SOME TESTS ASSOCIATED WITH THE EXPONENTIAL DISTRIBUTION

by

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ABSTRACT

The thesis, which is in three parts, is concerned with tests of hypotheses in which the exponential distribution plays a part.

In Part I, a sample $x_1, x_2, \ldots, x_n$ is assumed to be exponentially distributed. A test statistic, T_n, is proposed to test this hypothesis based on the ordered sample values. The distribution and other properties of T_n are derived. The test statistic is shown to be asymptotically normal and some further approximations to its distribution are investigated. The asymptotic relative efficiency of T_n with respect to the asymptotically most powerful test against the alternative of gamma distributed intervals is obtained. Some comments are made on the application of the test to several independent sets of data. Finally, a test for an incomplete sample is outlined.

In Part II, tests of separate families of hypotheses are considered. Cox gave general results for these and in the case of the log-normal distribution versus the exponential distribution derived test statistics and their asymptotic distributions. We give

closer approximations to the distributions of the statistics and derive power functions of the tests. Cox's general methods are then used to derive tests for the log-normal distribution versus the gamma distribution. Asymptotic distributions of the test statistics are given and the tests are applied to the distribution of wool fibre-diameter.

In Part III, the power of the statistic T_n (Part I), for a log-normal alternative, is compared with that of the more specific separate family test of the exponential distribution versus the log-normal distribution.

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CONTENTS

PART I

An Analysis of Departures from the Exponential Distribution

1. Introduction

In a number of statistical problems it may be required to test the assumption that a set of observations comes from a Poisson series, i.e. that the intervals between events are exponentially distributed. Anderson and Darling (1952) give a unified theory of non-parametric tests which can be used to test the assumption of exponentiality. Darling (1953) and Epstein (1960) have surveyed tests for exponentiality, the latter in connection with life testing. The asymptotic relative efficiency of some of these tests has been found by Bartholomew (1957) for various alternatives. Lewis (1965) has proposed a new test and given its asymptotic relative efficiency against a gamma alternative. Finally, Cox and Lewis (1966) discuss tests and other methods connected with series of events.

A further test, based on the ordered values of the observations, is proposed here. Some analytical properties of this test are obtained and other results are indicated by Monte Carlo experiments.

Suppose the random variables $X_1, X_2, \ldots, X_n$ are from an exponential distribution, p.d.f. $\lambda e^{-\lambda x}$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the ascending ordered values. Then it is well known that

$$E(X_{(r)}) = \left(\sum_{i=1}^{r} \frac{1}{n-i+1} \right)/\lambda = t_{r,n}/\lambda \qquad (r = 1,2,\ldots,n). \tag{1.1}$$

If the observed order statistics are plotted against t_{rn} a straight line through the origin is to be expected, but if the population is of non-exponential form a curved plot is to be expected. Shapiro and Wilk (1965) in their test for normality found that the ordered observations should be weighted by constants $\underline{a}' \propto \underline{m}' \underline{V}^{-1}$ where $\underline{V}$ is the covariance matrix and $\underline{m}$ the expected values of the ordered variables. Empirical investigations show that $\underline{m}$ is a good approximation to $\underline{a}$. These general remarks suggest a test based on $\sum t_{rn} X_{(r)}$ normalized to remove dependence on the nuisance parameter λ. It is this statistic which is considered in the rest of Part I of the thesis.

2. The Test Statistic T_n

Consider the n ordered sample values from an exponential distribution with $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$. The differences $X_{(1)}, X_{(2)}-X_{(1)}, X_{(3)}-X_{(2)}, \dots$ are mutually independent but non-identically distributed. The transformation

$$V_1/(n\lambda) = X_{(1)}, \quad V_r/\{(n-r+1)\lambda\} = X_{(r)} - X_{(r-1)} \qquad (r = 2, 3, \dots, n) \quad (2.1)$$

gives identically distributed random variables, and each V has the unit exponential density, e^{-v}.

From (2.1),

$$X_{(r)} = \frac{1}{\lambda}\left(\frac{V_1}{n} + \frac{V_2}{n-1} + \cdots + \frac{V_r}{n-r+1} \right), \quad r = 1,2,\dots, n$$

and

$$\sum_{r=1}^{n} X_{(r)} = \sum_{r=1}^{n} X_r = \left(\sum_{r=1}^{n} V_r \right)/\lambda \, .$$

We take as our test statistic

$$T_n = \{\sum_{r=1}^{n} X_{(r)}\ t_{r,n}\}/\{\sum_{r=1}^{n} X_{(r)}\} \qquad (2.2)$$

$$= \{\frac{V_1}{n}\frac{1}{n} + (\frac{V_1}{n} + \frac{V_2}{n-1})(\frac{1}{n} + \frac{1}{n-1}) + \ldots + (\frac{V_1}{n} + \ldots + \frac{V_n}{1})(\frac{1}{n} + \ldots + 1)\}$$

$$\div \{\Sigma V_r\}$$

$$= \{\sum_{r=1}^{n} C_r\ V_r\}/\{\sum_{r=1}^{n} V_r\} \qquad (2.3)$$

where C_r are constants depending on r and n. By equating the last two expressions for T_n we find that

$$C_1 = 1, \quad \text{all } n$$

and

$$C_r = 1 + \frac{1}{n} + \ldots + \frac{1}{n-r+2}, \qquad r = 2,3,\ldots,n.$$

In terms of $t_{r,n}$

$$C_r = 1 + t_{r-1,n} \qquad r = 1,2,\ldots,n \quad (2.4)$$

where we define $t_{o,n} \equiv 0$, all n.

Since T_n is scale invariant, the nuisance parameter, λ, is eliminated and there is no loss of generality if we take the observations as coming from the unit exponential distribution.

By a well known result for weighted means (Hardy et al, 1934, p.14)

$$\min\ a_v \leq \left(\frac{\sum p_v a_v^r}{\sum p_v}\right)^{\frac{1}{r}} \leq \max\ a_v \qquad v = 1,2,\ldots,n$$

where $\sum p_v$ = constant and $a_v, p_v \geq 0$.

Hence $\min C_i \leq T_n \leq \max C_i$, $\quad i = 1, 2, \ldots, n$.

Since $1 = C_1 < C_2 < \ldots < C_n = t_{n,n} = \log n + \gamma$,

where $\gamma = 0.5772$ is Euler's constant, we have that

$$1 \leq T_n \leq C_n. \tag{2.5}$$

The minimum is attained when all the intervals are of equal length and $X_1 \neq 0$. The maximum is attained when $X_{(1)} = X_{(2)} = \ldots = X_{(n-1)} = 0$ and $X_{(n)} \neq 0$. Thus both tails of T_n can be used to test departures from the exponential distribution.

For a gamma alternative the departure is away from the lower tail of the distribution of T_n, we thus use the lower tail for tests of significance.

3. Independence of T_n and $\sum V_i = W$

<u>Theorem</u>: $T_n = \dfrac{\sum C_i V_i}{\sum V_i}$ and $W = \sum V_i$ are statistically independent.

<u>Proof</u>: $V_1, V_2, \ldots, V_n$ are independent and each is a unit exponential. Let $W = \sum_1^n V_i$.

The joint density of the V's is

$$P_{\underline{v}}(v_1, v_2, \ldots, v_n) = e^{-\sum_1^n v_i} .$$

Also the p.d.f. for $W = \sum V_i$ is

$$P_{\underline{w}}(\sum V_i = W) = W^{n-1} e^{-W}/(n-1)!$$

Thus $P_{\underline{v}}(V_1, \ldots, V_n | W = w = \sum V_i) = (n-1)!/w^{n-1}$ (3.1)

Now transform the V's as follows:

$$U_1 = V_1$$

$$U_2 = V_1 + V_2$$

$$\vdots$$

$$U_{n-1} = V_1 + V_2 + \ldots + V_{n-1}$$

$(U_n = V_1 + \ldots + V_n = w$, a constant) .

Then $U_1, U_2, \ldots, U_{n-1}$ form (n-1) ordered random variables with the Jacobian of the transformation being unity, and we have from (3.1) that

$$P_{\underline{u}}(U_1, U_2, \ldots, U_{n-1} | W = w) = (n-1)!/w^{n-1} \quad , (3.2)$$

i.e. $U_1, U_2, \ldots, U_{n-1}$ are order statistics from a rectangular distribution over (0, w).

Now any linear combination $\sum_1^n C_i V_i$ can be written

as $bw + \sum_{i=1}^{n-1} b_i U_i$, and so

$$T_n = \sum_1^n C_i V_i / \sum_1^n V_i = b + (\sum_1^{n-1} b_i U_i)/W \quad .$$

Conditionally on $W = w$, T_n has a distribution not involving w, because each U_i/W is rectangular over (0, 1). Hence T_n and $W = \sum_1^n V_i$ are independent.

Corollary.

Some properties of T_n can be found using the facts that

a) $\sum C_i V_i = T_n (\sum V_i)$, where the factors on the right are independent,

and

$$\text{b)} \quad T_n = b + \sum_{i=1}^{n-1} b_i U_i^* , \tag{3.4}$$

where U_i^* are ordered values from the rectangular distribution $R(0, 1)$, and the b_i are constants.

Equating the two expressions for T_n, we have that

$$b = C_n, \quad b_i = C_i - C_{i+1}, \qquad i = 1,2,\ldots, n-1.$$

Thus (3.4) can be written as

$$T_n = C_n + \sum_{i=1}^{n-1} (C_i - C_{i+1}) U_i^*$$

$$= C_n - \sum_{i=1}^{n-1} \frac{U_i^*}{n-i+1} \tag{3.5}$$

where $0 \leq U_1^* \leq U_2^* \ldots \leq U_{n-1}^* \leq 1$.

It is awkward to use the form b) to obtain properties of T_n (except the first moment) since this involves products of the C_i's.

The independence of T_n and $\sum V_i$ follows also from a result of Pitman (1937). We suppose that $x_1, x_2, \ldots, x_n$ are independent random variables and x_i is a gamma variate with p.d.f. $\{e^{-x_i} x_i^{m_i-1}\}/\{\Gamma(m_i)\}$. $F(x_1, x_2, \ldots, x_n)$ is a function of x independent of scale, i.e. $F(kx_1, kx_2, \ldots, kx_n) \equiv F(x_1, x_2, \ldots, x_n)$. Then Pitman's result is that

$\sum x_i$ and $F(x_1, x_2, \ldots x_n)$ are independent.

This is proved by considering the characteristic function, $\phi(u, v)$, of the joint distribution of $\sum x_i$ and F. It is shown that $\phi(u, v)$ factorises into a function of u and a function of v, and hence the independence of $\sum x_i$ and F.

4. Null Hypothesis Distribution of T_n

4.1 Some preliminary results

We require the power sums of the $t_{r,n}$ and C_r for the moment calculations.

Let $S_{in} = \sum_{r=1}^{n} t_{rn}^{i}$.

There are recurrence relations like

$$S_{1n} = S_{1,\,n-1} + 1, \quad S_{2n} = S_{2,n-1} + (2/n)\, S_{1,n-1} + 1/n$$

between the S_{in}. Solving these, we obtain

$$\left.\begin{aligned} S_{1n} &= n, \; S_{2n} = 2n - t_{nn}, \\ S_{3n} &= 6n - 3t_{nn} + \sum_{1}^{n}(1/r^2) - 3\sum_{1}^{n}(t_{rr}/r), \\ S_{4n} &= 24n - 12t_{nn} + 4\sum_{1}^{n}(1/r^2) + 3\sum_{1}^{n}(1/r^3) + \ldots \end{aligned}\right\} \quad (4.1)$$

Using these results to obtain power sums for the C_i we have that since $C_i = 1 + t_{i-1,n}$, $\quad i = 1,2,\ldots, n$

$$\left.\begin{aligned} \sum_{1}^{n} C_i &= 2n - t_{nn} \\ \sum_{1}^{n} C_i^2 &= S_{2n} + 3n - 2t_{nn} - t_{nn}^2 \\ &= 5n - t_{nn}^2 - 3t_{nn} \\ \sum_{1}^{n} C_i^3 &= 3S_{1n} + 3S_{2n} + S_{3n} + n - 3t_{nn} - 3t_{nn}^2 - t_{nn}^3 \end{aligned}\right\} \quad (4.2)$$

$$= 16n - t_{nn}^3 - 3t_{nn}^2 - 9t_{nn} + \sum_1^n \frac{1}{r^2} - 3 \sum_1^n \frac{t_{rr}}{r}$$

and

$$\sum_1^n c_i^4 = 65n - t_{nn}^4 - 4t_{nn}^3 - 6t_{nn}^2 - 34t_{nn} + \dots \quad . \tag{4.2}$$

4.2 <u>Moments of T_n.</u>

All moment calculations are done using the fact that T_n and $\sum V_i$ are independent.

Since each V_i has p.d.f. e^{-v_i} $(v_i \geq 0)$

$$E(T_n)\, E(\sum V_i) = E\{\sum c_i\, V_i\}$$

and $\mu = E(T_n) = \frac{1}{n} \sum c_i$

$$= 2 - t_{nn}/n = 2 - \frac{\log n + \gamma}{n} + O(\frac{\log n}{n^2}), \tag{4.3}$$

where $\gamma = 0.5772$ is Euler's constant.

For the second moment we have

$$E(T_n^2)\, E\{(\sum V_i)^2\} = E\{(\sum c_i\, V_i)^2\} \quad ,$$

i.e. $E(T_n^2) E\{\sum_i V_i^2 + 2 \sum_{i>j} V_i V_j\}$

$$= E\{ \sum C_i^2 V_i^2 + 2 \sum_{i>j} C_i C_j V_i V_j \}$$

and $\mu_2' = E(T_n^2) = \{2\sum C_i^2 + 2 \sum_{i>j} C_i C_j\}/\{n(n+1)\}$

$$= \{\sum C_i^2 + (\sum C_i)^2\}/\{n(n+1)\} .$$

Therefore

$$\mu_2 = \text{var}(T_n) = \frac{\sum C_i^2}{n(n+1)} - \frac{(\sum C_i)^2}{n^2(n+1)} \tag{4.4}$$

$$= \{n + t_{nn} - t_{nn}^2(1+\tfrac{1}{n})\} / \{n(n+1)\}$$

$$= \frac{1}{n} - \frac{t_{nn}^2 - t_{nn} + 1}{n^2} - \frac{t_{nn} - 1}{n^3} + O(\frac{\log^2 n}{n^4}) . \tag{4.5}$$

By a similar argument

$$E\{(\sum V_i)^3\} = E\{\sum V_i^3 + 3\sum_{i\neq j} V_i^2V_j + \sum_{i\neq j\neq k} V_iV_jV_k\}$$

$$= n(n+1)(n+2).$$

Also

$$E\{(\sum c_iV_i)^3\} = E\{\sum c_i^3V_i^3 + 3\sum_{i\neq j} c_i^2c_jV_i^2V_j + \sum_{i\neq j\neq k} c_ic_jc_kV_iV_jV_k\}$$

$$= 6\sum c_i^3 + 6\sum_{i\neq j} c_i^2c_j + \sum_{i\neq j\neq k} c_ic_jc_k$$

$$= 2\sum c_i^3 + 3(\sum c_i)(\sum c_i^2) + (\sum c_i)^3 .$$

These lead to

$$\mu_3 = \frac{2\sum c_i^3}{n(n+1)(n+2)} - \frac{6(\sum c_i)(\sum c_i^2)}{n^2(n+1)(n+2)} + \frac{4(\sum c_i)^3}{n^3(n+1)(n+2)} . \quad (4.6)$$

The exact value of γ_1, the coefficient of skewness, is obtained from (4.4) and (4.6). Using the results of section 4.1 we obtain the following approximations:-

$$\mu_3 = \frac{4}{n^2} - \frac{2(t_{nn}^3 - 3t_{nn}^2 + 6)}{n^3} + O(\frac{\log^3 n}{n^4}) \quad (4.7)$$

$$\text{and} \quad \gamma_1 = \mu_3/\mu_2^{\frac{3}{2}} = \frac{4}{\sqrt{n}} \left(1 - \frac{t_{nn}^3 - 6t_{nn}^2 + 3t_{nn} + 3}{2n}\right) + O\left(\frac{\log^3 n}{n^{\frac{5}{2}}}\right).$$

(4.8)

For the fourth moment, similar arguments show that

$$E\{(\sum V_i)^4\} = n(n+1)(n+2)(n+3)$$

and

$$E\{(\sum c_i V_i)^4\} = 23 \sum c_i^4 + 20 \sum_{i \neq j} c_i^3 c_j + 9 \sum_{i \neq j} c_i^2 c_j^2$$

$$+ 6 \sum_{i \neq j \neq k} c_i^2 c_j c_k + (\sum c_i)^4$$

$$= (\sum c_i)^4 + 6(\sum c_i)^2(\sum c_i^2) + 8(\sum c_i)(\sum c_i^3) + 3(\sum c_i^2)^2$$

$$+ 6 \sum c_i^4 .$$

From these we find that

$$\mu_4 = \frac{3}{n^4(n+1)(n+2)(n+3)} \Big[(n-6)(\sum c_i)^4 - 2n(n-6)(\sum c_i)^2(\sum c_i^2)$$

$$+ n^3(\sum c_i^2)^2 - 8n^2(\sum c_i)(\sum c_i^3) + 2n^3 \sum c_i^4 \Big] .$$

(4.9)

We can find an approximation to the kurtosis by expressing (4.9) in inverse powers of n and using the results of section 4.1. Then

$$\beta_2 = \mu_4/\mu_2^2 = 3 - \frac{18}{n} + \frac{6}{n^2}\left\{11+8t_{nn}-8t_{nn}^2+7t_{nn}^3-t_{nn}^4\right\}$$

$$+ \; O\left(\frac{\log^4 n}{n^3}\right) \quad . \qquad (4.10)$$

From (4.8) and (4.10), $\gamma_1 \longrightarrow 0$ and $\beta_2 \longrightarrow 3$ as $n \longrightarrow \infty$. These suggest that the distribution of T_n tends asymptotically to normality. This will be proved formally in section 5.

The approximations to the mean and variance of T_n are very good even for small n; but for γ_1 and γ_2, n has to be large to give great accuracy.

4.3 Exact Distribution Function of T_n

This section is based on a result by Gurland (1948). This states that if $X_1, X_2, \ldots, X_n$ have joint distribution function $F(x_1, x_2, \ldots, x_n)$ with corresponding characteristic function $\phi(t_1, t_2, \ldots, t_n)$

and $G(x)$ is the distribution function of $(a_1 X_1 + \dots + a_n X_n)/(b_1 X_1 + \dots + b_n X_n)$, $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ real numbers, then if

$$P\{\sum_1^n b_j x_j \le 0\} = 0$$

$$G(x) + G(x-0) = 1 - \frac{1}{\pi i} \int \frac{\phi\{t(a_1 - b_1 x), \dots, t(a_n - b_n x)\}}{t} dt \qquad (4.11).$$

T_n satisfies the above conditions and in this case if

$$P\{T_n \le x\} = F(x)$$

$$F(x) + F(x-0) = 1 - \frac{1}{\pi i} \int \frac{\phi\{t(C_1 - x), \dots, t(C_n - x)\}}{t} dt \qquad (4.12).$$

The characteristic function of $x_1, x_2, \dots, x_n$ is

$$\phi(t_1, t_2, \dots, t_n) = \prod_{j=1}^{n} \{1/(1 - it_j)\} .$$

We need to evaluate

$$I = \oint \frac{dt}{t[1-it(C_1-x)][1-it(C_2-x)] \dots [1-it(C_n-x)]} .$$

This is done by considering a semi-circular contour in the upper half plane and indented at the origin. The number of poles inside this contour depends on the range of x and after some calculation of residues we obtain the distribution function

$$F(x) = \sum_k \frac{(C_{kn}-x)^{n-1}}{\prod_{j=1}^{n}{}'(C_{kn}-C_{jn})} = \sum_k \frac{(x-C_{kn})^{n-1}}{\prod_{1}^{n}{}'(C_{jn}-C_{kn})} \qquad (4.13)$$

where π' means that $j = k$ is omitted and C_{kn} is written for C_k to emphasise the dependence on n. In (4.13) the summation over k is continued as long as $x - C_{kn} > 0$, $k = 1, 2, \dots, n-1$. Writing (4.13) a little more specifically we get

$$F(x) \equiv \begin{cases} 0 & , \; x \leq 1 \\[2ex] \dfrac{(x-1)^{n-1}}{(C_{2n}-C_{1n})(C_{2n}-C_{1n})\ldots(C_{nn}-C_{1n})} = F_1(x), & 1 \leq x \leq C_{2n} \\[2ex] F_1(x) + \dfrac{(x-C_{2n})^{n-1}}{(C_{1n}-C_{2n})(C_{3n}-C_{2n})\ldots(C_{nn}-C_{2n})} = F_2(x), & C_{2n} \leq x \leq C_{3n} \\ \vdots & \\ F_{n-2}(x) + \dfrac{(x-C_{n-1,n})^{n-1}}{(C_{1n}-C_{n-1,n})(C_{2n}-C_{n-1,n})\ldots(C_{nn}-C_{n-1,n})} & \\[2ex] \qquad = F_{n-1}(x), & C_{n-1,n} \leq x \leq C_{n,n} \\[2ex] 1 & , \; x \geq C_{nn} \,. \end{cases} \tag{4.14}$$

An alternative form of the distribution function can be obtained by considering a semi circular contour in the lower half plane. In this case, for say $C_{n-1n} < x < C_{nn}$, there is only one pole inside the contour and $F_{n-1}(x)$ has only one term instead of $n-1$ as in (4.14). Then

$$F_{n-1}(x) = 1 - \frac{(C_{nn}-x)^{n-1}}{(C_{nn}-C_{1,n})\ldots(C_{nn}-C_{n-1,n})} ,$$

$$C_{n-1,n} \leq x \leq C_{nn} . \qquad (4.15)$$

Similarly, $F_{n-2}(x)$ has two terms instead of $n-2$, and so on.

The form of $F(x)$ makes it difficult to obtain the percentage points of T_n when n is large.

The distribution function $F(x)$ can also be obtained from results of Anderson (1942) on the serial correlation coefficient of lag 1,

$$r = (x_1x_2 + x_2x_3 + \ldots + x_nx_1)/(x_1^2 + \ldots + x_n^2).$$

Pyke (1965) shows that if $y_1, \ldots, y_n$ are independent exponential random variables with $E(y_i) = 1$ and

$$S = y_1 + \ldots + y_n, \text{ and } D_i = y_i/S$$

then $(D_1, D_2, \ldots, D_n)$ is distributed as the set of n spacings determined by $n-1$ independent random variables uniformly distributed over $(0,1)$. Durbin (discussion of Pyke, 1965) indicates that

$$r = \sum_{1}^{n/2} a_i D_i \quad , \text{ where } a_i \text{ are constants.}$$

Hence $F(x)$ can be obtained from Anderson's results on the distribution of r.

5. Asymptotic Normality of T_n

Theorem:

T_n is asymptotically normal, $N(2, \frac{1}{n})$.

Proof:

We employ Lindeberg's form of the Central Limit Theorem which states that if $\zeta_1, \zeta_2, \ldots, \zeta_n$ are non-identically distributed random variables then, $\zeta = \zeta_1 + \ldots + \zeta_n$ is asymptotically normal if

$$\lim_{n\to\infty} \frac{\rho}{\sigma} = 0,$$

where $\rho^3 = \sum \rho_i^3$, $\sigma^2 = \sum \sigma_i^2$, and $\rho_i^3 = E\,|\zeta_i - \mu_i|^3$,

$$\sigma_i^2 = E(\zeta_i - \mu_i)^2.$$

Let $\zeta = \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} \zeta_i$.

Then $\mu(\zeta) = \sum c_i = 2n - t_{nn}$, $\sigma^2(\zeta) = \sum c_i^2 = 5n - t_{nn}^2 - 3t_{nn}$.

The third moment of ζ_i is ρ_i^3 where

$$\rho_i^3 = E\,|\zeta_i - c_i|^3 = c_i^3\, E\{|v_i - 1|^3\} .$$

$$= c_i^3 \int_0^{\infty} e^{-v_i} |v_i - 1|^3 \, dv_i$$

$$= (12e^{-1} - 2) c_i^3 .$$

Therefore $\rho^3 = \sum \rho_i^3 = (12e^{-1} - 2) \sum c_i^3$

$$= (12e^{-1} - 2)(9n - t_{nn}^3 - 3t_{nn}^2 - 9t_{nn} - \ldots) .$$

Hence $$\left(\frac{\rho}{\sigma}\right)^3 = (12e^{-1} - 2) \frac{\left[9n - t_{nn}^3 - 3t_{nn}^2 - \ldots\right]}{\left[5n - t_{nn}^2 - 3t_{nn}\right]^{3/2}}$$

$$\longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Thus Lindeberg's condition is satisfied and $\sum C_i V_i$ tends to normality with mean $\sum C_i$ and variance $\sum C_i^2$.

Now consider

$$\xi_n = \sum C_i V_i - (\sum C_i)(\sum V_i/n).$$

Then $E(\xi_n) = 0$, $\sigma^2(\xi_n) = \sum C_i^2 - (\sum C_i)^2/n$.

Since $\sum C_i V_i$ tends to normality and $\sum V_i/n$ tends in probability to 1, ξ_n tends to normality and

$$\xi_n' = \frac{\sum C_i V_i - (\sum C_i)(\sum V_i/n)}{\sqrt{\{\sum C_i^2 - (\sum C_i)^2/n\}}} \xrightarrow{D} N(0, 1) \qquad (5.1)$$

where $\xrightarrow{D}$ means that the random variable tends in distribution to the stated distribution.

Let $\eta_n = \sum_1^n (V_i/n)$. Then $E(\eta_n) = 1$, and $\eta_n \longrightarrow 1$ in probability as $n \longrightarrow \infty$.

If $F(\xi_n')$ is the distribution function of ξ_n', then $F(\xi_n') \longrightarrow F(\xi') \equiv N(0, 1)$ as $n \longrightarrow \infty$.

Let $Z_n = \xi_n'/\eta_n$. Then by a theorem of Cramér (1946, p.254), the distribution function of $Z_n \longrightarrow F(\xi')$, i.e.

$$\frac{\sum c_i V_i - (\sum c_i)(\sum V_i/n)}{(\sum V_i/n)\sqrt{\left\{\sum c_i^2 - (\sum c_i)^2/n\right\}}} \xrightarrow{p} N(0,\ 1)$$

or $$T_n = \frac{\sum c_i V_i}{\sum V_i} \xrightarrow{p} N\left(\frac{\sum c_i}{n},\ \frac{\sum c_i^2 - (\sum c_i)^2/n}{n^2}\right).$$

Letting $n \longrightarrow \infty$ $\quad T_n \xrightarrow{p} N(2,\ \frac{1}{n})$,

or $\sqrt{n}(T_n - 2) \xrightarrow{p} N(0,\ 1)$.

The asymptotic normality of T_n can also be shown to follow from some general results of Pyke (1965) on the limiting distributions of functions of spacings.

6. Approximations to the Distribution of T_n

The rate of convergence of T_n to normality is slow $\left[\gamma_1 = 4/\sqrt{n} + O(n^{-\frac{3}{2}})\right]$ and the distribution function of T_n is in an open form, hence various approximations to T_n were tried. An Edgeworth Series approximation proved very good, but others were adequate and so are briefly outlined here.

Suppose a random variable X is the ratio of two random variables U, W; then

$$\Pr\{X = U/W < v\} = \Pr\{U - vW < 0\}$$

$$\sim \Phi\left(-\frac{\mu_u - v\,\mu_w}{\sqrt{\{\sigma_u^2 - 2v\,\mathrm{cov}(U,W) + v^2\sigma_w^2\}}}\right) \qquad (6.1)$$

where Φ is the standard normal probability integral and μ_u, σ_u^2 are the mean and variance of U etc.

If $T_n' = \dfrac{\sum c_i v_i - (\sum c_i)(\sum v_i/n)}{(\sum v_i/n)\sqrt{\{\sum c_i^2 - (\sum c_i)^2/n\}}}$, then

$T_n' \xrightarrow{p} N(0,1)$. From (6.1) and a little manipulation we find that

$$\Pr\{T_n' < v\} \sim \Phi\left(\frac{v}{\sqrt{\{1+v^2/n\}}}\right) \quad . \qquad (6.2)$$

This gives a better approximation to the distribution of T_n' than $N(0,1)$ does.

Another useful idea is the relation between the coefficient of variation, C, and the skewness, γ_1. For T_n, $\gamma_1 \doteqdot 8C$, and for $(T_n - 1)$, $\gamma_1 \doteqdot 4C$.

For a log-normal distribution, $\gamma_1 = 3C + C^3$. This suggests that a log-normal might be an adequate representation of (T_n-1). This is specially so for small n and an empirical investigation is done on this in section 7.

A chi-squared approximation was also tried. Let $T_n = A\chi^2_v + b$, where A, v, b are constants. We obtain these constants by fitting the first three moments of T_n and χ^2_v. A few values were plotted and compared with the empirical distribution of T_n. The fit was quite good and would be adequate for most practical purposes. However, if great accuracy is desired this approximation would entail a lot of calculation since **v**, the degrees of freedom of χ^2, is fractional.

Evaluation of the exact values of the skewness and kurtosis of T_n showed that these values were small for sample values as small as 5. Thus an Edgeworth Series expansion would give a good approximation to the distribution of T_n. In this case, **if**

$$x = \frac{T_n - E(T_n)}{\sqrt{(\text{var } T_n)}}$$

we can represent the p.d.f. of T_n by

$$g(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \{1+\frac{\gamma_1}{6} H_3 + \frac{\gamma_2}{24} H_4 + \frac{\gamma_1^2}{72} H_6 + \ldots\} \qquad (6.3)$$

where $H_r(x)$ $(r = 1,2,\ldots)$ are Hermite polynomials. We can use a Cornish-Fisher expansion to express x in terms of a standard normal variate, ξ, and vice-versa. If α is a one-sided per cent point of T_n we have

$$\alpha = \int_x^\infty g(u)du = \int_\xi^\infty \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}} du$$

and

$$x = \xi + \frac{\gamma_1}{6} (\xi^2-1) + \frac{\gamma_2}{24} (\xi^3-3\xi) - \frac{\gamma_1^2}{36}(2\xi^3- 5\xi)+\ldots \qquad (6.4)$$

to order n^{-1} at least, since $\gamma_1 = O(n^{-\frac{1}{2}})$, $\gamma_2 = O(n^{-1})$.

Also

$$\xi = x- \frac{\gamma_1}{6} (x^2-1)- \frac{\gamma_2}{24} (x^3-3x)+ \frac{\gamma_1^2}{36}(4x^3-7x)+\ldots \qquad (6.5)$$

to order n^{-1} at least.

Equation (6.4) is useful in calculating percentage points of T_n and (6.5) can be used if T_n is known and the corresponding percentage point is required using standard normal integral tables. A comparison using these methods was done and the results are summarised in Table 7.3.

7. Empirical Results.

Variates from an exponential distribution mean unity were generated on a computer. The procedure was that used by Clark and Holtz (1960).

The statistic, T_n, was calculated from a sample of n exponential variates. For each n, 5000 samples of T_n were then generated for different values of n and the moments and frequency distribution were obtained. A summary of the sampling moments together with the exact values is given in table 7.1. Frequency histograms were drawn and from these a normal approximation was quite good for about $n \geq 30$. For small n the distribution was skew so a logarithmic transformation was made, $T' = \log(T-1)$. The empirical distribution of T' was studied for n = 5, 10 using sample sizes of 1000. The results are given in table 7.2. Empirical percentage values of T_n were also obtained by plotting the empirical distribution function of T_n on normal probability paper and using graphical interpolation. A few of these values are given in table 7.3.

Table 7.1. Moments of T_n

n		Mean (μ)	Var (μ_2)	Skewness (γ_1)	Kurtosis (β_2)
5	Exact	1.5433	0.0342	0.3268	2.8393
	Sample	1.5409	0.0326	0.3319	2.9241
10	Exact	1.7071	0.0318	0.4015	3.1532
	Sample	1.7053	0.0312	0.4417	3.2259
30	Exact	1.8668	0.0188	0.3762	3.2505
	Sample	1.8673	0.0189	0.3373	3.1066
50	Exact	1.9100	0.0133	0.3368	3.2182
	Sample	1.9091	0.0134	0.3417	3.3665
100	Exact	1.9481	0.0077	0.2761	3.1576
	Sample	1.9485	0.0078	0.2396	3.0866

Table 7.2. Moments of $T'_n = \log(T_n - 1)$

n		Mean	Var	Skewness	Kurtosis
5	'Exact'	-0.7105	0.2308		
	Sample	-0.6806	0.1361	-0.7879	3.9714
10	'Exact'	-0.3907	0.0880		
	Sample	-0.3656	0.0635	-0.2727	3.1016

Moments of T_n are from samples of 5000, and those for T'_n are from samples of 1000.

In table 7.2, the 'exact' moments are approximate analytical solutions obtained from the exact moments for T_n.

From table 7.1 agreement between sampling and analytical moments is very good. For γ_1 for instance, the sampling error is roughly $\sqrt{6/N}$, which is 0.035 for $N = 5000$.

The closeness of the sampling moments to the exact ones suggests that the empirical percentage values of T_n obtained from the sampling distribution will be quite reliable as estimates of the exact values.

The log transformation gives for $n = 10$, a reasonably good normal fit but for $n = 5$ the fit seems to be worse than the simple normal fit for T_n. This suggests that perhaps for n between 10 and say 25, a log-normal approximation is adequate. Values of n less than 10 will, in practice, not give enough information for a satisfactory inference on the underlying distribution.

A comparison of the empirical percentage points of T_n, the asymptotic distribution of T_n, $N(2, \frac{1}{n})$, and the normal approximation using the exact mean and variance was done by plotting on probability paper. It was found that $N(2, \frac{1}{n})$ was a good representation of T_n for values of n greater than 100. Cornish-Fisher series approximations using (i) two moments, (ii) three moments, (iii) four moments were obtained and the results are given in table 7.3, together with some exact values of the percentage points of T_n which are generated by an iterative procedure on a computer.

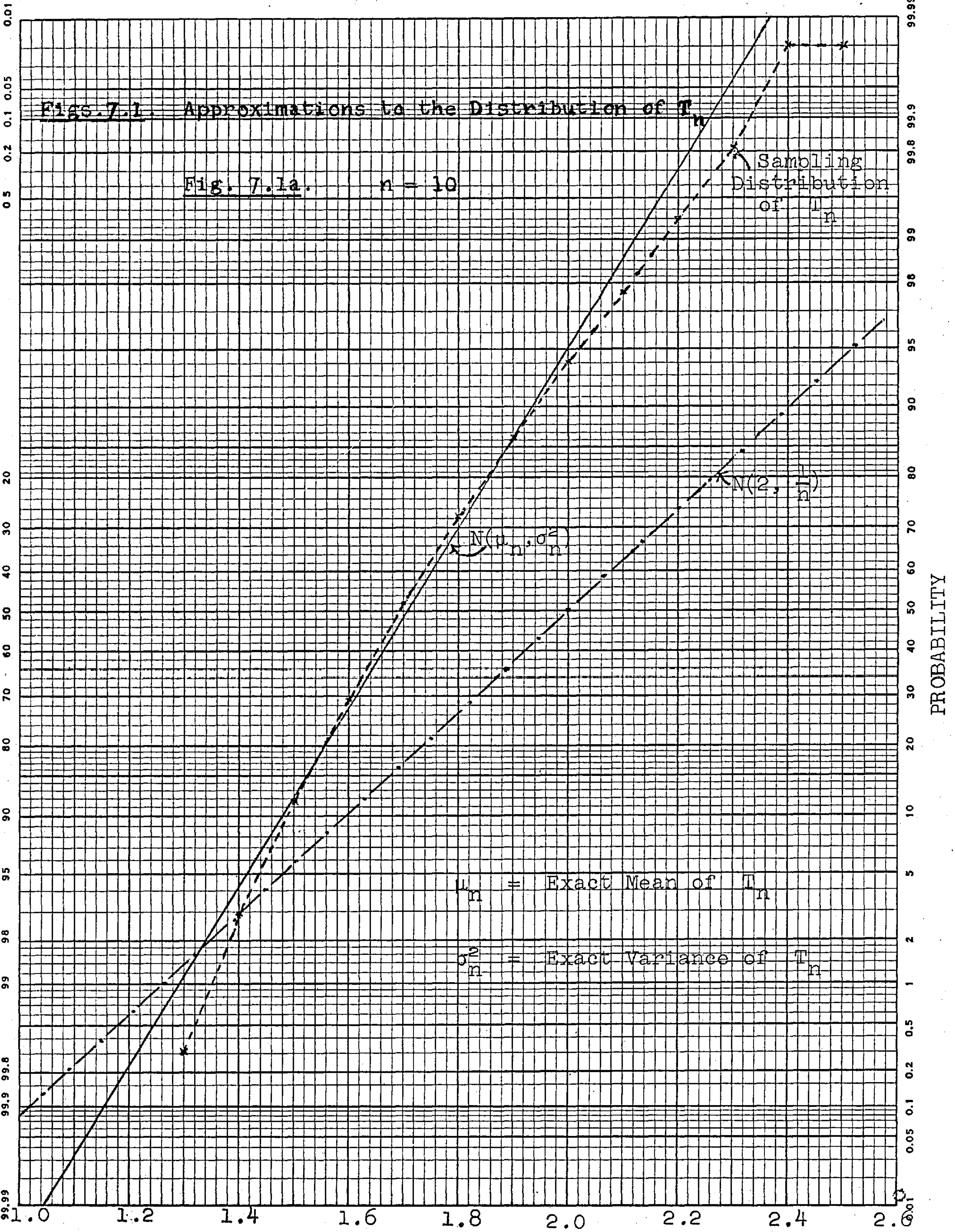
A few graphs of some of these comparisons are given at the end of this section in Figs. 7.1.

Table 7.3. Comparison of Various Percentage Point Approximations of T_n.

n	p	Cornish-Fisher Series (i) 2 moment	(ii) 3 moment	(iii) 4 moment	Empirical	Exact
5	.01	1.113	1.157	1.172	1.180	1.173
	.05	1.239	1.256	1.256	1.264	1.260
	.50	1.543	1.533	1.533	1.528	1.531
	.95	1.848	1.865	1.865	1.860	1.871
	.99	1.974	2.018	2.004	1.993	2.007
10	.01	1.293	1.345	1.351	1.348	1.348
	.05	1.414	1.434	1.435	1.437	1.431
	.50	1.707	1.695	1.695	1.695	1.695
	.95	2.000	2.021	2.020	2.022	2.021
	.99	2.122	2.174	2.168	2.172	2.172
30	.01	1.548	1.586	1.585	1.586	1.584
	.05	1.641	1.656	1.657	1.658	1.657
	.50	1.867	1.858	1.858	1.860	1.858
	.95	2.093	2.107	2.106	2.107	2.106
	.99	2.186	2.224	2.225	2.217	2.224
50	.01	1.642	1.671	1.670	1.666	
	.05	1.721	1.732	1.732	1.734	
	.50	1.910	1.904	1.904	1.905	
	.95	2.100	2.111	2.110	2.108	
	.99	2.178	2.207	2.208	2.212	
100	.01	1.744	1.763	1.762	1.758	
	.05	1.804	1.811	1.811	1.812	
	.50	1.948	1.944	1.944	1.944	
	.95	2.093	2.100	2.100	2.102	
	.99	2.153	2.172	2.172	2.176	

A few things emerge clearly from table 7.3. For $n \geq 30$ (possibly, even for smaller n) there is a difference of not more than one in the third decimal place between the three and four moment Cornish-Fisher Series. Except for the upper and lower one per cent points, the three moment series is good even for n as low as 5. The exact values for $n = 5$ show good agreement with the four moment approximation and the exact values for $n = 10$ agree well with the three moment series. We can infer, therefore, that the three moment approximation will become even better for larger n, as $n = 30$ shows, and hence will be accurate enough for all purposes. However, for most statistical tests the two moment approximation (i.e. the normal approximation using the exact mean and variance of T_n) will be adequate.

The empirical values agree very well with the exact values, especially when $n = 10$. This is not too surprising since sample sizes of 5000 were used. For this sample size and $n = 10$, the standard error of the mean is about 0.0025, the variance has a much smaller error, γ_1 has a standard error of about 0.035 and γ_2 an error of about 0.07. Presumably some of these errors cancel out to produce the very good agreement. Similar remarks also hold when $n = 30$.


Figs. 7.1. Approximations to the Distribution of T_n
Fig. 7.1a. n = 10
Sampling Distribution of T_n
$N(2, \frac{1}{n})$
$N(\mu_n, \sigma_n^2)$
μ_n = Exact Mean of T_n
σ_n^2 = Exact Variance of T_n
PROBABILITY
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6

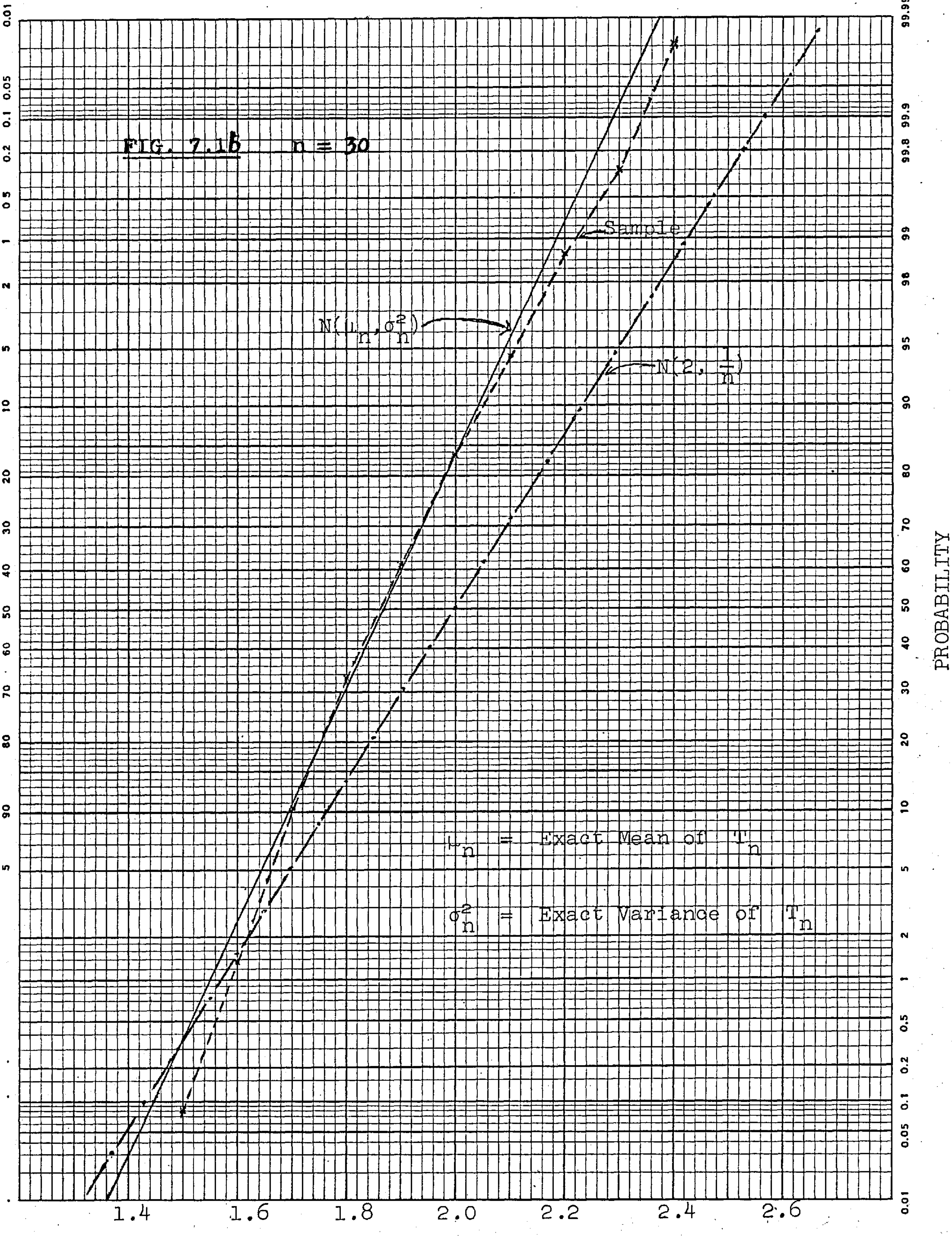
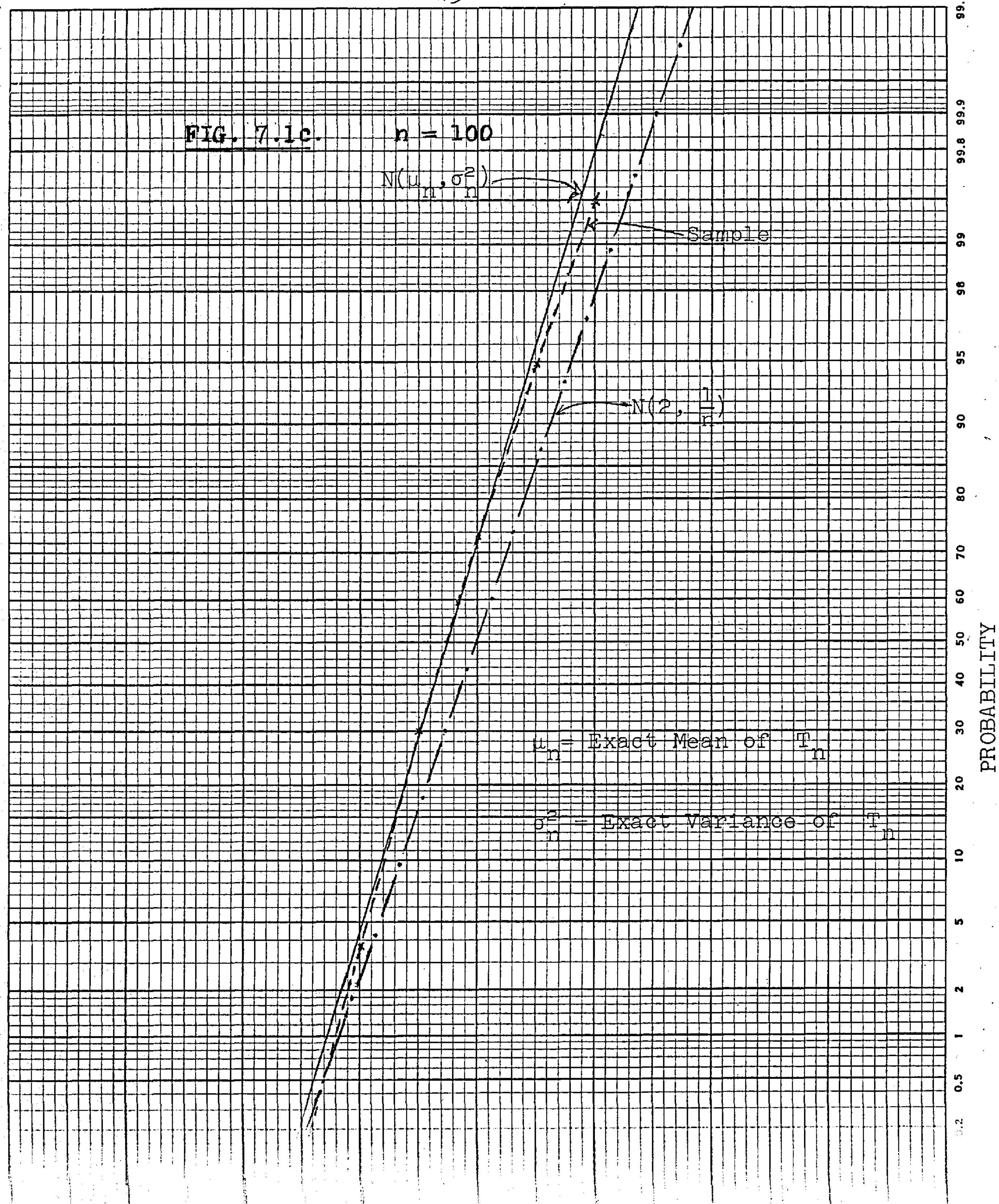


FIG. 7.1b n = 30
Sample
N(μn, σ²n)
N(2, 1/n)
μn = Exact Mean of Tn
σ²n = Exact Variance of Tn
PROBABILITY
1.4
1.6
1.8
2.0
2.2
2.4
2.6
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99

FIG. 7.1c. $n = 100$


N(μn, σ²n)
Sample
N(2, 1/n)
μn = Exact Mean of Tn
σ²n = Exact Variance of Tn
PROBABILITY
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2

8. Illustrative Example and Table of Coefficients.

An example illustrating the test is now given. The test statistic is

$$T_n = \{ \sum_{r=1}^{n} X_{(r)} t_{rn} \}/\{ \sum_{r=1}^{n} X_{(r)} \} . \qquad (8.1)$$

The data are operating hours between successive failures of air conditioning equipment in aircraft (Proschan, 1963) and are ordered here: n = 30

1 3 5 7 11 11 11 12 14 14 14 16 16 20 21 23 42 47 52 62 71 71 87 90 95 120 120 225 246 261.

From (8.1), $T_{30} = 2.0747$.

The standardised variate is

$$\xi = \frac{T_{30} - \mu}{\sigma} = 1.516.$$

Taking γ_1, the skewness, into account, the normal deviate is $Z = \xi - \frac{\gamma_1}{6}(\xi^2 - 1) = 1.435$.

Referring to normal probability integral tables, this is at the 7.6 per cent level of significance for a one-sided test. Hence the data is reasonably consistent with an underlying exponential distribution, as Proschan also concluded.

In this example the calculations, including those of the coefficients, $t_{r,n}$, were done on a computer. In this case the exact values of the moments of T_n can be evaluated as well. The $t_{r,n}$ have been tabulated for $n = 1,2,\ldots, 10$ by Gupta (1960). For the purposes of this test and the incomplete sample test outlined in section 10 a short table of coefficients, for n up to 30, is given in Table 8.1.

TABLE 8.1. THE COEFFICIENTS $t_{rn} = \sum_{i=1}^{r} \frac{1}{n - i + 1}$

n/r	1	2	3	4	5	6	7	8	9	10
1	1.0000									
2	0.5000	1.5000								
3	0.3333	0.8333	1.8333							
4	0.2500	0.5833	1.0833	2.0833						
5	0.2000	0.4500	0.7833	1.2833	2.2833					
6	0.1667	0.3667	0.6167	0.9500	1.4500	2.4500				
7	0.1429	0.3095	0.5095	0.7595	1.0929	1.5929	2.5929			
8	0.1250	0.2679	0.4345	0.6345	0.8845	1.2179	1.7179	2.7179		
9	0.1111	0.2361	0.3790	0.5456	0.7456	0.9956	1.3290	1.8290	2.8290	
10	0.1000	0.2111	0.3361	0.4790	0.6456	0.8456	1.0956	1.4290	1.9290	2.9290
11	0.0909	0.1909	0.3020	0.4270	0.5699	0.7365	0.9365	1.1865	1.5199	2.0199
	3.0199									
12	0.0833	0.1742	0.2742	0.3854	0.5104	0.6532	0.8199	1.0199	1.2699	1.6032
	2.1032	3.1032								
13	0.0769	0.1603	0.2512	0.3512	0.4623	0.5873	0.7301	0.8968	1.0968	1.3468
	1.6801	2.1801	3.1801							
14	0.0714	0.1484	0.2317	0.3226	0.4226	0.5337	0.6587	0.8016	0.9682	1.1682
	1.4182	1.7516	2.2516	3.2516						
15	0.0667	0.1381	0.2150	0.2984	0.3893	0.4893	0.6004	0.7254	0.8682	1.0349
	1.2349	1.4849	1.8182	2.3182	3.3182					
16	0.0625	0.1292	0.2006	0.2775	0.3609	0.4518	0.5518	0.6629	0.7879	0.9307
	1.0974	1.2974	1.5474	1.8807	2.3807	3.3807				
17	0.0588	0.1213	0.1880	0.2594	0.3363	0.4197	0.5106	0.6106	0.7217	0.8467
	0.9896	1.1562	1.3562	1.6062	1.9396	2.4396	3.4396			
18	0.0556	0.1144	0.1769	0.2435	0.3150	0.3919	0.4752	0.5661	0.6661	0.7773
	0.9023	1.0451	1.2118	1.4118	1.6618	1.9951	2.4951	3.4951		

TABLE 8.1 (Continued)

19	0.0526	0.1082	0.1670	0.2295	0.2962	0.3676	0.4445	0.5279	0.6188	0.7188
	0.8299	0.9549	1.0977	1.2644	1.4644	1.7144	2.0477	2.5477	3.5477	
20	0.0500	0.1026	0.1582	0.2170	0.2795	0.3462	0.4176	0.4945	0.5779	0.6688
	0.7688	0.8799	1.0049	1.1477	1.3144	1.5144	1.7644	2.0977	2.5977	3.5977
21	0.0476	0.0976	0.1503	0.2058	0.2646	0.3271	0.3938	0.4652	0.5421	0.6255
	0.7164	0.8164	0.9275	1.0525	1.1954	1.3620	1.5620	1.8120	2.1454	2.6454
	3.6454									
22	0.0455	0.0931	0.1431	0.1957	0.2513	0.3101	0.3726	0.4393	0.5107	0.5876
	0.6709	0.7618	0.8618	0.9730	1.0980	1.2408	1.4075	1.6075	1.8575	2.1908
	2.6908	3.6908								
23	0.0435	0.0889	0.1366	0.1866	0.2392	0.2947	0.3536	0.4161	0.4827	0.5542
	0.6311	0.7144	0.8053	0.9053	1.0164	1.1414	1.2843	1.4510	1.6510	1.9010
	2.2343	2.7343	3.7343							
24	0.0417	0.0851	0.1306	0.1782	0.2282	0.2809	0.3364	0.3952	0.4577	0.5244
	0.5958	0.6727	0.7561	0.8470	0.9470	1.0581	1.1831	1.3260	1.4926	1.6926
	1.9426	2.2760	2.7760	3.7760						
25	0.0400	0.0817	0.1251	0.1706	0.2182	0.2682	0.3209	0.3764	0.4352	0.4977
	0.5644	0.6358	0.7127	0.7961	0.8870	0.9870	1.0981	1.2231	1.3660	1.5326
	1.7326	1.9826	2.3160	2.8160	3.8160					
26	0.0385	0.0785	0.1201	0.1636	0.2091	0.2567	0.3067	0.3593	0.4149	0.4737
	0.5362	0.6029	0.6743	0.7512	0.8345	0.9255	1.0255	1.1366	1.2616	1.4044
	1.5711	1.7711	2.0211	2.3544	2.8544	3.8544				
27	0.0370	0.0755	0.1155	0.1572	0.2006	0.2461	0.2937	0.3437	0.3963	0.4519
	0.5107	0.5732	0.6399	0.7113	0.7882	0.8716	0.9625	1.0625	1.1736	1.2986
	1.4415	1.6081	1.8081	2.0581	2.3915	2.8915	3.8915			
28	0.0357	0.0728	0.1112	0.1512	0.1929	0.2364	0.2818	0.3294	0.3794	0.4321
	0.4876	0.5464	0.6089	0.6756	0.7470	0.8240	0.9073	0.9982	1.0982	1.2093
	1.3343	1.4772	1.6438	1.8438	2.0938	2.4272	2.9272	3.9272		
29	0.0345	0.0702	0.1072	0.1457	0.1857	0.2274	0.2708	0.3163	0.3639	0.4139
	0.4665	0.5221	0.5809	0.6434	0.7101	0.7815	0.8584	0.9418	1.0327	1.1327
	1.2438	1.3688	1.5117	1.6783	1.8783	2.1283	2.4617	2.9617	3.9617	
30	0.0333	0.0678	0.1035	0.1406	0.1790	0.2190	0.2607	0.3042	0.3496	0.3972
	0.4472	0.4999	0.5554	0.6143	0.6768	0.7434	0.8149	0.8918	0.9751	1.0660
	1.1660	1.2771	1.4021	1.5450	1.7117	1.9117	2.1617	2.4950	2.9950	3.9950

9. Power of T_n against a Gamma Alternative.

If the intervals between events, X_i, are independent and distributed as gamma variates with density

$$f(x) = \{\lambda^a x^{a-1} e^{-\lambda x}\}/\Gamma(a) \quad , \quad (a > 0) \tag{9.1}$$

it is possible to obtain the asymptotic relative efficiency (A.R.E.) of the test T, with respect to the asymptotically most powerful test (Moran, 1951) based on the statistic

$$M = -2 \sum_{i=1}^{n} \log(X_i/\overline{X}),$$

where $\overline{X}$ is the sample mean of the X_i.

By conditional distribution arguments (Lewis, 1965) it can be shown that since the distribution of $X/(X + Y)$ is independent of $(X + Y)$ where X, Y are gamma variables

$$E\{X_{(i)}/(\sum_{i=1}^{n} X_i)\} = E\{X_{(i)}\}/\{n\,E(X_i)\}$$

$$= E\{X_{(i)}\}/\{n\,a\} ,$$

where the X_i are from a standardised gamma distribution. Using the density function for order statistics, we have

$$n\ a\ E(T;\ a) = \sum_{i=1}^{n} t_{i,n}\ E(X_{(i)})$$

$$= \int_0^{\infty} x\ f(x) \left[\sum_{i=1}^{n} t_{i,n} \frac{n!}{(i-1)!(n-i)!} \{F(x)\}^{i-1} \{1-F(x)\}^{n-i} \right] dx$$

$$= \int_0^{\infty} x\ f(x)\ \frac{1}{p} \sum_{i=0}^{n} i\ t_{i,n}\ b(n:\ i,p)\ dx \quad , \qquad (9.2)$$

where $b(n:\ i,\ p)$ is the probability of i successes in n binomial trials, and $p \equiv F(x)$ is the probability of success.

Equation (9.2) can be evaluated by using the approximation

$$t_{i,n} = -\log(1 - i/n) + O(1/n^2)$$

and taking the r^{th} moment of the binomial to be approximately $n^r p^r$ $(r = 1,2,\ldots)$.

Hence $p^{-1} \sum_{i=0}^{n} i\ t_{i,n}\ b(n:\ i,p) \doteqdot n(p + \frac{p^2}{2} + \frac{p^3}{3} + \ldots)$.

Now

$$\frac{d}{da}\left\{\int_0^\infty xf(x)\left[F(x)\right]^r dx\right\}\Bigg|_{a=1}$$

$$= \int_0^\infty x(1-e^{-x})^{+r}\,(e^{-x}\log x + \gamma e^{-x})dx$$

$$+ r\int_0^\infty xe^{-x}(1-e^{-x})^{r-1}\int_0^x (e^{-u}\log u + \gamma e^{-u})du\,dx,$$

where $\gamma = 0.5772$.

This can be evaluated and from (9.2) we have that

$$\frac{d}{da}E(T;\ a)\Big|_{a=1} = -0.5$$

to a first order.

The variance of T at $a = 1$ is $\frac{1}{n}$ asymptotically, and using results of Bartholomew (1957), we obtain

$$A.R.E(T;\ M) = \lim_{n\to\infty}\left\{\frac{\left[\frac{d}{da}E(T|a)\Big|_{a=1}\right]^2}{V(T|a=1)} \Bigg/ \frac{\left[\frac{d}{da}E(M|a)\Big|_{a=1}\right]^2}{V(M|a=1)}\right\}$$

$$= 0.388 \qquad (9.3)$$

The above result is obtained assuming that T is asymptotically normal under the alternative gamma hypothesis; this follows from results of Proschan et al (1964). Bartholomew (1957) found the A.R.E. of tests for $a = 1$ based on the statistics

$$S = \sum_{i=1}^{n} \{X_i/(n\bar{X})\}^2$$

and $$\hat{\varpi} = \sum_{i=1}^{n} |X_i - \bar{X}|/(2n\bar{X}),$$

with respect to the test based on M, to be 0.39 and 0.63, respectively. Lewis (1965) found

$$\text{A.R.E.}(S', M) = 0.694$$

where $$S' = 2n - 2\sum_{i=1}^{n} i\, X_{(i)}/(n\bar{X}).$$

These results seem to show that T is not very powerful, at least against a gamma alternative. A closer look at (9.3) reveals that the variance of T tends to its limiting value slowly and this is a major factor in producing the low value of 0.388. If we replace the asymptotic variance of T by the exact value it is found, as in table 9.1 below, that the relative efficiency of T is much higher for small and moderate values of n. The other factors in (9.3) also increase the value of the notional relative efficiency, though not to the same extent as the variance of T. From Table 9.1 the efficiency (e) is given approximately by

$$e = 0.388 + 9.45/n. \tag{9.4}$$

Table 9.1. Notional Relative Efficiency of T_n.

n	5	10	20	30	50	100	∞
Limiting Variance / Exact Variance	5.848	3.145	2.099	1.771	1.507	1.295	1
'Relative Efficiency'	2.269	1.220	0.815	0.687	0.585	0.502	0.388

An empirical comparison of the powers of the tests T and S' was done, for a gamma alternative ($a = 2$, $\lambda = 1$ in (9.1)). For various values of n, T and S' were calculated for the same sample and 100 different sets for each value of n were generated.

The empirical distributions of T and S' were then plotted on arithmetical probability paper (Figs. 9.1). These plots give a description of the behaviour of the two tests under the alternative gamma distribution for all possible significance levels. At any chosen level of significance the power of either test can be read off. A short summary is given in table 9.2 below. The general conclusion from the probability plots is that for values of n up to about 35 T has higher power than S'. Beyond this value of n, S' takes over. These results are for a gamma alternative, but presumably similar results hold for other alternatives. With the well known objections to the use of the statistic M, particularly its sensitivity to recording errors for short intervals, the above is a good point in favour of T.

The statistic, T, has been applied to some other data and these suggest that T might have high power against a wide class of stationary alternatives, in particular renewal alternatives.

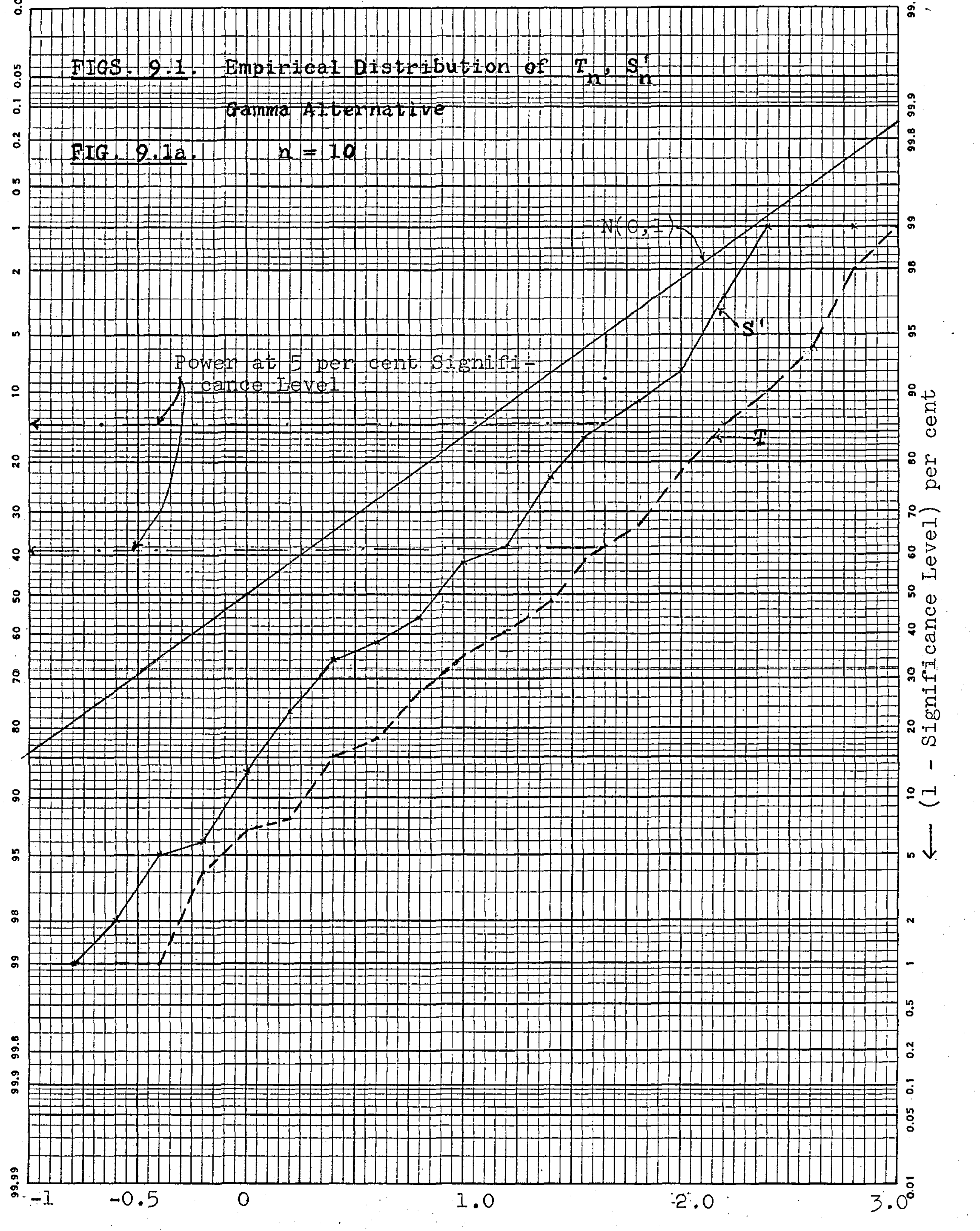
Remarks of Durbin (discussion of Pyke (1965)) on the power of the S' test apply equally to T.

Table 9.2. Empirical Power of T and S'.

N	10		20		30		40	
Significance level	5%	1%	5%	1%	5%	1%	5%	1%
T	.40	.12	.56	.20	.71	.41	.78	.50
S'	.15	.02	.43	.11	.69	.34	.83	.55

The Probability Plots.

The test statistics T_n and S_n' were scaled so that under the null hypothesis each was $N(0,1)$. Samples from the gamma distribution $(a = 2, \lambda = 1)$, i.e. with p.d.f. xe^{-x}, were generated and T_n, S_n' calculated for the same sample of $n (n = 10,20,30,40)$. For each n, 100 values of T_n and S_n' were computed and the empirical cumulative probability obtained for intervals of 0.2. These were then plotted. We expect the better test statistic to have a plot farther away from the null $N(0,1)$ line.


FIGS. 9.1. Empirical Distribution of T_n, S_n'
Gamma Alternative
FIG. 9.1a. n = 10
N(0,1)
S'
T
Power at 5 per cent Significance Level
← (1 - Significance Level) per cent
-1
-0.5
0
1.0
2.0
3.0

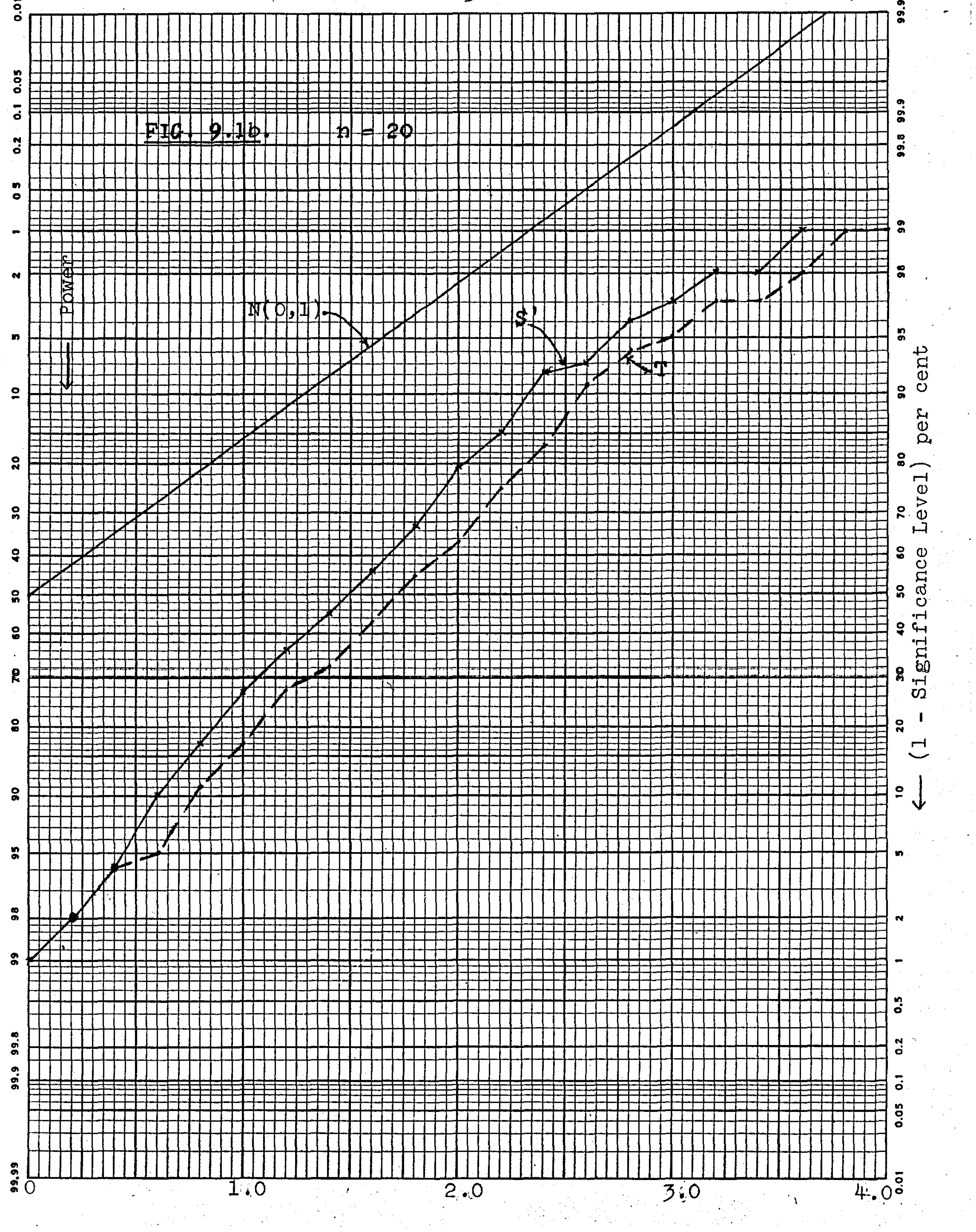


FIG. 9.1b.
n = 20
Power
N(0,1)
S
T
(1 - Significance Level) per cent
0
1.0
2.0
3.0
4.0
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99

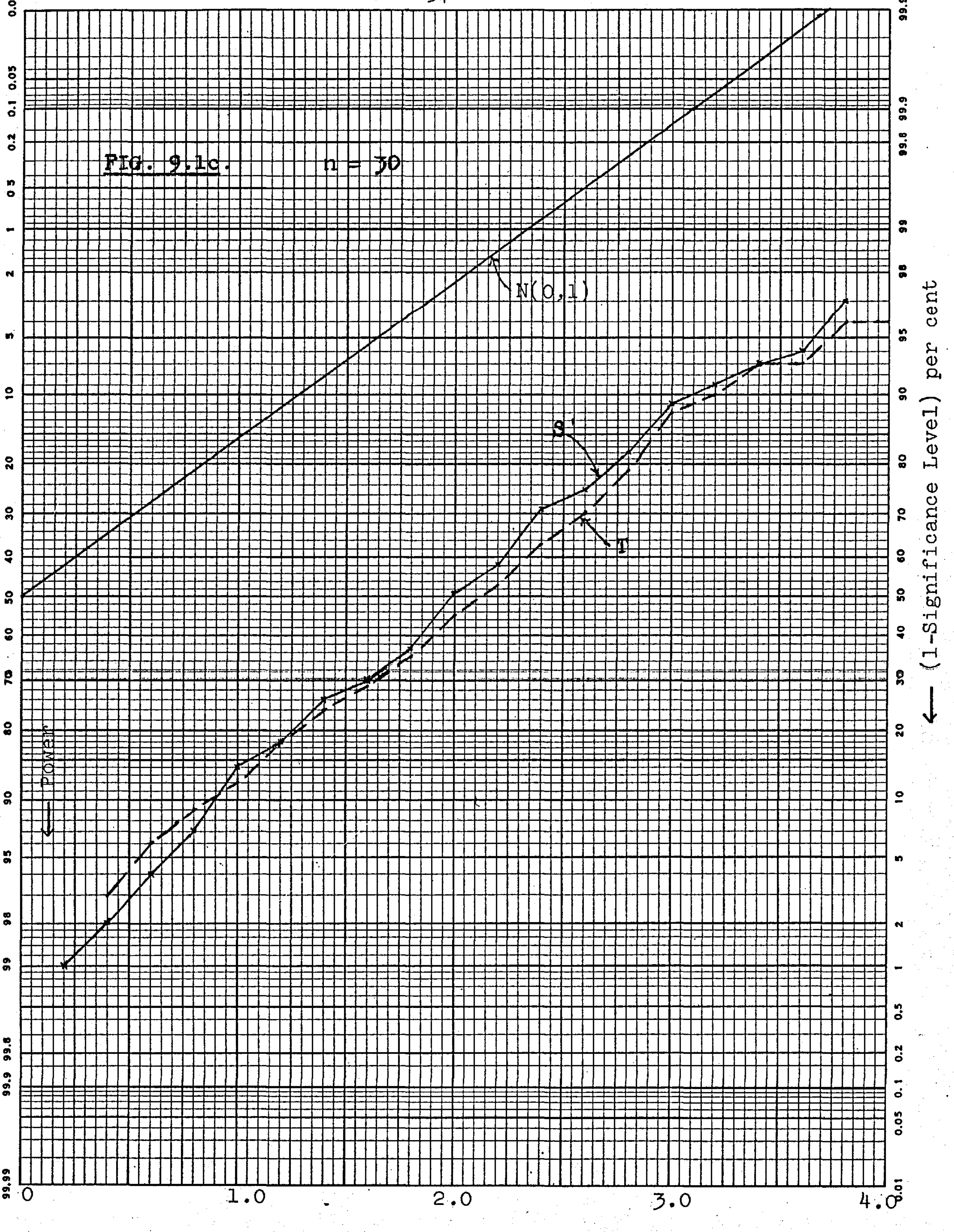


FIG. 9.1c.
n = 30
N(0,1)
S
T
← Power
← (1-Significance Level) per cent
0
1.0
2.0
3.0
4.0
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99

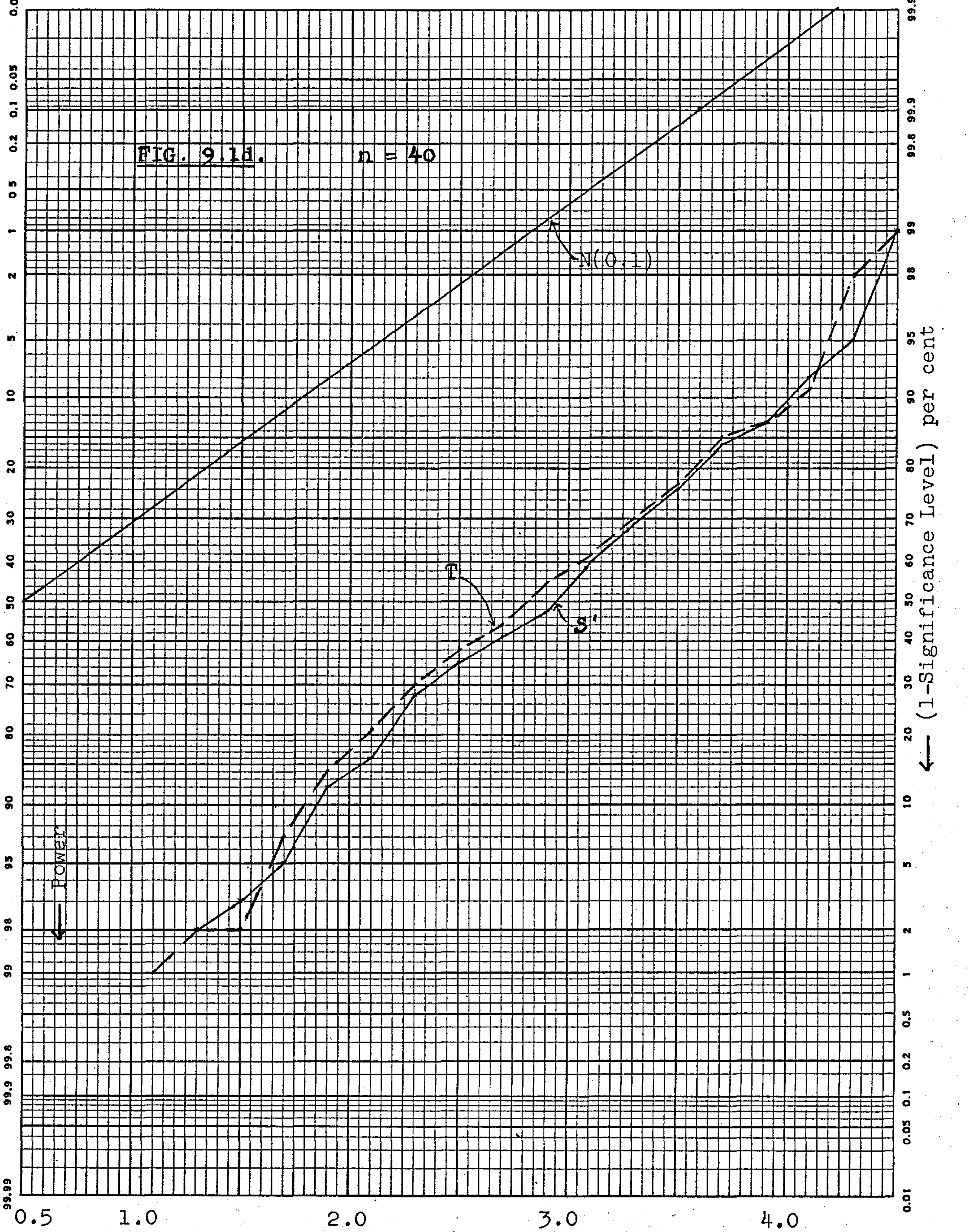

n = 40
N(0,1)
T
S'
Power
(1-Significance Level) per cent
0.5
1.0
2.0
3.0
4.0
0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 30 40 50 60 70 80 90 95 98 99 99.8 99.9 99.99
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01

FIG. 9.1d.

10. Extensions of the Test, T_n.

10.1. Combination of Tests of Significance.

The fact that the sum of chi-squared variables is also distributed as chi-squared can be used to combine the results of tests on several independent sets of data. The tests could be about the same hypothesis, with observations taken at different periods of time.

Suppose we have r sets of data containing $n_1, n_2, \ldots, n_r$ observations. Let $T_1, T_2, \ldots, T_r$ be the values of the statistic, normalised by the appropriate means and variances, so that

$$T_i \doteqdot N(0,1) \qquad (i = 1, \ldots, r).$$

Then a simple combination of the r separate tests is

$$\chi_r^2 = \sum_{i=1}^{r} T_i^2 .$$

A test using this can be performed in the usual way. We obtain a more sensitive test against a common alternative by using

$$\chi_1^2 = (T_1 + T_2 + \ldots + T_r)^2/r \qquad (10.1)$$

since this takes account of the signs of the T's. Now consider a more general form of (10.1). Take

$$T_W = \{W_1 T_1 + \ldots + W_r T_r\} / \{(\sum W_i^2)^{\frac{1}{2}}\} \qquad (10.2).$$

By an argument similar to that used in finding the asymptotic relative efficiency it turns out that, to a first order, the optimum weights required are independent of $n_i (i = 1, \ldots, r)$. Hence (10.1) is the best test within the family (10.2). Further, since $\{\sum T_i / \sqrt{r}\}$ is a standard normal variate, we can also use normal distribution tables in the usual way.

If all the sets of data except one come from an exponential distribution, a good test statistic will be either Max (T_i) or Min (T_i). Tables of percentage points for the extreme deviate are provided by Pearson and Hartley in their Biometrika Tables (Table 25).

10.2. Incomplete Sample Test, T^*.

In life testing and other situations, n items might be put on test and the experiment stopped when r observations are made. If the intervals between failures $X_1, X_2, \ldots, X_r$ have an exponential distribution of unknown mean, θ, the maximum likelihood estimate of θ is

$$\hat{\theta} = \{ \sum_{i=1}^{r} X_{(i)} + (n-r)\, X_{(r)} \}/r, \quad (r = 1,2,\ldots, n) \qquad (10.3)$$

where $X_{(i)}$ $(i = 1,\ldots, r)$ are the ordered values of the X_i. To test the assumption that the first r observations come from an exponential distribution we take as our test statistic

$$T^{*} = \{ \sum_{i=1}^{r} t_{i,n}\, X_{(i)} \}/\{ \sum_{i=1}^{r} X_{(i)} + (n-r)\, X_{(r)} \} \qquad (10.4)$$

$$= \{ \sum_{i=1}^{r} c_{i,n}^{(r)}\, V_i \}/\{ \sum_{i=1}^{r} V_i \}, \qquad (10.5)$$

where, as before, $V_i = (n-i+1)\theta\left[X_{(i)} - X_{(i-1)} \right]$,

and the constants $c_{i,n}^{(r)}$ now depend on r as well as i and n.

Equating coefficients in (10.4) and (10.5) we obtain

$$n\, c_{1,n}^{(r)} = t_{1,n} + \ldots t_{r,n} \qquad (10.6)$$

$$= \frac{r}{n} + \frac{r-1}{n-1} + \ldots + \frac{1}{n-r+1}$$

and

$$(n-i+1)C^{(r)}_{i,n} = (n-i+2)C^{(r)}_{i-1,n} - (\frac{1}{n} + \dots + \frac{1}{n-i+2})$$

$$= nC^{(r)}_{1,n} - \frac{1}{n} - (\frac{1}{n} + \frac{1}{n-1}) \dots -(\frac{1}{n} + \dots + \frac{1}{n-i+2}).$$

This simplifies to give

$$C^{(r)}_{i,n} = t_{i-1,n} + \frac{nC^{(r)}_{1,n} - i+1}{n-i+1}, \qquad i = 1,2,\dots, r$$

$$= C_{i,n} - \frac{n(1-C^{(r)}_{1,n})}{n-i+1} \tag{10.7}$$

where the $C_{i,n}$ are the coefficients of V_i for the complete sample test given in (2.7).

As before, the independence of T^* and $\sum V_i$ follow and by the same method the moments of T^* can be obtained as

$$\mu = E(T^*) = \frac{1}{r} \sum_{i=1}^{r} C^{(r)}_{i,n} \tag{10.8}$$

$$= 1 + \frac{n}{r}C^{(r)}_{1,n} - \frac{t_{r,n}}{r} - \frac{n}{r} \; t_{r,n} \left[1-C^{(r)}_{1,n}\right]$$

and

$$\mu_2 = V(T^*) = \frac{\sum_{i=1}^{r} (C_{i,n}^{(r)})^2}{r(r+1)} - \frac{\left[\sum_{i=1}^{r} C_{i,n}^{(r)}\right]^2}{r^2(r+1)} \qquad (10.9).$$

It is difficult to obtain explicit expressions for the variance and other moments of T^* but in view of the results on the power of the T test it is plausible that the T^* test also has high power for small and moderate values of n, against similar alternatives. Most of the results on the T test would apply to the T^* test with proofs following similar lines. Thus, T^* is a normal variate with mean and variance as above. It can be used to test incomplete samples from the exponential distribution and table 10.1 is provided for this purpose.

As an illustration of the T^* test an example given by Epstein (1960, Part II) is considered. Twenty items are placed on test and the test is discontinued after 11 failures occur. The times between failures in ascending order are 1, 3, 3, 5, 5, 5, 7, 9, 13, 13, 16. $n = 20$, $r = 11$.

The test statistic is

$$T^{*}_{11,20} = \{ \sum_{i=1}^{11} t_{i,20} X_{(i)} \} / \{ \sum_{i=1}^{11} X_{(i)} + 9X_{(11)} \}$$

$$= 0.1827 .$$

From table 10.1, the mean and variance are 0.1886 and 0.0128, and the standardised normal variate is thus

$$Z = -0.4589.$$

The sample is thus accepted as coming from an exponential distribution. Epstein reached the same conclusion.

TABLE 10.1. MEAN AND VARIANCE OF T^*

n/r	1	2	3	4	5	6	7	8	9	10
2	0.2500	1.2500								
	0.0000	0.1443								
3	0.1111	0.4028	1.3889							
	0.0000	0.0080	0.1712							
4	0.0625	0.2014	0.5255	1.4792						
	0.0000	0.0040	0.0166	0.1814						
5	0.0400	0.1212	0.2854	0.6258	1.5433					
	0.0000	0.0051	0.0097	0.0235	0.1850					
6	0.0278	0.0811	0.1808	0.3612	0.7095	1.5917				
	0.0000	0.0045	0.0095	0.0136	0.0291	0.1856				
7	0.0204	0.0581	0.1253	0.2382	0.4294	0.7807	1.6296			
	0.0000	0.0038	0.0083	0.0128	0.0166	0.0337	0.1847			
8	0.0156	0.0437	0.0921	0.1697	0.2922	0.4907	0.8422	1.6603		
	0.0000	0.0031	0.0070	0.0112	0.0152	0.0190	0.0375	0.1829		
9	0.0123	0.0340	0.0706	0.1274	0.2131	0.3428	0.5461	0.8960	1.6857	
	0.0000	0.0026	0.0059	0.0096	0.0135	0.0171	0.0210	0.0408	0.1806	
10	0.0100	0.0273	0.0558	0.0992	0.1628	0.2548	0.3899	0.5964	0.9436	1.7071
	0.0000	0.0022	0.0050	0.0082	0.0118	0.0153	0.0185	0.0227	0.0436	0.1782
11	0.0083	0.0224	0.0453	0.0796	0.1286	0.1976	0.2947	0.4338	0.6423	0.9860
	1.7255									
	0.0000	0.0019	0.0042	0.0071	0.0102	0.0135	0.0167	0.0197	0.0243	0.0459
	0.1756									
12	0.0069	0.0187	0.0375	0.0653	0.1043	0.1580	0.2315	0.3326	0.4748	0.6843
	1.0243	1.7414								
	0.0000	0.0016	0.0036	0.0061	0.0089	0.0119	0.0149	0.0179	0.0207	0.0257
	0.0480	0.1730								
13	0.0059	0.0158	0.0316	0.0545	0.0863	0.1294	0.1871	0.2642	0.3685	0.5131
	0.7231	1.0589	1.7554							
	0.0000	0.0014	0.0032	0.0053	0.0077	0.0104	0.0133	0.0161	0.0188	0.0215
	0.0270	0.0497	0.1705							

TABLE 10.1 (Continued)

14	0.0051	0.0136	0.0269	0.0462	0.0727	0.1080	0.1546	0.2156	0.2958	0.4027
	0.5489	0.7589	1.0904	1.7677						
	0.0000	0.0012	0.0028	0.0046	0.0068	0.0092	0.0117	0.0144	0.0170	0.0195
	0.0223	0.0281	0.0513	0.1680						
15	0.0044	0.0118	0.0232	0.0397	0.0621	0.0916	0.1300	0.1795	0.2433	0.3261
	0.4351	0.5826	0.7921	1.1194	1.7788					
	0.0000	0.0011	0.0024	0.0041	0.0060	0.0081	0.0104	0.0129	0.0153	0.0178
	0.0201	0.0229	0.0292	0.0526	0.1655					
16	0.0039	0.0103	0.0203	0.0345	0.0536	0.0787	0.1109	0.1520	0.2041	0.2703
	0.3552	0.4659	0.6142	0.8230	1.1460	1.7887				
	0.0000	0.0010	0.0022	0.0036	0.0053	0.0072	0.0093	0.0115	0.0138	0.0161
	0.0184	0.0206	0.0235	0.0302	0.0538	0.1631				
17	0.0035	0.0091	0.0178	0.0302	0.0468	0.0683	0.0958	0.1304	0.1738	0.2281
	0.2964	0.3831	0.4951	0.6440	0.8519	1.1706	1.7977			
	0.0000	0.0009	0.0019	0.0032	0.0048	0.0065	0.0083	0.0103	0.0125	0.0146
	0.0168	0.0189	0.0211	0.0241	0.0312	0.0549	0.1608			
18	0.0031	0.0081	0.0158	0.0267	0.0412	0.0599	0.0836	0.1132	0.1500	0.1954
	0.2516	0.3217	0.4099	0.5230	0.6722	0.8790	1.1935	1.8058		
	0.0000	0.0008	0.0017	0.0029	0.0043	0.0058	0.0075	0.0093	0.0113	0.0133
	0.0153	0.0173	0.0194	0.0215	0.0246	0.0320	0.0558	0.1586		
19	0.0028	0.0072	0.0141	0.0238	0.0366	0.0530	0.0736	0.0993	0.1308	0.1694
	0.2166	0.2745	0.3461	0.4356	0.5495	0.6988	0.9044	1.2148	1.8133	
	0.0000	0.0007	0.0016	0.0026	0.0039	0.0052	0.0068	0.0084	0.0102	0.0121
	0.0140	0.0159	0.0178	0.0197	0.0219	0.0251	0.0328	0.0566	0.1564	
20	0.0025	0.0065	0.0127	0.0213	0.0327	0.0472	0.0654	0.0878	0.1151	0.1483
	0.1886	0.2374	0.2968	0.3697	0.4602	0.5748	0.7241	0.9283	1.2347	1.8201
	0.0000	0.0006	0.0014	0.0024	0.0035	0.0048	0.0062	0.0077	0.0093	0.0110
	0.0128	0.0146	0.0164	0.0182	0.0201	0.0222	0.0256	0.0336	0.0573	0.1544

The first line (or two lines) for each n, represents the mean values, and the next line (or two lines) the variance of T^{*}.

PART II

TESTS OF SEPARATE FAMILIES OF HYPOTHESES

Tests of Separate Families of Hypotheses.

11. Introduction.

In two recent papers Cox (1961, 1962) proposed tests for separate families of hypotheses. It is desired to test a composite null hypothesis, H_f say, against an alternative hypothesis H_g which is separate from H_f in the sense that an arbitrary simple hypothesis in H_f cannot be obtained as a limit of simple hypotheses in H_g.

Suppose $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed and have p.d.f. $f(\underline{Y}, \underline{\alpha})$ under H_f and $g(\underline{Y}, \underline{\beta})$ under H_g. Cox proposed tests based on the logarithm of the likelihood ratio

$$L_{fg} = \log \frac{f(\underline{Y}, \hat{\underline{\alpha}})}{g(\underline{Y}, \hat{\underline{\beta}})} ,$$

where $\hat{\underline{\alpha}}, \hat{\underline{\beta}}$ denote the maximum likelihood estimates of the parameters under H_f and H_g respectively.

If H_f is the null hypothesis and H_g is the alternative, the test statistic considered was

$$T_f = L_{fg} - E_{\hat{\underline{\alpha}}}(L_{fg}),$$

where $E_{\hat{\underline{\alpha}}}(L_{fg})$ is the expected value under the p.d.f. $f(\underline{Y}, \underline{\alpha})$. Now under H_f, $\hat{\underline{\beta}}$ converges in probability to $\underline{\beta}_{\alpha}$, and

$$E_{\underline{\alpha}}\left[\frac{\partial\{\log g(\underline{Y}, \underline{\beta}_{\underline{\alpha}})\}}{\partial\beta_i}\right] = 0 \quad , \quad \underline{\beta} = \{\beta_i\},\ \underline{\alpha} = \{\alpha_i\} .$$

Writing

$$F = \log f(\underline{Y},\underline{\alpha}),\ F_{\alpha_i} = \frac{\partial \log f(\underline{Y},\underline{\alpha})}{\partial\alpha_i},\ F_{\alpha_i\alpha_j} = \frac{\partial^2 \log f(\underline{Y},\underline{\alpha})}{\partial\alpha_i\ \partial\alpha_j} ,$$

$$G = \log g(\underline{Y},\underline{\beta}),\ G_{\beta_i} = \frac{\partial \log g(\underline{Y},\underline{\beta})}{\partial\beta_i},\ G_{\beta_i\beta_j} = \frac{\partial^2 \log g(\underline{Y},\underline{\beta})}{\partial\beta_i\ \partial\beta_j} ,$$

etc.

Cox showed that T_f is asymptotically normally distributed with zero mean and variance

$$V_{\underline{\alpha}}(T_f) = n\{V_{\underline{\alpha}}(F-G) - \sum_i \frac{C^2_{\underline{\alpha}}(F-G, F_{\alpha_i})}{V_{\underline{\alpha}}(F_{\alpha_i})}\} .$$

When H_g is the null hypothesis and H_f the alternative the test statistic is

$$T_g = L_{gf} - E_{\hat{\underline{\beta}}}(L_{gf}).$$

This is again asymptotically normally distributed with zero mean and variance

$$V_{\underline{\beta}}(T_g) = n\{V_{\underline{\beta}}(G-F) - \sum_i \frac{C^2_{\underline{\beta}}(G-F, G_{\beta_i})}{V_\beta(G_{B_i})}\} .$$

Here $\hat{\underline{\alpha}}$ converges in probability to $\underline{\alpha}_\beta$.

We can therefore consider

$$T'_f = T_f/\{V(T_f)\}^{\frac{1}{2}} , \quad T'_g = T_g/\{V(T_g)\}^{\frac{1}{2}} ,$$ as

approximately N(0,1) variates and perform tests in the usual way.

A large negative value of T'_f indicates departure from H_f in the direction of H_g. Similarly, a large negative T'_g indicates departure from H_g in the direction of H_f. A large negative value of T'_f (or T'_g) and a large positive T'_g (or T'_f) would indicate that the sample is consistent with neither H_f nor H_g.

In the specific case when H_f is the hypothesis that the p.d.f. is log-normal and H_g that the p.d.f. is exponential, test statistics T_f and T_g were derived and their large sample variances obtained.

We now investigate the adequacy of the asymptotic results for the log-normal versus exponential case. We also derive power functions of the tests T_f, T_g when the other hypothesis serves as the alternative. The methods indicated above are then used to derive tests when H_f is the hypothesis that the p.d.f. is log-normal and H_g is that the p.d.f. is gamma. Finally, we apply the tests to some data on wool fibre-diameter considered by Monfort (1964).

12. The Log-Normal Distribution versus the Exponential Distribution.

12.1 Adequacy of Asymptotic Null Distributions of T_f and T_g.

In this section we use Taylor Series expansions to obtain corrections to the asymptotic results obtained by Cox (1961). We derive power functions of the tests T_f, T_g and give some empirical results.

Suppose $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed. The null hypothesis H_f is that the p.d.f. is log-normal and the alternative, H_g is that the p.d.f. is exponential, i.e.

$$H_f: \quad f(y,\underline{\alpha}) = \frac{1}{y\sqrt{2\pi\alpha_2}} \exp\left\{-\frac{(\log y - \alpha_1)^2}{2\alpha_2}\right\},$$

$$H_g: \quad g(y, \beta) = \beta^{-1} e^{-y/\beta} .$$

Here $\hat{\alpha}_1, \hat{\alpha}_2$ are the sample mean and variance of $\log Y_i$ respectively, and $\hat{\beta}$ is the sample mean of the Y_i. Under H_f $\hat{\beta}$ converges to $\beta_{\underline{\alpha}} = e^{\alpha_1 + (1/2)\alpha_2}$.

For H_f we have the test statistic

$$T_f = \frac{\sqrt{n}\,\log(\hat{\beta}/\beta_{\hat{\underline{\alpha}}})}{\left[e^{\hat{\alpha}_2} - 1 - \hat{\alpha}_2 - \frac{1}{2}\hat{\alpha}_2^2\right]^{\frac{1}{2}}} = \sqrt{n}\, f(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}) \qquad (12.1).$$

Asymptotically, T_f has a standard normal distribution, $N(0,1)$. For the distribution of T_f in finite samples we require closer approximation to the mean and variance.

Writing $A = \left[e^{\hat{\alpha}_2} - 1 - \hat{\alpha}_2 - \frac{1}{2}\hat{\alpha}_2^2\right]^{-\frac{1}{2}}$, (12.2)

we have from (12.1) that

$$\frac{\partial f}{\partial \hat{\alpha}_1} = -A, \quad \frac{\partial f}{\partial \hat{\beta}} = \frac{A}{\hat{\beta}}, \quad \frac{\partial f}{\partial \hat{\alpha}_2} = -\frac{A}{2} - \frac{A^3}{2}(e^{\hat{\alpha}_2}-1-\hat{\alpha}_2)\log(\hat{\beta}/\beta_{\hat{\alpha}}),$$

$$\frac{\partial^2 f}{\partial \hat{\alpha}_1^2} = \frac{\partial^2 f}{\partial \hat{\beta}\partial \hat{\alpha}_1} = 0, \quad \frac{\partial^2 f}{\partial \hat{\beta}^2} = -A/(\hat{\beta}^2), \quad \frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2} = \frac{A^3}{2}(e^{\hat{\alpha}_2}-1-\hat{\alpha}_2),$$

$$\frac{\partial^2 f}{\partial \hat{\alpha}_2 \partial \hat{\beta}} = -\frac{A^3}{2\hat{\beta}}(e^{\hat{\alpha}_2}-1-\hat{\alpha}_2), \quad \frac{\partial^2 f}{\partial \hat{\alpha}_2^2} = \frac{A^3}{2}(e^{\hat{\alpha}_2}-1-\hat{\alpha}_2)$$

$$+ \frac{3A^5}{4}(e^{\hat{\alpha}_2}-1-\hat{\alpha}_2)^2 \log(\hat{\beta}/\beta_{\hat{\alpha}}) - \frac{1}{2}A^3(e^{\hat{\alpha}_2}-1)\log(\hat{\beta}/\beta_{\hat{\alpha}}).$$

(12.3)

Now expanding $f(\cdot)$ about $\alpha_1, \alpha_2, \beta_\alpha$ and leaving out zero terms and those of order three and higher we have

$$f(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}) \equiv f(\alpha_1,\alpha_2,\beta_\alpha) + (\hat{\alpha}_1-\alpha_1)\frac{\partial f}{\partial \hat{\alpha}_1} + (\hat{\alpha}_2-\alpha_2)\frac{\partial f}{\partial \hat{\alpha}_2} + (\hat{\beta}-\beta_\alpha)\frac{\partial f}{\partial \beta}$$

$$+ (\hat{\alpha}_1-\alpha_1)(\hat{\alpha}_2-\alpha_2)\frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2} + \frac{1}{2}(\hat{\alpha}_2-\alpha_2)^2 \frac{\partial^2 f}{\partial \hat{\alpha}_2^2}$$

$$+ (\hat{\alpha}_2-\alpha_2)(\hat{\beta}-\beta_\alpha)\frac{\partial^2 f}{\partial \hat{\alpha}_2 \partial \hat{\beta}} + \frac{1}{2}(\hat{\beta}-\beta_\alpha)^2 \frac{\partial^2 f}{\partial \hat{\beta}^2} + \dots \quad . \qquad (12.4)$$

In the derivatives of the function $f(\cdot)$ we replace $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}$ by α_1, α_2, β_α. Taking expectations in (12.4) and noting that $E(\hat{\alpha}_1) = \alpha_1$, $E(\hat{\alpha}_2) = \frac{n-1}{n}\alpha_2$, $E(\hat{\beta}) = \beta_\alpha$, we have from (12.3) and results of Cox (1961, p.115) that

$$E\{f(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta})\} = -\frac{A}{2n}(e^{\alpha_2}-1-\alpha_2) + O(n^{-2}).$$

Hence
$$E(T_f) = -\frac{1}{2\sqrt{n}} \frac{e^{\alpha_2}-1-\alpha_2}{\{e^{\alpha_2}-1-\alpha_2-\frac{1}{2}\alpha_2^2\}^{\frac{1}{2}}} + O(n^{-\frac{3}{2}}) \quad .$$

(12.5)

A graph of this correction to the mean of T_f is given in Fig. 12.1.

The Taylor Series expansion for the variance of $f(\cdot)$ is rather complicated and appears to converge slowly. The result is therefore not very useful and is not given here. Instead empirical results on the variance of T_f are given in section 12.3.

Now suppose H_f and H_g change roles so that the null distribution is the exponential and the log-normal the alternative.

We now have the test statistic

$$T_g = \frac{\{\hat{\alpha}_1+\frac{1}{2}\log \hat{\alpha}_2-\log \hat{\beta}\} -\psi(1)-\frac{1}{2}\log\psi'(1)}{0.532/\sqrt{n}} \tag{12.6}$$

where $\hat{\alpha}_1 \longrightarrow \alpha_{1,\beta} = \log \beta + \psi(1)$, $\hat{\alpha}_2 \longrightarrow \alpha_{2,\beta} = \psi'(1)$

$\hat{\beta} \longrightarrow \beta$, $\psi(x) = \frac{d}{dx}\{\log \Gamma(x)\}$ and $\psi'(x), \psi''(x)$

etc., are derivatives of $\psi(x)$.

T_g is also a standard normal statistic.

Let $g(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}) \equiv \hat{\alpha}_1 + \frac{1}{2} \log \hat{\alpha}_2 - \log \hat{\beta}$.

Then

$$\frac{\partial g}{\partial \hat{\alpha}_1} = 1, \quad \frac{\partial g}{\partial \hat{\alpha}_2} = \frac{1}{2\hat{\alpha}_2}, \quad \frac{\partial g}{\partial \hat{\beta}} = -\frac{1}{\hat{\beta}},$$

$$\frac{\partial^2 g}{\partial \hat{\alpha}_1^2} = \frac{\partial^2 g}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2} = \frac{\partial^2 g}{\partial \hat{\alpha}_1 \partial \hat{\beta}} = \frac{\partial^2 g}{\partial \hat{\alpha}_2 \partial \hat{\beta}} = 0, \tag{12.7}$$

$$\frac{\partial^2 g}{\partial \hat{\alpha}_2^2} = -\frac{1}{2\hat{\alpha}_2^2}, \qquad \frac{\partial^2 g}{\partial \hat{\beta}^2} = \frac{1}{\hat{\beta}^2}.$$

Now expanding $g(\cdot)$ about $(\alpha_{1,\beta}, \alpha_{2,\beta}, \beta)$ to 2^{nd} order and leaving out zero terms we get

$$g(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}) \equiv g(\alpha_{1,\beta}, \alpha_{2,\beta}, \beta) + (\hat{\alpha}_1 - \alpha_{1,\beta}) \frac{\partial g}{\partial \hat{\alpha}_1}$$

$$+ (\hat{\alpha}_2 - \alpha_{2,\beta}) \frac{\partial g}{\partial \hat{\alpha}_2} + (\hat{\beta} - \beta) \frac{\partial g}{\partial \hat{\beta}} + \frac{1}{2}(\hat{\alpha}_2 - \alpha_{2,\beta})^2 \frac{\partial^2 g}{\partial \hat{\alpha}_2^2} \tag{12.8}$$

$$+ \frac{1}{2}(\hat{\beta} - \beta)^2 \frac{\partial^2 g}{\partial \hat{\beta}^2} + \ldots$$

where the derivatives are evaluated at $\alpha_{1,\beta}$, $\alpha_{2,\beta}$ and β. Taking expectations in (12.8) we obtain

$$E\{g(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta})\} = \psi(1)+\tfrac{1}{2}\log \psi'(1) - \frac{1}{2n} - \frac{\psi'''(1)}{4n\{\psi'(1)\}^2} + o(\frac{1}{n^2}).$$

Hence from (12.6)

$$E(T_g) = -\frac{1}{0.532\sqrt{n}}\{0.5 + \frac{\psi'''(1)}{4[\psi'(1)]^2}\} + o(\frac{1}{n^{3/2}})$$

$$= -\frac{2.0666}{\sqrt{n}} + o(\frac{1}{n^{3/2}}) . \qquad (12.9)$$

Here again the analytical result for the variance of T_g is not useful. However, some empirical results are given in section 12.3.

12.2. Power Functions of the Test Statistics.

It is possible to obtain the power of the test T_f against the alternative hypothesis, H_g, that the p.d.f. is exponential. Similarly, the power of T_g against the alternative, H_f, can be obtained. We assume that the distributions of T_f and T_g under the respective alternatives are normal.

We require the mean and variance of T_f when the p.d.f. is exponential. Using series expansions and the notation introduced earlier, we have

$$E\{f(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta})\} = E\{f(\alpha_{1,\beta},\alpha_{2,\beta},\beta)+\ldots\}$$

$$= -A(\psi+\tfrac{1}{2}\psi')+\frac{A\psi'}{2n} - \frac{1}{2n}\{A^3\psi'(\psi+\tfrac{1}{2}\psi')(e^{\psi'}-1-\psi')\}$$

$$+ \frac{1}{2n}\{A^3\psi''(e^{\psi'}-1-\psi')\} - \frac{A}{2n}$$

$$+ \{\frac{\psi'''+2(\psi')^2}{8n}\}\{2A^3(e^{\psi'}-1-\psi')-3A^3(e^{\psi'}-1-\psi')^2(\psi+\tfrac{1}{2}\psi')$$

$$+ 2A^3(e^{\psi'}-1)(\psi+\tfrac{1}{2}\psi')\} + 0(n^{-2})$$

$$= -0.2255 + \frac{1.7221}{n} + 0(n^{-2})$$

where $A = \{e^{\psi'}-1-\psi'-\tfrac{1}{2}(\psi')^2\}^{-\frac{1}{2}}$.

Therefore

$$E(T_f|H_g) = -0.2255\sqrt{n} + \frac{1.7221}{\sqrt{n}} + 0(n^{-\frac{3}{2}}) . \quad (12.10)$$

The series expansion method for the variance appears to be successful, even to first order, and we have

$$V\{f(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta})\} \equiv V\{f(\alpha_{1,\beta},\alpha_{2,\beta},\beta)+(\hat{\alpha}_1-\alpha_1)\frac{\partial f}{\partial \hat{\alpha}_1}+\dots\}$$

$$= \frac{\psi'}{n}\left(\frac{\partial f}{\partial \hat{\alpha}_1}\right)^2 + \frac{\psi'''+2(\psi')^2}{n}\left(\frac{\partial f}{\partial \hat{\alpha}_2}\right)^2 + \frac{\beta^2}{n}\left(\frac{\partial f}{\partial \hat{\beta}}\right)^2$$

$$+ \left(\frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2}\right)^2 \left\{\mu_{220} - \frac{(\psi'')^2}{n^2}\right\}$$

$$+ \frac{1}{4}\left(\frac{\partial^2 f}{\partial \hat{\alpha}_2^2}\right)^2 \left[\mu_{040} - \frac{\{\psi'''+2(\psi')^2+\dots\}^2}{n^2}\right]$$

$$+ \left(\frac{\partial^2 f}{\partial \hat{\alpha}_2 \partial \hat{\beta}}\right)^2 \cdot \frac{\beta^2\{\psi'''+2(\psi')^2+(\psi')^2/n\}}{n^2}$$

$$+ \frac{1}{4}\left(\frac{\partial^2 f}{\partial \hat{\beta}^2}\right)^2 \{\mu_{004} - \beta^4/n^2\} + \dots +$$

$$V\{f(\cdot)\} \doteq + \ldots + 2\frac{\psi''}{n}\left(\frac{\partial f}{\partial \hat{\alpha}_1}\right)\left(\frac{\partial f}{\partial \hat{\alpha}_2}\right) + \frac{2\beta}{n}\left(\frac{\partial f}{\partial \hat{\beta}}\right)\left(\frac{\partial f}{\partial \hat{\alpha}_1}\right) + 2\mu_{210}\frac{\partial f}{\partial \hat{\alpha}_1}\frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2}$$

$$+ \mu_{120}\frac{\partial f}{\partial \hat{\alpha}_1}\frac{\partial^2 f}{\partial \hat{\alpha}_2^2} + 2\mu_{111}\frac{\partial f}{\partial \hat{\alpha}_1}\frac{\partial^2 f}{\partial \hat{\alpha}_2 \partial \hat{\beta}} + \mu_{102}\frac{\partial f}{\partial \hat{\alpha}_1}\frac{\partial^2 f}{\partial \hat{\beta}^2}$$

$$+ 2\left(\mu_{120} - \frac{\psi'\psi''}{n^2}\right)\left(\frac{\partial f}{\partial \hat{\alpha}_2}\frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2}\right) + \mu_{030}\frac{\partial f}{\partial \hat{\alpha}_2}\frac{\partial^2 f}{\partial \hat{\alpha}_2^2}$$

$$+ 2\mu_{111}\frac{\partial f}{\partial \hat{\beta}}\frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2} + \mu_{003}\frac{\partial f}{\partial \hat{\beta}}\frac{\partial^2 f}{\partial \hat{\beta}^2}$$

$$+ \left(\frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2}\frac{\partial^2 f}{\partial \hat{\alpha}_2^2}\right)\left[\mu_{130} - \frac{\psi''\{\psi''' + 2(\psi')^2 + (\psi')^2/n\}}{n^2}\right]$$

$$+ 2\mu_{121}\frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2}\frac{\partial^2 f}{\partial \hat{\alpha}_2 \partial \hat{\beta}} + \left(\mu_{112} - \frac{\psi''\beta^2}{n^2}\right)\left(\frac{\partial^2 f}{\partial \hat{\alpha}_1 \partial \hat{\alpha}_2}\frac{\partial^2 f}{\partial \hat{\beta}^2}\right) + \ldots$$

$$= \frac{B}{n} + \frac{C}{n^2} + \ldots \qquad (12.11)$$

where $\mu_{220} = E\{(\hat{\alpha}_1 - \alpha_1)^2(\hat{\alpha}_2 - \alpha_2)^2\}$ etc., and

$$B = A^2\Big[1+\psi'+\tfrac{1}{4}\{\psi'''+2(\psi')^2\}$$

$$\{1-2A^2(\psi+\tfrac{1}{2}\psi')(e^{\psi'}-1-\psi')+A^4(\psi+\tfrac{1}{2}\psi')^2(e^{\psi'}-1-\psi')^2\}\Big]$$

$$+A^2\Big[\psi''-2-A^2\psi''(\psi+\tfrac{1}{2}\psi')(e^{\psi'}-1-\psi')\Big]$$

$$= 0.1473.$$

The expression for C is rather complicated and difficult to evaluate. However, neglecting the term in C does not seem to affect the power calculations very much.

Hence to a first order,

$$V(T_f \mid H_g) = 0.1473 + 0(\tfrac{1}{n}) \quad . \qquad (12.12)$$

Now suppose that the standard normal deviate corresponding to a level of significance α is λ_α. This is in fact negative since under H_g, T_f is negative and we consider one sided significance levels. Then the power of the test T_f under H_g is

$$P_f = \Phi\left(\frac{\lambda_\alpha - \mu}{\sigma}\right) \text{, for a level of significance } \alpha$$

where μ and σ^2 are given by (12.11) and (12.12), and where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du$.

Simplifying the last expression, we get

$$P_f \equiv P_f(n) = \Phi(2.606\,\lambda_\alpha + 0.5875\sqrt{n} - 4.487/\sqrt{n}) \tag{12.13}$$

In particular, if we take $\alpha = 0.05$, then $\lambda_\alpha = -1.64$, and

$$P_f = \Phi(-4.273 + 0.5875\sqrt{n} - 4.487/\sqrt{n}) \tag{12.14}$$

Table 12.1 gives this function.

A point of interest is the value of n which gives 50 per cent power. For a 5 per cent level of significance we have that if n^* is the appropriate sample number

$$E(T_f \mid H_g,\ n^*) = -1.64.$$

From (12.10)

$$n^* = 67.$$

The distribution of T_g under the log-normal hypothesis can now be obtained. For this we have

$$E\{g(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta})\} = E\{g(\alpha_1,\alpha_2,\beta_\alpha) + \ldots\}$$

$$= \frac{1}{2}(\log \alpha_2 - \alpha_2) + \frac{e^{\alpha_2} - 3}{2n} + o\left(\frac{1}{n^2}\right).$$

Thus

$$E(T_g | H_f) = \frac{-\sqrt{n}}{0.532}\{\overset{\circ}{\psi} + \frac{1}{2}\log\psi' + \frac{1}{2}(\alpha_2 - \log\alpha_2) + \frac{1}{2n}(3 - e^{\alpha_2})\} + O(\frac{1}{n^{3/2}})$$

$$= -1.8783\sqrt{n}\,\{\frac{1}{2}(\alpha_2 - \log\alpha_2) - 0.3283 + \frac{1}{2n}(3 - e^{\alpha_2})\} + O(\frac{1}{n^{3/2}}) \qquad (12.15).$$

We also have that

$$V\{g(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta})\} = V\{g(\alpha_1, \alpha_2, \beta_\alpha) + (\hat{\alpha}_1 - \alpha_1)\frac{\partial g}{\partial\hat{\alpha}_1} + \ldots\}$$

$$= \frac{1}{n}(e^{\alpha_2} - \frac{1}{2} - 2\alpha_2) + O(\frac{1}{n^2}),$$

and

$$V(T_g | H_f) = \frac{n}{0.2834}\; V(g)$$

$$= 3.5290(e^{\alpha_2} - \frac{1}{2} - 2\alpha_2) + O(\frac{1}{n}) \qquad (12.16).$$

Under the log-normal hypothesis T_g is negative and if

λ_α is the normal deviate corresponding to a level of significance α, the power of T_g against this alternative is

$$P_g(n) = \Phi(U_n) \qquad (12.17)$$

where
$$U_n = \frac{\lambda_\alpha + 0.9392\sqrt{n}(\alpha_2 - \log\alpha_2 - 0.6566)}{\sqrt{\{3.5290(e^{\alpha_2} - \frac{1}{2} - 2\alpha_2)\}}}$$

and μ, σ^2 are given by (12.15) and (12.16).
If $\alpha = 0.05$ then $\lambda_\alpha = -1.64$, and we get

$$P_g(n) = \Phi\left(\frac{-1.64 + 0.9392\sqrt{n}(\alpha_2 - \log\alpha_2 - 0.6566)}{\sqrt{\{3.5290(e^{\alpha_2} - \frac{1}{2} - 2\alpha_2)\}}}\right). \qquad (12.18)$$

Table 12.2 gives this function for various values of α_2 and n.

In the present case we obtain 50 per cent power for T_g at the 5 per cent level by noting that $E(T_g | H_f, n^*) = -1.64$, where n^* is the sample number required to achieve this power.

Substituting this in (12.15) gives

$$\sqrt{n^*} = \frac{0.8725 \pm \sqrt{\{0.7613 - (3-e^{\alpha_2})(\alpha_2 - \log \alpha_2 - 0.6566)\}}}{\alpha_2 - \log \alpha_2 - 0.6566} .$$

A graph of n^* is given in Fig.12.2. From this there is a local maximum, $n^* = 25$ at $\alpha_2 = 1.15$. For $\alpha_2 < 1$ n^* decreases steeply and for $\alpha_2 > 3$ n^* increases sharply. This apparently paradoxical fact can be explained since for large α_2 the log-normal and exponential get quite 'close' to each other and very large sample sizes would be required to distinguish one from the other.

Power Functions of Tests (5 per cent Significance Level)

Table 12.1. Null: Log-Normal, Alternative: Exponential; T_f

n	20	50	70	100	120	150	170
Power ($P_f(n)$)	0.004	0.227	0.544	0.875	0.960	0.995	0.999

Table 12.2. Null: Exponential, Alternative: Log-Normal, T_g

$\alpha_2 \backslash n$	20	30	50	100	150
0.5	0.802	0.939	0.996	1.000	1.000
1.0	0.409	.556	.764	0.964	0.996
1.5	0.540	.629	.752	.907	.966
2.0	0.633	.702	.800	.919	.966
3.0	0.698	.755	.831	.926	.966

In table 12.1 the rather low value, 0.004 for P_f when $n = 20$ arises because under the alternative, the variance of T_f is small, 0.1473, and the mean of T_f is not 'far' from that under the null.

Tables 12.1 and 12.2 indicate that T_g has high power, against the stated alternative, even for small n (and especially for small α_2, $\alpha_2 < 1$ say). The power of T_f is low for small n but rises steadily, and from $n = 120$ or so it generally has higher power than T_g.

12.3. Empirical Results

In this section empirical investigations are made into the adequacy of the asymptotic distributions of T_f and T_g, as well as the power functions of these tests, H_f and H_g serving as null and alternative hypotheses, and vice versa.

The test statistics are

$$T_f = \sqrt{n} \quad \frac{\log \hat{\beta} - \hat{\alpha}_1 - \frac{1}{2}\hat{\alpha}_2}{(e^{\hat{\alpha}_2} - 1 - \hat{\alpha}_2 - \frac{1}{2}\hat{\alpha}_2^2)^{\frac{1}{2}}} \quad ,$$

$$T_g = \frac{-\sqrt{n}}{0.532} \{\psi(1) + \frac{1}{2} \log \psi'(1) - \hat{\alpha}_1 - \frac{1}{2}\log \hat{\alpha}_2 + \log \hat{\beta}\}$$

$$= + 1.8783 \sqrt{n} \{0.3283 + \hat{\alpha}_1 + \frac{1}{2}\log \hat{\alpha}_2 - \log \hat{\beta}\}.$$

Random deviates, u, from the standard normal distribution (i.e. $\alpha_1 = 0$, $\alpha_2 = 1$) were generated on a computer. Taking $y = e^u$ gave random deviates, y, from a log-normal distribution. Using these we have

$$\hat{\alpha}_1 = \frac{\sum U_i}{n} , \quad \hat{\alpha}_2 = \frac{\sum U_i^2}{n} - (\frac{\sum U_i}{n})^2 , \quad \hat{\beta} = \frac{\sum y_i}{n} ,$$

where n is the sample size.

T_f and T_g were then calculated under the log-normal hypothesis, H_f. 500 trials for various sample values n were obtained and from these the first four moments of T_f and T_g were found. Since the shape of the log-normal curve depends on α_2 another value was tried; $\alpha_2 = 2$ was obtained by multiplying the deviates from the standard normal by $\sqrt{2}$. From these normal deviates we therefore got results on the null distribution of T_f and that of T_g under the alternative.

Random deviates from the unit exponential distribution, y, $(\beta = 1)$ were then generated. Letting $v = \log y$,

$$\hat{\alpha}_1 = \frac{\sum v_i}{n}, \quad \hat{\alpha}_2 = \frac{1}{n} \sum v_i^2 - (\hat{\alpha}_1)^2, \quad \hat{\beta} = \frac{\sum y_i}{n} .$$

Here again 500 trials were obtained for various n and the moments of T_f and T_g were calculated under the exponential hypothesis, H_g. From these we obtained results on the null distribution of T_g and the distribution of T_f under the alternative.

The results are summarised in Tables 12.3 -12.6 below together with some of the analytical results obtained in sections 12.1 and 12.2.

Table 12.3. Null Distribution of T_f

n		$E(T_f \mid H_f)$		$V(T_f \mid H_f)$		$\gamma_1(T_f \mid H_f)$		$\beta_2(T_f \mid H_f)$	
		$\alpha_2=1$	$\alpha_2=2$	$\alpha_2=1$	$\alpha_2=2$	$\alpha_2=1$	$\alpha_2=2$	$\alpha_2=1$	$\alpha_2=2$
20*	Empirical	-0.098	-0.149	0.490	0.293	0.822	0.930	4.215	4.687
	Analytical	-0.172	-0.318						
50	Empirical	-0.074	-0.125	0.647	0.436	0.938	1.156	4.185	4.723
	Analytical	-0.109	-0.201						
100	Empirical	-0.110	-0.148	0.693	0.505	0.591	0.944	3.635	4.553
	Analytical	-0.077	-0.142						
150	Empirical	-0.074	-0.114	0.752	0.564	0.534	0.806	3.823	4.277
	Analytical	-0.063	-0.116						
200	Empirical	-0.027	-0.072	0.862	0.680	0.580	0.785	3.903	4.038
	Analytical	-0.055	-0.100						

* The results for $n = 20$ are from 1000 trials. The others are from 500 trials.

Table 12.4. Null Distribution of T_g.

n		$E(T_g \mid H_g)$	$V(T_g \mid H_g)$	$\gamma_1(T_g \mid H_g)$	$\beta_2(T_g \mid H_g)$
20*	Empirical	-0.414	0.701	0.438	4.933
	Analytical	-0.462			
50	Empirical	-0.258	0.869	0.609	3.901
	Analytical	-0.292			
100	Empirical	-0.183	0.906	0.688	4.251
	Analytical	-0.207			
150	Empirical	-0.135	0.996	0.481	3.896
	Analytical	-0.169			

* The results for $n = 20$ are from 1000 trials. The others are from 500 trials.

Table 12.5. Distribution of T_f under alternative H_g

n		$E(T_f \mid H_g)$	$V(T_f \mid H_g)$	$\gamma_1(T_f \mid H_g)$	$\beta_2(T_f \mid H_g)$
20*	Empirical	-0.837	0.170	1.400	9.286
	Analytical	-0.623	0.147		
50	Empirical	-1.471	0.140	0.220	3.444
	Analytical	-1.351	0.147		
100	Empirical	-2.155	0.130	-0.010	3.455
	Analytical	-2.083	0.147		
150	Empirical	-2.688	0.143	0.427	4.011
	Analytical	-2.621	0.147		

* The results for $n = 20$ are from 1000 trials. The others are from 500 trials.

Table 12.6. Distribution of T_g under alternative H_f

n		$E(T_g \mid H_f)$		$V(T_g \mid H_f)$		$\gamma_1(T_g \mid H_f)$		$\beta_2(T_g \mid H_f)$	
		$\alpha_2=1$	$\alpha_2=2$	$\alpha_2=1$	$\alpha_2=2$	$\alpha_2=1$	$\alpha_2=2$	$\alpha_2=1$	$\alpha_2=2$
20*	Empirical	-1.614	-2.399	0.558	4.112	-1.289	-2.060	6.882	8.831
	Analytical	-1.506	-1.809	0.770	10.245				
50	Empirical	-2.379	-4.029	0.590	5.845	-1.152	-1.579	6.169	6.403
	Analytical	-2.319	-3.735	0.770	10.245				
100	Empirical	-3.246	-5.815	0.581	6.241	-0.710	-1.050	4.532	4.797
	Analytical	-3.254	-5.695	0.770	10.245				
150	Empirical	-3.992	-7.307	0.649	7.713	-0.668	-1.009	4.487	4.438
	Analytical	-3.970	-7.312	0.770	10.245				
200	Empirical	-4.625	-8.592	0.727	8.713	-0.708	-0.921	4.494	4.128
	Analytical	-4.580	-8.346	0.770	10.245				

* The results for $n = 20$ are from 1000 trials, the others are from 500 trials.

Some of the empirical investigations were repeated using different sets of random deviates and the results were generally in agreement with those quoted above.

In all cases the analytic mean values agree with the empirical ones allowing for sampling errors. The variance of T_f in the null case seems to approach unity rather slowly while that of T_g in the null case is reasonably fast. Agreement between empirical and analytic results is good for the distributions of T_f and T_g under the respective alternatives, except when α_2 is large (in which case n has to be large, say $n \geq 100$, for good agreement). This means that the power functions given will be generally reliable.

A number of broad conclusions can be drawn from the foregoing results. For both T_f and T_g the corrections to the means given in section 12.1 are good and can be used when the nature of the test requires greater accuracy than the asymptotic results. The empirical results on the variances also appear to be quite reliable. For most purposes however, the asymptotic results for T_f would seem adequate for n as low as 50. For T_g the correction to the mean is relatively large and can reasonably be ignored only for $n \geq 100$, say; however, the asymptotic null variance seems adequate for all n. However, the sample size at which the asymptotic results are adequate will generally depend on the degree of accuracy desired for the tests.

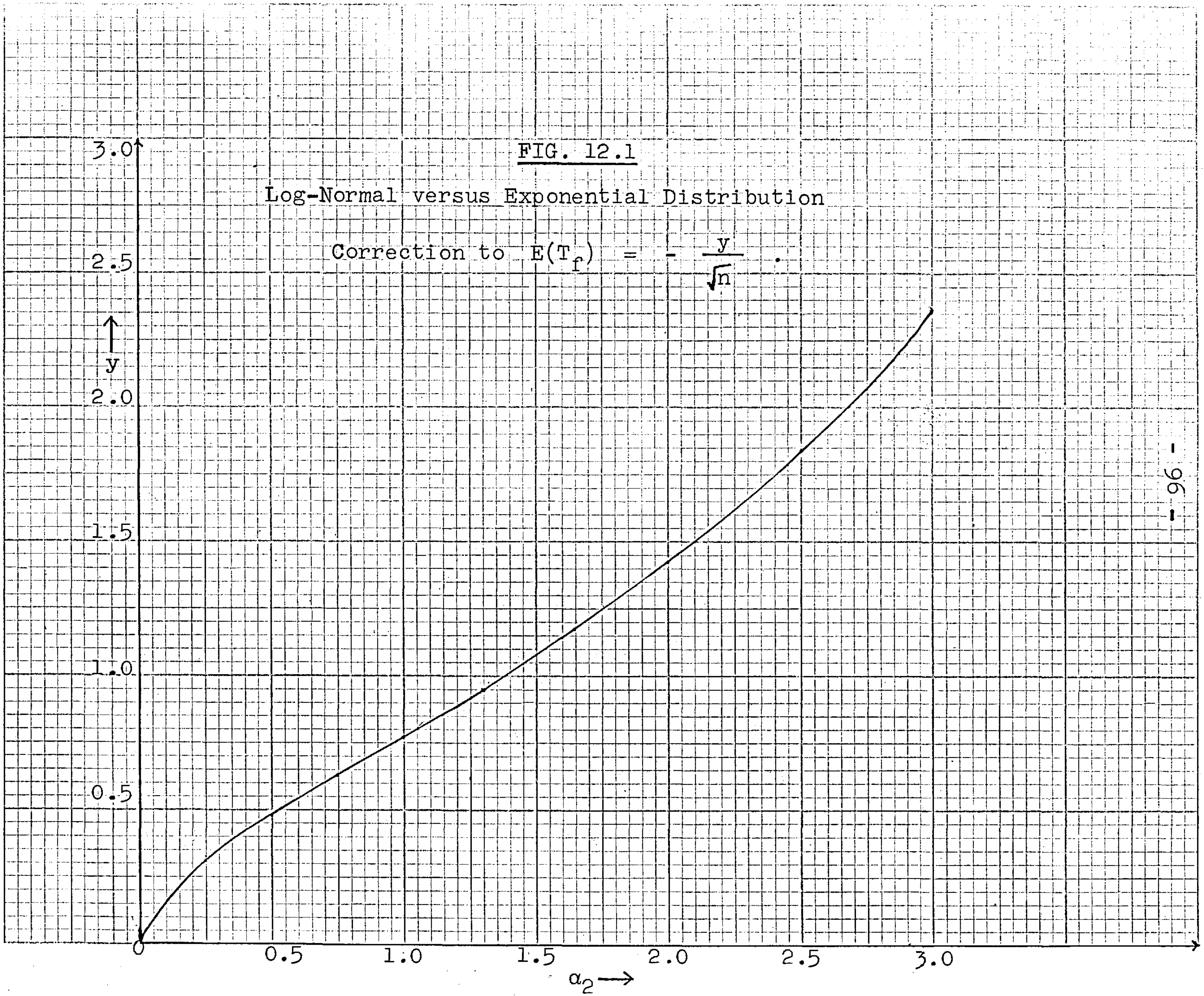


FIG. 12.1

Log-Normal versus Exponential Distribution

Correction to $E(T_f) = -\frac{y}{\sqrt{n}}$.

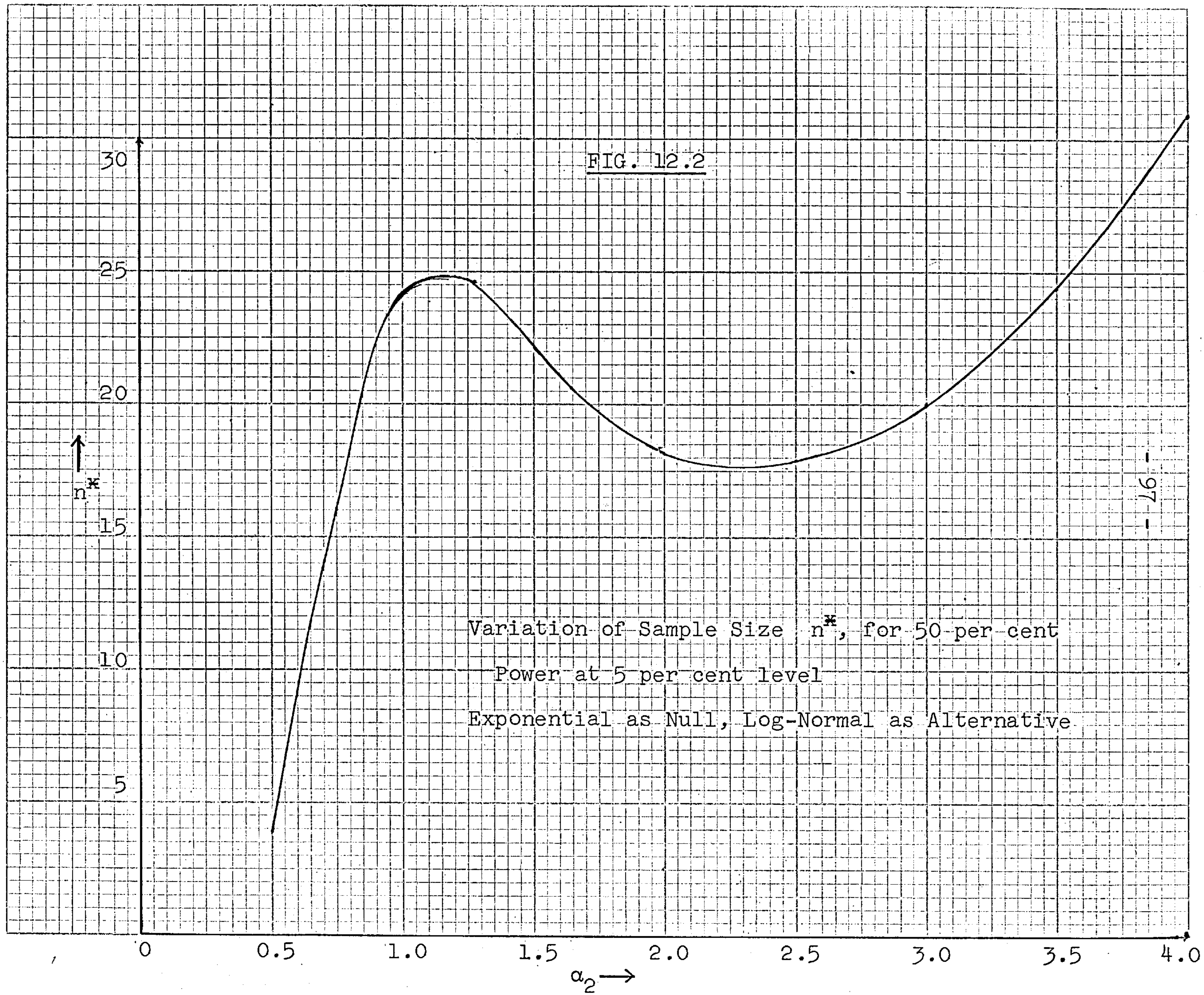


FIG. 12.2

Variation of Sample Size n^*, for 50 per cent Power at 5 per cent level

Exponential as Null, Log-Normal as Alternative

13. The Log-Normal Distribution versus the Gamma Distribution.

13.1. The Test Statistics and Their Distributions.

We now use the general methods proposed by Cox (1961, 1962) to derive a test of the log-normal distribution with the gamma distribution serving as an alternative against which high power is desired.

Suppose $Y_1, \ldots, Y_n$ are independent and identically distributed. Let the null hypothesis H_f be that the p.d.f. is log-normal, namely,

$$f(y, \underline{\alpha}) = \frac{1}{y\sqrt{(2\pi\alpha_2)}} \exp\{ -\frac{(\log y - \alpha_1)^2}{2\alpha_2} \} \tag{13.1}$$

and let H_g be the hypothesis that p.d.f. is gamma, namely,

$$g(y, \underline{\beta}) = \frac{\beta_2}{\beta_1 \Gamma(\beta_2)} \left(\frac{\beta_2 y}{\beta_1}\right)^{\beta_2 - 1} \exp\left(-\frac{\beta_2 y}{\beta_1}\right) \tag{13.2}.$$

Now $\hat{\alpha}_1$, $\hat{\alpha}_2$ are the sample mean and variance of the log Y_i and we obtain $\hat{\beta}_1$, $\hat{\beta}_2$ from the maximum likelihood equations for the gamma distribution.

From the likelihood function under the gamma distribution we obtain

$\hat{\beta}_1 = \bar{y}$, the sample mean of the Y_i,

and $\hat{\beta}_2$ is given by

$$\log \hat{\beta}_2 - \psi(\hat{\beta}_2) = \log \hat{\beta}_1 - \hat{\alpha}_1 \qquad (13.3)$$

where $\psi(x) = \frac{d}{dx}\{\log \Gamma(x)\}$.

The log likelihoods are

$$L_f(\hat{\underline{\alpha}}) = -\frac{n}{2}\log(2\pi\hat{\alpha}_2) - \frac{n}{2} - n\hat{\alpha}_1$$

and

$$L_g(\hat{\underline{\beta}}) = n\hat{\beta}_2 \log\left(\frac{\hat{\beta}_2}{\hat{\beta}_1}\right) - n\log\Gamma(\hat{\beta}_2) + n\hat{\alpha}_1(\hat{\beta}_2 - 1) - n\hat{\beta}_2.$$

The log-likelihood ratio is

$$L_f(\hat{\underline{\alpha}}) - L_g(\hat{\underline{\beta}}) = -\frac{n}{2}\log(2\pi\hat{\alpha}_2) - \frac{n}{2} - n\hat{\beta}_2\{\log(\frac{\hat{\beta}_2}{\hat{\beta}_1}) - 1\} + n\log\Gamma(\hat{\beta}_2) - n\hat{\alpha}_1\hat{\beta}_2 \qquad (13.4).$$

Under H_f, the log-normal hypothesis,

$$\hat{\beta}_1 \longrightarrow \beta_{1,\alpha} = e^{\alpha_1+\frac{1}{2}\alpha_2}, \quad \hat{\alpha}_1 \longrightarrow \alpha_1, \quad \hat{\alpha}_2 \longrightarrow \alpha_2 \quad \text{and}$$

$$\hat{\beta}_2 \longrightarrow \beta_{2,\alpha} \quad \text{where} \quad \log \beta_{2,\alpha} - \psi(\beta_{2,\alpha}) = \log \beta_{1,\alpha} - \alpha_1$$

$$= \tfrac{1}{2}\alpha_2 \qquad (13.5).$$

Now

$$\log \frac{f(y,\underline{\alpha})}{g(y,\underline{\beta}_\alpha)} = -\tfrac{1}{2}\log(2\pi\alpha_2) - \frac{(\log y-\alpha_1)^2}{2\alpha_2} - \beta_2\log(\frac{\beta_2}{\beta_1})$$

$$+ \log\Gamma(B_2) - \beta_2 \log y + \frac{\beta_2 y}{\beta_1},$$

where β_1, β_2 are to be evaluated at $\beta_{1,\alpha}$, $\beta_{2,\alpha}$.

We now take expectations under the log-normal distribution to obtain

$$E_{\underline{\alpha}}\left[\log \frac{f(y,\underline{\alpha})}{g(y,\underline{\beta}_\alpha)}\right] = -\tfrac{1}{2}\log(2\pi\alpha_2) - \tfrac{1}{2} - \beta_{2,\alpha}\log(\frac{\beta_{2,\alpha}}{\beta_{1,\alpha}})$$

$$+ \log\Gamma(\beta_{2,\alpha}) + \beta_{2,\alpha}(1-\alpha_1).$$

Thus under H_f, the test statistic is

$$T_f = L_{fg} - E_{\hat{\underline{\alpha}}}(L_{fg})$$

$$= n\{\beta_{2,\hat{\alpha}}(\log \frac{\beta_{2,\hat{\alpha}}}{\beta_{1,\hat{\alpha}}} + \hat{\alpha}_1 - 1) - \hat{\beta}_2(\log \frac{\hat{\beta}_2}{\hat{\beta}_1} - 1) - \hat{\alpha}_1\hat{\beta}_2$$

$$+ \log \frac{\Gamma(\hat{\beta}_2)}{\Gamma(\beta_{2,\hat{\alpha}})}\} \qquad (13.6)$$

or equivalently

$$T_f/n = \beta_{2,\hat{\alpha}}(\log \beta_{2,\hat{\alpha}} - \tfrac{1}{2}\hat{\alpha}_2 - 1) - \hat{\beta}_2(\log \frac{\hat{\beta}_2}{\hat{\beta}_1} + \hat{\alpha}_1 - 1) + \log \frac{\Gamma(\hat{\beta}_2)}{\Gamma(\beta_{2,\hat{\alpha}})} \qquad (13.7).$$

We now require the asymptotic variance of T_f. To do this we write $F \equiv \log f(\underline{Y}, \underline{\alpha})$, $G \equiv \log g(\underline{Y}, \underline{\beta})$ so that

$$F_{\alpha_j} = \frac{\partial}{\partial \alpha_j}\{\log f(\underline{Y}, \underline{\alpha})\} \quad \text{etc.}$$

Then,

$$F - G = -\frac{1}{2}\log(2\pi\alpha_2) - \frac{(\log Y - \alpha_1)^2}{2\alpha_2} - \beta_2 \log(\beta_2/\beta_1) + \log\Gamma(\beta_2)$$

$$-\beta_2 \log Y + \frac{\beta_2 Y}{\beta_1} \qquad (13.8)$$

and $F_{\alpha_1} = \dfrac{\log Y-\alpha_1}{\alpha_2}$, $F_{\alpha_2} = -\dfrac{1}{2\alpha_2} + \dfrac{(\log Y-\alpha_1)^2}{2\alpha_2^2}$.

In (13.8) β_1, β_2 are to be replaced by

$\beta_{1,\alpha} = e^{\alpha_1+\frac{1}{2}\alpha_2}$ and $\beta_{2,\alpha}$ is given by (13.5).

Under the lognormal hypothesis, H_f, the variances are given by

$$V(F_{\alpha_1}) = 1/\alpha_2, \qquad V(F_{\alpha_2}) = 1/(2\alpha_2^2) ,$$

$$V_\alpha(F-G) = \beta_{2,\alpha}^2 (e^{\alpha_2} - 1 - \alpha_2) - \alpha_2\beta_{2,\alpha} + \frac{1}{2} .$$

The covariances are

$$C_\alpha\left[F-G,\ F_{\alpha_1}\right] = E_\alpha\left[(F-G)F_{\alpha_1}\right]$$

$$= 0 ,$$

$$C_\alpha\left[F-G,\ F_{\alpha_2}\right] = E_\alpha\left[(F-G)F_{\alpha_2}\right]$$

$$= \frac{\alpha_2\beta_{2,\alpha}-1}{2\alpha_2} .$$

Hence the asymptotic variance of T_f is

$$V(T_f) = n\{V_\alpha(F-G) - \frac{C^2_{\alpha_1}(F-G,\ F_{\alpha_1})}{V_\alpha(F_{\alpha_1})} - \frac{C^2_{\alpha_2}(F-G,\ F_{\alpha_2})}{V_\alpha(F_{\alpha_2})}\}$$

$$= n\{\beta^2_{2,\alpha}(e^{\alpha_2}-1-\alpha_2-\tfrac{1}{2}\alpha_2^2)\}. \qquad (13.9).$$

Thus to a first approximation we can carry out a test by treating $T_f/\{V(T_f)\}^{1/2}$ as having a standard normal distribution $N(0,1)$ under H_f, negative values being expected under H_g.

The roles of H_f and H_g are now interchanged so that the gamma distribution is the null hypothesis. Here, $\hat{\alpha}_1 \longrightarrow \alpha_{1,\beta} = E(\log y)$, $\hat{\alpha}_2 \longrightarrow \alpha_{2,\beta} = V(\log y)$,

$\hat{\beta}_1 \longrightarrow \beta_1$, $\hat{\beta}_2 \longrightarrow \beta_2$ and all moments are evaluated

under H_g, the gamma hypothesis. Thus we have

$$\alpha_{1,\beta} = \psi(\beta_2) - \log(\beta_2/\beta_1), \qquad \alpha_{2,\beta} = \psi'(\beta_2). \qquad (13.10).$$

Thus $L_g(\hat{\underline{\beta}})-L_f(\hat{\underline{\alpha}})$ is given by (13.4) and

$$\log \frac{f(y,\underline{\alpha}_\beta)}{g(y,\underline{\beta})} = -\frac{1}{2}\log(2\pi\alpha_2) - \frac{(\log y-\alpha_1)^2}{2\alpha_2} - \beta_2\log(\beta_2/\beta_1)$$

$$+ \log \Gamma(\beta_2) - \beta_2 \log y + \frac{\beta_2 y}{\beta_1} .$$

Therefore

$$E_\beta\left[\log \frac{f(y,\underline{\alpha}_\beta)}{g(y,\underline{\beta})}\right] = -\frac{1}{2}\log(2\pi\alpha_{2,\beta}) - \frac{1}{2} - \beta_2 \log(\beta_2/\beta_1)$$

$$+ \log\Gamma(\beta_2) - \beta_2\alpha_{1,\beta} + \beta_2 .$$

The test statistic is

$$T_g = L_{gf} - E_{\hat{\underline{\beta}}}(L_{gf})$$

$$= \frac{n}{2}\log\left(\frac{\hat{\alpha}_2}{\alpha_{2,\hat{\beta}}}\right) + n\hat{\beta}_2(\hat{\alpha}_1-\alpha_{1,\hat{\beta}}) \qquad (13.11)$$

or equivalently,

$$T_g/n = \frac{1}{2}\log\left(\frac{\hat{\alpha}_2}{\alpha_{2,\hat{\beta}}}\right) + \hat{\beta}_2(\hat{\alpha}_1-\alpha_{1,\hat{\beta}}) . \qquad (13.12)$$

To obtain the asymptotic variance of T_g we require $(G-F)$ which is given by (13.8). We also have that

$$G_{\beta_1} = -\beta_2/\beta_1 + \frac{\beta_2 Y}{\beta_1^2}, \quad G_{\beta_2} = \log(\frac{\beta_2 Y}{\beta_1}) - \frac{Y-\beta_1}{\beta_1} - \psi(\beta_2).$$

Under the gamma hypothesis, these give

$$V_\beta(G_{\beta_1}) = \beta_2/\beta_1^2, \quad V_\beta(G_{\beta_2}) = \psi'(\beta_2) - 1/\beta_2,$$

$$V_\beta(G-F) = \beta_2^2\,\psi'(\beta_2) + \frac{\psi'''(\beta_2)}{4\{\psi'(\beta_2)\}^2} + \frac{\beta_2\,\psi''(\beta_2)}{\psi'(\beta_2)} + \frac{1}{2} - \beta_2.$$

Also, $C_\beta\left[G-F, G_{\beta_1}\right] = E_\beta\left[(G-F)G_{\beta_1}\right]$

$$= 0,$$

and

$C_\beta(G-F, G_{\beta_2})$

$$= C_\beta\left[(\log Y - Y/\beta_1),\ \frac{(\log Y-\alpha_1)^2}{2\alpha_2} + \beta_2 \log Y - \frac{\beta_2 Y}{\beta_1}\right]$$

$$= \frac{\psi''}{2\,\psi'} + \beta_2\,\psi' - 1.$$

Hence the asymptotic variance of T_g is

$$V(T_g) = n\{\frac{\psi'''(\beta_2)}{4\{\psi'(\beta_2)\}^2} - \frac{\beta_2\{\psi''(\beta_2)\}^2}{4\{\psi'(\beta_2)\}^2\{\beta_2\psi'(\beta_2)-1\}} + \frac{1}{2}\}$$

$$= n\,\phi(\beta_2), \text{ say.} \qquad (13.13)$$

Thus, under H_g, $T_g/\{n\phi(\beta_2)\}^{\frac{1}{2}}$ has asymptotically a standard normal distribution, and negative values are expected under H_f.

The above results are a generalisation of the tests between the log-normal and exponential distributions given by Cox (1961, 1962) and considered in Section 12, the gamma distribution having two parameters to be estimated as against one for the exponential distribution.

Special Case - One parameter gamma.

If instead of the two parameter gamma, we have a standardised gamma distribution parameter m, where

$$g(y, m) = \frac{e^{-y}\,y^{m-1}}{\Gamma(m)}$$

the T_f test statistic is much simpler. The results follow from those above on putting $\beta_1 = \beta_2 = m$. The maximum likelihood estimate of m is given by

$$\psi(\hat{m}) = \hat{\alpha}_1 .$$

Under H_f, $\hat{m} \longrightarrow m_{\alpha}$ where $\psi(m_{\alpha}) = \alpha_1$.

The test statistic, with log-normal as null is

$$T_f = n \log \frac{\Gamma(\hat{m})}{\Gamma(m_{\hat{\alpha}})} + n(1-\hat{\alpha}_1)\ (\hat{m}-m_{\hat{\alpha}})$$

with an asymptotic normal distribution of zero mean and variance

$$n\{ m_{\alpha}^2 (e^{\alpha_2} - 1 - \alpha_2 - \tfrac{1}{2}\alpha_2^2)\} .$$

With the gamma distribution as null, the test statistic is

$$T_g = \frac{n}{2} \log \frac{\hat{\alpha}_2}{\alpha_{2,\hat{m}}} + n\ \hat{m}(\hat{\alpha}_1 - \alpha_{1,\hat{m}})$$

with asymptotic normal distribution of zero mean and variance

$$n\left[\frac{\psi'''(m)}{4\{\psi'(m)\}^2} - \frac{m\{\psi''(m)\}^2}{4\{\psi'(m)\}^2\{m\,\psi'(m)-1\}} + \frac{1}{2}\right] .$$

13.2. Approximations to Functions Required for the Tests.

In order to carry out the above tests values of $\log \Gamma(x)$ and its derivatives, the polygamma functions, are required. These functions are tabulated by Davis (1933, 1935) and Brownlee (1923). The tables are given at intervals not always suitable for the application of the tests. We therefore give series expansions for the functions required. Generally,

$$\log \Gamma(x) = \frac{1}{2}\log(2\pi) + (x-\frac{1}{2})(\log x) - x + \sum_{n=1}^{m} \frac{B_{2n}\, x^{1-2n}}{2n(2n-1)} + O(x^{-2m-1})$$

$$\psi(x) = \log x - \frac{1}{2x} - \sum_{n=1}^{m} \frac{B_{2n}}{2n} x^{-2n} + O(x^{-2m-2}) \qquad (13.14)$$

$$\frac{d^n}{dx^n}\{\psi(x)\} = (-)^{n-1}\{\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{m=1}^{\infty}(-)^{m-1}\frac{B_m(2m+n-1)!}{(2m)!x^{2m+n}}\}, \quad n > 0 \qquad (13.15)$$

where B_r and $B_r(x)$ are Bernouilli numbers and functions, respectively.

We find that it is adequate to take

$$\log \Gamma(x) = 0.91895 + (x - \tfrac{1}{2})\log x - x + \frac{1}{60x} \qquad (13.16)$$

and for $\psi(x), \psi'(x), \psi''(x)$ and $\psi'''(x)$ to take the first three terms of (13.14) and (13.15) with the appropriate values of n.

We also obtain an approximation to the variance of T_g, $n\,\phi(\hat{\beta}_2)$ from

$$\phi(x) = \frac{1}{2}\left[1 - \frac{1 + \frac{2}{3x} - \frac{1}{18x^2} - \frac{4}{9x^3} + \frac{5}{36x^4} + \ldots}{\{1 + \frac{1}{2x} + \frac{1}{6x^2} - \frac{1}{30x^4} + \ldots\}^2}\right] . \qquad (13.17)$$

In table 13.1, below, a comparison of these approximations with the exact values is given. In Figs. 13.1 - 13.3 we give graphical representations of some of the functions. Table 13.1 shows that the approximations are good even for small x, say 5. The graphs simplify the use of the tests since $\phi(x)$, $\beta_{2,\alpha}$ and some of the other functions required can be read off directly.

Table 13.1. Comparison of Approximations.

x		$\log\Gamma(x)$	$\psi(x)$	$\psi'(x)$	$\psi''(x)$	$\psi'''(x)$	$\phi(x)$
5	Exact	3.178	1.506	0.2213	-0.0488	0.0214	0.0395
	Approx	3.165	1.509	0.2213	-0.0488	.0214	.0397
8	Exact	8.518	2.016	0.1331	-0.0177	0.0047	.0235
	Approx	8.517	2.017	0.1331	-0.0179	.0047	.0233
10	Exact	12.802	2.252	0.1052	-0.0111	0.0023	.0185
	Approx	12.795	2.252	0.1052	-0.0111	.0023	.0183

13.3. An Application of the Tests.

We now apply the above results to fibre-diameter measurements on wool tops, (Monfort, 1964). Monfort had fibre-diameter results on eight lots of combed slivers comprising reference wools of the International

Wool Textile Organisation. He obtained the means, coefficients of variation (C.V) and other parameters necessary for fitting either a log-normal or gamma distribution to each lot. Two methods were then used to assess how well the two distributions fitted. The first was a χ^2 test and the second was Cox's graphical method (Monfort, 1964) in which plots of γ_1 versus C.V , γ_2 versus C.V. are made and compared to the null plots for the log-normal and gamma distributions.

In order to apply the tests given in Section 13.1, logarithms of the observations were taken and the means and variances $\hat{\alpha}_1$, $\hat{\alpha}_2$ obtained from them. We give some details of the calculations for the first lot, A, and summarise the other results below.

Lot A.

Values of the parameters are given in Table 13.2. From these we have that under the log-normal hypothesis H_f,

$$T_f = 0.001928.$$

The estimated standard error of T_f is 4.725×10^{-3}, leading to an equivalent normal deviate of 0.408. This indicates good agreement with the log-normal distribution. Under the gamma hypothesis,

H_g, we obtain

$$T_g = -0.009556.$$

The estimated standard error is 3.652×10^{-3}, and we have an equivalent normal deviate of -2.616. There is thus quite strong evidence of a departure from the gamma distribution in the direction of the log-normal.

We give the values of the necessary parameters as well as the normal equivalent deviates and significance levels attained by the test statistics in the tables below.

Table 13.2. Estimates of Parameters for Wool Tops

Top	$\hat{\alpha}_1$	$\alpha_{1,\hat{\beta}}$	$\hat{\alpha}_2$	$\alpha_{2,\hat{\beta}}$	$\hat{\beta}_2/\hat{\beta}_1$	$\hat{\beta}_2$	$\beta_{2,\hat{\alpha}}$
A	2.8876	2.8874	0.04579	0.04716	1.182	21.71	21.87
B	3.0194	3.0193	.04998	.04999	0.9763	20.49	20.04
C	3.1369	3.1347	.06343	.07054	0.6167	14.67	15.80
D	3.2066	3.2070	.07447	.07377	0.5486	14.05	13.46
E	3.3436	3.3436	.07168	.07165	0.4927	14.45	13.98
F	3.4116	3.4110	.07712	.07815	0.4223	13.29	13.00
G	3.4375	3.4376	.07555	.07566	0.4247	13.71	13.27
H	3.5803	3.5836	.07990	.06884	0.4034	15.02	12.55

In all cases where the log-normal is accepted $\beta_{2,\hat{\alpha}} > \hat{\beta}_2$ while for the gamma $\beta_{2,\hat{\alpha}} < \hat{\beta}_2$. These seem to be true as long as $(\hat{\beta}_2 - \beta_{2,\hat{\alpha}})$ is not too large. For top H the difference appears to be too large and neither distribution fits the observations.

Table 13.3. Significance Tests for the Wool Tops

Top	No. of observations n	Log-Normal, H_f Normal Deviate	Log-Normal, H_f Level of Significance	Gamma, H_g Normal Deviate	Gamma, H_g Level of Significance	Log Likelihood L_{fg}	Max Likelihood Ratio e^L
A	600	+0.408	.682	-2.616	**.008	+3.372	29
B	600	-2.998	**.003	+0.635	.528	-3.612	1/37
C	600	+0.936	.348	-4.772	**<.001	+9.294	10,830
D	600	-5.511	**<.001	-0.153	.880	-3.126	1/23
E	450	-2.452	*.015	+0.104	.916	-2.871	1/18
F	450	-2.468	*.014	+0.317	.749	-3.659	1/39
G	450	-2.146	*.032	-0.446	.652	-1.733	1/6
H	450	-7.239	**<.001	+4.960	**<.001	-13.901	$\doteq 1\text{x}10^{-6}$

In table 13.3 the significance levels given are two sided and we use * and ** to indicate significance at the 5 per cent and 1 per cent levels respectively.

From table 13.3 we conclude that tops A, C belong to a log-normal distribution, tops B, D, E, F, G belong to a gamma distribution and top H belongs to neither. These conclusions agree generally with those obtained from the graphical plots of Monfort, though in that case they are not as clear cut.

Top H illustrates the point made by Cox (1961) that one hypothesis, H_f say, serves as null and the other, H_g serves as a possible alternative. In this case the T_f test shows that there is a departure from the log-normal in the direction of the gamma; this apparently goes too far and the T_g test indicates a departure from the gamma away from the log-normal distribution.

The above procedure is more sensitive than that of χ^2 for estimating agreement between two neighbouring distributions. To illustrate this we quote a table, 13.4 given by Monfort (1964) and compare this with table 13.3 above.

Table 13.4. χ^2 Tests for the Wool Tops.

Top	Degrees of Freedom	Fitting: Log-Normal χ^2	Log-Normal Level of Significance	Gamma χ^2	Gamma Level of Significance
A	8	8.64	0.50-0.30	11.89	0.20-0.10
B	9	7.52	> 0.50	7.07	0.70-0.50
C	11	10.57	0.50-0.30	19.84	* 0.05-0.01
D	12	14.51	0.30-0.20	3.60	> 0.50
E	13	18.99	0.20-0.10	15.05	0.50-0.30
F	14	18.38	0.20-0.10	15.44	0.50-0.30
G	14	17.62	0.30-0.20	18.12	0.30-0.20
H	18	60.56	** < 0.01	38.55	** < 0.01

From this table, a log-normal fit only is accepted for C and H belongs to neither distribution, but the other tops A, B, D, E, F, G are taken as having distributions consistent with both the log-normal and the gamma.

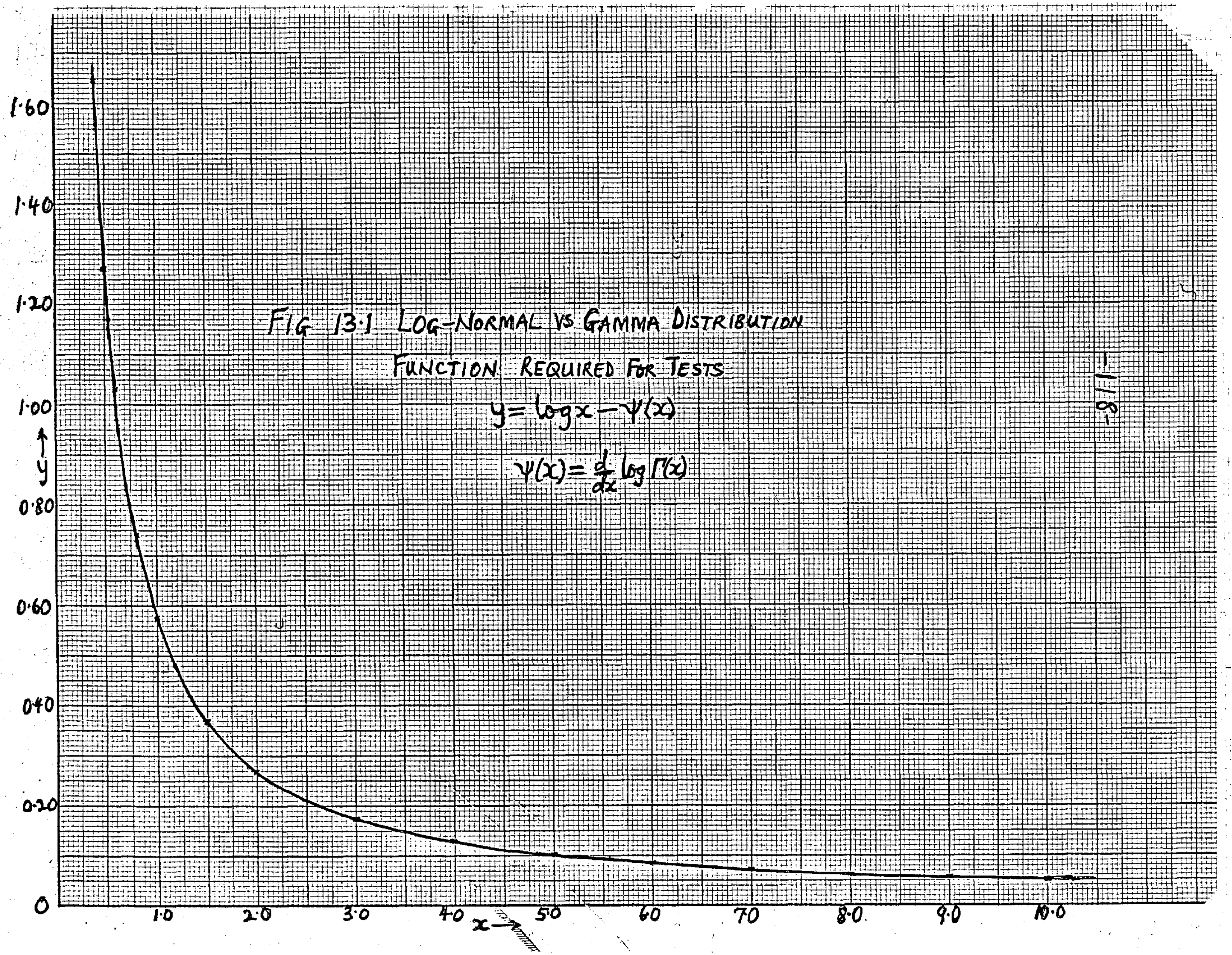


FIG 13·1 LOG-NORMAL VS GAMMA DISTRIBUTION

FUNCTION REQUIRED FOR TESTS

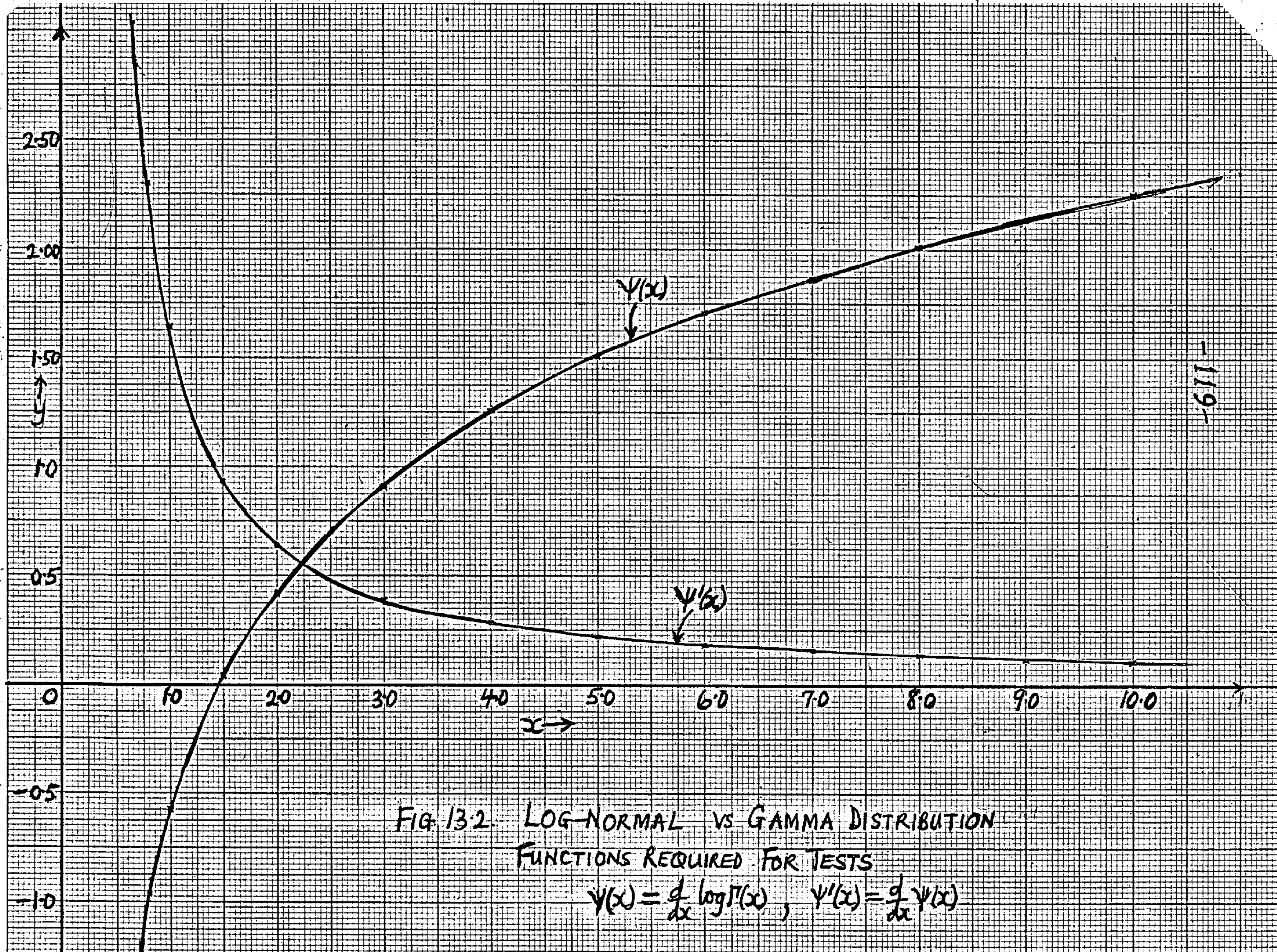


FIG 13.2 LOG-NORMAL VS GAMMA DISTRIBUTION

FUNCTIONS REQUIRED FOR TESTS

$\psi(x) = \frac{d}{dx}\log\Gamma(x)$, $\psi'(x) = \frac{d}{dx}\psi(x)$

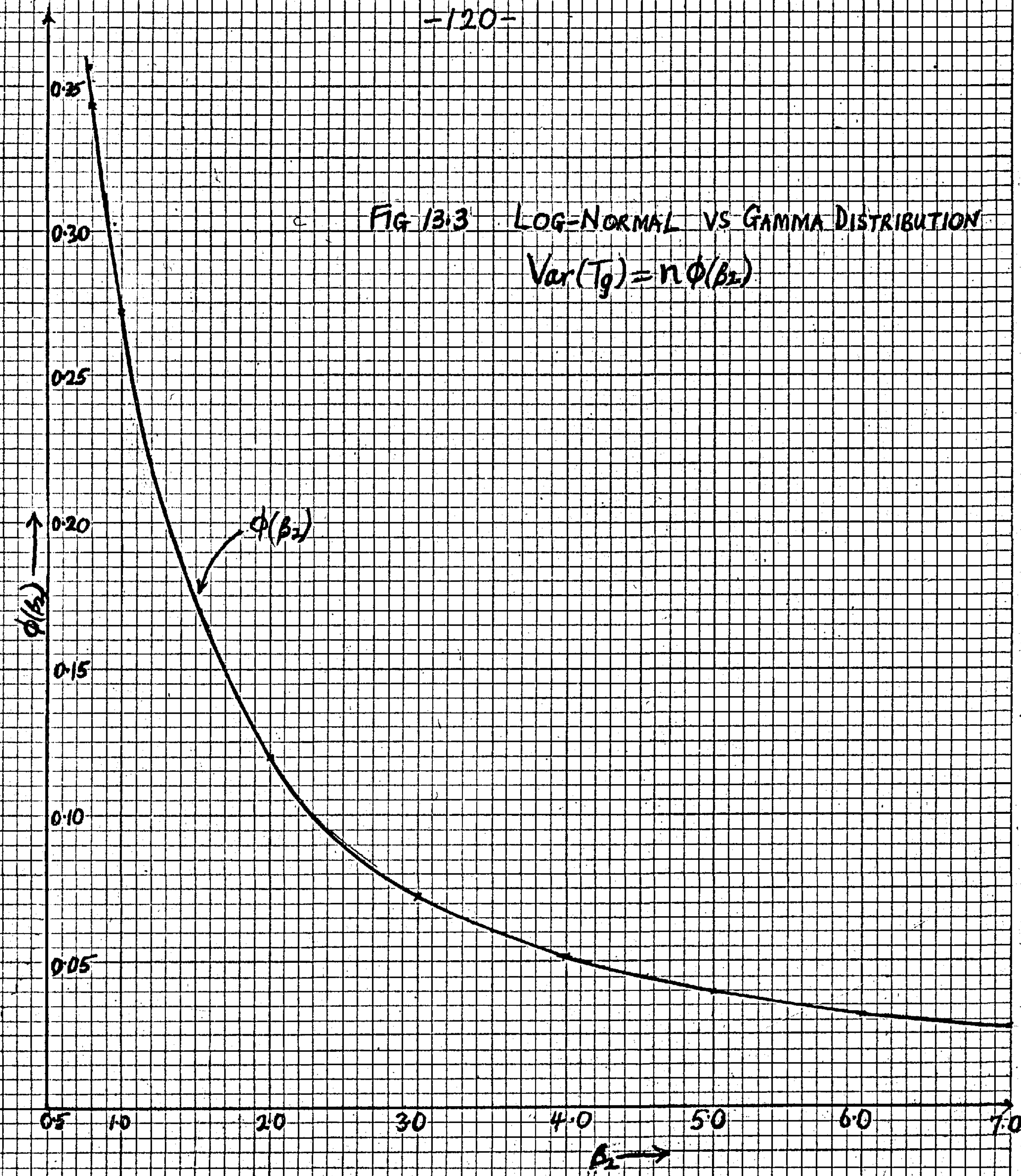


FIG 13·3 LOG-NORMAL VS GAMMA DISTRIBUTION

PART III

COMPARISON OF TESTS

14. Comparison of a 'Separate Families' Test and T_n.

A point of some interest is the comparison of the power of T_g, Cox's likelihood ratio test (Section 12) and that of T, the order statistic test (Section 2), against a log-normal alternative. Since T_g is based specifically on the exponential distribution as null and the log-normal as alternative we would expect it to be more powerful than T which is constructed with only vague alternatives in mind.

For a log-normal alternative, the usual analytic methods for calculating the power (or A.R.E.) of T either break down or are difficult unless drastic approximations are used. We therefore do a simulation experiment for the empirical power.

We consider the log-normal distribution with p.d.f.

$$f(y) = \frac{1}{y\sqrt{(2\pi\alpha_2)}} \exp\left\{-\frac{(\log y - \alpha_1)^2}{2\alpha_2}\right\}, \quad y > 0.$$

Deviates from this distribution are generated for $\alpha_1 = 0$ and different values of α_2, the shape parameter. For these values of α_2 and some values of n, the sample size, the statistics, T and T_g are calculated. For each combination of α_2 and n, 100 sets are generated and T and T_g calculated. In order to compare the performance of T to that of other 'vague alternative' tests Lewis's statistic S' (Section 9) and Moran's M statistic (which is asymptotically most powerful for a gamma alternative; Section 9) are computed for the same sets of data. The cumulative probabilities for the various tests are then plotted on arithmetic probability paper to obtain the power for all possible significance levels (Figs. 14).

Probability Plots

As in the case of the power for T_n (Section 9) the power of each test can be read off for any significance level. The further away the cumulative plot is from the null, $N(0,1)$, line the more powerful the test under the alternative.

For a 5 per cent significance level the power of the various tests are given in the table below. The statistics are

$$T_g = 1.8783\sqrt{n}\,\{0.3283 + \hat{\alpha}_1 + \tfrac{1}{2}\log\hat{\alpha}_2 - \log\hat{\beta}\}$$

$$T_n = \{\sum_{r=1}^{n} t_{r,n} X_{(r)}\}/\{\sum_{r=1}^{n} X_r\}$$

$$S' = 2n - \{2 \sum_{r=1}^{n} r X_{(r)}\}/\{\sum_{r=1}^{n} X_r\}$$

$$M = -2 \sum_{r=1}^{n} \log (X_r/\overline{X}), \quad \overline{X} = \frac{1}{n} \sum X_r.$$

All the statistics were scaled so that they were N(0,1) under the null.

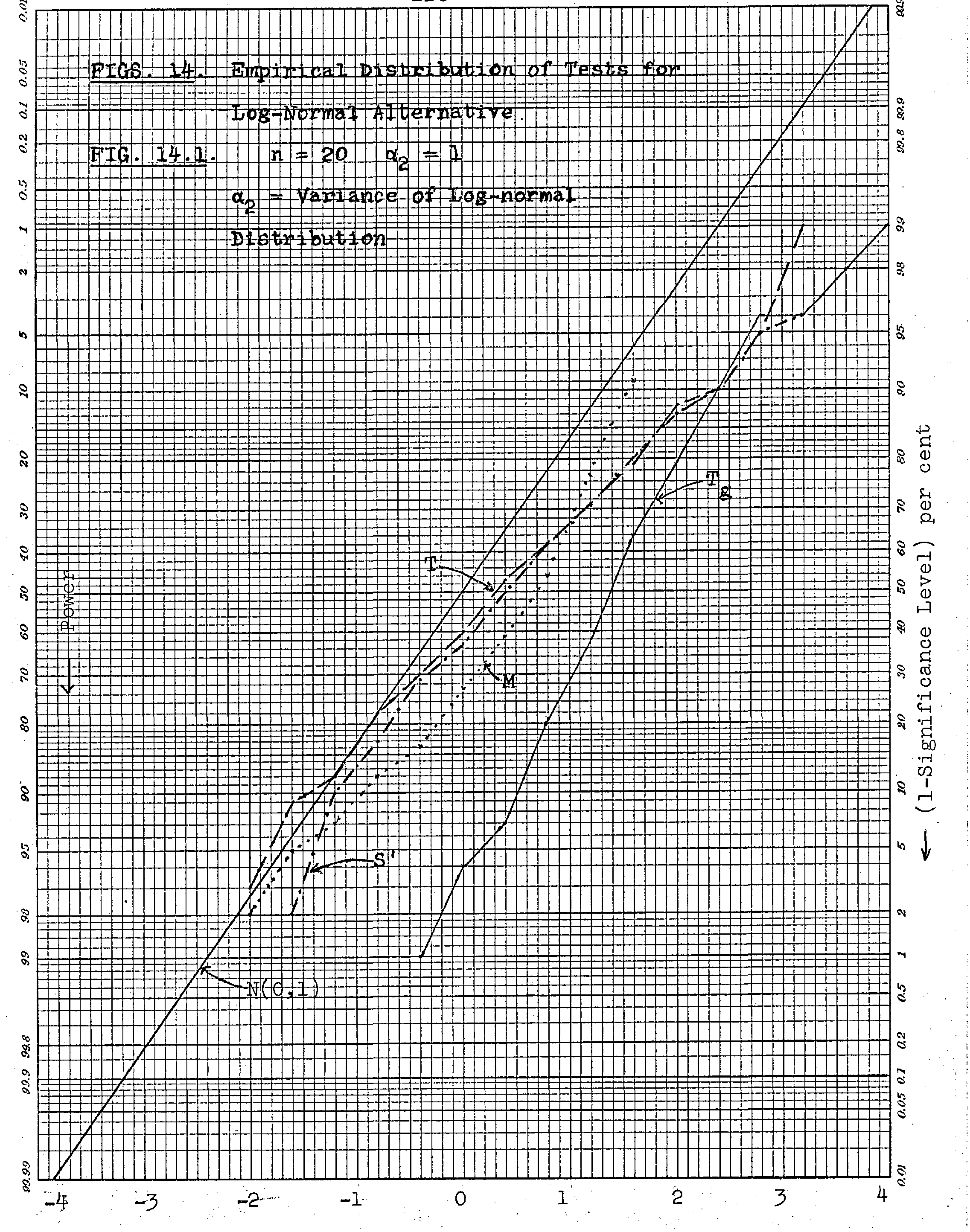
Table 14.1. Power of Tests Against Log-Normal Alternative - 5 per cent Significance Level

	$\alpha_2 = 1$			$\alpha_2 = 2$			$\alpha_2 = 3$		
n	20	50	100	20	50	100	20	50	100
T_g	.34	.83	>.98	.49	.87	>.98	.71	>.98	>.98
T_n	.20	.36	.56	.61	.92	>.98	.79	.97	.90
S'	.19	.28	.36	.68	.95	>.98	.90	>.98	>.98
M	.09	.17	.21	.54	.76	.97	.80	>.98	>.98

The power of the tests investigated above depend on α_2, the shape parameter of the log-normal distribution. From Fig. 12.2 the value of α_2 (in the range 0-3.5) requiring the largest sample size to achieve 50 per cent power is approximately $\alpha_2 = 1$. For this value of α_2 $(\alpha_2 = 1)$ T_g is a lot more powerful than T_n or either of S' and M, as would be expected. However, when $\alpha_2 = 2$ or $\alpha_2 = 3$, T_g does not do so well. For $n = 20$, T_n and S' seem to do much better than T_g. For $n \geq 50$, the difference in power could be accounted for by sampling errors.

On the whole T_n does much better than M which is the asymptotically most powerful test against a gamma alternative.

For the log-normal alternative, the performance of T_n compared to S' gets better as n increases, particularly when $\alpha_2 = 1$. This is the reverse of what happens for the gamma alternative (Section 9) and suggests that, at least for some alternatives, T_n will be a good test statistic even when n is large.


FIGS. 14. Empirical Distribution of Tests for Log-Normal Alternative
FIG. 14.1. n = 20 α2 = 1
α2 = Variance of Log-normal Distribution
Power
(1-Significance Level) per cent
T
Tg
M
S'
N(0,1)
-4
-3
-2
-1
0
1
2
3
4
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99

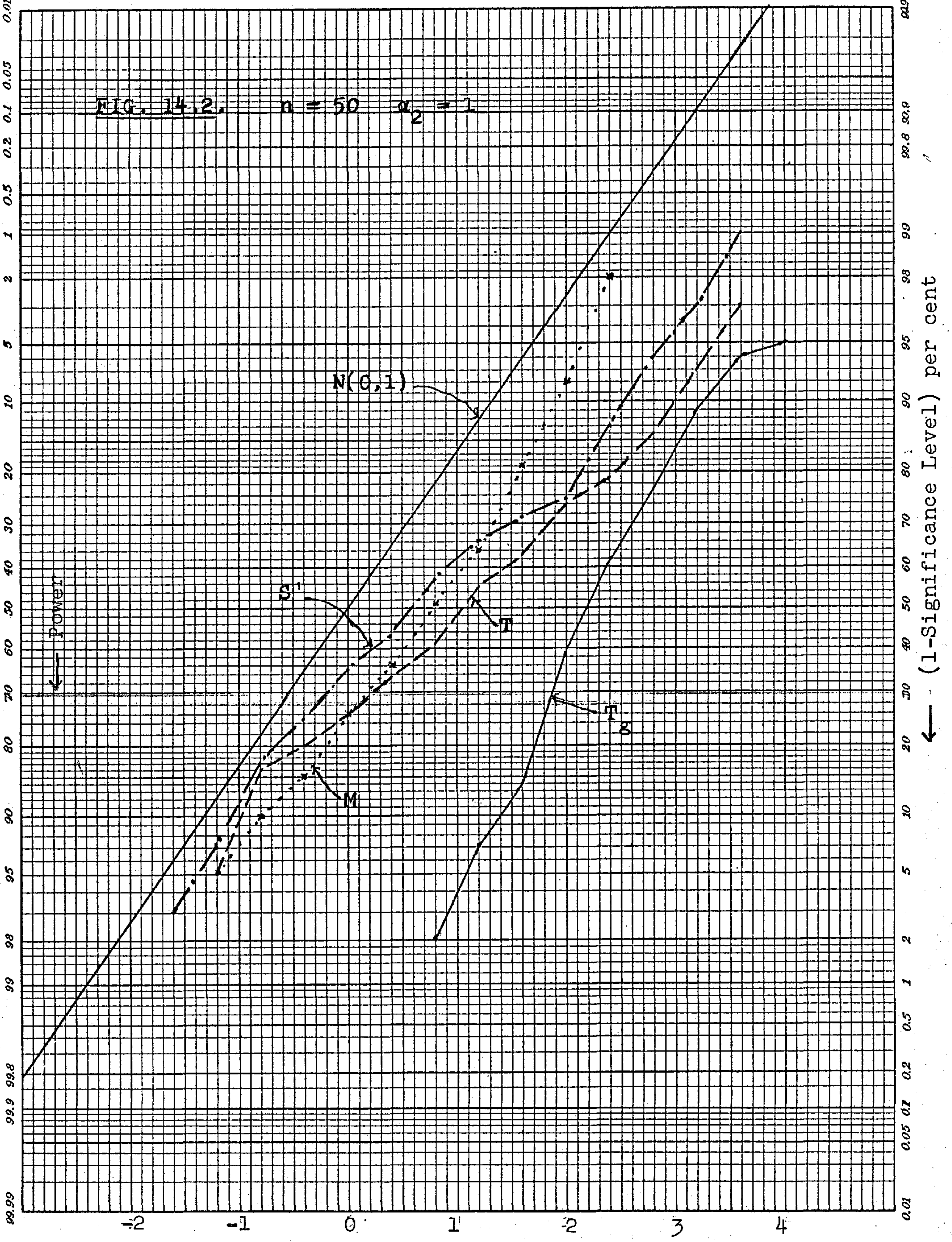


FIG. 14.2. n = 50 α_2 = 1
N(0,1)
S'
T
T_g
M
← Power
← (1-Significance Level) per cent
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99
-2
-1
0
1
2
3
4

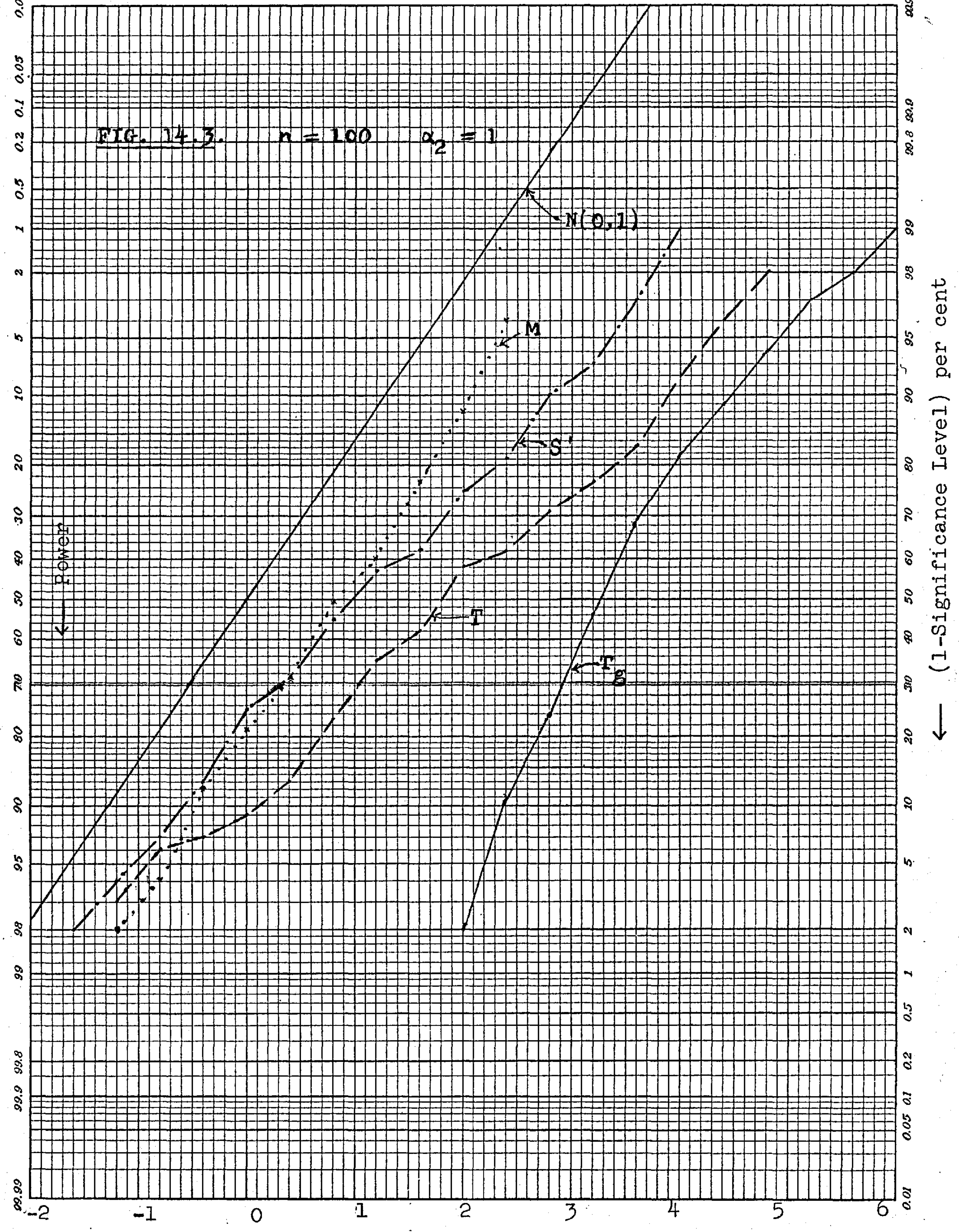


FIG. 14.3. n = 100 α_2 = 1
N(0,1)
M
S
T
T_g
← Power
← (1-Significance Level) per cent
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99
-2
-1
0
1
2
3
4
5
6

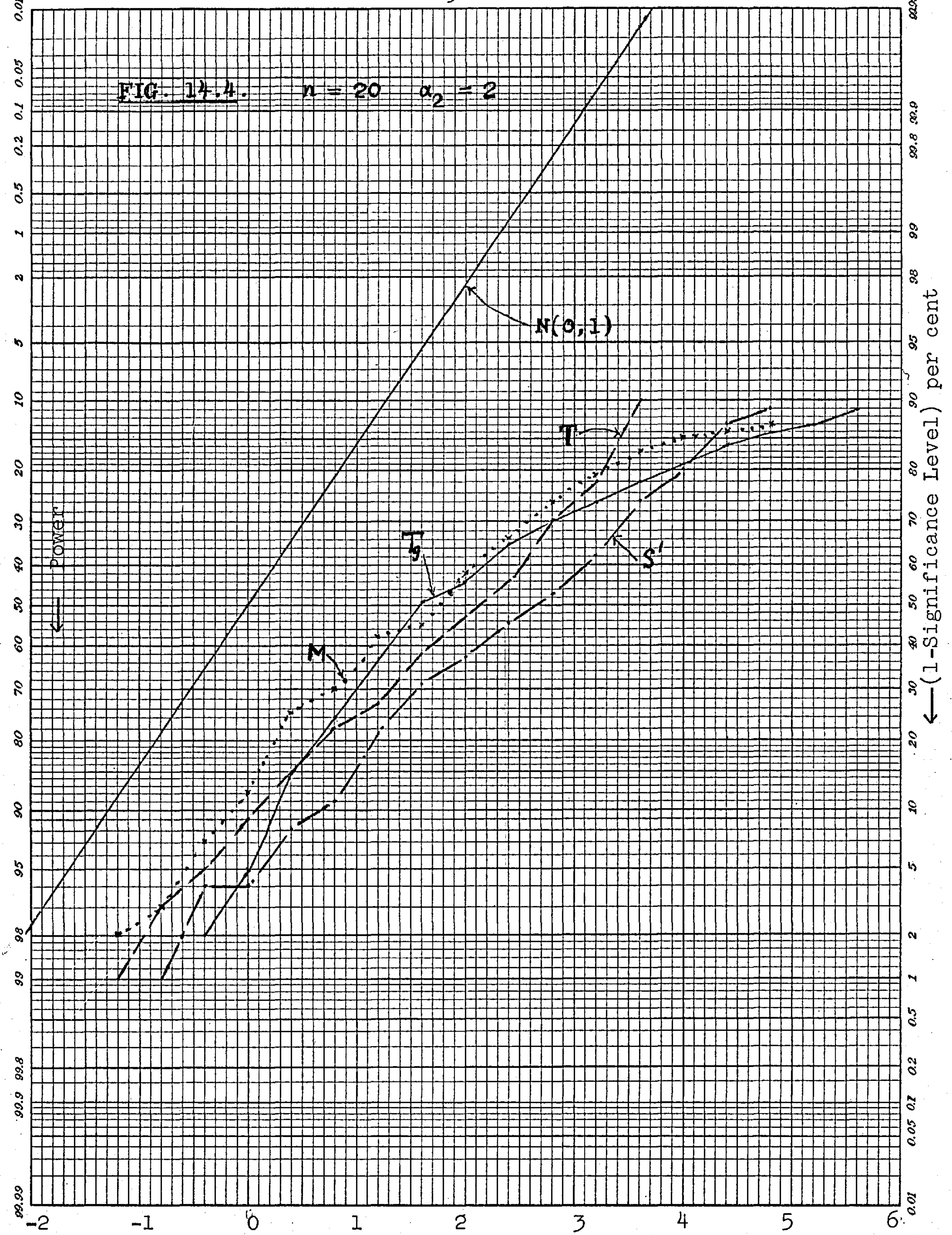


FIG. 14.4. n = 20 α2 = 2
N(0,1)
T
Tg
M
S'
Power
(1-Significance Level) per cent
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99
-2
-1
0
1
2
3
4
5
6

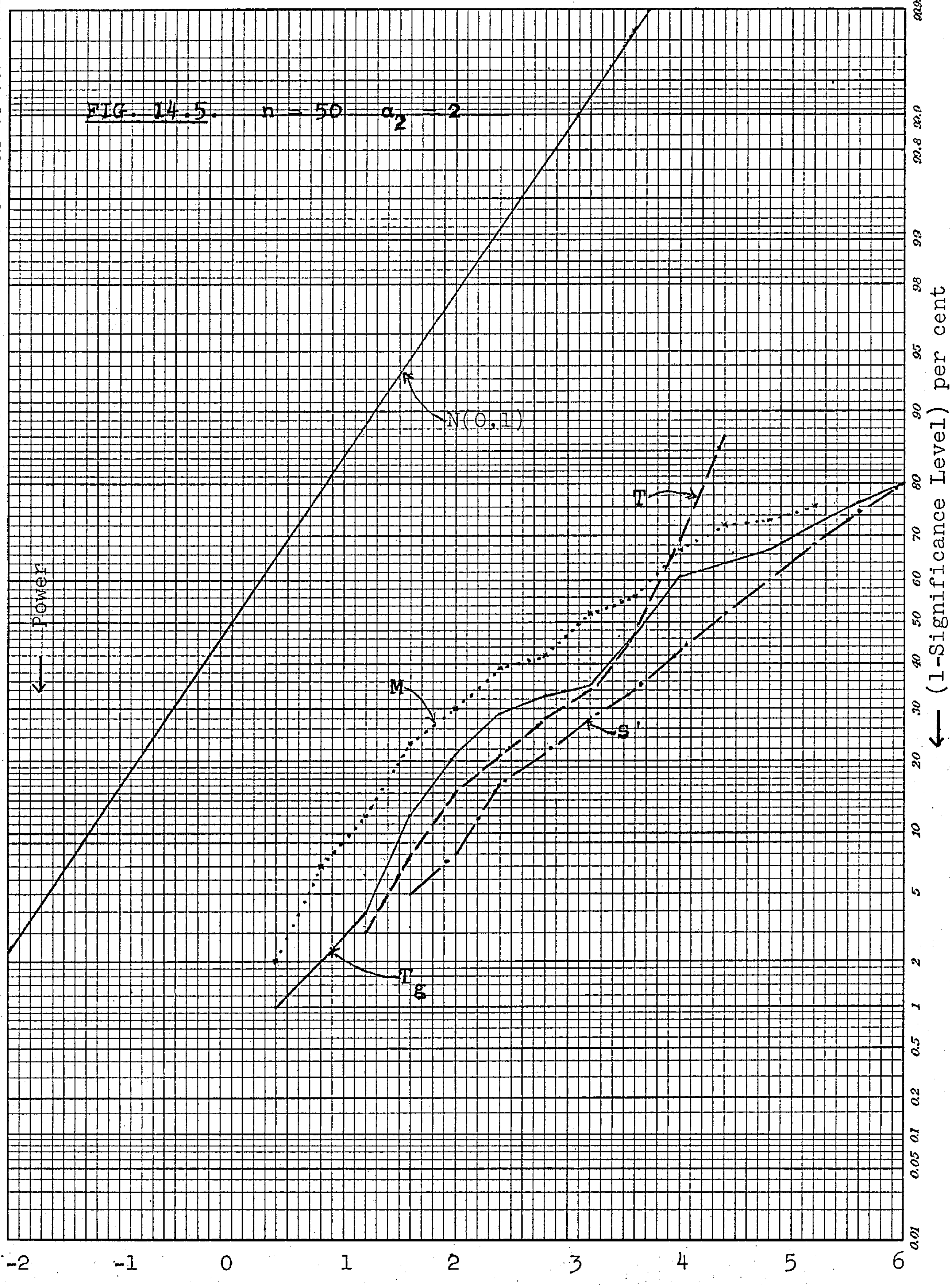


FIG. 14.5. n = 50 α_2 = 2
N(0,1)
T
M
S
T_g
Power
(1-Significance Level) per cent
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
-2
-1
0
1
2
3
4
5
6

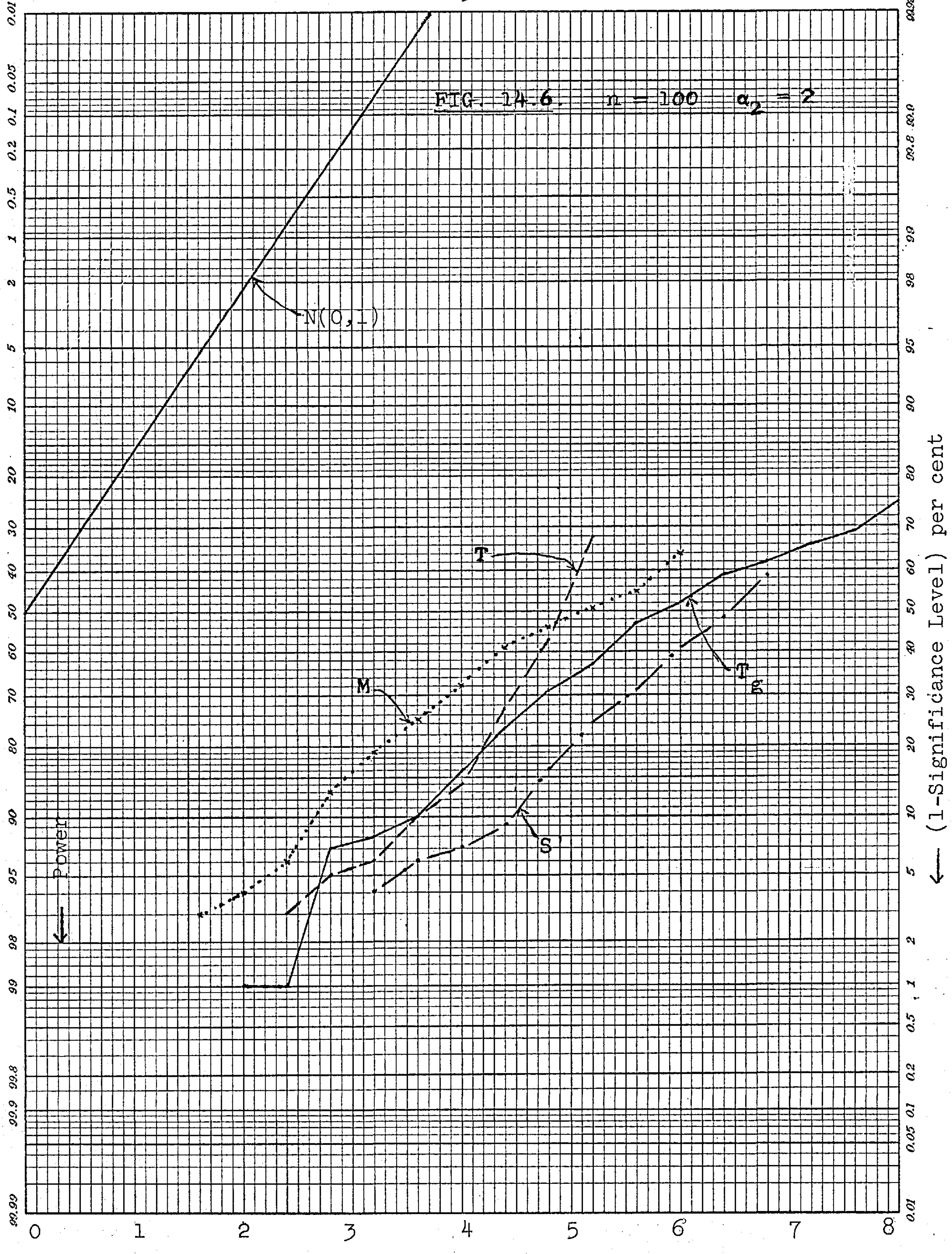


FIG. 14.6 n = 100 α_2 = 2
N(0,1)
T
M
T_g
S
Power
(1-Significance Level) per cent
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99
99.9
99.8
99.5
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0
1
2
3
4
5
6
7
8

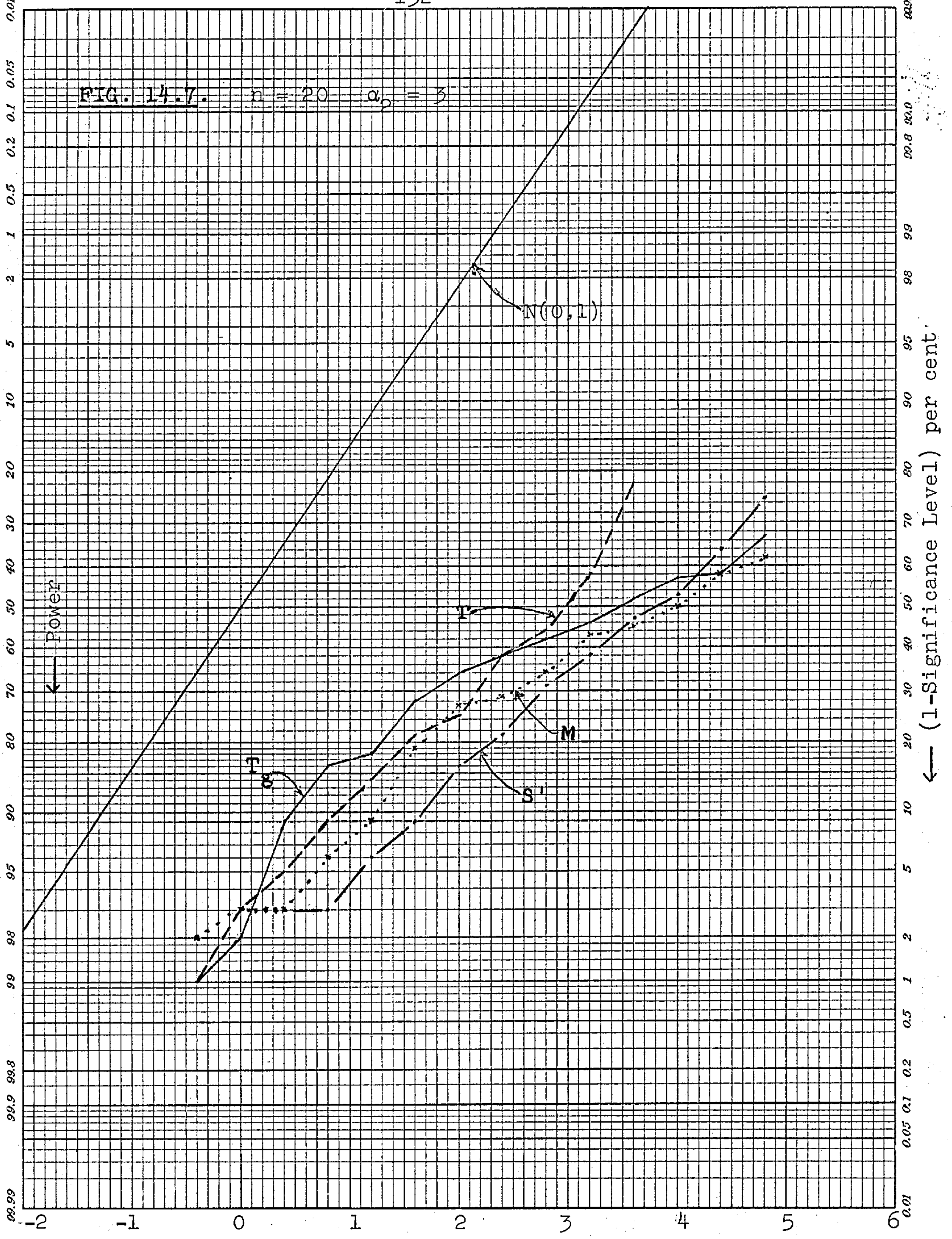


FIG. 14.7. $n = 20$ $\alpha_2 = 3$

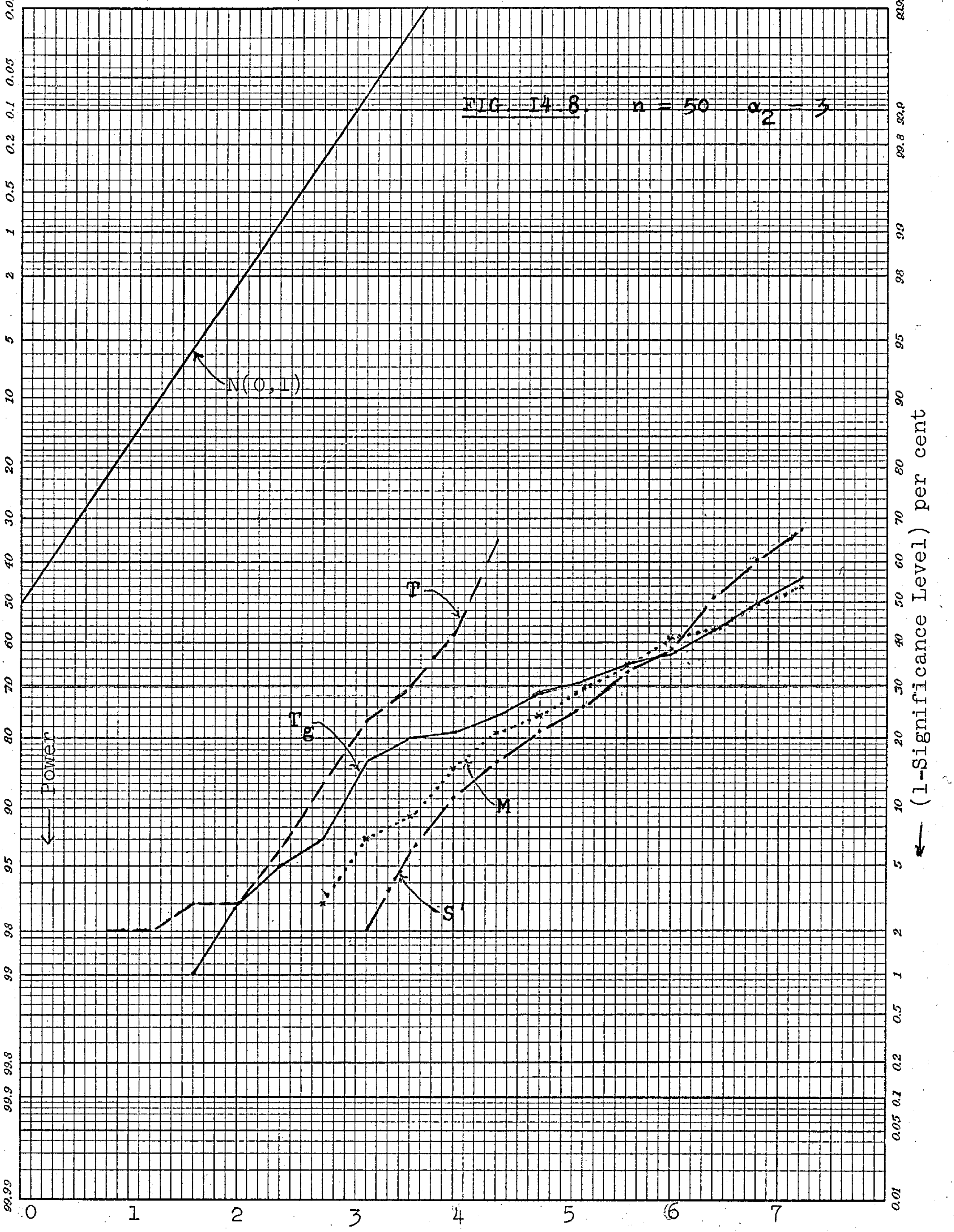


FIG. I4.8 n = 50 α_2 = 3
N(0,1)
T
T_g
M
S
← Power
(1-Significance Level) per cent ←
0
1
2
3
4
5
6
7

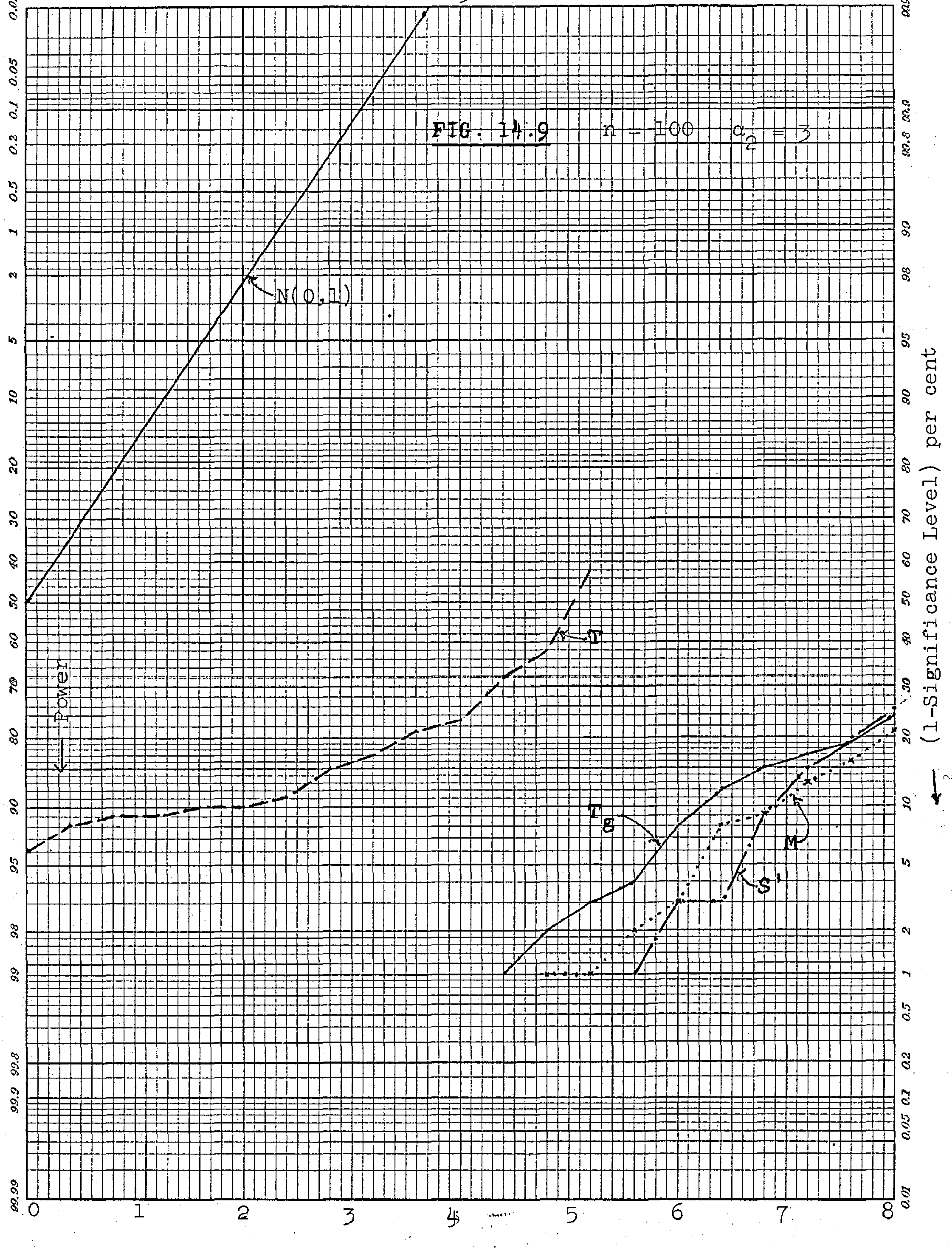


FIG. 14.9 n = 100 $\alpha_2 = 3$

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