"STABILITY OF NETWORKS OF NONLINEAR ELEMENTS WITH LOGICAL PROPERTIES"

by

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ABSTRACT

A time-continuous model of finite linear threshold gates is postulated and compared to some models of neural function. Such a model realizes a normalized threshold function under specified conditions.

Networks of such gates are shown, in a limiting case, to be analogous to nonlinear electrical networks. A topological analysis of two-gate networks is given. The stability of dynamical systems of threshold gates by characterization of singular points allows conclusions regarding realization of finite linear threshold nets, and the enumeration of all singular points of threshold systems is possible under a weak restriction on weight magnitudes.

The reduction of large nets to smaller subnets for purposes of analysis is possible by inspection of the weight matrix, and the rank of the matrix is shown to influence net dynamics.

Stability in the large by the second method of Lyapunov is investigated, using quadratic potential functions and the generalized Popov criterion. This analysis enables some conditions for the existence of limit-cycles to be given.

ACKNOWLEDGEMENTS

The author would like to express his gratitude to his supervisor, Mr. J. J. Sparkes, for his advice to "Write it down." He would like to thank Professor Jack Cowan of the University of Chicago and Mr. Peter Johannesma of the University of Nymegen for much encouragement and mathematical criticism. Without the discussions with his fellow inhabitants of the Electronics Laboratory at Imperial College the occupation of Research Student certainly would have been less amusing. Much appreciation goes also to the citizens of London, that ancient and incomparable city, for many unforgettable sights and sounds.

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LIST OF PRINCIPAL SYMBOLS

The general rule used is that matrices are represented by capital letters and scalars by lower-case letters. An attempt has been made to keep the meaning of symbols constant throughout the thesis. Where this has not been possible the meaning is obvious from context, and this is also hoped to be true for symbols not in the list below.

Symbol	Definition
α_{ij}	an element of the weight matrix
α_{io}	a constant input to gate i
β_i	the nonlinearity input constant
γ_i	an element of Γ
Γ	principal singular point
δ	a positive constant
∇	the vector differential operator
ϵ_i	constant term of gate excitation
η	a constant
λ	general polynomial variable
μ	a constant
μ_i	principal point of y_i
ν	a parameter
ζ_i	element of $\{a_i, b_i\}$
τ_i	a time constant
ϕ	the transformed nonlinearity
ψ	the general gate nonlinearity

Ω	a region in state space
ω	frequency variable
a_i	a binary constant of realization
A	the weight matrix
A_o	vector of elements α_o
b_i	a binary constant of realization
B	diagonal matrix of elements β_i
E	vector of elements ϵ_i
$\mathcal{E}_i$	excitation of gate i
F	a vector function
$F(j\omega)$	a complex matrix
G	a linear impulse response
$G(s)$	a linear transfer function
H	the matrix of the linearized system
I	the unit vector
$\mathcal{J}$	the set of network elements
$\mathcal{L}$	the Laplace transform
$\mathcal{M}$	vector of elements μ_i
n	the number of gates in a network
p_{iq}	element of external input weight matrix
P	matrix of a quadratic form
$\mathcal{R}(j\omega)$	a complex matrix
s	the Laplace operator
T	diagonal matrix of elements τ_i
u_q	an external input

U	the Heaviside step function
U_k	k^{th} input vector U
$U(t)$	general input function
v_i	an element of V
V	a state vector
W	a potential function
x_i	output variable of gate i
X	state variable of outputs
X^o	a singular point in X-space
y_i	nonlinear function input
Y	vector of elements y_i
z_i	an element of Z
$Z(t)$	a transformed state vector

LOCATION OF FIGURES

CHAPTER 1

DYNAMICAL MODELS OF LINEAR THRESHOLD FUNCTION

1.1 INTRODUCTION

The analysis to be presented in this thesis is an attempt to describe the behavior of certain mathematical models of physical systems. These systems are of two distinct types, although it will be shown that they exhibit many similarities. They are:

1. A dynamical model by which there is reason to believe certain functional activity in the central nervous system of man and lower creatures may be represented.

2. Networks of logical gates of the type usually found in computing machines.

These two systems may be regarded for certain purposes as control systems, and such a formulation will be used here. It is well known that the brain is a very complicated control system (if not much more) and it is clear that the switching networks to be considered may also be described in terms of control systems. However, the essential theme of this thesis is as follows: A new description of the class of switching circuits characterized by threshold functions has been developed precisely, by which the behavior of such circuits at all instants of time may be predicted. In other words, a time-continuous model which is characterized by differential equations is used. This model possesses the two necessary attributes of physical realizability: dissipation and nonlinearity. Little attempt will be made to quantify the faithfulness of the modeling with respect to any particular circuits because the results apply to quite general hardware realizations with only a few restrictions.

The Cowan model which is closely related to the gate equation will also be considered in some detail. No attempt will be made to justify it in terms of neurophysiology, since this is the responsibility of its originator, and also since this equation sheds a great deal of light on the dynamics of switching circuits, justification enough in the present context. However part of Chapter 1 is devoted to an introduction to the neurological implications of the equation.

It would appear that nonlinear differential equations have not been used to any degree as descriptions of logical circuits. There are, of course, equations for the output of, say, transistors, but the author has been unable to find a unified approach to the transient behavior of networks of switching circuits in the general literature. The obvious explanation is that the nonlinearities present formidable obstacles to analysis. Modern developments, particularly in the field of automatic control, do allow certain conclusions about the stability of equations even though they cannot be solved in terms of elementary functions. A certain amount of work has been done on control systems containing relays, that is, instantaneous switches, but this has not been applied to logical gates. Rather than extend this work, equations will be developed for the finite transients existing in most switching circuits, and the instantaneous results will then be limiting conditions.

At the time of writing a great deal of research is being done on various topics related to the anlysis presented here. The first and second methods of Lyapunov are of course well known, but the amount of research being done on the generation of Lyapunov functions even for linear systems is considerable. Recently extensions of the Popov stability criterion have appeared in the literature and it is expected that future developments will extend the analysis of systems similar to those considered here.

1.1.1 HISTORICAL NOTE

The work which is presented here was done at Imperial College from late 1964 to 1967. It was at first the author's intention to analyse some of the logical or statistical properties of threshold devices, but when the author analysed certain aspects of the behavior of the Cowan equation (to be discussed later) it quickly became clear that with simple modifications this equation would model switching circuits, and furthermore, that it is simple enough to yield to at least some analytical techniques.

Except where reference is made to published material, the results reported in this thesis were obtained independently by the author, and at the time of writing are believed to be original. Certain results regarding the existence of constants of motion were arrived at independently at about the same time by J. D. Cowan in his analysis of the statistical mechanics of nervous activity. The principle of the use of Lyapunov functions for analysis of stability is not new. However, it does not appear to have received much attention in the literature of switching circuits. A list of the original contributions to be presented is given below:

1. Detail analysis of the Cowan equation for the deterministic dynamic behavior of neural models.

2. Proof that under certain conditions the gate equation is equivalent to a physical realization of a threshold function. This leads to the important result that this type of equation gives a reasonable representation of the transient behavior of this class of threshold gates.

3. A topological (trajectory) analysis of low-order networks of threshold gates.

4. An investigation of the conditions for the dynamical stability and instability of this class of switching network. This allows an investigation of networks which exhibit stable oscillations.

5. A method for reducing the stability analysis of systems of high complexity to the analysis of systems with lower complexity.

6. An electrical network analogue of small switching networks and an analogue of an energy measure for a restricted class of switching systems.

7. Evaluation of the first and second methods of Lyapunov, as well as the Popov criterion, with respect to generalized networks of threshold gates.

8. Analogue and digital computer simulation of low order networks of threshold gates.

1.2 ORGANIZATION OF THE THESIS

Chapter 1 is a brief outline of a few of the significant results in neural modelling, and is intended to show that the stability of threshold networks is relevant to previous work in this field. The dynamical model which will be used throughout the rest of this work is also presented.

Chapter 2 contains an exposition of some of the major results in the static design of threshold functions, in an attempt to make the point that the results to be given later on may be generalized without great difficulty to different models of linear threshold gates. A theorem is also presented to show that the dynamical model of chapter 1 is a "reasonable" model of a real threshold gate. Finally the model is related to time-discrete representations. This chapter is also an introduction to the discussion of chapter 6, which relates the results of the theory to some practical problems.

Chapter 3 contains the preliminary analysis of the dynamical model. An analogy to an electrical network is given, and an example of a limiting case of the model. A large part of the chapter is devoted to a topological analysis of two-gate networks. Such an analysis is an introduction to, and contains examples of most of the results of larger networks.

Chapter 4 deals with stability of arbitrary networks near their singular points. The simplicity of the model allows this type of analysis to yield several conclusions about the stability of arbitrary systems. An important conclusion is also drawn about the concept of the rank of a system.

In chapter 5 the second method of Lyapunov is used to show results of "stability in the large." This type of analysis is particularly useful in finding "next stable states" of a system and in calculating the immunity of a network to "noise."

Chapter 6 contains a discussion of the relevance of the previous work to questions which may be reasonably asked of a mathematical model such as the one we use. This chapter perhaps could be read before chapter 3, as an introduction to the mathematical detail of chapters 3, 4, and 5.

Chapter 7 is a discussion of some of the conclusions which can be drawn from the previous 6 chapters, and a list of some suggestions for future work.

To make the reading of the thesis easier, each chapter contains an introduction and, at the end, a brief summary discussion. Also to this end, certain of the statements in the mathematical treatment have been given the name "assertion." An assertion is, in this context, a statement of a result which can be proven easily or is evident from the discussion. A theorem is a fundamental or important assertion. One or two lemmas and corollaries are also included, and have their usual relations to the theorems.

1.2.1 ORGANIZATION OF CHAPTER 1.

A short description of the purposes and techniques of neural modeling will be given, then an outline of two historically-important models, and finally the model with which this analysis is closely connected will be explained.

In section 1.6 a new and precise description of a class of dynamic logical threshold gates is presented, with emphasis on the form of nonlinearity. The development of a new description of dynamic networks of threshold gates in terms similar to those used for automatic control systems is in section 1.7. Similarities to linear systems are also mentioned.

1.3 PURPOSE AND TECHNIQUES OF NEURAL MODELING

The interest in mathematical neural modeling[4] in recent years has grown out of one fundamental idea: the behavior of complex machines[1] is to a degree brain-like, and the behavior of brains is to a degree machine-like. The understanding of functional aspects of nervous activity has benefited from knowledge of control systems and automata theory, and the desire for machines which change their behavior, that is, learn with experience, and which function despite failure of unreliable components[3], has grown with the knowledge that these two attributes exist in many forms of living matter. The purpose of neural modeling has been to increase understanding in these two areas.

The formulation of the descriptions or models at all levels of activity takes place by two different processes. In the first, a very large number of observables are reproduced with high accuracy. This has inevitably resulted in extreme complication of the resulting model, but the hope has been that in comparing it with experimental results, it will be reasonable to simplify to a significant degree, making the model more amenable to analysis. In the second a smaller set of parameters is chosen initially, and it is assumed that the use of the resulting description will point out areas where more complication is necessary. This approach has resulted in systems which are relatively easy to analyse, but whose behavior is very unlike the experimental observations of real systems.

Studies of the nervous system and its information-processing capabilities can be divided roughly into the following five categories. The precision of available techniques tends to decrease from beginning to end of the list.

1. Chemical and electrochemical processes at the subcellular level. It would appear that membrane mechanics is most relevant to the functional behavior of cells.

2. Cellular input-output functions. Deterministic stimulus-response functions tend to be either inaccurate or over-complicated, and therefore statistical descriptions are at present better for purposes of analysis.

3. Interactions between cells in groups. Such groups are often studied using probabilistic models of interconnexion.

4. Gross electrical and chemical activity of living nervous

tissue.

5. Attempts to understand entire organisms in relation to their surroundings.

Modelling techniques have generally depended on the complexity of the chosen neural system. Formal mathematical description with the power of its methods is generally satisfactory only for systems with a small number of parameters. Larger numbers of parameters and systems with complicated nonlinearities can often be handled by analogue computers which also have the advantage of giving the best insight into element interactions. A third technique, digital computer simulation, is the most flexible, although present digital computers are inherently inefficient in solving many-variable dynamical systems. Speed, storage capacity and ease of obtaining state pictures of the system at instants of time make this the most common technique. The fourth technique, electrochemical modelling, is of mainly historical importance. The most famous such experiment is the iron-wire model of Lillie[5].

1.4 ANATOMICAL CONSIDERATIONS

The brain is essentially tiered in structure; it can be categorized roughly into three regions: the cortex, mid-brain, and brain stem with branching formations to all parts of the body. Brain tissue is seen to consist of discrete units which will be classed here into two main types: neurons and other types of cells, mostly of the kind known as glial cells. Extreme diversity exists in the microstructure of nervous tissue but on a macroscopic level varying degrees of explicitness can be observed. That is, in parts of the central nervous system—the brain itself—the physical structure appears nearly random but in other regions it is more systematic. Mathematical theories of information processing have not in general concerned specifically restricted regions since certain features are common to all types of neurons and collections of neurons, and sufficient data is not available to describe actual systems adequately.

1.4.1 MORPHOLOGICAL FEATURES OF THE NEURON

It has been established that the following conditions may be taken to be true for the purposes of mathematical study of the neuron[6]:

1. A nerve cell has the general form of Fig. 1.1. The nucleus is in the cell body or soma from which branch a large number of dendrites, and a projection called the axon, which may also branch. Axons end at junctions, or synapses, near the dendrites or soma of a cell body (see Fig. 1.2).

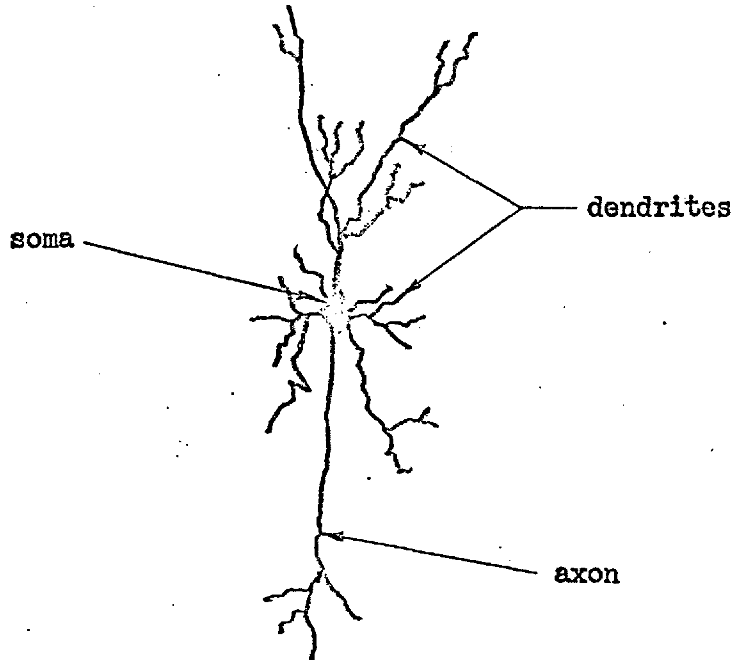

Fig. 1.1 A Neuron

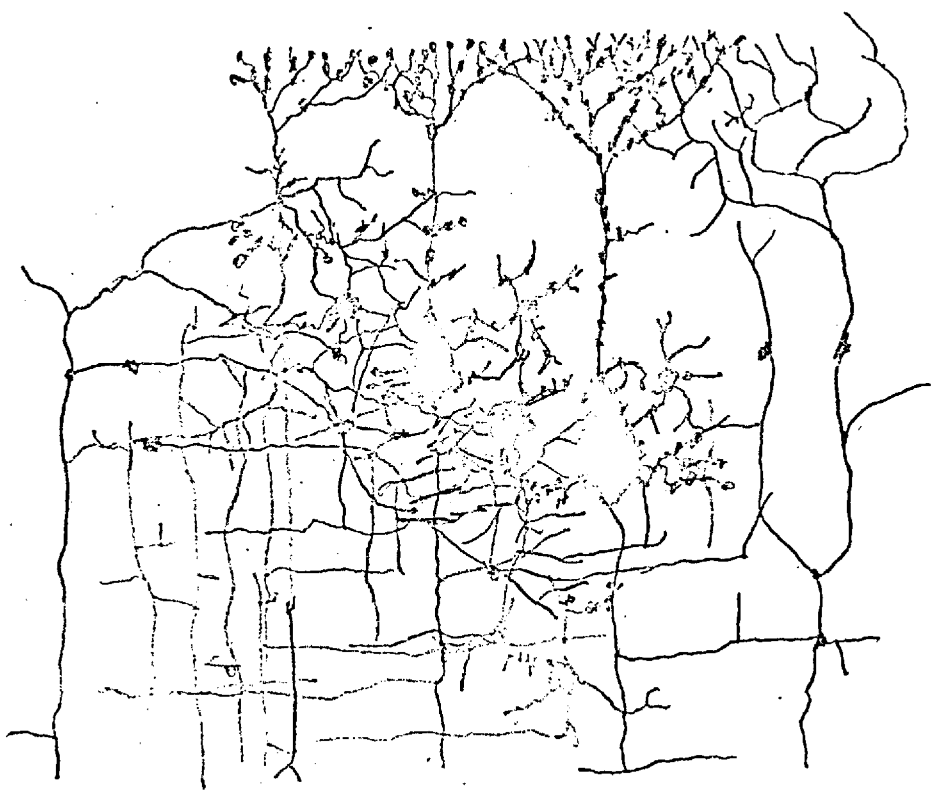

Fig. 1.2 A Neural Network

In the following work it will also be assumed that an axon is not restricted to interract only with other cells, but may also affect the cell from which it emanates. Generally axons may end at other neurons or at effector cells such as muscles or glands.

2. Axons transmit impulses only in the centrifugal direction, except for the case of "primary afferent axons" which appear to carry impulses from remote receptive structures to the cell body.

3. Very great functional significance is attached to the surface membrane of the neuron, although the interior processes are not fully understood.

4. In general terms the electrochemical processes produce intermittent action: a neuron can be observed to propagate "spikes" or electrochemical impulses along the axon to the synapses. The mechanism by which such impulses cross synaptic junctions is not fully understood.

5. Impulses travel along axons without significant attenuation. Once an impulse crosses a synapse, however, its shape changes. Generally the farther the synapse from the cell body, the more the afferent pulses are attenuated.

6. There is a significant time delay between transmission of a pulse from one cell body to reception at another.

1.5 FEATURES OF MATHEMATICAL NEURON MODELS

1.5.1 RELEVANT MODEL TYPES

The mathematical model which will be the basis for further chapters is a description of the functional behavior of a "neuron". It is a mathematical abstraction intended to explain the functional behavior of nerve cells in vivo without doing gross injustice to the known physical structure of electrical characteristics. As will be clarified, the McCulloch-Pitts type of cell has explicit functional properties but its dynamical properties are not defined. The differential equations of the Hodgkin-Huxley type are good descriptions of single-neuron behavior but formidable to use as descriptions of nerve nets. The Cowan model is midway between these two extremes.

1.5.2 OBSERVATION OF ELECTRICAL BEHAVIOR

Three basic types of electrical observations of nervous systems are commonly made:

1. Electroencephalography is a well-developed clinical method of measuring the electrical activity of large portions of the brain. Various characteristic waveforms can be observed* and have been correlated with factors such as illness, personality type, intensity of thought, and stages of sleep. The filtering properties of the tissue through which the signals prop-

* Figs. 1.3 and 1.4.

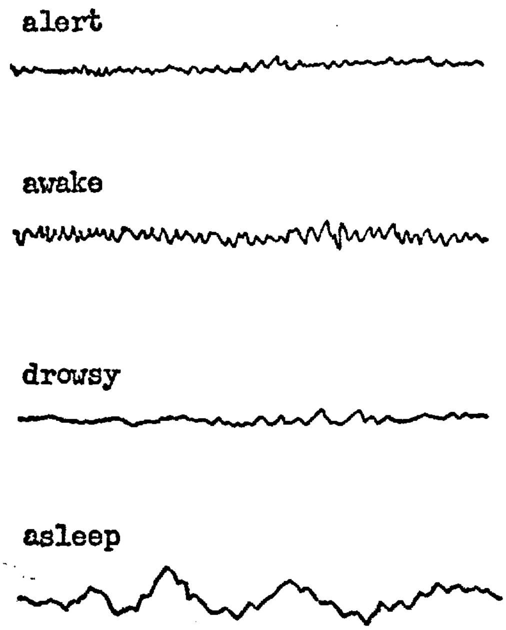

Fig. 1.3 Normal EEG Patterns

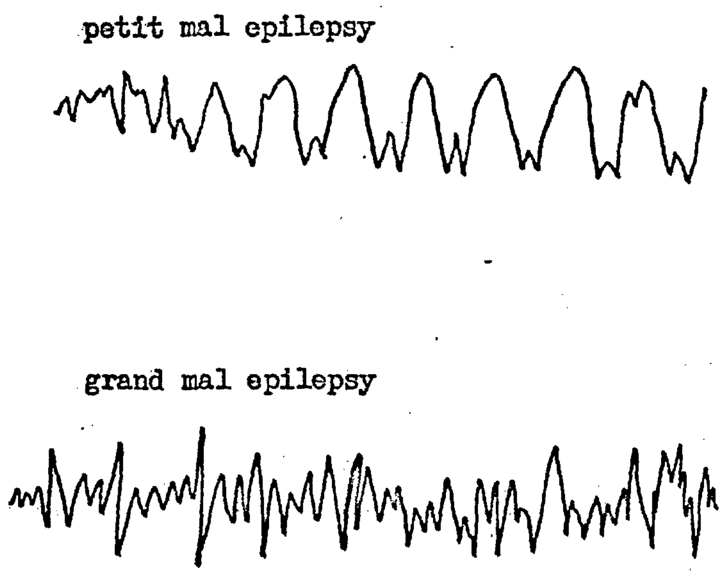

Fig. 1.4 Abnormal EEG Patterns

agate may significantly affect the observed waveform.

2. The behavior of regions of tissue of the order of one cubic millimetre can be observed by the insertion of electrodes which compare local voltages with over-all levels. This type of record is given the name "electrocorticogram."

3. Techniques involving the use of microelectrodes enable the observation of the input and output voltage waveforms of single neurons in living tissue. Thus the response of individual cells to specific sensory inputs or artificial stimulation can be observed.

1.5.3 THE RESPONSE OF SINGLE NEURONS

It is generally assumed that the response of a single cell can be categorised by considering its dendritic tree to be the signal input and the axon to be its output; that is, the flow of "information" through a cell is unidirectional. Since the shape of the waveform of the electrical potential of the cell body is clearly a train of spikes, the problem is one of analysing the impulse response of the cell including its dendrites and axon. The following general observations can be made:

1. A cell has a minimum time between firing (assuming continuous strong excitation) and thus a maximum firing frequency. This mimimum time is called the refractory period.

2. The effect of spikes at the synapses can be to cause the

cell to increase or decrease its rate of firing; the effect is either excitatory or inhibitory.

3. The response of a single cell may vary in time. The number of afferent pulses at synapses required to cause the cell to fire depends on the recent activity of the cell.

1.5.4 THE McCULLOCH-PITTS MODEL

Until this model was suggested there was no basis for the assumption that the logical or computational behavior of nerve nets could be specified mathematically. The McCulloch-Pitts model[18] (Fig. 1.5) provided such a basis by building on the following assumption: every neuron can be classified into one of two states during the course of time. At any instant it is either firing or not firing, and thus the algebra of Boole can be used to analyse the logical behavior of nets of such elements. Furthermore, the first-order approximation of the transfer function of a cell may be taken to be as follows:

Assume that in a network of n cells, those which are in the firing state are given the value 1 and those in the non-firing state are given the value 0. Then, the effect of the n cells acting through the synapses and dendrites of the i^{th} neuron is as follows: Consider the sum (Fig. 1.6)

$$\sum_{j=1}^{n} \alpha_{ij} x_j = S_i \qquad 1.1$$

where the α_{ij} are constant weighting coefficients. Then the relation

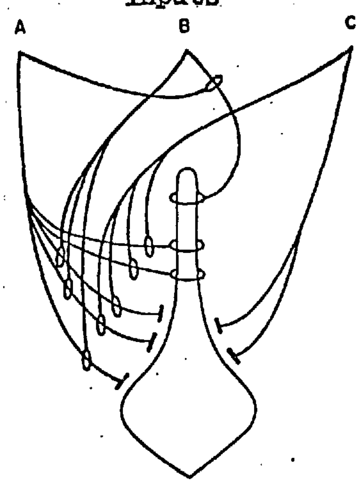

Fig. 1.5 A McCulloch-Pitts Formal Neuron

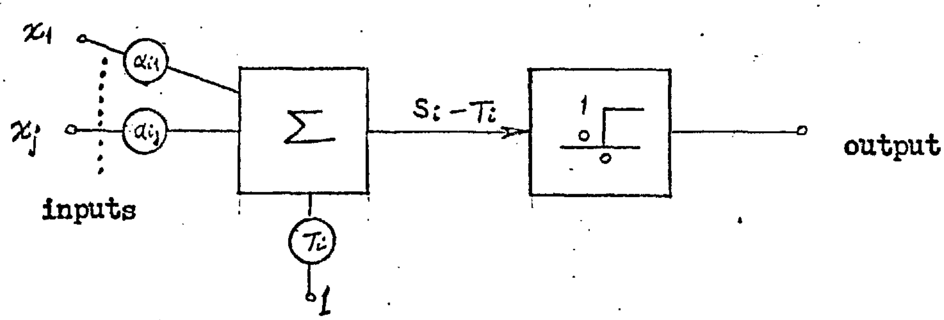

Fig. 1.6 Functional Diagram of a Formal Neuron

$$x_i = \begin{cases} 1, & S_i \geqslant T_i \\ 0, & S_i < T_i \end{cases} \qquad 1.2$$

holds, where T_i is a threshold value. The dendritic structure may then be said to sum the outputs of the n neurons, the magnitude of the coefficient α_{ij} depending on the nature of the dendritic structure through which the output of the j^{th} neuron passes to the soma of the i^{th}, and the sign depending on the effect of the j^{th} input. If the effect of cell j is excitatory, $\alpha_{ij} > 0$, and if inhibitory, $\alpha_{ij} < 0$.

Clearly if some cell, say k, has no effect on cell i, then $\alpha_{ik} = 0$. That is, there is no synapse between the axonal structure of cell k and the dendritic or somatic structure of cell i. Three important points emerge:

1. The state of such a system can be completely described by Boolean algebra.

2. Since the outputs of a neuron are not mapped one-to-one on the set of inputs (there are two outputs and 2^n possible input combinations) and since knowledge of the output does not imply knowledge of the input combination, the cell is performing the process of <u>computation</u>, rather than mere signal transmission.

The class of functions which may be computed by such a cell is given the name "linearly separable" and a great deal of work has been done in the analysis of such functions.

3. All possible Boolean functions can be computed by a network containing elements which compute linearly-separable functions. This will be demonstrated in Chapter 2.

This statement means that provided the basic assumption of two-level activity of cells holds, the logical behavior of nerve nets can be described mathematically, and conversely any logical function which can be unambiguously defined can be computed by nets of McCulloch-Pitts neurons.

The main difficulty with a two-level description of any real network of neurons is that it is obviously an approximation. Clearly a finite time must elapse while any cell switches from the 0 state to the 1 state during which it is in neither. The assumption usually made is that such a system is "clocked" or that a stroboscopic analysis is sufficient to specify its behavior. The two states of behavior of a cell might be taken to be that in which rapid spike emission occurs versus that corresponding to slow spike emission. The difficulty in this case is that cells often exhibit a continuum of firing rates between the minimum and maximum and two-level analysis is not entirely valid.

Two additional objections to the use of a binary analysis arise, even for the construction of a model of information processing rather than a model of the physiological processes involved. The first is that an arbitrary network of <u>real</u> gates may have stable states which cannot be defined by a two-level algebra. Consider Fig. 1.7. This

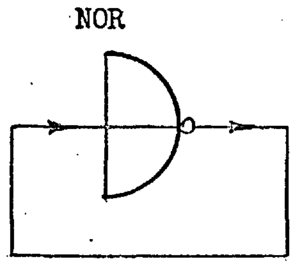

Fig. 1.7 Symbolic 1-Element NOR Network

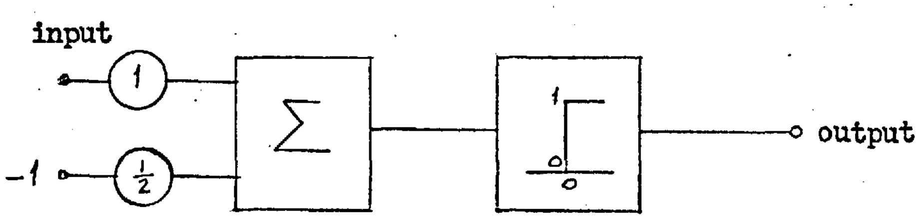

Fig. 1.8 Threshold Realization of a NOR Gate

represents a one-element network of NOR-gates. It will be shown (see Fig. 1.8) in chapter 2 that such a gate can be constructed from a linearly-separable function and therefore is of a type that we are considering. Clearly the Boolean function

$$x = \bar{x} \qquad 1.3$$

is not valid. The physical system, however, is easily realizable, and if the one-input NOR-gate is taken to be a very high-gain saturating amplifier with heavy negative feedback, the expected equilibrium value for the continuous variable x may be somewhere between the zero level and the one level.

The second need for a more valid model than a two-level one is that a network of real cells can be described by a set of differential equations, and therefore is a dynamical system. It may exhibit a transient behavior, and indeed, a never-dying oscillation may exist. If such behavior is of interest, then clearly a set of differential equations must be used.

1.5.5 THE HODGKIN-HUXLEY MODEL

This model[7] is a set of differential equations which approximate rather well the behavior of certain neurons. It will be described briefly here since it is the classic example of detailed mathematical modeling of cell mechanisms.

It is assumed that a typical section of cell membrane can be represented as in Fig. 1.9. The current across the membrane is produced

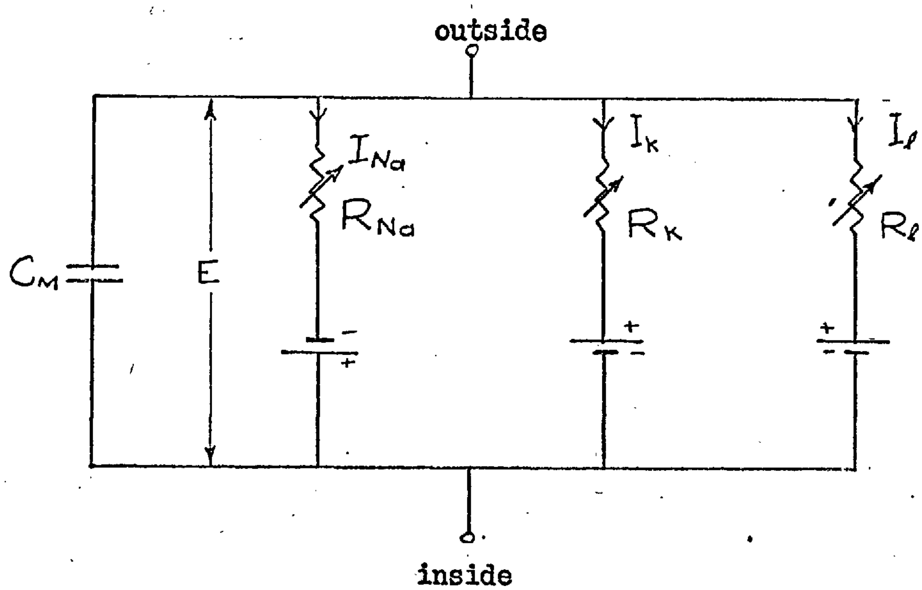

Fig. 1.9 Hodgkin-Huxley Equivalent Circuit of Neural Membrane

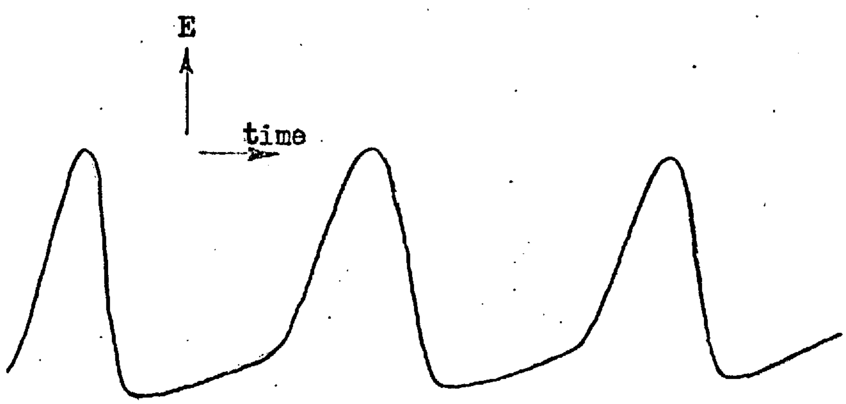

Fig. 1.10 Typical Waveform of Fig. 1.9

by the movement of ions through it. The variable conductances are functions of time and the polarising voltage E. All other parameters are constant. Then the equations for the propagation of an electrical disturbance along a cylindrical fibre of radius a, filled with a fluid of specific resistance R_2 are

$$\frac{a}{2R_2\theta^2}\frac{d^2V}{dx^2} = C_M\frac{dV}{dt} + \bar{g}_k n^4(V - V_k) + \bar{g}_{Na}m^3h(V - V_{Na}) + g_\ell(V - V_\ell) \qquad 1.4$$

$$\frac{dn}{dt} = \alpha_n(1 - n) - \beta_n n \qquad 1.5$$

$$\frac{dm}{dt} = \alpha_m(1 - m) - \beta_m m \qquad 1.6$$

$$\frac{dh}{dt} = \alpha_h(1 - h) - \beta_h h \qquad 1.7$$

where θ is the velocity of propagation and typical α and β parameters have the following forms:

$$\alpha_n = \frac{0.01(V + 10)}{\exp\left[\frac{V+10}{10}\right] - 1} \qquad 1.8$$

$$\beta_n = 0.125\exp\left[\frac{V}{80}\right] \qquad 1.9$$

$$\alpha_m = \frac{0.1(V + 25)}{\exp\left[\frac{V+25}{25}\right] - 1} \qquad 1.10$$

$$\beta_m = 4\exp\left[\frac{V}{18}\right] \qquad 1.11$$

$$\alpha_h = 0.007\exp\left[\frac{V}{20}\right] \qquad 1.12$$

$$\beta_h = \frac{1}{\exp\left[\frac{V+30}{10}\right] + 1} \qquad 1.13$$

Clearly such a set of equations is not amenable to analytic solution although analogue and digital computer solution is possible. Fig. 1.10 is a typical pulse train which can be shown to be one of the solutions of the equations and which is representative of observed action potentials.

1.5.6 THE COWAN MODEL

The Cowan model is essentially a simplification and interpretation of two generalized phenomena: the cell exhitation mechanism and its firing mechanism. It has recently been published in a thesis (reference 3) and the following has been extracted from it (with permission):

"Our starting point is J. C. Eccles' well-known[9,10] lumped equivalent circuit for the generation of post-synaptic potentials ...., and the .... distributed equivalent circuit or "cable" equations for neuronal membrane of W. Rall[8]. ....Since we are interested in modelling the responses of large networks, we are concerned only with the crudest possible nontrivial way of writing down an equation for the responses of a single neuron....

"...Let us now approximate the effects of the electrotonic decrement represented in the cable equations, merely by an attenuation and a delay! ....

"...We now have to consider the way in which a response to such an excitation is elicited in the cell. ....R. Fitzhugh[11] has shown how such [Hodgkin-Huxley] equations give rise to threshold phenomena consistent with "all-or-none" responses. ...

"...However such is the complexity of the requisite computations (and it is by no means clear that the equations cover all the relevant phenomena) that we have chosen to approximate the response by empirical curves obtained from experiments on the responses of nerve membrane to stimulating currents.....It often seems to be the case that ....the slope of the....rate-intensity curve changes in such a way that it can be reasonably fitted by a sigmoidal function.....

"...There is no fundamental reason for choosing the curve....which is of course the well-known Logistic curve of demography. It happens to be a convenient and tabulated sigmoidal function. What we are interested in studying is the qualitative nature of the results issuing from this choice of a sigmoidal function, not the exact quantitative aspects."

We interject at this point that in this thesis more emphasis will be placed on quantitative aspects, since we are interested in modelling deterministic physical devices rather than statistical functions, and since the detailed behavior is important for some purposes.

"Let us introduce the variable

$$X(t) = 1 - \lambda v(t) \qquad [1.14]$$

[where λ is the refractory period and v is the firing rate.] Then we [have] a difference equation relating the neuronal output to its inputs:

$$\ln\frac{X_r(t)}{1 - X_r(t)} = \epsilon_r + \frac{1}{\beta_r}\sum_s \alpha_{sr} X_s(t - \tau_{sr}) \qquad [1.15]$$

[where r is the neuron index, ϵ_r is a constant input, β_r a slope parameter for the sigmoidal function, the α_{sr} are coupling coefficients, and τ_{sr} is the delay from the sth to the rth neuron.]

"The variable X can be interpreted as the fraction of time when a neuron that is continually emitting spikes is not refractory, i.e., the fraction of time when it is "sensitive" to stimuli....

"...We now wish to convert the fundamental difference equation into a differential equation.....We make the assumption that all the delays are the same, so that we have $\tau_{sr} = \tau_{rs}$:

$$\ln\frac{X_r(t + \tau_r)}{1 - X_r(t + \tau_r)} = \epsilon_r + \frac{1}{\beta_r}\sum_s \alpha_{sr} X_s(t). \qquad [1.16]$$

"...For sufficiently small τ_r we have the equation:

$$\tau_r \frac{d}{dt}\left[\ln\frac{X_r(t)}{1 - X_r(t)}\right] + \ln\frac{X_r(t)}{1 - X_r(t)} = \epsilon_r + \frac{1}{\beta_r}\sum_s \alpha_{sr} X_s(t)." \qquad [1.17]$$

Various linearizations are then considered, and then the equation similar to the one above, but without the damping term, and with a change of time-scale:

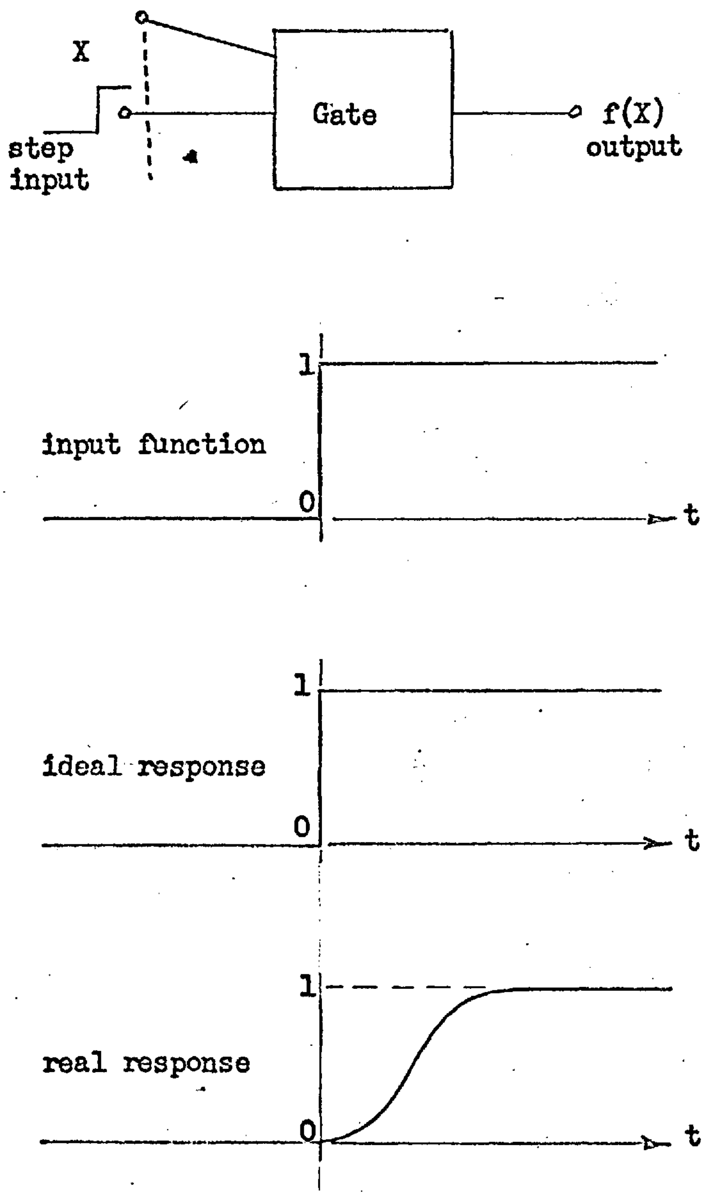

Fig. 1.11 Gate Step Response

$$\frac{d}{dT}\ln\frac{X_r}{1 - X_r} = \gamma_r + \frac{1}{\beta_r}\sum_s \alpha_{sr} X_s \; . \qquad 1.18$$

This equation forms the core of Cowan's work on statistical mechanics of neural nets.

1.6 A DYNAMIC MODEL OF THRESHOLD GATES

The design of threshold gates and their logical properties will be discussed precisely in the next chapter. Here we will be concerned with their transient behavior.

Consider Fig. 1.11. In the design of logic systems by Boolean algebra the response of the ideal gate $f(X)$ to a step input at one of its input terminals is assumed to be a step also. The actual response may be more like that of the "real" gate in the figure. The ideal gate is assumed to be free from "noise," that is, it has two completely stable outputs, 0 and 1. Any real gate, however, will always be subject to small fluctuations at its output.

The central assumption made in this thesis is that the response of a threshold gate is similar to that shown in Fig. 1.11. Other responses are possible, and may be accounted for by various changes in the analysis, but this will not be done here.

In the design of practical circuits, one of two assumptions is usually made. The first is that the output of a gate is a perfect or nearly perfect step which occurs after a finite delay from the time of the input change. The second is that the transition shape is of the form shown in Fig. 1.11. The fact that actual realizations do not have the ideal response leads to complications in the design of networks of several interconnected gates, and these problems will be discussed later. We note here only that two simplified analyses are used to describe sequential (dynamic) behavior. General gates are treated this way, not only threshold gates:

1. A three-level algebra is commonly used, in which the binary 0 and 1 states are represented by 0 and 1 respectively, and in addition a third "don't know" state is represented by $\frac{1}{2}$. References 12 to 15 deal with this method.

2. A pure delay is associated with each element, which is assumed to be in the 0 or 1 state at the end of the delay time. Numerous papers make this assumption, but the classic one is reference 16.

Both of these approaches are useful, but are necessarily accurate only when the simplifying assumptions allow correct prediction of network behavior. The continuous approach used here yields more information about the detailed behavior of switching networks but requires more computation.

1.6.1 SPECIFICATION OF THE THRESHOLD GATE MODEL.

We define a model of the i^{th} gate in a network of n gates, referring to Fig. 1.12a. The object is to define the simplest model which conforms to the above discussion, and to certain behavior to be

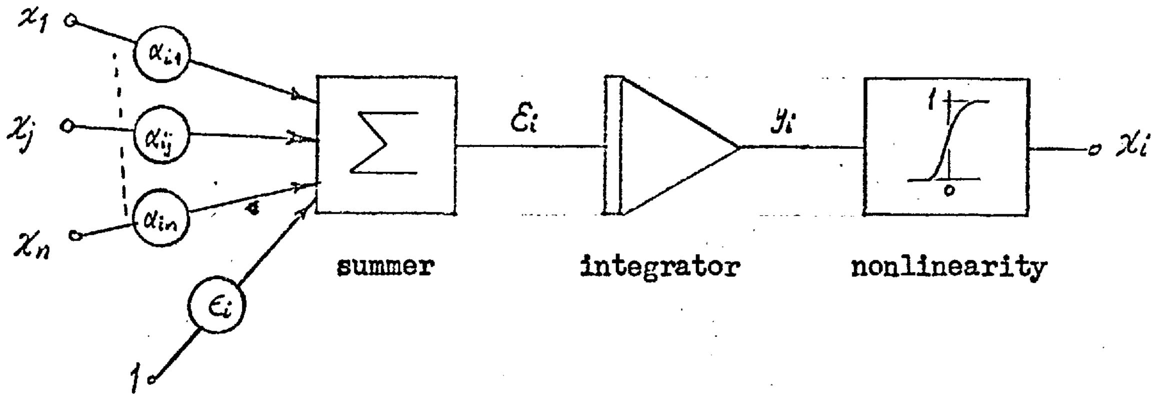

Fig. 1.12a The Threshold Gate Model

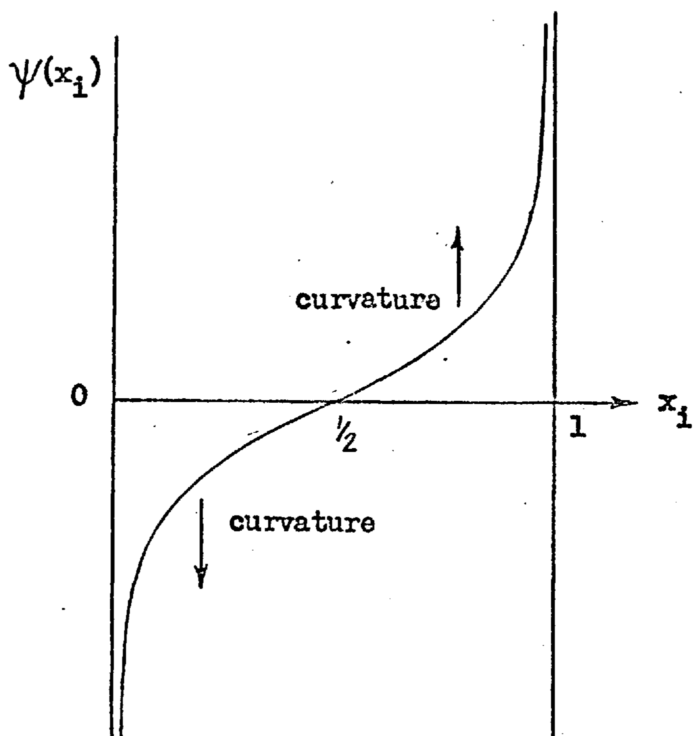

Fig. 1.12b A Suitable Nonlinearity

discussed later:

1. The inputs are the real variables x_i, $i = 1, 2, \ldots, n$ which are weighted by the real constants α_{ij}, $j = 1, 2, \ldots, n$ and summed, along with a real quantity ϵ_i. The inputs to the network are constant in a finite time interval and are included in the terms ϵ_i. That is, if element i has an "external" input, it may be written

$$\epsilon_i = \alpha_{io} + \sum_q p_{iq} u_q \quad , \tag{1.19}$$

where α_{io} and p_{iq} are constants, u_q is the q^{th} input, and q ranges over the set of input variables. Thus the total summed input is

$$\epsilon_i + \sum_j \alpha_{ij} x_j \quad . \tag{1.20}$$

2. The output of a gate is the variable x_i, and the outputs of the network are any set of the x_i, $1 \leq i \leq n$.

3. The system is autonomous except for changes in the input variables. During the finite time intervals in which the inputs are constant the system is completely autonomous. This precludes the usual situation in which "adaptation" or "accommodation" affect the short-time interval characteristics of an element. A change in inputs corresponds to a new set of initial conditions for the next time interval.

4. Each gate is a first-order dynamical system, such that x_i may be taken as the state variable. "Ringing" and overshoot responses to step inputs are thus eliminated.

5. The state equation of the i^{th} element is of the form

$$(\tau_i \frac{d}{dt} + 1)\beta_i \psi(x_i) = \epsilon_i + \sum_{j=1}^{n} \alpha_{ij} x_j \tag{1.21}$$

where τ_i and β_i are positive real constants. This equation exhibits behavior consistent with assumptions 3 and 4.

6. The function $\psi(x_i)$ in 1.21 is
   - single-valued with a single-valued inverse $\psi^{-1}(\,.\,)$,
   - strictly positive monotonic in x_i,
   - analytic for $0 < x_i < 1$,
   - asymptotic to the values $x_i = 0$ and $x_i = 1$,
   - symmetric about $x_i = \frac{1}{2}$.

The last two of these assumptions are simplifying normalizations, and are not necessary, but may be used with no loss of generality.

We assume further that $\psi(x_i)$ has exactly one point of inflexion at $x_i = \frac{1}{2}$. Fig. 12b is a diagram of a suitable function.

1.6.2 THE OPERATIONAL APPROACH

Consider Fig. 1.13. This is a lossy integrator feeding a function generator, and having as inputs the quantity

$$\mathcal{E}_i = \epsilon_i - y_i + \sum_{j=1}^{n} \alpha_{ij} x_j \,. \qquad 1.22$$

We assume that the variables x_j and y_i are normalized voltages. Then, if the operational amplifier is perfect,

$$\frac{1}{R_{io}} + \sum_j \frac{x_j}{R_{ij}} - \frac{y_i}{R_f} - C\,\frac{dy_i}{dt} = 0 \,, \qquad 1.23$$

or

$$(CR_f \frac{d}{dt} + 1)(\frac{y_i}{R_f}) = \frac{1}{R_{io}} + \sum_j \frac{x_j}{R_{ij}} \,. \qquad 1.24$$

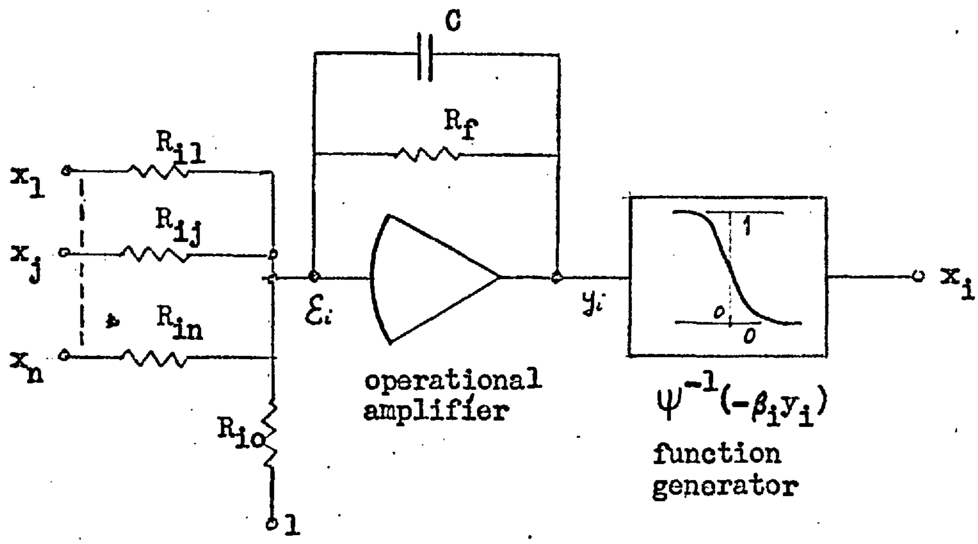

Fig. 1.13 Analogue Computer Diagram

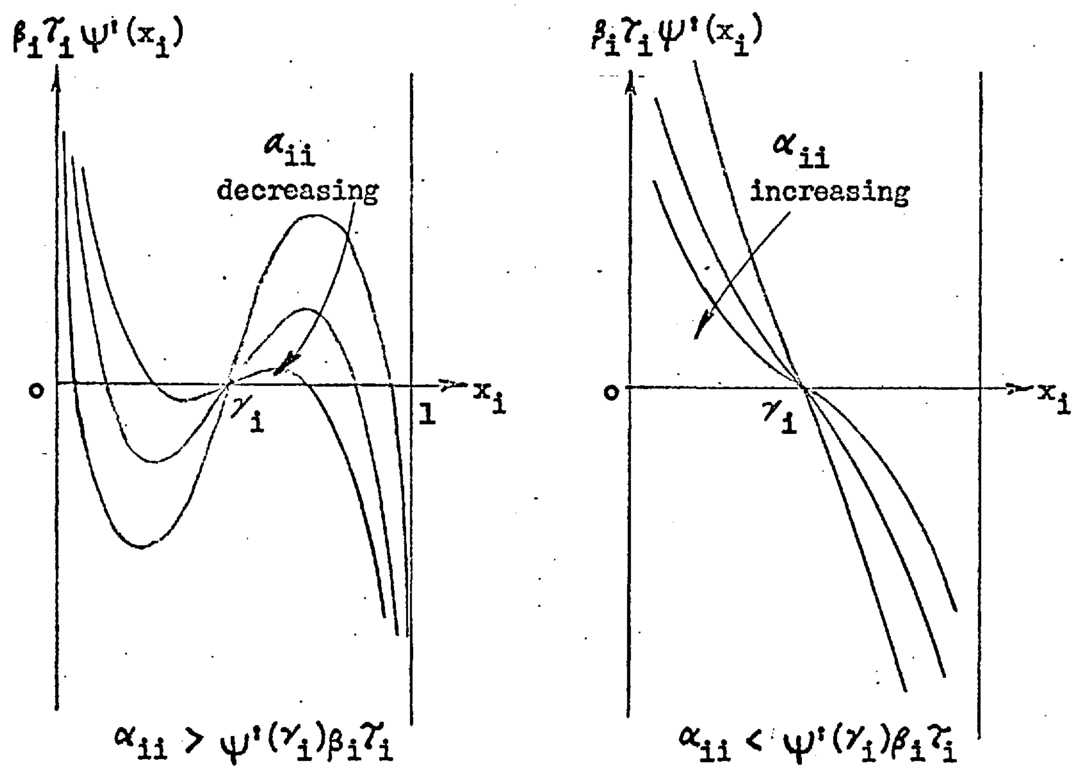

Fig. 1.14 Excitation Diagrams

Clearly, if the following equations hold

$$\epsilon_i = \frac{1}{R_{io}}$$

$$\alpha_{ij} = \frac{1}{R_{ij}}$$

$$\beta_i = \frac{1}{R_f} \qquad 1.25$$

$$\tau_i = CR_f$$

and in addition, the output of the function generator is $x_i = \psi^{-1}(\beta_i y_i)$, then 1.24 is exactly equation 1.21. Therefore it is possible to simulate 1.21 using the computing elements of Fig. 1.13.

From Fig. 1.13 it can be seen that no matter what finite value the variable y_i may reach, the function generated is in the open region $(0,1)$, that is, for all time

$$x_i \in (0,1) \; . \qquad 1.26$$

This result is due of course to the choice of the function $\psi(x_i)$.

It is possible to rewrite 1.21 as follows. Since

$$\frac{d\psi(x_i)}{dt} = \psi'(x_i)\frac{dx_i}{dt} \; , \qquad 1.27$$

we may write

$$\dot{x}_i = \frac{1}{\beta_i \tau_i \psi'(x_i)}\left[\epsilon_i - \beta_i \psi(x_i) + \sum_j \alpha_{ij} x_j\right] \qquad 1.28$$

which is valid for all x_i in $(0,1)$. Assume for the moment that all variables except x_i are constant. Then we have

$$\beta_i \gamma_i \psi'(x_i)\dot{x}_i = k - \beta_i \psi(x_i) + \alpha_{ii} x_i \qquad 1.29$$

where k includes all constant terms. The right-hand side of this equation (the "excitation" of the element) may have the general forms illustrated in Fig. 1.14a and 1.14b which differ only in the parameter α_{ii}. The function $\psi'(x_i)$ is always positive and non-zero, and at the limits 0 and 1 goes to plus infinity. It can be seen therefore that if the vertical axes in Fig. 1.14 are labelled $\beta_i \gamma_i \psi'(x_i)\dot{x}_i$ then x_i tends to one point in Fig. 1.14b and to two in 1.14a depending on the initial point. The exact behavior in Fig. 1.14a cannot be specified since the function

$$L(s) = \lim_{x \to s} \frac{\psi(x)}{\psi'(x)} \qquad 1.30$$

has not been specified for $s = 0$ or 1. The limit of the behavior as $\alpha_{ii} \to 0$ is an infinite-gain amplifier, and since the term $\alpha_{ii} x_i$ is a feedback term it has a large effect on the element stability.

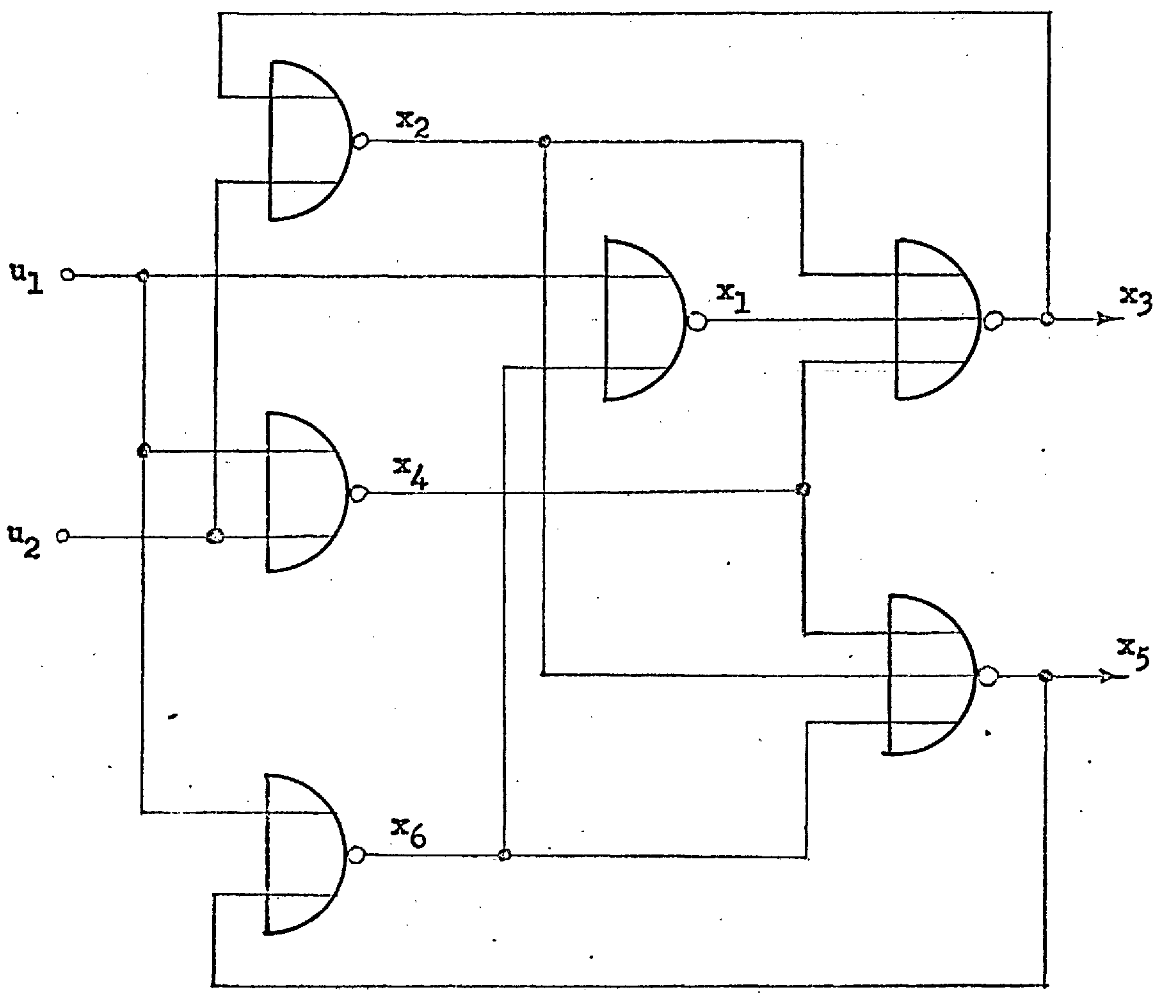

Fig. 1.15 NOR-Gate J-K Flip-Flop

1.7 THE SYSTEMS APPROACH

In section 1.6.1 the characteristics of one gate in a network of n gates were defined. The object here is to give a description of such a network in a manner similar to that of a nonlinear automatic control system. Consider Fig. 1.15. This is a functional diagram of a network of NOR gates, realizing a function known as a J-K bistable. (The equivalence of threshold gates to NOR gates will be discussed in Chapter 2.) Such a network may be represented by the following equation:*

$$(TD + I)BY = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} X + \begin{bmatrix} 1/2 \\ 1/2 \\ 3/2 \\ 1/2 \\ 3/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ -1 & -1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad 1.31$$

where I is the unit matrix, Y and X are n-dimensional variable vectors, and

$$B = \mathrm{diag}\left[\beta_i\right], \ \beta_i > 0 \qquad 1.32$$

$$T = \mathrm{diag}\left[\tau_i\right]. \ \tau_i > 0 \qquad 1.33$$

D is the diagonal differential operator of order n. In general, a system of threshold gates may be represented by the equation

$$(TD + I)BY = AX + A_o + PU_k \qquad 1.34$$

---

* No comments will be made as to the uses of this particular circuit.

where A_o is a column matrix, P is an n by q matrix, and U_k is the q-dimensional input vector. Recall from section 1.6.1 that we are assuming that U_k is constant in a finite time interval. We use the subscript k to denote the k^{th} such interval.

Equation 1.34 may be rewritten as follows. Let the vector Γ be the solution of the equation

$$A\Gamma + A_o + P\hat{U} - B\eta = 0 , \qquad 1.35$$

in which $\hat{U}$ is a constant vector, and $\eta = [\mu_i]$ is related to $\Gamma = [\gamma_i]$ by the equations

$$\begin{aligned} \eta &= \Psi(\Gamma) , \\ \Gamma &= \Psi^{-1}(\eta) , \end{aligned} \qquad 1.36$$

because of the relation

$$y_i = \psi(x_i) . \qquad 1.37$$

Now combining 1.34 and 1.35, we have

$$(TD + I)B(Y - \eta) = A(X - \Gamma) + P(U_k - \hat{U}) . \qquad 1.38$$

Taking Laplace transforms, 1.38 becomes

$$(Ts + I)V(s) = TV(0) + PU(s) + AZ(s) , \qquad 1.39$$

where

$$V(t) = B(Y(t) - \eta) , \qquad 1.40$$

$$Z(t) = X(t) - \Gamma , \qquad 1.41$$

$$U(t) = U_k - \hat{U} . \qquad 1.42$$

From 1.39,

$$V(s) = (Ts + I)^{-1}TV(0) + (Ts + I)^{-1}PU(s) + (Ts + I)^{-1}AZ(s). \quad 1.43$$

The inverse transform of the first-order transfer function is

$$\hat{G}_{\delta}(t) = \mathcal{L}^{-1}\left\{(Ts + I)^{-1}\right\} = \mathcal{L}^{-1}\left\{\mathrm{diag}\left[\frac{1/\tau_i}{s + 1/\tau_i}\right]\right\} = T^{-1}e^{-tT^{-1}}, \quad 1.44$$

and thus the inverse transform of 1.43 is

$$V(t) = e^{-tT^{-1}}V(0) + \int_0^t \hat{G}_{\delta}(t - \tau)PU(\tau)d\tau + \int_0^t G_{\delta}(t - \tau)Z(\tau)d\tau , \quad 1.45$$

where

$$G_{\delta}(t) = \mathcal{L}^{-1}\left\{(Ts + I)^{-1}A\right\} = T^{-1}e^{-tT^{-1}}A \quad 1.46$$

is the impulse response of the linear transfer function.

Equation 1.45 is similar in form to an equation usually considered in stability analyses of certain control systems. We now recall that we have assumed that U_k is constant. Therefore if $\hat{U} = U_k$, the input function in 1.45 is zero, and this equation is then the <u>zero-input</u> state equation.

1.7.1 SIMILARITIES TO LINEAR SYSTEMS

We make one further comment about the form of 1.38. The standard state-space representation[19] of linear systems is given in the following two equations:

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

in which X is the state vector, U the input vector, and Y is the output vector. The matrices B, C, and D need not be square. If, along with 1.38, we define an output vector $W(t)$ and a matrix C, we have

$$(TD + I)V = AZ + PU , \qquad 1.38$$

$$W = CZ , \qquad 1.47$$

where of course the second term of the second equation is zero. These two equations bear a strong resemblance to those for the linear system. The difference is that $V(t)$ is a highly nonlinear function of $Z(t)$ and therefore linear theory is of only limited relevance.

1.8 SUMMARY

Three mathematical models of neural function are given to illustrate the range of such models. The McCulloch-Pitts model is a fundamental hypothesis about static neural function, the Hodgkin-Huxley equations are the classical example of detailed membrane description, and the Cowan equations are intended to exhibit the function of the first with some of the dynamics of the second.

A precise model of real linear threshold gates resembles the Cowan model closely.

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CHAPTER 2

REALIZATION OF LINEAR THRESHOLD FUNCTIONS

2.1 INTRODUCTION

In recent years much effort has been expended on the analysis of logical elements based on the principle of the linear threshold function, and considerable use of such elements has been made in studies of learning machines[1] and in actual computing machines[7]. In this chapter some of the important features of this analysis will be discussed, with emphasis on the definition of a threshold function, its construction, and its circuit realization.

Section 2.2 contains definitions of functional elements and functional gates. In this section and throughout the thesis the word "Boolean" means logical, whereas "binary" means two-valued or "nearly" two-valued. (This distinction is not always made in the literature.) Normalizations of threshold functions are discussed, since the analysis to be presented later is of a set of equations normalized to the interval (0,1). The logical equivalence of normalized threshold functions to other threshold functions is demonstrated.

Section 2.4 presents the fundamental motivation of this thesis. We give a theorem which relates the dynamic model of Chapter 1 to the logical functions defined in the first part of this chapter. The result is extended in a corollary to the special case of "lossless" gates similar to equation 1.18. Convergence times and end-points are discussed.

Since the model we have used is a dynamical model it is possible to relate the continuous-time dynamics to discrete-time systems. A definition is introduced in section 2.5 which may be used to characterize systems which obey discrete-time equations at only a finite number of points. Properties of such systems are discussed.

2.1.1 DEFINITIONS

In the following work, use will be made of certain definitions:

Definition 2.1: A logical threshold element is an ideal logical element with a finite number of Boolean inputs and one Boolean output, whose operation can be described by reference to an arithmetic valuation of the inputs called a threshold function (see Fig. 2.1).

Definition 2.2: A threshold gate or threshold logic unit[1] is a physical realization of a logical threshold element. It has n (a positive integer) binary inputs and one binary output which is determined by an evaluation of the appropriate threshold function, or a suitable approximation of the appropriate function. It is often convenient to describe the logical behavior of such a gate by its Boolean function provided the output is a reasonable approximation to a two-valued variable (see Fig. 2.2).

Definition 2.3: A threshold function Φ is a logical function of n Boolean variables $u_1, u_2, \ldots, u_n$. If each variable is mapped onto a set of two numbers, $u_i \rightarrow x_i$, $1 \leq i \leq n$, such that if u_i is true, $x_i = b$, and if u_i is false, $x_i = a$, $b \neq a$, $1 \leq i \leq n$, and if the single-valued function $\mathcal{E}(x_1, x_2, \ldots, x_n, \epsilon) \equiv \mathcal{E}(X, \epsilon)$ is formed, then the Boolean output is

$$\underline{\text{true}} \quad \text{if} \quad \mathcal{E}(X,\epsilon) \geqslant 0 \tag{2.1}$$

$$\underline{\text{false}} \quad \text{if} \quad \mathcal{E}(X,\epsilon) < 0$$

where ϵ is a finite constant. That is,

$$\phi \rightarrow \begin{cases} b, & \mathcal{E}(X,\epsilon) \geqslant 0 \\ a, & \mathcal{E}(X,\epsilon) < 0 . \end{cases} \tag{2.2}$$

We use the arithmetic function

$$f(s) = \begin{cases} b, & s \geqslant 0 \\ a, & s < 0 . \end{cases} \tag{2.3}$$

The function $f(\mathcal{E}(X,\epsilon)) \triangleq f(X,\epsilon)$ given above is a valid description of the state of the binary gate as a Boolean function provided the output is a "reasonable" approximation to a two-level function for all time of interest.

Definition 2.4: A linear threshold function (L.T.F.) of n inputs x_i is a threshold function of the following form:

$$f(X,\epsilon) \triangleq f(\epsilon + \sum_{i=1}^{n} \alpha_i x_i) \tag{2.4}$$

where the constants α_i are not all zero. Defining

$$x_0 \triangleq b , \tag{2.5}$$

the equation becomes (Fig. 2.3)

$$f(X) = f(\mathcal{E}(X)) \triangleq f(\sum_{i=0}^{n} \alpha_i x_i) . \tag{2.6}$$

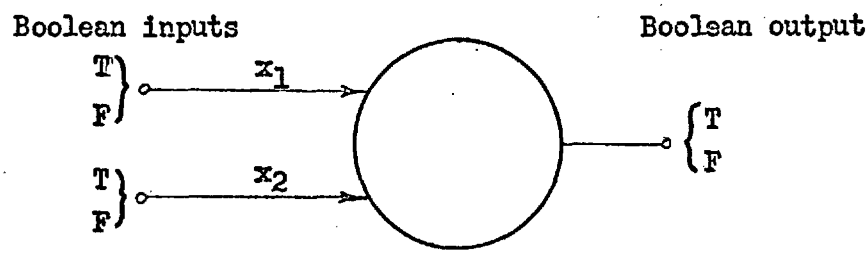

Fig. 2.1 A Boolean Element

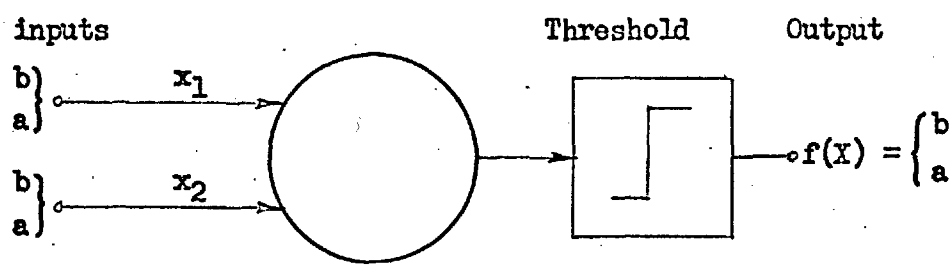

Fig. 2.2 A threshold Gate

2.2 NORMALIZED FUNCTIONS

In the literature of learning machines and pattern recognition devices, the input vector $X = (x_1, x_2, \ldots, x_n)$ is called the measurement vector. If n is a finite number and the values of x_i, $1 \leq i \leq n$ are a finite set, the set of possible vectors X can be represented as a finite number of points in an n-dimensional measurement space[3]. The equation

$$\mathcal{E}(X) = 0 \qquad 2.7$$

defines a surface in X-space which separates the points on either side of it into the regions corresponding to <u>true</u> and <u>false</u> values. This corresponds to division of the measurement space into two pattern classes A and B. If A and B are finite sets there must exist a $\theta > 0$ such that for some $\mathcal{E}$, not necessarily linear,

$$\mathcal{E}(\underline{a}) < -\theta < 0 < \theta < \mathcal{E}(\underline{b}) . \qquad 2.8$$

That is, if we consider the surface

$$\mathcal{E}(X) = 0, \quad X = (x_o, x_1, \ldots, x_n) \qquad 2.9$$

we can always find $\mathcal{E}$ such that there is a finite distance between any element $\underline{b}$ of B on one side, and any element $\underline{a}$ of A on the other. If A and B are convex sets, then a linear function $\mathcal{E}$ is sufficient to perform this classification.

If $\mathcal{E}(X)$ is a linear function, clearly it is possible to find a linear transformation

$$X = (x_1, x_2, \ldots, x_n) \rightarrow U = (u_1, u_2, \ldots, u_n) \qquad 2.10$$

such that for some finite ξ, the linear function $\mathcal{F}(U, \xi)$ is equal to $\mathcal{E}$ and therefore logically equivalent. The transformation is as

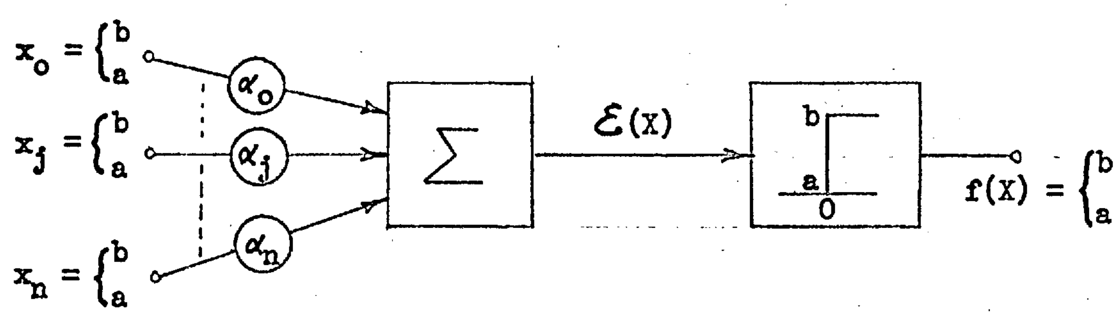

Fig. 2.3 A linear Threshold Gate

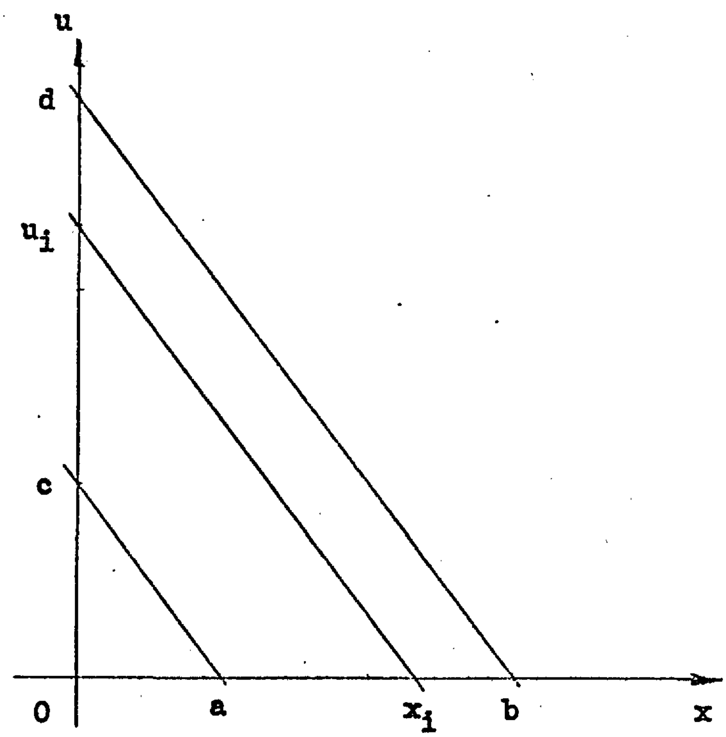

Fig. 2.4 A linear Transformation

follows:

Consider the linear transformation of Fig. 2.4:

$$\frac{x_i - a}{b - a} = \frac{u_i - c}{d - c}, \quad d \neq c \;. \qquad 2.11$$

Then,

$$\mathcal{E}(X,\epsilon) = \epsilon + \sum_{i=1}^{n} \alpha_i x_i = \epsilon + \sum_{i=1}^{n} \alpha_i \left[\frac{b-a}{d-c}(u_i - c) + a\right] . \qquad 2.12$$

Let

$$\frac{b - a}{d - c} = \eta \,. \qquad 2.13$$

Then

$$\mathcal{E}(X,\epsilon) = \epsilon + \sum_{i=1}^{n} \alpha_i (a - \eta c) + \sum_{i=1}^{n} \alpha_i \eta u_i \;. \qquad 2.14$$

Define:

$$\omega_i = \eta \alpha_i \,, \quad 1 \leq i \leq n \,, \qquad 2.15$$

$$\xi = \epsilon + \sum_{i=1}^{n} \alpha_i (a - \eta c) \;. \qquad 2.16$$

Then

$$\mathcal{E}(X,\epsilon) = \xi + \sum_{i=1}^{n} \omega_i u_i = \mathcal{F}(U,\xi) \qquad 2.17$$

is a linear function.

The linear threshold function can thus be normalized by a linear transformation. One such normalization is to give the variables x_i the algebraic values 0 and 1, $1 \leq i \leq n$, and to transform the values ϵ, θ such that

$$f(X) = \begin{cases} 1, & \sum_{i=1}^{n} \alpha_i x_i \geqslant T \\ 0, & \sum_{i=1}^{n} \alpha_i x_i \leqslant T - 1 \end{cases} \qquad 2.18$$

where T is a threshold and the term -1 is introduced to allow for a region of ambiguity in physical devices. A second normalization is

$$x_i \in \{-1, 1\}, \quad 0 \leqslant i \leqslant n . \qquad 2.19$$

2.3 DETERMINATION OF LINEAR THRESHOLD FUNCTIONS FROM LOGICAL FUNCTIONS

If a threshold gate is to be used in a logical machine it is essential to have an algorithm for calculating the constants α_i, $0 \leqslant i \leqslant n$ from the given Boolean function. Clearly for n binary inputs, 2^n possible input vectors $X = (x_0, x_1, \ldots, x_n)$, $x_0 = 1$ are possible. Clearly also, if each vector X is to be classified into one of two sets, A and B, there are exactly 2^{2^n} possible rules for performing the classification. The function f(X) specifies this classification rule.

Consider a normalized three-input threshold gate as in Fig. 2.3. In this case

$$x_i \in \{0,1\}, \qquad 2.20$$

and

$$\alpha_2 = \alpha_3 = 1 . \qquad 2.21$$

Thus, if $\alpha_1 = 1$, f(X) corresponds to the Boolean function

$$f(X) = x_2 + x_3 \ , \tag{2.22}$$

and if $\alpha_1 = 0$, $f(X)$ corresponds to

$$f(X) = x_2 x_3 . \tag{2.23}$$

If $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = -2$, $f(X)$ corresponds to the NOR function

$$f(X) = \overline{x_2 + x_3} \ . \tag{2.24}$$

Thus linear threshold functions can be used to make AND, OR and NOT (one-input NOR) gates. Therefore we can state

<u>Theorem 2.1</u>: Any Boolean function can be realized using one or more linear threshold functions.

The above famous result[8] is the basis for a large amount of work concerning the realization of given Boolean functions using linear threshold gates. Usually it is desired to use the minimum number of gates by allowing more inputs to each gate. Unfortunately, all the possible functions of $n > 1$ variables cannot be realized with one n-variable linear threshold function. A complete set of necessary and sufficient conditions under which an arbitrary Boolean function can be realized with one linear threshold function is not known in general. The test for separability (linear or nonlinear) therefore is usually to go through all or part of a design procedure[4,6] or "training" procedure[5] which is guaranteed to produce a zero-error solution if one exists.

2.3.1 DESIGN PROCEDURES

If $x_i \in \{-1,1\}$, $0 \leq i \leq n$, and a Boolean function is specified by a truth table in which the "don't cares" are given the value 0, one design prodecure[4] for the function

$$\mathcal{E} = \sum_{k=0}^{2^n-1} b_k r_k \qquad 2.25$$

is to compute the values b_k by considering the desired (correct) outputs $f^{(j)} \in \{-1,1\}$, $0 \leq j \leq 2^n - 1$, for all 2^n input combinations as follows:

$$b_k = \frac{1}{2^n}\sum_{j=0}^{2^n-1} f^{(j)} r_k^{(j)}, \quad 0 \leq k \leq 2^n - 1 \qquad 2.26$$

where the r_k are

$$\begin{aligned} r_o &= x_o = 1 \\ r_1 &= x_1 \\ &\vdots \\ r_n &= x_n \\ r_{n+1} &= x_1 x_2 \\ &\vdots \\ r_{2^n-1} &= x_1 x_2 \ldots x_n \,. \end{aligned} \qquad 2.27$$

Of course the product terms x_1x_2 and the like make the function $\mathcal{E}(X)$ nonlinear in general. This procedure always results in values of b_k which give the correct logical function of the variables

r_k provided all terms to $k = 2^n - 1$ are used. If, on the other hand, only the first $n + 1$ terms are allowed, a linear function results, but it does not necessarily give the correct desired function. Two procedures[6] are possible:

1. The weights α_i, $1 \leq i \leq n$ of the correct function are approximated by an equation of the form $p\alpha_i - q\alpha_i^3 = b_i$ where p and q are constants. Other forms are also possible. The weight α_0 and some of the others may have to be adjusted by further approximations or trial-and-error techniques.

2. The absolute values of the coefficients b_k are arranged in descending order. The result is the $n+1$ element characteristic[6] vector which has been tabulated for low orders of n. The correct values of the α_i's are then obtained from the tables by permuting or negating them to make them correspond in sign and magnitude to the sequence of b's.

The first of the above procedures yields approximate values of the α_i, and may not produce the required function for large numbers of arguments. Tables of characteristic vectors and the corresponding weights for threshold functions of more than seven variables are not available since the number of characteristic vectors increases very rapidly with the number of inputs n.

2.3.2 ERROR-CORRECTION PROCEDURES

Nearly all error-correction or "training" procedures for realizing threshold functions are based on the Rosenblatt[5] training procedure for linear devices. An n-degree polynomial function can be converted to a $2^n - 1$ variable linear function as in 2.27 to allow the use of the procedure for linear functions.

Let the threshold function output be

$$f(X) = \begin{cases} 1 & \text{if } \sum_{i=0}^{n} \alpha_i x_i \geq \theta \\ 0 & \text{if } \sum_{i=0}^{n} \alpha_i x_i \leq -\theta \end{cases} \qquad 2.28$$

for some finite $\theta \geq 0$. Let $\tau = \{X_1, X_2, \ldots\}$ be any sequence of vectors chosen from the set of all possible input vectors, and let $\{f^{(1)}, f^{(2)}, \ldots\}$ be the corresponding sequence of the values of the required binary function. We consider the input vector $X_j = (x_{oj}, x_{1j}, \ldots, x_{nj})$, the weight vector $A_j = (\alpha_{oj}, \alpha_{1j}, \ldots, \alpha_{nj})$ and their scalar product (which is the linear function 2.25)

$$(A_j, X_j) = \sum_{i=0}^{n} \alpha_{ij} x_{ij} \, . \qquad 2.29$$

In the literature of adaptive networks τ is referred to as the training sequence of input vectors. Then the sequence of weight vectors A_j, $j = 1, 2, \ldots$ is chosen as follows:

1. A_o is an arbitrary finite vector.
2. $$A_j = \begin{cases} A_{j-1} + X_j & \text{if } (A_{j-1}, X_j) \leq \theta \text{ and } f^{(j)} = 1 \\ A_{j-1} & \text{if } (A_{j-1}, X_j) > \theta \text{ and } f^{(j)} = 1 \\ A_{j-1} - X_j & \text{if } (A_{j-1}, X_j) \geq -\theta \text{ and } f^{(j)} = 0 \\ A_{j-1} & \text{if } (A_{j-1}, X_j) < -\theta \text{ and } f^{(j)} = 0 \end{cases} \qquad 2.30$$

Then, provided there is a linear function which will compute the desired logical function f without error,

> Theorem 2.2: The sequence $A_o, A_1, A_2, \ldots$ converges. There is an integer N (depending on f, A_o, and θ) such that $A_N = A_{N+1} = \ldots.$ If τ has the property that every possible input vector occurs an infinite number of times, then A_N is a solution for computing the function f.

This theorem due to Rosenblatt is often modified[5] by the absolute correction procedure (and others), which requires that for each input vector X_j, the weight vector A_j is changed by the factor

$$A_j = A_{j-1} \pm \lambda X_j \qquad 2.31$$

where $\lambda > 0$ is large enough to correct the functional error for that input vector.

2.4 FUNCTIONAL BEHAVIOR OF THE DYNAMIC MODEL

In this section the dynamic model introduced in section 1.6 will be related to the static functional systems of the first part of Chapter 2. From the operational model of Fig. 1.12a it is evident that provided y_i changes rapidly enough from a value sufficiently "high" to a value sufficiently "low," then the output x_i will be "near" the ideal response of Fig. 1.11. The essential difficulty is to define the permitted deviation from a perfect step, and to relate this to the parameters of the model.

Definition 2.5: A linear threshold function realizes a given logical function if the output of the threshold function is identical to the logical function for every possible input.

This is the usual definition of logical realization. We shall use in addition

Definition 2.6: A physical device with n-element input vector $X(t)$ and output x_i realizes a logical function f in the interval (t_o, t_1) if there exist two values, a and b, such that for $t_o < t < t_1$, the input vector is mapped onto one of these two values: $\xi = f(X(t)) \in \{a,b\}$ by the threshold function, and if, in addition, the inequality

$$|x_i(t) - \xi| < \mu \qquad 2.32$$

holds for μ arbitrarily small.

We recall that the state of the dynamical model specified in section 1.6.1 is given by equation 1.21, repeated here:

$$(\tau_i \frac{d}{dt} + 1)\beta_i \psi(x_i) = \epsilon_i + \sum_{j=1}^{n} \alpha_{ij} x_j \qquad 1.21$$

where τ_i and β_i are finite positive constants. We now state

Theorem 2.3: For any t_o and arbitrarily small $\mu > 0$, the numbers $\delta(\mu) > 0$ and T exist, such that equation 1.21 realizes a linear threshold function in the interval $(t_o + T, \infty)$ provided the inequality

$$|\epsilon_i + \sum_{j=1}^{n} \alpha_{ij} x_j| > \delta \qquad 2.33$$

holds for all $t > t_o$. Furthermore, the linear threshold function is specified by the constants α_{ij}, $j = 1, 2, \ldots, n$ and the value ϵ_i.

Proof: From 1.21 and 2.33,

$$\tau_i \beta_i \frac{dy_i}{dt} + \beta_i y_i = \epsilon_i + \sum_j \alpha_{ij} x_j \triangleq g_i(t)\delta, \qquad 2.34$$

where $y_i(t) = \psi(x_i(t))$ and $|g_i(t)| > 1$. We omit the subscripts for simplicity and rewrite this equation as follows:

$$\frac{dy}{dt} + \frac{1}{\tau} y = \frac{\delta}{\beta\tau} g(t) \ . \qquad 2.35$$

The well-known[9] solution of 2.35 is

$$y(t) = y(t_o) e^{-\frac{t-t_o}{\tau}} + \frac{\delta}{\beta} e^{-\frac{t-t_o}{\tau}} \int_{t_o}^{t} \frac{g(s)}{\tau} e^{\frac{s-t_o}{\tau}} \, ds \ . \qquad 2.36$$

Assume for the moment that $g(t) > 1$. Then

$$\int_{t_o}^{t} \frac{g(s)}{\tau} e^{\frac{s-t_o}{\tau}} \, ds > \int_{t_o}^{t} \frac{1}{\tau} e^{\frac{s-t_o}{\tau}} \, ds = e^{\frac{t-t_o}{\tau}} - 1 \qquad 2.37$$

in which case

$$\psi(x(t)) = y(t) > \left[y(t_o) - \frac{\delta}{\beta} \right] e^{-\frac{t-t_o}{\tau}} + \frac{\delta}{\beta} \ . \qquad 2.38$$

$\psi(s)$ and $\psi^{-1}(s)$ are positive monotonic functions, and furthermore the relation

$$\psi^{-1}(s) = 1 - \psi^{-1}(-s) \qquad 2.39$$

holds, since ψ^{-1} is a symmetric function. We may rewrite 2.38 as follows:

$$x(t) > 1 - \psi^{-1}\left\{\left[\frac{\delta}{\beta} - y(t_o)\right] e^{-\frac{t-t_o}{\tau}} - \frac{\delta}{\beta}\right\}, \qquad 2.40$$

and thus

$$1 - x(t) < \psi^{-1}\left\{\left[\frac{\delta}{\beta} - y(t_o)\right] e^{-\frac{t-t_o}{\tau}} - \frac{\delta}{\beta}\right\}. \qquad 2.41$$

We now choose δ to satisfy the inequality

$$\psi^{-1}\left\{-\frac{\delta}{\beta}\right\} < \mu, \qquad 2.42$$

in which case there must exist a $T > 0$ such that

$$1 - x(T) < \psi^{-1}\left\{\left[\frac{\delta}{\beta} - y(t_o)\right] e^{-\frac{T-t_o}{\tau}} - \frac{\delta}{\beta}\right\} \leq \mu. \qquad 2.43$$

In a similar way, if we had assumed that $g(t) < -1$, we would have, instead of 2.43,

$$x(T) < \psi^{-1}\left\{\left[\frac{\delta}{\beta} + y(t_o)\right] e^{-\frac{T-t_o}{\tau}} - \frac{\delta}{\beta}\right\} \leq \mu, \qquad 2.44$$

for δ satisfying 2.42. If we choose T to be the minimum value which satisfies both 2.43 and 2.44, then inequality 2.33 implies 2.32, with

$$\zeta = \begin{cases} b = 1 & \text{for } \epsilon_i + \sum_{j=1}^{n} \alpha_{ij} x_j > \delta > 0 \\ a = 0 & \text{for } \epsilon_i + \sum_{j=1}^{n} \alpha_{ij} x_j < -\delta < 0 \end{cases}. \qquad 2.45$$

Equation 2.45 is of the form specified in equations 2.3 and 2.6 of section 2.2 and thus the values of ϵ_i and α_{ij} specify a linear threshold function. This completes the proof of the theorem.

We make the following remarks about the above theorem:

1. The number δ specifies what is usually called a gap, or region of uncertainty, and a threshold gate may only be satisfactory in the time intervals in which the magnitude of the sum of the inputs exceeds this value, as in inequality 2.33. In addition the dynamic model we use requires that inequality 2.42 also be satisfied, and in any other physically-realizable model a criterion of the nature of 2.42 must be satisfied, because of the imperfections of real threshold elements such as relays, etc. The parameter β_i is related to the slope of the sigmoid function and thus is a measure of goodness of a given function. It is clear that for any value of δ we can satisfy 2.42 by changing β_i, that is by improving the device, and we can satisfy 2.33 by varying the values of α_{ij} and ϵ_i. It is obvious that since 2.33 is an inequality, the parameters of any realization are not unique to a given threshold function. The usual convention of specifying these parameters is to use the smallest integers which satisfy 2.18, in which case, $\delta = \frac{1}{2}$.

2. If the feedback resistor R_f in Fig. 1.13 is removed, the time-constant τ becomes infinite, and (again assuming a perfect amplifier) the transfer-function becomes a pure integration. Of course this situation is never achieved completely in practice, but corresponds, in the mathematical model, to the limit as β_i approaches zero with the product $\beta_i \tau_i$ remaining finite. The limiting equation is

$$\beta_i \tau_i \frac{d}{dt} \psi(x_i) = \epsilon_i + \sum_j \alpha_{ij} x_j \qquad 2.46$$

which is identical in form to the Cowan equation 1.18, provided we specify ψ:

$$\psi(x_i) \equiv \log \frac{x_i}{1 - x_i} \quad .$$

This special case is the basis of the following

Corollary 2.1: If equation 2.46 replaces 1.21, then Theorem 2.3 is valid for arbitrarily small $\delta > 0$, provided inequality 2.33 holds.

Proof: With $\beta_i = 0$ and $\delta > 0$, inequality 2.42 becomes

$$\psi^{-1}\{-\infty\} = 0 < \mu \qquad 2.47$$

which is always true for $\mu > 0$. Thus 2.43 and 2.44 are always true for some T and the theorem holds.

3. If a maximum input is specified as well as the minimum, then the two values a and b need not be taken as 0 and 1. In such a case we can specify $0 < a < b < 1$ as follows*:

Corollary 2.2: For any t_0 and arbitrarily small $\mu > 0$, the numbers $\delta_1(\mu) > \delta(\mu) > 0$ and T exist, such that if condition 2.33 is replaced by

$$\delta_1 > \left|\epsilon_i + \sum_{j=1}^{n} \alpha_{ij} x_j\right| > \delta \qquad 2.48$$

then Theorem 2.3 holds for numbers a and b, with $0 < a < b < 1$, such that 2.45 is replaced by

* see Fig. 2.5

$$\zeta = \begin{cases} b & \text{for } \delta_1 > \epsilon_i + \sum_j \alpha_{ij} x_j > \delta > 0 \\ a & \text{for } -\delta_1 < \epsilon_i + \sum_j \alpha_{ij} x_j < -\delta < 0 . \end{cases} \qquad 2.49$$

<u>Proof</u>: The argument is similar to the proof of the theorem, except as follows: For positive inputs, we have, instead of 2.41,

$$\psi^{-1}\left\{\left[y(t_o) - \frac{\delta_1}{\beta}\right] e^{-\frac{t-t_o}{\tau}} + \frac{\delta_1}{\beta}\right\} > x(t) > \psi^{-1}\left\{\left[y(t_o) - \frac{\delta}{\beta}\right] e^{-\frac{t-t_o}{\tau}} + \frac{\delta}{\beta}\right\}, \quad 2.50$$

and for negative inputs,

$$\psi^{-1}\left\{\left[y(t_o) + \frac{\delta_1}{\beta}\right] e^{-\frac{t-t_o}{\tau}} - \frac{\delta_1}{\beta}\right\} < x(t) < \psi^{-1}\left\{\left[y(t_o) + \frac{\delta}{\beta}\right] e^{-\frac{t-t_o}{\tau}} - \frac{\delta}{\beta}\right\}. \quad 2.51$$

We therefore choose a number b to satisfy the inequalities

$$\begin{aligned} & b - \psi^{-1}\left\{\frac{\delta}{\beta}\right\} < \mu \\ & \psi^{-1}\left\{\frac{\delta_1}{\beta}\right\} - b < \mu \end{aligned} \qquad 2.52$$

and a to satisfy

$$\begin{aligned} & a - \psi^{-1}\left\{-\frac{\delta_1}{\beta}\right\} < \mu \\ & \psi^{-1}\left\{-\frac{\delta}{\beta}\right\} - a < \mu . \end{aligned} \qquad 2.53$$

We then choose T to be the smallest value which, for any permitted initial condition, satisfies both inequalities

$$\begin{aligned} & |x_i(T) - b| \leq \mu \\ & |x_i(T) - a| \leq \mu \end{aligned} \qquad 2.54$$

and the proof is complete, except for the observation that inequalities 2.52 and 2.53 require that $0 < a < b < 1$.

It is clear from the above proof that by allowing the values of a and b to be within the interval $(0,1)$ the minimum input may be smaller than that for the theorem, for given convergence time T. Specifying a maximum input is not usually a severe restriction.

4. We remark that other definitions of realization are possible. For example, we could specify that the functional output be outside the region $[a,b]$ for some time interval, instead of converging to $\xi \in \{a,b\}$. It is expected, however, that definition 2.8 has the greatest practical significance.

5. The initial condition $x_i(t_o)$ must always be in the interval $(0,1)$ and thus $y_i(t_o) = \psi(x_i(t_o))$ must always be finite. For equation 2.46, however, as t becomes large, $y(t)$ becomes large, and a change of input will require a large time T to take effect. This is an unrealizable situation as mentioned previously, but is very interesting because it corresponds to a gate with an infinite memory. In practice, the value of β_i is finite and the memory of any gate is finite. Equation 1.21 has a maximum convergence time as well as a minimum, provided the range of initial conditions is limited. It is reasonable to limit the initial conditions to those values obtainable from the allowed inputs.

Corollary 2.3: If the conditions of Theorem 2.3 and Corollary 2.2 are satisfied, then there exists a number $T_M(\delta_1, \mu)$ such that $T < T_M$ provided the inequality

$$\psi^{-1}\left\{-\frac{\delta_1}{\beta}\right\} < x(t_o) < \psi^{-1}\left\{\frac{\delta_1}{\beta}\right\} \tag{2.55}$$

holds.

Rather than prove this corollary, we shall point out that it is obvious that the longest convergence time occurs either when the minimum positive input follows a maximum negative input, or vice versa. In such a case,

$$y(t_o) = \pm\frac{\delta_1}{\beta}, \tag{2.56}$$

and T_M is the minimum value for which the following inequalities are simultaneously satisfied:

$$\begin{aligned} b - \psi^{-1}\left\{\left[-\frac{\delta_1}{\beta} - \frac{\delta}{\beta}\right] e^{-\frac{T_M - t_o}{\tau}} + \frac{\delta}{\beta}\right\} &\leq \mu \\ \psi^{-1}\left\{\left[\frac{\delta_1}{\beta} + \frac{\delta}{\beta}\right] e^{-\frac{T_M - t_o}{\tau}} - \frac{\delta}{\beta}\right\} - a &\leq \mu . \end{aligned} \tag{2.57}$$

6. The final remark is that the symmetry of the function $\psi^{-1}(\ .\ )$ allows the gap to be symmetrical and 2.48 to apply, but that it is conceivable that in certain situations an asymmetrical gap might be required. In such a situation the above theorem and corollaries would require changes in detail but not in form.

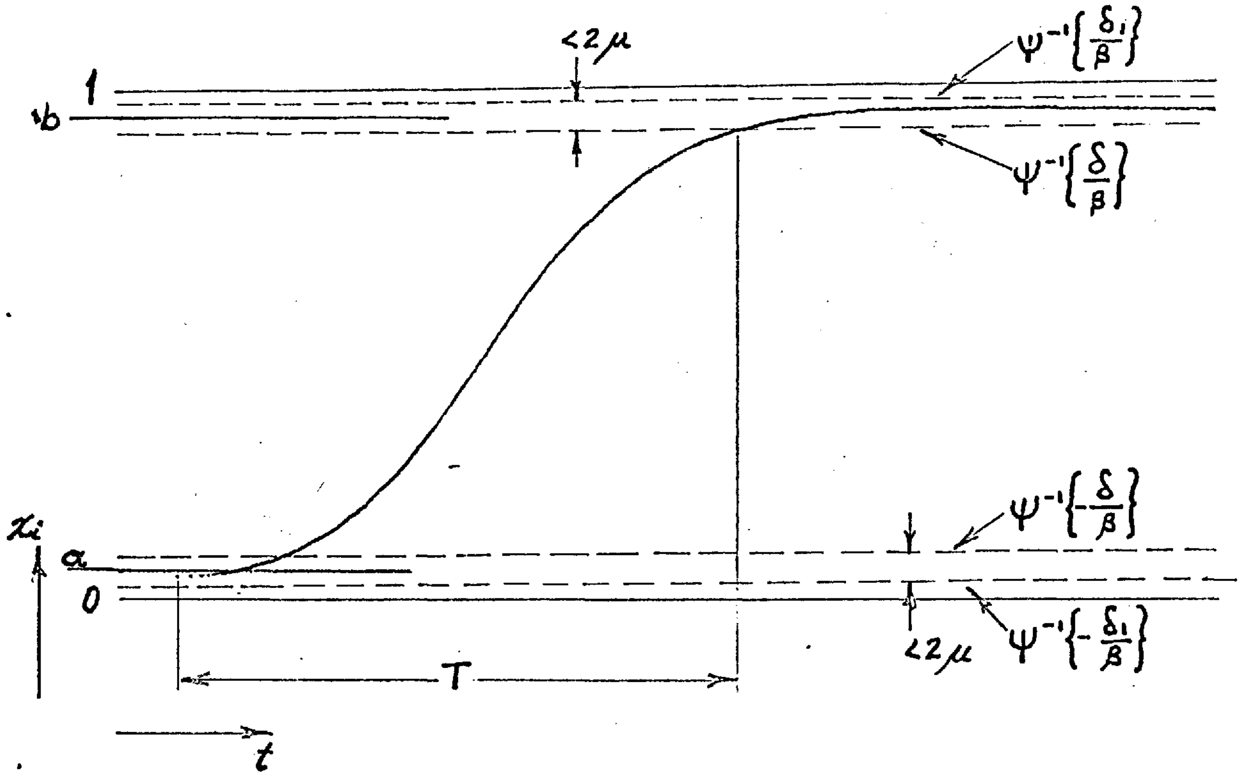

Fig. 2.5 Element Transient Response

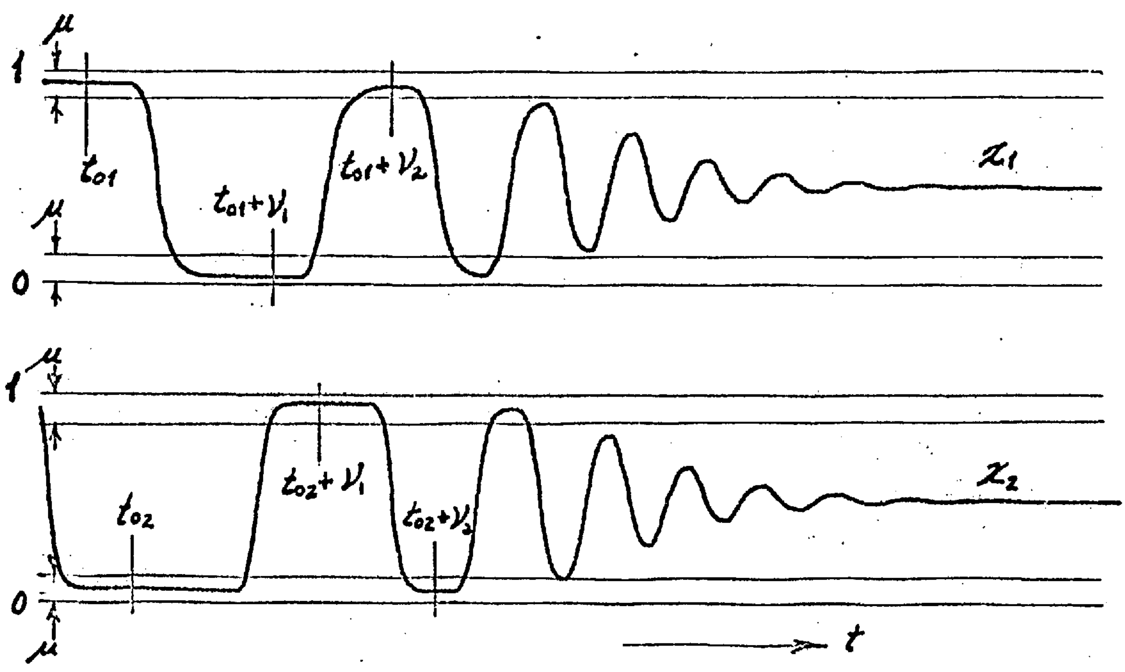

Fig. 2.6 A 2-Element 3-Discrete System

2.5 TIME-DISCRETE THRESHOLD NETWORKS

The development to this point has dealt exclusively with a differential equation model of a threshold gate. Simulation and implementation of such gates often requires a time-discrete representation of their dynamical behavior. We consider here the transition from continuous to discrete systems and the validity of a discrete-time functional representation of the dynamic model.

Assume that the inputs ϵ_i and x_j, $j = 1, 2, \ldots, n$ to gate i are continuous and therefore that $x_i(t)$ is continuously differentiable. We write

$$\frac{d\,\psi(x_i(t))}{dt} = \psi'(x_i)\,\frac{dx_i}{dt} = \lim_{\nu \to 0} \frac{\psi(x_i(t+\nu)) - \psi(x_i(t))}{\nu} \qquad 2.58$$

which depends on the above assumption and on the assumed properties of $\psi(x_i)$. Then for some small ν we may write the approximation

$$\frac{d\psi}{dt} \cong \frac{\psi(x_i(t+\nu)) - \psi(x_i(t))}{\nu} \qquad 2.59$$

and combining this with equation 1.21 and rearranging, we have

$$x_i(t + \nu) \simeq \psi^{-1}\left\{(1 - \frac{\nu}{\tau_i})\,\psi(x_i(t)) + \frac{\nu}{\beta_i \tau_i}(\epsilon_i + \sum_j \alpha_{ij} x_j(t))\right\} . \qquad 2.60$$

This trivial approximation will always have arbitrarily small error provided ν may be made arbitrarily small. Keeping ν finite and allowing β_i to approach zero, 2.60 becomes

$$x_i(t + \nu) \simeq U\left\{\epsilon_i + \sum_j \alpha_{ij} x_j(t)\right\} \qquad 2.61$$

where $U\{\,.\,\}$ is the step function. This is nontrivial and may not hold for <u>any</u> ν, but there are cases where x_i switches quickly from near 0 to near 1, and 2.61 is valid, even though the inputs may not be continuous.

<u>Definition 2.7</u>: If, in a system of n gates, we can find numbers t_{oi}, $1 \leq i \leq n$, and a number $\nu > 0$, such that for all i, the relation (see Fig. 2.6)

$$\left| x_i(t_{oi} + \nu) - U\left\{\epsilon_i + \sum_j \alpha_{ij} x_j(t_{oi})\right\}\right| < \mu \qquad 2.62$$

holds, for $\mu > 0$ arbitrarily small, then the system is first-order time-discrete, and may be termed 1-discrete. If there is an integer $m > 0$ and the relation

$$\left| x_i(t_{oi} + \nu_{r+1}) - U\left\{\epsilon_{ir} + \sum_j \alpha_{ij} x_j(t_{oi} + \nu_r)\right\}\right| < \mu \qquad 2.63$$

with $\nu_o = 0$ and $\nu_{r+1} > \nu_r$ holds for all $i = 1, 2, \ldots, n$ and all $r = 1, 2, \ldots, m$, then the system may be termed m-discrete.

Of course the order of discreteness depends on the initial conditions $x_i(t_{oi})$, $i = 1, 2, \ldots, n$.

Provided 2.63 holds we can generalize 2.61 for an m-discrete system:

$$X_{r+1} = U\left\{E_r + AX_r\right\}, \quad r = 0, 1, \ldots, m,$$

and adding the output equation

$$W_r = CX_r \qquad 2.65$$

we have two equations which resemble those for a linear machine[10], except, of course, that the step function in 2.64 is highly nonlinear. This difficulty may be overcome for some purposes for finite values of m by using polynomial interpolation methods.

It may happen that 2.63 holds where U is not a step between 0 and 1 but between two numbers a and b, with

$$U(t) = \begin{cases} b, & t \geq 0 \\ a, & t < 0 \end{cases} \qquad 2.66$$

and $0 < a < b < 1$. With a suitable transformation of coordinates definition 2.7 remains valid. Indeed a and b need not be between 0 and 1 but we shall only deal with this case in what follows.

2.6 SUMMARY

In order to compute a logical function with a real device, the distinction between "logical," and "binary" functions is made. It can be shown that binary functions on the set $\{0,1\}$ are equivalent to arbitrary binary functions, hence continuous functions in this context need only be considered in the interval $(0,1)$. Methods of designing threshold functions result in non-unique sets of functional parameters. Questions of realization of binary functions with the dynamic model can be answered precisely, both in a continuous time-scale and at discrete points in time.

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CHAPTER 3

PRELIMINARY ANALYSIS OF SOME GATE NETWORKS

3.1 INTRODUCTION

The purpose of this chapter is to serve as an introduction to the stability analyses to follow. The uniqueness of the solutions of the gate equations will be shown. Two cases will then be treated: an electrical network analogy, and a system of lossless integrators with perfect switches. The equilibrium or singular solutions of the general network and of the special Cowan equation will be discussed, and a topological (trajectory) analysis of two-unit systems will follow. A special case similar to Volterra's equations[2] of population dynamics is discussed in Appendix C.

3.2 EXISTENCE, UNIQUENESS, AND CONTINUITY

We write the i^{th} equation in a system of n simultaneous ordinary differential equations in the general form:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n) \triangleq f_i(X) \text{ , } i = 1, 2, \ldots, n \text{ .} \quad 3.1$$

Sufficient conditions for the function f_i to satisfy a Lipschitz condition[1] in a domain D are that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ $1 \leq j \leq n$ be bounded and continuous in D.

If the n functions $f_1, f_2, \ldots, f_n$ satisfy Lipschitz conditions in D, the Cauchy-Lipschitz theorem states that each equation 3.1 has a unique solution $x_i = x_i(t)$, $1 \leq i \leq n$, defined in the neighborhood of $t = t_o$ such that $x_i(t_o) = \xi_i$, $1 \leq i \leq n$. The theorem thus guarantees the existence of solutions with prescribed initial conditions and asserts that the initial conditions determine the solution uniquely.

If the n functions $f_i(X)$ satisfy Lipschitz conditions, then the solutions $x_i(t, t_o, \xi_1, \ldots, \xi_n)$ are analytic in all of their arguments in a neighborhood of t_o, ξ_i, $1 \leqslant i \leqslant n$.

The integral curves of 3.1 for specified functions f_i define a family of trajectories in the n-dimensional X-space, and the theorem applied to this autonomous system has that through every point in the X-space there passes one and only one integral curve.

In chapter 1 we have written the gate equation in the form of 3.1:

$$\frac{dx_i}{dt} = f_i(X) = \frac{1}{\beta_i \tau_i \psi'(x_i)}\left[\epsilon_i - \beta_i \psi(x_i) + \sum_j \alpha_{ij} x_j\right] . \qquad 1.28$$

The derivative

$$\frac{\partial f_i}{\partial x_i} = \frac{\alpha_{ii}}{\beta_i \tau_i \psi'(x_i)} - \frac{1}{\tau_i} - \frac{\psi''(x_i)}{\psi'(x_i)} f_i \qquad 3.2$$

is continuous and bounded since ψ is analytic and strictly monotonic. The other derivatives

$$\frac{\partial f_i}{\partial x_j} = \frac{\alpha_{ij}}{\beta_i \tau_i \psi'(x_i)} \quad i \neq j \qquad 3.3$$

also satisfy these conditions.

Thus the model we have chosen is well-behaved in the usual sense.

3.3 A NETWORK ANALOGUE OF A SPECIAL CASE

A model will now be derived which, under certain conditions, is described by differential equations identical to the special equation 1.18.

Consider the nonlinear passive conservative inductor of Fig. 3.1a. The current-flux linkage characteristic is

$$\frac{\lambda - \lambda_o}{\lambda_S} = \log \frac{I}{I_S - I} = \log \frac{I_o + I_L}{I_S - I_o - I_L} \,. \qquad 3.4$$

By definition[3], the magnetic energy function of this element is

$$W_m = \int_{\lambda^*}^{\lambda} I_L(u)du \qquad 3.5$$

where

$$I_L(\lambda^*)\lambda^* = 0 \,. \qquad 3.6$$

The inverse of 3.4 is

$$I_L = \frac{I_S}{1 + \exp\left[\frac{-(\lambda - \lambda_o)}{\lambda_S}\right]} - I_o \qquad 3.7$$

and therefore the initial point of integration λ^* must satisfy the equation

$$\left\{I_S\left\{1 + \exp\left[\frac{-(\lambda^* - \lambda_o)}{\lambda_S}\right]\right\}^{-1} - I_o\right\}\lambda^* = 0 \,. \qquad 3.8$$

The two solutions of 3.8 are both $\lambda^* = 0$ as might be expected. Thus the energy function is

$$W_m = \int_0^{\lambda}\left\{\frac{I_S}{1 + \exp\left[\frac{-(u - \lambda_o)}{\lambda_S}\right]} - I_o\right\} du$$

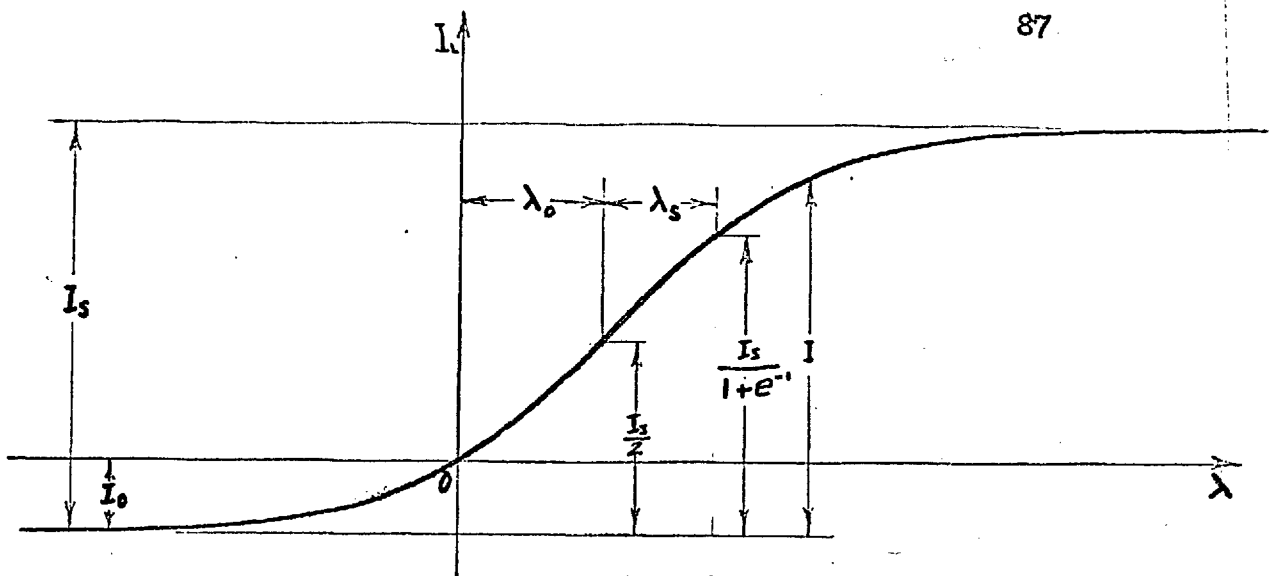


Fig. 3.1a Nonlinear Inductor

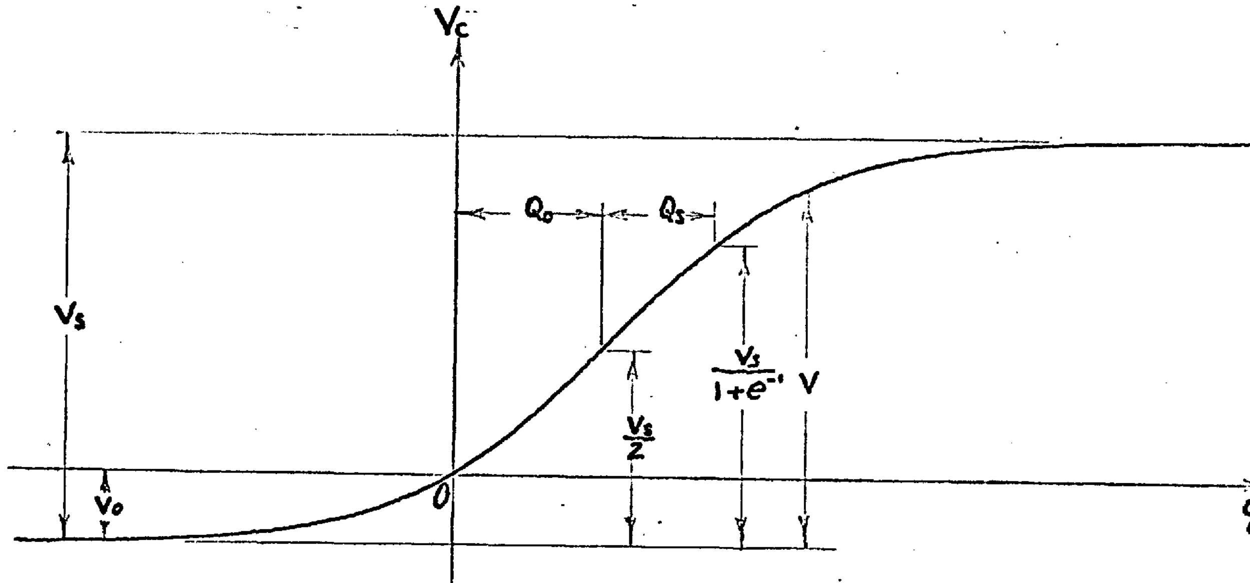


Fig. 3.1b Nonlinear Capacitor

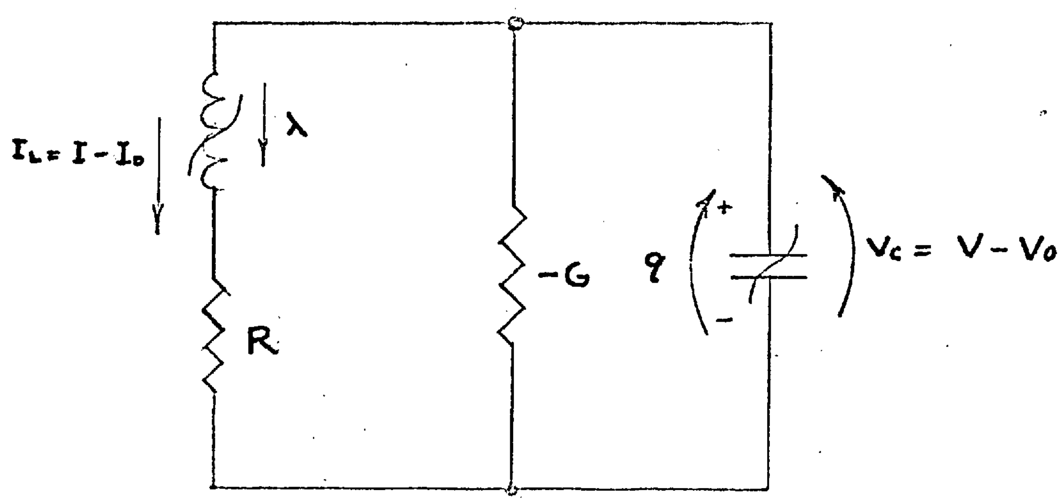


Fig. 3.2 Equivalent Circuit

$$= \lambda_S I_S \log \frac{I_S - I_o}{I_S - I} + \lambda_S I_o \log \frac{I_o(I_S - I)}{(I_S - I_o)I} \,. \qquad 3.9$$

Finally, the voltage across the terminals of this inductor is

$$V_L = \dot{\lambda} = \frac{\lambda_S \dot{I}_S}{I(I_S - I)} \,. \qquad 3.10$$

By a similar process, the energy function of the nonlinear capacitor of Fig. 3.1b is

$$W_E = Q_S V_S \log \frac{V_S - V_o}{V_S - V} + Q_S V_o \log \frac{V_o(V_S - V)}{(V_S - V_o)V} \,, \qquad 3.11$$

and the current flowing into the capacitor is

$$I_C = \dot{q} = \frac{Q_S V_S \dot{V}}{V(V_S - V)} \,. \qquad 3.12$$

These two elements are arranged in a circuit with a conductance G and a resistance R, as in Fig. 3.2. From Kirchoff's laws,

$$I - I_o + G(V - V_o) + \frac{Q_S V_S \dot{V}}{V(V_S - V)} = 0 \,, \qquad 3.13$$

$$(I - I_o)R + \frac{\lambda_S I_S \dot{I}}{I(I_S - I)} = V - V_o \,. \qquad 3.14$$

Thus, from 3.13

$$\frac{\dot{V}}{V_S} = \frac{V}{V_S}\left[1 - \frac{V}{V_S}\right]\left[\frac{I_o + GV_o}{Q_S} - \frac{GV_S}{Q_S}\frac{V}{V_S} - \frac{I_S}{Q_S}\frac{I}{I_S}\right] , \qquad 3.15$$

and from 3.14

$$\frac{\dot{I}}{I_S} = \frac{I}{I_S}\left[1 - \frac{I}{I_S}\right]\left[\frac{I_o R - V_o}{\lambda_S} + \frac{V_S}{\lambda_S}\frac{V}{V_S} - \frac{RI_S}{\lambda_S}\frac{I}{I_S}\right] . \qquad 3.16$$

These equations, by substituting x_1 for $\frac{V}{V_S}$ and x_2 for $\frac{I}{I_S}$

and equating the corresponding constants, are made identical to the following two-dimensional system:

$$\dot{x}_1 = x_1(1 - x_1)(\epsilon_1 + \alpha_{11}x_1 + \alpha_{12}x_2)$$
$$\dot{x}_2 = x_2(1 - x_2)(\epsilon_2 + \alpha_{21}x_1 + \alpha_{22}x_2) \qquad 3.17$$

which is a particular set of equations obtained from 1.18.

Since there is an analogy between 3.17 and an electrical network, we expect that the behavior of 3.17 will be similar to such a network. In particular we can define a quantity for 3.17 analogous to the stored energy in the network of Fig. 3.2, but first a brief digression will be made to show a property of trajectories in the X-space of equation 1.18, and thus in the x_1-x_2 plane of equations 3.17, which will be written as follows:

$$\dot{Y} = E + AX . \qquad 3.18$$

Let Γ be the solution of the equation

$$E + AX = 0 . \qquad 3.19$$

Then,

$$\Gamma = - A^{-1}E \qquad 3.20$$

and thus from 3.18

$$X(t) = A^{-1}\dot{Y} - A^{-1}E = A^{-1}\frac{dY}{dt} + \Gamma . \qquad 3.21$$

We wish to find the mean value of $X(t)$, which comes directly from 3.21:

$$\bar{X} = \lim_{a\to\infty} \frac{1}{a}\int_0^a (A^{-1}\frac{dY}{dt} + \Gamma)dt = \Gamma + A^{-1}\lim_{a\to\infty}\frac{1}{a}\left[Y(a) - Y(0)\right] \quad . \quad 3.22$$

For a finite solution to exist, $Y(a)$ must be finite for all values of a and the second term in 3.22 is zero. Thus if the solution of 3.17 is finite the mean value of the solution vector is Γ. Of course if the solution vector $X(t)$ ever coincides with Γ its time-derivative is zero and thus an oscillatory solution never takes on the value Γ in finite time.

We write 3.20 explicitly for the two-dimensional case:

$$\Gamma = [\gamma_i] = -A^{-1}E = \frac{1}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\begin{bmatrix}\epsilon_1\alpha_{22} - \epsilon_2\alpha_{12} \\ \epsilon_1\alpha_{21} - \epsilon_2\alpha_{11}\end{bmatrix} \quad 3.23$$

and, comparing 3.17 with 3.15 and 3.16 and using 3.23, we can write

$$V = x_1 V_S \quad 3.24a \qquad I = x_2 I_S \quad 3.25a$$

$$Q_S = -\frac{I_S}{\alpha_{12}} \quad 3.24b \qquad \lambda_S = \frac{V_S}{\alpha_{21}} \quad 3.25b$$

$$G = \frac{\alpha_{11} I_S}{\alpha_{12} V_S} \quad 3.24c \qquad R = -\frac{\alpha_{22} V_S}{\alpha_{21} I_S} \quad 3.25c$$

$$V_o = \gamma_1 V_S \quad 3.24d \qquad I_o = \gamma_2 I_S \; . \quad 3.25d$$

From these equations and the above discussion we conclude that an oscillation of the electric circuit example must be about the point (V_o, I_o).

Consider the sum of the magnetic and electric energy functions:

$$W = W_m + W_E \ . \tag{3.26}$$

The derivation of W_m and W_E ensures that W is a positive definite function of I_L and V_C, and therefore its zero corresponds to the points I_o and V_o in the $I - V$ plane. We have shown that any oscillation must be about the point (I_o, V_o), and thus a decaying oscillation corresponds to an energy function decaying to zero at the steady-state solution (I_o, V_o).

In terms of the general parameters, the energy function W becomes

$$\frac{W}{I_S V_S} = \frac{1}{-\alpha_{12}\alpha_{21}}\left[-\alpha_{12}\log\left(\frac{\gamma_2}{x_2}\right)^{\gamma_2}\left(\frac{1-\gamma_2}{1-x_2}\right)^{1-\gamma_2} + \alpha_{21}\log\left(\frac{\gamma_1}{x_1}\right)^{\gamma_1}\left(\frac{1-\gamma_1}{1-x_1}\right)^{1-\gamma_1}\right], \tag{3.27}$$

where I_S and V_S are arbitrary positive constants.

3.3.1 CONSTANTS OF MOTION

The development in the preceding section is a particular case of a general problem of characterization of dynamic systems: that of finding constants of motion. Clearly if an expression for a constant of motion can be obtained in the form

$$W(X) = C \tag{3.28}$$

where X is the state vector and C is a constant, then this is an additional constraint on the system, which is said to be conservative, and the solution $X(t)$ is restricted to some region of the phase space. In certain circumstances, then, knowledge of the function $W(X)$ is equivalent to a solution of the state equation.

We remark that the function $W(X)$ in 3.29 is called a Hamiltonian in classical mechanics, and has the units of energy. In other situations, however, such as in the example we have treated, energy can only be found by analogy.

A further consideration arises: the constant of motion $W(X)$ is not unique for any system. In linear mechanics this only involves arbitrary constants, but this is not the case for nonlinear systems. In the example of the previous section the function 3.27 is only constant for $\alpha_{11} = \alpha_{22} = 0$, that is, for the resistors in the circuit model zero. However, it will be conjectured later that this is not a necessary condition for conservative oscillations to exist.

It is well-known[3] that a necessary and sufficient condition for conservative motion is that a constant of motion, sometimes called a first integral, exists, but it is not generally possible to write down such a constant in closed form.

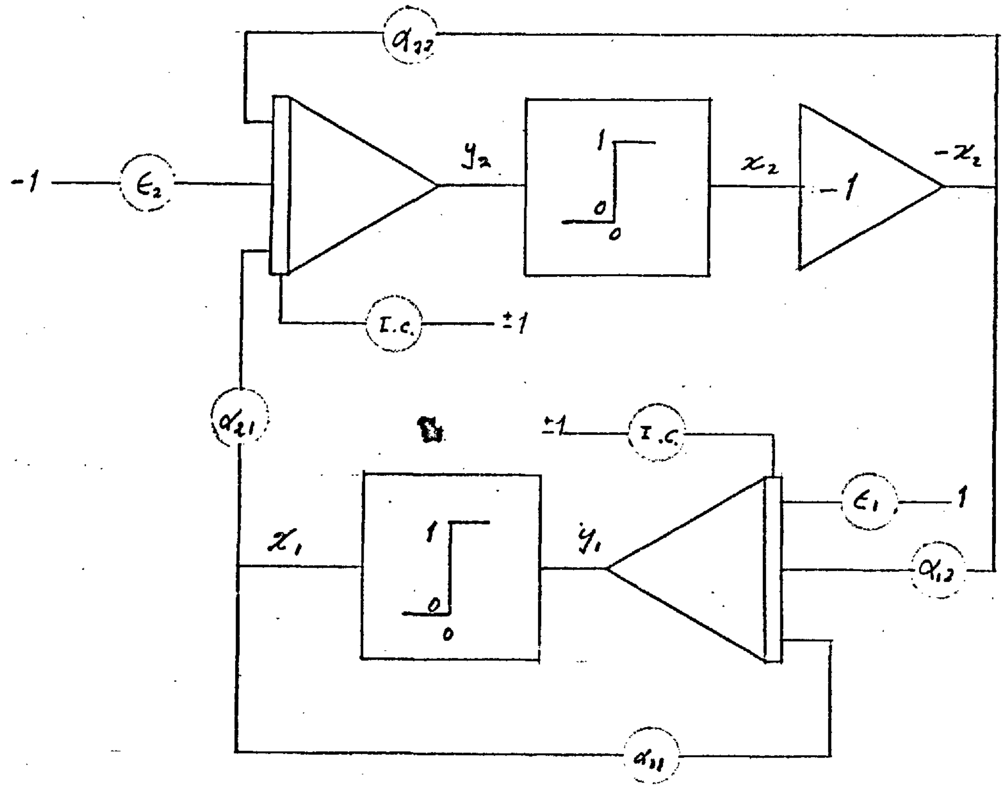


Fig. 3.3 Perfect Switch System

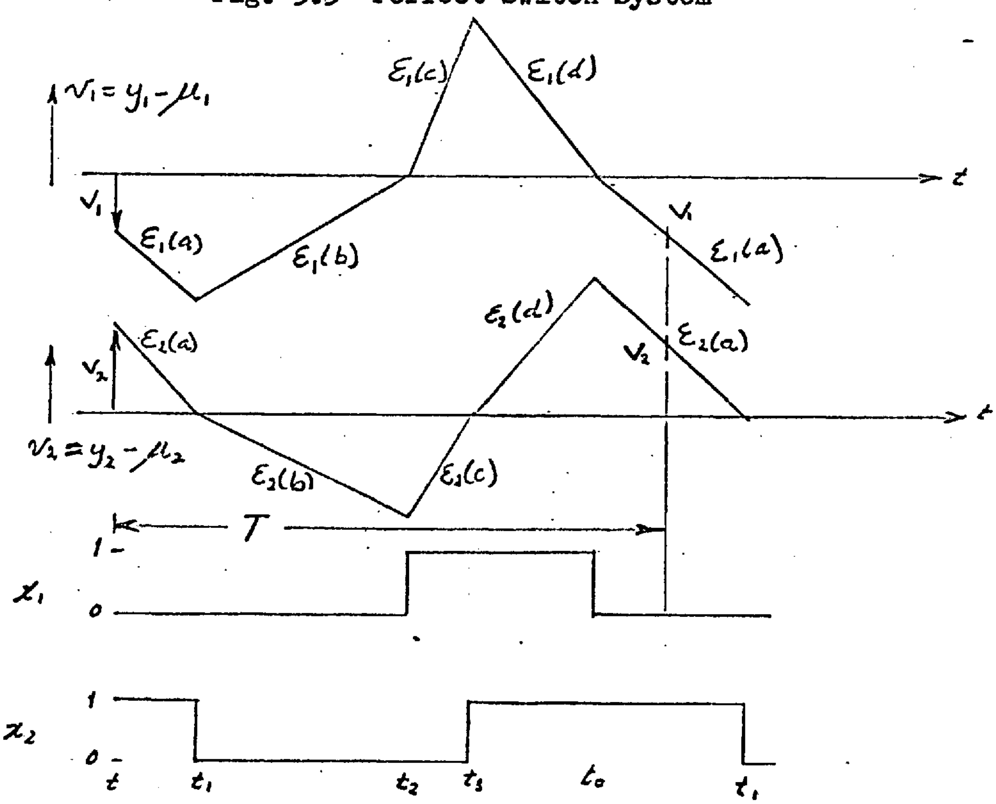


Fig. 3.4 Wave-forms of Fig. 3.3

3.4 THE PERFECT SWITCH

Consider Fig. 3.3. This contains two integrators and two perfect switches connected to solve the equations

$$\dot{y}_1 = \mathcal{E}_1(t) = \epsilon_1 + \alpha_{11}x_1 + \alpha_{12}x_2$$
$$\dot{y}_2 = \mathcal{E}_2(t) = \epsilon_2 + \alpha_{21}x_1 + \alpha_{22}x_2$$
$$x_1 = U(y_1) \qquad 3.28$$
$$x_2 = U(y_2)$$

in which the constants may in general be of either sign. If the constants are chosen so that a continuing oscillation exists, then the period T of this oscillation contains four distinct divisions, as in Fig. 3.4. The initial conditions are labelled V_1 and V_2 as in the diagram. The slopes of the $v - t$ graphs are $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$.

Cyclic behavior of the system admits the following four linear homogeneous equations:

$$\mathcal{E}_1(a)\, T_a + \mathcal{E}_1(b)\, T_b = 0$$
$$\mathcal{E}_1(c)\, T_c + \mathcal{E}_1(d)\, T_d = 0 \qquad 3.29$$
$$\mathcal{E}_2(a)\, T_a + \mathcal{E}_2(d)\, T_d = 0$$
$$\mathcal{E}_2(b)\, T_b + \mathcal{E}_2(c)\, T_c = 0\,.$$

Their equivalent in the continuous (finite switching-time) situation is the set of integral equations

$$\int_{t_0}^{t_2} \mathcal{E}_1(t)dt = 0 , \qquad \int_{t_2}^{t_0} \mathcal{E}_1(t)dt = 0 ,$$
$$\int_{t_1}^{t_3} \mathcal{E}_2(t)dt = 0 , \qquad \int_{t_3}^{t_1} \mathcal{E}_2(t)dt = 0 . \qquad 3.30$$

Equations 3.29 are easily reduced to the constraint

$$\frac{\mathcal{E}_1(a)\,\mathcal{E}_1(c)\,\mathcal{E}_2(b)\,\mathcal{E}_2(d)}{\mathcal{E}_1(b)\,\mathcal{E}_1(d)\,\mathcal{E}_2(a)\,\mathcal{E}_2(c)} = 1 . \qquad 3.31$$

Clearly, given any one time interval it is possible to solve for the other three. Equations 3.29 give the recursive formulae

$$\begin{aligned} T_b &= -\frac{\mathcal{E}_1(a)}{\mathcal{E}_1(b)} T_a \\ T_c &= -\frac{\mathcal{E}_2(b)}{\mathcal{E}_2(c)} T_b \\ T_d &= -\frac{\mathcal{E}_1(c)}{\mathcal{E}_1(d)} T_c \\ T_a &= -\frac{\mathcal{E}_2(d)}{\mathcal{E}_2(a)} T_d , \end{aligned} \qquad 3.32$$

and knowledge of the initial conditions may be used to specify one such interval. In Fig. 3.4 the equation

$$T_a = \frac{V_1(t)}{\mathcal{E}_1(a)} - \frac{V_2(t)}{\mathcal{E}_2(a)} > 0 \qquad 3.33$$

holds, but in interval b we would have

$$T_b = -\left\{\frac{V_1(t)}{\mathcal{E}_1(b)} - \frac{V_2(t)}{\mathcal{E}_2(b)}\right\} > 0 . \qquad 3.34$$

This difference in sign and the form of 3.32 make it necessary to observe the portion of the period in which $V_1(t)$ and $V_2(t)$ are specified before labelling the intervals. Intervals c and d yield results identical in sign to a and b respectively and therefore only two distinct initial configurations need be distinguished.

We carry this example further to the case for which $\alpha_{11} = \alpha_{22} = 0$, in which

$$\mathcal{E}_1(a) = \mathcal{E}_1(d) , \qquad \mathcal{E}_1(b) = \mathcal{E}_1(c) ,$$
$$\mathcal{E}_2(a) = \mathcal{E}_2(b) , \qquad \mathcal{E}_2(c) = \mathcal{E}_2(d) . \qquad 3.35$$

It is possible to derive using the recursion formulae 3.32 and relations 3.23, 3.28 and 3.35 the following general formula:

$$T = \frac{V_1 \mathcal{E}_2(t) - V_2 \mathcal{E}_1(t)}{-\alpha_{12}\alpha_{21}\gamma_1\gamma_2(1-\gamma_1)(1-\gamma_2)} \qquad 3.36$$

which is a positive-definite function.

Equation 3.36 is valid for the limiting case described by 3.28, and bears some resemblance to expression 3.27, the energy function of the network analogy. Consider the following limiting expressions for the x_1-portion of 3.27:

$$\lim_{x_1 \to 0} \alpha_{21} \log\left(\frac{\gamma_1}{x_1}\right)^{\gamma_1}\left(\frac{1-\gamma_1}{1-x_1}\right)^{1-\gamma_1} = \alpha_{21}(-\gamma_1)\log\frac{x_1(1-\gamma_1)}{(1-x_1)\gamma_1}$$
$$+\alpha_{21}\log(1-\gamma_1)$$

$$= \mathcal{E}_2(t)\, V_1 + \alpha_{21}\log(1 - \gamma_1) \; , \qquad 3.37$$

$$\lim_{x_1 \to 1} \alpha_{21}\log\left(\frac{\gamma_1}{x_1}\right)^{\gamma_1}\left(\frac{1-\gamma_1}{1-x_1}\right)^{1-\gamma_1}$$

$$= \alpha_{21}(1-\gamma_1)\log\frac{x_1(1-\gamma_1)}{(1-x_1)\gamma_1} + \alpha_{21}\log\gamma_1$$

$$= \mathcal{E}_2(t)V_1 + \alpha_{21}\log\gamma_1 \; . \qquad 3.38$$

Similar expressions obtain for the x_2-portion and hence expression 3.36 for the period T is linearly related to the limiting case of the network energy function. This example will be discussed further in chapter 5.

3.5 SINGULAR SOLUTIONS OF THE SYSTEM EQUATIONS

We consider the general system of n simultaneous equations 3.1:

$$\frac{dx_i}{dt} = f_i(X) \; . \qquad 3.1$$

Definition 3.1: A singular point[3] is a point $X^o = (x_1^o, x_2^o, \ldots, x_n^o)$ for which, from 3.1, $f_i(X^o) = 0$ for all $i = 1, 2, \ldots, n$.

Thus, from 1.28, the vectors X which satisfy the equation

$$\frac{1}{\beta_i \tau_i \psi'(x_i)}\left[\epsilon_i - \beta_i \psi(x_i) + \sum_j \alpha_{ij}x_j\right] = 0 \qquad 3.39$$

simultaneously for all $i = 1, 2, \ldots, n$ are the singular points of the threshold system. When written in matrix form, 3.29 is

$$\mathrm{diag}\left[\frac{1}{\beta_i \tau_i \, \psi'(x_i)}\right](E - B\Psi(X) + AX) = 0 \qquad 3.40$$

and one of the solutions of this equation, namely that for which the second term equals zero, has been used implicitly in the development of the system equations in section 1.7.

3.5.1 SIMPLIFYING ASSUMPTIONS

In order to reduce the number of separate cases to be considered, it is reasonable at this point to make some assumptions regarding realizations of the general gate equation. Theorem 2.3 and Corollaries 2.1 to 2.3 provide the necessary background.

1. We assume that the nonlinear curve is specified by the logistic function as in equation 1.18. That is,

$$\psi(x_i) = \log\frac{x_i}{1 - x_i} \, . \qquad 3.41$$

This function satisfies the assumptions of section 1.61 and will be used throughout the remainder of this thesis when numerical results are required. By specifying the function $\psi(\,\cdot\,)$ we have also specified the limit of equation 1.30:

$$L(s) = \lim_{x \to s} \frac{\psi(x)}{\psi'(x)} = \lim_{x \to s} x(1 - x)\log\frac{x}{1 - x} \qquad 3.42$$

and from this equation it can easily be shown that $L(0) = L(1) = 0.0$.

2. We assume that the conditions of Corollary 2.2 hold, in

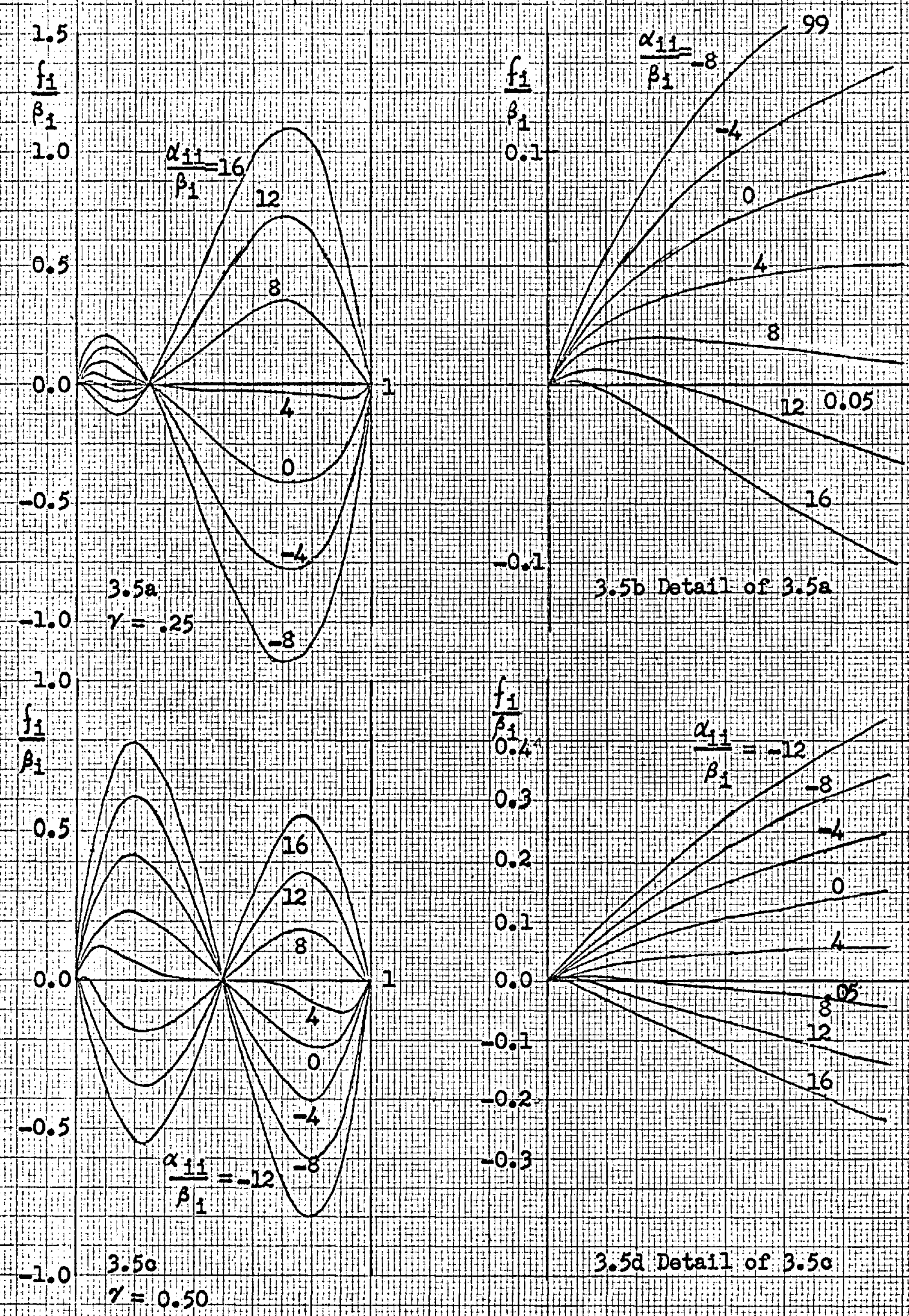


Fig. 3.5 Single-Element Phase-Plane Diagrams

other words, that the inputs to all gates are finite. This assumption is completely reasonable for real gates and provides an important simplification of the problem of enumerating the singular solutions of a network. Consider Fig. 3.5a to 3.5d. These are phase-plane diagrams of the normalized equation

$$\frac{\dot{x}_i}{\beta_i} = \frac{f_i}{\beta_i} = x_i(1 - x_i)\left(\frac{\epsilon_i}{\beta_i} + \frac{\alpha_{ii}}{\beta_i}x_i - \log\frac{x_i}{1 - x_i}\right) \qquad 3.43$$

for various values of $\frac{\epsilon_i}{\beta_i}$ and $\frac{\alpha_{ii}}{\beta_i}$. There are as many as five singularities of 3.43 in the general case: $x_i = 0$ and $x_i = 1$ which are precluded by the assumption, and exactly one or three solutions for which the other term in 3.43 is zero. The assumption we have made allows a simplified phase diagram to be used since the two extreme singularities may be ignored. We plot the function

$$\frac{\dot{y}_i}{\beta_i} = \frac{d}{dt}\frac{\psi(x_i)}{\beta_i} = \frac{\epsilon_i}{\beta_i} + \frac{\alpha_{ii}}{\beta_i}x_i - \log\frac{x_i}{1 - x_i} \qquad 3.44$$

as Fig. 1.14 and therefore are concerned with one or three singularities per equation.

3. We assume that when the conditions of Theorem 2.3 and Corollary 2.2 are fulfulled for nonvanishing $\delta(\mu)$, the quantity $\frac{\delta}{\beta_i}$ is "large." This requires that the values a and b of the corollary are "near" 0 and 1 respectively. From Theorem 2.3,

$$\left|\frac{\epsilon_i}{\beta_i} + \frac{1}{\beta_i}\sum_j \alpha_{ij}x_j\right| > \frac{\delta}{\beta_i}, \qquad 3.45$$

and if a and b are specified we may solve the equation

$$\frac{\delta}{\beta_i} - \max\left|\log\frac{\xi}{1-\xi}\right| = 0 , \qquad 3.46$$

where ξ has the values a and b. For 3.46 to hold for separate state vectors X, the parameters α_{ij} must satisfy certain constraints. Assume, for example, that x_j changes from $a \simeq 0$ to $b \simeq 1$, and that this change causes the gate to switch. Then the condition

$$\left|\frac{\alpha_{ij}}{\beta_i} b - \frac{\alpha_{ij}}{\beta_i} a\right| > \frac{2\delta}{\beta_i} \qquad 3.47$$

must hold, or, since β_i is positive,

$$|\alpha_{ij}| \; (b - a) > 2\delta . \qquad 3.48$$

We have assumed that $\frac{\delta}{\beta_i}$ is "large" and therefore $b - a \simeq 1$ and 3.48 becomes

$$|\alpha_{ij}| > 2\delta . \qquad 3.49$$

This is the well-known constraint that the input weights must exceed the gap magnitude for proper threshold operation. A further restriction for 3.45 to hold is of course that ϵ_i have a suitable value.

4. We assume that in case $\beta_i = 0$, that is, in case 2.46 describes the gate dynamic behavior, the output x_i may be arbitrarily close to the singularities 0 and 1, but may never be outside the interval (0,1). In this case the phase diagram is of the function

$$\dot{x}_i = x_i(1 - x_i)(\epsilon_i + \sum_j \alpha_{ij}x_j) \qquad 3.50$$

which is a third degree equation and a system of order n therefore has, in general, 3^n possible solutions.

5. We assume that, in certain cases, the parameters α_{ij} have the same sign for a given set of values of i. Fig. 3.6 shows a simple threshold gate containing one transistor and a number of resistors. Obviously if all resistors are positive, the values of the α_{ij} in the describing equation will all be negative since a transistor has negative gain. Other circuits may be used to give positive gain, but transistorized versions, at least, will often also exhibit greater delays and "ringing" and would be better described by several equations rather than one. We comment that the dynamic equation 1.21 is reasonably descriptive of a single transistor but may be less useful with circuits containing, for example, ferrites with significant hysteresis. If we allow one equation per transistor, then, the values of α_{ij} for all i and j are negative. The circuit in Fig. 3.6 corresponds to a positive value of ϵ_i due to the negative bias voltage but this condition will not be true in general.

6. We assume that if the parameter β_i is nonzero it has the value $+1$. This normalization has the effect of changing time-scaling only, as may be seen from equations 3.43 or 3.44, and does not change the character of the resulting trajectory diagrams.

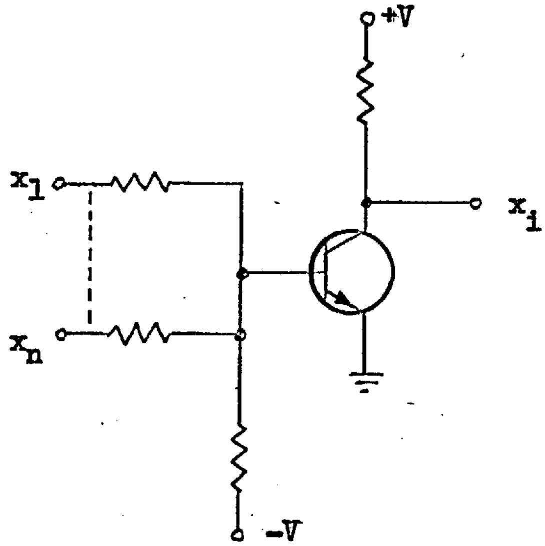


Fig. 3.6 A Transistorized Gate

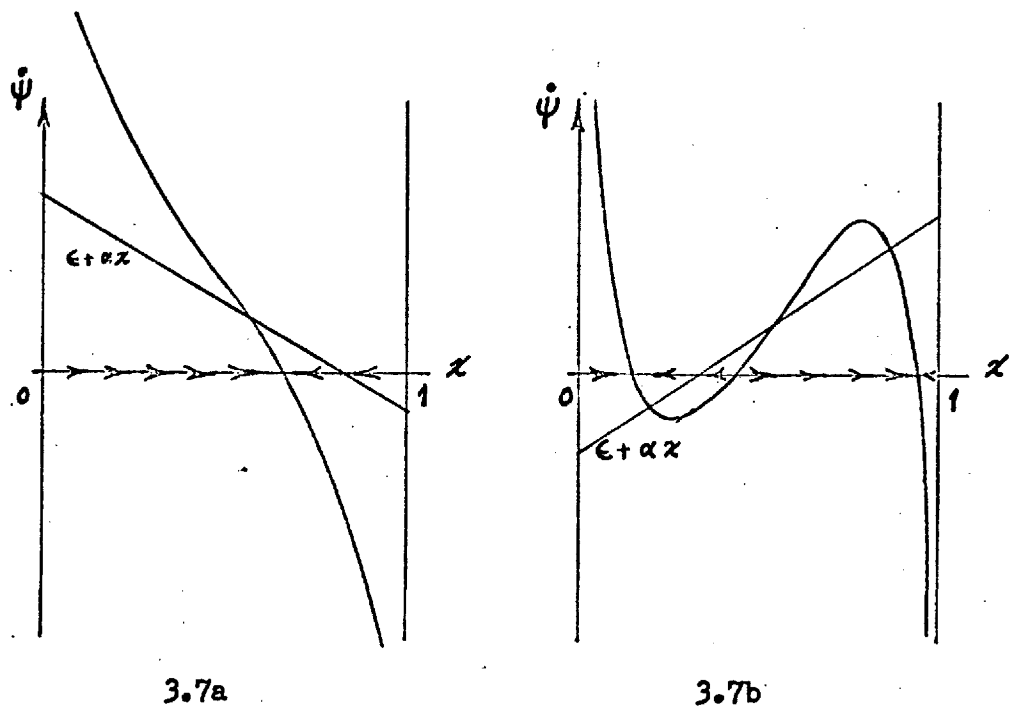


Fig. 3.7 Phase Plane of the Single Gate

3.6 TRAJECTORIES IN TWO DIMENSIONS

Graphical trajectory methods are of most use in two dimensions, because the state-space may be drawn on paper. We consider a two-dimensional general equation 3.1, and form the matrix of first-order derivatives:

$$H = \left[h_{ij}\right] = \left[\frac{\partial f_i(X^o)}{\partial x_j}\right] \qquad 3.51$$

where X^o is a singular point of system 3.1.

Definition 3.2: A singular point is simple[3] if the determinant of the matrix H defined above is nonzero.

A consequence of a singularity being simple is that it is isolated in the obvious sense.

To characterize a singularity X^o, the characteristic equation of matrix H is written:

$$\left|H - \lambda I\right| = 0 \qquad 3.52$$

and solved for the values λ_1 and λ_2. Then the point X^o is said to be

- a node if λ_1, λ_2 are real and of the same sign
- a saddle point if λ_1, λ_2 are real and of opposite signs
- a spiral point if λ_1, λ_2 are complex conjugates with nonzero real part
- a vortex point if λ_1, λ_2 are pure imaginary.

Thus knowledge of the types of the singular points of a system enables

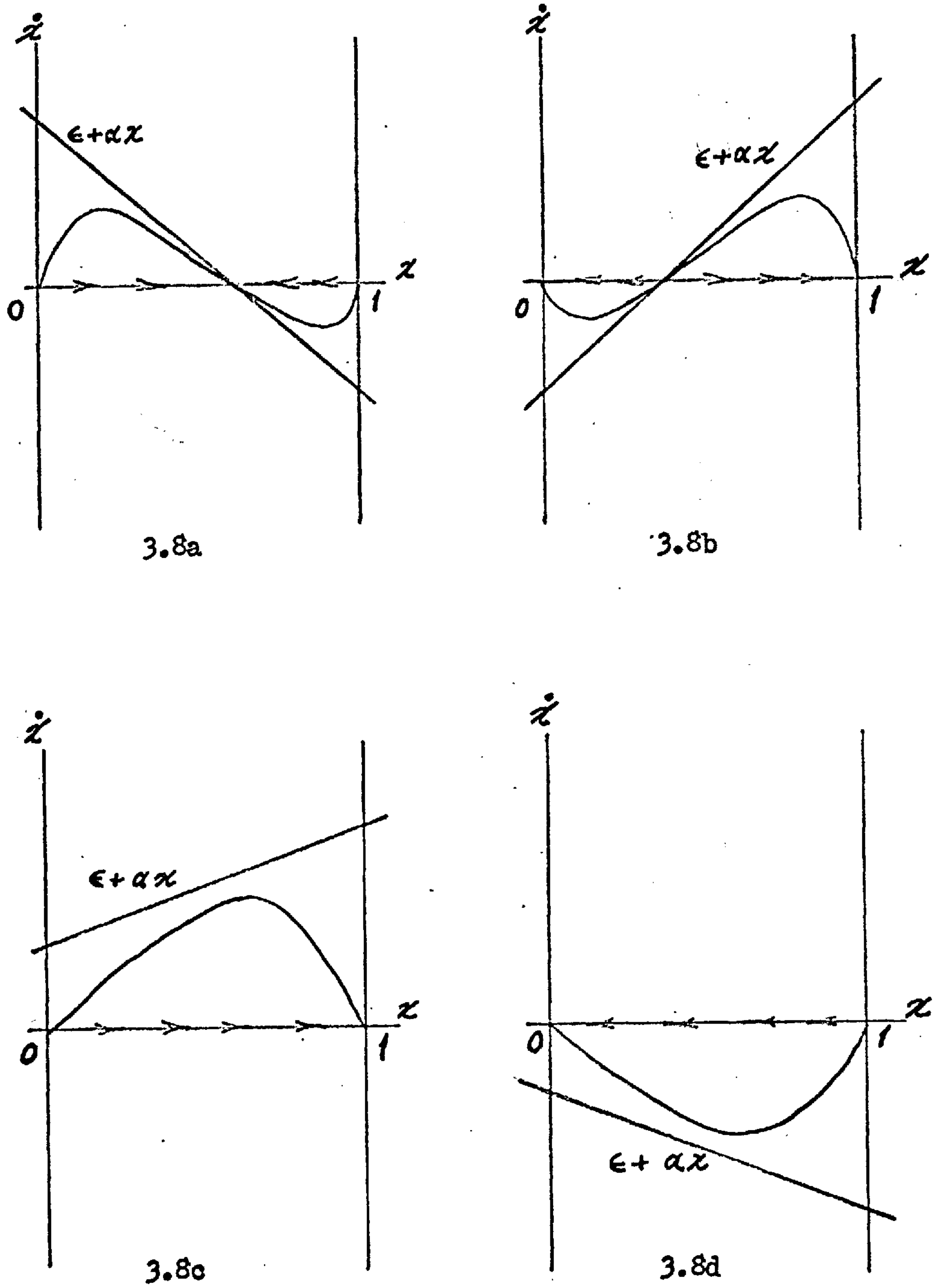


Fig. 3.8 Phase Planes of the Special Case

one to draw approximate solution curves near each singular point.

An additional method easily applied to two-dimensional systems is to calculate those curves for which f_1 and f_2 are respectively zero. Each curve partitions the state-plane into two regions, in which the function is positive or negative. Thus the direction of motion of the system solutions can be easily deduced near either of the curves.

The above two techniques have been used in the following sections, in which 3.44 has been used for the gate equation 1.21 and 3.50 has been used for special case 2.46.

3.6.1 SINGLE GATES

The dynamics of single gates is completely specified by the analyses of Chapter 2; here we only recall that the $x - \dot{x}$ plane or the $x - \dot{\psi}$ plane may be used to give two-dimensional diagrams.

Figures 3.7a and 3.7b show the two possible phase diagrams for the general gate 1.21. Fig. 3.8 shows the equivalent diagrams for the special case 2.46. In each diagram the graph of the linear function $\epsilon + \alpha x$ (subscripts have been removed for simplicity) is drawn.

Of course the diagrams only show representative members of families of curves; we include here and henceforth only those members which represent a change in the number or type of singularities.

Figs. 3.8a and 3.8b are comparable with 3.7a and 3.7b respectively. The general gate has exactly one or three singularities, and when three occur the outer two are stable. The special case may have two singularities if the line $\epsilon + \alpha x$ does not cross the axis in (0,1), in which case one of the singular points is stable.

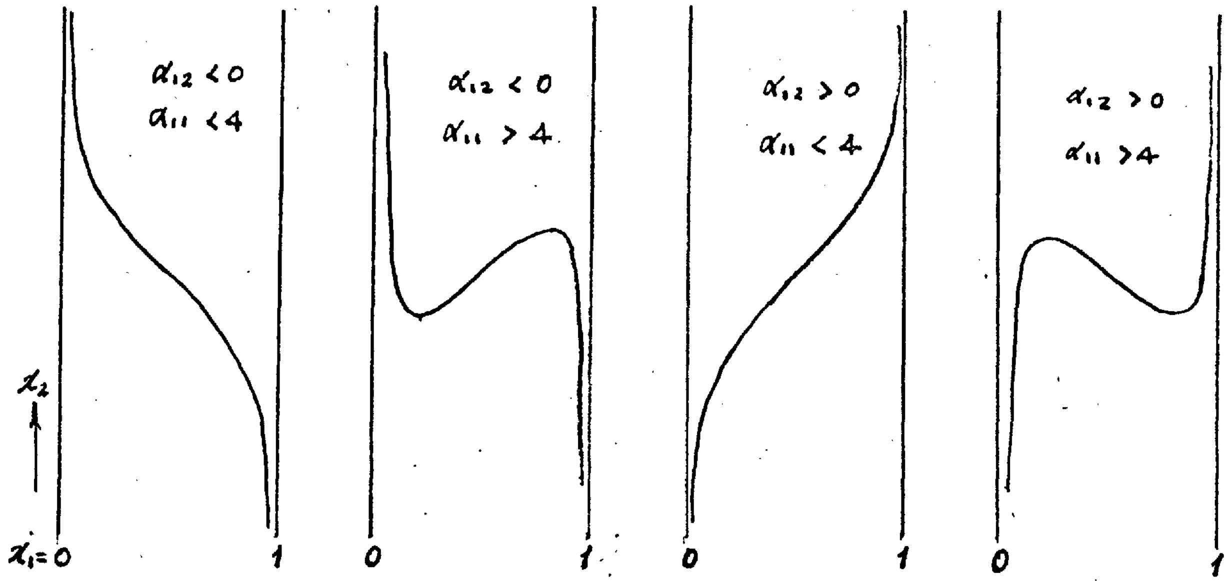


Fig. 3.9 Forms of Equation 3.54
$\mathcal{E}_1(X) = 0$

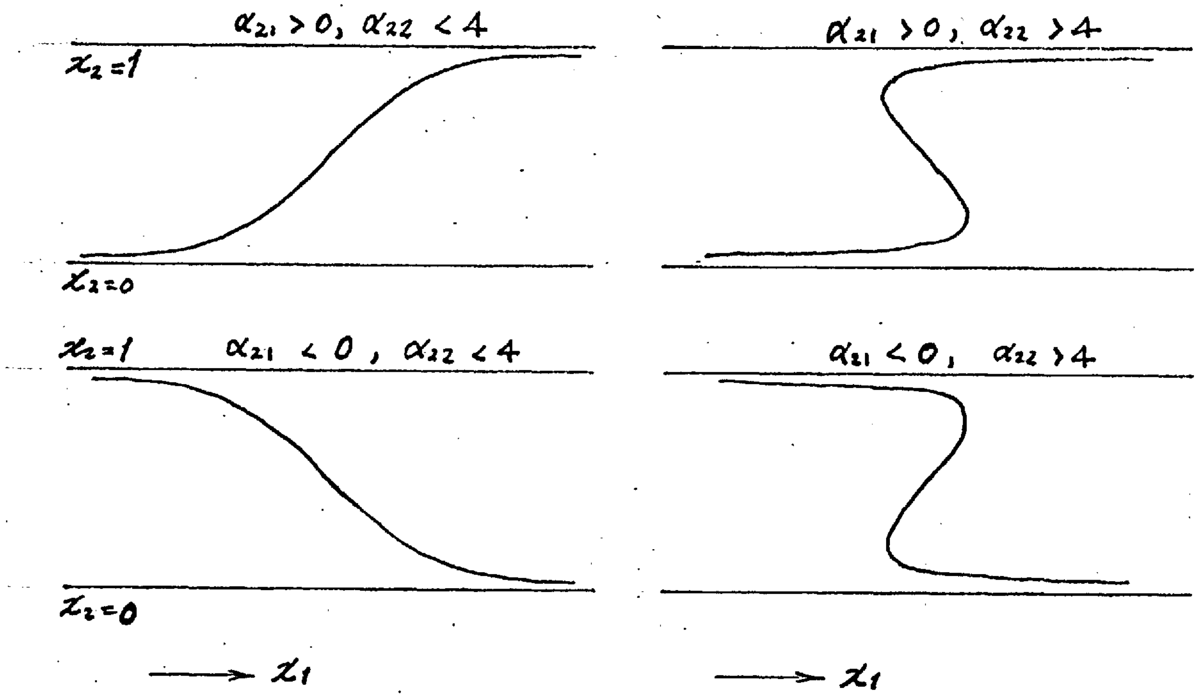


Fig. 3.10 Forms of Equation 3.55
$\mathcal{E}_2(X) = 0$

3.6.2 TWO-GATE NETWORKS: THE GENERAL CASE

State-plane diagrams of two-gate networks are constructed by combining the curves of the previous section in the $x_1 - x_2$ plane. The equation

$$\mathcal{E}_1(X) = \epsilon_1 + \alpha_{11}x_1 + \alpha_{12}x_2 - \log\frac{x_1}{1 - x_1} \qquad 3.53$$

is solved for x_2:

$$x_2 = \frac{1}{\alpha_{12}}\left[\log\frac{x_1}{1 - x_1} - \epsilon_1 - \alpha_{11}x_1\right] . \qquad 3.54$$

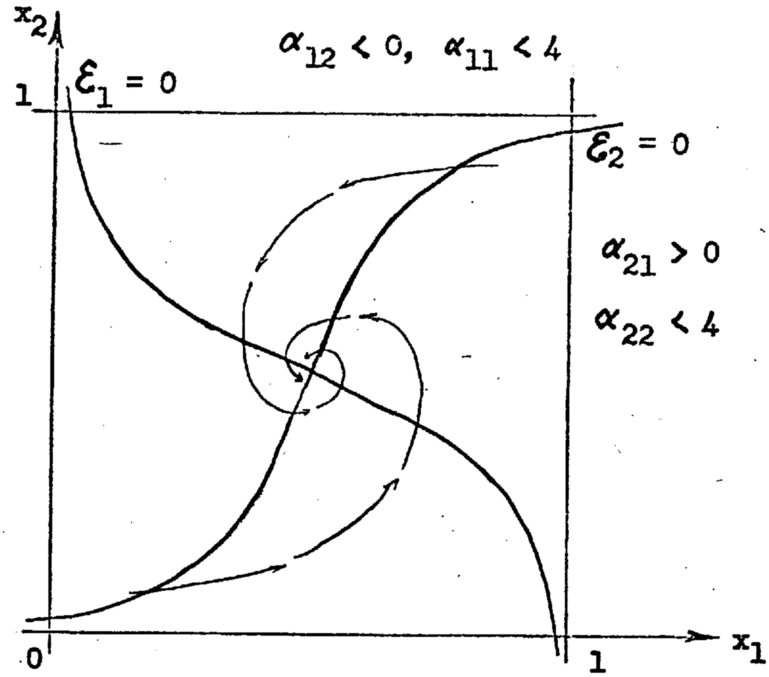
The possible forms for the solutions of this equation are shown in Fig. 3.9. No vertical origin or scale has been shown since these depend on the choice of the constants ϵ_1 and the value $\frac{\alpha_{11}}{\alpha_{12}}$ respectively. The curves have symmetry properties because of the symmetry of the function $\psi(\, . \,)$; if other functions had been used, the symmetry would not necessarily remain.

Corresponding curves exist for x_1 in the second equation $\mathcal{E}_2(X) = 0$, which is similar to 3.54:

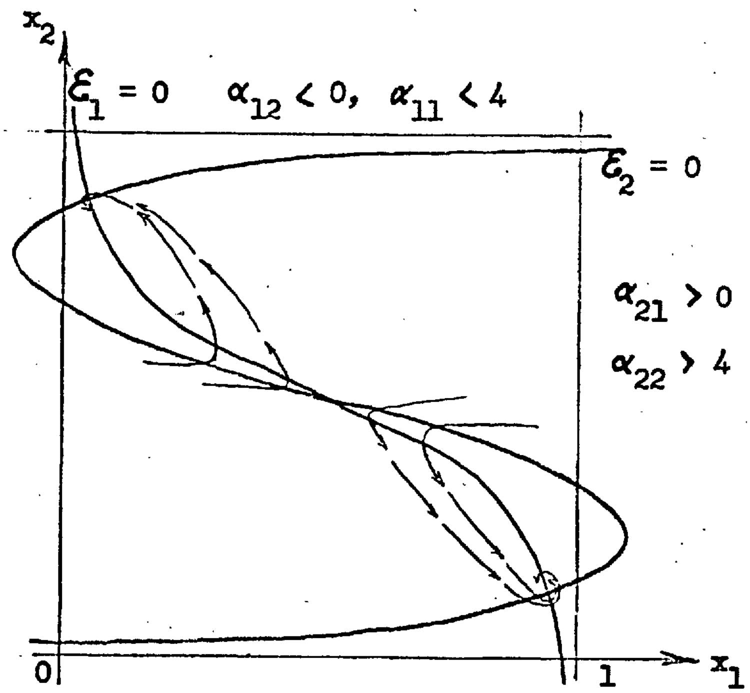
$$x_1 = \frac{1}{\alpha_{21}}\left[\log\frac{x_2}{1 - x_2} - \epsilon_2 - \alpha_{22}\right] , \qquad 3.55$$

and these are shown in Fig. 3.10, for which corresponding remarks may be made.

The object is to combine the relevant curve of Fig. 3.9 with one from Fig. 3.10. The points at which the curves intersect will then be the realizable singularities of the system.

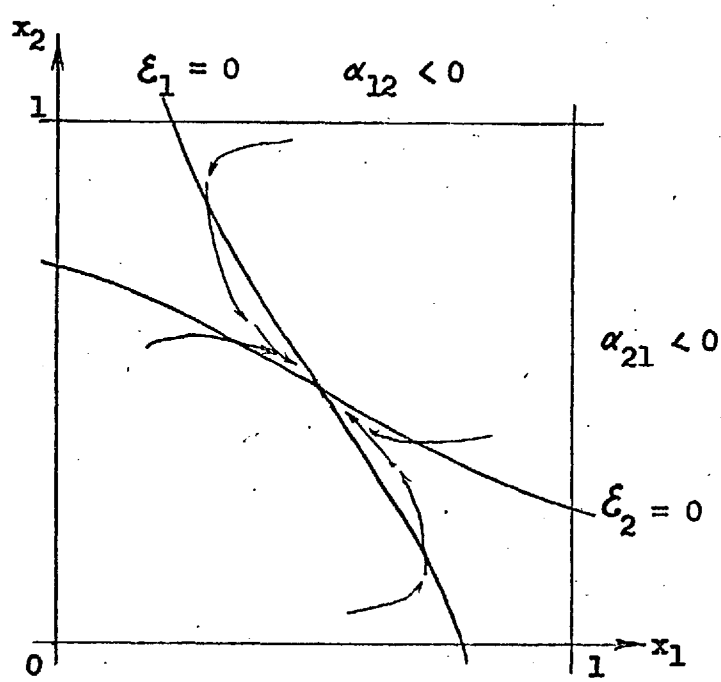


3.11a

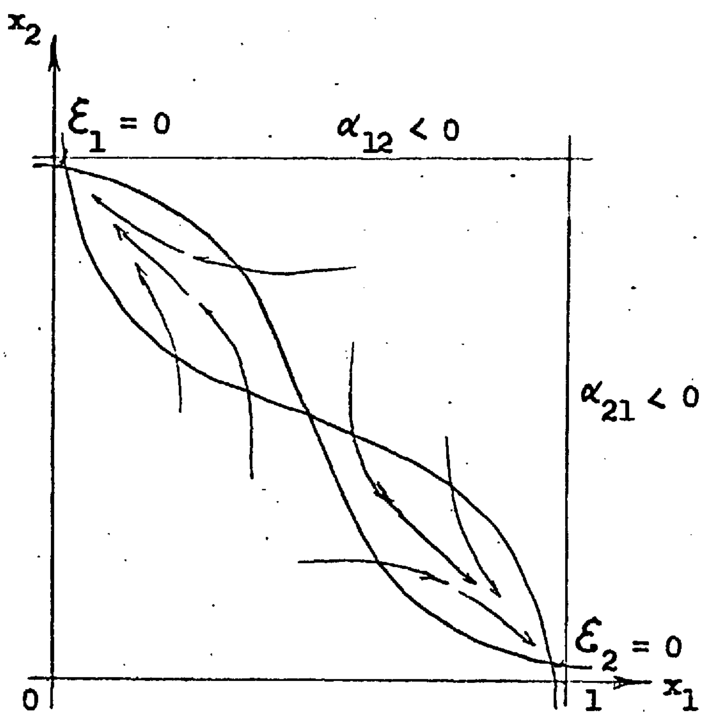


3.11b

Fig. 3.11 Forms of Network Trajectories

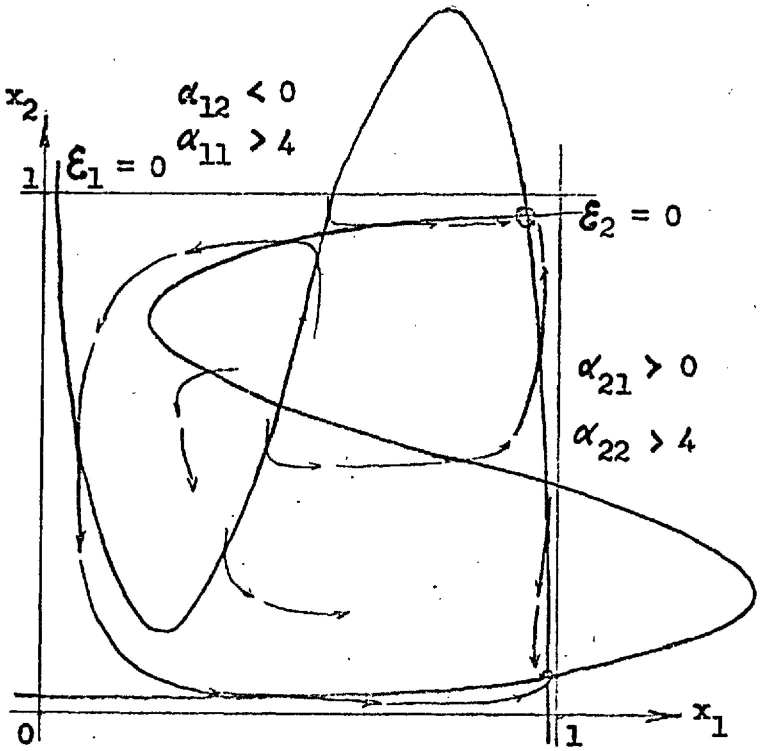


3.11c

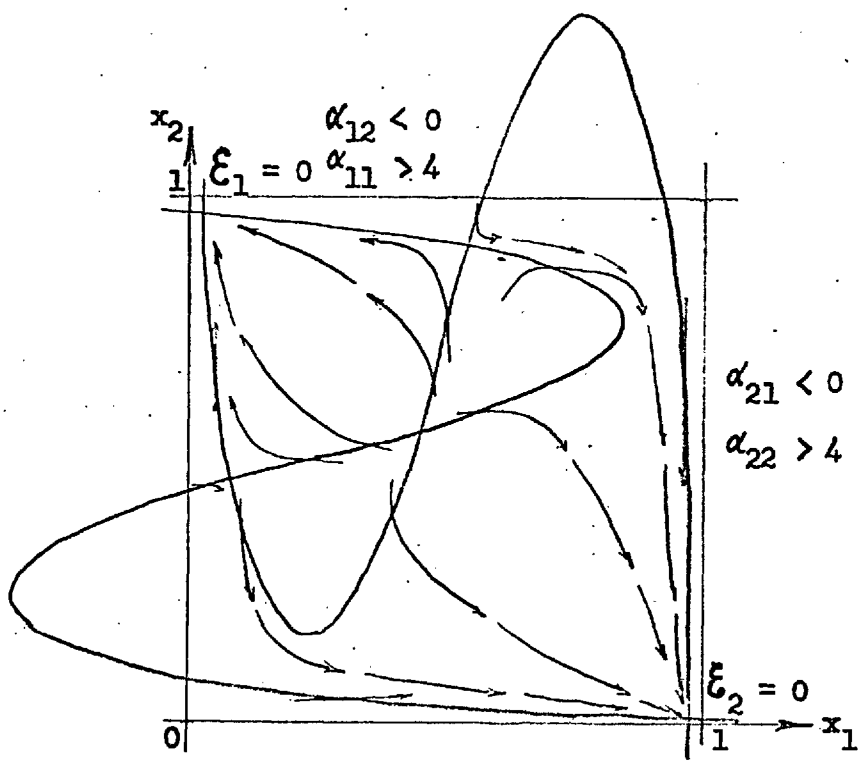


3.11d

Fig. 3.11 (continued)

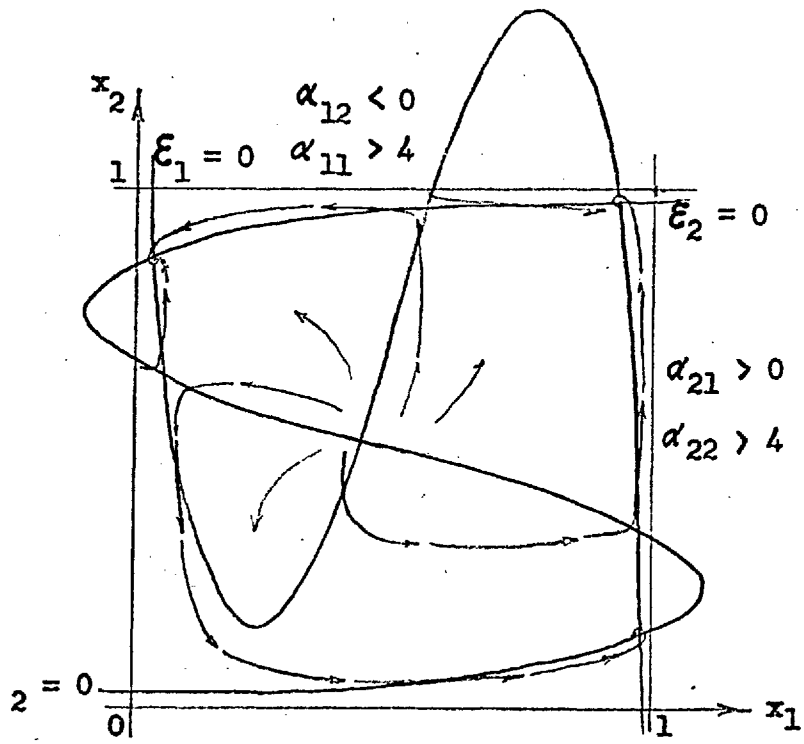


3.11è

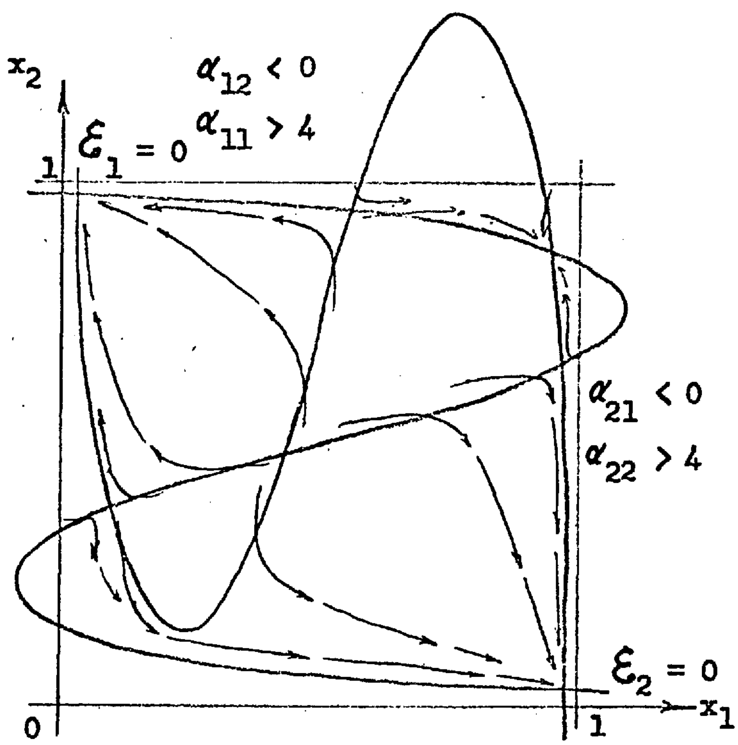


3.11f

Fig. 3.11 (continued)

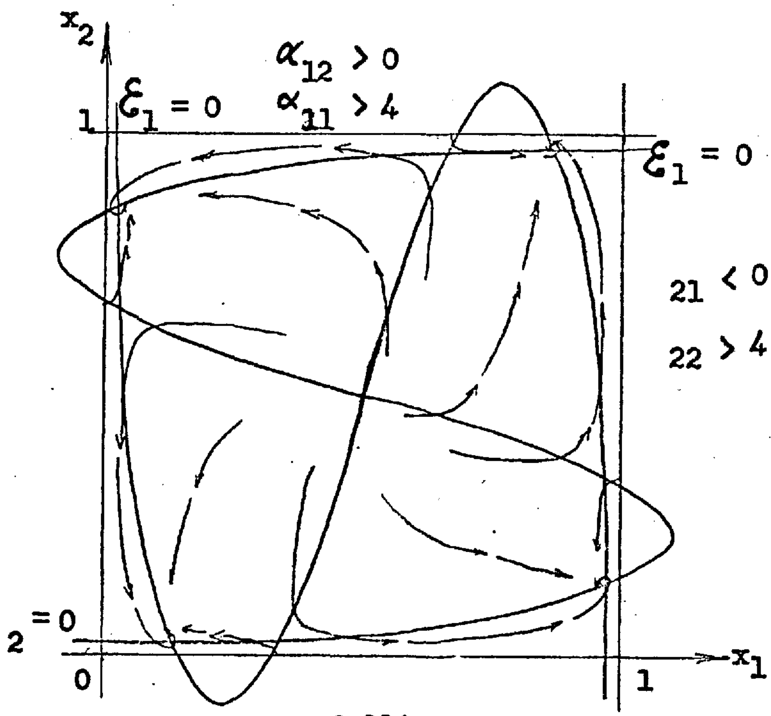


3.11g

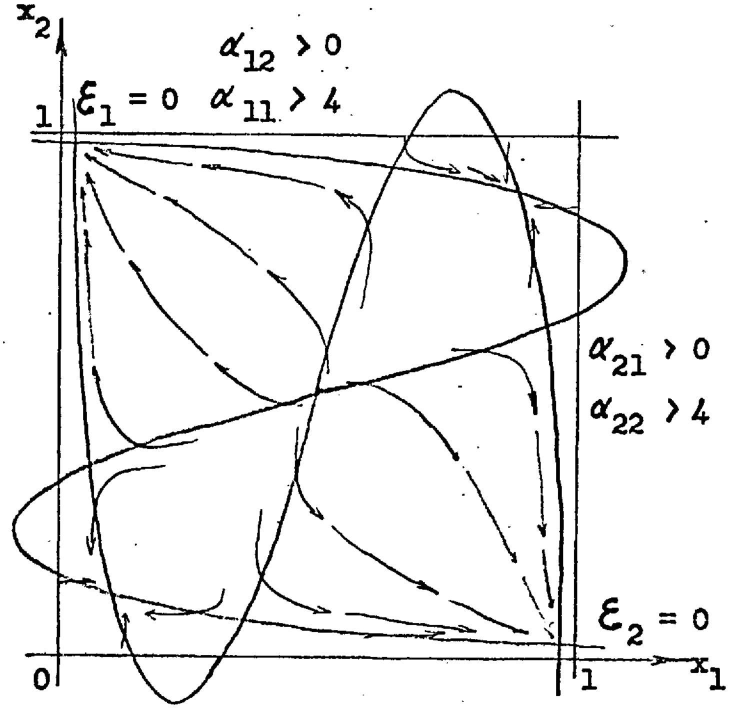


3.11h

Fig. 3.11 (continued)



3.11i



3.11j

Fig. 3.11 (continued)

Figs. 3.11a to 3.11j are representative diagrams of all configurations of two-gate networks. Those diagrams which may be obtained by renumbering the elements are not included since the forms would not change. Dashed lines represent typical trajectories. Fig. 3.12 is a table which contains a summary of some results which may be obtained by inspection of Fig. 3.11. We make the following remarks about these results:

1. Since the curves of Fig. 3.9 and 3.10 extend from plus infinity to minus infinity, the number of singularities must always be odd. Since the curves cross the axes three times at most, the largest possible number of singularities of the two-gate system is 9. By an extension of this argument, the largest possible number of singularities in a system of n gates is 3^n.

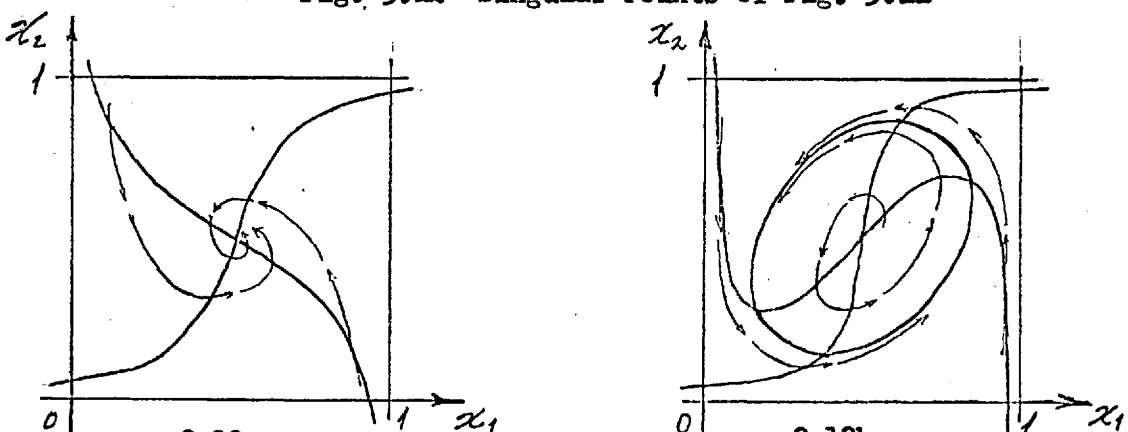
2. Fig. 3.12 shows that all configurations may be grouped in pairs with equal numbers of singular points. The members of each pair have an equal number of saddle points, which are always unstable. The nodes of one member correspond to the spiral points of the other, and the number of stable points is equal for the two members, except for the case in which only one singularity exists.

3. No diagram contains more than two types of singularities.

4. One special case exists, that corresponding to Fig. 3.11a. In this figure and in Fig. 3.13a the singular point is shown as a stable spiral point, but Fig. 3.13b has similar topology, that is, an equal number of singularities of the same types, but has instead an unstable spiral point, and in addition a stable trajectory.

Diagram	Number of Singular Points	Spiral Points Number	Spiral Points Stable	Spiral Points Unstable	Nodes Number	Nodes Stable	Nodes Unstable	Saddle Points
a	1	1	-	-				
c	1				1			
b	3	2	2					1
d	3				2	2		1
e	5	3	2	1				2
f	5				3	2	1	2
g	7	4	3	1				3
h	7				4	3	1	3
i	9	5	4	1				4
j	9				5	4	1	4

Fig. 3.12 Singular Points of Fig. 3.11



3.13a 3.13b

Fig. 3.13 The Special Case

5. It is a general rule of realizable Lipschitzian systems that all possible trajectories begin and end either at singular points or are asymptotic to closed curves known as limit-cycles, such as in Fig. 3.13b. More will be said about limit-cycles in Chapter 5. The question then arises: If the other diagrams in Fig. 3.11 do not contain limit-cycles, from which unindicated singularities do trajectories originate? The anxwer is, of course, that the points $x_1, x_2 \in \{0,1\}$ which we have previously designated unrealizable are also terminations of trajectories.

6. Only two classes of diagrams, represented by Fig. 3.11c and d, are possible for two-gate networks with negative parameters α_{ij}. Fig. 3.11d contains exactly two stable nodes, and corresponds to the same circuit, but to a condition in which "loop-gain" is less than unity.

7. It is remarkable that a system of two threshold gates may have three or four stable points. Positive parameters α_{ij} are required for this situation, however.

3.6.3 TWO-GATE NETWORKS: THE SPECIAL CASE

Consider a system of order n, that is, one which contains n gates. We shall show some results for general system, and the system of order 2 will be a special case. From section 3.5.1, assumption 4, the state equation is

$$\dot{X} = \mathrm{diag}\left[x_i(1 - x_i)\right](E + AX) . \qquad 3.56$$

This equation has at most 3^n distinct singular points, which may be characterized as follows:

1. One point, which we call the <u>principal singularity</u> Γ, is the solution of equation 3.19, provided the solution exists:

$$E + AX = 0 . \qquad 3.19$$

2. Exactly 2^n points exist for which all x_i are either 0 or 1.

3. Exactly $3^n - 2^n - 1$ points are solutions of 3.19 with some of the x_i constrained to be either 0 or 1. That is, if q unknowns x_i are specified as 0 or 1 a new equation,

$$\overline{E} + \overline{AX} = 0 \qquad 3.57$$

must be solved, the order of which is $n - q$.

It is not always true that there are 3^n distinct singular points. A multiple singularity exists when any two solutions of 3.57 obtained for different specifications of the $x_i \in \{0,1\}$, or when any

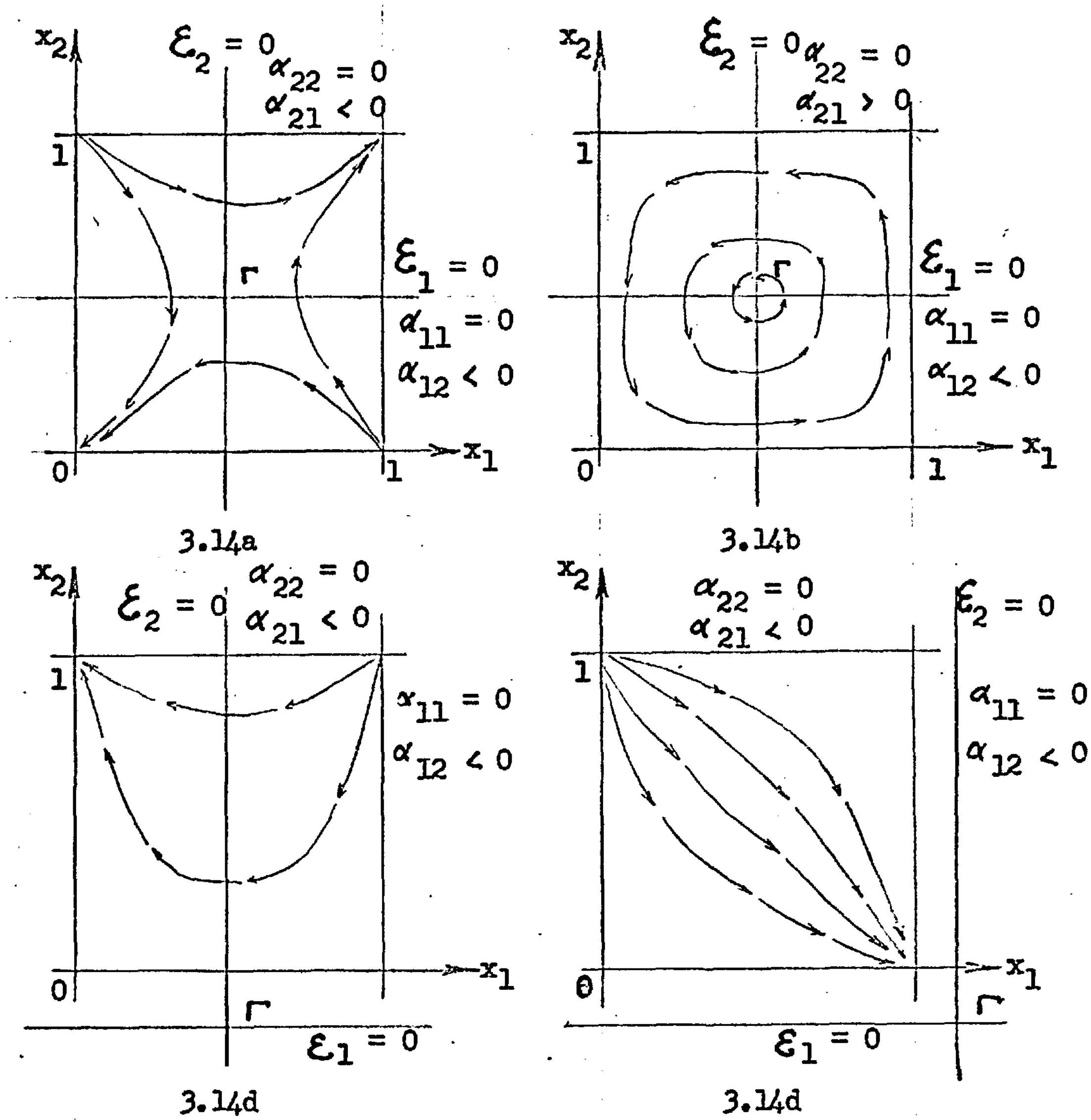


Fig. 3.14 The Special Trajectories

Diagram	Singular Points	Nodes Stable	Nodes Unstable	Saddle Points
d	4	1	1	2
c	4	1	1	2
b	5			4
a	5	2	2	1

Fig. 3.15 Singularities of Fig. 3.14

unspecified solution of 3.57 is 0 or 1.

A root of 3.57 which falls outside the unit hypercube (the unit square in the case $n = 2$) is an unrealizable singular point.

In the case $n = 2$, there are at most 9 singular points, 4 of which are solutions of equations of the form of 3.57. In this case, if the lines

$$\mathcal{E}_i(X) = \epsilon_i + \alpha_{ii}x_i + \alpha_{ij}x_j = 0, \quad i,j = 1,2 \qquad 3.58$$

are plotted, their intersection is Γ. The intersection of the first equation with the lines $x_2 = 0$ and $x_2 = 1$, and the second with the lines $x_1 = 0$ and $x_1 = 1$ are also singular points. Finally, the four verteces of the unit square are singular points.

The condition that $\alpha_{11} = \alpha_{22} = 0$ which was mentioned in Section 3.3.1 is a special case and is shown in Fig. 3.14. The point Γ is shown as $\left[\frac{1}{2},\frac{1}{2}\right]$ but this need not be true in general. In this situation there are at most 5 finite singular points. Only representative diagrams are shown. All others may be obtained by reversing arrows or axis labels or both. Fig. 3.14b is analogous to the electrical network discussed in Section 3.3, and was shown to possess a constant of motion (equation 3.27) and therefore motion is along closed trajectories about Γ. Fig. 3.14a is a bistable system similar to Fig. 3.11d, and is in fact the limiting case of Fig. 3.14d if the parameters β_i are allowed to approach zero with $\beta_i \tau_i$ constant and finite. Fig. 3.15 is a table listing the nodes and saddle points of Fig. 3.14. No spiral points exist.

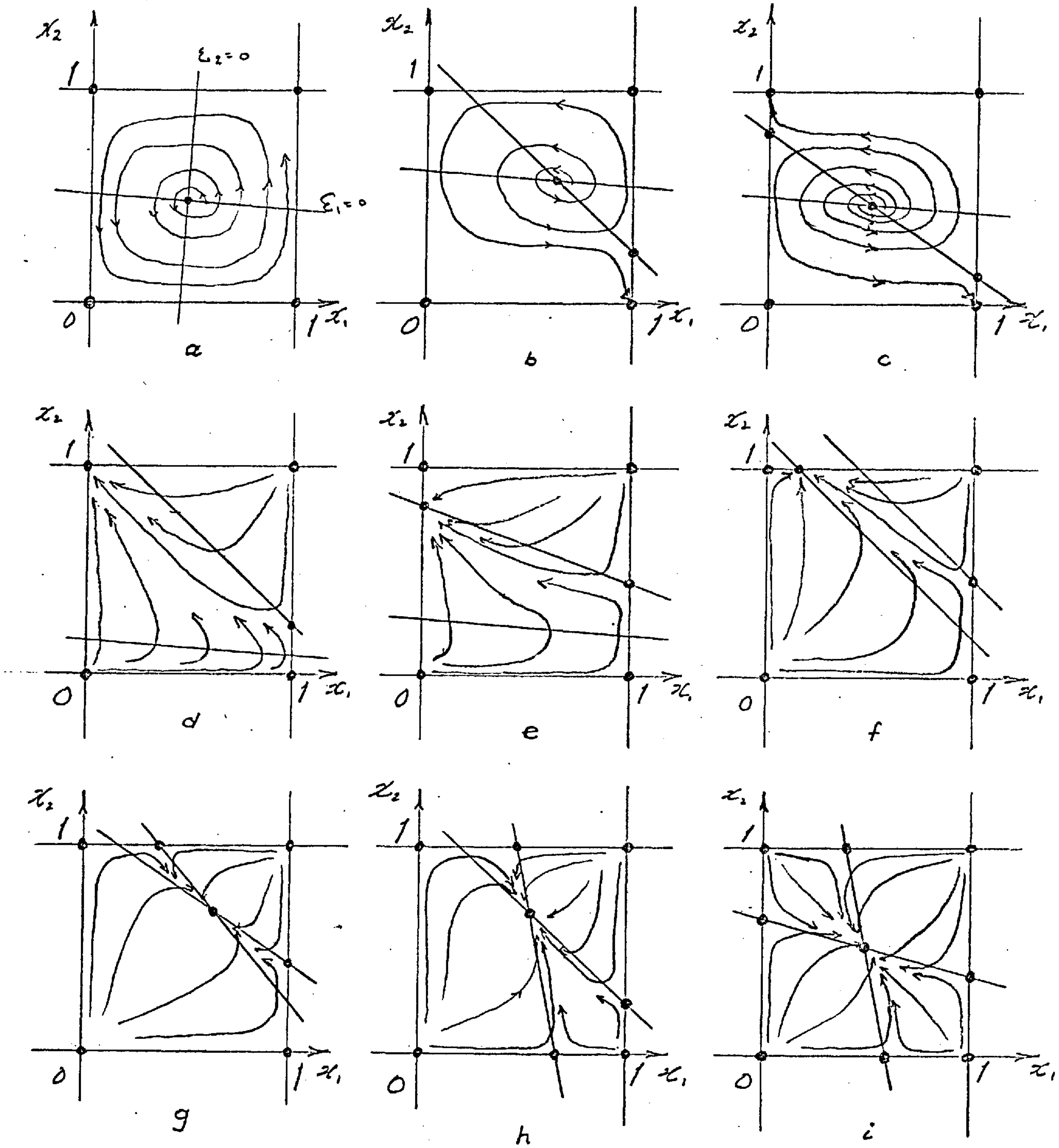


Fig. 3.16 Trajectories of 3.58

Fig. 3.16 shows representative diagrams of system 3.58 in the case $\alpha_{11} \neq 0$, $\alpha_{22} \neq 0$. As before, only representative diagrams are shown. A special case is Fig. 3.16a, which is shown to contain an unstable node. In fact, depending on the slopes of the lines $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 0$ the principal singularity may be a stable node or, in the special case of a conservative system, a centre. The results for these systems are summarized in the table of Fig. 3.17.

3.7 ENUMERATION OF SINGULARITIES: THE SPECIAL CASE.

The linear form of equations 3.19 and 3.57 make the enumeration of singularities of n-dimensional special systems particularly easy, at least for low n.

Let the binary number k take on the values of the 2^n - term sequence

$$k = 0, 1, \ldots, 2^n-1 .$$

Suppose for some particular value, say k, there are q binary digits in k which are 1, the rest being 0. Then if all values x_j are specified (as 0 or 1) for which the j^{th} digit is 1, there are exactly 2^q unique specifications of the x_j for this value of k. These 2^q specifications may be made according to a binary progression. Consider an example. Suppose $n = 4$, and $k = 3$, that is, $k = 1100$ if the least significant digit is on the left. We then specify x_1 and x_2 to be, in order, (0,0), (1,0), (0,1), (1,1). This process is

Diagram	Spiral Points		Nodes		Saddle Points
	Stable	Unstable	Stable	Unstable	
a	-	-			4
d			1	2	2
b		1	1		4
f			1	2	3
e			1	2	3
c		1		2	4
g			1	2	4
h			1	3	4
i			1	4	4

Fig. 3.17 Singularities of Fig. 3.16

```
  E          A MATRIX
1.000        0.          -2.000
1.000      -2.000         0.

     GAMMA
          0.5000     0.5000     UNSTABLE

     POINTS Q
R
1         0.                    NO SOLUTION
2         1.0000                NO SOLUTION
3                    0.         NO SOLUTION
4                    1.0000     NO SOLUTION

     POINTS P
R
0         0.         0.         UNSTABLE
1         1.0000     0.         STABLE
2         0.         1.0000     STABLE
3         1.0000     1.0000     UNSTABLE
```

Fig. 3.18 Example Enumeration of Singularities

E	A MATRIX		
3.000	-2.000	-4.000	1.000
3.000	-2.000	-4.000	-1.000
3.000	-4.000	-2.000	0.

GAMMA

0.5000	0.5000	0.	SEMISTABLE

POINTS Q

R				
1	0.	1.5000	-3.0000	NOT IN (0,1)
2	1.0000	-0.5000	3.0000	NOT IN (0,1)
3	0.7500	-0.	-1.5000	NOT IN (0,1)
4	0.2500	1.0000	1.5000	NOT IN (0,1)
5	0.	0.		NO SOLUTION
6	1.0000	0.		NO SOLUTION
7	0.	1.0000		NO SOLUTION
8	1.0000	1.0000		NO SOLUTION
9			0.	ARBITRARY
10			1.0000	NO SOLUTION
11	-0.	0.7500	0.	UNSTABLE
12	1.0000	0.2500	0.	SEMISTABLE
13	-0.	0.5000	1.0000	UNSTABLE
14	1.0000	-0.	1.0000	REDUNDANT
15	1.5000	-0.	0.	NOT IN (0,1)
16	-0.5000	1.0000	0.	NOT IN (0,1)
17	2.0000	-0.	1.0000	NOT IN (0,1)
18	-0.	1.0000	1.0000	REDUNDANT

POINTS P

R				
0	0.	C.	0.	UNSTABLE
1	1.0000	0.	0.	UNSTABLE
2	0.	1.0000	0.	UNSTABLE
3	1.0000	1.0000	0.	UNSTABLE
4	0.	0.	1.0000	UNSTABLE
5	1.0000	0.	1.0000	UNSTABLE
6	0.	1.0000	1.0000	UNSTABLE
7	1.0000	1.0000	1.0000	UNSTABLE

Fig. 3.19. Example Enumeration of Singularities

an algorithm for finding all singular points in a particular arbitrary order. The case $k = 0$ is of course that for which no x_j's are to be specified, and the corresponding singularity is Γ. In the case $k = 2^n-1$ all values are specified, no equation 3.57 need be solved, and the singular points are all vertices of the n-dimensional hypercube.

Let K_k be the q-element set of values of j for which all x_j are to be specified for a particular value of k. Then to arrive at 3.57 from 3.19, write the equations in the form

$$\epsilon_i + \sum_{j \in K_k} \alpha_{ij} x_j + \sum_{j \notin K_k} \alpha_{ij} x_j = 0 \ , \ i \notin K_k \ , \qquad 3.59$$

and since the first two terms in this equation are constant, we have defined 3.57.

Some practical considerations arise:

1. Solution of linear equation 3.57 depends on the inversion of the matrix $\bar{A}$. If $\bar{A}$ is singular, then either no solution exists with that particular $\bar{E}$, or solutions are linearly dependent, in which case there is a continuum of solution of at least one dimension, and the singular point is not simple. The test for linear dependency is to substitute the column $\bar{E}$ for each column of $\bar{A}$ in turn and to find the determinant of the resulting matrix. If all such determinants are zero the solutions are linearly dependent.

If, for any given k and specification of the numbers x_j, $j \in K_k$, the determinant of $\bar{A}$ is zero, then it is zero for any other specification since $\bar{A}$ is determined by k and not by any of the 2^q

specifications corresponding to that k.

2. Solutions of 3.57 which are identical to solution for some other specification are redundant.

Figs. 3.18 and 3.19 are examples of enumerations according to the above algorithm, in which solutions of 3.57 for which $k = 2^n - 1$ are called P, those for $0 < k < 2^n - 1$ are called Q, and the remainder is Γ. A remark is included about the existence and stability of each singular point. Fig. 3.18 corresponds to Fig. 3.14a.

3.8 SUMMARY

The dynamical model is analytically "well-behaved." A special case can be related by analogy to electrical networks, and another special case can be solved exactly.

A complete topological analysis of two-gate networks is possible, and the singular points of a limiting case can be solved in closed form. Enumeration of such singular points of networks of arbitrary n requires the solution of sets of linear equations.

3.9 REFERENCES

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2. Volterra, V., "Theorie Mathematique de la lutte pour la vie," Gauthier-Villars, Paris, 1931.

3. Stern, T.E., "Theory of Nonlinear Networks and Systems," Addison-Wesley, Reading, Mass., 1965.

CHAPTER 4

STABILITY IN THE NEIGHBORHOOD OF SINGULAR POINTS

4.1 INTRODUCTION

This chapter concerns several aspects of the stability of the systems described previously. Much of the mathematics has already been done; here it is applied to networks of arbitrary size.

Some precise definitions will be given, and then some well-known theorems of the stability of nonlinear systems. The final portion of the chapter concerns application of linearization techniques and location of singularities of the general system.

The methods of first-order approximation used here and the general uses Lyapunov functions are well-known and are found in many references. Reference 1 contains a chapter emphasizing first-order techniques, references 2 and 3 are classic treatments of nonlinear systems, and reference 4 is a useful summary of many of the theorems and methods relating to the uses and generation of Lyapunov functions.

4.2 DEFINITIONS

We assume throughout that only Lipschitzian systems are considered. Proof that the gate model satisfies this condition was given in Section 3.2. The discussion in Section 1.6 specifies the gate network to be a nonlinear stationary system in the usual sense. Provided the inputs are constant in a finite interval the system is also free or unforced, and a system satisfying this and the stationarity condition is called autonomous.

We have described the general nonlinear dynamic system by n equations 3.1

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n) \triangleq f_i(X) \qquad 3.1$$

which becomes in vector notation

$$\dot{X} = \frac{dX}{dt} = F(X) \qquad 4.1$$

where the nonlinear functions $f_i(X)$ are specified by equation 1.28. From 3.18,

$$\dot{Y} = E + AX \triangleq G(Y) \qquad 4.2$$

which also describes the system and is of the form of 4.1. In the following we consider a general system 4.1 although the theorems also apply to 4.2.

<u>Definition 4.1</u>: The <u>Euclidean</u> <u>norm</u> of an n-dimensional vector $X = [x_i]$ is given by

$$\|X\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \qquad 4.3$$

and an often used <u>non-Euclidean</u> <u>norm</u> is denoted by

$$|X| = \sum_{i=1}^{n} |x_i| \ . \qquad 4.4$$

<u>Definition</u> <u>4.2</u>: Let $X^o(t)$ be a solution of 4.1. Then $X^o(t)$ is <u>stable</u> if for every real number $\delta > 0$ and solution $X(t)$ of 4.1 there exists a real number $\eta(\delta) > 0$,

such that

$$\|X(t_o) - X^o(t_o)\| < \delta \qquad 4.5$$

implies

$$\|X(t) - X^o(t)\| < \eta \qquad 4.6$$

for all $t > t_o$. If $X^o(t)$ is stable and in addition

$$\|X(t) - X^o(t)\| \to 0 \qquad 4.7$$

as $t \to \infty$, $X^o(t)$ is asymptotically stable.

Of course $X^o(t)$ may be a singular point X^o as discussed in Section 3.5.

Definition 4.3: A solution $X^o(t)$ of 4.1 is asymptotically stable in the large if it is stable, and if every motion $X(t)$ converges to $X^o(t)$ as $t \to \infty$.

It is important to note that the above definitions do not apply to all possible situations where the intuitive concept of stability applies. Stability as defined above is usually known as "stability in the sense of Lyapunov."

4.3 STABILITY OF AUTONOMOUS SYSTEMS

Consider the system defined by

$$\dot{X} = HX + \bar{F}(X) \qquad 4.8$$

where H is a constant matrix and $\bar{F}(X)$ is a nonlinear vector function of X. Equations 4.1 and 4.2 may be rewritten in this form by the use of a Taylor series about some singular point X^o,

$$F(X) = F(X^o) + \left[\frac{\partial F(X^o)}{\partial X}\right](X-X^o) + \sum_{r=2}^{\infty}\frac{1}{r!}\left\{\sum_{j=1}^{n}(x_j - x_j^o)\frac{\partial}{\partial x_j}\right\}^r F(X^o) \qquad 4.9$$

which converges in some neighborhood of X^o. Since by definition $F(X^o) = 0$ this equation will be in the correct form if the transformation

$$Z(t) = [z_i(t)] \triangleq X(t) - X^o \qquad 4.10$$

is made, giving

$$\dot{Z} = HZ + \sum_{r=2}^{\infty}\frac{1}{r!}\left\{\sum_{j} z_j\frac{\partial}{\partial z_j}\right\}^r F(X^o) \equiv HZ + \bar{F}(Z) \qquad 4.11$$

where H is the constant matrix of first derivatives. This matrix is identical to that discussed in section 3.6, except that in this case it is of order n.

Consider the first-order approximation to 4.11, that is,

$$\dot{Z} = HZ \, . \qquad 4.12$$

<u>Theorem 4.1</u>: Solutions of 4.12 are asymptotically stable at the point $Z = 0$ if and only if all the eigenvalues of the matrix H have negative real parts.

The above result is well-known. The eigenvalues are the solutions of the characteristic equation

$$|H - \lambda I| = 0 \, , \qquad 3.52$$

which may be rewritten

$$\lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n = 0 \, . \qquad 4.13$$

The criterion in the theorem may be established by solving equation 3.53 or by the well-known Hurwitz procedure[5] or the Routh[6] test which is preferable for hand computation.

<u>Theorem 4.2</u>[1]: Solutions of 4.11 are asymptotically stable at $Z = 0$ if every solution of 4.12 is asymptotically stable in the large, if $\bar{F}(Z)$ is continuous in some region about $Z = 0$, if $\frac{|\bar{F}(Z)|}{|Z|} \to 0$ as $|Z| \to 0$, and if $|Z(0)|$ is sufficiently small.

The Taylor series representation ensures that the theorem is satisfied since $\bar{F}(Z)$ is of second- and higher-order terms, provided of course the linearized system is asymptotically stable by theorem 5.1. A linear system which is asymptotically stable is always asymptotically stable in the large.

Theorem 4.2 applies only in an arbitrarily small neighborhood of the singularity X^o, that is near $Z = 0$. To show behavior in a finite region, a result due to Lyapunov is used[7]:

<u>Theorem 4.3</u>: Let Ω be a bounded closed region about $Z = 0$ in the state space of system 4.11. Let Ω have the property that every solution $Z(t;Z(t_o),t_o)$ that starts in Ω remains for all future time in Ω. If there exists a

scalar function $W(Z)$ which is continuous and has continuous first partial derivatives in Ω and such that in Ω, $W(0) = 0$, $W(Z) > 0$ for all $Z \neq 0$, $\dot{W}(Z) \leq 0$ for all Z (i.e. $\dot{W}(Z)$ negative-semidefinite), and $\dot{W}\{Z(t;Z(t_o),t_o)\}$ does not vanish identically in $t \geq t_o$ for any t_o and $Z(t_o) = 0$, then every solution starting in Ω is asymptotically stable to $Z = 0$.

The selection of $W(Z)$ which is called a Lyapunov function may require much effort in general. Several methods for finding Lyapunov functions for certain classes of systems exist, and the next chapter is concerned with their application.

<u>Corollary 4.3.1</u>: If there exists a scalar function $W(Z)$ which satisfies the theorem, except that $\dot{W}(Z) \geq 0$ for all Z in Ω and $\dot{W}$ does not vanish identically along a trajectory, then $Z = 0$ is unstable.

The time derivative $\dot{W}$ along a trajectory is calculated as follows:

$$\dot{W} = \sum_{j=1}^{n} \frac{\partial W}{\partial x_j}\dot{x}_j = \sum_j \frac{\partial W}{\partial x_j} f_j(X) = (\nabla W)^T F(X) \; . \qquad 4.14$$

Clearly if $\dot{W} = \frac{dW}{dt}$ is sign-definite, then $-\frac{dW}{dt} = \frac{dW}{d(-t)}$ is also sign-definite and of the opposite sign. Thus if system 4.1 is stable, then the system defined by

$$-\dot{X} = F(X) \qquad 4.15$$

is unstable, and vice-versa. Hence all stability theorems may be used to show instability by reversing time (at least for Lipschitzian systems which have unique trajectories as $t \to +\infty$ or $t \to -\infty$).

We note that if it is possible to find a scalar function W whose derivative as calculated in 4.14 is identically zero along a trajectory then W is a constant of motion (section 3.3.1) and the system, under this condition said to be conservative, satisfies Liouville's theorem[8].

Finally we quote a result for linear systems, for which it is always possible to find Lyapunov functions in closed form.

Theorem 4.4: The equilibrium state $Z = 0$ of the linear system 4.12 is asymptotically stable if and only if given any symmetric, positive-definite matrix Q there exists a symmetric, positive-definite matrix P which is the unique solution of the $n(n+1)/2$ linear equations

$$PH + H^T P = -Q \quad . \qquad 4.16$$

Moreover $Z^T PZ$ is a Lyapunov function for the system.

Equation 4.16 is known as the Lyapunov Matrix Equation. If Q isthe zero matrix then $Z^T PZ$ is a constant of motion and has the units of energy in usual physical situations.

Lyapunov functions are not unique for given dynamic systems. Clearly if W is a Lyapunov function then αW where α is a positive constant is also a Lyapunov function. A given nonlinear system may also have Lyapunov functions of different forms.

4.4 SERIES EXPANSION OF THE GATE EQUATIONS

Equations 4.1 will be expanded about a singular point X^o, using assumption 1 of section 3.5.1; hence $\psi(x_i)$ is specified, and 3.1 becomes

$$\frac{dx_i}{dt} = f_i(X) = \frac{1}{\beta_i \gamma_i} x_i(1 - x_i)(\epsilon_i + \sum_j \alpha_{ij} x_j - \beta_i \log\frac{x_i}{1 - x_i}) \qquad 4.17$$

which is zero by definition at all singularities X^o. The matrix H is defined by

$$\frac{\partial f_i(X^o)}{\partial x_i} = \frac{1}{\beta_i \gamma_i}\left\{(1 - 2x_i)(\epsilon_i + \sum_j \alpha_{ij} x_j - \beta_i \log\frac{x_i}{1 - x_i}) + x_i(1 - x_i)\alpha_{ii} - \beta_i\right\}_{X=X^o}$$

$$= \frac{1}{\beta_i \gamma_i}\left\{x_i^o(1 - x_i^o)\alpha_{ii} - \beta_i\right\} \qquad 4.18$$

and

$$\frac{\partial f_i(X^o)}{\partial x_j} = \frac{1}{\beta_i \gamma_i} x_i^o(1 - x_i^o)\alpha_{ij} \qquad 4.19$$

which correspond to general forms 3.2 and 3.3. The higher derivatives may be found by further differentiation of 4.18 and 4.19. No further comments on the resulting series will be made except the following: If $\beta_i \to 0$ with $\beta_i \gamma_i$ finite, 4.17 is an equation of degree three, and therefore all derivatives of order four and higher will be zero.

The alternate form 4.2 may be written from 1.21:

$$g_i(Y) = \frac{dy_i}{dt} = \frac{1}{\beta_i \tau_i}\left\{\epsilon_i - \beta_i y_i + \sum_j \frac{\alpha_{ij}}{1 + e^{-y_j}}\right\} \qquad 4.20$$

and the matrix H is defined in this case by

$$\frac{\partial g_i}{\partial y_i} = \frac{1}{\beta_i \tau_i}\left\{-\beta_i + \frac{\alpha_{ii} e^{-y_i}}{\left(1 + e^{-y_i}\right)^2}\right\}_{Y=Y^o}$$

$$= \frac{1}{\beta_i \tau_i}\left\{-\beta_i + \alpha_{ii} x_i (1 - x_i)\right\}_{X=X^o} \qquad 4.21$$

and

$$\frac{\partial g_i}{\partial y_j} = \frac{1}{\beta_j \tau_j}\left\{\frac{\alpha_{ij} e^{-y_i}}{\left(1 + e^{-y_i}\right)^2}\right\}_{Y=Y^o} = \frac{1}{\beta_i \tau_i}\left\{\alpha_{ij} x_i (1 - x_i)\right\}_{X=X^o} \qquad 4.22$$

which are identical to 4.18 and 4.19.

4.5 ENUMERATION OF SINGULARITIES: THE GENERAL CASE

In order to apply theorems 4.1, 4.2 or 4.3 in any specific case it is necessary to be able to locate all singular points X^o. The case $\beta_i = 0$ for all i was treated in the last chapter and was seen to require the solution of at most n simultaneous linear equations for each singular point. An algorithm was given for enumerating all 3^n possible singular points, some of which may be unrealizable. In the general case n nonlinear equations must be solved, written

$$E - B\Psi(X) + AX = 0 \qquad 4.23$$

from 3.40.

It has been demonstrated in section 3.6.2 that because of the form of the nonlinearity there are at most 3^n solutions to 4.23. From the finiteness assumption the solutions of this equation are the only realizable singularities of the system.

Equation 4.23 is transcendental and it is not possible to write its solution in closed form. Solution of such equations is a standard computational problem for which relaxation or other methods may be used. It is a general requirement, however, that an initial guess of the solution must be made sufficiently near the actual solution for the method to succeed. We take refuge in simplifying assumption 3, section 3.5.1 to ensure that such a guess may always be made, subject to a further constraint.

Assertion 4.1: Let the matrix A in 4.23 be nonsingular. Let the principal singularity $\Gamma = [\gamma_i]$ be defined as that point which by definition satisfies 4.23, and in addition

$$\|\Psi(\Gamma)\| < \|\Psi(X^o)\| \qquad 4.24$$

for all other singular points X^o. For any $\mu > 0$ there exists a number $M(\mu) < \infty$ such that for some real square matrix $C = [c_{ij}]$, if

$$\max_{i,j} |c_{ij}| < M \qquad 4.25$$

then

$$\|\Gamma - \hat{X}\| < \mu \qquad 4.26$$

where $\hat{X}$ is the solution of the equation

$$E + AX = 0 \,. \qquad 3.19$$

Proof: From 4.23 and 3.19,

$$A(\Gamma - \hat{X}) + B\Psi(\Gamma) = 0 \qquad 4.27$$

and thus

$$\Gamma - \hat{X} = A^{-1}B\Psi(\Gamma) \qquad 4.28$$

since A is nonsingular. From the phase plane analysis it is clear that at least one point Γ exists for which $\Psi(\Gamma)$ is finite. Thus if $C = A^{-1}$ which is finite, we can find a finite number M which allows 4.25 for any $\mu > 0$ in 4.26.

Assertion 4.2: Let Γ be the principal singularity of 4.23. Consider the linear equation

$$(E - B\Psi(\Gamma) + A\Gamma) + (A - \text{diag}[\beta_i\, \psi'(\gamma_i)])(X - \Gamma) = 0 . \qquad 4.29$$

obtained from the Taylor series 4.9 with second-degree and higher terms omitted. Let $\hat{X}_o, \hat{X}_1, \ldots, \hat{X}_{3^n}$ be the sequence of singularities of 4.29 enumerated according to, say, the algorithm of section 3.7. Then for every singular point X^o of 4.23 and $\mu > 0$ there exists a number $M < \infty$ such that if either

$$|\alpha_{ii}| > M|\beta_i \psi'(\gamma_i)| \qquad 4.30$$

or

$$|\epsilon_i + \sum_j \alpha_{ij}\hat{x}_j^r| > M \qquad 4.31$$

hold for any $i = 1, 2, \ldots, n$, and some $r \leq 3^n$ in the sequence, then

$$||X^o - \hat{X}^r|| < \mu . \qquad 4.32$$

Proof: If 4.31 holds for gate i and M satisfies theorem 2.3, gate i realizes a linear threshold function and $|x_i^o - \hat{x}_i^r|$ may be specified as arbitrarily small. If 4.30 holds for gate k, say, then for arbitrarily large M the kth equation

$$(\epsilon_k - \beta_k \psi(\gamma_k) + \sum_j \alpha_{kj}\gamma_j) + \left\{\sum_j \alpha_{kj}(x_j - \gamma_j) - \beta_k \psi'(\gamma_k)(x_k - \gamma_k)\right\} = 0 \qquad 4.33$$

must have a solution arbitrarily near $\hat{x}_k^r$ since the ψ and ψ' terms are arbitrarily small compared to α_{ii}.

We remark that these assertions specify conditions which the network must satisfy for good approximate solutions of 4.23 to be found in a specified order. Obviously since the unit hypercube is a bounded set it is always possible to find an unordered sequence of approximate solutions simply by generating sequences of uniformly-spaced points in the set. A closed-form solution for an ordered sequence of good approximations to the singular points of an arbitrary network is not known.

The two-gate diagrams of Fig. 3.11 may be used to illustrate the restriction of assertion 4.2, which admits 3.11 a, b, c, d, i and j and their equivalents but no others.

4.6 STABILITY OF THE LINEARIZED GENERAL SYSTEM

Theorem 2.3 is a specification of the conditions for a gate described by equation 1.21 to realize a linear threshold function. Definition 2.7 and the discussion in section 2.5 related the system to a time-discrete representation. We shall show that theorems 4.1 and 4.2 are sufficient to establish certain conditions under which a collection of gates realizes a linear threshold net, the functional behavior of which is defined for time-discrete[9] representations. Here we treat realization as a generalization of the single-gate realization criteria. The time-continuous behavior of our model of networks of threshold gates has been shown previously to be described by equation 1.38, repeated here (U is assumed zero in the time interval of interest):

$$(TD + I)B(Y - \eta) = A(X - \Gamma) + PU \qquad 1.38$$

<u>Definition 4.4</u>: A physical system described by 1.38 realizes a linear threshold net if, for any integer M there exists an integer $m \geq M$ such that the system is m-discrete, and in addition there exists a number $T < \infty$ such that

$$t_{oi} < T \tag{4.34}$$

for all i, $1 \leq i \leq n$.

Obviously, at any singular point of 1.38 which is near enough to a vertex of the unit hypercube so that every gate realizes a linear threshold function, the system realizes a linear threshold net for all time subsequent to the initial time, that is for any $T > t_o$, provided the system remains at the singular point.

Theorem 4.5: Let a system of n gates be described by 1.38 and let X^o be a singular point. Then there exists a set of numbers $\mu_i > 0$, $1 \leq i \leq n$ such that if either

$$|x_i^o| < \mu_i \tag{4.35}$$

or

$$|1 - x_i^o| < \mu_i \tag{4.36}$$

hold for $1 \leq i \leq n$, all solutions with initial points $X(t_o)$ in a region Ω_1 containing X^o approach X^o as $t \to \infty$.

Proof: The system will be shown to satisfy theorem 4.2. From 3.2 and 3.3 the linearized system obtained from the Taylor series 4.9 at the point X^o has the matrix

$$H = \operatorname{diag}\left[\frac{1}{\beta_i \tau_i \psi'(x_i^o)}\right] A - \operatorname{diag}\left[\frac{1}{\tau_i}\right] . \tag{4.37}$$

Some results in matrix theory will be quoted:

Lemma 4.1[10]: Let H be a real square matrix. The quadratic form $Q = Z^T HZ$ is equal to the quadratic form of the symmetric part of H, i.e.

$$Q = \tfrac{1}{2}Z^T(H + H^T)Z \; . \tag{4.38}$$

Lemma 4.2[11,12]: A real square matrix $H(Z)$ for which

$$h_{ii} - \sum_{j \neq i} |h_{ij}| \geq \mu > 0 \tag{4.39}$$

is said to be uniformly Hadamard. If H is uniformly Hadamard and symmetric, then it is positive-definite, i.e.

$$Z^T HZ > 0 \tag{4.40}$$

for all $Z \neq 0$.

Consider the matrix $G = -H$. If G is positive-definite then H is negative-definite. The symmetric part of G is

$$G_s \triangleq \mathrm{diag}\left[\frac{1}{\gamma_i}\right] - \mathrm{diag}\left[\frac{1}{2\beta_i \gamma_i \psi'(x_i^o)}\right](A + A^T) \; . \tag{4.41}$$

Since G_s is a constant matrix it is uniformly Hadamard provided

$$\frac{1}{\gamma_i} - \frac{1}{2\beta_i \gamma_i \psi'(x_i^o)}\left\{2\alpha_{ii} + \sum_{j \neq i} |\alpha_{ij} + \alpha_{ji}|\right\} > 0, \; 1 \leq i \leq n \; . \tag{4.42}$$

Let 4.42 hold. Then by lemma 4.2 G_s is positive-definite, and by lemma 4.1 G is also positive-definite, and hence H is negative-definite. Consider the positive-definite scalar function

$$W = \tfrac{1}{2}Z^TZ \ . \tag{4.43}$$

The total derivative is

$$\dot{W} = (\nabla W)^T\dot{Z} = Z^THZ \tag{4.44}$$

which is negative-definite in an unbounded region Ω containing the origin and thus the linear system

$$\dot{Z} = HZ \equiv H(X - X^o) \tag{4.45}$$

is asymptotically stable in the large.

The high-order terms of the Taylor series obey the equation

$$\lim_{|Z|\to 0} \frac{|\overline{F}(Z)|}{|Z|} \to 0 \tag{4.46}$$

in a region Ω_1 containing $Z = 0$ which corresponds to $X = X^o$ and thus theorem 4.2 is true for this linear system provided 4.42 is true. Now from 4.42

$$\psi'(x_i^o) > \frac{1}{2\beta_i}\left\{2\alpha_{ii} + \sum_{j\neq i}|\alpha_{ij} + \alpha_{ji}|\right\} \tag{4.47}$$

and since (from section 1.6.1) $\psi'(x_i) \to +\infty$ as $x_i \to 0$ or $x_i \to 1$ there exists a set of numbers $\mu_i > 0$ for which 4.47 is satisfied when one of 4.35 or 4.36 is satisfied. This completes the proof.

The following comments may be made about the theorem:

1. It provides a proof that singular points sufficiently near to the vertices of the unit hypercube are stable. Examples are provided in Fig. 3.11; all singular points "near" corners of the unit square are stable.

2. Initial conditions sufficiently close to appropriate stable singular points imply that after a time the system realizes a linear threshold net. This will be expressed as

Corollary 4.5.1: Let there exist numbers a_i and b_i with $0 < a_i < b_i < 1$ and singular point X^o which satisfies, for arbitrary $\mu > 0$,

$$|x_i^o - \zeta_i| < \mu, \ 1 \leq i \leq n \qquad 4.48$$

where ζ_i equals either a_i or b_i. Let X^o satisfy theorem 4.5. Then there exists a number T such that if the initial point $X(t_o)$ is in Ω_1 the system realizes a linear threshold net in (T, ∞).

Proof: From theorem 4.5 if $X(t_o) \in \Omega_1$, $X(t) \to X^o$ as $t \to \infty$. Hence there exists T such that

$$|x_i(t) - \zeta_i| < \mu, \ 1 \leq i \leq n \qquad 4.49$$

for all $t > T$ and thus both definitions 2.6 and 4.4 are satisfied in (T,∞), i.e. the system realizes a linear threshold net in (T,∞).

3. The theorem and corollary specify sufficient conditions for a system to realize a linear threshold net at or near a singular point. It is important to observe that there is no reason to suppose that constraint 4.47 under which the theorem is true is also a necessary condition.

A theorem of instability will be quoted:

Theorem 4.64: The singular point $Z = 0$ of 4.11 is unstable if $\bar{F}(Z)$ is continuous in a region about $Z = 0$, if $\frac{|\bar{F}(Z)|}{|Z|} \to 0$ as $|Z| \to 0$, and if the matrix H possesses at least one eigenvalue with positive real part.

We now consider singularities of the system which do not satisfy theorem 4.5. Attention will first be given to those "near" the edges of the unit hypercube.

Theorem 4.7: Let $\mathcal{I} = \{i_1, \ldots, i_n\}$ be the set of n threshold gates in system 1.38. Let $\mathcal{I}_1$ be a proper subset of $\mathcal{I}$ and let X^o be a singular point. There exist $\mu > 0$ and M such that if $i \in \mathcal{I}$,

$$|x_i^o(1 - x_i^o)| < \mu \qquad 4.50$$

for all $i \in \mathcal{I}_1$ and

$$\alpha_{ii} > M \qquad 4.51$$

for all $i \notin \mathcal{I}_1$, then X^o is unstable.

Proof: It will be established that the system satisfies theorem 4.6. Consider the matrix of the linearized system given by 4.37.

Let the number μ approach 0. Then for 4.50 to hold, the quantity $\beta_i \tau_i \psi'(x_i^0)$ must approach $+\infty$. Hence for any positive number, μ_1 say, there exists $\mu > 0$ such that 4.50 holds and furthermore

$$\sum_{i \in \mathcal{J}_1} \frac{\alpha_{ii}}{\beta_i \tau_i \psi'(x_i^0)} < \mu_1 , \qquad 4.52$$

and thus

$$\sum_{i \in \mathcal{J}_1} h_{ii} < \mu_1 - \sum_{i \in \mathcal{J}_1} \frac{1}{\tau_i} . \qquad 4.53$$

The quantity $\beta_i \tau_i \psi'(x_i^0)$ is finite. Hence for any positive number, μ_2 say, a number M exists which satisfies

$$\sum_{i \notin \mathcal{J}_1} \left\{ \frac{M}{\beta_i \tau_i \psi'(x_i^0)} - \frac{1}{\tau_i} \right\} - \mu_2 = 0 , \qquad 4.54$$

in which case

$$\sum_{i \notin \mathcal{J}_1} h_{ii} - \mu_2 > 0 . \qquad 4.55$$

Since μ_1 may be chosen arbitrarily small, there exists a number μ_2 which satisfies the inequality

$$\sum_{i \notin \mathcal{J}_1} h_{ii} > \mu_2 > \sum_{i \in \mathcal{J}_1} \frac{1}{\tau_i} - \mu_1 \qquad 4.57$$

in which case

$$\sum_{i \in \mathcal{J}} h_{ii} > 0 . \qquad 4.58$$

Now consider the characteristic equation 4.13 of H. From[13] elementary algebra the sum of the n roots of 4.13 is equal to tr(H), i.e. if $c_0 \equiv 1$,

$$c_1 = -\mathrm{tr}(H) = -\sum_{i=1}^{n} \lambda_i \qquad 4.59$$

and thus if the trace is positive, at least one root must have positive real part. Equation 4.58 and 4.9 thus imply that the system satisfies theorem 4.6 and is unstable.

The final result of this section concerns the point Γ, which generally does not satisfy theorem 4.5, in which case it must be unstable for the system to realize a linear threshold net. We show a necessary restriction on the B-matrix.

<u>Theorem</u> <u>4.8</u>: There exist numbers $\hat{\beta}_i > 0$ such that if

$$\beta_i > \hat{\beta}_i \qquad 4.60$$

for all $i = 1, 2, \ldots, n$, all solutions with initial conditions in a neighborhood Ω_1 of Γ approach Γ as $t \to +\infty$.

<u>Proof</u>: The argument is similar to the proof of theorem 4.5. The matrix H of the linearized system at Γ is to be shown to be negative-definite.

By definition the quantity $\tau_i \psi'(\gamma_i)$ is finite. Therefore a set of numbers $\hat{\beta}_i$ exists for which

$$\frac{1}{\tau_i} - \frac{1}{2\hat{\beta}_i \tau_i \psi'(\gamma_i)} \left\{ 2\alpha_{ii} + \sum_{j \neq i} |\alpha_{ij} + \alpha_{ji}| \right\} > 0, \; 1 \leq i \leq n \qquad 4.61$$

and therefore from lemma 4.1 and 4.2, if 4.60 and 4.61 hold, the scalar function 4.43 has total derivative 4.44 which is negative-definite. Hence theorem 4.2 is satisfied and Γ is stable.

Some comments will be made:

1. Theorems 4.5, 4.7 and 4.8 establish sufficient conditions for their respective stability or instability results. The proofs are existence proofs and do not give tight bounds for the theorems to be true. It is quite possible, for example, that a characteristic polynomial whose second coefficient (c_1) is positive will have roots in the right half-plane. The most useful test in any specific case is to find the characteristic polynomial and test it for roots with positive real part.

2. The theorems allow a conclusion about the realization of linear threshold nets:

Assertion 4.3: If a threshold system has the properties

(a) all singular points satisfy theorem 4.5 or 4.7,

(b) all eigenvalues of the matrix H of the linearized system are real at Γ and at points satisfying theorem 4.5,

(c) Γ is unstable,

then the system realizes a linear threshold net.

Proof: Property (b) ensures that no stable closed trajectories (stable limit-cycles) exist, since if one did exist then it would be possible to find a linear approximation which has complex eigenvalues. Properties (a) and (c) ensure that all stable points realize a linear threshold net. Therefore a system which satisfies these properties must approach a singular point at which it realizes a linear threshold net.

Fig. 3.11 b and d both satisfy the assertion, and systems of higher order may also satisfy it. It is straightforward to show for example that a system whose coupling coefficients α_{ij}, $j \neq i$ all have the same sign has real eigenvalues and satisfies the assertion provided it has properties (a) and (c).

4.7 STABILITY OF THE LINEARIZED SPECIAL SYSTEM

We consider the n-gate special system described by the equation

$$BT\frac{d}{dt}\Psi(X) = E + AX \qquad 4.62$$

which is obtained from 2.46. Solutions will be examined in the neighborhood of singular points X^o enumerated as in section 3.7. The equation of first variation at X^o may be obtained from 4.18 and 4.19, in which $\psi(x_i)$ has been specified:

$$h_{ii} = \frac{\partial f_i(X^o)}{\partial x_i} = \frac{1}{\beta_i \gamma_i}\left\{(1 - 2x_i^o)(\epsilon_i + \sum_j \alpha_{ij} x_j^o) + x_i^o(1 - x_i^o)\alpha_{ii}\right\} \qquad 4.63$$

$$h_{ij} = \frac{\partial f_i(X^o)}{\partial x_j} = \frac{1}{\beta_i \gamma_i} x_i^o(1 - x_i^o)\alpha_{ij} \ . \qquad 4.64$$

The eigenvalues of the matrix H depend on the relative values of the quantities $\beta_i \gamma_i \psi'(x_i^o)$ and the elements of A, whereas in the general case the absolute value of the parameters β_i also influence system stability.

The principal singularity Γ of the special system has been defined as the solution of the equation

$$E + AX = 0 \text{ .} \qquad 3.19$$

It will be assumed here that such a solution exists and is unique; systems which do not have unique solutions will be discussed later. Provided Γ is in the unit hypercube its stability has physical meaning. The matrix is

$$H = \text{diag}\left[\frac{1}{\beta_i \gamma_i \psi'(\gamma_i)}\right] A \text{ .} \qquad 4.65$$

Little is known about the behavior of eigenvalues with respect to Γ for arbitrary matrices A.

Singularities at which $x_i \in \{0,1\}$ for all i realize linear threshold nets by definition, provided they are stable. In this case the matrix is

$$H = \text{diag}\left[\frac{-1}{\beta_i \gamma_i}(1 - 2x_i^o)(\epsilon_i + \sum_j \alpha_{ij} x_j^o)\right] \qquad 4.66$$

and the eigenvalues are the nonzero elements of H. The system is stable provided all eigenvalues are negative. Obviously if x_i^o is 1 and $\epsilon_i + \sum_j \alpha_{ij} x_j^o$ is positive then the solution $x_i(t)$ will tend to x_i^o provided the initial point is near enough to x_i^o. A similar situation exists for $x_i^o = 0$, in which case the excitation quantity must be negative for stability. The terms $(1 - 2x_i^o)$ in 4.66 confirm that in both cases the eigenvalues are negative. The choice of the logistic curve gives this simple expression.

<u>Assertion 4.4</u>: If the special system is unstable at a vertex X^o of the unit hypercube and no solution γ_i of 3.19 equals x_i^o, then the general system shich satisfies assertions 4.1 and 4.2 has no realizable singularity arbitrarily close to X^o.

<u>Proof</u>: Assertions 4.1 and 4.2 ensure that no more than one singularity of the general system is near X^o provided no γ_i equals x_i^o. If such a singularity exists it is stable by theorem 4.5. If it is stable then the singularity X^o of the special case must also be stable. We show this as follows: Assume x_i^o is 1. Let the singular point of the general system be denoted by X^1. If $x_i^1 \rightarrow 1$ the quantity $-\beta_i \psi(x_i^1)$ is negative and the quantity $\epsilon_i + \sum_j \alpha_{ij} x_j^1$ must be positive. Hence the quantity $\epsilon_i + \sum_j \alpha_{ij} x_j^o$ is also positive and x_i^o is stable. A similar argument applies near $x_i^o = 0$.

Points X^o for which some x_i^o are 0 or 1 and others are in (0,1) have the character of the point Γ and of the vertex points and must therefore be treated individually.

4.8 REDUCTION OF THE ORDER OF A NETWORK

We consider some of the situations which may be encountered in the analysis of general linear threshold nets. Let A be the weight matrix of an arbitrary net $\mathcal{J}$ of order n.

Assertion 4.5: Let $\alpha_{ij} = 0$ for all j. Let $\hat{A}$ be the matrix obtained from A by removing the i^{th} row and i^{th} column. Then the asymptotic stability of the system $\hat{\mathcal{J}}$ characterized by $\hat{A}$ is identical to the system $\mathcal{J}$ characterized by A.

Proof: If A has a zero row it has an element whose inputs are the "external" input set U_k. Thus its output x_i is

(a) asymptotically stable since we assume that U_k is constant over the time interval of interest, and

(b) an input to system $\hat{\mathcal{J}}$.

Hence if $\hat{\mathcal{J}}$ is asymptotically stable, $\mathcal{J}$ is also asymptotically stable, and if $\hat{\mathcal{J}}$ is unstable, $\mathcal{J}$ is unstable.

Assertion 4.6: Let $\alpha_{ij} = 0$ for all i. Let $\hat{A}$ be the matrix A with the j^{th} column and the j^{th} row removed. Then the asymptotic stability of system $\hat{\mathcal{J}}$ characterized by $\hat{A}$ is identical to that of $\mathcal{J}$ characterized by A.

Proof: If A has a zero column it has an element j whose output does not affect the other gates in the system. Its inputs are the external input set and the variables x_i, $1 \leq i \leq n$, i.e. gate j is dynamically dependent on the rest of the system, which is $\hat{\mathcal{J}}$. Hence the stability of the reduced system $\hat{\mathcal{J}}$ determines the stability of $\mathcal{J}$.

Assertion 4.7: Let $\hat{\mathcal{J}}$ be a proper subset of $\mathcal{J}$. If $\alpha_{ij} = 0$ for all $i \in \hat{\mathcal{J}}$ and $j \notin \hat{\mathcal{J}}$ or for all $i \notin \hat{\mathcal{J}}$ and $j \in \hat{\mathcal{J}}$ then the system stability is determined by the distinct networks $\hat{\mathcal{J}}$ and $\mathcal{J} - \hat{\mathcal{J}}$.

Proof: Let $\alpha_{ij} = 0$ for all $i \in \hat{\mathcal{J}}$ and $j \notin \hat{\mathcal{J}}$. This constraint specifies that system $\hat{\mathcal{J}}$ may affect the rest of the system if some $\alpha_{ji} \neq 0$ for $i \in \hat{\mathcal{J}}$ and $j \notin \hat{\mathcal{J}}$, but $\hat{\mathcal{J}}$ is independent of $\mathcal{J} - \hat{\mathcal{J}}$. Hence if $\hat{\mathcal{J}}$ is stable, $\mathcal{J}$ is stable if $\mathcal{J} - \hat{\mathcal{J}}$ is stable, and if $\hat{\mathcal{J}}$ is unstable then $\mathcal{J}$ is unstable. A similar argument holds for the alternate constraint.

Assertions 4.5 and 4.6 are special cases in which the rank of A is less than n, the order of the system. It has been argued[14] that time-discrete threshold networks are characterized by the rank of the weight matrix, rather than the order. Here this result will be generalized and proved for our continuous model. We shall use

Definition 4.5: Two networks are logically similar if there exists a nonsingular transformation such that every stable singular point of one network is mapped one-to-one onto a stable singular point of the other, and in addition every point on a stable trajectory of one is mapped one-to-one onto a stable trajectory of the other.

Theorem 4.9: Let A be the $n \times n$ weight matrix of an n-gate linear threshold net. Let the rank of A be $r \leq n$. Then the threshold net is logically similar to a network of r gates (not necessarily threshold gates).

Proof: An elementary result[15] in matrix theory will be quoted:

Lemma 4.3: Let A be a square matrix of order n and rank $r \leq n$. There exists a nonsingular matrix Q such that

$$AQ = C \qquad 4.67$$

which is of the form

$$C = \left[\begin{array}{cccc|c} c_{11} & & & 0 & \\ \cdot & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ c_{r1} & \cdot & \cdot & c_{rr} & 0 \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ c_{n1} & \cdot & \cdot & c_{nr} & \end{array}\right]. \qquad 4.68$$

Consider the zero-input system 1.38 with transformations 1.40 and 1.41. The state equation is

$$(TD + I)V = AZ\ . \qquad 4.69$$

From lemma 4.3 there exists a nonsingular matrix Q such that, by the linear transformation

$$Z = Q\Theta\ , \qquad 4.70$$

the equation becomes

$$(TD + I)V = AQ\Theta \qquad 4.71$$

where AQ is of the form of 4.68. The solution of 4.71 is

$$V(t) = e^{-tT^{-1}}V(0) + \int_0^t G_\delta(t - \tau)\Theta(\tau)d\tau \qquad 4.72$$

where

$$G_\delta = \mathcal{L}^{-1}\left\{(Ts - I)^{-1}AQ\right\} \qquad 4.73$$

which is also of the form of 4.68. Equation 4.71 defines the dynamical behavior of a certain network with output vector Θ. Since X is the output of a network of threshold gates, the state-space of which is a cube of n dimensions, the state-space of system 4.71 is a linear transformation of the X-space, and is a parallelpiped of n dimensions. But from 4.72 and 4.73 it is clear that after an initial transient (after the first term on the right-hand side of 4.72 has become negligible) the first r elements of V are independent of the rest, and the last $n - r$ elements are dependent on, i.e. have as inputs, only the first r elements. Thus as $t \to \infty$, trajectories in V-space are constrained to a manifold of r dimensions, and thus there are exactly r degrees of freedom in the Θ-space. In such a case there are at most 3^r stable singularities of this system and since transformation 4.70 is nonsingular, there are at most 3^r stable singular points of 4.69. Hence network 4.69 is logically equivalent to a network of order r.

Only if assertion 4.5, 4.6 or 4.7 holds is it clear that the threshold network is similar to one or more smaller threshold nets. Nevertheless theorem 4.9 specifies a fundamental limitation on the logical behavior of threshold nets with singular weight matrices.

4.9 SUMMARY

The behavior of autonomous systems in the neighborhood of singular points is determined by finding the eigenvalues of the matrix of the linearized system at each point. Such techniques may be applied to the gate network model, provided the singular points can be found.

Each singular point is near a point of a system similar to that discussed in the previous chapter, provided the magnitudes $|\alpha_{ij}|$ are larger than a certain finite number.

The analysis of the stability of linearized systems enables conclusions to be reached about the realization of linear threshold nets by groups of threshold gates. Further, it leads to methods of reducing large networks to groups of subnetworks for purposes of analysis.

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CHAPTER 5

THE SECOND METHOD OF LYAPUNOV APPLIED TO THE NETWORK MODEL

5.1 INTRODUCTION

The stability analyses of chapter 4 are limited exclusively to linearizations of the threshold system at singular points. Such linearizations approximate the nonlinear system with accuracy within small neighborhoods of the singularities, but it is also desirable to predict the global behavior. Usually the initial point of a system will not be within such a neighborhood, and it is necessary to predict the stable singular point at which the system will come to rest if, in fact, there is such a point. There may be a stable limit-cycle, in which case the network exhibits a continuous stable oscillation of constant period (in the absence of perturbations of the system) and therefore may realize a linear threshold net in a manner which cannot be predicted using the theorems of the previous chapter. Finally the Lyapunov functions which are required in determining global stability are often of the form of potenial functions and may be analogous to energy in physical dynamics, and may therefore provide insight into the behavior of the network model.

The method for the analysis of a general network is to proceed according to the discussion in chapter 4. The system may be reduced to one or more minimal nonsingular systems, and these may then be considered separately for stability analyses. The nature of the singular points should be investigated to determine the local stability near them. Finally if more information is required the techniques to be discussed in this chapter may be used.

We shall attempt to present some general results using various techniques for the generation of Lyapunov functions. Relevance to previous discussion will also be mentioned where appropriate.

5.2 LYAPUNOV FUNCTIONS OF THE GENERAL SYSTEM

We shall work with equation 4.69, which is a concise description of the threshold system:

$$(TD + I)V = AZ \ . \qquad 4.69$$

An alternative representation is the equation

$$\dot{Z} = \mathrm{diag}\left[\frac{1}{\beta_i \tau_i \psi'(x_i)}\right](AZ - V) \qquad 5.1$$

which obtains directly from 1.38 or 4.69.

By definition these equations have singular points at $V = Z = 0$, and the vectors V and Z are mutually dependent.

Some properties of Lyapunov-type functions in a neighborhood Ω of singular points will be recalled:

- Positive-definiteness or semidefiniteness: $W(X^o) = 0$, and $W(X) \geqslant 0$, $X \neq X^o$, $X \in \Omega$.
- Uniqueness: If W is unique, it satisfies

$$\frac{\partial^2 W(X)}{\partial x_i \partial x_j} = \frac{\partial^2 W(X)}{\partial x_j \partial x_i}, \quad i \neq j \qquad 5.2$$

for all X in Ω. This is equivalent to a generalized curl equation, i.e.

$$\nabla \times \nabla W = 0 \ . \qquad 5.3$$

- If

$$\dot{W}(X) \leq -\nu W(X) \qquad 5.4$$

where ν is an arbitrary positive constant, then

$$W(X(t)) \leq W(X(0))e^{-\nu t} \qquad 5.5$$

which approaches 0 as $t \to \infty$. Since W is continuous and positive-definite this equation implies that $X \to 0$ as $t \to \infty$. Let $W(X) = c \neq 0$ be the smallest value of W for which $\dot{W} = 0$. Then all solution trajectories with initial conditions within the region Ω bounded by $W(X) = c$ remain within Ω for all $t > 0$, and the system satisfies theorem 4.3 and is therefore asymptotically stable in the large. If ν is an arbitrary negative constant then the above discussion applies to a system which is unstable, and whose solution trajectories leave Ω as $t \to \infty$. If

$$\dot{W} \equiv 0 \qquad 5.6$$

for all X in a region containing 0 the system is said to be conservative.

Finally we quote a theorem which guarantees the existence of Lyapunov functions for systems such as ours, described in general by 4.11.

Theorem 5.1[1]: Let 4.11 describe an autonomous Lipschitzian system, and let $Z = 0$ be asymptotically stable in the large. Then there exists a Lyapunov function $W(Z)$ which is infinitely differentiable with respect to Z.

A consequence of the theorem is that every unstable system has a suitable function W whose time-derivative is negative-definite, and every conservative system such a function with derivative identically zero.

Unfortunately no general rules exist for writing down expressions for Lyapunov functions. Methods do exist[2] but suitable functions are not guaranteed. We shall apply several such methods, beginning with the assumption that stability in a neighborhood of each singular point has been established by the methods of the previous chapter, and that the system is in minimal form.

It is a general requirement that the region Ω be as large as possible in order that the strongest possible conditions for system behavior may be established.

5.2.1 VARIABLE GRADIENT METHODS

We assume that the gradient of W is of the form

$$\nabla W(Z) = PZ \qquad 5.7$$

or

$$\nabla W(V) = PV \, . \qquad 5.8$$

The function W is required to satisfy 5.2 and be sign-definite, i.e. it is required to satisfy 5.3 and 5.4 or 5.6. P need not be a constant matrix. We are thus required to find a matrix P and, if possible the matrix which gives the largest region Ω.

Consider a constant matrix P and a quadratic form

$$W(V) = \tfrac{1}{2}V^{T}PV \, , \qquad 5.9$$

for which the curl equation is

$$\frac{\partial \nabla_i W(V)}{\partial v_j} = \frac{\partial}{\partial v_j} \sum p_{ik} v_k = p_{ij} = \frac{\partial \nabla_j W(V)}{\partial v_i} = p_{ji} \;, \qquad 5.10$$

i.e. P is symmetric. If $W(V)$ is to be a Lyapunov function P must also be positive-definite. We quote some useful results from matrix theory[3], recalling lemma 4.1, which states that the symmetric part of a matrix determines its quadratic form.

<u>Definition 5.1</u>: For a symmetric matrix $A = [a_{ij}]$, the leading principal minors are:

$$p_o = 1,\ p_1 = a_{11},\ p_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \ldots,\ p_n = |A| \;. \qquad 5.11$$

<u>Theorem 5.2</u>: A real symmetric matrix A of order n is positive-definite if and only if its rank is n and all leading principal minors are positive.

Consider system 4.69, which may be rewritten

$$\dot{V} = -T^{-1}V + T^{-1}AZ \;. \qquad 5.12$$

We require the derivative

$$\dot{W} = -V^T P T^{-1} V + V^T P T^{-1} A Z \qquad 5.13$$

to be sign-definite in a region Ω which depends on P, and furthermore we require that solutions enter or remain in Ω for positive t.

Assertion 5.1: Let $V = Z = 0$ be an asymptotically stable point of 5.12. Then there exists a constant, symmetric, positive-definite matrix P for which 5.13 is at least negative semidefinite in a region Ω which may be vanishingly small. If $V = Z = 0$ is unstable there exists a positive-definite matrix P for which 5.13 is at least positive-semi-definite in Ω, which may be vanishingly small.

Proof: If $V = Z = 0$ is asymptotically stable then its linear approximation

$$\dot{V} = HV \qquad 5.14$$

is a valid description of the system in a region Ω containing 0 which may be vanishingly small. Thus by theorem 4.4 there exists a constant matrix P which satisfies the theorem in Ω. The proof of the unstable case follows directly from the stable case.

The following considerations arise:

1. It is desired to make the region of stability Ω finite, and as large as possible, that is, we wish to find Ω such that if the initial point is outside Ω, system behavior is different in character from behavior for initial conditions in Ω.

2. A constant matrix P may not give rise to a satisfactory region Ω.

3. P need not be a constant matrix. However it is extreme-

ly difficult in the general case to find a non-constant matrix which satisfies the curl equation, the positive-definiteness criterion, and which has a sign-definite derivative.

4. In general the easiest method for ensuring that trajectories remain in Ω is to specify that the boundary of Ω is a level contour $W = c$.

5. There are exactly $n(n+1)/2$ independent elements of the matrix P which may be adjusted so that Ω is suitable.

An elementary example will be used to illustrate the above discussion. Consider the completely symmetrical bistable circuit containing two gates described by the matrices

$$T = I , \quad B = I ,$$
$$E = \begin{bmatrix} 10 \\ 10 \end{bmatrix} , \quad A = \begin{bmatrix} 0 & -20 \\ -20 & 0 \end{bmatrix} . \qquad 5.15$$

The phase diagram for this system is Fig. 3.11d, which is repeated in Fig. 5.1a. Obviously since the system is symmetric all trajectories below the diagonal approach the singular point near (1,0) and all trajectories above the diagonal approach the singular point near (0,1). The point $\Gamma = (\frac{1}{2},\frac{1}{2})$ is a saddle point. The system is to be transformed to the form of 4.69, taking, for example, the singular point near (1,0) as the origin. We have

$$\begin{aligned} 10 - \psi(x_1^o) - 20x_2^o &= 0 \\ 10 - \psi(x_2^o) - 20x_1^o &= 0 \end{aligned} \qquad 5.16$$

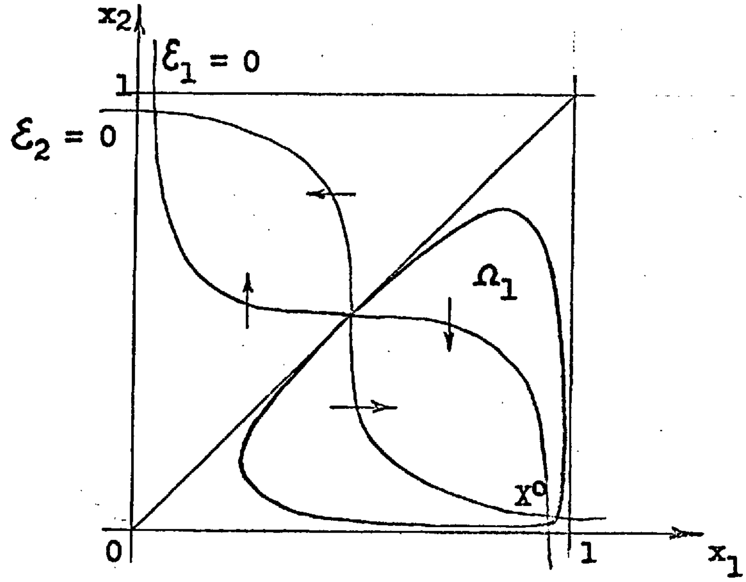

Fig. 5.1a A Symmetric Bistable

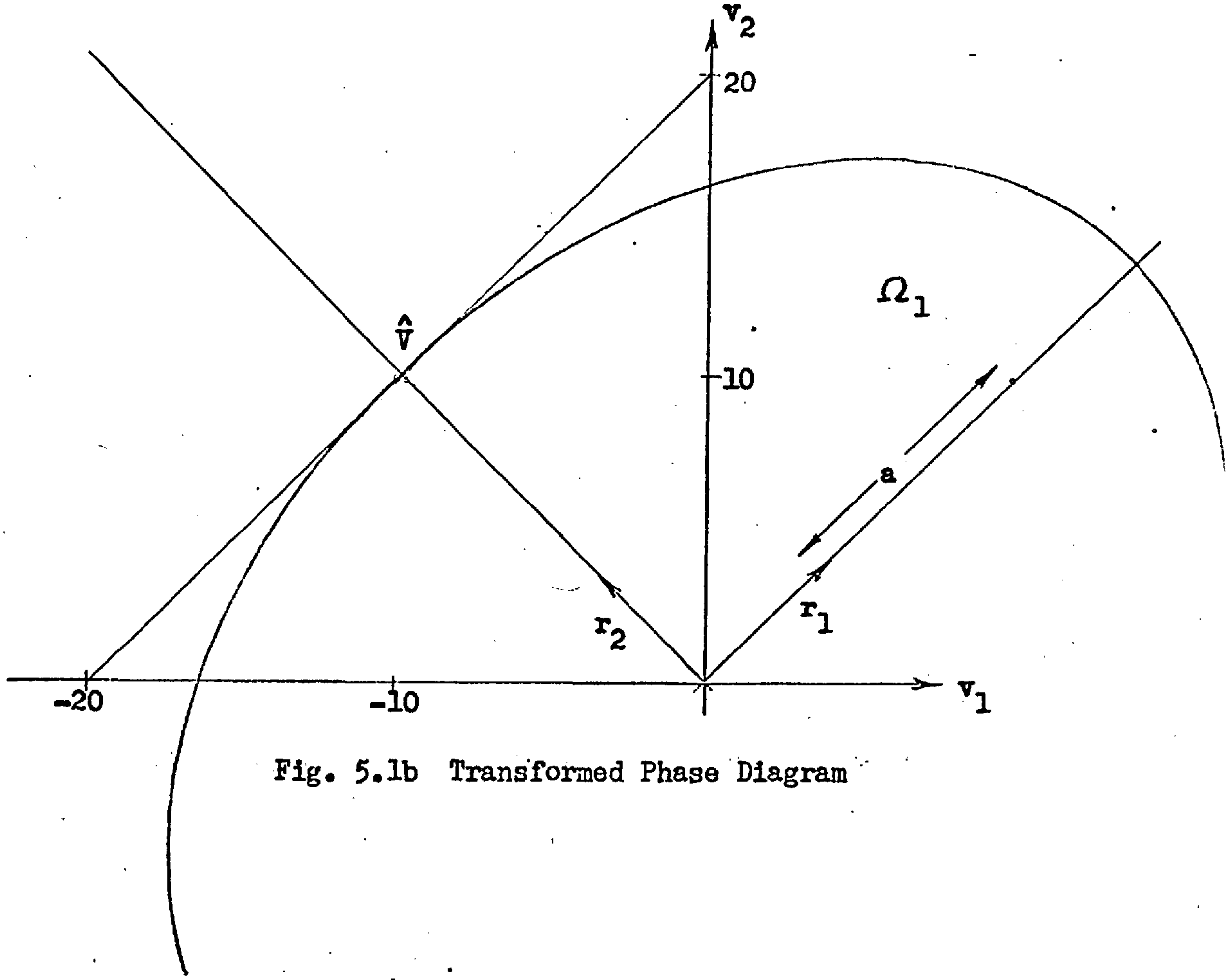

Fig. 5.1b Transformed Phase Diagram

the solution of which is

$$\begin{aligned} x_1^o &= 1 - \eta \\ x_2^o &= \eta \end{aligned} \qquad 5.17$$

where η is a small positive number (of the order of 4.5×10^{-5}). Hence

$$\begin{aligned} \psi(x_1^o) &= 10 - 20\eta \\ \psi(x_2^o) &= -10 + 20\eta \end{aligned} \qquad 5.18$$

from which

$$\begin{aligned} v_1 &= \psi(x_1) - 10 + 20\eta \\ v_2 &= \psi(x_2) + 10 - 20\eta \\ z_1 &= x_1 - 1 + \eta \\ z_2 &= x_2 - \eta \end{aligned} \qquad 5.19$$

and these transformations give the required form.

The unit square is mapped one-to-one onto the V-plane which is shown in Fig. 5.1b. The point nearest the origin at which equation 5.13 is zero is the point $\hat{V} = (-10+20\eta,\ 10-20\eta)$ which is the image of Γ. We choose Ω to be an ellipse with centre at the origin, minor axis $0 - \hat{V}$ and arbitrarily large major axis a, say. That is, in a set of orthogonal coordinates r_1, r_2, the equation of the ellipse is

$$\frac{r_1^2}{a^2} + \frac{r_2^2}{\|\hat{V}\|^2} \equiv R^T \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{\|\hat{V}\|^2} \end{bmatrix} R \equiv R^T \Lambda R = 1 . \qquad 5.20$$

To write this equation in terms of v_1, v_2 we find an orthogonal transformation $R = SV$, for example

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad 5.21$$

from which we have

$$V^T S^T \Lambda S V = \tfrac{1}{2} V^T \begin{bmatrix} \frac{1}{a^2} + \frac{1}{\|\hat{V}\|^2} & \frac{1}{a^2} - \frac{1}{\|\hat{V}\|^2} \\ \frac{1}{a^2} - \frac{1}{\|\hat{V}\|^2} & \frac{1}{a^2} + \frac{1}{\|\hat{V}\|^2} \end{bmatrix} V = 1 \,. \qquad 5.22$$

An ellipse, Ω_1 say, defined by this quadratic form transforms into the region indicated in Fig. 5.1a. In the limit we may take the major axis a to be infinitely large, in which case the stable region approaches the total area below the diagonal.

The procedure used above may be generalized to arbitrary networks and dimensions in a straightforward manner. A symmetric positive-definite matrix P always results from the choice of an ellipsoid as the region Ω. One or more dimensions of the ellipsoid may be arbitrarily large, in which case P is positive semidefinite.

5.2.2 FREQUENCY-PLANE STABILITY CRITERIA

Generalizations of the Popov[4] frequency-plane criterion to systems with many nonlinearities of a certain form are available[5,6,7]. We quote a result which gives necessary and sufficient conditions for the existence of Lyapunov functions of the form

$$W = \int_0^V Z^T dV + V^T P V \qquad 5.23$$

for the threshold system, where P is symmetric. Define the function $\phi_i(v_i)$ to be

$$z_i = \phi_i(v_i) \triangleq \psi^{-1}\left\{\frac{v_i}{\beta_i} + \psi(x_i^o)\right\} - x_i^o \equiv \psi^{-1}(y_i) - x_i^o \qquad 5.24$$

where x_i^o is the singular point to be investigated. From the shape of the function $\phi_i(v_i)$ it is clear that the integral in 5.23 is always positive. Thus W is positive definite if, but not only if P is positive semidefinite.

Consider 5.2a, which is a block diagram of a type of nonlinear multivariate control system in which R, V, Z and S are n-vectors. Φ is a set of n time-invariant, inertialess nonlinear gain elements whose outputs are $\phi_i(v_i)$ where

$$0 \leq v_i\phi_i(v_i) \leq k_i v_i^2 . \qquad 5.25$$

This inequality is referred to by stating that ϕ_i is confined to the sector $[0,k_i]$. G is a linear, time-invariant subsystem and the closed-loop system is described by the equation

$$-V(t) = -V_o(t) + \int_0^t G_\delta(t-\tau)Z(\tau)d\tau \qquad 5.26$$

where $V_o(t)$ is a linear function of the zero-input response of G and the input $R(t)$ which is assumed zero or a small disturbance which eventually dies away. It is assumed that the following conditions apply for $1 \leq i,j \leq n$:

1. For all inputs and initial states

(a) v_i is bounded,

(b) $v_{oi}(t) \to 0$ as $t \to \infty$,

(c) $v_{oi}, \dot{v}_{oi} \in L_2(0,\infty)$ where the notation $s(t) \in L_r(a,b)$ means

$$\left\{\int_a^b |s(t)|^r dt\right\}^{1/r} < \infty. \qquad 5.27$$

2. $g_{ij} \in L_1(0,\infty)$.

Systems of the above description are said to be in the Lur'e form[9], and for purposes of analysis are often transformed into equivalent canonical systems by expanding the linear functions $g_{ij}(s)$ in partial fractions. Note that the first-order form assumed for g_{ij} implies that the threshold system model is in a canonic form, but the following theorem applies for higher-order rational transfer functions with all poles in the left half-plane.

Let

$$Q = \mathrm{diag}[q_i] \qquad 5.28$$

$$K = \mathrm{diag}[k_i] \qquad 5.29$$

be real $n \times n$ matrices and let

$$\mathcal{R}(j\omega) = [I + j\omega Q]G(j\omega) + K^{-1} \qquad 5.30$$

$$F(j\omega) = \mathcal{R}(j\omega) + \mathcal{R}^*(j\omega) \qquad 5.31$$

where $\mathcal{R}^*$ is the complex conjugate transpose of $\mathcal{R}$. The elements k_i of K are to satisfy 5.25.

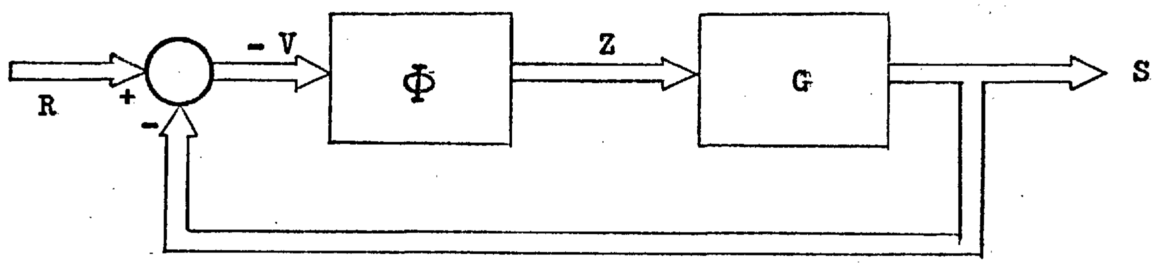

Fig. 5.2a A Multivariate Control System

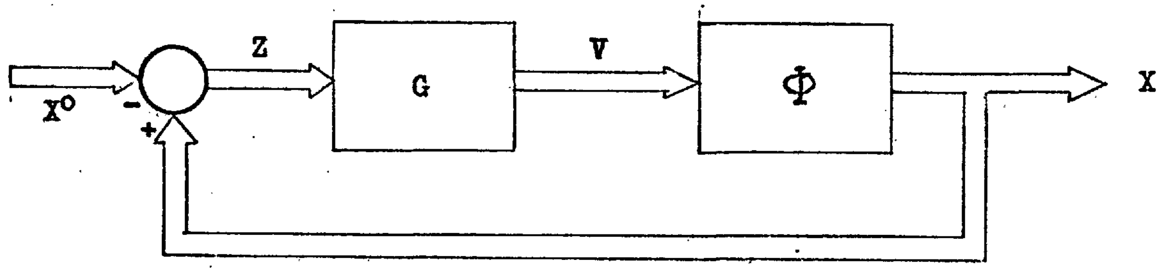

Fig. 5.2b A Threshold Network

Theorem 5.3[8]: Let the system of Fig. 5.2a satisfy the above conditions. If there exists a real diagonal matrix Q such that for all $\omega \geq 0$, $F(j\omega)$ is positive-definite, then the elements of $V(t)$ are bounded elements of $L_2(0,\infty)$ and tend to zero as $t \to \infty$.

Fig. 5.2b shows the threshold system in block-diagram form similar to the control system of Fig. 5.2a. When X^o is constant the state $Z = V = 0$ is stable provided the control system is stable. This is true since 5.26 is identical in form to 1.45 which, with zero input, describes Fig. 5.2b. Hence theorem 5.3 may be applied to find the maximum sectors $[0,k_i]$ in which a Lyapunov function 5.23 exists. Replacing the functions $\phi_i(v_i)$ by $k_i v_i$ an ellipsoidal region Ω_2 exists within the surface defined by

$$V^T(P + K)V = c \ , \qquad 5.32$$

and the region Ω inside $W(V) = c$ is greater than that defined by this equation, but smaller than Ω_1 defined by

$$V^T(P + K_1) = c \qquad 5.33$$

where $K_1 = [k_{1i}]$ is a diagonal matrix whose elements satisfy

$$v_i \phi_i(v_i) \geq k_{1i} v_i^2 \ . \qquad 5.34$$

It may not always be possible to specify such a matrix with nonzero elements. Thus we have

$$\Omega_2 \subset \Omega \subseteq \Omega_1 \qquad 5.35$$

where Ω_1 and Ω_2 are quadratic forms. Hence if Ω_1 is chosen to be the largest ellipsoid in V-space for which $\dot{W}$ is negative-definite, $\dot{W}$ is also negative-definite in Ω. Indeed Ω_1 may be chosen larger than this, so long as $\dot{W} \leq 0$ in Ω, but Ω_1 may be considerably easier to find.

As to the matrices Q, $\mathcal{R}$ and F, these are obtained as follows: From Fig. 5.2b which is described by equation 1.45 or 4.69,

$$-V(t) = -V_0(t) + \int_0^t -G_\delta(t - \tau)Z(\tau)d\tau \qquad 5.36$$

and the linear transfer function matrix is

$$G(s) = \mathcal{L}\left\{G_\delta(t)\right\} = (sT + I)^{-1}A = \left[\frac{\alpha_{ij}}{1 + s\tau_i}\right] . \qquad 5.37$$

Hence,

$$\mathcal{R}(j\omega) = \text{diag}\left[1 + j\omega q_{ii}\right]\left\{-G(j\omega)\right\} + K^{-1} = \left[\frac{-\alpha_{ij}(1 + j\omega q_i)}{1 + j\omega\tau_i}\right] + K^{-1} \quad 5.38$$

$$F(j\omega) = \left[\frac{-\alpha_{ij}(1 + j\omega q_i)}{1 + j\omega\tau_i} - \frac{\alpha_{ji}(1 - j\omega q_j)}{1 - j\omega\tau_j}\right] + 2K^{-1} . \quad 5.39$$

The usual test for the positive-definiteness of $F(j\omega)$ is to find the leading principal minors and apply theorem 5.2. Now let

$$\mathcal{R}(j\omega) = \mathcal{R}e(\mathcal{R}) + j\mathcal{I}m(\mathcal{R}) . \qquad 5.40$$

Then

$$F(j\omega) = \mathcal{R}e(\mathcal{R}) + [\mathcal{R}e(\mathcal{R})]^T + j\mathcal{I}m(\mathcal{R}) - j[\mathcal{I}m(\mathcal{R})]^T \quad . \quad 5.41$$

It is easy to show that lemma 4.1 applies to complex as well as real matrices; $F(j\omega)$ is positive-definite if and only if its symmetric part is also positive-definite. Thus

$$F_s(\omega) = \mathcal{R}e\{F(j\omega)\} \qquad 5.42$$

is to be investigated. From 5.37,

$$F_s(\omega) = \left[-\frac{\alpha_{ij}(1 + \omega^2 q_i \tau_i)}{1 + \omega^2 \tau_i^2} - \frac{\alpha_{ji}(1 + \omega^2 q_j \tau_j)}{1 + \omega^2 \tau_i^2}\right] + 2K^{-1}. \qquad 5.43$$

When $\omega \to 0$,

$$F_s(0) = \left[-\alpha_{ij} - \alpha_{ji}\right] + 2K^{-1} \qquad 5.44$$

and when $\omega \to \infty$,

$$F_s(\infty) = \left[-\alpha_{ij}\frac{q_i}{\tau_i} - \alpha_{ji}\frac{q_j}{\tau_j}\right] + 2K^{-1}. \qquad 5.45$$

For 5.43 to be positive-definite,

$$\frac{1}{k_i} > \alpha_{ii} \qquad 5.46$$

for all i, and if 5.45 is positive-definite,

$$|q_i| \leq \tau_i \qquad 5.47$$

for all i, if α_{ii} is nonzero. Equations 5.44 and 5.45 are of course only necessary conditions, unless Q is made equal to T, in which case 5.44 is also sufficient.

Before proceeding further with mathematical detail, it may be useful to discuss the significance of the development of frequency-plane methods so far. If theorem 5.3 is satisfied, i.e. if 5.43 is positive-definite, then the origin of the V-plane is asymptotically stable, provided the quantities $\frac{\phi_i(v_i)}{v_i}$ do not exceed k_i at any time. Furthermore, 5.23 is the form of a suitable Lyapunov function W. The theorem is suually employed merely to specify the sectors $[0,k_i]$ within which the nonlinear functions must lie. It is used less often to specify regions of stability since the constraint on $\frac{\phi_i(v_i)}{v_i}$ is often unnecessarily severe, and stable systems may be rejected. This may also be true of the threshold network. The use which may be made of the above development is to find extreme points of Ω, which is to be described by forms similar to 5.23.

Consider the following example, illustrated in Fig. 5.3, which is of the form of 3.11i: Such a system is described by matrices such as

$$T = I, \quad B = I,$$

$$E = \begin{bmatrix} -17 \\ -19 \end{bmatrix}, \quad A = \begin{bmatrix} 36 & -2 \\ 2 & 36 \end{bmatrix}, \qquad 5.48$$

and we shall investigate the point X^o near $(1,1)$. In this case from the diagram it is clear that X^o is stable with region Ω at least as large as the rectangle indicated passing through the singular points X^1 and X^2. Using assertion 4.3, that is, by assuming that the curves $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 0$ are piecewise linear within the unit square, it is

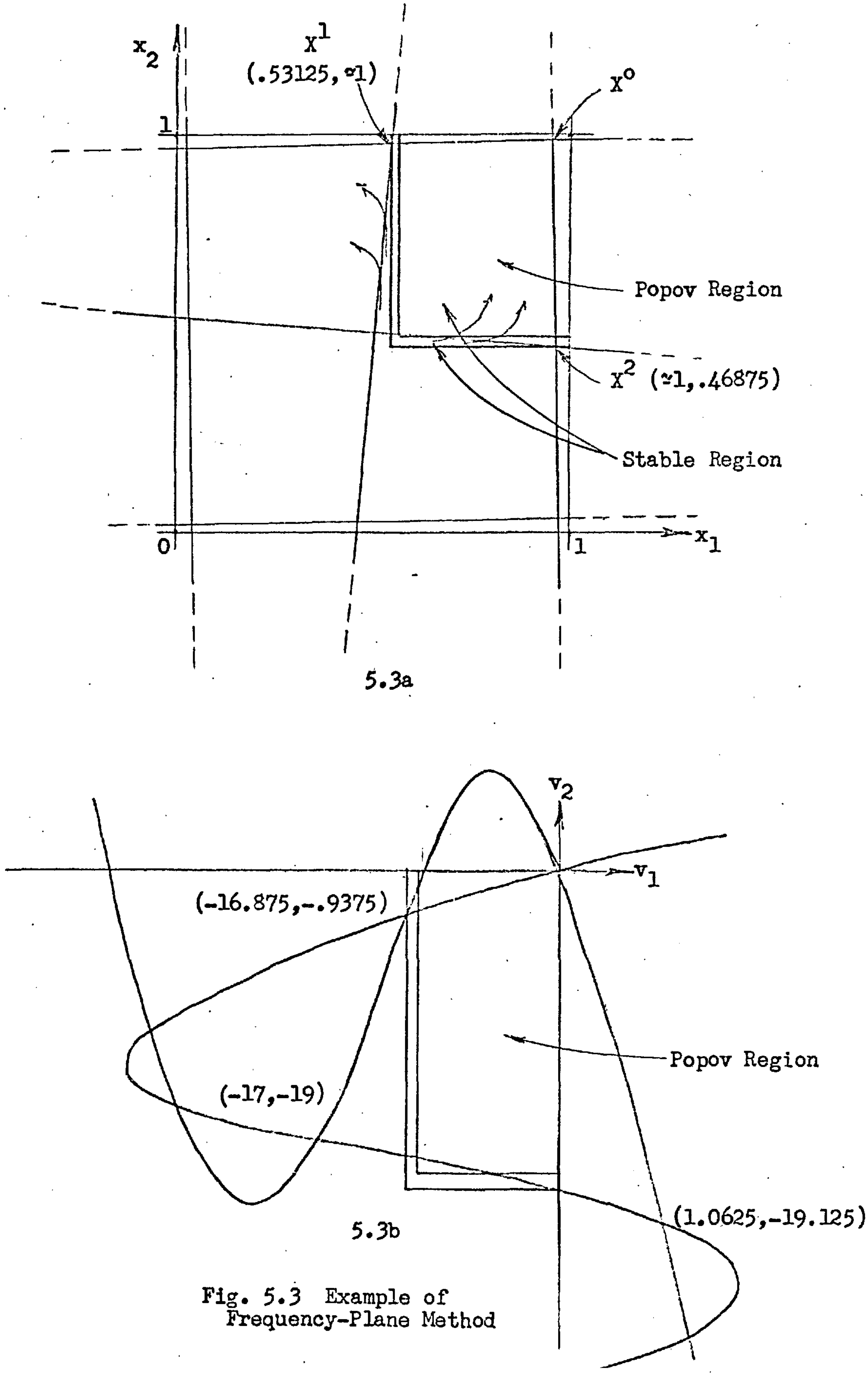

Fig. 5.3 Example of Frequency-Plane Method

possible to find the coordinates of the singularities as shown. We shall demonstrate that in this easily verifiable case the extended Popov criterion results in a region nearly equal to the rectangle through the singular points.

Selecting q_i to equal τ_i, equation 5.43 results in 5.44 for all frequencies, and using 5.48, we have

$$\begin{bmatrix} -72 + \frac{2}{k_1} & 0 \\ 0 & -72 + \frac{2}{k_2} \end{bmatrix} \geqslant 0 \qquad 5.49$$

which is to be positive-definite. Using theorem 5.2, this matrix is positive-definite if

$$k_i \leqslant \frac{1}{36} \qquad 5.50$$

where $i = 1, 2$. Solving the equation

$$x_i + k_i v_i = x_i + \frac{17 - \psi(x_i)}{36} = 1 \qquad 5.51$$

we arrive at

$$x_i = 0.532 , \qquad 5.52$$

$$v_1 = -16.872 , \qquad 5.53$$

$$v_2 = -18.872 , \qquad 5.54$$

which, as shown in the diagram, results in a region slightly smaller than the region we have already obtained.

Another example in which the Popov method results in a suitable region will be given later.

Note that the restrictions imposed by 5.43 always result in a rectangular region which, provided no trajectories leave it, is a region of stability. It is necessary in general to check that trajectories remain in this region, and in such cases it may be easier to use the form 5.23 so that the boundary of Ω is a level-surface of W.

5.3 LYAPUNOV FUNCTIONS OF THE SPECIAL SYSTEM

We apply variable-gradient methods to the system described by the equation

$$\dot{V} = AZ \qquad 5.55$$

which arises directly from 4.62 or 4.69 when all $\beta_i \to 0$ and $\beta_i \tau_i$ is finite. We now take v_i to equal $\beta_i \tau_i \{\psi(x_i) - \psi(x_i^o)\}$. A function is required which satisfies the description of section 5.2.1. Suppose there is a matrix P(Z), such that

$$\nabla W = P(Z)Z \qquad 5.56$$

and suppose that the curl equation is satisfied, i.e.

$$\frac{\partial \nabla_i W}{\partial v_k} = \frac{p_{ik}}{\beta_k \psi'(x_k)} + \sum_j z_j \frac{\partial p_{ij}}{\partial v_k} = \frac{p_{ki}}{\beta_i \psi'(x_i)} + \sum_j z_j \frac{\partial p_{kj}}{\partial v_i} . \qquad 5.57$$

One way to satisfy this equation is to specify that P be a constant diagonal matrix. Then we would have

$$\dot{W} = Z^T PAZ . \qquad 5.58$$

Observe that if 5.58 is identically zero and W is positive-definite, then W is a constant of motion, i.e. the system is conservative and the results of statistical mechanics may be applied. We rewrite this result as

> <u>Assertion 5.2</u>: If $\beta_i = 0$, $1 \leq i \leq n$ and there exists a diagonal matrix P with positive nonzero elements such that the product PA is skew-symmetric then the threshold system described by 5.55 is conservative.

<u>Proof</u>: It will be demonstrated that W is positive-definite. From 5.55,

$$\frac{\partial W}{\partial v_i} = p_{ii} z_i \ . \tag{5.59}$$

Hence

$$W = \sum_i p_{ii} \int_0^{v_i} z_i dv_i = \int_0^V Z^T P dV \tag{5.60}$$

which is always positive for $V \neq 0$ and $p_{ii} > 0$ because from 5.24 the function $z_i = \phi_i(v_i)$ is strictly positive-monotonic and passes through the origin. Now since P in 5.58 is a positive diagonal matrix and PA is skew-symmetric the assertion follows directly.

The above result is a minor generalization of the criterion for the existence of constants of motion mentioned by Cowan[10].

The function which results from the integral 5.60, when the explicit form of ψ is used, is

$$W = \sum_i p_{ii} \log\left(\frac{x_i^o}{x_i}\right)^{x_i^o}\left(\frac{1 - x_i^o}{1 - x_i}\right)^{1-x_i^o} \qquad 5.61$$

which is identical in form to 3.27, the function derived for the R-L-C electrical network analogue.

Of course this assertion specifies only a sufficient condition for a constant of motion. Results of simulation of the system as discussed in appendices A and B suggest that the system is conservative (within experimental error) for any A-matrix with pure imaginary eigenvalues. That is, if there exists a symmetric positive-definite matrix P for which PA is skew-symmetric, then the system would appear to be conservative. An explicit form for a constant of motion is not available, however.

We now consider special systems for which the matrix of the linearized system at a singular point has eigenvalues with negative real parts. Only the point Γ will be considered since the other singular points are never reached in finite time. Consider a system of the form

$$(TD + I)B(Y - \eta) = AZ \qquad 5.62$$

which comes directly from 1.38 and is equivalent to 5.55.

<u>Assertion 5.3</u>: Let 5.62 describe the special system as $\beta_i \to 0$ with $\beta_i \tau_i$ finite, and let Γ be stable. Then Γ is asymptotically stable in the large.

<u>Proof</u>: The extended Popov method of section 5.2.2 will be applied. The theorem is satisfied if Γ is stable because the linear transfer function has poles in the left half-plane. Straightforward

application of the method results in the matrix

$$F_s(\omega) = \left[\frac{-\alpha_{ij}(1 + \omega^2 q_i \tau_i)}{\beta_i(1 + \omega^2 \tau_i^2)} - \frac{\alpha_{ji}(1 + \omega^2 q_j \tau_j)}{\beta_j(1 + \omega^2 \tau_j^2)}\right] + 2K^{-1} . \quad 5.63$$

Consider the quantity

$$\frac{1 + \omega^2 q_i \tau_i}{\beta_i(1 + \omega^2 \tau_i^2)} = \frac{\frac{1}{\tau_i} + \omega^2 q_i}{\beta_i(\frac{1}{\tau_i} + \omega^2 \tau_i^2)} . \quad 5.64$$

In the limit $\beta_i \to 0$ and $\tau_i \to \infty$ such that $\beta_i \tau_i$ remains finite. The limiting value of this quantity is $\frac{q_i}{\beta_i \tau_i}$. Thus if all q_i are specified to be 0, $F_s(\omega)$ is positive-definite for all positive values of k_i, and the system is asymptotically stable in the large.

A consequence of the above result is that if Γ is unstable, the vector $Y - \eta$ becomes infinitely large as $t \to \infty$.

5.4 LIMIT-CYCLES IN THRESHOLD NETWORKS

In chapter 4 several conclusions were reached about realization of threshold nets, using only analyses of singular points and a knowledge of the topology of the state-space of threshold networks. The results are summarized in assertion 4.3, which specifies that the eigenvalues of the linearized system must be real at Γ for the results to apply, in which case this point is a node or saddle point. If Γ is a centre geometrical methods may be used, but with less specific results. Using the global techniques of this chapter the existence of limit-cycles may also be shown, and furthermore, bounds of the regions in which they must exist may be established.

Consider a general threshold system of n gates with connexion matrix A of rank n. Assume that there is exactly one realizable singularity, denoted Γ. We shall use

Definition 5.2: A limit-cycle is an isolated closed trajectory.

Consider a closed trajectory in the state-plane, or its projection onto an arbitrary plane if n is greater than 2.

Theorem 5.4[11]: A closed trajectory surrounds at least one singular point.

When $n > 2$ the theorem is interpreted to mean that a closed trajectory surrounds the image of at least one singular point.

By half-trajectory is meant a solution trajectory from t_0 to ∞ or $-\infty$. The Poincaré-Bendixon theorem[11] states:

Theorem 5.5: If a half-trajectory remains in a finite region Ω without approaching any singularities, then either the trajectory is closed or it approaches a closed trajectory.

We now state an application to the general threshold network:

Assertion 5.4: Solution half-trajectories of the threshold net approach either a singular point within the unit hypercube or a closed curve within the unit hypercube.

Proof: It is sufficient to demonstrate that solutions remain bounded in the V-space, since the boundaries of the unit hypercube correspond to infinity in the V-space. We show that there exists a finite region Ω_2 and a positive-definite function $W(V)$ with negative derivative everywhere outside Ω_2, and on the boundary of Ω_2. Consider the equation

$$(\tau_i \frac{d}{dt} + 1)v_i - \alpha_{ii}z_i = \sum_{j \neq i} \alpha_{ij} z_j \qquad 5.65$$

which describes the i^{th} gate. For all $j \neq i$, let

$$x_j = z_j + x_j^o \in \{0,1\} \qquad 5.66$$

such that the right-hand side of 5.65 is a maximum:

$$\sum_{j \neq i} \alpha_{ij} z_j = M_i \qquad 5.67$$

and let $z_{iM} = \phi_i(v_{iM})$ be the solution of the equation

$$v_i - \alpha_{ii} z_i = M_i \qquad 5.68$$

If a trajectory ever reaches v_{iM}, since $0 < x_i < 1$ for all i,

$$\frac{dv_{iM}}{dt} = \frac{1}{\tau_i}\left\{-M_i + \sum_{j \neq i} \alpha_{ij} z_j\right\} < 0 \ . \qquad 5.69$$

Now select all x_j, $j \neq i$ in 5.66 such that

$$\sum_{j \neq i} \alpha_{ij} z_j = m_j \qquad 5.70$$

is a minimum, and solve the following equation for z_{im}, v_{im}:

$$v_{im} - \alpha_{ii} z_{im} = m_i \ . \qquad 5.71$$

Now,

$$\frac{dv_{im}}{dt} = \frac{1}{\tau_i}\left\{-m_i + \sum_{j\neq i}\alpha_{ij}z_j\right\} > 0 \qquad 5.72$$

if a solution trajectory reaches v_{im}. Such trajectories pass through these points only during an initial transient, after which, for all time,

$$m_i < v_i - \alpha_{ii}z_i < M_i \ . \qquad 5.73$$

Let $W = \frac{1}{2}V^TV$. There exists a region Ω_2 bounded by the planes $v_i - \alpha_{ii}z_i = m_i$ and $v_i - \alpha_{ii}z_i = M_i$ for all i, such that on the boundary,

$$\dot{W} = \sum_i v_i\dot{v}_i < 0 \qquad 5.74$$

and the proof is complete.

The above result has been used implicitly in the original choice of 1.21 as a model of physical devices. The assertion gives a justification of the model in semi-formal terms.

<u>Assertion 5.5</u>: Let Γ be the sole finite singular point of a threshold network. If Γ is unstable then there exist bounded regions Ω_1, Ω_2 and $\Omega = \Omega_2 - \Omega_1$ such that trajectories within Ω remain in Ω as $t \to \infty$.

<u>Proof</u>: It was established in the proof of assertion 5.4 that a bounded region Ω_2 exists, such that trajectories enter the region at every point on its boundary. Furthermore Γ must be inside Ω_2

since Γ is finite, in which case m_i is negative and M_i positive in 5.73. If Γ is unstable there is by definition a region Ω_1 containing Γ, and from which solutions enter Ω. Now since Γ is the only finite singular point the assertion has been proven.

It may be remarked that it is usually possible to find a much larger region Ω_1 which satisfies the assertion, as follows:

Consider the network defined by the matrices

$$T = I\ , \qquad B = I\ ,$$

$$E = \begin{bmatrix} -4 \\ -12 \end{bmatrix}, \quad A = \begin{bmatrix} 28 & -20 \\ 20 & 4 \end{bmatrix}. \qquad 5.75$$

Fig. 5.4 shows the two state-planes of such a system. The matrix of the linearized system

$$\dot{V} = HV \qquad 5.76$$

at $\Gamma = (\frac{1}{2},\frac{1}{2})$ from 4.37 is found to be

$$H = \begin{bmatrix} 6 & -5 \\ 5 & 0 \end{bmatrix} \qquad 5.77$$

which has eigenvalues in the right half-plane and therefore Γ is unstable. A quadratic form $W = V^TPV = c$ may be found by specifying a symmetric positive-definite matrix P and a constant c for which the derivative

$$\dot{W} = V^TP(-T^{-1}V + T^{-1}AZ) \qquad 5.13$$

is positive everywhere in Ω_1 bounded by $W = c$. In this case P

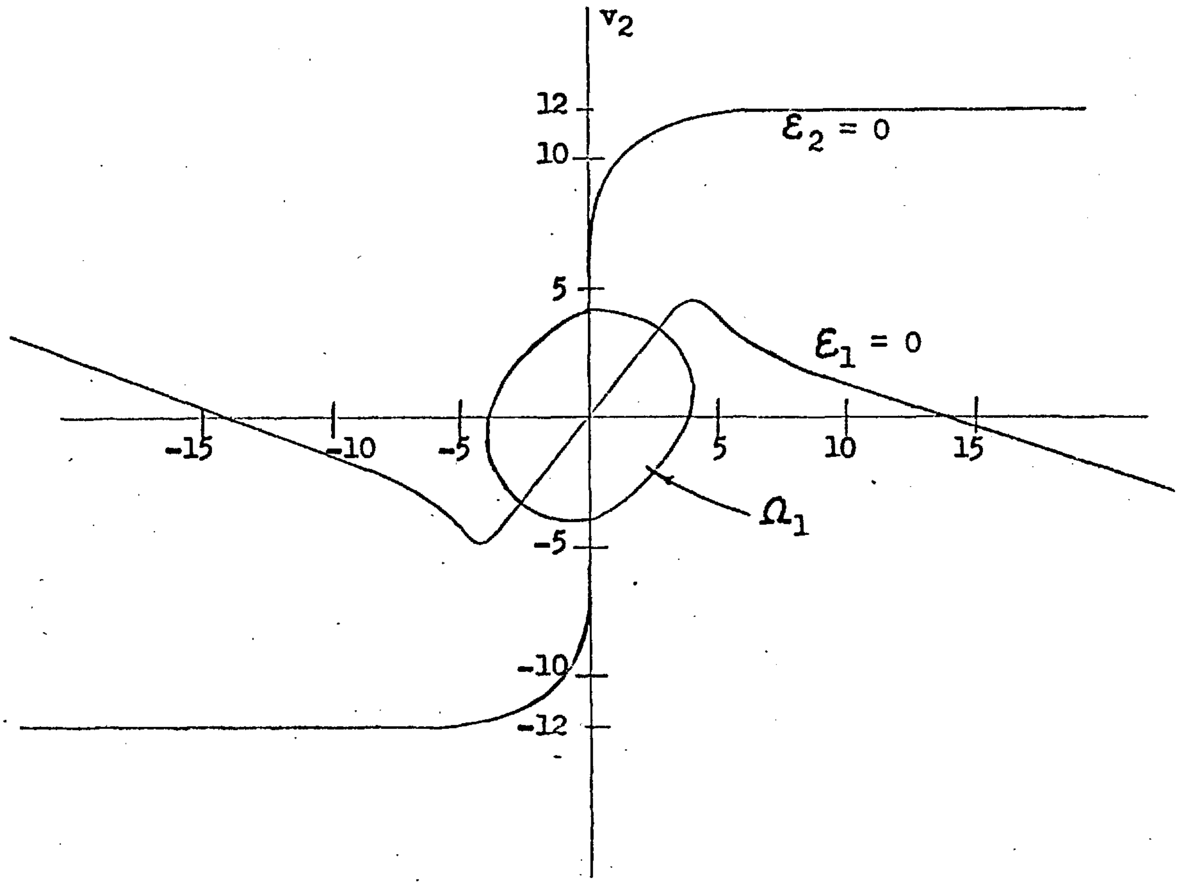

Fig. 5.4a Example V-Plane

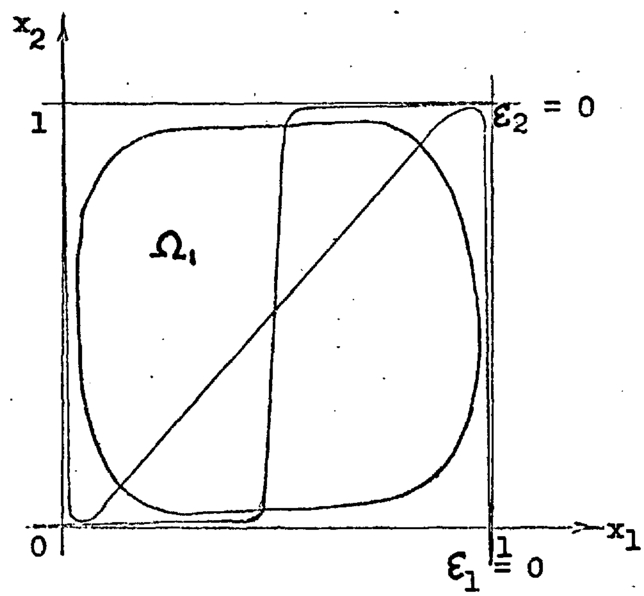

Fig. 5.4b Sketch of Example X-Plane

cannot be diagonal because $\dot{v}_2 < 0$ everywhere on the v_2-axis except 0, and it would be impossible to find a nonzero constant c for which $\dot{W}$ is positive in a finite region. The matrix P may be assumed to be of the form

$$P = \begin{bmatrix} b & -a \\ -a & 5 \end{bmatrix} \qquad 5.78$$

where p_{22} has been chosen as an arbitrary positive number. We assume that Ω_1 cuts the v_1- and v_2-axis far enough from the origin so that $\dot{V}$ may be approximated by a linear equation, i.e. if $v_1 = 0$, $\dot{W} \geqslant 0$,

$$-a(-4 + 14 - 20) + 5(-12 - v_2 + 10 + 4) \geqslant 0 \; ,$$

$$v_2 \leqslant 2(a + 1) \; , \qquad 5.79$$

and if $v_2 = 0$, $\dot{W} \geqslant 0$,

$$b(-4 - v_1 + 28 - 10) - a(-12 + 20 + 2) \geqslant 0 \; ,$$

$$v_1 \leqslant 14 - \frac{10a}{b} \; . \qquad 5.80$$

The values a and b must be such that P is positive-definite, and in addition the boundary of Ω_1 must cut the axes inside the values specified by 5.79 and 5.80. If, for example, P is

$$P = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \qquad 5.81$$

and if c is chosen to be 79 then the axes are cut at (0,3.97) and at (3.97,0) which are inside the constraints $v_2 \leqslant 4$ from 5.79 and $v_1 \leqslant 12$ from 5.80. It is easy to verify that $\dot{W}$ is indeed positive everywhere inside this region.

In general a quadratic form is available and defines a finite region Ω_1, provided some other suitable function defines a finite region, as stated in the following

Theorem 5.6[12]: Let a system be represented by the form

$$\dot{Z} = HZ + \bar{F}(Z) \qquad 4.11$$

in which H is an $n \times n$ matrix, and $\bar{F}$ is a vector function of Z whose components are power series of $z_1, \ldots, z_n$, convergent for all $\|Z\| < \delta$ for some $\delta > 0$. Then, if there exists a positive-definite function $W(Z)$ such that $\dot{W}(Z)$ is negative-definite, there also exists a $W(Z)$, a quadratic form in the variables $z_1, \ldots, z_n$ with the same properties. Moreover, this quadratic $W(Z)$ satisfies the conditions

$$W(Z) \geq a^2 \sum_i z_i^2 , \qquad \dot{W}(Z) \leq -b^2 \sum_i z_i^2 \qquad 5.82$$

for some a, $b > 0$.

Obviously the theorem applies to functions with negative derivatives.

The final result of this chapter is

Theorem 5.7: Let Γ be the only finite singular point of a threshold network of n gates, and let Γ be unstable such that for no gate i does the value $|x_i - \gamma_i|$ tend to zero as $t \to \infty$. Then there exists a number $T < \infty$ and numbers t_{oi}, $1 \leq i \leq n$ such that at $t_{oi} < T$ every gate i realizes a linear threshold function, and furthermore the system is m-discrete where m is an arbitrarily large integer.

<u>Proof</u>: Since Γ is unstable and the sole finite singular point, by assertion 5.4 there must be a closed curve C which all trajectories approach as $t \to \infty$. Therefore for any number $\mu > 0$ there exist points a_i and b_i on C and numbers $t_{oi} < \infty$ such that for all i, the quantity $|x_i(t_{oi}) - \zeta_i|$ is less than μ where $\zeta_i \in \{a_i, b_i\}$. Furthermore the existence of a closed trajectory implies that there exist numbers $\nu_1 < \nu_2 < \ldots$ such that the above condition holds at each time $t_{oi} + \nu_r$. Now it is always possible to choose a_i and b_i so that at every time $t_{oi} + \nu_r$, if $\epsilon_i + \sum_j \alpha_{ij} x_j > 0$ then $\dot{x}_i > 0$, and if $\epsilon_i + \sum_j \alpha_{ij} x_j < 0$ then $\dot{x}_i < 0$ and by definition 2.7 the system is m-discrete and by definition 4.4 it realizes a linear threshold net.

This theorem establishes that the existence of a limit-cycle in the dynamic system corresponds to a condition in switching circuits usually called simply a cycle[13].

5.5 SUMMARY

The behavior of the gate network may be predicted by applying the second method of Lyapunov, which requires the existence of an explicit form for a positive-definite function W with sign-definite derivative in a region of the system state-space. The method of variable gradients may be applied; for arbitrary matrices A it does not result in a unique function W unless the gradient of W is assumed to be a linear function, in which case W is quadratic. Other forms are possible, but except for a case to be mentioned below, have not been obtained.

It is possible to find a constant of motion W, analogous to the Hamiltonian of classical mechanics, for the Cowan system with a restriction on the matrix A. It is conjectured, however, that this restriction need not apply.

As to finding the best quadratic Lyapunov function, this problem requires an exact knowledge of the state topology, but appropriate approximate methods for systems with large weights are available. The generalized Popov method also applies to the dynamic gate system, and provides a straightforward method of finding regions of convergence in certain cases. The form of the resulting function is of a quadratic plus the integral of nonlinear terms.

The ability to generate Lyapunov functions for the system allows a conclusion to be reached about the existence of limit-cycles, and hence about the existence of cycles in switching circuits.

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CHAPTER 6

APPLICATIONS OF THE DYNAMICAL MODEL

6.1 INTRODUCTION

This chapter concerns applications of the model proposed in section 1.6 and analysed in chapters 3 to 5. Some of the questions which it can reasonably be used to answer or partly answer are given here.

6.2 SIMULATION OF NETWORKS

The obvious application of a differential equation model is to solve it. If closed-form solution is not possible then analogue or digital computers must be used. Four examples are presented here.

Fig. 6.1 is a plot produced by analogue computer of the transient step response of a single gate for various step heights, and a constant initial point. Such plots could, for example, be used in specifying gap tolerance limits (see Fig. 3.5) or switching times.

At a slightly greater order of complexity, Fig. 6.2 shows typical oscillation waveforms of a two-element system with weight matrix

$$A = \begin{bmatrix} 0 & \alpha_{12} \\ \alpha_{21} & 0 \end{bmatrix} \qquad 6.1$$

and various initial conditions. In this case the constants β_i are assumed negligible (β_i was zero in the simulation).

Fig. 6.3 is a state-plane diagram generated by digital computer. The circuit, described by

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad T = I ,$$

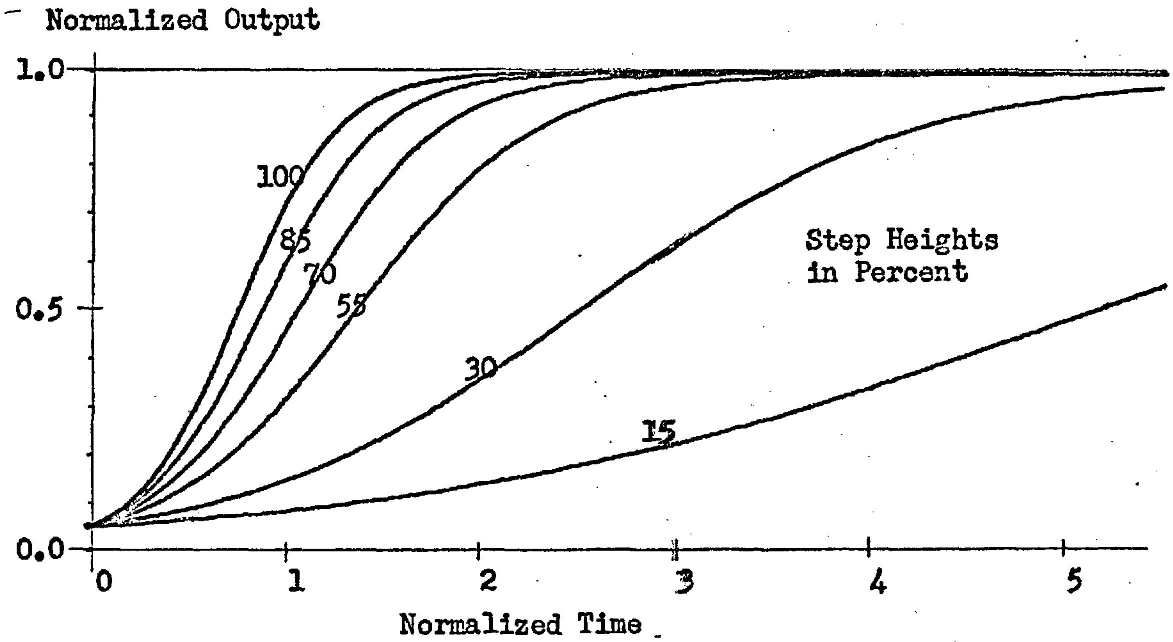


Fig. 6.1 Gate Transient Response

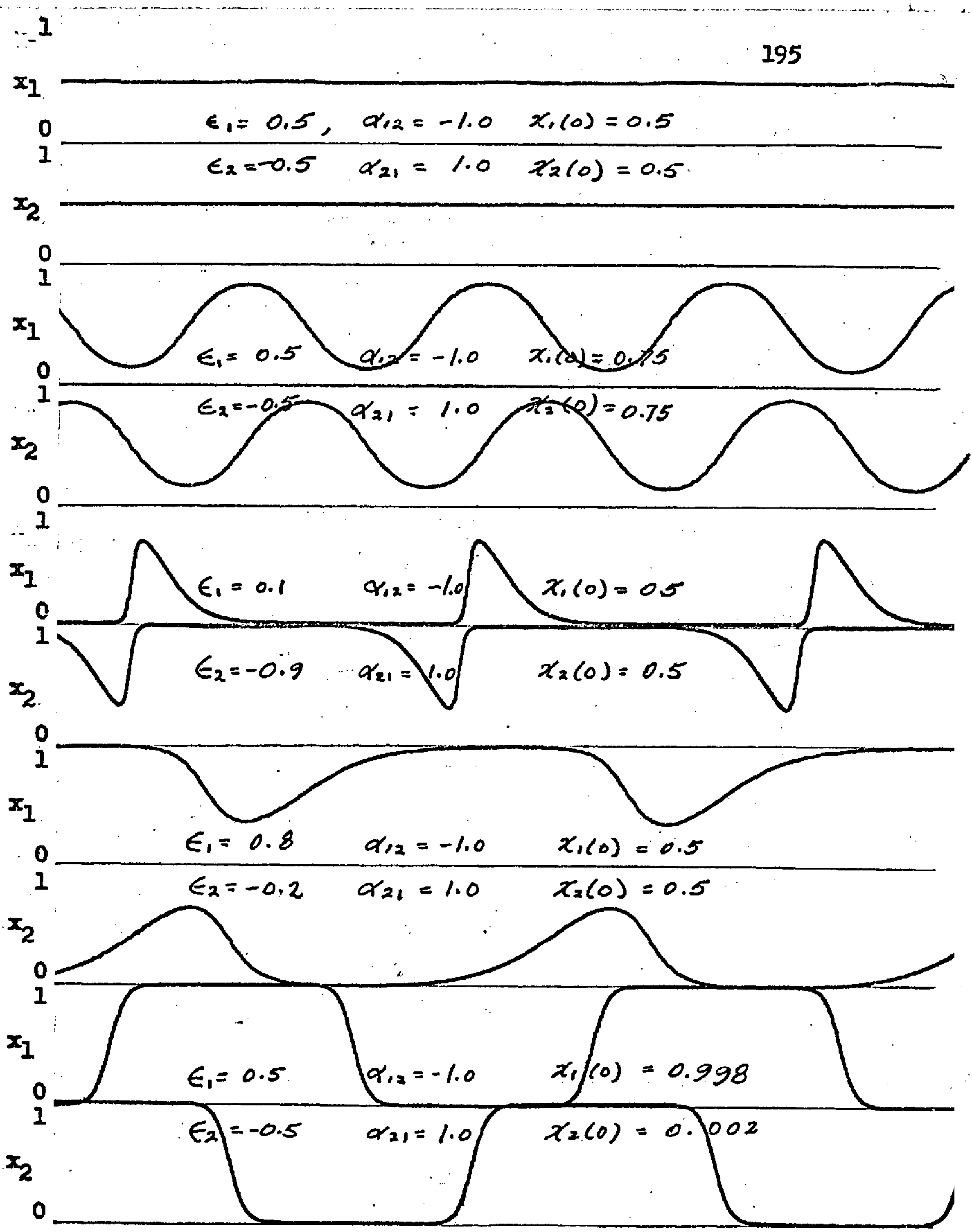


Fig. 6.2 Typical Oscillations

$$E = \begin{bmatrix} 10.4 \\ -9.6 \end{bmatrix}, \quad A = \begin{bmatrix} 28 & -36 \\ 36 & -8 \end{bmatrix} \qquad 6.2$$

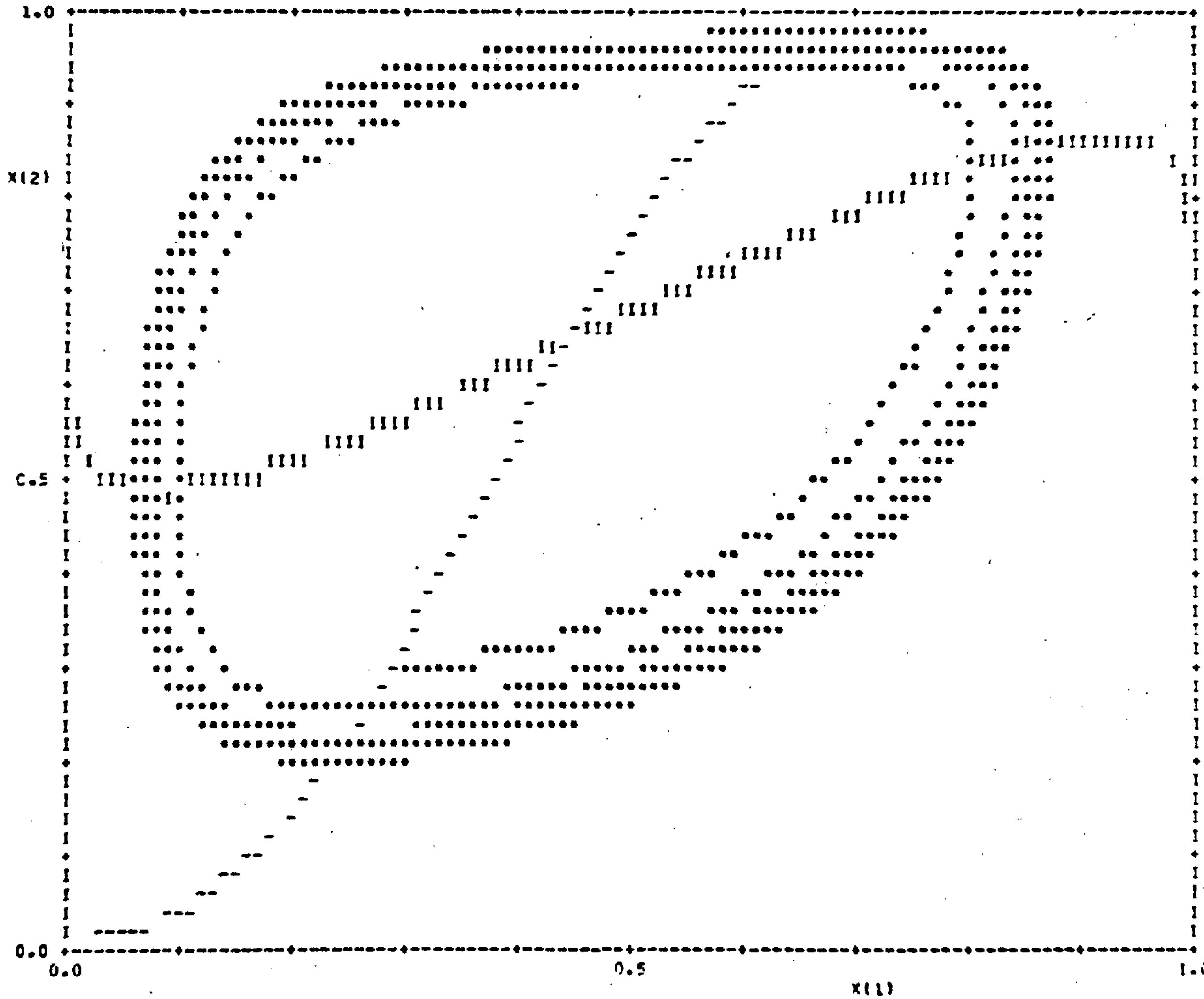
exhibits a limit-cycle. The lines $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 0$ are also shown.

Fig. 6.5 shows a digital computer solution of the master-slave flip-flop of Fig. 6.4.* In this case ten gates were simulated, the first two as a clock wave-form generator, and the remainder in the flip-flop itself. Of course much more accurate digital computer plots could be produced easily using a computer-controlled plotter, but the outputs shown give a reasonable view of waveforms, and at a lower cost in time and complexity. The solution shown required 1.2 minutes on an IBM 7090 computer.

6.3 NOISE IN LOGIC SYSTEMS

A problem which occurs in the design of logic circuits is the amount of "noise" a circuit will tolerate[1] before either giving an incorrect output or coming to rest at an incorrect stable state. The techniques of chapter 5 are directly applicable to this problem, and the following result can be stated: If a singular point is stable, then any disturbance which does not take the system out of the maximum region Ω, defined by either 5.9 or 5.25 and for which there exists a Lyapunov function, does not make the system move to a new stable state. In the simpler case of a temporary change of the output of a single gate, theorem 2.3 may be applied, with the addition of a noise signal, N(t).

* In the diagrams RHO refers to the elements β_i, and in Fig. 6.5 GAMMA is the solution of $E + AX = 0$.



GAMMA	RHO	E	A	
.4CC00	2.00	10.40	28.00	-36.00
.6C000	2.00	-9.60	36.00	-8.00

Fig. 6.3
Two-Gate Network
State Plane

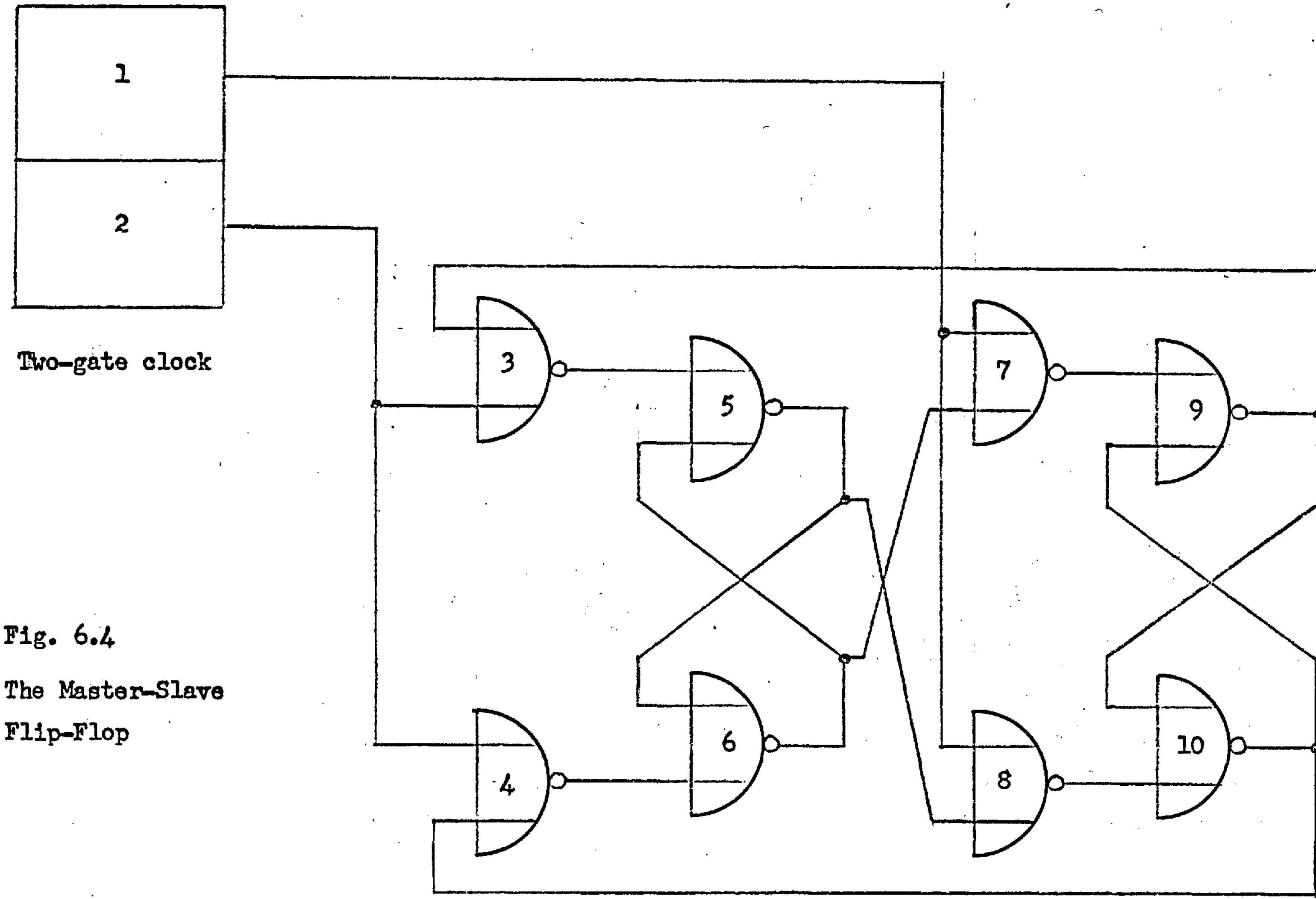


Fig. 6.4

The Master-Slave Flip-Flop

GAMMA	RFC	E	A MATRIX									
0.50	1.00	2.50	10.00	-15.00	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.
0.50	1.00	-12.50	15.00	10.00	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.
0.30	1.00	20.00	-0.	-30.00	-0.	-0.	-0.	-0.	-0.	-0.	-30.00	-0.
0.30	1.00	20.00	-0.	-30.00	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-30.00
0.17	1.00	20.00	-0.	-0.	-50.00	-0.	-0.	-30.00	-0.	-0.	-0.	-0.
0.17	1.00	20.00	-0.	-0.	-0.	-30.00	-30.00	-0.	-0.	-0.	-0.	-0.
0.30	1.00	20.00	-30.00	-0.	-0.	-0.	-0.	-30.00	-0.	-0.	-0.	-0.
0.30	1.00	20.00	-30.00	-0.	-0.	-0.	-30.00	-0.	-0.	-0.	-0.	-0.
0.17	1.00	20.00	-0.	-0.	-0.	-0.	-0.	-0.	-50.00	-0.	-0.	-30.00
0.17	1.00	20.00	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-50.00	-30.00	-0.

TMAX SEC.	INITIAL CONDITIONS									
15.0	.1000000	.9000000	.1000000	.1000000	.1000000	.9000000	.1000000	.9000000	.9000000	.1000000

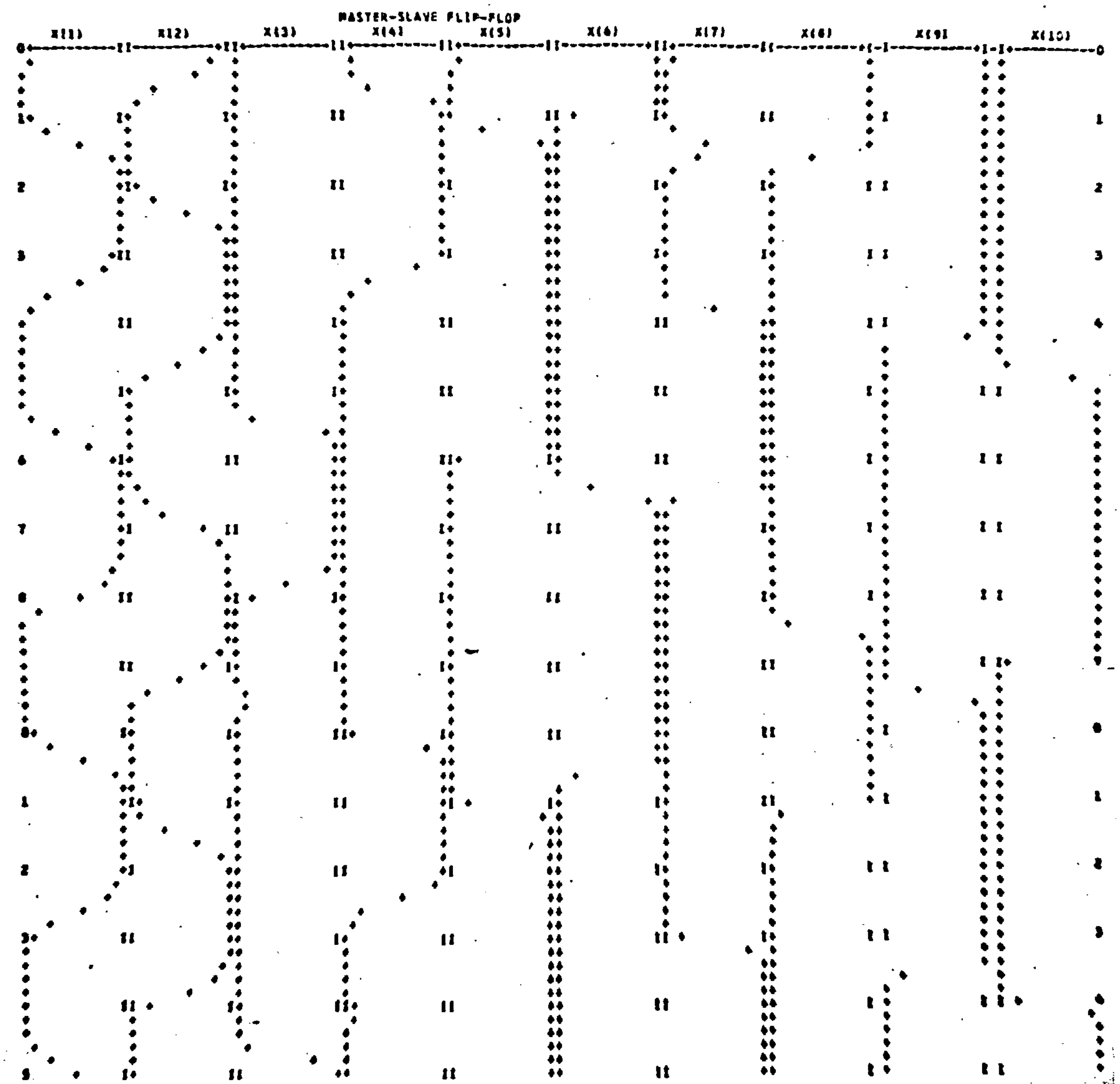


Fig. 6.5 Simulation of Master-Slave Flip-Flop

Inequality 2.33 is replaced by

$$\left|N(t) + \epsilon_i + \sum_{j=1}^{n} \alpha_{ij} x_j\right| > \delta \qquad 6.3$$

and as long as this remains true the element realizes a threshold function.

6.4 THE RANK OF A NETWORK

In section 4.8 is a fundamental result which applies to design as well as analysis of logic circuits. Theorem 4.8 states that the rank of the connexion matrix A determines the logical character of a network. For example, if the maximum number of stable states[2] of a circuit of n gates is k, then adding an additional gate will not increase the number of stable states unless the rank of the new matrix is $n + 1$. The applicability of the results in section 4.8 to the reduction of large networks of non-maximal rank is mentioned in that section.

6.5 DESIGN OF SEQUENTIAL CIRCUITS

A result which relates to the discussion of section 6.2 will be stated. Let a logic circuit be partitioned into circuits of maximal rank. Let the state-space of a maximal circuit be partitioned into stable regions Ω_i. In general these regions depend upon the values γ_i which are functions of ϵ_i and therefore of the circuit inputs. Then any input which changes the state topology so that the state vector is in a stable region Ω_i causes the next stable point to be in Ω_i. The input may be external, or may be an action from one or more of the other maximal circuits in the network.

The above is not intended as a new method for the design of sequential logic, because the computation required in such an analysis of stable regions is large. However, when applied to small circuits, this type of analysis could be used, for example, in specifying minimum input levels and their required duration.

6.6 BOOLEAN MEMORIES

It is a safe assumption that most logical memories not using a particular physical effect (ferromagnetism for example) are designed using bistable circuits as basic units. Recently memory circuits containing more general feedback connections have been mentioned in the literature[3,4,5]. A time-continuous model has not appeared, however. The enumeration of singular points of threshold circuits as discussed in chapter 3 is of direct relevance to this problem because a method has been presented for finding every singular point and testing its stability. The only restriction in this case is that the weights α_{ij} all have a magnitude large inough to guarantee that the singular points (except Γ) are sufficiently near edges of the unit hypercube. It must be emphasized that this is not a restriction on the logical function of a threshold gate, but only on its transient behavior. This is easily seen by noting that the multiplication of ϵ_i and α_{ij}, $j = 1, \ldots, n$ by a positive constant does not change the gate function. This is a linear transformation as discussed in chapter 2.

6.7 HAZARDS IN SEQUENTIAL CIRCUITS

The term "hazard" is used with a certain lack of precision in the literature of sequential circuits. The definitions of Unger[6] will be used here:

A transient hazard is a momentary false output of a circuit.

A steady-state or essential hazard is the condition in which a circuit may enter a "wrong" stable state after certain input changes.

Other definitions than the above wordings are available.[7] Transient hazards will be taken to mean the following in the context of the continuous model. Suppose that all inputs $u_q(t)$ are constant in the interval (t_o, t_5), and that gate i satisfies

$$|x_i - a_i| < \mu \qquad 6.4$$

in (t_o, t_1),

$$|x_i - b_i| < \mu \qquad 6.5$$

in (t_2, t_3) and the first inequality again in (t_4, t_5) with $t_o < t_1 < t_2 < t_3 < t_4 < t_5$. Now from theorem 2.3 it is evident that the point γ_i lies between a_i and b_i. Therefore, if a transient hazard exists the output x_i must pass through γ_i at least twice. The following conclusion arises directly from the topological analysis: If all eigenvalues of the linearized system are real at Γ then the singularity is a saddle-point, and no variable x_i passes through γ_i more than once.

Therefore no transient hazard exists.

A conclusion about essential hazards may also be reached, although it is not a very useful one. The following argument is used: If an essential hazard exists the system is not completely controllable[8]. The condition that a system be completely controllable is that the external input weight matrix P (see equation 1.34) have as many nonzero rows as A has rows, that is, as there are gates in the circuit. If the elements of P have large enough magnitude then every stable state may be reached from any other with exactly one input change. The concept of controllability belongs to the theory of automatic control systems. This result has not been proven explicitly in the analysis but is evident from theorem 2.3.

The above discussion is only relevant when every stable state of the threshold system is a realization of a threshold net (by definition 4.4). The discussion of essential hazards is applicable as a definition in control-system terminology but is not very useful in the design of circuits because it requires an external input to every gate.

6.8 CYCLES AND LIMIT-CYCLES

Assertion 4.4 and theorem 5.7 demonstrate that the dynamic model can exhibit oscillatory behavior under certain conditions. The theorem states that under such conditions the system realizes a linear threshold net in a set of intervals, and therefore "limit-cycles" in the dynamic system correspond to "cycles" in logic networks. The following argument applies: Let assertion 4.7 and theorem 4.9 be applied to divide the system into independent subnetworks. Cyclic behavior in an m-discrete subnetwork implies that the trajectory (or, as stated previously, its projection on some plane) must enclose the principal singularity Γ. In such a case, (a) Γ must be unstable, (b) no output x_i of the subnetwork approaches γ_i as $t \to \infty$, and (c) at least two of the eigenvalues of the linearized system at Γ must be complex. Therefore, if all eigenvalues of the linearized system are real at Γ, the network contains no cycles. This conclusion applies independently to each maximal network within a system, but account must be taken of the interaction between subnetworks, i.e. the existence or shape of a cyclic waveform depend on the stability of interacting subnetworks within the system. The action of an independent network on gate i has the effect of making ϵ_i non-constant if the independent network is cyclic.

6.9 REALIZATION OF THRESHOLD NETS

The question as to whether a given circuit computes a logical function is non-trivial if it must be answered a priori, that is, before all possible inputs are presented to it. Definitions 2.6 and 4.4 relate logical function to the dynamic model, and assertion 4.3 and theorem 5.7 summarize the analysis. The conclusion is that a network conforming to the model does compute a logical function provided (a) Γ is unstable, and (b) the magnitudes $|\alpha_{ij}|$ are large enough. Necessary conditions, however, have not been discussed. If theorem 5.7 applies then the threshold net realized contains a cycle.

6.10 APPLICATION OF THE SPECIAL SYSTEM

As stated in chapter 1, only applications to switching systems are considered here. Two uses of the special system may be made. Solutions for the singular points of such a system are available in closed form; only solution of sets of linear equations is required. As stated in chapter 4, the singular points of the general system approach those of the special system as the magnitudes of α_{ij} increase. Furthermore, from assertion 4.4, if the special system is unstable at a vertex of the unit hypercube, then the general system does not have a singularity near this vertex. These two results are not very profound from the theoretical point of view but greatly simplify the computations required in analysing the general system.

6.11 SUMMARY

The dynamical model is of a form which easily may be solved by computer. It is asserted that such simulations predict the behavior of switching circuits more accurately than binary models. They are particularly useful when circuits are operated near their limits of speed.

The problem of the noise immunity of logical circuits has a fundamental relation to the second method of Lyapunov. The analysis of chapter 5 therefore applies to this problem.

Another fundamental result is that the rank of the weight matrix is indicative of the system dynamic behavior. This, together with the topological conclusion, is useful in sequential design.

The discussion of the enumeration of singular points relates to recent developments regarding "Boolean memories."

The results on realization allow conclusions about hazards in sequential circuits, and also provide a framework for determining the functional behavior of an arbitrary circuit.

Finally, the special system is useful from a computational point of view.

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CHAPTER 7

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

7.1 INTRODUCTION

A mathematical model of a physical device can only be judged subjectively, according to the results which it is used to predict. This thesis is intended as a preliminary exploration of a model and some of the questions which reasonably may be asked of it. This, the final chapter, contains a discussion of the validity of the model, some conclusions which may be reached from and about it, and suggestions for future work.

7.2 THE PLACE OF THE MODEL IN A FUNCTIONAL HIERARCHY

A model is used to predict the behavior of the object being modelled. The assumption in this work is that we have a model permitting a reasonable amount of detailed prediction of a certain kind at a reasonable cost of complication and computation.

Consider an "object" with a finite number q of "input terminals" and one "output terminal." A mathematical model of such an object is a function which states a relation between two sets: a set of inputs or domain, and a set of outputs or range. Let Q be the set of input terminals at which inputs U, which are elements of the domain set, appear, and which cause an output x, an element of the range set. Fig. 7.1 is an attempt to present the dynamical model in a hierarchy of models of logical function.

Two questions are to be answered: (a) how can a device be

	DOMAIN	INPUT	FUNCTION TYPE	RANGE
(1)	$U \in \mathcal{U}(Q)$ The set of subsets of Q	$U = \{i \mid u_i = \text{True}\}$	Logical (Boolean) $x = f_L(U)$	$x \in L = \{\text{True}, \text{False}\}$
(2)	$U \in S^q$ A q-dimensional vector space	as (1) or $U = [u_1, \ldots, u_q]$ $u_i \in B$	Binary (Two-level) $x = f_B(U)$	$x \in B = \{a, b\}$ a,b distinct
(3)	as (1) or (2)		Threshold $x = \begin{cases} b, & \mathcal{E}(U) \geq 0 \\ a, & \text{otherwise} \end{cases}$ $\mathcal{E}$ unique with ordered range	as (2)
(4)	$U \in S^q$ usually real	as (1) or $U = [u_1, \ldots, u_q]$ $u_i \in \{a, b\}$	Linear Threshold $x = \begin{cases} b, & \mathcal{E}(U) \geq 0 \\ a, & \text{otherwise} \end{cases}$ $\mathcal{E}$ linear	as (2), with a,b real
(5)	$U \in$ {triple partitions on Q}	$U = \{U_1, U_2\}$ $U_1 = \{i \mid u_i = d\}$ $U_2 = \{i \mid u_i = c\}$	Ternary (Three-level) $x = f_D(U)$	$x \in D = \{a, b, d\}$ a,b,d distinct
(6)	$U \in R^q$ q-dimensional real space	$U = [u_1, \ldots, u_q]$ $u_i \in R$	Continuous $x = f_C(U)$	$x \in C = (a, b)$ an open interval on the real line
(7)	Physical quantities		Device	Physical measurements

Fig. 7.1 A Hierarchy of Functional Models

designed so that it "computes" a logical function, and (b) given a device, what logical function does it compute, if any? The levels of the hierarchy in the figure are various levels of abstraction. Logical functions are the essence of computation. Binary functions are used mainly for convenience in representing logical functions, and are equivalent to logical functions. Further, it can easily be demonstrated that every binary function can be computed by a threshold function, and vice-versa. Similarly, every linear threshold function is a threshold function, but not every threshold function is a linear threshold function. Much work has been done in devising ways of using one or more linear threshold function to compute arbitrary logical functions. A linear threshold function also "resembles" physical devices in that it is comparatively easy to "realize" a linear threshold function using real objects. Obvious examples are gates encorporating transistors, diodes, ferromagnetic materials, and of course, neurons.

One common characteristic of the functions discussed so far is the following: without exception they are defined only for a discrete time scale, and only for finite output sets. Clearly no physical event above the level of quantum mechanics is discrete in time or any other measurement.

The behavior of asynchronous logic circuits in time is accounted for by various techniques, mostly to do with "state-assignments" which are intended to ensure that a system is completely stable at the proper states. An important restriction in most cases is that only one input to a network is allowed to change at a time for proper operation to be guaranteed.

The principal use of ternary functions in logical design and analysis is the detection of hazard conditions. Although the functions themselves are not explicit functions of time their proper use enables hazard-free design of large circuits with two or more simultaneous input changes. However, questions about wave-shape detail cannot be answered.

As to the time-scale of continuous models, this need not be the set of real numbers, but only the set of integers, say, because the limiting behavior of a time-discrete function as the interval between points decreases is by definition the time-continuous behavior. This subject was dealt with in section 2.5.

In the light of the above discussion the conclusion may be reached that predictions which concern infinite output sets must be made using continuous functional models.

7.3 FUTURE RESEARCH PROBLEMS

More unsolved problems than solutions result from an investigation such as the one presented here. In this section some of the more important and obvious problems will be mentioned, divided into three categories: problems arising from or generated by the analysis given in previous chapters, algebraic problems relevant to the theory of automata, and engineering problems relevant to the analysis and design of sequential circuits.

7.3.1 PROBLEMS ARISING FROM THE ANALYSIS

1. A general problem is the tightening of bounds given in the proofs of several of the theorems and assertions, which, for the most part, are existence proofs. Particular cases where such improvements would be useful are assertions 4.1 and 4.2, and theorem 4.7.

2. A useful computational result would be an efficient algorithm for locating all the singular points of an arbitrary threshold system. The restrictions on weight magnitudes mentioned previously are not severe, but do restrict the class of systems satisfactorily treated.

3. The discussion of regions of stability in chapter 5 points to the need for an algorithm for generating the maximum stability region enclosing a singular point. Such a region might be defined by a quadratic form or the form of 5.23.

4. The result quoted in theorem 5.3 is valid for linear transfer functions of arbitrary order. An extension of the first-order model to one containing a transfer function, the nonlinearity, and another transfer function at the output of the nonlinearity would result in a model which is in the Lur'e form, provided the restrictions on the transfer functions hold. Such a model could account for more detailed behavior of threshold gates, and would be necessary where, for example, "ringing" transients are encountered.

5. An extension of the electric network analogy would be useful, both in the theory of electric networks and in the theory of threshold nets. Such an extension would result in electric networks with nonreciprocal elements, mutual inductance and capacitance, and coupling between

inductors and capacitors. Some useful results in the stability of such networks might arise. This extension with item 4 above would be useful in the large-signal characterization of arbitrary networks containing active elements which may be represented in the Lur'e form.

6. An extension of assertion 6.2 to positive-definite symmetric matrices would be useful, both for applications in neurodynamics, and as a potential function for electric networks such as discussed in item 5.

7.3.2 ALGEBRAIC PROBLEMS

Theorem 4.9 demonstrates that threshold network dynamics are characterized by the square matrix A. It is to be expected that the rich body of knowledge about matrices should contribute further results.

1. A useful result would be necessary and sufficient conditions under which two matrices characterize logically similar systems according to definition 4.5. A possible extension of this definition is as follows: Two networks might be defined to be logically equivalent if they are logically similar and if, in addition, they are both threshold networks. Then necessary and sufficient conditions for two matrices to describe logically equivalent networks would be desirable because it would allow generation of canonical networks.

2. The conditions under which threshold functions may be said to be equivalent are well-known[1,2,3,4]. In particular, tabulations are available of canonical weight-sets of all threshold functions of up to seven variables. It is thus possible to generate all logically equivalent threshold nets in a systematic way. However, the number of equivalent

nets so generated is astronomical unless an algorithm is available for generating only distinct (non-equivalent) nets, or the distinct nets plus a few more which may be easily tested for equivalence to previously generated nets.

Enumerations are easier at lower levels of sophistication. It is easier, for example, to identify nets with complex eigenvalues at Γ.

A useful point is that given n sets of weights for an n x n matrix representing an n-gate network, the various interconnexions may be generated by permuting elements within rows, and interchanging rows. Such a system of rows is known in the jargon of the theory of finite groups[5] as an imprimitive transitive permutation group, and obeys the laws of such groups. Indeed an algebraic approach to the problem of establishing equivalences in threshold nets may be of great value.

3. An algebraic defintion of finite linear threshold nets has been published[6], and a result given regarding transition or next-state functions. It would be useful to have a clear connexion between the restrictions on such functions, and the conditions for realization of threshold nets by the continuous model.

4. The dynamic behavior of linear systems[7,8,9] is characterized by matrices or polynomial factors. Bearing in mind the discussion of item 1 and sections 1.7.1 and 2.5, it would be extremely useful to know exactly which results of linear theory apply to threshold networks and which do not. A reason for optimism is that, as has been

stated, the threshold network is dependent on its weight matrix.

7.3.3 PROBLEMS IN SEQUENTIAL-CIRCUIT ENGINEERING

From the engineering point of view it would be useful to relate the dynamic model more closely to specific electric circuits and to problems encountered when designing them. A better understanding of such relations would undoubtedly result in research problems. A few such problems will be mentioned here.

1. The most general use of a system model is to construct real systems which are optimal in some sense. The construction of systems with the minimum number of gates is a classical problem, and requires the solution of some of the problems mentioned previously in this chapter. However, the model is evidently useful in finding networks less sensitive to noise, say, or which have shorter signal propagation times.

2. A useful result would be the listing of the static and transient behavior of common logic gates. Consideration would also need to be given to their interactions, for example in an integrated circuit. Cases in which the model is inadequate would show how it needs to be extended.

3. Although it is certainly true that there is no perfect delay in nature, cases may arise where a model incorporating pure delay gives reasonable predictions easier than one which does not. Two examples are contact networks and those in which signal propagation time between circuits is comparable to switching times. Results for such situations

exist in the control-systems literature[10], and could be incorporated profitably into the results presented here.

4. An effect mentioned in section 6.5 needs some clarification. A basic assumption in this work has been that external system inputs are constant during the intervals of interest. It would be extremely helpful to sequential design if the results given here were extended to non-constant inputs, or at least if discrete input changes could be related to system topology.

7.4 CONCLUSIONS

Many of the conclusions to be reached from this work have been mentioned previously. Here only the main points will be summarized.

Many good reasons exist for the consideration of continuous models of switching circuits. From the discussion of chapters 1 and 2 it may be concluded that the dynamic model which has been developed does to a degree represent the class of switching circuits known as linear threshold nets, and that it is also relevant to models, one in particular, of neurons. A method exists for relating this time-continuous model to others defined only in discrete time.

It may be concluded that the topological analysis of chapter 3 is suitable to indicate the behavior of two-gate networks, and that several of the results for such networks are directly extendable to networks of higher order. A special case is directly analogous to an electrical network.

The stability analysis of chapter 4 is based upon linearization of the nonlinear model in neighborhoods of its singular points. It is concluded that this type of analysis is useful for specifying the conditions under which a collection of threshold gates realizes a threshold function. Furthermore, methods for dividing arbitrary networks into smaller subnetworks for analysis have been given.

Questions of global stability discussed in chapter 5 are useful in determining network behavior. Two forms of positive-definite functions may be used in defining regions of convergence of the system. These results apply to the determination of conditions under which the system exhibits cyclic behavior.

The model which has been developed is useful in the understanding and analysing of several problems connected with logic circuit design, as demonstrated in chapter 6.

7.5 SUMMARY

It is concluded that the system model, based upon a function which has a continuous output set, explicates behavior predicted by simpler functions, predicts behavior which they cannot, and is necessary for answering several questions about logical circuits. A quotation from the literature[11] is relevant:

> "It is well known that electronic asynchronous circuits are rather sickly specimens when exposed to the full gambit of diseases which arise from variations in the response times of active elements and variations in the propagation times of signals between active elements."

This thesis is an attempt to explain and alleviate some of the difficulties which arise in the analysis and design of such systems.

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APPENDIX A

SIMULATION OF THE MODEL BY ANALOGUE COMPUTER

A TR-48 analogue computer was used to solve equations 4.17:

$$\frac{dx_i}{dt} = x_i(1 - x_i)(\epsilon_i + \sum_j \alpha_{ij}x_j - \beta_i \log\frac{x_i}{1 - x_i}) \quad . \qquad 4.17$$

In an orthodox solution of such a system the number of equations to be solved simultaneously is limited by the number of computing units available, usually the number of multipliers. A maximum of four equations of the special system could be solved, and Fig. A.1 is an operational diagram of such a solution.

This method produces reasonable accuracy, but is limited by the resolution of the multipliers used. When any input approaches 0 they are inaccurate.

To overcome these problems and to enable a larger number of equations to be solved using the same number of operational amplifiers, transistorized function generators were constructed. The procedure is to perform the functional operations of Fig. 1.12a.

The computation requires the following function to be calculated:

$$x_i = \frac{e^{y_i}}{1 + e^{y_i}} \quad . \qquad A.1$$

This can be accomplished using the principle of the long-tailed pair amplifier. Consider the circuit in Fig. A.2. To a good approximation, the collector current in a transistor is an exponential function of emitter-base voltage. Thus

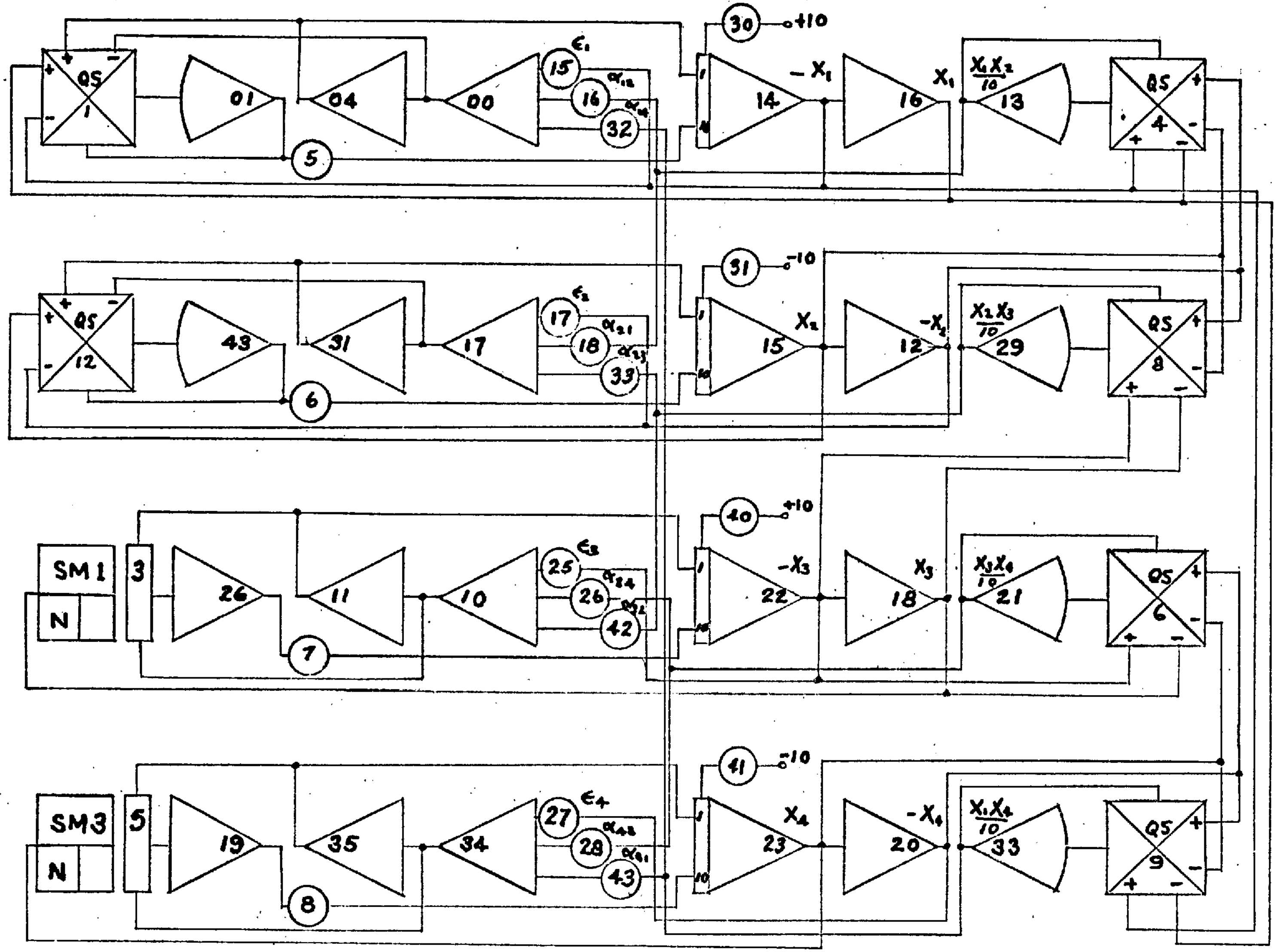


FIG. A.1 COMPUTER DIAGRAM

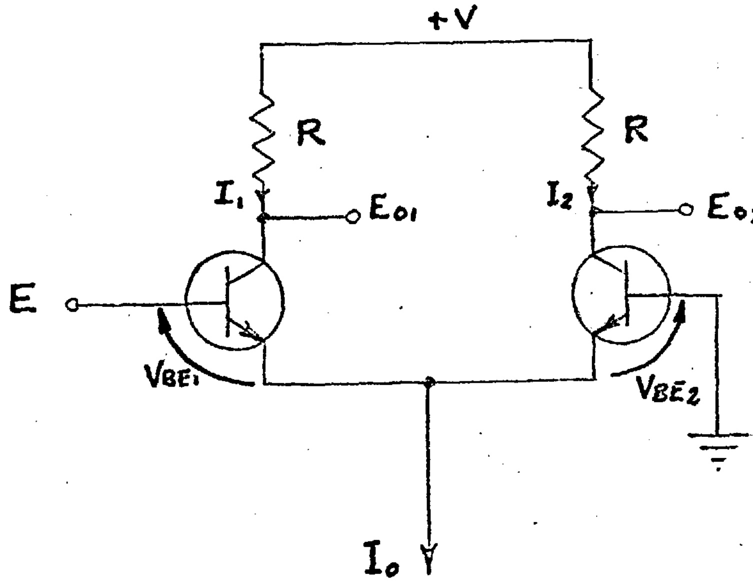


FIG A.2: LONG-TAILED PAIR

FUNCTION No.	1	2	3	4	5	6	7	8
a	37.7	38.2	37.9	37.9	37.5	36.3	37.9	38.2
b	-.045	-.021	-.004	-.001	-.023	-.053	-.041	-.013

$$\log_e \frac{X_i}{5 - X_i} = a Y_i + b$$

$$\bar{a} = 37.8$$

FIG. A.3: FUNCTION GENERATOR CONSTANTS

$$I_1 = \exp\{a_1 V_{BE_1} - b_1\} \ , \qquad \text{A.2}$$

$$I_2 = \exp\{a_2 V_{BE_2} - b_2\} \ . \qquad \text{A.3}$$

The parameter a_i is of the order of $\frac{q}{kT}$, and b_i is largely determined by I_{CBO}. Also to a good approximation, assuming the current gains are high,

$$I_1 + I_2 = I_o \ . \qquad \text{A.4}$$

From the diagram,

$$E - V_{BE_1} + V_{BE_2} = 0 \qquad \text{A.5}$$

$$E_{o1} = V - I_1 R \qquad \text{A.6}$$

$$E_{o2} = V - I_2 R \ . \qquad \text{A.7}$$

Combining equations A.2—A.5,

$$I_2 = \exp\left\{a_2\left[\frac{\log(I_o - I_2) + b_1}{a_1} - E\right] - b_2\right\} \qquad \text{A.8}$$

Define:

$$a_1 = a \ , \qquad a_2 = a + \Delta a \ ,$$
$$b_1 = b \ , \qquad b_2 = b + \Delta b \ . \qquad \text{A.9}$$

Then

$$I_2 = \exp\left\{(a + \Delta a)\left[\frac{\log(I_o - I_2) + b}{a} - E\right] - (b + \Delta b)\right\}$$

$$= \exp\left\{\log(I_o - I_2) + b - aE - b + \frac{\Delta a}{a}\log(I_o - I_2) + b \div \Delta aE - \Delta b\right\} \qquad \text{A.10}$$

If the transistors are chosen to have nearly identical values of a, so that $\frac{\Delta a}{a} \rightarrow 0$, the equation becomes

$$I_2 = (I_o - I_2)\exp\left\{-aE - \Delta b\right\} \qquad \text{A.11}$$

or

$$\frac{I_2}{I_o - I_2} = \exp\left\{-a(E + \frac{\Delta b}{a})\right\} . \qquad \text{A.12}$$

A similar equation obtains for I_1, except for a sign reversal:

$$\frac{I_1}{I_o - I_1} = \exp\left\{a(E + \frac{\Delta b}{a})\right\} . \qquad \text{A.13}$$

From A.12 and A.13 we can write

$$I_1 = \frac{I_o\exp\left\{a(E + \frac{\Delta b}{a})\right\}}{1 + \exp\left\{a(E + \frac{\Delta b}{a})\right\}} \qquad \text{A.14}$$

$$I_2 = \frac{I_o\exp\left\{-a(E + \frac{\Delta b}{a})\right\}}{1 + \exp\left\{-a(E + \frac{\Delta b}{a})\right\}} .$$

Combining equations A.6 and A.7 with equations A.14:

$$E_{o1} = V - \frac{I_oR\exp\left\{a(E + \frac{\Delta b}{a})\right\}}{1 + \exp\left\{a(E + \frac{\Delta b}{a})\right\}} ,$$

$$E_{o2} = V - \frac{I_oR\exp\left\{-a(E + \frac{\Delta b}{a})\right\}}{1 + \exp\left\{-a(E + \frac{\Delta b}{a})\right\}} . \qquad \text{A.15}$$

It can be seen that the second terms in these equations have the required forms, except that the input value E is multiplied by a constant, and is offset by the voltage $\frac{\Delta b}{a}$. This circuit can therefore be used to perform the necessary function generation provided the transistors are chosen so that the following conditions hold:

1. The function $I_C(V_{BE})$ must be logarithmic in both transistors.

2. The transistors must have nearly identical exponential factors a.

3. The base current of both transistors must be negligible compared to the emitter current.

Transistors 2N2926 have been found to satisfy the above requirements. Typical values of a and b are listed in Fig. A.3, and Fig. A.4 is a typical experimental plot of I_C against V_{BE}. These transistors have been used in the circuit of Fig. A.5.

The stabilized power supplies of the TR-48 are used. P_1 is adjusted so that $X_1 = 0$ for open-circuit input and the switch in the (+) position. The switch is then changed to (-) and P_2 used to adjust X_1 to +5.00 volts. This makes $I_oR = 5.00$ in A.15.

Fig. A.6 is a graph of the transfer function of a typical circuit, and Fig. A.7 is a graph of $\log\frac{X_1}{1 - X_1}$ for the function of Fig. A.6. It shows a nearly linear characteristic.

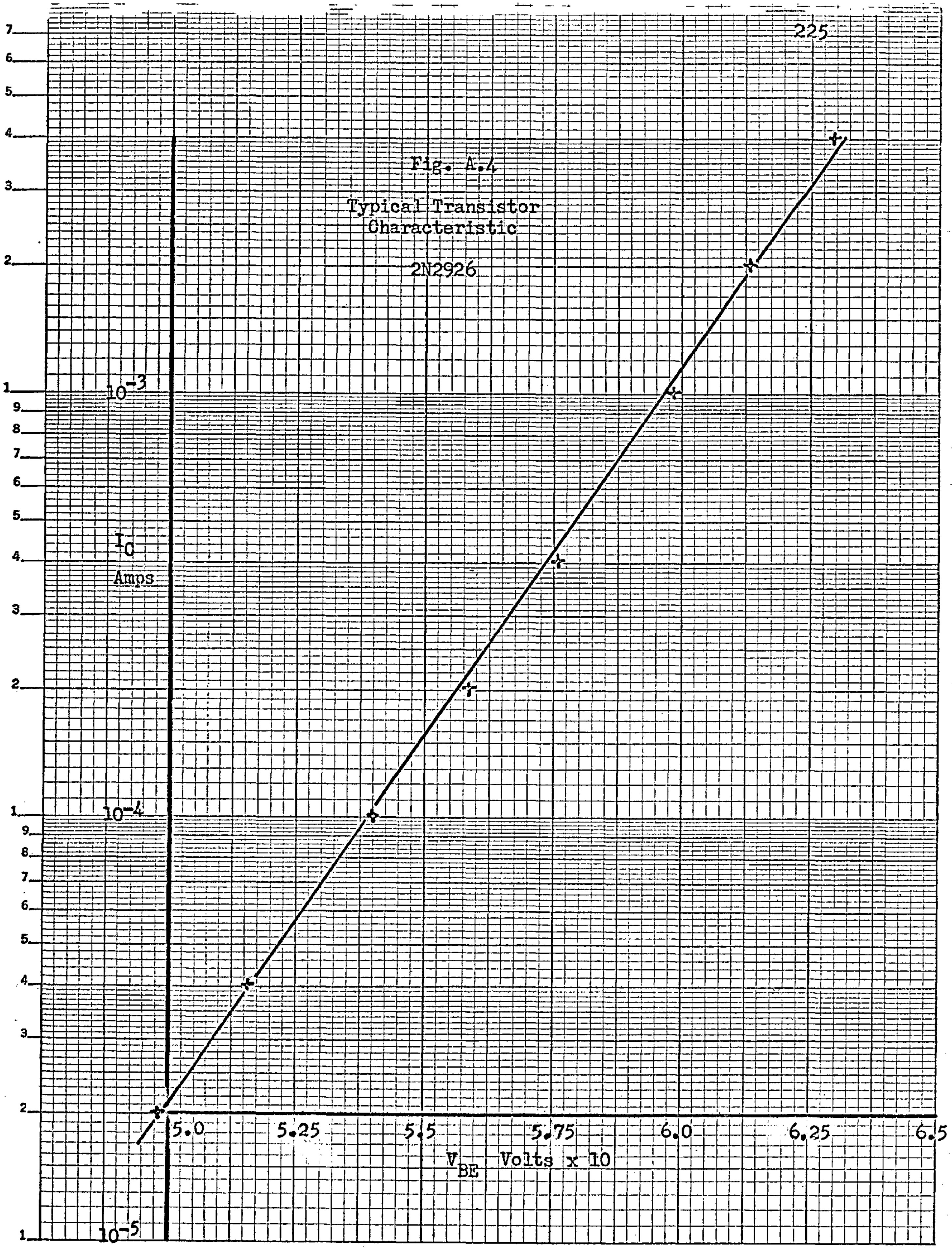


Fig. A.4
Typical Transistor
Characteristic
2N2926
I_0
Amps
10^{-3}
10^{-4}
10^{-5}
5.0
5.25
5.5
5.75
6.0
6.25
6.5
V_{BE} Volts x 10

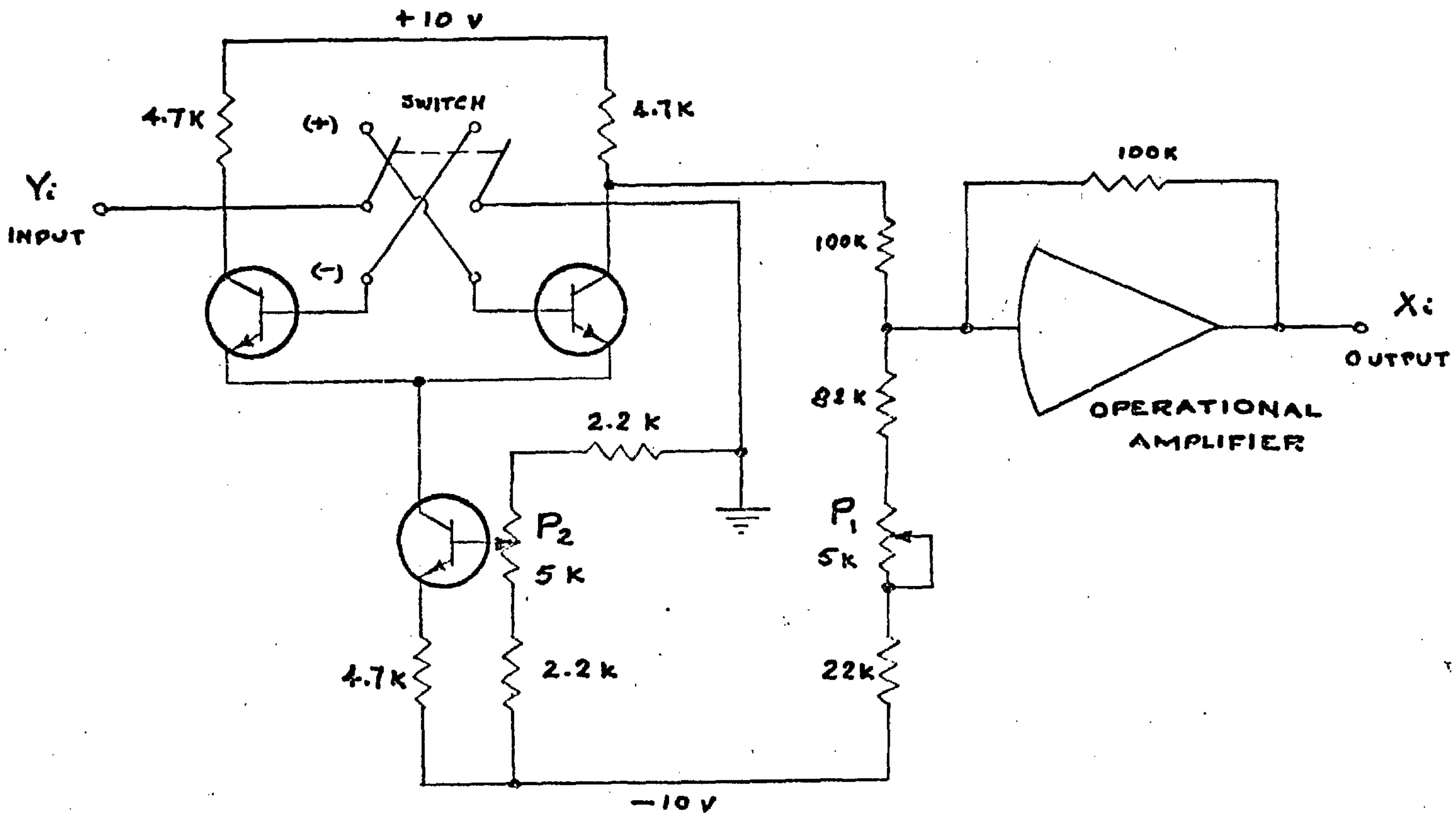


Fig. A.5 Function Generator

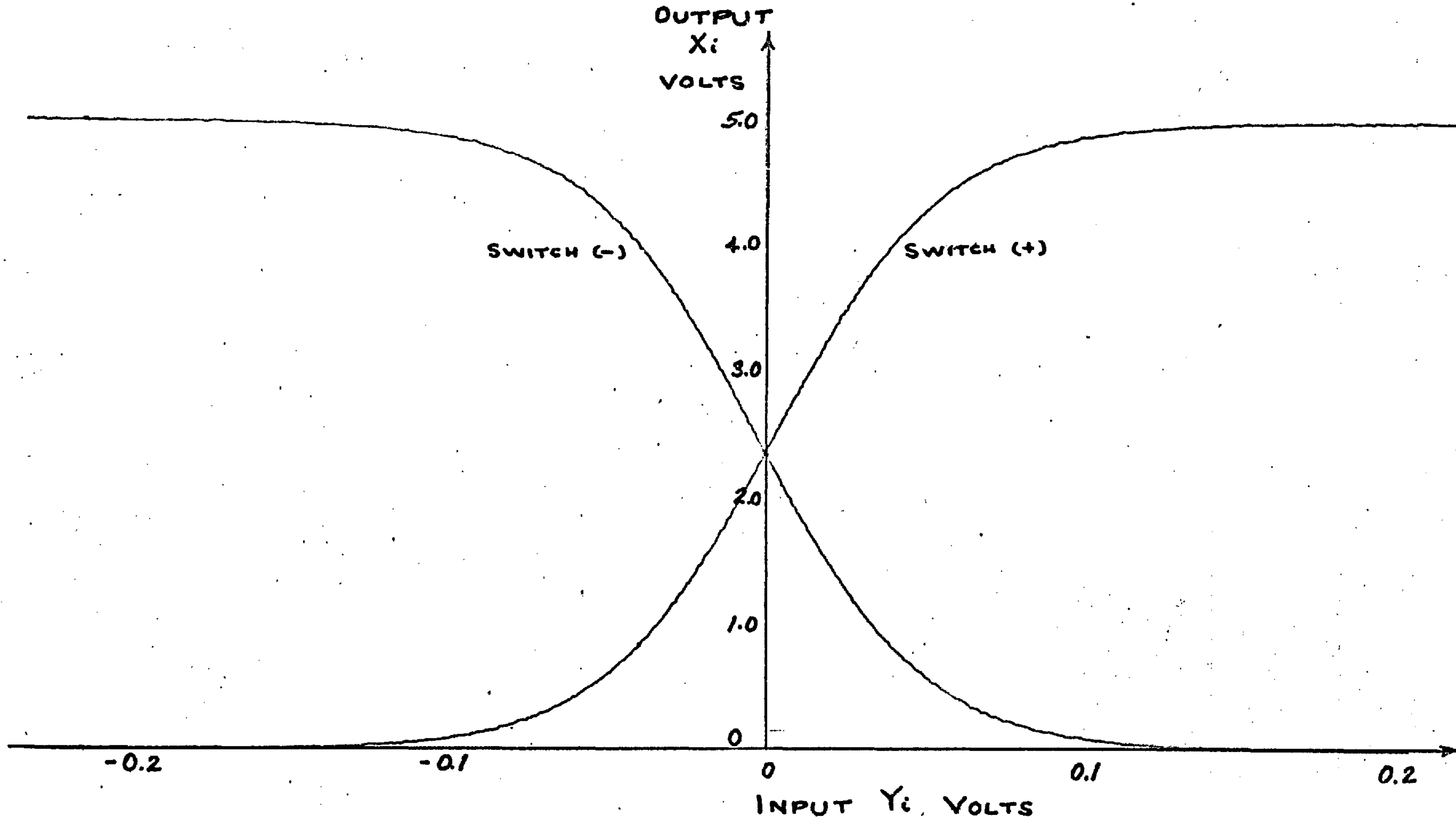


FIG. A.6 FUNCTION GENERATOR TRANSFER FUNCTION

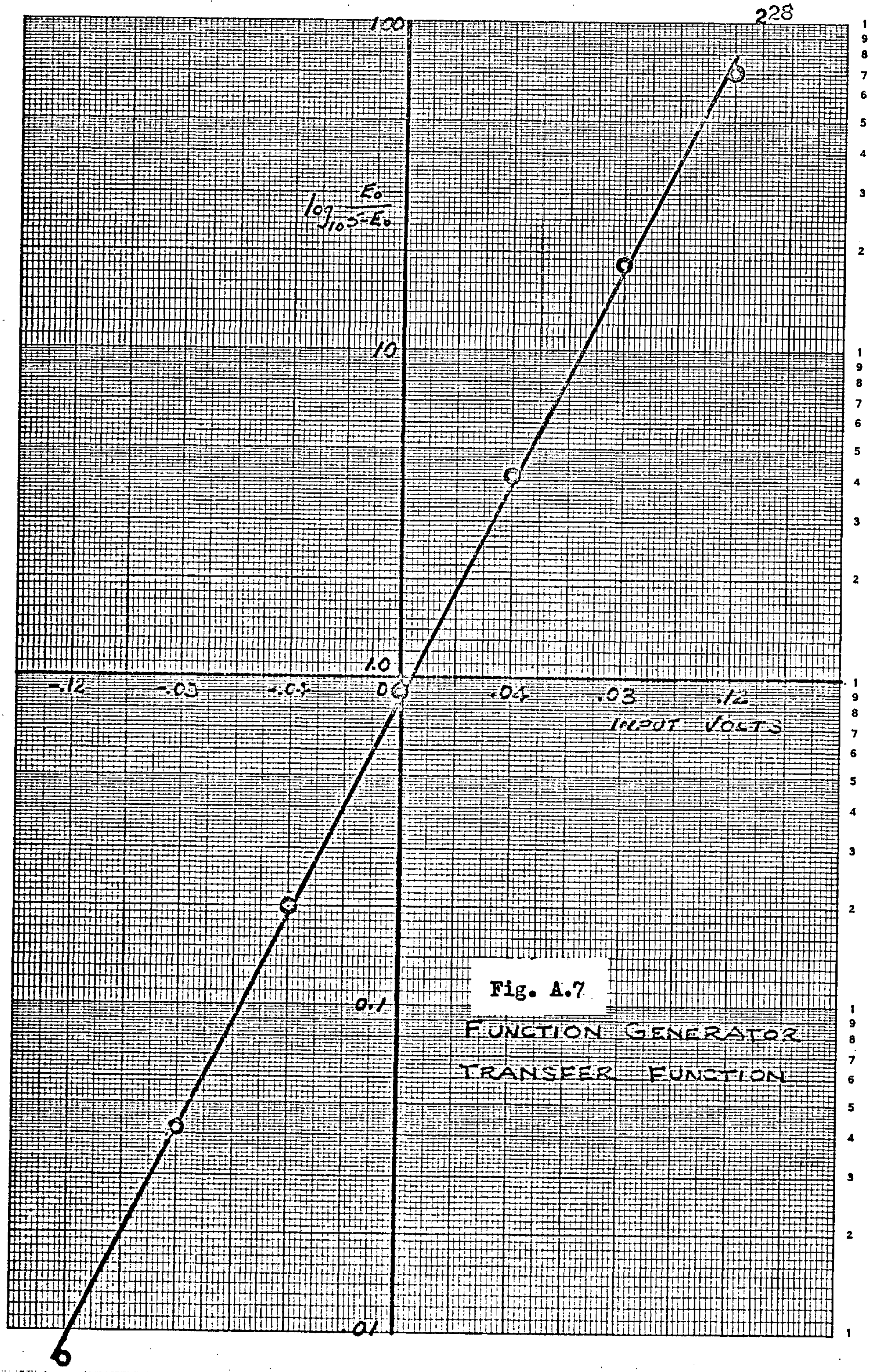


Fig. A.7

FUNCTION GENERATOR TRANSFER FUNCTION

APPENDIX B

SIMULATION OF THE MODEL BY DIGITAL COMPUTER

Details of the digital simulation will not be discussed, because the solution of the equation $\dot{X} = F(X)$ is a standard computing problem. It will only be mentioned that care must be taken whenever a value x_i approaches 0 or 1. Standard integration techniques are of course time-discrete, and it is possible to set x_i outside the interval (0,1). Such a condition may have several results, a common one being a negative argument in the function $\log\frac{x_i}{1 - x_i}$, in which case execution is stopped or becomes meaningless. This type of fault is avoided by ensuring always that initial conditions are within (0,1), that x_i is never set outside this interval, and that error tests are more stringent near 0 or 1. Another expedient is to solve the equations in terms of the variables V, in which case this difficulty is avoided. However, care must be taken to have small step-lengths near $x_i = \gamma_i$, as trajectories tend to have sharp changes of direction in this neighborhood.

APPENDIX C

GRAPHICAL SOLUTION OF A SPECIAL CASE

Consider the equations

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1)(\epsilon_1 + \alpha_{12}x_2) , \\ \dot{x}_2 &= x_2(1 - x_2)(\epsilon_2 + \alpha_{21}x_1) , \end{aligned} \qquad \text{C.1}$$

which are a special case of 3.17. In the same notation, Volterra's equations (chapter 3, reference 2) are

$$\begin{aligned} \dot{x}_1 &= x_1(\epsilon_1 + \alpha_{12}x_2) , \\ \dot{x}_2 &= x_2(\epsilon_2 + \alpha_{21}x_1) . \end{aligned} \qquad \text{C.2}$$

It is possible to obtain the equation of the phase trajectories, that is, the solution of the equation

$$\frac{dx_2}{dx_1} = \frac{x_2(1 - x_2)(\epsilon_2 + \alpha_{21}x_1)}{x_1(1 - x_1)(\epsilon_1 + \alpha_{12}x_2)} . \qquad \text{C.3}$$

This can be achieved following the method of Volterra as follows: Since x_1 and x_2 satisfy equations C.1, we can write

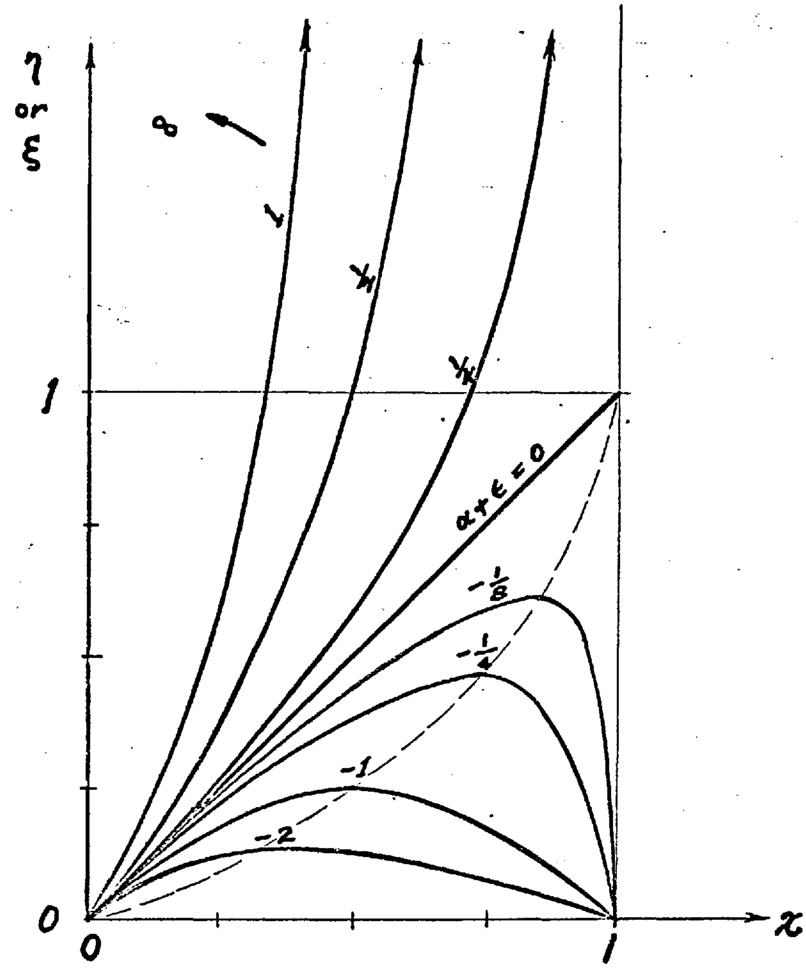
$$\frac{dx_2(\epsilon_1 + \alpha_{12}x_2)}{x_2(1 - x_2)} = \frac{dx_1(\epsilon_2 + \alpha_{21}x_1)}{x_1(1 - x_1)} . \qquad \text{C.4}$$

Separating into partial fractions:

$$\frac{\epsilon_1 dx_2}{x_2} + \frac{(\alpha_{12} + \epsilon_1)dx_2}{1 - x_2} = \frac{\epsilon_2 dx_1}{x_1} + \frac{(\alpha_{21} + \epsilon_2)dx_1}{1 - x_1} . \qquad \text{C.5}$$

Integrating this equation, we obtain

$$\frac{x_2^{\epsilon_1}}{(1 - x_2)^{\alpha_{12}+\epsilon_1}} = \frac{kx_1^{\epsilon_2}}{(1 - x_1)^{\alpha_{21}+\epsilon_2}} \qquad \text{C.6}$$



$$\eta \text{ or } \xi = \frac{x^{\epsilon}}{(1-x)^{\alpha+\epsilon}}, \quad \epsilon = 1$$

FIG. C.1 η or ξ vs. x

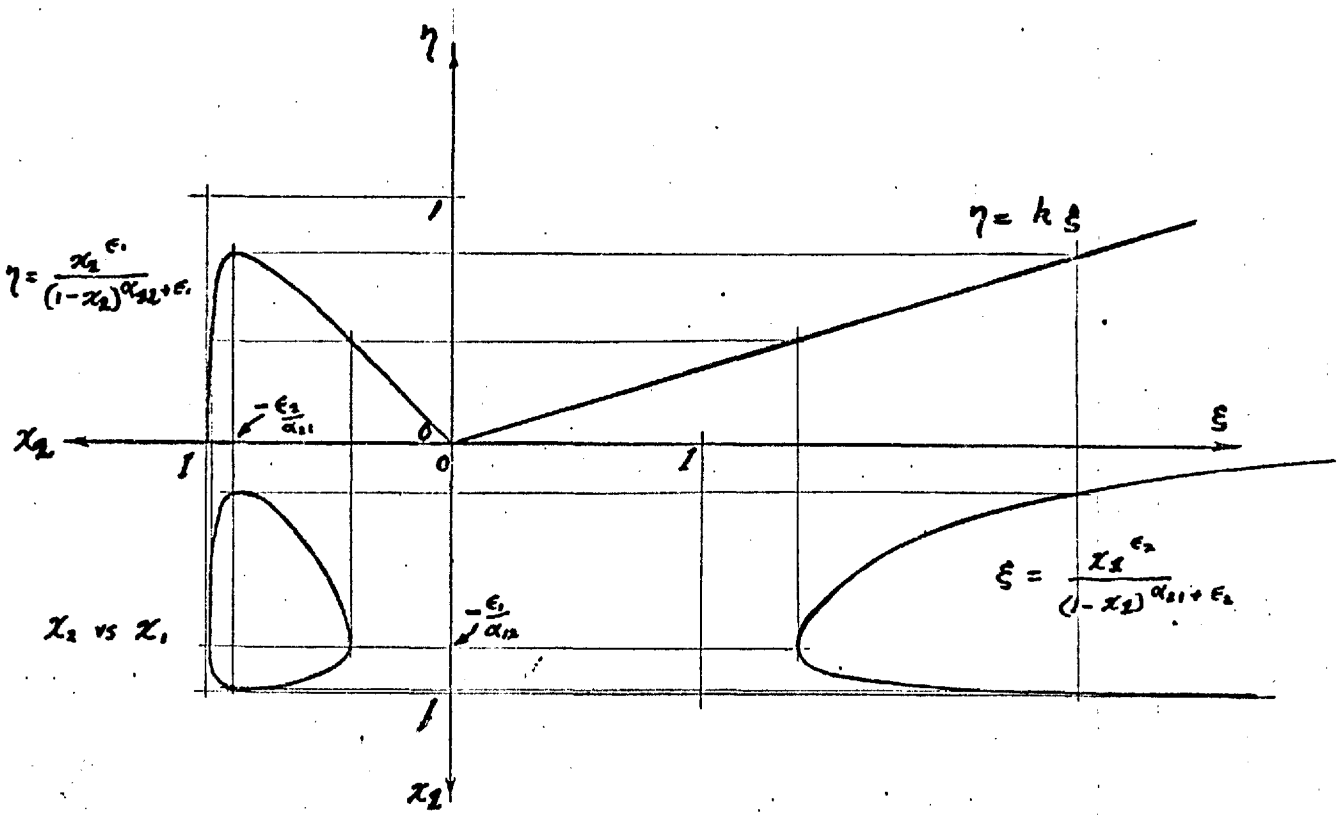


Fig. C.2 Phase Plane Diagram

where k is an arbitrary constant. Solving for k in terms of $x_1(0)$ and $x_2(0)$:

$$\frac{x_2(0)^{\epsilon_1}(1 - x_2(0))^{\alpha_{21} + \epsilon_2}}{x_1(0)^{\epsilon_2}(1 - x_1(0))^{\alpha_{12} + \epsilon_1}} \quad \text{C.7}$$

Equation C.6 is a functional relationship between x_1 and x_2. A graphical method devised by Volterra can be used to determine trajectory points most easily. Let

$$\eta = \frac{x_2}{(1 - x_2)^{\alpha_{12}+\epsilon_1}}, \quad \text{C.8}$$

$$\xi = \frac{x_1}{(1 - x_1)^{\alpha_{21}+\epsilon_2}}. \quad \text{C.9}$$

Typical graphs of η or ξ for $0 < x < 1$ are shown in Fig. C.1.

η is plotted with respect to x_1 and ξ with respect to x_2 as in Fig. C.2. The line in the $\eta - \xi$ quadrant has slope k. Then for any point on η a corresponding point or pair of points can be found on ξ and on the phase trajectory if such points exist.

Fig. C.2 illustrates the particularly interesting case for which $\epsilon_1 = A$, $\epsilon_2 = -B$, $\alpha_{12} = -C$, $\alpha_{21} = D$ where A, B, C and D are positive constants, and $B > A$, $D > C$. In this case the trajectory is a closed curve and the values $x_1(t)$ and $x_2(t)$ are bounded nonlinear oscillations.