Asymptotic solutions displaying the effects of gravity and viscosity on certain flows with free boundaries.

by

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A thesis submitted for the degree Ph.D. of the University of London.

Preface

In this thesis we consider three problems in two-dimensional fluid flow. All are characterized by the fact that the fluid considered is, at least partially, bounded by free streamlines. The difficulties associated with flows of this kind are, in addition to the normal non-linearities in the field equations and boundary conditions, that the boundaries themselves are unknown, being determined by the nature of the flow they contain.

We overcome this last difficulty in the first problem (the flow in an inviscid waterfall) by expressing the problem in streamline co-ordinates, that is, by using as independent variables the stream function and a co-ordinate forming an orthogonal net with it. The difficulties raised by the non-linear boundary conditions are resolved by employing a perturbation scheme. As this is singular, we resort to the " method of matched asymptotic expansions ".

The third problem (the flow under gravity in a jet of viscous liquid) also involves a singular perturbation, and so we again resort to the above method. One of the expansions is derived after expressing the problem in streamline co-ordinates. The other expansion is derived by using a technique, developed in the second problem, involving a complex variable formalism. It is shown that the Airy stress function is what might be called the " biharmonic conjugate " of the stream function, and this relationship proves to be very useful.

The second problem (the flow of a viscous fluid in the neighbourhood of separation at an edge) is included mainly as a vehicle for developing the complex variable formalism. The " neighbourhood " mentioned above is defined as being the region in which the non-linear terms in the field equations are negligibly small and where the free streamline can be considered to be rectilinear. This disposes of both chief difficulties in the problem.

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Part 1

On two-dimensional inviscid flow in a waterfall

Preface

The material for this section has been published in the form of a paper. This paper is included, with one correction, and constitutes the majority of Part 1; we make one addition. Since this paper was written, Markland (1965) has published some work on the same problem, though like Southwell and Vaisey (1946) (see below, pages 360 and 369), his method of attack is numerical rather than analytical. One advantage that Markland's work has over that of Southwell and Vaisey, from our point of view, is that like us he works with stream-line co-ordinates and treats the cases for which our analysis is suitable, that is , cases in which the Froude number is greater than unity. We have used Markland's results to compare with our own. These comparisons are presented in figures 6, 7, 8 and 9.

The pagination of our paper is retained. We make a slight correction on page 362. In equations (3.3), (i) and (ii) should read

(i) $u_1 = 0$ on $s_2 = 0$, $s_1 > 1$;

(ii) $u_1 = -\frac{1}{2}$ on $s_2 = 0$, $0 \leqslant s_1 < 1$;

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On two-dimensional inviscid flow in a waterfall

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This paper is concerned with the two-dimensional flow in a free waterfall, falling under the influence of gravity, the fluid being considered to be incompressible and inviscid. A parameter ϵ, such that $2/\epsilon$ is the Froude number based on conditions far upstream, is defined and considered to be small. A flowline co-ordinate system is used to overcome the difficulty that the boundary geometry is not known in advance. An asymptotic expansion based on ϵ is constructed as an approximation valid upstream and near the edge, but singular far downstream. Another asymptotic expansion, based upon the thinness of the fall, is constructed as an approximation valid far downstream, but failing to satisfy the conditions upstream. The two expansions are then matched to give a solution covering the whole flow field. The shapes of the free streamlines are shown for a number of values of ϵ for which the solutions are seemingly valid.

1. Introduction

An inviscid, incompressible fluid flows over a horizontal bed until it falls over an edge under the influence of gravity. The flow is considered to be plane and steady. Far upstream the fluid is of depth h and has a uniform horizontal velocity U_0, and gravity is acting vertically downwards (see figure 1). The problem is one of finding the velocity potential Φ and the stream function Ψ as functions of position. Both Φ and Ψ must satisfy the Laplace equation subject to certain non-linear boundary conditions, namely zero pressure on the free streamlines and zero normal velocity on the bed. The basic non-dimensional parameter appearing in the problem is $\epsilon = 2gh/U_0^2$, and this is assumed to be small in most of this paper.

This problem involves a singular perturbation, the singularity occurring far downstream. As such, it lends itself to the technique of 'inner and outer expansions'. Kaplun & Lagerström (1957) and Erdélyi (1961) give a general account of this technique and also cite further references. In the present paper an expansion, which is derived to satisfy the conditions in that part of the flow which is not far downstream, will be known as the inner expansion, and the region in which it is valid, as the inner region. Similarly, the outer expansion satisfies the conditions far downstream, and is valid in the outer region.

The inner expansion is constructed by a perturbation scheme, in which all lengths are referred to h and all velocities to U_0; this scheme may be regarded as a perturbation for weak gravity. The first approximation is therefore a uniform horizontal stream. In the outer region, all lengths are referred to $U_0^2/2g$ and velocities again to U_0, and so in this region the perturbation may be regarded as one for small width of the fall. Here the first approximation, being hydraulic, is of the well-known parabolic form.

It is to be expected that the inner and outer regions overlap to some extent. By matching the inner and outer expansions in the overlap region, the unknown constants in the outer expansion are found, and the combined solutions then cover the whole flow field.

Southwell & Vaisey (1946) found a result for the case $\epsilon = 2$ by relaxation techniques, and their solution has been used for comparison purposes in figure 5. Keller & Weitz (1957) also found a solution in the outer region, though by an approach different from the one given in this paper. This solution was found to agree with ours to the first approximation.

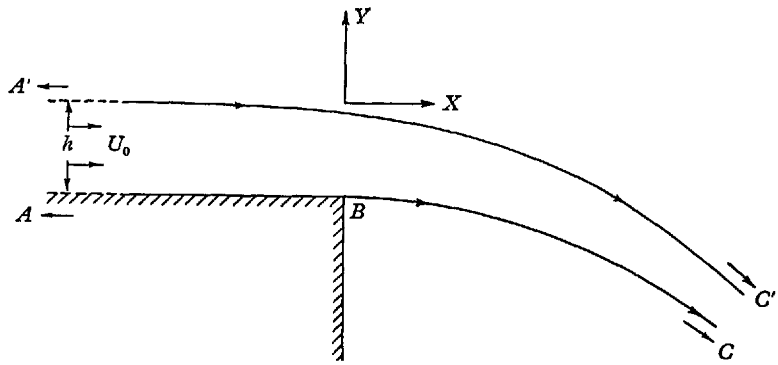

FIGURE 1. Notation.

2. Formulation

We denote the fluid velocity by $\mathbf{Q} = \nabla\Phi$, and consider a co-ordinate system $Z = X+iY$, in which the bed is described as $Y = -h$; $X \leqslant 0$. Gravity is acting in the direction of Y decreasing. The problem is to find the complex potential $F = \Phi + i\Psi$ satisfying $(\partial^2/\partial X^2 + \partial^2/\partial Y^2)F = 0$, subject to: (i) zero pressure on the free streamlines, (ii) zero normal velocity on the bed. The free streamlines are unknown in terms of X and Y, but are known in terms of Ψ. This suggests inverting the problem to one of finding Z as a function of F, that is, of finding Z satisfying $(\partial^2/\partial\Phi^2 + \partial^2/\partial\Psi^2)Z = 0$, subject to the same boundary conditions.

To find the boundary conditions explicitly, we make use of Bernoulli's equation:

$$P/\rho + \tfrac{1}{2}Q^2 + gY = \text{constant} = \tfrac{1}{2}U_0^2,$$

where the density ρ is constant throughout the fluid, and the constant on the right has been evaluated from the conditions far upstream on the upper free streamline.

We define non-dimensional variables by

$$p = P/\rho U_0^2; \quad q = |\mathbf{Q}|/U_0; \quad z = Z/h; \quad f = F/U_0 h;$$

and Bernoulli's equation becomes

$$2p + q^2 + \epsilon y = 1. \tag{2.1}$$

Therefore the boundary conditions are

$$\left.\begin{array}{llll}\text{(i)} & q^2 = 1-\epsilon y & \text{on} & \psi = 0, \quad \text{all } \phi; \\ \text{(ii)} & q^2 = 1-\epsilon y & \text{on} & \psi = -1, \quad \phi \geqslant 0; \\ \text{(iii)} & \operatorname{Im}\left(\dfrac{dz}{df}\right)^{-1} = 0 & \text{on} & \psi = -1, \quad \phi \leqslant 0.\end{array}\right\} \quad (2.2)$$

If we consider the problem to be in the complex f-plane, then the field equations are satisfied by any complex function $z(f)$. The problem is then, to find such a function $z(f)$ which satisfies (2.2).

3. The inner expansion

We pose that

$$z(f) = z_0(f) + \epsilon z_1(f) + \epsilon^2 z_2(f) + \dots. \quad (3.1)$$

To find the $z_n(f)$ we substitute (3.1) into (2.2) and, comparing coefficients of ϵ, obtain a sequence of linear problems in each of the $z_n(f)$ in turn.

$z_0(f)$ is simply the solution in the case when $\epsilon = 0$, and so $z_0(f) = f$.

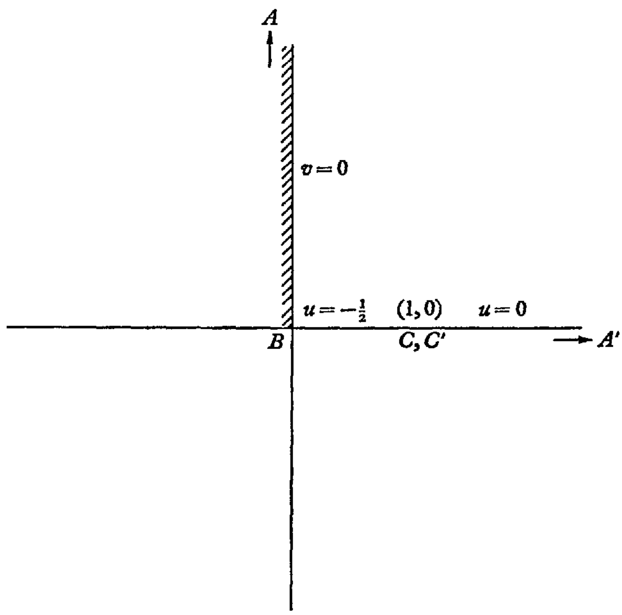

FIGURE 2. The complex s-plane, showing the boundary values of the first-order problem.

On substituting (3.1) into (2.2), and comparing first-order coefficients, we find that $x_{1\phi} = \frac{1}{2}\psi$ on the free streamlines, where the subscript ϕ denotes differentiation with respect to ϕ. We therefore seek $w_1 = u_1 + iv_1 = x_{1\phi} + iy_{1\phi}$, subject to

$$\left.\begin{array}{llll}\text{(i)} & u_1 = 0 & \text{on} & \psi = 0, \quad \text{all } \phi; \\ \text{(ii)} & u_1 = -\frac{1}{2} & \text{on} & \psi = -1, \quad \phi \geqslant 0; \\ \text{(iii)} & v_1 = 0 & \text{on} & \psi = -1, \quad \phi \leqslant 0.\end{array}\right\} \quad (3.2)$$

To solve this mixed boundary-value problem, we map the infinite strip, $0 \geqslant \psi \geqslant -1$, in the f-plane on to the upper right-hand quadrant of the $s = s_1 + is_2$ plane by the mapping: $s = \sqrt{(1+e^{-\pi f})}$. The boundary conditions then are as given by (3·3), and as shown in figure 2.

$$\left.\begin{array}{llll} \text{(i)} & u_1 = 0 & \text{on} \quad s_2 = 0, & s_1 \geqslant 1; \\ \text{(ii)} & u_1 = -\tfrac{1}{2} & \text{on} \quad s_2 = 0, & 0 \leqslant s_1 \leqslant 1; \\ \text{(iii)} & v_1 = 0 & \text{on} \quad s_1 = 0, & s_2 \geqslant 0. \end{array}\right\} * \tag{3.3}$$

This problem is familiar, in that it is analogous to that of finding the complex potential of an inviscid flow, covering the entire plane, with a pair of vortices situated at (1,0) and $(-1, 0)$. The solution is well known:

$$w_1(s) = \frac{i}{2\pi} \log\left(\frac{s-1}{s+1}\right). \tag{3.4}$$

However, it will be more helpful to solve a more general mixed boundary-value problem, as this more general solution may be used in the higher-order problems.

Consider a complex function $w_n = u_n + iv_n$, analytic in $s_1 \geqslant 0$, $s_2 > 0$, with u_n prescribed on the positive real axis. Following Woods's (1961) account, we assume that

(i) $v_n = 0$ on $s_1 = 0$, $s_2 \geqslant 0$;

(ii) $w_n(s) \sim O(s^{-1})$ as $|s| \uparrow \infty$;

(iii) $w_n(s)$ is integrable in the ordinary (Riemann) sense on any finite arc of the positive real axis. (Unlike Woods we do not allow w_n to have singularities of the Cauchy type.) (3.5)

If we consider the problem to be in the whole of the upper half plane, with u_n now also prescribed on the negative real axis, then the solution is well known,

$$w_n(s) = -\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{u_n(\sigma)}{\sigma - s}\, d\sigma. \tag{3.6}$$

To ensure that $v_n = 0$ on the positive imaginary axis, we have that

$$u_n(\sigma) = u_n(-\sigma),$$

and using this
$$w_n(s) = \frac{i}{\pi}\int_0^{\infty} u_n(\sigma)\left\{\frac{1}{s+\sigma} + \frac{1}{s-\sigma}\right\} d\sigma. \tag{3.7}$$

We return to the first-order problem. The boundary conditions satisfy (3.5), and so using this method we recover (3.4).

By the restriction (3.5, (iii)) we have excluded terms in $w_n(s)$ of the form

$$i\left(\frac{1}{s-1} + \frac{1}{s+1}\right),$$

which may be added to any solution without violating the boundary conditions except at the singular point $s = 1$. This is because we accept only the weakest possible singularity for $s \to 1$, a policy justified later by the matching procedure. Therefore the solution to the first-order problem is given by (3.4), which in terms of the original variables, becomes

$$z_{1f} = \frac{i}{2\pi} \log\left\{\frac{(1+e^{-\pi f})^{\frac{1}{2}} - 1}{(1+e^{-\pi f})^{\frac{1}{2}} + 1}\right\}. \tag{3.8}$$

* see note at foot of page 6.

When we come to the matching procedure, we will require an expression for $z_1(f)$ as $f \uparrow \infty$; this is then from (3.8)

$$z_1(f) \sim -\frac{i}{4}f^2 - \left(\frac{i}{\pi}\log 2\right)f + \text{const.} + O(e^{-\pi f}). \tag{3.9}$$

We now turn our attention to the second-order coefficient, z_2. On inserting (3.1) into (2.2), and comparing second-order terms, we have that

$$x_{2\phi} = \tfrac{1}{2}(y_1 + 3x_{1\phi}^2 - y_{1\phi}^2)$$

on the free streamlines. We therefore seek $w_2 = u_2 + iv_2 = x_{2\phi} + iy_{2\phi}$, subject to

$$\left.\begin{array}{llll} \text{(i)} & u_2 = \tfrac{1}{2}(y_1 - y_{1\phi}^2) & \text{on} \quad \psi = 0, & \text{all}\,\phi; \\ \text{(ii)} & u_2 = \tfrac{1}{2}(y_1 - y_{1\phi}^2) + \tfrac{3}{8} & \text{on} \quad \psi = -1, & \phi \geqslant 0; \\ \text{(iii)} & v_2 = 0 & \text{on} \quad \psi = -1, & \phi \leqslant 0. \end{array}\right\} \tag{3.10}$$

Mapping the f-plane onto the s-plane, we find that on $s_2 = 0$, u_2 has a finite discontinuity, and singularities of the nature $\log^2|s_1 - 1|$ and $\log|s_1 - 1|$ at $s_1 = 1$, but has no singularities elsewhere. Hence u_2 satisfies (3.5, (iii)). Also

$$u_2 = O(s^{-2}) \quad \text{as} \quad |s| \uparrow \infty,$$

and so all the conditions in (3.5) are satisfied. Therefore, using (3.7), the solution is given by

$$w_2(s) = -\frac{3i}{8\pi}\log\left(\frac{s-1}{s+1}\right) + \frac{i}{\pi}\int_0^\infty G(\sigma)\left\{\frac{1}{s-\sigma} + \frac{1}{s+\sigma}\right\}d\sigma, \tag{3.11}$$

where $G(\sigma) = \tfrac{1}{2}(y_1 - y_{1\phi}^2)$ on $s = \sigma$, σ real. The behaviour of $G(\sigma)$ near $\sigma = 1$, is given by

$$G(\sigma) = -\frac{1}{4\pi^2}\log^2|\sigma - 1| + \frac{1}{2\pi^2}\log 2\log|\sigma - 1| + \tfrac{1}{8}H(1-\sigma) + J(\sigma),$$

where H is the Heaviside unit function, and $J(\sigma)$ is regular at $\sigma = 1$. To remove this singularity from within the integral, we define the complex function $\gamma(s) = \alpha(s_1, s_2) + i\beta(s_1, s_2)$ by

$$\gamma(s) = -\frac{1}{4\pi^2}\log^2\left(\frac{s-1}{s+1}\right) + \frac{i}{8\pi}\log\left(\frac{s-1}{s+1}\right) - \frac{1}{4\pi^2}(s-1)\log(s-1)$$
$$+ \frac{1}{4\pi^2}(s+1)\log(s+1) - \frac{1}{2\pi^2}\log(s+i) - \frac{1}{2\pi^2}. \tag{3.12}$$

The function $\gamma(s)$ satisfies the conditions (3.5), and $G(\sigma) - \alpha(\sigma, 0) = \Omega(\sigma)$, where $\Omega(\sigma)$ and $d\Omega(\sigma)/d\sigma$ are continuous in $0 \leqslant \sigma \leqslant \infty$. From (3.7)

$$\gamma(s) = \frac{i}{\pi}\int_0^\infty \alpha(\sigma, 0)\left\{\frac{1}{s-\sigma} + \frac{1}{s+\sigma}\right\}d\sigma. \tag{3.13}$$

Subtracting (3.13) from (3.11) we have

$$w_2(s) = \frac{-3i}{8\pi}\log\left(\frac{s-1}{s+1}\right) + \gamma(s) + \frac{i}{\pi}\int_0^\infty \Omega(\sigma)\left\{\frac{1}{s-\sigma} + \frac{1}{s+\sigma}\right\}d\sigma. \tag{3.14}$$

The behaviour of $z_2(f)$ as $f \uparrow \infty$, is then

$$z_2(f) \sim -\tfrac{1}{12}f^3 + \frac{1}{2}\left(\frac{i}{4} - \frac{1}{\pi}\log 2\right)f^2 - \left(\frac{1}{\pi^2}\log^2 2 + \frac{1}{2\pi^2} + \frac{i}{8\pi} - \frac{i}{\pi}\log 2\right)f + \text{const.} + O(e^{-\pi f}). \tag{3.15}$$

It is worth noting here that on the lower streamline near the edge

$$z_1 \sim -\tfrac{1}{2}\phi - i\frac{2}{3\sqrt{\pi}}\phi^{\frac{3}{2}} + O(\phi^{\frac{5}{2}}),$$

and

$$z_2 \sim A_0\phi + i\frac{2A_1}{3}\phi^{\frac{3}{2}} + O(\phi^{\frac{5}{2}}),$$

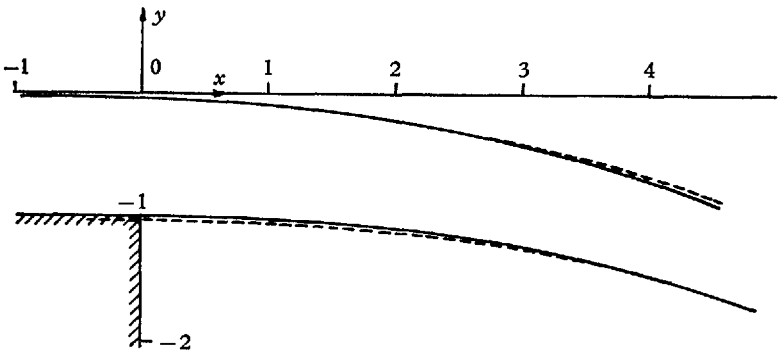

FIGURE 3. Case of $\epsilon = 0{\cdot}1$; ———, inner solution; - - - -, outer solution.

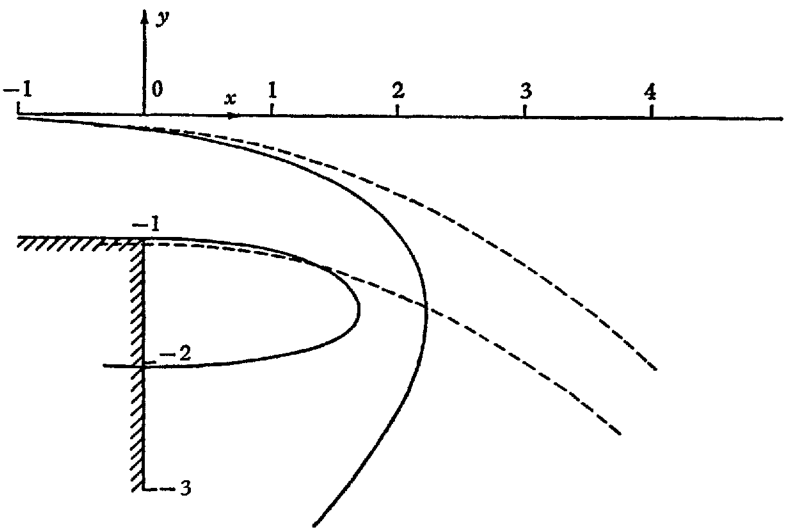

FIGURE 4. Case of $\epsilon = 0{\cdot}5$; ——, inner solution, - - - -, outer solution.

where A_0, A_1 are real finite constants. In both these expressions, the leading singular terms are of order $\phi^{\frac{3}{2}}$. This shows that the singularity in the first-order term does not give rise to a more singular term in the second-order expression. It would appear, then, that at the edge, $z(f)$ has no worse a singularity than that contained in $z_1(f)$.

The functions x_1, x_2, y_1, y_2, have been evaluated numerically for the upper and lower free streamlines in the range $-5 \leqslant \phi \leqslant +5$, and the results have been used in the construction of the figures 3 and 4. The higher-order terms in (3.1) could be derived in a similar manner, but we terminate the inner expansion after the second-order term.

4. The outer expansion

Defining the complex velocity by $qe^{-i\theta}$, we know that $qe^{-i\theta} = \text{fn}(\phi + i\psi; \epsilon)$, but we do not know the manner in which ϵ enters this function for large values of ϕ. However, if we take $U_0^2/2g$ as reference length, and U_0 as reference velocity, this makes the width of the fall of order ϵ. That is, if ψ^+ is the new non-dimensional stream function, then the flow is bounded by the streamlines $\psi^+ = 0$, $\psi^+ = -\epsilon$. This narrowness is useful so long as $\partial/\partial\psi^+ \sim O(1)$, for then we may assume little change across the fall. We have from the boundary conditions on the free streamlines

$$[q^2]_{\psi=-1}^{\psi=0} = -\epsilon[y(\phi, \psi)]_{\psi=-1}^{\psi=0} \quad (\phi > 0). \tag{4.1}$$

This indicates that $\partial/\partial\psi^+ \sim O(1)$ far downstream, and we therefore adopt $U_0^2/2g$ and U_0 as the reference length and velocity in this region. Then $z^+ = z^+(f^+; \epsilon)$, where

$$z^+ = \frac{2gZ}{U_0^2} = \epsilon z \quad \text{and} \quad f^+ = \frac{2gF}{U_0} = \epsilon f = \phi^+ + i\epsilon\psi.$$

We define the outer limit to be

$$\epsilon \downarrow 0, \quad \text{with} \quad \phi^+, \psi \quad \text{fixed} \quad \phi^+ > 0; \quad \text{applied to} \quad z^+(\phi^+ + i\epsilon\psi; \epsilon),$$

whereas the inner limit was

$$\epsilon \downarrow 0, \quad \text{with} \quad \phi, \psi \quad \text{fixed,} \quad \phi < \infty; \quad \text{applied to} \quad z(\phi + i\psi; \epsilon).$$

It will be noted that ϵ does not appear in the boundary conditions, but in the actual boundary $\psi^+ = 0$, $\psi^+ = -\epsilon$.

The expression $z^+ = z^+(f^+; \epsilon)$ suggests that we could expand z^+ in a power series of the form

$$z^+ = z_0^+(\phi^+ + i\epsilon\psi) + \epsilon z_1^+(\phi^+ + i\epsilon\psi) + \ldots, \tag{4.2}$$

and with direct substitution of (4.2) into the boundary conditions; $q^2 = 1 - y^+$ on $\psi^+ = 0$ and $\psi^+ = -\epsilon$, we would obtain a sequence of non-linear, ordinary differential equations for $x_n^+(\phi^+, 0)$ and $y_n^+(\phi^+, 0)$, which could be solved.

However, we approach the problem from a different viewpoint. The following derivation is more satisfactory in that it is simpler, sheds more light on the physical problem, and leads to a series valid not only under the outer limit previously defined, but also under two other limits.

First, we change to a less cumbersome notation, writing

$$z^+ = \zeta = \xi + i\eta; \quad f^+ = \tau; \quad \phi^+ = \sigma.$$

We have then that $\log q - i\theta = \text{fn}(\tau; \epsilon)$, and therefore, by the Cauchy–Riemann relations,

$$\epsilon q^{-1} q_\sigma = -\theta_\psi, \tag{4.3}$$

$$q^{-1} q_\psi = \epsilon \theta_\sigma, \tag{4.4}$$

Also from the definitions of ϕ and ψ,

$$d\xi = q^{-1}(\cos\theta\, d\sigma - \epsilon \sin\theta\, d\psi), \tag{4.5}$$

$$d\eta = q^{-1}(\sin\theta\, d\sigma + \epsilon\cos\theta\, d\psi). \tag{4.6}$$

The boundary conditions are

$$q^2 = 1-\eta \quad \text{on} \quad \psi = 0, \quad \psi = -1. \tag{4.7}$$

By considering momentum flux in the ξ-direction, it can be shown that

$$\int_{-1}^{0} (p/q+q)\cos\theta\, d\psi = \text{const.} = 1+\tfrac{1}{4}\epsilon = E, \tag{4.8}$$

where the flow conditions far upstream have been used to evaluate the constant on the right, and p is given by

$$p = \tfrac{1}{2}(1-\eta-q^2). \tag{4.9}$$

In terms of the variable ψ, the width of the fall is $O(1)$, and derivatives with respect to ψ are $O(\epsilon)$, and so we may take as a first approximation that q and θ are independent of ψ, and also that $q^2 \gg p$. Then from (4.8) we have

$$q_0 \sim E\sec\theta_0, \tag{4.10}$$

where the subscript '0' denotes the value taken on $\psi = 0$. Also from (4.7) and (4.6), $q_0^2 = 1-\eta_0$ and

$$\left(\frac{d\eta}{d\sigma}\right)_0 = q_0^{-1}\sin\theta_0,$$

which, with (4.10) give

$$\tfrac{2}{3}(q_0^2-E^2)^{\frac{3}{2}} + 2E^2(q_0^2-E^2)^{\frac{1}{2}} \sim \sigma - \Lambda(\epsilon),$$

where $\Lambda(\epsilon)$ is a constant of integration.

We define $\lambda(\sigma,\epsilon)$ by $\theta_0 = -\lambda$; then

$$\sigma - \Lambda(\epsilon) \sim 2E^3(\tan\lambda + \tfrac{1}{3}\tan^3\lambda). \tag{4.11}$$

We may now take λ, rather than σ, to be the independent variable, and express all other quantities in terms of λ, equation (4.11) providing the link with the original variable. In this case we then have

$$q_0 \sim E\sec\lambda, \tag{4.12}$$

$$\eta_0 \sim -E^2\sec^2\lambda + 1, \tag{4.13}$$

$$\xi_0 \sim \Delta(\epsilon) + 2E^2\tan\lambda. \tag{4.14}$$

(4.14) and (4.13) clearly show the parabolic form of the fall, to the first approximation. (4.14) was constructed by using (4.5).

We express q, η, ξ and θ in the form of Taylor series about $\psi = 0$, viz.;

$$q = q_0 + (q_\psi)_0 . \psi + \dots.$$

Using (4.3)–(4.6), we can show that

$$q \sim E\sec\lambda - (\epsilon\psi\cos^3\lambda)/2E^2, \tag{4.15}$$

$$\theta \sim -\lambda - (\epsilon\psi\cos^3\lambda\sin\lambda)/2E^3, \tag{4.16}$$

$$\eta \sim -E^2\sec^2\lambda + 1 + (\epsilon\psi\cos^2\lambda)/E, \tag{4.17}$$

$$\xi \sim \Delta(\epsilon) + 2E^2\tan\lambda + (\epsilon\psi\sin\lambda\cos\lambda)/E. \tag{4.18}$$

To find a second approximation, we put $q_0 = E\sec\lambda + q_1$, and neglect all terms of $O(q_1^2)$, so that $\eta_0 = -E^2\sec^2\lambda + 1 - 2E\sec\lambda q_1$. On substituting these values into (4.15) and (4.17), and then using the new values of (4.15) and (4.17) in (4.8), we find that

$$q_1 = -(\epsilon\cos\lambda)/4E^2.$$

It is then easily shown that the full second-order approximations are

$$q \sim E\sec\lambda - (\epsilon\cos\lambda)|4E^2 - (\epsilon\psi\cos^3\lambda)|2E^2, \tag{4.19}$$ *

$$\theta \sim -\lambda - (\epsilon\psi\cos^3\lambda\sin\lambda)|2E^3, \tag{4.20}$$

$$\eta \sim -E^2\sec^2\lambda + 1 + \epsilon|2E + (\epsilon\psi\cos^2\lambda)|E, \tag{4.21}$$

$$\xi \sim \Delta(\epsilon) + 2E^2\tan\lambda + (\epsilon\psi\sin\lambda\cos\lambda)|E, \tag{4.22}$$

and

$$\frac{d\lambda}{d\sigma} \sim \cos^4\lambda/[2E^3(1-\epsilon\cos^2\lambda|4E^3)], \tag{4.23}$$

so that

$$\sigma - \Lambda(\epsilon) \sim 2E^3(\tan\lambda + \tfrac{1}{3}\tan^3\lambda) - \tfrac{1}{2}(\epsilon\tan\lambda). \tag{4.24}$$

The equations (4.19)–(4.22) have the appearance of asymptotic expansions under three different limits, namely

(i) $\epsilon \downarrow 0$ with ψ, λ fixed, and $\lambda > 0$,

(ii) $\epsilon \uparrow \infty$ with ψ, λ fixed, and $\lambda > 0$,

(iii) $\lambda \uparrow \frac{1}{2}\pi$ with ϵ, ψ fixed, and $\epsilon \geqslant 0$.

It should be noted that in the case of the limit (ii) $E = 1 + \frac{1}{4}\epsilon \sim \frac{1}{4}\epsilon$, and so $\epsilon/E \ll E$. However only in the limit (i) can the unknown constants, $\Delta(\epsilon)$ and $\Lambda(\epsilon)$, be determined by matching.

5. The matching procedure

We consider the limiting process, $\epsilon \downarrow 0$ for $f = m(\epsilon)f_m$, with f_m fixed, and $1 \ll m(\epsilon) \ll \epsilon^{-1}$, where the notation $a(\epsilon) \ll b(\epsilon)$ means $a/b \downarrow 0$, as $\epsilon \downarrow 0$; $a, b \geqslant 0$. f_m is called an intermediate variable because;

$$f = m(\epsilon)f_m \uparrow \infty, \text{ as } \epsilon \downarrow 0 \text{ with } f_m \text{ fixed,}$$

and

$$\tau = \epsilon m(\epsilon)f_m \downarrow 0, \text{ as } \epsilon \downarrow 0 \text{ with } f_m \text{ fixed.}$$

We now assume that the set of intermediate order functions $m(\epsilon)$ defines an overlap region in which the inner expansion, the outer expansion and the exact solution are all asymptotically equal. Therefore we express the inner expansion, in terms of the intermediate variables, for $f \uparrow \infty$, and the outer expansion, also in terms of the intermediate variable, for $\tau \downarrow 0$, and compare the two resulting expansions.

We have, from § 3, the result that for $f \uparrow \infty$

$$z = m(\epsilon)f_m + \epsilon\left[-\frac{i}{4}m^2(\epsilon)f_m^2 - \frac{i}{\pi}\log 2m(\epsilon)f_m + \text{const}\right] + O(\epsilon^2 m^3(\epsilon)). \tag{5.1}$$

If in (4.21) and (4.22), we express λ in a double series in ϵ and σ, and making use of (4.23), we can put the outer expansion into the form,

$$\begin{aligned} z = {} & \epsilon^{-1}[(\Delta_0 + 2\tan c_0 + i\tan^2 c_0) + \epsilon m(\epsilon)f_m\cos^2 c_0(1 - i\tan^2 c_0) \\ & + \tfrac{1}{2}\epsilon^2 m^2(\epsilon)f_m^2(-\cos^3 c_0\tan c_0 + \tfrac{1}{2}i\cos^4 c_0(2\sin^2 c_0 - 1))] + O(\epsilon^2 m^3(\epsilon)) \\ & + [(\Delta_1 + \tfrac{1}{2}\tan c_0 - \tfrac{1}{2}\sin c_0\cos c_0 + 2a_1\cos^2 c_0 + i(\tfrac{1}{2}\sin^2 c_0 - 2a_1\sin c_0\cos c_0))] \\ & + \epsilon m(\epsilon)f_m[(\tfrac{1}{2}\sec^2 c_0 + \tfrac{1}{2}\sin^2 c_0 - \tfrac{1}{2}\cos^2 c_0 - 4a_1\cos c_0\sin c_0) + i(2\sin c_0\cos c_0 \\ & - 2a_1\cos 2c_0)]\tfrac{1}{2}\cos^4 c_0 + O(\epsilon^2 m^2) + O(\epsilon), \end{aligned} \tag{5.2}$$

* for | read /.

where c_0 is the value of λ for $\epsilon = 0$ and $\sigma = 0$, and $\Delta(\epsilon) = \Delta_0 + \epsilon\Delta_1 + \ldots$. a_1 is a constant to be determined and is related to Λ_1, where

$$\Lambda(\epsilon) = \Lambda_0 + \epsilon\Lambda_1 + \ldots.$$

On comparing (5.1) and (5.2), we find that, from first-order terms

$$\Delta_0 + 2\tan c_0 = 0, \tag{5.3}$$

$$\tan^2 c_0 = 0, \tag{5.4}$$

and we can thus deduce from these that

$$c_0 = 0, \quad \Delta_0 = 0.$$

We can also deduce, from the size of the terms we have neglected, that

$$1 \ll m(\epsilon) \ll \epsilon^{-1},$$

and so for matching to one term, the overlap region is defined by

$$f = m(\epsilon) f_m; \quad 1 \ll m(\epsilon) \ll \epsilon^{-1}, \quad 0 < f_m < \infty.$$

On putting c_0 and Δ_0 to zero in (5.2), we have

$$z = m(\epsilon) f_m - \frac{i}{4}\epsilon m^2(\epsilon) f_m^2 + (\Delta_1 + 2a_1) - ia_1 \epsilon m(\epsilon) f_m + O(\epsilon^2 m^3(\epsilon)) + O(\epsilon^2 m^2(\epsilon)) + O(\epsilon). \tag{5.5}$$

On comparing (5.5) and (5.1), the first two terms in each are the same, and from the other terms we have that

$$\Delta_1 + 2a_1 = 0, \tag{5.6}$$

$$a_1 = \frac{1}{\pi}\log 2, \tag{5.7}$$

and so

$$\Delta_1 = -\frac{2}{\pi}\log 2. \tag{5.8}$$

Also from the neglected terms, we can deduce that

$$1 \ll m(\epsilon) \ll \epsilon^{-\frac{1}{2}}.$$

Therefore, for matching to two terms, the overlap region is defined by

$$f = m(\epsilon) f_m, \quad 1 \ll m(\epsilon) \ll \epsilon^{-\frac{1}{2}}, \quad 0 < f_m < \infty.$$

From (5.7) and (4.24), we find that

$$\Lambda_0 = 0, \quad \Lambda_1 = -\frac{2}{\pi}\log 2.$$

Therefore by matching we have found that

$$\Delta(\epsilon) = -\frac{2}{\pi}\log 2 \,.\, \epsilon + O(\epsilon^2),$$

and

$$\Lambda(\epsilon) = -\frac{2}{\pi}\log 2 \,.\, \epsilon + O(\epsilon^2).$$

Also, the fact that the two expansions do have the same asymptotic form in the overlap region, provides a strong indication that our assumptions, as to the form the expansions should take, were correct.

The determination of the constants in the outer solution provides a complete solution covering the whole flow field. Figures 3 and 4 show this solution, for the upper and lower streamlines, in the cases when $\epsilon = 0{\cdot}1$ and $\epsilon = 0{\cdot}5$. In the latter case, the inner solution displays a tendency towards a reversal

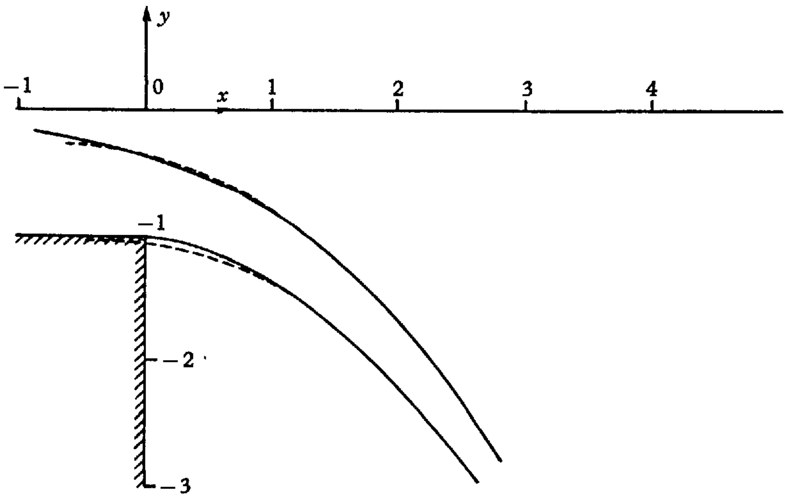

FIGURE 5. Case of $\epsilon = 2{\cdot}0$; comparison between our outer solution and the solution of Southwell & Vaisey (1946). ----, Outer solution; ———, Southwell & Vaisey solution.

in the direction of the flow, a tendency which becomes more severe with increasing ϵ. In figure 5, the outer solution is shown to be in close agreement with the Southwell & Vaisey solution for $\epsilon = 2$, though, for this case, the inner solution is such that it does not coincide with the outer solution before reversal occurs.

This work was done while the author was at the Mathematics Department, Imperial College, London. I am indebted to Mr L. E. Fraenkel for suggesting this problem, for his considerable guidance and encouragement during the course of this investigation, and for his advice on the presentation of this paper. I am also grateful to the Department of Scientific and Industrial Research for a grant during the period of this research.

REFERENCES

ERDELYI, A. 1961 An expansion procedure for singular perturbations. *Atti. Accad. Sci. Torino*, **95**, 1.

KAPLUN, S. & LAGERSTRÖM, P. A. 1957 Asymptotic expansions of Navier–Stokes solutions for small Reynolds number. *J. Math. Mech.* **6**, 585.

KELLER, J. B. & WEITZ, M. L. 1957 Theory of thin jets. *Proceedings of the ninth International Congress of Applied Mechanics, Brussels.*

SOUTHWELL, R. & VAISEY, G. 1946 Fluid motions characterized by free streamlines. *Phil. Trans.* A, **240**, 117.

WOODS, L. C. 1961 *The Theory of Subsonic Plane Flow*. Cambridge University Press.

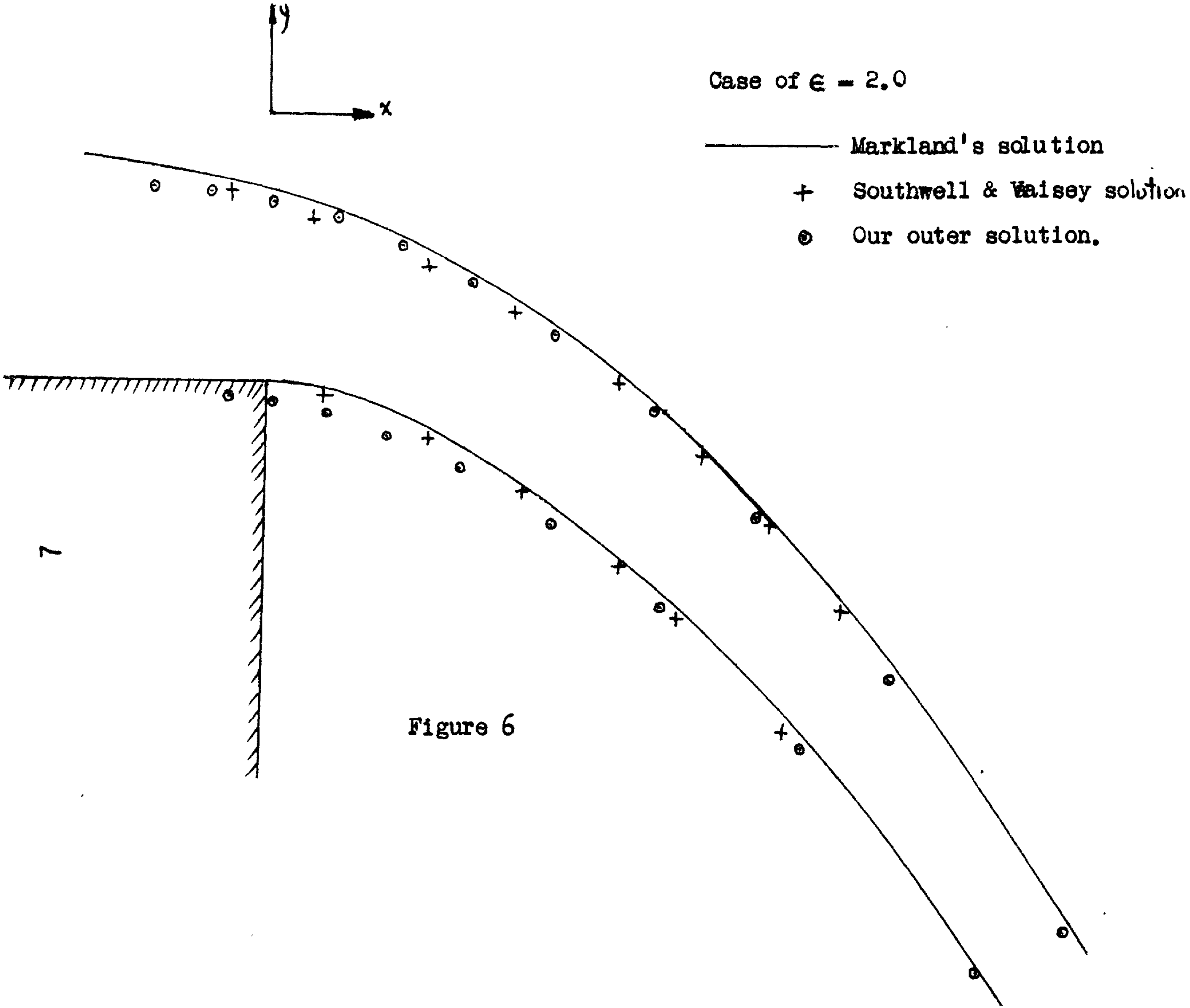
y
x
Case of $\epsilon = 2.0$
Markland's solution
Southwell & Vaisey solution
Our outer solution.

Figure 6

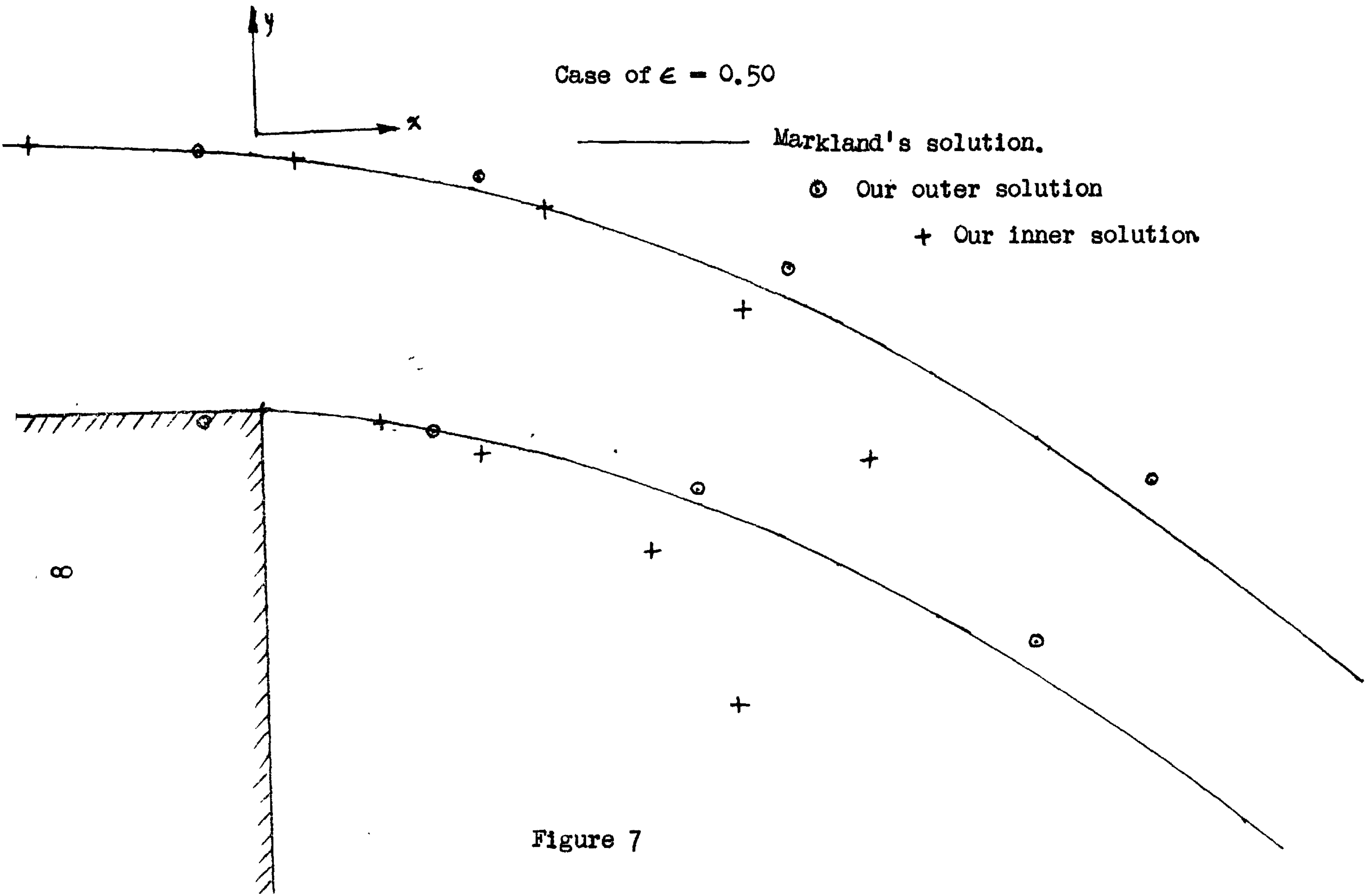
y
x
Case of $\epsilon = 0.50$
Markland's solution.
Our outer solution
Our inner solution
∞

Figure 7

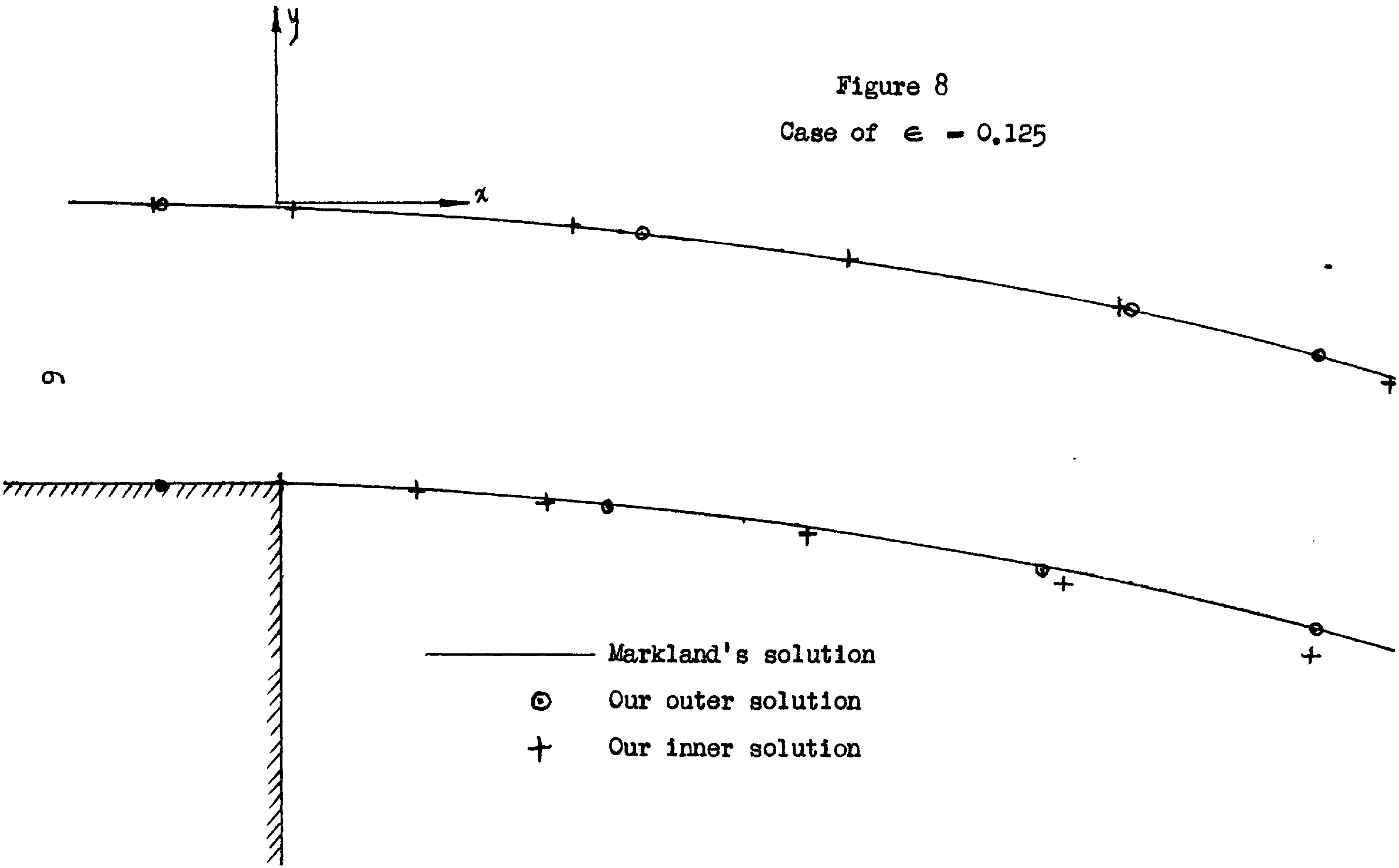

Figure 8

Case of $\epsilon = 0.125$

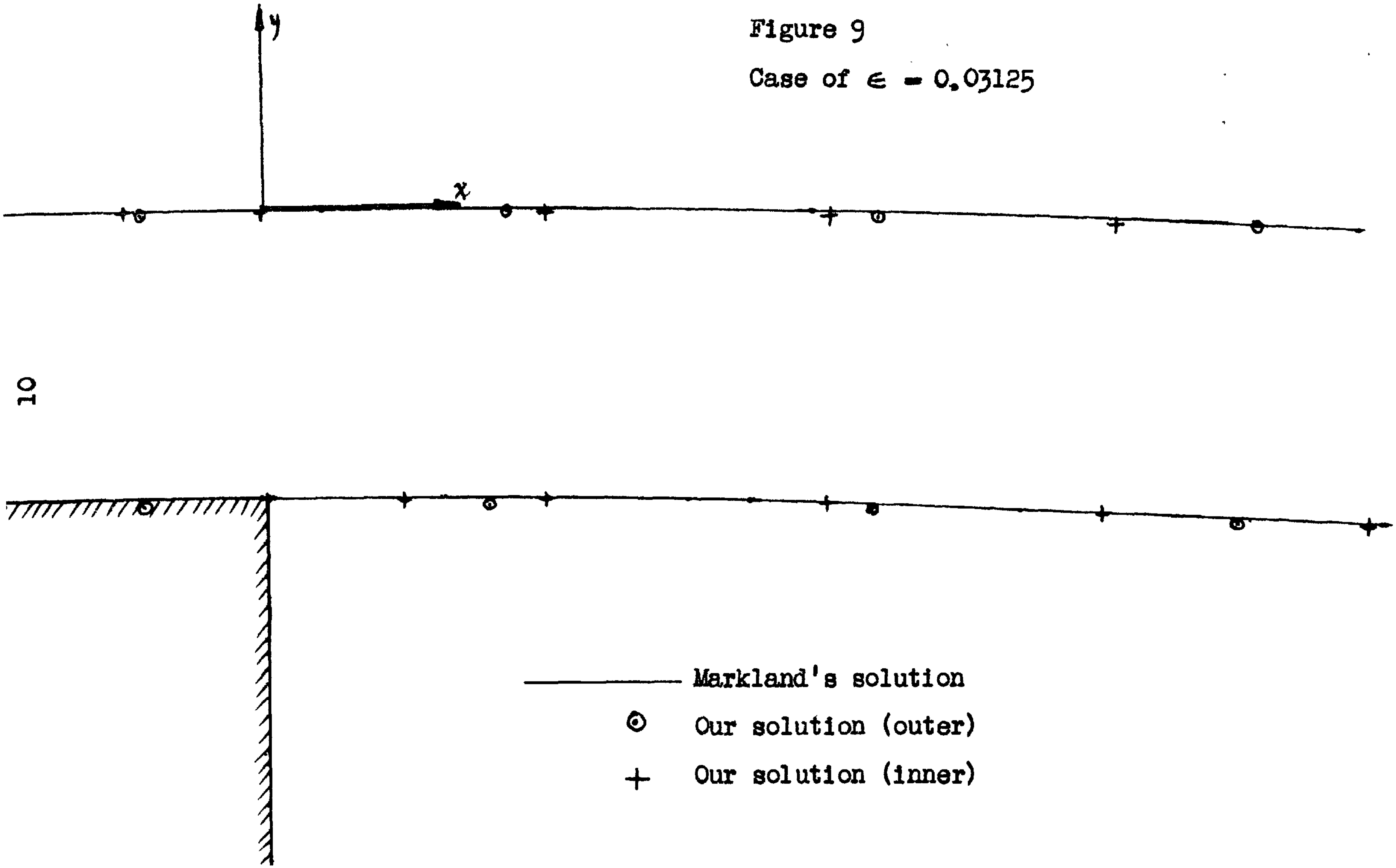

Figure 9

Case of $\epsilon = 0.03125$

In figures 6-9 we compare Markland's results with our own for the cases $\epsilon = 2.0$ (also included in this figure is the Southwell and Vaisey (1946) solution), $\epsilon = 0.50$, $\epsilon = 0.125$ and $\epsilon = 0.03125$ respectively. The agreement between the solutions can be seen to be very close for small ϵ, though a little disappointing for the cases $\epsilon = 0.5$ and $\epsilon = 2.0$, especially in the light of the close agreement between our solution and that of Southwell and Vaisey for the latter case.

REFERENCE

MARKLAND E. 1965 Calculation of flow at a free overfall by relaxation method.
Proc. Instn. civ. Engrs. 31, 78.

Part 2

The separation of a viscous fluid at an edge

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Part 2

The separation of a viscous fluid at an edge

1. Introduction

We consider the separation that a viscous fluid experiences when it encounters an abrupt change in the geometry of a solid boundary. Instances of such a change occur at the mouth of a nozzle, at the brink of a waterfall or at the trailing edge of a flat plate. The problem is considered to be two-dimensional and the fluid to be incompressible.

The problem will be idealised as follows; a viscous fluid flows along a plane, inclined at an angle β to the direction in which gravity is acting, until it encounters the end of the plane. The fluid then breaks away and is bounded below thereafter by a free streamline, initially inclined to the plane at an unknown angle α. As we shall consider only the flow in the immediate neighbourhood of the separation point, both the solid plane and the free streamline may be considered to be straight lines (see figure 1).

Michael (1958) treated this problem by separating the variables in the governing biharmonic equation, expressed in polar co-ordinates, and thereby found a number of possible solutions. We adopt an entirely different procedure. The problem is reformulated into one of finding a pair of complex functions satisfying certain boundary values. This technique has considerable generality, and will be utilised in Part 3 in a problem of greater originality. The present problem is to be seen more as an illustrative example of the technique.

Moisil (1955), and following him Langlois (1964), allude to this reformulation, though their approach is quite different from the one presented here.

2. Formulation

We use rectangular cartesian co-ordinates, taking the separation point to be the origin and the free streamline to be the x-axis. u and v are to be the components of the velocity in the x and y-directions. As we are considering only the flow very near the separation point, we shall neglect inertia effects. Accordingly, the equations of motion will be the Stokes equations.

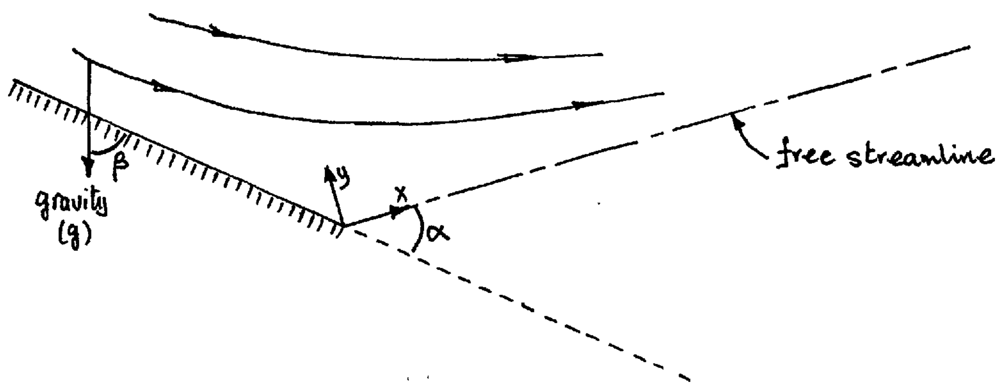


Figure 1.

The equations of motion, together with continuity, are

$$\frac{\partial p}{\partial x} = -\frac{\partial W}{\partial x} + \mu \nabla^2 u \ , \tag{2.2.1}$$

$$\frac{\partial p}{\partial y} = -\frac{\partial W}{\partial y} + \mu \nabla^2 v \ , \tag{2.2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ , \tag{2.2.3}$$

where W is the potential of a conservative body force, in our case

$$W = -\rho g \{ x \cos(\alpha+\beta) - y \sin(\alpha+\beta) \} \ . \tag{2.2.4}$$

(2.2.1) and (2.2.2) may be expressed together in terms of the stress tensor p_{ij}, as

$$p_{ij,j} - W_{,i} = 0 \ , \tag{2.2.5}$$

where

$$p_{11} = -p + 2\mu \frac{\partial u}{\partial x} \ , \tag{2.2.6}$$

$$p_{12} = p_{21} = \mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) , \tag{2.2.7}$$

$$p_{22} = -p + 2\mu \frac{\partial v}{\partial y} \ . \tag{2.2.8}$$

From (2.2.5) we infer the existence of an Airy stress function, χ , such that

$$p_{11} = \frac{\partial^2 \chi}{\partial y^2} + W \ , \tag{2.2.9}$$

$$p_{12} = p_{21} = -\frac{\partial^2 \chi}{\partial x \partial y} \ , \tag{2.2.10}$$

$$p_{22} = \frac{\partial^2 \chi}{\partial x^2} + W \,, \qquad (2.2.11)$$

and from (2.2.3) we infer the existence of a stream function ψ such that

$$\frac{\partial \psi}{\partial y} = u \quad ; \quad \frac{\partial \psi}{\partial x} = -v \,. \qquad (2.2.12)$$

If we subtract (2.2.6) from (2.2.8), and (2.2.9) from (2.2.11) and compare the two resulting equations, and compare (2.2.7) with (2.2.10), we have

$$\frac{\partial^2 \chi}{\partial x^2} - \frac{\partial^2 \chi}{\partial y^2} = -4\mu \frac{\partial^2 \psi}{\partial x \partial y} \,, \qquad (2.2.13)$$

$$\frac{\partial^2 \chi}{\partial x \partial y} = \mu\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right) . \qquad (2.2.14)$$

We now change the independent variables to $z(= x + iy)$ and $\bar{z}(= x - iy)$ (ordinarily z and $\bar{z}$ are not independent; however if we consider the more general case where x and y are both complex variables, then z and $\bar{z}$ are independent. We treat our situation as a special case of this, in which the imaginary parts of both x and y reduce to zero. In this sense z and $\bar{z}$ are independent), then (2.2.13) and (2.2.14) become

$$\frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^2 \chi}{\partial \bar{z}^2} = -i2\mu\left(\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial \bar{z}^2}\right) , \qquad (2.2.15)$$

$$\frac{\partial^2 \chi}{\partial z^2} - \frac{\partial^2 \chi}{\partial \bar{z}^2} = -i2\mu\left(\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \bar{z}^2}\right) . \qquad (2.2.16)$$

Adding these two equations we obtain

$$\frac{\partial^2}{\partial \bar{z}^2}(\chi + i2\mu\psi) = 0, \tag{2.2.17}$$

and therefore

$$\chi + i2\mu\psi = z\overline{F(z)} + \overline{G(z)}, \tag{2.2.18}$$

where the analytic functions $F(z)$ and $G(z)$ are to be determined from their boundary values.

It is worth noting here that as $\nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$, then

$$\overline{F'(z)} = \frac{1}{4}\nabla^2(\chi + i2\mu\psi) = -\frac{1}{2}(p + W + i\mu\omega),$$

where ω is the vorticity, this provides a physical interpretation of $F(z)$.

The boundary conditions to be imposed are, quite generally, that on a solid boundary, the velocity reduces to zero, and on a free streamline, the shear stress, the normal stress and the normal velocity all vanish. Mathematically these may be expressed as:

on a free streamline $p_{ij}n_j = 0$ and $v_j n_j = 0$;

on a solid boundary $v_i = 0$,

where n_i is the unit outward normal vector. In general, the location of the free streamline will be unknown and the streamline itself will be curvilinear. To overcome this difficulty, we will denote it by the equation $y = S(x)$ and treat S as another unknown dependent variable. In the particularly simple case we are considering in this section, the free streamline is rectilinear and is given by $S = 0$, though the angle α remains unknown.

The stress conditions are written

$$\left.\begin{aligned} p_{11}n_1 + p_{12}n_2 &= 0 \\ p_{21}n_1 + p_{22}n_2 &= 0 \end{aligned}\right\} \quad \text{on } y = S(x) ,$$

and the outward normal vector is $\left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$, where s is some length-like parameter measured along the free streamline. These equations are then expressed in differential form ;

$$\left.\begin{aligned} \left(\frac{\partial^2\chi}{\partial y^2} + W_s\right) dy + \left(\frac{\partial^2\chi}{\partial x\partial y}\right) dx &= 0 \\ \left(\frac{\partial^2\chi}{\partial x\partial y}\right) dy + \left(\frac{\partial^2\chi}{\partial x^2} + W_s\right) dx &= 0 \end{aligned}\right\} \quad \text{on } y = S(x) ,$$

where W_s denotes the value W takes on the free streamline. On integrating these equations (we ignore any constants of integration, as we may add any linear expression to χ without contradicting its definition), we obtain

$$\left.\begin{aligned} \frac{\partial\chi}{\partial y} &= - \int W_s . S'(x)\, dx \\ \frac{\partial\chi}{\partial x} &= - \int W_s . dx \end{aligned}\right\} \quad \text{on } y = S(x) .$$

We combine these into one complex condition

$$\frac{\partial \chi}{\partial x} + i\frac{\partial \chi}{\partial y} = -\int W_s(1+iS'(x))\,dx. \tag{2.2.19}$$

The velocity condition on the free streamline is equivalent to the condition that the stream function is a constant there. Hence

$$0 = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}.dy \quad \text{on } y = S(x),$$

and so

$$\operatorname{Im}\left\{(1-iS'(x))\left(\frac{\partial \psi}{\partial y} - i\frac{\partial \psi}{\partial x}\right)\right\} = 0 \quad \text{on } y = S(x). \tag{2.2.20}$$

On a solid boundary

$$\frac{\partial \psi}{\partial y} - i\frac{\partial \psi}{\partial x} = 0. \tag{2.2.21}$$

We now express these conditions in terms of $F(z)$ and $G(z)$. On differentiating (2.2.18) seperately with respect to x and y, and then adding and subtracting the resulting equations, we have

$$\left(\frac{\partial \chi}{\partial x} - i\frac{\partial \chi}{\partial y}\right) + i2\mu\left(\frac{\partial \psi}{\partial x} - i\frac{\partial \psi}{\partial y}\right) = 2\overline{F(z)},$$

$$\left(\frac{\partial \chi}{\partial x} + i\frac{\partial \chi}{\partial y}\right) + i2\mu\left(\frac{\partial \psi}{\partial x} + i\frac{\partial \psi}{\partial y}\right) = 2\left(z\overline{F'(z)} + \overline{G'(z)}\right).$$

Taking the conjugate of the first of these, and then adding and subtracting this with the second, we find that

$$\frac{\partial \chi}{\partial x} + i\frac{\partial \chi}{\partial y} = F(z) + z\overline{F'(z)} + \overline{G'(z)},$$

$$2\mu\left(\frac{\partial \psi}{\partial y} - i\frac{\partial \psi}{\partial x}\right) = F(z) - z\overline{F'(z)} - \overline{G'(z)}.$$

In the general problem, then, we have to find the analytic functions $F(z)$ and $G(z)$ and the function $S(x)$ which satisfy the following conditions:

(i) on a free streamline, denoted by $y = S(x)$

$$F(z) + z\overline{F'(z)} + \overline{G'(z)} = -\int W_s\,(1+i\,S'(x))\,dx \; ; \qquad (2.2.22)$$

$$\mathrm{Im}\left\{ (1-i\,S'(x))(F(z) - z\overline{F'(z)} - \overline{G'(z)}) \right\} = 0 \; ; \qquad (2.2.23)$$

(ii) on a solid boundary

$$F(z) - z\overline{F'(z)} - \overline{G'(z)} = 0 \; . \qquad (2.2.24)$$

3. The solution in the vicinity of the edge

As previously mentioned, in this case $S(x)$ is identically zero, and this of course greatly simplifies the problem. If we write $z = r.\exp(i\theta)$, then the free streamline is denoted by $\theta = 0$, and the solid plane by $\theta = \pi - \alpha = \gamma$. The conditions to be applied, are

$$F(z) + z\overline{F'(z)} + \overline{G'(z)} = \tfrac{1}{2}\rho g \cos(\alpha+\beta)\, x^2 \text{ on } \theta = 0 \; ; \qquad (2.3.1)$$

$$\mathrm{Im}\left\{ F(z) \right\} = 0 \quad \text{on} \quad \theta = 0 \; ; \qquad (2.3.2)$$

$$F(z) - z\overline{F'(z)} - \overline{G'(z)} = 0 \quad \text{on} \quad \theta = \gamma \; . \qquad (2.3.3)$$

In deriving (2.3.2) we have used (2.3.1) as well as (2.2.23).

We shall confine our interest to the region where r is small and assume that in this region $F(z)$ and $G(z)$ are algebraically small functions of z. We therefore pose that

$$F(z) = A.z^{\lambda} + \text{ smaller terms } .$$

An examination of (2.3.3) shows that $G(z)$ must be of the form

$$G(z) = B.z^{\lambda+1} + \text{ smaller terms.}$$

We can see that if the gravity terms are comparable with the viscous terms, then $\lambda = 2$. We shall show, however, that possible solutions exist for $0 < \lambda < 2$ (in fact we shall show that they must exist if we are to obtain a sensible solution). This means that the gravity terms are unimportant in the region very near the edge. For $\lambda < 2$, the boundary conditions will be homogeneous and so only the ratio B/A will be determined, the absolute values being dependent upon the flow outside the region of validity of our field equations. In this respect, and in the final solution, there is of course a great similarity with the solution for the flow in the neighbourhood of the leading edge of a semi-infinite flat plate in an unbounded fluid as given by Carrier and Lin (1948).

Therefore considering λ for $0 < \lambda < 2$ we see from (2.3.2) that A is real, and from (2.3.1) that B is also real and that $B = -A$. For other than a trivial solution (2.3.3) gives

$$e^{i\lambda\gamma} - \lambda e^{-i(\lambda-2)\gamma} + (\lambda+1)e^{-i\lambda\gamma} = 0.$$

That is

$$e^{i2\lambda\gamma} - \lambda e^{i2\gamma} + \lambda + 1 = 0 \quad .$$

Separating the real and imaginary parts of this equation:

$$\cos 2\lambda\gamma - \lambda \cos 2\gamma + (\lambda + 1) = 0 \quad , \tag{2.3.4}$$

$$\sin 2\lambda\gamma - \lambda \sin 2\gamma = 0 \quad , \tag{2.3.5}$$

(2.3.4) can be written in the form

$$\cos^2 \lambda\gamma + \lambda \sin^2 \gamma = 0 \quad . \tag{2.3.6}$$

With $\lambda > 0$, this is positive definite on the left hand side and hence we conclude that

$$\cos \lambda\gamma = 0 \quad ; \quad \sin \gamma = 0 \quad .$$

Therefore by (2.3.4) it is necessary that

$$\lambda\gamma = (2n+1)\frac{\pi}{2} \quad ; \quad \gamma = \pm\pi, \pm 2\pi, \cdots \quad ,$$

and by (2.3.5) these are sufficient. This means that

$$\alpha = 0, -\pi, \pm 2\pi, \pm 3\pi, \cdots \quad .$$

Of these, only $\alpha = 0, -\pi$ have any physically realiseable significance and so we have only the two possible sets of solutions:

$$\alpha = 0 \quad ; \quad \lambda = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \quad , \tag{2.3.7}$$

$$\alpha = -\pi \quad ; \quad \lambda = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots \quad . \tag{2.3.8}$$

We can reject the modes $\lambda = \frac{3}{2}, \frac{5}{2}, \cdots$ and $\lambda = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots$ for in these cases there exists at least one θ_j in $0 < \theta_j < \pi - \alpha$ such that $\psi(\theta_j) = 0$ which implies a situation such as that illustrated in figure 2.

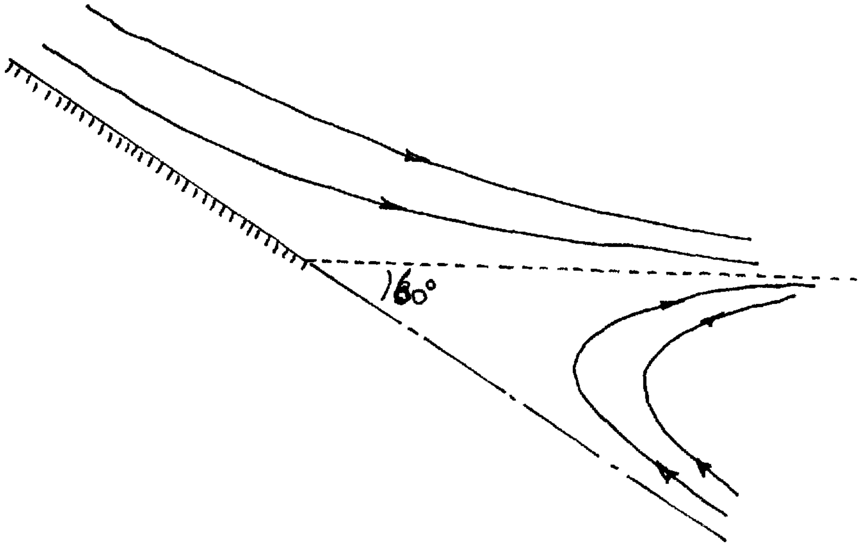


Figure 2 Qualitative diagram of the flow corresponding to the mode $\alpha = 0$, $\lambda = 3/2$.

The solution corresponding to $\lambda = 2$ (that is, the solution forced by gravity) is also unsuitable as a leading term for the following reasons. For $\lambda = 2$, α can assume three physically possible values $0, +\frac{\pi}{2}, -\frac{\pi}{2}$. In the last case, the flow region is divided into sectors such as in figure 2. The first two cases give the solutions

$$\alpha = 0 \quad ; \quad \psi = -\frac{g\cos\beta}{6\nu} y^3 ,$$

$$\alpha = +\frac{\pi}{2} \quad ; \quad \psi = \frac{g\sin\beta}{6\nu} y x^2 .$$

The first of these can be seen to represent a flow in the opposite direction to the one proposed. The second case vanishes for $\beta=0$ (i.e. when the solid plane is parallel to the direction of gravity). It is for this reason that we cannot admit this solution as a leading term.

All the rejected modes may of course appear as higher-order terms, but the mode $\lambda=\frac{1}{2}, \alpha=0$ must be present and dominant for $r \downarrow 0$.

For the mode $\alpha=0, \lambda=\frac{1}{2}$ we have

$F(z) = Az^{1/2} + \cdots \quad ; \quad G(z) = -Az^{3/2} + \cdots$, and so

$$\chi + i2\mu\psi = A\{z\bar{z}^{1/2} - \bar{z}^{3/2}\} + \cdots \quad . \tag{2.3.9}$$

The velocity components are, then, in polar co-odinates

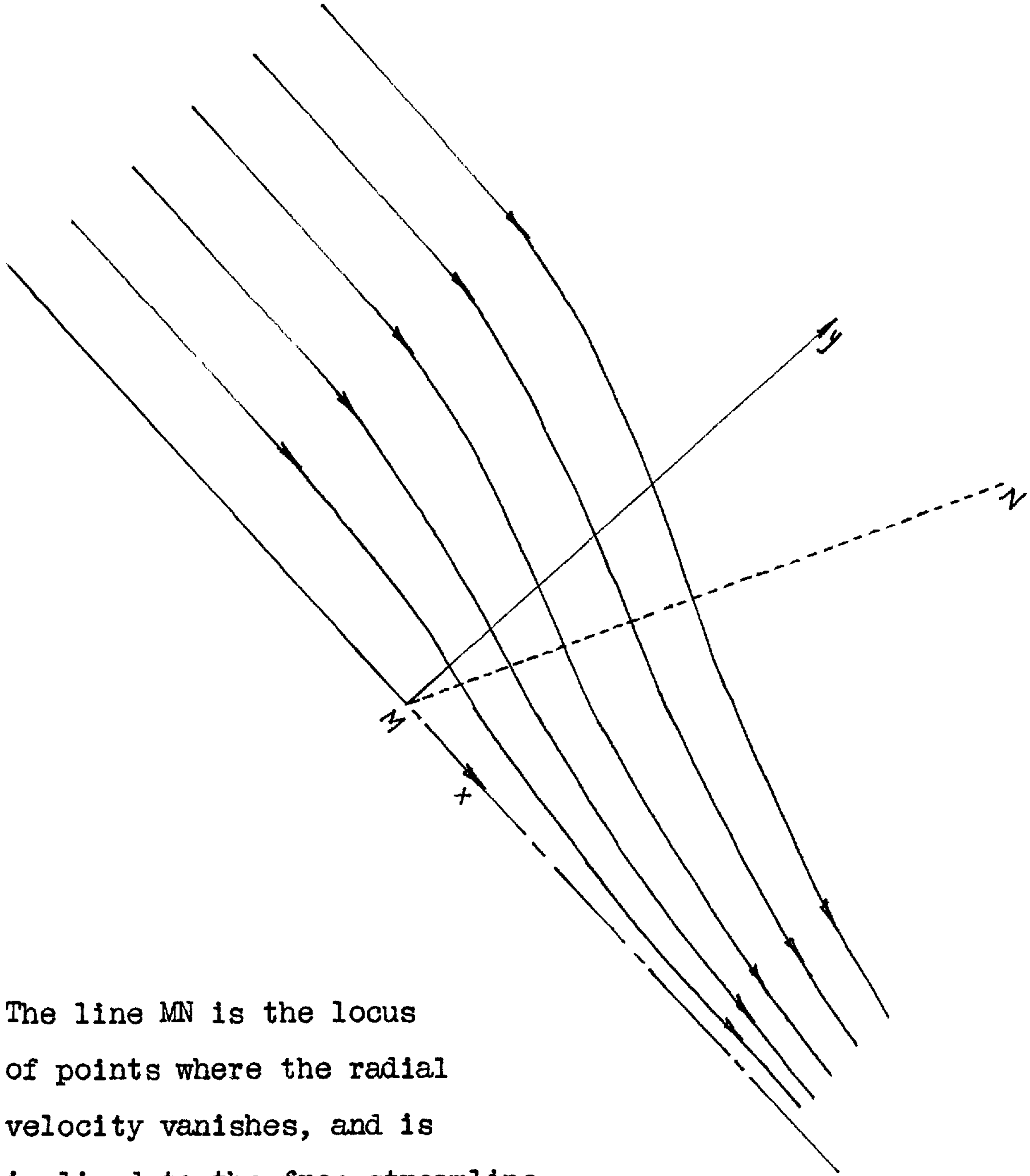
$$V_r = \frac{A}{4\mu} r^{1/2}\left(\cos\theta/2 + 3\cos 3\theta/2\right) , \tag{2.3.10}$$

$$V_\theta = -\frac{3A}{4\mu} r^{1/2}\left(\sin\theta/2 + \sin 3\theta/2\right) , \tag{2.3.11}$$

and the shear stress on the solid boundary is given by

$$\tau = -Ar^{-1/2} \quad . \tag{2.3.12}$$

The flow pattern is given qualitatively in figure 3.



The line MN is the locus of points where the radial velocity vanishes, and is inclined to the free streamline at an angle $2\cos^{-1}\sqrt{\frac{2}{3}}$.

Figure 3

References

CARRIER G. F. & LIN C. C. 1948 On the nature of the boundary layer near the leading edge of a flat plate. Quart. Appl. Math. 6, 63.

LANGLOIS W. E. 1964 Slow Viscous Flow, Collier-Macmillan.

MICHAEL D. H. 1958 The separation of a viscous liquid at a straight edge. Mathematika 5.

MOISIL Gr. C. 1955 Comunicarile, Academiei Republicii Populare Romine. Tomul V, Nr 10.

Part 3

The two-dimensional flow under gravity in a jet of viscous liquid

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The two-dimensional flow under gravity in a jet of viscous liquid

1. Introduction

An incompressible viscous fluid passes through a two-dimensional orifice and then falls vertically, and symmetrically, under the influence of gravity, bounded by two free streamlines (see figure 1). At some stage below the orifice we take a section AB, across the jet, and confine our interest to the region of the flow below this section. We leave the manner in which the fluid crosses AB (i.e. the precise nature of the velocity and stress distribution on AB) as arbitrary.

We would, of course, have liked to solve the problem in the whole of the fluid region. It was formulated and considered in some detail, but appeared to be intractable, even for Stokes flow, because of the difficulty arising from the mixed non-linear boundary conditions (the unknown function $z(t)$ describing the boundary enters these conditions in a non-linear way).

Gravity will accelerate the fluid and so by continuity there will be a contraction of the jet, thereby giving rise to viscous stresses, which will in turn produce an effect upon the velocity field. Eventually, however, we expect the jet to be extremely thin and each fluid particle to be falling as a solid body, i.e. with the inertia effects dominating the viscous effects.

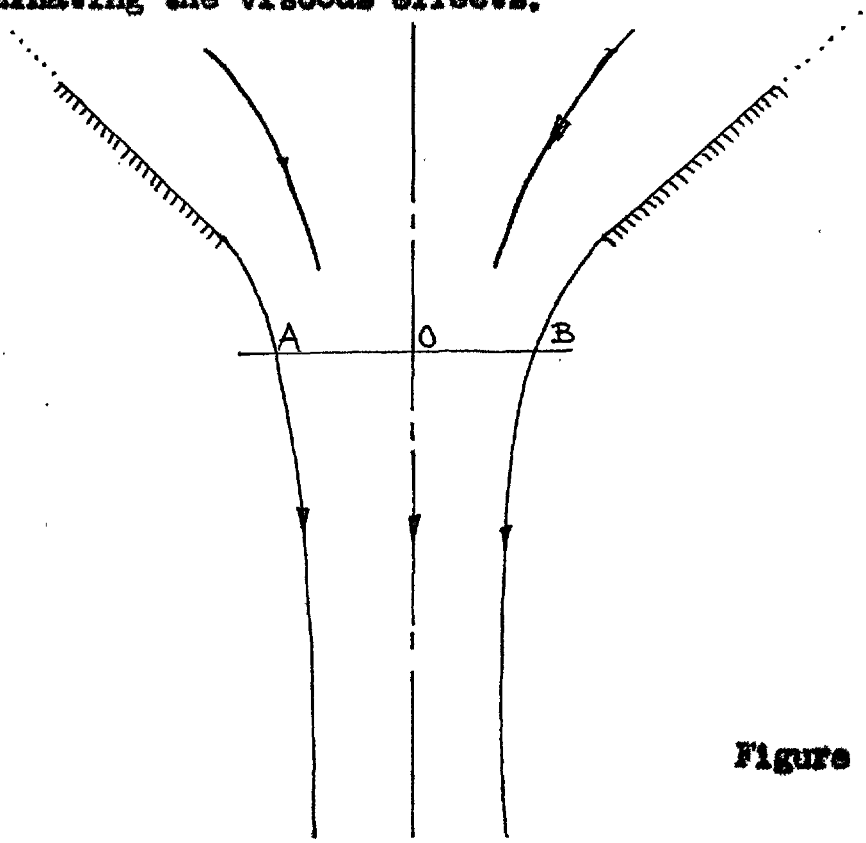

Figure 1

We take the mass flux across AB to be 2Q, and so the only parameters appearing in the problem are Q, g, and ν. Therefore the only dimensionless parameter is Q/ν : accordingly we define the Reynolds number R by $R = Q/\nu$. We shall consider R to be small. The basic length and velocity scales involving Q, g, and ν are $(\nu Q/g)^{1/3}$ and $(gQ^2/\nu)^{1/3}$.

If we non-dimensionalise the field equations with these quantities, and employ a perturbation scheme, based upon the smallness of R, then the solution so obtained will be incorrect far downstream. We will therefore use the method of matched asymptotic expansions, the perturbation parameter far downstream being the thinness of the jet coupled with the finite velocity gradients across it. The terms " inner " and " outer " have the same significance as they did in Part 1.

The inner expansion is derived by using the complex variable formalism developed in Part 2 for the Stokes equation: and then iterating to take account of the inertia effects. We were only able to express this inner expansion in a form that is valid at substantial distances from the initial station AB; this is, of course, sufficient to furnish us with the missing boundary conditions for the outer expansion. The outer expansion is derived in a manner very similar in concept to the derivation of the outer expansion in Part 1, although, because of the far more complicated algebra, the procedure is formalised. In order to utilise the thinness of the jet, distances and velocities are made non-dimensional with respect to $\nu^{2/3} g^{-1/3}$ and $\nu^{1/3} g^{1/3}$.

Strictly speaking, the two expansions should be developed side by side and matched at each stage before proceeding to the subsequent stage. However, in view of the fact that the two expansions are derived by such different methods, several terms of the inner expansion are derived here before the outer expansion is considered. Where a step in one expansion is dependent upon the previous stage in the other expansion, this will be noted and explained in the text.

Brown (1961) gave details of some experimental work on viscous sheets and in an appendix to this paper Taylor gives a derivation of the equation of motion of a one-dimensional viscous jet under gravity. This equation is the same equation as the one we derive for the leading term of our outer expansion, though the methods of derivation are very dissimilar. Maruo (1958) solved Taylor's equation numerically, though by somewhat imprecise methods. His solution is virtually indistinguishable numerically from the correct solution but in one region it is constructed upon a conceptually false basis. Here we solve the equation analytically and, on examination of this general solution, find that there are serious difficulties associated with any numerical solution.

2. The inner expansion

2.1. Formulation

We take O, the mid-point of AB, to be the origin of co-ordinates, and using rectangular cartesian co-ordinates we take the X-axis to be in the direction of gravity (i.e. along the line of symmetry) and the Y-axis in the direction OB. We denote the components of the velocity in the X and Y directions by U and V. Non-dimensional variables are defined by

$$(X,Y) = (\nu Q/g)^{1/3}(x,y) \quad : \quad (U,V) = (gQ^2/\nu)^{1/3}(u,v) \quad :$$

$$P = \rho g(\nu Q/g)^{1/3}p, \qquad (3.2.1)$$

where P is the pressure, normalised so that $P = 0$ outside the fluid region. ρ , g and ν are respectively, the fluid density, the acceleration due to gravity and the kinematic viscosity, all assumed to be constant. We define, as the only dimensionless parameter appearing in the problem, a Reynolds number $R = Q/\nu$ and consider this to be small. If we denote the velocity vector by $\underline{q} = (u,v)$ then we express the Navier-Stokes equations in terms of the non-dimensional variables

$$R(\underline{q}.\nabla)\underline{q} + \underline{\nabla} p = \underline{i} + \nabla^2 \underline{q} \quad , \qquad (3.2.2)$$

Where $\underline{i}$ is the gravity term, equivalent to (1,0). The condition for continuity becomes

$$\nabla.\underline{q} = 0. \qquad (3.2.3)$$

The boundary conditions to be applied are that the shear stress, the normal stress and the normal velocity are all zero on the free streamlines. As the free streamlines are also unknown we shall denote them, in view of symmetry, by $y = \pm S(x;R)$.

2.2. The zero-order approximation

The dependent variables are functions of x,y and R. We assume that we can expand $\underline{q}$, p and S in a series in R (or in functions of R) and that the first term in each will be independent of R. We obtain the zero-order field equations formally by putting R = 0 in (3.2.2) and (3.2.3) :

$$\underline{\nabla} p_0 = \underline{i} + \nabla^2 \underline{q}_0 \quad , \tag{3.2.4}$$

$$\nabla \cdot \underline{q}_0 = 0 \quad . \tag{3.2.5}$$

The equations (3.2.4) are, of course, the Stokes equations.

We now recall the technique in Part 2, and on the basis of this we construct a dimensionless Airy function χ_0 and a dimensionless stream function ψ_0 , and hence two analytic functions $F_0(z)$ and $G_0(z)$ such that

$$\chi_0 + i2\psi_0 = z\overline{F_0(z)} + \overline{G_0(z)} \quad . \tag{3.2.6}$$

By using the results of Part 2(2), our problem is to find

$F_o(z)$, $G_o(z)$ and a curve $y = S_o(x)$ such that

$$F_0(z) + z\overline{F_0'(z)} + \overline{G_0'(z)} = \tfrac{1}{2}x^2 + i\int x S_0'(x)\,dx \,, \qquad (3.2.7)$$

$$\operatorname{Im}\left\{\left[1 - iS_0'(x)\right]\left[F_0(z) - z\overline{F_0'(z)} - \overline{G_0'(z)}\right]\right\} = 0 \,, \qquad (3.2.8)$$

on $y = S_o(x)$.

We can see from (3.2.7) and (3.2.8) that, if a solution exists, then for large values of x we must have

$$F_o(z) \sim O(z^2) \quad ; \quad G_o(z) \sim O(z^3) \quad ; \quad S_o(x) \sim O(x^{-2}) \,.$$

The last of these, though not at first obvious, may be obtained by putting $S_o(x) \sim O(x^{-n})$ into (3.2.8) (choosing the negative power to ensure continuity) and finding that $n = 2$. Hence on putting

$$F_o(z) = a_o z^2 + \text{smaller terms} \,,$$
$$G_o(z) = b_o z^3 + \text{smaller terms} \,,$$
$$S_o(x) = c_o x^{-2} + \text{smaller terms} \,,$$

and inserting these into (3.2.7) and (3.2.8), we find that $a_o = 3/8$ and $b_o = -5/24$. c_o remains undetermined, as (3.2.8) and the imaginary part of (3.2.7), from which c_o could be determined, are both homogeneous in c_o. However, by the way in which the quantities were made dimensionless, the free streamlines are denoted by $\psi = \pm 1$, and so by considering the mass flux across any section we require that

$$\int_0^{S_0(x)} u_0(x,y)\,dy = 1 \,. \qquad (3.2.9)$$

Recalling that $u_0 + iv_0 = \frac{1}{2}(F_0 - z\overline{F_0'} - \overline{G_0'})$, we may use this in (3.2.9) to find that $c_0 = 8$.

The error involved in neglecting the smaller terms is of order x^{-6} times the retained terms, and so for a second approximation we put

$$F_0(z) = a_0 z^2 + a_{01} z^{-4} + \text{smaller terms},$$
$$G_0(z) = b_0 z^3 + b_{01} z^{-3} + \text{smaller terms},$$
$$S_0(x) = C_0 x^{-2} + c_{01} x^{-8} + \text{smaller terms}.$$

Again, on inserting these into (3.2.7) and (3.2.8), we find

$$a_{01} = -76/5 \quad : b_{01} = 548/15 \quad : c_{01} = 1792/15.$$

The error again is $O(x^{-6})$ times the smallest of the retained terms. Further terms in the expansion may be calculated in the same way, but we shall not do so here.

As can be seen, there are no arbitrary terms involved in this solution at any stage and yet we have not specified any initial conditions. This objection may be overcome by interpreting this solution as a "particular" solution forced by gravity to which we may add "complementary" solutions (by analogy with the terms as used in the theory of linear boundary-value problems; as remarked previously this problem is non-linear because of the form of the boundary conditions). For large values of x, these complementary solutions must be of lower order than the particular solution. If we introduce into F_0 a term $\alpha_0 z$, and into G_0 a term $\beta_0 z^2$, we find from

(3.2.7) and (3.2.8) that $\beta_0 = -\alpha_0$ and that a term $-64\alpha_0 X^{-3}$ must be introduced into $S_0(x)$. In addition it gives rise to terms $2\alpha_0^2$ and $-2\alpha_0^2 z$ in F_0 and G_0 and a term $384\alpha_0^2 X^{-4}$ in S_0 and so on. α_0 remains arbitrary and may only be determined from the initial conditions. The solution to the zero-order problem, written in the form of an expansion for large x, is then

$$F_0(z) = \frac{3}{8}z^2 + \alpha_0 z + 2\alpha_0^2 - \frac{76}{5}z^{-4} + \frac{1026}{5}\alpha_0 z^{-5} + \cdots\cdots \quad , \qquad (3.2.10)$$

$$G_0(z) = -\frac{5}{24}z^3 - \alpha_0 z^2 - 2\alpha_0^2 z + \frac{548}{15}z^{-3} - \frac{1026}{5}\alpha_0 z^{-4} + \cdots \quad , \qquad (3.2.11)$$

$$S_0(x) = 8X^{-2} - 2^6\alpha_0 X^{-3} + 3.2^7\alpha_0^2 X^{-4} - 2^{11}\alpha_0^3 X^{-5} + 5.2^{11}\alpha_0^4 X^{-6} - 3.2^{14}\alpha_0^5 X^{-7} + \left(7.2^{15}\alpha_0^6 + \frac{1792}{15}\right)X^{-8} - \left(2^{20}\alpha_0^7 - 2^7\,\frac{749}{15}\alpha_0\right)X^{-9} + \cdots \quad . \qquad (3.2.12)$$

When we come to match the inner and outer expansions we shall want a form of u_0 for large x on the line of symmetry (the matching for v_0 and the off-centre terms of u_0 is automatically accomplished when we match u_0 on $y = 0$). This is then, from (3.2.10) and (3.2.11)

$$u_0 \sim \frac{1}{8}X^2 + \alpha_0 X + 2\alpha_0^2 + \cdots \quad . \qquad (3.2.13)$$

2.3. The first-order approximation

At this stage we could calculate the dominant term of the outer expansion, matching it to (3.2.13) in order to fix an arbitrary constant in the outer expansion. It would then appear that, if the two expansions are to match to higher orders, the inner expansions for $\underline{q}$ and p must be of the form

$$\underline{q} = \underline{q}_0 + R^{1/3}\underline{q}_1 + R^{2/3}\underline{q}_2 + R\underline{q}_3 + R^{4/3}\underline{q}_4 + \cdots, \quad (3.2.14)$$

$$p = p_0 + R^{1/3}p_1 + R^{2/3}p_2 + Rp_3 + R^{4/3}p_4 + \cdots .$$

Also it is only the matching that can force the existance of $\underline{q}_1$ and p_1. As we shall see, $\underline{q}_1$ and p_1 vanish identically. We shall ignore these terms, rather than include them and their manifestations in the higher-order approximations, only for their effects to be eradicated upon formal matching. With $\underline{q}_1$ and p_1 absent, it is again only the matching that forces the existence of $\underline{q}_2$ and p_2. These terms, we shall see, are present. These assertions will be elaborated later: we turn now to the second-order approximations.

2.4. The second-order approximation

By (3.2.14) the field equations for the second-order problem are

$$\underline{\nabla} p_2 = \nabla^2 \underline{q}_2 \quad ,$$

and $$\nabla \cdot \underline{q}_2 = 0 \quad .$$

Therefore, again employing the technique in Part 2, we construct χ_2 and ψ_2, together with $F_2(z)$ and $G_2(z)$ such that

$$\chi_2 + i\,2\psi_2 = z\overline{F_2(z)} + \overline{G_2(z)} .$$

The boundary will experience a shift of order $R^{2/3}$ (i.e. S has the form

$$S(x;R) = S_0(x) + R^{1/3}S_1(x) + R^{2/3}S_2(x) + RS_3(x) + \cdots \qquad),$$

and so in the boundary conditions we must also include the zero-order terms as they will contribute some terms of order $R^{2/3}$ to the stress and velocity conditions on the new boundary. The total boundary conditions are then

$$\left\{F_0 + z\overline{F_0}' + \overline{G_0}'\right\} + R^{2/3}\left\{F_2 + z\overline{F_2}' + \overline{G_0}'\right\} = \tfrac{1}{2}x^2 + i\int x\left(S_0' + R^{2/3}S_2'\right)dx,$$

$$\operatorname{Im}\left\{\left[1 - i\left(S_0' + R^{2/3}S_2'\right)\right]\left[\left(F_0 - z\overline{F_0}' - \overline{G_0}'\right) + R^{2/3}\left(F_2 - z\overline{F_2}' - \overline{G_2}'\right)\right]\right\} = 0, \qquad (3.2.15)$$

on $y = S_0(x) + R^{2/3}S_2(x)$.

Using hints from the matching we look for a solution of the form

$$F_2(z) = a_2 z^4 + \text{smaller terms},$$

$$G_2(z) = b_2 z^5 + \text{smaller terms},$$

$$S_2(x) = c_2 + \text{smaller terms}.$$

Substituting these values into (3.2.15), taking care to absorb all the terms from the zero-order terms which contain $R^{2/3}.S_2(x)$, we find that $b_2 = -a_2$ and $c_2 = -64a_2$ where a_2 remains undetermined. For matching purposes the second-order contribution to u on the line of symmetry is $R^{2/3}(a_2x^4+\ldots\ldots)$. a_2 is determined from the outer expansion by matching. The terms, smaller in z, may be found in the same way as before.

2.4. The third-order approximation

Here the inertia terms make their first appearance. The field equations are

$$(\underline{q}_0.\nabla)\underline{q}_0 + \underline{\nabla} p_3 = \nabla^2 \underline{q}_3 \;,$$

$$\nabla.\underline{q}_3 = 0 \;.$$

In Appendix A we develop a procedure which allows us to extend the complex variable formalism of Part 2 to the n^{th} iteration. By the results of Appendix A, we construct χ_3 and ψ_3 such that

$$\frac{\partial^2}{\partial z^2}(\chi_3 + i2\psi_3) = \frac{1}{4}(u_0 - iv_0)^2$$

$$= \frac{1}{2^{10}}(5z^2 - 6z\bar{z} + 3\bar{z}^2)^2 \;,$$

i.e. $\chi_3 + i2\psi_3 = z\overline{F_3(z)} + \overline{G_3(z)} + \frac{1}{2^{10}}\left(\frac{5}{6}z^6 - 3z^5\bar{z} + \frac{11}{2}z^4\bar{z}^2 - 6z^3\bar{z}^3 + \frac{9}{2}z^2\bar{z}^4\right)$,

where again $F_3(z)$ and $G_3(z)$ are unknown analytic functions. We have retained only the leading terms of $u_0 - iv_0$ as only these will contribute to the leading terms of F_3 and G_3.

The total boundary conditions to the order (in R) required are then

$$\left\{F_0+z\overline{F_0'}+\overline{G_0'}\right\}+R^{2/3}\left\{F_2+z\overline{F_2'}+\overline{G_2'}\right\}+R\left\{F_3+z\overline{F_3'}+\overline{G_3'}\right\}+R2^{-11}\left\{-6z^5+40z^4\bar{z}\right.$$
$$\left.+12z^3\bar{z}^2+80z^2\bar{z}^3-30z\bar{z}^4+10\bar{z}^5\right\}=\tfrac{1}{2}x^2+i\int x\left(S_0'+R^{2/3}S_2'+RS_3'\right)dx, \tag{3.2.16}$$

$$\operatorname{Im}\left\{\left[1-i\left(S_0'+R^{2/3}S_2'+RS_3'\right)\right]\left[\left(F_0-z\overline{F_0'}-\overline{G_0'}\right)+R^{2/3}\left(F_2-z\overline{F_2'}-\overline{G_2'}\right)\right.\right.$$
$$\left.\left.+R\left(F_3-z\overline{F_3'}-\overline{G_3'}\right)+R2^{-11}\left(6z^5-4z^4\bar{z}+8z^2\bar{z}^3-30z\bar{z}^4+10\bar{z}^5\right)\right]\right\}=0,$$

on $y = S_0(x) + R^{2/3}S_2(x) + RS_3(x)$.

The solutions are here forced by the inertia terms and so we look for solutions of the form

$$\begin{aligned} F_3(z) &= a_3 z^5 + \text{smaller terms}, \\ G_3(z) &= b_3 z^6 + \text{smaller terms}, \\ S_3(x) &= c_3 x + \text{smaller terms}. \end{aligned}$$

Substituting these into (3.2.16), and again collecting all terms of order R from the lower-order terms arising from the boundary shift, we find that

$$a_3 = -2^{10} \quad ; \quad b_3 = -\frac{5}{6}\,2^{-10} \quad ; \quad c_3 = -1/8 \quad .$$

The leading contribution to u on the line of symmetry is

$$R\,2^{-9}x^5$$

We could in principle pursue our calculations further, but in view of the similarity of method and the absence of any salient results we will divert our interest to the outer expansion.

3 The outer expansion

3.1. Formulation

Far downstream we expect the jet to become very thin and the variations in the velocities and stresses across it to become small.

We define non-dimensional variables by

$$(X,Y) = \nu^{2/3} q^{-1/3}(\hat{x},\hat{y}) \quad : \quad (U,V) = \nu^{1/3} q^{1/3}(\hat{u},\hat{v}) \quad : \qquad (3.3.1)$$
$$P = \rho \nu^{2/3} q^{2/3}\, \hat{p} \quad : \quad \Psi = R\nu\psi \quad .$$

Whence $(x,y) = R^{-1/3}(\hat{x},\hat{y})$, $(u,v) = R^{-2/3}(\hat{u},\hat{v})$ and $p = R^{-1/3}\hat{p}$.

The quantities denoted by capital letters are as in the inner problem. Ψ , the dimensional stream function, is made non-dimensional as in (3.3.1) so that the free streamlines are again denoted by $\psi = \pm 1$. $R = Q/\nu$ is the Reynolds number, as defined in the inner problem. The way in which we make Y non-dimensional does not, of course, make $\hat{y}$ of order unity in the region considered (unlike $\hat{x}$), but this is immaterial as we shall be treating $\hat{y}$ as a dependent variable in what follows. We shall now omit the symbol $\hat{()}$ for convenience, and restore it when we come to match the two expansions formally.

As before the free streamlines are unknown in terms of x and y, but are given by $\psi = \pm 1$. We will therefore consider the problem to be in the ζ-plane, where $\zeta = (\varphi, R\psi)$. Here φ is defined by:

$\varphi = x$ on the line of symmetry, and the lines φ = constant are everywhere orthogonal to the lines $R\psi$ = constant. That is, φ and $R\psi$ constitute an orthogonal curvilinear co-ordinate system.

If we put $q = (u^2 + v^2)^{\frac{1}{2}}$, then the velocity components with respect to $(\varphi, R\psi)$ are $(q, 0)$ whereas those with respect to (x, y) were $(u, v) = (q\cos\theta, -q\sin\theta)$; this defines the angle θ .

The two planes are linked by the following transformation

$$x = \varphi + R\int_0^{\psi} \frac{\sin\theta}{q}\, d\psi' , \qquad (3.3.2)$$

$$y = R\int_0^{\psi} \frac{\cos\theta}{q}\, d\psi' . \qquad (3.3.3)$$

The arc length parameter associated with $R\psi$ is q^{-1} and that with φ , we denote by h. The formal definition of h is $h = \left\{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2\right\}^{1/2}$, and so h could be derived in terms of q and θ from (3.3.2) and (3.3.3). This is, however, an arduous task and not particularly illuminating: we adopt an alternative approach. Consider a constant vector (without any loss of generality we shall use the vector $\underline{\nabla} x$, which in this problem is representative of gravity). In the ζ-plane

$$\underline{\nabla} x = \left(\frac{1}{h}\frac{\partial x}{\partial \varphi}, \frac{q}{R}\frac{\partial x}{\partial \psi}\right) = \left(\frac{1}{h}\frac{\partial x}{\partial \varphi}, \sin\theta\right) .$$

As the modulus of $\underline{\nabla} x$ is unity, $\frac{1}{h}\frac{\partial x}{\partial \varphi} = \cos\theta$ (the sign being implied by (3.3.2)), and therefore

$$\underline{\nabla} x = (\cos\theta, \sin\theta) \quad .$$

We know that $\underline{\nabla}\wedge\underline{\nabla} x$ and $\nabla\cdot\underline{\nabla} x$ both vanish identically, and therefore expressing these in the ζ -plane we have

$$\frac{\partial}{\partial\varphi}\left(\frac{\cos\theta}{q}\right) + \frac{1}{R}\frac{\partial}{\partial\psi}\left(h\sin\theta\right) = 0 \quad , \qquad (3.3.4)$$

$$\frac{\partial}{\partial\varphi}\left(\frac{\sin\theta}{q}\right) - \frac{1}{R}\frac{\partial}{\partial\psi}\left(h\cos\theta\right) = 0 \quad . \qquad (3.3.5)$$

From these equations we may deduce, quite simply, that

$$h = Rq_\varphi/(q^2\theta_\psi) \quad , \qquad (3.3.6)$$

$$h_\psi = R\theta_\varphi/q \; , \qquad (3.3.7)$$

where the subscripts denote differentiation with respect to the variable indicated.

An interesting comparison may be drawn here with the outer problem of Part 1. Equations (3.3.2) and (3.3.3), defining streamline co-ordinates, are direct parallels with those of Part 1 (equations (4.5) and (4.6)), and (3.3.4) and (3.3.5) bear a very strong resemblance to the Cauchy-Riemann equations (equations (4.3) and (4.4) of Part 1). However we shall not have a momentum integral equation because of the symmetry of the present problem, and so we shall have to use the full Navier-Stokes equations. These become in the ζ-plane

$$q\frac{\partial q}{\partial \varphi} + \frac{\partial p}{\partial \varphi} = h\cos\theta + R^{-2} qh \frac{\partial}{\partial \psi}\left\{ \frac{q}{h}\frac{\partial}{\partial \psi}(qh) \right\} , \qquad (3.3.8)$$

$$-\frac{q^2}{h}\frac{\partial h}{\partial \psi} + \frac{\partial p}{\partial \psi} = \frac{R\sin\theta}{q} - \frac{1}{qh}\frac{\partial}{\partial \varphi}\left\{ \frac{q}{h}\frac{\partial}{\partial \psi}(qh) \right\} . \qquad (3.3.9)$$

The equation of continuity is automatically satisfied by the use we have made of the stream function as an independent variable. We have then, the four equations (3.3.6)-(3.3.9) for the four unknowns q, h, p and θ.

The boundary conditions are particularly simple : from the stress tensor we can calculate that the zero shear and normal stress conditions to be applied on $\psi = \pm 1$, are

$$R^{-1} qh \frac{\partial}{\partial \psi}(q/h) = 0 , \qquad (3.3.10)$$

$$p = -\frac{2}{h}\frac{\partial q}{\partial \varphi} ; \qquad (3.3.11)$$

or, in view of the fact that we shall not be considering the region in which q could vanish on the free streamline, and as h is non-zero, (3.3.10) becomes

$$\frac{\partial}{\partial \psi}(q/h) = 0 . \qquad (3.3.12)$$

From symmetry considerations, it can be seen that q, h, and p will be even functions of ψ, and θ will be an odd function of ψ, and hence of $R\psi$ (as in the field equations R and ψ always appear in conjunction with one another). We therefore pose that

$$q = q_0 + R^2\psi^2 q_2 + R^4\psi^4 q_4 + \cdots \quad , \quad (3.3.13)$$

$$h = 1 + R^2\psi^2 h_2 + R^4\psi^4 h_4 + \cdots \quad , \quad (3.3.14)$$

$$p = p_0 + R^2\psi^2 p_2 + R^4\psi^4 p_4 + \cdots \quad , \quad (3.3.15)$$

$$\theta = R\psi\,\theta_1 + R^3\psi^3\theta_2 + \cdots \quad . \quad (3.3.16)$$

Although R and ψ attend one another in the field equations, R does not appear explicitly with ψ in the designation of the boundary. We therefore assert that the coefficients of $R\psi$ in (3.3.13)-(3.3.16) will be functions of φ and R. When we match the outer expansion and the inner expansion, it becomes apparent that the coefficients must have the form, to take a typical example

$$q_0(\varphi;R) = q_{00}(\varphi) + R^{1/3}q_{01}(\varphi) + R^{2/3}q_{02}(\varphi) + \cdots \quad . \quad (3.3.17)$$

3.2. The derivation of the equations for q_{ij} etc.

We use the equations (3.3.6) and (3.3.7) and the boundary conditions (3.3.11) and (3.3.12) to obtain relationships between the coefficients in the expansions of the four dependent variables. The expressions (3.3.13)-(3.3.16) together with (3.3.17) and its counterparts are substituted into (3.3.6) and (3.3.7) and coefficients of R compared.

We find on some rearrangement that

(i) $\theta_{10} = q'_{00}/q^2_{00}$,

(ii) $\theta_{11} = q'_{01}/q^2_{00} - 2q_{01}q'_{00}/q^3_{00}$ etc.,

(iii) $h_{20} = \frac{1}{2}q''_{00}/q^3_{00} - q'^2_{00}/q^4_{00}$,

(iv) $h_{21} = \frac{1}{2}q''_{01}/q^3_{00} - \frac{3}{2}q''_{00}q_{01}/q^4_{00} - 2q'_{00}q'_{01}/q^4_{00} + 4q'^2_{00}q_{01}/q^5_{00}$ etc.

Here ()' denotes differentiation with respect to φ. Similarly using (3.3.11) and (3.3.12) we find that

(v) $p_{00} = -2q'_{00}$,

(vi) $p_{01} = -2q'_{01}$ etc.,

(vii) $q_{20} = q_{00}h_{20}$,

(viii) $q_{21} = q_{01}h_{20} + q_{00}h_{21}$ etc.

We now substitute the values given in (i)-(viii) into (3.3.13)-(3.3.17) and thence into (3.3.8) (and at a later stage into (3.3.9)) and comparing coefficients of R, we find that

$$4q''_{00} - 4q'^2_{00}/q_{00} - q_{00}q'_{00} + 1 = 0 \quad , \qquad (3.3.18)$$

$$4q''_{0n} - (q_{00} + 8q'_{00}/q_{00})q'_{0n} + (4q'^2_{00}/q^2_{00} - q'_{00})q_{0n} = M_n(\varphi), \qquad (3.3.19)$$

where $$\left.\begin{aligned} &M_1 = 0 \quad , \\ &M_2 = q_{01}q_{01}'\left(1-8q_{00}'/q_{00}^2\right)+4q_{00}'^2q_{01}^2/q_{00}^3+4q_{01}'^2/q_{00} \quad , \\ &\text{etc} \end{aligned}\right\} \qquad (3.3.20)$$

3.3. The solutions of the equations for q_{ij} etc.

By the relationships (i)-(viii) and their higher-order counterparts, we may express the coefficients of $R\psi$ in the expansions (3.3.13)-(3.3.16) solely in terms of the functions $q_{01}(\varphi)$, and these functions are given by the solutions of the differential equations (3.3.18)-(3.3.20). Therefore, by solving these equations we will obtain a solution to the problem in the outer region. We now derive exact solutions for q_{00} and q_{01} by analytic means and indicate how the higher-order functions may be obtained.

Firstly we will make a slight transformation of the variables to simplify the arithmetic. We put $\varphi = 4^{2/3}.\sigma$ and $q_o = 4^{1/3}.f(\sigma)$. In terms of these variables, (3.3.18) becomes

$$f_0'' - f_0'^2/f_0 - f_0 f_0' + 1 = 0 \quad , \qquad (3.3.21)$$

where ()' now denotes differentiation with respect to σ. This is the form in which Taylor (see Brown (1961)) expressed his equation. Similarly (3.3.19) becomes

$$f_n'' - (f_0 + 2f_0'/f_0)f_n' + (f_0'^2/f_0^2 - f_0')f_n = M_n(\sigma) \quad , \qquad (3.3.22)$$

the $M_n(\sigma)$ here being the direct transformation of the $M_n(\varphi)$ in (3.3.19).

We first consider equation (3.3.21), which is of a type discussed by Ince (1926 p.325). By the substitution $f_o = w^{-1}$, the equation is reduced to the canonical form

$$w'' = w'^2/w + w'/w + w^2 \quad . \qquad (3.3.23)$$

Ince gives a first integral of this equation as

$$(w'+1)^2 = 2w^2(w+\sigma+k) \quad ,$$

where k is a constant of integration. We change the independent variable for temporary convenience by writing

$\sigma = t - k$ giving

$$\left(\frac{dw}{dt}+1\right)^2 = 2w^2(w+t) \quad .$$

Putting $w+t = \tilde{w}^2$, we have

$$\frac{d\tilde{w}}{dt} = \pm\frac{1}{\sqrt{2}}(\tilde{w}^2 - t) \quad ,$$

which is a form of Ricatti's equation. Following the standard procedure for solving this equation,we put

$\tilde{w} = \mp\sqrt{2}\,\frac{1}{\tilde{v}}\frac{d\tilde{v}}{dt}$, and then $t = 2^{1/3}r$ to give

$$\frac{d^2\tilde{v}}{dr^2} - r\tilde{v} = 0 \quad ,$$

which is Airy's equation, and therefore

$$\tilde{V} = C\,Ai(r) + D\,Bi(r) .$$

Retracing our steps through the substitutions, we find that the general solution of (3.3.21) is given by

$$f_0 = \frac{1}{(\sigma+k)}\left\{\frac{2^{1/3}}{(\sigma+k)}\left[\frac{C Ai'[2^{-1/3}(\sigma+k)] + D Bi'[2^{-1/3}(\sigma+k)]}{C Ai[2^{-1/3}(\sigma+k)] + D Bi[2^{-1/3}(\sigma+k)]}\right]^2 - 1\right\}^{-1} , \qquad (3.3.24)$$

where the ()' associated with the Airy functions denotes differentiation with respect to the arguments of those functions.

For the particular solution which satisfies the conditions of our problem we must assign values to k and the ratio C:D. From the matching we must impose the condition that $f_o = 0$ at $\sigma = 0$. We must have then, from (3.3.24), that

$$\left[\frac{C Ai'(k^*) + D Bi'(k^*)}{C Ai(k^*) + D Bi(k^*)}\right]^2 - k^* \qquad \text{is infinite,}$$

where $k^* = 2^{-1/3}k$. The only way in which this expression can become infinite (other than for infinite k^*, a case which we disregard) is for

$$C Ai(k^*) + D Bi(k^*) = 0 \quad ,$$

therefore

$$f_0 = 2^{-1/3}\Big/\left\{\left[\frac{Bi(k^*)Ai'(r) - Ai(k^*)Bi'(r)}{Bi(k^*)Ai(r) - Ai(k^*)Bi(r)}\right]^2 - r\right\} \; ; \quad r = 2^{-1/3}(\sigma+k) .$$

If we assume, for the time being, that k^* is such that $Ai(k^*)$ is non-zero, then on examining the asymptotic behaviour of $f_0(k^*, r)$ for small and large values of σ we find that

$f_0(k^*, r) > 0$ for small positive σ, and

$f_0(k^*, r) < 0$ for large positive σ.

Also we may easily deduce from (3.3.24) that

$$f_0' = f_0^2 \left(1 + \frac{2^{2/3}}{f_0} \sqrt{r + \frac{2^{-1/3}}{f_0}} \right).$$

That is, for values of σ for which $f_0 = 0$ we have that $f_0' = 0$, and using (3.3.21), that $f_0'' = +1$. Hence we can see that f_0 can never cut the r-axis as all intersections are tangential and points of local minima. We therefore conclude that f_0 has a singularity in the region $0 < \sigma < \infty$. In this problem we do not admit singularities of q in the finite part of the field. We therefore assert that

$$Ai(k^*) = 0 . \tag{3.3.25}$$

Then f_0 is given by

$$f_0 = \frac{2^{-1/3} Ai^2(r)}{Ai'^2(r) - r Ai^2(r)} . \tag{3.3.26}$$

From (3.3.25) we can see that k may take any of an infinite set of negative values k_0, k_1, k_2, etc.. f_0, as given by (3.3.26), is qualitatively as in figure 2.

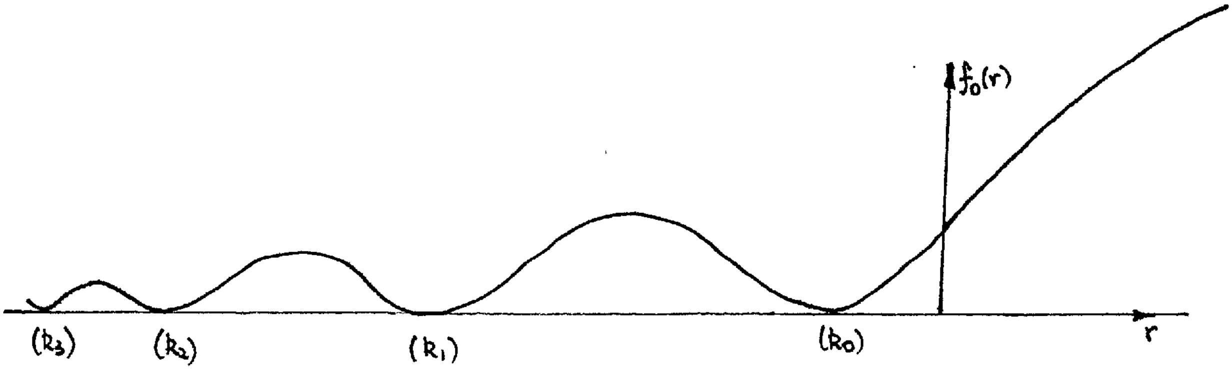

Figure 2

To exclude the oscillatory behaviour in this particular problem we will choose k to be k_0 ($k_0 \simeq -2.94583....$). For small values of σ, f_0 has the form

$$f_0 \sim \tfrac{1}{2}\sigma^2 + \tfrac{1}{12}k_0\sigma^4 + \tfrac{1}{8}\sigma^5 + \cdots \quad . \tag{3.3.27}$$

It is from the second term of this expression that, on matching, we can assert the form the inner expansion must take. That is, it is the term in φ^4 of $\hat{q}_{00}$ which forces the existence of the term $R^{2/3}q_2$ of the inner expansion. For large values of σ

$$\hat{q}_{00} \sim + \sqrt{2\varphi} + \cdots \quad ,$$

showing that the fluid does indeed fall ultimately as a solid body.

A discussion of the singular solution is given in Appendix B, together with a criticism of Maruo's numerical solution.

We now turn to consider the solutions of (3.3.22). Firstly consider the homogeneous equation

$$f_n'' - (f_0 + 2f_0'/f_0)f_n' + (f_0'^2/f_0^2 - f_0')f_n = 0 . \qquad (3.3.28)$$

As $M_1 = 0$, (3.3.28) is in fact the complete equation for f_1; and as f_1 is a perturbation of f_0, and the equation for f_0 does not contain σ explicitly, we expect f_0' to be a solution of (3.3.28). It is easily verified by substitution that this is in fact the case.

On putting $f_n = f_0'.T_n$ and using the foregoing property of f_0', (3.3.28) becomes

$$T_n'' + \left\{2f_0''/f_0' - f_0 - 2f_0'/f_0\right\}T_n' = 0 . \qquad (3.3.29)$$

We can express (3.3.26) in the form

$$f_0 = -\frac{d}{d\sigma}\log\left[Ai'^2(r) - rAi^2(r)\right] ,$$

and therefore a first integral of (3.3.29) is

$$T_n' = f_0^2/\left[f_0'^2(Ai'^2 - rAi^2)\right] .$$

We ignore constants of integration as we are seeking only particular solutions of the equation, and will multiply these solutions by arbitrary constants when considering the general solution of the equation.

Taking the logarithm of the expression for f_0

$$\log f_0 = -\tfrac{1}{3}\log 2 + 2\log Ai - \log(Ai'^2 - rAi^2) ,$$

and differentiating this with respect to r, we have that

$$f_0'/f_0 = 2^{2/3}Ai'/Ai + 2^{-1/3}Ai^2/(Ai'^2 - rAi^2) ,$$

and therefore

$$\frac{dT_n}{d\sigma} = 2^{2/3}Ai^2(Ai'^2 - rAi^2)/[2Ai'(Ai'^2 - rAi^2) + Ai^3]^2 .$$

On integrating this equation, we have

$$T_n = -2Ai'/[2Ai'(Ai'^2 - rAi^2) + Ai^3] ,$$

or on rearranging terms

$$T_n = -2Ai' f_0^2/Ai^3 f_0' .$$

Therefore two linearly independent solutions of the homogeneous equation are $v_1 = f_0'$ and $v_2 = Ai'.f_0^2/Ai^3$. The first of these is of order $\sigma^{-1/2}$ for large σ but the second is exponentially large at infinity.

Returning to the inhomogeneous equation; we do not admit solutions that are exponentially large at infinity, and so following Courant and Hilbert (1953) we define a Green's function as follows

$$G(\sigma,\xi) = \begin{cases} -v_1(\sigma)v_2(\xi)/W(\xi) & ; \quad \sigma \geq \xi \\ -v_1(\xi)v_2(\sigma)/W(\xi) & ; \quad \sigma \leq \xi , \end{cases}$$

where W is the Wronskian $(v_1v_2' - v_1'v_2)$.

The most general solution of (3.3.22) with the correct behaviour at infinity is given by

$$f_n(\sigma) = \int_0^\infty G(\sigma,\xi) M_n(\xi)\, d\xi + A_n f_0'(\sigma) . \quad (3.3.30)$$

The general solution of the first-order equation is

$$f_1(\sigma) = A_1 f_0'(\sigma) , \quad (3.3.31)$$

where A_1 is to be determined from the matching.

We should note here that the outer expansion is not really restricted to the case of vanishingly small Reynolds number, for it also has the appearance of an asymptotic expansion for $\varphi \uparrow \infty$. It is, of course, only in the former case that we can match it with the present inner expansion. Also in the present problem it seems physically reasonable to exclude any oscillatory behaviour in the solution. If, however, we were considering a situation in which the inner problem was quite different, then the rejection of oscillatory solutions would be open to question.

4. The matching procedure

We are relying on the matching to provide the boundary conditions for the equations for $\hat{q}_{01}$ of the outer expansion, and to show the necessity of g_2 in the inner expansion. We have seen from the outer expansion that a knowledge of the velocity on the line of symmetry uniquely determines the entire flow field in the outer region and so we only need to match the two expansions on that line.

Following the procedure as expounded in section 5 of Part 1, we consider the limiting process, $R \downarrow 0$ for $x = m(R)x_m$, with x_m fixed and $1 \ll m(R) \ll R^{-1/3}$. x_m is called an intermediate variable because

$x = m(R)x_m \uparrow \infty$ with x_m fixed and $R \downarrow 0$, and
$\sigma = 2^{-4/3} R^{1/3} m(R) x_m \downarrow 0$ with x_m fixed and $R \downarrow 0$.

We now express the inner and outer expansions in terms of the intermediate variable, the former for $x \uparrow \infty$ and the latter for $\sigma \downarrow 0$, and then compare the two resulting expansions. From section 2, we have for $x \uparrow \infty$

$$
\begin{aligned}
u = {} & \left[\tfrac{1}{8} m^2(R) X_m^2 + \alpha_0 m(R) X_m + 2\alpha_0^2 + \cdots \right] \\
& + R^{2/3}\left[a_2 m^4(R) X_m^4 + \cdots\cdots \right] \\
& + R \left[2^{-9} m^5(R) X_m^5 + \cdots \right] \\
& + \cdots
\end{aligned}
\qquad (3.4.1)
$$

As yet we do not know the form for u as given by the outer expansion because we do not know the boundary condition to be applied at $\sigma = 0$. However, we assert that it must have the same leading term as that provided by the inner expansion. That is, in terms of the intermediate variable, the outer expansion has a leading term $\frac{1}{8}m^2(R)x_m^2$. Rephrasing this in the outer variables, we have that

$$f_0 \sim \tfrac{1}{2}\sigma^2 \quad \text{as} \quad \sigma \downarrow 0 \quad ,$$

and hence we have the boundary condition for f_0 (anticipated in 3.3.)

$$f_0 = 0 \quad \text{at} \quad \sigma = 0 \quad .$$

On using this condition to solve the equation for f_0, we have (equation (3.3.27)) that, for small

$$f_0 \sim \tfrac{1}{2}\sigma^2 + \tfrac{1}{12}k_0\sigma^4 + \tfrac{1}{8}\sigma^5 + \cdots \quad . \tag{3.4.2}$$

From the terms in (3.4.1) that we neglected, we find that for matching to one term the overlap domain is defined by

$$X = m(R)X_m \; ; \quad 1 << m(R) << R^{-1/3} \quad ; \quad 0 < X_m < \infty \quad .$$

We may now express the outer expansion in terms of the intermediate variables, using (3.4.2) and the terms from f_1;

$$u = \left[\tfrac{1}{8}m^2(R)X_m^2 + \tfrac{1}{3}2^{-20/3}k_0R^{2/3}m^4(R)X_m^4 + 2^{-9}Rm^5(R)X_m^5 + \cdots\right] \tag{3.4.3}$$

$$+ \left[2^{-2/3}A_1\,m(R)X_m + \cdots \right]$$

$$+ \cdots\cdots$$

The term in the second line arises from f_1. We now compare (3.4.3) with (3.4.1). Firstly we see, as expected, that the leading terms in each are identical. The next largest terms will be those of order m(R) in $1 \ll m(R) \ll R^{-2/9}$, and those of order $R^{2/3}m^4(R)$ in $R^{-2/9} \ll m(R) \ll R^{-1/3}$; equating the former terms we have

$$A_1 = 2^{2/3}\alpha_0 \ . \qquad (3.4.4)$$

From the neglected terms we can show that for matching the terms of orders $m^2(R)$ and m(R), there is an overlap domain defined by

$$X = m(R)X_m \ ; \quad 1 \ll m(R) \ll R^{-2/9} \ ; \quad 0 < X_m < \infty \ .$$

It is from the matching at this order that we are justified in posing the forms (3.3.17) and that the terms $\hat{q}_{01}$ etc. are directly dependent upon the initial conditions of the inner problem.

The next largest terms (in $1 \ll m(R) \ll R^{-2/9}$) will be those of order $R^{2/3}m^4(R)$ and so on comparing (3.4.1) and (3.4.3) we find that

$$a_2 = \tfrac{1}{3}\, 2\tau^{20/3} k_0 \ , \qquad (3.4.5)$$

and again from the neglected terms, the overlap domain for matching to three terms is defined by

$$X = m(R)X_m \ ; \quad 1 \ll m(R) \ll R^{-1/6} \ ; \quad 0 < X_m < \infty \ .$$

The matching at this order provides the justification for the comments and the forms posed in section 2.3..

In figure 3 we show $f_o(\sigma)$ and compare it with $\sqrt{2\sigma}$, to which it asymptotes. It can be seen that f_o does in fact approach $\sqrt{2\sigma}$ very slowly. Also in figure 3 we show $g_o(\sigma)$ which is essentially $\hat{q}_{20}$, normalised for comparison purposes by $\hat{q}_{20} = 4^{1/3} g_o$ and then expressed in terms of known functions as follows:

from the relationships (iii) and (vii) of section 3.2. we have that

$$\hat{q}_{20} = \frac{1}{2} \hat{q}''_{00} / \hat{q}^2_{00} - \hat{q}'^2_{00} / \hat{q}^3_{00} ,$$

and so
$$g_o(\sigma) = \frac{1}{32}\left[f_o''/f_o^2 - 2 f_o'^2 / f_o^3 \right]$$

$$= -\frac{1}{32} \frac{d^2}{d\sigma^2}(1/f_o) = -\frac{2^{1/3}}{64} \frac{d^2}{dr^2}(1/f_o) ,$$

and as
$$f_o^{-1} = 2^{1/3}\left[(Ai'/Ai)^2 - r \right] ,$$

$$g_o = -\frac{2^{2/3}}{64} \frac{d^2}{dr^2}\left(\frac{Ai'}{Ai}\right)^2 = \frac{2^{-2/3}}{16} \frac{d}{dr}\left[\frac{Ai'}{Ai} \cdot \frac{1}{f_o}\right]$$

$$= -\frac{2^{-2/3}}{16}\left[\frac{2^{-1/3}}{f_o^2} - 2^{1/3} \frac{Ai'}{Ai} \frac{f_o'}{f_o^2} \right] .$$

Therefore

$$g_o = -\frac{1}{32} \frac{1}{f_o^2}\left[1 + 2^{2/3} \frac{Ai'}{Ai} f_o' \right] ,$$

and hence, using the result that

$$\frac{f_o'}{f_o} = 2^{2/3} \frac{Ai'}{Ai} + f_o ,$$

we may evaluate g_o. Also, from the last expression we may evaluate $f_o'(\sigma)$, a constant multiple of which gives $\Omega(\sigma)$.

This is shown in figure 4, together with $1/\sqrt{2\sigma}$ to which it asymptotes.

From figure 3, we can see that g_0, which is the leading term showing the variation of velocity across the flow, has become insignificant, as compared with f_0, for values of σ greater than 3. This shows that, even for substantial values of R, the velocity distribution across the jet rapidly becomes uniform. The fact that $g_0(\sigma)$ and other higher-order terms are singular at the origin is not unexpected; it shows that the outer expansion, as well as the inner expansion, results from a singular perturbation. We expect the inner expansion to be applicable in the region where the singularities of the outer expansion have a significant effect. Figure 4 provides an illustration that the leading term, f_0, also dominates on the line of symmetry not only for $R \downarrow 0$ but also for $\sigma \uparrow \infty$.

From (3.3.2) and (3.3.3) we have, on terminating the expansions immediately prior to the first appearance of terms containing any arbitrary constants depending upon initial conditions, that

$$\hat{x} = 2^{4/3}\left[\sigma + \tfrac{1}{32}R^2\psi^2 f_0'/f_0^3\right],$$

$$\hat{y} = 2^{-2/3}\left[R\psi/f_0\right].$$

Figures 5 and 6 show the shape of one of the free streamlines

(the other being the mirror image in the $\hat{x}$-axis) for the cases R = 1 and R = 0.5 . The expressions 1-term and 2-term outer indicate whether we have omitted or included the R-dependent term in the expression for $\hat{x}$. The upstream singularity in the outer expansion manifests itself in the R-dependent terms. Also in figures 5 and 6 is displayed the first term of the inner expansion, suitably expressed in terms of the outer variables. Unfortunately the only form in which we were able to express the inner expansion is also unsuitable for small x. This fact, as we have previously remarked, reduces the inner expansion to playing the role of providing boundary conditions for the outer expansion.

In the preparation of these figures, considerable use has been made of the tables of Airy functions by Miller (1946).

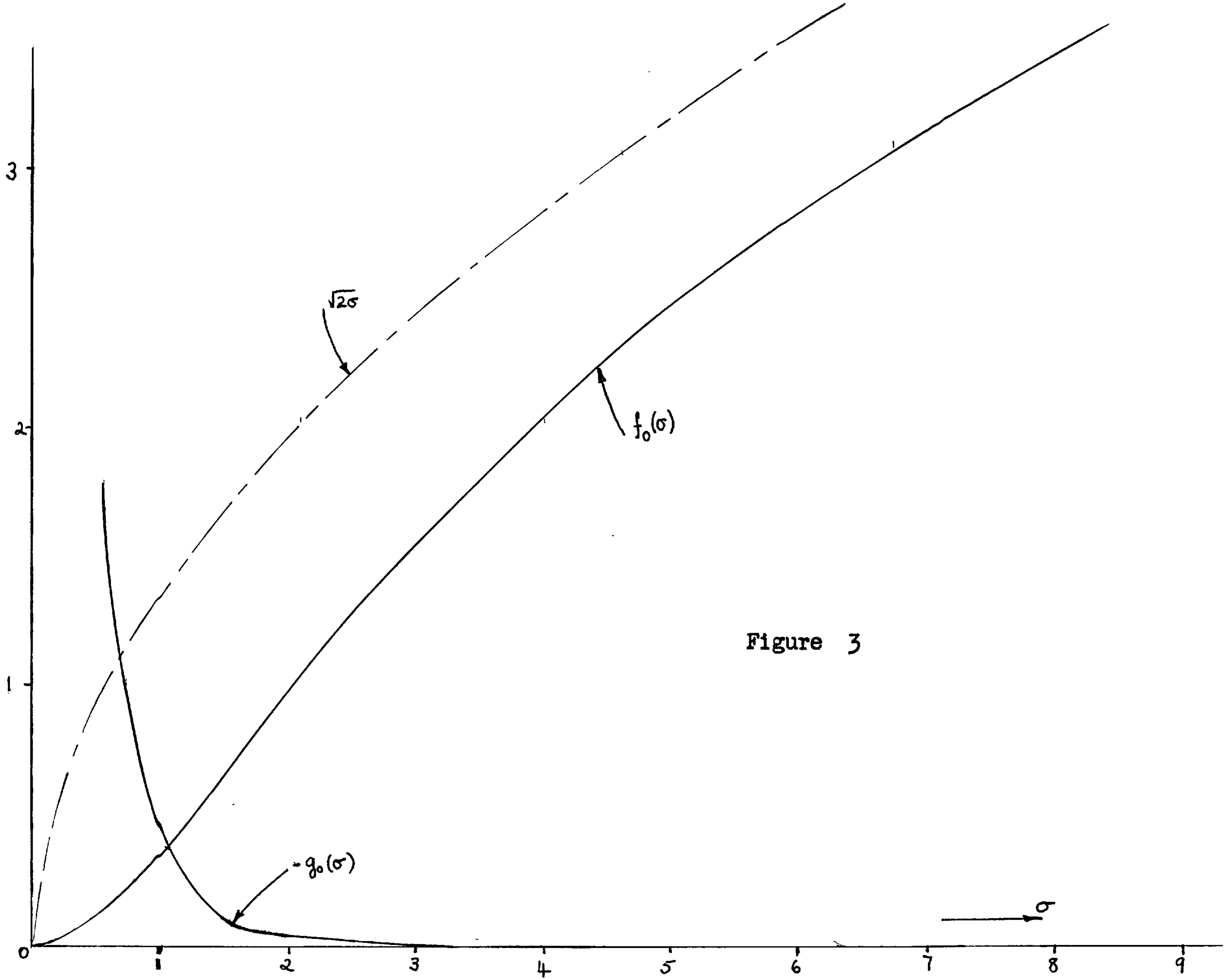

Figure 3

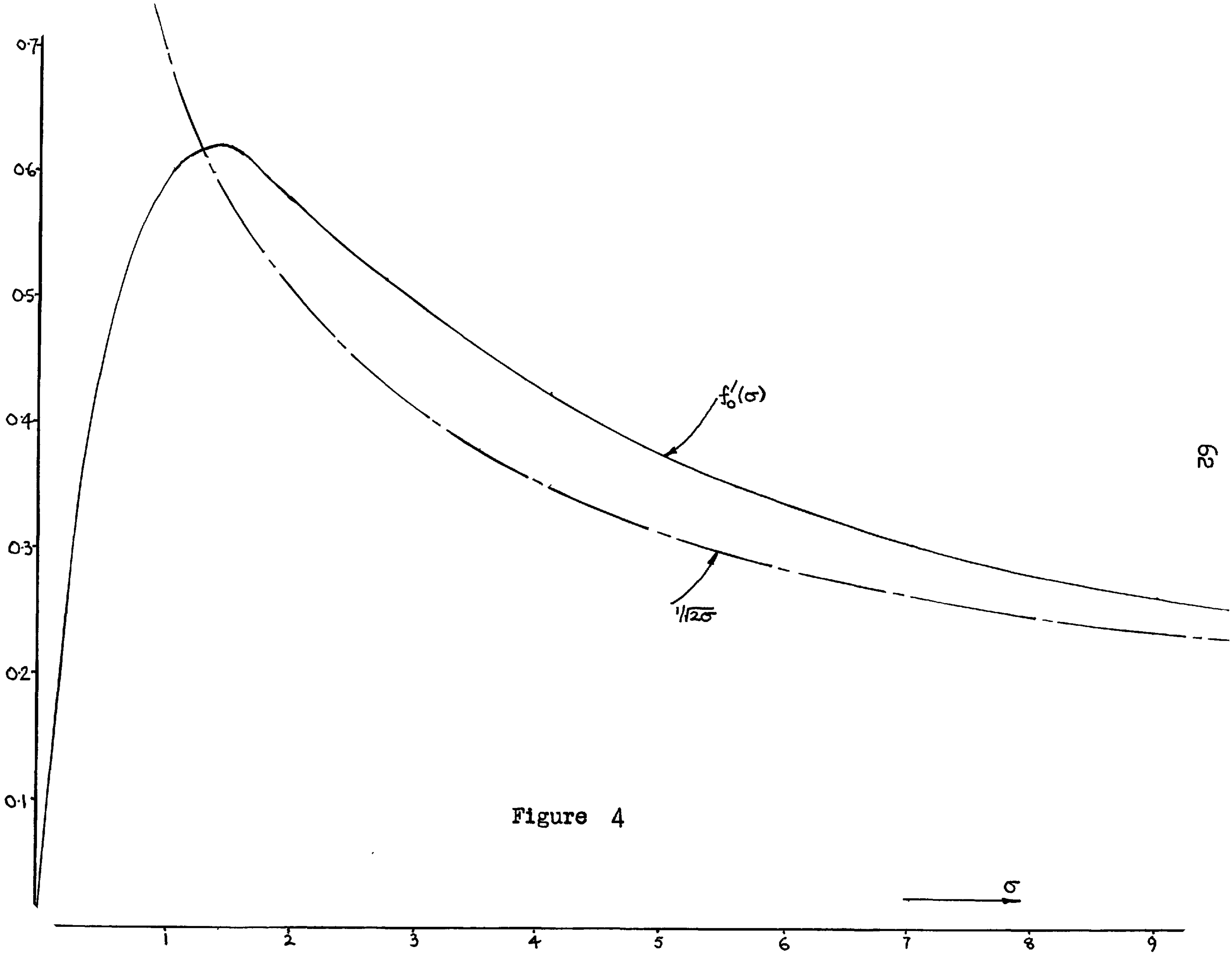

Figure 4

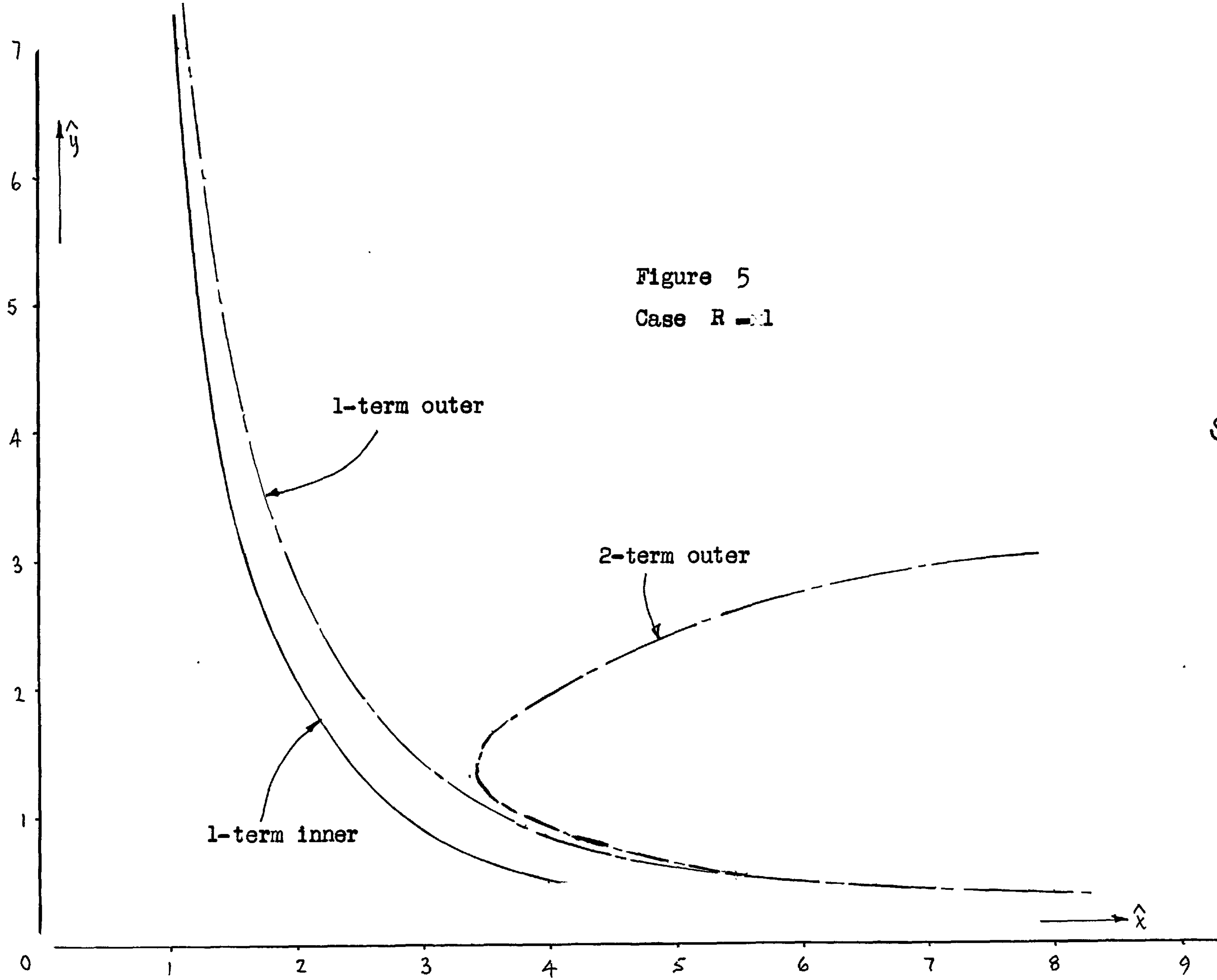

Figure 5

Case R = 1

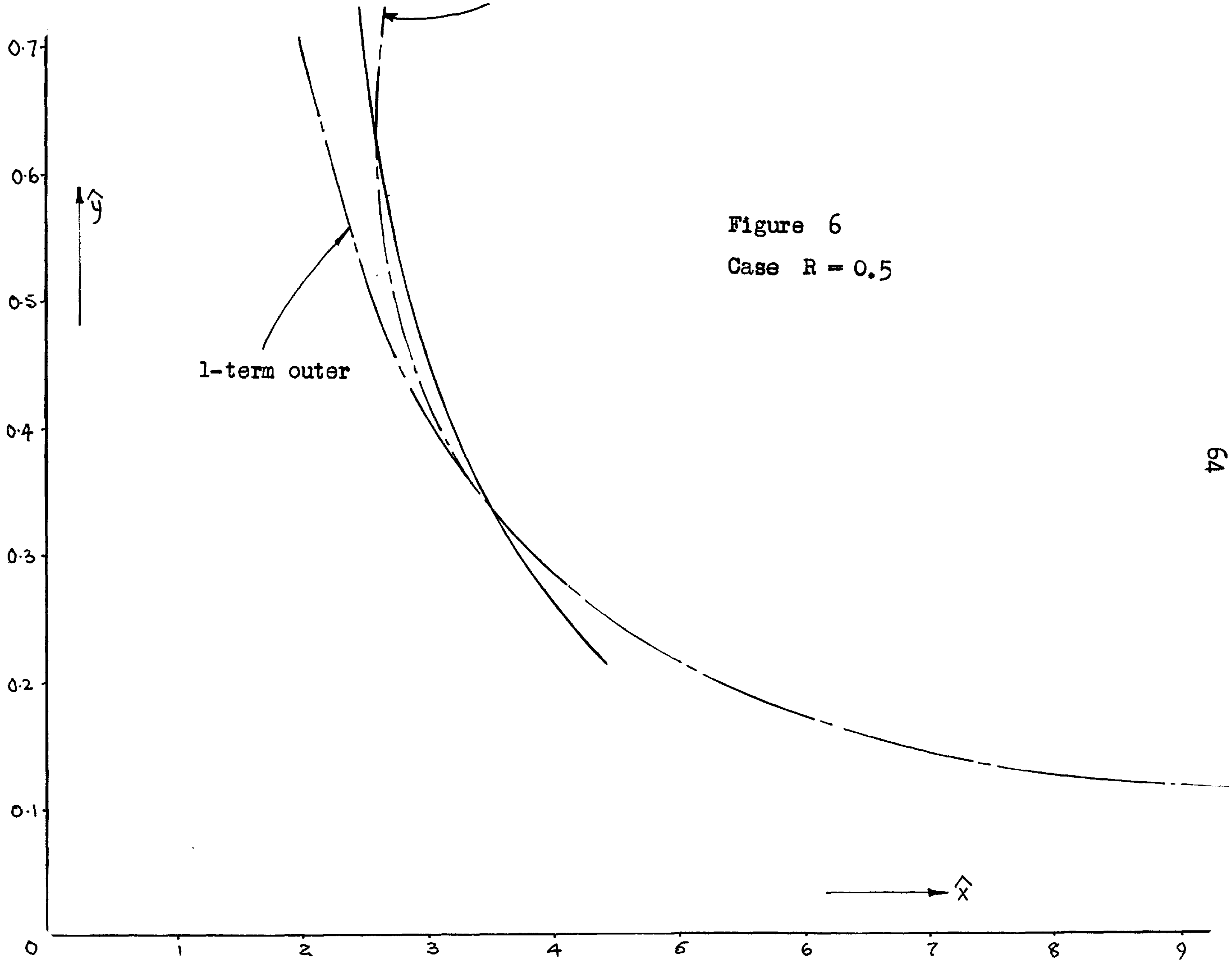

Figure 6

Case R = 0.5

Appendix A

Extension of the complex variable formalism to the n^{th} iteration

We extend the technique we developed in Part 2 for the Stokes equation to take account of the inertia terms. This extension is then used in section 2.4. of Part 3.

The Navier-Stokes equations are given by (3.2.2)

$$R(\underline{q}.\nabla)\underline{q} + \underline{\nabla} p = \underline{i} + \nabla^2 \underline{q} \quad ,$$

and continuity by $\quad \nabla . \underline{q} = 0 \quad .$

If we denote $\underline{i}$ by $-\nabla W$, then writing the Navier-Stokes equations in full, we have

$$R\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] + \frac{\partial p}{\partial x} = -\frac{\partial W}{\partial x} + \nabla^2 u \quad ,$$

$$R\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] + \frac{\partial p}{\partial y} = -\frac{\partial W}{\partial y} + \nabla^2 v \quad ,$$

and using continuity, these may be written

$$R\left[\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\right] + \frac{\partial p}{\partial x} = -\frac{\partial W}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \; ,$$

$$R\left[\frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y}\right] + \frac{\partial p}{\partial y} = -\frac{\partial W}{\partial y} + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) ,$$

and therefore

$$\frac{\partial}{\partial x}\left\{-p + 2\frac{\partial u}{\partial x} - Ru^2 - W\right\} + \frac{\partial}{\partial y}\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - Ruv\right\} = 0 \; , \qquad \text{(A1)}$$

$$\frac{\partial}{\partial x}\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - Ruv\right\} + \frac{\partial}{\partial y}\left\{-p + 2\frac{\partial v}{\partial y} - Rv^2 - W\right\} = 0 \; . \qquad \text{(A2)}$$

Following Muskhelishvili (1963 p.104), equation (A1) is the necessary and sufficient condition for the existence of a function $\Omega_1(x,y)$ such that

$$\frac{\partial \Omega_1}{\partial y} = -p + 2\frac{\partial u}{\partial x} - W - Ru^2 , \tag{A3}$$

$$\frac{\partial \Omega_1}{\partial x} = -\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + Ruv . \tag{A4}$$

Similarly the equation (A2) is the necessary and sufficient condition for the existence of a function $\Omega_2(x,y)$ such that

$$\frac{\partial \Omega_2}{\partial y} = -\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + Ruv , \tag{A5}$$

$$\frac{\partial \Omega_2}{\partial x} = -p + 2\frac{\partial v}{\partial y} - W - Rv^2 . \tag{A6}$$

Comparing (A4) and (A5) we have

$$\frac{\partial \Omega_1}{\partial x} = \frac{\partial \Omega_2}{\partial y} ,$$

and hence the existence of a function $\chi(x,y)$ such that

$$\frac{\partial \chi}{\partial y} = \Omega_1 \quad ; \quad \frac{\partial \chi}{\partial x} = \Omega_2 .$$

Therefore we have from (A3), (A4) and (A6) that

$$\frac{\partial^2 \chi}{\partial y^2} = -p + 2\frac{\partial u}{\partial x} - W - Ru^2 ,$$

$$-\frac{\partial^2 \chi}{\partial x \partial y} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - Ruv ,$$

$$\frac{\partial^2 \chi}{\partial x^2} = -p + 2\frac{\partial v}{\partial y} - W - Rv^2 .$$

We can see that χ is an Airy stress function very similar to those in Parts 2 and 3 but with the inertia terms on the right.

We now proceed as in Part 2, changing the independent variables to z and $\bar{z}$ and derive the new form for the field equations

$$\frac{\partial^2}{\partial z^2}(\chi + i2\psi) = \frac{1}{4} R (u - iv)^2 , \tag{A7}$$

or

$$\frac{\partial^2}{\partial z^2}(\chi + i2\psi) = - R \left(\frac{\partial \psi}{\partial z}\right)^2 . \tag{A8}$$

As in the main text we pose that χ and ψ be expressed by the formal expansions

$$\chi = \sum_{j=0}^{\infty} R^{j/3} \chi_j \quad ; \quad \psi = \sum_{j=0}^{\infty} R^{j/3} \psi_j .$$

This gives for the $(n+3)^{th}$-order equations

$$\frac{\partial^2}{\partial z^2}(\chi_{n+3} + i2\psi_{n+3}) = - \sum_{j=0}^{n} \frac{\partial \psi_j}{\partial z} \cdot \frac{\partial \psi_{n-j}}{\partial z} .$$

In section 2.4. of Part 3 we require this result for the case when n = 0, that is

$$\frac{\partial^2}{\partial z^2}(\chi_3 + i2\psi_3) = - \left(\frac{\partial \psi_0}{\partial z}\right)^2 = \frac{1}{4}(u_0 - iv_0)^2 .$$

Appendix B

The singular solutions of (3.3.21) and Maruo's numerical solution

We have seen from section 3.3. that the general solution of (3.3.21) which takes the value $f_0 = 0$ at $\sigma = 0$, is given by

$$f_0 = 2^{-1/3} \Big/ \left\{ \left[\frac{Bi(k^*)Ai'(r) - Ai(k^*)Bi'(r)}{Bi(k^*)Ai(r) - Ai(k^*)Bi(r)} \right]^2 - r \right\} ,$$

with k^* and r as defined in section 3.3.. We have also shown that if $Ai(k^*)$ is other than zero, then f_0 has a singularity in $0<\sigma<\infty$, the exact location depending on the value assigned to k. For small values of σ , f_0 has the asymptotic form

$$f_0 \sim \tfrac{1}{2}\sigma^2 + \tfrac{1}{12}k\sigma^4 + \tfrac{1}{8}\sigma^5 + \cdots \qquad \text{(B1)}$$

This is the same form as for the non-singular solution, though in this case the coefficient, k, of σ^4 is necessarily different. It is apparent that if we had attempted to obtain a solution by numerical integration, starting at $\sigma = 0$, then, because of even the slightest of rounding errors, we could never keep to the non-singular solution we wanted, but would always veer onto a singular solution.

For large values of σ , $f_0 \sim -\sqrt{2\sigma}$ regardless of the value of k (other than those satisfying $Ai(k^*) = 0$). The singular solutions of f_0 are qualitatively as in figure 7, though the origin of co-ordinates and the sharpness of the peaks will

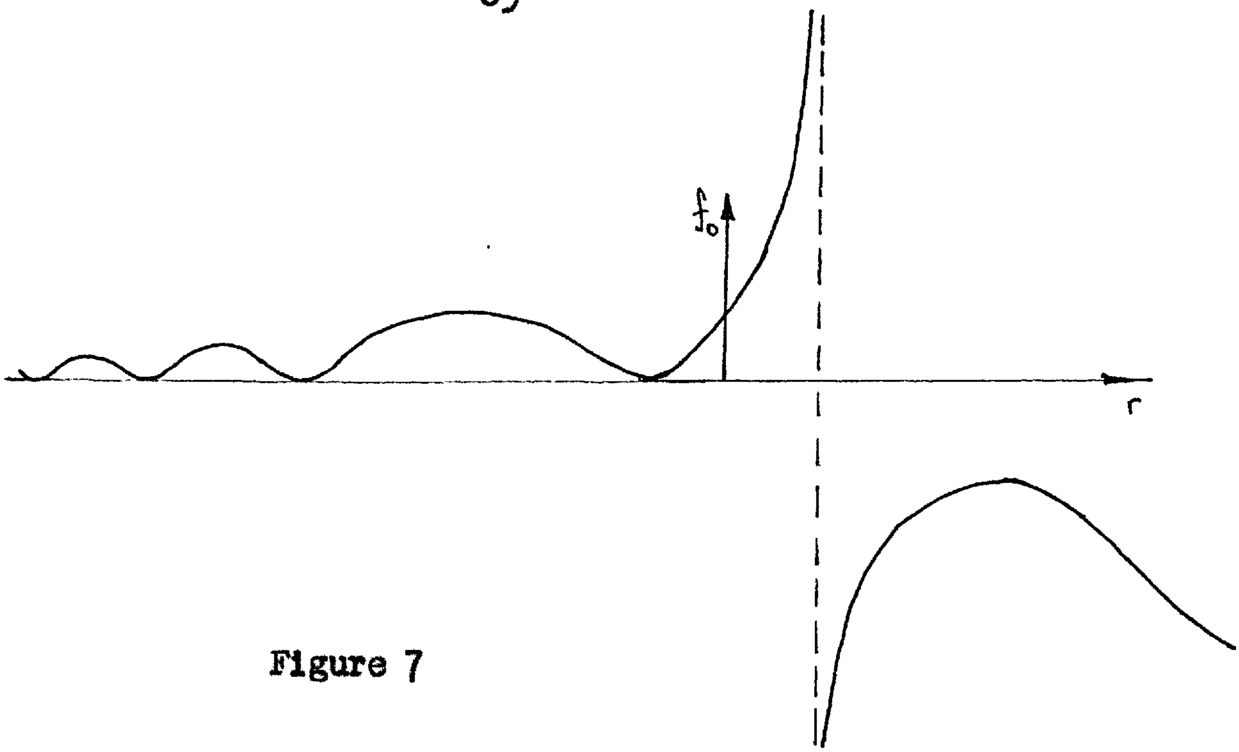

Figure 7

vary with k.

We can now see that by refusing to admit a singularity, we have effectively refused to admit the negative possibility in the choice of behaviour at infinity.

Maruo tried to integrate numerically forwards from $\sigma = 0$, but encountered the singularity, (although he never mentions boundary conditions, the details of his working indicate that the condition $f_o = 0$ at $\sigma = 0$ was tacitly assumed for so long as it was convenient). He then derives a series solution for small σ, similar in essence to (B1), though omitting the possibility of a term in σ^4. At large σ he assumes $+\sqrt{2\sigma}$ to be the leading term and iterating on this derives a series solution for large σ. Starting at some substantial value of σ

he integrates numerically backwards towards small σ. On experienceng some difficulty in patching his numerical solution onto the series solution for small σ, he relaxes the unspoken boundary condition at the origin and arrives at a solution resembling that in figure 8.

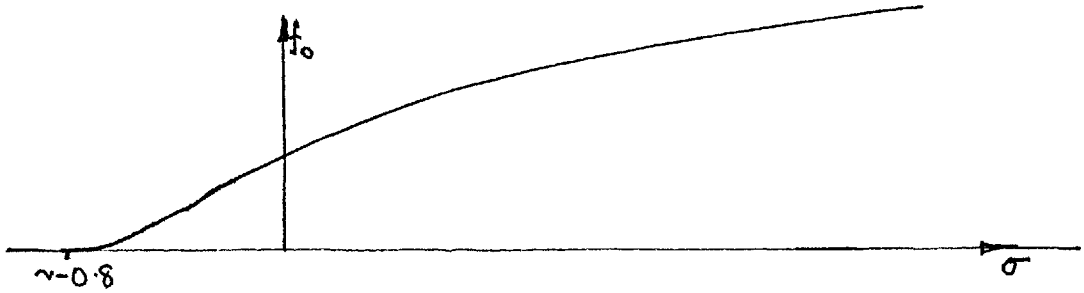

Figure 8

As we have remarked earlier, in this particular problem, choosing the behaviour at infinity is the same in practice (though not in principle) as rejecting solutions with singularities in $0<\sigma<\infty$. In the absence of an analytic solution we could have refined Maruo's procedure, starting at infinity and integrating backwards to $\sigma = 0$. We could have avoided the difficulty that Maruo experienced in satisfying the boundary conditions at the origin by noting that the leading term for large σ is $+\sqrt{2(\sigma+c)}$ where c is an arbitrary constant, depending on the conditions at the origin. The independent variable can then be changed to $t = \sigma + c$, and then the backward integration carried out until we reach the value $f_o = 0$. The value of c can then be chosen so that $f_o = 0$ at $\sigma = 0$. Considerable accuracy must be maintained throughout this integration as we would still have to find the value for k from this numerical solution.

References

BROWN D. R. 1961, A study of the behaviour of a thin sheet of moving liquid.
J. Fluid Mech. 10, p 297.

COURANT R. & HILBERT D. 1953, Methods of Mathematical Physics, vol 1. Interscience Publishers.

INCE E. L. 1926, Ordinary Differential Equations, Longmans, Green and Co.

MARUO H. 1958, Private communication with G. I. Taylor.

MILLER J. C. P. 1946, The Airy Integral, British Association for the Advancement of Science, Mathematical Tables, Part volume B, C. U. P.

MUSKHELISHVILI N. I. 1963, Some Basic Problems of the Mathematical Theory of Elasticity.
P. Noordhoff.