ALMOST LOCAL FIELD THEORY

by

MLADEN MARTINIS

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Department of Physics,
Imperial College of
Science and Technology.

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PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, London, between January 1964 and January 1966 under the supervision of Professor P.T. Matthews.

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Except where stated in the text, the work described is original and has not been submitted in this or any other University for any other degree.

ABSTRACT

In Chapters II - IV we give a brief survey of the quantum field theory necessary here. Particular attention is given to the Haag-Ruelle Collision theory. The theory developed in Chapters II - IV is applied to a particular model of almost local field theory. In fact, we set up an approximation scheme by requiring that the theory is relativistically equivalent to the presumably correct local field theory. Then, an almost local field possesses the Haag expansion.

In Chapter V we demonstrate the possibility of imposing the condition of almost locality on a 4-point matrix element. This is done in order to obtain certain restrictions on the functions F appearing in the Haag expansion of an almost local field. Disregarding possible end-point singularities we have been able to show that in a certain finite energy region the functions $(p^2 - m^2)F$ satisfy equations similar to "physical unitarity". Assuming that the F-functions appearing in the Haag expansion possess analytic properties, we have been able to find a model (Chapter VI) which explicitly shows how the end-point singularity can be cancelled by the threshold behaviour if the energy region is restricted to the elastic region.

The whole of Chapter VII is spent in showing how the

bound state problem may be incorporated into the spirit of almost local field theory. The problem is analysed in the case of AB elastic scattering where B represents a two-particle bound state of A .

The last Chapter, or conclusion, indicates the difficulties which are met when the 6-point function is examined.

C O N T E N T S

C O N T E N T S (Contd.)

I. GENERAL INTRODUCTION

It is the ideal of particle physics to arrive at a unified description of all kinds of particles and all possible interactions. The attempts to understand the strongly interacting particles and their couplings represent another partial approach in this direction.

Obviously all information we have about particles has been derived directly from experiment and using the language of particles. This so-called particle concept is characterized by giving the mass m, energy momentum four vector p with $p^2 = m^2$, spin s and internal quantum numbers α. It became clear, when the list of particles and resonances swelled up as the energy available had been increased, that there must be something more fundamental than the observed particles. Thus, in the current theoretical discussion of elementary particle physics, quantum field theory is the most sophisticated concept presently available. It is in terms of fields that we attempt to construct the basic theory. Then, there is a hope that by starting from a few fields we may, at least qualitatively, calculate the existence of various kinds of particles and the variety of their dynamical properties. So far, no fully satisfactory and simple theoretical model is known, from which the existence of particles, that is, their mass spin values and other experimental properties can be accurately evaluated.

There are various approaches to field theory, each of which has advantages in certain situations. Their common purpose is the desire to exploit to as full an extent as possible the mathematical consequences of a few physical principles.

We may distinguish three main divisions:

a) Lagrangian field theory(1). The theory deals with a certain function called Lagrangian density L which is specified as a function of "bare" fields ψ and their gradients. The parameters which are involved are "bare" masses and coupling constants. This formalism leads to a definite equation of motions which is then solved by perturbation methods or some modification thereof. Of course, it is not always possible to find exact mathematical solutions to the complicated equations one writes down in this way. The S-matrix is calculated by the method of Feynman diagrams(2).

b) A second approach is that of Lehman, Symanzik and Zimmermann (LSZ)(3), Wightman(4), Haag-Ruelle(5) and others. This approach is adopted here and is treated in most details.

c) The third approach is that of dispersion theory[6] or now extensively called analytic S-matrix theory[7]. By imposing directly relativity, unitarity and other symmetries together with analyticity to the S-matrix one hopes to have a complete theory which is able to describe all the properties of elementary particles.

A good feature of the dispersion theory approach is that one works with quantities that are observables or nearly so. It should also be emphasized that the analytic S-matrix theory considers the analytic properties of S-matrix elements <u>on the mass shell</u> in all their variables.

In none of the above-mentioned kinds of field theory have we any assurance that solutions of the equations actually exist or describe nature accurately. To make progress with field theory at present requires setting up phenomenologically based models or approximation schemes that reflect the general properties of the underlying fundamental theory.

It is the aim of this thesis to obtain a reasonable approximation to a local field theory, and still have a complete dynamical theory with at least a partial particle interpretation. The way to achieve it is to consider an almost local field[8] having the Haag expansion[9] in terms of free fields which are complete.

The Haag expansion introduces an infinite set of generalized F_{mn} functions (potentials), then, the condition that the field $B(x)$ is almost local requires certain mathematical restrictions on F_{mn}. For finite energy it is possible to smear $B(x)$ with such a test function that only a finite number of F_{mn}'s occur in the Haag expansion for $B(x)$. Thus, the idea of the approximation is to approximate the almost local field $B(x)$ by a finite number of F_{mn}. The S-matrix and the partial particle interpretation is given by the Haag-Ruelle collision theory(5). Since we deal only with finite energy, it must be possible to modify the theory at any stage so that the physical interpretation can be extended to higher energies. The most general set of F_{mn}'s with the "almost local" conditions is supposed to provide a parametrization of high energy physics. In this thesis we mainly consider the four-point truncated Wightman functions(5), for the "almost local" condition is best understood in terms of them. The energy domain is extended to the threshold for the $n + 1$ particle production. We find as a consequence of this (see Chapter V) that a certain function $T_2 = (p^2-m^2)F_{21}$ related to a $2 \longrightarrow 2$ scattering amplitude, when restricted to the mass shell, satisfies an equation similar to the physical unitarity up to the $n + 1$ particle production. The solution to such a unitary equation possess several isolated singularities[10] on the positive real energy axis known as physical threshold singularities. It is then shown on the model how the two-particle threshold branch point can be cancelled with an end-point singularity when the analysis of Chapter V, is applied in the elastic region. The derived formalism allows incorporation of bound states as well. To show this, we consider the bound state problem in Chapter VII in the case of two-particle bound state. The whole problem is then analysed on AB elastic scattering where B is a bound state of A.

The conclusion summarizes the obtained results, pointing out the difficult points in the theory. It also contains a brief sketch about the higher approximation, i.e. the 6-point function which is supposed to be connected with the 3-particle scattering region and therefore perhaps with the 3-particle unitarity as well.

For the convenience of the reader and the sake of completeness, we give at the beginning (Chapters II-IV) a necessary review of the theory, especially the Haag-Ruelle scattering theory, which will be used here. The review given here is by no means complete and is therefore supplied with a number of references to which the interested reader may refer.

II. AXIOMS IN QUANTUM FIELD THEORY AND WIGHTMAN FUNCTIONS

(a) Introduction

In this chapter we give a brief review of certain important features of relativistic field theories necessary here. This survey is not intended to be complete in any way. The starting point is the definition of the axiomatic approach to the quantum field theory contained in studying the consequences of a set of a few fundamental postulates on the theory. These postulates are stated in terms of a condition on operators called fields in a Hilbert space. Some of them, as Lorentz covariance[11] and statements concerning the structure of the energy momentum spectrum[11], are adopted by practically all authors. Apart from this the requirement that such a theory is also physically reasonable brings a number of other necessary properties of the fields (depending upon the treated model). Among these, one which we consider as a very important requirement, is the "asymptotic condition"[3,5]. This condition, of course, imposes further mathematical restrictions on the field operators, but the necessity for having it is to make a particle interpretation possible at all, as will be seen later.

For simplicity, we treat the case of a single neutral scalar field interacting with itself. All of the investigations which have been carried out make the following general assumption about the theory:

(b) Axioms in Quantum Field Theory

I. The usual postulates of quantum mechanics are valid, i.e., the states of the systems are represented by the vectors of a (separable) Hilbert space $\mathcal{H}$ with positive definite metric[12]. The "field operator" is introduced in the following way: for every point x of space-time there exists a bilinear form A(x) in the Hilbert space, such that for any two vectors $\Psi, \Phi \in D$ (D is a dense domain in $\mathcal{H}$) $(\Psi, A(x)\Phi)$ is a finite number which, of course, depends linearly on Φ, antilinearly on Ψ. Furthermore, one requires that D is a linear set containing vacuum Ψ_0. It was also recognized that the components of fields are in general more singular than ordinary functions in their dependence on a space-time point. This suggests that only smeared fields[5,11] could yield to well-defined operators. However, examples show that even after smearing, the fields are still unbounded operators which are not defined on every vector in a natural way. In spite of this difficulty one defines the smearing as a requirement that if $\phi(x)$ is any real function of space-time, which belongs to class $\mathcal{S}$[13], i.e., which is arbitrarily often differentiable and vanishes faster than any power for large $\|x\|$, then

$$A_\phi = \int A(x)\phi(x)d^4x \qquad \text{(II.1)}$$

is an (unbounded) self-adjoint operator defined on a dense linear subset D of Hilbert space $\mathfrak{H}$. In this case we shall say that the theory is specified by a Hilbert space and a linear and weakly continuous mapping[12] $\phi(x) \longrightarrow A_\phi$ from a suitable test-function space into the set of closed linear operators in $\mathfrak{H}$.

Although the functions of class $\mathscr{S}$ are very well behaved there are several known field theoretic models[14] in which the functions are not good enough to make A_ϕ an operator. In such a model one has to restrict oneself to test functions $\phi(x)$ whose Fourier transform differs from zero only in a finite region of momentum space. As smeared fields can still have an arbitrarily large expectation value in a suitably chosen state (i.e., if A_ϕ is an unbounded operator) we are obliged to make some assumptions about the domain of vectors in which the smeared fields are definable. We say that an operator A_ϕ is defined on $D(\overline{D} = \mathfrak{H})$ such that $A_\phi D \subset D$ and such that for $\Psi, \Phi \varepsilon D$, $(\Psi, A_\phi \Phi) \varepsilon \mathscr{S}$.

II. The theory is invariant under inhomogeneous Lorentz transformation[15]. This says that the relativistic transformation law of the states is given by a continuous unitary representation of the inhomogeneous SL(C, 2):

$$\{a, \Lambda\} \longrightarrow U(a, \Lambda) \qquad \text{(II.2)}$$

so that $\mathfrak{H}$ is invariant under unitary operators $U(a, \Lambda)$.

The unitary irreducible representations of the translation group, i.e., $\{a, 1\} \to U(a, 1)$ can be written in the form[12]

$$U(a, 1) = \exp(iP^{\mu} a_{\mu}) \qquad (II.3)$$

as it is a four-parameter abelian group whose unitary irreducible representations are all one-dimensional. P^{μ} are of course, infinitesimal generators of the group and are unbounded hermitian operators interpreted as the energy momentum operators of the theory. The specification of the irreducible representations of the group can be given in terms of the eigenvalues of the so-called Casimir operators[16], (scalars of the group). One such Casimir operator is definitely $P^2 = P^{\mu} P_{\mu}$ so that an irreducible representation can thus be partially spacified by giving the eigenvalue of this operator. For <u>physical</u> states we restrict this eigenvalue to be non-negative and interpret it as a square of the mass.

Furthermore, the sign of p_o (the eigenvalue of the operator P_o) may also be specified, since it cannot be reversed by the operations of the group. Thus a partial labelling which we require is given by

$$P^{\mu} P_{\mu} = P^2 = m^2 > 0 ; \quad p_o > 0 \qquad (II.4)$$

This condition is known as a <u>spectral condition</u>[8].
Physically it is clear that we have selected only those states corresponding to a given mass and non-negative energy.

The wave function is thus defined on the hyperboloid $p^2 = m^2$. This hyperboloid is called an orbit.(17)

III. There exists a unique (strictly speaking, up to a constant phase factor) invariant vacuum state Ψ_0, characterized by

$$U(a,\Lambda)\Psi_0 = \Psi_0 \tag{II.5}$$

IV. The transformation rule for the field is defined by

$$U(a,\Lambda)A_\phi U^{-1}(a,\Lambda) = A_{\{a,\Lambda\}\phi} \tag{II.6}$$

where

$$(\{a,\Lambda\}\phi)(x) = \phi(\Lambda^{-1}(x - a)) \tag{II.7}$$

and

$$U(a,\Lambda)D \subset D \tag{II.8}$$

The other usual requirements on the theory as for example:

Completeness: the Hilbert space is irreducible with respect to the algebra generated by the set of operators, i.e., it does not contain any invariant subspace.

Causality: This is also known as a condition that the theory is local, i.e., that a field observable at a point x commutes with a field observable at x', if the distance between x and x' is space-like and the

asymptotic condition: the condition that the field observables $A(x)$, contain particle observables and also that the usual formulation of S-matrix theory is possible. It is also a requirement that a field theory has an interpretation in terms of asymptotic observables corresponding to particles of definite mass and charge,

will be treated separately later. The reason is that they could be relaxed or even replaced with a less strong condition. For example, the question whether physical particles and their interactions can be described only in terms of local operators is answered with no, since one can quite well formulate the theory using so-called almost local field operators (Haag-Ruelle collision theory)(5). The asymptotic condition $t \to \pm \infty$ can also be translated into the asymptotic conditions for large spacial separations(5), if the effective interaction between the particles is at least of short range. This leads to a very important cluster decomposition property of the S-matrix. On the other hand, the completeness, although a good physical requirement which implies that the S-matrix is unitary, is rather a difficult one to handle or satisfy in practice.

So far there are only, in general, two different methods of reducing the above stated conditions on field operators to a set of functions. One of them is initiated by Lehmann, Symanzik and Zimmermann (LSZ)(3) and the other by Wightman(4).

The former considers the vacuum expectation values of either time ordered or retarded products of field operators giving rise to the set of τ- or r-functions respectively, whereas the latter considers the vacuum expectation values of simple products giving rise to the set of Wightman Functions. From the practical (and also physical) point of view the LSZ method is very useful as it includes the asymptotic condition and extremely helpful reduction formulae which gives the possibility of studying the analytic properties of the S-matrix elements, together with the system of equations for τ - and r- expressing essentially the unitarity of S. On the other hand, from the purely axiomatic point of view and for mathematical rigour the Wightman method seems to be more adaptable and also has a few important applications to problems of a general nature.

We shall follow the method of Wightman and first discuss briefly the Wightman functions[4,8].

(c) Wightman functions. Let us consider the following vacuum expectation value of the product of field operators

$$W_n(x_1 \ldots x_n) = (\Psi_0, A(x_1)\, A(x_2) \ldots A(x_n) \Psi_0) \qquad (II.9)$$

together with the equivalent ones where the operators are smeared out and $A_\phi\, D \subset D$

$$W_n(\phi_1 \times \ldots \times \phi_n) = (\Psi_0, A_{\phi_1} \ldots A_{\phi_n} \Psi_0) \qquad (II.10)$$

where

$$(\phi_1 \otimes \ldots \otimes \phi_n)(x_1 \ldots x_n) = \phi_1(x_1) \ldots \phi_n(x_n) \quad (II.11)$$

is considered as an element of $\mathcal{S}_{\otimes n}$, with $\otimes n$ as a space of n-tuple Minkowski vectors $(x_1 \ldots x_n)$. Since $\mathcal{S}_{\otimes n}$ is dense in $\mathcal{S}'_n$, the W's may be uniquely extended to distributions in $\mathcal{S}'_n$ by Schwartz's nuclear-theorem[19]. According to the properties $(\Psi, A_\phi \Phi) \varepsilon \mathcal{S}$ for $\Psi, \Phi \varepsilon D$ we conclude that $W_n \varepsilon \mathcal{S}'_n$ i.e., W functions are tempered distributions. Now, as we have a rigorous methematical definition of W's, the study of their properties can be done in an unambiguous, precise fashion. We shall not go into this study any further as it may be found elsewhere.

Briefly, we can only say that the properties of the Wightman functions equivalent to the aforementioned postulates are the positive-demidefiniteness, the hermicity, the relativistic coveriance, the locality and the spectral condition.

For example the spectral conditions imply that the Wightman functions are boundary values of analytic functions in complex coordinate space, and the locality says that some of these analytic functions are identical. The relativistic covariance, of course, enlarges the domain of analyticity due to the theorem of Bargman, Hall and Wightman[20]. Except for positive definiteness the other properties of Wightman functions are known as linear properties, since they connect only a finite number of the Wightman functions.

III. CLUSTER DECOMPOSITION AND TRUNCATED VACUUM EXPECTATION VALUE

The cluster decomposition properties, or sometimes called the connectedness structure, of the S-matrix elements are generally considered true in all scattering theories whether or not there exists a local, or non-local quantum field theory. Such a property is usually postulated. However, it seems that the S-matrix decomposition properties are in some way related to the approximate locality or "short range" of particle interactions. This approximate locality could be made by stating the observed fact that experiments sufficiently separated in space or time are mutually independent. That is to say, the outcome of a scattering or production process between massive particles is asymptotically independent of the presence of other particles. Of course, these asymptotic properties for $t \longrightarrow \pm\infty$ are a consequence of the vanishing of the effective interaction between two subsystems as their separation becomes infinite. A way to translate the asymptotic properties for large separation at finite times into asymptotic properties for large t has been developed and described by Haag.(5) The physical idea behind this assumption is the following. Consider the simple case where $x_1 \ldots x_p$ are concentrated in a finite region $\mathcal{R}_a$ and $x_{p+1}, \ldots x_n$ to a finite region $\mathcal{R}_b$. Let the distance between $\mathcal{R}_a$ and $\mathcal{R}_b$ tend to infinity. We introduce a (partial) physical interpretation of the field quantity $A(x)$ by saying that the change, caused by $A(x)$, on the state on which it operaties is concentrated near the point x, so that $A(x_1) \ldots A(x_p)\Psi_o$ cannot be distinguished from the vacuum except "near the region $\mathcal{R}_a$". The same conclusion follows for the product $A(x_{p+1}) \ldots A(x_n)\Psi_o$ which is different from the vacuum only "near the region $\mathcal{R}_b$". In other words, $A(x_1) \ldots A(x_p)\Psi_o$ is the state which is experimentally localized in $\mathcal{R}_a$, while $A(x_{p+1}) \ldots A(x_n)\Psi_o$ is the state experimentally localized in $\mathcal{R}_b$. Therefore as the distance between $\mathcal{R}_a$ and $\mathcal{R}_b$ tends to infinity, the vacuum expectation value tends to

$$(\Psi_o, A(x_1) \ldots A(x_n)\Psi_o) \longrightarrow (\Psi_o, A(x_1) \ldots A(x_p)\Psi_o) \cdot (\Psi_o, A(x_{p+1}) \ldots A(x_n)\Psi_o) \quad \text{(III.1)}$$

Repeating this kind of heuristic argument we arrive at cluster decomposition properties. This kind of cluster decomposition can be proved mathematically rigorously from the conventional postulates of relativistic quantum field theory. The difficulties which could come from the high energy behaviour are expected not to be essential in the behaviour at large distances. The rigorous proof of the cluster decomposition is based on the fact that D is a <u>Gårding domain</u>(21) for the infinitesimal generators of the Lorentz group, or in other words, since D is a Gårding domain

$$(\Phi, U(a,1)\Psi) \text{ converges to } (\Phi,\Psi_0)(\Psi_0,\Psi) \quad \text{(III.2)}$$

as $\|\vec{a}\| \to \infty$. Using this fact one can prove that

$$\|\vec{a}\|^n W_{n+m}(x_1 \ldots x_n, y_1 + a, \ldots, y_m + a) - W_r(x_1 \ldots x_n) W_m(y_1 \ldots y_m) \to 0 \quad \text{(III.3)}$$

as $\vec{a} \to \infty$

and the convergence is in the range of distributions[19].

In order to get a neat statement of the required cluster decomposition property it is helpful to introduce the notion of the truncated part of a vacuum expectation value. Before giving the definition of the truncation itself, let us recall that according to the spectral condition the support of a distribution $\tilde{W}_n$ (i.e., the Fourier transform of W_n which exists since W_n is a tempered distribution) is contained in a forward cone

$$\left\{ p ; p_1 + \ldots + p_i \in \bar{V}_+^m \cup \{ C \} \text{ for } i = 1,\ldots,n, \sum^n p_i = 0 \right\} \quad \text{(III.4)}$$

where $V_+^m = (p | p \in V_+, p^2 > m^2)$ and the bar on V means the closure.

(III.4) includes also the contribution from vacuum intermediate states so that at $p_1 + \quad + p_i = 0$ (22)

$$\tilde{W}_n(p_{ij} \ldots p_n) = \tilde{W}(p_1 \ldots p_i) \tilde{W}(p_{i+1} \ldots p_n) \quad \text{(III.5)}$$

We shall, however, in what follows use the stronger requirement that there exists a positive lowest mass m in the theory. If we consider now the vacuum expectation value itself, then the occurrence of the vacuum intermediate states as in (III.5) will hide the existence of the positive smallest mass m of the theory. To remedy this situation we subtract the contribution due to the vacuum intermediate states from the Wightman functions in a symmetric and systematic way, with respect to the permutation of the n field operators. Thus, one defines "truncated" vacuum expectation value by induction

$$W(x_1) = W_T(x_1)$$

$$W(x_1 x_2) = W_T(x_1 x_2) + W_T(x_1) W_T(x_2)$$

$$\begin{aligned} W(x_1 x_2 x_3) = {} & W_T(x_1 x_2 x_3) + W_T(x_1) W_T(x_2 x_3) \\ & + W_T(x_1 x_2) W_T(x_3) + W_T(x_2) W_T(x_1 x_3) \\ & + W_T(x_1) W_T(x_2) W_T(x_3) \end{aligned}$$

and so on. In general we have

$$W(x_1 \ldots x_n) = W_T(x_1 \ldots x_n) + \sum_A W(x_{i_1 i_2 \ldots}) \ldots W(x_{j_1 j_2 \ldots}) \ldots$$

The sum $\sum_A$ is taken over all possible partitions A of the indices $1, \ldots, n$ in distinct classes $i_1 i_2, \ldots;$ $j_1 j_2 \ldots;$ $\ldots$, and the order of operators inside $W_T(\ldots)$ is the same as that on the left hand side.

With the above definition of truncation Haag has assumed that if all x_i had the same time components x_i^0, then as the diameter d of the three-dimensional set of points $\bar{x}_i$ tends to infinity $W_T(x_1 \ldots x_n)$ tends to zero faster than any power of d. The first rigorous proof of this so-called space-like asymptotic condition was given by Ruelle.[5]

It is interesting to mention that the truncated part calculated in perturbation theory is just the sum of all connected diagrams. Therefore in analogy with the perturbation theory we can say that the cluster decomposition properties are stated as: the truncated parts go to zero as their arguments separate.

At the end of this chapter we can briefly mention that the truncated functions have the same properties as those of the Wightman functions, except for the smaller support in momentum space. We have also seen (although without proof) that truncated functions have a better property than the Wightman function, at an infinite separation of their arguments.

IV. ALMOST LOCAL FIELDS AND HAAG-RUELLE COLLISION THEORY.

(a) Introduction

In the previous chapter it has been mentioned that the cluster decomposition properties may be related to the approximate locality due to the assumption of short range effective interaction, which leads to the space-like asymptotic properties of the S-matrix. It is important to see the significance of the cluster decomposition properties for the theory of collisions. Although the collision theory has been developed by Haag on the basis of a spacial asymptotic condition, the work by Ruelle has put Haag's arguments on a rigorous mathematical foundation. It is well known that if a field theory is to be useful it must have a well-defined physical interpretation in terms of asymptotic scattering states. In other words, the general concept of collision theory is based on the relationship between an initial asymptotic configuration of particles and the corresponding final asymptotic configuration.

To talk about the asymptotic configuration of the particles one has to define the notion of localized states, i.e. a criterion which tells whether a state Φ is at time t localized in a region V or not. If one considers two states, one localized in V_1 and another one in V_2 at time t, then the third state which one may hope to find to describe simultaneously the situation in V_1 and V_2 need not be very well defined if V_1 and V_2 are close together. But, asymptotically when V_1 and V_2 are far apart from each other the third state will have unambiguous physical meaning. (For example, this state is simply the direct product of the states localized in V_1 and V_2).

Since such an asymptotic product has a physical significance, it should only enter into the final formulae. One has also to remember that if the localization volumes of Φ_1 and Φ_2 , say, are far apart then the interaction between the two subsystems is assumed to be negligible and the asymptotic product between Φ_1 and Φ_2 is a well defined finite vector in Hilbert space. The limit in which such an asymptotic product is taken will depend upon the explicit assumption made about the vanishing of the interaction for large distances.

In a field theory the basic quantities which describe the physical situation are the set $A(x)$ (one quantity A for every point x in space-time). As $A(x)$ cannot be a proper observable (i.e. an operator in the Hilbert space) both for physical and mathematical reasons one introduces the regularized or smeared field(5),

$$A_\phi(x) = \int d^4y\phi(y-x)\,A(x) = U(x,1)\int d^4y\phi(y)\,A(y)\,U(x,1)^{-1} \qquad \text{(IV.1)}$$

with $\phi \in \mathcal{S}$.

Then one can investigate the time dependence of $(\Psi, A_{\phi}(t)\,\Phi)$ on the basis of the dispersion of a wave-packet, where Ψ and Φ are asymptotic states.

From the physical point of view one requires that particles behave as free ones at time $t \longrightarrow \pm\infty$ and that the free particle states are described by the field operators $A^{ex}(x)$, which satisfy the free field equation of motion

$$(\Box + m^2)A^{ex}(x) = 0 \qquad \text{(IV.2)}$$

We use the symbol ex which consistently replaces out or in. Defining a quantity

$$A^{ex}\,(t;f) = i\int \left(A^{ex}(x)\,\frac{\partial f}{\partial x_0} - \frac{\partial A^{ex}(x)}{\partial x_0}\,f(x)\right)d^3x \qquad \text{(IV.3)}$$

one can easily see that it is independent of t as it is a scalar product of two solutions of the Klein-Gordon equation.

The operators $A^{in}(f)$ and $A^{out}(f)$ and their adjoints are of course defined on "in" and "out" states which span two subspaces of Hilbert space $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ respectively. There is no assurance that $\mathcal{H}_{in} = \mathcal{H}_{out}$, unless we require that TCP theorem holds. Of course, this does not mean that if $\mathcal{H}_{in} = \mathcal{H}_{out}$ then $\mathcal{H}_{in} = \mathcal{H}$; In fact, examples show that the asymptotic states need not be complete at all. If we talk about the

S-matrix which connects "out" states with "in" ones, then the S-operator is only a unitary mapping of Φ_{out} onto Φ_{in}, but is generally undefined on these vectors of which are not in Φ_{out}.

The condition that $A^{ex}(t)$ approximates $A(t)$ at $t \longrightarrow \pm\infty$, written symbolically as

$$\lim_{x_0 \longrightarrow \pm\infty} (A(x) - A^{out,in}(x)) \longrightarrow 0 \qquad (IV.4)$$

is known as the "asymptotic condition" in the theory where the meaning of the passage to the limit is not yet clarified. If this limit somehow exists, we shall say that the particles behave as free ones at $t \longrightarrow \pm\infty$ and that this behaviour may be described by the operators $A^{out,in}(x)$. The idea of introducing the asymptotic condition is first of all the requirements which relate the mathematical object $A(x)$ with the quantities of physical interest, as for example cross-sections by collision processes and, of course, the possible particle interpretation of the theory. On the other hand, the particle interpretation without the asymptotic condition is possible if one starts from certain assumptions about the behaviour of vacuum expectation values where all times are equal and the space distances large. In this direction recently D. Ruelle(5) has succeeded in proving these assumptions provided that there is no particle with vanishing rest-mass in the theory. We are now left with mainly two problems:

one, to calculate the S-matrix (or scattering amplitude) from the knowledge of $A^{ex}(x)$ and the second, the existence proof of the limit in (IV.4) under, of course, a suitable assumption for the interaction.

So far there exists two main approaches to the asymptotic condition in "axiomatic quantum field theory". One is due to Lehmann, Symanzik and Zimmermann (LSZ)[3] in which they postulate the convergence of field matrix elements to matrix elements of free fields. Expressed mathematically the LSZ-asymptotic condition requires that

$$\lim_{t \to \pm\infty} (\Psi, A(t;f)\Phi) \longrightarrow (\Psi, A^{out,in}(t;f)\Phi) \quad (IV.6)$$

exists for all Ψ, Φ in the domain of the operators. The limit is zero if m is not one of the masses of the stable particle described by the theory. We notice that only the matrix element of $A(t;f)$ between two fixed states are assumed to converge whereas the vectors $A(t;f)\Phi$ need not approach any limit as $t \longrightarrow \pm\infty$. Using mathematical language we say that only "weak convergence"[3] is required, i.e. only weak limits of $A(t;f)$ exist for $t \longrightarrow \pm \infty$.

The other approach to the asymptotic condition is due to Haag[5]. Haag's main idea is that it is possible to construct asymptotic "in" and "out" states as a <u>strong limit</u> (limit in the norm) in Hilbert space, if a certain "space like asymptotic" condition holds.

Of course, we are already familiar with the definition and required properties of the "space like asymptotic condition" which was introduced in Chapter III. Haag's programme, as was mentioned earlier, was carried through the rigorous mathematical framework of the Wightman axioms by Ruelle [5]. He has introduced, in addition to the standard Wightman axioms, a new postulate - completeness of the asymptotic states and spectral conditions connected to this. Again, for the sake of completeness we shall only briefly recall this, now usually quoted as the Haag-Ruelle collision theory, in the form in which it will be used here.

(b) Almost Local Fields

Consider a quantized field $A(x)$ satisfying the axioms (I - IV) of Chapter I, plus locality, i.e.

$$[A(x), A(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0 \tag{IV.7}$$

As we are dealing with the theory of a neutral scalar field with cyclic vacuum the physical spectrum must be additive. The theorem actually says: if p_1 and p_2 are in the spectrum, then $p_1 + p_2$ is also in the spectrum. The proof due to Wightman[5] is: Consider two open domains V_1 and V_2 of p_1 and p_2 respectively. Then by choosing field operators $B^{(1)}(x)$ and $B^{(2)}(x)$ satisfying

$$U(a,1)\, B^{(i)}(x)\, U^{-1}(a,1) \;=\; B^{(i)}(x + a) \qquad \text{(IV.8)}$$

and test functions ϕ_j with supports in V_1 and V_2 we require that

$$B^{(1)}_{\phi_1} \Psi_o \neq 0 \text{ and } B^{(2)}_{\phi_2} \Psi_o \neq 0 \qquad \text{(LV.9)}$$

The energy-momentum spectra of these vectors is, of course, in V_1 and V_2 respectively as

$$U(a,1) B^{(j)}_{\phi_j} \Psi_o \;=\; B^{(j)}_{\{a,1\}\phi_j} \Psi_o \,, \qquad \text{(IV.10)}$$

where $(\{a,1\}\phi_j)(x) \;=\; \phi_j(x - a)$.

To get the required B's we consider the closed subspace $\mathcal{H}_{\overline{T}_i}$ of $\mathcal{H}$, where $\overline{T}_i \subset V_i$ and all the vectors whose spectra lie in $\overline{T}_i$. Since Ψ_o is cyclic there exist vectors of the form

$$\sum_{n=1}^{N} \int \cdots \int h_n^{(j)}(x_1 \ldots x_n)\, A(x_1) \ldots A(x_n) d^4x_1 \ldots dx_n \Psi_o \qquad \text{(IV.11)}$$

which are not orthogonal to $\mathcal{H}_{\overline{T}_j}$. The coefficient functions $h_n(x_1 \ldots x_n)$ have a compact support in the time coordinates and vanish faster than any power of $\|\vec{x}_i\|$ when any of the $\vec{x}_i$ gets large.

Naturally we define our $B^{(j)}$ to be

$$B^{(j)}(x) \;=\; \sum_{n=1}^{N} \int \cdots \int h_n^{(j)}\,(x-x_n, \ldots, x-x_n) \,.\, A(x_1), \ldots, A(x_n)\, dx_1, \ldots d^4x_n \,, \qquad \text{(IV.12)}$$

and they definitely satisfy (IV.8). Now, the vectors

$$B^{(1)}_{\phi_1}\, U(a,1)\; B^{(2)}_{\phi_2}\, \Psi_o \qquad (IV.13)$$

have their support in $V_1 + V_2$ by the same argument as before.

The norm of this vector is

$$\| B^{(1)}_{\phi_1}\, U(a,1)\, B^{(2)}_{\phi_2} \Psi_o \|^2 = (\Psi_o, B^{(2)*}_{\phi_2}\, U(a,1)^*\, B^{(1)*}_{\phi_1} \cdot$$

$$\cdot\; B^{(1)}_{\phi_1}\, U(a,1)\, B^{(2)}_{\phi_2} \Psi_o) \qquad (IV.14)$$

Now, as $a \longrightarrow \infty$ in a space-like direction the above norm converges to

$$(\Psi_o, B^{(2)*}_{\phi_2} B^{(2)}_{\phi_2} \Psi_o)(\Psi_o, B^{(1)*}_{\phi_1} B^{(1)}_{\phi_1} \Psi_o) \qquad (IV.15)$$

as a consequence of cluster decomposition properties.

This completes the proof since the norm cannot be zero for all a as it would require that either $B^{(2)}_{\phi_2} \Psi_o = 0$ or $B^{(1)}_{\phi_1} \Psi_o = 0$.

We shall now pay special attention to the fields which can be written as a polynomial $B_N(x)$ in the basic field operators, i.e., in the following form

$$B_N(x) = \sum_{n=1}^{N} \int d^4x_1 \ldots d^4x_n\; h_n(x-x_1, \ldots, x-x_n) A(x_1) \ldots A(x_n) \qquad (IV.16)$$

such that $B_N \Psi_o$ is not orthogonal to $\mathcal{H}^{(N)}_m$. If m is an eigenvalue of the mass operator $M = \sqrt{p^2} = \int m\, dE(p)$ then there exists a projection operator P_m such that

$P_m \mathcal{H} = \mathcal{H}_m = \oplus_N \mathcal{H}_m^{(N)}$. where $\mathcal{H}_m^{(N)}$'s are irreducible representations of the Lorentz group.

After regularizing A's, (IV.16) becomes

$$\underset{N}{B}(x) = \sum_{i=1}^{N} \int d^4x_1 \ldots d^4x_i \; h_i(x-x_1, \ldots, x-x_i), \\ A_{\phi_1}(x_1), \ldots, A_{\phi_i}(x_i) \quad . \qquad \text{(IV.17)}$$

Thus, the field $B_N(x) = U(x,1) B_N \, U^{-1}(x,1)$ has support in the momentum space restricted to a neighbourhood of $\{ p \mid p^2 = m^2 \}$. There is, however, one additional difficulty connected with the Lorentz invariance of B's, that is to say the Lorentz invariance is hard to establish for them.

In future we will drop the index N and write only B. It is also easy to prove that if $\Phi \in D$ and $f \in \mathcal{S}_3$, then

$$\int B(t,\vec{x}) \, f(\vec{x}) d^3x \; \Phi \qquad \text{(IV.18)}$$

exists, and is continuous in f and C^∞ in t. In what follows we shall need to know a bit more about the solution of the KG equation.

The smooth positive frequency solutions of the KG equation with mass m, $(\Box + m^2)f = 0$ are of the form

$$f(x) = (2\pi)^{-2} \int \delta(p^2-m^2)\theta(p_o) e^{-i(p,x)} g(\vec{p}) d^4p \qquad \text{(IV.19)}$$

where $g(\vec{p}) \in \mathcal{D}_{\vec{p}}^{(25)}$. This follows from the fact that

$$f(x) = (2\pi)^{-2} \int e^{-i(p,x)} \tilde{f}(p) d^4p \qquad \text{(IV.19')}$$

is a solution of $(\Box + m^2)f = 0$ provided

$$(p^2 - m^2)\, \tilde{f}(p) = 0 \qquad \text{(IV.19'')}$$

Then it is easy to check that the following $\tilde{f}(p)$ satisfies (IV.19'')

$$\tilde{f}(p) = \delta(p^2-m^2)\left[\theta(p^0)g_+(\vec{p}) + \theta(-p^0)g_-(\vec{p})\right] \qquad \text{(IV.19''')}$$

where

$$\theta(p^0) = +1 \quad \text{if} \quad p^0 > 0$$
$$= 0 \quad \text{if} \quad p^0 < 0 .$$

Thus, if f is a smooth solution of the KG equation then for every t

$$f(t; \vec{x}) \varepsilon\, \mathcal{S}_4 \qquad \text{(IV.20)}$$

i.e. $\lim_{\|x\| \to \infty} \|x^{\ell}\| f^{(n)}(x) = 0$ exists for all ℓ, m .

There is also a very important theorem giving the asymptotic behaviour of the smooth solutions of the KG equation. We shall only quote it without the proof(5).

It says: if f is a smooth solution of the KG equation, then

1. For all t we have $f(t, \vec{x}) \varepsilon\, \mathcal{S}_3$.

2. There exist constants A and B independent of t such that

$$\text{(a)} \quad \max_{\vec{x}} |t|^{3/2} f(t, \vec{x}) < A$$

$$\text{(b)} \quad \int f(t, \vec{x})\, d^3x < B\,(1 + |t|^{3/2}) \qquad \text{(IV.21)}$$

Let us now consider the vacuum expectation value of the product of basic field operators $A(x)$

$$W(\phi) = (\Psi_o, A_{\phi_1} \ldots A_{\phi_n} \Psi_o) \qquad (IV.22)$$

where $\phi = \phi_1 \otimes \ldots \otimes \phi_n \in \mathcal{S}_{4n}$

by using Schwartz nucleon theorem and define the quantity

$$W(a,\phi) = W(a_1,\ldots a_n,\phi) = \int d^4x_1 \ldots d^4x_n \; \phi(x_1 \ldots x_n) W(x_1+a_1,\ldots x_n+a_n) \qquad (IV.23)$$

Let π be the element (permutation) of the symmetric group on n objects such that $\pi(1\ldots n) = (i_1 \ldots i_n)$, so that

$$W^{\pi}(a,\phi) = \int d^4x_1 \ldots d^4x_n \phi(x_1,\ldots x_n) W^{\pi}(x_{i_1}+a_{i_1},\ldots x_{i_n}+a_{i_n}) \qquad (IV.24)$$

In general we take a_i to be pure space-like: $a_i = (0,\vec{a}_i)$ and the diameter $d(\vec{a})$ of the set $\{\vec{a}_1,\ldots,\vec{a}_n\}$ is then given by

$$d \equiv d(a) = \max_{i,j} \| \vec{a}_i - \vec{a}_j \|^2 \qquad (IV.25)$$

The translation invariance of $A(x)$ leads to the translation invariance of $W(a,\phi)$, that is

$$W(\vec{a}_1,\ldots, \vec{a}_n; \phi) = \overline{W}(\vec{\alpha}_1,\ldots, \vec{\alpha}_{n-1}; \phi) \qquad (IV.26)$$

$\vec{a}_i = \vec{a}_1 - \vec{a}_{i+1}$. Furthermore, since $\phi \in \mathcal{S}_{4n}$ and $W(x + a)$ is a tempered distribution $\overline{W}(\vec{a}_1, \phi)$ is at least in C^{∞} and is polynomially bounded.

Let us denote the truncated vacuum expectation values (TVEV's) corresponding to (IV.23) and (IV.24) by $W_T(a, \phi)$ and $W_T^{\pi}(a, \phi)$ respectively.

Consider the partition of $\vec{a}_i$'s into two disjoint sets, and then define the partitions of $(1 \ldots n)$ into two subsets X and X' such that $X \cup X' = (1, \ldots, n)$, $X \cap X' = \phi$. $X \equiv (i_1, \ldots, i_m)$; $X' \equiv (i'_1, \ldots, i'_{m'})$, $m + m' = n$, and both sets are in natural order. The distance between $(\vec{a}_i)_{i \in X}$ and $(\vec{a}_{i'})_{i' \in X'}$ is $\delta(X) = \min_{i, i'} \| \vec{a}_i - \vec{a}_{i'} \|$.

Now, if

$$I(1, \ldots, n) = (1, \ldots, n) \quad \text{identity permutation}$$

$$J(1, \ldots, n) = (i_1, \ldots, i_m; i'_1, \ldots, i'_{m'}) \tag{IV.27}$$

then for any integer N

$$\lim_{d \to \infty} d^N (W_T^I(\vec{a} ; \phi) - W_T^J(\vec{a} ; \phi)) = 0 \tag{IV.28}$$

if the configuration of the $\vec{a}_i$ remains the same, i.e., $i \in X$ and $i' \in X'$. We shall omit the detailed proof (5) of (IV.28) as it is quite long and can be found elsewhere. We should also mention that here $W - W^{\pi}$ is a tempered distribution, and ϕ decreases faster than any power. The following inequality holds

$$d^N \left| W_T(\vec{a}, \phi) - W_T^{\pi}(\vec{a}, \phi) \right| < C_N \tag{IV.29}$$

for any N, where C_N depends on the specific partition.

Thus, it is possible quite generally to show (and also to prove) that, $W(\vec{a}, \phi) \in \mathcal{S}_{\vec{\sigma}}$ and $|d|^N \tilde{W}(\vec{a}; \phi) \longrightarrow 0$ as $d \to \infty$, at least when $\vec{a}_i$'s are separated into two clusters whose space-like separation distance becomes increasingly larger as $d \to \infty$.

The field operators $A_\phi(x)$ in the above procedure can be replaced by an arbitrary polynomial $B_\phi(x)$ in the field operators $A(x)$. Then, it follows that

$$(\Psi_0, B_{\phi_1}(t_1 \vec{x}_1) \dots B_{\phi_n}(t_n, \vec{x}_n)\Psi_0)_T \in \mathcal{S}_{\xi_k} \qquad (IV.30)$$

$$\xi_k = \vec{x}_k - \vec{x}_{k+1}$$

is for fixed $t_1, \dots, t_n$ a tempered distribution of rapid decrease at ∞ in the variables $\vec{\xi}_k = \vec{x}_k - \vec{x}_{k+1}$ $k = 1, \dots, n-1$. We will call such a fixed $B_\phi(x_1)$ according to Haag's definition, "<u>almost local</u>"(8).
In other words we say that the field is called "almost local" if the truncated functions decrease faster than any power of the distance between the points with increasing separation in spacelike directions, and they do not increase for large separations in time-like directions.

The "almost locality" may also be seen from the equations

$$|a|^N \| [B(ax), B(0)] \underset{\sim}{\Phi} \| \longrightarrow 0 \qquad \text{(IV.31)}$$

as $|a| \longrightarrow \infty$ for all N if $x^2 < 0$ and $\underset{\sim}{\Phi} \varepsilon D$.

Since,

$$W_T(x_1, \ldots, x_n) = W_T(\xi_1, \ldots, \xi_{n-1}) =$$

$$(\underset{\sim}{\Psi}_o, B(t, \vec{x}_1) \ldots B(t, \vec{x}_n) \underset{\sim}{\Psi}_o)_T \qquad \text{(IV.32)}$$

is a distribution strongly decreasing at infinity, i.e.,

$$\lim_{d \to \infty} d^N \quad (\underset{\sim}{\Psi}_o, B(t, \vec{x}_1) \ldots B(t, \vec{x}_n) \underset{\sim}{\Psi}_o)_T = 0 \qquad \text{(IV.33)}$$

it can be represented as a finite sum of derivatives of continuous functions[19].

$$W_T(\xi) = \sum D^{m_k} F_k(\xi)$$

where (IV.34)

$$|F_k(\xi)| \leqslant \frac{C_n}{(1 + \|\xi\|)^2}$$

Furthermore (IV.33) together with (IV.34) imply that the following integral

$$\int (\underset{\sim}{\Psi}_o, B(t, \vec{x}_1) \ldots B(t, \vec{x}_n) \underset{\sim}{\Psi}_o)_T \, d^3x_2, \ldots, d^3x_n \qquad \text{(IV.35)}$$

is a constant and independent of $\vec{x}_1$.

Since translation invariance holds for $B(t, \vec{x})$ and Ψ_o is translation invariant, we have

$$(\Psi_o, B(t, \vec{x}_1)\ldots B(t, \vec{x}_n)\Psi_o)_T = (\Psi_o, B(0, \vec{x}_1)\ldots B(0, \vec{x}_n)\Psi_o)_T .$$

It therefore follows that

$$\left| \int (\Psi_o, B(t, \vec{x}_1)\ldots B(t, \vec{x}_n)\Psi_o)_T \, f_1(t,\vec{x}_1)\ldots f_n(t,\vec{x}_n) d^3x_1 \ldots d^3x_n \right|$$

$$\leq \prod_{i=2}^{n} \max_{\vec{x}} |f_i(t, \vec{x}_i)| \int |f_1(t, \vec{x}_1)| \, d^3x_1 \; .$$

$$\int (\Psi_o, B(0,\vec{x}_1) \ldots B(0, \vec{x}_n)\Psi_o)_T \, d^3 x_2 \ldots d^3 x_n$$

$$\leq \prod_{i=2}^{n} A_i \; t^{-3/2(n-1)} \; A_1(1 + |t|)^{3/2} \; . \text{ const.}$$

$$\leq \text{ const. } t^{-3/2(n-2)} \; . \qquad \text{(IV.36)}$$

(c) <u>Haag's Theorem on Strong Convergence</u>

In order to proceed with the actual construction of the asymptotic states and the S-matrix, it is desirable that for an irreducible representation contained in $U(a,\Lambda)$ of mass m, there exists an almost local field such that $B(x)\Psi_o$ lies in the subspace of that irreducible representation. That is to say, we require that the one-particle state of mass m and spin 0 is generated by the application of $B(x)$ to the vacuum. This requirement is known as the "solution of the one-body problem." In a physically reasonable theory this should follow from spectral properties connected with the stability condition and selection rules.

In general, we do not know how to "solve the one-body problem." To solve it means actually to construct and find the almost local field $B(x)$ such that $B(x)\Psi_0$ is a one particle state.

It is clear that under some circumstances it can always be done. For example, if the discrete mass state in question is isolated in the mass spectrum, then the required $B(x)$ could be an appropriate polynomial in the basic local fields $A(x)$. Therefore we shall assume the discrete mass state is isolated in the mass spectrum, so that one can "solve the one-body problem" exactly. Now define

$$\rho(\vec{p}) = \frac{1}{8\pi\omega_p^2} \int d^3(x-y)(\Psi_0, B(0,\vec{x})B(0,\vec{y})\Psi_0)_T \, e^{-i\vec{p}(\vec{x}-\vec{y})} \tag{IV.37}$$

with

$$\omega_p = (\vec{p}^2 + m^2)^{\frac{1}{2}}$$

so that $(\Psi_0, B(0,\vec{x}) B(0,\vec{y})\Psi_0)_T$ can be written

$$(\Psi_0, B(0,\vec{x})B(0, \vec{y})\Psi_0)_T$$

$$= (2\pi)^{-2} \int d^3p(2\omega_p)^2 \rho(\vec{p}) \exp\left[i\vec{p}(\vec{x} - \vec{y})\right] \tag{IV.38}$$

Let $f_i(t, \vec{x}_i)$ be the smooth solutions of the KG equation, then

$$\int (\Psi_o, B(t, \vec{x})\, B(t, \vec{y}) \Psi_o)_T\, f_1(t, \vec{x})\, f_2(t, \vec{y}) d^3x\, d^3y =$$

$$(2\pi)^{-2} \int d^3p (2\omega_p)^2 \rho(\vec{p}) \int e^{i\vec{p}(\vec{x}-\vec{y})}\, f_1(t, \vec{x}) f_2(t, \vec{y}) d^3x\, d^3y \,. \tag{IV.39}$$

Consider the integral

$$\int e^{i\vec{p}(\vec{x}-\vec{y})}\, f_1(t, \vec{x})\, f_2(t, \vec{y})\, d^3x\, d^3y =$$

$$(2\pi)^2 \int dq^o\, dq^{o'}\, e^{-it(q^o + q^{o'})}\, \tilde{f}_1(q^o, -\vec{p})\, \tilde{f}_2(q^o, \vec{p}) \tag{IV.40}$$

Since

$$\tilde{f}_i(q^o, \vec{q}) = \delta(q^2 - m^2) \left[\theta(q^o) g_i^+(\vec{q}) + \theta(-q^o) g_i^-(\vec{q}) \right] \,.$$

or

$$\tilde{f}_i(q^o, \vec{q}) = \frac{1}{2\omega_q} \left[\delta(q^o - \omega_q) g_i^+(\vec{q}) + \delta(q^o + \omega_q) g_i^-(\vec{q}) \right] \tag{IV.41}$$

we have a term in (IV.40) which is completely independent of time. It is of the following form

$$\frac{(2\pi)^2}{(2\omega_p)^2} \left[g_1^+(-\vec{p})\, g_2^-(\vec{p}) + g_1^-(-\vec{p})\, g_2^+(\vec{p}) \right] \tag{IV.42}$$

and will only remain when the limit $t \longrightarrow \pm \infty$ is taken.

Finally, we are left with

$$\int (\Psi_o, B(t,\vec{x})\ B(t,\vec{y})\Psi_o)\ f_1(t,\vec{x})\ f_2(t,\vec{y})\ d^3x\ d^3y \rightarrow$$
$$\rightarrow \int d^3p\ \rho(\vec{p}) \left[g_1^+(-\vec{p})\ g_2^-(\vec{p}) + g_1^-(-\vec{p})\ g_2^+(\vec{p}) \right] \qquad \text{(IV.42)}$$

as $t \longrightarrow \pm\infty$.

The representation of the inhomogeneous Lorentz group for a theory of a free field of mass m and spin 0 can be reduced to the direct integral of the irreducible representations labelled by $[m, 0]$. The corresponding Hilbert space can be written as a direct sum

$$\mathcal{H} = \bigoplus_{0}^{\infty} \mathcal{H}_n \qquad \text{(IV.43)}$$

where H_o is a one-dimensional subspace corresponding to to the identity representation and is taken to be proportional to the vacuum.

$\mathcal{H}_1$ is the (separable) Hilbert space of functions square integrable with measure

$d\mu(p) = \Theta(p)\ \delta(p^2 - m^2)\, d^4p$, and consists of one-particle states corresponding to the irreducible representation $[m, 0]$. Furthermore, as we have already indicated, suppose that the spectrum of P^2 in $\mathcal{H}_1^{\perp}$ is disjoint from m^2, so that $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \left\{ g(\vec{p}) \; ; \; \int \frac{d^3p}{2\omega_p} |g(\vec{p})|^2 < \infty \; , \quad \omega_p^2 = \vec{p}^2 + m^2 \right\} \tag{IV.44}$$

with a scalar product

$$(g_1, g_2) = \int \frac{d^3p}{2\omega_p} \; g_1^*(\vec{p}) \; g_2(\vec{p}) \tag{IV.45}$$

The one-particle projection operator

$E_1 = E(\{ p \,|\, p^2 = m^2, \; p^0 > 0 \})$ has, of course, the property that

$$E_1 \mathcal{H} = \mathcal{H}_1 \quad . \tag{IV.46}$$

Now consider

$$B(t, f_i) = \int B(t, \vec{x}) \; f_i(t, \vec{x}) \; d^3x \tag{IV.47}$$

with

$$(\Psi_o, B(t,f_i)\Psi_o) = 0 \tag{IV.48}$$

and

$$E_1 \; B(t, f_i)\Psi_o = B(t, f_i)\Psi_o \tag{IV.49}$$

where $f(t,\vec{x})$ is the smooth positive frequency solution of the KG equation (i.e., $g_i^-(\vec{p}) = 0$).

The equality (IV.49) says that $B(t, f_i)\Psi_o$ is a one-particle state. We shall prove that the one-particle state (IV.49) is time independent. To see this we take (IV.47) and rewrite it as

$$B(t, f_i)\Psi_o = \int d^3x\; f_i(t,\vec{x})\; U(x,1)\; B(0)\Psi_o$$
$$= \int d^3x\; f_i(t,\vec{x})\; U(x,1)\; E_1\; B(0)\Psi_o \qquad \text{(IV.50)}$$

The unitary representation $U(x,1)$ of the translation group is represented by the continuous integral

$$U(x, 1) = \int e^{i(qx)}\, d^4E(q) \qquad \text{(IV.51)}$$

with a unique projection valued measure $E(q)$ whose support is in the forward cone (i.e., $\bar{V}_+ = \{ p \mid p^2 \geqslant 0,\; p^o > 0 \}$). It is the consequence of Stone's theorem for the representation of Abelian groups by unitary operators. If, however, we had restricted ourselves to the states in Hilbert space, orthogonal to the vacuum Ψ_o, then $E(q)$ would have had its support in $\bar{V}_+^m = \{ p \mid p^2 \geqslant m^2,\; p^o > 0 \}$.

Thus we have

$$B(t,f_i)\Psi_o = \int dE(q)\; \left(\int d^3x\; f_i(t,\vec{x})e^{i(qx)}\right)\; E_1B(0)\Psi_o$$
$$= (2\pi)\int dE(q)\; \frac{1}{2\omega_q}\; g(\vec{q})e^{it(q^o-\omega_q)}\, E_1B(0)\Psi_o$$

(IV.52)

and it is time independent, since $E(q)E_1$ has its support in $\{ q \mid q^2 = m^2,\; q^o > 0 \}$ thus giving $q^o = \omega_q = (\vec{q}^2+m^2)^{\frac{1}{2}}$. Furthermore, $B(x)\Psi_o$ satisfies the KG equation

$$(\Box + m^2)\, B(x)\Psi_o = (\Box + m^2) U(x,1) E_1 B(0)\Psi_o$$

$$= (\Box + m^2) \int dE(p) e^{i(px)}\; E_1 B(0)\Psi_o$$

$$= -\int dE(p)\; (p^2 - m^2) e^{i(px)} E_1 B(0)\Psi_o = 0$$

(IV.53)

for the same reason as before.

The properties of B's listed above enable us to prove the existence of strong convergence in $\mathcal{H}$, as $t \longrightarrow \pm \infty$ for the following vectors

$$\Phi_{f_1 \ldots f_n}(t) = B(t,f_1)\ldots B(t,f_n)\Psi_o = \Phi(t)$$

(IV.54)

In other words $\lim_{t \to \pm \infty} \Phi_{f_1 \ldots f_n}(t)$ exists in the norm and is written $\Phi^{out,in}_{f_1 \ldots f_n}$, thus defining asymptotic states. It follows from the definition of $\Phi_{f_1 \ldots f_n}(t)$ that

$$\frac{d}{dt}\Phi_{f_1 \ldots f_n}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\Phi_{f_1 \ldots f_n}(t+\Delta t) - \Phi_{f_1 \ldots f_n}(t)\right]$$

(IV.50)

exists, and the limit is taken in the norm. Then, one estimates $\left\| \frac{d\Phi_{f_1 \ldots f_n}(t)}{dt} \right\|$ by expanding it into a sum of products of TVEV's.

First we notice that

$$\frac{d\Phi(t)}{dt} = \sum_j B(t, f_1)\ldots \frac{dB(t, f_j)}{dt} \ldots B(t, f_n)\Psi_o \qquad \text{(IV.55)}$$

gives

$$\left\|\frac{d\Phi(t)}{dt}\right\|^2 = \sum_{i,j} \Big(B(t,f_1)\ldots \frac{dB(t,f_i)}{dt}\ldots B(t,f_n)\Psi_o,\ B_1(t,f_1) \ldots \frac{dB(t,f_j)}{dt} \ldots B(t,f_n)\Psi_o\Big) \qquad \text{(IV.56)}$$

The TVEV' expansion of (IV.56) has the following behaviour

a) Terms with one-point functions vanish:

$$(\Psi_o,\ B(t,f)\Psi_o) = 0, \quad (\Psi_o,\ \frac{d}{dt} B(t,f)\Psi_o) = 0 .$$

b) Purely quadratic terms in B vanish because it has the form

$(\ldots \frac{d}{dt} B(t,f)\Psi_o)$, and, as we have shown $B(x)\Psi_o$ satisfies the KG equation.

c) Terms that only contain truncated two point functions including $(\Psi_o,\ B(t,f_i)\ B(t,f_j)\Psi_o)$ and $(\Psi_o,\ B^*(t,f_k)\ B^*(t,\ f_\ell)\Psi_o)$ vanish because f is the positive frequency solution of the KG equation.

d) Terms that contain a truncated function of order $n \geqslant 4$, or two cubic terms in B, vanish as $|t|^{-3}$ at least, (see (IV.36)).

Hence,

$$\left\|\frac{d\Phi(t)}{dt}\right\| < \text{const.}\ |t|^{-3/2} , \qquad \text{(IV.57)}$$

at worst as $t \rightarrow \pm\infty$

Then,

$$\|\Phi(t_1) - \Phi(t_2)\| \leq \int_{t_1}^{t_2} \left\|\frac{d\Phi(t)}{dt}\right\| dt < \text{const.}(|t_1|^{-\frac{1}{2}} - |t_2|^{-\frac{1}{2}})$$

and this can be made arbitrarily small for sufficiently large t_1 and t_2. So the limit exists. In fact, (IV.58) shows that $\Phi(t_m)$ is a Cauchy sequence with respect to the norm in Hilbert space. Therefore there always exists a limit vector (Hilbert space is complete), which we denote by $\Phi^{out,in}$ according to whether $t \to +\infty$ or $-\infty$.

We should also mention that

$$\lim_{t \to \pm\infty} \|\Phi_{f_i \ldots f_n}(t)\|^2 \not\longrightarrow 0 \tag{IV.59}$$

since the TVEV expansion of (IV.59) contains the following products

$$\sum_{\pi(j)} (\Psi_0, B^*(t,f_i^*) B(t,f_{j_1})\Psi_0)_T \, (\Psi_0, B^*(t,f_2^*) \, B(t,f_{j_2})\Psi_0)_T \ldots (\Psi_0, B^*(t,f_n^*) \, B(t,f_{j_n})\Psi_0)_T \tag{IV.60}$$

The sum runs over all possible permutations of $j_1 \ldots j_n$, i.e. $\pi(1 \ldots n) = (j_1,,,,j_n)$.

Now, as f's are positive frequency solutions of the KG equation, and $[B(t,f_i)]^* = B^*(t,f_i^*)$ it follows that

$$(\Psi_0, B^*(t,f_i^*)\, B(t,f_{j_\ell})\Psi_0)_T =$$

$$= \int (\Psi_0, B^*(t, \vec{x})\, B(t,\vec{y})\Psi_0)_T\, f_i^*(t,\vec{x}) f_{j_\ell}(t,\vec{y}) d^3x d^3y$$

$$\longrightarrow \int d^3p\, \rho(\vec{p})\, g_i^*(\vec{p}) g_{j_\ell}(\vec{p}) \qquad \text{(IV.60)}$$

as $t \rightarrow \pm\infty$

has a time independent part which remains after limit $t \longrightarrow \pm \infty$ is taken. The relation (IV.60) can be still more simplified by noting that

$$B(t,\vec{x})\Psi_0 = U(x,1)\, B(0)\Psi_0 = U(x,1) E_1 B(0)\Psi_0$$

is a one-particle state, satisfying the KG equation. Then for the representation of $U(x,1)$ we have

$$(U(x,1)f_B)(\vec{p}) = e^{i(px)}\, f_B(\vec{p})\ ;\quad p^0 = (\vec{p}^2 + m^2)^{1/2} \qquad \text{(IV.61)}$$

where

$$E_1\, B(0)\Psi_0 \sim f_B(\vec{p}) \qquad \text{(IV.62)}$$

Thus,

$$(\Psi_0, B^*(t,\vec{x})B(t,\vec{y})\Psi_0) = \int \frac{d^3p}{2\omega_p}\, e^{i(\vec{x}-\vec{y})\vec{q}} f_B^*(\vec{q}) f_B(\vec{q}) \qquad \text{(IV.63)}$$

and one easily finds that

$$\rho(\vec{p}) = \frac{(2\pi)^2}{(2\omega_p)^3}\, f_B^*(\vec{p})\, f_B(\vec{p}) \qquad \text{(IV.64)}$$

By redefining g_i and g_{j_ℓ} in (IV.60) in the following way

$$\tilde{g}_i(\vec{p}) = \frac{2\pi}{2\omega_p} f_B(\vec{p})\ g_i(\vec{p})$$
$$\tilde{g}_{j_\ell}(\vec{p}) = \frac{2\pi}{2\omega_p} f_B(\vec{p})\ g_{j_\ell}(\vec{p}) \qquad \text{(IV.65)}$$

we obtain

$$(\Psi_o, B^*(t, f_i^*)B(t, f_{j_\ell})\Psi_o)_T \to \int \frac{d^3p}{2\omega_p} \tilde{g}_i^*(\vec{p})\ \tilde{g}_{j_\ell}(\vec{p})$$
$$\text{as} \quad t \to \pm\infty. \qquad \text{(IV.66)}$$

When we were estimating $\left\| \frac{d\Phi(t)}{dt} \right\|^2$ one of the important requirement was $(\Psi_o, B(t,f)\Psi_o) = 0$. This is a consequence of the assumption that the "one-body problem" can be solved exactly, i.e. $E_1 B(0)\Psi_o = B(0)\Psi_o$. Let us now consider this problem from a slightly different point of view. Consider

$$\phi(-x) = (2\pi)^{-2} \int e^{-i(qx)}\ \tilde{\phi}(q) d^4\phi \quad \varepsilon\ \mathcal{S} \qquad \text{(IV.67)}$$

and

$$B_\phi = \int B(x)\ \phi(-x) d^4x \quad . \qquad \text{(IV.68)}$$

Then

$$B_\phi \Psi_o = \int d^4x\ \phi(-x)\ U(x,1) \quad B(0)\Psi_o$$
$$= (2\pi)^2 \int \tilde{\phi}(p)\ dE(p)\ B(0)\Psi_o$$
$$= (2\pi)^2\ \tilde{\phi}\ (P)\ B(0)\Psi_o \qquad \text{(IV.69)}$$

where

$$\tilde{\phi}(P) = \int \tilde{\phi}(p)\ dE(p) \qquad \text{(IV.70)}$$

(IV.70) follows from the fact, that the translation operator P has spectral resolution of the form

$$P = \int p \, dE(p) \tag{IV.71}$$

Now if at the same time

$$\operatorname{supp} \tilde{\phi} \cap \operatorname{supp} P = \{ p \mid p^2 = m^2 \} \tag{IV.72}$$

then

$$E_1 B_\phi \Psi_0 = B_\phi \Psi_0 \quad \text{for all } \phi \tag{IV.77}$$

Finally, we shall only mention a few important properties of $\Phi^{out,in}_{f_1 \ldots f_n}$ without giving a detailed proof of them, as they can easily be worked out.

a) $\Phi^{out,in}_{f_1 \ldots f_n}$ is independent of the Lorentz frame used to define it.

b) $\Phi^{out,in}_{f_1 \ldots f_n}$ is independent of the choice of $B(t,f)$. In other words, the collision states are unique. Suppose we choose two different sets of B's, say B and $\hat{B}$, such that the two one-particle states

$$B(t,f)\Psi_0 \quad \text{and} \quad \hat{B}(t,f)\Psi_0 \in \mathcal{H}_1,$$

are equal, and B and $\hat{B}$ are almost local with respect to each other (i.e. $B^{out,in} = \hat{B}^{out,in}$)

then,

$$\lim_{t \to \pm \infty} \| [B(t,f) - \hat{B}(t,f)] \Phi_{f_1 \ldots f_n}(t) \| = 0 \tag{IV.78}$$

To see this we expand (IV.78) in TVEV's and let $t \to \infty$. The only remaining terms are purely quadratic and all contain a factor

$$\left[B(t,f) - \hat{B}(t,f)\right]\Psi_0 = 0 \qquad \text{(IV.79)}$$

for B, $\hat{B}$ both give a one-particle state with the same amplitude.

c) $\Phi^{out,in}_{f_1 \ldots f_n}$ depends only on $f_1 \ldots f_n$, whose support is on the mass shell and on these in a symmetrical way, that is

$$\lim_{t \to \pm\infty} \left\| \Phi_{f_1 \ldots f_n}(t) - \Phi_{f_{i_1} \ldots f_{i_n}}(t) \right\| \to 0; \quad f_n = f_{i_n} \qquad \text{(IV.80)}$$

If we now want to introduce an S-matrix, we have to ensure that

$$\mathcal{H}^{in} = \mathcal{H}^{out} \qquad \text{(IV.81)}$$

with

$$\mathcal{H}^{out,in} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{out,in}_n \quad \text{and}$$

$$\mathcal{H}^{out,in}_n = \overline{L}\left\{ \Phi^{out,in}_{f_1 \ldots f_n} \right\} \qquad \text{(IV.82)}$$

where $\overline{L}\{\ \}$ denotes the closed linear hull of the vectors $\left\{ \Phi^{out,in}_{f_1 \ldots f_n} \right\}$.

One way to do this is to postulate $\mathcal{H}^{in} = \mathcal{H}$.

Then $\mathcal{H}^{in} = \mathcal{H} = \mathcal{H}^{out}$ by TCP invariance.

The postulate $\mathcal{H}^{in} = \mathcal{H}$, called "the completeness of asymptotic states" pose new and difficult problems although immediately implies that S is a unitary operator on $\mathcal{H}$. The asymptotic completeness, too, enables us to prove that $B^{out,in}$ is completely determined by $B\Psi_o$. If we have $\mathcal{H}^{in} = \mathcal{H}^{out}$ only (which is true by requiring TCP) then the S operator is a unitary mapping of $\mathcal{H}^{out}$ onto $\mathcal{H}^{in}$, but is undefined on those vectors of $\mathcal{H}$ which are not in $\mathcal{H}^{out}$.

V. UNITARITY AS A CONDITION FOR ALMOST LOCALITY.

(a) Introduction

The basic concept in any elementary particle theory is that of a free particle. This so-called stable particle concept is usually characterized by giving the mass of the particle, four vector k with $k^2 = m^2$, spin s and internal quantum numbers, α, say.

All these quantities, except mass, are the results of the symmetry of space-time or of the particle. Thus, k and s follow from relativistic invariance, the quantum numbers α from the invariance properties in the internal space of the particle. Furthermore, having several particles one makes the wave packets out of these states so that in the limit where the wave packets are not overlapping (assuming the forces between particles are of a finite range, that is to say, we exclude massless particles from the theory) they can be observed independently. Then, one distinguishes two sets of wave packets - those which are coming together and subsequently interact, and those which are going out signifying that the interaction is supposed to have already taken place and the wave packets are receding. The elements of the S-matrix are then the amplitudes for finding in the "in" state defined by the incoming beam the various "out" components which are defined by the detecting telescope. Thus, the S-matrix is a function of the properties of a number of incoming and a number of outgoing particles. Physically, it is defined in such a way that the absolute square of one of its elements gives the transition probability from an initial to a final state. One important property of the S-matrix which we have only indirectly mentioned before comes from the requirement that the sum of the probabilities for all the final and initial states must be exactly equal to one (probability conservation requirement). This leads to the unitarity of the S-matrix. However, we have already seen that one way of ensuring it is to require that both the final and the initial states form a complete orthonormal set of states. It is then hoped that all experimentally observable quantities can be calculated from matrix elements of S. All that we have said so far is more or less the experimental requirement on the S-matrix. Any physically reasonable quantum field theory usually tries to incorporate the above-mentioned experimental facts. The difficulties which are then met could be overcome by introducing slightly relaxed axioms. For example, as we have already noticed, the space-like asymptotic condition is quite closely connected with almost locality, and is simply understood and interpreted as the cluster decomposition properties of the S-matrix.

Experimentally, we know that a given collection of particles, each initially out of range of the force of all the others, may interact in two or more distinct groups so that the corresponding S-matrix elements split up into the sum of terms.

Since the particle interpretation is still possible, even if the theory is not strictly local (due to Haag and Ruelle), we assume from now on that our fields are almost local in the sense of the definition given in Chapter IVb.

Furthermore, the unitarity of the S-matrix may be assured by requiring asymptotic completeness. Another immediate consequence of asymptotic completeness is that the relativistic transformation law $U(a, \Lambda)$ of an asymptotically complete theory is unitary equivalent to that of the theory of free fields. Therefore it is naturally expected to require that a representation $U(a, \Lambda)$ of the Poincare group is unitary equivalent to the representation $U_0(a, \Lambda)$ in a theory of free particles, instead of asymptotic completeness. Of course, a theory relativistically equivalent to the free fields describing the same particles might not be, by any reason, asymptotically complete, even if it has a collision theory. The unitary equivalence between $U(a, \Lambda)$ and $U_0(a, \Lambda)$ implies that the field has a Haag expansion(9) in terms of a free field, which is complete.

As a result of the equivalence between $U(a,\Lambda)$ and $U_0(a,\Lambda)$ we expect to obtain certain relations very similar to physical unitarity.

Before proceeding further, let us for the sake of completeless, list the usual requirement on the theory we will be using here for spin-zero particles with mass m.

A) For each test function $\phi(x) \varepsilon \mathcal{S}_4$

$B_\phi = \int B(x)\phi(x)d^4x$ is an (unbounded) operator in a Hilbert space $\mathcal{H}$, defined on vectors in $D \subset \mathcal{H}$ and $B_\phi D \subset D$.

B) $B(x)$ transforms

$$U(a,\Lambda)B(x)\,U(a,\Lambda)^{-1} = B(\Lambda x + a)$$

and the spectrum of the energy-momentum operator P_μ is assumed to lie in the forward light cone.

C) $B(x)$ is an "almost local" field.

D) A non-degenerate one-particle state, i.e., a discrete mass state in question is isolated in the mass spectrum.

E) A representation $U(a,\Lambda)$ of the Poincare group is unitary equivalent to the representation $U_0(a,\Lambda)$ in a theory of free particles.

b) <u>Haag expansion.</u>

The theory satisfying axions A) - E) has a particle interpretation given by the Haag-Ruelle collision theory. The first step in the Haag-Ruelle collision theory of "strong convergence" in Hilbert space is the construction of almost local fields B(x). The concept is best understood in terms of the truncated functions, as we have already seen in Chapter IVc.

Then from the property (IV.30)

$$(\Psi_o, B_{\phi}(t,\vec{x}_1)\ldots B_{\phi_n}(t_n\ \vec{x}_n)\Psi_o)_T \ \varepsilon \mathcal{S}_{\vec{\xi}} = \vec{x}_k - \vec{x}_{k+1} \tag{V.1}$$

it follows that the Fourier transform of (V.1) as a distribution in $\mathcal{S}_{\vec{\xi}}$ is an infinitely differentiable function (C^{∞}) which increases no faster than a polynomial at ∞. The set of all C^{∞} functions with all derivatives bounded by polynomials at ∞ is usually denoted by O_M. That is to say

$f \ \varepsilon \ O_M$ if

(a) $f \ \varepsilon \ C^{\infty}$

(b) for given m there exists k_m such that $D^m f/(1+\|x\|^2)^{k_m/2}$ is bounded.

Thus, we have an alternative definition for almost locality which states that $B_{\phi}(t, \vec{x})$ is almost local if

$$W_{\phi}^{T}(t_1\vec{p}_1; \ldots\ldots t_n\vec{p}) \in O_M(\vec{p}_1, \ldots, \vec{p}_n) \qquad (V.3)$$

where $\delta^{(3)}(\sum \vec{p}_i)$. W_{ϕ}^{T} is the Fourier transform of (V.1) in $\vec{p}_1, \ldots, \vec{p}_n$, and $\phi = \phi_1 \otimes \ldots \otimes \phi_n$.

An interesting question which arises is to ask what restrictions on the field give the condition for almost locality for a particular model chosen to ensure that the S-matrix (which follows from the Haag-Ruelle theory) is different from unity. The first work along this line has been done by Streater[26].

The starting point is the Haag expansion of an almost local field $B(x)$ given by

$$B(x) = B^{o}(x) + (2\pi)^{-2} \sum \int \frac{1}{m!n!} F_{mn}(p; q) \exp i(\sum^{m} p_i - \sum^{n} q_j)$$
$$\prod^{m} \Delta^{(+)}(p_i) d^4p_i \prod^{n} \Delta^{(+)}(q_j) d^4q_j \; a^{+}(\vec{p}_m) \ldots a^{+}(\vec{p}_m)$$
$$a(\vec{q}_1) \ldots\ldots a(\vec{q}_n) \qquad (V.4)$$

where

$$F_{oo} = F_{1o} = F_{o1} = 0 \qquad (V.5)$$
$$F_{mn}(p; q) \equiv F_{mn}(p1, \ldots, p_m; q_n)$$
$$B^{o}(x) = (2\pi)^{-2} \int d^4p \Delta^{+}(p) \left\{ a^{+}(\vec{p}) \exp[i(px)] + a(\vec{p}) \exp[-i(px)] \right\}$$
$$\Delta^{+}(p) = \theta(p^o)\delta(p^2 - m^2) = (2\omega_p)^{-1}\delta(p^o - \omega_p)$$
$$\omega_p = (\vec{p}^2 + m^2)^{1/2} .$$

In (V.4) $a^{+}(\vec{p})$ and $a(\vec{q})$ are the creation and annihilation operators respectively for free spinless boson particles satisfying the following commutation relations:

$$[a(\vec{q}),\ a^+(\vec{p})] = 2\omega_p\, \delta^{(3)}(\vec{p} - \vec{q})$$
$$[a(\vec{q}),\ a(\vec{q}\,')] = [a^+(\vec{p}),\ a^+(\vec{p}\,')] = 0 \quad . \qquad \text{(V.6)}$$

The field $B^o(x)$ is a given free field. It contains two parts, $B^o_+(x)$ and $B^o_-(x)$, corresponding to the creation and annihilation operators respectively. This notion enables us to write

$$B^o(x) = B^o_+(x) + B^o_-(x)$$

with an obvious identification.

One could have started the expansion (V.4) either with $B^{in}(x)$ or $B^{out}(x)$ according to which of them is assumed to exist. Then, our theory would be asymptotically complete, at least for either $t \to -\infty$ or $t \to +\infty$. In a local theory the completeness of $B(x)$ for $t \to -\infty$ say, implies the completeness of $B(x)$ for $t \to +\infty$ by the PCT theorem. Although $B^{in}(x)$ and $B^{out}(x)$ are also free fields, they are a priori not in any way related to $B^o(x)$. The Haag expansion introduced here for an almost local field $B(x)$ starting out with $B^o(x)$ does not involve any loss of generality since we assume that $B(x)$ can create the one particle state with one application to the vacuum. We see that Haag expansion (V.4) supplies us with an infinite set of, in general, unknown generalized functions F_{mn} which determine an interpolating almost local field $B(x)$. In local field theory these functions may be related to S-matrix elements.

Furthermore, it is clear from the expansion (V.4) that only mass shell values $(p_i^2 = m^2, \; q_j^2 = m^2)$ of F_{mn}'s enter into the definition of the field B(x). Off the mass shell these functions can be chosen in arbitrary manner. In other words, if B(x) is given, the functions $F_{mn}(p\,;\,q)$ are not uniquely determined. Apart from that it is possible to have many different fields giving rise to the same S-matrix, as a reflection of the arbitrariness which exists in the extrapolation of F_{mn} off the mass shell. It is important to mention that in a complete local field theory F_{mn}'s have analyticity properties which come from locality, spectrum and Lorentz invariance. Since our theory is not local, the equivalent analyticity properties of F_{mn} are not known. The property that might be used here is the invariance of B(x) under arbitrary inhomogeneous Lorentz transformation (property B) (Chapter Va). This implies that

$$F_{mn}(p_1, \ldots, p_m; \; q_1, \ldots, q_n) = F_{mn}(\Lambda p_1, \ldots, \Lambda p_m; \Lambda q_1, \ldots \Lambda q_n) \tag{V.7}$$

To prove (V.7) one assumes that the functions $F_{mn}(p; q)$ are sufficiently well behaved so that they have Fourier transforms: $\widetilde{F}_{mn}(x; y)$ defined by

$$\tilde{F}_{mn}(x_1,\ldots,x_m; y_1,\ldots,y_n) = (2\pi)^{-2(m+n)}\int \prod^m d^4p_i \prod^n d^4q_j$$

$$F_{mn}(p_1,\ldots,p_m;q_1,\ldots,q_n)\exp i(\sum^m p_ix_i - \sum^n q_jy_j)$$

(V.7)

Then, expansion for B(x) may be put in the following form

$$B(x) = B^0(x) + (2\pi)^{-2}\sum \frac{1}{m!n!} \tilde{F}_{mn}(x-x_1,\ldots,x-x_m;x-y_1,\ldots,x-y_n)$$

$$\prod^m d^4x_i \prod^n d^4y_j\ B^0_+(x_1)\ldots B^0_+(x_m)B^0_-(y_1)\ldots B^0_-(y_n)$$

(V.8)

that is suitable for the examination of the Lorentz invariance of $F_{mn}(p,q)$.

We shall consider only models with a finite number of terms in the Haag expansion. This can be avhieved by smearing the field B(x) with an appropriate test function whose Fourier transform is zero outside a certain region Δ in the momentum space.

Let Δ_f be a small region in the momentum space including mass shell $p^2 = m^2$ (Fig. 1), and let $\mathcal{S}(\Delta_f)$ be the space of test functions $\tilde{f} \in \mathcal{S}_4$ with supp $\tilde{f} \subset \Delta_f$.

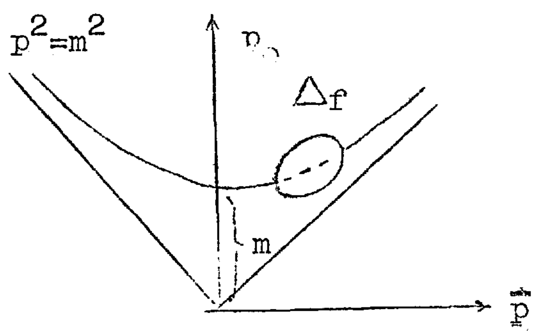

Figure 1.

Then we consider for any $\tilde{f} \varepsilon \mathcal{S}_4(\Delta_f)$ the well-defined operator

$$B(t; f) = \int d^4x \, B(x)f(x,t) \tag{V.9}$$

where

$$f(x,t) = (2\pi)^{-2} \int d^4p \, \tilde{f}(p) e^{i(p^0-\omega_p)t} \, e^{-ipx} \tag{V.10}$$

Defining the Fourier transform of B(x) to be

$$B(x) = (2\pi)^{-2} \int e^{ipx} \, \tilde{B}(p) d^4p \tag{V.11}$$

the relation (V.9) may be written in the following form as well

$$B(t; f) = \int \tilde{B}(p) \, \tilde{f}(p) e^{i(p^0-\omega_p)t} \, d^4p \tag{V.12}$$

It is clear from (V.4) that $B(t,f)\Psi_0$ is a one-particle state $\Phi_{\hat{f}} = |\hat{f}\rangle \in \mathcal{H}^t_m$ with a wave function $\hat{f}(\vec{p}) = \tilde{f}(\omega_p, \vec{p}) \varepsilon \mathcal{S}_3$. To see this we apply the vacuum Ψ_0 to (V.12) and by using the expansion (V.4) we arrive at

$$B(t;f)\Psi_0 = \int d^4p e^{i(p^0-\omega_p)t} \, \tilde{f}(p) \, \tilde{B}^0(p)\Psi_0 \tag{V.13}$$

All other terms are equal to zero since the support of $\tilde{f}(p)$ is concentrated around the mass-shell $p^2 = m^2$ (Fig. 1). By this we mean that the smaller the support of $\tilde{f}$, the more correct the equation (V.13) is. In V.13)
$\tilde{B}^0(p) = \Delta^{(+)}(p) \, a^+(\vec{p}) = (2\omega_p)^{-1} \, \delta(p^0 - \omega_p) a^+(\vec{p})$
so that it follows

$$B(t;f)\Psi_o = a^+(\hat{f})\Psi_o = \int \frac{d^3p}{2\omega_p} \tilde{f}(\omega_p,\vec{p})\Phi_{\vec{p}} \qquad (V.14)$$

with

$$a^+(\vec{p})\Psi_o \equiv \Phi_{\vec{p}} \quad .$$

We should also point out that any other function $\tilde{f}'$, say, with smaller support which coincides with $\tilde{f}$ in the neighbourhood (or even only on) the mass-shell will give rise to the same state (V.14). Thus although B(x) has to be almost local for all $\tilde{f}$, we get a consistent scattering theory provided only that B(x) is almost local for a sequence of $\tilde{f}$ coinciding on the mass-shell and with smaller and smaller supports, i.e. for the germ of $\tilde{f}(\omega_p, \vec{p})$. The advantage gained by choosing the $\tilde{f}$'s with small support is that their product of the fields have then nearly the same maximum energy as the corresponding asymptotic state, and only a finite number of terms enter in the product, of course, if energy is finite.

Using (V.13) one can construct asymptotic states. Haag and Ruelle (Chapter IV) have shown that for $\tilde{f}_i \in \mathcal{S}(\Delta_{f_i})$ and B(x) almost local

$$\lim_{t \to \pm\infty} \prod_{i=1}^{n} B(t;f_i)\Psi_o \qquad (V.15)$$

exists in the strong sense, i.e., in the norm. Now we realise that the requirement that B(x) is almost local will depend heavily upon the chosen functions F_{mn}. It is the purpose of this chapter to find out what these conditions on F_{mn} are to make the field $B(x)$ almost local.

Since we are dealing with the asymptotic limits of the type (V.15) thus defining an "in" and "out" state, we must choose F_{mn}'s so that the "in" and "out" states differ. For otherwise, the S-matrix will turn out to be unity. To illustrate this statement we recall that by using test functions $\tilde{f}(p)$ which are zero outside a small neighbourhood Δ_f of the mass hyperboloid $p^2 = m^2$ we can have many different fields giving rise to the same S-matrix. According to the general theory(5) (Chapter IVc; equation (IV.78)), a creation operator $B(t,f)$ leads to the same S-matrix as a creation operator $\hat{B}(t,f)$ if

$$\lim_{t \to \pm\infty} \| [B(t,f) - \hat{B}(t,f)] \Phi_{f_1 \ldots f_n}(t) \| = 0 \qquad \text{(IV.78)}$$

where B and $\hat{B}$ are almost local with respect to each other, and the two one-particle states $B(t,f)\Psi_0$ and $\hat{B}(t,f)\Psi_0$ are equal. Then, it follows that (IV.78) holds provided that the following norm of the state

$$[B(x) - \hat{B}(x)] \, B(x_1) \ldots B(x_n)\Psi_0 \qquad \text{(V.16)}$$

is rapidly decreasing in space-like directions.

We can also express it in a different way by saying that the Fourier transform of (V.15) is ∞-differentiable (C^∞) as a function of the spacial $\vec{p}$. Since our $B(x)$ is given as an infinite series in terms of the free fields

$B^o(x)$ we can assure the Fourier transform of (V.16) to be C^∞ by choosing F_{mn}'s to be C^∞ themselves. However, this would lead to $S = 1$. In order to get scattering we must choose F_{mn}'s with "singularities" which cancel in the truncated function but not in the matrix element (V.16), or similar ones.

c) Unitarity.

Consider an almost local field with the expansion (V.4) and choose test functions $\tilde{f}_1 \varepsilon \mathcal{S}_4(\Delta_{f_1})$ and $\tilde{f}_2 \varepsilon \mathcal{S}_4(\Delta_{f_2})$ such that the largest momentum component of the state

$$\Phi_{f_1 f_2}(t) = B(t; f_1)B(t; f_2)\Psi_o \qquad \text{(V.17)}$$

is less than the threshold for n+1 particle production. We assume that domains Δ_{f_1} and Δ_{f_2} contain the mass shell, i.e.,

$$\Delta_{f_1(f_2)} \cap \bar{V}_m^+ \neq \emptyset \qquad \text{(V.18)}$$

where

$$\bar{V}_m^+ \equiv \{p; p_o > 0, \quad p^2 = m^2\} \qquad \text{(V.19)}$$

and the bar on V_m^+ means the closure of V_m^+. We shall also consider the domains $\Delta_{f_1}, \Delta_{f_2}$ to be mutually disjoint

$$\Delta_{f_1} \cap \Delta_{f_2} = \emptyset \qquad \text{(V.20)}$$

Since $B(t; f_1)\Psi_o$ is the one-particle state (by assumption) the only terms in the expansion of $B(t; f_1)$ that will contribute to $\Phi_{f_1 f_2}(t)$ have no more than one annihilation operator in them.

For that reason $B(t; f)$ may be considered to have the following form

$$B(t; f) = B^o(t; f) + \sum_{\alpha=1}^{n-1} B_\alpha(t; f) \qquad (V.21)$$

where

$$B_\alpha(t; f) = \frac{(2\pi)^{-2}}{(\alpha+1)!} \int F_{\alpha+1}(p_1 \ldots p_{\alpha+1}; q)\, \Delta^{(+)}(q) d^4q$$

$$\prod_{j=1}^{\alpha+1} \Delta^{(+)}(p_j) d^4p_j \exp\left[i\left(\sum^{\alpha+1} \omega_{p_k} - \omega_q - \omega_{\sum p_k - q}\right)t\right]$$

$$\tilde{f}\left(\sum^{\alpha+1} p_k - q\right) a^+(\vec{p}_1) \ldots a^+(\vec{p}_{\alpha+1}) a(\vec{q}) \qquad (V.22)$$

The other ignored terms in the expansion of $B(t; f)$ will not effectively contribute to a state $\Phi_{f_1 f_2}(t)$.

We now construct the following 4-point truncated function

$$W_T(t_1, \ldots, t_4) = (\Psi_o, \overset{*}{B}(t_1; \overset{*}{f_1})\, \overset{*}{B}(t_2; \overset{*}{f_2}) B(t_3; f_3) B(t_4; f_4)\Psi_o)$$

$$= W(t_1, \ldots, t_4) - W(t_1 t_3) W(t_2 t_4) - W(t_1 t_4) W(t_2 t_3)$$

$$(V.23)$$

Substituting (V.21) into (V.23) we arrive at the equation[27]

$$W_T(t_1\ldots t_4) = (\Psi_o, B^o(t_1f_1)^* B_1^o(t_2f_2)^* B^o(t_3f_3)B^o(t_4f_4)\Psi_o)$$
$$+ (\Psi_o, B^o(t_1f_1)^* B^o(t_2f_2)^* B_1(t_3f_3)B^o(t_4f_4)\Psi_o)$$
$$+ \sum_{\alpha=1}^{n-1}(\Psi_o, B^o(t_1f_1)^* B_\alpha(t_2f_2)^* B_\alpha(t_3f_3)B^o(t_4f_4)\Psi_o) \qquad (V.24)$$

Before going further, let us define the Fourier transform of the truncated function

$$W_T(x_1\ldots x_4) = (\Psi_o, B(x_1)^* B(x_2)^* B(x_3)B(x_4)\Psi_o)_T \qquad (V.25)$$

which we write as

$$W_T(x_1\ldots x_4) = (2\pi)^{-8}\int \prod_{i=1}^{4} d^4p_i\, \delta^{(4)}(p_1+p_2 - p_3-p_4)\hat{W}_T(p_1\ldots p_4)$$
$$\exp\left[-i(p_1x_1 + p_2x_2 - p_3x_3 - p_4x_4)\right] \qquad (V.26)$$

The B(x) is an almost local field (by the assumption C) so that the truncated function (V.26) in x-space is strongly decreasing when the distance between the points with increasing separation in space-like direction is large. Then the Fourier transform $\tilde{W}_T(p_1,\ldots,p_4)$ of (V.26) is of the form $\delta^{(4)}(\Sigma p)\hat{W}_T(p_1\ldots p_4)$ with $\hat{W}_T$ being C^∞ and at most of a polynomial increase in $\vec{p}_2,\ldots,\vec{p}_4$, when integrated over $p_2^o,\ldots,p_4^o$ with the test function from $\mathcal{S}_4$.

Therefore it follows that (V.26) becomes

$$(\Psi_o, B(t_1f_1)^* B(t_2f_2)^* B(t_3f_3) B(t_4f_4) \Psi_o)_T =$$

$$\int \prod_{i=1}^{4} d^3p_i \; \exp\left[i(\omega_{p_1} t_1 + \omega_{p_2} t_2 - \omega_{p_3} t_3 - \omega_{p_4} t_4)\right]$$

$$\delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \; W_T(t_1\vec{p}_1, \ldots, t_4\vec{p}_4) \qquad (V.27)$$

where

$$W_T(t_1,\vec{p}_1, \ldots, t_4\vec{p}_4) = \int \prod_{i=1}^{4} dp_i^o \; \delta(p_1^o + p_2^o - p_3^o - p_4^o) \; \tilde{f}^*(p_1)\tilde{f}^*(p_2)$$

$$\cdot \tilde{f}(p_3)\tilde{f}(p_4) \; \exp\left[-i(p_1^o t_1 + p_2^o t_2 - p_3^o t_3 - p_4^o t_4)\right]$$

$$\cdot \; \widetilde{W}_T(p_1, \ldots, p_4) \qquad (V.28)$$

is then the required C^∞ function which behaves at worst like a polynomial at ∞. In (V.28) $\vec{p}_i$'s always satisfy the condition $\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4$. that comes from (V.27).

As it was already pointed out, in order to avoid $S = 1$, we ought to assume certain singularities for F_α's in the neighbourhood of the mass shell. These singularities must be not only of such a kind that $S \neq 1$ but also such that the C^∞ condition for (V.28) holds. In other words, they have somehow to cancel themselves in (V.28). The condition that the assumed singularities for F_α's should cancel in (V.28) will supply us with a certain relation which should be satisfied between F_α's in order to preserve the C^∞ property of (V.28) in $\vec{p}_2, \ldots, \vec{p}_4$, for fixed times. What kind of singularities shall we assume for F_α's to make $S \neq 1$? Everybody will agree that the above question is completely open and there is no simple way of deciding which chosen singularity for F_α is the best one.

As is well known from perturbation theory and local field theory with $B^o(x) = B^{in}(x)$, the F_α's have the retarded singularity $[(p^o + i\varepsilon)^2 - \omega_p^2]^{-1}$. Perhaps we would not make a mistake to assume for F α's δ-singularities of the type $\delta(p^2 - m^2)$ or even more principle value singularities like $P \frac{1}{p^2-m^2}$.

In the local field theory the above mentioned singularities are expected to be contained in the vacuum expectation value of the products of local field operators(28).

Recently, Hepp(29) has proved that also in the framework of the Haag-Ruelle collision theory 1-particle singularities exist in the physical region of any connected scattering amplitude. They occur with the causal propagator $(p^2 - m^2 + i\varepsilon)^{-1}$ in the dominant term and have a residue, which factors into the product of two connected amplitudes for subprocesses. Furthermore the remainder of the amplitude is infinitely often differentiable in the critical variable. Using the graphical language our F_α's will represent only the connected irreducible graphs.

It is clear that F_α's contain the whole complications of dynamics which $B(x)$ has. Since we expect that F_α is related to the $(\alpha+2)$-leg scattering function with $(\alpha+1)$-legs on the mass shell and one off, a suitable $(p^2 - m^2)$ factor when applied to F_α and then restricted to the mass shell will turn $(p^2 - m^2)F_\alpha$ to the connected part of the S-matrix for the process involving $\alpha+2$ particles.

Thus, we assume the following retarded singularities as possible ones for F_α's

$$\left[(p^0 + i\varepsilon)^2 - \vec{p}^2 - m^2\right] F_\alpha(p_1, \dots, p_\alpha; q) = T_\alpha(p_1, \dots, p_\alpha; q) \tag{V.29}$$

where

$$p = \sum_{i=1}^{\alpha} p_i - q .$$

With this choice of singularities for F_α (V.28) has the following form

$$W_T(t_1\vec{p}_1, \dots, t_4\vec{p}_4) = (W_1+W_2+W_3)(t_1\vec{p}_1, \dots, t_4\vec{p}_4)$$

$$\begin{aligned} W_1(\dots) = {} & f_1^*(\omega_1, \vec{p}_1) f_2^*(-\omega_1+\omega_3+\omega_4, \vec{p}_2) f_3(\omega_3, \vec{p}_3) f_4(\omega_4, \vec{p}_4) \\ & \cdot \exp i\left[-\omega_1 t_1 - (-\omega_1+\omega_3+\omega_4)t_2 + \omega_3 t_3 + \omega_4 t_4\right] \\ & \cdot (8\omega_1\omega_3\omega_4)^{-1} T_2^*(p_3, p_4; p_1)\left[(\omega_3+\omega_4-\omega_1-i\varepsilon)^2 - \omega_2^2\right]^{-1} \end{aligned} \tag{V.31}$$

$$\begin{aligned} W_2(\dots) = {} & f_1^*(\omega_1, \vec{p}_1) f_2^*(\omega_2, \vec{p}_2) f_3(\omega_1+\omega_2-\omega_4, \vec{p}_3) f_4(\omega_4, \vec{p}_4) \\ & \cdot \exp i\left[-\omega_1 t_1 - \omega_2 t_2 + (\omega_1+\omega_2-\omega_4)t_3 + \omega_4 t_4\right] \\ & \cdot (8\omega_1\omega_2\omega_4)^{-1} T_2(p_1, p_2; p_4)\left[(\omega_1+\omega_2-\omega_4+i\varepsilon)^2 - \omega_3^2\right]^{-1} \end{aligned}$$

$$\begin{aligned} W_3(\dots) = {} & f_1^*(\omega_1, \vec{p}_1) f_4(\omega_4, \vec{p}_4) \exp i\left[-\omega_1 t_1 + (\omega_1-\omega_4)t_3 + \omega_4 t_4\right] \\ & (4\omega_1\omega_4)^{-1} \int \frac{A(p_2^0) f_2^*(p_2^0, \vec{p}_2) f_3(p_2^0+\omega_1-\omega_4, \vec{p}_3) \exp i(t_3-t_2)p_2^0}{\left[(p_2^0-i\varepsilon)^2 - \omega_2^2\right]\left[(p_2^0+\omega_1-\omega_4+i\varepsilon)^2 - \omega_3^2\right]} dp_2^0 \end{aligned} \tag{V.32}$$

with

$$A(p_2^0) = \sum_{\alpha=1}^{n=1} \int d^4q_1 \cdots d^4q_{\alpha+1} \prod_{i=1}^{\alpha+1} \Delta^+(q_i)\, \delta^{(4)}(p_1+p_2 - \sum_{i=1}^{\alpha+1} q_i)$$

$$\cdot\, T^*_{\alpha+1}(q_1,\ldots,q_{\alpha+1};p_1)\, T_{\alpha+1}(q_1,\ldots,q_{\alpha+1};p_4) \qquad (V.33)$$

and $\omega_i \equiv \omega_{p_i}$.

Let us consider the integral over p_2^0 in W_3. Simple examination shows that the contour of p_2^0 integration is pinched with two coincident zeros of the denominator which are

$$p_2^0 - \omega_2 - i\varepsilon = 0$$

$$p_2^0 + \omega_1 - \omega_3 - \omega_4 + i\varepsilon = 0$$

when $\omega_1 + \omega_2 = \omega_3 + \omega_4$ and limit $\varepsilon \longrightarrow 0$ is performed. If we assume the integrand in W_3 to be an analytic function then the well-known pinch analysis[(30)] can be applied. This analysis consists essentially in replacing the pinched p_2^0-integration contour by the two contours. One of them encircles one of the singularities and another is taken away from the pinch, so that it is regular there. If the integrand is not analytic in p_2^0, but is differentiable we can use the standard function analysis instead of the pinch analysis (Appendix I). There is also another difficulty which may occur when the end points of p_2^0 integration coincide with different threshold branch points of $A(p_2^0)$.

The end points of p_2^0 integration depend upon the size and position of the domains Δ_{f_2} and Δ_{f_3} in the momentum space as is seen from the form of W_3. Let us for the moment forget end point singularities and consider which condition should hold in order to cancel a pole arising from the pinch.

It is now quite easy to see the following equation should hold

$$T_2^*(p_3,p_4;p_1) - T_2(p_1,p_2;p_4) =$$

$$2\pi i \sum_{\alpha=2}^{n} \int d^4q_1\cdots d^4q_\alpha \prod_{i=1}^{\alpha} \Delta^+(q_i)\delta^{(4)}(p_1+p_2-\sum_{i=1}^{\alpha} q_i)$$

$$\cdot\, T_\alpha^*(q_1,\ldots q_\alpha;\, p_1)T_\alpha(q_1,\ldots,q_\alpha;\, p_4) \qquad (V.34)$$

for $p_1 + p_2 = p_3 + p_4$ and $p_1^2 = m^2$ if we want to cancel the pole singularity coming from the pinch of the p_2^0 integration contour in W_3. Solutions to this equation may have several branch points corresponding to the different thresholds for the production of two, three,... etc. particles. It is known and it has been proved(10) that from the analyticity and unitarity hypothases for a general transition amplitude, it follows the conclusion that the general transition amplitude has no singularities on the positive real energy axis other than isolated singularities at physical thresholds. Now we realise the role that end point singularities of the integral over p_2^0 may play here.

They must in fact cancel with the threshold branch points which can appear according to the position of the domains $\triangle_{f_i}$ in the momentum space. This is because at the branch points the functions in question are usually not differentiable in the momentum space $\vec{p}_i$. To show this cancellation explicitly one must find particular models for T_α's or at least their analytic behaviour near each threshold singularity. In the next Chapter we shall show that it is really possible to find such a model, at least in the elastic region, i.e., below three particle production. One model has already been found by Streater(26) but it seems too restrictive as it eliminates both threshold and end-point singularities.

However, this model does show that one can have a form of "macroscopic causality", namely, almost locality, in a theory whose S-matrix is not the boundary value of an analytic function. One gets more physical models by assuming that $A(p_2^0)$ has analytic properties, and this is done in the next chapter.

VI. MODEL

The method and the result of the previous Chapter will be applied here only in the elastic region (below the three-particle production). The aim is to find a relatively simple model which will explicitly show how the end point singularities in W_3 may be cancelled with the threshold behaviour in W_1 and W_2. Having this in mind we restrict the total energy of the state to be below the three-particle threshold. In other words, if

$p_1 \varepsilon \Delta_{f_1}$ and $p_2 \varepsilon \Delta_{f_2}$ the following inequality holds

$$(p_1 + p_2)^2 < 9m^2 \qquad \text{(VI.1)}$$

This restriction reduces the number of terms in $A(p_2^o)$ to one, which is

$$A(p_2^o) = \int d^4q_1 d^4q_2 \, \Delta^+(q_1)\Delta^+(q_2)\delta^{(4)}(p_1+p_2-q_1-q_2) \cdot T_2^*(q_1, q_2; p_1)T_2(q_1, q_2; p_4) \qquad \text{(VI.2)}$$

In order to put (IV.2) in a more transparent form we shall work in a particular Lorentz frame (since T's are Lorentz invariant functions) where $\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 = 0$. After simple and straight forward manipulation we obtain (VI.2) in the form

$$A(p_2^0) = \frac{\left[(p_2^0+\omega_1)^2 - 4m^2\right]^{1/2}}{8(p_2^0 + \omega_1)} \int d^2\hat{q}\left[T_2(\vec{q},-\vec{q};p_1)T_2(\vec{q},-\vec{q};p_4)\right]_{q^2=\lambda}$$

(VI.3)

where $\lambda = (1/4)\left[(p_2^0+\omega_1)^2 - 4m^2\right]$ and $p_1^2 = p_4^2 = m^2$, where the remaining integral is over angular variables only. It is clear from (VI.3) that $A(p_2^0)$ has a branch point at $(p_2^0+\omega_1)^2 = 4m^2$. Using restriction (VI.1), we find easily that the lower limit of p_2^0 integration in W_3 is m, since $p_2^0 > 0$, $p_2^0 + \omega_1 - \omega_4 > 0$ and $m \leq \omega_i < 2m$. Therefore, there is a possibility[26] that the lower end point of p_2^0 integration coincides with the branch point of (VI.3) when $\omega_1 = m$. According to the well-known prescription for dealing with end point singularities[30] we should assume that we are far enough away from the pinch so that $\omega_1 + \omega_2 \neq \omega_3 + \omega_4$. In that case the integrand in W_3 has two poles at $p_2^0 = \omega_2$ and $p_2^0 = \omega_3 + \omega_4 - \omega_1$.

Let us now take T_2 to have a square root branch point. We can exhibit this property completely by writing

$$T_2(p_1,p_2;p_4) = d(p_1,p_2;p_4) - i\frac{\left[(\omega_1+\omega_2)^2 - 4m^2\right]^{1/2}}{\omega_1 + \omega_2}\, a(p_1,p_2;p_4)$$

($p_i^2 = m^2$ is always assumed) (VI.4)

where d and a have no branch point at $(\omega_1 + \omega_2)^2 = 4m^2$ and are furthermore C^{∞} in the considered elastic region.

Taking the residue of the pole at $p_2^0 = \omega_2$, say, we find that the condition for cancelling threshold square root branch point is

$$a(p_1,p_2;p_4) = (\pi/4)\int d^2\hat{q}\left[T_2(\vec{q},-\vec{q};p_1)T_2(\vec{q},-\vec{q};p_4)\right]_{q=0} \qquad (IV.5)$$

at the threshold.

Thus, the unitarity relation and the model (VI.4) with (VI.5) are enough to show that T_2 may be interpreted as the usual scattering amplitude at least in the elastic region.

Finally, we should also mention that this model has been invented and used before by Oehme(31) although for completely different reasons.

VII. THE BOUND STATE PROBLEM

a) Introduction

In the conventional formulation of quantum field theory each of the "elementary" particles is described by a basic field operator whereas the composite particles appear as the stable bound states of the system. The distinction between what to call elementary and what composite particles has not yet been cleared up. For example, considering strong interactions only, it has recently been suggested[32] that perhaps there are no "elementary" particles; all baryons and mesons being bound states of one another. It is also very well known that most of the so-called elementary particles are unstable or composite ones. In practice it would be almost impossible to find directly from experiment whether or not a given particle was elementary. Thus, the definition of a composite particle usually depends on the formalism or on the model used to describe it.[33] On the other hand, however, we must be aware that there is no generally satisfactory theory for treating the scattering of composite systems. The reasons seems to be in the complexity of the many-particle system in dynamics.

For the description of composite particles in field theory it is important to define a field B corresponding to a composite particle which satisfies the required asymptotic condition, in the sense that B^{in} and B^{out}, which are the limits of B when $t \rightarrow \mp \infty$, do exist.

In the local field theory the treatment of the bound state problem has been carried out by Zimmermann[34] in the spirit of the LSZ[3] formalism. Zimmermann's result may be briefly stated, that the bound state can be described by a local and invariant field operator, and the S-matrix derived using the LSZ reduction technique. According to a rather similar situation in almost local field theory, where the LSZ weak asymptotic condition is replaced by the Haag strong asymptotic condition, there is no reason not to believe that it is also possible to derive the same results here as Zimmermann has found in the local field theory.

The advantage which our model gains is that there are no difficulties with reduction formulae, since the Haag expansion, in terms of free fields, for an almost local field has been assumed to exist.

b) The Bound State Problem

We consider a model where an almost local field $A(x)$ describes just two kinds of particles, an elementary one of mass m_A and a composite one of mass m_B, both of spin zero. Note that $A(x)$ is the same almost local field which has been denoted by $B(x)$ in previous chapters. Being more precise we assume the existence of a stable two particle bound state with mass m_B which satisfies the following inequality

$$m_A < m_B < 2m_A \qquad \text{(VII.1)}$$

The so-called fundamental almost local field $A(x)$ belongs to the mass m_A. This means that

$$A(h)\Psi_o = \int d^4x \, h(x) \, A(x)\Psi_o = \Phi_A \qquad \text{(VII.2)}$$

is "A" one-particle state with h being a test function with finite support in the momentum space which, of course contains mass shell $p^2 = m_A^2$. Further, we require that

$$A(h^f) \, A(h^g)\Psi_o = B(h)\Psi_o = \Phi_B \; ; \quad h = h^f \otimes h^g \qquad \text{(VII.3)}$$

is a "B" one-particle state. The inequality (VII.1) and the condition (VII.2) tell us that the test function h^f must have its support in an unphysical region in the momentum space, i.e. below the mass shell $p^2 = m_A^2$ (Fig. 2) if

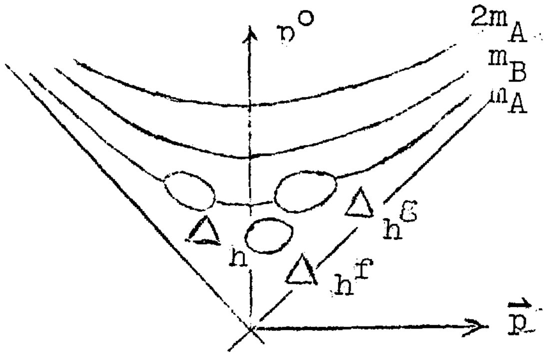

Figure 2.

we want to satisfy (VII.3). The problem which we wish to analyse in the spirit of the almost local field theory is the AB scattering. The vectors that we are interested in are

$$\begin{aligned} B(h_1)A(h_2)\Psi_0 &= |BA\rangle \\ A(h_2)B(h_1)\Psi_0 &= |AB\rangle \end{aligned} \qquad \text{(VII.4)}$$

The above problem cannot be considered in its full generality as we have not yet derived the condition that the 6-point function $W(x_1,\ldots,x_6)$ is almost local in a three particle scattering region.

Therefore we shall discuss only the elastic region. For that reason it is required that domains Δ_h, Δ_h^f, Δ_h^g, outside which the corresponding test functions vanish in momentum space, satisfy the following relations

a) Δ_h, Δ_h^f and Δ_h^g are mutually disjoint (as in Fig. 2)

b) Δ_h and Δ_h^g contain part of the mass shell $p^2 = m_A^2$ but Δ_h^f does not (Fig. 2).

c) $m_A^2 < (p_f + p_g)^2 < 4m_A^2$

d) $(p_h + p_f + p_g)^2 < 9m_A^2$

where

$p_f \in \Delta_h^f$, $p_g \in \Delta_h^g$ and $p_h \in \Delta_h$.

e) It is always possible to find a vector $p_f \varepsilon \Delta_h f$ and a vector $p_g \varepsilon \Delta_g h$ such that

$$(p_f + p_g)^2 = m_B^2 \quad .$$

According to the Haag-Ruelle collision theory if $A(t;h_1)$ and $B(t;h_2)$ are almost local fields which create corresponding one-particle states from vacuum Ψ_o then

$$\| (A(t;h_1)B(t;h_2) - B(t;h_2)A(t;h_1))\Psi_o \| \rightarrow 0 \quad \text{as } t \rightarrow \infty \tag{VII.5}$$

where $A(t;h_1) = i\int A(x) \overleftrightarrow{\partial}_o h_1(x) d^3x$ and similarly $B(t;h_2)$. Thus, the relation (VII.5) shows that there are in fact only three essentially different Wightman functions $\langle B^* A^* AB \rangle$, $\langle A^* B^* BA \rangle$ and $\langle A^* B^* AB \rangle$ in our problem. Their truncated functions will fall off rapidly at large space like distances.

Consider the fundamental field $A(x)$ satisfying (VII.3). Then the Haag expansion for $A(x)$ must be slightly modified by introducing creation and annihilation operators for the F-state. Having this in mind and the domain restrictions a)...e) we may consider $A(x)$ to have the following expansion

$$A(x) = A^o(x) + \sum_{i=1}^{4} A_i(x) \tag{VII.6}$$

where

$$A_1(x) = (2\pi)^{-2}\tfrac{1}{2}\int T(p_1,p_2;p_3)e^{i(p_1+p_2-p_3)x}\prod_{i=1}^{3}\Delta_A^+(p_i)d^4p_i$$
$$\cdot a^+(\vec{p}_1)a^+(\vec{p}_2)a(\vec{p}_3)$$

$$A_2(x) = (2\pi)^{-2}\int V(p_1;p_2)e^{i(p_1-p_2)x}\Delta_B^+(p_1)\Delta_A^+(p_2)d^4p_1d^4p_2$$
$$\cdot b^+(\vec{p}_1)a(\vec{p}_2)$$

$$A_3(x) = (2\pi)^{-2}\int M(p_1,p_2;p_3)e^{i(p_1+p_2-p_3)x}\Delta_A^+(p_1)\Delta_B^+(p_2)$$
$$\Delta_B^+(p_3)d^4p_1d^4p_2d^4p_3a^+(\vec{p}_1)b^+(\vec{p}_2)b(\vec{p}_3)$$

$$A_4(x) = (2\pi)^{-2}\tfrac{1}{2}\int G(p_1,p_2,p_3;p_4)e^{i(p_1+p_2-p_3-p_4)x}\Delta_B^+(p_1)d^4p_1$$
$$\prod_{i=2}^{4}\Delta_A^+(p_i)d^4p_ib^+(\vec{p}_1)a^+(\vec{p}_2)a(\vec{p}_3)a(\vec{p}_4)$$

(VII.7)

with $\Delta_{A\,(B)}^+ = \theta(p)\,\delta(p^2 + m_{A(B)}^2)$ and $\overset{o}{A}(x)$ equals the free field introduced in Chapter V.b. In (VII.7) b^+ and b are respectively creation and annihilation operators for the B-state, with the following commutation relations

$$\left[b(\vec{p}),\ b^+(\vec{q})\right] = 2\omega_p^B\,\delta^{(3)}(\vec{p}-\vec{q})$$
$$\left[b(\vec{p}),\ b(\vec{p}\,')\right] = \left[b^+(\vec{q}),\ b^+(\vec{q}\,')\right] = 0 \qquad \text{(VII.8)}$$

where $\omega_p^B = (\vec{p}^{\,2} + m_B^2)^{\frac{1}{2}}$.

As before we have ignored the terms in the Haag expansion that do not contribute to $|AB\rangle$ or $|BA\rangle$. By comparing the terms in the above expression for $A(x)$ with corresponding ones in the local field theory we arrive at the conclusion that $\widetilde{T} = (p^2 - m_A^2)T$ may be regarded as the $AA \rightarrow AA$ scattering amplitude and similarly $\widetilde{V}$ as $B \rightarrow 2A$ decay amplitude (or simply the vertex function). $\widetilde{M}$ as $AB \rightarrow AB$ sattering amplitude and $\widetilde{G}$ as $AB \rightarrow 3A$ production amplitude. The validity of this conclusion is not yet beyond doubt in the case of G for the same reason as before in the three-particle region. In our case G will be defined only in the unphysical region.

Before going into final considerations of the almost locality condition, let us estimate which of the functions T, V, M and G will contribute to $AB\Psi_0$ and $BA\Psi_0$ vectors. It is then quite easy to check that the following terms are only relevant ones in the expansion of A and B, for the vectors $|AB\rangle$ and $|BA\rangle$: a^+, (a^+b^+b) in the expansion for A and b^+, (b^+a^+a) in the expansion for B. The corresponding coefficients are

$$f_A(p) = (2\pi)^2\, h(p) \qquad \text{for} \quad a^+$$

$$F_A(p_1p_2;p_3) = (2\pi)^2\, h(p)\, M(p_1p_2;p_3) \quad \text{for}\ (a^+b^+b)$$

$$f_B(p) = (2\pi)^2 \int d^4\alpha\; h^f(\alpha)\, h^g(p-\alpha)\Delta_A^+(p-\alpha)\, V(p;p-\alpha) \ \text{for}\ b^+ \qquad \text{(VII.9)}$$

and F_B which is the coefficient of (b^+a^+a) as a rather complicated function of V, T and G is given in the Appendix II. Now we can impose the condition that the corresponding 4-point matrix elements are almost local by using the same procedure as in Chapter VI. Consider the following functions

$$W_T^{(I)}(t_1\vec{p}_1\ldots,t_4\vec{p}_4) = \int \delta^{(1)}(p_1^o+p_2^o-p_3^o-p_4^o)e^{-i(p_1^o t_1+p_2^o t_2-p_2^o t_3-p_4^o t_4)}$$

$$\times \tilde{W}_T^{(I)}(p_1,\ldots,p_4)dp_1^o,\ldots,dp_4^o \qquad \text{(VII.10)}$$

which must be C^∞ as a function of the spatial $\vec{p}$, where $I = \{a,b,c\}$ corresponds to

$$W_T^{(a)}(\ldots) \equiv W_T(B^* A^* AB)$$

$$W_T^{(b)}(\ldots) \equiv W_T(A^* B^* BA)$$

$$W_T^{(c)}(\ldots) \equiv W_T(A^* B^* AB)$$

In order to avoid $S = 1$ we again choose the retarded singularities as possible ones for F_A and F_B i.e.,

$$\left[(p_4^o+i\varepsilon)^2-\omega_4^{A2}\right]F_A(p_1p_2;p_3) = \tilde{F}_A(p_1p_2;p_3)$$

$$\left[(p_4^o+i\varepsilon)^2-\omega_4^{B2}\right]F_B(p_1p_2;p_3) = \tilde{F}_B(p_1p_2;p_3) \qquad \text{(VII.12)}$$

where $p_4 = p_1 + p_2 - p_3$.

Proceeding as before (Chapter VI) we find that the following relations

$$\tilde{F}^*_A(p_3p_4;p_1)f_A(\omega^A_3,\vec{p}_3) - f^*_A(\omega^A_2,\vec{p}_2)\,\tilde{F}_A(p_2p_1;p_4) =$$

$$2\pi i\int d^4qd^4q'\ \tilde{F}^*_A(qq';p_1)\,\tilde{F}_A(qq';p_4)\Delta^+_A(q)\Delta^+_A(q')\delta^{(4)}(q+q'-p_1-p_2)$$

(VII.13a)

$$\tilde{F}^*_B(p_4p_3;p_2)\ f_B(\omega^B_4\ \vec{p}_4) - f^*_B(\omega^B_1,\vec{p}_1)\,\tilde{F}_B(p_1p_2;p_3) =$$

$$2\pi i\int d^4qd^4q'\ \tilde{F}^*_B(qq';p_2)\tilde{F}_B(qq';p_3)\Delta^+_B(q)\,\Delta^+_A(q')\delta^{(4)}(q+q'-p_1-p_2)$$

(VII.13b)

$$\tilde{F}^*_B(p_4,p_3;p_2)\ f_A(\omega^A_3,\vec{p}_3) - f^*_B(\omega^B_1,\vec{p}_1)\,\tilde{F}_A(p_2,p_1;p_4) =$$

$$2\pi i\int d^4qd^4q'\ \tilde{F}^*_B(q,q';p_2)\tilde{F}_A(q,q;p_4)\Delta^+_B(q)\,\Delta^+_A(q')\delta^{(4)}(q+q'-p_1-p_2)$$

(VII.13c)

when $\vec{p}_1+\vec{p}_2 = \vec{p}_3+\vec{p}_4$ and $\omega^B_1+\omega^A_2 = \omega^A_3+\omega^B_4$, must hold if we want (IV.11) to be C^∞ as a function of the spatial $\vec{p}_i$. The relations (VII.13) are only the condition for cancelling pole like singularities. The end point singularities which are also present here may be treated as before.

There is, however, one more interesting question which we would like to touch upon here. For example, one may naturally ask what is the reason for taking singularities for F_A and F_B in the form (VII.12).

In the case of F_A this is a pure analogy with local field theory where F_A has retarded singularity. We may, of course, assume some other more complicated singularities as principal value, for example. The problem which then remains is to prove that $S \neq 1$ and the condition for almost locality is satisfied as well.

In the case of F_B it seems at first glance that there is no such simple analogy with local field theory. F_B is given in terms of V, T and G so that it is quite hard to find a rodel which will produce the desired singularity in F_B. However, a satisfactory explanation could be drawn from the Zimmermann result in local field thecry which says that the bound state can be described by a local and invariant field operator. If this is so then F_B would have the mentioned retarded singularities in the local field theory. Thus, there should exist a model connecting (roughly speaking) V, T and G which would be able to produce the desired singularity.

The simplest one which can be constructed is the following factorization for G

$$G(p_1,p_2;p_3,p_4) \sim \frac{V(p_3+p_5;p_3)M(p_2,p_1;p_3+p_5)}{(p_3^0+p_5^0+i\varepsilon)^2-(\vec{p}_3+\vec{p}_5)^2-m_B^2} + \text{regular terms} \qquad (VII.14)$$

where $p_1+p_2 = p_3+p_4+p_5$.

Then by using the formulae in the Appendix II, we find without difficulty that

$$f_A(p_3)\widetilde{F}_B(p_1,p_2;p_3) \sim f_B(p_4)\widetilde{F}_A(p_2,p_1,p_4) \qquad (VII.15)$$

With (4.15) the relations (4.13) are trivially satisfied and reduce to a single one for F_A only, i.e., to (VII.13a).

Since $\tilde{F}_A \sim \tilde{M}$ the equation (VII.13a) is the requirement that $\tilde{M}$ satisfied elastic unitarity in the elastic region. Thus we have indirectly established that in fact $\tilde{M}$ may be considered as a scattering amplitude for $AB \longrightarrow AB$ elastic scattering, which has certain analytic properties.

Finally, we may say that one could also consider the bound state composed of more than two elementary particles described by a field $A(x)$. In that case the field $B(x)$ describing the bound state will be a certain polynomial in the basic field $A(x)$. The treatment presented here is easily applied to these cases as well.

VIII. CONCLUSION

So far all serious attempts to describe elementary particle phenomena in mathematical terms, by using different approaches to field theory for the purpose of interpreting any experimental result in high energy physics, have been rather limited. We cannot deny that some of the approaches to field theory have advantages in certain situations, but none of them is able to describe nature with enough accuracy. The reason is that the full dynamical problem is practically impossible to solve at present. Thus, any model which one could probably imagine must of necessity be extremely complicated. It will require well defined equations with a unique solution, which can be computed by reliable approximation methods, and that by such calculations we can predict experimental results.

Even regardless of the ultimate form of the theory, we know that we have to deal with an infinite set of functions which are interrelated. This set of functions may be, for instance, the S-matrix elements of all possible scattering and creation processes, or it may be the set of "Green's Functions" of a field theory, or it may be something else. Since there are very few problems which can now be solved completely either analytically or numerically, the consequences are that whenever someone suggests a specific dynamical scheme in high energy physics, it is extremely hard to find out what the scheme predicts.

It is perhaps worthwhile to stress that even although the field theoretical approach to elementary particles is not the best one, the advantages gained by choosing it are related to the possibility of setting up approximation schemes. These approximations may be regarded as taking account of all contributions involving up to some n-particles in intermediate states. In graphical language the approximation assumes that the connected graphs with few external lines dominate in a particular energy region. The usual requirements in an approximation scheme to the strong interactions are, of course, physical reasonableness and numerical solubility. It is possible to argue that the above-mentioned complications may be connected with the locality condition, asymptotic completeness and so on. Thus, some physicists believe that to find a theory of the particles we must violate some of the postulates of field theory. If we dropped the locality condition, we could construct any number of quantized fields with any spin we like, by using the well-known Haag expansion of a field in terms of a free-field. Such a "theory" is again fully determined only if we know an infinite set of so-called generalized potentials, F_{mn}, containing all the complication of dynamics. Of course, we expect that F_{mn} are such that it is possible to define the scattering of incoming states to outgoing states. For the exact F_{mn}'s in a local quantum field theory, there are formulae which give the F_{mn}'s in terms of the multiple retarded commutator functions. But, as we have seen, a much weaker condition also allows us to define an S-matrix, and this is that the field shall be "almost local" in the sense of Haag. Our programme was to write down the condition on the functions F_{mn} which ensure that the field be almost local. These conditions have been written in terms of the Wightman functions for the field $B(x)$. (Chapter VI). Then the "connected part" (truncated part) of the Wightman functions, being the many particle correlation function, should decrease in space-like directions according to the way the potential does, i.e. exponentially. Thus, examining successively the connected parts of 2,3,4,... point functions we obtain the necessary conditions on F_{mn} which make the field $B(x)$ almost local up to certain order in the Haag expansion.

At each stage we have not got, precisely, an almost local field, but it is possible to make it if F_{mn} satisfy certain relations. It is also interesting to notice that at each stage for a finite energy, only a <u>finite</u> number of functions F_{mn} enter; so that we have a feasible approximation scheme.

There is still the question to what extent an almost local field is a good approximation to the local one. It is possible to look upon it in such a way as to ensure that the theory is relativistically equivalent to the presumably correct local field theory. Unfortunately, the exact relationships of these two descriptions (of Wightman field and Haag almost local field) have not yet been fully explored.

Coming back to our programme, of setting up an approximation scheme for an almost local field having Haag expansion, we consider the almost locality condition beyond the elastic region for 4-point matrix element, but still for finite energy. It is found that almost locality conditions require F_{mn}'s to satisfy an equation similar to "physical" unitarity. The different threshold branch points are supposed to appear in a final solution of that unitarity equation. They probably could be cancelled with the end-point singularity if the functions involved are analytic. This is explicitly shown here (Chapter VI), if the energy is restricted to the elastic region only. If the functions in questions are not analytic, then a model which removes both threshold and end-point singularities seems to be necessary in order to satisfy the condition for almost locality and have $S \neq 1$.

The bound state problem, which is very involved in elementary particle physics, may also be considered in almost local field theory. The approach can go along the same lines as in local field theory. Here we suppose that the B particle may be regarded as composed of two A particles. Then, the AB scattering in the elastic region is treated exactly in the same way as the case of AA elastic scattering. The generalization to a B particle composed of several A particles is straightforward. The difficulties which may appear are connected with the 3-particle scattering region. This region is fairly important because of the relationship which it has with the 3-particle relativistic theory.

The problem that remains is to find the condition that the 6-point function $W(x_1, \ldots, x_6)$ is almost local at least when the test functions are chosen to have support below the 4-particle threshold in momentum space. The new feature which will appear then is that the defined asymptotic (in and out) 3-particle states need not span the corresponding Hilbert space. This is because the irreducible representations of the Poincare group enter with infinite multiplicity in this energy region.

The proof that the 6-point function $W(x_1, \ldots, x_6)$ is possible to make almost local, in a sense that

$$(\Psi_o, B_\phi(t_1, \vec{x}_1) \ldots B_\phi(t_6, \vec{x}_6) \Psi_o)_T \in \mathcal{S}_{\vec{\xi}_k} = \vec{x}_k - \vec{x}_{k+1}$$

for fixed $t_1, \ldots, t_6$,

has not yet been completed. Thus, the question whether or not in our approximation scheme the three particle states exist as a strong limit, is still open. When we say in our approximation scheme we mean that the supports of the test functions are chosen such that the energy momentum spectra of the vectors $B_{\phi_1} B_{\phi_2} B_{\phi_3}\Psi_o$ are below the threshold for the 4-particle production process, i.e., $(p_1 + p_2 + p_3)^2 < 16m^2$. The finite number of terms which then come into the Haag expansion for $B(x)$ is three, i.e. a^+, $F_{21}(a^+a^+a)$ and $F_{32}(a^+a^+a^+aa)$.

The corresponding truncated function will contain twenty four topologically different terms in the sense that we count F and F^* as two different functions. Assuming retarded singularities for F_{21} and F_{32} it seems at present very difficult to prove that the condition $\sum_{i=1}^{24} W_i$ is C^∞ requires only that $(p^2 - m^2)F_{32}$ satisfies 3-particle unitarity. There is, however, the problem of determining whether the higher order conditions (or approximations) reflect back on the lower order ones we have already solved, or whether <u>any</u> solution, say, of the 4-point function, is a possible solution of the coupled 4 and 6-point functions. This problem is, as we have seen, closely connected with the composite particle models.

APPENDIX I

Consider the function

$$W(a,b;x_1,x_2) = \int_{x_1}^{x_2} f(x)\left[(x-a+i\varepsilon)(x-b-i\varepsilon)\right]^{-1} dx \qquad (I.1)$$

where $f(x)$ is a continuous scalar function in the closed interval (x_1,x_2) defined on the real axis. Here we consider only the case where both a and b are points in the interval (x_1, x_2). If the derivative $f'(x)$ exists at every point interior to the interval (x_1, x_2) we can split the function $W(\ldots)$ into two parts. One part will have a pole at $a = b$ (when limit $\varepsilon \to 0$ is taken) and the other will be regular there, and moreover C^{∞}

When $a \neq b$ we can write

$$\left[(x-a+i\varepsilon)(x-b-i\varepsilon)\right]^{-1} = (a-b-2i\varepsilon)^{-1}\left[(x-a+i\varepsilon)^{-1} - (x-b-i\varepsilon)^{-1}\right] \qquad (I.2)$$

Using the following identity

$$(x-b-i\varepsilon)^{-1} = (x-a-i\varepsilon)^{-1}\left[1 - (a-b)(x-b-i\varepsilon)^{-1}\right] \qquad (I.3)$$

we have (I.1) in the form

$$W(\ldots) = (a-b)^{-1} \int_{x_1}^{x_2} f(x)\left[(x-a+i\varepsilon)^{-1} - (x-a-i\varepsilon)^{-1}\right]dx + \int_{x_1}^{x_2} f(x)\left[(x-a-i\varepsilon)(x-b-i\varepsilon)\right]^{-1} dx \qquad (I.4)$$

In the neighbourhood of $a \sim b$ (I.4) becomes

$$W(\ \dots\ ) \sim 2\pi i(a-b)^{-1}\, f(a) + \int_{x_1}^{x_2} f(x)(x-a-i\varepsilon)^{-2} dx =$$

$$2\pi i(a-b)^{-1}\, f(a) + P \int_{x_1}^{x_2} f(x)(x-a)^{-2}\, dx + i\pi f'(a) \tag{I.5}$$

APPENDIX II

Consider $B(x;h)$ defined by the relation

$$B(x;h) = U(x,1)B(h)U^{-1}(x,1) = A(x;h^f)A(x;h^g) \quad (II.1)$$

By taking the Fourier transform of (II.1) we obtain

$$B(p;h) = (2\pi)^{-2} \int e^{ipx}B(x;h)\, d^4x =$$
$$(2\pi)^{-2} \int \tilde{A}(\alpha;h^f)\, \tilde{A}(p-\alpha;h^g)\, d^4\alpha \quad (II.2)$$

where $\tilde{A}(.;.)$ is the Fourier transform of $A(.;.)$. We now take the expansion for $A(x)$ given in Chapter VII and rewrite the indicated product $\tilde{A}\tilde{A}$, (II.2) in the form of usual Wick product thus obtaining the coefficient F_B of (b^+a^+a) to be

$$F_B(p_1,p_2;p_3) = (F_V + F_{VT} + F_G + F_{GT})(p_1,p_2;p_3) \quad (II.3)$$

where

$$F_V(\dots) = -(2\pi)^2 h^f(p-p_2)h^g(p_2)V(p_1;p_3)$$

$$F_{VT}(\dots) = (2\pi)^2 \int d^4_\alpha\, h^f(\alpha)h^g(p-\alpha)V(p_1;p_1-\alpha)$$
$$\cdot \Delta^+_A(p_1-\alpha)T(p_1-\alpha,p_2;p_3)$$

$$F_G(\dots) = (2\pi)^2 \int d^4\alpha h^f(\alpha)h^g(p-\alpha)\Delta^+_A(p-\alpha)G(p_1,p_2;p-\alpha,p_3)$$

$$F_{GT}(\dots) = (1/2)(2\pi)^2 \int d^4\alpha h^f(\alpha)h^g(p-\alpha)G(p_1,p_2;q,q')\Delta^+_A(q)\Delta^+_A(q')$$
$$\cdot\ \delta^{(4)}(p_1+p_2-q-q'-\alpha)\ T(q,q';p_3)d^4q d^4q'$$
$$(II.4)$$

The four vectors p_i and p are connected by the relation

$$p = p_1 + p_2 - p_3 .$$

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$$||\phi||_{r,s} = \sum_{|k| \leq r} \sum_{|\ell| \leq s} \sup_x |x^k D^\ell \phi(x)| < \infty$$

for all integers r, s, where

$$x^k = x_1^{k_1}, \ldots, x_n^{k_n} \quad \text{and} \quad D^k = \frac{\partial^{|k|}}{(\partial x_1)^{k_1}, \ldots, (\partial x_n)^{k_n}} .$$

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$$n_u - n_b = \frac{1}{\pi}\left[\delta(\infty) - \delta(0)\right]$$

where, n_u is the number of unstable particles of a given quantum number, n_b the number of bound states and $\delta(\infty)$ and $\delta(0)$ are the phase shifts at infinity and at threshold. The above relation is nothing but Levinson's Theorem.

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