \quad (5.32)$$

$$\bar{\lambda}_3 = \frac{(1-\nu^2)}{EhK_2 l_1 l_2} \lambda_3 \quad (5.33)$$

With $\bar{Z} = 1$ equations (5.25) to (5.30) inclusive are the equations used in conjunction with the Lagrangian multiplier method

As before the non-dimensional and actual values of the displacements, stress-resultants and stress-couples are obtained from tables 5.1 and 5.2 respectively.

Interpretation of the Lagrangian multipliers

The Lagrangian multipliers provide the generalised reactive force associated with the corresponding constraint condition.

Then for the symmetric case considered:

$$n_{11}(o,\alpha_2) = \lambda_1(\alpha_2) \tag{5.34}$$

$$n_{22}(\alpha_1,o) = \lambda_2(\alpha_1) \tag{5.35}$$

$$Q(o,o) = \lambda_3 \tag{5.36}$$

where Q is the normal reactive force at a corner of the shell and is positive when acting in the (-y) direction.

In non-dimensional form equations (5.34), (5.35) and (5.36) reduce to:

$$\bar{n}_{11}(o,\beta_2) = \bar{\lambda}_1(\beta_2) = \sum_e \bar{\lambda}_1^e L_1^e(\beta_2) \tag{5.37}$$

$$\bar{n}_{22}(\beta_1,o) = \bar{\lambda}_2(\beta_1) = \sum_e \bar{\lambda}_2^e L_2^e(\beta_1) \tag{5.38}$$

$$\bar{Q}(o,o) = \bar{\lambda}_3 \tag{5.39}$$

Alternative expressions for $\bar{n}_{22}(o,\beta_2)$ and $\bar{n}_{11}(\beta_1,o)$ will now be derived.

From table 5.1, the following expressions are obtained:

$$\bar{n}_{22} - \nu\,\bar{n}_{11} = (1-\nu^2)b_{ij}u_2^iU_{2,2}^j - (1-\nu^2)c_{pq}w_pW_q \qquad (5.40)$$

$$\bar{n}_{11} - \nu\,\bar{n}_{22} = (1-\nu^2)a_{mn}u_{1,1}^mU_1^n - c(1-\nu^2)c_{pq}w_pW_q \qquad (5.41)$$

At the boundary $\beta_1 = 0$, equation (5.40), after substitution of equation (5.37) becomes:

$$\bar{n}_{22}(o,\beta_2) = (1-\nu^2)b_{ij}u_2^i(o)U_{2,2}^j - (1-\nu^2)c_{pq}w_p(o)W_q + \nu\bar{\lambda}_1(\beta_2) \qquad (5.42)$$

At the boundary $\beta_2 = 0$, equation (5.41), after substitution of equation (5.38) becomes:

$$\bar{n}_{11}(\beta_1,o) = (1-\nu^2)a_{mn}u_{1,1}^m\,U_1^n(o) - c(1-\nu^2)c_{pq}w_pW_q(o) + \nu\bar{\lambda}_2(\beta_1) \qquad (5.43)$$

Equations (5.37), (5.38), (5.39), (5.42) and (5.43) provide alternative* boundary values to those based on the displacement derivatives. The actual values are obtained, as before, from table 5.2.

This matter will be discussed further in section (5.5) in conjunction with numerical examples.

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*As discussed in section (3.1.3), the values based on the Lagrangian multipliers are generally different from the corresponding values based on the displacement derivatives.

5.2 Boundary Conditions and Approximating Functions

The boundary conditions to be considered in this chapter are given in table 5.3.

Only boundary conditions which are symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$ are considered.

In chapter 4 various types of approximating functions were considered. In view of these results and subsequent discussion, the functions chosen to specify a particular boundary condition are given in table 5.4. Details of the approximating functions are given in table 3.1.

In table 5.4 two separate sets of functions are associated with each boundary condition:

(a) functions which satisfy all the boundary conditions

(b) functions which violate the condition $u_1(o,a_2) = O$ or $u_2(a_1,o) = O$ but satisfy the remaining conditions on a boundary.*

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*Only when normal slides (1) are considered along all boundaries, is the constraint condition $w(o,o) = O$ considered in conjunction with the Lagrangian multiplier method.

Case (b) is considered in conjunction with the Lagrangian multiplier method.

Any combination of the boundary conditions given in table 5.4 may be specified.

5.3 Reduction to a System of Linear Algebraic Equations

For a particular set of approximating functions, equations (5.9), (5.10) and (5.11), with $\overline{Z} = 1$, reduce, on integration, to a system of linear algebraic equations, which in matrix form are:

$$\begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} \\ \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} \\ \underline{A}_{31} & \underline{A}_{32} & \underline{A}_{33} \end{bmatrix} \begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} + \begin{bmatrix} \underline{O} \\ \underline{O} \\ \underline{g} \end{bmatrix} = \begin{bmatrix} \underline{O} \\ \underline{O} \\ \underline{O} \end{bmatrix} \qquad (5.44)$$

or, more compactly:

$$\underline{A} \; \underline{\bar{a}} + \underline{\bar{g}} = \underline{O} \qquad (5.45)$$

where $\underline{\bar{a}} = \text{col}\left\{\underline{a} \;\; \underline{b} \;\; \underline{c}\right\}$

$\underline{\bar{g}} = \text{col}\left\{\underline{O} \;\; \underline{O} \;\; \underline{g}\right\}$

Typical elements of the submatrices in equations (5.44) are given in table 5.5a.* The relevant integration formulae are given in Appendix 2.

Since the notation used in defining the submatrices in table 5.5a is a departure from the usual matrix notation, typical examples will be given to illustrate the pattern of the matrices.

In table 5.5a typical elements of $\underline{A}_{12}$ and $\underline{b}$ were given as $a^{12}_{mn,kl}$ and $b_{kl,1}$ respectively. Assuming, for example that m, n, k and l each range over the values 1 and 2, then the respective matrix patterns are:

$$\underline{A}_{12} = \begin{bmatrix} a^{12}_{11,11} & a^{12}_{11,12} & a^{12}_{11,21} & a^{12}_{11,22} \\ a^{12}_{12,11} & a^{12}_{12,12} & a^{12}_{12,21} & a^{12}_{12,22} \\ a^{12}_{21,11} & a^{12}_{21,12} & a^{12}_{21,21} & a^{12}_{21,22} \\ a^{12}_{22,11} & a^{12}_{22,12} & a^{12}_{22,21} & a^{12}_{22,22} \end{bmatrix}$$

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*The comma notation used in defining a typical matrix element in tables 5.5a and 5.5b (e.g. $a^{11}_{mn,ij}$) does not represent differentiation. However, the comma notation used in the expression corresponding to a typical element represents differentiation with respect to β_1 and β_2

$$\underline{b} = \text{col}\left\{ b_{11} \quad b_{12} \quad b_{21} \quad b_{22} \right\}$$

Similarly for the other submatrices in table 5.5a.

5.3.1 Modification for the Lagrangian Multiplier Method

When the Lagrangian multiplier method is applied, the modified form given by equations (5.25) to (5.30) inclusive is used. These equations may also be reduced to a system of linear algebraic equations, which in matrix form are:

$$\begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} & \underline{D}_1 & \cdot & \cdot \\ \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} & \cdot & \underline{D}_2 & \cdot \\ \underline{A}_{31} & \underline{A}_{32} & \underline{A}_{33} & \cdot & \cdot & \underline{d}_5 \\ \underline{D}_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \underline{D}_4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \underline{d}_6^T & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \\ \bar{\underline{\lambda}}_1 \\ \bar{\underline{\lambda}}_2 \\ \bar{\lambda}_3 \end{bmatrix} + \begin{bmatrix} \cdot \\ \cdot \\ \underline{g} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \underline{O} \quad (5.46)$$

Typical elements of the submatrices $\underline{A}_{ij}$ $(i,j = 1,2,3)$, $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{g}$ are, as before, given in table 5.5a. Typical elements of the remaining submatrices in equations (5.46) are given in table 5.5b.

The matrix notation used in table 5.5b will be illustrated by typical examples.

In table 5.5b typical elements of $\underline{D}_1$, $\underline{D}_3$, $\underline{D}_4$, $\underline{d}_5$ and $\overline{\underline{\lambda}}_1$ were given as $(d^1_{mn,e})$, $(d^3_{j,ij})$, $(d^4_{k,kl})$, $(d^5_{mn,1})$ and $\overline{\lambda}^e_1$ respectively. Assuming, for example that m, n, i, j, k, l and e range over the values 1 and 2, then the respective matrix patterns are:

$$\underline{D}_1 = \begin{bmatrix} d^1_{11,1} & d^1_{11,2} \\ d^1_{12,1} & d^1_{12,2} \\ d^1_{21,1} & d^1_{21,2} \\ d^1_{22,1} & d^1_{22,2} \end{bmatrix}$$

$$\underline{D}_3 = \begin{bmatrix} d^3_{1,11} & \cdot & d^3_{1,21} & \cdot \\ \cdot & d^3_{2,12} & \cdot & d^3_{2,22} \end{bmatrix}$$

$$\underline{D}_4 = \begin{bmatrix} d^4_{1,11} & d^4_{1,12} & \cdot & \cdot \\ \cdot & \cdot & d^4_{2,21} & d^4_{2,22} \end{bmatrix}$$

$$\underline{d}_5 = \text{col}\left\{ d^5_{11} \quad d^5_{12} \quad d^5_{21} \quad d^5_{22} \right\}$$

$$\underline{\bar{\lambda}}_1 = \text{col}\left\{ \bar{\lambda}^1_1 \quad \bar{\lambda}^2_1 \right\}$$

Similarly for the other submatrices defined in table 5.5b.

It has been established that the Lagrangian multipliers $\lambda_1(a_2)$ and $\lambda_2(a_1)$ provide alternative values for $n_{11}(o,a_2)$ and $n_{22}(a_1,o)$ respectively (refer to section 5.1.1). Then the functions L^e_1 and L^e_2 should be chosen such that the condition on n_{11} or n_{22} in the corner of the shell is satisfied. A suitable set of functions is IA (refer to table 5.4), which correctly satisfies the zero condition on n_{11} or n_{22} in the corner of the shell for all combinations of the boundary conditions considered.

For the boundary conditions and approximating functions considered in this chapter (table 5.4):

$$L^e_1 = U^e_1 \qquad (5.47)$$

$$L^e_2 = u^e_2 \qquad (5.48)$$

which, on substitution in the expressions for typical elements of $\underline{D}_1$ and $\underline{D}_2$ given in table 5.4b, yields respectively:

$$d^1_{mn,e} = 2u_1^m(o)\int_o^1 U_1^e U_1^n d\beta_2 \quad (5.49)$$

$$d^2_{mn,e} = 2U_2^n(o)\int_o^1 u_2^e u_2^m d\beta_1 \quad (5.50)$$

Since the functions chosen for U_1^e, U_1^n, u_2^e and u_2^m are, in fact, sine functions, then the non-zero elements of $\underline{D}_1$ and $\underline{D}_2$, after integrating the expressions in equations (5.49) and (5.50) are respectively:

$$d^1_{mn,n} = u_1^m(o) \quad (5.51)$$

$$d^2_{mn,m} = U_2^n(o) \quad (5.52)$$

Hence

$$\underline{D}_1 = \underline{D}_3^T \quad (5.53)$$

$$\underline{D}_2 = \underline{D}_4^T \quad (5.54)$$

If S is the number of functions chosen to represent <u>each</u> of the displacements u_1, u_2 and w in <u>each</u> of the directions β_1 and β_2,* then the order of

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*It is not essential to adopt the same value of S for each displacement.

the respective submatrices is tabulated below:

Submatrix	Order
$\underline{A}_{ij}$ (i,j = 1,2,3)	$S^2 \times S^2$
$\underline{D}_1$, $\underline{D}_2$	$S^2 \times S$
$\underline{D}_3$, $\underline{D}_4$	$S \times S^2$
$\underline{a}, \underline{b}, \underline{c}, \underline{g}, \underline{d}_5, \underline{d}_6$	$S^2 \times 1$
$\underline{\bar{\lambda}}_1$, $\underline{\bar{\lambda}}_2$	$S \times 1$
$\bar{\lambda}_3$	1×1

Then there are $3S^2$ constants to be determined by equations (5.44) and $(3S^2 + 2S + 1)$ constants to be determined by equations (5.46).

However, since functions ID have been chosen to be used in conjunction with the Lagrangian multiplier method, the number of constants to be determined by equations (5.46) may be reduced . For the symmetric problem chosen here, functions ID for u_1^i (say) are:

$(1-2\beta_1)$, $\sin 2\pi\beta_1$, $\sin 4\pi\beta_1$, $\sin 6\pi\beta_1$, ..., $\sin 2i\pi\beta_1$,.....

where $i = 1,2,3,\ldots\ldots,(S-1)$ and function $(1-2\beta_1)$ corresponds to $i = O$. Substituting these functions in the constraint condition given by equation (5.28) yields:

$$a_{oj} \quad = \quad O \qquad\qquad (5.55)$$

where $\qquad j = 1,2,3,\ldots\ldots,S.$

The S constants given by equation (5.55) are set to zero before solving equations (5.46). In this way the number of constants has been reduced by S. A similar argument applies to the constraint condition given by equation (5.29).

A further advantage in following the procedure outlined above is that it avoids any numerical difficulties that may arise when using functions ID. (This matter was discussed in detail in section 4.4.2).

The solution of equations (5.46) forms the basis of the numerical results presented in this chapter. When no Lagrangian multipliers are used these equations reduce to equations (5.44).

5.4 Overall Equilibrium Check

In chapter 4, the numerical results were compared with available exact solutions. In the problems considered in this chapter, no such exact results are available. It is therefore necessary to apply some check on the solution.

Since the indirect methods discussed in this thesis attempt to satisfy equilibrium, a suitable check is one of overall equilibrium.

5.4.1. Geometry and Assumptions

From figure (2.4) the equation of the middle surface of a translational shell is given by:

$$z = \frac{K_2}{2}\left[c(x_1^2 - l_1x_1) + (x_2^2 - l_2x_2)\right] \qquad (5.56)$$

where $$K_2 = -\frac{8f_2}{l_2^2} \qquad (5.57)$$

The slopes of the middle surface in the x_1 and x_2 directions are respectively:

$$z,_1 = \frac{K_2}{2}\left[c(2x_1 - l_1)\right] \qquad (5.58)$$

$$z,_2 = \frac{K_2}{2}\left[2x_2 - l_2\right] \qquad (5.59)$$

Substitution for K_2 given by equation (5.57) in equations (5.58) and (5.59) yields:

$$z_{,1} = 4cr\frac{f_2}{l_2}\left(1 - \frac{2x_1}{l_1}\right) \tag{5.60}$$

$$z_{,2} = 4\frac{f_2}{l_2}\left(1 - \frac{2x_2}{l_2}\right) \tag{5.61}$$

The assumptions relating to the shallow curved plate theory (chapter 2) imply that the products of the slopes $z_{,1}$ and $z_{,2}$ may be neglected as small compared with unity.

Similarly it may be assumed that

$$z_{,i} \; (i = 1 \text{ or } 2) \simeq \tan \Theta_i \simeq \sin \Theta_i \simeq \Theta_i \tag{5.62}$$

$$\cos \Theta_i \simeq 1.0 \tag{5.63}$$

Within the limits of the curved plate approximation $\frac{x_1}{l_1}$ and $\frac{x_2}{l_2}$ may be replaced by β_1 and β_2 respectively and equations (5.60) and (5.61) become:

$$z_{,1} = 4cr\frac{f_2}{l_2}(1 - 2\beta_1) \tag{5.64}$$

$$z_{,2} = 4\frac{f_2}{l_2}(1 - 2\beta_2) \tag{5.65}$$

5.4.2. Resolution of the Load and Stress-Resultants in the Directions x_1, x_2 and z.

Since the loading and boundary conditions are symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$, only one quarter of the shell need be considered (figure 5.1).

Let E_1, E_2 and E_3 be the errors in equilibrium for one quarter of the shell (figure 5.1) measured positive in the directions x_1, x_2 and z respectively.

Then resolving the load and stress resultants in the directions x_1, x_2 and z respectively and allowing for the assumptions previously made, i.e. ignoring terms containing products of slopes of the middle surface and assuming that the relations given by equations (5.62) and (5.63) hold, yields the following three equations respectively:

$$E_1 = \int_0^{\frac{l_1}{2}} \left[-n_{12}(\alpha_1,o) + q_2(\alpha_1,o)(z,_1)_{x_2=o} \right] d\alpha_1 + \int_0^{\frac{l_2}{2}} \left[-n_{11}(o,\alpha_2) + n_{11}(\tfrac{l_1}{2}, \alpha_2) + q_1(o,\alpha_2)(z,_1)_{x_1=o} \right] d\alpha_2 - \int_0^{\frac{l_1}{2}} \int_0^{\frac{l_2}{2}} Z(z,_1) d\alpha_1 d\alpha_2 + Q(z,_1)_{\substack{x_1=o \\ x_2=o}} \qquad (5.66)$$

$$E_2 = \int_0^{\frac{l_1}{2}} \left[-n_{22}(\alpha_1, o) + n_{22}(\alpha_1, \frac{l_2}{2}) + q_2(\alpha_1, o)(z_{,2})_{x_2=o} \right] d\alpha_1 +$$

$$+ \int_0^{\frac{l_2}{2}} \left[- n_{12}(o, \alpha_2) + q_1(o, \alpha_2)(z_{,2})_{x_1=o} \right] d\alpha_2 -$$

$$- \int_0^{\frac{l_1}{2}} \int_0^{\frac{l_2}{2}} Z(z_{,2}) d\alpha_1 d\alpha_2 + Q(z_{,2})_{\substack{x_1=o \\ x_2=o}} \qquad (5.67)$$

$$E_3 = - \int_0^{\frac{l_1}{2}} \left[n_{22}(\alpha_1, o)(z_{,2})_{x_2=o} + n_{12}(\alpha_1, o)(z_{,1})_{x_2=o} + q_2(\alpha_1, o) \right] d\alpha_1 -$$

$$- \int_0^{\frac{l_2}{2}} \left[n_{11}(o, \alpha_2)(z_{,1})_{x_1=o} + n_{12}(o, \alpha_2)(z_{,2})_{x_1=o} + q_1(o, \alpha_2) \right] d\alpha_2 -$$

$$- Q + \int_0^{\frac{l_1}{2}} \int_0^{\frac{l_2}{2}} Z d\alpha_1 d\alpha_2 \qquad (5.68)$$

From table 5.4 and equations (5.4) and (5.5) the following are obtained:

$$n_{ij} = \frac{Z}{K_2}\bar{n}_{ij} = -\frac{Zl_2^2}{8f_2}\bar{n}_{ij}$$

$$q_i = Zl_1\bar{q}_i$$

$$Q = Zl_1l_2\overline{Q}$$

$$da_i = l_i d\beta_i \quad \text{(not summed)}$$

where i and j range over the values 1 and 2. Substituting for n_{ij}, q_i, Q and da_i by the above expressions and for $(z,_1)$ and $(z,_2)$ by equations (5.64) and (5.65) in equations (5.66), (5.67) and (5.68) yields:

$$E_1 = Zl_1l_2(\overline{E}_1) \tag{5.69}$$

$$E_2 = Zl_1l_2(\overline{E}_2) \tag{5.70}$$

$$E_3 = Zl_1l_2(\overline{E}_3) \tag{5.71}$$

where the non-dimensional forms $\overline{E}_1$, $\overline{E}_2$ and $\overline{E}_3$ are given by:

$$\overline{E}_1 = \frac{1}{8}\cdot\left(\frac{l_2}{f_2}\right)\int_o^{\frac{1}{2}}\left[\bar{n}_{12}(\beta_1,o) + 32cr^2\left(\frac{f_2}{l_2}\right)^2\bar{q}_2(\beta_1,o)(1-2\beta_1)\right]d\beta_1 +$$

$$+ \frac{1}{8}\left(\frac{l_2}{f_2}\right)\int_o^{\frac{1}{2}}\left[\frac{1}{r}\bar{n}_{11}(o,\beta_2) - \frac{1}{r}\bar{n}_{11}(0.5,\ \beta_2) + 32cr\left(\frac{f_2}{l_2}\right)^2\bar{q}_1(o,\beta_2)\right]d\beta_2 +$$

$$+ 4cr\left(\frac{f_2}{l_2}\right)\overline{Q} - \tfrac{1}{2}cr\left(\frac{f_2}{l_2}\right) \tag{5.72}$$

$$\bar{E}_2 = \frac{1}{8}\left(\frac{l_2}{f_2}\right) \int_0^{\frac{1}{2}} \left[\bar{n}_{22}(\beta_1, o) - \bar{n}_{22}(\beta_1, \; 0.5) + 32r\left(\frac{f_2}{l_2}\right)^2 \bar{q}_2(\beta_1, o) \right] d\beta_1 \; +$$

$$+ \; \frac{1}{8}\left(\frac{l_2}{f_2}\right) \int_0^{\frac{1}{2}} \left[\frac{1}{r} \bar{n}_{12}(o, \beta_2) + 32 \left(\frac{f_2}{l_2}\right)^2 \bar{q}_1(o, \beta_2)(1 - 2\beta_2) \right] d\beta_2 \; +$$

$$+ \; 4\left(\frac{f_2}{l_2}\right)\bar{Q} - \frac{1}{2} . \left(\frac{f_2}{l_2}\right) \qquad (5.73)$$

$$\bar{E}_3 = \frac{1}{2} \int_0^{\frac{1}{2}} \left[\bar{n}_{22}(\beta_1, o) + rc\bar{n}_{12}(\beta_1, o)(1 - 2\beta_1) - 2r\bar{q}_2(\beta_1, o) \right] d\beta_1 \; +$$

$$+ \; \frac{1}{2} \int_0^{\frac{1}{2}} \left[c\bar{n}_{11}(o, \beta_2) + \frac{1}{r} \cdot \bar{n}_{12}(o, \beta_2)(1 - 2\beta_2) - 2\bar{q}_1(o, \beta_2) \right] d\beta_2 \; -$$

$$- \; \bar{Q} + 0.25 \qquad (5.74)$$

Equations (5.72), (5.73) and (5.74) are the equations used to check overall equilibrium for a shallow curved plate. $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ are the errors in equilibrium expressed as a factor of $(Zl_1 l_2)$ and measured positive in the directions x_1, x_2 and z respectively.

For the overall equilibrium check it is necessary to define a further parameter $(\frac{l_2}{f_2})$, which is a measure of the shallowness of the shell, to determine $\bar{E}_1$ and $\bar{E}_2$. Note that $\bar{E}_3$ is independent of $(\frac{l_2}{f_2})$.

5.5 Convergence Study of the Approximating Functions

In this section combinations of the boundary conditions given in table 5.4 will be applied to particular numerical examples. In the computer program developed, provision is made for the symmetry of the problem by choosing the functions given in table 5.4 such that:

(a) u_2^m, w_m, L_2^e are symmetric functions about $\beta_1 = 0.5$

(b) u_1^m is an antimetric function about $\beta_1 = 0.5$

(c) U_1^n, W_n, L_1^e are symmetric functions about $\beta_2 = 0.5$

(d) U_2^n is an antimetric function about $\beta_2 = 0.5$

5.5.1. Numerical Examples

The examples to be studied are given in table 5.6. The corresponding results* are presented in tables 5.7 to 5.12 inclusive and

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*The results in the tables are presented in floating point notation e.g. 1.234, +3 means 1.234×10^3

figures 5.2a, 5.2b and 5.2c.

Displacements, stress-resultants and stress-couples are presented in non-dimensional form (table 5.1). The actual values are obtainable from the expressions given in table 5.2.

In tables 5.7 to 5.12 inclusive the values marked with an asterisk (*) are based on the Lagrangian multipliers and the corresponding values in brackets are based on the displacement derivatives.

In the overall equilibrium check, E_1 and E_2 are presented in their non-dimensional forms $\bar{E}_1$ and $\bar{E}_2$, whilst E_3 is expressed as a percentage error ($= 400\ \bar{E}_3$).*

S is the number of functions used to represent each displacement in each of the directions β_1 and β_2, due allowance being made for the symmetry of the problem in the selection of these functions.

5.5.2. Discussion

The convergence of the displacements was good for all combinations of the boundary conditions considered i.e. clamped, hinged and normal slide (1).

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*This check was not incorporated in the computer program and the integrations in equations (5.72), (5.73) and (5.74) were performed numerically using Simpson's rule.

When a hinged boundary was used, the convergence of n_{11} (or n_{22}) along this boundary was slow, but satisfactory within the shell (note example 5.2A). This was reflected in the large errors in the overall equilibrium check. Application of the Lagrangian multiplier method in conjunction with functions ID (case (b) in table 5.4) greatly improved the boundary value of n_{11} (or n_{22}) and reduced the errors in equilibrium (compare example 5.2B with example 5.2A). The boundary value of n_{11} (or n_{22}) based on the Lagrangian multiplier again provided a more accurate estimate than the corresponding value based on the displacement derivative (refer to section 4.4.2 where this matter was discussed in detail). It was previously noted (section 4.4.2) that functions ID when used in the manner described in this thesis may be effective in improving a particular stress-resultant, which is slowly convergent on the boundary but satisfactory elsewhere. The examples studied in this section are a further illustration of this.

With clamped and normal slide (1) boundaries, the convergence of the membrane stress-resultants was good and, although provision was made in the computer program, it was unnecessary to apply the Lagrangian multiplier method (case (b) in table 5.4).

When normal slides (1) were considered along all boundaries (example 5.3, table 5.9, figures 5.2a, 5.2b and 5.2c), the moments (stress-couples) were generally very small except in the region of the corners where convergence was slow. In this example the Lagrangian multiplier gave the normal reaction in the corner. Example 5.3, in some respects, is similar to example A7 in section (4.4.2). In example A7 functions IB were used to represent W_m and the Lagrangian multiplier method applied in conjunction with the constraint condition $W_m = 0$ at $\beta_2 = 0,1$. The boundary moment in this case was very slowly convergent (note, in particular, figure 4.6). For comparison, the solution for a shell with the same parameters but with all edges clamped, is also given in figures 5.2a, 5.2b and 5.2c.

With other combinations of clamped, hinged and normal slides (1), the moments were generally converging satisfactorily.

Normal shears (stress-resultants) on the boundary were generally slowly convergent, which undoubtedly contributed to the errors in equilibrium, particularly if the shears were of a significant magnitude.

5.6 Comparison with Other Available Solutions

Example 5.7: Consider an elliptic paraboloid with the following data:

$$r = 1.0$$

$$\rho_T = 0.0152928$$

$$c = 0.5$$

$$\nu = 0.25$$

and boundary conditions:

clamped at $\alpha_1 = 0, l_1$ and $\alpha_2 = 0, l_2$.

The convergence of this solution was studied in example 5.1.

This example was also solved by Noor and Veletsos[15], using a Rayleigh-Ritz analysis and a modified finite difference technique. A comparison is given in table 5.13 and figures 5.3a and 5.3b. The solutions show good agreement.

In the Rayleigh-Ritz analysis used in reference (15), a w - ϕ formulation is used and functions IE and IC have been chosen to represent respectively w and ϕ in each of the directions β_1 and β_2. Note that the boundary values of the moments so obtained, are smaller than for the other solutions (table 5.13 , figure 5.3a). It has been previously noted (section 3.1.4), that the functions IE impose the additional boundary conditions of zero normal shear and zero Kirchhoff shear, and this undoubtedly is reflected in the slower convergence of the boundary moments observed for this case (refer also to example A1 in section 4.4.2).

Example 5.8: Consider a cylinder with the following data:

$l_1 = 600$ in. $l_2 = 497.4$ in.* $h = 4$ in.

$K_1 = 0$ $K_2 = 1.8519, -3$ in^{-1}

$E = 3.0, +6$ lbs/in^2 $\nu = 0$ $Z = 0.555$ lbs/in^2

and boundary conditions:

clamped at $\alpha_1 = 0, l_1$ and $\alpha_2 = 0, l_2$.

This example was also solved by Gunasekera[6] and by Lu[5], using an extended Levy method of solution. The solution is compared with that given by Gunasekera in table 5.14.

The solution for m_{11} on the boundary is less than that given by Gunasekera by approximately 10%. Otherwise the solutions show good agreement.

---

*This is the arc length corresponding to a plan length of 480 inches. In the shallow curved plate theory discussed in this thesis, no distinction is made between the arc length and the plan length. However, in order to compare the indirect solutions of this thesis with other available solutions it is sometimes necessary to use the arc length.

Example 5.9: Consider a hyperbolic paraboloid with the following data:

$l_1 = 51.32$ ft.* $\qquad L_2 = 61.59$ ft.* $\qquad h = 2.5$ in.

$K_1 = -1.5385, -2 \text{ ft}^{-1} \qquad K_2 = 1.2821, -2 \text{ ft}^{-1}$

$E = 4.5, +8 \text{ lbs/ft}^2 \qquad \nu = 0.15 \qquad Z = 50 \text{ lbs/ft}^2$

and boundary conditions:

hinged at $\alpha_1 = 0, l_1$ and $\alpha_2 = 0, l_2$.

This example was solved by Gunasekera[(6)]** and a comparison is given in table 5.15.

Note that the Lagrangian multiplier method is used in an attempt to improve the boundary values of n_{11} and n_{22}. Functions corresponding to case (b) in table 5.4 are used.

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*These are arc lengths corresponding to the plan lengths 50 ft. and 60 ft. respectively.

**The results presented by Gunasekera were at $\frac{1}{8}$th points. The results presented here are at $\frac{1}{10}$th points. This example was re-run for this latter output using Gunasekera's computer program.

The solutions generally show good agreement. Note the good agreement of the boundary values of n_{11} and n_{22} based on the Lagrangian multipliers with those given by Gunasekera.

5.7 Further Solutions - Variation of the Shell Parameters

The non-dimensional form of equations (5.9), (5.10) and (5.11) shows that the translational shell is completely defined by the parameters r, ρ_T, c and ν. Such a representation permits the behaviour of translational shells to be conveniently studied by the variation of these parameters.

The examples considered and the particular parameter being varied are given in table 5.16. The corresponding results are given in tables 5.17 to 5.22 inclusive.

All results are presented in non-dimensional form (table 5.1), the actual values being obtained from the expressions given in table 5.2.

The Lagrangian multiplier method is used in example 5.14 (case (b) in table 5.4). In this case the boundary values of $\bar{n}_{11}$ given in table 5.21 are based on the Lagrangian multiplier.

$S = 8$ has been chosen in each of these examples.

5.7.1 Discussion

Variation of ρ_T: The parameter ρ_T, defined by equation (5.13), varies with shell thickness and shallowness. This thesis is concerned with the study of thin shallow shells and ρ_T should be interpreted accordingly. The thin flat plate is recovered from $\rho_T = \infty$.

For comparison the following flat plate solutions for $r = 1$ and $\nu = 0.15$ are given*:

(i) all boundaries clamped:

$$\bar{w}(0.5,\ 0.5) = 1.265,\ -3$$

$$\bar{m}_{11}(0,\ 0.5) = -5.084,\ -2$$

$$\bar{m}_{11}(0.5,\ 0.5) = 2.021\ \ -2$$

(ii) all boundaries hinged (simply supported):

$$\bar{w}(0.5,\ 0.5) = 4.062,\ -3$$

$$\bar{m}_{11}(0.5,\ 0.5) = 4.234,\ -2$$

---

*These values are in non-dimensional form and were obtained from the computer program developed by using a very large value of ρ_T. The actual values follow from table 5.2. S = 8 was again adopted.

(iii) all boundaries with normal slides:

$\bar{w}$ (0.5, 0.5)	=	5.771, -3
$\bar{w}$ (0, 0.5)	=	4.323, -3
$\bar{m}_{11}$(0,0)	=	-2.322, -1*
$\bar{m}_{11}$ (0, 0.5)	=	-1.956, -2
$\bar{m}_{11}$ (0.5, 0.5)	=	3.159, -2
$\bar{Q}$	=	2.500, -1

As the shell becomes shallower, i.e. as ρ_T increases, tables 5.17, 5.19 and 5.20 show that $\bar{w}$, $\bar{m}_{11}$ and $\bar{Q}$ (example 5.15 only) also increase, slowly approaching the solution for a thin flat plate. When ρ_T = 0.03, which corresponds to the values $(\frac{h}{l_1}) = \frac{1}{100}$ and $(\frac{l_2}{r_2})$ = -24, the solution for the shell, although very shallow, is still very different from the corresponding flat plate solution.

On the other hand, $\bar{n}_{11}$ decreases very slowly with increasing shallowness and is still of significant value even for a very shallow shell (ρ_T = 0.10), particularly in example 5.15. Similarly $\bar{n}_{12}$ is decreasing slowly with increasing shallowness, but it is of smaller magnitude than $\bar{n}_{11}$.

---

*This value is slowly convergent.

Similar remarks apply to increasing thickness, which also corresponds to an increasing ρ_T, but any comments are restricted to thin shells.*

Note that only the effect on the non-dimensional values has been considered. The actual values follow from table 4.2, which gives:

$$w = \frac{12\ Zl_1(1-\nu^2)}{E}\left(\frac{l_1}{h}\right)^3 \bar{w}$$

$$n_{11} = Zl_1\left(-\frac{1}{8r} \cdot \frac{l_2}{f_2}\right)\bar{n}_{11} = \frac{Zl_1^2}{h}(\rho_T\bar{n}_{11})$$

Then if increasing ρ_T is interpreted as increasing shallowness the effect on the actual stress-resultant n_{11} is dependent on $(\rho_T\bar{n}_{11})$. Referring to

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*Vlasov (page 337, reference 9) restricts thin shells to the range:

$$h\left|K_{max}\right| \leq \frac{1}{30} \qquad \text{(a)}$$

where $\left|K_{max}\right|$ is, numerically, the maximum undeformed curvature.

If $\left|K_2\right| > \left|K_1\right|$, then (a) becomes:

$$\left(\frac{h}{l_1}\right)\left(\left|\frac{f_2}{l_2}\right|\right) \leq \frac{1}{240\,r}$$

and if $\left|K_1\right| > \left|K_2\right|$ then (a) becomes:

$$\left(\frac{h}{l_1}\right)\left(\left|\frac{f_2}{l_2}\right|\right) \leq \frac{1}{240\,r\,|c|}$$

the tables, it will be noted that the product $(\rho_T \bar{n}_{11})$ increases with increasing ρ_T. However, beyond the range considered here, the product begins to decrease with increasing ρ_T and approaches zero as ρ_T becomes very large.

If increasing ρ_T is interpreted as increasing thickness, but restricted to thin shells, the effect on the actual displacement w is dependent on $(\frac{l_1}{h})^3 \bar{w}$.

The non-dimensional presentation of the tables given in this section covers, very compactly, the solutions for a wide range of thin shallow shells.

<u>Variation of c:</u> Only the case with all boundaries clamped is considered and the results are presented in table 5.18.

Since ρ_T remains constant, the variation of c represents, in effect, the variation of K_1 with all other data fixed.

As c increases from -2.0 to +2.0:

(a) $\bar{w}$ and $\bar{m}_{11}$ initially increase, reaching their maximum values, within the limits of the results presented here, at c = 0 after which they begin to decrease.

(b) $\bar{m}_{22}$ initially increases, reaching its maximum value at c = -0.5, after which it begins to decrease.

(c) $\bar{n}_{11}$ changes sign, reaching its maximum positive value at $c = -1.0$, and its maximum negative value at $c = +1.0$.

(d) $\bar{n}_{22}$ initially increases, reaching its maximum value when $c = 0$, after which it begins to decrease.

(e) $\bar{n}_{12}$ changes sign, reaching its maximum positive value when $c = -2.0$ and its maximum negative value when $c = 0.5$.

Variation of r: Only the case with all boundaries clamped is considered. The results are presented in table 5.19.

Since ρ_T remains constant, the variation of r represents, in effect, the variation of l_2 with all other data fixed.

As r increases from 0.5 to 5.0:

(a) the maximum value of $\bar{w}$ along $\beta_1 = 0.5$ increases, reaching its highest value, within the limits of the results presented here, at $r = 2.0$, after which it begins to decrease; the location of this maximum value moves towards the centre of the shell.

(b) the maximum value of $\bar{w}$ along $\beta_2 = 0.5$ increases, reaching its highest value at $r = 2.0$, after which it begins to decrease; the location of this maximum value moves away from the centre of the shell.

(c) the maximum value of $\bar{m}_{11}$ along the boundary $\beta_1 = 0$ increases, reaching its highest value at r = 3.0, after which it begins to decrease;, the location of this maximum value moves towards the centre of the boundary.

(d) $\bar{m}_{22}$ along the boundary $\beta_2 = 0$ increases, the location of its maximum value remaining unchanged.

(e) $\bar{n}_{11}$ along the boundary $\beta_1 = 0$ decreases, whilst along $\beta_2 = 0.5$ it increases reaching a maximum at r = 2.0, after which it begins to decrease.

(f) $\bar{n}_{22}$ along $\beta_1 = 0.5$ decreases, whilst along the boundary $\beta_2 = 0$ it initially increases, reaching a maximum at r = 1.0, after which it begins to decrease.

Variation of ν : Only the case with all boundaries clamped is considered. The results are presented in table 5.20.

As ν increases from 0 to 0.30:

(a) $\bar{w}$ decreases

(b) the magnitude of $\bar{m}_{11}$ generally decreases

(c) $\bar{n}_{11}$ increases slightly

(d) $\bar{n}_{12}$ decreases.

The actual displacement w is given by (table 5.2):

$$w = \frac{Zl_1^4}{D}\,\bar{w} = \frac{12\ Zl_1^4}{Eh^3}(1 - \nu^2)\bar{w}$$

which also decreases with increasing ν .

5.8 Discussion of the Computer Programs

The single computer program developed for translational shells is limited to uniformly distributed normal loading and to boundary conditions which are symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$. However, any symmetric combination of clamped, hinged or normal slide (1) boundary conditions (table 5.4) may be specified. Provision is also made to apply the Lagrangian multiplier method in conjunction with $u_1 = 0$ (along $\beta_1 = 0,1$) or $u_2 = 0$ (along $\beta_2 = 0,1$) for each of the boundary conditions specified (refer to case (b) in table 5.4).

The approximating functions are selected in accordance with the symmetry of the problem (section 5.5). The same value of S for each displacement in each of the directions β_1 and β_2 is considered.

Input, and therefore output, could be either in non-dimensional form or in terms of the actual dimensions.

In order to economise on computer storage, the system of linear algebraic equations was solved by partitioning the equations into their submatrix form (equations 5.44). Note that when the modified form given by equations (5.46) was used, the S constants given by equation (5.55) were pre-set to zero and the equations rearranged in the form given by equations (5.44).

Further details of the computer programs are available at Imperial College.[69]

The computer programs were written in EXCHLF Autocode[70],[71] for the University of London Atlas Computer.

CHAPTER 6

APPLICATION OF THE INDIRECT METHODS TO RULED SURFACE SHELLS

In this chapter the proposed indirect methods will be applied to ruled surface hyperbolic paraboloid shells (figure 2.5), for which no exact solutions are available.

Only uniformly distributed normal loading (Z) will be considered.

A u_1-u_2-w formulation will be used after a short discussion of its merits in comparison with a w-ϕ formulation.

The Galerkin equations, in terms of w and ϕ are:

$$\int_0^{l_1}\int_0^{l_2}\left[D\nabla^4 w + 2K_{12}\phi,_{12} - Z\right]\delta w \, da_1 da_2 = 0$$

$$\int_0^{l_1}\int_0^{l_2}\left[\nabla^4\phi - 2EhK_{12}w,_{12}\right]\delta\phi \, da_1 da_2 = 0$$

In the Galerkin method the functions for w and ϕ must be chosen such that all boundary conditions are satisfied.

The first of these equations can be derived from the principle of minimum total potential energy and the second from the principle of minimum complementary energy (refer to section 2.3). Note that the Pücher stress-resultant function, ϕ, automatically satisfies the relevant equations of equilibrium. A variational treatment along these lines is given in reference 15.

Consider the case when the shell is supported on all boundaries by normal gables (Table 6.3), which correspond to the boundary conditions $w = o = m_{11}$ and $u_2 = o = n_{11}$ at $\alpha_1 = o,\ l_1$ and $w = o = m_{22}$ and $u_1 = o = n_{22}$ at $\alpha_2 = o,\ l_2$. In terms of w and ϕ, the boundary conditions become $w = o = w,_{11}$ and $\phi = o = \phi,_{11}$ at $\alpha_1 = o,\ l_1$ and $w = o = w,_{22}$ and $\phi = o = \phi,_{22}$ at $\alpha_2 = o,\ l_2$. The obvious functions for w and ϕ, which will satisfy all the boundary conditions, are sine functions (IA in Table 3.1) such that w is symmetric and ϕ antimetric about the centre of the rectangular plan-form.

The application of these functions yielded results which compared unfavourably with other available solutions.[(19),(25)] The use of the Pücher stress-resultant function, in this case, inhibits the selection of an approximating function which simultaneously yields realistic distributions of the corresponding three stress-resultants i.e. n_{11}, n_{22} and n_{12} .

The u_1-u_2-w formulation allows greater freedom in the distribution of the membrane stress-resultants and a more direct formulation for the boundary conditions. A suitable form for these equations follows.

6.1 Non-Dimensional Form of Equations

Let the displacement distributions assume the following forms:

$$u_1 = l_1 \sum_m \sum_n a_{mn} u_1^m(\beta_1) U_1^n(\beta_2) \tag{6.1}$$

$$u_2 = l_2 \sum_m \sum_n b_{mn} u_2^m(\beta_1) U_2^n(\beta_2) \tag{6.2}$$

$$w = \frac{1}{K_{12}} \sum_m \sum_n c_{mn} w_m(\beta_1) W_n(\beta_2) \tag{6.3}$$

where

$$\beta_1 = \frac{\alpha_1}{T_1} \tag{6.4}$$

$$\beta_2 = \frac{\alpha_2}{T_2} \tag{6.5}$$

a_{mn}, b_{mn} and c_{mn} are constants to be determined, u_1^m, u_2^m, w_m, U_1^n, U_2^n and W_n represent sets of independent kinematically admissible functions, and m and n are positive integers.

The corresponding displacement variations may be selected in the following forms:

$$\delta u_1 = l_1 \sum_m \sum_n u_1^m(\beta_1) U_1^n(\beta_2) \delta a_{mn} \qquad (6.6)$$

$$\delta u_2 = l_2 \sum_m \sum_n u_2^m(\beta_1) U_2^n(\beta_2) \delta b_{mn} \qquad (6.7)$$

$$\delta w = \frac{1}{K_{12}} \sum_m \sum_n w_m(\beta_1) W_n(\beta_2) \delta c_{mn} \qquad (6.8)$$

where δa_{mn}, δb_{mn} and δc_{mn} are arbitrary variations in the constants a_{mn}, b_{mn} and c_{mn} respectively.

In the following derivation only the boundary integrals corresponding to n_{12} will be retained. In all other cases the boundary integrals will be assumed to vanish by virtue of the chosen functions.

Then the variational equation (2.90) after:

(a) setting X_1, X_2 and the applied boundary loads to zero,

(b) non-dimensionalising the co-ordinates to the (β_1, β_2) set defined by equations (6.4) and (6.5),

(c) setting K_{11} and K_{22} to zero, and (d) substitution of equations (6.1), (6.2), (6.3), (6.6), (6.7) and (6.8),

reduces to the following three independent equations, since δa_{mn}, δb_{mn} and δc_{mn} are arbitrary:

$$\int_0^1 \int_0^1 \Big[-a_{ij} u^i_{1,11} U^j_1 - \frac{(1-\nu)}{2} r^2 a_{ij} u^i_1 U^j_{1,22} - \frac{(1+\nu)}{2} b_{kl} u^k_{2,1} U^l_{2,2} + $$

$$+ r(1-\nu) c_{pq} w_p W_{q,2} \Big] u^m_1 U^n_1 d\beta_1 d\beta_2 - 2r \int_0^1 \bar{n}_{12}(\beta_1, o) u^m_1 U^n_1(o) d\beta_1 = 0 \quad (6.9)$$

$$\int_0^1 \int_0^1 \Big[-\frac{(1+\nu)}{2} a_{ij} u^i_{1,1} U^j_{1,2} - b_{kl} u^k_2 U^l_{2,22} - \frac{(1-\nu)}{2r^2} b_{kl} u^k_{2,11} U^l_2 + $$

$$+ \frac{(1-\nu)}{r} c_{pq} w_{p,1} W_q \Big] u^m_2 U^n_2 d\beta_1 d\beta_2 - \frac{2}{r} \int_0^1 \bar{n}_{12}(o, \beta_2) u^m_2(o) U^n_2 d\beta_2 = 0 \quad (6.10)$$

$$\int_0^1 \int_0^1 \Big[-r(1-\nu) a_{ij} u^i_1 U^j_{1,2} - \frac{(1-\nu)}{r} b_{kl} u^k_{2,1} U^l_2 + \frac{p^2}{12} (c_{pq} w_{p,1111} W_q + $$

$$+ 2r^2 c_{pq} w_{p,11} W_{q,22} + r^4 c_{pq} w_p W_{q,2222}) + 2(1-\nu) c_{pq} w_p W_q - $$

$$- \bar{Z} \Big] w_m W_n d\beta_1 d\beta_2 = 0 \quad (6.11)$$

where $r = \frac{l_1}{l_2}$ (6.12)

$$\rho_R = \frac{h}{l_1} \cdot \frac{1}{K_{12} l_1} = \frac{h}{l_1} \cdot \frac{l_2}{f} = -\frac{1}{2}\frac{h}{l_1} \cdot \frac{l_2}{\bar{f}} \qquad (6.13)$$

$$\bar{Z} = \frac{Z(1-\nu^2)}{EhK_{12}} \qquad (6.14)$$

i, j, k, l, m, n, p, q are positive integers and $\bar{n}_{12}$ is a function of β_1 and β_2 and is the non-dimensional form (given in Table 6.1) of n_{12}. In equations (6.9), (6.10), (6.11) and table (5.1), the Einstein summation convention is adopted and comma notation is used to represent differentiation with respect to β_1 and β_2.

In deriving expressions for the boundary integrals, the boundary conditions were assumed symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$. If this were not the case equations (6.9), (6.10) and (6.11) would be modified in the following way:

replace $\left[- 2r \int_0^1 \bar{n}_{12}(\beta_1, o) u_1^m U_1^n(o) d\beta_1\right]$ by

$$\left[+ r \int_0^1 \bar{n}_{12}(\beta_1, \beta_2) u_1^m U_1^n(\beta_2) d\beta_1\right]_{\beta_2=0}^{\beta_2=1}$$

replace $\left[-\frac{2}{r} \int_o^1 \bar{n}_{12}(o,\beta_2) u_2^m(o) U_2^n d\beta_2 \right]$ by

$$\left[+\frac{1}{r} \int_o^1 \bar{n}_{12}(\beta_1,\beta_2) u_2(\beta_1) U_2^n d\beta_2 \right]_{\beta_1=0}^{\beta_1=1}$$

It is evident from the foregoing that the problem is specified through the non-dimensional parameters ρ_R, r and ν .*

With $\bar{Z}$ = 1 equations (6.9), (6.10) and (6.11 are the equations used for the solutions presented in this chapter.

The actual values of the displacements, stress-resultants and stress-couples, for any uniformly distributed normal loading Z, are obtained from the non-dimensional forms given in table 6.1 by the factors given in table 6.2.

Equations (6.9), (6.10) and (6.11) are the Galerkin equations modified by expressions corresponding to the relevant boundary integrals in equation (2.90).

*The single parameter ρ_R could have been replaced by the separate parameters $(\frac{h}{l_1})$ and $(\frac{l_2}{r})$. However, the use of ρ_R covers a wider range of shells.

6.1.1 Modification for the Lagrangian Multiplier Method

In this section only the following homogeneous kinematic conditions will be considered:

$$u_1 = 0 \quad \text{at} \quad \alpha_2 = 0,\ l_2 \qquad (6.15)$$

$$u_2 = 0 \quad \text{at} \quad \alpha_1 = 0,\ l_1 \qquad (6.16)$$

$$w = 0 \quad \text{at} \quad (\alpha_1, \alpha_2) = (o,o),(l_1,o),(o,l_2),(l_1 l_2) \qquad (6.17)$$

Assume that the conditions given by equations (6.15), (6.16) and (6.17) are now applied as constraint conditions.

Then the following procedure described in section (3.1.3) and assuming that the boundary conditions are symmetric about $\alpha_1 = 0.5\ l_1$ and $\alpha_2 = 0.5\ l_2$, the variational equation (2.90) is modified to:

Left hand side of equation (2.90) +

$$+ \ 2\int_o^{l_1} \lambda_1(\alpha_1)\delta u_1(\alpha_1, o)d\alpha_1 \ +$$

$$+ \ 2\int_o^{l_2} \lambda_2(\alpha_2)\delta u_2(o, \alpha_2)d\alpha_2 \ +$$

$$+ \ 4\ \lambda_3\ \delta w(o,o) \ = \ 0 \qquad (6.18)$$

where $\lambda_1(\alpha_1)$, $\lambda_2(\alpha_2)$ and λ_3 (a constant) are the Lagrangian multipliers corresponding to the displacements u_1, u_2 and w respectively.

The constraint conditions are:

$$u_1(\alpha_1, o) = O \tag{6.19}$$

$$u_2(o, \alpha_2) = O \tag{6.20}$$

$$w(o,o) = O \tag{6.21}$$

Equations (6.18) to (6.21) inclusive completely define the problem. Expressing

$$\lambda_1(\alpha_1) = \sum_e \lambda_1^e L_1^e(\alpha_1) \tag{6.22}$$

$$\lambda_2(\alpha_2) = \sum_e \lambda_2^e L_2^e(\alpha_2) \tag{6.23}$$

where λ_1^e and λ_2^e are constants

$L_1^e(\alpha_1)$ and $L_2^e(\alpha_2)$ represent sets of independent functions,

and e is a positive integer,

and proceeding as in sections (6.1) and (3.1.3), equations (6.18) to (6.21) inclusive reduce to the following:

Left hand side of equation (6.9) + $2r\bar{\lambda}_1^e U_1^n(o) \int_o^1 L_1^e u_1^m \, d\beta_1 = 0$ (6.24)

Left hand side of equation (6.10) + $\frac{2}{r}\bar{\lambda}_2^e u_2^m(o) \int_o^1 L_2^e U_2^n \, d\beta_2 = 0$ (6.25)

Left hand side of equation (6.11) + $4\bar{\lambda}_3 w_m(o) W_n(o) = 0$ (6.26)

$$a_{ij} \; U_1^j(o) = 0 \qquad (6.27)$$

$$b_{kl} \; u_2^k(o) = 0 \qquad (6.28)$$

$$c_{pq} w_p(o) W_q(o) = 0 \qquad (6.29)$$

where

$$\bar{\lambda}_1^e = \left(\frac{1-\nu^2}{Eh}\right)\lambda_1^e \qquad (6.30)$$

$$\bar{\lambda}_2^e = \left(\frac{1-\nu^2}{Eh}\right)\lambda_2^e \qquad (6.31)$$

$$\bar{\lambda}_3 = \frac{(1-\nu^2)}{EhK_{12}l_1 l_2}\lambda_3 \qquad (6.32)$$

With $\bar{Z} = 1$ equations (6.24) to (6.29) inclusive are the equations used in conjunction with the Lagrangian multiplier method.

As before the non-dimensional and actual values of the displacements, stress-resultants and stress-couples are obtained from tables 6.1 and 6.2 respectively.

Interpretation of the Lagrangian multipliers

The Lagrangian multipliers provide the generalised reactive force associated with the corresponding constraint condition.

Then for the symmetric case considered:

$$n_{12}(\alpha_1, o) = \lambda_1(\alpha_1) \tag{6.33}$$

$$n_{12}(o, \alpha_2) = \lambda_2(\alpha_2) \tag{6.34}$$

$$Q(o, o) = \lambda_3 \tag{6.35}$$

where Q is the normal reactive force at a corner of the shell and is positive when acting in the $(-\gamma)$ direction.

In the non-dimensional form equations (6.33), (6.34) and (6.35) reduce to:

$$\bar{n}_{12}(\beta_1, o) = \bar{\lambda}_1(\beta_1) = \sum_e \bar{\lambda}_1^e L_1^e(\beta_1) \tag{6.36}$$

$$\bar{n}_{12}(o, \beta_2) = \bar{\lambda}_2(\beta_2) = \sum_e \bar{\lambda}_2^e L_2^e(\beta_2) \tag{6.37}$$

$$\bar{Q}(o, o) = \bar{\lambda}_3 \tag{6.38}$$

Equations (6.36), (6.37) and (6.38) provide alternative* boundary values to those based on the displacement derivatives. The actual values are obtained as before from table 6.2.

This matter will be discussed further in section (6.5) in conjunction with numerical examples.

6.2 Boundary Conditions and Approximating Functions

The boundary conditions to be considered in this chapter are given in table 6.3.

Only boundary conditions which are symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$ are considered.

For the reasons given in section (5.2), the functions chosen to specify a particular boundary condition are given in table 6.4. Details of the approximating functions are given in table 3.1.

In table 6.4 two separate sets of functions are associated with the hinged and normal gable boundary conditions:

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*As discussed in section (3.1.3), the values based on the Lagrangian multipliers are generally different from the corresponding values based on the displacement derivatives.

(a) functions which satisfy <u>all</u> the boundary conditions

(b) functions which violate the conditions $u_1(a_1, o) = 0$ or $u_2(o, a_2) = 0$ but satisfy the remaining conditions on a boundary.*

Case (b) is considered in conjunction with the Lagrangian multiplier method.

Any combination of the boundary conditions given in table 6.4 may be specified.

6.8 Reduction to a System of Linear Algebraic Equations

For a particular set of approximating functions, equations (6.9), (6.10) and (6.11), with $\overline{Z} = 1$, reduce on integration to a system of linear algebraic equations, which in matrix form are:

$$\begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} \\ \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} \\ \underline{A}_{31} & \underline{A}_{32} & \underline{A}_{33} \end{bmatrix} \begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{g} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \end{bmatrix} \qquad (6.37)$$

*Only when normal slides (1 or 2) are considered along all boundaries, is the constraint condition $w(o,o) = 0$ considered in conjunction with the Lagrangian multiplier method.

or, more compactly:

$$\underline{A} \quad \bar{\underline{a}} + \bar{\underline{g}} = \underline{O} \qquad (6.40)$$

where $\bar{\underline{a}} = \text{col}\left\{ \underline{a} \quad \underline{b} \quad \underline{c} \right\}$

$\bar{\underline{g}} = \text{col}\left\{ \underline{o} \quad \underline{o} \quad \underline{g} \right\}$

Typical elements of the submatrices in equation (6.39) are given in table 6.5a.* The relevant integration formulae are given in Appendix 2.

The examples given in section (5.3) to illustrate the matrix notation adopted are equally applicable here.

6.3.1 Modification for the Lagrangian Multiplier Method

When the Lagrangian multiplier method is applied, the modified form given by equations (6.24) to (6.29) inclusive is used. These equations may also be reduced to a system of linear algebraic equations which in matrix form are:

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*The comma notation used in defining a typical matrix element in tables 6.5a and 6.5b (e.g. $a^{11}_{mn,ij}$) does not represent differentiation. However, the comma notation used in the expression corresponding to a typical element represents differentiation with respect to β_1 and β_2.

$$\begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} & \underline{D}_1 & \cdot & \cdot \\ \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} & \cdot & \underline{D}_2 & \cdot \\ \underline{A}_{31} & \underline{A}_{32} & \underline{A}_{33} & \cdot & \cdot & \underline{d}_5 \\ \underline{D}_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \underline{D}_4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \underline{d}_6^T & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \underline{a} \\ \underline{b} \\ \underline{c} \\ \underline{\lambda}_1 \\ \underline{\lambda}_2 \\ \underline{\lambda}_3 \end{bmatrix} + \begin{bmatrix} \cdot \\ \cdot \\ \underline{g} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \underline{O} \qquad (6.41)$$

Typical elements of the submatrices $\underline{A}_{ij}$ $(i,j = 1,2,3)$, $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{g}$ are, as before, given in table 6.5(a). Typical elements of the remaining submatrices in equations (6.41) are given in table 6.5(b).

The examples given in section (5.3.1) to illustrate the matrix notation adopted are equally applicable here.

The Lagrangian multiplier method is considered only in conjunction with a hinged or normal gable boundary (table 6.4).

It has been established that the Lagrangian multipliers provide alternative values for n_{12} on the boundary (refer to section 6.1.1). Then the functions L_1^e and L_2^e should be chosen such that the condition on n_{12} in the corner of the shell is satisfied. In Table 6.4 there is a choice of functions (IA or IB) for L_1^e and L_2^e when hinged boundaries are considered. If the boundaries $\beta_1 = 0,1$ (say) are hinged and the boundaries $\beta_2 = 0,1$ are either:

(i) clamped, hinged, normal slides (1)

or (ii) normal gables, normal slides (2)

then the functions chosen for L_2^e will be IA for (i) and IB for (ii), satisfying correctly the zero and non-zero conditions on n_{12} in the corner of the shell respectively. Similarly for L_1^e by considering hinged boundaries at $\beta_2 = 0,1$.

If S is the number of functions chosen to represent each of the displacements u_1, u_2 and w in each of the directions β_1 and β_2, then the order of the submatrices in equations (6.39) and (6.41) are identical to those given for translational shells in section (5.3.1). Also the subsequent remarks made in section (5.3.1) about functions ID, when used in conjunction with the Lagrangian multiplier method, are equally applicable here.

The solution of equations (6.41) forms the basis of the numerical results presented in this chapter. When no Lagrangian multipliers are used these equations reduce to equations (6.39).

6.4 Overall Equilibrium Check

6.4.1 Geometry and Assumptions

In figure 2.5 two equations, defining the middle surface of a ruled surface hyperbolic paraboloid shell, were given. The surface in figure 2.5b is symmetric about either diagonal whereas the surface in figure 2.5a is not. Since, in later derivations, only one-quarter of the shell will be considered, the surface defined by figure 2.5b will be adopted.

From figure 2.5b the equation of the middle surface is given by:

$$z = K_{12}(-\tfrac{1}{2}\, l_2 x_1 - \tfrac{1}{2}\, l_1 x_2 + x_1 x_2) \tag{6.42}$$

where

$$K_{12} = -\frac{2\bar{f}}{l_1 l_2} \tag{6.43}$$

The slopes of the middle surface in the x_1 and x_2 directions are respectively:

$$z,_1 = K_{12}(-\frac{l_2}{2} + x_2) \tag{6.44}$$

$$z,_2 = K_{12}(-\frac{l_1}{2} + x_1) \tag{6.45}$$

Substituting for K_{12} given by equation (6.43) in equations (6.44) and (6.45) yields:

$$z,_1 = \frac{1}{r} \frac{\overline{f}}{T_2}(1 - \frac{2x_2}{T_2}) \qquad (6.46)$$

$$z,_2 = \frac{\overline{f}}{T_2}(1 - \frac{2x_1}{T_1}) \qquad (6.47)$$

The assumptions relating to the shallow curved plate theory (Chapter 2) imply that the products of the slopes $z,_1$ and $z,_2$ may be neglected as small compared with unity.

Similarly it may be assumed that:

$$z,_i \ (i = 1 \text{ or } 2) \simeq \tan \Theta_i \simeq \sin \Theta_i \simeq \Theta_i \qquad (6.48)$$

$$\cos \Theta_i \simeq 1.0 \qquad (6.49)$$

Within the limits of the curved plate approximation $\frac{x_1}{T_1}$ and $\frac{x_2}{T_2}$ may be replaced by β_1 and β_2 respectively and equations (6.46) and (6.47) become:

$$z,_1 = \frac{1}{r} \cdot \frac{\overline{f}}{T_2}(1 - 2\beta_2) \qquad (6.50)$$

$$z,_2 = \frac{\overline{f}}{T_2}(1 - 2\beta_1) \qquad (6.51)$$

6.4.2 Resolution of the Load and Stress-Resultants in the Directions x_1, x_2 and z

Since the loading and boundary conditions are symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$, only one quarter of the shell need be considered (figure 6.1).

Let E_1, E_2 and E_3 be the errors in equilibrium for one quarter of the shell (figure 6.1) measured positive in the directions x_1, x_2 and z respectively.

Then resolving the load and stress-resultants in the directions x_1, x_2 and z and allowing for the assumptions previously made, i.e. ignoring terms containing products of slopes of the middle surface and assuming that the relations given by equations (6.48) and (6.49) hold, yields the following three equations respectively:

$$E_1 = \int_o^{\frac{l_1}{2}} \left[-n_{12}(a_1,o) + n_{12}(a_1,\frac{l_1}{2}) + q_2(a_1,o)(z_{,1})_{x_2=o} \right] da_1 +$$

$$+ \int_o^{\frac{l_2}{2}} \left[-n_{11}(o,a_2) + q_1(o,a_2)(z_{,1})_{x_1=o} \right] da_2 -$$

$$- \int_o^{\frac{l_1}{2}} \int_o^{\frac{l_2}{2}} Z(z_{,1}) da_1 da_2 + Q(z_{,1})_{\substack{x_1=o \\ x_2=o}} \qquad (6.52)$$

$$E_2 = \int_o^{\frac{l_1}{2}} \left[-n_{22}(\alpha_1, o) d\alpha_1 + q_2(\alpha_1, o)(z,_2)_{x_2=o} \right] d\alpha_1 +$$

$$+ \int_o^{\frac{l_2}{2}} \left[-n_{12}(o, \alpha_2) + n_{12}(\frac{l_2}{2}, \alpha_2) + q_1(o, \alpha_2)(z,_2)_{x_1=o} \right] d\alpha_2 -$$

$$- \int_o^{\frac{l_1}{2}} \int_o^{\frac{l_2}{2}} Z(z,_2) d\alpha_1 d\alpha_2 - Q(z,_2)_{\substack{x_1=o \\ x_2=o}} \qquad (6.53)$$

$$E_3 = - \int_o^{\frac{l_1}{2}} \left[n_{12}(\alpha_1, o)(z,_1)_{x_2=o} + n_{22}(\alpha_1, o)(z,_2)_{x_2=o} + q_2(\alpha_1, o) \right] d\alpha_1 -$$

$$- \int_o^{\frac{l_2}{2}} \left[n_{12}(o, \alpha_2)(z,_2)_{x_1=o} + n_{11}(o, \alpha_2)(z,_1)_{x_1=o} + q_1(o, \alpha_2) \right] d\alpha_2 -$$

$$- Q + \int_o^{\frac{l_1}{2}} \int_o^{\frac{l_2}{2}} Z \, d\alpha_1 d\alpha_2 \qquad (6.54)$$

From Table 6.4 and equations (6.4) and (6.5) the following are obtained:

$$n_{ij} = \frac{Z}{K_{12}} \bar{n}_{ij} = -\frac{Zl_1 l_2}{2\Gamma} \bar{n}_{ij}$$

$$q_i = Zl_1 \bar{q}_i$$

$$Q = Zl_1 l_2 \bar{Q}$$

$$da_i = l_i d\beta_i \quad \text{(not summed)}$$

where i and j range over the values 1 and 2. Substituting for n_{ij}, q_i, Q and da_i by the above expressions and for $(z,_1)$ and $(z,_2)$ by equations (6.50) and (6.51) in equations (6.52), (6.53) and (6.54) yields:

$$E_1 = Zl_1 l_2(\bar{E}_1) \tag{6.55}$$

$$E_2 = Zl_1 l_2(\bar{E}_2) \tag{6.56}$$

$$E_3 = Zl_1 l_2(\bar{E}_3) \tag{6.57}$$

where the non-dimensional forms $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ are given by:

$$\bar{E}_1 = \int_0^{\frac{1}{2}} \left[\frac{r}{2}\left(\frac{l_2}{\bar{r}}\right) \left\{ \bar{n}_{12}(\beta_1, o) - \bar{n}_{12}(\beta_1, 0.5) \right\} + \right.$$

$$\left. + \left(\frac{\bar{r}}{l_2}\right)\bar{q}_2(\beta_1, o) \right] d\beta_1 +$$

$$+ \int_0^{\frac{1}{2}} \left[\tfrac{1}{2}\left(\frac{l_2}{\bar{r}}\right)\bar{n}_{11}(o, \beta_2) + \frac{1}{r}\left(\frac{\bar{r}}{l_2}\right)\bar{q}_1(o, \beta_2)(1 - 2\beta_2) \right] d\beta_2 +$$

$$+ \frac{1}{r} \cdot \left(\frac{\bar{r}}{l_2}\right)\bar{Q} - \frac{1}{8r}\left(\frac{\bar{r}}{l_2}\right). \qquad (6.58)$$

$$\bar{E}_2 = \int_0^{\frac{1}{2}} \left[\frac{r}{2}\left(\frac{l_2}{\bar{r}}\right)\bar{n}_{22}(\beta_1, o) + r\left(\frac{\bar{r}}{l_2}\right)\bar{q}_2(\beta_1, o)(1 - 2\beta_1) \right] d\beta_1 +$$

$$+ \int_0^{\frac{1}{2}} \left[\tfrac{1}{2}\left(\frac{l_2}{\bar{r}}\right) \left\{ \bar{n}_{12}(o, \beta_2) - \bar{n}_{12}(0.5, \beta_2) \right\} + \left(\frac{\bar{r}}{l_2}\right)\bar{q}_1(o, \beta_2) \right] d\beta_2 +$$

$$+ \left(\frac{\bar{r}}{l_2}\right)\bar{Q} - \frac{1}{8}\left(\frac{\bar{r}}{l_2}\right) \qquad (6.59)$$

$$\bar{E}_3 = \frac{1}{2}\int_0^{\frac{1}{2}} \left[\bar{n}_{12}(\beta_1,o) + r\,\bar{n}_{22}(\beta_1,o)(1-2\beta_1) - 2r\,\bar{q}_2(\beta_1,o) \right] d\beta_1 \; +$$

$$+ \; \frac{1}{2}\int_0^{\frac{1}{2}} \left[\bar{n}_{12}(o,\beta_2) + \frac{1}{r}\bar{n}_{11}(o,\beta_2)(1-2\beta_2) - 2\bar{q}_1(o,\beta_2) \right] d\beta_2 \; -$$

$$- \; \bar{Q} + 0.25 \qquad (6.60)$$

Equations (6.58), (6.59) and (6.60) are the equations used to check overall equilibrium for a shallow curved plate. $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ are the errors in equilibrium expressed as a factor of (Zl_1l_2) and measured positive in the directions x_1, x_2 and z respectively.

For the overall equilibrium check it is necessary to define a further parameter $(\frac{l_2}{f})$, which is a measure of the shallowness of the shell, to determine $\bar{E}_1$ and $\bar{E}_2$. Note that $\bar{E}_3$ is independent of $(\frac{l_2}{f})$

6.5 Convergence Study of the Approximating Functions

In this section combinations of the boundary conditions given in Table 6.4 will be applied to particular numerical examples. In the computer program developed, provision is made for the symmetry of the problem by choosing the functions given in Table 6.4 such that:

(a) u_1^m, w_m, L_1^e are symmetric functions about $\beta_1 = 0.5$

(b) u_2^m is an antimetric function about $\beta_1 = 0.5$

(c) U_2^n, W_n, L_2^e are symmetric functions about $\beta_2 = 0.5$

(d) U_1^n is an antimetric function about $\beta_2 = 0.5$.

6.5.1 Numerical Examples

The examples to be studied are given in table 6.6. The corresponding results* are presented in tables 6.7 to 6.18 inclusive and figures 6.2 to 6.5 inclusive.

Displacements, stress-resultants and stress-couples are presented in non-dimensional form (table 6.1). The actual values are obtainable from the expressions given in table 6.2.

In tables 6.7 to 6.18 inclusive, the values marked with an asterisk (*) are based on the Lagrangian multipliers and the corresponding values in brackets are based on the displacement derivatives.

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*The results in the tables are presented in floating point notation, e.g. 1.234, +3 means 1.234×10^3.

In the overall equilibrium check, E_1 and E_2 are presented in their non-dimensional forms $\overline{E}_1$ and $\overline{E}_2$, whilst E_3 is expressed as a percentage error (= 400 E_3).*

S is the number of functions used to represent each displacement in each of the directions β_1 and β_2, due allowance being made for the symmetry of the problem in the selection of these functions.

6.5.2 Discussion

The convergence of the displacements was good for all combinations of the boundary conditions considered i.e. clamped, hinged, normal slide (1), normal gable or normal slide (2).

When hinged or normal gable boundaries were used, the convergence of $\bar{n}_{12}$ along these boundaries was slow, but satisfactory within the shell (note, in particular, examples 6.2A and 6.3A). Application of the Lagrangian multiplier method in conjunction with functions ID (case (b) in Table 6.4) improved the boundary value of $\bar{n}_{12}$ and reduced the errors in equilibrium (compare examples 6.2B and 6.3B with examples 6.2A and

*This check was not incorporated in the computer program and the integrations in equations (6.58), (6.59) and (6.60) were performed numerically using Simpson's rule.

6.3A respectively). Note, however, that the errors $\bar{E}_1$ and $\bar{E}_2$ in example 6.3B are of opposite sign and are of similar magnitude to example 6.3A. The boundary value of $\bar{n}_{12}$ based on the Lagrangian multiplier again provided a more accurate estimate than the corresponding value based on the displacement derivative (refer to sections 4.4.2 and 5.5.2 where this matter was also discussed).

As an alternative to example 6.3B functions IC were considered in place of functions ID in Table 6.4 for a normal gable boundary. A separate computer program was written for this case and only a maximum of S = 6 was considered.* The results are presented in figure 6.2 and compared with examples 6.3A and 6.3B. These results illustrate the good agreement obtained for $\bar{n}_{12}$ using either functions IC or ID in conjunction with the Lagrangian multiplier method. However, the solution for $\bar{n}_{12}$ in example 6.3A shows poor agreement with the other solutions on the boundary, but good agreement within the shell.

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*No numerical difficulties were observed when using these functions for this maximum value of S. Refer to section 4.4.2 where this matter was discussed in detail.

Functions ID again illustrate that, when used with the Lagrangian multiplier method in the manner described in this thesis, they effectively improve the boundary value of a particular stress resultant, which is slowly convergent on the boundary, but satisfactory elsewhere.

With clamped, normal slide (1) or normal slide (2) boundaries the convergence of $\bar{n}_{12}$ was good. Also $\bar{n}_{11}$ and $\bar{n}_{22}$ generally showed good convergence for all combinations of the boundary conditions considered.

When normal slides (1 or 2) were considered along all boundaries (examples 6.4, 6.5 and 6.13) the moments in the region of the corners were very slowly convergent (note figures 6.3, 6.4 and 6.5). This effect was also noted with example 5.3 (see Table 5.4) and the remarks made in section 5.5.2 in reference to this example are also relevant here.

With all other combinations of the boundary conditions considered, the moments were generally converging satisfactorily.

Normal shears on the boundary were generally slowly convergent which undoubtedly contributed to the errors in equilibrium, particularly if the shears were of a significant magnitude.

Note the form of the solution in examples 6.7, 6.10 and 6.11, where $\bar{u}_1$, $\bar{n}_{11}$, $\bar{n}_{22}$, $\bar{m}_{12}$ and $\bar{q}_2$ are zero <u>throughout</u> the shell and

$\bar{w}$, $\bar{u}_2$, $\bar{m}_{22}$, $\bar{m}_{11}$ and $\bar{q}_1$ are constant in the β_2 direction i.e. in the direction of the normal slide (2) boundaries. Note also that examples 6.10 and 6.11 yielded identical results. For normal slide (2) boundaries at $\beta_2 = 0,1$ functions IB were chosen for U_2^n and W_n and functions IA for U_1^n (Table 6.4). Of these, only the first function i.e. the constant function, of U_2^n and W_n had any effect on the solution. The displacement, stress-resultant and stress-couple distributions were therefore reduced to a single series, which converged rapidly. Note that the solutions of these examples are very similar to a membrane solution where $\bar{n}_{12} = -0.5$ and $\bar{n}_{11}$ and $\bar{n}_{22}$ are zero throughout the shell.

6.6 Comparison with Other Available Solutions

Example 6.14. Consider a shell with the following data:

$l_1 = 12.92$ in. $\quad l_2 = 12.92$ in. $\quad h = 0.25$ in.

$K_{12} = -3.1247, -2\ \text{in}^{-1} \qquad \nu = 0.39$

$E = 5.0, +5\ \text{lbs/in}^2 \qquad Z = 1\ \text{lb/in}^2$

and boundary conditions:

clamped at $\alpha_1 = 0, l_1$ and $\alpha_2 = 0, l_2$. This data corresponds to the following shell parameters:

$$r = 1.0 \qquad\qquad \nu = 0.39$$

$$\rho_R = -0.0479258$$

The convergence of this solution was studied in example 6.1.

This example was also solved by Chetty[25] and by Gunasekera[6] and a comparison is given in figures 6.6a and 6.6b.

The solutions show good agreement.

Example 6.15. Consider a shell with the following data:

$$l_1 = 360 \text{ in.} \qquad l_2 = 360 \text{ in.} \qquad h = 2.5 \text{ in}$$

$$K_{12} = -1.1111, -3 \text{ in}^{-1} \qquad \nu = 0.16$$

$$E = 3.0, +6 \text{ lbs/in}^2 \qquad Z = 50 \text{ lbs/ft}^2$$

and boundary conditions:

normal gables at $\alpha_1 = 0, l_1$ and $\alpha_2 = 0, l_2$. This data corresponds to the following shell parameters:

$$r = 1.0 \qquad\qquad \nu = 0.16$$

$$\rho_R = -0.0173603.$$

The convergence of this solution was studied in examples 6.3A and 6.3B.

This example was also solved by Chetty[25] and by Mohtraz and Schnobrich[19] and a comparison is given in figure 6.7.

Note that the results corresponding to example 6.3B are presented in figure 6.7. This example uses functions ID in conjunction with the Lagrangian multiplier method (case (b) in Table 6.4). For comparison, a further solution for n_{12} is given using functions IC in place of functions ID (refer to section 6.5.2) and figure 6.2).

The solutions generally show satisfactory agreement. However, it should be noted that the boundary value of n_{12} is somewhat different from the solutions given in the references. The convergence study in example 6.3B showed that the application of the Lagrangian multiplier method reduced the error in vertical equilibrium, and that the boundary value of n_{12} using either functions IC or ID was virtually the same (refer to figure 6.2).

<u>Example 6.16.</u> Consider a shell with the following data:

$l_1 = 80$ ft. $\quad l_2 = 60$ ft. $\quad h = 0.25$ ft.

$K_{12} = 5.0, -3$ ft.$^{-1}$ $\quad \nu = 0.15$

$E = 4.5, +8$ lbs/ft^2 $\quad Z = 50$ lbs/ft^2

and boundary conditions

hinged* at $\alpha_1 = 0, l_1$

clamped at $\alpha_2 = 0, l_2$

This example was also solved by Gunasekera**(6) and a comparison is given in Table 6.19.

The solutions for the displacement w and the moments show satisfactory agreement. The solutions for the membrane stress-resultants show satisfactory agreement near the central region of the shell but poorer agreement near the corner. Note, in particular, that n_{11} and n_{22} in the corner should, by virtue of the boundary conditions, be zero.

---

*The Lagrangian multiplier method was used with this boundary condition, i.e. case (b) in Table 6.4.

**The results presented by Gunasekera were at $\frac{1}{5}$th points. The results presented in Tables 6.19 and 6.20 are at $\frac{1}{10}$th points. Examples 6.16 and 6.17 were re-run for this latter output using Gunasekera's computer program.

Example 6.17. Consider a shell with the following data:

$l_1 = 50$ ft. $\qquad l_2 = 50$ ft. $\qquad h = 0.25$

$K_{12} = -8.0, -3\ \text{ft}^{-1}$ $\qquad \nu = 0.15$

$E = 4.5, +8\ \text{lbs/ft}^2$ $\qquad Z = 50\ \text{lbs/ft}^2$

and the boundary conditions:

hinged* at $\alpha_1 = 0, l_1$ and $\alpha_2 = 0, l_2$.

This example was also solved by Gunasekera[6] and a comparison is given in Table 6.20.

The solutions for n_{12}, except near the corner of the shell, show good agreement. However, the solutions for n_{11} are quite different. Similarly the solutions for w differ. Note, in particular, that n_{11} and n_{22} in the corner should, by virtue of the boundary conditions, be zero.

Note also that a similar comparison was made when considering translational shells viz., example 5.9 in section 5.6. In this case

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*The Lagrangian multiplier method was used with these boundary conditions i.e. case (b) in Table 6.4.

similar functions were used and the solutions generally showed good agreement.

6.7 Further Solutions - Variation of the Shell Parameters

The non-dimensional form of equations (6.9), (6.10) and (6.11) shows that the ruled surface hyperbolic paraboloid is completely defined by the parameters r, ρ_2 and ν. Such a representation permits the behaviour of ruled surface shells to be conveniently studied by the variation of these parameters.

The examples considered and the particular parameter being varied are given in Table 6.21. The corresponding results are given in Tables 6.22 to 6.27 inclusive.

All results are presented in non-dimensional form (Table 6.1), the actual values being obtained from the expressions given in Table 6.2.

The Lagrangian multiplier method is used in examples 6.18, 6.19, 6.20 and 6.22 (case (b) in Table 6.4). In these cases the boundary values of $\bar{n}_{12}$ given in the corresponding tables are based on the Lagrangian multiplier.

S = 8 has been chosen in each of these examples.

6.7.1 Discussion

Variation of ρ_R: The parameter ρ_R, defined by equation (6.13), varies with shell thickness and shallowness. This thesis is concerned with the study of thin shallow shells and ρ_R should be interpreted accordingly. The thin flat plate is recovered from $\rho_R = \infty$.

For comparison, flat plate solutions for r = 1 and $\nu = 0.15$ and for all boundaries clamped, simply supported and normal slides are given in section 5.7.1.

As the shell becomes shallower, i.e. as ρ_R increases, Tables 6.22, 6.25, 6.26 and 6.27 show that $\bar{w}$, $\bar{m}_{11}$ and $\bar{G}$ (example 6.23 only) increase also, approaching the solution for a thin flat plate at a faster rate than for the elliptic paraboloids (translational shells) considered in examples 5.10, 5.14 and 5.15. For example, if $\rho_R = 0.10$ which corresponds to the values $(\frac{h}{T_1}) = \frac{1}{100}$ and $(\frac{l_2}{T})= -20$, the values for $\bar{w}$ and $\bar{m}_{11}$ vary from approximately 11% for example 6.23 to approximately 45% for example 6.21 of the corresponding flat plate solution. For a similar value of $(\frac{l_2}{T_2})$ in examples 5.10, 5.14 and 5.15 to the value of $(\frac{l_2}{f})$ above (for $\rho_T = 0.03$ say), the corresponding variation is approximately 10% of the above (refer to Tables 5.17, 5.21 and 5.22).

On the other hand $\bar{n}_{12}$ decreases slowly with increasing shallowness and is still of significant value when the shell is very shallow. In the steeper range ($\rho_R = 0.01$) the values of $\bar{n}_{12}$ are very close to the membrane solution ($\bar{n}_{12} = -0.5$), particularly when normal slides (2) are on all boundaries (Table 6.27). Similarly $\bar{n}_{11}$ decreases with increasing shallowness.

Similar remarks apply to increasing thickness which also corresponds to an increasing ρ_R, but any comments are restricted to <u>thin</u> shells.*

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*The note on thin shells in section 5.7 may be extended, such that:

$$h\,\left|K_{12}\right| \leq \frac{1}{30}$$

Since $\left|K_{12}\right| = \left|\frac{2f}{l_1 l_2}\right|$ (figure 2.5b), this becomes

$$\left(\frac{h}{l_1}\right)\left(\left|\frac{f}{l_2}\right|\right) \leq \frac{1}{60}$$

Note that only the effect on the non-dimensional values has been considered. The actual values follow from Table 6.2, which gives:

$$w = \frac{12\,Z\,l_1(1-\nu^2)}{E}\left(\frac{l_1}{h}\right)^3 \bar{w}$$

$$n_{12} = Zl_1\left(-\frac{2F}{l_1 l_2}\right)\bar{n}_{12} = \frac{Zl_1^2}{h}(\rho_R \bar{n}_{12})$$

Then if increasing ρ_R is interpreted as increasing shallowness, the effect on the actual stress-resultant n_{12} is dependent on $(\rho_R \bar{n}_{12})$. It will be noted that if all boundaries are clamped (example 6.21) the product $(\rho_R \bar{n}_{12})$ begins to decrease between $\rho_R = 0.10$ and 0.20. For the other examples (6.16, 6.22 and 6.23), the product $(\rho_R \bar{n}_{12})$ increases with increasing ρ_R. However, beyond the range considered here, this product begins to decrease with increasing ρ_R and approaches zero as ρ_R becomes very large.

If increasing ρ_R is interpreted as increasing thickness, but restricted to thin shells, the effect on the actual displacement w is dependent on $\left(\frac{l_1}{h}\right)^3 \bar{w}$.

The non-dimensional presentation of the tables given in this section covers, very compactly, the solutions for a wide range of thin shallow shells.

Variation of r : Only the case with normal gables along all boundaries is considered. The results are presented in table 6.23.

Since ρ_R remains constant, the variation of r represents, in effect, the variation of I_2 with all other data fixed.

As r increases from 0.5 to 5.0:

(a) the value of $\bar{w}$ at the centre of the shell increases, reaching its maximum value, within the limits of the results presented here, at r = 1.5, after which it begins to decrease,

(b) the maximum value of $\bar{m}_{11}$ along $\beta_2 = 0.5$ increases, reaching its highest value at r = 3.0, after which it begins to decrease; the location of this maximum value moves away from the centre of the shell,

(c) the maximum value of $\bar{m}_{22}$ along $\beta_1 = 0.5$ increases, reaching its highest value at r = 3.0, after which it begins to decrease; the location of this maximum value moves towards the centre of the shell.

(d) $\bar{n}_{12}$ generally decreases.

Variation of ν : Only the case with normal gables along all boundaries is considered. The results are presented in table 6.24.

As ν increases from 0 to 0.30:

(a) $\bar{w}$ increases

(b) $\bar{m}_{11}$ increases

(c) the effect on $\bar{n}_{12}$ is small but variable, increasing at some points and decreasing at others.

(d) $\bar{n}_{11}$ increases.

The actual displacement w is given by (Table 6.2):

$$w = \frac{Zl_1^4}{D}\bar{w} = \frac{12Zl_1^4}{Eh^3}(1 - \nu^2)\bar{w}$$

which, from Table 6.24, also increases with increasing ν.

It should be noted that in this example, increasing ν has the opposite effect on $\bar{w}$ and $\bar{m}_{11}$ than for the elliptic paraboloid considered in example 5.13 (Table 5.20).

6.8 Discussion of the Computer Programs

The single computer program developed for ruled surface hyperbolic paraboloid shells is limited to uniformly distributed normal loading and to boundary conditions which are symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$. However, any symmetric combination of clamped, hinged, normal gable, normal slide (1) or normal slide (2) boundary conditions (Table 6.4) may be specified. Provision is also made to apply the Lagrangian multiplier method in conjunction with $u_1 = O$ (along $\beta_2 = O,1$) or $u_2 = O$ (along $\beta_1 = O,1$) for hinged and normal gable boundaries (refer to case (b) in Table 6.4).

The approximating functions are selected in accordance with the symmetry of the problem (section 6.5). The same value of S for each displacement in each of the directions β_1 and β_2 is considered.

Input, and therefore output, could be either in non-dimensional form or in terms of the actual dimensions.

In order to economise on computer storage, the system of linear algebraic equations was solved by partitioning the equations into their submatrix form (equations 6.39). Note that when the modified form given by equations (6.41) was used, the equations were rearranged in the form given by equations (6.39), in the manner discussed in section (5.8).

A separate program was written for the solution of example 6.3A. For this special case a fixed value of r = 1 was chosen, so that allowance could be made for symmetry in the solution for the constants $\underline{a}$, $\underline{b}$ and $\underline{c}$, i.e. $a_{ij} = b_{ji}$ and $c_{pq} = c_{qp}$. The equations were again solved by partitioning. In this way a larger value of S could be considered.

Further details of the computer programs are available at Imperial College.(69)

The computer programs were written in EXCHLF Autocode(70),(71) for the University of London Atlas Computer.

CHAPTER 7

APPLICATION OF THE METHOD OF LINES TO TRANSLATIONAL AND RULED SURFACE SHELLS

A brief discussion of the method of lines was given in section (3.2). In the following, equations (2.76) will be reduced to a system of linear first order ordinary differential equations with constant coefficients. The co-ordinates will be non-dimensionalised. All other quantities will retain their dimensional form.

Only uniformly distributed normal loading (Z) will be considered.

7.1 Form of Equations

After non-dimensionalising the co-ordinates such that

$$\beta_1 = \frac{\alpha_1}{l_1} \tag{7.1}$$

$$\beta_2 = \frac{\alpha_2}{l_2} \tag{7.2}$$

and setting X_1 and X_2 to zero, equations (2.76) become:

$$\frac{\partial}{\partial\beta_2}\begin{bmatrix} n_{22} \\ n_{12} \\ m_{22} \\ r_2 \\ u_2 \\ u_1 \\ \theta \\ w \end{bmatrix} + \begin{bmatrix} \cdot & A_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{21} & \cdot & \cdot & \cdot & \cdot & A_{26} & \cdot & A_{28} \\ \cdot & \cdot & \cdot & A_{34} & \cdot & \cdot & A_{37} & \cdot \\ A_{41} & A_{42} & A_{43} & \cdot & \cdot & A_{46} & \cdot & A_{48} \\ A_{51} & \cdot & \cdot & \cdot & \cdot & A_{56} & \cdot & A_{58} \\ \cdot & A_{62} & \cdot & \cdot & A_{65} & \cdot & \cdot & A_{68} \\ \cdot & \cdot & A_{73} & \cdot & \cdot & \cdot & \cdot & A_{78} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A_{87} & \cdot \end{bmatrix}\begin{bmatrix} n_{22} \\ n_{12} \\ m_{22} \\ r_2 \\ u_2 \\ u_1 \\ \theta \\ w \end{bmatrix} + \begin{bmatrix} l_2 X_2 \\ l_2 X_1 \\ \cdot \\ l_2 Z \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \underline{O} \qquad (7.3)$$

where $A_{12} = \frac{1}{r} \cdot \frac{\partial}{\partial\beta_1} = A_{65}$

$$A_{21} = \frac{\nu}{r} \cdot \frac{\partial}{\partial\beta_1} = A_{56}$$

$$A_{26} = \frac{Eh}{r l_1} \frac{\partial^2}{\partial\beta_1{}^2}$$

$$A_{28} = -\frac{EhK_{11}}{r} \frac{\partial}{\partial\beta_1} = -A_{46}$$

$$A_{34} = -l_2 = A_{87}$$

$$A_{37} = -\frac{Eh^3}{6(1+\nu)rl_1} \frac{\partial^2}{\partial\beta_1{}^2}$$

$$A_{41} = (K_{22} + \nu K_{11})l_2 = -A_{58}$$

$$A_{42} = 2K_{12}l_2 = -A_{68}$$

$$A_{43} = \frac{\nu}{rl_1} \frac{\partial^2}{\partial\beta_1{}^2} = A_{78}$$

$$A_{48} = \left[-EhK_{11}{}^2 l_2 - \frac{Eh^3}{12rl_1{}^3} \frac{\partial^4}{\partial\beta_1{}^4}\right]$$

$$A_{51} = -\frac{(1-\nu^2)l_2}{Eh}$$

$$A_{62} = \frac{-2(1+\nu)l_2}{Eh}$$

$$A_{73} = \frac{12(1-\nu^2)l_2}{Eh^3}$$

$$r = \frac{l_1}{l_2}$$

$$\Theta = \frac{\partial w}{\partial\alpha_2}$$

or, more compactly,

$$\underline{Y}_{,2} + \underline{A}\,\underline{Y} + \underline{\bar{L}} = \underline{O} \qquad (7.4)$$

where $\underline{Y} = \text{col}\left\{ n_{22} n_{12} m_{22} r_2 u_2 u_1 \theta\ w \right\}$

$\underline{L} = \text{col}\left\{ .\ \ .\ \ .\ Zl_2 .\ .\ .\ . \right\}$

and comma notation is used to represent partial differentiation with respect to β_2.

The equations corresponding to the translational shell are obtained by setting K_{12} to zero and replacing K_{11} and K_{22} by K_1 and K_2 respectively.

The equations corresponding to the ruled surface shell are obtained by setting K_{11} and K_{22} to zero.

In the following, when reference is made to a shell, <u>either</u> a translational shell <u>or</u> a ruled surface shell is implied.

7.2 Boundary Conditions

Only boundary conditions which are symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$ will be considered.

Throughout this chapter the boundary conditions at $\beta_1 = 0,1$ will be assumed clamped ($u_1 = o = u_2$, $w = o = w_{,1}$).

The boundary conditions at $\beta_2 = 0,1$ will be either free ($n_{22} = o = n_{12}$, $m_{22} = o = r_2$) or clamped ($u_1 = o = u_2$, $w = o = \Theta$).

7.3 Finite Difference Formulae

Since either translational or ruled surface shells will be considered, the solution will be symmetric about $\beta_1 = 0.5$ and $\beta_2 = 0.5$.

Then only forward and central difference expressions will be required for the derivatives.

Let y represent the variable whose derivative is required.

The 5-point central and forward difference expressions for the derivatives of y used in this chapter are given in table 7.1. Only an equal width, a, between points has been considered. These expressions retain the same order of differences and were obtained by application of Taylor's theorem.[(67)]

Note that some of the forward differences have been given in terms of a fictitious point (-1).

7.4 Reduction to a System of Linear First Order Ordinary Differential Equations with Constant Coefficients

Let the region of the shell (translational or ruled surface) bounded by $\beta_1 = 0$ and $\beta_1 = 1$ be divided into 2N equal divisions by the lines

β_1^{o}, $\beta_1^{1}, \ldots, \beta_1^{k}, \ldots, \beta_1^{2N}$, which are in the same direction as β_2 (figure 7.1). The boundaries $\beta_1 = 0$ and $\beta_1 = 1$ correspond respectively to the lines β_1^{o} and β_1^{2N}.

Let the notation y^{k}, where y represents a displacement, stress-resultant or stress-couple, denote the value of y along the line β_1^{k}.

Because of symmetry only the region bounded by $\beta_1 = 0$ and $\beta_1 = 0.5$ is considered i.e. the lines $\beta_1^{o}, \beta_1^{1}, \ldots, \beta_1^{k}, \ldots, \beta_1^{N}$.

Only the derivatives with respect to β_1 in equations (2.76) will be replaced by the corresponding finite difference expressions.

Since the boundary $\beta_1 = 0$ is clamped and using comma notation to represent differentiation with respect to β_1 or β_2:

$$u_1^{o} = 0 = u_2^{o} \qquad (7.5)$$

$$w^{o} = 0 = w_{,1}^{o} \qquad (7.6)$$

Then it follows that:

$$\theta^{o} = 0 = \theta_{,1}^{o} \qquad (7.7)$$

$$u_{1,2}^{o} = 0 = u_{2,2}^{o} \qquad (7.8)$$

$$w_{,22}^{o} = 0 = w_{,222}^{o} \qquad (7.9)$$

Equations (7.5) to (7.9) inclusive and equations (2.63), (2.64), (2.66) and (2.61) yield the following:

$$n^o_{22} = \frac{\nu Eh}{l_1(1-\nu^2)} u^o_{1,1} \tag{7.10}$$

$$n^o_{12} = \frac{Eh}{2l_1(1+\nu)} u^o_{2,1} \tag{7.11}$$

$$m^o_{22} = \frac{-\nu Eh^3}{12(1-\nu^2)l_1^{\,2}} w^o_{,11} \tag{7.12}$$

$$r^o_2 = \frac{-Eh^3(2-\nu)}{12(1-\nu^2)l_1^{\,2}} \theta^o_{,11} \tag{7.13}$$

Using the formulae given in table 7.1, finite difference expressions for the derivatives in equations (7.10) to (7.13) inclusive will be obtained.

From equation (7.6) and table 7.1:

$$w^o_{,1} = 0 = \frac{1}{12a}\left[-3w^{-1} + 18w^1 - 6w^2 + w^3\right]$$

which gives:

$$w^{-1} = 6w^1 - 2w^2 + \frac{1}{3}w^3 \tag{7.14}$$

Similarly from equation (7.7) and table 7.1:

$$\theta^{-1} = 6\theta^{1} - 2\theta^{2} + \frac{1}{3}\theta^{3} \qquad (7.15)$$

From table 7.1 and equations (7.5), (7.6), (7.7), (7.14) and (7.15) the required derivatives are obtained:

$$u^{o}_{1,1} = \frac{1}{12a}(48u^{1}_{1} - 36u^{2}_{1} + 16u^{3}_{1} - 3u^{4}_{1}) \qquad (7.16)$$

$$u^{o}_{2,1} = \frac{1}{12a}(48u^{1}_{2} - 36u^{2}_{2} + 16u^{3}_{2} - 3u^{4}_{2}) \qquad (7.17)$$

$$w^{o}_{,11} = \frac{1}{18a^{2}}(108w_{1} - 27w_{2} + 4w_{3}) \qquad (7.18)$$

$$\theta^{o}_{,11} = \frac{1}{18a^{2}}(108\theta_{1} - 27\theta_{2} + 4\theta_{3}) \qquad (7.19)$$

Then the actions at $\beta_1 = 0$ are given by equations (7.10) to (7.13) inclusive, where the derivatives are defined by equations (7.16) to (7.19) inclusive.

The solution for the dependent variables along the line β_1^{o} is therefore known and the equations need only be applied along the lines $\beta_1^1, \beta_1^2, \ldots, \beta_1^k, \ldots, \beta_1^N$, where line β_1^N corresponds to $\beta_1 = 0.5$.

Note that in deriving the finite difference expressions given by equations (7.18) and (7.19) use was made of a fictitious line β_1^{-1} in the forward difference expressions. In this way a more accurate representation for the derivatives near the boundary is possible. The values of w and Θ along this fictitious line are given by equations (7.14) and (7.15). Such values are not available for u_1 and u_2 for the clamped conditions considered and less accurate forward difference expressions are used for the derivatives of u_1 and u_2 near the boundary.

In a similar manner, finite difference expressions for the derivatives with respect to β_1 in equations (7.4) can be obtained.

Let the column matrices $\underline{n}_{22}$, $\underline{n}_{12}$, $\underline{m}_{22}$, $\underline{r}_2$, $\underline{u}_2$, $\underline{u}_1$, $\underline{\Theta}$, $\underline{w}$ and $\underline{Z}$ be defined by:

$$\underline{n}_{22} = \text{col}\left\{ n_{22}^1 \; n_{22}^2 \ldots n_{22}^k \ldots n_{22}^N \right\} \qquad (7.20)$$

$$\underline{n}_{12} = \text{col}\left\{ n_{12}^1 \; n_{12}^2 \ldots n_{12}^k \ldots n_{12}^N \right\} \qquad (7.21)$$

$$\underline{m}_{22} = \text{col}\left\{ m_{22}^1 \; m_{22}^2 \ldots m_{22}^k \ldots m_{22}^N \right\} \qquad (7.22)$$

$$\underline{r}_2 = \text{col}\left\{ r_2^1 \; r_2^2 \ldots r_2^k \ldots r_2^N \right\} \qquad (7.23)$$

$$\underline{u}_2 = \text{col}\left\{ u_2^1 \;\; u_2^2 \;\; \ldots \;\; u_2^k \;\; \ldots \;\; u_2^N \right\} \tag{7.24}$$

$$\underline{u}_1 = \text{col}\left\{ u_1^1 \;\; u_1^2 \;\; \ldots \;\; u_1^k \;\; \ldots \;\; u_1^N \right\} \tag{7.25}$$

$$\underline{\theta} = \text{col}\left\{ \theta^1 \;\; \theta^2 \;\; \ldots \;\; \theta^k \;\; \ldots \;\; \theta^N \right\} \tag{7.26}$$

$$\underline{w} = \text{col}\left\{ w^1 \;\; w^2 \;\; \ldots \;\; w^k \;\; \ldots \;\; w^N \right\} \tag{7.27}$$

$$\underline{Z} = \text{col}\left\{ Z^1 \;\; Z^2 \;\; \ldots \;\; Z^k \;\; \ldots \;\; Z^N \right\} \tag{7.28}$$

where Z^k is the line load corresponding to line k.

For the clamped conditions at $\beta_1 = 0, 1$, equations (7.3), after:

(a) substitution of the relevant finite difference expressions for the derivatives with respect to β_1, and

(b) substituting for n^o_{22}, n^o_{12}, m^o_{22} and r^o_2, whenever necessary, by equations (7.10) to (7.13),

reduce to a system of linear first order ordinary differential equations with constant coefficients, which in matrix form are:

$$\frac{d}{d\beta_2}\begin{bmatrix}\underline{n}_{22}\\ \underline{n}_{12}\\ \underline{m}_{22}\\ \underline{r}_{2}\\ \underline{u}_{2}\\ \underline{u}_{1}\\ \underline{\theta}\\ \underline{w}\end{bmatrix} + \begin{bmatrix} \cdot & \underline{A}_{12} & \cdot & \cdot & \underline{A}_{15} & \cdot & \cdot & \cdot \\ \underline{A}_{21} & \cdot & \cdot & \cdot & \cdot & \underline{A}_{26} & \cdot & \underline{A}_{28} \\ \cdot & \cdot & \cdot & \underline{A}_{34} & \cdot & \cdot & \underline{A}_{37} & \cdot \\ \underline{A}_{41} & \underline{A}_{42} & \underline{A}_{43} & \cdot & \cdot & \underline{A}_{46} & \cdot & \underline{A}_{48} \\ \underline{A}_{51} & \cdot & \cdot & \cdot & \cdot & \underline{A}_{56} & \cdot & \underline{A}_{58} \\ \cdot & \underline{A}_{62} & \cdot & \cdot & \underline{A}_{65} & \cdot & \cdot & \underline{A}_{68} \\ \cdot & \cdot & \underline{A}_{73} & \cdot & \cdot & \cdot & \cdot & \underline{A}_{78} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \underline{A}_{87} & \cdot \end{bmatrix}\begin{bmatrix}\underline{n}_{22}\\ \underline{n}_{12}\\ \underline{m}_{22}\\ \underline{r}_{2}\\ \underline{u}_{2}\\ \underline{u}_{1}\\ \underline{\theta}\\ \underline{w}\end{bmatrix} + \begin{bmatrix}\cdot\\ \cdot\\ \cdot\\ I_2\underline{Z}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix} = \underline{O} \qquad (7.29)$$

where the elements of the non-zero submatrices $\underline{A}_{ij}$ $(i,j = 1,2,3,\ldots,8)$ are defined by the expressions given in table 7.2 for the lines β_1^1, β_1^2 and β_1^k $(k \geq 3)$.

In the computer program developed, allowance is made for symmetry about $\beta_1 = 0.5$ by adjusting matrices $\underline{A}_{ij}$ for the following conditions:

(a) for translational shells:

n_{22}, m_{22}, r_2, u_2, θ, w are symmetric about $\beta_1 = 0.5$

n_{12}, u_1 are antimetric about $\beta_1 = 0.5$

(b) for ruled surface shells:

$n_{12}, m_{22}, r_2, u_1, \Theta, w$ are symmetric about $\beta_1 = 0.5$

n_{22}, u_2 are antimetric about $\beta_1 = 0.5$

$$\text{Let } \underline{X} \equiv \underline{X}(\beta_2) = \text{col}\left\{ \underline{n}_{22}, \underline{n}_{12}, \underline{m}_{22}, \underline{r}_2 \right\} \quad (7.30)$$

$$\underline{U} \equiv \underline{U}(\beta_2) = \text{col}\left\{ \underline{u}_2, \underline{u}_1, \underline{\Theta}, \underline{w} \right\} \quad (7.31)$$

$$\underline{F} \equiv \underline{F}(\beta_2) = \text{col}\left\{ \underline{X}, \underline{U} \right\} \quad (7.32)$$

$$Z = \text{col}\left\{ . \; . \; . \; I_2\underline{Z} \; . \; . \; . \; . \right\} \quad (7.33)$$

then equations (7.29) can be written in the form

$$\underline{F}_{,2} + \underline{A}\,\underline{F} + \underline{\overline{Z}} = \underline{O} \quad (7.34)$$

Note that $\underline{F}$ and $\underline{\overline{Z}}$ are (8N x 1) matrices and $\underline{A}$ is an (8N x 8N) matrix.

7.5 Integration of Equations (7.34) Using the Matrix Progression Method

7.5.1 General Solution

Since the coefficients of $\underline{A}$ and $\underline{\overline{Z}}$ are constant, the general solution of equations (7.34) can be written in the following form:

$$\underline{F} = \underline{G}(\beta_2)(\underline{F}_o - \underline{F}^{(p)}) + \underline{F}^{(a)} \qquad (7.35)$$

where

$$\underline{G}(\beta_2) = e^{-\underline{A}\,\beta_2} \qquad (7.36)$$

$$\underline{F}^{(p)} = -\underline{A}^{-1}\underline{Z} \qquad (7.37)$$

and where the following notation has been adopted:

$$\underline{F}_q \equiv \underline{F}\Big|_{\beta_2=q} \qquad (7.38)$$

The matrix $\underline{G}(\beta_2)$ is referred to as the "distribution matrix"(65) in the matrix progression method. It is also referred to as a "transfer matrix"(63) and can be determined by the following series, which always converges: (68)

$$\underline{G}(\beta_2) = \underline{I} - \underline{A}\beta_2 + \frac{\underline{A}^2\beta_2^{\,2}}{2!} - \frac{\underline{A}^3\beta_2^{\,3}}{3!} + \frac{\underline{A}^4\beta_2^{\,4}}{4!} - \ldots \qquad (7.39)$$

where $\underline{I}$ is the unit matrix.

$\underline{F}^{(p)}$ is the particular solution and is constant for the constant loading selected.

Let $\underline{F}^{(p)}$ be partitioned in the following way:

$$\underline{F}^{(p)} = \text{col}\left\{\underline{X}^{(p)},\ \underline{U}^{(p)}\right\} \qquad (7.40)$$

where $\underline{X}^{(p)} = \text{col}\left\{ \underline{n}_{22}^{(p)}, \underline{n}_{12}^{(p)}, \underline{m}_{22}^{(p)}, \underline{r}_{2}^{(p)} \right\}$ (7.41)

$$\underline{U}^{(p)} = \text{col}\left\{ \underline{u}_{2}^{(p)}, \underline{u}_{1}^{(p)}, \underline{\theta}^{(p)}, \underline{w}^{(p)} \right\} \quad (7.42)$$

The notation $n_{22}^{k(p)}$ will denote the particular solution for n_{22} along the line β_1^k. Similarly for the other dependent variables.

$$\text{Let } \underline{C} \equiv \underline{C}(\beta_2) = \underline{F} - \underline{F}^{(p)} \quad (7.43)$$

$$\text{and } \underline{C}_o = \underline{F}_o - \underline{F}^{(p)} \quad (7.44)$$

then equation (7.35) may be written in the following form:

$$\underline{C} = \underline{G}(\beta_2)\underline{C}_o \quad (7.45)$$

Equation (7.45) will be useful in determining the final solution at intervals along β_2 (refer to section 7.7.5).

The boundary conditions at $\beta_2 = 0,1$ will determine $\underline{F}_o$, i.e. the initial values of the dependent variables.

For the symmetric problem considered $\underline{F}_o$ can be determined in the following ways:

(i) direct application of the boundary conditions at $\beta_2 = 0,1$ in equation (7.35)

(ii) direct application of the boundary conditions at $\beta_2 = 0$ and the symmetry conditions about $\beta_2 = 0.5$ in equation (7.35)

(iii) application of a stiffness method which segments the path of integration.

In the following each of these approaches will be discussed in conjunction with the boundary conditions:

(a) clamped at $\beta_2 = 0,1$

and (b) free at $\beta_2 = 0,1$.

It will be convenient to partition $\underline{G}(\beta_2)$ in the following way:

$$\underline{G}(\beta_2) = \begin{bmatrix} G_{11}(\beta_2) & G_{12}(\beta_2) \\ G_{21}(\beta_2) & G_{22}(\beta_2) \end{bmatrix}^* \qquad (7.46)$$

7.5.2 Direct Application of the Boundary Conditions at $\beta_2 = 0,1$ in Equation (7.35)

(a) Clamped at $\beta_2 = 0,1$

The conditions to be satisfied are:

$$\underline{U}_o = \underline{O} \qquad (7.47)$$

$$\underline{U}_1 = \underline{O} \qquad (7.48)$$

---

*Each submatrix $G_{ij}(\beta_2)(i,j = 1,2)$ is of order (4N x 4N).

which on application to equation (7.35) yields

$$\underset{\sim o}{F} = \begin{bmatrix} \underset{\sim o}{X} \\ \\ \underset{\sim o}{U} \end{bmatrix} = \begin{bmatrix} \underline{G}_{21}^{-1}(1)\left[\underline{G}_{22}(1) - \underline{I}\right] \underline{U}^{(p)} + \underline{X}^{(p)} \\ \\ \underline{O} \end{bmatrix} \qquad (7.49)$$

(b) Free at $\beta_2 = 0, 1$

The conditions to be satisfied are:

$$\underset{\sim o}{X} = \underline{O} \qquad (7.50)$$

$$\underline{X}_1 = \underline{O} \qquad (7.51)$$

which on application to equation (7.35) yields:

$$\underset{\sim o}{F} = \begin{bmatrix} \underset{\sim o}{X} \\ \\ \underset{\sim o}{U} \end{bmatrix} = \begin{bmatrix} \underline{O} \\ \\ \underline{G}_{12}^{-1}(1)\left[\underline{G}_{11}(1) - \underline{I}\right] \underline{X}^{(p)} + \underline{U}^{(p)} \end{bmatrix} \qquad (7.52)$$

7.5.3 Direct Application of the Boundary Conditions at $\beta_2 = 0$ and the Symmetry Conditions about $\beta_2 = 0.5$ in Equation (7.35)

The symmetry conditions about $\beta_2 = 0.5$ are given by

$$\underline{J}\,\underline{F}_{0.5} = \underline{O} \qquad (7.53)$$

where $\underline{J}$ is a (4N x 8N) matrix and is given by:

$$\underline{J} = \begin{bmatrix} \cdot & \underline{I} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \underline{I} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \underline{I} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \underline{I} & \cdot \end{bmatrix} \qquad (7.54)$$

for translational shells, and by:

$$\underline{J} = \begin{bmatrix} \underline{I} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \underline{I} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \underline{I} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \underline{I} & \cdot \end{bmatrix} \qquad (7.55)$$

for ruled surface shells.

Each submatrix of $\underline{J}$ is of order $(N \times N)$. $\underline{I}$ is the unit matrix.

<u>(a) Clamped at</u> $\beta_2 = 0, 1$

The conditions to be satisfied are:

$$\underline{U}_o = \underline{O} = \underline{J}\,\underline{F}_{0.5} \qquad (7.56)$$

Let $\underline{H}$, a $(4N \times 8N)$ matrix, be defined by:

$$\underline{H} = \underline{J}\,\underline{G}(0.5) = \left[\underline{H}_1 \, , \, \underline{H}_2\right] \qquad (7.57)$$

where $\underline{H}_1$ and $\underline{H}_2$ are $(4N \times 4N)$ matrices. Application of the conditions given by equation (7.56) to equation (7.35) yields

$$\underset{\sim o}{F} = \begin{bmatrix} \underset{\sim o}{X} \\ \\ \underset{\sim o}{U} \end{bmatrix} = \begin{bmatrix} \underline{H}_1^{-1}\left[\underline{H}_2\underline{U}^{(p)} - \underline{J}\,\underline{F}^{(p)}\right] + \underline{X}^{(p)} \\ \\ \underline{O} \end{bmatrix} \tag{7.58}$$

<u>(b) Free at</u> $\beta_2 = 0, 1$

The conditions to be satisfied are:

$$\underset{\sim o}{X} = \underline{O} = \underline{J}\,\underline{F}_{0.5} \tag{7.59}$$

which on application to equation (7.35) yields:

$$\underset{\sim o}{F} = \begin{bmatrix} \underset{\sim o}{X} \\ \\ \underset{\sim o}{U} \end{bmatrix} = \begin{bmatrix} \underline{O} \\ \\ \underline{H}_2^{-1}\left[\underline{H}_1\underline{X}^{(p)} - \underline{J}\,\underline{F}^{(p)}\right] + \underline{U}^{(p)} \end{bmatrix} \tag{7.60}$$

7.5.4 Solution which Segments the Path of Integration - Stiffness Method

As discussed in section (3.2), a solution which segments the path of integration may become necessary when roundoff errors become significant.

Let the region bounded by $\beta_2 = 0$ and $\beta_2 = 1$ be divided into M segments by lines which are in the same direction as β_1 (figure 7.2).

These lines will be referred to as segment lines and will be numbered o,1,2,.....,m,.....,M. Segment lines o and M are the boundaries $\beta_2 = O$ and $\beta_2 = 1$ respectively.

Consider one such segment, m, bounded by the segment lines $\beta_2 = d$ and $\beta_2 = d + b$ which will be referred to as "edges 1 and 2" respectively. (Figure 7.3).

Let $\underline{\overline{X}}_1^m$ and $\underline{\overline{X}}_2^m$ represent the actions, and $\underline{\overline{U}}_1^m$ and $\underline{\overline{U}}_2^m$ the displacements at edges 1 and 2 of segment m respectively (figure 7.4), and let these be defined by:

$$\underline{\overline{X}}_1^m = \text{col}\left\{\underline{n}_{22}, \underline{n}_{12}, \underline{m}_{22}, \underline{r}_2\right\}_1^m \qquad (7.61)$$

$$\underline{\overline{X}}_2^m = \text{col}\left\{\underline{n}_{22}, \underline{n}_{12}, \underline{m}_{22}, \underline{r}_2\right\}_2^m \qquad (7.62)$$

$$\underline{\overline{U}}_1^m = \text{col}\left\{\underline{u}_2, \underline{u}_1, \underline{\theta}, \underline{w}\right\}_1^m \qquad (7.63)$$

$$\underline{\overline{U}}_2^m = \text{col}\left\{\underline{u}_2, \underline{u}_1, \underline{\theta}, \underline{w}\right\}_2^m \qquad (7.64)$$

Note that in this section the superscript refers to the segment and not to an individual line as discussed in section (7.4).

Defining

$$\underline{\bar{F}}_1^m = \text{col}\left\{ \underline{\bar{X}}_1^m, \underline{\bar{U}}_1^m \right\} \tag{7.65}$$

$$\underline{\bar{F}}_2^m = \text{col}\left\{ \underline{\bar{X}}_2^m, \underline{\bar{U}}_2^m \right\} \tag{7.66}$$

and

$$\underline{\bar{G}} = \underline{\bar{G}}(b) \tag{7.67}$$

then, from equation (7.35), the solution for segment m is given by:

$$\underline{\bar{F}}_2^m = \underline{\bar{G}} \left[\underline{\bar{F}}_1^m - \underline{F}^{(p)} \right] + \underline{F}^{(p)} \tag{7.68}$$

The particular solution $\underline{F}^{(p)}$ is constant across the segment for the constant loading selected (equation 7.38).

<u>Stiffness Matrix of a Segment</u>

With the load set to zero equation (7.68) becomes:

$$\underline{\bar{F}}_2^m = \underline{\bar{G}}\,\underline{\bar{F}}_1^m \tag{7.69}$$

Let $\underline{S}_m$ be the stiffness matrix of the segment m and let its partitioned form be:

$$\underline{S}^m = \begin{bmatrix} \underline{S}_{11}^m & \underline{S}_{12}^m \\ \underline{S}_{21}^m & \underline{S}_{22}^m \end{bmatrix} \tag{7.70}$$

where the order of each submatrix $\underline{S}_{ij}^m$ is (4N x 4N).

Then, by definition, $\underline{S}_{ij}^m$ (i,j = 1,2) are the actions $\underline{\overline{X}}_i^m$ produced by unit displacements $\underline{\overline{U}}_j^m$ with all other displacements zero.

Equation (7.69) is written in the form:

$$\begin{bmatrix} \underline{\overline{X}}_2^m \\ \underline{\overline{U}}_2^m \end{bmatrix} = \begin{bmatrix} \underline{\overline{G}}_{11} & \underline{\overline{G}}_{12} \\ \underline{\overline{G}}_{21} & \underline{\overline{G}}_{22} \end{bmatrix} \begin{bmatrix} \underline{\overline{X}}_1^m \\ \underline{\overline{U}}_1^m \end{bmatrix} \tag{7.71}$$

With $\underline{\overline{U}}_1^m = \underline{I}$ and $\underline{\overline{U}}_2^m = \underline{O}$, equation (7.71) becomes:

$$\begin{bmatrix} \underline{\overline{X}}_2^m \\ \underline{O} \end{bmatrix} = \begin{bmatrix} \underline{\overline{G}}_{11} & \underline{\overline{G}}_{12} \\ \underline{\overline{G}}_{21} & \underline{\overline{G}}_{22} \end{bmatrix} \begin{bmatrix} \underline{\overline{X}}_1^m \\ \underline{I} \end{bmatrix} \tag{7.72}$$

from which:

$$\underline{\overline{X}}_1^m = -(\underline{\overline{G}}_{21})^{-1}\underline{\overline{G}}_{22} = \underline{S}_{11}^m \quad \text{by definition} \tag{7.73}$$

$$\underline{\overline{X}}_2^m = -\underline{\overline{G}}_{11}(\underline{\overline{G}}_{21})^{-1}\underline{\overline{G}}_{22} + \underline{\overline{G}}_{12} = \underline{S}_{21}^m \quad \text{by definition} \tag{7.74}$$

With $\underline{\overline{U}}_2^m = \underline{I}$ and $\underline{\overline{U}}_1^m = \underline{O}$, equation (7.71) becomes:

$$\begin{bmatrix} \bar{\underline{X}}_2^m \\ \\ \underline{I} \end{bmatrix} = \begin{bmatrix} \bar{\underline{G}}_{11} & \bar{\underline{G}}_{12} \\ \\ \bar{\underline{G}}_{21} & \bar{\underline{G}}_{22} \end{bmatrix} \begin{bmatrix} \bar{\underline{X}}_1^m \\ \\ \underline{O} \end{bmatrix} \tag{7.75}$$

from which:

$$\bar{\underline{X}}_1^m = (\bar{\underline{G}}_{21})^{-1} = \underline{S}_{12}^m \quad \text{by definition} \tag{7.76}$$

$$\bar{\underline{X}}_2^m = \bar{\underline{G}}_{11}(\bar{\underline{G}}_{21})^{-1} = \underline{S}_{22}^m \quad \text{by definition} \tag{7.77}$$

Then the stiffness matrix of the segment $\underline{S}^m$ is given by:

$$\underline{S}^m = \begin{bmatrix} -(\bar{\underline{G}}_{21})^{-1}\,\bar{\underline{G}}_{22} & (\bar{\underline{G}}_{21})^{-1} \\ \\ -\bar{\underline{G}}_{11}(\bar{\underline{G}}_{21})^{-1}\bar{\underline{G}}_{22} + \bar{\underline{G}}_{12} & \bar{\underline{G}}_{11}(\bar{\underline{G}}_{21})^{-1} \end{bmatrix} \tag{7.78}$$

Segment Clamped Edge Solution for the Loading $\bar{\underline{Z}}$

Let $\bar{\underline{X}}_1^{om}$ and $\bar{\underline{X}}_2^{om}$ represent the actions corresponding to the clamped edge* solution for the loading $\underline{Z}$ at edges 1 and 2 of segment m respectively and let these be defined by:

---

* The term "clamped edge" implies that the edges 1 and 2 of a segment are clamped.

$$\underline{\overline{X}}_1^{om} = \text{col}\left\{ \underline{n}_{22}, \underline{n}_{12}, \underline{m}_{22}, \underline{r}_2 \right\}_1^{om} \qquad (7.79)$$

$$\underline{\overline{X}}_2^{om} = \text{col}\left\{ \underline{n}_{22}, \underline{n}_{12}, \underline{m}_{22}, \underline{r}_2 \right\}_2^{om} \qquad (7.80)$$

where the superscript o denotes the clamped edge solution.

The solution for $\underline{\overline{X}}_1^{om}$ and $\underline{\overline{X}}_2^{om}$ corresponds respectively to the solution for $\underline{\overline{X}}_1^{m}$ and $\underline{\overline{X}}_2^{m}$ from equation (7.68) with $\underline{\overline{U}}_1^{m} = \underline{O} = \underline{\overline{U}}_2^{m}$ and is given by:

$$\underline{\overline{X}}_1^{om} = \underline{\overline{G}}_{21}^{-1}\left[\underline{\overline{G}}_{22} - \underline{I}\right]\underline{U}^{(p)} + \underline{X}^{(p)} \qquad (7.81)$$

$$\underline{\overline{X}}_2^{om} = \underline{\overline{G}}_{11}\underline{\overline{G}}_{21}^{-1}\left[\underline{\overline{G}}_{22} - \underline{I}\right]\underline{U}^{(p)} - \underline{\overline{G}}_{12}\underline{U}^{(p)} + \underline{X}^{(p)} \qquad (7.82)$$

Another way of obtaining this clamped edge solution is to apply the stiffness matrix of the segment. The particular solution actions are $\underline{X}^{(p)}$ and the particular solution displacements are $\underline{U}^{(p)}$ (equation 7.40).

To reduce the displacements at edges 1 and 2 of the segment to zero, displacements $-\underline{U}^{(p)}$ are applied. In this way the clamped edge solution is obtained in the following form:

$$\begin{bmatrix} \underline{\overline{X}}_1^{om} \\ \underline{\overline{X}}_2^{om} \end{bmatrix} = \begin{bmatrix} \underline{S}_{11}^{m} & \underline{S}_{12}^{m} \\ \underline{S}_{21}^{m} & \underline{S}_{22}^{m} \end{bmatrix} \begin{bmatrix} -\underline{U}^{(p)} \\ -\underline{U}^{(p)} \end{bmatrix} + \begin{bmatrix} \underline{X}^{(p)} \\ \underline{X}^{(p)} \end{bmatrix} \qquad (7.83)$$

Substituting for $\underline{S}_{ij}^{m}$ from equation (7.78) in equation (7.83) yields solutions for $\overline{\underline{X}}_{1}^{om}$ and $\overline{\underline{X}}_{2}^{om}$ which are identical to those given by (7.81) and (7.82).

Assembled Stiffness Matrix for the Shell

Let the M segments which subdivide the region bounded by $\beta_2 = 0$ and $\beta_2 = 1$ (figure 7.2) be numbered 1,2,3,.....,m,.....,M (figure 7.5a).

Let $\underline{U}_o$, $\underline{U}_1$, $\underline{U}_2$,.....,$\underline{U}_M$ represent respectively the displacements at the segment junctions 0,1,2,.....,M (figure 7.5a). The elements of $\underline{U}$ are given by equation (7.31).

The sign convention adopted for the actions and displacements at each junction is given in figure 7.5b.

The sign convention adopted for the actions and displacements for each segment is given in figure 7.4.

Assembling the segments into the original shell form, yields equations of equilibrium at the junctions, which in matrix form are (figure 7.6):

$$\begin{bmatrix} -\underline{S}_{11}^{1} & -\underline{S}_{12}^{1} & \cdot & \cdot & \cdot \\ \underline{S}_{21}^{1} & (\underline{S}_{22}^{1}-\underline{S}_{11}^{2}) & -\underline{S}_{12}^{2} & \cdot & \cdot \\ \cdot & \underline{S}_{21}^{2} & (\underline{S}_{22}^{2}-\underline{S}_{11}^{3}) & -\underline{S}_{12}^{3} & \cdot \\ \hline \cdot & \cdot & \underline{S}_{21}^{m} & (\underline{S}_{22}^{m}-\underline{S}_{11}^{m+1}) & -\underline{S}_{12}^{m+1} \\ \hline & & & \underline{S}_{21}^{M} & \underline{S}_{22}^{M} \end{bmatrix} \begin{bmatrix} \underline{U}_{o} \\ \underline{U}_{1} \\ \underline{U}_{2} \\ \hline \underline{U}_{m} \\ \hline \underline{U}_{M} \end{bmatrix} + \begin{bmatrix} -\underline{\overline{X}}_{1}^{o1} \\ \underline{\overline{X}}_{2}^{o1}-\underline{\overline{X}}_{1}^{o2} \\ \underline{\overline{X}}_{2}^{o2}-\underline{\overline{X}}_{1}^{o3} \\ \hline \underline{\overline{X}}_{2}^{om}-\underline{\overline{X}}_{1}^{o(m+1)} \\ \hline \underline{\overline{X}}_{2}^{oM} \end{bmatrix} = \underline{O} \qquad (7.84)$$

Let

$$\underline{B} = \begin{bmatrix} \underline{B}_{o} \\ \underline{B}_{1} \\ \underline{B}_{2} \\ \hline \underline{B}_{m} \\ \hline \underline{B}_{M} \end{bmatrix} = \begin{bmatrix} -\underline{\overline{X}}_{1}^{o1} \\ \underline{\overline{X}}_{2}^{o1}-\underline{\overline{X}}_{1}^{o2} \\ \underline{\overline{X}}_{2}^{o2}-\underline{\overline{X}}_{1}^{o3} \\ \hline \underline{\overline{X}}_{2}^{om}-\underline{\overline{X}}_{1}^{o(m+1)} \\ \hline \underline{\overline{X}}_{2}^{oM} \end{bmatrix} \text{ and } \underline{\overline{U}} = \begin{bmatrix} \underline{U}_{o} \\ \underline{U}_{1} \\ \underline{U}_{2} \\ \hline \underline{U}_{m} \\ \hline \underline{U}_{M} \end{bmatrix} \qquad (7.85)$$

Then equation (7.84) becomes

$$\underline{S}\,\underline{\bar{U}} + \underline{B} = \underline{O} \tag{7.86}$$

where $\underline{S}$ is the assembled stiffness matrix for the shell.

Each submatrix of $\underline{S}$ is of order (4N x 4N) and therefore $\underline{S}$ is of order $\left[4N(M+1) \times 4N(M+1)\right]$.

Solution of Equations (7.84) by Tri-diagonalisation

The assembled stiffness matrix $\underline{S}$ partitions directly into a tri-diagonal submatrix form, i.e. into a diagonal submatrix, a superdiagonal submatrix and a subdiagonal submatrix. This form is convenient for solution by an elimination process.

Let

$$\underline{S}_o = -\underline{S}_{11}^{1} \; * \tag{7.87}$$

$$\underline{S}_m = (\underline{S}_{22}^{m} - \underline{S}_{11}^{m+1}) \tag{7.88}$$

$$\underline{S}_M = \underline{S}_{22}^{M} \; * \tag{7.89}$$

where m = 1,2,3,.....,(M-1).

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*A flexible beam at β_2 = O can be handled by adding to $\underline{S}_o$ the beam stiffness corresponding to $\underline{U}_o$.

Similarly $\underline{S}_M$ can be modified for a beam at $\beta_2 = 1$.

Then, from equation (7.84), the equations of equilibrium at junctions O and 1 are respectively:

$$\underline{S}_o \underline{U}_o - \underline{S}^1_{12} \underline{U}_1 + \underline{B}_o = \underline{O} \qquad (7.90)$$

$$\underline{S}^1_{21} \underline{U}_o + \underline{S}_1 \underline{U}_1 - \underline{S}^2_{12} \underline{U}_2 + \underline{B}_1 = \underline{O} \qquad (7.91)$$

Eliminating $\underline{U}_o$ from equations (7.90) and (7.91) yields:*

$$\overline{\underline{S}}_1 \underline{U}_1 - \underline{S}^2_{12} \underline{U}_2 + \overline{\underline{B}}_1 = \underline{O} \qquad (7.92)$$

where

$$\overline{\underline{S}}_1 = \underline{S}_1 + \underline{S}^1_{21} \underline{S}^{-1}_o \underline{S}^1_{12} \qquad (7.93)$$

$$\overline{\underline{B}}_1 = \underline{B}_1 - \underline{S}^1_{21} \underline{S}^{-1}_o \underline{B}_o \qquad (7.94)$$

Note that equation (7.92) is of the same form as equation (7.90).

From equation (7.84), the equation of equilibrium at junction 2 is:

$$\underline{S}^2_{21} \underline{U}_1 + \underline{S}_{22} \underline{U}_2 - \underline{S}^3_{12} \underline{U}_3 + \underline{B}_2 = \underline{O} \qquad (7.95)$$

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*This assumes that $\underline{U}_o$ is not specified. If any of the displacements of $\underline{U}_o$ are specified, equations (7.90) and (7.91) should be adjusted accordingly. For a clamped boundary ($\underline{U}_o = \underline{O}$), the elimination procedure would commence at junction 1.

Eliminating $\underline{U}_1$ from equations (7.92) and (7.95) yields:

$$\overline{\underline{S}}_{22}\,\underline{U}_2 - \underline{S}^3_{12}\,\underline{U}_3 + \overline{\underline{B}}_2 = \underline{O} \tag{7.96}$$

where

$$\overline{\underline{S}}_{22} = \underline{S}_{22} + \underline{S}^2_{21}\,\overline{\underline{S}}^{-1}_{11}\,\underline{S}^2_{12} \tag{7.97}$$

$$\overline{\underline{B}}_2 = \underline{B}_2 - \underline{S}^2_{21}\,\overline{\underline{S}}^{-1}_{11}\,\overline{\underline{B}}_1 \tag{7.98}$$

In general, the following expression is obtained:

$$\overline{\underline{S}}_m\,\underline{U}_m - \underline{S}^{m+1}_{12}\,\underline{U}_{m+1} + \overline{\underline{B}}_m = \underline{O} \tag{7.99}$$

where:

$$\overline{\underline{S}}_m = \underline{S}_m + \underline{S}^m_{21}\,\overline{\underline{S}}^{-1}_{m-1}\,\underline{S}^m_{12} \tag{7.100}$$

$$\overline{\underline{B}}_m = \underline{B}_m - \underline{S}^m_{21}\,\overline{\underline{S}}^{-1}_{m-1}\,\overline{\underline{B}}_{m-1} \tag{7.101}$$

and $m = 1,2,3,\ldots,(M-1)$.

Proceeding in this way, the equilibrium equation at junction M, i.e. $\beta_2 = 1$, becomes:

$$\overline{\underline{S}}_M\,\underline{U}_M + \overline{\underline{B}}_M = \underline{O} \tag{7.102}$$

where $\bar{\underline{S}}_M$ and $\bar{\underline{B}}_M$ are respectively given by equations (7.100) and (7.101) with m replaced by M.

Then

$$\underline{U}_M = -\bar{\underline{S}}_M^{-1} \bar{\underline{B}}_M \quad * \qquad (7.103)$$

From equation (7.99) the following is obtained by replacing m by (m-1):

$$\underline{U}_{m-1} = \bar{\underline{S}}_{m-1}^{-1} \underline{S}_{12}^{m} \underline{U}_m - \bar{\underline{S}}_{m-1}^{-1} \bar{\underline{B}}_{m-1} \qquad (7.104)$$

The displacements $\underline{U}_{M-1}$, $\underline{U}_{M-2}$, ..., $\underline{U}_o$ are then obtained from equation (7.104) by back substitution.

The matrices $\bar{\underline{S}}_{m-1}^{-1} \underline{S}_{12}^{m}$, of order (4N x 4N), and $\bar{\underline{S}}_{m-1}^{-1} \bar{\underline{B}}_{m-1}$, of order (4N x 1), are determined during the application of equation (7.99) in the elimination procedure. In the computer program, these matrices are stored and used in the back substitution procedure.

In the computer program developed, provision is made for a solution using either 4 or 8 equal segments, i.e. M = 4 or 8. Then the stiffness matrix for each segment is the same. Also for the symmetric problems considered, the elimination procedure is terminated at $\beta_2 = 0.5$ by

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*This assumes that $\underline{U}_M$ is not specified. If any of the displacements of $\underline{U}_M$ are specified, equation (7.103) should be adjusted accordingly. For a clamped boundary ($\underline{U}_M = \underline{O}$), the elimination procedure would be terminated at junction (M-1).

allowing for the following symmetry conditions:

(a) for translational shells:

(i) $\underline{u}_2$ and $\underline{\theta}$ are antimetric about $\beta_2 = 0.5$

i.e. $\underline{u}_2 = \underline{0} = \underline{\theta}$ at $\beta_2 = 0.5$

(ii) $\underline{u}_1$ and $\underline{w}$ are symmetric about $\beta_2 = 0.5$.

(b) for ruled surface shells:

(i) $\underline{u}_1$ and $\underline{\theta}$ are antimetric about $\beta_2 = 0.5$

i.e. $\underline{u}_1 = \underline{0} = \underline{\theta}$ at $\beta_2 = 0.5$

(ii) $\underline{u}_2$ and $\underline{w}$ are symmetric about $\beta_2 = 0.5$.

7.6 Determination of Displacements, Stress-resultants and Stress-couples

The values of the dependent variables for the lines 1,2,...,N along β_2 are given by equation (7.35) or the alternative form given by equation (7.45) (refer to section 7.7.5).

The boundary values of the dependent variables are given by equations (7.10) to 7.13) inclusive and (7.16) to (7.19) inclusive.

It remains to determine suitable expressions for n_{11}, m_{11}, m_{12}, q_1, q_2 and r_1.

Equations (2.58), (2.60), (2.61), (2.65), (2.66), (2.67), (2.68) and (2.70), after non-dimensionalising the co-ordinates, yield the following:

$$n_{11} = Eh(\frac{1}{l_1} u_{1,1} - K_{11} w) + \nu n_{22} \quad (7.105)$$

$$m_{11} = \frac{-Eh^3}{12l_1^2} w,_{11} + \nu m_{22} \quad (7.106)$$

$$m_{12} = \frac{-Eh^3}{12(1+\nu)l_1} \theta,_1 \quad (7.107)$$

$$q_1 = \frac{-Eh^3}{12(1+\nu)l_1^3} w,_{111} + \frac{1}{l_1} m_{22,1} \quad (7.108)$$

$$q_2 = r_2 - \frac{Eh^3}{12(1+\nu)l_1^2} \theta,_{11} \quad (7.109)$$

$$r_1 = \frac{-Eh^3(1-\nu)}{12(1+\nu)l_1^3} w,_{111} + \frac{(2-\nu)}{l_1} m_{22,1} \quad (7.110)$$

where comma notation is used to represent partial differentiation with respect to β_1.

Using the finite difference formulae given in table 7.1 and allowing for the clamped conditions at $\beta_2 = 0,1$, as discussed in section (7.4), the required finite difference expressions are obtained for equation (7.105) to (7.110) inclusive. Details are given in table 7.3 for the lines 0,1 and k, where $k \geq 2$.

In the computer program developed allowance is made for symmetry about $\beta_1 = 0.5$ in the expressions given in table 7.3.

7.7 Some Notes on the Numerical Computations

7.7.1 The $\underline{G}(\beta_2)$ Matrix

(a) $\underline{G}(\beta_2)$ is determined from the series given by equation (7.39). However, for better accuracy, particularly for a large number of terms, $\underline{G}(\beta_2)$ is best computed from:

$$\underline{G}(\beta_2) = \underline{I} - \underline{A}\,\beta_2\left(\underline{I} - \frac{\underline{A}\beta_2}{2}\left(\underline{I} - \frac{\underline{A}\beta_2}{3}\left(\underline{I} - \frac{\underline{A}\beta_2}{4}\left(\ldots - \frac{\underline{A}\beta_2}{n-1}\left(\underline{I} - \frac{\underline{A}\beta_2}{n}(\ldots)\right)\right.\right.\right.\right. \qquad (7.111)$$

(b) A useful property of $\underline{G}(\beta_2)$ is:

$$\underline{G}(a).\underline{G}(b) = \underline{G}(a+b) \qquad (7.112)$$

In the computer program $\underline{G}(\frac{1}{16})$ is determined and $\underline{G}(\beta_2)$, for other values of β_2 (multiples of $\frac{1}{16}$), is obtained by application of equation (7.112). By determining $\underline{G}(\beta_2)$ in this way, fewer terms in the series are required.

7.7.2 The Particular Solution $\underline{F}^{(p)} = -\underline{A}^{-1}\underline{Z}$

In the computer program, the solution for $\underline{F}^{(p)}$ is determined by partitioning the $\underline{A}$ matrix into its submatrices given by equation (7.29). However, for the symmetric case considered, $\underline{A}_{12}$ is singular for ruled surface shells and $\underline{A}_{65}$ is singular for translational shells (refer to table 7.2). In the partitioning procedure the particular solution $u_2^{k(p)}$ is obtained by inverting $\left[\underline{A}_{12}\,\underline{A}_{62}^{-1}\,\underline{A}_{65} - \underline{A}_{15}\right]$, which, referring above and to table 7.2, is singular for both shell types. However, because of symmetry and since $\underline{F}^{(p)}$ is constant, the value of $u_2^{N(p)}$ i.e. at $\beta_2 = 0.5$, is zero.* Adjusting the above matrix accordingly eliminates the singularity.

Note that $\underline{F}^{(p)}$ is independent of l_2.

7.7.3 Singularity in the Matrix $\underline{G}_{12}(\beta_2)$ for Ruled Surface Shells

For the symmetric case considered the matrix $\underline{G}_{12}(\beta_2)$ for ruled surface shells is singular.

Then the solution for free conditions at $\beta_2 = 0, 1$, whichever approach is used (section 7.5.2, 7.5.3 or 7.5.4), is not possible.

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*For translational shells, $u_2^{k(p)}$, because of symmetry, is zero along all lines k. (k = 1,2,3,....,N).

However, because of symmetry, u_2^N is zero for ruled surface shells. By modifying the matrices accordingly, the singularity is removed.

7.7.4 Roundoff Errors in the Solution

In order to investigate any accumulation of errors the solution for each of the methods given in sections (7.5.2), (7.5.3) and (7.5.4) was determined from $\beta_2 = 0$ to $\beta_2 = 1$. In this way the effect of any errors on the symmetry in the solution could be observed.

When applying the conditions at $\beta_2 = 1$ (section 7.5.2), the roundoff errors for a shell with $N = 4$ and $r = 1$ were only slight. However, with $r = 1$, increasing N yielded very serious errors and distorted the solution, particularly at $\beta_2 = 1$. On the other hand, increasing r (> 1 only) with N constant, reduced the roundoff errors. Similarly for application of the conditions at $\beta_2 = 0.5$ (section 7.5.3).

Segmenting the path of integration (section 7.5.4), which, in effect, uses a value of $r > 1$, greatly improved the solution.

To investigate this matter further, solutions were obtained for different values of $(\frac{N}{r})$ using the approach given in section (7.5.2). It was noted that when $(\frac{N}{r})$ was less than 3 the errors, if any, were very small. For values of $(\frac{N}{r})$ greater than 4, the errors were very serious and completely distorted the solution at $\beta_2 = 1$. For values of $(\frac{N}{r})$ between 3 and 4, there was some evidence of errors.

However, these values of $(\frac{N}{r})$ are only a guide and more specific values would require further investigation.

Chetty[25] applied the method of lines to ruled surface shells using $N = 5$ and $r = 1$ only and noted that errors were accumulating. To overcome this problem Chetty suggested using a computer which handled more significant digits. This is undoubtedly beneficial but a more satisfactory approach would be to segment the path of integration.

7.7.5 The Determination of $\underline{F}$ along β_2

Let the interval $\beta_2 = 0$ to $\beta_2 = 1$ be divided into equal divisions of width e.

Then, using the property given by equation (7.112), equation (7.45) can be expressed as the following recurrence relation:

$$\underline{C}_{n+1} = \underline{G}(e)\, \underline{C}_n \qquad (7.113)$$

where n represents a point along β_2.

$\underline{F}$ then follows from equation (7.43). Equation (7.113) is useful for the determination of the solution at constant intervals along β_2.

7.8 Translational Shells

In this section numerical examples will be given for translational shells.

The solutions will be presented for N = 4,6,8 or 10. Since r = 1 in each example considered, the solution is obtained by segmenting the path of integration (section 7.5.3). Four segments (i.e. M = 4) are used when N = 4 and eight segments (i.e. M = 8) are used when N = 6, 8 or 10.

All results are presented in floating point notation and in ft.lb. units unless otherwise stated.

7.8.1 Convergence Study - Numerical Examples

Example 7.1. Consider an elliptic paraboloid with the following data:

l_1 = 50 ft. l_2 = 50 ft. h = 0.25 ft.

K_1 = 1.0, -2 ft^{-1} K_2 = 1.0, -2 ft^{-1}

E = 4.5, + 8 lb/ft^2 ν = 0.15 Z = 50 lb/ft^2

and boundary conditions:

clamped at $\alpha_1 = o, l_1$ and $\alpha_2 = o, l_2$.

The results are presented in tables 7.4a and 7.4b.

It will be noted from table 7.4a that n_{11} along the boundary $\beta_1 = O$ and u_1 are slowly convergent. Otherwise the solution is generally converging satisfactorily.

In table 7.4b the results corresponding to N = 8 are presented for $\beta_1 = O.5$ and $\beta_2 = O.5$. Because of symmetry w, n_{22} and m_{22} at $\beta_1 = O.5$ should be the same as w, n_{11} and m_{11} at $\beta_2 = O.5$. This provides a check on the solution. It will be noted from table 7.4b that, except for n_{11} at $\beta_1 = O$, there is generally good agreement.

These results show that the finite difference representation of n_{11} at the boundary $\beta_1 = O$ is poor (refer to equation (7.105) and table 7.3).

This could be improved by adopting a closer spacing of the lines adjacent to the boundary.

Example 7.2. Data as for example 7.1 but with boundary conditions:

clamped at $\alpha_1 = o, l_1$

free at $\alpha_2 = o, l_2$.

The results are presented in tables 7.5a and 7.5b, and figure 7.7.

It will be noted from table 7.5a and figure 7.7 that n_{11} along $\beta_1 = O$ is again slowly convergent.

Note that the corner* values of m_{11}, n_{12} and q_1 are also slowly convergent. The remainder of the solution is converging satisfactorily.

For reference, a detailed solution for $\beta_1 = 0.5$ and $\beta_2 = 0.5$ is given in table 7.5b. This solution corresponds to N = 8.

7.8.2 Comparison With Other Available Solutions

Example 7.3. The shell in example 7.1 was solved using the computer program described in chapter 5. Functions corresponding to case (a) in table 5.4 were used. Also S = 8 was adopted.

A comparison with the line solution using 8 segments (i.e. M = 8) and N = 8 is made in figure 7.8 for $\beta_1 = 0.5$. The solutions show good agreement.

7.8.3 Comparative Study of Different Boundary Conditions

Example 7.4. Data as for example 7.1 but with boundary conditions:

clamped at $\alpha_1 = o, l_1$

and (i) clamped at $\alpha_2 = o, l_2$

(ii) free at $\alpha_2 = o, l_2$

---

*These values should be interpreted as being at a point very close to the corner, but on the clamped boundary ($\beta_1 = 0$).

(iii) normal slide (1) at $\alpha_2 = o, l_2$.

Cases (i) and (ii) correspond to examples 7.1 and 7.2 respectively. Case (iii) is obtained from the computer program described in chapter 5 using S = 8 and functions corresponding to case (a) in table 5.4.

A comparison is made in figures 7.9a and 7.9b for w, n_{22} and m_{22}.

These results show that the normal slide (1) boundary is comparatively stiff. The value of m_{22} at $\beta_1 = 0.5$ is virtually zero for normal slide (1) boundaries.

7.9 Ruled Surface Shells

In this section numerical examples will be given for ruled surface shells.

As in section (7.8), the solutions are obtained by segmenting the path of integration (section 7.5.3). Four segments (i.e. M = 4) are used when N = 4 and eight segments (i.e. M = 8) are used when N = 6, 8 or 10.

All results are presented in floating point notation and in ft.lb units unless otherwise stated.

7.9.1 Convergence Study - Numerical Examples

Example 7.5: Consider a ruled surface hyperbolic paraboloid with the following data:

$l_1 = 37.50$ ft. $l_2 = 37.50$ ft. $h = 0.25$ ft.

$K_{12} = +4.444, -3$ ft.$^{-1}$ $\nu = 0.15$

$E = 4.5, +8$ lbs/ft^2 $Z = 50$ lbs/ft^2.

and boundary conditions:

clamped at $\alpha_1 = o, l_1$ and $\alpha_2 = o, l_2$.

The results are presented in tables 7.6a and 7.6b respectively.

It will be noted from table 7.6a the the solution is converging satisfactorily even along the clamped boundary $\beta_1 = 0$.

In table 7.6b the results corresponding to $N = 8$ are presented for $\beta_1 = 0.5$ and $\beta_2 = 0.5$.

Because of symmetry w, n_{12} and m_{22} at $\beta_1 = 0.5$ should be the same as w, n_{12} and m_{11} at $\beta_2 = 0.5$. This provides a check on the solution. It will be noted from table 7.6b that there is generally good agreement.

Example 7.6: Data as for example 7.5 but with boundary conditions:

clamped at $\alpha_1 = o, l_1$

free at $\alpha_2 = o, l_2$

The results are presented in tables 7.7a and 7.7b, and figures 7.10a and 7.10b, from which it will be noted that the solution is, in general, converging satisfactorily.

For reference, a detailed solution for $\beta_1 = 0.5$ and $\beta_2 = 0.5$ is given in table 7.7b. This solution corresponds to $N = 8$.

7.9.2 Comparison with Other Available Solutions

Example 7.7: Consider a ruled surface hyperbolic paraboloid with the following data:

$l_1 = 12.92$ in. $l_2 = 12.92$ in. $h = 0.25$ in.

$K_{12} = -3.1247, -2$ in.$^{-1}$ $\nu = 0.39$

$E = 5, +5$ lb/in^2 $Z = 1$ lb./in^2

and boundary conditions:

clamped at $\alpha_1 = o, l_1$ and $\alpha_2 = o, l_2$.

This example was also solved by Chetty[25], using a mixed Kantorovitch-Galerkin procedure and by Gunasekera[6]*, using an extended Levy procedure.

A comparison with the line solution using 8 segments (i.e. $M = 8$) and $N = 8$ is made in table 7.3, from which it will be noted that there is good agreement.

7.9.3 Comparative Study of Different Boundary Conditions

Example 7.8: Data as for example 7.5 but with boundary conditions:

clamped at $\alpha_1 = o, l_1$

and (i) clamped at $\alpha_2 = o, l_2$

(ii) free at $\alpha_2 = o, l_2$

(iii) normal slide (1) at $\alpha_2 = o, l_2$

(iv) normal slide (2) at $\alpha_2 = o, l_2$

Cases (i) and (ii) correspond to examples 7.5 and 7.6 respectively. Cases (iii) and (iv) are obtained from the computer program described in chapter 6 using $S = 8$ and functions corresponding to case (a) in table 6.4.

---

*Gunasekera used a slightly different value for K_{12}. The particular results presented in table 7.3 were obtained from Gunasekera's computer program using the above value of K_{12}.

A comparison is made in figures 7.11a and 7.11b for w, n_{12} and m_{22}.

Note that the difference between (iii) and (iv) is that normal slide (1) has $u_2 = 0$ at $\alpha_2 = o, l_2$ and normal slide (2) has $n_{22} = 0$ at $\alpha_2 = o, l_2$. However, the results show that both normal slide (1) and normal slide (2) are comparatively stiff, with a small value for m_{22} at $\beta_1 = 0.5$ and almost the membrane solution for n_{12}.

7.10 Discussion of the Computer Programs

A single computer program was developed to solve either translational or ruled surface shells. The program is limited to clamped conditions at $\beta_1 = 0, 1$ and to either clamped or free conditions at $\beta_2 = 0, 1$. Only uniformly distributed normal loading (Z) is considered.

A minimum value of N = 4 is considered.

The solution can be determined in any one of the following ways:

(a) application of the boundary conditions at $\beta_2 = 0, 1$ (section 7.5.2),

(b) application of the boundary conditions at $\beta_2 = 0$ and the symmetry conditions at $\beta_2 = 0.5$ (section 7.5.3),

(c) segmenting the path of integration into 4 or 8 equal segments (i.e. M = 4 or 8) and terminating the tri-diagonal elimination procedure at $\beta_2 = 0.5$ (section 7.5.4).

Further relevant comments have been made in section (7.7).

Further details of the program are available at Imperial College.[69]

The program was written in EXCHLF Autocode[70],[71] for the University of London Atlas computer.

CHAPTER 8

CLOSURE

The use of a Levy-type solution was convenient for studying the application of the indirect methods (Rayleigh-Ritz, Galerkin and Lagrangian multiplier methods) in conjunction with various approximating functions (tables 3.1 and 4.5). This study showed that:

(i) the Rayleigh functions (IIA)* for clamped boundaries were converging satisfactorily

(ii) the Filonenko-Boroditch functions (IE) were somewhat slower to converge than functions IIA

(iii) functions IF, obtained by modifying functions IE, converged rapidly but could cause numerical difficulties (refer to section 4.4.3)

(iv) the mixed cosine and sine set (IC), whenever used, converged rapidly but could also cause numerical difficulties (refer to section 4.4.3)

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*The functions have been classified in table 3.1

(v) using sine functions (IA) the membrane stress-resultants at a hinged boundary were slowly convergent

(vi) the Lagrangian multiplier method was effective in improving the slow convergence of a boundary action

(vii) functions ID, used in conjunction with the Lagrangian multiplier method, were effective in improving a particular stress-resultant which was slowly convergent on the boundary but satisfactory within the shell.

Application of the indirect methods to translational shells with combinations of clamped, hinged or normal slide (1) conditions on two opposite boundaries showed that, using the functions given in table 5.4:

(i) n_{11} (or n_{22}) at a hinged boundary was slowly convergent and was effectively improved using the Lagrangian multiplier method in conjunction with functions ID

(ii) the moments near the corner of a shell with normal slides (1) on all boundaries were slowly convergent

(iii) the normal shears were slowly convergent.

Otherwise the solutions were converging satisfactorily for all combinations of the boundary conditions considered.

Application of the indirect methods to ruled surface shells with combinations of clamped, hinged, normal slide (1), normal gable or normal slide (2) conditions on two opposite boundaries showed that, using the functions given in table 6.4:

(i) n_{12} at a hinged or normal gable boundary was slowly convergent and was effectively improved using the Lagrangian multiplier method in conjunction with functions ID

(ii) the moments near the corner of a shell with normal slides (1 or 2) on all boundaries were slowly convergent

(iii) the normal shears were slowly convergent.

Otherwise the solutions were converging satisfactorily for all combinations of the boundary conditions considered.

For all cases considered, the Lagrangian multiplier yielded a more accurate estimate of a boundary action than the corresponding displacement derivative. However, the solution adjacent to the shell boundary is based on these derivatives and, depending on the selected approximating functions, could be less satisfactory.

Varying the non-dimensional shell parameters showed that, for translational and ruled surface shells:

(i) the normal displacement and the moments increased with increasing shallowness, slowly approaching the solution for a thin flat plate

(ii) the membrane stress-resultants decreased slowly with increasing shallowness.

For ruled surface shells in the steeper range, the solution for n_{12} is similar to the membrane solution. Further points are discussed in sections (5.7.1) and (6.7.1).

The method of lines was applied to translational and ruled surface shells for clamped conditions at $\alpha_1 = o, l_1$ and clamped or free conditions at $\alpha_2 = o, l_2$. This analysis showed that:

(i) for translational shells n_{11} at $\alpha_1 = 0$ was slowly convergent due to the inaccuracy of the finite difference representation for the derivatives of u_1 at the boundary; otherwise the solution was converging satisfactorily.

(ii) for ruled surface shells the solutions were converging satisfactorily

(iii) roundoff errors became significant and at times distorted the solution as the ratio $(\frac{N}{r})$ increased much beyond 3

(iv) the roundoff errors were offset by segmenting the shell and restoring equilibrium at the segment junctions using a stiffness method.

The slow convergence of n_{11} at $\alpha_1 = 0$ observed with translational shells could be improved by adopting a closer spacing of the lines adjacent to the boundary. This would lead to a more accurate finite difference representation for the derivatives of u_1.

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APPENDIX I

RAYLEIGH FUNCTIONS

Rayleigh functions[(35),(37)] are derived from the engineering theory of free undamped transverse vibrations of a uniform slender beam and are of the form:

$$F_m \equiv F_m(\beta_i) = A_m \sin a_m\beta_i + B_m \sinh a_m\beta_i + C_m \cos a_m\beta_i + \\ + \cosh a_m\beta_i \qquad (1)$$

where A_m, B_m, C_m and a_m are constants,* m is a non-zero positive integer and i can have the value 1 or 2. The constants are determined by the two boundary conditions specified at each of $\beta_i = 0$ and $\beta_i = 1$.

The derivatives of F_m are given by:

$$F'_m = a_m\left[A_m \cos a_m\beta_i + B_m \cosh a_m\beta_i - C_m \sin a_m\beta_i + \sinh a_m\beta_i\right] = \\ = a_m \Theta_m \qquad (2)$$

---

*In equation (1) the function has been divided throughout by a constant associated with cosh $a_m\beta_i$ (say D_m). This is permissible only when D_m is non-zero.

$$F''_m = a_m^2 \left[-A_m \sin a_m\beta_i + B_m \sinh a_m\beta_i - C_m \cos a_m\beta_i + \cosh a_m\beta_i \right] = a_m^2 \phi_m \qquad (3)$$

$$F'''_m = a_m^3 \left[-A_m \cos a_m\beta_i + B_m \cosh a_m\beta_i + C_m \sin a_m\beta_i + \sinh a_m\beta_i \right] = a_m^3 \chi_m \qquad (4)$$

$$F''''_m = a_m^4 \left[A_m \sin a_m\beta_i + B_m \sinh a_m\beta_i + C_m \cos a_m\beta_i + \cosh a_m\beta_i \right] = a_m^4 F_m \qquad (5)$$

where a prime denotes differentiation. Note that a summation is not implied in equations (2) to (5) inclusive.

Two sets of boundary conditions will be considered and will be referred to as the:

(a) clamped-clamped case

(b) free-free case

(a) <u>Clamped-Clamped Case</u>

The boundary conditions satisfied at $\beta_i = 0, 1$ are:

$$F_m = 0 = F'_m \qquad (6)$$

and the Rayleigh function reduces to:

$$F_m = \cosh a_m\beta_i - \cos a_m\beta_i - A_m(\sinh a_m\beta_i - \sin a_m\beta_i) \qquad (7)$$

where a_m and A_m are obtained from the relations:

$$\cosh a_m \cos a_m = 1 \qquad (8)$$

$$A_m = \frac{\sinh a_m + \sin a_m}{\cosh a_m - \cos a_m} = \frac{\cosh a_m - \cos a_m}{\sinh a_m - \sin a_m} \qquad (9)$$

As m increases solution for these constants by equations (8) and (9) involves small differences of large numbers, causing considerable numerical difficulty. A method discussed by Rayleigh (page 277 in reference (37)) could help to overcome this problem.

The values of a_m, A_m, F_m, Θ_m, ϕ_m and χ_m used in this thesis were provided by Kuo [(59)] for values of m up to 27. These have been reproduced, for odd values of m only, in table A1.1 for a_m and A_m and in table A1.2 for F_m, Θ_m, ϕ_m and χ_m. Note that table A1.2 has been reproduced from the computer program, in which these values have been tabulated.

(b) Free-Free Case

The boundary conditions satisfied at $\beta_i = 0, 1$ are:

$$F''_m = 0 = F'''_m \tag{10}$$

The Rayleigh function for this case is the same as the second derivative of the Rayleigh function for the clamped-clamped case.

Then

$$F_m(\text{free-free case}) = \phi_m \ (\text{clamped-clamped case}) \tag{11}$$

For both cases (a) and (b) the Rayleigh functions are orthogonal functions

i.e.

$$\int_0^1 F_m F_n d\beta_i \neq 0 \quad \text{for } m = n \tag{12a}$$

$$= 0 \quad \text{for } m \neq n \tag{12b}$$

Refer to the integration formulae given in Appendix 2.

APPENDIX 2

INTEGRATION FORMULAE

In the following formulae, F_m denotes a Rayleigh function. The functions Θ_m, ϕ_m and X_m and the constant a_m are defined in Appendix 1.

f_n denotes a Rayleigh function which satisfies different boundary conditions to F_m. The functions $\bar{\Theta}_n$, $\bar{\phi}_n$ and $\bar{X}_n$ and the constant $\bar{a}_n$ are defined in the same way as Θ_m, ϕ_m, X_m and a_m respectively.

The integer i can have the value 1 or 2.

The integration formulae used in this thesis can be summarised in the following way:

$$\int_0^1 \sin m\pi\beta_i \cos n\pi\beta_i d\beta_i = + \frac{2m}{\pi(m^2-n^2)}, \text{ for } |m-n| \text{ odd} \quad (1a)$$

$$= 0, \qquad \text{for } |m-n| \text{ even} \quad (1b)$$

$$\int_0^1 \sin m\pi\beta_i \sin n\pi\beta_i d\beta_i = + \tfrac{1}{2}, \quad \text{for } m = n \quad (2a)$$

$$= 0, \quad \text{for } m \neq n \quad (2b)$$

$$\int_o^1 \cos m\pi\beta_i \quad \cos n\pi\beta_i d\beta_i \;=\; +1, \text{ for } m = n = o \tag{3a}$$

$$= +\tfrac{1}{2}, \text{ for } m = n > o \tag{3b}$$

$$= O, \quad \text{for } m \neq n \tag{3c}$$

$$\int_o^1 (1-2\beta_i)^2 d\beta_i \;=\; +\frac{1}{3} \tag{4}$$

$$\int_o^1 (1-2\beta_i) \sin m\pi\beta_i \;=\; +\frac{2}{m\pi}, \text{ for } m \text{ even} \tag{5a}$$

$$= O, \text{ for } m \text{ odd} \tag{5b}$$

$$\int_o^1 (1-2\beta_i) \cos m\pi\beta_i \;=\; \frac{4}{(m\pi)^2}, \text{ for } m \text{ odd} \tag{6a}$$

$$= O, \text{ for } m \text{ even} \tag{6b}$$

$$\int_o^1 F_m F_n d\beta_i^{\,*} = \frac{1}{4a_m}\Big[3F_m \chi_m - \phi_m \Theta_m\Big]_{\beta_i=O}^{\beta_i=1} \; +$$

$$+\; \frac{1}{4}\Big[F_m^2 - 2\Theta_m \chi_m + \phi_m^2\Big]_{\beta_i=1}, \text{ for } m = n \tag{7a}$$

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*For derivations of these expressions see references (35) and (37)

$$= \frac{1}{(a_m^4 - a_n^4)} \Big[a_m^3 \chi_m F_n - a_m^2 a_n \phi_m \Theta_n +$$

$$+ a_m a_n^2 \Theta_m \phi_n - a_n^3 F_m \chi_n \Big]_{\beta_i=0}^{\beta_i=1} \text{, for } m \neq n \qquad (7b)$$

$$\int_0^1 F'_m F_n d\beta_i = 0 \text{, for } m = n \qquad (8a)$$

$$= \frac{a_m}{(a_m^4 - a_n^4)} \Big[a_m^3 F_m F_n - a_m^2 a_n \chi_m \Theta_n +$$

$$+ a_m a_n^2 \phi_m \phi_n - a_n^3 \Theta_m \chi_n \Big]_{\beta_i=0}^{\beta_i=1} \text{, for } m \neq n \qquad (8b)$$

$$\int_0^1 F''_m F_n d\beta_i = \Big[a_m \Theta_m F_m \Big]_{\beta_i=0}^{\beta_i=1} - \frac{a_m}{4} \Big[3\Theta_m F_m - \chi_m \phi_m \Big]_{\beta_i=0}^{\beta_i=1} -$$

$$- \frac{a_m^2}{4} \Big[\Theta_m^2 + \chi_m^2 \Big]_{\beta_i=1} \text{, for } m = n \qquad (9a)$$

$$= \frac{a_m^2}{(a_m^4 - a_m^4)} \Big[a_m^3 \Theta_m F_n - a_m^2 a_n F_m \Theta_n +$$

$$+ a_m a_n^2 \chi_m \phi_n - a_n^3 \phi_m \chi_n \Big]_{\beta_i=0}^{\beta_i=1} \text{, for } m \neq n \qquad (9b)$$

$$\int_0^1 F_m f_n d\beta_i = \frac{1}{(a_m^4 - \bar{a}_n^4)} \Big[a_m^3 \chi_m f_n - a_m^2 \bar{a}_n \phi_m \bar{\Theta}_n + $$

$$+ a_m \bar{a}_n^2 \Theta_m \bar{\phi}_n - \bar{a}_n^3 F_m \bar{\chi}_n \Big]_{\beta_i=0}^{\beta_i=1} \qquad (10)$$

$$\int_0^1 F'_m f_n d\beta_i = \frac{a_m}{(a_m^4 - \bar{a}_n^4)} \Big[a_m^3 F_m f_n - a_m^2 \bar{a}_n \chi_m \bar{\Theta}_n + $$

$$+ a_m \bar{a}_n^2 \phi_m \bar{\phi}_n - \bar{a}_n^3 \Theta_m \bar{\chi}_n \Big]_{\beta_i=0}^{\beta_i=1} \qquad (11)$$

$$\int_0^1 (1-2\beta_i) F_m d\beta_i = \frac{1}{a_m^2} \Big[a_m \chi_m - 2\beta_i a_m \chi_m + 2\phi_m \Big]_{\beta_i=0}^{\beta_i=1} \qquad (12)$$

where a prime denotes differentiation and i can have the value 1 or 2. Note that equations (5a), (5b), (6a) and (6b) are particular cases of equation (12), when f_n denotes a trigometric function. Also equation (1a) is a particular case of equation (10). However, in equations (10) and (11) a_m must be different from $\bar{a}_n$.

When F_m is the Rayleigh function corresponding to the clamped-clamped case (refer to Appendix 1) and f_n is, in particular, a trigonometric function, the following are obtained:

$$\int_o^1 F_m F_n d\beta_i \;=\; 1, \text{ for } m = n \tag{13a}$$

$$= \; O, \text{ for } m \neq n \tag{13b}$$

$$\int_o^1 F''_m F_n d\beta_i \;=\; (2a_m A_m - a_m^2 A_m^2), \quad \text{for } m = n \tag{14a}$$

$$= \frac{8a_m^2 a_n^2}{(a_m^4 - a_n^4)} \left[a_m A_m - a_n A_n \right], \quad \text{for } m \neq n, \; \left| m-n \right| \text{ even} \tag{14b}$$

$$= \; O, \quad \text{for } m \neq n, \; \left| m-n \right| \text{ odd} \tag{14c}$$

$$\int_o^1 F_p \cos q\pi\beta_i d\beta_i \;=\; \frac{4A_p}{a_p}, \text{ for } q = o \tag{15a}$$

$$= \frac{4a_p^3 A_p}{(a_p^4 - q^4\pi^4)}, \quad \text{for } q = 2,4,6,\ldots \tag{15b}$$

$$\int_o^1 F_p \sin k\pi\beta_i d\beta_i \;=\; \frac{4a_p^2 k\pi}{(a_p^4 - k^4\pi^4)}, \quad \text{for } k = 1,3,5,\ldots \tag{16a}$$

$$\int_o^1 F'_p \sin j\pi\beta_i d\beta_i = \frac{-4j\pi a_p^3 A_p}{(a_p^4 - j^4\pi^4)}, \quad \text{for } j = 2,4,6,\ldots \qquad (16b)$$

where $m,n = 1,2,3,\ldots\ldots$

$p = 1,3,5,\ldots\ldots$