GROUP THEORETIC METHODS IN

ELEMENTARY PARTICLE PHYSICS

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Ph.D. Degree

June 1967.

PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, between October 1963 and December 1966 under the supervision of Professor P. T. Matthews. The author wishes to thank him for his guidance and assistance.

Except where stated in the text the work described is original and has not been submitted in this or any other University for any other Degree.

The author gratefully acknowledges the financial support provided by Shell Oil (N.Z.Ltd.) for the duration of his studies.

CONTENTS

ABSTRACT

Chapter 3 of this thesis contains tables giving the reduction of the SU(6) direct products $\underline{56}\otimes\underline{\overline{56}}$, $\underline{56}\otimes\underline{35}$, $\underline{35}\otimes\underline{35}$ and the partial reduction of $\underline{70}\otimes\underline{\overline{70}}$.

Chapter 3 and the preceding chapters also treat, in detail, the structure of the SU(6) algebra and associated problems such as subgroup reduction as well as more technical matters such as phase conventions.

Chapter 4 uses special techniques to investigate the effect of $\underline{35}$-like symmetry breaking on the predictions of $SU(6)_W$ for two body scattering; we are unable to avoid some of the poor results found in the limit of exact symmetry.

The fifth chapter presents a model for the weak interactions of baryons based upon the mixing of $\underline{56}$(L=0) and $\underline{70}$(L=1) irreducible representations of SU(6) x O(3).

INTRODUCTION

During the past five or six years group theory, extended from the long established SU(2) isospin symmetry, has provided an amazingly flexible and fertile base from which to initiate forays into the elementary particle physics battle field; in most such essays the compact SU(3) symmetry has played a central role[1]. One of the major reasons for success thus gained has been the organizational or unifying power of the group theoretic approach allowing as it does an at least partially integrated view of strong, electromagnetic and weak forces; indeed it might now be claimed that the true contribution of groups or algebras is a coherent but coarse organisation of large areas of data - the earlier ideas of generalised gauge invariance[2] contrasted with, for example, a later suggestion that (especially higher) symmetry schemes may arise phenomenologically as a result of unknown interactions at a fundamental level[3] indicates perhaps a changing attitude to the question of whether or not group theory itself says any thing about primary dynamics.

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1. M. Gellmann, Phys.Rev., 125, 1067 (1962)
2. See, e.g. J.Schwingers 1962 Trieste seminar lectures and also the 1963 I.C. thesis of P.A. Rowlett.
3. J. Schwinger 1965 Trieste seminar (unpublished)

Again the idea of an internal symmetry has often been easily wedded to other independent physical concepts, although at a deeper level, the more significant union of internal and space time symmetries in a non trivial fashion has produced only deep conflict, not yet resolved.

A third reason for the popularity of group and algebraic methods has been their capacity to sustain wide ranging and detailed calculations - although these are usually of a comparative nature and always limited in their success by the inherent approximations. It is with the mechanics of calculation and some of their results that this thesis is concerned - we are further restricted to compact symmetry schemes, or more precisely schemes where multiplets contain a finite number of particles. Non-compact systems employing representations containing an infinite number of particles have also been studied, but are not noted for ease of computation.

The SU(6) group studied here was first seen as a direct extension of the supermultiplet theory of Wigner giving a partial non relativistic combination of internal, now SU(3), and space time symmetries[4]. A crucial departure from Wigners work was in multiplet anignments. The low lying baryous and mesons did not occur in the fundamental group representation (also the case for

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4. F. Gursey and L.A. Radicati. Phys.Rev.Letts., 13, 173 (1964) A. Pais ibid, 13, 175 (1964). B.Sakita, Phys.Rev., 136, B.1756,1964

SU(3))[5] and the successes of SU(6) consequently gave considerable impetus to the quark – or composite – model of elementary particles which does then relate the SU(6) irreducible representation $\underline{6}$ to postulated physical states, quarks.

Our first three chapters present the mathematics of this group and culminate in the reduction of the direct product for the most important physical multiplets – a new feature of SU(6), carefully elaborated here, was the increased involvement of the underlying symmetric or permutation groups due to the subgroup decomposition required by the physics. We also take care with the notorious and elusive question of phases and phase conventions.

Given the SU(6) scheme an immediate problem, urgently attacked, was that of finding a relativistic counterpart, corresponding to the incorporation of the Poincare group, and not just one of its little groups, with SU(3). It was hoped that, for example, a clearer understanding of symmetry breaking might result, since the mass operator, now to be included in the symmetry scheme holds in its non degenerate spectrum perhaps the key to this problem. Unfortunately O'Raifeartaigh's[6] theorem on

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5. M.Gellmann, Phys.Letts., 8, 314 (1964)

. G.Zweig, unpublished Cern. notes.

6. L. O'Raifeartaigh, Phys.Rev., 139, B.1052 (1965)

the impossibility of finding a discrete mass spectrum in an exact Poincare group containing symmetry brought this work to a negative conclusion.

Neglecting this shortcoming, considerable development of the relativistic theory occurred. It was found moreover that many of the predictions of such schemes could be deduced from a study of certain <u>compact</u> subgroups. One of these was again SU(6), the so called $SU(6)_W$ and in Chapter 4 we present our own calculations of the predictions of this group for some two body scattering processes in the presence of various types of symmetry breaking.

A parallel line of development also suggested by Gellmann[1] regards the symmetry properties of transition operators to be more basic or at least simpler than those of the single particle states between which they operate and on which was built the group theoretic approach. Perhaps the most noted success in this field, that of current algebras, was the calculation of the weak axial vector to vector coupling constant ratio[7]. Essentially the compact algebra SU(2) x SU(2) was employed so that the transition operators involved had irreducible SU(2) x SU(2) transformation properties whilst the (nuclear) states used were algebraically irreducible only under the isostin subalgebra, and were in fact infinitely reducible under SU(2) x SU(2). From this success arose attempts

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7. S.I Adler, Phys.Rev.Ltees., <u>14</u>, 1051 (1965)

W.I.Weisberger, Phys.Rev.Letts., <u>14</u>, 1047 (1965) to approximate states as a combination of a finite number (usually two!) of irreducible representations, and in Chapter 5 we show an application of the SU(6) algebra and some of our Tables to this question of representation mixing.

CHAPTER 1

ALGEBRAIC AND OTHER PRELIMINARIES

This chapter comprises a miscellany of results and definitions needed in the sequence. We emphasize that there is no attempt at either an elementary or a complete discussion of the material - there are far too many treatments already in existence to justify such delay. Discussion of phase conventions and of the full nature of the dual relationship between continuous and symmetric group is perhaps less readily available and on these two questions we give more detail.

1.1 Canonical forms for simple Lie algebras

The above remarks apply especially in this section; in particular we have not thought it necessary to provide a glossary of Lie algebra terminology. The book by Jacobson treats the subject in full mathematical rigour[1,2], whilst there also exist many well known reviews[3,4].

We shall regard a Lie algebra as a set of matrices which is closed under (a) commutation of any two elements, (b) addition and subtraction of any two elements, (c) multiplication by arbitrary elements from a base field; the specification of this field is essential in the transition from algebra to group and receives brief mention below (§ 1.2). Given a (matrix) basis for the algebra all elements may be obtained by the operations (a), (b), (c). Then,

(1) There exists the following canonical form for the commutation relations (C.R.) of a simple complex (i.e. over the complex field) Lie algebra L of rank l :- L contains a Cartan subalgebra $\mathfrak{H}$ with l linearly independent mutually commuting elements. The remaining ('non diagonal') operators (N-l in number if L has order N) may be split into two sets, raising operators E_α and lowering operators $E_{-\alpha}$ (also collectively denoted shift or ladder operators), and the C.Rs are

$$[H_i, H_j] = 0, \quad i,j = 1, \ldots, l \qquad H \in \mathfrak{H} \tag{1.1a}$$

$$[H_i, E_{\pm\alpha}] = \pm\, r_i(\alpha) E_{\pm\alpha} \tag{1.1b}$$

$$[E_\alpha, E_{-\alpha}] = 2\sum_i r_i(\alpha) H_i \tag{1.1c}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}\ , \quad \text{for any } \pm\alpha,\ \pm\beta \tag{1.1d}$$

where $r_i(\alpha)$ is the i^{th} component of the root vector $r(\alpha)$, and $N_{\alpha\beta}$ is a c-number, equal to zero unless $r(\alpha) + r(\beta)$ is a root.

(2) The complete algebra is generated by a subset of elements associated with the simple roots (see especially Dynkin[2]). These are the generators $E_{\pm\alpha}$ where $r(\alpha)$ is a simple root (there are l of these) and for this system we may write, with $h_\alpha = r(\alpha)\cdot H$

$r(\alpha)$ a simple root

$$[E_\alpha, E_{-\beta}] = 2\,\delta_{\alpha\beta}\, h_\alpha \qquad (r(\alpha) \text{ a simple root}) \tag{1.2a}$$

$$[h_\alpha, E_{\pm\beta}] = \pm\ r(\alpha)\cdot r(\beta)\ E_{\pm\beta} \quad (r(\alpha) \text{ a simple root}) \qquad 1.2b$$

and

$$[E_\alpha, E_\beta] = N_{\alpha\beta}\ E_{\alpha+\beta} \quad \text{as before} \qquad 1.2c$$

(Equation 1.2c prevents 1.2a, 1.2b from forming a subalgebra) Equations 1.2a 1.2b illustrate the remark that a simple Lie algebra of rank l can be viewed as l non-orthogonal SU(2) Lie algebras. (In the following we often represent an algebra by its customary physical symbol, e.g. SU(n) rather than A_{n-1} [3] - the same term may also describe a related group.) We have introduced both H and h since this is closer to physics, e.g. in SU(3) we have I_3 and Y for H_1 and H_2 whereas $h_1 = \frac{1}{2}H_1 + \sqrt{3}/2\ H_2$ $h_2 = \frac{1}{2}H_1 - \sqrt{3}/2\ H_2$. cf. § 1.3.

(3) The numerical factors occurring in the commutation relations (the structure constants) are all real. The numbers $r(\alpha)$, $r(\beta)$ are completely determined, for a given simple L, only upto an overall normalization constant. The commutation relations determine only $N^2_{\alpha\beta}$ - for each α,β $\pm N_{\alpha\beta}$ must be chosen consistent with the C.R. and consistently adhered to (ch. Behrends et al ref. [3]). Different allowed choices produce isomorphic Lie algebras.

(4) Corresponding to the sets E_α, $E_{-\alpha}$, we have the positive roots $r(\alpha)$ and the negative roots $r(-\alpha) = -r(\alpha)$. The concept of a positive root is defined with respect to a certain arbitary ordering of elements H_i in the Carton subalgebra and is extended also to

weights in a arbitrary irreducible representation. (Recall that a root vector is a weight vector of the regular representation.) It is important to state the ordering in defining highest weight, and differences exist in the literature on SU(3), where the common choice is the order (I_3, Y) used by DeSwart [5] and by Behrends et al [3] but (Y, I_3) has also been used, e.g. Salam [3], Rashid [4]. See also § 1.9.

(5) The l simple roots have non positive scalar products whose values may be displayed in a Dynkin diagram [2]; further if $r(\beta)$ is any positive root then $r(\beta) = \sum_{a=1}^{l} n_a r^a(\alpha)$ where n_a is a non negative integer and $r^a(\alpha)$ $a = 1, .., l$ are the simple roots. A positive root $r(\alpha) = \sum n_a r^a(\alpha)$ is said to belong to the k^{th} layer, where $k = \sum_a n_a$; similarly a negative root $r(\beta) = \sum m_a r^a(\alpha)$ (m_a negative integers) belongs to the l^{th} layer $l = \sum_a |m_a|$. The commutator of two generators E_α, E_β where $r(\alpha)$, $r(\beta)$ belong to layers k, l, if non zero is in the $\pm k \pm l$ layer where we take +,(-) for positive (negative) roots.

The concept of layer is also extended to weights in an arbitary IR.

(6) The complex Lie algebra L comprises arbitrary linear combinations, with complex coefficients, of the N generators in eqn. 1.1. For the above structural theorems the use of the complex field is essential [1]. However, for the discussion of associated

groups it is often more convenient to regard this system (eqs. 1.1 over the complex field) as one with 2N real degrees of freedom. When we talk about a real simple Lie algebra L we shall mean the system of eqs. 1.1 with N non complex (i.e. either real or imaginary) parameters.

Much of this thesis is concerned with the finding and subsequent utilization of matrix solutions to the system, eqs. 1.1 of non linear algebraic equations. Standing alone the equations do not have uniquely defined solutions - in order to ensure uniqueness (upto unitary equivalence) and to further specify the nature of the solution additional conditions will have to be imposed. The first of these, the hermeticity conditions, have little computational importance (once we have settled on finite dimensional irreducible representations that is) but considerable theoretical significance, whilst the second, the choice or specification of phase convention has no theoretical significance but is of prime importance when it comes to numerical calculation.

1.2 Hermeticity conditions

These arise when we attempt to pass from a representation of a Lie algebra to a representation of an associated Lie group by a process of exponentiation. We have the complex Lie algebra L represented by a set of matrices $\{E,H\}$ over the complex field, an arbitrary element has the form $L = \sum_{\alpha,i} X_\alpha E_\alpha + Y_i H_i$, $X_i , Y_i \in \mathcal{C}$

For certain ranges of the parameters x, y it is possible to exponentiate L (the exponential converges) and we write $U(x,y) = \exp(iL)$ for the representative of some element, parametrised by the set x,y of some Lie group. For infinitesimal values of these parameters (i.e. in the neighbourhood of the group identity) the equation assumes the form $U(x,y) = 1 + iL$. The particular group thus generated may be compact or non-compact and its associated irreducible representations unitary or non-unitary. An important example is provided by the representations of compact groups, for which we have the following two theorems (1).

I. Every finite dimensional representation of a compact group is equivalent to a unitary representation.

II. In the above canonical form the compact subalgebra of L (i.e. the subalgebra whose elements exponentiate to elements of the compact group) is generated by the set $(E_\alpha + E_{-\alpha})$, $i(E_\alpha - E_{-\alpha})$, H_i taken over the <u>real</u> numbers ($i=1, \ldots, l$; α ranges over the positive roots). Thus also the compact subalgebra is real.

Writing now $U = \exp iL$,

$$L = (x_\alpha(E_\alpha + E_{-\alpha}) + y_\alpha i(E_\alpha - E_{-\alpha}) + u_i H_i),$$

$$x,y,u \text{ real}$$

$$= (Z_\alpha E_\alpha + \bar{Z}_\alpha E_{-\alpha} + u_i h_i) \quad Z = x + iy$$

the condition that U be unitary gives $L = L^+$ (+ denotes hermition adjoint) or

$$E_\alpha + E_{-\alpha} = E_\alpha^+ + E_{-\alpha}^+$$

$$i(E_\alpha - E_{-\alpha}) = -i(E_\alpha^+ - E_{-\alpha}^+)$$

$$H_i = H_i^+$$

i.e. $E_{\pm\alpha}^+ = E_{\mp\alpha}$, $H_i^+ = H_i$ 1.3

Thus to obtain representations of an algebra which exponentiate to unitary representations of the associated unique compact group, the commutation relations must be solved subject to these hermeticity conditions - in this form the hermeticity conditions assume parametrization of the compact group elements with real parameters x,y,u. Keeping the hermeticity conditions and the commutation relations fixed and allowing some of the associated real parameters to become imaginary (we still have a 'real' Lie algebra) may provide a finite dimensional representation of a non-compact group. In this way one representation of the algebra will lead to representations of several non-homomorphic Lie groups. Changing the hermeticity conditions with respect to a given parametrization of a group will then alter the nature of the representation. For example, starting from the compact case and changing some parameters (here unspecified) from real to imaginary can lead to a finite dimensional non-unitary representation of a non-compact group; changing now the hermeticity condition on the associated generators from $E_{\pm\alpha}^+ = E_{\mp\alpha}$ to

$E^{+}_{\pm\alpha} = -E_{\mp\alpha}$ will impose a unitary representation of the group. From a well known theorem it then follows that the associated commutation relations have only infinite dimensional solutions.

The solutions obtained in the next chapters will always be subject to eqs. 1.3; and these then define the appropriate hermeticity conditions in a non-canonical basis.

From the above relationship of group and algebra it is clear that the one representation space may serve for both - we may speak of a group transformation of a basis element, or of an algebra transformation. A space which is irreducible for the group will also be irreducible for the algebra, and so on.

We have adopted a view point for this discussion which has enabled us to emphasize (i) the same solutions of the same C.Rs. of a given (complex) Lie algebra may serve, with multiplicative factors real or imaginary, to provide representations of several Lie groups; (ii) the difference between these groups may be thought to lie in the group parameter space; (iii) keeping the C.Rs fixed the hermeticity conditions are very important in determining the nature of the solution.

There is an alternative viewpoint, also in frequent use,[17] which does not make the factorization: element of L = basis element of complex algebra X real or imaginary number (i.e. all numbers are instead taken either all real or all imaginary) and in this case the commutation relations become characteristic of the associated group as also do the hermeticity conditions. For example consider the complex Lie algebra of SU(2): (we work in a non-canonical basis for which the hermeticity conditions are more easily handled).

From the first point of view we have the commutation relations

$$[S_1\ S_2] = iS_3 \quad [S_2\ S_3] = iS_1 \quad [S_3\ S_1] = iS_2$$

to be solved with hermeticity conditions $S_i^+ = S_i$. Then an ~~external~~ element $L = \sum_i \theta_i S_i$ θ_i real will exponentiate to a unitary representation of the compact SU(2) - whereas if we take θ_1, θ_2 imaginary we get a finite dimensional representation of O(2,1). Changing now to $S_1^+ = -S_1$, $S_2^+ = -S_2$, but $S_3^+ = S_3$ (θ_1, θ_2 imaginary, θ_3 real) will give, with the same parameters, a unitary representation of O(2,1).

Alternatively one keeps the parameters always real, the C.R.s for O(2,1) become $[S_1 S_2] = -iS_3$, the others as above, and a finite dimensional IR will have $S_1^+ = -S_1$, $S_2^+ = -S_2$ $S_3^+ = S_3$ whilst a unitary IR has $S_i^+ = S_i$. From this point of view it makes sense to talk of non compact or compact generators etc. cf. [17].

1.3 Generalities on irreducible representations

The solutions to eqs. 1 will be realised on finite dimensional vector spaces where the matrices act as linear transformations.

Infinite dimensional solutions of Lie algebra equations are also of importance in physics, but will not be discussed here. We shall further focus attention on irreducible solutions, i.e. given the system of matrices L and the vector space M, the representation is formally considered as a linear mapping of L x M onto M. Irreducibility then implies that there is no subspace $M' \subset M$ such that $L \times M' \subset M'$ where C signifies strict containment. Such an irreducible matrix system we shall denote IR. The solutions or representations are to be constructed as follows:-

(1) We choose a basis in the vector space M. This involves labelling or identifying each of the basis vectors and can be achieved by demanding that the basis vectors form a set of orthonormal eigenvectors of some set of matrices (some of which may belong to L) such that no two basis vectors belong to the same eigenvalue of each labelling matrix. (The problem is to find, and to establish the spectra of such a set - usually one looks at subgroups of the given group.) As usual the eigenvalues are to be associated with physical attributes, quantum numbers, drawn from the physical system which our equations are attempting to describe; it will be a matter of interest that the physical labelling so defined need not coincide with the purely mathematical solution, and moreover may not be even a complete or sufficient alternative.

(2) Subject to a given ordering scheme on the weight components,

the finite dimensional IR's of a Lie algebra are characterised by a single vector known as the highest weight. From this vector an appropriately labelled basis may be derived or rather defined using the ladder operators of § 1.1. Alternatively, see § 1.5 the IR may be specified with the aid of a Young tableau; these are defined and discussed in G.Murtaza and M.A.Rashid[4] as well as in the book by Hamermesh[6].

Again, in place of either of these, an IR may be specified by giving the values attained in that IR by the Casimir operators - of which there are l independent ones in a simple Lie algebra of rank l. (It is worth remarking that this method fails for infinite dimensional representations.) Labelling by Casimir operators is not much used in the physical literature.

A further property of the highest weight labelling is:-

(3) In the canonical scheme there are l IRs called 'basic irreducible modules' in[1] - these basic IRs have highest weight vector U^a say, of the form $U^a_i = \begin{bmatrix} 0, & i \neq a \\ \frac{1}{2}, & i = a \end{bmatrix}$ $a, i = 1, \ldots, l$ (the factor $\frac{1}{2}$ results from the factor 2 in eq.1.2a); the eigen-value equation is $h_i U^a = \frac{1}{2} \delta^a_i U^a$ (no summation). Any highest weight now has the unique form

$$U = \sum_a n_a U^a \qquad 1.3a$$

where n_a is a non negative integer. Note also that

$U^l > U^2 > \cdot > U^1$, where $>$ means higher than.

In the following we always denote a basis vector of any representation by the ket symbol $|\alpha\rangle$ where the set α specified quantum or labelling numbers. (An arbitary state or vector will be exapnded in terms of these basis.) Often it will be convenient to write for α that particle which conventionally has the quantum numbers α, e.g. $|\pi^+\rangle$ is the $I = I_3 = 1$ $Y = 0$ basis state of the SU(3) octet. We emphasize that physical states may be associated with $\pm$ a basis state. See §§ 2.2, 3.2. An alternative form for α is (μ,ν) where μ is the dimension of the IR and ν represents other labels.

Having obtained various IR's of the algebra we shall employ them in the standard group theoretic process of reducing the inner or direct product.

1.4 Inner and Outer products

(1) Inner product: Here we take two IR's of a group and ask which IR's (of the same group) appear in their inner, or direct, or Kronecker product - in this way the inner product is expressed as a direct sum: formally we have the Clebsch-Gordan series,

$$\mu_1 \otimes \mu_2 = \sum_i \oplus \mu_i$$

where μ_j labels the IRs of some group (or algebra), $\otimes$ indicates direct product, and $\oplus$ direct sum. Specialising to basis vectors we write

$$|\mu_1\nu_1\rangle \otimes |\mu_2\nu_2\rangle = \sum_i \begin{pmatrix} \mu_1 & \mu_2 & \mu_i \\ \nu_1 & \nu_2 & \nu_i \end{pmatrix} |\mu_i,\nu_i\rangle \qquad 1.4$$

where we have introduced the Clebsch-Gordan coefficient (C.G.c.) $\begin{pmatrix} \mu_1 & \mu_2 & \mu_i \\ \nu_1 & \nu_2 & \nu_i \end{pmatrix}$ - we use the same notation as deSwart[5], but entended to an arbitrary group or algebra; usually the group or algebra to which the CGc refers to will be quite clear. We depart from the deSwart notation in one respect, viz his symmetric and antisymmetric CGc's used in the SU(3) direct product $\gamma \otimes \gamma$ are denoted γ_s, γ_a rather than γ_1, γ_2 respectively. The suffices 1, 2, are reserved for another role. Chapter 3 is concerned with the calculation of CGc's.

(2) Outer product: Here we take two IR's of different groups and ask which IR's of a third group appear in their outer product;

$$\mu_1 \times \mu_2 \uparrow \sum_i \mu_i$$

x denotes outer product and the symbol $\uparrow$ emphasises that this is an embedding and one is really enlarging the representation space in going to the outer product.

We shall also use the symbol x alone, without $\uparrow$. Then, as is usual, it will merely indicate the independent existence of the two component groups or algebras.

1.5 Symmetric group

The symmetric group S_r is the group of permutations on r objects, it is a finite group of order r! In the following we assume acquaintance with Young Tableaux (Y.T.) (cf. Rashid and Murtaza[4, 6, 7, 10]) and their role in defining the IRs of Sr.

A YT shall be denoted $[\lambda]$ corresponding to the partition $\lambda_1 \geqslant \lambda_2, \ldots \geqslant \lambda_p$ of r, or by the familiar array of boxes.

$[1] = \square$; $[2] = \square\square$; $[1^2] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$; $[21] = \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ etc.

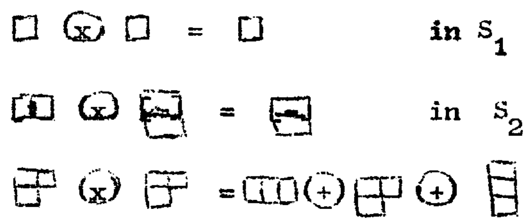
Now take r objects labelled 1,2,...r and form the r! different permutations; a given YT then constitutes a shorthand way of stating which sets of linear combinations of these elements are invariant under permutation, i.e. the YT enables a direct construction of a complete basis for the associated IR. We give some examples:-

Group	IR	dim	Basis function
S_1	[1]	1	$\|1\rangle$
S_2	[2]	1	$\sqrt{\tfrac{1}{2}}\,(\|12\rangle + \|21\rangle)$
	$[1^2]$	1	$\sqrt{\tfrac{1}{2}}\,(\|12\rangle - \|21\rangle)$
S_3	[3]	1	$\sqrt{\tfrac{1}{6}}\,(\|123\rangle + \|231\rangle + \|312\rangle + \|213\rangle + \|321\rangle + \|132\rangle)$
S_3	[21]	2	$\tfrac{1}{2}(\|123\rangle + \|213\rangle - \|321\rangle - \|231\rangle)$
			$\tfrac{1}{2}(\|132\rangle + \|312\rangle - \|321\rangle - \|231\rangle)$
	[21]	2	$\tfrac{1}{2}(\|132\rangle - \|312\rangle + \|231\rangle - \|321\rangle)$
			$\tfrac{1}{2}(\|123\rangle - \|213\rangle + \|321\rangle - \|231\rangle)$
	$[1^3]$	1	$\sqrt{\tfrac{1}{6}}(\|123\rangle + \|231\rangle + \|312\rangle - \|213\rangle - \|321\rangle - \|132\rangle)$

The entries under basis function have been obtained using the Young operator, a different operator for each basis function (this incidentally provides a labelling). A recent full discussion of the method can be found in ref.[8]. Note that in S_3 the IR[21] is of dimension 2 and we find two equivalent orthogonal sets of basis functions; this is an example of a general result, which states

$$\sum_{\lambda} n_{\lambda}^{2} = g$$

where the summation is over the dimensions n of different IRs of a finite group of order g. Notice also that the basis functions found by this YT prescription are not orthogonal within an IR - for physical applications this generally is a disadvantage of this system, cf. also[8]. We can combine IRs of Sr according to the inner product. Some simple Clebsch-Gordan series are



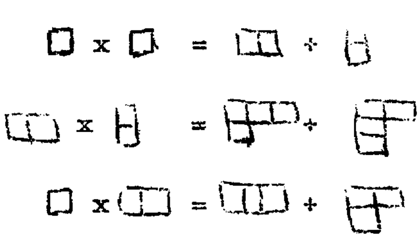
in S_1

in S_2

in S_3

The product of dimensions on the left is equal to the sum of dimensions on the right. Some CGc's for the lower S_r are given in [6].

Some outer products are



$S_1 \times S_1 \uparrow S_2$

$S_R \times S_2 \uparrow S_4$

$S_1 \times S_2 \uparrow S_3$

We shall now see that these define inner products in SU(n) and with respect to this group the product of dimensions of the two IRs on the left is equal to the sum of those on the right.

1.6 Tensorial realisations of SU(n)

To avoid complication, in the following we refer to SU(n), however the results may be adapted to others of the so called classical[9] series of groups.

The unitary finite dimensional IRs of SU(n) may be realised on a tensor space which is a direct product, p times, of the n dimensional fundamental or defining representation space, A; the (defining) group matrices are nxn, unitary, and unimodular. The product space $A^1 \times A^2 \ldots \times A^p$ of p^{th} rank tensors is reducible, its reduction is accomplished with the aid of the Young operators or symmetrisers (cf. Rashid[4,6]) which act on the indices 1,...p of the product spaces to produce tensors of definite symmetry type. (These we shall often denote $T[\lambda]$ - thus $T_{\tiny\yng(2,1)}$ represents a 3rd rank tensor with [21] symmetry.) Corresponding to the appearance of $[\lambda]$ n times in the outer product $\square \times \square \ldots \times \square$ of $S_1 \times S_1 \ldots \times S_1 \uparrow S_p$ the tensor space can carry n orthogonal equivalent IRs $[\lambda]$ of SU(n).

To obtain a basis for the IR$[\lambda]$ of SU(n) construct a tensor $T_{i_1 \ldots i_p}$ of symmetry $[\lambda]$; each i_j ranges from 1 to n and the SU(n) transformations change this index value for each i_j. Now consider all allowed sets of index values (if the tensor is anti-symmetric in i_j, i_k we cannot have $i_j = i_k$ for any i=1,...,n) and for each such set construct basis functions for the IR$[\lambda]$ of S_p - in general a set of index values will not support a complete IR of S_p due to equalities amongst the index values. In this way one obtains a labelled basis for the IR$[\lambda]$ of SU(n), as used by Weyl[9] in his work on the classical groups; however the basis is non orthogonal and the labels have no direct physical interpretation.

The inner product in SU(n) multiplies two irreducible tensors, rank r,s say to produce a reducible tensor of rank r+s - the original IR tensors were defined with the aid of S_r S_s, clearly the reduction of their direct product will involve S_{r+s} and we have the correspondence: inner product in SU(n) $\sim$ outer product in symmetric group. Rules for the formation of Clebsch-Gordan series in SU(n) are thus those for the formation of symmetric group outer product[6,10]. Of course these rules must be supplemented when the symmetric group does not completely reduce the continuous group as, for example, when it is possible to form traces. For SU(n) as is well known (cf. Rashid[4]) any tensor can be written with covariant indices only so that the removal of traces in this case can be avoided. For example, in SU(3) the direct product $\underline{8}$ (x) $\underline{8}$ corresponds to [21] $\otimes$ [21] and is evaluated in this way, see eq.[6] p.252, to give $\underline{8} \otimes \underline{8} = \underline{1} \oplus \underline{8}^2 \oplus \underline{10} \oplus \underline{\overline{10}} \oplus \underline{27}$

Alternatively in SU(3) $T^{\square}_{\square} \sim 8 \oplus 1$ and

$$T^{\square}_{\square} \otimes T^{\square}_{\square} = T^{\square\square}_{\square\square} \oplus T^{\square\square}_{\substack{\square\\\square}} \oplus T^{\substack{\square\\\square}}_{\square\square} \oplus T^{\substack{\square\\\square}}_{\substack{\square\\\square}}$$

$$(8 \oplus 1) \otimes (8 \oplus 1) = (27 \oplus 8 \oplus 1) \oplus (\overline{10} \oplus 8) \oplus (10 \oplus 8) \oplus (8 \oplus 1).$$

We shall call the outer product in SU(n) that process whereby IRs of SU(p) and SU(q) are combined to form IRs of SU(pq)

$$SU(p) \times SU(q) \uparrow SU(pq)$$

$SU(p) \times SU(q)$ is a maximal subgroup of SU(pq), i.e. there is no subgroup G of SU(pq) such that we have the following scheme of strict containment: $SU(pq) \supset G \supset SU(p) \times SU(q)$. See Dynkin[2]. Now take rth rank IR tensors of SU(p) and SU(q) say $P_{i_1 \ldots i_r}$, $Q_{j_1 \ldots j_r}$ and define a set of index values in SU(pq) by $k_s = i_s j_s$ (no sum) $k = 1,\ldots,pq$ $i = 1,\ldots,p$, $j=1,\ldots,q$, $s=1,\ldots,r$. From the point of view of S_r the IR tensor P has, for each set of index values, definite permutation symmetry on r objects (the objects being the underlying product spaces), similarly for the tensor Q. Such tensors can thus be combined according to the inner product in S_r; in so doing we create an rth rank tensor with indices k_s, $s=1,\ldots,r$. Thus the process of outer product in SU(n) corresponds to that of inner product in S_r[11,12].

In passing we note there are two ways in which one might decompose SU(n) according to its unitary subgroups. In the fundamental IR $\underline{n}$ these correspond to splitting the representation space into (i) a direct sum, so that say the first p indices belong to SU(p), the last q belong to SU(q) with $n = p+q$;

[illegible] product, so that the fundamental tensor in SU(n) to SU(p), the last q belong to SU(q) with $n = p+q$.

(ii) a direct product, so that the fundamental tensor in SU(n) is a product of fundamental tensors in SU(p) and SU(q).

$$T_k = T_p T_q \quad , \quad pq = n$$

Having introduced tensors we can also, outline the relation between the highest weight and YT labelling of SU(n) IRs. of § 1.3.

Suppose the ordering scheme is defined so that $T_1 > T_2 \ldots > T_n$ where by T_i is meant the ith basis vector of the fundamental IR in SU(n). Hence we can arrive at the highest weight in an arbitrary $[\lambda]$ by filling the first row with indices 1, the ith with indices i (cf. Rashid[4]). Now view the YT column wise - each column on its own denotes an IR and the state of each IR specified is in each case the highest weight. Thus the highest weight in $[\lambda]$ is obtained by summing over highest weights for YT of the form $[1^r]$ $1 \leq r \leq n-1$. (Recall that in SU(n) $[1^n]$ is the scalar IR, Rashid[4].) We thereforehave a 1:1 correspondence between the n-1 YTs $[1^r]$ and the $1 \leq (n-1)$ basic IRs of SU(n), which is explicitly formulated as, for $[\lambda]$ $[\lambda_1, \lambda_2, \ldots \lambda_{n-1}]$, and using eq. 1.3a,

$$n_a = \lambda_a - \lambda_{a+1} \qquad 1.4a$$

1.7 Canonical labelling scheme for SU(n)

The decomposition (i) above is important for the case $p=n-1$, $q=1$, since the chain

$SU(n) \to SU(n-1) \times U(1)$, $SU(n-1) \to SU(n-2) \times U(1)$, ..., $SU(2) \to U(1)$

can be used to provide a complete set of labels for an arbitrary IR of SU(n). For example in SU(3) each basis vector is given definite I^2, I_3, Y eigenvalue, and this suffices to distinguish states in an IR - I^2, Y correspond to the step SU(3) $\rightarrow$ SU(2) x U(1), whilst I_3 labels the (equivalent) IRs of U(1) occurring in SU(2) $\rightarrow$ U(1). From the point of view of § 1.3(i), the diagonal labelling operators are the Casimir operators of SU(n), SU(n-1),. SU(2) (these are not elements of the SU(n) Lie algebra) and the n-1 independent U(1)'s, which are elements of the algebra and span its Cartan subalgebra. Instead of using the eigenvalues of the Casimir operators as labels one may alternatively give the corresponding Young tableau - this allows a concise tabular representation of basis vectors in terms of a 'Gelfand pattern'. A full discussion occurs in a series of papers by G.E.Baird and L.C.Biedenharn[13] where this labelling scheme is exploited to derive the matrix elements of the SU(n) generators in an arbitrary IR. For explicit calculation this canonical scheme was thus more tractable than that of Weyl, and further, in the case of SU(3), the labels could be identified with physical labels. Unfortunately this is not so for the physical SU(6). See Chapter 2.

1.8 Irreducible tensor operators

These are defined and discussed in many places[5,13,14], as for CGc we shall use the definitions of deSwart[5]. The fundamental results are the transformation law

$$T(\mu_1\nu) \rightarrow U(\alpha)\ T(\mu_1\nu)\ U^{-1}(\alpha) \ = \ \sum_{\nu'} D^{*\mu}_{\nu'\nu}(\alpha)\ T(\mu_1\nu')$$

and the Wigner Eckart theorem cf.[5] §13.2 $T(\mu_1\nu)$ is an irreducible tensor operator, belonging to the IR $\underline{\mu}$, α are the parameters of an arbitrary group transformation and $D^{\mu}_{\nu'\nu}(\alpha)$ is the IR $\underline{\mu}$ representative of that transformation (* denotes complex conjugate). Writing $U(\alpha) = \exp.(i\alpha_i F_i)$, the algebraic form of the defining relation becomes (using $F_i^{+} = F_i$)

$$[F_i,\ T(\mu_1\nu)] = \sum_{\nu'} (F_i)_{\nu\nu'}\ T(\mu_1\nu'). \tag{1.5a}$$

This equation is linear in the generators, and will hold for any set obtained by a linear transformation of the set $[F_i]$. It is also clear that the generators themselves form an irreducible set, and for them, in an abritrary tensor basis for example,

$$[T(\mu_1\nu),\ T(\mu_1\lambda)] = \sum_{\lambda'} T(\mu_1\nu)_{\lambda\lambda'}\ T(\mu_1\lambda') \tag{1.5b}$$

where $$T(\mu_1\nu)_{\lambda\lambda'} \equiv \langle(\mu_1\lambda')\,|\,T(\mu_1\nu)\,|\,(\mu_1\lambda)\rangle \tag{1.6a}$$

$$= \begin{pmatrix}\mu & \mu & \mu \\ \lambda & \nu & \lambda'\end{pmatrix} \langle \| T \| \rangle \tag{1.6b}$$

and we have used the Wigner Eckart theorem; $\langle \| T \| \rangle$ is the reduced matrix element of the generators in the regular representation.

The equation 1.6a also defines the matrix elements of the generator $T(\mu_1\nu)$ in a general representation, $\underline{\mu}'(\mu$= regular representation).

$$T(\mu_1\nu)_{\lambda\lambda'} = \langle(\mu',\lambda')|T(\mu_1\nu)|(\mu_1'\lambda)\rangle$$
$$=\begin{pmatrix}\mu' & \mu & \mu' \\ \lambda & \nu & \lambda'\end{pmatrix}\langle\mu'\|T\|\mu'\rangle \qquad 1.6c$$

$\begin{pmatrix}\mu' & \mu & \mu' \\ \lambda & \nu & \lambda'\end{pmatrix}$ is the Clebsch Gordan coefficient for the reduction $u' \otimes \mu \rightarrow \mu' \oplus \ldots$ and $\langle\mu'\|T\|\mu'\rangle$ is the reduced matrix element of the generators in the IR μ'. It may happen that μ' appears more than once in the product $\mu' \otimes \mu$, then of course only one of the associated sets of coefficients will be related by eq. 1.6c to the matrix elements of the generators. In any case eq. 1.6c exhibits one of the many roles of the CGc's, in this case that of providing essentially the matrix elements of the generators in a given representation. A more exhaustive list of their varied functions is given in the Boulder 1962 lectures of Biedenharn[3].

The transformation law for basis states is, after deSwart,

$$|(\mu_1\nu)\rangle \xrightarrow{U(\alpha)} |(\mu_1\nu)\rangle' = \sum_{\nu'} D^*_{\nu'\nu}\,|(\mu_1\nu')\rangle$$

or for the algebra

$$|(\mu_1\nu)\rangle \xrightarrow{T(\mu_1\lambda)} |(\mu_1\nu)\rangle' = \sum_{\nu'} T(\mu_1\lambda)_{\nu\nu'}\,|(\mu_1\nu')\rangle\ .$$

In the regular or adjoint representation the generators themselves provide the basis: $T(\mu_1\nu)$ has the matrix representative $T(\mu_1\nu)_{\lambda\lambda'}$, where

$$[T(\mu_1\nu),\ T(\mu_1\lambda)] = \sum_{\lambda'} T(\mu_1\nu)_{\lambda\lambda'}\ T(\mu_1\lambda').$$

We can thus set up a mapping generator $\longleftrightarrow$ basis state, with

$$[T(\mu_1\nu),\ T(\mu_1\lambda)] \rightarrow T(\mu_1\nu)|(\mu_1\lambda)\rangle$$
$$= \sum_{\lambda'} T(\mu_1\nu)_{\lambda\lambda'}|(\mu_1\lambda')\rangle \qquad 1.7$$

To get numerical factors correct in the mapping it is best to work from the irreducible tensor basis, where we can take $T(\mu_1\nu) \rightarrow C|(\mu_1\nu)\rangle$, and we may or may not take the overall constant C=1 cf Table 2, next Chapter. In other bases the numerical factor occurring in the mapping may vary from state to state. Eq.1.7 is very useful for obtaining generator matrix elements from the commutation relations, cf.§ 2.1.

1.9 Phase conventions

The question of phase conventions is not unique to the Lie algebra commutation relations; it also arises for example in deriving the representations of the symmetric group[6]; it results from a freedom of choice analogous to the choice between right-handed and left-handed co-ordinate systems in the representations of O(3), the Euclidean group of 3 dimensional rotations. A transition from right-handed to left-handed system requires reversing the definition of the positive direction on one of the three Cartesian axes.

We begin with the example of SU(2) :

Commutation relations : $[S_+S_-] = 2S_3$, $[S_3S_\pm] = \pm S_\pm$ 1.8a

Hermeticity conditions: $(S_\pm)^+ = S_\mp$ $S_3^+ = S_3$ 1.8b

Labelling operators: $S^2 = \frac{1}{2}(S_+S_- + S_-S_+) + S_3^2$ and S_3

with Spectra: $S^2 = S(+1)\ 1 \qquad S = 0, \frac{1}{2}, 1, \ldots$

$$S_3 = \text{diagonal } (S, S-1, \ldots -S+1, -S)$$

and one fixed value of S for each IR.

The only unknown matrices are $S_{\pm}$ and the equations 1.8a, 1.8b are easily solved to find

$$\langle S, S_3 \pm 1 | S_{\pm} | S, S_3 \rangle = \pm\sqrt{(S \mp S_3)(S \pm S_3 + 1)} \qquad 1.9$$

all other matrix elements vanish. A unique solution is now chosen by taking always the plus sign in eq. 1.9 - this is the universally used Condon and Shortley phase convention: $S_{\pm}$ have non negative matrix elements in every IR.

From a computational view point we start with the highest state $|(S, S_3=S)\rangle$ (this may be constructed by symmetrising in a direct product of 2S+1 fundamental spaces, or may occur as a vector in the direct product of two IRs) and the problem is then to obtain the remaining 2S basis functions. The operator S_- is used to produce these 2S states which have different weight (i.e. the weights are simple) and are thus orthogonal; since orthoronality does not determine relative signs there are 2^{2S+1} different sets of basis functions, and 2^{2S} different solutions for $S_{\pm}$ (an overall minus sign has no effect on matrix elements hence $2^{2S} = 2^{2S+1}/2$).

Within the canonical scheme a phase convention for general

SU(n) has been suggested by Baird and Biedenharn[13] and adopted by a number of other workers[15]. We now describe this convention and relate it in particular to the structure given in §1.1 :-

The positive roots r(ij), or in short (ij), $i < j$ (it is convenient to replace α by ij here) and their associated generators may be schematically displayed as follows:

(12) (23) (n-2,n-1) (n-1,n) 1st layer

(13) (n-2,n) 2nd layer

(1,n) $(n-1)^{th}$ layer

The simple roots are (i,i+1) and the positive roots satisfy

$$(i,j) = (i,i+1) + (i+1,i+2) + \dots + (j-1,j) \qquad 1.10$$

From §1.1 (2) the generators $E_{i,i+1}$ generate the whole algebra when we include hermitian conjugation, and hence our phase convention is sufficient if it uniquely specifies this set.

The labelling subgroups can be embedded so that the simple roots of SU(i) are (1,2), ..., (i-1,2) - when we go from SU(n) to SU(n-1) x U(1) the generators excluded are $E_{\pm(n-1,n)}$ i=1,...n-1. As will be seen in the SU(6) example, the commutation relations amongst the Eij generators take the form $[E_{ij},E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}$ and from this it follows that E_{kl}, l fixed k=1,...,l-1 form a set of irreducible tensor operators (apart from a phase which is important,)transforming like the defining IR of SU(j-1).

Suppose the representatives of the generators of SU(m), $m \leq n-1$ are know. A basis for an arbitrary IR$[\lambda]$ of SU(n) is obtained by assembling the different IRs of SU(n-1) specified in the decomposition SU(n) $\rightarrow$ SU(n-1) x U(1) - in the IR$[\lambda]$ the matrices of SU(n-1) x U(1) then assume block diagonal form and correspondingly the representation space $[\lambda]$ is a direct sum of subspaces irreducible under SU(n-1) x U(1). Adjoining any one of the generators $E_{n-1,n}$ to the SU(n-1) x U(1) set the CRs now close on the algebra of SU(n), or equivalently this same operator acting on any basis vector of any invariant [under SU(n-1) x U(1)] subspace must lead on repeated application to all other such invariant subspaces. Hence $E_{n-1,n}$ may be used to define the relative signs of the different subspaces, or more aptly from our point fo view, introducing overall signs between different invariant subspaces will allow variation of the signs of the matrix elements of $E_{n-1,n}$. For general SU(n) the operator $E_{n-1,n}$ acting on a basis vector in some invariant subspace $[\alpha]$ say may lead to more than one basis vector in another subspace $[\beta]$. But the relative signs of basis vectors within the same SU(n-1) x U(1) are fixed (by conventions adopted to get this far) and we may not nominate independently the sign of each matrix element of $E_{n-1,n}$. What we must do is remove the dependence on explicit SU(n-1) x U(1) states using the Wigner-Eckart theorem :

$$E_{n-1,n}|[\alpha],\alpha\rangle = \sum_{\substack{\beta \\ [\beta]}} \begin{pmatrix} [\alpha] & [1] & [\beta] \\ \alpha & n-1 & \beta \end{pmatrix} |[\beta],\beta\rangle . \langle \| [1] \| \rangle . \qquad 1.11$$

The SU(n-1) CGc is known; the phase ambiguity now resides solely in the reduced matrix element and can be resolved there, e.g. Baird and Biedenharn[13] find equations for the squares of these reduced matrix elements. Including also the SU(n-1) CGc's from eq.1.11 they are able to adjust the signs of the reduced matrix elements so that the operator $E_{n-1,n}$ has positive matrix elements, and this then constitutes a general phase convention for SU(n).

It is evident that in proceeding from SU(j) to SU(j+1) we may in fact take any of the operators $E_{j+1-i,j}$ $i=1, \ldots, j$ to have non negative matrix elements, but the most obvious choice, canonically, seems to be the simple generator $E_{j,j+1}$.

For SU(3), where we have the positive root scheme

$$\begin{array}{ccc} (12) & & (23) \\ & (13) & \end{array}$$

de Swart[5] has defined highest weight by the order (I_3, Y), and one is forced to identify E_{13} with the isospin operator I_+, since the corresponding root vector $|\pi^+\rangle \sim (1,0)$ is now highest weight in the adjoint representation. In order that the Condon and Shortley phase convention hold for isospin de Swart takes E_{13}, and then $E_{12}(K_+)$ to have non-negative matrix elements. In going from SU(2) to SU(3) the identification of isospin matrices has changed; this could be avoided by adopting the ordering (Y, I_3) when $|K_+\rangle$ would be highest root, and $|\pi^+\rangle$, $|K^0\rangle$ would be simple roots, We must next discuss the consistency of phase conventions, evidently a convention will be consistent if it produces the same representation for the generators no matter how the particular basis is produced, i.e. whether it occurs in the direct product of various paris of the IRs or whether it be constructed directly from the fundamental representation.

A related consistency requirement arises in the definition of complex conjugate representation. In fact if general algebraic solutions are found subject to some phase-convention then obviously that convention can be applied in any given IR - however in the absence of general solutions one should always check that the choice made may be consistently applied to the complex conjugate IR.

As is discussed, e.g. in ref.[6] § 5.4, given one IR of a group, others may be constructed from it, not only by forming the direct product, but also by taking the complex conjugate, or inverse transpose matrices and these are irreducible. For groups of unitary matrices $UU^{+}=1$ implies $(U^{-1})^{+} = U^{*}$ and complex conjugate & inverse transpose are trivially equivalent, more generally Hammermesh[6] shows that these are always equivalent if the group supports an invariant non-singular hermition form. This is the case for non compact forms of SU(n) eq. SU(p,n-p). In detail, the transformation $\psi^{\mu}_{\nu} \rightarrow D^{*\mu}_{\nu',\nu}\psi^{\mu}_{\nu'}$, becomes, under complex conjugation, $\psi^{*\mu}_{\nu} \rightarrow D^{\mu}_{\nu',\nu}\psi^{*\mu}_{\nu'}$, or for the matrices of the algebra, in a hermition basis, we have

$$U(\alpha) \sim 1 + iL, \quad L^{+} = L \qquad \bar{U}(\alpha) \quad 1 - i\bar{L} = 1 + i(-L^{T})$$

$$\text{or} \quad L \rightarrow -L^{T}$$

However the direct use of $L \rightarrow -L^{T}$ in our solutions of eqs. 1.1 is not allowed due to the phase convention we have adopted on the matrix elements of some E_{ij} viz that certain of these be positive.

We add that the transformation $L \rightarrow -L^{T}$ on the algebra is clearly non trivial; for example all the weights will be reversed in sign, corresponding to the change in sign of the diagonal matrices and associated with this raising and lowering operators interchange their functions. For the canonical phase convention, where $E_{i,i+1}$ are to have non negative matrix elements, we can now make a second trivial (phase) transformation by changing the signs of all matrices E_{ij} belonging to an odd root layer. This is easily seen to be consistent with the CRs eq. 1.1 - the only one needing checking is 1.1d-and we thereby recover a solution subject to the required phase convention. In general one should always check, by inspecting the CRs, that the transformation

$$E_{ij} \rightarrow -E_{ij} \qquad E_{ij} \text{ phase determining matrix}$$

is consistent with them, and that the phase convention may therefore be extended to the complex conjugate IR.

We illustrate the consequences for basis vectors with some examples:

a) SU(2) IR$\underline{2}$ basis vectors $|p\rangle$, $|n\rangle$; the complex conjugation transformation takes $|p\rangle \longrightarrow |p^*\rangle$, $|n\rangle \longrightarrow |n^*\rangle$ and the change in sign of E_{12} in the IR $\underline{2}^*$ is then accomplished by taking as basis states $|p^*\rangle$, $-|n^*\rangle$ i.e. in more customary notation

$$\begin{pmatrix} p \\ n \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}$$

Clearly we could also have taken $\begin{pmatrix} -\bar{n} \\ \bar{p} \end{pmatrix} = -1\begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}$ as basis states.

b) SU(3) IR$\underline{3}$ $\begin{pmatrix} p \\ \lambda \\ n \end{pmatrix}$ The deSwart [5] phase convention gives the matrices connecting $(p \leftrightarrow n)$ and $(p \leftrightarrow \lambda)$ positive matrix elements. Under complex conjugation

$$\begin{pmatrix} p \\ \lambda \\ n \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{n} \\ \bar{\lambda} \\ \bar{p} \end{pmatrix} \qquad (\bar{n} > \bar{\lambda} > \bar{p})$$

and to preserve positive matrix elements where required we can take as basis states

$$\begin{pmatrix} \bar{n} \\ \bar{\lambda} \\ -\bar{p} \end{pmatrix}$$

CHAPTER 2

THE STRUCTURE OF THE SU(6) ALGEBRA

In this chapter the various techniques and theorems displayed in the preceding chapter are used to investigate the structure of the SU(6) algebra and its irreducible representations. We try to be especially careful with regard to that notorious bugbear of numerical calculations, plus and minus signs. Our aims are

(1) to define a set of generators, their commutation relations and their subgroup structure for SU(6) - as a byproduct we give in detail the regular 35 dimensional representation;

(2) to pay special attention to the setting up of a phase convention with a specified SU(3) x SU(2) convention;

(3) to analyse the SU(3) x SU(2) structure of the SU(6) IRs;

(4) to emphasize the difficulties which arise in the use of a non canonical labelling scheme for SU(6).

2.1 Commutation relations of SU(6)

We proceed by employing three different methods to write down equivalent sets of generators and their commutation relations.

(1) One formulation of the SU(2) and SU(3) CR's is

$$\mathrm{SU}(2):\quad S_\alpha,\quad \alpha = 1,2,3\ [S_\alpha, S_\beta] = iE_{\alpha\beta\gamma}S_\gamma \qquad 2.1a$$

$$\mathrm{SU}(3):\quad F_i,\quad i = 1,\dots,8\ [F_i, F_j] = if_{ijk}F_k \qquad 2.1b$$

where $E_{\alpha\beta\gamma}$ is the usual permutation symbol on three letters and the f_{ijk} are given by Gell-Mann[16]. In the fundamental representations additional relations hold:

$$SU(2): \; S_\alpha \longrightarrow \tfrac{1}{2}\sigma_\alpha : [\sigma_\alpha \sigma_\beta]_+ = 2\delta_{\alpha\beta} \qquad \alpha,\beta = 1,2,3 \qquad 2.2a$$

$$SU(3): \; F_i \longrightarrow \tfrac{1}{2}\lambda_i : [\lambda_i \lambda_j]_+ = 2d_{ijk}\lambda_k + \tfrac{4}{3}\delta_{ij}1 \qquad 2.2b$$

where σ_α are the Pauli matrices, λ_i, d_{ijk} are given by Gell-Mann[16] $[\;\;]_+$ denotes anticommutator. Solutions of eqs. 2.1a,b in general will <u>not</u> satisfy eqs. 2.2a(b) - only the commutator of a Lie algebra has invariant significance. Note however that equations similar in form to eqs. 2a,2b do hold for the fundamental representation of any SU(n) algebra.

Now define $\lambda_0 = \sqrt{2/3}\;1_{(3x3)}$ $d_{ojk} = \sqrt{2/3}\;\delta_{jk}$ $f_{ojk} = 0$ and consider the matrix system

$$\lambda_i \times 1_{2x2}, \quad 1_{3x3} \times \sigma_\alpha, \quad \alpha = 1,2,3 \quad i=1,\ldots,8$$

$$2.3a$$

$$\lambda_i \times \sigma_\alpha \qquad \alpha = 1,2,3 \quad i=0,1,\ldots 8$$

x signifies matrix direct product. Using the identity

$$[A \times B, \; C \times D] = \tfrac{1}{2}\,[AC] \times [BD]_+ + \tfrac{1}{2}[AC] \times [BD]_+ \qquad 2.4$$

and eqs. 2.1a,b, 2.2a,b one arrives at a set of commutation relations for the system 2.3

$$[1 \times \sigma_\alpha, \; 1 \times \sigma_\beta] = 2iE_{\alpha\beta\gamma}\, 1 \times \sigma_\gamma$$

$$[\lambda_i \times 1, \; \lambda_j \times 1] = 2if_{ijk}\lambda_k \times 1$$

$$[\lambda_i \times 1, \sigma_\alpha \times 1] = 0$$

$$[1 \times \sigma_\alpha, \lambda_j \times \sigma_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\lambda_j \times \sigma_\gamma \qquad 2.5a$$

$$[\lambda_i \times 1, \lambda_j \times \sigma_\beta] = 2if_{ijk}\lambda_k \times \sigma_\beta$$

$$[\lambda_i \times \sigma_\alpha, \lambda_j \times \sigma_\beta] = 2i\delta_{\alpha\beta}f_{ijk}\lambda_k \times 1 + 2i\varepsilon_{\alpha\beta\gamma}d_{ijk}\lambda_k \times \sigma_\gamma$$

Notice that we could have taken $i\lambda_i \times \sigma_\alpha$ etc and this would alter the last CR in 2.5a above.

We now have a system of 35 6x6 matrices closed under commutation (by an identity similar to 2.4 the set is also closed under anticommutation when we adjoin the unit matrix $1_{3\times3} \times 1_{2\times2}$). The theory of maximal subalgebras Dynkin[2] tells us this must be the fundamental representation of the SU(6) algebra, and consequently we may define any representation of this algebra to be given by a set of 35 matrices, in 1:1 correspondence with the set 2.3, which satisfy the commutation relations 2.5. An alternative way to identify the SU(6) algebra is given below (eq. 2.10). In the form eq. 2.3 the commuting SU(3) and SU(2) subgroups are clearly displayed - we emphasize that in a general representation those matrices corresponding to $\lambda_i \times \sigma_\alpha$ will be different from the direct product of SU(3) and SU(2) representative matrices. This is exactly the difference between the SU(3) x SU(2) and the SU(6) algebras or groups.

(2): In this second formulation we write the commutation relations in the usual [illegible] diagonal [illegible] basis. [illegible] [illegible] is an irreducible [illegible] for the algebra whose basis [illegible] the generator [illegible] basis vector of regular representation in the usual particle diagonal or spherical basis. This then leads directly to an irreducible tensor basis for the algebra whose basis vector image under the generator $\longrightarrow$ basis vector of regular representation mapping is a physical particle, i.e. a state with pure I^2, I_3 and S_3 as well as being SU(3) x SU(2) irreducible. Our procedure is standard.

(a) SU(2). With S_α as in eq. 1a define $S_{\pm} = S_1 \pm iS_2$. This gives the canonical CR. $[S_3 S_{\pm}] = \pm S_{\pm}$ $[S_+, S_-] = 2S_3$. Now transform from Cartesian to spherical basis by setting

Spherical basis		Cartesian basis
$\vert(3,0)\rangle$	=	$+ \ \vert 3\rangle$

From the known matrix elements of $S_{\pm}$ (subject to the Condon and Shortley phase convention), and the regular representation $(S_k)_{ij}$ $\rightarrow - iE_{ijk}$ follow immediately

Spherical basis		Cartesian basis
$\vert(3,1)\rangle$	=	$-\sqrt{\tfrac{1}{2}}\ (\vert 1\rangle + i\ \vert 2\rangle)$
$\vert(3,-1)\rangle$	=	$\sqrt{\tfrac{1}{2}}\ (\vert 1\rangle - i\ \vert 2\rangle)$.

The desired transformation is thus accomplished by the unitary matrix E_i^α

	$\vert 1\rangle$	$\vert 2\rangle$	$\vert 3\rangle$
$\vert(3,1)\rangle$	$-1/\sqrt{2}$	$-i/\sqrt{2}$	0
$\vert(3,-1)\rangle$	0	0	1
$\vert(3,-1)\rangle$	$1/\sqrt{2}$	$-i/\sqrt{2}$	0

2.6a

$$|(3,\nu)\rangle = \sum_i E_i^{\nu} |i\rangle$$

The unitarity relations are

$$\sum_{\alpha} E_i^{\alpha} E_j^{\alpha *} = \delta_{ij} \qquad \sum_i E_i^{\alpha} E_i^{\beta *} = \delta_{\alpha\beta}$$

and the irreducible tensor basis for the generator is

$$S(3,\nu) = E_i^{\nu} S_i \;;\; S(3,1) = -\sqrt{\tfrac{1}{2}}\, S_+,\; S(3,-1) = \sqrt{\tfrac{1}{2}}\, S_-, S(3,0) = S_3 \qquad 2.7a$$

From $\langle (3,0) | S_- | (3,1) \rangle = \sqrt{2}$ we obtain the reduced matrix element:

$$\sqrt{2} = \langle (3,0) | S_- | (3,1) \rangle = \sqrt{2}\, \langle (3,0) | S(3,-1) | (3,1) \rangle = \sqrt{2} \begin{pmatrix} 3 & 3 & 3 \\ 1 & -1 & 0 \end{pmatrix} \langle \| S \| \rangle$$

so that $\langle \| S \| \rangle = \sqrt{2}$.

The commutation relations become

$$[S(3,\alpha), S(3,\beta)] = \sqrt{2} \sum_{\gamma} \begin{pmatrix} 3 & 3 & 3 \\ \beta & \alpha & \gamma \end{pmatrix} S(3,\gamma)$$

$$= -\sqrt{2} \sum_{\gamma} \begin{pmatrix} 3 & 3 & 3 \\ \alpha & \beta & \gamma \end{pmatrix} S(3,\gamma) \qquad 2.8a$$

and from this is deduced

$$E_i^{\alpha} E_j^{\beta} E_k^{\gamma *} E_{ijk} = i\sqrt{2} \begin{pmatrix} 3 & 3 & 3 \\ \alpha & \beta & \gamma \end{pmatrix}$$

The analogue of 2.2a in this basis becomes

$$\text{IR } \underline{\underline{2}} : [S(3,\alpha)\; S(3,\beta)]_+ = -\sqrt{3} \begin{pmatrix} ' & ' & 0 \\ \alpha & \beta & 0 \end{pmatrix} \qquad 2.9a$$

whence $\sum_i E_i^{\alpha} E_i^{\beta} = -\sqrt{3} \begin{pmatrix} ' & ' & 0 \\ \alpha & \beta & 0 \end{pmatrix}$

(b) SU(3): Defining the highest state $|\rho^+\rangle$ of the SU(3) octet spherical basis by $|\Pi^+\rangle = -\sqrt{\tfrac{1}{2}}\,(|1\rangle + i\,|2\rangle)$, using the regular representation $(F_i)_{jk} \longrightarrow -if_{ijk}$ of eqs. 2.1b, and the deSwart[5] definitions of spherical generators $I_{\pm}$, $K_{\pm}$ etc and his phase conventions for their matrix elements now leads to the unique unitary transformation e_i^u exactly analogous to eq. 2.6a

$\mu \backslash i$	1	2	3	4	5	6	7	8
$\vert\pi^+\rangle$	$-1/\sqrt{2}$	$-i/\sqrt{2}$						
$\vert\pi^0\rangle$			1					
$\vert\pi^-\rangle$	$1/\sqrt{2}$	$-i/\sqrt{2}$						
$\vert\eta^0\rangle$								1
$\vert K^+\rangle$				$-1/\sqrt{2}$	$-i/\sqrt{2}$			
$\vert K^0\rangle$						$-1/\sqrt{2}$	$-i/\sqrt{2}$	
$\vert \bar{K}^0\rangle$						$-1/\sqrt{2}$	$i/\sqrt{2}$	
$\vert K^-\rangle$				$1/\sqrt{2}$	$-i/\sqrt{2}$			

2.6b

cf. B.W.Lee[17].

The irreducible tensor operator set of generators is

$$Q(8,\nu) = \sum_i e_i^u F_i \qquad 2.7b$$

The reduced matrix element $\langle 8 \| Q \| 8 \rangle = \sqrt{3}$ and analogously to the SU(2) case we have :-

$$[Q(8,\mu),\ Q(8,\nu)] = -\sqrt{3}\begin{pmatrix} 8 & 8 & 8 \\ \mu & \nu & \lambda \end{pmatrix}_a Q(8,\lambda) \qquad 2.8b$$

IR3: $$[Q(8,\mu),\ Q(8,\ )]_+ = \sqrt{5/3}\begin{pmatrix} 8 & 8 & 8 \\ \mu & \nu & \lambda \end{pmatrix}_s Q(8,\lambda) - \sqrt{8/3}\begin{pmatrix} 8 & 8 & 1 \\ \mu & \nu & 0 \end{pmatrix} 1_{3\times 3} \qquad 2.9b$$

with the supplementary relations: cf. B.W.Lee[17]

$$e_i^u\, e_j^\nu\, e_k^{\lambda *}\, f_{ijk} = i/\sqrt{3}\ \begin{pmatrix} 8 & 8 & 8 \\ \mu & \nu & \lambda \end{pmatrix}_a$$

$$e_i^u\, e_j^\nu\, e_k^{\lambda *}\, d_{ijk} = \sqrt{5/3}\begin{pmatrix} 8 & 8 & 8 \\ \mu & \nu & \lambda \end{pmatrix}_s$$

$$\sum_i e_i^\mu e_j^\nu = 2\sqrt{3}\begin{pmatrix}8 & 8 & 1\\ \mu & \nu & 0\end{pmatrix}$$

Defining now the 6 x 6 matrices

$$F_{\nu,\alpha} = \begin{cases} 1_{3x3} \times S(3,\alpha) \\ \\ Q(8,\nu) \times 1_{2x2} \\ \\ Q(8,\nu) \times S(3,\alpha) \end{cases} \qquad 2.3b$$

will now lead exactly as before to the particle diagonal set of CRs, defined to hold for any IR :

$$[F_{o,\alpha}, F_{o\beta}] = -\sqrt{2}\begin{pmatrix}3 & 3 & 3\\ \alpha & \beta & \gamma\end{pmatrix} F_{o\gamma}$$

$$[F_{\mu,o}, F_{\nu,o}] = -\sqrt{3}\begin{pmatrix}8 & 8 & 8\\ \mu & \nu & \lambda\end{pmatrix}_a F_{\lambda,o}$$

$$[F_{\mu,o} F_{o,\beta}] = 0 \qquad 2.5b$$

$$[F_{\mu,o} F_{\nu,\alpha}] = -\sqrt{3}\begin{pmatrix}8 & 8 & 8\\ \mu & \nu & \lambda\end{pmatrix}_a F_{\lambda,\alpha}$$

$$[F_{o,\alpha} F_{\nu,\beta}] = -\sqrt{2}\begin{pmatrix}3 & 3 & 3\\ \alpha & \beta & \gamma\end{pmatrix} F_{\nu,\gamma}$$

$$\begin{aligned}[F_{\mu,\alpha} F_{\nu,\beta}] = {} & 3/4\begin{pmatrix}3 & 3 & 1\\ \alpha & \beta & 0\end{pmatrix}\begin{pmatrix}8 & 8 & 8\\ \mu & \nu & \lambda\end{pmatrix}_a F_{\lambda o} \\ & -\sqrt{5/6}\begin{pmatrix}3 & 3 & 3\\ \alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}8 & 8 & 8\\ \mu & \nu & \lambda\end{pmatrix}_s F_{\lambda\gamma} \\ & + 2/3\begin{pmatrix}3 & 3 & 3\\ \alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}8 & 8 & 1\\ \mu & \nu & 0\end{pmatrix} F_{o\gamma}\end{aligned}$$

We emphasize that the set $F_{\nu,\alpha}$ eq. 2.3b are not yet an irreducible tensor basis for SU(6); it is necessary to include numerical factors with the different (μ,ν) components, so that for example reduced matrix elements calculated from the thereby modified eqs. 2.5b, are the same for each (μ,σ) cpt. of $F_{\nu,\alpha}$. Indeed using the canonical set of CRs given below and anticipating the result eqn 2.15, we have in fact for the SU(6) embedding

$$S+ = -\sqrt{2}\ S(3,1) \longrightarrow -\sqrt{3}\ \mathbb{T}(\rho(1))$$
$$I+ = -\sqrt{2}\ Q(8,\pi^{+}) \longrightarrow -\sqrt{2}\ \mathbb{T}(\pi^{+})$$
$$S+.I+ = 2\ S(3,1)\ Q(8,\pi^{+}) \longrightarrow \mathbb{T}(\rho^{+}(1))$$

where $\mathbb{T}(\nu)$ denotes the appropriate SU(6) irreducible tensor generator basis; so we may write

$$T(\beta) = \begin{cases} \sqrt{2/3}\ \underline{1} \times S(3,\alpha) \\ Q(8,\nu) \times \underline{1} \\ 2\ Q(8,\nu) \times S(3,\alpha) \end{cases} \qquad 2.3c$$

as the correctly normalised relation in the IR$\underline{6}$ between SU(2) x SU(3) and SU(6) irreducible tensor bases.

(3) Finally we relate the canonical and SU(3) x SU(2) diagonal forms. Keeping in mind the product nature of the SU(3) x SU(2) subalgebra we begin by choosing a basis for the Cartan subalgebra in the form $[Q_i] = (I_3, Y, S_3, I_3S_3, YS_3)$ (we use conventional notation for the generators) where by $I_3S_3(YS_3)$ we mean that operator which has weight equal to $I_3S_3(YS_3)$ in the fundamental representation. (This relationship will not hold in any other representation.) The operators I_3S_3, YS_3 evidently have simple spin and isospin transformation properties, and are the usual choice e.g. Pais[18] but of course any two operators which complete the Cartan subalgebra, $\mathcal{H}$, could be used. The set $[Q_i]$ certainly do not correspond to a

canonical basis for $\mathcal{H}$; as we shall see this is because the embedded SU(3) x SU(2) algebras have simple roots which are not simple roots of SU(6), nor do the eigenvalues of the Q_i in the regular representation form normalised roots. However the positivity property implicit in the 'ordering scheme' (§ 1.1) is clearly invariant under change of scale; so we choose the order $I_3, Y, S_3, I_3S_3, YS_3$ and determine a set $[H_i]$, $H_k = q_kQ_k$ (no summation) with respect to which the weights in the IR$\underline{35}$ become normalised roots. The particular order chosen above is convenient since then the highest weight is that highest SU(3) vector which has highest S_3 component. The weights m_i with respect to Q_j; i = 1, ..., 6 in the fundamental IR$\underline{6}$, and the non zero positive weights $m_{ij} = m_i - m_j$, $i < j$ in $\underline{35}$ are now easily found by inspection. (We change our notation slightly from § 1.1, converting generator labels $\pm\alpha$ on $E_{\pm\alpha}$ to ij, $i \lessgtr j$ on E_{ij}. The hermeticity condition is then $(E_{ij})^+ = E_{ji}$.)

TABLE 1(a)

Weight vector	Physical label	I_3	Y	S_3	I_3S_3	YS_3
m_1	p	1/2	1/3	1/2	1/4	1/6
m_2	$\hat{p}$	1/2	1/3	-1/2	-1/4	-1/6
m_3	λ	0	-2/3	1/2	0	-1/3
m_4	$\hat{\lambda}$	0	-2/3	-1/2	0	1/3
m_5	n	-1/2	1/3	1/2	-1/4	1/6
m_6	n	-1/2	1/3	-1/2	1/4	-1/6
	$m_1 > m_2 > \cdots > m_6$					
m_{16}	$p\bar{\hat{n}}$	1	0	1	0	1/3
m_{15}	$p\bar{n}$	1	0	0	1/2	0
m_{26}	$\hat{p}\bar{\hat{n}}$	1	0	0	-1/2	0
m_{25}	$\hat{p}\bar{n}$	1	0	-1	0	-1/3
m_{14}	$p\bar{\hat{\lambda}}$	1/2	1	1	1/4	-1/6
m_{13}	$p\bar{\lambda}$	1/2	1	0	1/4	1/2
m_{24}	$\hat{p}\bar{\hat{\lambda}}$	1/2	1	0	-1/4	-1/2
m_{23}	$\hat{p}\bar{\lambda}$	1/2	1	-1	-1/4	1/6
m_{36}	$\lambda\bar{\hat{n}}$	1/2	-1	1	-1/4	-1/6
m_{35}	$\lambda\bar{n}$	1/2	-1	0	1/4	-1/2
m_{46}	$\hat{\lambda}\bar{\hat{n}}$	1/2	-1	0	-1/4	1/2
m_{45}	$\hat{\lambda}\bar{n}$	1/2	-1	-1	1/4	1/6
m_{12}	$p\bar{\hat{p}}$	0	0	1	1/2	1/3
m_{34}	$\lambda\bar{\hat{\lambda}}$	0	0	1	0	-2/3
m_{56}	$n\bar{\hat{n}}$	0	0	1	-1/2	1/3
	$m_{16} > m_{15} > \cdots > m_{56}$					

p, n, Λ are spin $+\frac{1}{2}$ quarks, $\hat{p}$ $\hat{n}$, $\hat{\lambda}$ spin $-\frac{1}{2}$; the bar denotes the complex conjugate state having diagonal quantum numbers negatives of the unbarred states.

Notice that already in $\underline{35}$ (and in $\underline{6}^*$) (YS_3) (I_3S_3) do not have eigenvalues equal to the product of those of $S_3 \otimes Y$, $(I_3)(S_3)$ By inspection one establishes the relationship characteristic of the SU(6) algebra:

$$m_{ij} = m_{i,i+1} + m_{i+1,i+2} + \dots + m_{j-1,j} \qquad 2.10$$

The simple roots are related to the $m_{i,i+1}$ and are now obtained by introduction of appropriate scale and normalization factors:- In the IR$\underline{6}$ it is easily seen that the equation $[Q,E_{ij}] = m_{ij}E_{ij}$ (no sum) requires $E_{ij}\ \alpha\ e_{ij}$ where $(e_{ij})_{ab} = \delta_{ia}\delta_{jb}$ is the familiar generator form used by Weyl[9] cf. also Rashid[4] (remember we are solving CRs of the general form eqs. 1.1); with $H_k = q_k Q_k$ and correspondingly $(r_{ij})_k = q_k(m_{ij})_k$ (no sum, k labels components) the equations $[E_{ij}, E_{ji}] = 2\,r_{ij}$ .H defining H + r may be used to solve for the factors q_k. (The equations are actually for q_k^2 but the negative solutions are discussed since they would conflict with the ordering scheme.) We thus obtain

$$H_1 = \sqrt{\tfrac{1}{2}}\ I_3 \quad H_2 = \tfrac{1}{2}\sqrt{3/2}\ Y \quad H_3 = \sqrt{1/3}\ S_3 \quad H_4 = \sqrt{2}\ I_3S_3 \quad H_5 = \sqrt{3/2}\,2YS_3 \qquad 2.11$$

and correspondingly a set of normalised root vectors

$$(r_{i,i+1}, r_{j,j+1}) = \begin{cases} 1 & j = i \\ -\tfrac{1}{2} & j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

The complete set of positive roots is obtained by introducing the numerical factors in eq. 2.11 into the set m_{ij}.

TABLE 1(b) - System of positive roots for SU(6)

Root Vector	H_1	H_2	H_3	H_4	H_5
r(16)	$\sqrt{1/2}$	0	$\sqrt{1/3}$	0	$\sqrt{1/6}$
r(15)	$\sqrt{1/2}$	0	0	$\sqrt{1/2}$	0
r(26)	$\sqrt{1/2}$	0	0	$-\sqrt{1/2}$	0
r(25)	$\sqrt{1/2}$	0	$-\sqrt{1/3}$	0	$-\sqrt{1/6}$
r(14)	$\frac{1}{2}\sqrt{1/2}$	$\frac{1}{2}\sqrt{3/2}$	$\sqrt{1/3}$	$\frac{1}{2}\sqrt{1/2}$	$-\frac{1}{2}\sqrt{1/6}$
r(13)	$\frac{1}{2}\sqrt{1/2}$	$\frac{1}{2}\sqrt{3/2}$	0	$\frac{1}{2}\sqrt{1/2}$	$3/2\sqrt{1/6}$
r(24)	$\frac{1}{2}\sqrt{1/2}$	$\frac{1}{2}\sqrt{3/2}$	0	$-\frac{1}{2}\sqrt{1/2}$	$-3/2\sqrt{1/6}$
r(23)	$\frac{1}{2}\sqrt{1/2}$	$\frac{1}{2}\sqrt{3/2}$	$\sqrt{1/3}$	$-\frac{1}{2}\sqrt{1/2}$	$\frac{1}{2}\sqrt{1/6}$
r(36)	$\frac{1}{2}\sqrt{1/2}$	$-\frac{1}{2}\sqrt{3/2}$	$\sqrt{1/3}$	$-\frac{1}{2}\sqrt{1/2}$	$-\frac{1}{2}\sqrt{1/6}$
r(35)	$\frac{1}{2}\sqrt{1/2}$	$-\frac{1}{2}\sqrt{3/2}$	0	$\frac{1}{2}\sqrt{1/2}$	$-3/2\sqrt{1/6}$
r(46)	$\frac{1}{2}\sqrt{1/2}$	$-\frac{1}{2}\sqrt{3/2}$	0	$-\frac{1}{2}\sqrt{1/2}$	$3/2\sqrt{1/6}$
r(45)	$\frac{1}{2}\sqrt{1/2}$	$-\frac{1}{2}\sqrt{3/2}$	$-\sqrt{1/3}$	$\frac{1}{2}\sqrt{1/2}$	$1/2\sqrt{1/6}$
r(12)	0	0	$\sqrt{1/3}$	$\sqrt{1/2}$	$\sqrt{1/6}$
r(34)	0	0	$\sqrt{1/3}$	0	$-2\sqrt{1/6}$
r(56)	0	0	$\sqrt{1/3}$	$-\sqrt{1/2}$	$\sqrt{1/6}$

At this stage the algebra is not completely defined since those structure constants corresponding to $N_{\alpha\beta}$ in Chapter 1 have not been given. We arbitrarily fix these by choosing for all generators

E_{ij} the representative e_{ij} in $\underline{6}$. With $[e_{ij}, e_{rs}] = e_{is}\delta_{rj} - e_{rj}\delta_{is}$ this gives

$$N_{ij,jk} = 1 = -N_{ji,kj} \tag{2.12}$$

all other $N_{ij,mn} = 0$

This choice leads to $N_{\alpha\beta}$ for the subgroup SU(3) the same as chosen by deSwart[5].

The complete set of commutation relations are now

$$[E, E_{ij}] = r(ij)\, E_{ij} \quad (r(ij) = -r(ji))$$

$$[E_{ij}, E_{ji}] = 2r(ij)\,.H \quad i < j \tag{2.5c}$$

$$[E_{ij}, E_{rs}] = N_{ij,rs}\, E_{ij+rs}$$

It is now an easy matter to locate the diagonal SU(3) and SU(2) subalgebras in terms of the canonical generators:

SU(2) : Noting that $r(12) + r(34) + r(56)$ has component only along H_3 we take

$$S_+ = E_{12} + E_{34} + E_{56} \qquad S_- = E_{21} + E_{43} + E_{65}$$

then $2S_3 = [S_+, S_-] = 2(r(12) + r(34) + r(56)).\ H$ 2.13

$$= 2\sqrt{3}\, H_3 \text{ as expected cf. eq. 2.11}$$

Similarly we can easily locate and identify the commuting SU(3) subalgebra :

$$I_+ = E_{15} + E_{26} \qquad K_+ = E_{13} + E_{24}$$

$$I_- = E_{51} + E_{62} \qquad K_- = E_{31} + E_{42} \tag{2.14}$$

$$
\begin{aligned}
I_3 &= \sqrt{2}\, H_2 & L_+ &= E_{53} + E_{64} \\
\sqrt{3/2}\, Y = M &= \sqrt{2}\, H_3 & L_- &= E_{35} + E_{46}
\end{aligned}
\qquad 2.14
$$

We have used the SU(3) generators defined by deSwart[5]; one can check that the E_{ij} forms given for the generators do indeed lead to his commutators. Using the relations 2.13, 2.14 and eqs. 2.5a in the IR$\underline{6}$ (where the representative matrices have the simple multiplication rules:

$$e_{ij}e_{jk} = e_{ik}, \quad e_{ji}e_{kj} = e_{ki},$$

other products vanish (no summations)

$$e_{ij}H = H_{jj}e_{ij} \quad He_{ij} = H_{ii}e_{ij},$$

H diagonal matrix) the products I_+S_+ in this IR may be evaluated in terms of the e_{ij}, and these relations then defined to hold in an arbitrary IR.

We now map the generators onto basis states: it is clear that within an SU(2) x SU(3) submultiplet, once we have mapped one generator into one basis state the known SU(2) and SU(3) matrix elements determine the remainder. Hence we have three overall constants x, y, z to solve for, associated with, example:

$$S_+ \longrightarrow x|\varphi(1)\rangle \qquad I_+ \longrightarrow y|\pi^+\rangle \qquad I_+S_+ \longrightarrow z|\rho^+(1)\rangle$$

The equation $[E_{16}, E_{61}] = 2r(16).H = \sqrt{2}H_1 + 2\sqrt{3}\,H_3 + \sqrt{2/3}\,H_5$ becomes $E_{16}.z.|\rho^-(-1)\rangle = -y\sqrt{\tfrac{1}{2}}|\pi^0\rangle + x\sqrt{2/3}|\varphi(0)\rangle + z\sqrt{1/3}|\omega(0)\rangle$

using also the hermition conjugate form of the following:-

$$E_{61}|\pi^{o}\rangle \longrightarrow -\frac{2}{y}[E_{61},H_1] \longrightarrow -\sqrt{2}\,\frac{z}{y}\,|\rho^{-}(-1)\rangle$$

and

$$E_{61}|\varphi(0)\rangle \text{ — } -\frac{\sqrt{6}}{x}[E_{61},H_3] \text{ — } -\sqrt{2}\,\frac{z}{x}\,|\rho^{-}(-1)\rangle$$

we obtain

$$x^2 : y^2 : z^2 = 3:2:1$$

or

$$x : y : z = +\sqrt{3} : +\sqrt{2} : +1 \qquad 2.15$$

The relative signs are in fact fixed by the phase convention, which is defined and discussed in the next section. The solutions for x,y,z above correspond to the factors required in equations 2.3b, 2.5b, for the tensorial set, eq. 2.3c. The complete set of relations are given in Table 2.

The Table gives the image of the generators under the generator basis vector mapping for different labelling schemes. We have omitted a column containing the irreducible tensor labelling, since as emphasized elsewhere this mapping is trivial, characterised by an overall constant.

TABLE 2

SU(3) x SU(2)	Canon	Basis Vector	SU(3) x SU(2)	Canon	Basis Vector
S_+	$E_{12}+E_{34}+E_{56}$	$-\sqrt{3}\,\lvert\varphi(1)\rangle$	K_+S_+	E_{14}	$\lvert K^{+}(1)\rangle$
S_3	$\sqrt{3}\,H_3$	$\sqrt{\frac{3}{2}}\lvert\varphi(0)\rangle$	K_+S_3	$\frac{1}{2}(E_{13}-E_{24})$	$-\sqrt{\frac{1}{2}}\,\lvert K^{+}(0)\rangle$
S_-	$E_{21}+E_{43}+E_{65}$	$\sqrt{3}\,\lvert\varphi(-1)\rangle$	K_+S_-	E_{23}	$-\lvert K^{+}(-1)\rangle$
I_+	$E_{15}+E_{26}$	$-\sqrt{2}\,\lvert\pi^+\rangle$	K_-S_+	E_{32}	$-\lvert K^{-}(1)\rangle$
I_3	$\sqrt{2}\,H_1$	$\lvert\pi^0\rangle$	K_-S_3	$\frac{1}{2}(E_{31}-E_{42})$	$\sqrt{\frac{1}{2}}\,\lvert K^{-}(0)\rangle$
I_-	$E_{51}+E_{62}$	$\sqrt{2}\,\lvert\pi^-\rangle$	K_-S_-	E_{41}	$\lvert K^{-}(-1)\rangle$
$M=\frac{\sqrt{3}}{2}Y$	$\sqrt{2}\,H_2$	$\lvert\eta^0\rangle$			
K_+	$E_{13}+E_{24}$	$-\sqrt{2}\,\lvert K^+\rangle$	L_+S_+	E_{54}	$\lvert K^{0}(1)\rangle$
K_-	$E_{31}+E_{42}$	$\sqrt{2}\,\lvert K^-\rangle$	L_+S_3	$\frac{1}{2}(E_{53}-E_{64})$	$-\sqrt{\frac{1}{2}}\,\lvert K^{0}(0)\rangle$
L_+	$E_{53}+E_{64}$	$-\sqrt{2}\,\lvert K^0\rangle$	L_+S_-	E_{63}	$-\lvert K^{0}(-1)\rangle$
L_-	$E_{35}+E_{46}$	$-\sqrt{2}\,\lvert\bar{K}^0\rangle$	L_-S_+	E_{36}	$\lvert\bar{K}^{0}(1)\rangle$
I_+S_+	E_{16}	$\lvert\rho^+(1)\rangle$	L_-S_3	$\frac{1}{2}(E_{35}-E_{46})$	$-\sqrt{\frac{1}{2}}\,\lvert\bar{K}^{0}(0)\rangle$
I_+S_3	$\frac{1}{2}(E_{15}-E_{26})$	$-\sqrt{\frac{1}{2}}\,\lvert\rho^+(0)\rangle$	L_-S_-	E_{45}	$-\lvert\bar{K}^{0}(-1)\rangle$
I_+S_-	E_{25}	$-\lvert\rho^+(-0)\rangle$			
			MS_+	$\frac{2}{\sqrt{3}}(E_{12}$ [illegible] $-2E_{34}+E_{56})$	$-\sqrt{\frac{1}{2}}\,\lvert\omega(1)\rangle$
I_3S_+	$\frac{1}{2}(E_{12}-E_{56})$	$-\sqrt{\frac{1}{2}}\,\lvert\rho^0(1)\rangle$	MS_3	$\frac{1}{\sqrt{2}}H_5$	$\frac{1}{2}\lvert\omega(0)\rangle$
I_3S_3	$\frac{1}{\sqrt{2}}H_4$	$\frac{1}{2}\lvert\rho^0(0)\rangle$	MS_-	$\frac{2}{\sqrt{3}}(E_{21}-2E_{43}+E_{65})$	$\sqrt{\frac{1}{2}}\,\lvert\omega(-1)\rangle$
I_3S_-	$\frac{1}{2}(E_{21}-E_{65})$	$\sqrt{\frac{1}{2}}\,\lvert\rho^0(-1)\rangle$			
I_-S_+	E_{52}	$-\lvert\rho^-(1)\rangle$			
I_-S_3	$\frac{1}{2}(E_{51}-E_{62})$	$\sqrt{\frac{1}{2}}\,\lvert\rho^-(0)\rangle$			
I_-S_-	E_{61}	$\lvert\rho^-(-1)\rangle$			

We have used standard notation for the quantum numbers but omitting the redundant * on $k^{+*}(1)$ etc, with $|\rho(1)\rangle$ being the SU(6) $\underline{35}$, SU(3) x SU(2) (1,3), $S_3 = 1$ element and $|\omega(0)\rangle$ the $\underline{35}$, (8,3), $Y = I^2 = S_3 = 0$ element etc.

2.2 Phase conventions for SU(6)

It should be clear that to completely determine the matrix solutions of eqs. 5, consistent with the embedded SU(3) x SU(2) solutions, we have only to find some way of determining the relative signs of SU(2) x SU(3) subspaces occurring in the decomposition of a given SU(6) IR.

Incorporating the SU(2) Condon and Shortley and the SU(3) deSwart phase conventions we must have the following operators (and their hermition conjugates) with non negative matrix elements

$$S_+ = E_{12} + E_{34} + E_{56}$$

$$I_+ = E_{15} + E_{26}$$

$$K_+ = E_{13} + E_{24}$$

It is evident that the canonical solution discussed in 1.9 cannot apply. (It is necessary to 'embed' the SU(3) x SU(2) phase conventions, if only because of the consequent simplification in the construction and use of C.G.c tables. See next chapter.)

We can suggest two possible phase conventions

(i) As will be seen in Chapter 3 the SU(2) x SU(3) decomposition facilities explicit construction of basic vectors in SU(3) x SU(2) submultiplets, in terms of the fundamental $\underline{6}$ and $\underline{6}^*$ entities, i.e. quarks and antiquarks. A given IR may, however, be constructed in various ways, corresponding to the addition of multiples of the variously named SU(6) scalar, or inert, unit $[1^6]$ or the traced T^{α}_{α}. Allowing for the presence of these factors, it is always easy to make a direct comparison of the same IR constructed in different ways and hence obtain a consistent phase convention. Essentially this method was employed in ref.[11]; its main defect is that it is not easily communicable, i.e. not based on any well defined operator; however, it was computationally very simple to apply. A second phase convention could be set up as follows:

(ii) The SU(3) x SU(2) algebra is maximal in SU(6) - as may be verified directly, the addition of any operator SU(6) external to the SU(3) x SU(2) set leads to the whole algebra. Again, as in § 1.9, we may use any one such operator to define the relative signs of the invariant subspaces. We choose H_5 which applied to some state $|\lambda, \mu I I_3 Y; \; S S_3\rangle$ gives

$$0 = \Delta\lambda = \Delta I = \Delta I_3 = \Delta Y = \Delta S_3 \; ; \quad \Delta\mu \subset 8 \otimes \mu \qquad 2.16$$
$$\Delta S \subset 3 \otimes S$$

(Notice that H_5, a member of the Cartan subalgebra, is not diagonal.) It is clear from the IR$\underline{6}$ that we cannot demand H_5 to have only non negative matrix elements (one can further check in the $\underline{35}$ that no member of the (8,3) set can be chosen to have solely non negative matrix elements) and we have to proceed as follows:

(a) Within an SU(3) x SU(2) multiplet it is possible to define a unique ordering, according to (I_3, Y, S_3), supplemented by total isospin where necessary, i.e. States of the same (I_3,Y,S_3) are ordered, highest first, according to decreasing isospin.

(b) By definition the highest weight of the SU(6) IR is also a highest weight of some SU(3) x SU(2) IR, and from the uniqueness of highest weights in SU(3) and SU(2) we may order distinct SU(3) x SU(2) multiplets according to their highest weights. (This fails when an SU(3) x SU(2) submultiplet occurs more than once in an SU(6) IR - we discuss this situation when it arises § 3.4).

(c) Applying now the operator H_5 to a decreasing, ordered, set of basis vectors in the highest SU(3) x SU(2) submultiplet will lead into other submultiplets. Let (μ_1,σ_1) be the highest submultiplet and let (μ_j, σ_j) be some other submultiplet. Then we define H_5 to have positive matrix elements between the highest possible state in (μ_1 σ_1) and the highest of the so determined states in (μ_j σ_j) (in general of course this will not be the highest state of (μ_j, σ_j),) i.e. in this sequence the first non zero matrix element of H_5 is taken positive by convention.

If $\langle(\mu_j\sigma_j)|H_5|(\mu_1,\sigma_1)\rangle = 0$ for all basis vectors in (μ_1 σ_1) we proceed to the next highest submultiplet and so on. In this way from the irreducibility of the SU(6) IR, and the maximality of SU(3) x SU(2), all relative signs can be determined.

This method was first employed (tho' not fully described) by Schulke;[19] however it is there erroneously stated that both H_4 and H_5 may be treated in this way. A third paper on SU(6) Clebsch-Gordan coefficients[20], does not state the phase convention used.

In detail for the <u>35</u> we have :

The highest state leading out of (8,3) is $|K^+(0)\rangle$ (octet states are ordered $\pi^+, K^+, \bar{K}^0, \pi^0, \eta^0, K^0, K^-, \pi^-$). Then mapping $F_{16} \longrightarrow +|\rho^+(1)\rangle$

$$H_5 |K^+(0)\rangle \longrightarrow [H_5, -\sqrt{\tfrac{1}{2}}(E_{13}-E_{24})] = -\frac{\sqrt{3}}{4}(E_{13}+E_{24})$$

$$\longrightarrow + \tfrac{1}{2}\sqrt{\tfrac{3}{2}}\,|K^+\rangle \text{ if we have } (E_{13}+E_{24}) \longrightarrow -\sqrt{2}\,|K^+\rangle$$

This fixes the (8,1) overall phase. For the (1,3) we must choose $|\omega(1)\rangle$ in (8,3), and then, on defining $E_{12}+E_{34}+E_{56} \longrightarrow -\sqrt{3}\,|\varphi(1)\rangle$

$$H_5 |\omega(1)\rangle \longrightarrow [H_5, -\sqrt{\tfrac{1}{6}}(E_{12}-2E_{34}+E_{56})]$$

$$= -\frac{1}{6}(E_{12}+4E_{34}+E_{56})$$

$$= -\frac{1}{6}[2(E_{12}+E_{34}+E_{56}) - (E_{12}-2E_{34}+E_{56})]$$

$$\longrightarrow \sqrt{\tfrac{1}{3}}\,|\varphi(1)\rangle - \sqrt{\tfrac{1}{6}}\,|\omega(1)\rangle$$

We close this section with a discussion of the IR<u>6</u>* although since this contains only one SU(3) x SU(2) multiplet its structure is already decided by SU(2) and SU(3) phase conventions .

The weights of the IR<u>6</u> are shown in Table 1a - the basis states are denoted by the customary quark label. Under complex conjugation these six states become $\bar{p} < \bar{\hat{p}} < \bar{\lambda} < \bar{\hat{\lambda}} < \bar{n} < \bar{\hat{n}}$. The Condon and Shortley convention will require the introduction of a relative minus sign between each spin doublet and the deSwart convention will require relative minus sign between p and n, states, cf. § 1.9. Hence we take as basis states in $\underline{6}^*$

$$\bar{p},\quad -\bar{\hat{p}},\quad -\bar{\lambda},\quad \bar{\hat{\lambda}},\quad -\bar{n},\quad \bar{\hat{n}} \tag{2.17}$$

It is between these states then, that the operators $\underline{S_\pm}$, $\underline{I_\pm}$, $\underline{\pm}$ will have non negative matrix elements; the complex conjugation operation changes the sign of the basis elements in the Cartan subalgebra, § 1.9, and further it is not possible to eliminate this sign change with a second trivial phase transformation such as $E_{ij} \longrightarrow -E_{ij}$. Thus

(a) In a non self conjugate IR these matrix elements of M_5 which are positive by definition will be negative by definition in the complex conjugate IR. cf. § 2.4.

(b) In a self conjugate IR, special care must be taken since then the convention will require both positive and negative matrix elements in the same IR. cf. § 2.4.

The eqs. 2.17 may be summarised by

$$|\bar{q}\rangle = (-1)^{\Lambda_0 - \Lambda} \quad |q\rangle^* \tag{2.18}$$

where $\Lambda_0 = Q + S_3$ of the highest antiquark state ($Q = I_3 + \frac{1}{2}Y$)

$\Lambda = Q + S_3$ of the antiquark state q

Due to the additivity of the weights I_3, Y, S_3 equation 2 then becomes a general rule for constructing the complex conjugate basis state, although of course the freedom of an overall minus sign still remains. Taking this plus one in $\underset{\sim}{6}$ we see that for quark states we may omit the $| \rangle$ indicating basis vector (by definition) but we may not do so for antiquark states, $|\underset{\sim}{6}^* \hat{p}^* \rangle = -\bar{p}$ etc.

The manner in which these (-1) phase factors arise can also be seen thus: given any basis $\alpha, \beta, \ldots \delta$ for a unitary IR, then unitarity implies that $\alpha\bar{\alpha} + \beta\bar{\beta} + \ldots + \delta\bar{\delta}$ is a scalar, (where - denotes complex conjugate) so that any generator applied to this expression gives zero. Choose in particular a generator postulated by convention to have all its matrix elements non negative - then clearly since the action of this generator in the IR containing $\alpha, \beta, \ldots$ introduces only + signs (or zero), then the action, (or representative) of the same generator in the IR containing $\bar{\alpha}, \bar{\beta} \ldots$ must introduce minus signs, i.e. $\bar{\alpha}, \bar{\beta}, \ldots$ must be related to a new basis, consistent with the given phase convention, by some factors -1.

2.3 SU(3) x SU(2) decompositions in SU(6)

Following the ideas outlined in § 1.6 the SU(3) x SU(2) structure of an SU(6) IR derives from the symmetric group ... Clebsch-Gordan series. This was pointed out in ref.[11] and has also been given in [21]. In practice one complements this method with others; we give some examples :

(i) $\underline{6}$ — $(3,2)$ $\quad \bar{\underline{6}}$ — $(\bar{3},2)$

$$\underline{6} \otimes \bar{\underline{6}} = \underline{35} \oplus \underline{1} \longleftrightarrow (3,2) \otimes (\bar{3},2) = (8 \oplus 1,\ 3 \oplus 1)$$

so that $\underline{35} \longrightarrow (8,3) \oplus (8,1) \oplus (1,3)$

(ii) $\underline{70}$ has YT $[21]$

Using the C.G. series for S_3 :

$$[21] \subset [3] \otimes [21],\quad [21] \otimes [21],\quad [21] \otimes [1^3]$$

$$\underline{70} \longrightarrow (10,2) \oplus (8,4) \oplus (8,2) \oplus (1,2)$$

$$[3] \otimes [21] \qquad [21] \otimes [3] \qquad [21] \otimes [21] \qquad [1^3] \otimes [3]$$

(iii) $T^{[2]}_{[2]} \supset \underline{405} \oplus \underline{35} \oplus \underline{1}$

$$\therefore\ T^{[2]}_{[2]} \sim Q^{[2]}_{[2]} \times S^{[2]}_{[2]} \oplus Q^{[1^2]}_{[1^2]} \times S^{[1^2]}_{[1^2]}$$

$$\oplus\ Q^{[2]}_{[1^2]} \times S^{[2]}_{[1^2]} \oplus Q^{[1^2]}_{[2]} \times S^{[1^2]}_{[2]}$$

T is an SU(6), Q, S SU(3) and SU(2), tensors

$$\sim (27 \oplus 8 \oplus 1;\ 5 \oplus 3 \oplus 1) \oplus (8 \oplus 1;\ 1)$$

$$\oplus\ (\overline{10} \oplus 8;\ 3) \oplus (10 \oplus 8;\ 3)$$

Whence subtracting out $35 \oplus 1$ we obtain

$$\underline{405} \longrightarrow (27,\ 5 \oplus 3 \oplus 1) \oplus (10,3) \oplus (\overline{10},3) \oplus 2(8,3) \oplus (8,1) \oplus (1,5 \oplus 1)$$

Notice that $(8,3)$ occurs twice in $\underline{405}$; thus already the SU(3) x SU(2) labels are insufficient for labelling states in an SU(6) IR.

We give below at the SU(3) x SU(2) content of a number of SU(6) IRs. Further tabulation is given in H.Ruegg et al[21].

TABLE 3 :

SU_6 I.R.:[Y.T]	(μ, σ) content
2695; $[63^4]$	(64; 7+5+3+1), (35; 5+3), ($\overline{35}$; 5+3), (27; 7), 2(27; 5), 3(27; 3), (27; 1), (10; 5+3+1), ($\overline{10}$; 5+3+1), (8; 7), 2(8; 5), 2(8; 3), (8; 1),(1; 7+3)
1134; $[421^3]$	(35; 4+2), (27; 6), 2(27; 4), 2(27; 2), (10; 6), 2(10;4), 2(10;2), ($\overline{10}$;4+2), (8;6), 3(8;4), 3(8;2), (1;4+2)
700; $[51^4]$	(35;6+4), (27;4+2), (10;6+4+2), ($\overline{10}$;2), (8;4+2)
405; $[42^4]$	(27;5+3+1), (10,3), ($\overline{10}$;3), (8;5), 2(8;3), (8;1),(1;5+1)
280; $[31^3]$	(27;3), (10;5+3+1), ($\overline{10}$;1), (8;5), 2(8;3), (8;1), (1;3)
$\overline{280}$; $[3^2 2^3]$	(27;3), (10;1), ($\overline{10}$;5+3+1), (8;5), 2(8;3), (8;1), (1;3)
189; $[2^2 1^2]$	(27;1), (10;3), ($\overline{10}$;3), (8;5), 2(8;3), (8;1), (1;5+1)
70; [21]	(10;2), (8;4+2), (1;2)
56; [3]	(10;4), (8;2)
35; $[21^4]$	(8;3+1), (1;3)
1;	(1;1)

2.4 Construction of SU(3) x SU(2) submultiplets

Using again the symmetric group we can construct explicit SU(3) x SU(2) states appearing in an SU(6) IR. This is exploited further in the next chapter, here we obtain the SU(6) 3 quark states, and quark-antiquark states. We employ the H_5 phase convention, but revert to the alternative in Chapter 3. p', n', λ' denote SU(3) quarks, p, n, λ and $\hat{p}\ \hat{n}\ \hat{\lambda}$ the spin up or spin down SU(6) quarks. We shall have to 'multiply' SU(3) and SU(2) basic states together to construct SU(6) entities and this we represent by e.g. $p'\uparrow\ = p;\quad p'\downarrow\ = \hat{p}$ etc.
When several factors occur it is of course important to preserve the order. Thus $p'n'\lambda'\uparrow\downarrow\uparrow\ = p\,\hat{n}\,\lambda$, $p'n'\lambda'\uparrow\uparrow\downarrow\ = p\,n\,\hat{\lambda}$ etc (it is this order upon which the symmetric group operates).
We use the symmetric group basis functions given in § 1.5, but with p', n', λ' or $\uparrow, \downarrow$ replacing the numeric labelling.

Some important matrix elements are :-

$$\underline{6}:\quad I_+ n = p,\quad I_+ p, \lambda = 0\quad I_- p = n\quad I_- n, \lambda = 0$$
$$K_+ \lambda = p,\quad K_+ n, p = 0\quad K_- p = \lambda\quad K_- n, \lambda = 0$$

and similarly for spin down states. 2.19

$$S_+\hat{p} = p\quad S_+\hat{n} = n\quad S_+\hat{\lambda} = \lambda\quad S_+ p, n, \lambda = 0$$
$$S_- p = \hat{p}\quad S_- n = \hat{n}\quad S_- \lambda = \hat{\lambda}\quad S_- \hat{p}, \hat{n}, \hat{\lambda} = 0$$

$$\underline{\bar{6}}:\quad I_+\bar{p} = -\bar{n}\quad I_+\bar{n}, \bar{\lambda} = 0\quad I_-\bar{n} = -\bar{p}\quad I_-\bar{p}, \bar{\lambda} = 0$$
$$K_+\bar{p} = -\bar{\lambda}\quad K_+\bar{n}, \bar{\lambda} = 0\quad K_-\bar{\lambda} = -\bar{p}\quad K_-\bar{p}, \bar{n} = 0 \tag{2.20}$$

and similarly for spin up states

$$S_+ \bar{p} = -\hat{\bar{p}} \quad S_+ \bar{n} = -\hat{\bar{n}} \quad S_+ \bar{\lambda} = -\hat{\bar{\lambda}} \quad S_+ \hat{\bar{p}}, \hat{\bar{n}}, \hat{\bar{\lambda}} = 0 \text{ etc.}$$

(i) IR.$\underline{56}$:

$$[3] \subset [3] \otimes [3], [21] \otimes [21]$$

i.e. $\underline{56} = (10,4) \oplus (8,2)$

Highest weight $= |N^{*++}(\tfrac{3}{2})\rangle$

$$SU(3) \times SU(2) \quad \boxed{p'}\boxed{p'}\boxed{p'} \quad \times \boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}$$

$$= p'\,p'\,p' \,.\, \uparrow\uparrow\uparrow$$

i.e. $|N^{*++}(\tfrac{3}{2})\rangle = p\,p\,p$ which is manifestly SU(6) [3] symmetric.

The highest weight of the (8,2) multiplet is $|\Sigma^+(\tfrac{1}{2})\rangle$

$$|\Sigma^+(\tfrac{1}{2})\rangle \sim \begin{array}{|c|c|}\hline p' & p' \\ \hline \lambda' & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline\end{array}$$

$$= (p'\lambda' p' - \lambda' p' p') \times (\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$= p\hat{\lambda}p - \lambda\hat{p}p - \hat{p}\lambda p + \hat{\lambda}pp$$

We now apply the symmetriser corresponding to $\boxed{}\boxed{}\boxed{}$ to project out the $\underline{56}$,[3], component of [21] $\otimes$ [21].

Thus abc $\longrightarrow$ (abc) $\equiv$ abc + bca + cab + bac + cba + acb

aab $\longrightarrow$ 2(aab) $\equiv$ 2 aab + 2 aba + 2 baa

Similarly

$$p\hat{\lambda}p - \lambda\hat{p}p - \hat{p}\lambda p + \hat{\lambda}pp \longrightarrow 2(pp\hat{\lambda}) - (p\hat{p}\lambda) - (p\hat{p}\lambda) + 2(pp\hat{\lambda})$$

$$= 4(pp\hat{\lambda}) - 2(p\hat{p}\lambda)$$

(due to symmetrization, we would get the same result starting with the other [21] S_3 IR . cf § 1.5)

Normalizing

$$|\Sigma^{+}(\tfrac{1}{2})\rangle = \frac{\pm 1}{3\sqrt{2}} (2(p\hat{p}\lambda) - (pp\hat{\lambda}))$$

The choice of plus or minus sign is fixed by the phase convention.

Using eqs. 2.19 we find

$$|Y^{*+}(\tfrac{3}{2})\rangle = \sqrt{\tfrac{1}{3}}\,(pp\lambda)$$

$$|Y^{*+}(\tfrac{1}{2})\rangle = \frac{1}{3}\,((pp\hat{\lambda}) + (p\hat{p}\lambda))$$

Instead of M_5 it is convenient to use $6\sqrt{\tfrac{2}{3}}\,M_5$ since this has integral eigenvalues in $\underline{6}$ and $\underline{\bar{6}}$:

State / eigenvalue of $6\sqrt{\tfrac{2}{3}}M_5 = P$	p	$\hat{p}$	λ	$\hat{\lambda}$	n	$\hat{n}$
	1	-1	-2	2	1	-1

Writing $P = 6\sqrt{\tfrac{2}{3}}\,M_5$, the action of this operator on a product of quarks is found simply by adding up the P weights of those quarks:

$$P|Y^{*}(\tfrac{1}{2})\rangle = \frac{1}{3}\,(4\,(pp\hat{\lambda}) - 2\,(p\hat{p}\lambda))$$

Hence our phase convention requires

$$|\Sigma^{+}(\tfrac{1}{2})\rangle = +\frac{1}{3\sqrt{2}}(2\,(pp\hat{\lambda}) - (p\hat{p}\lambda))$$

The remaining $\underline{56}$ states may now be obtained with SU(3) and SU(2) operators.

(ii) $\underline{70}$

$$70 \longrightarrow (10,2) \oplus (8,4) \oplus (8,2) \oplus (1,2)$$

(a) $(10,2) \sim$ ×

SU(3) SU(2)

Highest weight $\sim$ $(p'p'p')$ x $\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow$

$$= p\,\hat{p}\,p - \hat{p}\,p\,p$$

Since in fact × = there is no need to apply an IR.$\underline{70}$ [21] symmetriser and we take, in an obvious notation:

$$|\underline{70}, N^{*}(\tfrac{1}{2})\rangle = \sqrt{\tfrac{1}{2}}\,[p\,\hat{p}]p \quad \text{where } [ab] \equiv ab - ba$$

(b) $(8,4) \sim$ ×

Highest weight $(p'\lambda'p' - \lambda'p'p')$ $\uparrow\uparrow\uparrow$

$$= [p\lambda]p$$

Again [21] symmetrization is unnecessary and we take

$$\underline{70}|\Sigma^{+}(\tfrac{3}{2})\rangle = \pm\sqrt{\tfrac{1}{2}}\,[p\lambda]p$$

$$\therefore |\underline{70}(8,4)\Sigma^{+}(\tfrac{3}{2})\rangle = \pm\sqrt{\tfrac{1}{6}}\,([p\lambda]\hat{p} + [p\hat{\lambda}]p + [\hat{p}\lambda]p)$$

(c) $(8,2) \sim$ ×

Highest weight $(p'\lambda'p' - \lambda'p'p')$ x $(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$

$$= p\hat{\lambda}p - \lambda\hat{p}p - \hat{p}\lambda p + \hat{\lambda}pp$$

This time we must apply the [21] projection operator :

with the prototype $abc \xrightarrow{[21]} [ab]c + [cb]a$ we get

$$-[p\hat{p}]\lambda - [p\lambda]\hat{p} + [p\hat{\lambda}]p$$

and we must take

$$|\underline{70}\,\Sigma^{+}(\tfrac{1}{2})\rangle = \pm\sqrt{\tfrac{1}{6}}\,([p\hat{p}]\lambda + [p\lambda]\hat{p} - [p\hat{\lambda}]p)$$

(d) (1,2) $\sim$

Highest weight $\sim$ $([p\lambda]n + [\lambda n]p + [np]\lambda) \times ((\uparrow\downarrow)\uparrow - 2\uparrow\uparrow\downarrow)$

$\sim [\hat{p}n]\lambda - [\acute{p}n]\lambda + [\lambda\hat{n}]p$

(where $(ab) \equiv ab + ba$)

so that the corresponding basis vector is $\pm\sqrt{\frac{1}{6}}$ times this.

We now fix $\pm$ signs with our phase convention:

For this we need the following basis state in $\underline{70}(10,2)$, obtained by the action of SU(3) operators on the highest weight.

$$|(10,2)\ Y^{*+}(\tfrac{1}{2})\rangle = \sqrt{\tfrac{1}{6}}\ ([p\hat{p}]\lambda + [p\hat{\lambda}]p + [\lambda\hat{p}]p)$$

then $P|(10,2)Y^{*+}(\frac{1}{2})\rangle = \sqrt{\frac{1}{6}}\ (-2\ [p\hat{p}]\lambda + 4[p\acute{\lambda}]p - 2[\lambda\hat{p}]p)$

so that we must take

$$|\underline{70}(,84)\Sigma^{+}(\tfrac{1}{2})\rangle = +\sqrt{\tfrac{1}{6}}\ ([p\lambda]\hat{p} + [p\hat{\lambda}]p + [\hat{p}\lambda]p)$$

and

$$|\underline{70}(8,2)\Sigma^{+}(\tfrac{1}{2})\rangle = -\sqrt{\tfrac{1}{6}}\ ([p\hat{p}]\lambda + [p\lambda]\hat{p} - [p\hat{\lambda}]p)$$

in order that these two basis states have positive scalar product with $P|(10,2)\ Y^{*+}(\frac{1}{2})\rangle$

From the selection rules eq. 2.16 for P we see that the relative sign of the (1,2) multiplet may be determined by applying P to the state $|(8,4)\ \Lambda^{0}(\frac{1}{2})\rangle$. We find

$$|(8,4)\ \Lambda^{0}(\tfrac{1}{2})\rangle = \tfrac{1}{3}\ (-\tfrac{1}{2}[p\lambda]\hat{n} - \tfrac{1}{2}[p\hat{\lambda}]n - \tfrac{1}{2}[\hat{p}\lambda]n + \tfrac{1}{2}[n\lambda]\hat{p} + \tfrac{1}{2}[n\hat{\lambda}]p$$
$$+ \tfrac{1}{2}[\hat{n}\lambda]p + [np]\hat{\lambda} + [n\hat{p}]\lambda + [\hat{n}p]\lambda)$$

and derive

$$\underline{70}(1,2)|S^{o}(\tfrac{1}{2})\rangle = +\sqrt{\tfrac{1}{6}}\,([p\hat{n}]\lambda - [\hat{p}n]\lambda + [\lambda\hat{n}]p)$$

Note: (1) Using the equivalent orthogonal S_3 IR a second equivalent orthogonal $\underline{70}$ IR can be constructed.

(2) Related to this care must be taken to ensure that one always constructs members of the same $\underline{70}$ as in (a),.. (d) above; hence the choice of spin function in (d). In general one can use orthogonality to decide which S_3 IR to use.

(iii) IR. $\underline{20} \longrightarrow (8,2) \oplus (1,4)$

corresponding to

$$[1^3] \subset [21] \otimes [21],\quad [1^3] \otimes [21]$$

(a) (8,2) highest weight $[p'\lambda']p'\ ((\uparrow\downarrow)\uparrow - 2\uparrow\uparrow\downarrow)$

$$= [\hat{p}\lambda]p + [\hat{p}\lambda]p - 2[p\lambda]\hat{p}$$

Applying the $[1^3]$ symmetiser: abc $\longrightarrow$ [abc] = abc + bca + cab − bac − cba − acb we get $|\underline{20}(8,2)\Sigma^{+}(\tfrac{1}{2})\rangle = \sqrt{\tfrac{1}{6}}\,[p\,\hat{p}\lambda]$.

We shall need the state $\Lambda^{o}(\tfrac{1}{2})$ for the phase convention and this is $|\underline{20}(8,2)\Lambda^{o}(\tfrac{1}{2})\rangle = \tfrac{1}{3}.([pn\hat{\lambda}] + \tfrac{1}{2}[\hat{p}\lambda n] - \tfrac{1}{2}[p\hat{n}\lambda])$.

(b) (1,4) highest weight $[p'\lambda'n']\uparrow\uparrow\uparrow$

$$[p\lambda n]$$

$\therefore$ spin $\tfrac{1}{2}$ c pt. ~ $\pm\tfrac{1}{3}\sqrt{\tfrac{1}{2}}\,([p\lambda\hat{n}] + [p\hat{\lambda}n] + [\hat{p}\lambda n])$

($[1^3]$ symmetrization is unnecessary).

$$P.\ |\underline{20}(8,2)\Lambda^{o}(\tfrac{1}{2})\rangle = \tfrac{1}{3}\,(4[pn\hat{\lambda}] - [\hat{p}\lambda n] + [p\hat{n}\lambda])$$

So we must take

$$|\, 20(1,4)\ \Sigma^{o}(\tfrac{3}{2})\, \rangle = -\sqrt{\tfrac{1}{6}}\ [p\lambda n]$$

(ix) The IR.$\underline{35}$

Highest weight $= |\, \Sigma^{+}(1)\rangle = p\ \bar{\hat{n}}$

Filling out the $(8,3)$ component the highest state which is not an eigenstate of P is $|\, K^{+}(0)\rangle = \sqrt{\tfrac{1}{2}}(-p\bar{\lambda} + \hat{p}\bar{\hat{\lambda}})$

$$P\, |\, K^{+}(0)\rangle = -3\ \sqrt{\tfrac{1}{2}}(p\bar{\lambda} + p\bar{\hat{\lambda}})$$

So we must take $|\, K^{+}\rangle = -\sqrt{\tfrac{1}{2}}(p\bar{\lambda} + \hat{p}\bar{\hat{\lambda}})$

The highest state in $(8,3)$ for which P can have non zero matrix element with $(1,3)$ is $|\omega(1)\rangle$. We find

$$|\, \omega(\phi)\rangle = \frac{-1}{\sqrt{6}}\ (p\bar{\hat{p}} - 2\lambda\bar{\hat{\lambda}} + n\bar{\hat{n}})$$

$$P|\omega(\phi)\rangle = \frac{-1}{\sqrt{6}}\ (2p\bar{\hat{p}} + 8\lambda\bar{\hat{\lambda}} + 2n\bar{\hat{n}})$$

$$= \frac{-1}{\sqrt{6}}\ [4(p\bar{\hat{p}} + \lambda\bar{\hat{\lambda}} + n\bar{\hat{n}}) - 2(p\bar{\hat{p}} - 2\lambda\bar{\hat{\lambda}} + n\bar{\hat{n}})]$$

So that we must take

$$|\, \varphi(1)\rangle = -\sqrt{\tfrac{1}{3}}\ (p\bar{\hat{p}} + \lambda\bar{\hat{\lambda}} + n\bar{\hat{n}})$$

(v) The IR.$\underline{1}$: This has YT $[1^{6}]$ and we may represent it as $[p\ \hat{p}\ \lambda\ \hat{\lambda}\ n\ \hat{n}]$, where [ ] denotes antisymetry under interchange of any two quarks.

Expanding

$$[p\hat{p}\lambda\hat{\lambda}n\hat{n}] = p[\hat{p}\lambda\hat{\lambda}n\hat{n}] - \hat{p}[\lambda\hat{\lambda}n\hat{n}p] + \lambda[\hat{\lambda}n\hat{n}p\hat{p}]$$

$$-\hat{\lambda}[n\hat{n}p\hat{p}\lambda] + n[\hat{n}p\hat{p}\lambda\hat{\lambda}] - \hat{n}[p\hat{p}\lambda\hat{\lambda}n]$$

Let $|\bar{p}\rangle = \sqrt{\frac{1}{5!}}[\acute{p}\acute{\lambda}\hat{n}\dot{n}]$, then using $S_\pm$ $I_\pm$ $K_\pm$ we obtain $|\hat{\bar{p}}\rangle = \sqrt{\frac{1}{5!}}\times$
$[p\hat{\lambda}\hat{n}\dot{n}]$

$$|\bar{n}\rangle = \sqrt{\frac{1}{5!}}[\hat{p}\hat{\lambda}\hat{p}\acute{n}] \qquad |\hat{\bar{n}}\rangle = \sqrt{\frac{1}{5!}}[\hat{p}\hat{\lambda}pn]$$

$$|\bar{\lambda}\rangle = \sqrt{\frac{1}{5!}}[\hat{p}p\hat{\lambda}n\hat{n}] \qquad |\hat{\bar{\lambda}}\rangle = \sqrt{\frac{1}{5!}}[\hat{p}p\lambda n\hat{n}]$$

So that

$$\begin{aligned} \underline{1} &= \sqrt{\tfrac{1}{6}}(|p\rangle|\bar{p}\rangle - |\hat{p}\rangle|\hat{\bar{p}}\rangle - |\lambda\rangle|\bar{\lambda}\rangle + |\hat{\lambda}\rangle|\hat{\bar{\lambda}}\rangle - |n\rangle|\bar{n}\rangle + |\hat{n}\rangle|\hat{\bar{n}}\rangle) \\ &= \sqrt{\tfrac{1}{6}}(p\bar{p} + \hat{p}\hat{\bar{p}} + \hat{\lambda}\hat{\bar{\lambda}} + \lambda\bar{\lambda} + n\bar{n} + \hat{n}\hat{\bar{n}}) \end{aligned} \qquad 2.18$$

In Table 4 we summarise the results of this section, giving the highest vector in each SU(3) x SU(2) multiplet of the five IRs discussed, together with those for the second orthogonal equivalent $\underline{70}$.

TABLE 4 :

SU(6) IR	SU(3) x SU(2) highest vector	Quark structure
56	$\|(10,4)\, N^{*++}(\frac{3}{2})\rangle$	ppp
	$\|(8,2)\, \Sigma^{+}(\frac{1}{2})\rangle$	$\frac{1}{3\sqrt{2}}\,[2(pp\hat{\lambda}) - (p\hat{p}\lambda)]$
70	$\|(10,2)\, N^{*+}(\frac{1}{2})\rangle$	$\sqrt{\frac{1}{2}}\,[p\hat{p}]p$
	$\|(8,4)\, \Sigma^{+}(\frac{3}{2})\rangle$	$\sqrt{\frac{1}{2}}\,[p\lambda]p$
	$\|(8,2)\, \Sigma^{+}(\frac{1}{2})\rangle$	$-\sqrt{\frac{1}{6}}\,([p\hat{p}]\lambda + [p\lambda]\hat{p} - [p\hat{\lambda}]p)$
	$\|(1,2)S^{0}(\frac{1}{2})\rangle$	$\sqrt{\frac{1}{6}}([p\hat{n}]\lambda - [\hat{p}n]\lambda + [\lambda\hat{n}]p)$
70'	$\|(10,2)\, N^{*+}(\frac{1}{2})\rangle$	$\sqrt{\frac{1}{6}}(2pp\hat{p} - (p\hat{p})p)$
	$\|(8,4)\, \Sigma^{+}(\frac{3}{2})\rangle$	$\sqrt{\frac{1}{6}}(2pp\lambda - (p\lambda)p)$
	$\|(8,2)\, \Sigma^{+}(\frac{1}{2})\rangle$	$\frac{1}{3\sqrt{2}}\,(2pp\hat{\lambda} + 2(\hat{p}\lambda)p - (p\hat{p})\lambda - (p\lambda)\hat{p} - (p\hat{\lambda})p)$
	$\|(1,2)S^{0}(\frac{1}{2})\rangle$	$\frac{1}{2\sqrt{3}}\,((-p\hat{\lambda})n + (p\hat{m})\lambda + (\hat{p}\lambda)n - (\hat{p}n)\lambda - (\lambda\hat{n})p + (\hat{\lambda}n)p)$
20	$\|(8,2)\Sigma^{+}(\frac{1}{2})\rangle$	$\sqrt{\frac{1}{6}}[p\hat{p}\lambda]$
	$\|(1,4)S^{0}(\frac{3}{2})\rangle$	$-\sqrt{\frac{1}{6}}\,[p\lambda n]$
35	$\|(8,3)\, \rho^{+}(1)\rangle$	$p\,\bar{n}$
	$\|(8,1)\, \pi^{+}\rangle$	$-\sqrt{\frac{1}{2}}\,(p\,\bar{n} + \hat{p}\,\bar{\hat{n}})$
	$\|(1,3)\, \varphi^{0}(1)\rangle$	$-\sqrt{\frac{1}{3}}\,(p\bar{p} + \lambda\bar{\lambda} + n\bar{n})$
1	$\|(1,1)\, X^{0}\rangle$	$\sqrt{\frac{1}{6}}\,(p\bar{p} + \hat{p}\bar{\hat{p}} + \lambda\bar{\lambda} + \hat{\lambda}\bar{\hat{\lambda}} + n\bar{n} + \hat{n}\bar{\hat{n}})$

2.5 Some special features of SU(6)

We conclude by emphasizing that the departure from a canonical formalism in SU(6) has introduced some special features.

(1) The Cartan subalgebra is not diagonal in the physical (SU(3) x SU(2)) basis used - this is highlighted by our use of an element of $\mathcal{H}$ as a ladder operator; associated with this, SU(6) has introduced no new quantum numbers of a linear or simply additive type. Of course it is still true that SU(6) invariance is much more restrictive than SU(3) x SU(2) invariance.

(2) The SU(3) x SU(2) subgroup labelling of states in an SU(6) IR is not sufficient to distinguish all states of all IRs - thus there may be more than one vector in an IR having a given SU(3) x SU(2) transformation, although of course such degenerate vectors will be differentiated if we include the complete SU(6) group of transformations. The general canonical solution to the labelling problem for SU(n) has been enunciated by Racah[2] and is also discussed in ref.[13]. Within a given IR the Cartan subalgebra is to supply, for SU(n), n-1 labelling operators and a further $\frac{1}{2}(n-1)(n-2)$ independent operators commuting with themselves and also with are then needed, $\frac{1}{2}n(n-1)$ in all. (The SU(n) Casimir operators provide a further n-1 labels sufficient to distinguish inequivalent IRs.) In SU(6) we need then 15 operators and our SU(3) x SU(2) provide only seven (being SU(3) and SU(2) Casimirs together with

I^2, I_3, Y, S_3) - a further eight operators and their spectra, commuting with these seven but not with any of the five SU(6) Casimirs are therefore required. Given these one could then attempt a general algebraic solution for SU(6) generator matrix elements and CGc's - such an undertaking seems academic from the physicists view point, the elementary techniques developed in this thesis are largely adequate for his needs. However it is interesting to note that there is now a partial solution to an identical problem for a different group viz SU(4) decomposed according to SU(2) xSU(2). See ref.[22]. We could also note that non canonical decompositions of SU(n) are perhaps the rule rather than the exception in physics. Thus the chain SU(n) $\rightarrow$ O(n) provides (an incomplete) labelling useful in nuclear physics[27]. Using methods described in this thesis it is easy to establish for SU(3) $\rightarrow$ O(3) that the generator embedding may be taken as $S_{\pm} = \sqrt{2}\ (K_{\pm} + L_{\mp})$ $S_3 = 2I_3$ for the generators of the O(3) subgroup, and that in this basis I_+ transforms like an $S_3 = 2$ $S = 2$ tensor. Eignestates of the pair S^2, S_3 in general are not eigenstates of Y (this is obvious from the form for $S_{\pm}$) and so in this decomposition we 'lose' one of the diagonal quantum numbers, in contrast to the situation for the SU(2) x U(1) decomposition.

This degeneracy does become more troublesome when mass formulae for the SU(6) group are discussed. There, clearly, it is essential to label (i.e. distinguish) all states in an IR. The practical solution has been, following Beg and Singh[23] to introduce a second decomposition chain $SU(6) \supset U(1) \times SU(2) \times SU(4) \supset U(1) \times SU(2) \times SU(2) \times SU(2)$ – the two chains together have sufficed, in those multiplets so far discussed[23].

Thus, in brief, it is the product rather than the sum, subgroup decomposition of SU(6) which introduces an unaccustomed aspect of an SU(n) group. In the next chapter we further exploit this novelty to calculate some SU(6) Clebsch-Gordan coefficients.

CHAPTER 3

CALCULATION OF THE CLEBSCH-GORDAN COEFFICIENTS

This chapter deals with the reduction of the direct or inner product in SU(6) in the following cases :-

i) $\underline{35} \otimes \underline{35} = \underline{1} \oplus \underline{35}_F \oplus \underline{35}_D \oplus \underline{189} \oplus \underline{280} \oplus \overline{\underline{280}} \oplus \underline{405}$

ii) $\underline{56} \otimes \underline{35} = \underline{56} \oplus \underline{70} \oplus \underline{1134} \oplus \underline{700}$

iii) $\underline{56} \otimes \overline{\underline{56}} = \underline{1} \oplus \underline{35} \oplus \underline{405} \oplus \underline{2695}$

iv) $\underline{70} \otimes \overline{\underline{70}} = \underline{1} \oplus \underline{35}_F \oplus \underline{35}_D \oplus \ldots$

(an extensive list of specific Clebsch-Gordan series for SU(6) may be found in H.Ruegg et al[21]). Complete tables for the series (i) (ii) and (iii) were first published in ref.[11]; for the series (iv) we extract only the coefficients associated with the [illegible]) octet parts of the two $\underline{35}$'s since only these will be needed in some work on representation mixing discussed in Chapter 5. The tables can be found in §3.[illegible] In §3.1 we introduce some notation and definitions and discuss the method of calculation of CGc's, and in §3.2 we gather together all phase conventions operative in our work. §3.3 deals with fundamental symmetry and orthogonality properties. §3.4 treats cases (i)-(iv) above and in §3.5 we add a brief note on use and application of the tables.

3.1 Some simple examples

Let

$$|X\rangle \equiv |\lambda_1, (\mu_1 \sigma_1)_i; \quad Y_1 I_1^2 I_{1_3}; \quad S_{1_3}\rangle, \; |Y\rangle \equiv |\lambda_2, (\mu_2 \sigma_2)_j; \quad Y_2 I^2_2, I_{2_3}, S_{2_3}\rangle$$

be two normalised basis vectors of two SU(6) IRs, λ_1, λ_2. We have exhibited all the necessary labels, note especially i, j which have no group theoretic definition and enter when the SU(3) x SU(2) labelling is not by itself unique. Then we write

$$|X\rangle \otimes |Y\rangle = \sum_{\substack{\gamma' \\ \mu^\gamma, \sigma, k \\ Y\, I^2 I_3 S_3}} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^{\gamma'} \\ (\mu_1 \sigma_1)_i & (\mu_2 \sigma_2)_j & (\mu^\gamma \sigma)_k \\ Y_1 I_1^2 J_{1_3} S_3 & Y_2 I_2^2 I_{2_3} S_3 & Y I^2 I_3 S_3 \end{pmatrix} \cdot |\lambda_{\gamma'} \; (\mu^\gamma \sigma)_k; \quad Y I^2 \; I_3 \; S_3 \rangle \qquad 3.1$$

as a detailed expression of the Clebsch-Gordan series

$$\lambda_1 \otimes \lambda_2 = \sum_{\nu} \oplus \; n_\nu \lambda_\nu \qquad n_\nu \geq 1 \qquad 3.2$$

μ_i, (σ_i) denote the dimension of the SU(3) (SU(2)) IR involved

γ, γ' are employed when the direct product is not simple reducible, e.g. for SU(6) γ' when n_ν is > 1 in eq. 3.2

$YI^2I_3S_3$ are the usual SU(3) and SU(2) quantum numbers; we shall often represent YI^2I_3 by the single symbol ν.

We require the numbers, Clebsch-Gordan coefficients,

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^{\gamma'} \\ (\mu_1\sigma_1)_i & (\mu_1\sigma_1)_j & (\mu^\gamma,\sigma)_k \\ Y_1I_1^2I_{1_3}S_{1_3} & Y_2I_2^2I_{2_3}S_3 & YI^2I_3S_3 \end{pmatrix}$$

appearing in eq. 3.1 in the four cases mentioned in § 3.0. As has already been remarked we may not employ general SU(n) solutions[24] since these are given in the wrong (canonical) basis - our calculations can be viewed as establishing a transformation from this basis to the SU(3) x SU(2) basis in some special cases. Since the states used are eignestates of SU(3) and SU(2) the SU(6) direct product must satisfy the SU(3) x SU(2) direct product relations. Thus for SU(3) x SU(2) alone we have

$$|(\mu_1\sigma_1);\ \nu_1 S_{1_3}\rangle \otimes |(\mu_2\sigma_2);\ \nu_2 S_{2_3}\rangle$$

$$= \sum_{\mu^\gamma,\nu,\sigma} \begin{pmatrix} \mu_1 & \mu_2 & \mu^\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma \\ S_{1_3} & S_{2_3} & S_3 \end{pmatrix} |(\mu^\gamma\sigma);\ \nu\ S_3\rangle \qquad 3.3$$

and the question then becomes: if the states on the left hand side are promoted to SU(6) eigenstates, how are the SU(3) x SU(2) states on the right hand side in turn distributed amongst the various terms of the SU(6) direct product. In brief, we may extract from an SU(6) CGc an SU(3) and SU(2) CGc and it is necessary to calculate only the residual quantity:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^{\gamma'} \\ (\mu_1\sigma_1)_i & (\mu_2\sigma_2)_j & (\mu^\gamma\sigma)_k \\ \nu_1 s_{1_3} & \nu_2 s_{2_3} & \nu\, s_3 \end{pmatrix} = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma \\ s_{1_3} & s_{2_3} & s_3 \end{pmatrix}\begin{pmatrix} \mu_1 & \mu_2 & \mu^\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix}\left(\begin{matrix} \lambda_1 & \lambda_2 \\ (\mu_1\sigma_1)_i & (\mu_2\sigma_2)_j \end{matrix}\right\Vert \left.\begin{matrix} \lambda^{\gamma'} \\ (\mu^\gamma\sigma)_k \end{matrix}\right) \qquad 3.4$$

We need the last factor,

$$\left(\begin{matrix} \lambda_1 & \lambda_2 \\ (\mu_1\sigma_1)_i & (\mu_2\sigma_2)_j \end{matrix}\right\Vert \left.\begin{matrix} \lambda^{\gamma'} \\ (\mu^\gamma\sigma)_k \end{matrix}\right)$$

which we call a unitary scalar factor (usf), and which is the new number given by SU(6). Only the usf need be tabulated, the full CGc can then be reconstructed with the aid of SU(3)[5,25] and SU(2) tables - unfortunately in computation we have had to calculate the full CGc.

We give two elementary examples :

(i) $6 \otimes \bar{6} = \underline{1} \oplus \underline{35}$

The coefficients may be written down immediately from Table 4.

From $\quad |\underline{35}\ (8,3)\ \rho^+(1)\rangle = p\,\bar{n} \equiv |p\rangle\,|\bar{n}\rangle$

we deduce $\quad \begin{pmatrix} 6 & \bar{6} & 35 \\ p & \bar{n} & \rho^+(1) \end{pmatrix} = 1$

(we use an obvious shorthand form for the labelling quantum numbers) by the factorization property it follows that

$$\begin{pmatrix} 6 & \bar{6} \\ (3,2) & (3,2) \end{pmatrix} \Big\| \begin{matrix} 35 \\ (8,3) \end{matrix} = \begin{pmatrix} 6 & \bar{6} & 35 \\ p & n & \delta^{+}(1) \end{pmatrix}_{SU(6)} \begin{pmatrix} 3 & \bar{3} & 8 \\ p & \bar{n} & \pi^{+} \end{pmatrix}^{-1}_{SU(3)}$$

$$\begin{pmatrix} 2 & 2 & 3 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}^{-1}_{SU(2)}$$

where the subscript indicates the relevant group; using now the SU(3) and SU(2) tables[25], $\begin{pmatrix} 6 & \bar{6} \\ (3,2) & (\bar{3},2) \end{pmatrix} \Big\| \begin{matrix} 35 \\ 83 \end{matrix} = 1$

Similarly from

$$|\underline{35}(8,1);\ \pi^{+}\rangle = -\sqrt{\tfrac{1}{2}}(p\bar{n} + \hat{p}\hat{\bar{n}})$$

$$-\sqrt{\tfrac{1}{2}}(|p\rangle|\bar{n}\rangle + |\hat{p}\rangle|\hat{\bar{n}}\rangle) \quad (\text{cf. } \S\ 2.4)$$

we have $\begin{pmatrix} 6 & \bar{6} & 35 \\ p & \bar{n} & \pi^{+} \end{pmatrix} = -\sqrt{\tfrac{1}{2}}$

so that $\begin{pmatrix} 6 & \bar{6} \\ (3,2) & (\bar{3},2) \end{pmatrix} \Big\| \begin{matrix} 35 \\ 81 \end{matrix} = -1$

From

$$|\underline{35}(13);\ \varphi(1)\rangle = -\sqrt{\tfrac{1}{3}}(p\,\hat{\bar{p}} + \lambda\hat{\bar{\lambda}} + n\,\hat{\bar{n}})$$

$$= -\sqrt{\tfrac{1}{3}}(-|p\rangle|\hat{\bar{p}}\rangle + |\lambda\rangle|\hat{\bar{\lambda}}\rangle + |n\rangle|\hat{\bar{n}}\rangle)$$

we have $\begin{pmatrix} 6 & \bar{6} & 35 \\ p & \hat{\bar{p}} & \varphi(1) \end{pmatrix} = \sqrt{\tfrac{1}{3}}$

and therefore $\begin{pmatrix} 6 & \bar{6} \\ (3,2) & 32 \end{pmatrix} \Big\| \begin{matrix} 35 \\ (1,3) \end{matrix} = +1$

Finally, from

$$|\underline{1}(1,1);\ X_o\rangle = \sqrt{\tfrac{1}{6}}(p\bar{p} + \hat{p}\hat{\bar{p}} + \lambda\bar{\lambda} + \hat{\lambda}\hat{\bar{\lambda}} + n\bar{n} + \hat{n}\hat{\bar{n}})$$

$$\sqrt{\tfrac{1}{6}}(|p\rangle|\bar{p}\rangle - |\hat{p}\rangle|\hat{\bar{p}}\rangle - |\lambda\rangle|\bar{\lambda}\rangle + |\hat{\lambda}\rangle|\hat{\bar{\lambda}}\rangle - |n\rangle|\bar{n}\rangle$$

$$-|\hat{n}\rangle|\hat{\bar{n}}\rangle)$$

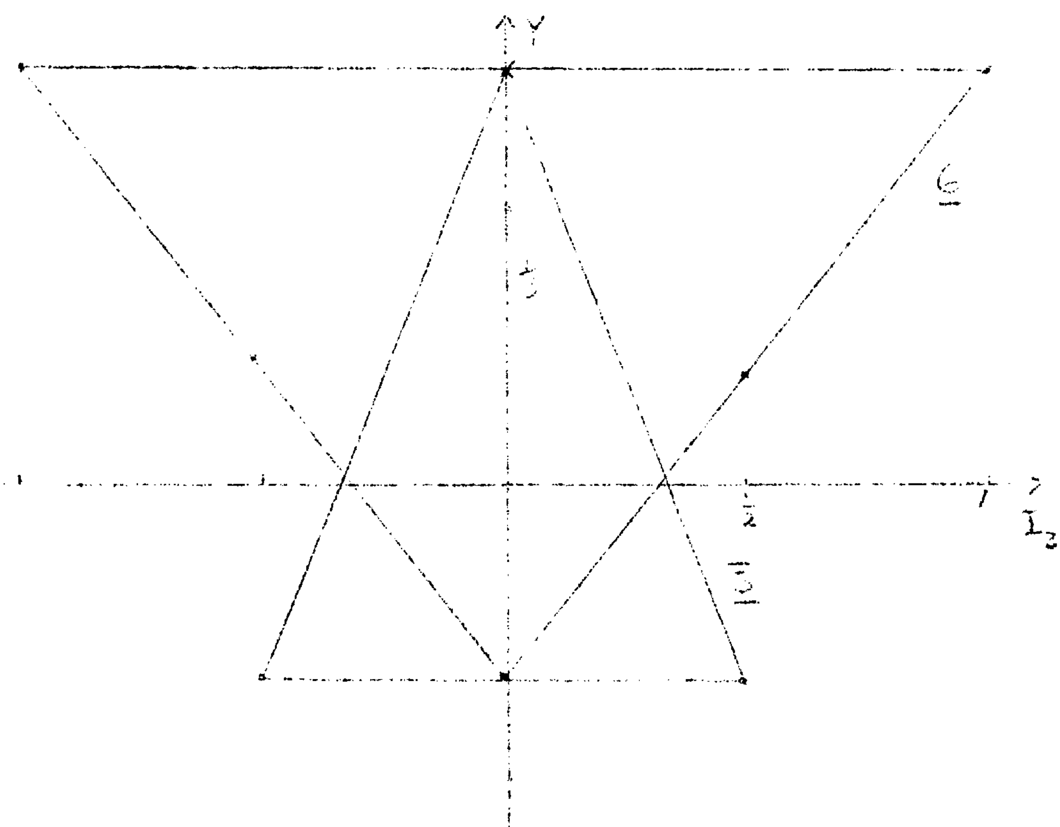
we find

$$\begin{pmatrix} 6 & \bar{6} & 1 \\ p & p & X^{o} \end{pmatrix} = \sqrt{\tfrac{1}{6}} \text{ so that } \left(\begin{matrix} 6 & \bar{6} \\ (3,2) & (\bar{3},2) \end{matrix} \right\| \left. \begin{matrix} 1 \\ (1,1) \end{matrix} \right) = +1.$$

Notice how important it is to employ basis vectors in the expansion in order to obtain the correct signs - Clebsch-Gordan coefficients always refer to basis states.

(ii) A second less trivial example is provided by $\underline{6} \otimes \underline{6} = \underline{21} \oplus \underline{15}$: $\underline{21}$ has YT [2] and SU(3) x SU(2) content $\underline{21} = (6,3) \oplus (\bar{3},1)$: $\underline{15}$ has YT $[1^2]$ and SU(3) x SU(2) content $\underline{15} = (6,1) \oplus (\bar{3},3)$

The weight diagrams are, neglecting spin degeneracy,



The highest state of $\underline{21}$ is the product of the highest states of the two factors 6. Using $\longrightarrow$ to denote application of a ladder operator and neglecting normalizations we now have

$$\underline{21}\text{ highest} = pp \xrightarrow{S_-} p\hat{p} + \hat{p}p \equiv (p\hat{p}) \xrightarrow{K_-} (p\hat{\lambda}) + (\hat{\lambda}p)$$

We can now use P (introduced in § 2.4) to transfer from $(6,3)$ to $(\bar{3},1)$ $(p\hat{\lambda}) + (\hat{p}\lambda) \xrightarrow{P} 3(p\hat{\lambda}) - 3(\hat{p}\lambda)$. This last state is orthogonal to $(p\hat{\lambda}) + (\hat{p}\lambda)$, has spin zero, I-spin $\frac{1}{2}$ and by our phase convention we thus take $|\,\underline{21}\ (\bar{3},1)\ {-\tfrac{1}{2}}\ 2\tfrac{1}{2};\ \tfrac{1}{2}\rangle = \frac{1}{2}((p\hat{\lambda}) - (\hat{p}\lambda))$.

We can thus calculate the two usfs

$$\left(\begin{matrix} 6 & 6 \\ (3,2) & (3,2) \end{matrix}\,\middle\|\,\begin{matrix} 21 \\ (6,3) \end{matrix}\right) = \begin{pmatrix} 6 & 6 & 21 \\ p & p & pp \end{pmatrix}_{SU(6)} \begin{pmatrix} 3 & 3 & 6 \\ p & p & pp \end{pmatrix}^{-1}_{SU(3)} \begin{pmatrix} 2 & 2 & 3 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}^{-1}_{SU(2)}$$

$$= 1.\ 1.\ 1 = +1$$

$$\left(\begin{matrix} 6 & 6 \\ (3,2) & (3,2) \end{matrix}\,\middle\|\,\begin{matrix} 21 \\ \bar{3}1 \end{matrix}\right) = \begin{pmatrix} 6 & 6 & 21 \\ p & \hat{\lambda} & (\bar{3},1)\text{highest} \end{pmatrix} \begin{pmatrix} 3 & 3 & \bar{3} \\ p & \lambda & \text{highest} \end{pmatrix}^{-1}_{SU(3)}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^{-1}_{SU(2)}$$

$$= \tfrac{1}{2}\ \sqrt{2}.\ \sqrt{2} = +1$$

One can check that the same usf results from the use of any term in the direct product expansion e.g. $\hat{p}\lambda$ in $\underline{21}(\bar{3},1)$ above.

For $\underline{15}$ we note that $p\hat{p} - \hat{p}p = [p\hat{p}]$ has the correct quantum numbers for the highest vector, and is orthogonal to all $\underline{21}$ states.

Taking $\underline{15}$ highest $= \sqrt{\tfrac{1}{2}}[p\hat{p}] \xrightarrow{K_-} \sqrt{\tfrac{1}{2}}([p\hat{\lambda}] + [\lambda\hat{p}]) \xrightarrow{P}$ $3\sqrt{\tfrac{1}{2}}([p\hat{\lambda}] - [\lambda\hat{p}])$. So we take

$|\,\underline{15}(\bar{3},3);\ \ p\lambda\, S_3=0\rangle = \tfrac{1}{2}([p\hat{\lambda}] + [\hat{p}\lambda])$, using S_+ we then have

$|\,15(\bar{3},3)\text{ highest}\rangle = \sqrt{\tfrac{1}{2}}[p\lambda]$.

The usfs follow

$$\begin{pmatrix} 6 & 6 & 15 \\ 32 & 32 & 61 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 15 \\ p & p & (pp) \end{pmatrix}_{SU(6)} \begin{pmatrix} 3 & 3 & 6 & -1 \\ p & p & pp \end{pmatrix}_{SU(3)} \begin{pmatrix} 2 & 2 & 1 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}_{SU(2)}$$

$$= -\sqrt{\tfrac{1}{2}}\,.\ \ 1\,.-\sqrt{2} \qquad = \ +1$$

$$\begin{pmatrix} 6 & 6 & 15 \\ 32 & 32 & \bar{3}3 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 15 \\ p & \lambda & [p\lambda] \end{pmatrix}_{SU(6)} \begin{pmatrix} 3 & 3 & \bar{3} & -1 \\ p & \lambda & p\lambda \end{pmatrix}_{SU(3)} \begin{pmatrix} 2 & 2 & 3 & -1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}_{SU(2)}$$

$$= \sqrt{\tfrac{1}{2}}\,.\ \ \ \sqrt{2}\,.\ 1 \qquad = +1$$

Alternatively we could have calculated basis vectors in $\underline{21}$ and $\underline{15}$ directly, without using ladder operators :

$\underline{21}(6,3)$ highest $\sim \square\square \sim Q_{\square\square} \times S_{\square\square} \sim (p'p')(\uparrow\uparrow) = pp$

$21(\bar{3},1)$ highest $\sim \square\square \sim Q_{\boxminus} \times S_{\boxminus} \sim [p'\lambda'][\uparrow\downarrow]$

$= [p\hat{\lambda}] - [\hat{p}\lambda]$, by direct multiplication

Similarly

$|\,\underline{15}(6,1)\text{ highest}\rangle \sim \boxminus \sim Q_{\square\square} \times S_{\boxminus} \sim (p'p')[\uparrow\downarrow]$

$= [p\,\hat{p}]$

$|\,\underline{15}(\bar{3},3)\text{ highest}\rangle \sim \boxminus \sim Q_{\boxminus} \times S_{\square\square} \sim [p'\lambda'](\uparrow\uparrow)$

$= [p\,\lambda]$

Normalising these basis vectors we obtain the same results as before - notice that this second direct method was very simple here since it was not necessary to apply symmetrisers for the SU(6) symmetry. Summarising, to calculate the usfs we had only to calculate one basis vector from each (μ,σ) multiplet in each SU(6) IR found in the direct product, the CGc then enter as normalization and orthogonality factors. In the first method we used ladder operators, drawn from the algebra and this is in fact the standard method mentioned by Racah[2] and employed e.g. by Rashid[4] - in the second method (which has been employed in the actual usf calculations which follow) direct construction has enabled us to dispense with such ladder operators.

The advantage of this latter method is only seen in more complicated situations; the important ladder operator P has SU(3) octet transformation properties with the consequent selection rules eq.2.16. It does not in general produce a pure SU(3) x SU(2) state e.g. acting on (8,3) in <u>405</u> it can produce a vector with non zero projection into every (μ,σ) submultiplet, and one must then use SU(3) and SU(2) ladder operators and orthogonality to isolate, in a straight forward but tedious way, the required pure (μ,σ) vector. Of course ladder operators with pure transformation properties, but not in the SU(6) algebra, can be found:

e.g. $P' = \sum_{\nu} T((8,3)\nu,0)\ T((8,3)\ -\nu,0)$ has Δ SU(3)=0, $\Delta S=2,0$

$P'' = \quad T((8,3)\nu,\alpha)\ T((1,3),0,\ -\alpha)$ has $\Delta S=0$

but again these are complicated objects with which to work.

On the other hand the orthodox method has the advantage that the correct phases within the SU(6) multiplet are a byproduct of the calculation whereas in our case, if we were to employ the P convention throughout, as in Ch.2.4, they must be adjusted after construction.

Again our method will always give directly the quark antiquark composition of a vector, whereas conventionally to reduce the direct product one needs only the matrix elements of the ladder operators in the factor or component, representations; however as mentioned in Ch.2.2 this explicit quark structure provides us with a simple alternative means of arriving at a consistent phase convention. Thus for example $\underline{35} \otimes \underline{35}$ the quark-antiquark structure of $\underline{405}$ is $T^{\square\square}_{\square\square}$, the lower boxes hold quarks, the upper antiquarks. One $\underline{35}$ appears as a trace of this $\underline{405}$, which we show symbolically by $T^{\square\square}_{\square\square}$. Comparing with the structure of $\underline{35}$ given in Table 4 we see that for the direct product we may take (the $\underline{35}$ tensor is clearly symmetric in its $\underline{35}$ components, hence the label D)

$$|35_D;(8,3)\ \text{highest}\rangle \sim \sum_\alpha + (p\alpha)(\hat{\bar n}\bar\alpha) \qquad 3.5a$$

$$|35_D;(8,1)\ \text{highest}\rangle \sim \sum_\alpha - (p\alpha)(\bar n\bar\alpha) - (\hat p\alpha)(\hat{\bar n}\bar\alpha) \qquad 3.5b$$

$$|35_D;(1,3)\ \text{highest}\rangle \sim \sum_\alpha - (p\alpha)(\hat{\bar p}\bar\alpha) - (\lambda\alpha)(\hat{\bar\lambda}\bar\alpha) - (n\alpha)(\hat{\bar n}\bar\alpha) \quad 3.5c$$

where the α summation is over all states in $\underline{6}$ and effects the trace. Since $\alpha\bar\alpha$ belongs to $\underline{1}$, and all operators in the algebra produce zero when operating upon it, it is clear that, once normalised, the above states will provide a consistent (in the sense of Ch.2.2) basis for $\underline{35}$. Notice that we have arranged signs so that $\sum_{\alpha} \alpha\bar{\alpha}$, eq. 2.18, is indeed the correct scalar quantity.

This then is the method we have adopted, looking at (i)- (iv) of Ch.3.0 we see that only the IRs $\underline{35}$, $\underline{56}$, $\underline{70}$ and $\underline{405}$ occur on both sides of the equations, so only for these was the method of obtaining consistency invoked.

3.2 Summary of phase conventions

In work with Clebsch-Gordan coefficients three different phase conventions enter.

(i) The convention determining the matrix elements of the generators - i.e. fixing the solution to eqs. 1.1. This has now been fully discussed in , 2.2, 3.1. For our SU(6) tables we have adopted the P convention for the IRs $\underline{35}$, $\underline{56}$, $\underline{70}$, $\underline{20}$; for all other IRs the relative signs of (μ,σ) vectors have been chosen arbitrarily but, in the case of 405, consistently in the two relevant cases. We have already remarked that against the advantage in computation afforded by this convention must be set the disadvantage of lack of communicability, i.e. it would not be easy for other workers to construct other SU(6) tables consistent with our own.

(ii) In each IR in the direct product there is still an overall sign to be fixed which can be considered as the relative sign between different SU(6) IRs occurring in the product space.

Thus e.g. for the highest states of $\underline{21}$ or $\underline{15}$ we could have taken $-pp$ or $-[p\hat{p}]$ respectively and this would then alter the CGc but not the matrices of the generators. To resolve this ambiguity we always take, in the highest state of the product IR, that CGc coupling highest $I_1\ I_2\ I_{13}\ S_1\ S_2\ S_{13}$, in that order, within highest $(\mu_1\bar{\mu}_1)$, $(\mu_2\bar{\mu}_2)$, to be positive. Labels 1, 2 refer to the page order of the factor states. Thus e.g. we take

$$\begin{pmatrix} 6 & 6 & 21 \\ p & p & \text{highest} \end{pmatrix} > 0 \quad \text{i.e. basis vector} + pp$$

and

$$\begin{pmatrix} 6 & 6 & 15 \\ p & \hat{p} & \text{highest} \end{pmatrix} > 0 \text{ (rather than } \begin{pmatrix} 6 & 6 & 15 \\ \hat{p} & p & \text{highest} \end{pmatrix} > 0)$$

i.e. basis vector $+\sqrt{\tfrac{1}{2}}\ [p\hat{p}]$.

This is a direct extension of the usual procedure, cf.[5].

(iii) A third phase convention enters when we assign physical particles to multiplets. Already we have seen that if $p, \hat{p}, \ldots \hat{n}$ are basis vectors for $\underline{6}$ then their antiparticles, defined to be the complex conjugate state may not be immediately taken as basis state for $\underline{\bar{6}}$. In self-conjugate representations the situation is more involved. Complex conjugation here maps basis states into basis states with a phase according to eq. 2.18. If we want the same operator also to map particle $\longrightarrow$ + antiparticle it is therefore evident that we may not take particle $\longrightarrow$ + 1 basis vector) for all particles. For example for $\underline{35}$, from Table 2, we can easily see that the mapping 'canonical' generator $\longrightarrow$ + particle does have the desired property particle $\longrightarrow$ + antiparticle under complex conjugation, and using the mapping canonical generator basis vector we can therefore determine the appropriate signs for particle $\longrightarrow$ basis vector. We emphasize that it clearly is not necessary to arrange this added convenience of behaviour under complex conjugation for single particle states - but it does help to avoid more book-keeping on $\pm$ signs.

3.3 Symmetry and orthogonality properties of C.G.c.

These are fully discussed in deSwart[5], Ch.14, and in the following we use his notation. Discussions for a general compact group are found in ref.[28]. We have

(i)

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^{\gamma'} \\ (\mu_1\sigma_1) & (\mu_2\sigma_2) & (\mu^{\gamma}\sigma) \\ \nu_1 s_{13} & \nu_2 s_{23} & \nu s_3 \end{pmatrix} = \zeta_1^{\gamma'} \begin{pmatrix} \lambda_2 & \lambda_1 & \lambda^{\gamma'} \\ (\mu_2\sigma_2) & (\mu_1\sigma_1) & (\mu^{\gamma},\sigma) \\ \nu_2 s_{23} & \nu_1 s_{13} & \nu s_3 \end{pmatrix} \qquad 3.6a$$

$$= \zeta_2^{\gamma'} \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}^{\gamma'} \\ (\bar{\mu}_1\sigma_1) & (\bar{\mu}_2\sigma_2) & (\bar{\mu}^{\gamma}\sigma) \\ -\nu_1 -s_{13} & -\nu_2 -s_{23} & -\nu, -s_3 \end{pmatrix} \qquad 3.6b$$

$$= \zeta_2'(-1)^{\bar{\Lambda}_1 - \Lambda_1} \sqrt{\frac{\lambda}{\lambda_2}} \begin{pmatrix} \lambda_1 & \bar{\lambda} & \bar{\lambda}_2^{\gamma'} \\ (\mu_1 \sigma_1) & (\bar{\mu}\sigma) & (\bar{\mu}_2^{\gamma}\sigma_2) \\ \nu_1 S_{13} & -\nu, -S_3 & -\nu_2, -S_{23} \end{pmatrix} \quad 3.6c$$

ζ_1', ζ_2', ζ_3', have values ± 1 and in general depend only upon $(\lambda_1, \lambda_2, \lambda)$ - an exception to this occurs for ζ_3' in the case of $(\mu_1\sigma)$ degeneracy, cf. Ch.3.4(1). In eq. 3.6c we have used λ, λ_2 to represent the dimensionality of the respective IRs whilst $\Lambda_1 = I_{13} + \frac{1}{2} Y_1 + S_{13}$ and $\bar{\Lambda}_1$ is the value of Λ_1 for the highest vector of $\underline{\lambda}_1$, cf. also [5]. Using the factorization property and equations analogous to 3.6 for SU(3) and SU(2) we may rewrite eqs. 3.6 in terms of usfs only: (setting $\sigma = 2j + 1$ etc.)

$$\left(\begin{matrix} \lambda_1 & \lambda_2 \\ (\mu_1\sigma_1) & (\mu_2\sigma_2) \end{matrix} \middle\| \begin{matrix} \lambda^{\gamma'} \\ (\mu^{\gamma}\sigma) \end{matrix} \right) = \zeta_1' \zeta_1 (-1)^{j_1+j_2-j} \left(\begin{matrix} \lambda_2 & \lambda_1 \\ (\mu_2\sigma_2) & (\mu_1\sigma_1) \end{matrix} \middle\| \begin{matrix} \lambda^{\gamma'} \\ (\mu^{\gamma}\sigma) \end{matrix} \right) \quad 3.7a$$

$$= \zeta_3' \zeta_3 (-1)^{j_1+j_2-j} \left(\begin{matrix} \bar{\lambda}_1 & \bar{\lambda}_2 \\ (\bar{\mu}_1\sigma_1) & (\bar{\mu}_2\sigma_2) \end{matrix} \middle\| \begin{matrix} \bar{\lambda}^{\gamma'} \\ (\bar{\mu}^{\gamma}\sigma) \end{matrix} \right) \quad 3.7b$$

$$= \zeta_2' \zeta_2 \sqrt{\frac{\sigma_2 \cdot \mu_2 \cdot \lambda}{\sigma \cdot \mu \cdot \lambda_2}} \left(\begin{matrix} \lambda_1 & \bar{\lambda} \\ (\mu_1\sigma_1) & (\bar{\mu}\sigma) \end{matrix} \middle\| \begin{matrix} \lambda_2^{\gamma'} \\ (\bar{\mu}_2\sigma_2) \end{matrix} \right) \quad 3.7c$$

In 3.7c we have absorbed in ζ_2' a constant factor $(-1)^{\bar{q}}$, $\bar{q}$ = charge of highest vector in λ_1 resulting from a slight deviation of our definition of ζ_2' from that of deSwart[5] whose corresponding phases are here represented by $\zeta_1, \zeta_2, \zeta_3$.

Given say the set of usfs for λ_1 (x) $\lambda_2 \longrightarrow \sum_{\lambda\gamma} \oplus \lambda_1^\gamma$ it is elementary to calculate the factors ζ', e.g.

By convention (i) of § 3.2

$$0 < \begin{pmatrix} \lambda_2 & \lambda_1 & \lambda^{\gamma'} \\ (\mu_2\sigma_2) & (\mu_1\sigma_1) & (\mu^\gamma\sigma) \\ \nu_2 s_{23} & \nu_1 s_{13} & \nu\, s_3 \end{pmatrix}$$

where $|(\mu^\gamma\sigma)\nu s_3\rangle$ is highest in $\lambda^{\gamma'}$ and $\nu_2 \geq \nu_1$ $\sigma_2 \geq \sigma_1$, $s_{23} \geq s_{13}$;

but this CGc = $\zeta_1 \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^{\gamma'} \\ (\mu_1\sigma_1) & (\mu_2\sigma_2) & (\mu^\gamma\sigma) \\ \nu_1 s_{13} & \nu_2 s_{23} & \nu\, s_3 \end{pmatrix}$

by definition of ζ_1. The latter CGc is known by hypothesis, whence we can determine the sign of ζ_1.

The tables we construct must be consistent with these symmetry properties which therefore provide some checks on our calculations - we include subsidiary tables giving some of these

TABLE 10 - Some factors ζ'_1

		ζ'_1	ζ'_2	ζ'_3
$35 \otimes 35$	$\underline{405}$	1		
	$\underline{280}$	-1		
	$\underline{280}$	-1		
	$\underline{189}$	1		
	$\underline{35}_F$	-1		
	$\underline{35}_D$	1		
	$\underline{1}$	1		
$\underline{56} \otimes \underline{35}$	$\underline{700}$	1		
	$\underline{1134}$	-1		
	$\underline{70}$	-1		
	$\underline{56}$	1		
$\underline{56} \otimes \overline{\underline{56}}$	$\underline{2695}$	1		
	$\underline{405}$	1		
	$\underline{35}$	1		
	$\underline{1}$	1		
$\underline{70} \otimes \overline{\underline{70}}$	$\underline{35}_F$	1	-1	1
	$\underline{35}_D$	1	-1	1

N.B. These factors may fail in the case of multiple (μ,σ) occurrence.

A second important general property (which also provides us with a check) is the orthogonality of the CGcs resulting from their constituting a real orthogonal transformation from one basis, that of the product, to another, that of its reduced or direct sum form. Again these are adequately discussed in deSwart, here we merely emphasize that the SU(3) and SU(2) internal summations (i.e. 'magnetic' quantum number summations) can be carried out to leave us with the simple relations for the usf;

$$\sum_{(\mu_1\sigma_1),(\mu_2\sigma_2)\gamma} \begin{pmatrix} \lambda_1 & \lambda_2 & \| & \lambda^{\gamma'} \\ (\mu_1\sigma_1) & (\mu_2\sigma_2) & \| & (\mu^{\gamma'}\sigma)_k \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \| & \lambda'^{\rho'} \\ (\mu_1\sigma_1) & (\mu_2\sigma_2) & \| & (\mu'\gamma\sigma')_{k'} \end{pmatrix} = \delta_{\lambda\lambda'}.\delta_{\gamma\rho'}.\delta_{kk'}.\delta_{\mu\mu'}.\delta_{\sigma\sigma'} \qquad 3.8a$$

$$\sum_{\lambda,\gamma',\mu,\sigma,k} \begin{pmatrix} \lambda_1 & \lambda_2 & \| & \lambda^{\gamma'} \\ (\mu_1\sigma_1) & (\mu_2\sigma_2) & \| & (\mu^{\gamma},\sigma)_k \end{pmatrix} \begin{pmatrix} \lambda'_1 & \lambda'_2 & \lambda^{\gamma'} \\ (\mu'_1\sigma'_1) & (\mu'_2\sigma'_2) & (\mu^{\rho}\sigma)_k \end{pmatrix} = \delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\delta_{\rho\gamma}\,\delta_{\mu_1\mu_1'}\delta_{\mu_2\mu_2'}\,\delta_{\sigma_1\sigma_1'}.\delta_{\sigma_2\sigma_2'} \qquad 3.8b$$

In brief these equations imply our tables shall consist of rows and columns of orthonormal vectors.

3.4 Details of the tabulation

(i) <u>35</u> (x) <u>35</u>

We represent <u>35</u> by the mixed second rank tensor notice that considered as $q'\bar{q}$ this is not really irreducible since

$$\text{e.g. } T^{D}_{D} \equiv D\bar{D} = \sqrt{\tfrac{1}{6}}|X_0\rangle + \sqrt{\tfrac{1}{6}}|\varphi(0)\rangle + \tfrac{1}{2}\sqrt{\tfrac{1}{3}}|\omega(0)\rangle + \tfrac{1}{2}\sqrt{\tfrac{1}{3}}|\eta\rangle + \tfrac{1}{2}|\rho^0(0)\rangle + \tfrac{1}{2}|\pi^0\rangle \qquad 3.9$$

(this equation is obtained by inverting the equations summarised in Table 4 for the quark structure of $\underline{35}$ and $\underline{1}$) and the term X_0 is a member of $\underline{1}$. Really $\underline{35}$ is represented by a traceless tensor $T^B_A(35) = T^B_A - \frac{1}{6}\delta^B_A \sum_C T^C_C$; $\sum_C T^C_C = \sqrt{6}\, X_0$, but in practise it is more convenient to omit the traces and simply ignore (i.e. put equal to zero) the factor X_0 wherever it occurs, instead of explicitly subtracting it out.

The direct product is now partially reduced by operating (i.e. applying the symmetric group outer product) independently on upper and lower indices:

$$T^{\square}_{\square} \otimes T^{\square}_{\square} = T^{\square\square}_{\square\square} \oplus T^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\square\square} \oplus T^{\square\square}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus T^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

We find $T^{\square\square}_{\square\square} \longrightarrow \underline{405} \oplus \underline{35} \oplus 1$

$T^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\square\square} \longrightarrow \underline{280} \oplus \underline{35}$

$T^{\square\square}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \longrightarrow \overline{\underline{280}} \oplus \underline{35}$

$T^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \longrightarrow \underline{189} \oplus \underline{35} \oplus \underline{1}$

The $\underline{35}$ traces appearing in the first and last tensors (they are equivalent) are obviously symmetric under interchange of constituent $\underline{35}$ states, whilst the remaining two (also equivalent) will be antisymmetric under such interchange. Further, this opposition in symmetry will automatically make the two <u>35</u> traces orthogonal and so we have already (in a standard fashion) dispensed with the problem of double occurrence of <u>35</u> in <u>35</u> $\otimes$ <u>35</u>.

Using Table 4 we can now construct 35_F (antisymmetric) and 35_D (symmetric) and <u>1</u> vectors in terms of quarks (an example of this already occurs eqs. 3.5) and using these equations such as eq. 3.9 derive the various usfs.

It remains to discuss <u>405</u>, <u>280</u>, $\overline{280}$ and <u>189</u>, for which the main feature is the double occurrence in each one of (8,3). We describe the construction of basic vectors in this case :-

SU(6) <u>IR</u>	Contributing SU(3) x SU(2) tensors
<u>405</u>	Q x S , Q x S , Q x S
<u>280</u>	Q x S , Q x S , Q x S
$\overline{280}$	Q x S , Q x S , Q x S
<u>189</u>	Q x S , Q x S , Q x S

This list demonstrates that we can in each case construct three linearly independent (8,3) basis vectors using our method; it is necessary to take traces on the Q and S tensors to arrive at the correct SU(3) x SU(2) transformation properties, e.q.

<u>8</u> Q ~ Q Q Q

and similarly for $\underline{3}$ in SU(2) e.g. in $\underline{405}$:

$$Q^{\bar{B}}_{\mathrm{IE}} \quad S^{B}_{\mathrm{IB}} \sim \sum_{q'} (p'q')(\uparrow\hat{\uparrow})[\bar{n}'\bar{q}'][\uparrow\downarrow\mathrm{I}]$$

$$= 2(p\,)((\bar{\hat{n}}\bar{p}) - (\bar{n}\hat{\bar{p}})) + (p\lambda)((\hat{\bar{n}}\lambda) - (\bar{n}\bar{\lambda})) \quad \text{etc.}$$

However none of these (8,3) vectors will be orthogonal to the (83) terms occurring in their $\underline{35}$ traces. The extraction of these traces is then effected by forming orthogonal combinations, e.g. in $\underline{405}$ we find that, symbolically,

$$Q^{\mathrm{ID}}_{B} \quad S^{\mathrm{ID}}_{B} - Q^{B}_{\mathrm{ID}} \quad S^{B}_{\mathrm{ID}} \qquad 3.10a$$

and

$$3Q^{\mathrm{ID}}_{\mathrm{IB}} \quad S^{\mathrm{ID}}_{\mathrm{IB}} - 5(Q^{\mathrm{ID}}_{B} \quad S^{\mathrm{ID}}_{B} + Q^{B}_{\mathrm{ID}} \quad S^{B}_{\mathrm{IB}}) \qquad 3.10b$$

are an orthogonal pair which are also orthogonal to the $\underline{35}$ (83) vectors already constructed. Such orthogonnalizations and normalizations are always most easily carried out using explicit quark structures of the basis vectors; when these have been obtained one then resubstitutes for a sufficient number of $\underline{35}_1$, $\underline{35}_2$ states to enable all the usfs to be extracted.

From eqs. 3.10 we can also see how the $\mathcal{S}_3$ symmetry property fails to hold in the case of multiple occurrence. Indeed under complex conjugation we generate a minus sign in 3.10a and a plus sign in 3.10b, one (8,3) multiplet is thus 'normal' and the other 'abnormal'. (In fact by calculation 3.10a is abnormal.) This

This contrasting behaviour represents, perhaps, the best possible resolution of the (8,3) ambiguity (it holds also in the cases of $\underline{280}$, $\overline{\underline{280}}$ and $\underline{189}$) - however when the multiplicity is greater than two such a procedure is inadequate.

The final results appear in Table 5.

(ii) $\underline{56} \otimes \underline{35}$

Here the tensor multiplication and partial reduction is given by

$$T_{\square\square\square} \otimes T^{\square}_{\square} = T^{\square}_{\square\square\square\square} \oplus T^{\square}_{\substack{\square\square\square \\ \square}}$$

with $T^{\square}_{\square\square\square\square} \sim \underline{700} \oplus \underline{56}$

$$T^{\square}_{\substack{\square\square\square \\ \square}} \sim \underline{1134} \oplus \underline{56} \oplus \underline{70}$$

Again the two IRs $\underline{56}$ are equivalent in SU(6) and they appear because we use $T^{\square}_{q}$ for the IR $\underline{35}$.

Once more one proceeds by calculating first the basis vectors associated with trace terms viz $\underline{56}$ and $\underline{70}$, and these may then be used when it comes to extracting traces in the $\underline{1134}$ and $\underline{700}$. IRs We reproduce the calculation for $\underline{56}$:

$$\underline{56} \sim T^{\square}_{\square\square\square\square} \quad \text{and } |56(10,4)\text{ highest}\rangle \sim +(ppp)$$

$\therefore$ in $T^{\square}_{\square\square\square\square}$ we represent this state by $+ \sum_{q} (pppq)\bar{q}$ where ( ) denotes complete symmetry corresponding to the Y.T.[4].

Expanding $\sim (ppp) \sum_{q} q\bar{q} + 3 \sum_{q} (ppq)p\bar{q}$

The first term has the factor $|X_{o}\rangle$ and is omitted. For the rest we obtain

$$\frac{1}{3}\sqrt{\frac{2}{5}}\left\{3(ppp)p\bar{p} + (pp\hat{p})p\hat{\bar{p}} + (pp\lambda)p\bar{\lambda} + (pp\hat{\lambda})p\hat{\bar{\lambda}} + (ppn)p\bar{n} + (pp\hat{n})p\hat{\bar{n}}\right\} \qquad 3.10$$

as normalised basis vector. (In computing the normalization it is important to remember that $p\bar{p}$ has norm $\sqrt{\frac{5}{6}}$, not 1, by our rule of ignoring X_o cf. eq. 3.9) Rewriting now eq. 3.10 in terms of <u>56</u> and <u>35</u> states (using Table 4) we obtain the usfs $\left(\begin{smallmatrix}56 \\ \mu_1\nu_1\end{smallmatrix}\ \begin{smallmatrix}35 \\ \mu_2\nu_2\end{smallmatrix}\ \middle\|\ \begin{smallmatrix}56 \\ (10,4)\end{smallmatrix}\right)$

Similarly $|56(8,2)$ highest$\rangle \sim +2(pp\hat{\lambda}) - (p\hat{p}\lambda)$ from Table 4, so in this direct product we take

$$|56(8,2)\ \text{highest}\rangle \sim +\sum_q 2(pp\hat{\lambda}q)\bar{q} - (p\hat{p}\lambda q)\bar{q}$$

Omitting complete factors $\sum\limits_q q\bar{q}$ we find

$$\frac{1}{9\sqrt{5}}\sum_q 2(ppq)\hat{\lambda}\bar{q} + 2(p\hat{\lambda}q)p\bar{\lambda} - (p\hat{p}q)\lambda\bar{q} - (p\lambda q)\hat{p}\bar{q} - (\hat{p}\lambda q)p\bar{q}$$

as the normalised basis vector, and from it we obtain the usfs

$\left(\begin{smallmatrix}56 \\ \mu_1\nu_1\end{smallmatrix}\ \begin{smallmatrix}35 \\ \mu_2\nu_2\end{smallmatrix}\ \middle\|\ \begin{smallmatrix}56 \\ 8\ \gamma_2\end{smallmatrix}\right)$.

The complete set of usfs are given in Table 6.

(iii) <u>56</u> (x) $\overline{\underline{56}}$

Here the tensorial multiplication is

$$T_{\square\square\square} \otimes T^{\square\square\square} = T^{\square\square\square}_{\square\square\square} \longrightarrow \underline{2695} + \underline{405} + \underline{35} + \underline{1}$$

with $|\underline{1}\rangle \sim \sum\limits_{qrs} (qrs)(\bar{q}\ \bar{r}\ \bar{s})$

$|\underline{35}$ highest$\rangle \sim + \sum\limits_{q,r} (qrp)(\bar{q}\ \bar{r}\ \hat{\bar{n}})$

$| \underline{405}$ highest$\rangle \sim + \sum_q (qpp)(\bar{q}\bar{n}\bar{n})$

$| \underline{2695}$ highest$\rangle \sim \quad (ppp)(\bar{n}\bar{n}\bar{n})$

We have been careful to construct $\underline{405}$ basis vectors consistent with those appearing in $\underline{35}$ (x) $\underline{35}$ and the complete set of results appears in Table. 7.

(iv) $\underline{70}$ (x) $\overline{\underline{70}}$

What we shall require for our work in Chapter 5 is in fact the matrix elements of the generators in the IR. $\underline{70}$. One way to arrive at these is to compute the coefficients for $\underline{70}$(x)$\overline{\underline{70}}$ $\longrightarrow$ 35 and then use the ξ_2 symmetry of ξ_3. to obtain the desired coefficients Corresponding to the occurrence of $\underline{35}$ twice in $\underline{70}$ (x) $\overline{\underline{70}}$ we can construct two $\underline{35}$ tensors in the tensor product viz.

$$(35_1)^D_C = T\,[(AB)C]\;T^{[(AB)D]}$$

$$(35_2)^D_C = T\,[(AC)B]\;T^{[(AB)D]}$$

where $[(AB)C] = (AB)C - (CB)A$.

We can proceed as in the above examples to extract two sets of usfs. Unfortunately the basis vectors we construct in doing this are not orthogonal or equivalently the orthogonality relation-eq.3.8a is not satisfied (since we are considering only the $\underline{35}$'s we are not able to test the relation eq.3.8b). Given two linearly independent vectors it is a simple matter to derive an orthogonal pair, but this is not the end since we shall require matrix elements of the generators; one of our orthogonal sets of usfs must serve as in equation 1.6c.

One way to arrive at the required sets of usfs is to solve the commutation relations: thus for example we have from Table 2 :

$$[T(\pi^+), T(\pi^-)] = -T(\pi^o) = [T(\rho^+(o)), T(\rho^-(o))]$$

We have two unknowns, the reduced matrix element $\langle 70 \| T \| 70 \rangle$ and the mixing angle θ by which we must mix our two orthogonal sets of usfs in order to obtain one set corresponding to generator matrix elements. Taking an appropriate matrix element, such that $\langle \alpha | T(\pi^o) | \beta \rangle = 0$ allows us to neglect the reduced matrix element and solve directly for the mixing angle.

In this way with $\underset{\sim}{a}$, $\underset{\sim}{b}$ labelling our two sets of orthogonal usfs found indirectly from $\underline{35}_1$, $\underline{35}_2$ and tabulated in Table 8, taking $\cos\theta$ a + $\sin\theta$ b as the generator set we get the equation

$$\left(\frac{\cos\theta}{\sqrt{2}} - \frac{5}{4}\sin\theta\right)^2 = 0$$

Whence we obtain the required sets given in Tables 9 once again we use F to denote the generator usfs.

An alternative method is to recall that we know the matrix elements of some generators e.g. charge, isospin etc. from the SU(3) x SU(2) decomposition. Again by taking linear combinations one can adjust a and b to reproduce the correct physical situation cf. [26].

In Table 9 we present the data for the product $\underline{70} \otimes \overline{\underline{70}} \longrightarrow \underline{35}_F \oplus \underline{35}_D + \ldots$ The ξ_2 factors required for $\underline{70} \times \underline{35} \longrightarrow \underline{70}$ may be found in Table 10.

TABLE 8 :-

TABLE 8

$\underline{70} \otimes \overline{\underline{70}}$	(8,1)			(8,3)			
	$(35)_1 \equiv \underset{\sim}{a}$	$(35)_2$	$\underset{\sim}{b}$	$(35)_1 \equiv \underset{\sim}{a}$	$(35)_2$	b	Υ
$(10,2)\,(\overline{10},2)$	$-\frac{1}{2}\sqrt{\frac{5}{3}}$	$-\frac{1}{3}\sqrt{\frac{5}{3}}$	$-\frac{1}{4}\sqrt{\frac{5}{6}}$	0	$-\frac{1}{3}\sqrt{\frac{5}{3}}$	$-\frac{1}{2}\sqrt{\frac{5}{6}}$	
$(10,2)\,(8,4)$	0	0	0	$-\frac{1}{2}\sqrt{\frac{5}{6}}$	0	$\frac{1}{8}\sqrt{\frac{5}{3}}$	
$(10,2)\,(8,2)$	$\frac{1}{4}\sqrt{\frac{5}{6}}$	$-\frac{1}{3}\sqrt{\frac{5}{6}}$	$-\frac{5}{16}\sqrt{\frac{5}{3}}$	$-\frac{1}{4}\sqrt{\frac{5}{6}}$	$-\frac{1}{3}\sqrt{\frac{5}{6}}$	$-\frac{3}{16}\sqrt{\frac{5}{3}}$	
$(10,2)\,(1,2)$	0	0	0	0	0	0	
$(8,4)\,(\overline{10},2)$	0	0	0	$\frac{1}{2}\sqrt{\frac{5}{6}}$	0	$-\frac{1}{8}\sqrt{\frac{5}{3}}$	-
$(8,4)\,(8,4)$	$-\frac{1}{4}\sqrt{\frac{5}{3}}$	$\frac{1}{3}\sqrt{\frac{5}{3}}$	$\frac{5}{8}\sqrt{\frac{5}{6}}$	$-\frac{5}{12}\sqrt{\frac{1}{3}}$	$\frac{1}{3}\sqrt{\frac{1}{3}}$	$\frac{25}{24}\sqrt{\frac{1}{3}}$	8_s
$(8,4)\,(8,4)$	$-\frac{\sqrt{3}}{4}$	$-\sqrt{\frac{1}{3}}$	$-\frac{3}{8}\sqrt{\frac{3}{2}}$	$-\frac{1}{4}\sqrt{\frac{5}{3}}$	$-\frac{1}{3}\sqrt{\frac{5}{3}}$	$-\frac{3}{8}\sqrt{\frac{5}{6}}$	8_a
$(8,4)\,(8,2)$	0	0	0	$-\frac{1}{3}\sqrt{\frac{5}{6}}$	$-\frac{2}{9}\sqrt{\frac{5}{6}}$	$-\frac{1}{12}\sqrt{\frac{5}{3}}$	8_s
$(8,4)\,(8,2)$	0	0	0	0	$\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{2}\sqrt{\frac{1}{3}}$	8_a
$(8,4)\,(1,2)$	0	0	0	$-\frac{1}{6}\sqrt{\frac{1}{6}}$	$-\frac{4}{9}\sqrt{\frac{1}{6}}$	$-\frac{7}{24}\sqrt{\frac{1}{3}}$	
						contd.	

TABLE 8 cont'd.

(8,2) $(\overline{10},2)$	$-\frac{1}{4}\sqrt{\frac{5}{6}}$	$\frac{1}{3}\sqrt{\frac{5}{6}}$	$\frac{5}{16}\sqrt{\frac{5}{3}}$	$\frac{1}{4}\sqrt{\frac{5}{6}}$	$\frac{1}{3}\sqrt{\frac{5}{6}}$	$\frac{3}{16}\sqrt{\frac{5}{3}}$	
(8,2) (8,4)	0	0	0	$-\frac{1}{3}\sqrt{\frac{5}{6}}$	$-\frac{2}{9}\sqrt{\frac{5}{6}}$	$-\frac{1}{12}\sqrt{\frac{5}{3}}$	8_s
(8,2) (8,4)	0	0	0	0	$\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{2}\sqrt{\frac{1}{3}}$	8_a
(8,2) (8,2)	0	0	0	$-\frac{1}{6}\sqrt{\frac{5}{6}}$	$\frac{2}{9}\sqrt{\frac{5}{6}}$	$\frac{5}{24}\sqrt{\frac{5}{3}}$	8_s
(8,2) (8,2)	$\sqrt{\frac{1}{6}}$	$\frac{2}{3}\sqrt{\frac{1}{6}}$	$\frac{1}{4}\sqrt{\frac{1}{3}}$	$\frac{1}{2}\sqrt{\frac{1}{6}}$	0	$-\frac{1}{8}\sqrt{\frac{1}{3}}$	8_a
(8,2) (1,2)	$+\frac{1}{4}\sqrt{\frac{1}{6}}$	$-\frac{1}{3}\sqrt{\frac{1}{6}}$	$-\frac{5}{16}\sqrt{\frac{1}{3}}$	$\frac{5}{12}\sqrt{\frac{1}{6}}$	$\frac{1}{9}\sqrt{\frac{1}{6}}$	$+\frac{1}{48}\sqrt{\frac{1}{3}}$	
(1,2) $(\overline{10},2)$	0	0	0	0	0	0	
(1,2) (8,4)	0	0	0	$-\frac{1}{6}\sqrt{\frac{1}{6}}$	$-\frac{4}{9}\sqrt{\frac{1}{6}}$	$-\frac{7}{24}\sqrt{\frac{1}{3}}$	
(1,2) (8,2)	$\frac{1}{4}\sqrt{\frac{1}{6}}$	$-\frac{1}{3}\sqrt{\frac{1}{6}}$	$-\frac{5}{16}\sqrt{\frac{1}{3}}$	$\frac{5}{12}\sqrt{\frac{1}{6}}$	$\frac{1}{9}\sqrt{\frac{1}{6}}$	$-\frac{1}{48}\sqrt{\frac{1}{3}}$	
(1,2) (1,2)	0	0	0	0	0	0	

TABLE 9 - Unitary Scalar Factors for $70 \otimes \overline{70} \rightarrow (35_1 \oplus 35_2$

(Octet parts)

	35_F		35_D		
$70 \otimes \overline{70}$	(8,1)	(8,3)	(8,1)	(8,3)	γ'
$(10,2)\,(\overline{10},2)$	$\sqrt{\frac{5}{11}}$	$\frac{1}{3}\sqrt{\frac{5}{11}}$	$-\frac{1}{4}\sqrt{\frac{5}{22}}$	$\frac{5}{6}\sqrt{\frac{5}{22}}$	
(10,2) (8,4)	0	$\frac{1}{3}\sqrt{\frac{10}{11}}$	0	$\frac{-13}{24}\sqrt{\frac{5}{11}}$	
(10,2)(8,2)	0	$\frac{1}{3}\sqrt{\frac{10}{11}}$	$\frac{1}{16}\sqrt{55}$	$\frac{7}{48}\sqrt{\frac{5}{11}}$	
(10,2) (1,2)	0	0	0	0	
$(8,4)\,(\overline{10},2)$	0	$-\frac{1}{3}\sqrt{\frac{10}{11}}$	0	$\frac{13}{24}\sqrt{\frac{5}{11}}$	
(8,4) (8,4)	0	0	$-\frac{1}{8}\sqrt{\frac{55}{2}}$	$\frac{-5}{24}\sqrt{\frac{11}{2}}$	s
(8,4) (8,4)	$\frac{2}{\sqrt{11}}$	$\frac{2}{3}\sqrt{\frac{5}{11}}$	$\frac{7}{8}\sqrt{\frac{1}{22}}$	$\frac{7}{24}\sqrt{\frac{5}{22}}$	a
(8,4) (8,2)	0	$\frac{1}{3}\sqrt{\frac{10}{11}}$	0	$-\frac{1}{12}\sqrt{\frac{5}{11}}$	s
(8,4) (8,2)	0	$-\frac{1}{3}\sqrt{\frac{2}{11}}$	0	$\frac{-5}{6}\sqrt{\frac{1}{11}}$	a
(8,4) (1,2)	0	$+\frac{1}{3}\sqrt{\frac{2}{11}}$	0	$+\frac{3}{8}\sqrt{\frac{1}{11}}$	
$(8,2)\,(\overline{10},2)$	0	$-\frac{1}{3}\sqrt{\frac{10}{11}}$	$-\frac{1}{16}\sqrt{55}$	$\frac{-7}{48}\sqrt{\frac{5}{11}}$	
(8,2) (8,4)	0	$\frac{1}{3}\sqrt{\frac{10}{11}}$	0	$\frac{-1}{12}\sqrt{\frac{5}{11}}$	s
(8,2) (8,4)	0	$-\frac{1}{3}\sqrt{\frac{2}{11}}$	0	$-\frac{5}{6}\sqrt{\frac{1}{11}}$	a
(8,2) (8,2)	0	0	0	$-\frac{1}{24}\sqrt{55}$	s
(8,2) (8,2)	$-\sqrt{\frac{2}{11}}$	$-\frac{1}{3}\sqrt{\frac{2}{11}}$	$\frac{1}{4}\sqrt{\frac{1}{11}}$	$\frac{13}{24}\sqrt{\frac{1}{11}}$	a
(8,2) (1,2)	0	$-\frac{1}{3}\sqrt{\frac{2}{11}}$	$+\frac{1}{16}\sqrt{11}$	$+\frac{5}{16}\sqrt{\frac{1}{11}}$	
$(1,2)\,(\overline{10},2)$	0	0	0	0	
(1,2) (8,4)	0	$+\frac{1}{3}\sqrt{\frac{2}{11}}$	0	$+\frac{3}{8}\sqrt{\frac{1}{11}}$	
(1,2) (8,2)	0	$-\frac{1}{3}\sqrt{\frac{2}{11}}$	$+\frac{1}{16}\sqrt{11}$	$+\frac{5}{16}\sqrt{\frac{1}{11}}$	
(1,2) (1,2)	0	0	0	0	

3.5 Application of the Tables

The only new feature of the Tables, and one which might cause confusion in their application, is that associated with the unresolved labelling problem. However, to illustrate their rather tortuous, if elementary, use we first reduce the direct product of a proton-like (spin up) state and a neutral-pion-like state

$$|p^+(\tfrac{1}{2})\rangle \otimes |\pi^o\rangle = \sum_{\substack{\lambda\,\gamma' \\ \mu\gamma\sigma\,\nu S_3}} \begin{pmatrix} 56 & 35 & \lambda^{\gamma'} \\ 8,2 & 8,1 & (\mu^\gamma,\sigma) \\ p^+(\tfrac{1}{2}) & \pi^o & \nu S_3 \end{pmatrix} |\lambda^{\gamma'}(\mu^\gamma\sigma)\nu S_3\rangle$$

(we use the static SU(6) π^o assignment)

$$= \sum \begin{pmatrix} 2 & 1 & 2 \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} \end{pmatrix} \begin{pmatrix} 8 & 8 & \mu^\gamma \\ 1\tfrac{1}{2}\tfrac{1}{2} & 010 & 1I\tfrac{1}{2} \end{pmatrix} \left(\begin{array}{cc|c} 56 & 35 & \lambda^{\gamma'} \\ 82 & 81 & (\mu^\gamma,2) \end{array}\right)$$

$$\cdot\ |\lambda^{\gamma'}(\mu^\gamma,2);\ 1I\tfrac{1}{2};\ 2\tfrac{1}{2}\rangle$$

We must sum over λ = $\underline{56}$, $\underline{70}$, $\underline{1134}$, $\underline{700}$

μ^γ = 27 10 $\overline{10}$ 8_s 8_a

$I = \frac{3}{2}, \frac{1}{2}$

The relevant CGc's are

SU(2) $\begin{pmatrix} 2 & 1 & 2 \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} \end{pmatrix} = 1$

SU(3)		27	10	$\overline{10}$	8_a	8_a
$\begin{pmatrix} 8 & 8 & \mu^{\gamma} \\ 1\frac{1}{2}\frac{1}{2} & 010 & 1I\frac{1}{2} \end{pmatrix}$:	$I = \frac{3}{2}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$			
	$I = \frac{1}{2}$	$\sqrt{\frac{1}{60}}$		$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{12}}$

For the unitary scalar factors we look at the 82 81 rows of Table 6: the following are relevant:-

	(27,2)	(10,2)	(10,2)	$(8_s,2)$	$(8_a,2)$
700	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{4}$	$-\frac{1}{4}\sqrt{\frac{5}{3}}$
$(1134)_1$	$\frac{1}{2}$	0	$-\sqrt{\frac{3}{2}}$	$-\frac{3}{2}\sqrt{\frac{3}{22}}$	$-\frac{1}{2}\sqrt{\frac{15}{22}}$
$(1134)_2$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{\frac{3}{2}}$		$-\frac{1}{2}\sqrt{\frac{3}{11}}$	$\sqrt{\frac{3}{55}}$
$(1134)_3$				$-\frac{3}{4}\sqrt{\frac{1}{2}}$	$\frac{9}{4}\sqrt{\frac{1}{10}}$
70				$-\frac{1}{4}\sqrt{\frac{5}{2}}$	$-\frac{1}{4}\sqrt{\frac{1}{2}}$
56				0	$\sqrt{\frac{2}{15}}$

For example the $I = \frac{1}{2}$, (27,2) component of the direct product is written, omitting redundant labels

$$\Big(|p\rangle \otimes |\pi^{o}\rangle\Big)_{(27,2)} = \sqrt{\frac{1}{60}}\ \Big[-\sqrt{\frac{1}{2}}\ |700\rangle + \frac{1}{2}|(1134)_1\rangle - \frac{1}{2}(1134)_2\rangle\Big)$$

In all, the product state has non zero components in 21 orthogonal states occurring in the reduction - clearly in expansions such as occur in the use of the Wigner Eckart Theorem in scattering relations the sheer labour involved is considerable. Fortunately other methods are availat .e, see next Chapter. Notice in the above that for different μ the vectors $(\mu_1 2)_i$ in 1134 with the same i are in no way especially related - they all occur in the same IR. 1134.

A second calculation, more amenable to Clebsch Gordan methods is extraction of the ratios of specific coupling constants - a famous example is the D:F ratio of the coupling of pseudoscalar pions to baryons. Using a 35(83) assignment for the pions, (one best explained by U spin, see next chapter) we calculate for the $P^+(\frac{1}{2}) - \pi^o - \bar{P}^+(\frac{1}{2})$ vertex :

$$\frac{D}{F} = \frac{\begin{pmatrix} 56 & \overline{56} & 35 \\ 82 & 82 & 8_s 3 \\ p^+(\frac{1}{2}) & \bar{p}^+(\frac{1}{2}) & \pi^o \end{pmatrix}}{\begin{pmatrix} 56 & \overline{56} & 35 \\ 82 & 82 & 8_a 3 \\ p^+(\frac{1}{2}) & \bar{p}^+(\frac{1}{2}) & \pi^o \end{pmatrix}}$$

$$= \frac{\begin{pmatrix} 2 & 2 & 3 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_s \\ p^+ & \bar{p}^+ & \pi^o \end{pmatrix}\begin{pmatrix} 56 & 56 & \| & 35 \\ 82 & 82 & \| & 8_s 3 \end{pmatrix}}{\begin{pmatrix} 2 & 2 & 3 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_a \\ p^+ & \bar{p}^+ & \pi^o \end{pmatrix}\begin{pmatrix} 56 & \overline{56} & \| & 35 \\ 82 & 82 & \| & 8_a 3 \end{pmatrix}}$$

$$= \frac{-\sqrt{\frac{3}{20}} \cdot + \frac{1}{3} \cdot \sqrt{\frac{5}{6}}}{\sqrt{\frac{1}{12}} \cdot - \frac{1}{3} \cdot \sqrt{\frac{2}{3}}} = \frac{3}{2}$$

Concerning the Wigner-Eckart Theorem, its use in two body 'elastic' scattering leads to the following equations (assuming the scattering matrix S to be an SU(6) scalar operator).

$$\langle \lambda_1, (\mu_1\sigma_1), \nu_1 : \lambda_2 (\mu_2\sigma_2) \nu_2 | S | \lambda_3 (\mu_3\sigma_3), \nu_3 : \lambda_4, (\mu_4\sigma_4), \nu_4 \rangle$$

$$= \sum_{\substack{\lambda^{\gamma'} \rho'_i \\ \mu, \gamma, \sigma, \nu}} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^{\gamma'} \\ \mu_1\sigma_1 & \mu_2\sigma_2 & (\mu^{\gamma}\sigma)_i \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \lambda_3 & \lambda_4 & \lambda^{\rho'} \\ \mu_3\sigma_3 & \mu_4\sigma_4 & (\mu^{\rho}\sigma)_i \\ \nu_3 & \nu_4 & \nu \end{pmatrix}$$

$$\times \langle \lambda^{\gamma'} \| S \| \lambda^{\rho'} \rangle$$

Since the scattering is elastic $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$ say - we emphasize now, that in the summation cross terms in the redundancy label, i are not included - one can imagine these $(\mu\sigma)_i$ as in fact distinguished by some operator which an SU(6) scalar must respect, i.e. in the above S cannot cause i — j transitions. This conclusion also points to a real difficulty of our labelling scheme - if the scattering is not elastic we have to ensure nonetheless that the $(\mu\sigma)_i$ appearing in a common product state are in fact always the same basis vector. One way to do this is to calculate the generators in the two equivalent IRs under comparison, alternatively the method adopted here (for 405 $(83)_i$) was again by a method of direct comparison of basis vectors.

TABLE V – *Unitary scalar factors* $\begin{pmatrix} 35 & 35 & \Big\| & \lambda_{\gamma'} \\ \mu_1\sigma_1 & \mu_2\sigma_2 & \Big\| & \mu_\gamma\sigma \end{pmatrix}$ *for the C.G. series.*

$$\underline{35} \otimes \underline{35} = \underline{1} + \underline{35}_D + \underline{35}_F + \underline{189} + \underline{280} + \underline{\overline{280}} + \underline{405}.$$

(27, 5)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	(405) $\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	1

(27, 3)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	(405)	(280)	$(\overline{280})$ $\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	0	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$
8, 3; 8, 1	$\sqrt{\frac{1}{2}}$	$\frac{1}{2}$	$+\frac{1}{2}$
8, 1; 8, 3	$\sqrt{\frac{1}{2}}$	$-\frac{1}{2}$	$-\frac{1}{2}$

(27, 1)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	(405)	(189) $\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$
8, 1; 8, 1	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$

(10, 5)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	(280) $\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	1

(10, 3)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	(405)	(280)	(189) $\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	$\sqrt{\frac{1}{2}}$	0	$\sqrt{\frac{1}{2}}$
8, 3; 8, 1	$\frac{1}{2}$	$\sqrt{\frac{1}{2}}$	$-\frac{1}{2}$
8, 1; 8, 3	$-\frac{1}{2}$	$\sqrt{\frac{1}{2}}$	$\frac{1}{2}$

(10, 1)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	(280)	$(\overline{280})$ $\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$
8, 1; 8, 1	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$

TABLE V. (*continued*).

$(\overline{10}, 5)$

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	$(\overline{280})$	$\lambda_{\gamma'}$ / μ_λ
8, 3; 8, 3	-1	

$(\overline{10}, 3)$

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(405)	$(\overline{280})$	(189)	$\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	$-\sqrt{\frac{1}{2}}$	0	$-\sqrt{\frac{1}{2}}$	
8, 3; 8, 1	$\frac{1}{2}$	$-\sqrt{\frac{1}{2}}$	$-\frac{1}{2}$	
8, 1; 8, 3	$-\frac{1}{2}$	$-\sqrt{\frac{1}{2}}$	$\frac{1}{2}$	

$(\overline{10}, 1)$

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(280)	$(\overline{280})$	$\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	
8, 1; 8, 1	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	

(8, 3)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	$(405)_1$	$(405)_2$	$(280)_1$	$(280)_2$	$(\overline{280})_1$	$(\overline{280})_2$	$(189)_1$	$(189)_2$	$(35)_D$	$(35)_F$	$\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	0	0	$-\frac{1}{2}\sqrt{\frac{2}{11}}$	$-\frac{5}{6}\sqrt{\frac{5}{11}}$	$-\frac{1}{2}\sqrt{\frac{2}{11}}$	$-\frac{5}{6}\sqrt{\frac{5}{11}}$	0	0	0	$-\frac{1}{3}\sqrt{\frac{5}{2}}$	8_s
8, 3; 8, 3	0	$\frac{1}{4}\sqrt{5}$	0	0	0	0	0	$\frac{1}{2}\sqrt{\frac{1}{2}}$	$\frac{3}{4}$	0	8_a
8, 3; 8, 1	0	$-\frac{1}{4}\sqrt{\frac{1}{2}}$	$-\frac{1}{2}\sqrt{\frac{1}{11}}$	$\sqrt{\frac{5}{22}}$	$\frac{1}{2}\sqrt{\frac{1}{11}}$	$-\sqrt{\frac{5}{22}}$	0	$-\frac{1}{4}\sqrt{5}$	$\frac{1}{4}\sqrt{\frac{5}{2}}$	0	8_s
8, 3; 8, 1	$\frac{1}{2}$	0	$\frac{1}{2}\sqrt{\frac{5}{11}}$	$\frac{1}{2}\sqrt{\frac{1}{22}}$	$\frac{1}{2}\sqrt{\frac{5}{11}}$	$\frac{1}{2}\sqrt{\frac{1}{22}}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	8_a
8, 1; 8, 3	0	$-\frac{1}{4}\sqrt{\frac{1}{2}}$	$\frac{1}{2}\sqrt{\frac{1}{11}}$	$-\sqrt{\frac{5}{22}}$	$-\frac{1}{2}\sqrt{\frac{1}{11}}$	$\sqrt{\frac{5}{22}}$	0	$-\frac{1}{4}\sqrt{5}$	$\frac{1}{4}\sqrt{\frac{5}{2}}$	0	8_s
8, 1; 8, 3	$-\frac{1}{2}$	0	$\frac{1}{2}\sqrt{\frac{5}{11}}$	$\frac{1}{2}\sqrt{\frac{1}{22}}$	$\frac{1}{2}\sqrt{\frac{5}{11}}$	$\frac{1}{2}\sqrt{\frac{1}{22}}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	8_a
8, 3; 1, 3	$\frac{1}{2}$	0	$-\frac{1}{2}\sqrt{\frac{5}{11}}$	$\frac{2}{3}\sqrt{\frac{2}{11}}$	$-\frac{1}{2}\sqrt{\frac{5}{11}}$	$\frac{2}{3}\sqrt{\frac{2}{11}}$	$-\frac{1}{2}$	0	0	$-\frac{1}{3}$	
1, 3; 8, 3	$-\frac{1}{2}$	0	$-\frac{1}{2}\sqrt{\frac{5}{11}}$	$\frac{2}{3}\sqrt{\frac{2}{11}}$	$-\frac{1}{2}\sqrt{\frac{5}{11}}$	$\frac{2}{3}\sqrt{\frac{2}{11}}$	$\frac{1}{2}$	0	0	$-\frac{1}{3}$	
8, 1; 1, 3	0	$-\frac{1}{4}\sqrt{5}$	$\frac{1}{2}\sqrt{\frac{10}{11}}$	$\frac{1}{2}\sqrt{\frac{1}{11}}$	$-\frac{1}{2}\sqrt{\frac{10}{11}}$	$-\frac{1}{2}\sqrt{\frac{1}{11}}$	0	$\frac{1}{2}\sqrt{\frac{1}{2}}$	$\frac{1}{4}$	0	
1, 3; 8, 1	0	$-\frac{1}{4}\sqrt{5}$	$-\frac{1}{2}\sqrt{\frac{10}{11}}$	$-\frac{1}{2}\sqrt{\frac{1}{11}}$	$\frac{1}{2}\sqrt{\frac{10}{11}}$	$\frac{1}{2}\sqrt{\frac{1}{11}}$	0	$\frac{1}{2}\sqrt{\frac{1}{2}}$	$\frac{1}{4}$	0	

TABLE V. *(continued)*

(8, 5)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(405)	(280)	$(\overline{28}\overline{0})$	(189)	$\lambda_{\gamma'}$ \ μ_γ
8, 3; 8, 3	$-\sqrt{\frac{1}{6}}$	0	0	$-\sqrt{\frac{5}{6}}$	8_s
8, 3; 8, 3	0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	0	8_a
8, 3; 1, 3	$-\frac{1}{2}\sqrt{\frac{5}{3}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{\frac{1}{3}}$	
1, 3; 8, 3	$-\frac{1}{2}\sqrt{\frac{5}{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{\frac{1}{3}}$	

(1, 5)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(405)	(189)	$\lambda_{\gamma'}$ \ μ_γ
8, 3; 8, 3	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	
1, 3; 1, 3	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	

TABLE V. (*continued*).

(8, 1)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	(405)	(280)	(280)	(189)	$(35)_D$	$(35)_F$	$\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	$-\frac{7}{4}\sqrt{\frac{1}{6}}$	0	0	$-\frac{1}{4}\sqrt{\frac{1}{3}}$	$-\frac{1}{4}\sqrt{\frac{15}{2}}$	0	8_s
8, 3; 8, 3	0	$\frac{1}{2}\sqrt{\frac{1}{2}}$	$\frac{1}{2}\sqrt{\frac{1}{2}}$	0	0	$\frac{1}{2}\sqrt{3}$	8_a
8, 1; 8, 1	$-\frac{3}{4}\sqrt{\frac{1}{2}}$	0	0	$\frac{3}{4}$	$\frac{1}{4}\sqrt{\frac{5}{2}}$	0	8_s
8, 1; 8, 1	0	$\frac{1}{2}\sqrt{\frac{3}{2}}$	$\frac{1}{2}\sqrt{\frac{3}{2}}$	0	0	$-\frac{1}{2}$	8_a
8, 3; 1, 3	$\frac{1}{4}\sqrt{\frac{5}{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{\frac{5}{6}}$	$-\frac{1}{4}\sqrt{3}$	0	
1, 3; 8, 3	$\frac{1}{4}\sqrt{\frac{5}{3}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{\frac{5}{6}}$	$-\frac{1}{4}\sqrt{3}$	0	

(1, 3)

$\mu_1, \sigma_1; \mu_2, \sigma_2$	(280)	(280)	$(35)_D$	$(35)_F$	$\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	$\frac{1}{3}\sqrt{\frac{1}{2}}$	$\frac{1}{3}\sqrt{\frac{1}{2}}$	0	$\frac{2}{3}\sqrt{2}$	
8, 3; 8, 1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{2}}$	0	
8, 1; 8, 3	$\frac{1}{2}$	$-\frac{1}{2}$	$-\sqrt{\frac{1}{2}}$	0	
1, 3; 1, 3	$\frac{2}{3}$	$\frac{2}{3}$	0	$-\frac{1}{3}$	

TABLE V. (*continued*).

(1, 1)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(405)	(189)	(1)	$\lambda_{\gamma'}$ / μ_γ
8, 3; 8, 3	$\frac{5}{2}\sqrt{\frac{1}{21}}$	$\frac{1}{2}\sqrt{\frac{1}{15}}$	$2\sqrt{\frac{6}{35}}$	
8, 1; 8, 1	$\frac{3}{2}\sqrt{\frac{1}{7}}$	$\frac{3}{2}\sqrt{\frac{1}{5}}$	$-2\sqrt{\frac{2}{35}}$	
1, 3; 1, 3	$2\sqrt{\frac{2}{21}}$	$-2\sqrt{\frac{2}{15}}$	$-\sqrt{\frac{3}{35}}$	

TABLE VI. – *Unitary scalar factors* $\begin{pmatrix} 56 & 35 & \| & \lambda_{\gamma'} \\ \mu_1\sigma_1 & \mu_2\sigma_2 & \| & \mu_\gamma\sigma \end{pmatrix}$ *for the C.G. series.*

$$56 \otimes 35 = \underline{56} + \underline{70} + \underline{1134} + \underline{700}.$$

(35, 6)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	$\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 3	1	

(35, 4)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	(1134)	$\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 3	$\frac{1}{2}\sqrt{\frac{3}{2}}$	$\frac{1}{2}\sqrt{\frac{5}{2}}$	
10, 4; 8, 1	$\frac{1}{2}\sqrt{\frac{5}{2}}$	$-\frac{1}{2}\sqrt{\frac{3}{2}}$	

(35, 2)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(1134)	$\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 3	1	

(27.6)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(1134)	$\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 3	-1	

TABLE VI. (*continued*).

(27, 4)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	$(1134)_1$	$(1134)_2$	$\lambda_{\gamma'} \diagup \mu_\gamma$
10, 4; 8, 3	$-\frac{1}{2}\sqrt{\frac{5}{6}}$	$-\frac{1}{2}\sqrt{\frac{3}{2}}$	$-\frac{1}{2}\sqrt{\frac{5}{3}}$	
10, 4; 8, 1	$\frac{1}{2}\sqrt{\frac{1}{2}}$	$-\frac{1}{2}\sqrt{\frac{5}{2}}$	$\frac{1}{2}$	
8, 2; 8, 3	$-\sqrt{\frac{2}{3}}$	0	$\sqrt{\frac{1}{3}}$	

(27, 2)

$\mu_1, \sigma_2;\ \mu_2, \sigma_2$	(700)	$(1134)_1$	$(1134)_2$	$\lambda_{\gamma'} \diagup \mu_\gamma$
10, 4; 8, 3	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	0	
8, 2; 8, 3	$-\sqrt{\frac{1}{6}}$	$\frac{1}{2}\sqrt{\frac{1}{3}}$	$\frac{1}{2}\sqrt{3}$	
8, 2; 8, 1	$-\sqrt{\frac{1}{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	

(10, 6)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	(1134)	$\lambda_{\gamma'} \diagup \mu_\gamma$
10, 4; 8, 3	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
10, 4; 1, 3	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	

$(\overline{10}, 4)$

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(1134)	$\lambda_{\gamma'} \diagup \mu_\gamma$
8, 2; 8, 3	-1	

Table VI (*continued*).

(10, 4)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	$(1134)_1$	$(1134)_2$	(56)	$\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 3	$\frac{1}{6}\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$\frac{1}{2}\sqrt{\frac{1}{6}}$	$\frac{2}{3}$	
10, 4; 8, 1	$-\frac{1}{2}\sqrt{\frac{5}{6}}$	$-\sqrt{\frac{3}{10}}$	$\frac{3}{2}\sqrt{\frac{1}{10}}$	$2\sqrt{\frac{1}{15}}$	
10, 4; 1, 3	$-\frac{1}{3}\sqrt{2}$	0	$-\sqrt{\frac{2}{3}}$	$\frac{1}{3}$	
8, 2; 8, 3	$\frac{1}{3}\sqrt{5}$	$-\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{1}{15}}$	$\frac{2}{3}\sqrt{\frac{2}{5}}$	

(10, 2)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	$(1134)_1$	$(1134)_2$	(70)	$\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 3	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	0	$\sqrt{\frac{2}{3}}$	
10, 4; 1, 3	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{6}}$	
8, 2; 8, 3	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{2}{3}}$	$-\frac{1}{2}\sqrt{\frac{1}{2}}$	$\frac{1}{2}\sqrt{\frac{1}{6}}$	
8, 2; 8, 1	$\sqrt{\frac{1}{2}}$	0	$\frac{1}{2}\sqrt{\frac{3}{2}}$	$-\frac{1}{2}\sqrt{\frac{1}{2}}$	

TABLE VI (*continued*).

$(\overline{10}, 2)$

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	(1134)	$\lambda_{\gamma'} / \mu_\gamma$
8, 2; 8, 3	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	
8, 2; 8, 1	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	

(1, 4)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(1134)	$\lambda_{\gamma'} / \mu_\gamma$
8, 2; 8, 3	1	

(8,6)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(1134)	$\lambda_{\gamma'} / \mu_\gamma$
10, 4; 8, 3	-1	

(1, 2)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(1134)	(70)	$\lambda_{\gamma'} / \mu_\gamma$
8, 2; 8, 3	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	
8, 2; 8, 1	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	

TABLE VI (*continued*).

(8, 4)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	$(1134)_1$	$(1134)_2$	$(1134)_3$	(70)	λ_γ / μ_γ
10, 4; 8, 3	$-\frac{1}{2}\sqrt{\frac{5}{6}}$	0	$-\frac{1}{2}\sqrt{\frac{15}{14}}$	$-\frac{1}{4}\sqrt{\frac{1}{21}}$	$+\frac{5}{4}\sqrt{\frac{1}{3}}$	
10, 4; 8, 1	$\frac{1}{2}\sqrt{\frac{1}{2}}$	0	$\frac{3}{2}\sqrt{\frac{1}{14}}$	$\frac{3}{4}\sqrt{\frac{5}{7}}$	$+\frac{1}{4}\sqrt{5}$	
8, 2; 8, 3	$-\frac{1}{2}\sqrt{\frac{1}{6}}$	0	$\frac{3}{2}\sqrt{\frac{3}{14}}$	$-\frac{5}{4}\sqrt{\frac{5}{21}}$	$+\frac{1}{4}\sqrt{\frac{5}{3}}$	8_s
8, 2; 8, 3	$-\frac{1}{2}\sqrt{\frac{5}{6}}$	$\sqrt{\frac{2}{3}}$	$\frac{1}{2}\sqrt{\frac{5}{42}}$	$\frac{5}{4}\sqrt{\frac{1}{21}}$	$-\frac{1}{4}\sqrt{\frac{1}{3}}$	8_a
8, 2; 1, 3	$-\frac{1}{2}\sqrt{\frac{5}{3}}$	$-\sqrt{\frac{1}{3}}$	$\frac{1}{2}\sqrt{\frac{5}{21}}$	$\frac{5}{2}\sqrt{\frac{1}{42}}$	$-\frac{1}{2}\sqrt{\frac{1}{6}}$	

(8, 2)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(700)	$(1134)_1$	$(1134)_2$	$(1134)_3$	(70)	(56)	λ_γ / μ_γ
10, 4; 8, 3	$-\frac{1}{3}\sqrt{\frac{1}{2}}$	0	0	$-\frac{1}{2}\sqrt{\frac{1}{3}}$	$+\frac{1}{2}\sqrt{\frac{5}{3}}$	$+\frac{2}{3}$	
8, 2; 8, 3	$\frac{7}{12}$	$-\frac{1}{2}\sqrt{\frac{1}{22}}$	$-2\sqrt{\frac{1}{11}}$	$\frac{1}{4}\sqrt{\frac{1}{6}}$	$-\frac{1}{4}\sqrt{\frac{5}{6}}$	$+\frac{1}{3}\sqrt{2}$	8_s
8, 2; 8, 3	$\frac{1}{12}\sqrt{5}$	$\frac{1}{2}\sqrt{\frac{5}{22}}$	$\frac{9}{2}\sqrt{\frac{1}{55}}$	$-\frac{7}{4}\sqrt{\frac{1}{30}}$	$-\frac{5}{4}\sqrt{\frac{1}{6}}$	$+\frac{2}{3}\sqrt{\frac{2}{5}}$	8_a
8, 2; 8, 1	$-\frac{1}{4}\sqrt{3}$	$-\frac{3}{2}\sqrt{\frac{3}{22}}$	$-\frac{1}{2}\sqrt{\frac{3}{11}}$	$-\frac{3}{4}\sqrt{\frac{1}{2}}$	$-\frac{1}{4}\sqrt{\frac{5}{2}}$	0	8_s
8, 2; 8, 1	$-\frac{1}{4}\sqrt{\frac{5}{3}}$	$-\frac{1}{2}\sqrt{\frac{15}{22}}$	$\sqrt{\frac{3}{55}}$	$\frac{9}{4}\sqrt{\frac{1}{10}}$	$-\frac{1}{4}\sqrt{\frac{1}{2}}$	$+\sqrt{\frac{2}{15}}$	8_a
8, 2; 1, 3	$-\frac{1}{3}\sqrt{\frac{5}{2}}$	$\sqrt{\frac{5}{11}}$	$-2\sqrt{\frac{2}{55}}$	$\frac{1}{2}\sqrt{\frac{1}{15}}$	$-\frac{1}{2}\sqrt{\frac{1}{3}}$	$+\frac{1}{3}\sqrt{\frac{1}{5}}$	

TABLE VII—*Unitary scalar factors* $\begin{pmatrix} 56 & \overline{56} & \| & \lambda_{\gamma'} \\ \mu_1\sigma_1 & \mu_2\sigma_2 & \| & \mu_\gamma\sigma \end{pmatrix}$ *for the C. G. series.*

$$\underline{56}\otimes\underline{\overline{56}} = \underline{1} + \underline{35} + \underline{405} + \underline{2695}.$$

(64, 7)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
10, 4; $\overline{10}$, 4	1

(64, 5)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
10, 4; $\overline{10}$, 4	1

(64, 3)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
10, 4; $\overline{10}$, 4	1

(64, 1)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
10, 4; $\overline{10}$, 4	1

(35,5)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 2	1

(35, 3)

$\mu_1, \sigma_2;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 2	1

($\overline{35}$, 5)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
8, 2; $\overline{10}$, 4	1

($\overline{35}$, 3)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
8, 2; $\overline{10}$, 4	1

(27, 7)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695) $\lambda_{\gamma'}$ / μ_γ
10, 4; $\overline{10}$, 4	1

TABLE VII (cont'd)

(27, 5)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	$(2695)_1$	$(2695)_2$	(405)	$\lambda_{\gamma'}$ / μ_γ
$10, 4;\ \overline{10}, 4$	0	$2\sqrt{\frac{2}{15}}$	$\sqrt{\frac{7}{15}}$	
$10, 4;\ 8, 2$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{7}{30}}$	$-2\sqrt{\frac{1}{15}}$	
$8, 2;\ \overline{10}, 4$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{7}{30}}$	$2\sqrt{\frac{1}{15}}$	

(27, 3)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	$(2695)_1$	$(2695)_2$	$(2695)_3$	(405)	$\lambda_{\gamma'}$ / μ_γ
$10, 4;\ \overline{10}, 4$	0	0	$\frac{1}{3}\sqrt{2}$	$-\frac{1}{3}\sqrt{7}$	
$10, 4;\ 8, 2$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{10}}$	$\frac{1}{3}\sqrt{\frac{14}{5}}$	$+\frac{2}{3}\sqrt{\frac{1}{5}}$	
$8, 2;\ \overline{10}, 4$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$	$-\frac{1}{3}\sqrt{\frac{14}{5}}$	$-\frac{2}{3}\sqrt{\frac{1}{5}}$	
$8, 2;\ 8, 2$	0	$-2\sqrt{\frac{1}{5}}$	$-\frac{1}{3}\sqrt{\frac{7}{5}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	

(27, 1)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	(405)	$\lambda_{\gamma'}$ / μ_γ
$10, 4;\ \overline{10}, 4$	$\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{14}{15}}$	
$8, 2;\ 8, 2$	$-\sqrt{\frac{14}{15}}$	$-\sqrt{\frac{1}{15}}$	

(10, 5)

$\mu_1, \sigma_1;\ \mu_1, \sigma_2$	(2695)	$\lambda_{\gamma'}$ / μ_γ
$10, 4;\ 8, 2$	1	

TABLE VII (cont'd)

(10, 3)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	(405)	$\lambda_{\gamma'}$ / μ_γ
10, 4; 8, 2	$\sqrt{\frac{1}{5}}$	$2\sqrt{\frac{1}{5}}$	
8, 2; 8, 2	$+2\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{1}{5}}$	

(10, 1)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	$\lambda_{\gamma'}$ / μ_γ
8, 2; 8, 2	1	

$(\overline{10}, 5)$

$\mu_1, \sigma_1;\ \mu_1, \sigma_2$	(2695)	$\lambda_{\gamma'}$ / μ_γ
8, 2; $\overline{10}$, 4	-1	

$(\overline{10}, 3)$

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	(405)	$\lambda_{\gamma'}$ / μ_γ
8, 2; $\overline{10}$, 4	$-\sqrt{\frac{1}{5}}$	$-2\sqrt{\frac{1}{5}}$	
8, 2; 8, 2	$-2\sqrt{\frac{1}{5}}$	$+\sqrt{\frac{1}{5}}$	

$(\overline{10}, 1)$

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	$\lambda_{\gamma'}$ / μ_γ
8, 2; 8, 2	-1	

(8, 7)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	$\lambda_{\gamma'}$ / μ_γ
10, 4; $\overline{10}$, 4	1	

TABLE VII (cont'd)

(8, 5)

$\mu_1, \tau_1;\ \mu_2, \sigma_2$	$(2695)_1$	$(2695)_2$	(405)	$\lambda_{\gamma'}$ / μ_γ
$10, 4;\ \overline{10}, 4$	0	$\sqrt{\frac{1}{5}}$	$-2\sqrt{\frac{1}{5}}$	
$10, 4;\ 8, 2$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{2}{5}}$	$+\sqrt{\frac{1}{10}}$	
$8, 2;\ \overline{10}, 4$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{1}{10}}$	

(8, 3)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	$(2695)_1$	$(2695)_2$	$(405)_1$	$(405)_2$	(35)	$\lambda_{\gamma'}$ / μ_γ
$10, 4;\ \overline{10}, 4$	0	$\frac{1}{3}\sqrt{\frac{1}{3}}$	0	$-\sqrt{\frac{1}{2}}$	$-\frac{5}{3}\sqrt{\frac{1}{6}}$	
$10, 4;\ 8, 2$	$-\sqrt{\frac{1}{10}}$	$-\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{5}}$	$+\frac{1}{3}\sqrt{\frac{5}{3}}$	
$8, 2;\ \overline{10}, 4$	$\sqrt{\frac{1}{10}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\sqrt{\frac{1}{2}}$	$+\sqrt{\frac{1}{5}}$	$-\frac{1}{3}\sqrt{\frac{5}{3}}$	
$8, 2;\ 8, 2$	$2\sqrt{\frac{1}{5}}$	$-\frac{1}{3}\sqrt{\frac{1}{15}}$	0	$-\sqrt{\frac{1}{10}}$	$+\frac{1}{3}\sqrt{\frac{5}{6}}$	8_s
$8, 2;\ 8, 2$	0	$-\frac{5}{3}\sqrt{\frac{1}{3}}$	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	8_a

TABLE VII (cont'd)

(8, 1)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	(405)	(35)	$\lambda_{\gamma'} \diagup \mu_\gamma$
$10, 4;\ \overline{10}, 4$	$\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{5}{6}}$	
$8, 2;\ \ 8, 2$	$\sqrt{\frac{3}{5}}$	$+\sqrt{\frac{2}{5}}$	0	8_s
$8, 2;\ \ 8, 2$	$-\sqrt{\frac{1}{3}}$	$+\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{6}}$	8^a

(1, 7)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	$\lambda_{\gamma'} \diagup \mu_\gamma$
$10, 4;\ \overline{10}, 4$	1	

(1, 5)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(405)	$\lambda_{\gamma'} \diagup \mu_\gamma$
$10, 4;\ \overline{10}, 4$	1	

(1, 3)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(2695)	(35)	$\lambda_{\gamma'} \diagup \mu_\gamma$
$10, 4;\ \overline{10}, 4$	$\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{5}{3}\sqrt{\frac{1}{3}}$	
$8, 2;\ \ 8, 2$	$-\frac{5}{3}\sqrt{\frac{1}{3}}$	$\frac{1}{3}\sqrt{\frac{2}{3}}$	

(1, 1)

$\mu_1, \sigma_1;\ \mu_2, \sigma_2$	(405)	(1)	$\lambda_{\gamma'} \diagup \mu_\gamma$
$10, 4;\ \overline{10}, 4$	$-\sqrt{\frac{2}{7}}$	$\sqrt{\frac{5}{7}}$	
$8, 2;\ \ 8, 2$	$+\sqrt{\frac{5}{7}}$	$\sqrt{\frac{2}{7}}$	

CHAPTER 4

SU(6) AND W SPIN SYMMETRIES

This chapter contains a discussion of the $SU(6)_W$ higher symmetry group; noting that for 4 body processes with one notable (but unreliable see § 4.2) exception the group is poorly favoured by experiment, we investigate the consequences of relaxing the system by allowing some specified symmetry breaking. In particular we discuss the Johnson-Treiman relation and the $\Delta W \neq 2$ selection rule - we use tensor methods.

4.1 W-spin and higher symmetries

The $SU(6)_W$ scheme first materialised[29] as a subgroup of the higher group SU(6,6) or $\tilde{U}(12)$[30]. The origins of this relativistic system and in particular the use of SU(6,6) rather than the minimal (with respect to SU(3)) group SL(6,C) are discussed in refs.[31]. The commutation relations appropriate to SU(6,6) can be obtained, on adopting the sixteen 4 x 4 matrices, Γr, of the Dirac algebra as a basis for the fundamental IR of the U(2,2) subgroup, by constructing the fundamental twelve dimensional representation as a direct product $\Gamma r \times \lambda^{\alpha}$ exactly as in § 3.1.

It was first proposed[30] to use finite dimensional representations of SU(6,6) to accommodate physical particles, the representations then became unitary representations of an inhomogeneous SU(6,6), ISU(6,6), which was the semi-direct product of SU(6,6) and a space of 143 commuting 'translation' operators. The complete structure, including the SU(6,6) field equations ensuring a positive definite norm and defining independent particle states was closely analogous to that used in defining irreducible unitary representations of the Poincare group via the Lorentz group. An essential difference however lay in the occurrence of surplus momenta, 139 in total, which would be needed for writing invariant equations, and amplitudes, but for which there was neither physical evidence nor interpretation. Denoting an SU(6,6) 12 x 12 group element by S and $P_A^B = \sum_{r,\alpha} P_{r\alpha}(\Gamma_{r\alpha})_A^B$, $\Gamma_{r\alpha} = \Gamma_r \times \lambda^\alpha$, then the only physical momenta are $P_{\mu o}$ and one must consequently limit the allowed symmetry transformations to those which do not transform to an unphysical realm :

$$S \not{P} S^{-1} = \not{P}' \qquad \not{P} = P_\mu \gamma_\mu$$

and P' can be obtained from $\underline{P}$ by a Lorentz transformation.

Indeed the view point first adopted (e.g. first two papers of ref.[30]) was to ignore momentum completely - the SU(6,6) matrices merely transformed the field indices; this reflected the static SU(6) situation where spin was supposed completely decoupled from the orbital motion for a spin-$\frac{1}{2}$ particle. This occurs only for free particles and is demonstrated by the Foldy-Wouthuysen transformation. In this chapter our description will apply to the case in which the physical Poincaré group is a subgroup of ISU(6,6); we shall not discuss the alternative approaches (see also [23]) employing infinite dimensional unitary representations of a homogeneous higher symmetry group, cf. Ruegg et al[21], Fronsdal[17; 32].

We proceed now to exploit the analogy with the Poincaré group: single particle states are classified according to the appropriate 'little group' of ISU(6,6). The little group of a momentum vector P_A^B is that subgroup of ISU(6,6) for which $SPS^{-1}=P$ (i.e. transformations of the little group do not change the reference frame). For massive one particle states we can choose as 'standard vector' the usual rest frame four-momentum $(m,\underset{\sim}{0})$. The little group then satisfies $S\gamma_o S^{-1}=\gamma_o$ and can easily be located as, in γ terminology, $\frac{1}{2}(1\pm\gamma_o) \times G_a^b$ where G_a^b generate a subgroup of SU(6,6) which plays the role of, and is isomorphic to, non-relativistic $SU(6)_\sigma$. The little group is thus S(U(6) x U(6)) - it gives the space time degeneracy of an ISU(6,6) multiplet; the field equations are designed to preserve this degeneracy for moving one particle state. The corresponding little group in P, the Poincaré group, is of course SU(2).

For systems composed of two particles the little group, relating to the total four-momentum, will not be the same as that of the separate particles (although there may be isomorphism). It is clear that we can choose a frame (one in which the two 3-momenta are collinear, conventionally the '3' direction,) when the intersection of the two separate little groups is given by those S for which $S\gamma_o S^{-1}=\gamma_o$, $S\gamma_3 S^{-1}=\gamma_3$. This subgroup of U(6) x U(6) is $SU(6)_W$; under $SU(6)_W \longrightarrow SU(3) \times SU(2)_W$ the generators of the SU(3) are those of the physical SU(3) viz λ^α, whilst those of $SU(2)_W$ are, in γ terminology,

$W_1 = \frac{1}{2}i\gamma_o\gamma_2\gamma_3$, $W_2 = \frac{1}{2}i\gamma_o\gamma_3\gamma_1$, $W_3 = \frac{1}{2}i\gamma_1\gamma_2$, with $[W_i, W_j] = iE_{ijk}W_k$

The important property of $SU(6)_W$, and that which lead to its discovery[29], is that all the generators commute with the generator of Lorentz transformations in the collinear, '3' direction; this follows from

$$[\lambda^\alpha, \gamma_o\gamma_3] = 0 = [\underset{\sim}{W}, \gamma_o\gamma_3] \qquad \alpha = 0, .. 8$$

where $\frac{1}{2}i\gamma_o\gamma_3$ is the Lorentz generator. This property distinguishes $SU(6)_W$ from non-relativistic SU(6) , which contains SU(2) with generators $\frac{1}{2}i\gamma_2\gamma_3$, $\frac{1}{2}i\gamma_3\gamma_1$, $\frac{1}{2}i\gamma_1\gamma_2$, such that only $[S_3, \gamma_o\gamma_3] = 0$, and has two important consequences:

(i) Since the generators of $SU(6)_W$ are unchanged for arbitrary motion in the '3' direction we may couple in the usual direct product way the representations describing two particles in an arbitrary collinear state of motion. Such freedom does not exist for little groups, of P, where the coupling of two particles with spin is complicated by the occurrence of orbital angular momentum, or equivalently, the generators of the two little groups do not coincide in P. It is also evident that the matrix elements of $SU(6)_W$ generators satisfy the following equation:

$$\langle \lambda \nu \underset{\sim}{o} | G_W{}_a{}^b | \underset{\sim}{\lambda} \nu' o \rangle = \langle \lambda \nu p | G_W{}_a{}^b | \underset{\sim}{\lambda} \nu' p' \rangle \qquad 4.1$$

where $(\lambda \nu)$ $(\lambda' \nu')$ refer to $SU(6)_W$ labels of a state and p , p' , are two collinear momenta obtained by a given boost in the '3' direction. $G_W{}_a{}^b$ is any generator of $SU(6)_W$.

(ii) The application of a simpler symmetry in special cases leads to immediate, simpler, tests of the theory - assigning particles to irreducible representations of $SU(6)_W$ we can apply this group to a study of the vertex function, a function of two independent momenta, or to the case of forward or backward two body scattering processes, where again the one $SU(6)_W$ group will be relevant to both initial and final states; (the general procedure might be to be expand the amplitude in terms of U(6) x U(6) partial waves). It is the second example, that of scattering, which concerns us here.

We can again find an analogy in the Poincare group; there, for collinear motion, the intersection of the little groups of two massive particles is O_2, generated by J_3, the '3' component of total angular momentum. In a given frame we may classify states according to irreducible (one dimensional) unitary representations of O_2, labelled by m. However, under a space rotation the m value will change, according to the usual rotation matrices,

$$|jm\rangle \longrightarrow \sum_{m'} \mathcal{D}^{j}_{m'm} \; |jm'\rangle \qquad 4.2$$

so that a state $|jm\rangle$ belonging to the IR $\underline{m}$ of O_2 under rotation in general becomes a reducible sum of IRs $\underline{m}'$. O_2 invariance for the four point function implies of course J_3 conservation, and this holds for any direction of motion, but only in the collinear case are J_3 IR assignments invariant. Notice also that there is no conflict with unitarity which would relate the collinear scattering amplitude to a product of non collinear amplitudes through the symbolic equation:

$$\mathcal{I}_m \, T_{fi} \;=\; \sum_n T^*_{nf} \, T_{ni} \qquad 4.3$$

We prefer then to adopt this point of view of unitarity in $SU(6)_W$; this is similar to that of Pais[31] who considers it a 'matter of language whether or not $SU(6)_W$ is compatible with unitarity. The group makes no claims concerning non forward directions....'. But it should be noted that in the $SU(6)_W$ case the unit operator $\sum_n |n\rangle\langle n|$ occurring in the unitarity relation, viewed as a unit operator in ISU(6,6) is distorted due to the limitation of the sum $\sum_n$ to physical momenta.

Continuing with our discussion of $SU(6)_W$ we now briefly relate $SU(2)_W$ and $SU(2)$:

The maximal compact subgroup of SU(2,2) is SU(2) x SU(2), locally isomorphic to O(4), and contains as subgroups both SU(2) and $SU(2)_W$:

Group	Generators	
SU(2) x SU(2)	$\frac{1}{4}(1+\gamma_o)i\gamma_i\gamma_j \times \frac{1}{4}(1-\gamma_o)i\gamma_l\gamma_m \quad i,j,l,m=1,\ldots,3$	
	$\sim \quad \underset{\sim}{M} \times \underset{\sim}{N}$	4.4a
$SU(2)_s$	$\frac{1}{2}i\gamma_i\gamma_j$	4.4b
	$\underset{\sim}{M} + \underset{\sim}{N}$	
$SU(2)_W$	$\frac{1}{2}i\gamma_o\gamma_2\gamma_3, \quad \frac{1}{2}i\gamma_o\gamma_3\gamma_1, \quad \frac{1}{2}i\gamma_1\gamma_2$	4.4c

We see that

$$S_3 = W_3 = M_3 + N_3 \tag{4.5a}$$

$$S_{\pm} = M_{\pm} + N_{\pm} \tag{4.5b}$$

$$W_{\pm} = M_{\pm} - N_{\pm} \tag{4.5c}$$

The finite dimensional unitary irreducible representations of SU(2) x SU(2) are labelled by pairs of non negative integers and half integers (m,n) corresponding to the two commuting SU(2)'s $\underset{\sim}{M}$, $\underset{\sim}{N}$. As expalined by Lipkin[17] quarks (antiquarks) transform solely under $\underset{\sim}{M}(\underset{\sim}{N})$ - this is decided by the appearance of $\frac{1}{2}(1\pm\gamma_o)$ as a positive or negative energy projection operator. The S spin content of (m,n) is clearly (from eq.4.4b) s = m+n, m+n-1,.. m-n , and from eq. 4.5a it follows that this must also be the W-spin content.

From eqs. 4.5b,c, we see that for IRs of the form (m,o) there is no distinction between S and W spin eigenstates, but for IR's of the type (o,n) we have, under subgroup reductions,

$$|S_1 S_3\rangle = (-1)^{n-n_3} \; |W_1 W_3\rangle \qquad 4.6$$

where $S = W = n$ $S_3 = W_3 = n_3$ and we have arbitrarily chosen an overall phase by setting $|S,n\rangle = +|W,n\rangle$ (as usual $|\mu,\nu\rangle$ denotes + basis vector for the vector ν of the IR $\underline{\mu}$ of the group specified).

For a general IR (m,n) we may now write

$$
\begin{aligned}
|S_1 S_3\rangle &= \sum_{n_3} \begin{pmatrix} m & n & s \\ S_3-n_3 & n_3 & S_3 \end{pmatrix} |m,S_3-n_3\rangle\, |n_1 n_3\rangle \\
&= \sum_{n_3} (-1)^{n-n_3} \begin{pmatrix} m & n & S \\ S_3-n_3 & n_3 & S_3 \end{pmatrix} |m,S_3-n_3\rangle \left((-1)^{n-n_3} |n_1 n_3\rangle\right) \\
&= \sum_{W_1 n_3} (-1)^{n-n_3} \begin{pmatrix} m & n & S \\ S_3-n_3 & n_3 & S_3 \end{pmatrix} \begin{pmatrix} m & n & W \\ S_3-n_3 & n_3 & W_3=S_3 \end{pmatrix} |W,W_3\rangle
\end{aligned}
\qquad 4.7
$$

and a similar equation for $|W_1 W_3\rangle$

As an example we consider the $|1,0\rangle$ and $|0,0\rangle$ subgroup states of the IR$(\frac{1}{2},\frac{1}{2})$, corresponding to the quark-antiquark composite; by straight substitution in eq. 4.7 we obtain :-

S spin state		W spin state
$\|1,1\rangle$	$\equiv$	$\|1,1\rangle$
$\|1,-1\rangle$	$\equiv$	$-\|1,-1\rangle$
$\|1,0\rangle$	$\equiv$	$-\|0,0\rangle$
$\|0,0\rangle$	$\equiv$	$-\|1,0\rangle$

Further discussion of the relation between S and W spin occurs in[34,35] - the latter also arrives at a relation of the form eq. 4.7.

It should also be noted that it has subsequently been possible to arrive at the concept of W spin invariance as deriving from rotation and inversion invariance[36] - the only new contribution arises from transitions which are forbidden by $\Delta W=0$ (mod.2)[37] i.e. a process forbidden under W-spin invariance by $\Delta W=2$ would not be forbidden by rotation and purity invariance alone. Further discussion of this specific higher symmetry prediction is given in Ch. 4.4.

4.2 W-spin symmetry breaking

Our interest in W-spin and $SU(6)_W$ is to test the stability of some of the predictions of this group for the 4 point function under various modes of symmetry breaking. Here we investigate the type of symmetry breaking needed and also our method of calculation.

The computation of exact symmetry predictions may be thought to proceed by constructing all possible 'Lagrangian' terms relevant to the process under consideration and invariant under the given symmetry. For mass terms, e.g. exact symmetry predictions result from the scalar term in $\underline{\mu} \otimes \bar{\underline{\mu}}$ and this will always give equal masses - similarly for higher n-point functions where now generally more than one scalar term exists. In group-theoretic language we apply the Wigner-Eckart theorem to a scalar operator. In a broken symmetry one allows specific non-scalar transofrmation properties to the Lagrangian - either we may use the Wigner-Eckart theorem, or, as is more customary, we may introduce a 'spurion' and construct again scalar Lagrangians only one term of which will be physical. The question of whether the same spurion should apply to different n-point functions has been investigated for SU(3) e.g. in the work of Dashen and Frautschi[38] for n=3 and 4 - for SU(3) there is ample evidence that I=Y=0 $\underline{\mu}=8$ transformation properties give significant and dominant symmetry breaking contributions to processes with n=2,3,4,5 (for n=5 see e.g. [39]). Making the assumption of n-independence the easiest way to search for symmetry breaking terms is to try to fit n=2 mass terms with the various allowed spurions - since it is the mass terms which give the clearest and most accessible indications of symmetry breaking, although of course in the SU(6) scheme they supposedly refer to $SU(6)_{\sigma}$ rather than $SU(6)_{W}$.

A number of people have considered mass formulae in SU(6) - two papers especially relevant to our work are those by Harari and Lipkin, and Harari and Rashid[40] who conclude:

(i) It is not possible to fit the observed baryon and meson mass spectra with the same mass operator in each case. In particular the octet I=Y=0 part of an SU(6) $\underline{405}$ tensor is required in the baryon but not in the meson case.

(ii) The major contribution to SU(3) symmetry breaking in both casos comes from an I=Y=0 octet component of a 35.

(iii) Terms which break SU(6) but not SU(3) symmetry may be as important as those which break both symmetries.

Indeed, their considerations applied without adaption to baryon-meson scattering suggest that four types of spurion should be invoked viz (8,1) components in 35 and 405 and (1,1) components in 189 and 405. A moments reflection suggests that a completely general investigation would not be sufficiently predictive to warrant the labour necessitated - as a first investigation we consider the effect of 35 only spurions on some specific processes. Further in $SU(6)_W$ there does not seem to be any reason to consider only W=0 spurions - whereas in SU(6) J=0 was essential - in the following we use at different times three sorts of spurion :-

9 SU(6)	$SU(3) \times SU(2)_W$	I	W_3	
35	(8,1)	0	0	4.8a
35	(1,3)	0	0	4.8b
35	(8,3)	0	0	4.8c

There is one further form of symmetry breaking which we investigate below: SU(6) evolved by combining SU(3) and SU(2) in a minimal way; however SU(3) itself is not nearly as well satisfied as the isospin-hypercharge symmetry group $SU(2) \times U(1)$. Thus in testing the validity of combining a purely internal symmetry [SU(3)] with a space time symmetry [$SU(2)_{\sigma}$ or $SU(2)_W$] we should (in the face of bad predictions) try the effect of combination at the SU(2) x U(1) level i.e. we should investigate the consequences of exact $SU(4)_W \times SU(2)_W$ symmetry. This bears some similarity to an SU(6) $\lambda^8 \times \delta$ spurion. It is as well to note however that $SU(4)_W \times SU(2)_W$ is a subgroup of $SU(6)_W$ and therefore cannot deliver results which flatly disagree with the predictions of the larger group. In the following we shall be concerned to avoid a bad $SU(6)_W$ prediction for the ratio of two forward scattering amplitudes – it would seem that we may not expect $SU(4)_W \times SU(2)_W$ to predict a different ratio but at best that it may no longer be possible to form a ratio.

Finally the mode of calculation: the introduction of symmetry breaking via a spurion (or any other method) means we require CG tables sufficient for the direct product of five SU(6) IRs. The tables of Ch. 3 are not enough and instead we resort to tensor methods – in our opinion in any case more suited to calculations involving more than three SU(6) IRs. Thus we represent the meson $SU(6)_W$ $\underline{35}$ tensor as :-

$$M_A^B = \frac{1}{\sqrt{2}}\left[\delta_\alpha^\beta M_a^b + \frac{1}{\sqrt{2}}\,\sigma_\alpha^\beta\left(\lambda_a^{ob} W + V_a^b\right)\right] \qquad 4.9$$

where M,W,V represent (8,1), (1,3), (8,3) components of the $\underline{35}$ and

$$(V_0)^b_a = \begin{pmatrix} \sqrt{\tfrac{1}{6}}\eta + \sqrt{\tfrac{1}{2}}\,\pi^o & \pi^+ & K^+ \\ \pi^- & \sqrt{\tfrac{1}{6}}\eta - \sqrt{\tfrac{1}{2}}\,\pi^o & K^o \\ K^- & & -2\sqrt{\tfrac{1}{6}}\eta \end{pmatrix} \qquad 4.10$$

according to the W-S spin slip.

(We could note that a phase convention, additional to those at the SU(3) x SU(2) level, and corresponding to that discussed in Ch. 2.2,4 has been arbitrarily chosen here by fixing on relative plus signs between M, W and V; clearly any relative sign is allowed.)

Similarly one may write down the tensor wave function of the 56 (for a fairly complete tabulation of tensor wave functions see Ruegg et al[21]).

We represent our symmetry breaking by components $\delta^{\beta}_{\alpha}\lambda^{8b}_{a'}$, $\sigma^{3\beta}_{a}\delta^{b}_{a'}$, $\sigma^{\beta}_{3\alpha}\lambda^{8b}_{a}$ of the spurion tensor S^{B}_{A} for the three possibilities (8,1), (1,3), (8,3) respectively. The calculation then involves evaluating tensor contractions - although it will be seen that in some cases 'short cuts' do exist.

4.3 The Johnson-Treiman relations

An early success of the spin containing higher symmetries was the prediction of the following relation for the differences in total cross-sections for the scattering of pseudoscalar mesons on proton targets:

$$\Delta p\pi^{+} = \Delta pk^{o} = \tfrac{1}{2}\,\Delta pk^{+} \qquad 4.11$$

where $\Delta pm = \sigma_{tot}(p+m \longrightarrow p+m) - \sigma_{tot}(p+\bar{m} \longrightarrow p+\bar{m})$

p = proton, m = ps.meson.

These equations have come to be known as the Johnson-Treiman relations[42]. They were first obtained in static $SU(6)_{\sigma}$.

Using our tables in the direct channel for

$$\underline{56} + \underline{35} \longrightarrow \underline{56} + \underline{35}$$

since the IR $\underline{1}$ occurs four times in $\underline{56} \otimes \underline{35} \otimes \underline{56} \otimes \underline{35}$ there are four independent amplitudes or reduced matrix elements $A_{\lambda} = \langle \lambda \| T \| \lambda \rangle$ with $\lambda = \underline{56}, \underline{70}, \underline{1134}$ or $\underline{700}$ corresponding to $\underline{56} \otimes \underline{35} = \underline{56} \oplus \underline{70} \oplus \underline{1134} \oplus \underline{700}$. The Johnson-Treiman relations in $SU(6)_{\sigma}$ then follow via the optical theorem since the amplitude differences Δpm given in eqn. 4.11 are proportional to

$$A = 5/18\, A_{700} - \frac{3}{20}\, A_{1134} - \frac{1}{12}\, A_{70} - \frac{2}{45}\, A_{56}$$

In fact there is a simpler way to arrive at eqn. 4.11 which we shall describe and exploit below.

Since the same relation also holds in the W-spin formalism doubts about the applicability of $SU(6)_{\sigma}$ were relieved - the relation was first checked for incident meson momentum in the range 5-20 Bev/c. In any case there was a tendency for this prediction to be accepted as important evidence in favour of the $SU(6)_W$ and SU(6,6) symmetries. However the following points must be emphasized:

(i) The Johnson-Treiman relations can be derived in other models viz (a) exact SU(3) symmetry plus dominance of the meson baryon scattering amplitude at high energies by a purely F coupled vector meson Ruegge trajectory[43], (b) the quark model (SU(3) invariance is not assumed here)[44]. On the basis of either of these models it is perhaps surprising that eqn.4.11 hold (roughly) down to energies $\sim$ 10 Bev/c incident meson lab momentum. (But see below for comparison with experimental data.)

(ii) It has been very clearly emphasized by Harari[17] that any evaluation of the predictions of SU(3) containing symmetry schemes must allow breaking of SU(3). Further in the case of the baryon-meson system departures from exact symmetry may be as high as 20-30%. For eqn.4.11 Harari finds that exact SU(3) plus experimental information that in-elastic processes are small implies that $\Delta pm=0$ (experimentally $\Delta pm \sim 5$ mb) - the simplest way out of this is to conjecture that the SU(3) symmetry breaking is confined to the in-elastic amplitudes[45].

(iii) We have argued briefly above that $SU(6)_W$ may not flaunt unitarity as blatantly as has been suggested - however the simplest physical interpretation of the group theory, that one particle vector meson exchange (producing no conflict with unitarity) is obviously inadequate since such amplitudes have zero imaginary part. (This is not so for the Regge model where e.g. the ρ trajectory can produce an imaginary part.) It has been shown[45], by rather involved argument that the inclusion of two particle intermediate states (i.e. two intermediate $\underline{35}$'s in the crossed channel) does not affect the prediction if one assumes each 3 particle vertex invariant under its own $SU(6)_W$ - so that consistency with unitarity exists to a higher degree than is suggested by superficial examination. Notice that one might hope to pick up this result with a W-spin breaking spurion since non collinear 2 particle states are breaking W-spin conservation. We find below for a simple $W = 1$ $\underline{35}$ spurion this hope is not realised.

(iv) The comparison with experiment has been made in a number of places, Ruegg et al[21, 44, 46]. Quite apart from the difficulties of correlating symmetry predictions with experimental data the conclusion reached here depends upon the mode of comparison. The (pm) cross-sections themselves are of the order of 20-30 millibars their differences about 15% of this. Naturally if the differences are compared directly the agreement seems poorer[46] than if we rewrite the relations in terms of sums,[21,44]. Anyway the best possible figure seems to be about 3% departure from eqn. 4.11

(v) It is generally agreed that the relation

$$\Delta pk^{+} = \Delta p\pi^{+} + \Delta pk^{0} \qquad 4.12$$

is always better satisfied than eqn. 4.11. The above relation results from SU(3) alone on the assumption of octet dominance in the annihilation channel[47].

With (ii) above in mind we now compute the effects of 35 type symmetry breaking on the Johnson-Tr iman relations. We denote the spurion by S and find that there are eighteen different ways of forming an $SU(6)_W$ scalar from $\bar{B}$, B, M_1, M_2 and S by saturation of tensor indices; this agrees with the number calculated directly by comparing terms appearing in the direct products $\underline{56} \otimes \underline{56}$ and $\underline{35} \otimes \underline{35} \otimes \underline{35}$. (The higher direct product reductions needed for this are now tabulated in Ruegg et al[21].) The general amplitude now has the form $(\bar{B}BM_1M_2)_A^B$ where we suppose M_1 and B to absorb the initial meson and baryon, and M_2 abd $\bar{B}$ to create the final states, and the indices A, B depend on S. We can represent the effect of time reversal, T, on these amplitudes by $B_{ABC} \longleftrightarrow \bar{B}^{ABC}$, M_{1D}^{E} —— M_{2E}^{D} corresponding to the interchange of initial and final particles and their creation and annihilation operators. Notice that the transformation also changes momenta and spins so that one must take care in rejecting T antisymmetric combinations in $SU(6)_W$ space that T antisymmetric combinations in spin space may not be formed, cf.[48]. For the baryon meson system this is the case. Note also that we automatically have parity invariance.

In this way we find the following twelve amplitudes must be considered for baryon-meson scattering subject to 35-like symmetry breaking. (The number twelve compares with that of thrity-four which result - after time reversal invariance - for the same system in SU(3) with I=Y=0 octet symmetry breaking[49].

$$A : \bar{B}^{ABC} B_{DEF} M1_A^D \, M2_B^E \, S_C^F$$

$$B\pm : \bar{B}^{ABP} B_{CDP} M1_A^C \, [M_2 S]_{\pm B}^{\ D} + \bar{B}^{ABP} B_{CDP} M2_A^C \, [SM_1]_{\pm B}^{\ D}$$

$$C : \bar{B}^{ABP} B_{CDP} S_A^C [M_1 M_2]_B^D$$

$$D\pm : \bar{B}^{ABC} B_{DBC} ([M_1 M_2]_\pm S + S[M_1 M_2]_\pm)_A^D$$

$$E\pm : \bar{B}^{ABC} B_{DBC} (M_1 S M_2 \pm M_2 S M_1)_A^D$$

$$F : \bar{B}^{ABC} B_{DBC} [(M_1 S) M_{2A}^{\ D} + (M_2 S) M1_A^D]$$

$$G\pm : \bar{B}^{ABC} B_{ABC} [(M_1 M_2 S) \pm (M_2 M_1 S)]$$

$$H : \bar{B}^{ABC} B_{DBC} S_A^D \, (M_1 M_2)$$

where, $[M_1 M_2]_{\pm A}^{\ B} = M1_A^C \, M2_C^B \pm M2_A^C \, M1_C^B$

$$(M_1 M_2) = M1_A^B \, M2_B^A$$

By inspection we see of these twelve all except three A, B$\pm$, contain either the meson or baryon tensors coupled into a <u>35</u> or <u>1</u>, and also that only B-, D-, E-, G- are antisymmetric in M_1 and M_2.

But we now notice that only antisymmetric (in M_1 and M_2) amplitudes can contribute to Δpm,[43] since by our convention a term $\bar{p}p\pi_1^+\pi_2^-$ will contribute to $p+\pi^+ \longrightarrow p+\pi^+$ whereas $\bar{p}p\pi_1^-\pi_2^+$ contributes to $p+\pi^- \longrightarrow p+\pi^-$, we look now at the four antisymmetric terms.

Of these D- and E- have the factor $(\bar{B}B)^A_B$, and using the octet part of the <u>56</u> tensor

$$B_{ABC} \sim E_{abr}\ E_{\alpha\beta}\ N^r_{c\gamma} + \text{cycle } (abc,\ \alpha\beta\gamma) \qquad 4.13a$$

we find

$(\bar{B}B)^A_B$ (octet-octet parts)

$$(-2(N\bar{N})^a_b - 4(\bar{N}N)^a_b)\delta^\alpha_\beta + \delta^a_b\ (4(\bar{N}N)\ \delta^\alpha_\beta - 2\bar{N}^\alpha N_\beta)$$

$$+ 2(\,(\bar{N}N)^{a\alpha}_{b\beta} + 5(N\bar{N})^{a\alpha}_{b\beta}) \qquad 4.13b$$

(extracting traces via

$$\bar{N}N^{a\alpha}_{b\beta} = (\widetilde{\bar{N}N})^{a\alpha}_{b\beta} + \frac{1}{3}\delta^a_b(\widetilde{\bar{N}N})^\alpha_\beta + \frac{1}{2}(\widetilde{\bar{N}N})^a_b\,\delta^\alpha_\beta + \frac{1}{6}\delta^A_B\ (\bar{N}N)$$

where $\sim$ signifies zero trace on the free indices, we can arrive at the familiar results $\rightarrow$ D + $\frac{2}{3}$F coupling for $(\widetilde{N\bar{N}})^{a\alpha}_{b\beta}$ and pure F coupling for $(\widetilde{\bar{N}N})^a_b\,\delta^\alpha_\beta$.)

Similarly using the W spin identification of the pseudoscalar mesons we extract from M_1, M_2 factors $M_{1a}^{\ b}\ \sigma_{3\alpha}^{\ \beta}$, $M_{2a}^{\ b}\ \sigma_{3\alpha}^{\ \beta}$ (ab,αβ always label respectively, SU(3) and SU(2) vectors). Corresponding to our three possible spurions there are now two cases to consider

(i) the W=0 spurion gives a factor λ, as also does $(\sigma_3)^2$ from M_1, M_2 so the overall meson W spin factor is δ^{α}_{β} and this selects the W spin singlet (pure F) term from $\bar{B}B$.

(ii) for w=1 $(\sigma_3)^3 = \sigma_3$ and we get the vector part of $\bar{B}B$; hence in this case the SU(3) coupling does not have the form $[\bar{N}N]_-[M_1M_2]_-$ cf (i) but only $(\alpha[\bar{N}N]_+ + \beta[\bar{N}N]_-$ and so only the weak form, (v) above, of the Johnson-Treiman relation will result from D- and E- whilst G- gives no contribution.

Inserting now the λ^8 factor the following three types of SU(3) amplitude can occur in D-, E-, G- :

SU(3) amplitude	$\Delta p\pi^+$	Δpk^+	Δpk^0
$(N\bar{N})[(M_1M_2\lambda^8) - (M_2M_1\lambda^8)]$	0	6	6
$(N\bar{N}M_1\lambda^8M_2) - (N\bar{N}M_2\lambda^8M_1)$	2	-4	0
$(\bar{N}NM_1\lambda^8M_2) - (\bar{N}NM_2\lambda^8M_1)$	0	-2	-2

From the above ab ve it is clear that no linear combination of the three (SU(3)) amplitudes can give non-trivial ($\Delta pm \neq 0$) Johnson-Treiman relations, and this then follows also for D-, E-, G-.

We discuss the amplitude B- in more detail: expand the tensor wave function

$$\phi \equiv \bar{B}^{ABP} B_{CDP} M_1{}^C_A M_2{}^E_B M_3{}^D_E$$

retaining only spin-$\frac{1}{2}$ baryon - spin $\frac{1}{2}$ baryon terms.

$$
\begin{aligned}
\phi = {} & \left[(\bar{N}N)\{(M_1)(M_2M_3) - (M_1M_2M_3)\} \right] \times \left[\begin{array}{l} (\bar{N}N)(M_1)(M_2M_3) + (\bar{N}NM_2M_3M_1) + (\bar{N}NM_1M_2M_3) \\ -(\bar{N}N)(M_1M_2M_3) - (\bar{N}NM_1)(M_2M_3) - (\bar{N}NM_2M_3)(M_1) \end{array} \right] \\
& + \left[(\bar{N}M_2M_3M_1N) - (\bar{N}M_1N)(M_2M_3) \right] \times \left[\begin{array}{l} (\bar{N}M_2M_3M_1N) + (\bar{N}M_1NM_2M_3) \\ -(\bar{N}M_1N)(M_2M_3) - (\bar{N}M_2M_3)(NM_1) \end{array} \right] \\
& + \left[(\bar{N}M_1M_2M_3N) - (\bar{N}M_2M_3N)(M_1) \right] \times \left[\begin{array}{l} (\bar{N}M_1M_2M_3N) + (\bar{N}M_2M_3NM_1) \\ -(\bar{N}M_2M_3N)(M_1) - (\bar{N}M_1)(NM_2M_3) \end{array} \right] \\
& + \left[(\bar{N}M_1M_2M_3N) - (\bar{N}M_1N)(M_2M_3) \right] \times \left[\begin{array}{l} (\bar{N}M_1M_2M_3N) + (\bar{N}M_1NM_2M_3) \\ -(\bar{N}M_1N)(M_2M_3) - (\bar{N}M_1)(NM_2M_3) \end{array} \right] \\
& + \left[(\bar{N}M_1N)(M_2M_3) \right] \times \left[(\bar{N}M_1N)(M_2M_3) - (\bar{N}M_1NM_2M_3) \right] \\
& + \left[(\bar{N}M_1M_2M_3N) \right] \times \left[(\bar{N}M_1M_2M_3N) - (\bar{N}M_1)(NM_2M_3) \right] \\
& + \left[(\bar{N}M_2M_3M_1N) - (\bar{N}M_2M_3N)(M_1) \right] \times \left[\begin{array}{l} (\bar{N}M_2M_3M_1N) + (\bar{N}M_2M_3NM_1) \\ -(\bar{N}M_2M_3N)(M_1) - (\bar{N}M_2M_3)(NM_1) \end{array} \right] \\
& + \left[(\bar{N}M_2M_3M_1N) \right] \times \left[(\bar{N}M_2M_3M_1N) - (\bar{N}M_2M_3)(NM_1) \right] \\
& + \left[(\bar{N}M_2M_3N)(M_1) \right] \times \left[(\bar{N}M_2M_3N)(M_1) - (\bar{N}M_2M_3NM_1) \right]
\end{aligned}
$$

The above exhibits the $SU(2)_W \times SU(3)$ structure of the SU(6) scalar amplitude ϕ explicitly - the SU(2) factor is always written first, and evey factor is a tensor trace.

We now arrive at the predictions of B- by permuting on the meson labels 1,2,3, and so establish in a straightforward way that the spurions 48b,c give no contribution to Δpm, $M = \pi^+, K^+, K^0$ whilst 4.8a contributes to Δpk^+ only so that in this case neither relation holds.

We conclude ;

(i) Non trivial Johnson-Treiman relations do not survive under symmetry breaking of any of the three types listed above.

(ii) Their weak form[47] and (v) above does hold with a $W=1$ SU(3) singlet $\underline{35}$ spurion.

4.4 $\Delta W \neq 2$ Selection rule

We now turn to some predictions of exact $SU(6)_W$ first listed in [50] which were soon shown to be in gross contradiction with experiment[51].

In ref.[50] it was shown how a number of processes of the form baryon+meson $\longrightarrow$ baryon+meson proceed via just one $SU(6)_W$ amplitude. We may easily locate these processes by looking in the 'crossed channel' as above - of the four cross-amplitudes, $\underline{1}$, $\underline{35}$ (twice) and $\underline{405}$ only the latter allows transfer of quantum numbers charge $\ngtr 2$ or $|W(W+1)| \geqslant 6$, or both so that all processes characterized by such exchanges (in the crossed channel) must all be proportional to one amplitude, 405.

e.g. (a) $K^- + p \longrightarrow K^+ + \Xi^-$ has Q(crossed) = 2.

(b) reactions of the form $P + B \longrightarrow P + D$

when crossed become $P + P \longrightarrow \bar{B} + D$

with W spin couplings $1 \otimes 1 \rightarrow \frac{1}{2} \otimes \frac{3}{2}$

For ps mesons, P, the coupling CG is now $\begin{pmatrix} 3 & 3 & W \\ 0 & 0 & 0 \end{pmatrix}$ $W = 1,3,5$, but $\begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} = 0$; thus there is common to both sides only W=2 and so again only 405 exchange is possible.

It is clear that those processes excluded from a 1 or 35 channel in the exact symmetry case must again be excluded for a W=0, 8 spurion which conserves charge and W spin but that 35 channels, in the case (b), will be admitted by a W=1 spurion. We begin with the W=0 λ^8 spurion and thus need consider only A, $B\pm$. For $B\pm$ with $S^B_A = \delta^\beta_\alpha \; \lambda^{8b}_a$ the W spin parts are unmodified from $S^B_A = \delta^B_A$. Let

$$B(M_1 M_2) \equiv (\bar{3}B)^{AB}_{CD} \, M_{1C}^{\;A} \, M_{2B}^{\;D} \tag{4.14}$$

Writing $\Delta = 3\lambda^8 =$ diagonal $(1,1,-2)$ we find for the SU(3) parts

$$[M\,\Delta]_+ = 2M - 3 \quad x \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ K^- & \bar{K}^0 & \frac{-4}{\sqrt{6}}\eta \end{pmatrix}$$

$$= 2M-3\left[\begin{pmatrix} \sqrt{\frac{1}{6}}\,\eta' & 0 & K^+ \\ 0 & \sqrt{\frac{1}{6}}\,\eta' & K^0 \\ K^- & \bar{K}^0 & -2\sqrt{\frac{1}{6}}\,\eta' \end{pmatrix} - \begin{pmatrix} \sqrt{\frac{1}{6}}\,\eta' & & \\ & \sqrt{\frac{1}{6}}\,\eta' & \\ & & \sqrt{\frac{1}{6}}\,\eta' \end{pmatrix}\right]$$

(where $\eta' = \frac{4}{3}\eta$.)

$$\overset{\text{def.}}{=} 2M - 3M' + 3N \qquad 4.15$$

Similarly

$$[M\,\Delta]_- = -3 \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ -K^- & -\bar{K}^0 & 0 \end{pmatrix}$$

$$\overset{\text{def.}}{=} -3M'' \qquad 4.16$$

Then

$$B_+ = 4\,B(M_1M_2)-3\,B(M_1M_2')-3\,B(M_2M_1')+3\,B(M_1N)+3\,B(M_2N)$$

$$= 4\,B(M_1M_2)-3\,B(M_1M_2')-3\,B(M_2M_1') \qquad 4.17$$

since the factor N produces an SU(6) δ^C_A which vanishes against the $\underline{405}$ $(\bar{B}B)^{AB}_{CD}$

Similarly

$$B_- = -3\,B(M_1M_2'') +3\,B(M_2M_1'') \qquad 4.18$$

The overall effect of symmetry breaking is to replace one at a time in B of eq. 4.14 the SU(3) matrices Mi by M_2' or M_2'' - and thus can be computed directly from the exact symmetry amplitudes (which unfortunately were not published in [50], only their squares appearing). In this way, using notation of ref.[50] we obtain the following predictions:

No.	Process	$\lvert Amp.\rvert^2 \equiv \lvert\alpha\rvert^2$	No.	Process	$\lvert Amp.\rvert^2 \equiv \lvert\beta\rvert^2$
1	$\langle K^- p\mid \pi^+\Sigma^-\rangle$	3	2	$\langle K^- p\mid K^0\,\Xi^0\rangle$	3
6	$\langle K^- p\mid \rho^+\Sigma^-\rangle$	168	3	$\langle K^- p\mid K^+\,\Xi^-\rangle$	12
9	$\langle K^- p\mid \pi^- Y^{*+}\rangle$	24	4	$\langle K^- p\mid K^{*0}\,\Xi^0\rangle$	129
10	$\langle K^- p\mid \pi^+ Y^{*-}\rangle$	96	5	$\langle K^- p\mid K^{*+}\,\Xi^-\rangle$	48
11	$\langle K^- p\mid \pi^0 Y^{*0}\rangle$	54	7	$\langle K^- p\mid K^0\,\Xi^{*0}\rangle$	24
12	$\langle K^- p\mid \eta Y^{*0}\rangle$	2	8	$\langle K^- p\mid K^+\,\Xi^{*-}\rangle$	96
15	$\langle K^- p\mid \rho^+ Y^{*-}\rangle$	168	13	$\langle K^- p\mid K^{*0}\,\Xi^{*0}\rangle$	168
16	$\langle \pi^- p\mid K^+\Sigma^-\rangle$	3	14	$\langle K^- p\mid K^{*+}\,\Xi^{*-}\rangle$	168
17	$\langle \pi^- p\mid K^{*+}\Sigma^-\rangle$	129	30	$\langle \bar{K}^0 p\mid K^+\,\Xi^0\rangle$	3
22	$\langle \pi^- p\mid K^0 Y^{*0}\rangle$	12	31	$\langle \bar{K}^0 p\mid K^{*+}\,\Xi^0\rangle$	57
23	$\langle \pi^- p\mid K^+ Y^{*-}\rangle$	96	35	$\langle \bar{K}^0 p\mid K^+\,\Xi^{*0}\rangle$	24
25	$\langle \pi^- p\mid K^{*+} Y^{*-}\rangle$	168	36	$\langle \bar{K}^0 p\mid K^{*+}\,\Xi^{*0}\rangle$	240
26	$\langle \pi^+ p\mid K^+ Y^{*+}\rangle$	24			
32	$\langle \bar{K}^0 p\mid \eta Y^{*+}\rangle$	4			
33	$\langle \bar{K}^0 p\mid \pi^0 Y^{*+}\rangle$	12			
34	$\langle \bar{K}^0 p\mid \pi^+ Y^{*0}\rangle$	12			

No.	Process	$\|Amp.\|^2 \equiv \|Y\|^2$
18	$\langle \pi^- p \| \pi^- N^{*+} \rangle$	24
19	$\langle \pi^- p \| \gamma N^{+o} \rangle$	4
20	$\langle \pi^- p \| \pi^o N^{*o} \rangle$	108
21	$\langle \pi^- p \| \pi^+ N^{*} \rangle$	288
24	$\langle \pi^- p \| \rho^+ N^{*-} \rangle$	504
27	$\langle \pi^+ p \| \pi^+ N^{*+} \rangle$	24
28	$\langle \pi^+ p \| \pi^o N^{*++} \rangle$	36
29	$\langle \pi^+ p \| \gamma N^{*++} \rangle$	12

There are three distinct sets of amplitudes, each proportional to linear combinations of A, $B\pm$.

With

$$\begin{aligned} \alpha &= 2B+ \ +2B- \ -A \\ \beta &= -B+ \ +B- \ -A \\ \gamma &= 4B+ \ + \ A \end{aligned} \qquad 4.19$$

the proportionality (to α,β,γ) factors within each subset are then the same as in [50]. Notice a general sum rule $\alpha - 2\beta - \gamma = 0$ 4.20

Turning to the experimental data given in [51] we see that the symmetry breaking has not significantly changed the bad predictions. Data is given for 1,2,3,5,9,10,28 in the forward and backward directions - now only the subsets 2,3,5; 1,9,10; 28

afford comparison. But this still includes the predicted ratios 2:5, 2:3, 1:10 and 9:10 disagreeing by factors of $10 \sim 100$. Evaluating the sum rule 4-20 in the forward direction it is seen to be violated for the combinations

$$\gamma = 28 \qquad \beta = 1 \qquad \alpha = 2,3,5$$

$$\gamma = 28 \qquad \beta = 9 \qquad \alpha = 2.$$

One reason for these poor predictions has been pointed out in [51]. Consider for example the processes

$$9 : \quad K^- + p \longrightarrow \pi^- + Y^{*+}$$

$$10 : \quad K^- + p \longrightarrow \pi^+ + Y^{*-}$$

The first can proceed via K^* exchange as determined by the peripheral model known to have some validity in this situation, whilst the second involving $\Delta Q = -2$ is not peripheral in nature (no I spin 2 mesons are known to exist) and may be expected accordingly to be damped. Indeed we have

$$\left.\begin{aligned} \frac{\langle K^- p \mid \pi^- Y^{*+} \rangle}{\langle K^- p \mid \pi^+ Y^{*-} \rangle} &\sim \frac{8}{1} \qquad \text{experimentally} \\ \text{but} \qquad &\sim \frac{1}{4} \qquad \text{by exact } SU(6)_W \end{aligned}\right\} \qquad 4.21$$

The $\Delta W \neq 2$ selection rule does not seem to be valid in the forward direction - however if we were to allow a W=1 spurion some amplitudes are decoupled, e.g. in eqn. 4.21 above $\underline{35}$-like channels are opened for $K^- p \rightarrow \pi^- Y^{*+}$ but not for $K^- p \rightarrow \pi^+ Y^{*-}$. We have also checked that in this case it is no longer possible to form ratios for 1:10 and between 2,3, and 5.

As a further illustration subgroup reduction techniques, and a propros of some remarks in Ch.4.2, we evaluate the prediction of the $SU(4) \times SU(2)$ subgroup of $SU(6)_W$ for the ratio 9:10, eqn. 4.21. The decomposition appropriate to this case is defined in $\underline{6}$: $(p,\hat{p},n,\hat{n},\lambda,\hat{\lambda})$ to be that subgroup of unitary unimodular 6 x 6 matrices which act separately and independently on $(p,\hat{p},n,\hat{n})$ forming a $\underline{4}$ of $SU(4)$ and $(\lambda,\hat{\lambda})$ forming a $\underline{2}$ of $SU(2)$. In the language of eqn. 2.5a we replace the λ's by isospin τ's to get $SU(4)$ as a completion of $SU(2)_I \times SU(2)_W$, whilst the W spin matrices alone represent the remaining $SU(2)$ factor. Note that the decomposition of $\underline{6}$ is this time in the form of a tensor sum, rather than a tensor product. For the explicit decompositions we find

$$
\begin{array}{llll}
 & SU(6) & & SU(4) \times SU(2) \\
\text{Y.T.} & [3] & \longrightarrow & [3]. \oplus [2][1] \oplus [1][2] \oplus .[3] \\
\sim & \underline{56} & \longrightarrow & (20,1) \oplus (10,2) \oplus (4,3) \oplus (1,4)
\end{array}
$$

and for $\underline{35}$:

$$\underline{35} \longrightarrow (15,1) \oplus (1,3) \oplus (4,2) \oplus (4^*,2) \oplus (1,1)$$

To locate physical particles we need W spin-isospin properties. For $SU(4) \longrightarrow SU(2)_I \times SU(2)_W$ we have (under now a product decomposition)

$$\underline{10} = [2] \longrightarrow (3,3) \oplus (1,1)$$

$$\underline{4} = [1] \longrightarrow (2,2)$$

so that in SU(4) x SU(2)

$$[2][1] \longrightarrow [(3,3) \oplus (1,1)] \otimes (1,2) = (3,4 \oplus 2) \oplus (1,2)$$

$$[1][1] \longrightarrow (2,2) \otimes (1,2) \qquad = (2,\ 1 \oplus 3)$$

the last entry on the right gives the $(I,\underline{W})$ representation.

Hence we obtain

SU(4)x SU(2)	(I,W)	Particles
(20,1)	$(4,4) \oplus (2,2)$	N^*, N isoplets
(10,2)	$(3,\ 4 \oplus 2) \oplus (1,2)$	Y^*, Σ, Λ "
(4,3)	$(2,\ 4 \oplus 2)$	Ξ^*, Ξ "
(1,4)	(1,4)	Ω
(15,1)	$(3,\ 1 \oplus 3) \oplus (1,3)$	ρ, π, physical ω isoplets
(1,3)	(1,3)	physical φ "
(4,2)	$(2,\ 1 \oplus 3)$	K^*, K "
$(4^*,2)$	$(2,\ 1 \oplus 3)$	$\bar{K}^*$, $\bar{K}$ "
(1,1)	(1,1)	X^0

In the right hand column the physical , are defined as members of SU(4) x SU(2) IRs - this corresponds to the commonly accepted treatment of $\omega - \varphi$ mixing.

To count amplitudes for $Kp \longrightarrow \pi Y^*$ we consider the direct product

$$(4^*,2) \otimes (20,1) \otimes (\overline{10},2) \otimes (15,1)$$

The SU(2) part $\sim$ $2 \otimes 1 \otimes 2 \otimes 1$ contains one scalar. In SU(4) we have

$$\underline{4}^* \otimes \underline{20} = [1^3] \otimes [3] \quad = [41^2] \oplus [31^3] = \underline{70} \oplus \underline{10}'$$

$$\underline{10} \otimes \underline{15} = [2] \otimes [21^2]$$

$$= [41^2] \oplus [321] \oplus [31^3] \oplus [1^2]$$

$$= \underline{70} \oplus \underline{64} \oplus \underline{10}' \oplus \underline{6}$$

Therefore there are two SU(4), and so two SU(4) x SU(2) amplitudes.

We further write

$$(10,2) \sim Y_{(AB)\hat{C}} \equiv Y_{ab,\alpha\beta,\gamma}$$

$$\longrightarrow Y_{(ab)(\alpha\beta\gamma)} + \sqrt{\tfrac{1}{3}}\,[\varepsilon_{\alpha\gamma}\, Y_{(ab)\beta} + \varepsilon_{\beta\gamma}\, Y_{(ab)\alpha}\,]$$

$$(20,1) \sim P_{ABC}$$

$$\longrightarrow N_{(abc)(\alpha\beta\gamma)} + \frac{1}{3}\,(\varepsilon_{ab}\varepsilon_{\alpha\beta}N_{c,\gamma} + \varepsilon_{bc}\varepsilon_{\beta\gamma}N_{a,\alpha} + \varepsilon_{ca}\varepsilon_{\gamma\alpha}N_{b,\beta})$$

$$(15,1) \sim M_A^B \longrightarrow \sigma^i{}_a^b\, \sigma^{j\beta}{}_\alpha\, \rho_{ij} + \frac{\sqrt{3}}{2}\,\sigma^i{}_a^b\, \delta^\beta_\alpha\, \pi_i + \frac{\sqrt{3}}{2}\,\delta^b_a\, \sigma^{i\beta}{}_\alpha\, \varphi_i$$

$$(4^*,2) \quad T_B^A \longrightarrow K^{a,\alpha}{}_\beta + \sqrt{\tfrac{1}{2}}\,\delta^\alpha_\beta K^a$$

On the left above A, B, C etc, $\hat{A}, \hat{B}, \hat{C}$ etc are respectively SU(4) and SU(2) indices - we replace the SU(4) indices by $SU(2)_I$ x $SU(2)_W$ indices a, b, c and α, β, γ respectively and reduce to isospin x W spin normalised vectors, e.g.

$$Y_{(ab)(\alpha\beta\gamma)} \sim Y^* \text{ spin } \tfrac{3}{2} \text{ isospin 1 state}$$

$$Y_{ab,\beta} \sim \text{ spin } \tfrac{1}{2} \text{ isospin 1 state etc.}$$

An independent pair of tensor amplitudes are now seen to be:

$$A_1 : K^A_{\hat{B}}\, P_{(CDE)}\, \bar{Y}^{(CD)\hat{B}}\, \pi^E_A$$

$$A_2 : M^{A}_{\hat{B}} \; P_{(ACD)} \; \bar{Y}^{(CE)\hat{B}} \; \pi^{D}_{E}$$

Inserting now the factors, drawn from the above reductions, appropriate to eqn. 4.21 we find that A_1 does not contribute to either process whilst A_2 (as it now must) preserves the unwanted factor 4.

4.5 Summary and conclusions

In this Chapter we have glanced at $SU(6)_W$ symmetry breaking with two opposing hopes in mind - viz to find spurions which retain the symmetric Johnson-Treiman relations but destroy the class of predictions hinging on $\Delta W \neq 2$. To keep special significance for the idea of W spin invariance we might further have hoped to accomplish this by breaking only the SU(3) part of the scheme.

In none of their aims have we been successful - a W=1 SU(3) singlet 35 spurion proved most acceptable, giving the weak Johnson-Treiman relation and eliminating those predictions running contrary to the peripheral model and experiment - certainly the latter, for any semblance of agreement with experiment, demand that the interaction must be reducible in $SU(6)_W$. Of course, there is a precedent for such reducibility - in SU(3) the (strength) hierachy strong - medium strong - weak and electromagnetic forces corresponds to increasing reducibility of the lagrargian. But an analogous inference here that decuplet production processes are depressed in strength with respect to other, W spin conserving, reactions is clearly unacceptable. On the other hand a success of the $\Delta W \neq 2$ rule has been pointed out by Olsson[52] who observes that since reactions

$$\pi + N \longrightarrow \pi + \Delta \quad , \quad \Delta \sim \tfrac{3}{2} \text{ baryon resonance}$$

proceed through just one amplitude the ratio of isospin amplitudes is determined (there are two different isospin channels open $I = \frac{1}{2}$, $I = \frac{3}{2}$). Using

$$\frac{\langle \pi^- p | \pi^- N^{*+} \rangle}{\langle \pi^- p | \pi^0 N^{*0} \rangle} = -\frac{\sqrt{2}}{3} \quad \text{(by computation)}$$

and the Wigner-Eckart theorem for isospin :

$$\langle \pi^- p | \pi^- N^{*+} \rangle = \frac{1}{3} A_{\frac{1}{2}} - \frac{2}{3} \sqrt{\frac{2}{5}} A_{\frac{3}{2}}$$

$$\langle \pi^- p | \pi^0 N^{*0} \rangle = -\frac{1}{3} \sqrt{2} A_{\frac{1}{2}} + \frac{1}{3} \sqrt{\frac{1}{5}} A_{\frac{3}{2}}$$

one finds that $\dfrac{A_{\frac{1}{2}}}{A_{\frac{3}{2}}} = \sqrt{10}$

Olsson remarks that this agrees well with the value 3.4 ± 0.3 deduced from experiment - but we emphasize that this calculation is model dependent and in particular assumes S wave isobar production. We note also that this prediction is clearly invariant under the type of SU(3) symmetry breaking introduced in Ch.4.4 since the ratio $-\sqrt{2}/3$ is not thereby destroyed.

Despite this one success we feel forced to conclude that our main aim has not been achieved - the introduction of some SU(3) breaking into the $SU(6)_W$ scheme for 2 body scattering processes neither preserves the Johnson-Treiman relations nor invalidates the $\Delta W \neq 2$ disagreement with experiment.

CHAPTER 5

SU(6) AND CURRENT ALGEBRA

In §.5.1 we prepare the ground for the calculation, in §.5.2 of the following parameters associated with the weak interactions of the baryons.

(i) the renormalised weak axial vector coupling constant, gA and the D/F ratio of the weak current.

(ii) the baryon anomalous magnetic moments.

(iii) N-N* axial vector transition constant, G*

(iv) The N-N* M1 electromagentic transition moment.

The 'calculation' reduces to adjusting the amount of mixing of two $SU(6)_W \times O(3)$ IRs and the system is too flexible to allow any significant conclusion.

5.1 Current algebras and representation mixing

The concept of an algebra of currents has been central to the successful study of broken symmetries. That exact and broken symmetries could consistently share the same algebraic structure (i.e. commutation relations) was first emphasized by Gell Mann[53] and developed into a non relativistic theory of symmetry breaking by Fubini and Furlan[54] - relativistic formulations were soon available[55]. Without doubt the major success of this body of work has been the Adler-Weisberger (A-W) calculation[56] of the renormalization of the weak axial vector coupling constant g_A, defined in Adlers' notation by

$$\langle N(q)|J^{+}|N(q)\rangle = \frac{M_N}{q_0} G_V \bar{U}_N(q)(\gamma_\lambda + g_A \gamma_\lambda \gamma_5) J^{+} U_N(q) \qquad 5.1$$

where J is the weak baryon current responsible for $\Delta S=0$ leptonic decays.

For our own purposes we would like to emphasize the following points about this famous A-W calculation, summarised in the following equation

$$1 - \frac{1}{g_A^2} = \frac{4M_N^2}{2K^{NN\pi}(0)^2} \frac{1}{\pi} \int_{M_N+M}^{\infty} \frac{WdW}{W^2-M_N^2} [\sigma_0^{+}(W) - \sigma_0^{-}(W)] \qquad 5.2$$

where $\sigma_0^{\pm}(W)$ is the total cross-section for scattering of a zero mass $\pi^{\pm}$ on a proton at centre of mass energy W, $K^{NN\pi}$ is the pion form factor of the nudeon etc.

(i) The transition operators (chiralities in Adler's notation) whose matrix elements between proton and neutron give a measure of g_A are assumed to obey the (chiral) algebra of SU(2) x SU(2). However no statement is made about the SU(2) x SU(2) properties of the particle states p,n - only the conventional isospin subgroup assignments to irreducible representations are made.

(ii) This omission renders the algebra, alone, impotent; to complete the calculation information is drawn from experiment with the aid of the PCAC hypothesis, in Adler's notation

$$\partial_\lambda J_\lambda^{Aa} = \frac{-iM_N \mu_\pi^2 g_A}{g_r K^{NN\pi}(0)} \phi_\pi^a \qquad 5.3$$

where gr is the rationalised renormalised pion-nucleon coupling constant, φ_{π}^{a} is the renormalised pion-field, etc. This allows to relate the generator matrix elements (in what really is an infinitely reducible SU(2) x SU(2) IR) instead to experimentally measurable π-p total cross-sections (tho' an extrapolation to zero pion mass is required).

The injection of experimental numbers in this way, with the resultant degree of accuracy ($\sim$ 5% in the gA calculation) engendered considerable confidence in current algebra calculations forbroken symmetries.-

(iii) Since the symmetry is not exact the transition operators, its generators, become time dependent, and this was translated into an energy dependence or non-covariance of the gA sum rule. Following the suggestion in ref.[54] the sum rule was evaluated in the limit of infinite momentum of the external one particle state, and the use of this frame has later come to be seen as equivalent to a fully relativistic approach cf.[57].

On the basis of a quark model with an SU(3) triplet of spin-$\frac{1}{2}$ quarks the largest algebra one can envisage is that of U(12) - with current densities transforming as the appropriate quark bilinears. We must take care to distinguish Lorentz transformation properties from those of the compact algebra, the distinction being a γ_0 factor in the bilinear corresponding to the use of the anticommutator

$$\{\psi_\alpha(x), \psi^\dagger_\beta(x')\}_{t'=t} = \delta_{\alpha\beta}\delta^3(x-x')$$

rather than $\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}_{t'=t} = (\gamma_o)_{\alpha\beta}\delta^3(x-x')$.

When we take matrix elements between single particle states at infinite momentum two well known factors enter[57,58].

(i) Certain densities have their matrix elements damped by a factor E^{-1} and so do not appear when $p \longrightarrow \infty$. The rule is that those U(12) elements survive which commute with $\gamma_o\gamma_3$, the generator of Lorentz transformations in the p direction. Omitting the SU(3) factors there are

$$1,\ \gamma_1,\ \gamma_2,\ \gamma_1\gamma_2,\ \gamma_o\gamma_3,\ \gamma_5,\ \gamma_1\gamma_5,\ \gamma_2\gamma_5 \qquad 5.4a$$

(ii) Amongst these 72 (=8 x 9) 'good' charges of U(12) certain equalities appear for their matrix elements - essentially because $\gamma_o\gamma_3 \sim$ unit operator on infinite momentum States. Thus we get

$$\begin{array}{ccc} 1 & \sim & \gamma_o\gamma_3 \\ \gamma_1\gamma_2 & \sim & \gamma_5 \\ \gamma_1\gamma_5 & \sim & \gamma_2 \\ \gamma_2\gamma_5 & \sim & \gamma_1 \end{array} \qquad 5.4b$$

and the U(12) algebra thus degenerates into that of $U(6)_W$ represented in its space-time components by the left hand column cf. § 4.1.

In this way the $U(6)_W$ algebra contains the chiral algebra relevant to the calculation of gA.

Representation mixing presents an alternative at (ii) above. It was known that the pure 56 baryon assignment does provide an approximation to some parameters, e.g. it gives $gA = \frac{-5}{3}$, $d/f = \frac{3}{2}$ $G^* = \sqrt{\frac{8}{3}}$ (see [17]). (See following table for experimental estimates of these numbers.) Again it was observed that in the A-W equation 5.2 dominant contributions were made to the integral term (the renormalization correction) by some low lying resonances which might be fitted into a higher symmetry multiplet. In deciding which representations to mix one is thus guided by the SU(6) classification of the baryon resonances - it has been shown by Dalitz[59] that the bracket of negative parity resonances lying above the $N^*-\frac{3}{2}$ can be best fitted into a 70(L=1) multiplet. This classification uses the quark model and introduces an orbital quantum number L to cover the relative motion of the three constituent quarks. However difficulties arise when we attempt to relate this to an $SU(6)_W$ current algebra in the infinite momentum frame[57] since only in the rest frame can W spin and L couple like S and L to give total J thereby identifying the physical particles. Outside of the rest frame the non vector character of W may be expected to interfere with this simple procedure - it was first suggested[57] that possibly this was only an apparent complication and the coupling could be effected as in the static case. However Lipkin et al[62] have argued that we are indeed faced with a serious limitation here, and later work by Dashen and Gell-Mann[66] seems to confirm this. In any case it must be remarked that an additional orbital degree of freedom has to be introduced into the scheme if it is to allow non zero anamolous magnetic moments - this follows from the Cabibbo-Radicati[60] identification of the anomalous magentic moment operator in terms of the expectation value of the electric dipole operator between infinite momentum states (the Dirac moment term receives a damping factor 1/E).

An alternative representation of the magnetic moment operator uses the Lorentz tensor parts of the W spin vector and then, to make contact with electromagnetism, relates the tensor divergences to the vector field operators (PCTC) and uses the customary assumption of vector dominance of the electromagnetic form factors. This approach has been discussed by Gatto et al[62].

5.2 The $\underline{56}$(L=0) $\oplus$ $\underline{70}$(L=1) Mixing scheme

The $\underline{70}$(L=1) IR decomposes into the following ($\underline{\mu}$; 2J+1) multiplets : ($\underline{10}$; 4,2), ($\underline{8}$; $6,4^2,2^2$), ($\underline{1}$; 4,2) - note in particular the existence of two spin $-\frac{1}{2}$ octets so that including the $\underline{56}$ contribution we have three spin-$\frac{1}{2}$ octets requiring two mixing angles. This greatly reduces the predictive power rendering it impossible, e.g. to obtain a relation between g_A and D/F, one of the successes of other mixing schemes[62].

We write, for the octet baryon, in a helicity diagonal representation,

$$|B\rangle_{\frac{1}{2}} = \cos\theta\,|56\rangle_{\frac{1}{2}} + \sin\theta\,[\cos\varphi(\sqrt{\tfrac{1}{3}}|70(82)\rangle_{\frac{1}{2}} - \sqrt{\tfrac{2}{3}}|70(82)\rangle_{-\frac{1}{2}})$$

$$+ \sin\varphi(\sqrt{\tfrac{1}{2}}|70(8,4)\rangle_{\frac{3}{2}} - \sqrt{\tfrac{1}{3}}|70(8,4)\rangle_{\frac{1}{2}} + \sqrt{\tfrac{1}{6}}|70(8,4)\rangle_{-\frac{1}{2}})] \qquad 5.5$$

θ measures the $\underline{56}$-$\underline{70}$ mixing, φ that between the two $J=\frac{1}{2}$ octets occurring in the angular momentum direct products $\frac{3}{2} \otimes 1$, $\frac{1}{2} \otimes 1$ for $\underline{70}$(L=1). The suffix gives the U(6) contribution S_z to the helicity h ($\frac{1}{2}$ above as designated on the left) - the orbital contribution is defined by $l_z = h-S_z$. Using the tables of Chapter 3 and the Wigner-Eckart theorem for generators (which, as emphasized by Gell-Mann leaves us with no overall scale factor or unknown reduced matrix element - those of the generators which enter here are

$$\langle 70 \| 35_F \| 70\rangle = 3\sqrt{\tfrac{11}{6}} \qquad \langle 56 \| 35 \| 56\rangle = 3\sqrt{\tfrac{5}{2}} \qquad 5.6$$

we find for the $\underline{70}$ contributions

$$g_A \big|_{70} = \frac{1}{9}\left(2 - 8 \text{ Sin } 2\varphi - 3 \text{ Cos } 2\varphi\right) \qquad 5.7a$$

$$\frac{D}{D+F}\bigg|_{70} = -\frac{6 \text{ Sin } 2\varphi}{2 - 8 \text{ Sin } 2\varphi - 3 \text{ Cos } 2\varphi} \qquad 5.7b$$

$$G^*_A \big|_{70} = \frac{1}{3}\sqrt{\frac{8}{3}}\left(2 \text{ Cos}\varphi - \tfrac{1}{2} \text{ Sin}\varphi\right) \qquad 5.7c$$

These can now be combined with the $\underline{56}$ contributions as laid out in the following Table :-

TABLE 1

θ	φ	g_A	D/(D+F)	G^*_A
0	-	1.66	0.60	1.63 (= $\sqrt{8/3}$)
90	-30	0.83	0.73	± 1.08
90	-40	1.04	0.63	± 1.01
90	-50	1.16	0.57	± 0.97
38	-15	1.18	0.63	1.44
45	-25	1.18	0.64	1.37
55	-35	1.18	0.63	1.24
Experiment		1.18	0.65±0.05	1.1± 0.1

For a given $\underline{70}$ mixing angle 2φ in eqns. 5.7a,b we may employ either φ or $\varphi + \pi$ in eqn.5.7c, and this corresponds respectively to the $\pm$ spin for G^*_A at $\theta = 90$. For $\theta \neq 90$ we tabulate only those G^*_A resulting from the positive 70 contribution since they give a better fit. We insert an elementary remark on the calculation of D:F ratios such as in this Chapter, since confusion may arise due to alternative normalizations. Using the conventional notations we have

$$\langle i | F^k | j \rangle = F \; i f_{ijk} + D \; d_{ijk} \text{ defining } D{:}F.$$

The f's and d's play the role of unnormalised reduction coefficients since

$$\sum_{ij} f_{ijk} f_{ijl} = 3\delta_{kl}, \qquad \sum_{ij} d_{ijk} d_{ijl} = \frac{5}{3}\delta_{kl} .$$

In a higher symmetry scheme using CGcs we calculate

$$\langle i | F^k | j \rangle = (S + a) \times \langle \text{reduced matrix element} \rangle$$

where S, a are CGcs referring to normalised symmetric and anti-symmetric octet products

e.g.

$$S \sim \begin{pmatrix} \overset{56}{\underset{v_1}{82}} & \overset{35}{\underset{v_2}{83}} & \overset{56}{\underset{v_3}{8_s 2}} \end{pmatrix}$$

Clearly we have $\frac{s}{a} = \frac{D}{F}$ when $i f_{ijk} = d_{ijk}$ otherwise we must take care to convert from one scheme to the other, e.g.

	$i f_{ijk}$	d_{ijk}
$\langle p \mid j^{em} \mid p \rangle$	1	$\frac{1}{3}$
$\langle n \mid j^{em} \mid n \rangle$	0	$-\frac{2}{3}$
$\langle p \mid \pi^+ \mid n \rangle$	2	2

So in our case no conversion was necessary.

The anomalous magnetic moments are calculated assigning M, the magnetic moment operator, the U(6) transformation properties $35(8,1)_{1_z} = \pm 1$ (and in SU(3) $\sim Q$). We must calculate the matrix elements of $M \pm 1$ corresponding to transverse momentum transfer (the momentum is infinite in the collinear direction) neccessarily non zero according to

$$\underset{\sim}{M} \qquad \frac{i\partial}{\partial \underset{\sim}{q}} \times j^{em}(q) \Big|_{\underset{\sim}{q}=o} \tag{5.8}$$

$\underset{\sim}{q}$ = momentum transfer

Since $\underline{70}$ (x) $\underline{35}$ contains $\underline{70}$ twice there are in all three reduced matrix elements to consider $\langle 70 \| 35 \| 70 \rangle_1$, $\langle 70 \| 35 \| 70_2 \rangle$ and $\langle 70 \| 35 \| 56 \rangle$ (the fourth $\langle 56 \| 35 \| 36 \rangle$ does not enter by the Wigner-Eckart theorem on L, and represents one of the motives for allowing mixing with orbital excitation). We find

$$\mu_\Lambda \text{ (proton)} = - \mu_\Lambda \text{(neutron)} = \text{const.} \sin 2\theta \cos\varphi \langle 70 \| 35 \| 56 \rangle \tag{5.9a}$$

or equivalently

$$\alpha_M = 0.75 \tag{5.9b}$$

where $\alpha_M = \frac{D}{D+F}$ gives the D:F ratio of the anomalous magnetic moments from factor F_2 and experimentally has the value 0.774 [63]. Of course this result has been found earlier by current algebra methods[64].

We are not able to make any prediction concerning the N-N* M1 electromagnetic transition moment μ^* due to the presence of the elements 70 35 $70_{1,2}$. However an alternative scheme, cf. Lippin et al[62] involving no decuplet mixing, and giving $G_A^* = 1.15$ would predict

$$\mu^* \sim \frac{1}{10} - \frac{2\sqrt{2}}{3} \mu(\text{proton}) \qquad 5.10$$

to be compared with the experimental value $1.3 \frac{2\sqrt{2}}{3}$ μ proton[65].

We thus find the 56(L=0) (+) 70(L=1) system advocated by Dashen and Gell-Mann[57] adequate but inconclusive - however some decuplet mixing does seem to be necessary.

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