On univalent polynomials and related classes of functions.

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Chapter Index.

Abstract.

The behaviour of the coefficients of polynomials $p_n(z) = z + a_2z^2 + \dots + a_nz^n$ and $\mu_n(z) = \frac{1}{z} + b_1z + \dots + b_nz^n$ univalent in $|z| < 1$, $0 < |z| < 1$ respectively, has received surprisingly little attention. After a survey of those significant facts known about p_n and μ_n, bounds are established for a_n, b_n, a_{n-1}, b_{n-1}; several interesting results are obtained for special types of univalent polynomials when a_n and b_n are maximal. The correct order of growth with n of a_k (for fixed k), where the Bieberbach conjecture is assumed to hold for k, is established. The coefficient regions for p_3, p_4, μ_2, μ_3 are then studied, with complete results for p_3.

We conclude with the proof of a special case of a conjecture of L. Ilieff, some results from the theory of apolar polynomials, and several examples connected with a theorem of S. Bernstein.

All published papers with significant results on univalent polynomials appear in the list of references, marked*.

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Chapter 1. Preliminaries.

'A day's work - getting started'.

- Gaelic proverb

1. Introduction.

The class of polynomials univalent in $|z| < 1$ has been studied relatively little, and surprisingly few significant results are known concerning them.

Let us, first of all, introduce some of the notation which we shall use.

Definition 1.1.1. A function f(z) is univalent in a domain D if it is regular, single-valued, and does not take any value more than once in D.

Definition 1.1.2. A function f(z) is said to belong to the class S if it is of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and is regular and univalent in $|z| < 1$.

Definition 1.1.3. A function f(z) is said to belong to the class $\sum$ if it is of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

and is regular and univalent in $0 < |z| < 1$.

Definition 1.1.4. A polynomial $p_n(z)$ of the form:

$$p_n(z) = z + a_2 z^2 + \dots + a_n z^n$$

which is univalent in $|z| < 1$ is said to belong to the class P_n. The P_{n+1} contains P_n, since $p_n(z)$ is not necessarily of proper degree n.

Definition 1.1.5. A function $\mu_n(z)$ of the form:

$$\mu_n(z) = \frac{1}{z} + a_1 z + \dots + a_n z^n$$

which is univalent in $0 < |z| < 1$ is said to be a meromorphic univalent polynomial of degree n, belonging to the class M_n.

The following two theorems give important results concerning infinite sequences of polynomials in P_n and M_n respectively.

Theorem 1.1.1[21] Let $\left(f_n(z)\right)_1^{\infty}$ be a uniformly convergent sequence of functions regular and univalent in a domain D, and let f(z) be the limit function of the sequence. Then f(z) is either constant or univalent in D.

Proof. By the Weierstrass Limit Theorem, f(z) is regular in D. If f(z) is not univalent in D, there are two points z_1 and z_2 at which $w = f(z)$ takes the same value w_o. Describe, with z_1 and z_2 as centres, two circles which lie in D, do not overlap, and such that $f(z) - w_o$ does not vanish on either circumference (this is possible unless $f(z)$ is a constant).

Let m be the lower bound of $|f(z) - w_o|$ on the two circumferences. Then we can choose $\underline{n}$ so large that $|f(z) - f_n(z)| < m$ on the two circumferences. Hence, by Rouche's theorem, the function:

$$f_n(z) - w_o = \left(f(z) - w_o\right) + \left(f_n(z) - f(z)\right)$$

has as many zeros in the circles as $f(z) - w_o$, i.e. at least two. Hence $f_n(z)$ is not univalent, contrary to hypothesis. This proves the theorem.

In a similar way, we may prove:

<u>Theorem 1.1.2. Let</u> $\left(f_n(z)\right)_1^{\infty}$ <u>be a sequence of functions of the form</u>:

$$f_n(z) = \frac{1}{z} + \sum_{0}^{\infty} a_n z^n \quad ,$$

<u>regular and univalent in $0 < |z| < 1$, and uniformly convergent to a function $f(z)$ in any compact subset of $0 < |z| < 1$.</u> <u>Then $f(z)$ is regular and univalent in $0 < |z| < 1$.</u>

In view of Theorem 1.1.1, any function univalent in $|z| < 1$ may be approximated arbitrarily closely by a sequence of polynomials univalent in $|z| < 1$ (for example, renormalisations of partial sums of the original function). Consequently, it might be expected that many important results for the class S could be obtained as the limiting cases of the corresponding results for P_n. By Theorem 1.1.2, a similar relation holds between Σ and M_n.

Unfortunately, few of the usual techniques for dealing with univalent functions are of any value when we consider polynomials in P_n or M_n. For example, the application of the bilinear transformation to a polynomial in P_n or M_n does not generally yield another such polynomial. In addition, if $p_n(z) \in P_n$ and $\mu_n(z) \in M_n$, then $p_n(z^2)^{\frac{1}{2}}$ and $\mu_n(z^2)^{\frac{1}{2}}$ do not generally belong to the classes P_n and M_n.

However, if $p_n(z)$ and $\mu_n(z)$ are odd polynomials in P_n and M_n, then $p_n(z^{\frac{1}{2}})^2 \in P_n$ and $\mu_n(z^{\frac{1}{2}})^2 - 2\left[\frac{\mu_n(z)}{z} - \frac{1}{z^2}\right]_{z=0} \in M_n$. No other really useful variation for P_n or M_n is known at present.

Furthermore, in dealing with S it is often helpful to guess that the Koebe function $\frac{z}{(1-z)^2}$ may be extremal for whatever property we are investigating. Unfortunately, there are no such convenient 'possible extremals' known in P_n.

Consequently, it is necessary to develop new techniques for dealing with the classes P_n and M_n in order to obtain other than the simplest results.

Thus the consideration of the special subclasses P_n of S and M_n of $\sum$ does not appear to simplify the task of establishing such things as, for example, coefficients and maximum modulus estimates, but only makes it more difficult. This means that, in general, we do not expect to solve problems for S or $\sum$ by using the solutions of the corresponding problems for P_n or M_n. As a result, we will study the classes P_n and M_n mainly for their own independent interest.

2. The Dieudonné Criterion.

A fundamental result concerning univalent functions may be expressed in the following form:

Theorem 1.2.1.[21] *A function $w = f(z)$, regular on a domain containing a simple closed rectifiable curve C and its interior D, is univalent on D if it is univalent on C.*

<u>Proof</u>. The curve C corresponds to a curve C' in the w-plane. C' is closed, since $f(z)$ is single-valued; and it has no double points, since $f(z)$ does not take any value twice on C. Let D' be the region enclosed by C'.

Clearly $f(z)$ takes in D values other than those on C, say at z_o. Then if Δ_C denotes the variation round C,

$$\frac{1}{2\pi} \Delta_C \arg [f(z) - f(z_o)]$$

is equal to the number of zeros of $f(z) - f(z_o)$ in D, by the Argument Principle. It is therefore a positive integer, since there is at least one such zero. But it is also equal to:

$$\frac{1}{2\pi} \Delta_{C'} \arg (w - w_o)$$

where $w_o = f(z_o)$; and this is either 0, if w_o is outside C', or ± 1, if w_o is inside C', the sign depending on the direction in which C' is described. Hence it is equal to 1. Hence w_o lies inside C', C' is described in the positive direction, and $f(z)$ takes the value w_o just once in D. Thus D is mapped univalently onto D'.

Let us consider the radius of univalency R of polynomials:

$$p_n(z) = z + a_2z^2 + \dots + a_nz^n .$$

Then R is characterised, in view of Theorem 1.2.1, by the fact that, on the circumference $C : [\,|z| = R]$, the polynomial $p_n(z)$ assumes the same value α (say) at at least two points, distinct or coincident, and that no concentric circumference of smaller radius possesses this property.

The second part of this remark is obvious. To establish the first, it is sufficient to notice that, if for each point z_1 on C, the root z_2 of the equation $p_n(z) - p_n(z_1) = 0$ nearest to C always remains exterior to C, the same is true for the circumference $|z| = R + \varepsilon$ (by continuity arguments) for a sufficiently small positive number ε. Thus $p_n(z)$ is then univalent in $|z| < R + \varepsilon$, which contradicts the definition of R.

Let x be the midpoint of the smaller arc on C between the points z_1 and z_2, and let z_1 and z_2 subtend an angle 2θ at the origin. Then if, without loss of generality, we put $z_1 = xe^{i\theta}$, $z_2 = xe^{-i\theta}$, we may write the equation $p_n(z_1) = p_n(z_2)$ in the form:

$$\phi(x,\theta) = 1 + a_2x\,\frac{\sin 2\theta}{\sin\theta} + a_3x^2\,\frac{\sin 3\theta}{\sin\theta} + \dots + a_nx^{n-1}\,\frac{\sin n\theta}{\sin\theta}$$
$$= 0.$$

This equation in x is the associated equation of the polynomial $p_n(z)$.

We have, in fact, established the following:

Theorem 1.2.2. (Dieudonné Criterion).[5] The radius of univalency of the polynomial $p_n(z) = z + a_2z^2 + \ldots + a_nz^n$ is equal to the radius of the largest circle, centre the origin, which contains no root of the associated equation of $p_n(z)$:

$$\phi(x,\theta) = 1 + a_2x \frac{\sin 2\theta}{\sin\theta} + \ldots + a_nx^{n-1} \frac{\sin n\theta}{\sin\theta} = 0 ,$$

as θ varies from 0 to $\frac{\pi}{2}$.

Consequently, $p_n(z) \in P_n$ iff no root of the associated equation has modulus less than one for any such θ.

We can, similarly, establish the following:

Theorem 1.2.3. (Dieudonné Criterion). The radius of univalency of the polynomial $\mu_n(z) = \frac{1}{z} + a_1z + \ldots + a_nz^n$ is equal to the radius of the largest circle, centre the origin, which contains no root of the associated equation of $\mu_n(z)$:

$$\phi(x,\theta) = a_nx^{n+1} \frac{\sin n\theta}{\sin\theta} + a_{n-1}\, x^n \frac{\sin(n-1)\theta}{\sin\theta} \ldots + a_1x^2 - 1 = 0,$$

as θ varies from 0 to $\frac{\pi}{2}$.

Consequently, $\mu_n(z) \in M_n$ iff no root of the associated equation has modulus less than one for any such θ.

Note. It is sufficient to consider almost all θ in $[0, \frac{\pi}{2}]$; for if $\phi(x,\theta) = 0$ has no zeros in $|x| < 1$ outside a set of θ of measure zero (for example), it has no zeros in $|x| < 1$ for any θ in $[0, \frac{\pi}{2}]$ since the zeros of a polynomial are continuous functions of its coefficients.

3. A miscellany of results concerning polynomials.

First of all, we wish to establish a special case of the well-known Cohn Rule [14], which plays a fundamental role in most of our work.

Theorem 1.3.1. Suppose that $|C_o| > |C_n|$. Then the polynomial:

$$f(x) = C_o + C_1 x + \ldots\ldots\ldots + C_n x^n$$

(of degree n) has no zeros in $|x| < 1$ iff neither has the polynomial:

$$f_1(x) = \overline{C}_o(C_o + C_1 x + \ldots + C_n x^n) - C_n(\overline{C}_n + \overline{C}_{n-1} x + \ldots + \overline{C}_o x^n)$$

$$(3.1) \qquad = \sum_{k=0}^{n-1} (\overline{C}_o\ C_k - C_n\ \overline{C}_{n-k}) x^k$$

<u>(of degree n-1)</u>.

<u>Proof</u>. Let us associate with $f(x)$ the polynomial:

$$f^{*}(x) = x^n \,\overline{f}\,(\tfrac{1}{x}) = \overline{C}_o x^n + \overline{C}_1 x^{n-1} + \ldots + \overline{C}_n$$

$$= \overline{a}_o \prod_{j=1}^{n} (x - {}^1/\overline{x}_j),$$

whose zeros are the inverses of the zeros of $f(x)$ in the circle $|x| = 1$. Thus any zero of $f(x)$ on $|x| = 1$ is also a zero of $f^{*}(x)$; and if $f(x)$ has p zeros in $|x| > 1$, then $f^{*}(x)$ has p zeros in $|x| < 1$. Also:

$$f^{*}(e^{i\theta}) = \overline{C}_o \prod_{j=1}^{n} (e^{i\theta} - {}^1/\overline{x}_j)$$

$$= \frac{C_o \; e^{in\theta}(-1)^n}{\overline{x}_1 \; \overline{x}_2 \; \ldots . \; \overline{x}_n} \prod_{j=1}^{n} (e^{-in\theta} - \overline{x}_j)$$

$$= e^{in\theta} \,\overline{f}\, (e^{-i\theta}) \;,$$

and so:

$$\left| f^{*}(e^{i\theta}) \right| = \left| f(e^{i\theta}) \right| \quad .$$

Consequently:

$$\left| C_n \; f^{*}(x) \right| \; < \; \left| \overline{C}_o \; f(x) \right| \;\; (\text{on } |x| = 1).$$

Hence, by Rouchés theorem, $\overline{C}_0\ f(x)$ has as many zeros in $|x| < 1$ as has $\overline{C}_0\ f(x) - C_n f^*(x) = f_1(x)$. Thus the theorem is proved.

Theorem 1.3.1. appears to be the most powerful available method for extracting information concerning P_n and M_n from the Dieudonné Criterion, as we shall see in Chapters 2 and 3.

Finally, we give a new proof of the well-known Bernstein Theorem for polynomials in the unit circle. This depends on:

Lemma.[21] Let $F(z) = \sum_{k=0}^{\infty} a_k z^k$ be regular in $|z| \leq 1$, $S_n(z) = \sum_{k=0}^{n} a_k z^k$, and $\sigma_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} S_k(z)$. then $|F(z)| \leq M$ in $|z| \leq 1$ iff $|\sigma_n(z)| \leq M$ for all n and $|z| \leq 1$.

Theorem 1.3.3*.[3] Let $p(z) = \sum_{k=0}^{n} a_k z^k$. Then:

$$(3.9) \qquad M(1, p_n') \leq n\, M(1, p_n)$$

with equality iff $p(z) = a_n z^n$.

Proof. Let $q(z) = z^n p(1/z) = \sum_{k=0}^{n} a_{n-k} z^k$.

Then $M(1,p) = M(1,q)$. We now apply the lemma to the (Fejer) means σ of $q(z)$. Clearly:

$$\sigma_{n-1}(z) = \sum_{k=0}^{n-1} \frac{n-k}{n} a_{n-k} z^k$$

$$= \frac{1}{n} z^n p'(^1/z),$$

and so: $M(1,p') = n\,M(1,\sigma_{n-1})$.

Applying the lemma, we obtain:

$$M(1,p') \leq n\,M(1,q)$$

$$= n\,M(1,p).$$

It is easy to determine the unique extremal $p(z)$; for equality holds in the lemma iff $F(z) \equiv e^{i\alpha}M$.

4. Starlike and close-to-convex functions.

We will find it convenient to discuss two special subclasses of S and $\sum$, consisting of functions which satisfy certain analytic and geometric conditions.

Definition 1.4.1.[8] A domain D in the w-plane is said to be starlike with respect to a fixed point 0 in D, if for any point P in D the straight line segment OP also lies in D.

If $f(z) \in S$, and maps $|z| < 1$ onto a starlike domain with respect to $w = 0$, we shall call $f(z)$ starlike univalent. It is not difficult to establish the following:

Theorem 1.4.1.[8]. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then $f(z)$ is starlike univalent in $|z| < 1$ iff:

$$\mathrm{Re}\,(zf'(z)/f(z)) > 0 \qquad (|z|<1).$$

Theorem 1.4.2.[8]. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be starlike univalent in $|z| < 1$. Then $|a_n| \leq n$ $(n \geq 2)$, with equality iff $f(z) = \dfrac{z}{(1-\alpha z)^2}$ where $|\alpha| = 1$.

We now define a class of functions which contains starlike functions as a special subclass.

Definition 1.4.2.[12]. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then $f(z)$ is a close-to-convex function in $|z| < 1$ iff it maps each circle $|z| = r < 1$ $(z = re^{i\theta})$ onto a simple closed curve whose unit tangent vector T either rotates in an anticlockwise direction so θ increases, or else rotates clockwise in such a manner that the variation of arg T over all arcs of $|z| = r$ exceeds $-\pi$, as θ increases.

We may express this analytically in the following form:

Theorem 1.4.3.[12]. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then $f(z)$ is close-to-convex in $|z| < 1$ iff there exists a function $g(z)$ starlike univalent in $|z| < 1$ such that:

$$\operatorname{Re}(zf'(z)/g(z)) > 0 \qquad (|z| < 1).$$

Note. If $f(z)$ is close-to-convex in $|z| < 1$, and $g(z)$ is a starlike univalent function in $|z| < 1$ such that $\operatorname{Re}(zf'(z)/g(z)) > 0$, we shall say that $f(z) \in CTC(g(z))$.

5. Previous results concerning univalent polynomials.

In this section we will give a rapid survey of those known results concerning P_n which will not be proved in the following chapters. In view of the simplicity of the criterion in Theorem 1.4.1, many of these results deal with univalent starlike polynomials.

Theorem 1.5.1[1]. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $f(z)$ is starlike univalent in $|z| < 1$ if

$$\sum_{n=2}^{\infty} n|a_n| \leq 1.$$

Proof. It is sufficient to establish that $\arg f(re^{i\theta})$ is a non-decreasing function of θ, for $0 < r < 1$. Here:

$$\frac{\partial}{\partial\theta} \arg f(re^{i\theta}) = \frac{\partial}{\partial\theta} \arg \left(re^{i\theta} + \sum_{n=2}^{\infty} a_n r^n e^{in\theta}\right)$$

$$\geq \left|\frac{\partial}{\partial\theta} \arg(re^{i\theta})\right| - \sum_{n=2}^{\infty} \left|a_n \frac{\partial}{\partial\theta} \arg(r^n e^{in\theta})\right|$$

$$\geq 1 - \sum_{n=2}^{\infty} n\,|a_n|$$

$$\geq 0\,.$$

Hence the theorem is proved.

We now give three theorems for univalent polynomials, which are easily generalised to multivalent polynomials. The method gives best possible results, and is of very wide application.

<u>Theorem 1.5.2.</u>[16]. <u>Let $f(z) = z + \sum_{k=1}^{n} a_k z^{kq+1}$,</u> <u>and let α_i $(1 < i < n)$ be the zeros of the polynomial:</u>

$$(5.1)\qquad 1 + \sum_{k=1}^{n} a_k z^k = 0\,.$$

<u>Let S be the smallest positive zero of $f_o'(x)$, where:</u>

$$(5.2)\qquad f_o(x) = x(x^q - |\alpha_1|)\,\ldots\,(x^q - |\alpha_n|)\,.$$

<u>Then the polynomial $f(z)$ is starlike univalent in $|z| \leq S$.</u>

<u>Proof</u>. Let us put:

$$\begin{aligned} \Phi &= \arg f(z) \\ &= \arg a_n + \theta + \sum_{k=1}^{n} \arg(z^q - \alpha_k), \end{aligned} \tag{5.3}$$

where $z = re^{i\theta}$ $(r < S)$ and study the variation of Φ as θ increases from 0 to 2π. We also put:

$$z^q = \alpha_k u \quad , \quad u = \rho e^{i\phi} \; , \psi = \arg(u-1) .$$

Consequently:

$$\arg(z^q - \alpha_k) = \arg\alpha_k + \arg(u-1), \text{ and:}$$

$$\phi = q\theta - \arg\alpha_k \; , \; d\phi = q d\theta \; .$$

It is easy to show that:

$$\tan \psi = \frac{\rho \sin \phi}{\rho \cos\psi - 1} \; , \; \frac{d\psi}{d\phi} = \frac{\rho^2 - \rho \cos \phi}{\rho^2 - 2\rho \cos \phi + 1} \; ,$$

and so:

$$- \frac{\rho}{1-\rho} \leq \frac{d\psi}{d\phi} \leq \frac{\rho}{1+\rho} \; .$$

Hence, for $r^q < r_k = |\alpha_k|$, we have:

$$- \frac{qr^q}{r_k - r^q} \leq \frac{d}{d\theta} [\arg(z^q - \alpha_k)] \leq \frac{qr^q}{r_k + r^q} \tag{5.4}$$

If also $r^q < \min_k r_k$ in (5.3) and (5.4), we may at once deduce that:

$$\frac{d\Phi}{d\theta} \geq 1 - \sum_{k=1}^{n} \frac{qr^q}{r_k - r^q}$$

$$= r\left[\frac{1}{r} - \sum_{k=1}^{n} \frac{qr^{q-1}}{r_k - r^q}\right]$$

$$= \frac{rf_0'(r)}{f_0(r)} \geq 0 ,$$

for $r < S$. Hence Φ is an increasing function of θ for $r < S$.

Now suppose that $|a| < S$. Then $\arg[f(z) - f(a)]$ increases from 0 to 2π as θ increases from 0 to 2π. Consequently $f(z) - f(a)$ has only one zero $z=a$ in the circle $|z| < S$. Thus we have proved that $f(z)$ is starlike univalent in $|z| < S$.

Using arguments similar to the above, we may establish:

<u>Theorem 1.5.3</u>[17]. <u>Let</u>:

$$f(z) = z(z-\alpha_1)^{\mu_1}\ldots(z-\alpha_n)^{\mu_n}(z-\beta_1)^{\nu_1}\ldots(z-\beta_m)^{\nu_m} ,$$

<u>where μ and $-\nu$ are both positive, $r_k = |\alpha_k|$ $(1\leq k\leq n)$, $t_s = |\beta_s|$ $(1\leq s\leq m)$, and $g = \min_{k,s} (r_k, t_s)$. Let S be the</u>

smallest positive zero of $f_o'(x)$, where:

$$f_o(x) = x(x-r_1)^{\mu_1}\dots(x-r_n)^{\mu_n}(x+t_1)^{\nu_1}\dots(x+t_m)^{\nu_m}\ .$$

Then $f(z)$ is starlike univalent in $|z| < \min(g,S)$.

As an application of this theorem, we may deduce:

Theorem 1.5.4. Let $f(z) = z/(z-\beta_1)(z-\beta_2)$, where $|\beta_1| = |\beta_2| = 1$. Then $f(z)$ is starlike univalent in $|z| < 1$.

Furthermore, similar arguments may be used to establish the following two well-known theorems.

Theorem 1.5.5.[2]. Let $p(z)$ be a polynomial of degree n with no zeros in $|z| < 1$. If α is real and non-zero, then the function $f(z) = z[p(z)]^{\alpha/n}$ is starlike univalent in $|z| < 1$ if $-2 \leq \alpha < 0$, and in $|z| < \left|\frac{1}{1+\alpha}\right|$ otherwise. If $-2 \leq \alpha < 0$, $f(z)$ has the minimum radius of starlikeness and univalency iff $p(z)$ has at least one zero on $|z| = 1$; otherwise the minimum is attained iff all zeros of $p(z)$ lie at one point on $|z| = 1$.

Theorem 1.5.6.[2]. Let $p(z)$ be a polynomial of degree $(m + n)$, $mn \neq 0$, and let $f(z) = zp(z)$. Then, if m zeros of $p(z)$ lie in the annulus $0 < d \leq |z| \leq D$, and the remaining n zeros in $|z| \geq D$, the minimum radii of starlikeness and univalency of $f(z)$ are

$\frac{-y_o D}{1-y_o}$, where y_o is the greater root of the equation:

$$n(d-D)y^2 + (d(1-n) - D(1+m))y - d = 0.$$

This minimum is attained iff m zeros of $p(z)$ lie at one point on $|z| = d$, and n at one point on $|z| = D$, where the concentration points are collinear with the origin, and on the same side of it.

Most of the significant results concerning the coefficients of polynomials in P_n are consequences of the work of Dieudonné. The estimates established in his thesis have not been improved in the last thirty years, and so it is of value to quote them in full.

Theorem 1.5.7.[5]. Let $p_n(z) = z + \sum_{k=2}^{n} a_k z^k$. Then $p_n(z)$ is starlike univalent in $|z| \leq R$, where R is the positive root of the equation:

$$1 - \sum_{k=2}^{n} k \; |a_k| \; x^{k-1} = 0 .$$

This follows at once from Theorem 1.2.2 and Pellet's Theorem.

As an application of the Dieudonné criterion which we have already established (Theorem 1.2.2) and a well-known criterion of Schur [20] that a polynomial has no zeros in the unit disc, Dieudonné established:

Theorem 1.5.8.[5]. Let $p_n(z) = z + \sum_{k=2}^{n} a_k z^k \in P_n$.

Then, if $\left[\frac{n+1}{2}\right] \leq m \leq n$,

$$n^2 |a_n|^2 + \ldots + m^2 |a_m|^2 - (m-1)^2 |a_{m-1}|^2 - \ldots -1 \leq 0.$$

Consequently $|a_n| \leq 1/n$, $|a_{n-1}| \leq \sqrt{17}/(n-1)$, and $|a_{n-2}| \leq \sqrt{98}/(n-2)$.

Using standard inequalities for the coefficients of non-negative trigonometric polynomials [7], and the Dieudonné criterion, it is easy to establish:

Theorem 1.5.9.[5]. Let $p_n(z) = z + \sum_{k=2}^{n} a_k z^k \in P_n$, where the a_k are real. Then $|a_2| \leq 2\cos(\frac{\pi}{n+3}) < 2$, $|a_3 - 1| \leq 2$, and $|a_k - a_{k-2}| \leq 2$ for $k \geq 4$. In particular, $|a_k| \leq k$ $(2 \leq k \leq n)$.

Since the last estimate is independent of n, **we deduce that the well-known Bieberbach conjecture holds for functions in S with real coefficients,** by letting $n \longrightarrow \infty$.

Theorem 1.5.10.[5]. Let $p_{2n+1}(z) = z + \sum_{k=1}^{n} b_{2k+1} z^{2k+1}$ $\in P_n$, where the b_k are real. Then $|b_{2k+1}| < 2$, and $|b_{2k+1}| + |b_{2k-1}| \leq 2$ $(2 \leq k \leq n)$.

Using the Dieudonné criterion, and lengthy geometric arguments, we may also establish:

Theorem 1.5.11.[5]. The radius of univalency of the trinomial $f(z,a) = z + z^p + az^n$ $(n>p)$ is at most $p^{\frac{-1}{p-1}}$ if $(p-1)$ does not divide $(n-1)$, and $[\sin(\frac{\pi}{n})/\sin(\frac{p\pi}{n})]^{\frac{1}{p-1}}$ if $n - 1 = h(p - 1)$. The maximum is attained in the first case (with $a = 0$), and in the second case when h is above a certain integer $h_o(p) \leq 12$.

It is clear that polynomials in P_n are bounded functions in $|z| \leq 1$, and so we might expect to find a useful relationship between the classes of bounded functions and P_n. In this direction, we have Theorem 2.3.1., and the following:

Theorem 1.5.12.[5]. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be a regular function in $|z| < 1$, bounded by M. Then $f(z)$ is starlike univalent in:

$$|z| \leq M - \sqrt{M^2-1} .$$

Finally, let us state the well-known " area principle" for functions in $\sum$. Since M_n is a subset of $\sum$, and $\lim_n M_n = \sum$, this gives us some idea of the magnitude of the coefficients of polynomials in M_n (by Theorem 1.1.1).

Theorem 1.5.13.[8]. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \sum$.

Then $\sum_{n=1}^{\infty} n\,|a_n|^2 \leq 1$, with equality iff $|a_1| = 1$, $a_n = 0 (n > 1)$.

6. Sections of Power Series.

It is natural to consider whether renormalisations of sections of some of the standard power series in univalent function theory are extremal polynomials in P_n in any sense. Consequently we consider the radii of starlikeness and univalency of sections of $\frac{z}{1-z}$ and $\frac{z}{(1-z)^2}$ and some associated polynomials.

Theorem 1.6.1. The partial sums $s_n(z) = z\,\frac{1-z^n}{1-z}$ of the function $\frac{z}{1-z}$ are starlike univalent in $|z| < 1 - \frac{\log n}{n}(1 + o(1))$. In general the coefficient of $\frac{\log n}{n}$ cannot be replaced by any smaller constant.

Proof. Clearly:

$$\frac{z\,s_n'(z)}{s_n(z)} = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)(1-z^n)},$$

and so, if $C_A = (z : |z| = 1 - \frac{A\log n}{n})$, we have:

$$\frac{z\,s_n'(z)}{s_n(z)} = \frac{1 + O(n^{1-A})}{1-z} \quad \text{on } C_A\text{, and:}$$

$$\mathrm{Re}\left[\frac{z\ s_n'(z)}{s_n\ (z)}\right] \geq \tfrac{1}{2} + O(n^{1-A}) \quad \text{on} \quad C_A .$$

Hence, by Theorem 1.4.1, $s_n(z)$ is starlike univalent inside and on C_A for all $A > 1$ and sufficiently large n. The last part of the result follows since $s_n'(z)$ is real for $\mathrm{Im} z = 0$, $s_n'(0) = 1$, $s_n'(-1 + \frac{\log n}{n}) < 0$ for sufficiently large n ; consequently $s_n'(z) = 0$ at some point inside C_1 for sufficiently large n.

In a similar way we may establish:

Theorem 1.6.2. *The polynomial* $z(\frac{1-z^n}{1-z})^2$ *is starlike univalent in* $|z| < 1 - 2\frac{\log n}{n}(1 + o(1))$. *The constant 2 cannot be replaced by any smaller constant.*

Theorem 1.6.3. *The partial sums:*

$$s_n(z) = z + 2z^2 + \ldots + nz^n = \frac{z - (n+2)z^{n+2} + (n+1)z^{n+3}}{(1-z)^2}$$

of the function $\frac{z}{(1-z)^2}$ *are starlike univalent in* $|z| < 1 - 3\frac{\log n}{n}(1 + o(1))$. *The constant 3 cannot be replaced by any smaller constant.*

Proof. The result may be established as in Theorem 1.6.1. However we also give a new method of establishing the radius of univalency of such polynomials using the Dieudonné Criterion. Now:

$$z + 2z^2 + \ldots + nz^n = \frac{z - (n+2)z^{n+2} + (n+1)z^{n+3}}{(1-z)^2} .$$

If we now suppose that x is real, and $z = xe^{i\theta}$, we have:

$$(6.1)\ x \sin \theta + 2x^2 \sin 2\theta + \ldots + nx^n \sin n\theta$$

$$= \operatorname*{Im}_{z-xe^{i\theta}} \left[\frac{xe^{i\theta}-(n+2)x^{n+2}e^{i(n+2)\theta}+(n+1)x^{n+3}e^{i(n+3)\theta}}{(1-xe^{i\theta})^2} \right]$$

$$= \frac{1}{F} \operatorname{Im} [xe^{i\theta}-(n+2)x^{n+2}e^{i(n+2)\theta}+(n+1)x^{n+3}e^{i(n+3)\theta}].[1-xe^{-i\theta}]^2$$

$$= \frac{x}{F} \left[\begin{array}{l} (1-x^2)\sin \theta - (n+2)\sin(n+2)\theta\ x^{n+1}\ . \\ + (n+1)\sin(n+3)\theta\ x^{n+2} + 2(n+2)\sin(n+1)\theta\ x^{n+2} \\ - (n+2)\sin n\theta\ x^{n+3} - 2(n+1)\sin(n+2)\theta\ x^{n+3} \\ + (n+1)\sin(n+1)\theta\ x^{n+4}. \end{array} \right]$$

$$= \frac{x \sin\theta}{F} [(1-x^2) + g(x,n,\theta)],\ \text{say, where}$$

$F = |1-xe^{i\theta}|^4$.

Since (6.1) and (6.2) are identical for real x, they are identical also for complex x. Then, by the Dieudonné Criterion, the radius of univalency of $s_n(z)$ is the modulus of the smallest zero of:

$$1 + 2x \frac{\sin 2\theta}{\sin\theta} + \ldots.. + nx^{n-1} \frac{\sin n\theta}{\sin\theta}$$

and so of: $(1-x^2) + g(x,n;\theta)$.

If $C = (x: |x| = 1 - 3\frac{\log n}{n})$,

$|1 - x^2| > 6\frac{\log n}{n}(1+o(1))$ and $|g(x,n;\theta)| < \frac{8}{n}(1+o(1))$ for $x \in C$. Hence, by Rouchés theorem, $(1-x^2)+g(x;n,\theta)$ has the same number of zeros in C as $(1-x^2)$, i.e. none, for sufficiently large n. Finally, it is easy to show that $s_n'(z)$ has a zero inside $|z| = 1 - A\frac{\log n}{n}$ for any $A < 3$ and sufficiently large n, and so the theorem is proved.

We may apply both the above methods to establish:

Theorem 1.6.4. *The polynomial* $\sum_{k=1}^{n} k\frac{n-k}{n-1}z^k$ *is starlike univalent in* $|z| < 1 - 2\frac{\log n}{n}(1 + o(1))$. *The constant 2 cannot be replaced by any smaller constant.*

The preceding four theorems may be compared with:

Theorem 1.6.5.[19]. *Let* $s_n(z)$ *be the* n^{th} *partial sum of the function* $z + \sum_{k=2}^{\infty} a_k z^k$ *which is starlike univalent in* $|z| < 1$. *Then* $s_n(z)$ *is starlike univalent in* $|z| < 1 - 3\frac{\log n}{n}(1 + o(1))$.

Finally, we mention:

Theorem 1.6.6. *The* n^{th} *partial sum of the function* $\frac{1+z}{1-z}$ *has positive real part in* $|z| \leq 1 - \frac{\log n}{n}$ *for* sufficiently large n. The coefficient of $\frac{\log n}{n}$ cannot be replaced by any smaller constant.

This may be proved by the C_A-method of Theorem 1.6.1, and replaces the circle of radius $(1 - 2\frac{\log n}{n})$ of Robertson [19].

7. Apolar Polynomials.

Definition 1.7.1. Two polynomials $f(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k$ and $g(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$ are said to be apolar if:

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} (-1)^k = 0.$$

The importance of apolar polynomials stems from:

Theorem 1.7.1. (Grace's Theorem)[14]. If $f(z)$ and $g(z)$ are apolar polynomials, and if one of them has all its zeros in a circular region C, then the other will have at least one zero in C.

This may be applied in one direction to give:

Theorem 1.7.2.[14]. Let $A(z'_1, z'_2, \ldots, z'_n)$ be a linear symmetric function in the variables $z'_1, z'_2, \ldots, z'_n$, and let C be a circular region containing the points $z_1, z_2, \ldots z_n$. Then in C there exists at least one point z such that $A(z,z,\ldots,z)=A(z_1,z_2,\ldots,z_n)$.

Applying Grace's Theorem in a different direction, we may establish:

Theorem 1.7.3. (Szegö Convolution Theorem).[14] Let $f(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k$, $g(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$, and $h(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k$. If all the zeros of $f(z)$ lie in a circular region C, then every zero H of $h(z)$ has the form $H = -cG$, where c is a suitable point in C and G is a zero of $g(z)$.

This has the following simple corollary:

Theorem 1.7.4.[14]. Let $f(z)$, $g(z)$, $h(z)$ be defined as in Theorem 1.7.3. If all the zeros of $f(z)$, $g(z)$ lie in $|z| \geq R_1, R_2$ respectively, then all zeros of $h(z)$ lie in $|z| \geq R_1R_2$.

A lengthy and complicated application of Theorem 1.7.3, using polar derivatives, yields:

Theorem 1.7.5.[14]. Let $f(z) = \sum_{k=0}^{n} a_k z^k$, $g(z) = \sum_{k=0}^{m} b_k z^k$, and $h(z) = \sum_{k=0}^{n} a_k g(k) z^k$. If all the zeros of $f(z)$, $g(z)$ lie in $|z| \geq R$, $\text{Re } z \geq \frac{n}{2}$ respectively, then all the zeros of $h(z)$ lie in $|z| \geq R$.

Chapter 2. Univalent polynomials of arbitrary degree

'Because it is there'

- G.L. Mallory.

1. A particular subclass of P_n and M_n.

If the polynomial $p_n(z) = z + \sum_{k=2}^{n} a_k z^k \in P_n$,

$p_n'(z)$ cannot vanish in $|z| < 1$ (an elementary consequence of univalency). Consequently $|a_n| \leq \frac{1}{n}$. Similarly, if $\mu_n(z) = \frac{1}{z} + \sum_{k=1}^{n} b_k z^k \in M_n$, then $|b_n| \leq \frac{1}{n}$. In this section we will consider polynomials in P_n and M_n where the coefficient of z^n is $\frac{1}{n}$. Apart from the intrinsic interest of the results which we will obtain for such polynomials, we will indicate, in the next section, another reason for the importance of this particular subclass of P_n and M_n.

If $p_n(z) = z + \sum_{k=2}^{n-1} a_k z^k + \frac{1}{n} z^n \in P_n$, it is natural to wonder what conditions on the coefficients a_k are either necessary or sufficient for the univalency, close-to-convexity, or the starlikeness in $|z| < 1$ of $p_n(z)$. By elementary methods, it is clear that the polynomial $z + \frac{1}{n} z^n$ is univalent and starlike in $|z| < 1$. Some idea of the situation can be obtained from the following:

<u>Theorem 2.1.1.</u> <u>Let $p_n(z) = z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + \frac{1}{n} z^n$.</u>

(a) <u>If $p_n(z) \in P_n$, then:</u>

$$(n-k)a_{n-k} = (k+1)\bar{a}_{k+1} \quad (0 \leq k \leq n-1).$$

(b) <u>If $(n-k)a_{n-k} = (k+1)\bar{a}_{k+1}$, and a_{k+1} is sufficiently small, $1 \leq k \leq n-1$, then $p_n(z) \in P_n$.</u>

<u>Proof</u> (a) If $p_n(z) \in P_n$, and $a_n = \frac{1}{n}$, then

$$p_n'(z) = 1 + \sum_{k=2}^{n-1} k a_k z^{k-1} + z^{n-1}$$ must have all its zeros on $|z| = 1$. Since the zeros of $p_n'(z)$ are then inverse in $|z| = 1$, the required condition on the coefficients must certainly be satisfied.

(b) The polynomial $p_n(z) \in P_n$ iff:

$$1 + \sum_{k=1}^{n-2} a_{k+1} \frac{\sin(k+1)\theta}{\sin\theta} x^k + \frac{\sin n\theta}{n \sin\theta} x^{n-1} = 0$$

has no roots in $|x| < 1$, for $0 \leq \theta \leq \pi/2$.

Applying Cohn's Rule to this equation, since $\left|\sin n\theta / n \sin\theta\right| < 1$ for $0 < \theta \leq \frac{\pi}{2}$, we see that $p_n(z) \in P_n$ iff:

$$0 = 1 - \left(\frac{\sin n\theta}{n\sin\theta}\right)^2 + \sum_{k=1}^{n-2} x^k \left(a_{k+1}\frac{\sin(k+1)\theta}{\sin\theta} - \bar{a}_{n-k}\frac{\sin n\theta\sin(n-k)\theta}{n\sin^2\theta}\right)$$

$$= 1 - \left(\frac{\sin n\theta}{n\sin\theta}\right)^2 + \sum_{k=1}^{n-2} x^k a_{k+1}\left(\frac{\sin(k+1)\theta}{\sin\theta} - \frac{k+1}{n-k}\cdot\frac{\sin n\theta\sin(n-k)\theta}{\sin^2\theta}\right)$$

has no roots in $|x| < 1$ for $0 < \theta \leq \pi/2$.

Now each coefficient of $x^r (0 \leq r \leq n-2)$ has a double zero at $\theta=0$, and the constant is always positive otherwise. Consequently, if all the coefficients of x^r $(1 \leq r \leq n-2)$ are chosen sufficiently small, then (by Rouché's theorem) this equation has the same number of roots in $|x| < 1$ as has $1 - (\sin n\theta/n\sin\theta)^2 = 0$, i.e. none. Hence $p_n(z) \in P_n$.

A similar result holds for M_n:

Theorem 2.1.2. Let $\mu_n(z) = \frac{1}{z} + a_1 z + \ldots + a_{n-1}z^{n-1} + \frac{1}{n}z^n$.

(a) If $\mu_n \in M_n$, then:

$$(n-k)a_{n-k} = -(k-1)\bar{a}_{k-1} \quad (1 \leq k \leq n).$$

In particular $a_{n-1} = 0$.

(b) If $(n-k)a_{n-k} = -(k-1)\bar{a}_{k-1}$ $(1 \leq k \leq n)$, and a_k is chosen sufficiently small $(1 \leq k \leq n-2)$, then

$\mu_n(z) \in M_n$.

In addition, it is easy to show that $(\frac{1}{z} + \frac{z^n}{n})$ is starlike in $0 < |z| < 1$.

In 1915, Alexander [1] showed that the polynomials $\sum_{k=1}^{n} \frac{1}{k} z^k$ and $\sum_{k=0}^{n} \frac{z^k}{2k+1}$ were both univalent in $|z| < 1$. We put this result in a more general setting in the following:

Theorem 2.1.3. Let $p_n(z) = z + \sum_{k=2}^{n} a_k z^k$, and $q_{2n+1}(z) = z + \sum_{k=1}^{n} b_{2k+1} z^{2k+1}$.

(a) If ka_k decreases as k increases, then $p_n(z) \in CTC\ (\frac{z}{1-z})$.

(b) If $(2k+1)b_{2k+1}$ decreases as k increases, then $q_{2n+1} \in CTC\ (\frac{z}{1-z^2})$.

Proof We have that:

$$p_n'(z) = 1 + \sum_{k=2}^{n} ka_k z^{k-1}, \text{ and so:}$$

$$(1-z)\ p_n'(z) = 1 + (1-2a_2)z + \sum_{k=2}^{n-1} [ka_k-(k+1)a_{k+1}]z^k - na_n z^n .$$

Consequently, since ka_k is decreasing, we see that:

$$\operatorname*{Re}_{z \leq 1} \left((1-z)p_n'(z)\right) \geq 1 - (1-2a_2) - \sum_{k=2}^{n-1} [ka_k - (k+1)a_{k+1}] - na_n$$

$$= 0.$$

This may be written in the form:

$$\text{Re} \left(\frac{z\, p_n'(z)}{z/(1-z)} \right) \geq 0 \quad \text{in} \quad |z| \leq 1.$$

Consequently, by Theorem 1.4.3, the polynomial $p_n(z) \in$ CTC $(\frac{z}{1-z})$.

Similarly $q_n(z) \in$ CTC $(\frac{z}{1-z^2})$.

<u>Corollary.</u> <u>The polynomials $\sum_{k=1}^{n} \frac{z^k}{k}$ and $\sum_{k=0}^{n} \frac{z^{2k+1}}{2k+1}$ are both univalent and close-to-convex in $|z| < 1$.</u>

This is an immediate deduction from Theorem 2.1.3.

In spite of Theorems 2.1.1 and 2.1.3, and in spite of what might be regarded as a reasonable extension of Theorem 2.1.3, the following astounding result holds for starlike polynomials in P_n with $a_n = {}^1/n$.

<u>Theorem 2.1.4*.</u> <u>Let $p_n(z) = z + a_2z^2 + \ldots + a_{n-1}z^{n-1} + \frac{z^n}{n} \in P_n$. Then $p_n(z)$ is starlike in $|z| < 1$ iff</u>

$a_k = 0 \quad (2 \leq k \leq n-1)$

Proof. If $a_2 = a_3 = \ldots = a_{n-1} = 0$, it is easy to show that $p_n(z)$ is, in fact, starlike in $|z| < 1$. We therefore assume that $p_n(z)$ is starlike in $|z| < 1$, and then show that this implies that $a_k = 0$ $(2 \leq k \leq n-1)$.

Suppose $p_n(z)$ is starlike in $|z| < 1$. Then, by Theorem 1.4.1, we have that:

$$\frac{1 + 2a_2 z + \ldots\ldots + z^{n-1}}{1 + a_2 z + \ldots\ldots + \frac{1}{n} z^{n-1}} = \frac{zp_n'(z)}{p_n(z)}$$

$$= \frac{p_n'(z)}{h(z)} \quad (h(z) = \frac{p_n(z)}{z})$$

has positive real part in $|z| < 1$. Since $p_n(z) \in P_n$, we have that $(k+1)a_{k+1} = (n-k)\bar{a}_{n-k}$ $(1 \leq k \leq n-2)$ by Theorem 2.1.1. Consequently, on $|z| = 1$ with $z = e^{i\theta}$, we may define:

$$\alpha(\theta) = \frac{p_n'(z^2)}{z^{n-1}} \quad (z = e^{i\theta})$$

$$= \left(\frac{1}{z^{n-1}} + z^{n-1}\right) + \left(\frac{2a_2}{z^{n-3}} + 2\bar{a}_2\, z^{n-3}\right) + \ldots.$$

$$= 2[\cos(n-1)\theta + 2\, a_2 \cos((n-3)\theta - \phi_2) + \ldots.]$$

where $\phi_k = \arg a_k$ $(2 \leq k \leq \frac{1}{2}(n-1))$. Furthermore, on $|z| = 1$ with $z = e^{i\theta}$, we may define $\beta(\theta)$ and $\gamma(\theta)$

as follows:

$$\frac{p_n'(z^2)}{h\,(z^2)} = \frac{p_n'(z^2)}{z^{n-1}} \Big/ \frac{h(z^2)}{z^{n-1}} \qquad (z = e^{i\theta})$$

$$= \frac{\alpha(\theta)}{\beta(\theta) + i\gamma(\theta)}$$

$$= \frac{\alpha(\theta)\beta(\theta) - i\alpha(\theta)\gamma(\theta)}{\beta^2(\theta) + \gamma^2(\theta)} \quad .$$

No difficulty arises from the denominator, since the univalency of $p_n(z)$ in $|z| < 1$ ensures that:

$$\beta^2(\theta) + \gamma^2(\theta) = \left|h(e^{2i\theta})\right|^2 = \left|p_n(e^{2i\theta})\right|$$

$$> 0 \quad (0 \leq \theta \leq 2\pi).$$

We now show that $\alpha(\theta)$ can have only simple zeros for $0 \leq \theta \leq 2\pi$. Let ϕ be a zero of $\alpha(\theta)$. Now, with $z = e^{i\theta}$, we have that:

$$\alpha'(\theta) = \frac{\partial}{\partial\theta}\left[\frac{p_n'(z^2)}{z^{n-1}}\right]$$

$$= iz\frac{d}{dz}\left[\frac{p_n'(z^2)}{z^{n-1}}\right]$$

$$= iz\left[\frac{2z\,p_n''(z^2)}{z^{n-1}} - (n-1)\frac{p_n'(z^2)}{z^n}\right]$$

Now $\alpha(\phi) = 0$, so that $p_n'(z^2) = 0$ when $z = e^{i\phi}$.

Hence if $\alpha'(\phi)$ is also zero, we see that $p_n''(z^2)$ is zero at $z = e^{i\phi}$ as well. But then $p_n''(z) = 0$ at $z = e^{2i\phi}$. This, however, is impossible, since the existence of a double zero of $p_n'(z)$ on $|z| = 1$ is ruled out by the univalency of $p_n(z)$ in $|z| < 1$.

Now, the condition $\operatorname{Re}(zp_n'(z)/p_n(z)) \geq 0$ in $|z| < 1$ may be written in the form:

$$\alpha(\theta)\ \beta(\theta) \geq 0\ , \quad (0 \leq \theta \leq 2\pi).$$

in view of (1.1). Since the zeros of $\alpha(\theta)$ are simple, this in turn shows that, whenever $\alpha(\theta) = 0$, necessarily $\beta(\theta) = 0$. Now all of its $2(n-1)$ zeros lie in $0 \leq \theta \leq 2\pi$ (corresponding to the $(n-1)$ zeros of $p_n'(z)$ all on $|z| = 1$) in the case of $\alpha(\theta)$, and hence the same must be true of $\beta(\theta)$ since it is also a trigonometric polynomial of degree $(n-1)$. Since a polynomial which has its maximum number of zeros is determined by these zeros to within a constant factor, it follows that, for some constant λ, we must have:

$$\alpha(\theta) = \lambda\ \beta(\theta)\quad , \text{ or:}$$

$$\frac{p_n'(z^2)}{z^{n-1}} = \lambda \quad \operatorname{Re}\left(\frac{p_n(z^2)}{z^{n-1}}\right) \quad \text{on } |z| = 1.$$

This may be written in the form:

$$(1.2)\qquad \frac{1 + 2a_2z^2 + \ldots + (n-1)a_{n-1}z^{2n-4} + z^{2n-2}}{z^{n-1}} = \lambda \ \mathrm{Re}\left(\frac{1 + a_2z^2 + \ldots + a_{n-1}z^{2n-4} + \frac{1}{n}z^{2n-2}}{z^{n-1}} \right).$$

Equating the highest terms on both sides, with $z = e^{i\theta}$, we find that:

$$2 \cos (n-1)\theta = \lambda(1 + \tfrac{1}{n})\cos(n-1)\theta \ ,$$

and so: $$\lambda = \frac{2n}{n+1} \ .$$

Suppose that a_k is the first non-zero coefficient, with $k \neq \frac{1}{2}(n-1)$, so that $k < \frac{1}{2}(n-1)$. Comparing the terms on both sides of (1.2) of degree $(n - 2k + 1)$, we find that: $2k \left| a_k \right| \cos ((n-2k+1)\theta - \phi_k)$

$$= \lambda \left|a_k\right| (1 + \frac{k}{n-k+1})\cos((n-2k+1)\theta - \phi_k).$$

Since $a_k \neq 0$, and $\lambda = \frac{2n}{n+1}$, we deduce that:

$$\begin{aligned} 2k &= \frac{2n}{n-k+1} \\ &< \frac{2n}{\frac{1}{2}(n+3)} \quad (\text{since } k < \tfrac{1}{2}(n-1)) \\ &= \frac{4n}{n+3} < 4 , \end{aligned}$$

or: $k < 2$. But $2 \leq k < \frac{1}{2}(n-1)$, and so a_k must have been zero after all.

Now suppose that $a_k \neq 0$, $k = \frac{1}{2}(n-1)$. Then, comparing coefficients on both sides of (1.2) we find

that: $$a_k = \lambda a_k \ ,$$
since $a_k = a_{\frac{1}{2}(n-1)}$ must be real. But $\lambda \neq 1$, and so we have again arrived at an impossible situation.

Thus the assumption that $p_n(z)$ is starlike in $|z| < 1$ implies that $a_k = 0$ $(2 \leq k \leq n-1)$. This completes the proof of the theorem.

Similarly we may establish the following:

Theorem 2.1.5. Let $\mu_n(z) = \frac{1}{z} + a_1 z + \ldots + a_{n-1} z^{n-1} + \frac{z^n}{n} \in M_n$. Then $\mu_n(z)$ is starlike in $0 < |z| < 1$ iff $a_k = 0$ $(1 \leq k \leq n-1)$.

2. Some coefficient bounds for P_n and M_n.

In this section we will consider bounds for the $(n-1)^{th}$ coefficients of polynomials in P_n and M_n, and for the middle coefficient of a particular family of trinomials in P_{2n+1} and M_{2n+1}.

Theorem 2.2.1. The polynomial:

$$p_{2n+1}(z) = z + a\, z^{n+1} + \frac{z^{2n+1}}{2n+1} \in P_{2n+1} \quad \text{iff}$$

a is real, and:

$$(2.1) \qquad |a| \leq \operatorname{Min}_{[0,\frac{\pi}{2}]} \left(\frac{1 + \frac{\sin(2n+1)\theta}{(2n+1)\sin\theta}}{\left|\frac{\sin(n+1)\theta}{\sin\theta}\right|} \right) .$$

Note. By Theorem 2.1.1, a necessary condition for $p_{2n+1}(z) \in P_{2n+1}$ is that a is real. Also, since $p_{2n+1}(z) \in P_{2n+1}$ iff $z - az^{n+1} + \frac{z^{2n+1}}{2n+1} \in P_{2n+1}$, it is sufficient to prove the theorem for $a > 0$.

We will use the following:

Lemma. Let C and D be real, and $-1 < D < 1$. Then the equation:

$$(2.2) \qquad 1 + C\,y + D\,y^2 = 0$$

has no roots in $|y| < 1$ iff:

$$(2.3) \qquad 1 + D \geq |C| \quad .$$

Proof. Since C and D are both real, both roots of the equation have the same modulus; hence both lie in $|y| < 1$, or else both lie in $|y| \geq 1$. Applying Cohn's Rule to equation (2.2), it has the same number of zeros in $|y| < 1$ as has the equation:

$$(1-D^2) + Cy(1-D) = 0.$$

Consequently, (2.2) has no roots in $|y| \leq 1$ iff:

$$|C|(1-D) \leq 1 - D^2 ,$$

i.e. $$|C| \leq 1 + D ,$$

since $-1 < D < 1$. Hence the lemma is proved.

Proof of Theorem 2.2.1. We assume that $a>0$. Then, by the Dieudonné criterion for univalency (Theorem 1.2.2), $p_{2n+1}(z) \in P_{2n+1}$ iff the equation:

$$1 + \frac{a \sin(n+1)\theta}{\sin\theta} x^n + \frac{\sin(2n+1)\theta}{(2n+1)\sin\theta} x^{2n} = 0,$$

and, also, the equation:

$$(2.4) \qquad 1 + \frac{a \sin(n+1)\theta}{\sin\theta} y + \frac{\sin(2n+1)\theta}{(2n+1)\sin\theta} y^2 = 0$$

have no roots in $|y| = |x^n| < 1$, for $0 \leq \theta \leq \pi/2$.

Now $\left|\sin(2n+1)\theta/(2n+1)\sin\theta\right| < 1$ for $0 < \theta \leq \pi/2$. Hence, by the preceding lemma, (2.4) has no roots in $|y| < 1$ iff:

$$a \frac{|\sin(n+1)\theta|}{\sin\theta}\left(1 - \frac{\sin(2n+1)\theta}{(2n+1)\sin\theta}\right) \leq 1 - \left(\frac{\sin(2n+1)\theta}{(2n+1)\sin\theta}\right)^2, \text{ or:}$$

$$a \leq \frac{1 + \frac{\sin(2n+1)\theta}{(2n+1)\sin\theta}}{\left|\frac{\sin(n+1)\theta}{\sin\theta}\right|},$$

for $0 < \theta \leq \pi/2$, and so for $0 \leq \theta \leq \pi/2$. Hence the result of the theorem follows immediately.

Similarly, we may establish:

<u>Theorem 2.2.2</u>. <u>The polynomial</u>:

$$\mu_{2n+1}(z) = \frac{1}{z} + aiz^n + \frac{z^{2n+1}}{2n+1}$$

<u>$\in M_{2n+1}$ iff a is real, and:</u>

$$(2.5) \qquad |a| \le \operatorname*{Min}_{[0,\frac{\pi}{2}]} \left(\frac{1 + \frac{\sin(2n+1)\theta}{(2n+1)\sin\theta}}{\left|\frac{\sin n\theta}{\sin\theta}\right|} \right) .$$

As an immediate application of Theorems 2.2.1 and 2.2.2, it is easy to establish the following:

Corollary 1. The polynomials $p_3(z) = z + az^2 + \frac{z^3}{3}$ and $p_5(z) = z + bz^3 + \frac{z^5}{5}$ belong to P_3 and P_5 respectively iff a and b are real, and $|a| \le \sqrt{8/9}$, $|b| \le \frac{3}{5}$.

Corollary 2. The polynomials $\mu_3(z) = \frac{1}{z} + aiz + \frac{z^3}{3}$ and $\mu_5(z) = \frac{1}{z} + biz^3 + \frac{z^5}{5}$ belong to M_3 and M_5 respectively iff a and b are real, and $|a| \le \frac{2}{3}$, $|b| \le \sqrt{\frac{8}{25}}$.

We now establish the asymptotic values, for large n, of the bounds for the central coefficients given by (2.1) and (2.5).

Lemma. Let $A_n = \operatorname*{Min}_{0\le\theta\le\frac{\pi}{2}} f_n(\theta)$, where:

$$f_n(\theta) = \frac{1 + \frac{\sin(2n+1)\theta}{(2n+1)\sin\theta}}{\left|\frac{\sin(n+1)\theta}{\sin\theta}\right|} . \quad \text{Then:}$$

$$A_n = \frac{\pi}{4n}[1 + o(1)], \text{ for sufficiently large n.}$$

Proof. When $\theta > \frac{4\pi}{2n+1}$, we have:

$$\frac{\sin(2n+1)\theta}{(2n+1)\sin\theta} > \frac{-1}{(2n+1)\sin(\frac{4\pi}{2n+1})}$$

$$> \frac{-1}{4\pi}\,[1 + o(1)]\ , \text{ and so:}$$

$$f_n(\theta) > [1 - \frac{1}{4\pi} + o(1)].\left|\frac{\sin\,\theta}{\sin(n+1)\theta}\right|$$

$$> [1 - \frac{1}{4\pi} + o(1)].\ (\frac{4\pi}{2n+1}) \tag{2.6}$$

$$> 2\ .\ \frac{\sin(\frac{\pi}{2n+1})}{\sin(\frac{(n+1)\pi}{2n+1})} = \frac{2\pi}{2n+1}\,(1 + o(1)).$$

$$= 2\ f_n\ (\frac{\pi}{2n+1}) > 2\ A_n\ .$$

Consequently, if $A_n = f_n(\theta_n)$, then $0 \le \theta_n < \frac{4\pi}{2n+1}$ for sufficiently large n. Suppose that, at least on a sequence of n, $(2n+1)\theta_n \longrightarrow \alpha$ as $n \longrightarrow \infty$; here $0 \le \alpha \le 4\pi$.

Suppose, first, that $\alpha \neq 0$. Then:

$$A_n = f_n\ (\frac{\alpha + o(1)}{2n+1}) \quad \text{(for sufficiently large n)}$$

$$= \frac{1 + \frac{\sin\alpha}{\alpha}}{\left|\sin(\frac{\alpha}{2})\right|}\ .\ \frac{\alpha}{2n+1}\ .\ (1 + o(1)). \tag{2.7}$$

In view of (2.7), consider the function:

$$g(x) = \frac{1 + \frac{\sin 2x}{2x}}{\sin x}\ .\ x$$

$$= \frac{x}{\sin\,x} + \cos\,x \qquad (0 \le x \le 2\pi). \tag{2.8}$$

Clearly $g(x)$ is non-zero, and becomes infinite when $x = \pi,\ 2\pi$. Consequently the minimum of $|g(x)|$, in which we are interested in view of (2.7), is attained at some point where $g'(x) = 0$. This occurs when $\cos x = 0$ or $2x = \sin 2x$, and so:

$$\begin{aligned}\min_{0\le x\le 2\pi} |g(x)| &= \operatorname{Min}\left(g(0),\ g(\tfrac{\pi}{2}),\ |g(\tfrac{3\pi}{2})|\right)\\ &= \operatorname{Min}\left(2,\ \tfrac{\pi}{2},\ \tfrac{3\pi}{2}\right)\\ &= \pi/2\ .\end{aligned}$$

Returning to (2.7), this implies that $\alpha = \pi$, and:

$$\begin{aligned} A_n &= \frac{\pi}{2(2n+1)}\,[1 + o(1)] \\ &= \frac{\pi}{4n}\,[1 + o(1)]. \qquad (2.9)\end{aligned}$$

If, in fact, α were zero, then we would have:

$$\begin{aligned} A_n &= \frac{2}{n+1}\,[1 + o(1)] \\ &> \frac{\pi}{4n}\,[1 + o(1)].\end{aligned}$$

Hence α cannot be zero, and the result of the lemma follows at once.

Combining the lemma and Theorem 2.2.1, we deduce the following:

<u>Theorem 2.2.3.</u> <u>The polynomial:</u>

$$p_{2n+1}(z) = z + az^{n+1} + \frac{z^{2n+1}}{2n+1}$$

$\in P_{2n+1}$ iff a is real, and $|a| \leq A_n$, where $A_n \sim \frac{\pi}{4n}$ for large n.

In a similar way, we can establish:

Theorem 2.2.4. The polynomial:

$$\mu_{2n+1}(z) = \frac{1}{z} + aiz^n + \frac{z^{2n+1}}{2n+1}$$

$\in M_{2n+1}$ iff a is real, and $|a| \leq B_n$, where $B_n \sim \frac{\pi}{4n}$ for large n.

Let us now turn to the estimation of the $(n-1)^{th}$ coefficient of polynomials in P_n and M_n.

Theorem 2.2.5. Let $p_n(z) = z + a_2 z^2 + \ldots + a_n z^n \in P_n$. Then:

$$(2.10) \quad (n-1)|a_{n-1}| \leq 1 + 2|a_2| n |a_n| - n^2 |a_n|^2 < 4.$$

In particular, $(n-1)|a_{n-1}| \leq \begin{cases} 1 + |a_2|^2 & , \text{ if } |a_2| \leq 1. \\ 2|a_2| & , \text{ if } |a_2| \geq 1. \end{cases}$

Proof. By the Dieudonné criterion, since $p_n(z) \in P_n$, the equation:

$$1 + \sum_{k=1}^{n-1} a_{k+1} \frac{\sin(k+1)\theta}{\sin\theta} x^k = 0$$

has no roots in $|x| < 1$ for $0 < \theta \leq \pi/2$, and $|a_n| \leq \frac{1}{n}$.

Applying the Cohn Rule, we deduce that the equation:

$$1 - |a_n|^2\left(\frac{\sin n\theta}{\sin\theta}\right)^2 + \sum_{k=1}^{n-2} x^k\left(a_{k+1}\frac{\sin(k+1)\theta}{\sin\theta} - a_n\bar{a}_{n-k}\cdot\frac{\sin n\theta\sin(n-1)\theta}{\sin^2\theta}\right) = 0$$

has no zeros in $|x| < 1$ for $0 < \theta \le \pi/2$, and so for $0 \le \theta \le \pi/2$. Consequently:

$$1 - |a_n|^2\left(\frac{\sin n\theta}{\sin\theta}\right)^2 \ge \left|a_{n-1}\frac{\sin(n-1)\theta}{\sin\theta} - a_n\bar{a}_2\frac{\sin n\theta\sin 2\theta}{\sin^2\theta}\right|$$

$$\ge \left|a_{n-1}\frac{\sin(n-1)\theta}{\sin\theta}\right| - \left|a_na_2\frac{\sin n\theta\sin 2\theta}{\sin^2\theta}\right|$$

for $0 \le \theta \le \pi/2$. Substituting $\theta = 0$, we obtain:

$$1 - n^2|a_n|^2 \ge (n-1)|a_{n-1}| - n|a_n|\cdot|2a_2| \text{ or:}$$

(2.11) $(n-1)|a_{n-1}| \le 1 + 2|a_2|\cdot n|a_n| - n^2|a_n|^2$.

(a) Since $p_n(z) \in S$, $|a_2| < 2$ [8]. Thus:

$$(n-1)|a_{n-1}| \le 1 + 4n|a_n| - n^2|a_n|^2$$

$$\le 4 \quad (\text{since } n|a_n| \le 1).$$

(b) Now let $f(y) = 1 + 2|a_2|y - y^2$. Then $f'(y) = 2|a_2| - 2y = 0$ when $y = |a_2|$; also $f(y)$ is an increasing function of y for $y < |a_2|$, and a decreasing function of y for $y > |a_2|$.

Suppose, first, that $|a_2| \le 1$. Then

$\max\limits_{[0,1]} f(y) = f(|a_2|) = 1 + |a_2|^2$; and so, by (2.11), we obtain $(n-1)|a_{n-1}| \leq 1 + |a_2|^2$.

Suppose, next, that $|a_2| \geq 1$. Then $\max\limits_{[0,1]} f(y) = f(1) = 2|a_2|$; and so, by (2.11), we obtain $(n-1)|a_{n-1}| \leq 2|a_2|$.

Corollary. Let $p_n(z) = z + a_2z^2 + \ldots + a_nz^n \in P_n$.

(a) If $a_2 = 0$, then $(n-1)|a_{n-1}| \leq 1 - n^2|a_n|^2$.

(b) If $a_{n-1} = 0$, then $2|a_2| \leq \dfrac{1-n^2|a_n|^2}{n|a_n|}$. This is an immediate consequence of equation (2.11).

We now establish the following result, which will put Theorem 2.2.5 in its proper perspective.

Theorem 2.2.6. Let $p_n(z) = z + a_2z^2 + \ldots + a_nz^n \in P_n$, and let $A_{n-1} = \max\limits_{p_n} |a_{n-1}|$. Then:

$$(2.12)\quad \frac{2}{n-1}\cos\left(\frac{\pi}{n+1}\right) \leq A_{n-1} < \frac{4}{n-1}\,.$$

Proof. The right hand inequality of (2.12) follows from Theorem 2.2.5. We now construct a close-to-convex polynomial in P_n whose $(n-1)^{th}$ coefficient is $\frac{2}{n-1}\cos\left(\frac{\pi}{n+1}\right)$; all its coefficients will be real. Consequently the left hand inequality of (2.12) will follow.

Consider the function:

$$q(z) = \int_0^z \frac{1 + t^{n+1}}{1 - 2\cos(\frac{\pi}{n+1})t + t^2}\,dt. \tag{2.13}$$

Clearly $q(z)$ is regular in $|z| < 1$. From (2.13), we deduce, immediately, that:

$$(1 - 2\cos(\tfrac{\pi}{n+1})z + z^2)q'(z) = 1 + z^{n+1}$$

$$= \prod_{m=0}^{n} (1 - ze^{\frac{2m+1}{n+1}\pi}) \tag{2.14}$$

This latter product contains the factor:

$$(1 - ze^{\frac{\pi i}{n+1}})(1 - ze^{\frac{-\pi i}{n+1}}) = 1 - 2\cos(\tfrac{\pi}{n+1})z + z^2 ,$$

and so $q(z)$ is, in fact, a polynomial in z of degree n. In addition:

$$\mathrm{Re}\left(zq'(z) \Big/ \frac{z}{1 - 2\cos(\frac{\pi}{n+1})z+z^2} \right) = \mathrm{Re}(1 + z^{n+1})$$

$$\geq 0 \text{ in } |z| \leq 1.$$

By Theorem 1.5.4, the function $s(z) = z/(1-2\cos(\frac{\pi}{n+1})z + z^2)$ is starlike in $|z| < 1$. Consequently, $p_n(z) \in CTC(s(z))$, and so $p_n(z) \in P_n$.

Suppose that $q(z) = z + a_2z^2 + \ldots + a_nz^n$. Then, by (2.14),

$$1 + z^{n+1} = (1-2\cos(\tfrac{\pi}{n+1})z+z^2)(1+2a_2z+\ldots+(n-1)a_{n-1}z^{n-2}+ +na_nz^{n-1}).$$

Equating coefficients on both sides, we obtain $a_n = \frac{1}{n}$, and $a_{n-1} = \frac{2}{n-1}\cos(\frac{\pi}{n+1})$.

This completes the proof of the theorem.

Using the Dieudonné criterion for M_n and arguments similar to those of Theorem 2.2.5, we may establish the following:

<u>Theorem 2.2.7.</u> <u>Let $\mu_n(z) = \frac{1}{z} + a_1 z + \ldots + a_n z^n \in M_n$. Then $|a_n| \leq \frac{1}{n}$, and:</u>

(2.15) $$(n-1)\,|a_{n-1}| \leq 1 - n^2 |a_n|^2 .$$

3. <u>The second coefficient problem.</u>

In this section we establish estimates for the coefficients a_2 and b_3 of the polynomials $p_n(z) = z + a_2 z^2 + \ldots + a_n z^n \in P_n$ and $p_{2n+1}(z) = z + b_3 z^3 + b_5 z^5 + \ldots + b_{2n+1} z^{2n+1}$. Upper bounds for these have been known for a long time [15]; we will show that, surprisingly enough, these give the correct order of magnitudes of $\sup_{P_n}|a_2|$ and $\sup_{P_{2n+1}}|b_3|$.

<u>Theorem 2.3.1.[15]</u> <u>Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S$, and $|f(z)| \leq M$ in $|z| < 1$. Then:</u>

(3.1) $$|a_2| \leq 2(1 - \frac{1}{M}).$$

Proof. If $M=1$, $f(z) \equiv z$, and there is nothing to prove. Thus we suppose $M>1$.

Since $f(z) \in S$, clearly:

$$(3.2) \quad F(z) \equiv \frac{f(z)}{[1 - \frac{e^{i\alpha}}{M} f(z)]^2} \qquad (0 \leq \alpha \leq 2\pi).$$

$$= (z + a_2 z^2 + \ldots)(1 + \frac{2}{M} e^{i\alpha}(z + \ldots))$$

$$= z + (a_2 + \frac{2}{M} e^{i\alpha}) z^2 + \ldots \in S.$$

Thus, by a well-known theorem of Bieberbach [8], $\left| a_2 + \frac{2}{M} e^{i\alpha} \right| \leq 2$ for all $\alpha \in [0, 2\pi]$, and so:

$$\left| a_2 \right| \leq 2 \left(1 - \frac{1}{M}\right).$$

Thus our theorem is proved.

We now state, in the following sequence of lemmas, a number of well-known results which form the background to our present discussion.

Lemma 1.[8] Let $f(z) = z + a_2 z^2 + \ldots \in S$. Then $|a_3| \leq 3$, and $|a_4| \leq 4$; also $|a_n| < en$ for all $n>4$. If all the coefficients of $f(z)$ are real, then $|a_n| \leq n$ for all $n>1$.

Lemma 2.[8] The function $f(z) \in S$ if $f(z^2)^{\frac{1}{2}} \in S$; and if $g(z)$ is an odd function in S, then $g(z^{\frac{1}{2}})^2 \in S$. In particular, if $q_{2n+1}(z)$ is an odd polynomial in

P_{2n+1}, then $q_{2n+1}(z^{\frac{1}{2}})^2 \in P_{2n+1}$.

In view of the importance of this lemma, we make the following definition.

Definition 2.3.1. A polynomial $p_{2n+1}(z)$ is said to belong to the class $P^o{}_{2n+1}$ iff it belongs to the class P_{2n+1} and contains only odd powers of z.

Lemma 3. If $p_N(z) = z + a_2z^2 + \ldots + a_Nz^N \in P_N$, then:

$$\max_{P_n} |a_k| \leq \max_{P_{n+1}} |a_k| \qquad (2 \leq k \leq n < N).$$

Similarly, if $q_{2N+1}(z) = z + b_3z^3 + \ldots + b_{2N+1}z^{2N+1} \in P^o{}_{2N+1}$, then:

$$\max_{P^o{}_{2n+1}} |b_{2k+1}| \leq \max_{P^o{}_{2n+3}} |b_{2k+1}| \qquad (1 \leq k \leq n < N)$$

From Lemma 1 and Theorem 2.3.1, we deduce:

Theorem 2.3.2. Let $p_n(z) = z + a_2z^2 + \ldots + a_nz^n \in P_n$.

(a) If all the coefficients are real, or if $p_n(z)$ is close-to-convex in $|z| < 1$, then:

$$|a_k| \leq k\left(1 - \frac{A(k)}{n^2}\right) \qquad (2 \leq k \leq n-1)$$

where $A(k)$ is a constant depending on k, but not on n. If the well-known Bieberbach hypothesis[8] were true, this bound would be valid for all $p_n(z) \in P_n$.

(b) <u>In general,</u> $|a_3| \leq 3\,(1 - \frac{A}{n^2})$, <u>and</u> $|a_4| \leq 4$.
$(1 - \frac{B}{n^2})$, where A and B are absolute constants.

The proof follows from the fact that if $p_n(z) \in P_n$,

then $|p_n(z)| \leq |z| + \sum_{k=2}^{n} |a_k z^k| \leq 1 + \sum_{k=2}^{n} ek < en^2$

for $|z| \leq 1$.

Similarly we may establish:

<u>Theorem 2.3.3.</u> <u>Let $p_{2n+1}(z) = z + b_3 z^3 + \ldots + b_{2n+1} z^{2n+1}$ $\in P^o_{2n+1}$. Then $|b_3| \leq 1 - \frac{A}{n^2}$ where A is an absolute constant.</u>

We now state a striking result, which will be a fundamental tool in what follows.

<u>Lemma 4.</u> (<u>Fejer Representation Theorem</u>)[7] <u>Suppose</u>:

$$g(\theta) = \sum_{k=0}^{n} \lambda_k \cos k\theta + \sum_{k=1}^{n} \mu_k \sin k\theta .$$

<u>Then $g(\theta)$ is a non-negative trigonometric polynomial for $0 \leq \theta \leq 2\pi$ iff it is of the form $g(\theta) = |h(e^{i\theta})|^2$, where $h(z) = x_o + x_1 z + \ldots. + x_n z^n$.</u> <u>Then</u>:

$$\lambda_o = \sum_{\nu=0}^{n} |x_\nu|^2 \text{ , and}$$

$$(3.3) \qquad \lambda_\nu + i\mu_\nu = 2\sum_{r=0}^{n-\nu} x_r \bar{x}_{\nu+r} \qquad (1 \leq \nu \leq n).$$

First of all, in order to illustrate the methods we will use, without introducing too many complications, we will establish the following:

<u>Theorem 2.3.4</u>. (a) <u>The polynomial</u>:

$$q(z) = z + \sum_{k=1}^{n+1} \left(2k+1 - \frac{3k(k+1)}{2(n+1)}\right) \frac{z^{2k+1}}{2k+1}$$

$$(3.4) \qquad + \sum_{k=n+2}^{2n} \left(k^2 - (4n+3)k+2(n+2)(n+1)\right) \frac{z^{2k+1}}{2(n+1)(2k+1)}$$

$$= z + \frac{n}{n+1} z^3 + \ldots + \frac{z^{4n+1}}{(n+1)(4n+1)} \in P^o{}_{4n+1} .$$

(b) <u>The polynomial</u>:

$$p(z) = q(z^{\frac{1}{2}})^2$$

$$= z + 2\frac{n}{n+1} z^2 + \ldots + \frac{z^{4n+1}}{(n+1)^2(4n+1)^2} \in \underline{P_{4n+1}} .$$

Then, as an immediate consequence, we will have established:

<u>Theorem 2.3.5.</u>(a) <u>Let $p_n(z) = z + a_2 z^2 + \ldots + a_n z^n$ $\in P_n$. Then</u>:

$$2\left(1 - \frac{4}{n} + o(1/n)\right) \leq \max_{p_n} |a_2| \leq 2\left(1 - \frac{1}{en^2}\right)$$

(b) <u>Let $q_n(z) = z + b_3 z^3 + \ldots + b_{2n+1} z^{2n+1} \in P^o{}_{2n+1}$</u> .

Then:

$$(1 - \frac{2}{n} + o(\frac{1}{n})) \leq \max_{q_n} |b_3| \leqslant 1 - O(1/n^2).$$

Proof. The theorem follows, at once, from Theorems 2.3.1 and 2.3.4, and Lemma 1.

Proof of Theorem 2.3.4. We will apply Lemma 4 to the polynomial:

$$(3.6)\quad h(z) = \frac{1}{\sqrt{2(n+1)}}[1+z^2+\ldots+z^{2n} - z^{2n+2}(1+z^2+\ldots+z^{2n})]$$

$$= \sum_{r=0}^{4n+2} x_r z^r \text{ , say.}$$

Here $h(z)$ is an even polynomial of degree $(4n+2)$, normalised so that $\sum_{r=0}^{4n+2} x_r{}^2 = 1$.

Now let us put:

$$|h(e^{i\theta})|^2 = g(\theta)$$

$$= \operatorname*{Re}_{z=e^{i\theta}} (t(z)) ,$$

and examine the precise form of the coefficients of $t(z) = \sum_{k=0}^{2n+1} a_{2k} z^{2k}$. Obviously we then have $g(\theta)$ of the form $g(\theta) = \sum_{k=0}^{2n+1} a_{2k} \cos 2k\theta$. Then, by (3.3), we obtain:

$$a_o = 1, \; a_2 = 2\,\frac{2n-1}{2n+2}\;, \; a_4 = 2\,\frac{2n-4}{2n+2}\;, \; \ldots .$$

and: $a_{4n+2} = \dfrac{-2}{2n+2}\;, \; a_{4n} = \dfrac{-4}{2n+2}\;, \; a_{4n-2} = \dfrac{-6}{2n+2}\;, \; \ldots \; .$

In general, then: $a_{2k} = \begin{cases} 2\;(1 - \frac{3k}{2n+2}) & \text{if } 1 \leq k \leq n+1. \\ 2\;(\frac{k}{2n+2} - 1) & \text{if } n+2 \leq k \leq 2n+1. \end{cases}$

Thus:

$$(3.7) \quad t(z) = 1 + 2\sum_{k=1}^{n+1}(1 - \frac{3k}{2n+2})z^{2k} + 2\sum_{k=n+2}^{2n+1}(\frac{k}{2n+2} - 1)z^{2k}.$$

We have, in fact, constructed $t(z)$ in such a way that:

$$t(\pm 1) = g(^{0}_{\pi}) = |h(\pm 1)|^2 = 0.$$

Thus $t(z)$ has a factor $(1-z^2)$, and also $\operatorname{Re} t(z) \geq 0$ in $|z| \leq 1$.

We now determine the polynomial

$$q(z) = z + \sum_{k=1}^{2n} b_{2k+1} z^{2k+1}$$ such that:

$$t(z) = (1-z^2)q'(z)$$

$$= zq'(z)/(\frac{z}{1-z^2}).$$

Thus $q(z) \in CTC(\frac{z}{1-z^2})$, since $\frac{z}{1-z^2}$ is a starlike function in $|z| < 1$. Then we have:

$$1 + 2\sum_{k=1}^{n+1}\left(1 - \frac{3k}{2n+2}\right)z^{2k} + 2\sum_{k=n+2}^{2n+1}\left(\frac{k}{2n+2} - 1\right)z^{2k}$$

$$= t(z)$$

$$= (1-z^2)q'(z)$$

$$= 1 + (3b_3-1)z^2 + (5b_5-3b_3)z^4 + \dots$$
$$\dots\dots + [(4n+1)b_{4n+1} - (4n-1)b_{4n-1}]z^{4n} - (4n+1)\cdot b_{4n+1}z^{4n+2} .$$

Comparing coefficients in this equation, we obtain:

$$(3.8) \quad (2k+1)b_{2k+1} = \begin{cases} (2k+1) - \dfrac{3k(k+1)}{2(n+1)} , & \text{if } 1 \le k \le n+1. \\[2ex] \dfrac{k^2-(4n+3)k+(4n+2)(n+1)}{2(n+1)}, & \text{if } n+2 \le k \le 2n+1. \end{cases}$$

Consequently the polynomial $q(z)$, given by (3.4), is the polynomial given by (3.8), and the first part of the theorem is proved.

The second part follows at once, by Lemma 2.

Note. All the coefficients of the polynomials (3.4) and (3.5) are real and positive.

Corollary. Let $p_N(z) = z + a_2z^2 + \dots + a_Nz^N \in P_N$. Then:

$$(3.9) \quad \left(\frac{1}{4}\log 2\right)^2 N^2[1+o(1)] \le \max_{p_N} M(1;p_N) < \tfrac{1}{2}eN^2[1+o(1)].$$

Proof. The second inequality in (3.9) follows at once from Lemma 1, and we will use the polynomial $p(z)$ of Theorem 2.3.4 to prove the first inequality of (3.9) when $N \equiv 1 \pmod 4$. Then by Lemma 3, the corollary will have been proved.

Since the coefficients of the polynomial $p(z)$ in P_{4n+1} of Theorem 2.3.4 are positive,

$$M(1;p) = M(1;q)^2 = q(1)^2, \text{ where:} \tag{3.10}$$

$$q(1) = 1 + \sum_{k=1}^{n+1} \left(2k+1 - \frac{3k(k+1)}{2(n+1)}\right) \frac{1}{2k+1} \tag{3.11}$$

$$+ \sum_{k=n+2}^{2n} (k^2-(4n+3)k+2(n+1)(2n+1)) \Big/ 2(n+1)(2k+1).$$

Let us denote the two summations by $\sum_1$ and $\sum_2$ respectively. We estimate both of these to within an error term $o(n)$. Now:

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{k=1}^{n+1} \left(1 - \frac{3k(k+1)}{2(n+1)(2k+1}\right) \\
&= n+1 - \frac{3}{2(n+1)} \sum_{k=1}^{n+1} \left(\frac{k}{2} + \frac{k}{2(2k+1)}\right) \\
&= n+1 - \frac{3}{2(n+1)} \cdot \frac{1}{4}(n+1)(n+2) - \frac{3}{2(n+1)} \sum_{k=1}^{n+1} O(1) \\
&= \frac{5}{8} n + o(n) .
\end{aligned}$$

Examining the terms in $\sum_2$ separately, we have:

$$\sum_{k=n+2}^{2n} \frac{k^2}{2k+1} = \sum_{k=n+2}^{2n} \left(\frac{k}{2} + O(1) \right)$$

$$= \tfrac{1}{2} \sum_{k=1}^{2n} k - \tfrac{1}{2} \sum_{k=1}^{n+1} k + O(n)$$

$$= \frac{3}{4} n^2 + O(n) \text{ , and:}$$

$$\sum_{k=n+2}^{2n} \frac{k}{2k+1} = \tfrac{1}{2} \sum_{k=n+2}^{2n} 1 - \tfrac{1}{2} \sum_{k=n+2}^{2n} \frac{1}{2k+1}$$

$$= \tfrac{1}{2} n + O(1), \text{ and:}$$

$$\sum_{k=n+2}^{2n} \frac{1}{2k+1} = \tfrac{1}{2} \sum_{k=n+2}^{2n} \frac{1}{k} + o(1)$$

$$= \tfrac{1}{2}(\log(2n) - \log n) + o(1)$$

$$= \tfrac{1}{2} \log 2 + o(1).$$

Combining these estimates, we see that:

$$\sum\nolimits_2 = \frac{1}{2n} \cdot \frac{3}{4}n^2 - \frac{4n}{2n} \cdot \frac{n}{2} + \frac{4n^2}{2n} \cdot \tfrac{1}{2} \log 2 + o(n)$$

$$= (\log 2 - \tfrac{5}{8})\, n + o(n), \text{ and so by (3.10) and}$$

$$(3.11),\quad M(1,p) = \log^2 2\; n^2 + o(n^2)$$

$$= \frac{1}{16}\log^2 2\; N^2\;(1+o(1)),\quad \text{putting}$$

$N = 4n+1$. This completes the proof of the corollary.

We now proceed to give a slightly more sophisticated application of the Fejer Representation Theorem, which will give lower estimates for $\max\limits_{P_N} |a_2|$, $\max\limits_{P_N} |a_3|$, $\max\limits_{P_N} |a_4|$, and $\max\limits_{P^o_{2N+1}} |b_3|$. The corresponding upper estimates have been known for a long time, but no lower estimates existed previously apart from those connected with sections of power series (as in Theorem 1.6.5).

<u>Theorem 2.3.5</u>. <u>There exists a sequence of polynomials</u>: $p_N(z) = z + a_2z^2 + \ldots + a_Nz^N \in P_N$ <u>for all N with</u> $\frac{N-3}{2K} \equiv 1 \pmod 2$ <u>such that</u>:

$$(3.12)\qquad a_s > s - \frac{A(k)}{N^2}\quad (2 \le s \le k+1),$$

<u>where $A(k)$ is a positive constant depending on k, but not on N</u>.

<u>Proof</u>. We will apply Lemma 4 to the polynomial:

$$h(z) = C\sum_{r=0}^{n} \sin\left(\frac{r+1}{n+2}\pi\right) z^{2kr}\,(1+z^2+\ldots+z^{2k-2})(1-z^{2kn+2k+4}).$$

$$= \sum_{r=0}^{4kn+2k+4} x_r z^r\;,\ \text{say}.$$

Here $h(z)$ is an even polynomial of degree $(4kn+4k+2)$, normalised so that $\sum_r x_r^2 = 1$ by the choice:

$$c^{-2} = 2k \sum_{r=0}^{n} \sin^2(\tfrac{r+1}{n+2}\,\pi) = k(n+2) \ .$$

Now let us put:

$$|h(e^{i\theta})|^2 = g(\theta)$$

$$= \operatorname*{Re}_{z=e^{i\theta}} \ (t(z)),$$

and examine the precise form of the coefficients of $t(z) = 1 + \sum_{s=0}^{2kn+2k} d_{2s+2}\, z^{2s+2}$. Obviously we then have $g(\theta)$ of the form $g(\theta) = 1 + \sum_{s=0}^{2kn+2k} d_{2s+2} \cos(2s+2)\theta$.

Then, by (3.3), we obtain:

$$d_2 = 2c^2 \left[2(k-1)\sum_{r=0}^{n}\sin^2(\tfrac{r+1}{n+2}\pi) + 2\sum_{r=0}^{n-1}\sin(\tfrac{r+1}{n+2}\pi)\sin(\tfrac{r+2}{n+2}\pi)\right]$$

$$= \frac{2}{k(n+2)}\left[(k-1)(n+2) + (n+2)\cos(\tfrac{\pi}{n+2})\right]$$

$$= \frac{2}{k}\left[k-1 + \cos(\tfrac{\pi}{n+2})\right], \text{ and:}$$

$$d_4 = 2c^2 \left[2(k-2)\sum_{r=0}^{n}\sin^2(\tfrac{r+1}{n+2}\pi) + 2\sum_{r=0}^{n-1}\sin(\tfrac{r+1}{n+2}\pi)\sin(\tfrac{r+2}{n+2}\pi)\right]$$

$$= \frac{2}{k}\left[k-2 + 2\cos(\tfrac{\pi}{n+2})\right].$$

It is easy to verify, furthermore, that:

$d_{2s} = \frac{2}{k}\,[k - s + s\cos(\frac{\pi}{n+2})]$, if $1 \leq s \leq k$.

Also, $d_{4kn+4k+2} = -2C^2 \sin^2(\frac{\pi}{n+2})$

$$= \frac{-2}{k(n+2)} \sin^2(\frac{\pi}{n+2}).$$ Thus:

$$t(z) = 1 + \frac{2}{k}\sum_{s=1}^{k}[k-s+s\cos(\frac{\pi}{n+2})]z^{2s} + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots \frac{-2}{k(n+2)} \sin^2(\frac{\pi}{n+2}) z^{4kn+4k+2} .$$

We have, in fact, constructed $t(z)$ in such a way that:

$t(\pm 1) = g(^{0}_{\pi}) = |h(\pm 1)|^2 = 0.$

Thus $t(z)$ has a factor $(1-z^2)$, and also $\mathrm{Re}\, t(z) \geq 0$ in $|z| \leq 1$.

We now determine the polynomial $q(z) = z + \sum_{s=1}^{2kn+2k} b_{2s+1} z^{2s+1}$ such that: $t(z) = (1-z^2)q'(z)$

$$= zq'(z)/(\frac{z}{1-z^2}).$$

Thus $q(z) \in \mathrm{CTC}\ (\frac{z}{1-z^2})$, since $\frac{z}{1-z^2}$ is a starlike function in $|z| < 1$. Then we have:

$$1 + \frac{2}{k}\sum_{s=1}^{k}[k-s+s\cos(\frac{\pi}{n+2})]z^{2s} + \ldots - \frac{2}{k(n+2)}\sin^2(\frac{\pi}{n+2}). z^{4kn+4k+2}$$

$= t(z)$

$= (1-z^2)q'(z)$

$= 1 + (3b_3-1)z^2 + (5b_5-3b_3)z^4 + \ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots -(4kn+4k+1)b_{4kn+4k+1}\ z^{4kn+4k+1}$.

Comparing coefficients in this equation, we obtain:

$$(2s+1)b_{2s+1} - 1 = \frac{2}{k}\left[ks - \tfrac{1}{2}s(s+1)\left(1-\cos\frac{\pi}{n+2}\right)\right] \quad \text{for}$$

$1 \leq s \leq k$, and also:

$$(4kn + 4k + 1)b_{4kn+4k+1} = 2\sin^2\left(\frac{\pi}{n+2}\right)/k(n+2).$$

Consequently, the polynomial:

$$q(z) = z + \sum_{s=1}^{k}\left[1 - \frac{s(s+1)}{(2s+1)k}\left(1 - \cos\frac{\pi}{n+2}\right)\right]z^{2s+1} + \ldots\ldots + \frac{2\sin^2\left(\frac{\pi}{n+2}\right)}{k(n+2)(4kn+4k+1)}\, z^{4kn+4k+1}$$

$\in$ CTC $\left(\frac{z}{1-z^2}\right)$, and so $\in P^{o}_{N}$ where $N = 4kn + 4k + 1$.

It is clear that all coefficients of $q(z)$ are positive, and that: $b_{2s+1} = 1 - \frac{A(k)}{N^2}(1 + o(1))$ for $1 \leq s \leq k$, and sufficiently large N. In particular,

$$\begin{aligned} b_3 &= 1 - \frac{2}{3k} \cdot 2\sin^2\left(\frac{\pi}{2(n+1)}\right) \\ &= 1 - \frac{\pi^2}{3k\, n^2}(1 + o(1)). \\ &= 1 - \frac{16k\pi^2}{3N^2}(1 + o(1)). \end{aligned}$$

Applying Lemma 2 to the polynomial $q(z)$, we find that for a fixed k, and arbitrarily large N of the form $N = 4kn + 4k + 1$, there is a polynomial in P_N of the form:

$$p_N(z) = z + 2\left[1 - \frac{4}{3k}\sin^2\left(\frac{\pi}{2(n+1)}\right)\right]z^2 + \sum_{r=3}^{N-1} a_r z^r + \frac{4\sin^4\left(\frac{\pi}{n+2}\right)}{k^2(n+2)^2(4kn+4k+1)^2}\, z^{4kn+4k+1}.$$

Furthermore, the coefficients a_s satisfy the condition:

$$a_s > s - \frac{A(k)}{N^2}$$

for $2 \leq s \leq k$, as required. Hence the theorem has been proved.

From Theorem 2.3.5 we may deduce immediately the following two theorems.

<u>Theorem 2.3.6.</u> <u>Let $q_{2N+1}(z) = z + \sum_{s=1}^{N} b_{2s+1} z^{2s+1}$ $\in P^o_{2N+1}$.</u> <u>Then:</u> $1 - \frac{16\pi^2}{3N^2}(1+o(1)) \leq \max_{P^o_{2N+1}} |b_3| \leq 1 - \frac{A}{N^2}$ <u>for arbitrarily large N, and some absolute constant A</u>.

<u>Proof</u>. The left inequality follows from the construction in Theorem 2.3.5, and the right inequality from Theorem 2.3.3.

<u>Theorem 2.3.7.</u> <u>Let $p_n(z) = z + \sum_{k=2}^{n} a_k z^k \in P_n$</u> .

(a) <u>Then, for arbitarily large n,</u>

$$2 - \frac{32\pi^2}{3n^2}[1+o(1)] \leq \max_{P_n} |a_2| \leq \begin{cases} 2\cos\left(\frac{\pi}{n+3}\right), & \text{if all } a_k \text{ real.} \\ 2 - \frac{4}{en^2}, & \text{otherwise.} \end{cases}$$

(b) <u>Also, for arbitrarily large n,</u>

$$k - \frac{A_1^{(k)}}{n^2} \leq \max_{P_n} |a_k| \leq k - \frac{A_2^{(k)}}{n^2}$$

<u>where $k = 3$ and 4, and $A_1^{(k)}$ and $A_2^{(k)}$ are</u>

<u>constants depending on k but not on n</u>.

(c) <u>For a fixed value of k, and arbitrarily large n,</u>

$$\max_{p_n} |a_k| \geq K - \frac{A(k)}{n^2} ,$$ <u>where $A(k)$ is a constant depending on k but not on n</u>.

<u>Proof</u>. The left inequalities in (a) and (b), and also (c), follow from the construction in Theorem 2.3.5. The right inequalities in (a) and (b) follow from Theorems 2.3.1 and 2.3.2, together with Theorem 1.5.9.

Notice that (a) and (b) settle the correct order of magnitude of $(k - \max_{p_n} |a_k|)$ for large n, $k = 2,3,4$. The estimate in (b) clearly also gives the correct order of magnitude of $k - \max_{p_n} |a_k|$ for any value of k for which the Bieberbach hypothesis is satisfied; for example, in the class of polynomials of degree n having real coefficients.

Chapter 3. Coefficient regions for univalent polynomials of small degree.

'L'amore è tanto piu fervente quanto la
cognitione è piu certa'
- Leonardo da Vinci.

1. Coefficient regions for P_2 and P_3.

The polynomial $p_3(z) = z+a_2z^2+a_3z^3 \in P_3$ iff:

$$1 + a_2 \frac{\sin 2\theta}{\sin\theta} x + a_3 \frac{\sin 3\theta}{\sin\theta} x^2 = 0$$

i.e. $1 + 2a_2c\ x + a_3(4c^2-1)x^2 = 0$

has no roots in $|x| < 1$ for $0 \le \theta \le \frac{\pi}{2}$, $0 \le c = \cos\theta \le 1$, by Theorem 1.2.2. Thus $|a_3| \le \frac{1}{3}$; and then, by Cohn's Rule, $p_3 \in P_3$ iff:

$$1 - |a_3|^2(4c^2-1)^2 + x(2a_2c - 2\bar{a}_2ca_3(4c^2-1)) = 0$$

has no roots in $|x| < 1$, and so iff:

(1.1) $1 - |a_3|^2(4c^2-1)^2 \ge 2c\ |a_2 - \bar{a}_2a_3(4c^2-1)|$

for $0 \le c \le 1$, with $|a_3| \le 1/3$.

Now suppose that a_3 is real; and let us use the substitutions $a_3 = t$, $a_2 = x + iy$, where x,y,t are real numbers. Then (1.1) is equivalent to the inequality:

$$(1.2)\quad 1-t^2(4c^2-1)^2 \geq 2c\left| x(1-t(4c^2-1))+iy(1+t(4c^2-1))\right|$$

which may be rewritten as:

$$(1.3)\quad [1-t^2(4c^2-1)^2]^2 \geq 4c^2[x^2(1-t(4c^2-1))^2+y^2(1+t(4c^2-1))^2]$$

for all $0 \leq c \leq 1$, for points belonging to the coefficient body $(\mathrm{Re}\,a_2, \mathrm{Im}\,a_2, a_3)$. Clearly it is sufficient to consider the coefficient body in the first octant, due to its symmetry properties (as seen in (1.2)).

Suppose that $t=0$. Then (1.2) reduces to:

$$1 \geq 2c\left|x+iy\right| = 2c\left|a_2\right|$$

for all $0 \leq c \leq 1$. Thus $2\left|a_2\right| \leq 1$, and we have:

<u>Theorem 3.1.1</u> <u>$p_2(z) = z + a_2z^2 \in P_2$ iff $\left|a_2\right| \leq \frac{1}{2}$</u>.

(This is, of course, not difficult to prove directly).

Suppose, next, that $y=0$. Then (1.2) implies that:

$1 - t^2(4c^2-1)^2 \geq 2c\,(1 - t(4c^2-1))x$, or:

$$(1.4)\qquad 2x \leq \frac{1}{c}\,(1 + t(4c^2-1))\ ,\ 0 \leq c \leq 1.$$

It is easily verified that the right hand side of (1.4) attains its minimum when $c^2 = (1-t)/4t$ if $\frac{1}{5} \leq t \leq \frac{1}{3}$, and when $c = 1$ if $0 \leq t \leq \frac{1}{5}$. Hence:

$$(1.5)\qquad a_2 \leq \begin{cases} \frac{1}{2}(1+3a_3) & \text{if } 0 \leq a_3 \leq \frac{1}{5}\,. \\ 2\sqrt{a_3(1-a_3)} & \text{if } \frac{1}{5} \leq a_3 \leq \frac{1}{3}\,. \end{cases}$$

Thus if $a_3 = 0, \frac{1}{5}, \frac{1}{3}$, then $a_2 \leq \frac{1}{2}, \frac{4}{5}, \sqrt{8/9}$ respectively.

Suppose, next, that $x = 0$. Then, from (1.3), we obtain: $1-t^2(4c^2-1)^2 \geq 2yc\,(1 + t(4c^2-1))$, or:

(1.6) $$2y \leq \frac{1}{c}\,(1 - t(4c^2-1)), \quad 0 \leq c \leq 1.$$

The right hand side attains its minimum when $c = 1$, and so:

(1.7) $$\mathrm{Im}\,a_2 \leq \tfrac{1}{2}(1-3t) \; .$$

We next consider sections of the coefficient body with a fixed value of t. Then, by (1.3), the point (x,y) satisfies the inequality:

(1.8) $$1 \geq (1+d)\left[\frac{x^2}{(1+td)^2} + \frac{y^2}{(1-td)^2}\right]$$

where $d = 4c^2 - 1$, $-1 \leq d \leq 3$. Hence the point (x,y) belongs to the closed interior of all ellipses with centre the origin, major axis $a(d) = \frac{1+td}{\sqrt{1+d}}$, and minor axis $b(d) = \frac{1-td}{\sqrt{1+d}}$ $(-1 \leq d \leq 3)$. Let us denote the interior and boundary of the ellipse $1 = (1+d)[(\frac{x}{1+td})^2 + (\frac{y}{1-td})^2]$ by E_d .

Now $b(d)$ is a strictly decreasing function of d for $0 \leq t \leq {}^1/3$. However $\frac{\partial a}{\partial d}$ is zero when $d = \frac{1-2t}{t}$, which lies in the range $-1 \leq d \leq 3$ only when $\frac{1}{5} \leq t \leq {}^1/3$. In addition, $a(d)$ decreases if $d < \frac{1-2t}{t}$, and $a(d)$ increases if $d > \frac{1-2t}{t}$.

Consequently, if $0 \leq t \leq \frac{1}{5}$, $a(d)$ and $b(d)$ are both decreasing, and so $\bigcap_{-1 \leq d \leq 3} E_d = E_3$. Thus points in the coefficient body satisfy the inequality:

$$(1.9) \qquad 1 \geq 4\left[\left(\frac{x}{1+3t}\right)^2 + \left(\frac{y}{1-3t}\right)^2\right]$$

However, if $\frac{1}{5} \leq t \leq \frac{1}{3}$, the minimum value of $a(d)$ is $a\left(\frac{1-2t}{t}\right) = 2\sqrt{t(1-t)}$. Then, as above, points (x,y) in the coefficient body belong to $\bigcap_d E_d$, over $\frac{1-2t}{t} \leq d \leq 3$. It is clear that, for $\frac{1}{5} < t < {}^1/3$, the cross- section is not a single ellipse.

Moreover, if $0 \leq t \leq \frac{1}{5}$, the circle:

$$(1.10) \qquad x^2 + y^2 = \left(\frac{1-3t}{2}\right)^2$$

lies in the coefficient body; and if $\frac{1}{5} \leq t \leq {}^1/3$, so does the ellipse:

$$(1.11) \qquad 1 = \frac{x^2}{4t(1-t)} + \frac{4y^2}{(1-3t)^2} .$$

Also, if $0 \leq t \leq \frac{1}{5}$, the maximum value of (x^2+y^2) occurs when $y=0$, $x = \frac{1}{2}(1+3t)$; and for $\frac{1}{5} \leq t \leq \frac{1}{3}$, the maximum value of (x^3+y^2) occurs when $y=0$, $x = 2\sqrt{t(1-t)}$. Accordingly, for all t we see that $|a_2| \leq \sqrt{8/9}$, with equality only for $t = {}^1/3$ and real a_2.

We have now proved:

Theorem 3.1.2. Suppose $p_3(z) = z + a_2z^2 + a_3z^3$ where a_3 is real and positive, and $a_2 = x + iy$. Then:

(a) For $0 \leq a_3 \leq \frac{1}{5}$, $p_3(z) \in P_3$ iff:

$$\frac{1}{4} \geq \left(\frac{x}{1+3a_3}\right)^2 + \left(\frac{y}{1-3a_3}\right)^2$$

If $\frac{1}{5} \leq a_3 \leq \frac{1}{3}$, $p_3(z) \in P_3$ iff the inequality (1.8) is satisfied for all d such that $(1-2a_3)/a_3 \leq d \leq 3$; in particular, $p_3(z) \in P_3$ if:

$$1 \geq \frac{x^2}{4a_3(1-a_3)} + \frac{4y^2}{(1-3a_3)^2} .$$

(b) When a_2 is real, $p_3(z) \in P_3$ iff:

$$|a_2| \leq \begin{cases} \frac{1}{2}(1+3a_3) & \text{for } 0 \leq a_3 \leq \frac{1}{5} . \\ 2\sqrt{a_3(1-a_3)} & \text{for } \frac{1}{5} \leq a_3 \leq \frac{1}{3} . \end{cases}$$

(c) If $p_3(z) \in P_3$ then $|a_2| \leq \sqrt{8/9}$ with equality only for $p_3(z) = z \pm \sqrt{\frac{8}{9}}\, z^2 + \frac{1}{3} z^3$.

Note. Many of the results in this section have been established recently [4], under a different normalisation of $p_3(z)$. This other method, however, fails to give the results of the sections to follow.

2. Coefficient Regions for a subclass of P_4.

Due to the complexity of the situation, we restrict ourselves to the polynomials in P_4 having $a_4 = \frac{1}{4}$, since this is the most interesting subclass of P_4.

Hence, by Theorem 2.1.1, we consider polynomials of the form:

$$p_4(z) = z + az^2 + \frac{2}{3}\,\bar{a}z^3 + \frac{z^4}{4}$$

This polynomial $\in P_4$ iff the equation:

$$1 + a\,\frac{\sin 2\theta}{\sin\theta}\,x + \frac{2}{3}\,\bar{a}\,\frac{\sin 3\theta}{\sin\theta}\,x^2 + \frac{\sin 4\theta}{4\sin\theta}\,x^3 = 0, \text{ or:}$$

$$1 + 2acx + \frac{2}{3}\,\bar{a}\,(4c^2-1)x^2 + c(2c^2-1)x^3 = 0 \tag{2.1}$$

has no roots in $|x| < 1$ for $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq c = \cos\theta \leq 1$, by Theorem 1.2.2. Then, by Cohn's Rule, a necessary and sufficient condition for $p_4(z) \in P_4$ is that $\phi_1(x,c)$ has no zeros in $|x| < 1$, where:

$$\begin{aligned}\phi_1(x,c) &= 1 - 2acx + \frac{2}{3}\,\bar{a}\,(4c^2-1)x^2 \\ &\qquad -c(2c^2-1)\left[c(2c^2-1) + \tfrac{2}{3}a(4c^2-1)x + 2\bar{a}cx^2\right] \\ &= (1-c^2)\begin{bmatrix}(1+4c^4) + \frac{4}{3}ac(1+4c^2)x \\ \qquad + \frac{2}{3}\,\bar{a}(6c^2-1)x^2\end{bmatrix},\end{aligned}$$

or, alternatively, that:

$$(2.2) \quad \phi_2(x,c) = (1+4c^4) + \tfrac{4}{3}ac(1+4c^2)x + \tfrac{2}{3}\bar{a}(6c^2-1)x^2$$

has no zeros in $|x| < 1$ for $0 \le c \le 1$. In that case:

$$\tfrac{2}{3}|a| \le \frac{1+4c^4}{|6c^2-1|} \qquad (0 \le c \le 1,\ c^2 \ne \tfrac{1}{6})$$
$$= |f(c^2)|, \text{ say .}$$

Now if $f(x) = \frac{4x^2+1}{6x-1}$, then $f'(x) = 0$ when $x = \frac{1}{6}(1+\sqrt{10})$ in the range $0 \le x \le 1$. Hence:

$$\tfrac{2}{3}|a| \le \mathrm{Min}(|f(o)|, |f(1)|, |f(\tfrac{1+\sqrt{10}}{6})|)$$
$$= \mathrm{Min}(1, 1, \tfrac{2}{9}(1+\sqrt{10}))$$
$$= \tfrac{2}{9}(1+\sqrt{10}) \text{ , and so:}$$

$$(2.3) \quad |a| \le \tfrac{1}{3}(1+\sqrt{10}).$$

Now let us return to (2.2), assuming that $|a| \le \frac{1}{3}(1+\sqrt{10})$, and suppose that $\phi_2(x,c)$ has no zeros in $|x| < 1$ for $0 \le c \le 1$, $c^2 \ne \frac{1+\sqrt{10}}{6}$. Applying Cohn's Rule again to $\phi_2(x,c)$, we find that:

$$\phi_3(x,c) = (1+4c^4)^2 - \tfrac{4}{9}|a|^2(6c^2-1)^2$$
$$+ \tfrac{4}{3}c(4c^2+1)x[a(4c^4+1) - \tfrac{2}{3}\bar{a}^2(6c^2-1)]$$

can have no zeros in $|x| < 1$ for $0 \le c \le 1$. Thus:

$$(2.4)\quad (4c^4+1)^2 - \tfrac{4}{9}|a|^2(6c^2-1)^2 \geq \tfrac{4}{3}\, c(4c^2+1)\left|a(4c^4+1) - \tfrac{2}{3}\,\bar{a}^2(6c^2-1)\right|$$

for $0 \leq c \leq 1$. Substituting $c^2 = \frac{1}{6}$ in (2.4), we obtain that:

$$(2.5)\qquad |a| \leq \sqrt{3/2} < \tfrac{1}{3}(1+\sqrt{10}).$$

Consequently, the polynomial $p_4(z) \in P_4$ iff:

$$(2.6)\quad \min_{0\leq c\leq 1} \frac{(4c^4+1)^2 - \frac{4}{9}|a|^2(6c^2-1)^2}{\frac{4}{3}\, c(4c^2+1)\left|a(4c^4+1)-\frac{2}{3}\bar{a}^2(6c^2-1)\right|} \geq 1.$$

and, in particular, if $\arg(a)$ is a multiple of $\frac{\pi}{3}$, (2.6) becomes:

$$(2.7)\quad \min_{0\leq c\leq 1} \left| \frac{1+4c^4+\frac{2}{3}\, a(6c^2-1)}{\frac{4}{3}\, a\, c\, (4c^2+1)} \right| \geq 1 .$$

We now consider the implications of (2.7) for real $\underline{a}$. Suppose, first, that $\underline{a}$ is positive. Then $p_4(z) \in P_4$ iff: $1 + 4c^4 + \frac{2}{3}\, a(6c^2-1) \geq \frac{4}{3}\, ac(4c^2+1)$, for $0 \leq c \leq 1$, i.e. $\frac{2}{3}\, a(4c+1)(2c^2-2c+1) \leq (1+2c^2)^2 - (2c)^2$, or:

$$(2.8)\quad \tfrac{2}{3}\, a \leq \frac{2c^2+2c+1}{4c+1} = f(c)\ ,\ 0 \leq c \leq 1\ ,$$

since $(4c+1)$ and $(2c^2-2c+1)$ are positive for $0 \leq c \leq 1$. Now we can easily show that $f'(c) = 0$ in our range of c only when $c = \frac{1}{4}(\sqrt{5}-1)$. Thus we require that:

$$\frac{2}{3}a \leq \mathrm{Min}(f(o),\ f(1),\ f(\tfrac{\sqrt{5}-1}{4}))$$
$$= \mathrm{Min}(\ 1,\ 1,\ \tfrac{1}{4}(\sqrt{5}+1))$$
$$= \tfrac{1}{4}(\sqrt{5}+1),\text{ or:}$$

$$(2.9) \qquad a \leq \frac{3}{8}(\sqrt{5}+1) \doteqdot 1.2145.$$

Suppose, next, that <u>a</u> is negative, and let $b = -a > 0$. Then $p_4(z) \in P_4$ iff:

$$1 + 4c^4 - \frac{2}{3}b(6c^2-1) \geq \frac{4}{3}bc(4c^2+1)$$

i.e. $\frac{2}{3}b(4c-1)(2c^2+2c+1) \leq (2c^2+2c+1)(2c^2-2c+1)$,

for $0 \leq c \leq 1$, and so iff:

$$(2.10) \quad \frac{2}{3}b \leq \frac{2c^2-2c+1}{4c-1} \quad \text{for } \frac{1}{4} < c \leq 1,$$

$$= g(c)\text{ , say,}$$

since if $0 \leq c \leq \frac{1}{4}$ the condition is always satisfied. Now we can easily show that $g'(c) = 0$ in $\frac{1}{4} < c \leq 1$ only when $c = \frac{1}{4}(\sqrt{5}+1)$. Thus we require that:

$$\frac{2}{3}b \leq \mathrm{Min}\ (f(\tfrac{\sqrt{5}+1}{4})\ ,\ f(1))$$
$$= \mathrm{Min}\ (\tfrac{\sqrt{5}-1}{4}\ ,\ \tfrac{1}{3}),\text{ or:}$$

$$(2.11) \qquad b \leq \frac{3}{8}(\sqrt{5}-1).$$

Combining (2.9) and (2.11), we have:

Theorem 3.2.1. Let $p_4(z) = z + a_2z^2 + a_3z^3 + \frac{z^4}{4}$ where a_2 and a_3 are real. Then $p_4(z) \in P_4$ iff: $a_3 = \frac{2}{3} a_2$, and: $-\frac{3}{8}(\sqrt{5}-1) \leq a_2 \leq \frac{3}{8}(\sqrt{5}+1)$.

Remark. It is clear that if $p_4(z) = z + az^2 + \frac{2}{3}\bar{a}z^3 + \frac{z^4}{4} \in P_4$ then so do $p_4(ze^{2\pi i/3})e^{-2\pi i/3}$ and $p_4(ze^{4\pi i/3}) \cdot e^{-4\pi i/3}$.

Now suppose that, for all a satisfying $|a| \leq y$, $p_4(z) \in P_4$. We show that $y = \frac{3}{8}(\sqrt{5}-1)$. By (2.6), we have, for all $0 \leq c \leq 1$, that:

$$(1+4c^4)^2 - \frac{4}{9}y^2(6c^2-1)^2 \geq \frac{4}{3}c(4c^2+1)[y(1+4c^4) + \frac{2}{3}y^2(6c^2-1)]$$

for $\frac{1}{6} \leq c^2 \leq 1$, since, for any such c, we can choose a, with $|a| = y$, such that:

$$\left|a(1+4c^4) - \frac{2}{3}\bar{a}^2(6c^2-1)\right| = y(1+4c^4) + \frac{2}{3}y^2(6c^2-1).$$

Thus:

$1+4c^4 - \frac{2}{3}y(6c^2-1) \geq \frac{4}{3}yc(4c^2+1)$, $\frac{1}{6} \leq c^2 \leq 1$. This is equivalent to the condition:

$$(2.12) \quad \frac{2}{3}y \leq \frac{2c^2-2c+1}{4c-1}, \quad \frac{1}{6} \leq c^2 \leq 1,$$

(as in our previous calculations); and so, as before, we have that: $y \leq \frac{3}{8}(\sqrt{5}-1)$ (since $\frac{1}{8}(3+\sqrt{5}) > \frac{1}{6}$). Clearly the value $|a| = \frac{3}{8}(\sqrt{5}-1)$ is maximal only for $\arg a = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

Similarly, if $0 \leq c^2 \leq \frac{1}{6}$, we have, from (2.6), that:

$$(1+4c^4)^2 - \frac{4}{9}y^2(6c^2-1)^2 \geq \frac{4}{3}c(4c^2+1)[y(1+4c^4) - \frac{2}{3}y^2(6c^2-1)]$$

i.e. $1+4c^4 + \frac{2}{3} y(6c^2-1) \geq \frac{4}{3} y\, c(4c^2+1)$

i.e. $$\frac{2}{3} y \leq \frac{2c^2+2c+1}{4c+1} \quad \text{for} \quad 0 \leq c^2 \leq \frac{1}{6} .$$

The minimum of $\frac{2c^2+2c+1}{4c+1}$ is attained in $(0, 1/\sqrt{6})$ when $c = \frac{1}{4}(\sqrt{5}-1)$, and is $\frac{1}{4}(\sqrt{5}+1)$. Thus: $y \leq \frac{3}{8}(\sqrt{5}+1)$.

Consequently, we have proved:

Theorem 3.2.2. If $|a| \leq \frac{3}{8}(\sqrt{5}-1)$, then $p_4(z) = z + az^2 + \frac{2}{3}\bar{a}z^3 + \frac{z^4}{4} \in P_4$.

The constant $\frac{3}{8}(\sqrt{5}-1)$ is exact, and gives the maximum possible value of $|a|$ iff $\arg a = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

We now find an estimate for the maximum value A of $|a|$ for which a polynomial $p_4(z)$ of the above form $\in P_4$. We require, by (2.6), that:

$$\frac{(1+4c^4)^2 - \frac{4}{9}|a|^2(6c^2-1)^2}{\frac{4}{3}|a|c(4c^2+1)} \geq \left|(1+4c^4) - \frac{2\bar{a}^2}{3a}(6c^2-1)\right| \tag{2.13}$$

for $0 \leq c \leq 1$. Let us denote the left and right hand sides of (2.13) by L and R, respectively, Then, when $c = \frac{1}{4}(\sqrt{5}-1)$, $L = \frac{1}{8}(15-3\sqrt{5}) + \frac{1}{6}(3\sqrt{5}-5)\bar{a}^2/a$. If $a = \frac{3}{8}(\sqrt{5}+1)$, we have strict inequality in (2.13) unless $c = \frac{1}{4}(\sqrt{5}-1)$, in which case equality occurs.

Suppose that $L > (1+\varepsilon) R$ in (2.13) for $\left(\left|c - \frac{1}{4}(\sqrt{5}-1)\right| \geq \frac{1}{100},\ 0 \leq c \leq 1 \right)$ for some fixed positive ε. We will vary a so slightly that always $L > (1 + \frac{\varepsilon}{2}) R$ in (2.13), for $0 \leq c \leq 1$, $\left|c - \frac{1}{4}(\sqrt{5}-1)\right| \geq \frac{1}{100}$.

Now let $\arg a$ vary slightly, so that $a \longrightarrow a'$, $|a| = |a'| = \frac{3}{8}(\sqrt{5}+1)$. Then, for $\left| c - \frac{1}{4}(\sqrt{5}-1) \right| < \frac{1}{100}$, R is decreased, and L is unaltered. Thus (2.13), with a' in place of a, holds with strict inequality. Then for fixed $\arg a'$, we may increase $|a'|$ to $|b'|$, $\arg b' = \arg a'$, so that (2.13) again holds, and we have equality in it for some c (with b' in place of a in (2.13)).

Thus there exists some $p_4(z) \in P_4$ having $|a| > \frac{3}{8}(\sqrt{5}+1)$, i.e. $A > \frac{3}{8}(\sqrt{5}+1)$. Clearly we can, in a similar way to the above, always increase $|a|$ in $p_4(z)$, unless equality occurs in (2.13) when one of the following conditions is satisfied:

(2.14)
(a) $\bar{a}^2/a$ is real and negative, $c^2 < \frac{1}{6}$.
(b) $\bar{a}^2/a$ is real and positive, $c^2 > \frac{1}{6}$.
(c) $c^2 = \frac{1}{6}$.
(d) Equality occurs in (2.13) for some $c^2 < \frac{1}{6}$, and for some $c^2 > \frac{1}{6}$, for the same value of a, $|a| = A$.

The maximum value of $|a|$ is attained in one of these cases, and gives equality in (2.13) for some value of c.

(a) If $\bar{a}^2/a$ is real and negative, $c^2 < \frac{1}{6}$, then, by (2.13), with $|a| = A$,

$$(1+4c^4)^2 - \frac{4}{9}A^2(6c^2-1)^2 \geq \frac{4}{3}Ac(4c^2+1)[4c^4+1+\tfrac{2}{3}A(6c^2-1)],$$

i.e. $1+4c^4 - \frac{2}{3} A(6c^2-1) \geq \frac{4}{3} Ac(4c^2+1)$,

i.e. $$\frac{2}{3} A \leq \frac{2c^2 - 2c + 1}{4c - 1} = g(c), \text{ say,}$$

for $\frac{1}{4} < c \leq 1/\sqrt{6}$; and equality occurs for some such value of c. We can easily show that, in this range of c, g(c) is a decreasing function of c, and so:

$$\frac{2}{3} A \leq \left[\frac{2c^2 - 2c + 1}{4c - 1} \right]_{c = 1/\sqrt{6}}$$

$$= \sqrt{\frac{2}{3}} \text{ , or } A \leq \sqrt{\frac{3}{2}} .$$

Thus we cannot have equality in (2.13) in the range of c given by (a), and so this case does not arise.

(b) If $\bar{a}^2/a$ is real and positive , $c^2 > \frac{1}{6}$, then, by (2.13), with $|a| = A$,

$(1+4c^4)^2 - \frac{4}{9} A^2(6c^2-1)^2 \geq \frac{4}{3} Ac(4c^2+1)[1+4c^4 - \frac{2}{3} A(6c^2-1)]$.

i.e. $1+4c^4 + \frac{2}{3} A(6c^2-1) \geq \frac{4}{3} Ac(4c^2+1)$

i.e. $$\frac{2}{3} A \leq \frac{2c^2 + 2c + 1}{4c + 1} = g(c) \text{ , say,}$$

for $\frac{1}{\sqrt{6}} < c \leq 1$, and equality occurs for some such value of c. We can easily show that g(c) is an increasing function of c in $[\frac{1}{\sqrt{6}}, 1]$, and so, as before, we require $A \leq \sqrt{\frac{3}{2}}$; also the case cannot occur since we cannot have equality in $]\frac{1}{\sqrt{6}}, 1]$.

(c) If $c^2 = \frac{1}{6}$ when equality occurs in (2.13), then:

$$1+4c^4 = \frac{4}{3} A\, c(4c^2+1),\ c^2 = \frac{1}{6} ,$$

i.e. $$\frac{10}{9} = \frac{4}{3} A \cdot \frac{5}{3} \cdot \frac{1}{\sqrt{6}} ,$$

i.e. $$A = \sqrt{\frac{3}{2}} .$$

We now show that $A < \sqrt{\frac{3}{2}}$, and so case (c) cannot occur. Then we will have shown that:

$$\frac{3}{8} (\sqrt{5}+1) < A < \sqrt{\frac{3}{2}} ,$$

or A lies between 1.213 and 1.224 approx. The extremal case in (2.14) is then (d).

<u>Lemma. The polynomial $p_4(z) = z + az^2 + \frac{2}{3} \bar{a}z^3 + \frac{1}{4} z^4$ cannot belong to P_4 if $|a| = \sqrt{\frac{3}{2}}$.</u>

<u>Proof</u>. Suppose $a = \sqrt{\frac{3}{2}}\, e^{-i\theta/3}$. Then, by (2.13), $p_4 \in P_4$ iff:

$$(1+4c^4)^2 - \frac{4}{9}|a|^2(6c^2-1)^2 \geq \frac{4}{3}|a|c(4c^2+1)\left|1+4c^4 - \frac{2}{3} \cdot \frac{\bar{a}^2}{a}(6c^2-1)\right|$$

where $0 \leq c \leq 1$. Putting $x = c^2$, $0 \leq x \leq 1$, and squaring both sides, we obtain:

$$(1+4x^2)^4 - \frac{4}{3} (1+4x^2)(6x-1)^2 + \frac{4}{9} (6x-1)^4$$
$$\geq \frac{8}{3} x(4x+1)^2(4x^2+1)^2 + \frac{16}{9} x(4x+1)^2(6x-1)^2$$
$$- \frac{16}{3} \sqrt{\frac{2}{3}}.x(4x+1)^2(4x^2+1)(6x-1)\cos\theta .$$

This may be rewritten as:

$$(2.15)\quad (1+4x^2)^2[(1+4x^2)^2 - \tfrac{8}{3}x(4x+1)^2] + \tfrac{16}{3}\sqrt{\tfrac{2}{3}}.x(4x+1)^2 (4x^2+1)(6x-1)\cos\theta$$
$$\geq (6x-1)^2 \left[\frac{16}{9} x(4x+1)^2 - \frac{4}{9} (6x-1)^2 + \frac{4}{3} (1+4x^2)\right]$$

Let us denote the left hand side of (2.15) by L_1 .

Both sides of (2.15) have a zero at $x = {}^1/6$. Since the right hand side has a double zero at $x = {}^1/6$, and is positive near $x = {}^1/6$, L_1 must also have a double zero at $x = {}^1/6$. It is easy to check that the necessary condition for this is that $\cos\theta = \frac{7}{9}\sqrt{\frac{3}{2}}$. Substituting this value into L_1 , we find that:

$L_1 = \frac{1}{27}(4x^2+1)(6x-1)^2(48x^4-112x^3+232x^2+140x+27)$.

Consequently, we can take a factor $(6x-1)^2$ out of (2.15); and so (2.15) holds iff:

$$\frac{1}{27}(4x^2+1)(48x^4-112x^3+232x^2+140x+27)$$
$$\geq \frac{16}{9}x\,(4x+1)^2 - \frac{4}{9}(6x-1)^2 + \frac{4}{3}(1+4x^2),$$

which may be rewritten as:

$$(4x^2+1)(48x^4-112x^3+232x^2+140x-9)$$
$$\geq 48x\,(4x+1)^2 - 12(6x-1)^2$$
$$= 12\,[64x^3 - 4x^2 + 16x - 1]$$
$$= 12\,(16x-1)(4x^2+1) \text{ , or as:}$$

$$(2.16) \quad 48x^4 - 112x^3 + 232x^2 - 52x + 3 \geq 0$$

for $0 \leq x \leq 1$.

But when $x = \frac{1}{9}$, the left hand side of (2.16) is easily shown to be negative. Thus (2.16) does not hold. Consequently $p_4(z) \notin P_4$, and the lemma is proved.

We have now established:

<u>Theorem 3.2.3.</u> <u>Let $p_4(z) = z + az^2 + \frac{2}{3}\bar{a}z^3 + \frac{1}{4}z^4 \in P_4$</u>, <u>and $A = \max_{p_4} |a|$. Then:</u>

$$\underline{\frac{3}{8}(\sqrt{5}+1) < A < \sqrt{\frac{3}{2}}}\,.$$

3. <u>Coefficient Regions for M_1 and M_2.</u>

It is obvious from Theorem 1.2.3 that $\mu_1(z) = \frac{1}{z} + a_1 z \in M_1$ iff $|a_1| \leq 1$.

Suppose that $\mu_2(z) = \frac{1}{z} + a_1 z + a_2 z^2 \in M_2$. By Theorem 2.1.1, $a_1 = 0$ if $|a_2| = \frac{1}{2}$. We now assume $0 \leq a_2 < \frac{1}{2}$, and put $c = \cos\theta$. Then $\mu_2 \in M_2$, by Theorem 1.2.3, iff:

$$a_2 \frac{\sin 2\theta}{\sin\theta} x^3 + a_1 x^2 - 1 = 0\,,\ \text{or:}$$

$$(3.1)\quad 2ca_2 x^3 + a_1 x^2 - 1 = 0$$

has no roots in $|x| < 1$, for $0 \leq c = \cos\theta \leq 1$. Then, by Cohn's Rule, the same holds for $\phi(x,\theta) = 0$, where:

$$\phi(x,\theta) \equiv 1 - a_1 x^2 - 2ca_2 x^3 + 2ca_2(-2ca_2 - \bar{a}_1 x + x^3)$$

$$(3.2)\quad = (1 - 4c^2 a_2^2) - 2ca_2\bar{a}_1 x - a_1 x^2\,.$$

Thus $|a_1| \leq 1 - 4c^2 a_2^2$, for $0 \leq c \leq 1$, and so:

$$(3.3)\qquad |a_1| \leq 1 - 4a_2^2\,.$$

Applying Cohn's Rule to $\phi(x,\theta)$, we find that the polynomial:

$$(1-4c^2a_2^2)[1-4c^2a_2^2-2ca_2\bar{a}_1x]+a_1[-\bar{a}_1-2ca_2a_1x]$$

has no zeros in $|x| < 1$, and so:

$$(1-4c^2a_2^2)^2 - |a_1|^2 \geq \left|- 2ca_2\bar{a}_1(1-4c^2a_2^2) - 2ca_1^2a_2\right|$$

$$= 2a_2c\,|a_1|\cdot\left|1+a_1({}^{a_1}/\bar{a}_1) - 4c^2a_2^2\right|, \tag{3.4}$$

for $0 \leq c \leq 1$. Hence:

$$2a_2|a_1|c \leq \frac{(1+|a_1|-4c^2a_2^2)(1-|a_1|-4c^2a_2^2)}{\left|1 + a_1({}^{a_1}/\bar{a}_1) - 4c^2a_2^2\right|} \quad (0\leq c\leq 1). \tag{3.5}$$

Suppose this holds for all $|a_1| = y = y(a_2)$. Then $2a_2yc \leq 1 - y - 4c^2a_2^2$, or $y \leq 1-2ca_2$ for $0 \leq c \leq 1$. Thus:

$$y \leq 1 - 2a_2 . \tag{3.6}$$

Suppose, next, that $A = \max\limits_{\mu_2} |a_1|$ for a fixed value of real a_2. We prove that $A = 1 - 4a_2^2$, by showing that $\mu_2(z) = \frac{1}{z} + z(1-4a_2^2)e^{i\pi/3} + a_2z^2 \in M_2$. This satisfies (3.3); and the result follows if it satisfies (3.5). This is so if:-

$$2\, a_2\, |a_1|\, c \leq 1 + |a_1| - 4c^2a_2^2 ,$$

i.e. $(2a_2c-1)|a_1| \leq 1 - 4c^2a_2^2$,

which is certainly true. Thus $\mu_2(z) \in M_2$, and $A = 1 - 4a_2^2$.

Combining the above results, we have:

Theorem 3.3.1. Let $\mu_2(z) = \frac{1}{z} + a_1 z + a_2 z^2$.

(a) If $\mu_2(z) \in M_2$, then $|a_1| \leq 1-4|a_2|^2$, and this estimate is exact.

(b) If $|a_2| \leq \frac{1}{2}$, and $|a_1| \leq 1-2|a_2|$, then $\mu_2(z) \in M_2$.

4. Coefficient Regions for M_3.

Again, owing to the complexity of the situation, we consider only polynomials in M_3 of the form: $\mu_3(z) = \frac{1}{z} + a_1 z + a_3 z^3$, where a_3 is real and positive. By Theorem 1.2.3, $\mu_3(z) \in M_3$ iff:

$$a_3 \frac{\sin 3\theta}{\sin \theta} x^4 + a_1 x^2 - 1 = 0, \text{ and so:}$$

(4.1) $a_3(4c^2-1)x^2 + a_1 x - 1 = 0$

have no roots in $|x| < 1$, where $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq c = \cos\theta \leq 1$. Clearly, from (4.1), we must have:

(4.2) $\quad |a_3| \leq \frac{1}{3}$.

Applying Cohn's Rule to (4.1), we see that:

$$a_3(4c^2-1)[a_3(4c^2-1)+\bar{a}_1 x] + [a_1 x - 1] = 0,$$

i.e. $a_3^2.(4c^2-1)^2-1 + x[a_1 + a_3\bar{a}_1(4c^2-1)] = 0$,

has no roots in $|x| < 1$, and so:

$$1 - a_3{}^2(4c^2-1)^2 \geq \left| a_1 + a_3\bar{a}_1(4c^2-1) \right|$$

$$(4.3) \qquad = |a_1| \cdot \left| 1 + a_3\left(\frac{\bar{a}_1}{a_1}\right)(4c^2-1) \right|,$$

for $0 \leq c \leq 1$. Suppose, first, that (4.3) holds for all $|a_1| = y = y(a_2)$. Then, from (4.3), we have that:

$$(4.4) \qquad y \leq 1 - a_3(4c^2-1) \quad \text{for } \tfrac{1}{2} \leq c \leq 1,$$

and so: $\quad y \leq 1 - 3a_3$.

We next find $\max\limits_{\mu_3} |a_1|$ for a fixed positive a_3.

By (4.3), this occurs when a_1 is imaginary, and so when:

$$|a_1| \leq 1 + a_3(4c^2-1), \; 0 \leq c \leq 1, \text{ or:}$$

$$(4.5) \qquad |a_1| \leq 1 - a_3 \; .$$

It is easily verified, using (4.3), that:

$\mu_3(z) = \frac{1}{z} + (1-a_3)iz + a_3z^3 \in M_3$, and so the estimate (4.5) cannot be improved.

We have now established:

<u>Theorem 3.4.1.</u> <u>Let $\mu_3(z) = \frac{1}{z} + a_1z + a_3z^3$.</u>

(a) <u>If $|a_3| \leq \frac{1}{3}$, and $|a_1| \leq 1 - 3|a_3|$, then $\mu_3(z) \in M_3$</u> .

(b) <u>If $\mu_3(z) \in M_3$, then $|a_1| \leq 1 - |a_3|$, and this estimate is exact.</u>

Chapter 4. A Conjecture of Ilieff.

'Every man has a right to utter what he thinks truth, and every other man has a right to knock him down for it'

- Samuel Johnson.

We will here prove the following special case of an (unpublished) conjecture of L. Ilieff for a polynomial of degree n:

Theorem 4.1. If all zeros of the cubic polynomial $p_3(z)$ lie in $|z| \leq 1$, then at least one zero of $p_3'(z)$ lies in or on the boundary of a circle of radius unity around each zero of $p_3(z)$.

The form of the conjecture for a polynomial of degree n is clear if we replace $p_3(z)$ by $p_n(z)$ in Theorem 4.1. We will give two proofs of the theorem, one depending on the Cohn Rule alone, and one which uses the theory of apolar polynomials.

1. First proof of the theorem.

This proof depends essentially on:

Lemma 1. If $f(S) = T + US + VS^2$ is non-zero in

<u>$|S| \leq 1$, then:</u>

(1) $$|T| > |V| \text{ , } \underline{\text{and}}$$

(2) $$|T|^2 - |V|^2 > |\bar{T}U - V\bar{U}|.$$

This is a very simple case of the well-known Cohn Rule (Theorem 1.3.1).

<u>Lemma 2</u>. <u>If Theorem 4.1 is false, there exists a cubic polynomial having all its zeros on $|z| = 1$ for which the theorem is false</u>.

<u>Proof</u>. Suppose there exists a cubic polynomial $p_3(z)$ for which the theorem is false, which has its zeros z_1, z_2, z_3 not all on $|z| = 1$.

(a) Then the smallest circle containing z_1, z_2, z_3 may have z_1, z_2, z_3 on its boundary; suppose it has centre p ($p \leq 1$) and radius R ($R < 1$). Then the cubic polynomial $q_3(z) = p_3(\frac{z-p}{R})$ has all its zeros on $|z| = 1$, and the distances between zeros of the polynomial and its derivative have been magnified by a factor $^1/R$. Thus the theorem is false for $q_3(z)$, which gives the required result.

(b) Alternatively, the smallest circle containing z_1, z_2, z_3 may have z_1 and z_2 (say) at opposite ends of a diameter, and z_3 inside the circle.

As above, we may assume $|z_1| = |z_2| = 1$, $|z_3| < 1$; and, in fact, $z_1 = 1$, $z_2 = -1$. Then $p_3(z) = (z^2-1)(z-\alpha)$ does not satisfy the result of the theorem for some α with $|\alpha| < 1$. Hence Lemma 2 will be proved once we have established:

Lemma 3. Suppose $p(S) = S(S-2)(S-1-\alpha)$ for $|\alpha| < 1$. Then the theorem holds for $p(S)$.

Proof. We have:

$$p'(S) = 3S^2 - 2(3+\alpha)S + 2(1+\alpha).$$

Suppose that for some α, $|\alpha| < 1$, $p'(S)$ has no zeros in $|S| < 1$. Then, applying Lemma 1 with $T = 2(1+\alpha)$, $U = -2(3+\alpha)$, $V = 3$, we deduce from (1) that $2|1+\alpha| > 3$. Hence certainly $\operatorname{Re}\alpha > 0$. Substituting in (2), we obtain:

$$4|1+\alpha|^2 - 9 > |2(1+\bar{\alpha})2(3+\alpha) - 3.2(3+\bar{\alpha})|$$

$$\geq 4|1+\alpha|.|3+\alpha| - 6|3+\alpha|,$$

and so $2|1+\alpha| + 3 > 2|3+\alpha|$, since $2|1+\alpha| - 3 > 0$. Using the previous bound for $|1+\alpha|$, $6 > 2|3+\alpha|$, which is impossible for $\operatorname{Re}\alpha > 0$.

Also $p'(S)$ has no zero in $|S-1-\alpha| < 1$ iff $q(t) = 3t^2 + 2t(1+2\alpha-\alpha^2) + (\alpha^2-1)$ has none in $|t| < 1$, which is not the case. Hence Lemma 3 is proved.

Proof of the theorem. As discussed above, we may, without loss of generality, consider the polynomial:

(3) $\quad p(S) = S[1 + Ae^{i\phi}(1+S) + e^{2i\phi}(1+S)^2]$

with all its zeros on $|1+S| = 1$, where $0 \leq A \leq 2$, $0 \leq \phi \leq 2\pi$. Thus:

$$p'(S) = (1+Ae^{i\phi}+e^{2i\phi}) + 2Se^{i\phi}(A+2e^{i\phi}) + 3e^{2i\phi}S^2$$

This has a zero in $|S| \leq 1$ iff:

(4) $\quad p'(e^{-i\phi}S) = (1+Ae^{i\phi}+e^{2i\phi}) + 2(A+2e^{i\phi})S + 3S^2$

also has a zero in $|S| \leq 1$.

Suppose the theorem is false. Then there exist some A,ϕ such that $p'(e^{-i\phi}S)$ has no zero in $|S| \leq 1$. Thus, applying Lemma 1 to the polynomial (4) with $T = 1 + Ae^{i\phi} + e^{2i\phi}$, $U = 2(A+2e^{i\phi})$, $V=3$, we deduce from (1) that:

(5) $\quad A + 2\cos\phi > 3.$

Clearly this implies $1 < A \leq 2$, $\frac{1}{2} < \cos\phi \leq 1$. Now let us use the following notation:

(6) $\quad c = \cos\phi$, $d = 2c + A$, where:

(7) $\quad 3 < d \leq 4$, $\frac{1}{2} < c \leq 1$, $1 < d - 2c \leq 2$.

Substituting in (2) and simplifying both sides, we obtain:

$\frac{1}{2}(d^2-9) > \left|d^2 - 3dc + (6c^2-6) + i\sin\phi\,(6c-d)\right|.$

Squaring, expanding out the terms on the right, and rearranging, we have:

(8) $\quad f(c,d) < 0$, where:

$$f(c,d) = (\tfrac{3}{4}d^4 - \tfrac{13}{2}d^2 + \tfrac{63}{4}) + 6cd(4-d^2) + 4c^2(5d^2-9) - 24c^3 d.$$

Now we define $g(c,d) = \begin{cases} \dfrac{f(1,d) - f(c,d)}{1-c} & \text{if } c \neq 1, \\ \dfrac{\partial}{\partial c} f(1,d) & \text{if } c = 1. \end{cases}$

Then, for those (c,d) satisfying (7), we have:

(9) $g(c,d) = 6d(4-d^2) + 4(1+c)(5d^2-9) - 24d(1+c+c^2)$.

It is easily verified that $\frac{\partial}{\partial c} g(c,d)$ is zero only when $c = (5d^2-6d-9)/12d$, and that this point does not satisfy the last condition in (7). Let V be the region of variability of the point (c,d) subject to (7). In V, $\frac{\partial g}{\partial c}$ is non-zero, and so has the same sign as $\frac{\partial g}{\partial c}(1,4) = -4$. Thus $g(c,d)$ is a strictly decreasing function of c for fixed $\underline{d}$, and so is always strictly less than its value when $1 = d - 2c$ (from (7)); hence

in V: $$g(c,d) < \max_{\frac{1}{2} < c \leq 1} g(c, 1+2c)$$

$$= \max_{\frac{1}{2} < c \leq 1} [-4(1+c)(1-2c)^2] = 0 .$$

Consequently $g(c,d) < 0$ in V, and $f(1,d) < f(c,d)$ for $c \neq 1$; so if (8) is satisfied anywhere in V, it is satisfied when $c=1$. However:

$f(1,d) = \frac{3}{4}d^4 - 6d^3 + \frac{27}{2}d^2 - \frac{81}{4}$ is easily shown to be a strictly increasing function of d, and so $f(1,d) > f(1,3) = 0$ for $\underline{d}$ satisfying (7). Therefore

$f(c,d) > 0$ on V, which is a contradiction. Hence Theorem 4.1 is proved.

Note 1. An application of Cohn's Rule similar to Lemma 1 shows that if $p(S)$ is given by (3), then $p'(S)$ is non-zero in $|S| < 1$ only for that polynomial $p(S)$ corresponding to the point $(c=1, d=3)$, namely $p(S) = (1+S)^3 - 1$.

Note 2. The second proof, using results from the theory of apolar polynomials, is much shorter. The first proof, on the other hand, depends only on Lemma 1 (which is a simple deduction from Rouche's theorem), and so is of independent interest.

2. Second proof of the theorem [9].

This proof depends on Theorem 1.7.2, which we restate as:

Lemma 4. Let $z_1, z_2, \ldots, z_n$ be a system of numbers which satisfy the convolution equation:

$$A(z_1, \ldots, z_n) \equiv a_o\, s_o + a_1\, s_1 + \ldots + a_n\, s_n = 0$$

associated with the polynomial:

$$A(z) \equiv a_o + \binom{n}{1} a_1 z + \ldots + a_n z^n ,$$

where $s_o = 1$, $s_1 = z_1 + z_2 + \ldots + z_n$,

$s_2 = z_1z_2 + z_1z_3 + \dots + z_{n-1}z_n, \dots, s_n = z_1z_2 \dots z_n$.

Then $A(z)$ has at least one zero in each circular domain K which contains all the points $z_1, z_2, \dots, z_n$.

We now proceed to the second:

Proof of Theorem 4.1. Let $p(z) = (z-w_1)(z-w_2)(z-w_3)$ be a polynomial with all its zeros in $|z| \leq 1$. The result will follow if we can show that the equation:

$$\begin{aligned} A(z) &\equiv p'(z+w_1) \\ &\equiv 3z^2 + 2z(2w_1-w_2-w_3) + (w_1-w_2)(w_1-w_3) \\ &= 0 \end{aligned}$$

has at least one root in $|z| \leq 1$. The convolution equation associated with $A(z)$ is:

$$\begin{aligned} A(z_1, z_2) &\equiv 3z_1z_2 + (z_1+z_2)(2w_1-w_2-w_3) + (w_1-w_2)(w_1-w_3) \\ &= 0 . \end{aligned}$$

If $1, \gamma, \gamma^2$ are the roots of the equation $z^3-1=0$, then the numbers z_1, z_2 defined by:

$$z_1 = -\frac{1}{3}(w_1 + \gamma w_2 + \gamma^2 w_3),$$

$$z_2 = -\frac{1}{3}(w_1 + \gamma^2 w_2 + \gamma w_3)$$

satisfy the equation $A(z_1, z_2) = 0$, and $|z_1| \leq 1$, $|z_2| \leq 1$. Hence $A(z)$ has at least one zero inside the unit circle.

Hence Theorem 4.1 is again proved.

Note. The Ilieff conjecture is trivially true when $n=2$. However nothing is known about its validity for $n \geq 4$.

Chapter 5. Composition of Coefficients.

'If the Romans had been obliged to learn Latin, they would never have found time to conquer the world'

– Heine.

Using the Grace Apolarity Theorem (1.7.1) and the Dieudonné Criterion, we may easily obtain:

Theorem 5.1.[5] Let $f(z) = z + \sum_{k=2}^{n} a_k z^k \in P_n$, and $g(z) = 1 + \sum_{k=2}^{n} \binom{n-1}{k-1} b_k z^{k-1}$ be non-zero in $|z| < 1$. Then $h(z) = z + \sum_{k=2}^{n} a_k b_k z^k \in P_n$.

The only other such theorem known is:

Theorem 5.2.[18] Let $f(z) = \sum_{k=1}^{\infty} a_k z^k \in S$. Then $f(z)$ is starlike univalent in $|z| < 1$ iff

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k$$

is starlike univalent in $|z| < 1$ for all n. $V_n(z)$ is the de la Vallée Poussin mean of $f(z)$.

We will establish a number of composition theorems where the coefficients are generally larger than those produced by the above two theorems.

One of our main results is:

<u>Theorem 5.3.</u> <u>Let $p_n(z) = z + \sum_{k=2}^{n} a_k z^k \in P_n$</u> ,
<u>and all the zeros of $g(z) = 1 + \sum_{k=1}^{m} b_k z^k$ lie in</u>
<u>$\text{Re } z \geq \frac{1}{2}(n-1)$. Then $z + \sum_{k=2}^{n} a_k\, g(k-1) z^k \in P_n$</u> .

<u>Proof</u>. By the Dieudonné Criterion, since $p_n(z) \in P_n$, all zeros of $1 + \sum_{k=1}^{n-1} a_{k+1} \frac{\sin(k+1)\theta}{\sin\theta} x^k$ lie in $|x| \geq 1$ for any $\theta \in [0, \frac{\pi}{2}]$. Then, by Theorem 1.7.5, all zeros of $1 + \sum_{k=1}^{n-1} a_{k+1} \frac{\sin(k+1)\theta}{\sin\theta} g(k) x^k$ lie in $|x| \geq 1$, and so $z + \sum_{k=2}^{n} a_k g(k-1) z^k \in P_n$.

<u>Note</u>. By applying Theorem 1.7.3, we may show that if $p_n'(z)$ has a zero on $|z| = 1$, so has the derivative of $z + \sum_{k=2}^{n} a_k g(k-1) z^k$.

As a direct application of Theorem 5.3, we deduce:

<u>Theorem 5.4.</u> <u>Let $p_n(z) = \sum_{k=1}^{n} a_k z^k \in P_n$ $(a_1 = 1)$</u>.
<u>Then the following polynomials also belong to P_n</u> :

$$\sum_{k=1}^{n} a_k\left(1 - \frac{k-1}{v}\right) z^k , \quad \sum_{k=1}^{n} \frac{n+1-2k}{n-1} a_k z^k , \quad \sum_{k=1}^{n} \frac{n-k}{n-1} a_k z^k$$

<u>where $\text{Re } v > \frac{1}{2}(n-1)$</u>.

We now establish the well-known theorem of

Kakeya:

<u>Theorem 5.5.[11,15]</u> <u>Let $p_n(z) = z + \sum_{k=2}^{n} a_k z^k$.</u> <u>If $p_n'(z)$ has no zeros in $|z| < r$, then $p_n(z)$ is univalent in $|z| < r \sin(\frac{\pi}{n})$.</u>

<u>Proof</u>. Suppose:

$$f(z) = p_n'(z) = \sum_{k=0}^{n-1} (k+1)a_{k+1} z^k ,$$

which has no zeros in $|z| < r$, and:

$$g(z) = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\sin(k+1)\theta}{(k+1)\sin\theta} z^k \quad (0 \le \theta \le \tfrac{\pi}{2}) .$$

Then $g(z)$ has no zeros in $|z| < R$, where R is the radius of univalency of:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{z^{k+1}}{k+1} = \frac{1}{n}\left((1+z)^n - 1\right) .$$

A direct application of Definition 1.1.1 yields $R = \sin(\frac{\pi}{n})$. Hence, by Theorem 1.7.4, there are no zeros in $|z| < r\sin(\frac{\pi}{n})$ of the function:

$$\sum_{k=0}^{n-1} a_{k+1} \frac{\sin(k+1)\theta}{\sin\theta} z^k ,$$

and the result follows by the Dieudonné Criterion.

<u>Note</u>. It is easy to show that $r\sin(\frac{\pi}{n})$ is the radius of univalency only for $p_n(z)$ of the form:

$$p_n(z) = \frac{1}{\varepsilon r n}\left((1+\varepsilon rz)^n - 1\right) \qquad (|\varepsilon| = 1) .$$

Similarly we may prove:

Theorem 5.6.[15] Let $p_n(z) = z + \sum_{k=2}^{n} a_k z^k$. If $z^{-1}\, p_n(z)$ has no zeros in $|z| < r$, $p_n(z)$ is univalent in $|z| < \frac{r}{n}$.

Finally, we note that by a simple application of Theorem 1.7.3, we may also obtain:

Theorem 5.7. Let $\sum_{k=1}^{n} a_k z^k \in P_n$ $(a_1 = 1)$. Then $\sum_{\substack{k=1\\k\neq N}}^{n} a_k z^k$ is univalent in $|z| < \frac{1}{d}$, where $d(> 1)$ is the modulus of the smallest root of the equation $(x - 1)^{n-1} = \binom{n-1}{N-1} x^{N-1}$.

Chapter 6. The Theorem of Bernstein.

'Much may be made of a Scotchman, if he be caught young'

– Samuel Johnson.

We now return to discuss Theorem 1.3.3 for special types of polynomials, especially in view of:

Theorem 6.1.[13] *Let $p(z)$ be a polynomial of degree N with all its zeros on $|z| = 1$. Then:*

$$M(1, p') = \tfrac{1}{2} N\, M(1, p).$$

From this we may easily deduce:

Corollary. *Let $p(z) = \sum_{k=0}^{N} \varepsilon_k z^k$, $\varepsilon_0 = \varepsilon_N = 1$, $|\varepsilon_k| = 1$, where $\varepsilon_k = \bar{\varepsilon}_{N-k}$. Then:*

$$M(1, p') = \tfrac{1}{2} N . M(1, p).$$

[Note. For conciseness, let us denote $M(1, p')$ by M', and $M(1, p)$ by M.]

Consequently it is of interest to find upper and lower bounds for M'/NM for polynomials with coefficients of modulus unity, or simply ± 1.

Theorem 6.2.* *Given any $\varepsilon > 0$, there is an integer N and polynomials of degree N with coefficients ± 1*

such that (a) $M'/NM < \frac{1}{2\sqrt{2}} + \varepsilon$,

(b) $M'/NM > \frac{3}{4} - \varepsilon$.

Proof. Consider the polynomial:

$$p(z) = \frac{1-z^n}{1-z} + z^n \frac{1-z^{\alpha n}}{1+z}$$

of degree $N = n(1+\alpha) - 1$, where αn is an even integer. The terms in $p(z)$ attain their maximum moduli n and αn at $z=1$, $z=-1$, and are $O(1)$ when $\mathrm{Re}\, z < 0$, $\mathrm{Re}\, z > 0$ respectively ($|z| = 1$). Hence:

$$(6.1)\quad M(1, p) = \left[n+O(1)\right] \max(1,\alpha).$$

Further:

$$p'(z) = \frac{d}{dz}\left[\frac{1-z^n}{1-z}\right] + nz^{n-1}\left[\frac{1-z^{\alpha n}}{1+z}\right] + z^n \frac{d}{dz}\left[\frac{1-z^{\alpha n}}{1+z}\right].$$

The terms in $p'(z)$ attain their maximum moduli $\frac{1}{2}n^2\left[1+o(1)\right]$, αn^2, $\frac{1}{2}\alpha^2 n^2\left[1+o(1)\right]$ at $z=1$, $z=-1$, $z=-1$, and are $O(n)$ in $\mathrm{Re}\, z < 0$, $\mathrm{Re}\, z > 0$, $\mathrm{Re}\, z > 0$ respectively ($|z| = 1$). Hence:

$$(6.2)\quad M(1, p') = \tfrac{1}{2}n^2\left[1+o(1)\right] \max(1, \alpha^2 + 2\alpha).$$

Choosing α such that $\alpha n = 2[\frac{1}{2}(\sqrt{2}-1)n]$, we obtain:

$$\frac{M'}{NM} = \frac{1}{2\sqrt{2}} + o(1)$$

from (6.1) and (6.2); and, choosing $\alpha=1$ and n even, we obtain:

$$\frac{M'}{NM} = \frac{3}{4} + o(1) .$$

We may contrast Theorem 6.2 and the unique extremal polynomial in Theorem 1.3.3 with:

Theorem 6.3. Given any $\varepsilon > 0$, there is an integer N and polynomials of degree N with coefficients of modulus unity such that (a) $M'/NM < \varepsilon$,

(b) $M'/NM > 1-\varepsilon$.

Lemma. Let $p(z)$ be a polynomial of degree N , and $q(z) = z^N p(1/z)$. Then $M(1,p) = M(1,q) = M$, and:

$$M(1,p') + M(1,q') \geq NM .$$

Proof of Theorem 6.3. (a) Consider $p(z) = \sum_{r=0}^{m-1} p_r(z)$, where:

$$(6.3)\quad \begin{cases} p_r(z) = z^{\alpha(r,n)} \cdot \dfrac{1-z^{\beta(r,n)}}{1-w^r z} , \quad w = \exp(2\pi i/m), \\[2ex] \alpha(r,n) = m[n\sqrt{r/m}], \quad \beta(r,n) = \alpha(r+1,n) - \alpha(r,n) . \end{cases}$$

Then $p(z)$ is a polynomial of degree $N = m[n/\sqrt{m}] - 1$ with coefficients of modulus unity. We choose:

$$(6.4)\qquad n = 2^{5x} , \quad m = 2^x ,$$

where x is a large positive integer. Consequently $N = 2^{6x} - 1$, $\alpha(r,n) = 2^{5x}\sqrt{r}\,[1+o(1)]$, $\beta(r,n) = 2^{5x-1}\cdot[1+o(1)]/\sqrt{r}$ for sufficiently large x.

Firstly, we notice that:

(6.5) $M \geq |p(1)| = |p_o(1)| = 2^{5x}$.

We now consider $p'(z)$, and so:

$$p_r'(z) = \frac{\alpha(r,n)z^{\alpha(r,n)-1}(1-z^{\beta(r,n)}) - \beta(r,n)z^{\alpha(r,n)+\beta(r,n)-1}}{1 - w^r z}$$

$$+ w^r z^{\alpha(r,n)} \frac{1 - z^{\beta(r,n)}}{(1 - w^r z)^2} .$$

Then: $$M(1,p_r') = |p_r'(w^{-r})|$$

$$= \alpha(r,n)\beta(r,n) + \tfrac{1}{2}\beta(r,n)[\beta(r,n) - 1]$$

(6.6) $$= 2^{10x-1}[1 + o(1)]$$

for large x, and:

$$p_r'(z) = \frac{O[\alpha(r,n) + \beta(r,n)]}{1 - w^r z} + \frac{O(1)}{(1-w^r z)^2}$$

$$= \frac{O(2^{5x}\sqrt{r})}{1-w^r z} + \frac{O(1)}{(1-w^r z)^2}$$

for $\arg(w^r z) \geq 2\pi/m$. Consequently:

$$\underset{0<2\pi s/m\leq\theta\leq 2\pi(s+1)/m\leq\pi}{M(1,\ p_r'(w^{-r}e^{i\theta}))} = \frac{O(2^{5x}\sqrt{r})}{s/m} + \frac{O(1)}{(s/m)^2}$$

(6.7) $$= 2^{7x}\, O(\sqrt{r})/s .$$

From (6.6) and (6.7) we deduce that:

$$M' \le 2M(1,\ p'_{m-1}) + 2\sum_{r=1}^{m-1} 2^{7x}\ O(\sqrt{m-1-r})/r$$

$$= [2^{10x} + 2^{7x}\sum_{r=1}^{m-1} O(\log m \ .\ \sqrt{m})][1+o(1)]$$

$$= 2^{10x}\ [1 + o(1)]\ ,\ \text{and so:}$$

$$M'/NM \le 2^{-x}\ [1 + o(1)] = N^{-1/6}\ [1 + o(1)]\ .$$

This proves the first part of the theorem.

(b) By the lemma, the polynomial $z^N p(^1/z)$ satisfies the second part of the theorem.

<u>Note</u>. The polynomial $\sum_{r=0}^{k-1} z^{rn} \dfrac{1 - z^n}{1 - w^{(k-1)r}z}$ also satisfies the condition $M'/NM > 1 - o(1)$, where $w = \exp\ (2\pi i/k)$ and k divides n .

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'Odi profanum vulgus et arceo'.

— Horace.

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