SYMMETRIES IN ELEMENTARY PARTICLES

by

MUNEER AHMAD RASHID

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Department of Theoretical Physics,
Imperial College of Science and Technology,
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PREFACE.

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, London between October 1960 and December 1963 under the supervision of Professor Abdus Salam. The author wishes to thank him and Professor P.T. Matthews, Dr. T.W.B. Kibble, and Dr. A.R. Edmonds, for their continued help, guidance and encouragement.

Except where stated in the text, the work described is original and has not been submitted in this or any other University for any other degree. The thesis is based principally upon a paper by the author and two papers with the author as one of the contributors. The author is grateful to his collaborators for their permission to include the joint work in his thesis. He also wishes to express his acknowledgement to the Pakistan Atomic Energy Commission and the British Department of Technical Cooperation for a financial grant under the Colombo Plan.

CONTENTS

ABSTRACT

We study the group-theoretic techniques used in obtaining consequences of the Unitary Symmetry schemes for the strong interactions of elementary particles as proposed by Sakata and Gell-Mann and Neeman. The mathematics of the groups U(n) and SU(n) is discussed in Chapter I. We obtain the irreducible representations and, by decomposing the bases relative to one of the subgroups SU(n-1), we arrive at a reduction formula for calculating the dimensions of these representations. In particular, this method gives the (I,S) multiplets in a unitary supermultiplet.

In the second chapter, we elaborate the use of Young's tableaux in obtaining the Clebsch-Gordan series in the reduction of the product of two irreducible representations. We also give a technique which starts from first principles and obtain the complete reduction of the products of two and three octets. We then describe another method which does not use the form of the basis elements in terms of the fundamental fields. Some of the applications are also given. In particular, we are able to discard the Sakata model.

The third chapter is devoted to the consideration of the breakdown of the symmetry and, following Okubo, we generalize the mass formula to any order. We see that the lowest order formula fits nicely the baryon and pseudoscalar

meson octets. However, for vector mesons, neither φ nor ω masses can give a reasonable fit. Following Sukurai, we try to see whether these could be mixtures. The consequences can be experimentally verified.

In the fourth chapter, we follow the U-spin approach of Lipkin et al for the discussion of the electromagnetic interaction and obtain the results already arrived at by Cabibbo et al in a very simple fashion.

The concluding chapter is devoted to a critical examination of the present position of the octet version and its comparison with schemes based on other rank 2 groups.

INTRODUCTION

Gell-Mann and Nishijima[1] classified the strongly interacting particles (baryons and mesons) into isotopic multiplets (forming basis vectors for the irreducible representations of the isotopic spin group SU(2)) characterized by three quantum numbers I, S and N. On the basis of the conservation of these quantum numbers, one can write the strong interaction Lagrangian for the Yukawa type baryon-meson interaction in terms of 8 (strong) coupling constants. The search for higher symmetries was the natural outcome of the success of the above classification, the main purpose being to examine whether the strong interactions are more restricted or not. Historically, the quest started with the introduction of the Global Symmetry independently by Gell-Mann and Schwinger[2] with the basic assumption that all the 8 baryons are alike and so also their interactions with the π-mesons. The K-mesons being much heavier have an interaction with the baryons, the strength of which is an order of magnitude smaller than the corresponding one for the pions. We may, therefore, neglect the K-meson interactions while considering the very strong interactions. However, as the experiments indicated that the K-meson couplings are comparable to those of the pions, the scheme gave way to the consideration of many a priori choices of equations between the above coupling constants, some of which met limited success.[3]

Another reason for postulating higher symmetries is to understand the systematics of the vast number of strongly decaying resonances - in order not to have too many elementary entities, one would very much like to consider these as bound states in a scheme which also tells when and where to look for new resonances. (This sort of thinking might lead one to discarding the notion of elementarity attached to some or all of the commonly listed elementary particles. The latter approach, where every particle is considered to be a bound state of every other, is very much in fashion these days.[4] Unfortunately the presentation of such dynamical calculations is beyond the scope of the present thesis.) The answer to the question of which <u>could</u> be the elementary entities was first provided by Sakata[5] who suggested that we consider p, n, Λ and their anti-particles as the elementary ones. This proposal, being one of many possible solutions, has the virtue that the least massive baryons are considered elementary.[6] However, noting that p, n, and Λ have very nearly the same mass, we may think of them as forming the triplet representation of some symmetry group which has SU(2) as one of its subgroups. Apart from the phase group associated with the baryon number operator, the only solution is the group SU(3). This is how Ikeda, Ogawa and Ohnuki[7] suggested U(3) as a possible symmetry group for the strong interactions.

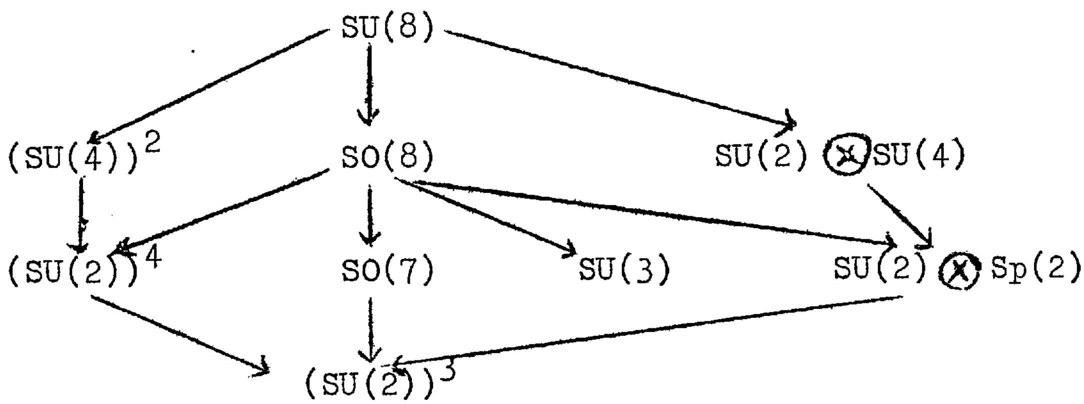
Now Σ has a mass very nearly the same as of Λ, but there is no place for Σ in the 3-dimensional representation. This is not very nice, particularly when the $\Sigma\Lambda$ parity appears to be even. We shall see later that resonances do not fit properly in this model. We shall see also that this model forbids

$$p + \bar{p} \rightarrow K_1^0 + K_2^0$$

Thus the Sakata model of unitary symmetry has to be abandoned.

Can we remedy the above defects within the context of the group U(3) or SU(3)? We have eight baryons and both U(3) and SU(3) have an 8-dimensional representation. However using strangeness as the quantum number, we find that these will not fit into it. But if we abandon the baryon number and consider the hypercharge in place of strangeness, then eight baryons can be placed in the 8-dimensional representation. On the other hand, if we use the other rank 2 groups like G_2 and C_2 (or B_2), we do not have any 8-dimensional representation to fit all the 8 baryons.

Coleman[8] and, independently, Speiser and Tarski[9] analysed all the Lie groups which have an eight-dimensional representation, which breaks up into isomultiplets so as to fit the 8 baryons. Clearly, if we want them to be in the same multiplet, the highest symmetry that we can think of is SU(8). The maximum symmetry existing in nature might correspond to just a subgroup of it. The analysis of these authors is summarized in the following diagram.



There are two ends of the chains pictured here, namely the groups SU(3) and $(SU(2))^3$. The second one of these corresponds to the Global Symmetry,[10] the predictions of which are not supported by experiment. Thus the only other choice to be tried is SU(3) which can give rise to an octet with correct isospin and hyperchange values. It is clear that in the above the analysis has not been restricted to rank 2 or even semi-simple Lie groups. In fact $SU(2) \otimes SU(2) \otimes SU(2)$ group of Global Symmetry is neither semi-simple nor of rank 2. It is also evident that the existence of a symmetry higher than SU(3) necessarily demands the existence of SU(3) at least as a symmetry. We must, nevertheless, emphasize that the analysis is based on the demand that we want to have an octet representation to fit in all the 8 baryons in the same super-multiplet. This is only justified by the existence of 8 baryons of the same spatial properties. If the spin of Ξ should turn out to be $\frac{3}{2}$, all this analysis would have to be abandoned.

If we now consider the Yukawa type couplings between the baryons and the mesons, we shall have to look for the mesons in the decomposition of the direct product of two octets which breaks up into the representations 1, 8_a, 8_s, 10, 10 , 27. The known pseudo-scalar and the vector mesons have the quantum numbers which can fit into the 8 as well as the 27-dimensional representation, though the second possibility requires the existence of many more such states. Thus Neeman[11] and Gell-Mann,[12] preferring the first choice, proposed the "8-fold way" wherein the baryons, the pseudo-scalar and vector mesons, were all considered to form octets.

The mathematics of U(3) and SU(3) can now be used to obtain the consequences of the Sakata as well as the Gell-Man Neeman models. This is the principal goal of our thesis. In Chapter I, we give as much of the mathematics of the groups U(n) and SU(n) as is necessary for our purpose.

The contents of this chapter appear in many texts on group theory.[13] Our main purpose is to emphasize the points relevant to the physics we are to consider.

In Chapter 2, we describe the Young tableau technique and reduce[14] the direct products $8 \otimes 8$, $8 \otimes 10$, $8 \otimes \overline{10}$, $8 \otimes 27$ modifying the work of the Japanese group.[7,15] The reduction of $8 \otimes 8$ has been carried out by Edmonds[16] and also forms part of Tarjanne's Ph.D. thesis.[27] Recently Tarjanne and, independently, de Swart,[18] obtained the reduction of the products $10 \otimes 10$, $10 \otimes \overline{10}$, $\overline{10} \otimes \overline{10}$, in addition to those carried out by the author.

The third chapter is devoted to the consideration of the breakdown of the symmetry and, following Okubo,[19] Diu and Ginibre,[20] we generalize the mass formula to any order. In the fourth chapter, we follow the U-spin approach of Lipkin et al[21] for the discussion of the electromagnetic interaction and obtain the results already arrived at by Cabibbo et al[22] in a very simple fashion.

In the concluding chapter, we review the present position of the octet model and compare it with schemes based on other rank 2 groups.

CHAPTER 1.

The Groups U(n) and SU(n).

i) Definitions.

The group U(n) is the group of all the unitary (generally complex) transformations in a space of n dimension. These are described by n x n matrices M which satisfy

$$MM^{\dagger} = 1 = M^{\dagger}M \qquad ....(1.1)$$

where M^{+} is the hermitean conjugate of M. As the matrix M has, in general, complex enterties, it can always be described by $2n^2$ real numbers. The condition (1.1) is equivalent to n^2 conditions on these $2n^2$ real numbers. Thus a basis for the nxn unitary matrices could be constructed in terms of n^2 nxn matrices.

If we impose the further condition

$$\det M = 1 \qquad ....(1.2)$$

we obtain a unitary unimodular matrix which corresponds to a unitary unimodular transformation. The group of such matrices is denoted by SU(n). Since (1.2) is one additional restriction ((1.1) implies that the determinant of the matrix M has unit magnitude), the set of unitary unimodular nxn matrices will have $n^2 - 1$ matrices in its basis.

In particular, the groups U(3) and SU(3) have 9 and 8 elements in their basis.

Considering infinitesimal transformations described by

$$M = 1 + i\alpha H \qquad \ldots.(1.3)$$

(α a real infinitesimal)

we see that the condition (1.1) ((1.2)) is satisfied if, and only if, H is hermitean (traceless). An H satisfying condition (1.1) (conditions (1.1) and (1.2)) can then be expressed as a linear combination

$$H = \alpha_i H_i \qquad \ldots.(1.4)$$

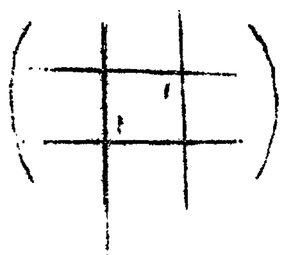
of n^2 (($n^2 - 1$)) hermitean (hermitean and traceless) matrices H_i with real coefficients α_i.

A particular set of such n^2 nxn hermitean matrices can be taken to consist of

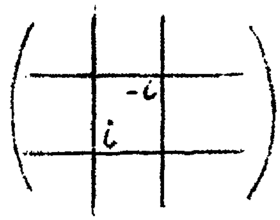
$$A_1 = I_n = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \end{pmatrix}$$

$$A_i = \begin{pmatrix} 1 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \cdot & \\ & & & & & & & & 0 \end{pmatrix}$$

($2 \leq i \leq n$) where -1 is in the ith position,

$A_{ij} =$ 

$B_{ij} =$



$(1 \leq i,\ j \leq n,\ i \neq j)$

where A_{ij} (B_{ij}) has 1 ($\mp$ i) in the ij and ji positions.

These n^2 matrices form a particular basis for the nxn hermitean H in (1.4) and thus generate the infinitesimal algebra corresponding to the group U(n). The infinitesimal algebra which corresponds to the group SU(n) has a particular basis consisting of all the above matrices with the neglect of the identity which is, in fact, the only matrix which is not traceless. As shown in (1.4), the coefficients that appear in the expression of a hermitean (or hermitean traceless) in terms of these bases are all real. This result is actually evident from the fact that these bases consist of <u>hermitean</u> matrices.

Another basis, which is called "canonical", consists of n^2 <u>real</u> matrices $A^i{}_j$ where

$$(A^i{}_j)_{lm} = \delta_{il} \cdot \delta_{jm} \qquad \ldots.(1.5)$$

i.e. $A^i{}_j$ has a unity in the position of ith row and jth column and zeros everythwere else. These matrices satisfy the commutation relations

$$[A^i{}_j, A^l{}_m] = \delta_{lj} A^i{}_m - \delta_{im} A^l{}_j \qquad \ldots.(1.6)$$

It is evident that an nxn hermitean matrix H expressed in terms of the $A^i{}_j$ as

$$H = \alpha_{ij} A^i{}_j$$

will no longer have <u>all</u> the α_{ij}'s as real. In fact the hermiticity requires

$$\alpha_{ij} = \alpha^*_{ji}$$

Now the groups U(n) and SU(n) are Lie groups. Thus their structures are related locally to the structures of their Lie algebras, i.e. their infinitesimal parts. Again the commutation relations (1.6) are sufficient to allow a reconstruction of the Lie algebra associated with these groups. Finally we also note that these groups are compact, i.e. their elements vary over a finite range as follows from the unitarity of the matrices representing the transformations belonging to these groups. Thus by a famous theorem of Weyl,[13,23] all the <u>finite-dimensional</u> irreducible representations of these groups are <u>equivalent</u> to unitary representations.

In fact, we only construct the finite-dimensional irreducible representations of the generators $A^i{}_j$ of the Lie algebras which satisfy the commutation relations (1.6) and then, by the usual techniques of exponentiation, we can construct the representations of the elements of the groups. Thus the problem of understanding the structures becomes very much simpler when we deal with the infinitesimal generators of the groups. The price we have to pay is however the fact that the determined structure is applicable only locally. The problem of global structure is also very interesting. We shall, nevertheless, not try to go any further on this point.

ii) Irreducible Representations.

In the last section we mentioned the fact that the groups U(n) and SU(n) are compact Lie groups, and as a consequence every finite-dimensional representation is equivalent to a unitary representation. In this section we shall enumerate the irreducible representations.[13,23]

Let $\underline{a} = (a_{ij})$ be any unitary nxn matrix operating on an n-dimensional vector space consisting of vectors $\underline{u}$ with components u^i $(1 \leq i \leq n)$ such that

$$u'^i = (A\underline{u})^i = a_{ij}\, u^j \qquad \ldots.(1.7)$$

Then the n^r quantities

$$u_{(1)}^{i_1}\, u_{(2)}^{i_2} \ldots.\, u_{(r)}^{i_r} \qquad \begin{pmatrix} 1 \leq i_k \leq n \\ 1 \leq k \leq r \end{pmatrix} \qquad \ldots.(1.8)$$

formed from r vectors

$$\underline{u}_{(1)}, \underline{u}_{(2)}, \ldots \underline{u}_r \qquad \ldots.(1.9)$$

transform as follows:

$$u^{i_1'}_{(1)} \ldots u^{i_r'}_{(r)} = a_{i_1 j_1} \ldots a_{i_r j_r} u^{j_1}_{(1)} \ldots u^{j_r}_{(r)} \qquad \ldots.(1.10)$$

Thus the product (1.9) of vectors transforms according to the direct product

$$\underline{a} \times \underline{a} \times \ldots \times \underline{a} \quad (r \text{ factors}) \qquad \ldots.(1.11)$$

If we consider $\underline{a}$ as an irreducible representation of U(n) with $\underline{u}$ as forming the corresponding basis vector, thenthe product (1.9) with components (1.8) acts as a <u>tensor space</u> for the operation of the direct product represention (1.11). We show below that this is, in general, a reducible representation of the group. For this purpose, we abbreviate the product in (1.8) as

$$u^{i_1 i_2 \ldots i_r} \qquad \ldots.(1.12)$$

and take its transformation law under $\underline{a}$ to be

$$u'^{i_1 i_2 \ldots i_r} = a_{i_1 j_1} \ldots a_{i_r j_r} u^{j_1 \ldots j_r} \qquad \ldots.(1.13)$$

In fact, any collection of n^r quantities of the type (1.12) which transform according to (1.13) is called an rth rank tensor under the considered group (here U(n) or SU(n)).

To each permutation

$$p = \begin{pmatrix} 1 & 2 & \ldots & r \\ 1' & 2' & \ldots & r' \end{pmatrix} \qquad \ldots.(1.14)$$

of the symmetric group S_r on r symbols, we associate an operator p which acts on the subindices $i_1\ i_2\ \ldots\ i_r$ of (1.12) and transforms them to $i_{1'}\ i_{2'}\ \ldots\ i_{r'}$. Thus

$$\underline{p}\ u^{i_1 i_2 \ldots i_r} = u^{i_{1'} \ldots i_{r'}} \qquad \ldots.(1.15)$$

As p operates on the subindices $i_1 \ldots i_r$, it transforms u and u' in (1.13) alike. Therefore applying (1.15) on both sides of (1.13), we obtain

$$u'^{i_{1'} i_{2'} \ldots i_{r'}} = a_{i_1 j_1}\ a_{i_2 j_2} \ldots a_{i_r j_r}\ u^{j_{1'} \ldots j_{r'}}$$

$$= a_{i_{1'} j_{1'}}\ a_{i_{2'} j_{2'}} \ldots a_{i_{r'} j_{r'}}\ u^{j_{1'} \ldots j_{r'}} \qquad \ldots.(1.16)$$

The last step follows from the fact that all the a_{ij}'s appearing in (1.16) commute. Thus we can rearrange them which is equivalent to applying the same permutation to both the first and the second indices.

We may rewrite (1.16) as

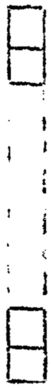
$$\underline{p}\ \underline{u}' = \underline{p}(\underline{a} \times \underline{a} \ldots \times \underline{a})\underline{u} = \underline{a} \times \underline{a} \times \ldots \times \underline{a}\ \underline{p}\ \underline{u} \qquad \ldots.(1.17)$$

This shows that the permutation operator p commutes with the group operations on the tensor space. This result is very important as it clearly indicates that tensors of a particular symmetry type remain tensor of the same symmetry type under operations of the group. The space of the tensors is, therefore, reducible into subspaces consisting of tensors of different symmetry types. This result, evidently, holds for any proper or improper subgroup of the group of linear transformations in n-dimensional space. These symmetry types are associated with Young's tableau which consist of a number of squares arranged in rows, each starting from the same vertical line such that the number of squares in the ith line is not less than the ones in the i+1st line. When we are considering any subgroup of linear transformation in n dimension, it is evident that the number of rows could be at most n. This is because each index can take only n values, and therefore a tensor which is anti-symmetric in more than n indices is identically zero. However, a diagram with more than n rows corresponds to a tensor with asymetry in more than n of its indices. Again the total number of squares in any diagram is equal to the rank of the tensor, i.e. the number of indices it has been constructed from.

Up till now, our discussion was confined to apply to any proper or improper subgroup of the group of linear transformations in n dimensions and we say that r-rank tensors of different symmetry types are associated with different Young's tableau consisting of r squares. These tensors reduce the full tensor space.

For the full linear group, this is the only operation for reduction of the tensor space, and so, to each Young's tableau there corresponds an irreducible tensor. We now show that these tensors are irreducible even when we restrict to the subgroups U(n) and SU(n) of the full linear group GL(n) in n dimensions. We first consider SL(n). Any matrix of GL(n) can be written as $\underline{a} = \alpha \underline{b}$ where det $\underline{b} = 1$ by setting $\alpha = (\det \underline{a})^{\frac{1}{2}}$. Thus, to each matrix $\underline{a}$ of GL(n), there corresponds a matrix $\underline{b}$ of SL(n). For reducibility under SL(n), some of the polynomials which are homogeneous in the elements of the matrix $\underline{b}$ must vanish. From $\underline{a} = \alpha \underline{b}$ these very polynomials will vanish for the elements of the matrix $\underline{a}$ which belongs to GL(n). Thus reducibility under SL(n) implies reducibility under GL(n) also. In other words, a representation irreducible under GL(n) remains so when we go over to the subgroup SL(n). To prove the result for U(n) and SU(n), we look at the infinitesimal generators. In equation (1.4) we expressed a generator of U(n) as a linear combination, with <u>real</u> coefficients, of n^2 hermitean matrices. Reducibility under U(n) will lead to some homogeneous polynomials in these coefficients vanish for arbitrary real values of the arguments. These very polynomials will therefore also vanish for complex values of these coefficients. Considering complex coefficients in (1.4) describes the full group GL(n). Thus reducibility under U(n) implies reducibility under GL(n). The argument for SU(n) is now essentially the same as given before for SL(n).

Thus we have seen that to each Young's tableau corresponds an irreducible tensor under the group U(n) or SU(n). However, for the group SU(n), all these tensors are not independent. This is clear from the fact that the matrix of transformation of an n-rank tensor which corresponds to the Young's tableau $[1^n]$



is just the determinant of the matrix $\underline{a}$. For SU(n), det $\underline{a} = 1$. Thus the representation matrices of two irreducible tensors which differ in their Young's tableau by the addition of some n-square columns only, are the same. These tensor spaces are therefore equivalent.

We can summarize above conclusions as follows:

The irreducible representations of the groups U(n) and SU(n) are given by n integers $f_1, f_2, \ldots, f_n$ where

$$f_1 + f_2 + \ldots + f_n = r \quad (r = 0, 1, 2 \ldots)$$

$$\text{and} \quad f_1 \geq f_2 \geq \ldots \geq f_n \geq 0 \qquad \ldots .(1.18)$$

The representations connected by

$$f_i = f'_i + e \qquad \text{....(1.19)}$$

where 'e' is some integer independent of the subindex 'i', have their representation matrices the same except for a factor $(\det \underline{a})^e$ in the case of U(n).

By means of (1.19), we can, for the irreducible representations of SU(n), just forget about the columns containing n squares. Thus we can replace the set of numbers

$$(f_1, \ldots, f_n)$$

characterizing any irreducible representation of SU(n) by

$$(f_1-f_n, f_2-f_n, \ldots, f_{n-1}-f_n) \qquad \text{....(1.20)}$$

The corresponding diagram will now consist of n-1 rows only.

iii) The Contragradient Representation.

As multiplications of matrices is preserved by complex conjugation, if some matrices form a representation of some group, so will the complex conjugates of these matrices. These two representations are called contragradient to each other. We took the vector with components u^i to form a basis for the representation matrices $\underline{a}$. Thus we had the equation

$$u'^i = a_{ij}\, u^j \qquad (1.7)$$

We define the space for operation of the contragradient representation to consist of vectors with components u_i. Thus

$$u'_i = a^*_{ij}\, u_j \qquad ....(1.21)$$

Then

$$u'_i\, u'^i = a^*_{ij}\, a_{ik}\, u_j\, u^k \qquad ....(1.22)$$

If the matrices a are a representation of the group U(n) (or SU(n)), they must be unitary. In other words,

$$a^*_{ij}\, a_{ik} = \delta_{jk}$$

with the help of which (1.22) becomes

$$u'_i\, u'^i = u_j\, u^j \qquad ....(1.23)$$

Thus we have obtained an invariant tensor in the space of the product of a contragradient and a cogradient vector. In general, if we have a certain number of upper and lower indices in a tensor of a certain rank, by "contracting" a number k of upper indces with the same number of lower indices, we shall obtain a tensor with its rank reduced by 2k. This result is well-known in relation to the various orthogonal groups, where there is no distinction between contravariant and covariant indices. The tensor obtained by contraction is called a trace.

Finally we consider the transformation law of a completely anti-symmetric $\overline{n-1}$ rank tensor, i.e. the one which corresponds to the Young's tableau

(n-1 squares)

It has only n independent components. These can be constructed as

$$u_i = [u^1 u^2 \ldots u^{i-1} u^{i+1} \ldots u^n] \qquad \ldots.(1.24)$$

where the square bracket signifies that we have to construct the completely anti-symmetric combination in the indices. Its transformation matrix is found to have as its elements just the cofactors of the matrix a which transforms each one of the u^i.

Thus

$$u'_i = (a^{-1})_{ji} \,(\det a)\, u_j \qquad \ldots.(1.25)$$

As $\underline{a} \in U(n)$ or $SU(n)$

$$a^{-1} = a^{\dagger} \qquad \ldots.(1.26)$$

$$\therefore \quad u'_i = a^*_{ij} \,(\det a)\, u_j \qquad \ldots.(1.21')$$

Thus relative to the group U(n), the completely asymmetric $\overline{n-1}$ rank tensor is equivalent to the contragradient vector except for the fact that the transforming matrices are multiplied by the determinants. For SU(n), it is clear from (1.21') that we have found another equivalence. In terms of the Young's tableau

$$(1^{n-1}) \equiv (1)^{*} \qquad \ldots.(1.27)$$

In fact, this result can be generalized by

$$(f_1, f_2, \ldots., f_{n-1}, f_n) \equiv (f_1-f_n, f_1-f_{n-1}, \ldots, f_1-f_2)^{*} \qquad \ldots.(1.28)$$

So far we have been considering every irreducible representation to be given by a set of +ve integers which are then directly associated with a Young's tableau. However, in (1.21') we have seen that the contragradient representation a^{*} is not completely equivalent to the one obtained by completely anti-symmetrizing the tensor space in $\overline{n-1}$ indices. To make them completely equivalent, we have just to apply the transformation

$$f_i \rightarrow f_i - 1$$

This transformation will remove (det a) from equation (1.21') with the consequence

$$(1,0,0,\ldots,0)^{*} \equiv a^{*} \equiv (1^{n-1})_{f_i \rightarrow f_{i-1}} = (0,0,\ldots,0,-1) \qquad \ldots.(1.27')$$

The general result (1.28) gets modified in the case of U(n) as

$$(f_1, f_2, \ldots, f_n)^{*} \equiv (f_1-f_n, f_1-f_{n-1}, \ldots, f_1-f_2, f_1-f_1)_{f_1-f_i \rightarrow -f_i}$$

$$= (-f_n, -f_{n-1}, \ldots, -f_1) \qquad \ldots.(1.28')$$

As the operation $f_i \to f_i + e$ leads to equivalent representations for SU(n), we may adopt the equation (1.28') as defining the irreducible representation contragradient to any representation in both these groups. Evidently, some of the integers may now be negative. However, since

$$f_1 \geq f_2 \geq f_3 \dots \geq f_n,$$

we necessarily have

$$-f_n \geq -f_{n-1} \geq \dots \geq -f_1$$

Now we shall have the same tableau associated with many irreducible representations of U(n) (these are equivalent from the point of view of the SU(n)) wherein we shall have the understanding of adjoining a certain number of (possibly -ve) columns, each containing n squares, to arrive at the correct representation matrices. Thus if we started from the representation

$$(f_1, \dots, f_n)$$

where $\qquad f_1 \geq f_2 \dots \geq f_n \geq 0,$

the corresponding contragradient representation

$$(-f_n, -f_{n-1}, \dots, -f_1)$$

with $\qquad 0 \geq -f_n \geq -f_{n-1} \dots \geq -f_1$

will have its associated Young's tableau as

$$(f_1 - f_n, f_1 - f_{n-1}, \dots, 0)$$

with the understanding that in the end we shall adjoin $-f_1$ columns, each containing n squares, to obtain the correct representation matrices.

iv) Specialization to the Groups U(3) and SU(3).

We state below the results arrived at in the last section as applied to the groups U(3) and SU(3):

The irreducible representations of these groups are characterized by sets of three integers (f_1, f_2, f_3) such that $f_1 \geq f_2 \geq f_3$. To each one of these corresponds a Young's diagram with f_1+e, f_2+e, f_3+e squares in the first, second and third rows respectively, such that $f_3+e \geq 0$ with the understanding that, to obtain the correct representation matrices, we shall further assume $-e$ columns of 3 squares each to be adjoined to the diagram. The representation contragradient to (f_1, f_2, f_3) is given by $(-f_3, -f_2, -f_1)$. The f's that are positive (negative) correspond to the contragradient (cogradient) indices in the corresponding tensor space.

v) Dimension of an irreducible Representation of U(3) or SU(3).

To obtain the dimension of the irreducible representation (f_1, f_2, f_3) we first subtract f_2 from each one of the three f_i to obtain another representation $(f_1-f_2, 0, -f_2+f_3)$. These two have the same dimension. Now $f_1-f_2 = \mu \geq 0$, while $-f_2+f_3 = -\nu \geq 0$. The corresponding tensor, therefore, has μ upper and ν lower indices, and is completely symmetric in all the upper as well as the lower indices. As we are working in dimension 3, each of the indices can take only 3 different values. Thus the number of independent components in such a tensor will be

$$\frac{(\mu+2)(\mu+1)(\nu+2)(\nu+1)}{2 \,.\, 2} \qquad \ldots.(1.29)$$

We note now that symmetrizing the upper and lower indices does not necessarily result in an irreducible tensor with respect to U(n) or SU(n). We ought now to remove the traces. This can be done by considering

$$\frac{(\mu+1)\mu(\nu+1)\nu}{4} \qquad \ldots.(1.30)$$

of the components in (1.29) to be identically zero.

Thus

$$d(f_1,f_2,f_3) = d(f_1-f_2,0,-f_2+f_3) = d(\mu,0,-\nu)$$

$$= \frac{(\mu+1)(\nu+1)}{4}[(\mu+2)(\nu+2) - \mu\nu]$$

$$= \frac{(\mu+1)(\nu+1)(\mu+\nu+2)}{2} \qquad \ldots.(1.31)$$

Clearly this method is not applicable to U(n) or SU(n) with n more than 3, as in such cases we cannot, in general, replace the representation $(f_1,f_2,\ldots,f_n)$ by $(\mu,0,0,\ldots,-\nu)$. However, wherever such a replacement is possible, the above procedure will be applicable.

vi) The Branching Law.

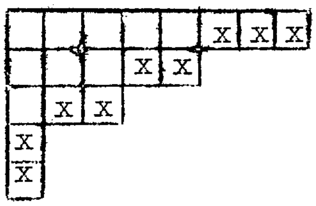
In this section we give a general method of decomposing an irreducible representation of SU(n) (or U(n)) relative to one of its subgroup SU(n-1) (or U(n-1)). This is clear that an irreducible representation of SU(n) will, in general, be reducible under any one of its subgroups. Also we note that SU(n) has many subgroups SU(n-1) and we choose one of these such that its matrices only operate on the first n-1 indices. For this purpose, we first see how we can write the linearly independent components for an irreducible tensor which corresponds to the tableau $(f_1,f_2,..,f_n)$. The technique is evident. We try to fill the squares of the tableau by the integers 1,2,....n to form "standard" patterns such that

(i) no integer is repeated in the same column,

(ii) the integers in the same row or column are in a non-decreasing order.

The number of such standard patterns is clearly the dimension of the irreducible representation, as one and only one independent component of the tensor corresponds to each one of these patterns.

On account of the above restrictions, it is evident that the integer 'n' can only appear in the 'overhung' squares crossed in the following diagram.



The set of independent components of the tensor which correspond to the standard patterns with some of these overhung squares containing the integer n form a basis for an irreducible representation of SU(n-1). For the corresponding tableau, we may, in addition, remove these squares altogether. Finally, by using the equivalence of the representations corresponding to the tableau differing by the addition of some complete columns, we arrive at the following branching law:

Each set of integers $(f_1', f_2', \ldots, f_{n-1}')$ such that

$$f_1 \geq f_1' \geq f_2 \geq f_2' \ldots \geq f_{n-1}' \geq f_n \qquad \ldots.(1.32)$$

corresponds to an irreducible representation of SU(n-1) contained in the irreducible representation $(f_1, \ldots, f_n)$ of SU(n). These sets complete the decomposition.

We also conclude that

$$d(f_1, f_2, \ldots, f_n) = \Sigma\, d(f_1', f_2', \ldots, f_{n-1}') \qquad \ldots.(1.33)$$

where the summation is over all the sets $(f_1', f_2', \ldots, f_{n-1}')$ in equation (1.32).

Now clearly the diagram (f,0) has dimension f+1 when considered as describing an irreducible representation of SU(2). It follows from equivalence that

$$d(f_1,f_2) = d(f_1-f_2,0) = f_1-f_2+1 \qquad \ldots.(1.34)$$

From (1.33) now

$$d(f_1,f_2,f_3) = \sum_{f_1'=f_2}^{f_1} \sum_{f_2'=f_3}^{f_2} (f_1' - f_2' + 1)$$

$$= \tfrac{1}{2}(f_1-f_2+1)(f_2-f_3+1)(f_1-f_3+2) \qquad \ldots.(1.31')$$

This remains the same when the transformation

$$f_i \to f_i + e$$

is applied to it.

Equation (1.31) is reproduced on writing

$$f_1 - f_2 = \mu$$
$$f_2 - f_3 = \nu \qquad \ldots.(1.35)$$

vii) <u>Eigenstates and eigenvalues for any irreducible representation</u>.

In this section we elaborate the connection of the groups U(3) and SU(3) to strong interaction physics. We use the familiar technique of considering the infinitesimal generators of these groups to be operators with some physical significance. Firstly we note the fact that the group U(3) (SU(3)) is of rank 3 (2). Thus we can construct only three (two) independent and mutually commuting linear operators from the infinitesimal generators. These operators can, therefore, be simultaneously diagonalized in any representation. Again we know two operators, namely the strangeness and a component of the isotopic spin operator which commute with each other. Thus we can choose two of the diagonal elements to be S and I_3. The choice of I_3 is purely conventional and just fixes the direction of quantization of the isotopic spin. The other components of isotopic spin do in fact commute with S but not with I_3. Thus they cannot be simultaneously diagonalized with I_3.

Now a basic distinction between theories based on U(3) and SU(3) will arise out of the existence of another operator in U(3) which commutes with both S and I_3. This is the identity operator and must therefore be associated with a quantum number which is to be shared equally by all the members of a supermultiplet. We could take it to be the baryon number as any super-multiplet resulting from strong interactions cannot consist of members with different baryon numbers. Thus the baryon number arises naturally in U(3) theories while it is to be imposed from the outside on SU(3) theories.

There is a difficulty which arises in the classification of eigenstates in a representation with <u>only</u> S and I_3 as diagonal. In some representations, more than one state will have the same I_3 and S eigenvalues. This incomplete classification is the result of the fact that the group SU(3) is of rank 2 and has 8 generators. In fact we need $\frac{1}{2}(8 - 3 \times 2)$, i.e. one more operator which commutes with S and I_3 but does not commute with <u>all</u> the generators of the group. For otherwise, by Schur's Lemma, it will be a multiple of the identity in every representation and will not help us to distinguish some of the eigenvectors within a representation (such operators are called Casimir operators and their eigenvalues can be used to characterize various irreducible representations). As the group has rank 2, we cannot have any other <u>linear</u> operator commuting with both S and I_3. Thus we must consider a <u>non-linear</u> operator. Fortunately we have $|\underline{I}|^2$ which commutes with I_3 and S and we can see from the commutation relations (next chapter) that it odes not commute with <u>all</u> the infinitesimal generators. Thus a complete classification of eigenvectors can be obtained by specifying for each eigenvector the corresponding I, I_3, Y eigenvalues. As said earlier, each representation will now be diagonal in these three operators.

The above-mentioned problem is even worse in G_2 and C_2 which have 14 and 10 generators respectively and therefore we shall require 4 and 2 such non-linear operators for a complete classification of the eigenstates in the bases for the irreducible representations of these groups. In passing we also state that in SU(2), there is no such problem, as it is of rank '1' and has three generators. This is why eigenvalues of only one operator I_3 are sufficient to distinguish various eigenvectors.

Now we determine the various (I,S) multiplets in the basis of an irreducible representation (f_1,f_2,f_3) of U(3) or SU(3). We have already seen that an irreducible representation of U(3) or SU(3) can be decomposed relative to one of the SU(2) subgroups these contain. This SU(2) subgroup we take as the isotopic spin group. By the technique we gave before, the subspaces have dimensions given by $f_1' - f_2' + 1$ which corresponds to the I-spin value of $\frac{f_1'-f_2'}{2}$. To determine the corresponding eigenvalue of the S operator, we have to define a basis vector for the 3-dimensional representation (1,0,0). By the branching law, it contains a doublet and a singlet and our decomposition was such that 1 and 2 components form the doublet and the third component is a singlet.

Thus we take

$$I_3 = \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad \ldots\ldots(1.36a)$$

$$I_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \ldots (1.36b)$$

and

$$I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \ldots (1.36c)$$

The strangeness operator in this representation must be diagonal and not just a multiple of the identity matrix. Further, its value for the indices 1 and 2 must be equal. It is therefore of the form

$$\begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix}$$

which is $aI + (b-a)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

As the fixation of the scale as well as the starting point is arbitrary, we can take it to be

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \ldots (1.36d)$$

Again the particles p, n and Λ satisfy all the above conditions. i.e. they have the correct values of the isotopic spin, its third component, and the strangeness. We have therefore obtained the Sakata model wherein $\begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$ form a basis for the 3-dimensional representation of the group U(3).

The matrix in (1.36d) is not traceless and is therefore not an SU(3) generator. The corresponding traceless operator is

$$\frac{1}{3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix} \qquad \ldots.(1.36e)$$

This is $\frac{1}{3}N + S$ for the basis $\begin{pmatrix}p\\ n\\ \Lambda\end{pmatrix}$ where

$$N = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \qquad \ldots.(1.36f)$$

Now the Young's tableau corresponding to the irreducible representation (f_1,f_2,f_3) has $f_1+f_2+f_3$ squares. In the branching into representations of SU(2), corresponding to (f'_1,f'_2), we have $(f_1+f_2+f_3) - (f'_1+f'_2)$ overhung squares, each containing the index 3. All the $(f_1+f_2+f_3)$ squares contribute 1 to the operator N (equation (1.36f)) and each of the $(f_1+f_2+f_3) - (f'_1+f'_2)$ squares containing index 3 contributes −1 to the operator S (equation (1.36d)). Thus the eigenvalues n and s of the operators N and S are given by

$$n = f_1 + f_2 + f_3 \qquad \ldots.(1.36f')$$

$$s = f'_1 + f'_2 - (f_1+f_2+f_3) \qquad \ldots.(1.36d')$$

For completeness, we also quote here

$$i = \frac{f'_1 - f'_2}{2} \qquad \ldots.(1.36g')$$

The operator $\frac{1}{3}N + S$ in equation (1.36e) has the eigenvalues

$$f_1' + f_2' - \frac{2}{3}(f_1+f_2+f_3) \qquad \text{....(1.36e')}$$

This combination of N and S is evidently invariant under the transformations

$$f_i \to f_i + e$$

$$f_i' \to f_i' + e$$

though N and S separately are not. Thus from the point of view of the SU(3) as a symmetry group, the operator (1.36e') should replace the S in (1.36d'). In addition, as the identity matrix is not a generator of SU(3), this group does not directly include the nucleon number operator.

We have seen before that the irreducible representations of the groups U(3) and SU(3) are the same. Thus if we agree to consider SU(3) supermultiplets to contain members with different baryon numbers, the consequences of schemes based on these two groups will be identical. This point was first emphasized by Okubo.[19]

We may however forget about the baryon number and consider the operator (1.36e) which arises quite naturally in SU(3). It is clear from (1.36e') that its eigenvalues will not all be <u>integers</u> in any irreducible representation. If, however, this is to be considered as strangeness or hypercharge operator, its eigenvalues must necessarily be integral.[24] This will restrict us to only those representations for which $f_1 + f_2 + f_3 \equiv 0 \pmod 3$. Indeed using the equivalence of the representations connected by $f_i' = f_i + e$, we may replace this condition by $f_1 + f_2 + f_3 = 0$. In such a case, we are not considering the full group, but rather its quotient group $SU(3)/C(3)$.

The Gell-Mann-Neeman model, where this condition is implicit, is based on this quotient group. In particular, the three dimensional representation does not exist in this model ($f_1 + f_2 + f_3 = 1 + 0 + 0 \not\equiv 0 \pmod 3$). The least-dimensional non-trivial representation is (1,0,-1) of dimension 8, which we shall see can fit the baryons as well as the pseudoscalar and vector mesons.

CHAPTER 2.

Reduction of the Direct Product of Irreducible Representations.

(i) The Clebsch-Gordan Series.

We have already considered the decomposition of an irreducible representation of the groups U(3) and SU(3) relative to the isotopic spin subgroup and we were able to give the (i,s) value of the various isotopic sub-multiples. Next we consider the direct product of two irreducible representations. Evidently this product is a representation, though, in general, reducible. The problem of reduction of a direct product of two irreducible representations has already been solved for all semi-simple groups and the graphical procedure is beautifully explained in the review article by Behrends, Dreitlein, Fronsdal and Lee.[25] For U(3) and SU(3), however, the Young's tableau technique is much simpler. We have seen that, for these groups, each Young's tableau corresponds to an irreducible representation. Thus each tableau in the product of the tableau for two irreducible representations will correspond to an irreducible representation contained in the direct product. For the multiplication of two tableau,[13] we first of all forget about the complete columns containing three squares each in one of the diagrams and fill in the squares in the remaining skeleton α's and β's in the first and second rows respectively. Next we adjoin these squares containing α's and β's, in this order, to the other diagram in a manner such that

(i) the final diagram has ≤ 3 rows,

(ii) when we finish adjoining squares containing α's (β's), it is a Young's tableau,

(iii) the adjoined α's and β's when read from the right, exhausting the first row first, and then the second, etc., form a lattice order, i.e. at each stage in this order, the number of α's is not less than the number of β's.

Corresponding to each of the manners that satisfies the above three conditions, there is a Young's tableau and consequently corresponding irreducible representation in the product. Complete columns can just be added at the end and due consideration also given to the -ve integers appearing in the characterization of some of the irreducible representations.

It must of course be noted that we are actually trying to consider the product of two simple characters and to express this product as a linear sum of simple characters. The problem can therefore also be solved by means of characters. The diagrammatic procedure is rather easy to work with.

Thus we may write

$$D(f_1',f_2',f_3') \otimes D(f_1'',f_2'',f_3'') = \sum_{(f)} \nu_{f_1,f_2,f_3} D(f_1,f_2,f_3) \qquad \ldots.(2.1)$$

where the direct sum on the right in equation (2.1) consists of representations which occur in the reduction of the product on the left and ν_{f_1,f_2,f_3} is the multiplicity of occurrence of such a representation.

The series on the right in the above equation is called the Clebsch-Gordan series.

(ii) The Clebsch-Gordan coefficients.

The basis vectors for any representation (f_1,f_2,f_3) with I, I_3, S diagonal are of the form

$$|(f_1,f_2,f_3) ; i, i_3, s\rangle$$

which we shall abbreviate as

$$|f ; i, i_3, s\rangle \qquad \ldots.(2.2)$$

In the space of the product of the two representations (f_1',f_2',f_3') and (f_1'',f_2'',f_3'') we may choose basis vectors as the direct products

$$|f'; i', i_3', s'\rangle \, | f''; i'', i_3'' s''\rangle \qquad \ldots.(2.3)$$

To obtain the basis wherein the matrices of the direct product representation reduce according to the decomposition (2.1), we shall have to operate upon the bases in (2.3) by a unitary transformation. This expresses the basis vectors

$$|f_\alpha;\ i,i_3,s\rangle$$

as linear combinations of the vectors in equation (2.3) in the form

$$|f_\alpha;\ i,i_3,s\rangle = \Sigma C(f_\alpha;\ i,i_3,s|f';\ i',i_3',s';;\ f'',i'',i_3'',s'')$$
$$|f';\ i',i_3',s'\rangle\,|\ f'';\ i'';i_3'',s''\rangle \quad ....(2.4)$$

where the summation on the right is over all the dummy variables $i',i_3',s';\ i'',i_3'',s''$ and the index α is for the enumeration of equivalent representations. For a particular f this takes the values 1, 2, ..., ν_f where ν_f is the multiplicity of the representation (f_1,f_2,f_3) appearing in equation (2.1).

From equation (2.4) we however see that

$$s = s' + s'' \quad ....(2.5a)$$

$$i_3 = i_3' + i_3'' \quad ....(2.5b)$$

and $$i = \Delta(i',i'') \quad ....(2.5c)$$

where $\Delta(i',i'')$ stands for any one of $i' + i''$, $i' + i'' - 1$, ..., $|i' - i''|$.

Thus we may write

$$C(f_\alpha;\ i,i_3,s\ |f';\ i',i_3',s';;\ f'';\ i'',i_3'',s'')$$

$$= \delta_{i_3 i_3'+i_3''}\ \delta_{s\,s'+s''}\ \delta_{i\Delta(i',i'')}\ X$$

$$C'(f_\alpha;\ i,i_3,s\ |f';\ i',i_3',s';;\ f'';\ i'',i_3'',s'')$$

The coefficients C in this equation are the Clebsch-Gordan coefficients for the groups U(3) or SU(3).

We can obtain the vectors $|f_\alpha; i, i_3, s>$ in two steps. First of all we select any pair of multiplets $(f'; i', s')$, $(f''; i'', s'')$ such that

$$s = s' + s'' \qquad \ldots.(2.5a)$$

$$i = \Delta(i', i'') \qquad \ldots.(2.5c)$$

and construct from the $(2i' + 1)(2i'' + 1)$ basis vectors in their products a vector which corresponds to the correct i and i_3 we require. The coefficients in this linear combination are the ordinary SU(2) Clebsch-Gordan coefficients and are independent of the irreducible representations of SU(3) to which (i', s') and (i'', s'') belong and also of the s', s'' values. To signify this independence, we do not write f', f'', s', s'' in the coefficients in the following equation:

$$|f'; i', s';; f'', i'', s''; i, i_3, s>$$

$$= \sum_{I_3 = I_3' + I_3''} C(i, i_3 \mid i', i_3'; i'', i_3'')$$

$$\times \; |f'; i', i_3', s'> | f''; i'', i_3'', s''> \qquad \ldots.(2.7)$$

We can construct as many vectors of the type appearing on the left of this equation as there are ways of choosing pairs of multiplets $(f'; i', s'')$, $(f''; i'', s'')$ which satisfy equations (2.5a) and (2.5c), the number being again equal to the number of representations (taking into consideration multiplicities also) on the right in equation (2.1) which contain this (i,s) multiplet.

Now to obtain $|f_\alpha;i,i_3,s>$, we have to express it as a linear combination of the vectors on the left in equation (2.7). Thus we may write

$$|f_\alpha;i,i_3,s> = \Sigma\, C(f_\alpha;i,i_3,s\,|\,f';i',i_3';;f'',i'',s'') \times |f';i',s';;f'';i'',s'';i,i_3,s> \qquad \ldots.(2.8)$$

The coefficients that appear in this equation are independent of i_3 as can be seen by operating with $I_\pm$ operators, and are therefore known as the "isoscalar factors".[26] In future we rewrite them without the i_3 and the C in the front. Combining this equation with the previous one, we arrive at

$$|f_\alpha;i,i_3,s> = \Sigma\,(f_\alpha;i,s\,|\,f';i',s';;f'';i'',s'') \times C(i,i_3\,|\,i',i_3';i'',i_3'')|f';i',i_3',s'>|f'';i'',i_3'',s''> \qquad \ldots.(2.9)$$

where the summation is over all the primed and double primed indices restricted by the equations (2.5). If we take into consideration the property (2.6), then we may extend the summation to all the values of these variables. We also define

$$(f_\alpha;i,s \mid f';i',s';;f'';i'',s'') = 0 \qquad \ldots.(2.10)$$

unless f_α is included in the decomposition of the direct product $f' \otimes f''$ when its value is determined from the above equation.

Remarks.

(i) In the reduction of the direct product, we have considered the eigenvectors to have an additional index which is the result of the fact that some representations occur more than once in the reduction. From the point of view of the group, these representations are equivalent and there is nothing within the group with the help of which we can distinguish these. For the cases that we shall consider, we will see that the R-operation of Gell-Mann, which does not belong to the group, is sufficient to distinguish the multiply-occurring representations (the multiplicity in the cases we consider is at most 2).

(ii) The matrix of transformation from the basis in the product space which diagonalizes the eigenvalues of the individual state vectors to the one that diagonalizes the total I, I_3 and S becomes unitary on normalization of the basis vectors. We may, however, make it real orthogonal by a special choice of phases, which in turn makes all the isoscalar coefficients real if we take all the SU(2) coefficients real.

(iii) What we have essentially done in equation (2.9) is to factorize the Clebsch-Gordan coefficients as a product of two - one of which is an ordinary SU(2) coefficient with the help of which we combine two isotopic spins, and the other which we called the isoscalar factors. This factorization is very useful for the purpose of tabulation, for it allows us to give a very small number of such coefficients. The SU(2) coefficients are already tabulated. The isoscalar coefficients multiply these to give the Clebsch-Gordan coefficients for the groups U(3) and SU(3).

(iii) Ladder Operators.

In section 2(vii) we obtained the eigenvalues of the operators N (= 0 in the Gell-Mann-Neeman model) and S for the basis vectors belonging to any irreducible representation of these groups. These results are sufficient to give all their matrix elements as the operators are diagonal in any representation. The matrix elements for the operators $I_{\pm}, I_3$ of the subgroup SU(2) can be read off from the angular momentum theory as for example we know[27]

$$I_{\pm} |i,i_3\rangle = \sqrt{(i\mp i_3)(i\pm i_3+1)} \; |i,i_3\pm 1\rangle$$
$$I_3 |i,i_3\rangle = I_3 |i,i_3\rangle \qquad \ldots.(2.11)$$

In our context we may rewrite these as

$$I_{\pm}|f;i,i_3,s\rangle = \sqrt{(i\mp i_3)(i\pm i_3+1)}\ |f;i,i_3\pm 1,s\rangle$$
$$I_3|f;i,i_3,s\rangle = I_3|f;i,i_3,s\rangle \qquad \ldots(2.11)$$

Thus if we know the highest member of any sub-multiplet, by repeated application of the operator I_-, we shall be able to complete the submultiplet. To complete the multiplet, we must determine some other operator (operators) which takes (take) one from one submultiplet to another.

For this purpose, we first of all write the commutation relations. These can be obtained in particular from the three-dimensional representation. In equation (1.6) we had the commutation relation

$$[\,A^i{}_j,\ A^l{}_m] = \delta_{lj}A^i{}_m - \delta_{im}A^l{}_j \qquad \ldots.(1.6)$$

for the nine matrices $A^i{}_j$ such that

$$(A^i{}_j)_{lm} = \delta_{il}\ \delta_{jm} \qquad \ldots.(1.5)$$

i.e. $A^i{}_j$ has a unity in the position of the ith row and jth column. These nine matrices form the generators of the group U(3). To obtain the generators for the group SU(3), we have just to make the three diagonal ones traceless which will leave only two of these as linearly independent. These can be taken to be

$$I_3 = \tfrac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} = \tfrac{1}{2}(A^1{}_1 - A^2{}_2) \qquad \ldots.(2.12)$$

$$S' = \frac{1}{3}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} = \tfrac{1}{3}N + S \qquad \ldots.(2.13)$$

where

$$N = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad \ldots.(2.14)$$

The off-diagonal generators are

$$A^1{}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = I_+ \qquad \ldots.(2.15a)$$

$$A^2{}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = I_- \qquad \ldots.(2.15b)$$

and $A^2{}_3$, $A^3{}_2$, $A^1{}_3$, $A^3{}_1$. Operating then on the basis $\begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$ for the 3-dimensional representation, we can see that

$$A^2{}_3 \begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \Lambda \\ 0 \end{pmatrix} \quad \text{etc.}$$

$A^2{}_3$, thus, increases (decreases) the strangeness (I_3) by 1 ($\tfrac{1}{2}$). The result is true whether we take S or S' as the strangeness and holds for <u>all</u> representations as is clear from the operation by both sides of the commutation relations

$$[A^2{}_3, S] = [A^2{}_3, -A^3{}_3] = -A^2{}_3$$

$$[A^2{}_3, I_3] = \frac{1}{2}[A^2{}_3, A^1{}_1, -A^2{}_2] = \frac{1}{2}A^2{}_3$$

on an eigenstate

$$|f; i, i_3, s>$$

of I, I_3, S with the corresponding eigenvalues i, i_3, s which leads to

$$S\, A^2{}_3 |f; i, i_3, s> = (s+1) A^2{}_3 |f; i, i_3, s> \qquad \text{....(2.16a)}$$

$$I_3 A^2{}_3 |f; i, i_3, s> = (i_3 - \tfrac{1}{2}) A^2{}_3 |f; i, i_3, s> \qquad \text{....(2.16b)}$$

By the same method, we can see that all the four operators $A^1{}_3$, $A^2{}_3$, $A^3{}_1$, $A^3{}_2$ change s (i_3) by 1 ($\frac{1}{2}$). The details will be presented in table I.

Again

$$\underline{I}^2 = \tfrac{1}{2}(I_+ I_- + I_- I_+) + I^2{}_3$$

$$= \tfrac{1}{2}(A^1{}_2 A^2{}_1 + A^2{}_1 A^1{}_2) + \tfrac{1}{4}(A^1{}_1 - A^2{}_2)^2 \qquad \text{....(2.17)}$$

Thus from (1.6)

$$[A^1{}_3, \underline{I}^2] = -A^1{}_3(I_3 + \tfrac{3}{4}) - A^2{}_3 A^1{}_2 \qquad \text{....(2.18a)}$$

$$[A^3{}_1, \underline{I}^2] = A^3{}_1(I_3 - \tfrac{3}{4}) + A^3{}_2 A^2{}_1 \qquad \text{....(2.18b)}$$

$$[A^2{}_3, \underline{I}^2] = A^2{}_3(I_3 - \tfrac{3}{4}) - A^1{}_3 A^2{}_1 \qquad \text{....(2.18c)}$$

$$[A^3{}_2, \underline{I}^2] = -A^3{}_2(I_3 + \tfrac{3}{4}) + A^3{}_1 A^1{}_2 \qquad \text{....(2.18d}$$

Operating both sides of the equation (2.18c) on the eigenstate

$$| f;i,i_3 = -i,s >$$

and noting that $A^2{}_1$ is I_- and

$$I_- |f;i,i_3 = -i,s> = 0$$

we obtain

$$\underline{I}^2 A^2{}_3 |f;i_3 = -i,s > = (i(i+1)+i+\tfrac{3}{4}) A^2{}_3 |f;i_3 = -i,s >$$

$$= (i+\tfrac{1}{2})(i+\tfrac{3}{2}) A^2{}_3 | f;i,i_3 = -i,s > \quad \ldots.(2.19)$$

Combining this with (3.16b), we see that the operator $A^2{}_3$ increases i by ½ when applied to the lowest member of an isotopic multiplet ($i_3 = -i$) in any irreducible representation. The operator $A^3{}_1$ has exactly the same effect. However $A^1{}_3$, $A^3{}_2$ increase i by ½ when applied to the highest member of a submultiplet ($i_3 = i$).

Finally

$$[A^1{}_2, S] = [A^1{}_2, \underline{I}^2] = [A^2{}_1, S] = [A^2{}_1, \underline{I}^2] = 0 \quad \ldots(2.20)$$

We summarize the above results in the following table:

Table I.

Effect of the operators on I, I_3, S eigenvectors.

Operator	Δi_3	Δs	New i
$N = \Sigma A^i{}_i$	0	0	i
$I_3 = A^1{}_1 - A^2{}_2$	0	0	i
$S = - A^3{}_3$	0	0	i
$S' = \frac{1}{3}N + S$	0	0	i
$I_+ = A^1{}_2$	1	0	i
$I_- = A^2{}_1$	−1	0	i
$A^1{}_3$	$\frac{1}{2}$	1	$i + \frac{1}{2}$ at $i = i_3$
$A^3{}_1$	$-\frac{1}{2}$	−1	$i + \frac{1}{2}$ at $i = - i_3$
$A^2{}_3$	$-\frac{1}{2}$	1	$i + \frac{1}{2}$ at $i = - i_3$
$A^3{}_2$	$\frac{1}{2}$	−1	$i + \frac{1}{2}$ at $i = i_3$

We now describe the method of the construction of the highest member of the supermultiplet (f_1, f_2, f_3), i.e. the one that has the highest i_3 amongst those which have the highest s value. In $\begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$, the highest is p, then comes n and then Λ. Thus we must obtain this highest member by putting in as many of the p's as is possible, then as many n's as are consistent with the symmetry of the representation, and take the remaining ones as Λ's. Evidently we can take at most f_1 p's, and, having made this choice, at most f_2 n's and the remaining f_3 Λ's. These, when arranged in accordance with the symmetry of the corresponding Young's tableau, give the highest eigenvector. This state then has

$$(i_3)_o = \tfrac{1}{2}(f_1 - f_2) \qquad \text{....(2.21a)}$$

and $$(s)_o = - f_3 \qquad \text{....(2.21b)}$$

We see from equation (1.28d) and

$$f_1 \geq f_1'' \geq f_2 \geq f_2' \geq f_3 \qquad \text{....(2.22)}$$

that $(s)_o = - f_3$ is indeed the highest strangeness in the supermultiplet, the corresponding i being $\frac{f_1 - f_2}{2}$ which shows that the $(i_3)_o$ given above is definitely the highest I_3 eigenvalue of the submultiplet which has the highest s. From equations (2.21 a,b) it is also evident that the <u>highest</u> eigenvector can be uniquely fixed for each representation by knowing only the eigenvalues of the <u>diagonal</u> generators.

We now want to use the above table to obtain all the eigenvectors. To do it, we shall first of all use $A^2{}_1 = I_-$ repeatedly to complete this very isotopic multiplet. We must now know an operator which will enable us to jump to other submultiplets. From table I, it is clear that we must use $A^3{}_2$ which increases the isotopic spin by $\frac{1}{2}$ and decreases the strangeness by 1 when applied to the highest member of any submultiplet. As i_3 also increases by $\frac{1}{2}$, we are actually lead to the highest member of another sub-multiplet. The repeated application of $A^2{}_1$ will complete this sub-multiplet.

We must now have an operator which can decrease the strangeness as well as the isotopic spin. $A^3{}_1$ does have the properties of decreasing i_3 and s but does not in fact reduce i in general. Rather it mixes the I values $i \pm \frac{1}{2}$.

We try to look for such an operator as a linear combination of $A^3{}_1 I_3$ and $A^3{}_2 I_-$ both of which have the same Δi_3 and Δs. From the commutation relations

$$[A^3{}_1 I_3, \underline{I}^2] = A^3{}_1 I_3 (I_3 - \tfrac{3}{4}) + A^3{}_2 I_- I_3$$

$$[A^3{}_2 I_-, \underline{I}^2] = - A^3{}_2 (I_3 + \tfrac{3}{4}) I_- + A^3{}_1 I_+ I_-$$

$$= - A^3{}_2 I_- (I_3 - \tfrac{1}{4}) + A^3{}_1 I_- I_+ + 2A^3{}_1 I_3$$

follows

$$[\alpha A^3{}_1 I_3 + A^3{}_2 I_-, \underline{I}^2] = A^3{}_1 I_3 (\alpha I_3 - \tfrac{3}{4}\alpha + 2) + A^3{}_2 I_- (\alpha I_3 - I_3 + \tfrac{1}{4}) + A^3{}_1 I_- I_+$$

We search for the solution of

$$\frac{\alpha I_3 - \frac{3}{4}\alpha + 2}{\alpha I_3 - I_3 + \frac{1}{4}} = \frac{\alpha}{1}$$

which is <u>independent</u> of I_3. This requires that we must have <u>simultaneously</u>

$$\tfrac{1}{4}\alpha = -\tfrac{3}{4}\alpha + 2 \quad \text{and} \quad \alpha(\alpha - 1) = \alpha$$

Thus $\alpha = 2$.

With this value of α, we obtain

$$[2A^3{}_1 I_3 + A^3{}_2 I_-, \underline{I}^2] = (2A^3{}_1 I_3 +, A^3{}_2 I_-)(I_3 + \tfrac{1}{4}) + A^3{}_1 I_- I_+$$

Operating both sides on an eigenstate of i, i_3, s with $i = i_3$, we arrive at

$$\underline{I}^2(2A^3{}_1 I_3 + A^3{}_2 I_-)|i, i_3 = i, s>$$

$$= (i + \tfrac{1}{2})(i - \tfrac{1}{2})(2A^3{}_1 I_3 + A^3{}_2 I_-)|i, i_3 = i, s>$$

Thus the operator[7,15]

$$\bar{A}^3{}_1 = 2A^3{}_1 I_3 + A^3{}_2 I_-$$

operating on the highest state of an isotopic submultiplet gives a pure I value equal to $i - \frac{1}{2}$, the corresponding S being $s - 1$. This operator is therefore the one we have been looking for which reduces the strangeness and simultaneously the isotopic spin.

It may be checked that the operators $A^3{}_2$ and $\bar{A}^3{}_1$ commute. They may therefore be applied in any order.

The above procedure is illustrated diagrammatically in the following (i,s) plot for an irreducible representation:

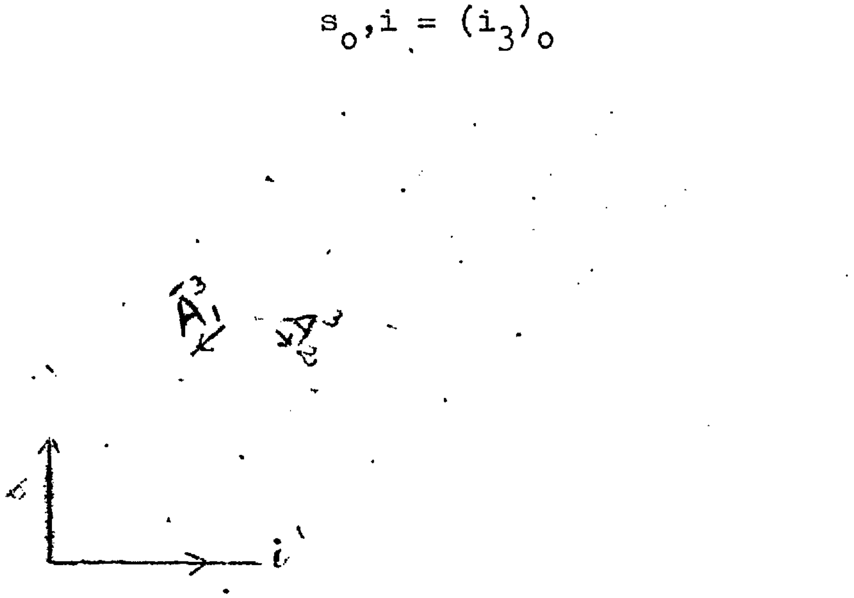


Fig. I: (i,s) plot for an irreducible representation.

Having obtained all the eigenvectors in any representation, we can determine all the matrix elements of the generators for the representation. We have just to operate upon the various eigenvectors by the generators and express the result in terms of the other eigenvectors. We shall, of course, have to use the 3-dimensional representation of these generators. What we essentially need are relations of the type

$$A^2{}_3\,\Lambda = n, \quad A^2{}_3\bar{n} = -\,\bar{\Lambda}$$

$$A^2{}_3 p(n,\bar{p},\bar{\Lambda}) = 0$$

In fact we need the matrix elements of only one of the four strangeness changing operators. The others can be obtained either by complex conjugation or by using SU(2) coefficients. These matrix elements have also been obtained as expressions valid for all the representations by using analysis which is similar to the one used in deriving the relations (3.11) in angular momentum theory.[28] The following tables for the basis vectors and the matrix elements of $A^3{}_1$ and $A^3{}_2$ for the representations (1,0,-1), (2,-1,-1), (2,0,-2) have been obtained by using our straightforward technique.

Table II.

Basis Vectors for the Representations (1,0,-1), (2,-1,-1), (2,0,-2).

Reps.	Dims.	$\vert i,i_3,s\rangle$	Basis Vectors	
(1,0,-1)	8	$\vert\frac{1}{2},\frac{1}{2},1\rangle$	$p\bar{\Lambda}$	K^+
		$\vert\frac{1}{2},-\frac{1}{2},1\rangle$	$n\bar{\Lambda}$	K^o
		$\vert 1,1,0\rangle$	$-p\bar{n}$	$-\pi^+$
		$\vert 1,0,0\rangle$	$\frac{p\bar{p}-n\bar{n}}{\sqrt{2}}$	π^o
		$\vert 1,-1,0\rangle$	$n\bar{p}$	π^-
		$\vert 0,0,0\rangle$	$-\frac{p\bar{p}-n\bar{n}+2\Lambda\bar{\Lambda}}{\sqrt{6}}$	$-\pi^{o\prime}$
		$\vert\frac{1}{2},\frac{1}{2},-1\rangle$	$-\Lambda\bar{n}$	$-\bar{K}^o$
		$\vert\frac{1}{2},-\frac{1}{2},-1\rangle$	$-\Lambda\bar{p}$	K^-
The last column is for future reference.				
(2,-1,-1)	10	$\vert\frac{3}{2},\frac{3}{2},1\rangle$	$\frac{1}{\sqrt{2}}\ pp[\bar{n}\bar{\Lambda}]$	
		$\vert\frac{3}{2},\frac{1}{2},1\rangle$	$\frac{1}{\sqrt{6}}((np)[\bar{n}\bar{\Lambda}] - pp[\bar{p}\bar{\Lambda}])$	
		$\vert\frac{3}{2},-\frac{1}{2},1\rangle$	$\frac{1}{\sqrt{6}}(nn[\bar{n}\bar{\Lambda}] - (np)[\bar{p}\bar{\Lambda}])$	
		$\vert\frac{3}{2},-\frac{3}{2},1\rangle$	$-\frac{1}{\sqrt{2}}\ nn[\bar{p}\bar{\Lambda}]$	

		$\|1,1,0\rangle$	$\frac{1}{\sqrt{6}}(p\Lambda)[\overline{n}\overline{\Lambda}] - pp[\overline{n}\overline{p}]$
		$\|1,0,0\rangle$	$\frac{1}{\sqrt{12}}((n\Lambda)[\overline{n}\overline{\Lambda}] - (p\Lambda)[\overline{p}\Lambda] - (pn)[\overline{n}\overline{p}])$
		$\|1,-1,0\rangle$	$\frac{1}{\sqrt{6}}(-(n\Lambda)[\overline{p}\overline{\Lambda}] - nn[\overline{n}\overline{p}])$
		$\|1,\frac{1}{2},-1\rangle$	$\frac{1}{\sqrt{6}}(\Lambda\Lambda[\overline{n}\overline{\Lambda}] - (p\Lambda)[\overline{n}\overline{p}])$
		$\|1,-\frac{1}{2},-1\rangle$	$\frac{1}{\sqrt{6}}(-\Lambda\Lambda[\overline{p}\overline{\Lambda}] - (n\Lambda)[\overline{n}\overline{p}])$
		$\|0,0,-2\rangle$	$-\frac{1}{\sqrt{2}}\Lambda\Lambda[\overline{n}\overline{p}]$
(2,0,-2)	27	$\|1,1,2\rangle$	$pp\,\overline{\Lambda}\overline{\Lambda}$
		$\|1,0,2\rangle$	$\frac{1}{\sqrt{2}}(pn)\overline{\Lambda}\overline{\Lambda}$
		$\|1,-1,2\rangle$	$nn\,\overline{\Lambda}\overline{\Lambda}$
		$\|\frac{3}{2},\frac{3}{2},1\rangle$	$-\frac{1}{\sqrt{2}}\,pp(\overline{\Lambda}\overline{n})$
		$\|\frac{3}{2},\frac{1}{2},1\rangle$	$\frac{1}{\sqrt{6}}(-(np)(\overline{\Lambda}\overline{n}) + pp(\overline{\Lambda}\overline{p}))$
		$\|\frac{3}{2},-\frac{1}{2},1\rangle$	$\frac{1}{\sqrt{6}}(-nn(\overline{\Lambda}\overline{n}) + (np)(\overline{\Lambda}\overline{p}))$
		$\|\frac{3}{2},-\frac{3}{2},1\rangle$	$\frac{1}{\sqrt{2}}nn(\overline{\Lambda}\overline{p})$
		$\|2,2,0\rangle$	$pp\overline{n}\overline{n}$
		$\|2,1,0\rangle$	$\frac{1}{2}((pn)\overline{n}\overline{n} - pp(\overline{p}\overline{n}))$
		$\|2,0,0\rangle$	$\frac{1}{\sqrt{6}}(nn\overline{n}\overline{n} - (np)(\overline{p}\overline{n}) + pp\overline{p}\overline{p})$
		$\|2,-1,0\rangle$	$\frac{1}{2}(-nn(\overline{p}\overline{n}) + (np)\overline{p}\overline{p})$
		$\|2,-2,0\rangle$	$nn\overline{p}\overline{p}$

		$\lvert\frac{1}{2},\frac{1}{2},1\rangle$	$\frac{1}{\sqrt{30}}(3(p\Lambda)\overline{\Lambda\Lambda} - 2pp(\bar{p}\bar{\Lambda}) - (pn)(\bar{n}\bar{\Lambda}))$
		$\lvert\frac{1}{2},-\frac{1}{2},1\rangle$	$\frac{1}{\sqrt{30}}(3(n\Lambda)\overline{\Lambda\Lambda} - (pn)(\bar{p}\bar{\Lambda}) - 2nn(\bar{n}\bar{\Lambda}))$
		$\lvert 1,1,0\rangle$	$\frac{1}{2\sqrt{5}}(-2(p\Lambda)(\bar{n}\bar{\Lambda}) + pp(\overline{pn}) + (pn)\overline{nn})$
		$\lvert 1,0,0\rangle$	$\frac{1}{\sqrt{10}}(-(n\Lambda)(\bar{n}\bar{\Lambda}) + (p\Lambda)(\bar{p}\bar{\Lambda}) - pp\overline{pp} + nn\overline{nn})$
		$\lvert 1,-1,0\rangle$	$\frac{1}{2\sqrt{5}}(2(n\Lambda)(\bar{p}\bar{\Lambda}) - (pn)\overline{pp} - nn(\overline{pn}))$
		$\lvert 0,0,0\rangle$	$\frac{1}{\sqrt{120}}\; 6\Lambda\Lambda\overline{\Lambda\Lambda} - 3(p\Lambda)(\bar{p}\bar{\Lambda}) + 2pp\overline{pp} - 3(\Lambda n)(\bar{\Lambda}\bar{n}) + 2\,nn\overline{nn} + (pn)(\overline{pn})$
		The basis vectors for -ve strangeness can be obtained from the ones for +ve strangeness by the operation $p \longleftrightarrow \bar{p}$, $n \longleftrightarrow \bar{n}$, $\Lambda \longleftrightarrow \bar{\Lambda}$	

Table III.

Matrix Elements of the operators $A^3{}_1$, $A^3{}_2$ for the Representations (1,0,−1), (2,−1,−1), (2,0,−2).

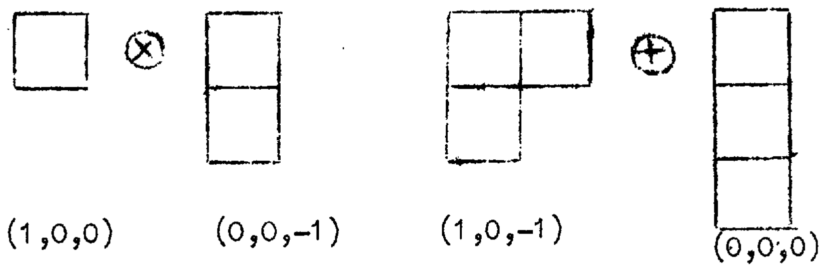
Reps.	$\|i,i_3,s\rangle$	$A^3{}_1$		$A^3{}_2$	
		$\Delta i_3 = -\frac{1}{2},\ \Delta s = -1$		$\Delta i_3 = +\frac{1}{2},\ \Delta s = -1$	
		$\Delta i = \frac{1}{2}$	$\Delta i = -\frac{1}{2}$	$\Delta i = \frac{1}{2}$	$\Delta i = -\frac{1}{2}$
(1,0,−1)	$\|\frac{1}{2},\frac{1}{2},1\rangle$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{6}$	1	0
	$\|\frac{1}{2},-\frac{1}{2},1\rangle$	−1	0	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{6}$
	$\|1,1,0\rangle$	0	1	0	0
	$\|1,0,0\rangle$	0	$\frac{1}{2}\sqrt{2}$	0	$\frac{1}{2}\sqrt{2}$
	$\|1,-1,0\rangle$	0	0	0	1
	$\|0,0,0\rangle$	$-\frac{1}{2}\sqrt{6}$	0	$\frac{1}{2}\sqrt{6}$	0
	$\|\frac{1}{2},\frac{1}{2},-1\rangle$	0	0	0	0
	$\|\frac{1}{2},-\frac{1}{2},-1\rangle$	0	0	0	0
(2,−1,−1)	$\|\frac{3}{2},\frac{3}{2},1\rangle$	0	$\sqrt{3}$	0	0
	$\|\frac{3}{2},\frac{1}{2},1\rangle$	0	$\sqrt{2}$	0	1
	$\|\frac{3}{2},-\frac{1}{2},1\rangle$	0	1	0	$\sqrt{2}$
	$\|\frac{3}{2},-\frac{1}{2},1\rangle$	0	0	0	$\sqrt{3}$
	$\|1,1,0\rangle$	0	2	0	0
	$\|1,0,0\rangle$	0	$\sqrt{2}$	0	$\sqrt{2}$
	$\|1,-1,0\rangle$	0	0	0	2
	$\|1,\frac{1}{2},-1\rangle$	0	$\sqrt{3}$	0	0
	$\|1,-\frac{1}{2},-1\rangle$	0	0	0	$\sqrt{3}$
	$\|0,0,-2\rangle$	0	0	0	0

(2,0,-2)	$\|1,1,2\rangle$	$-\frac{1}{3}\sqrt{6}$	$\frac{1}{3}\sqrt{30}$	$\sqrt{2}$	0
	$\|1,0,2\rangle$	$-\frac{2}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{15}$	$\frac{2}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{15}$
	$\|1,-1,2\rangle$	$-\sqrt{2}$	0	$\frac{1}{3}\sqrt{6}$	$\frac{1}{3}\sqrt{30}$
	$\|\frac{3}{2},\frac{3}{2},1\rangle$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{10}$	$\sqrt{2}$	0
	$\|\frac{3}{2},\frac{1}{2},1\rangle$	-1	$\frac{1}{3}\sqrt{15}$	$\frac{1}{2}\sqrt{6}$	$\frac{1}{6}\sqrt{30}$
	$\|\frac{3}{2},-\frac{1}{2},1\rangle$	$-\frac{1}{2}\sqrt{6}$	$\frac{1}{6}\sqrt{30}$	1	$\frac{1}{3}\sqrt{15}$
	$\|\frac{3}{2},-\frac{3}{2},1\rangle$	$-\sqrt{2}$	0	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{10}$
	$\|2,2,0\rangle$	0	$\sqrt{2}$	0	0
	$\|2,1,0\rangle$	0	$\frac{1}{2}\sqrt{6}$	0	$\frac{1}{2}\sqrt{10}$
	$\|2,0,0\rangle$	0	1	0	1
	$\|2,-1,0\rangle$	0	$\frac{1}{2}\sqrt{2}$	0	$\frac{1}{2}\sqrt{6}$
	$\|2,-2,0\rangle$	0	0	0	$\sqrt{2}$
	$\|\frac{1}{2},\frac{1}{2},1\rangle$	$-\frac{2}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{6}$	0
	$\|\frac{1}{2},-\frac{1}{2},1\rangle$	$-\frac{2}{3}\sqrt{6}$	0	$\frac{2}{3}\sqrt{3}$	2
	$\|1,1,0\rangle$	$-\frac{1}{6}\sqrt{30}$	$\frac{2}{3}\sqrt{6}$	$\frac{1}{2}\sqrt{10}$	0
	$\|1,0,0\rangle$	$-\frac{1}{3}\sqrt{15}$	$\frac{2}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{15}$	$\frac{2}{3}\sqrt{3}$
	$\|1,-1,0\rangle$	$-\frac{1}{2}\sqrt{10}$	0	$\frac{1}{6}\sqrt{30}$	$\frac{2}{3}\sqrt{6}$
	$\|\frac{3}{2},\frac{3}{2},-1\rangle$	0	$\sqrt{2}$	0	0
	$\|\frac{3}{2},\frac{1}{2},-1\rangle$	0	$\frac{2}{3}\sqrt{3}$	0	$\frac{1}{3}\sqrt{6}$
	$\|\frac{3}{2},-\frac{1}{2},-1\rangle$	0	$\frac{1}{3}\sqrt{6}$	0	$\frac{2}{3}\sqrt{3}$
	$\|\frac{3}{2},-\frac{3}{2},-1\rangle$	0	0	0	$\sqrt{2}$
	$\|0,0,0\rangle$	-2	0	2	0
	$\|\frac{1}{2},\frac{1}{2},-1\rangle$	$-\frac{1}{3}\sqrt{15}$	0	$\frac{1}{3}\sqrt{30}$	0
	$\|\frac{1}{2},-\frac{1}{2},-1\rangle$	$-\frac{1}{3}\sqrt{30}$	0	$\frac{1}{3}\sqrt{15}$	0
	$\|1,1,-2\rangle$	0	0	0	0
	$\|1,0,-2\rangle$	0	0	0	0
	$\|1,-1,-2\rangle$	0	0	0	0

Tables of the first type appear in Refs. 7, 15 in a different context. The matrix elements have only been tabulated for completion and for some future reference. Similar tables also appear in Ref. 18b . Many of the coefficients in the table of matrix elements are related by $I_-(I_+)$ operations which commutes with $A^3{}_1$ $(A^3{}_2)$.

iv. <u>Evaluation of the reduction coefficients for the products $3 \otimes \bar{3}$ and $8 \otimes 8$</u>.

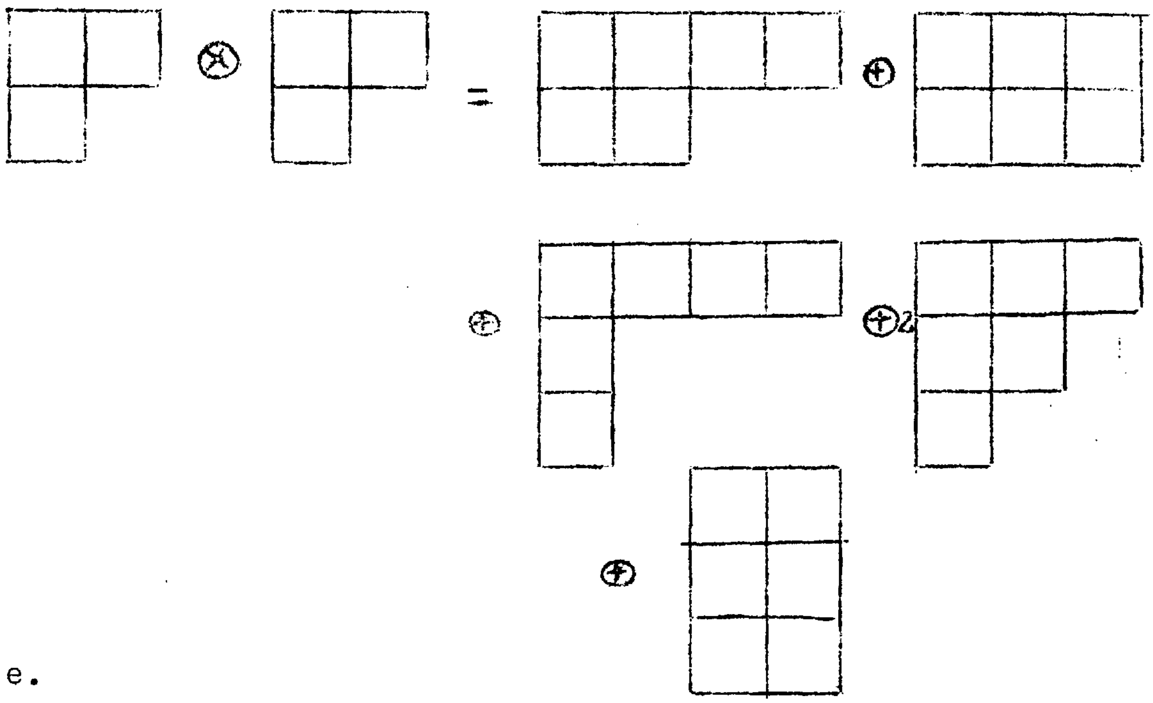
The Young tableau representation for the product $3 \otimes \bar{3}$ is



The suffix 1 under (0,0,0) is indicative of the fact that one trace is being taken. Now we have just to know the highest members of the representations (1,0,-1) and (0,0,0) in terms of the basis vectors for (1,0,0), (0,0,-1). We already know about (1,0,-1). In fact, in table II, we tabulated all its basis vectors. $(0,0,0)_1$ is just $\frac{p\bar{p} + n\bar{n} + \Lambda\bar{\Lambda}}{\sqrt{3}}$ which is orthogonal to the corresponding $|0,0,0\rangle$ state $\frac{-p\bar{p} - n\bar{n} + 2\Lambda\bar{\Lambda}}{\sqrt{6}}$ which appears in (1,0,-1).

This completes the reduction of the product 3   3.

Next we go on to $8 \otimes 8$. Here the product is represented as



i.e.

$$(1,0,-1) \otimes (1,0,-1) = (2,0,-2) \oplus (2,-1,-1) \oplus (1,1,-2) \oplus 2(1,0,-1)_1 \oplus (0,0,0)_2 \qquad \ldots.(2.23)$$

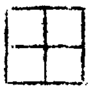
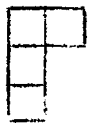
Taking into consideration the fact that the +ve and -ve integers appearing in the above correspond to the upper (cogradient) and lower (contragradient) indices, we see that (2,0,-2) is symmetric in both the upper and lower indices, (2,-1,-1) ((1,1,-2)) symmetric (anti-symmetric) in the upper and anti-symmetric (symmetric) in the lower, while (1,0,-1), (0,0,0) have no special symmetry as some traces have been taken.

Again, (1,0,-1) corresponds to a second rank tensor T^{α}_{β} such that $T^{\alpha}_{\alpha} = 0$. We can also symmetrize and anti-symmetrize the upper and lower indices in the product $T^{\alpha}_{\gamma} \otimes T^{\beta}_{\delta}$ to obtain

$$T^{\alpha}_{\gamma} \otimes T^{\beta}_{\delta} = T^{(\alpha\beta)}_{(\gamma\delta)} \oplus T^{(\alpha\beta)}_{[\gamma\delta]} \quad T^{[\alpha\beta]}_{(\gamma\delta)} \quad T^{[\alpha\beta]}_{[\gamma\delta]} \qquad \ldots.(2.24)$$

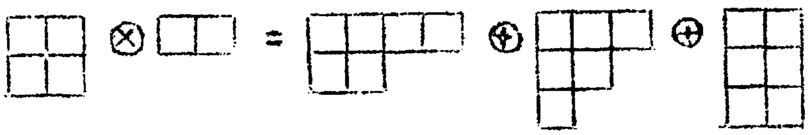
The tensors on the right hand side are still not <u>irreducible</u>. To decompose each one of these into irreducible parts, we have to take out the traces. Some of these traces are identically zero; to analyse it, we perform the above decomposition in another manner which corresponds to the equation (2.24). For this purpose, we again multiply the Young's tableau taking the following correspondences:

(i) $T^{(\alpha\beta)} \rightarrow$  , (ii) $T^{[\alpha\beta]} \longrightarrow$ 

(iii) $T_{(\gamma\delta)} \longrightarrow$  , (iv) $T_{[\gamma\delta]} \longrightarrow$ 

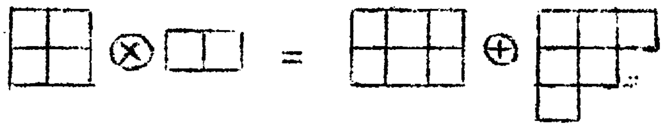
The symmetric and the anti-symmetric tensors above are 6 and 3 dimensional respectively.

Corresponding to $T^{(\alpha\beta)}_{(\gamma\delta)}$ we have the multiplication

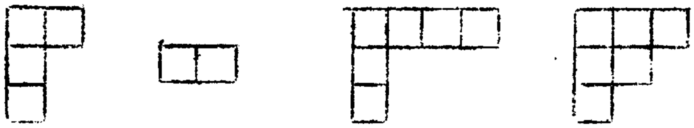


6 x 6 = 27 + 8 + 1

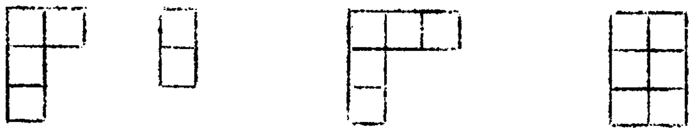
The other multiplications give



6 x 3 = 10 + 8

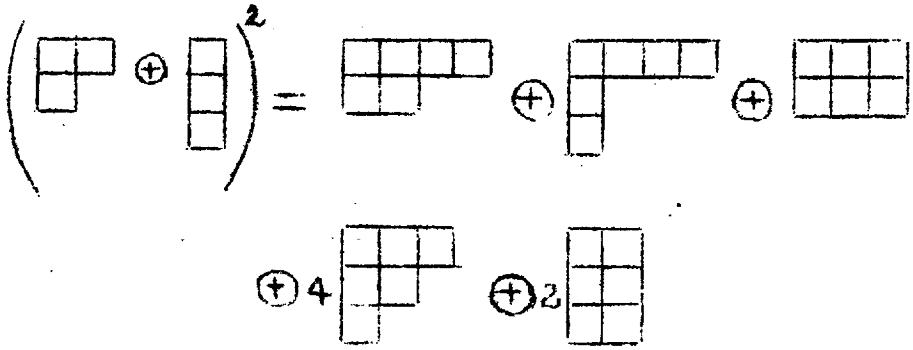


3 x 6 = 10 + 8



3 x 3 = 8 + 1

Neglecting how any diagram was obtained, we write the conclusion as



which represents

$$((1,0,-1) \oplus (0,0,0)) \otimes ((1,0,-1) \oplus (0,0,0))$$
$$= (2,0,-2)_0 \oplus (2,-1,-1)_9 \oplus (1,1,-2)_0 \oplus 4(1,0,-1)_1 \oplus (0,0,0)_2$$

....(2.25)

It is evident that 9 diagrams have been obtained as we did not impose the condition of tracelessness on the tensors T^{α}_{γ}, T^{β}_{δ}. This is actually the reason why the decomposition is difficult as there is no direct method of imposing this condition. However, we know that these conditions must reduce equation (2.25) to (2.23). Thus, under the conditions of tracelessness, the vectors we obtain from the above symmetry characters will not be linearly independent in the cases of 1 and 8 dimensional irreducible parts, and we must be able to pick a set of linearly independent ones which correspond to the equation (2.23). In the following table we give the highest eigenvectors which belong to the diagrams representing (2.25). These have been written by taking into account the symmetry of the tableau and the trace indices in equation (2.25).

Table IV.

Tensors	Symmetry of the Irreducible Parts	Highest eigenvectors
1) $T^{(\alpha\beta)}_{(\gamma\delta)}$	$(2,0,-2)_{ss}$	$pp\,\overline{\Lambda\Lambda}$
	$(1,0,-1)_{ss}$	$2pp(\bar{p}\bar{\Lambda}) + (np)(\bar{n}\bar{\Lambda}) + 2(\Lambda p)\bar{\Lambda}\bar{\Lambda}$
	$(0,0,0)_{ss}$	$2pp\bar{p}\bar{p} + 2nn\bar{n}\bar{n} + 2\Lambda\Lambda\bar{\Lambda}\bar{\Lambda} + (pn)(\bar{p}\bar{n}) + (p\Lambda)(\bar{p}\bar{\Lambda}) + (n\Lambda)(\bar{n}\bar{\Lambda})$
2) $T^{(\alpha\beta)}_{[\gamma\delta]}$	$(2,-1,-1)_{sa}$	$pp[\bar{n}\bar{\Lambda}]$
	$(1,0,-1)_{sa}$	$2pp[\bar{p}\bar{\Lambda}] + (np)[\bar{n}\bar{\Lambda}]$
3) $T^{[\alpha\beta]}_{(\gamma\delta)}$	$(1,1,-2)_{as}$	$[pn]\bar{\Lambda}\bar{\Lambda}$
	$(1,0,-1)_{as}$	$[pn](\bar{n}\bar{\Lambda}) + 2[p\Lambda]\bar{\Lambda}\bar{\Lambda}$
4) $T^{[\alpha\beta]}_{[\gamma\delta]}$	$(1,0,-1)_{aa}$	$[pn][\bar{n}]$
	$(0,0,0)_{aa}$	$[pn][\bar{p}\bar{n}] + [p\Lambda][\bar{p}\bar{\Lambda}] + [n\Lambda][\bar{n}\bar{\Lambda}]$

The first (second) index describes the symmetry or anti-symmetry in the upper (lower) indices.

Using the condition that

$$T^{\alpha}_{\alpha} = p\bar{p} + n\bar{n} + \Lambda\bar{\Lambda} = 0$$

we see that

$$(1,0,-1)_{ss} = -(1,0,-1)_{aa}$$

$$(1,0,-1)_{sa} = (1,0,-1)_{as} \qquad \ldots.(2.26)$$

$$(0,0,0)_{ss} = -(0,0,0)_{aa}$$

To complete the decomposition $(1,0,-1) \otimes (1,0,-1)$, we consider the association of particles to the 8-states. In both the unitary symmetry models, the 8 pseudoscalar mesons are collected into an octet. As this is a self-contragradient representation, the bosons appear in it with their anti-particles. Hence the choice of phases is important. We choose the phases such that π^+, K^+, K^0 have their anti-particles as π^-, K^-, $\bar{K}^0$. With this choice, we indeed get the correct spinors as $\begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$ and $\begin{pmatrix} -\bar{K}^0 \\ K^0 \end{pmatrix}$ and the isotopic vector is taken as $\begin{pmatrix} -\pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$.

We can collect the same 8 states in a 3 x 3 "traceless" matrix as

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi^{0\prime} & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi^{0\prime} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\pi^{0\prime} \end{pmatrix} \qquad \ldots.(2.27)$$

where in "components" form this expresses

$$\begin{pmatrix} \frac{2}{3}p\bar{p} - \frac{1}{3}(n\bar{n} + \Lambda\bar{\Lambda}) & p\bar{n} & p\bar{\Lambda} \\ n\bar{p} & \frac{2}{3}n\bar{n} - \frac{1}{3}(p\bar{p} + \Lambda\bar{\Lambda}) & n\bar{\Lambda} \\ \Lambda\bar{p} & \Lambda\bar{n} & \frac{2}{3}\Lambda\bar{\Lambda} - \frac{1}{3}(p\bar{p} + n\bar{n}) \end{pmatrix} \qquad \ldots.(2.28)$$

With these assignments, we can finally calculate the vectors in the decomposition of the direct product of two octets (starting from the highest member of an irreducible representation, we have to operate by I_-, $A^3{}_2$, $\bar{A}^3{}_1$ as described in Section 2(iii)). The results are presented in the following table wherein each vector has been normalized to unity to ensure unitarity of the reducing matrix.

Table V.

Eigenvectors in the Reduction of the Direct Product of Two Octets.

Irr. Rep.	Dim.	i	i_3	s	Eigenvector	Sym.
(2,0,-2)	27	1	1	2	K^+K^+	S
			0		$\frac{1}{\sqrt{2}}(K^+K^0 + K^0K^+)$	
		–	–1		K^0K^0	
		$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{1}{\sqrt{2}}(-\pi^+K^+ - K^+\pi^+)$	
			$\frac{1}{2}$		$\frac{1}{\sqrt{\;}}(\sqrt{2}\,\pi^0K^+ - \pi^+K^0 - K^0\pi^+ + \sqrt{2}\,K^+\pi^0)$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{6}}(\sqrt{2}\,\pi^0K^0 + \pi^-K^+ + K^+\pi^- + \sqrt{2}\,K^0\pi^0)$	
			$-\frac{3}{2}$		$\frac{1}{\sqrt{2}}(\pi^-K^0 + K^0\pi^-)$	

Irr. Rep.	Dim.	i	i_3	s	Eigenvector	Sym.
		$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{\sqrt{30}}\left[-(\frac{1}{\sqrt{2}}\pi^0+\frac{9}{\sqrt{6}}\pi^{0'})K^+ - K^+(\frac{1}{\sqrt{2}}\pi^0+\frac{9}{\sqrt{6}}\pi^{0'}) - \pi^+K^0 + K^0\pi^+\right]$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{30}}\left[(\frac{1}{\sqrt{2}}\pi^0-\frac{9}{\sqrt{6}}\pi^{0'})K^0 + K^0(\frac{1}{\sqrt{2}}\pi^0-\frac{9}{\sqrt{6}}\pi^{0'}) - \pi^-K^+ - K^+\pi^-\right]$	
		2	2	0	$\pi^+\pi^+$	
			1		$-\frac{1}{\sqrt{2}}(\pi^+\pi^0 + \pi^0\pi^+)$	
			0		$\frac{1}{\sqrt{6}}(-\pi^+\pi^- + 2\pi^0\pi^0 - \pi^-\pi^+)$	
			-1		$\frac{1}{\sqrt{2}}(\pi^-\pi^0 + \pi^0\pi^-)$	
			-2		$\pi^-\pi^-$	
		1	1	0	$\frac{1}{\sqrt{5}}(\frac{\sqrt{3}}{\sqrt{2}}\pi^+\pi^{0'} + \frac{\sqrt{3}}{\sqrt{2}}\pi^{0'}\pi^+ - K^+\bar{K}^0 - \bar{K}^0K^+)$	
			0		$\frac{1}{\sqrt{10}}(-\sqrt{3}\pi^0\pi^{0'} - \sqrt{3}\pi^{0'}\pi^0 - K^0\bar{K}^0 - \bar{K}^0K^0 + K^+K^- + K^-K^+)$	
			-1		$\frac{1}{\sqrt{5}}(-\frac{\sqrt{3}}{\sqrt{2}}\pi^-\pi^{0'} - \frac{\sqrt{3}}{\sqrt{2}}\pi^{0'}\pi^- + K^0K^- + K^-K^0)$	
		0	0	0	$\frac{1}{\sqrt{120}}(\pi^0\pi^0 + 9\pi^{0'}\pi^{0'} + \pi^+\pi^- + \pi^-\pi^+ - 3K^+K^- - 3K^-K^+ - 3K^0\bar{K}^0 - 3\bar{K}^0K^0)$	
		1	$\frac{1}{2}$	-1	$\frac{1}{\sqrt{30}}\left[-(\frac{1}{\sqrt{2}}\pi^0-\frac{9}{\sqrt{6}}\pi^0)\bar{K}^0 - \bar{K}^0(\frac{1}{\sqrt{2}}\pi^0-\frac{9}{\sqrt{6}}\pi^{0'}) + \pi^+K^- + K^-\pi^+\right]$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{30}}\left[-(\frac{1}{\sqrt{2}}\pi^0+\frac{9}{\sqrt{6}}\pi^{0'})\bar{K} - \bar{K}(\frac{1}{\sqrt{2}}\pi^0+\frac{9}{\sqrt{6}}\pi^{0'}) - \pi^-\bar{K}^0 - \bar{K}^0\pi^-\right]$	

Irr. Rep.	Dim.	i	i_3	s	Eigenvector	Sym.
		$\frac{3}{2}$	$\frac{3}{2}$	−1	$\frac{1}{\sqrt{2}}(\pi^+\bar{K}^0 + \bar{K}^0\pi^+)$	
			$\frac{1}{2}$		$\frac{1}{\sqrt{6}}(-\sqrt{2}\,\pi^0\bar{K}^0 - \sqrt{2}\,\bar{K}^0\pi^0 + \pi^-K^+ + K^+\pi^-)$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{6}}(\sqrt{2}\pi^0 K^- + \sqrt{2}\,K^-\pi^0 - \pi^-\bar{K}^0 - \bar{K}^0\pi^-)$	
			$-\frac{3}{2}$		$\frac{1}{\sqrt{2}}(\pi^- K^- + K^-\pi^-)$	
		1	1	−2	$\bar{K}^0\bar{K}^0$	
			0		$-\frac{1}{\sqrt{2}}(K^-\bar{K}^0 + \bar{K}^0K^-)$	
			−1		K^-K^-	
(2,−1,−1)	10	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{1}{\sqrt{2}}(\pi^+K^+ - K^+\pi^+)$	A
			$\frac{1}{2}$		$\frac{1}{\sqrt{6}}(-\sqrt{2}\pi^0K^+ + \sqrt{2}\,K^+\pi^0 + \pi^+K^0 - K^0\pi^+)$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{6}}(-\pi^-K^+ + K^+\pi^- - \sqrt{2}\pi^0K^0 - \sqrt{2}\,K^0\pi^0)$	
			$-\frac{3}{2}$		$\frac{1}{\sqrt{2}}(-\pi^-K^0 + K^0\pi^-)$	
		1	1	0	$\frac{1}{\sqrt{6}}\left(\left(\frac{1}{\sqrt{2}}\pi^0 + \frac{3}{\sqrt{6}}\pi^{0'}\right)\pi^+ - \pi^+\left(\frac{1}{\sqrt{2}}\pi^0 + \frac{3}{\sqrt{6}}\pi^{0'}\right) + \bar{K}^0K^+ - K^+\bar{K}^0\right)$	
			0		$\frac{1}{2\sqrt{3}}\left(-\sqrt{3}\pi^{0'}\pi^0 + \sqrt{3}\pi^0\pi^{0'} + \pi^-\pi^+ - \pi^+\pi^- - K^-K^+ + K^+K^- + \bar{K}^0K^0 - K^0\bar{K}^0\right)$	
			−1		$\frac{1}{\sqrt{6}}\left(\left(\frac{1}{\sqrt{2}}\pi^0 - \frac{3}{\sqrt{6}}\pi^{0'}\right)\pi^- - \pi^-\left(\frac{1}{\sqrt{2}}\pi^0 - \frac{3}{\sqrt{6}}\pi^{0'}\right) - K^-K^0 + K^0K^-\right)$	

Irr. Rep.	Dim.	i	i_3	s	Eigenvector	Sym.
		$\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{\sqrt{6}}\left[(\frac{1}{\sqrt{2}}\pi^{0}+\frac{3}{\sqrt{6}}\pi^{0\prime})\bar{K}^{0} - \bar{K}^{0}(\frac{1}{\sqrt{2}}\pi^{0}+\frac{3}{\sqrt{6}}\pi^{0}) + K^{-}\pi^{+} - \pi^{+}K^{-}\right]$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{6}}\left[(\frac{1}{\sqrt{2}}\pi^{0}-\frac{3}{\sqrt{6}}\pi^{0\prime})K^{-} - K^{-}(\frac{1}{\sqrt{2}}\pi^{0}-\frac{3}{\sqrt{6}}\pi^{0\prime}) + \pi^{-}\bar{K}^{0} - \bar{K}^{0}\pi^{-}\right]$	
		0	0	-2	$\frac{1}{\sqrt{2}}(K^{-}\bar{K}^{0} - \bar{K}^{0}K^{-})$	
(1,1,−2)	10^{*}	0	0	2	$\frac{1}{\sqrt{2}}(-K^{+}K^{0} + K^{0}K^{+})$	A
		$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{\sqrt{6}}\left[(\frac{1}{\sqrt{2}}\pi^{0}-\frac{3}{\sqrt{6}}\pi^{0\prime})K^{+} - K^{+}(\frac{1}{\sqrt{2}}\pi^{0}-\frac{3}{\sqrt{6}}\pi^{0\prime}) + \pi^{+}K^{0} - K^{0}\pi^{+}\right]$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{6}}\left[-(\frac{1}{\sqrt{2}}\pi^{0}+\frac{3}{\sqrt{6}}\pi^{0\prime})K^{0} + K^{0}(\frac{1}{\sqrt{2}}\pi^{0}+\frac{3}{\sqrt{6}}\pi^{0\prime}) - K^{+}\pi^{-} + \pi^{-}K^{+}\right]$	
		1	1	0	$\frac{1}{\sqrt{6}}\left[-(\frac{1}{\sqrt{2}}\pi^{0}-\frac{3}{\sqrt{6}}\pi^{0\prime})\pi^{+} + \pi^{+}(\frac{1}{\sqrt{2}}\pi^{0}-\frac{3}{\sqrt{6}}\pi^{0\prime}) + K^{+}\bar{K}^{0} - \bar{K}^{0}K^{+}\right]$	
			0		$\frac{1}{2\sqrt{3}}(-\sqrt{3}\pi^{0\prime}\pi^{0}+\sqrt{3}\pi^{0}\pi^{0\prime}+\pi^{+}\pi^{-}-\pi^{-}\pi^{+}-K^{+}K^{-} + K^{-}K^{+} + K^{0}\bar{K}^{0} - \bar{K}^{0}K^{0})$	
			-1		$\frac{1}{\sqrt{6}}\left[-(\frac{1}{\sqrt{2}}\pi^{0}+\frac{3}{\sqrt{6}}\pi^{0\prime})\pi^{-} + \pi^{-}(\frac{1}{\sqrt{2}}\pi^{0}+\frac{3}{\sqrt{6}}\pi^{0\prime}) - K^{0}K^{-} + K^{-}K^{0}\right]$	

Irr. Rep.	Dim.	i	i_3	s	Eigenvector	Sym.
		$\frac{3}{2}$	$\frac{3}{2}$	-1	$\frac{1}{\sqrt{2}}(-\pi^+\bar{K}^0 + \bar{K}^0\pi^+)$	
			$\frac{1}{2}$		$\frac{1}{\sqrt{6}}(\pi^+K^- - K^-\pi^+ + \sqrt{2}\pi^0\bar{K}^0 - \sqrt{2}\,\bar{K}^0\pi^0)$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{6}}(-\sqrt{2}\pi^0K^- + \sqrt{2}K^-\pi^0 + \pi^-\bar{K}^0 - \bar{K}^0\pi^-)$	
			$-\frac{3}{2}$		$\frac{1}{\sqrt{2}}(-\pi^-K^- + K^-\pi^-)$	
(1,0,−1)	8	$\frac{1}{2}$	$\frac{1}{2}$	1	$\sqrt{\frac{3}{10}}\Big(-\pi^+K^0 - K^0\pi^+ - K^+(\frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\pi^{0\prime})$ $- (\frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\pi^{0\prime})K^+\Big)$	S
			$-\frac{1}{2}$		$\sqrt{\frac{3}{10}}\Big((\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi^{0\prime})K^0 + K^0(\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi^{0\prime})$ $- K^+\pi^- - \pi^-K^+\Big)$	
		1	1	0	$\sqrt{\frac{3}{10}}\Big(\sqrt{\frac{2}{3}}\,\pi^+\pi^{0\prime} + \sqrt{\frac{2}{3}}\,\pi^{0\prime}\pi^+ + K^+\bar{K}^0 + \bar{K}^0K^+\Big)$	
			0		$\sqrt{\frac{3}{20}}\Big(\frac{2}{\sqrt{3}}\pi^0\pi^{0\prime} - \frac{2}{\sqrt{3}}\pi^{0\prime}\pi^0 + K^0\bar{K}^0 + \bar{K}^0K^0$ $- K^+K^- - K^-K^+\Big)$	
			-1		$\sqrt{\frac{3}{10}}\Big(-\sqrt{\frac{2}{3}}\pi^-\pi^{0\prime} - \sqrt{\frac{2}{3}}\,\pi^{0\prime}\pi^- - K^0K^- - K^-K^0\Big)$	
		0	0	0	$\frac{1}{2\sqrt{5}}\Big(2\pi^0\pi^0 + 2\,\pi^+\pi^- + 2\pi^-\pi^+ - 2\,\pi^{0\prime}\pi^{0\prime}$ $- K^+K^- - K^-K^+ - K^0\bar{K}^0 - \bar{K}^0K^0\Big)$	
		$\frac{1}{2}$	$\frac{1}{2}$	-1	$\sqrt{\frac{3}{10}}\Big[-(\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi^{0\prime})\bar{K}^0 - \bar{K}^0(\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\pi^{0\prime})$ $+ \pi^+K^- + K^-\pi^+\Big]$	
			$-\frac{1}{2}$		$\sqrt{\frac{3}{10}}\Big[-\pi^-\bar{K}^0 - \bar{K}^0\pi^- - (\frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\pi^0)K^-$ $- K^-(\frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\pi^{0\prime})\Big]$	

Irr. Rep.	Dim.	i	i_3	s	Eigenvector	Sym.
(1,0,-1)	8	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{\sqrt{6}}\Big[(\frac{1}{\sqrt{2}}\pi^{o} + \frac{3}{2}\pi^{o'})K^{+} - K^{+}(\frac{1}{\sqrt{2}}\pi^{o} + \frac{3}{2}\pi^{o'}) + \pi^{+}K^{o} - K^{o}\pi^{+}\Big]$	A
			$-\frac{1}{2}$		$\frac{1}{\sqrt{6}}\Big[-(\frac{1}{\sqrt{2}}\pi^{o} - \frac{3}{2}\pi^{o'})K^{o} + K^{o}(\frac{1}{\sqrt{2}}\pi^{o} - \frac{3}{2}\pi^{o'}) - K^{+}\pi^{-} + \pi^{-}K^{+}\Big]$	
		1	1	0	$\frac{1}{\sqrt{6}}(-\sqrt{2}\pi^{o}\pi^{+} + \sqrt{2}\pi^{+}\pi^{o} + \bar{K}^{o}K^{+} - K^{+}\bar{K}^{o})$	
			0		$\frac{1}{\sqrt{12}}(-2\pi^{-}\pi^{+} + 2\pi^{+}\pi^{-} - K^{-}K^{+} + K^{+}K^{-} + \bar{K}^{o}K^{o} - K^{o}\bar{K}^{o})$	
			-1		$\frac{1}{\sqrt{6}}(-\sqrt{2}\pi^{o}\pi^{-} + \sqrt{2}\pi^{-}\pi^{o} + K^{o}K^{-} - K^{-}K^{o})$	
		0	0	0	$\frac{1}{2}(K^{-}K^{+} - K^{+}K^{-} + \bar{K}^{o}K^{o} - K^{o}\bar{K}^{o})$	
		$\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{\sqrt{6}}\, -(\frac{1}{\sqrt{2}}\pi^{o} - \frac{3}{2}\pi^{o'})\bar{K}^{o} + \bar{K}^{o}(\frac{1}{\sqrt{2}}\pi^{o} - \frac{3}{2}\pi^{o'}) - K^{-}\pi^{+} + \pi^{+}K^{-}$	
			$-\frac{1}{2}$		$\frac{1}{\sqrt{6}}\, -(\frac{1}{\sqrt{2}}\pi^{o} + \frac{3}{2}\pi^{o'})K^{-} + K^{-}(\frac{1}{\sqrt{2}}\pi^{o} + \frac{3}{2}\pi^{o'}) - \pi^{-}\bar{K}^{o} + \bar{K}^{o}\pi^{-}$	
(0,0,0)	1	0	0	0	$\frac{1}{2\sqrt{2}}(\pi^{+}\pi^{-} + \pi^{-}\pi^{+} + \pi^{o}\pi^{o} + K^{+}K^{-} + K^{-}K^{+} + K^{o}\bar{K}^{o} + \bar{K}^{o}K^{o} + \pi^{o'}\pi^{o'})$	S

In the above, we have taken the following combinations in terms of quantities in table IV.

$$(2,0,-2)_S = (2,0,-2)_{ss}$$

$$(2,-1,-1)_A = (2,-1,-1)_{sa}$$

$$(1,1,-2)_A = -(1,1,-2)_{as}$$

$$(1,0,-1)_S = (1,0,-1)_{aa}$$

$$(1,0,-1)_A = (1,0,-1)_{sa}$$

$$(0,0,0)_S = (0,0,0)_{ss}$$

The above reduction of the direct product of two octets was first done by Edmonds.[16]

Considering the fact that in the decomposition of the product of two identical angular momenta, the various angular momenta that appear have the corresponding wave-functions as symmetrical or anti-symmetrical, we can try, by analogy, to understand the symmetries of the vectors in the above decomposition. For this purpose, we have to interchange simultaneously the upper and the lower indices. Evidently under such a transformation ss and aa combinations are symmetric while sa and as are anti-symmetric. The last column in table IV describes this.

In the above reduction of the direct product of two octets, the octet representation appears twice. There is no operator in the group which can distinguish these equivalent representations. In the case of identical particles forming the two octets, we have seen that symmetry of these octets under their exchange can distinguish the two. This can again work only when the multiplicity is 2. In the case of non-idential particles forming the octets we shall have to construct some other operation. We shall see that under the R-operation of Gell-Mann,[29] $8_S \to 8_S$ while $8_A \to -8_A$. This very operation is sufficient to distinguish the twice occurring representations in the reductions we consider. For higher multiplicities, some other mechanism will have to be developed. This R-operation is a reflection in the (i_3, s) space i.e. under R, $i_3 \to -i_3$, $s \to -s$. The operator therefore does not, in general, leave a representation invariant. In fact, only those representations which are symmetrical, i.e. contain the submultiplet $(i,-s)$ whenever they contain (i,s) are invariant under this operation. In the case of other representations, we shall obtain state vectors with strangenesses reversed. These new vectors will belong to the representation contragradient to the one we started with. To see this, we know that (f_1, f_2, f_3) and $(-f_3, -f_2, -f_1)$ are contragradient to each other. Corresponding to each (f_1', f_2') such that

$$f_1 \geq f_1' \geq f_2 \geq f_2' \geq f_3 \qquad \ldots.(2.22)$$

there exists a sub-multiplet (i,s) of (f_1, f_2, f_3) where

$$i = \frac{f_1' - f_2'}{2}, \quad s = f_1' + f_2' - (f_1+f_2+f_3) \qquad \ldots.(2,29)$$

Equation (2.22) can also be written as

$$-f_2 \geq -f_2' \geq -f_2 \geq -f_1' \geq -f_1$$

Thus (i',s') for $(-f_3,-f_2,-f_1)$ are given by

$$i' = \frac{f_1' - f_2'}{2}$$

$$s' = -f_1' - f_2' + (f_1+f_2+f_3)$$

Corresponding to any (i,s) in (f_1,f_2,f_3) there is an (i,-s) in $(-f_3,-f_2,-f_1)$. A representation will be symmetrical i.e. containing (i,±s) if, and only if, it is self-contragradient. In other words,

$$f_1 = -f_3,\; f_2 = -f_2,\; f_3 = -f_1$$

or it is of the form (f,0,-f). The dimension of such a representation can be found to be $(f_1 + 1)^3$ from equation (1.26'). Again from the fact that R does not leave a representation space invariant, we conclude that R is not a group operation. Thus invariance of the interaction Lagrangian under U(3) or SU(3) does not necessarily imply invariance under R and vice versa.

In terms of the 3-dimensional basis $\begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$, R is defined by

$$p \leftrightarrow \bar{p},\quad n \leftrightarrow -\bar{n},\quad \Lambda \leftrightarrow \bar{\Lambda}$$

From table II we see that, under R, the octet transforms as

$$K^+ \leftrightarrow K^-, \quad K^0 \leftrightarrow -\bar{K}^0, \quad \pi^+ \leftrightarrow -\pi^-, \quad \pi^{0\prime} \leftrightarrow \pi^{0\prime}$$

The effect on the various representations appearing in Table IV is as follows:

$$(2,0,-2) \to (2,0,-2)$$
$$(2,-1,-1) \to (1,1,-2)$$
$$(1,1,-2) \to (2,-1,-1)$$
$$(1,0,-1)_S \to (1,0,-1)_S$$
$$(1,0,-1)_A \to -(1,0,-1)_A$$
$$(0,0,0) \to (0,0,0)$$

We can summarize the operation R by

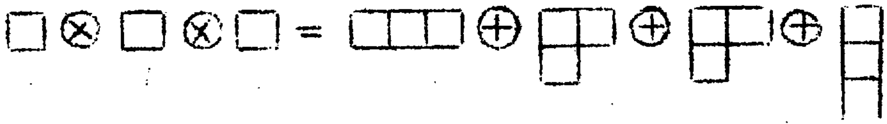
$$R\,|(f_1,f_2,f_3);\, i, i_3, s\rangle = \eta_f\,|(-f_3,-f_2,-f_1); i, -i_3, -s\rangle$$

where the phase-factor η_f depends only on the representation (f_1,f_2,f_3) and not on its basis vectors.

It is also evident from above that the eigenvectors in Table V for -ve strangeness could be obtained from those of +ve strangenesses by the application of the R-operation. We have, however, given the complete table on account of their extreme usefulness.

Product of Three Octets.

The procedure for reduction of the product of three octets is similar to the one for the product of two octets. However, this introduces new complications as might have been expected. Our first step is to consider various possible symmetries of the upper and lower indices. This can be obtained by considering the product



i.e. $\quad 3 \times 3 \times 3 = 10 + 8 + 8 + 1$

$$
\begin{aligned}
T^{\lambda}_{\alpha} \times T^{\mu}_{\beta} \times T^{\nu}_{\gamma} &= T^{\lambda\mu\nu}_{\alpha\beta\gamma} \\
&= T^{(\lambda\mu\nu)}_{(\alpha\beta\gamma)} + T^{\lambda[\mu\nu]}_{(\alpha\beta\gamma)} + T^{[\lambda\mu]\nu}_{(\alpha\beta\gamma)} + T^{[\lambda\mu\nu]}_{(\alpha\beta\gamma)} \\
&= T^{(\lambda\mu\nu)}_{\alpha[\beta\gamma]} + T^{\lambda[\mu\nu]}_{\alpha[\beta\gamma]} + T^{[\lambda\mu]\nu}_{\alpha[\beta\gamma]} + T^{[\lambda\mu\nu]}_{\alpha[\beta\gamma]} \\
&= T^{(\lambda\mu\nu)}_{[\alpha\beta]\gamma} + T^{\lambda[\mu\nu]}_{[\alpha\beta]\gamma} + T^{[\lambda\mu]\nu}_{[\alpha\beta]\gamma} + T^{[\lambda\mu\nu]}_{[\alpha\beta]\gamma} \\
&= T^{(\lambda\mu\nu)}_{(\alpha\beta\gamma)} + T^{\lambda[\mu\nu]}_{[\alpha\beta\gamma]} + T^{[\lambda\mu]\nu}_{[\alpha\beta\gamma]} + T^{[\lambda\mu\nu]}_{[\alpha\beta\gamma]}
\end{aligned}
$$

Each one of the sixteen tensors above except the ones in the last row and column are again reducible. The reduction is carried out by drawing Young's tableau, which now corresponds to taking out traces. The following table gives the results along with the vectors that correspond to the largest S and the largest I_3 for that value of S in the irreducible spaces obtained in this manner:

Table VI.

Decomposition into irreducible spaces.

Tensor	Corre-sponding product	Decomposition: irr. rep.	Decomposition: dimen-sion	Vectors corresponding to the largest S and the largest I_3 for that value of S
1) $T^{(\lambda\mu\nu)}_{(\alpha\beta\gamma)}$	[300][00$\bar{3}$]	[30$\bar{3}$]	64	(ppp)($\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$)
		[20$\bar{2}$]$_1$	27	(ppp)($\bar{p}\bar{\Lambda}\bar{\Lambda}$)+(npp)($\bar{n}\bar{\Lambda}\bar{\Lambda}$) + ($\Lambda$pp)($\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$)
		[10$\bar{1}$]$_1$	8	(ppp)($\bar{p}\bar{p}\bar{\Lambda}$) + 2(pnp)($\bar{p}\bar{n}\bar{\Lambda}$) +2(p$\Lambda$p)($\bar{p}\bar{\Lambda}\bar{\Lambda}$)+2(n$\Lambda$p)($\bar{n}\bar{\Lambda}\bar{\Lambda}$) +(nnp)($\bar{n}\bar{n}\bar{\Lambda}$)+($\Lambda\Lambda$p)($\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$)
		[000]$_1$	1	(ppp)($\bar{p}\bar{p}\bar{p}$)+3(ppn)($\bar{p}\bar{p}\bar{n}$) +3(ppΛ)($\bar{p}\bar{p}\bar{\Lambda}$)+3(pnn)($\bar{p}\bar{n}\bar{n}$) +6(pn$\Lambda$)($\bar{p}\bar{n}\bar{\Lambda}$)+3(p$\Lambda\Lambda$)($\bar{p}\bar{\Lambda}\bar{\Lambda}$) +(nnn)($\bar{n}\bar{n}\bar{n}$)+3(nn$\Lambda$)($\bar{n}\bar{n}\bar{\Lambda}$) +3(n$\Lambda\Lambda$)($\bar{n}\bar{\Lambda}\bar{\Lambda}$)+($\Lambda\Lambda\Lambda$)($\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$)
2) $T^{\lambda[\mu\nu]}_{(\alpha\beta\gamma)}$	[210][00$\bar{3}$]	[21$\bar{3}$]$_1$	35	p[pn][$\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$]
		[20$\bar{2}$]$_2$	27	p[np]($\bar{n}\bar{\Lambda}\bar{\Lambda}$)+p[$\Lambda$ p]($\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$)
		[11$\bar{2}$]$_1$	10	p[pn]($\bar{p}\bar{\Lambda}\bar{\Lambda}$)+p[$\Lambda$ n]($\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$) −n[np]($\bar{n}\bar{\Lambda}\bar{\Lambda}$)−n[$\Lambda$ p]($\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$)
		[10$\bar{1}$]$_2$	8	p[np]($\bar{p}\bar{n}\bar{\Lambda}$)+p[$\Lambda$ p]($\bar{p}\bar{\Lambda}\bar{\Lambda}$) +n[np]($\bar{n}\bar{n}\bar{\Lambda}$)+n[$\Lambda$ p]($\bar{n}\bar{\Lambda}\bar{\Lambda}$) +$\Lambda$[np]($\bar{\Lambda}\bar{n}\bar{\Lambda}$)+$\Lambda$[$\Lambda$ p]($\bar{\Lambda}\bar{\Lambda}\bar{\Lambda}$)
3) $T^{[\lambda\mu]\nu}_{(\alpha\beta\gamma)}$	[210][00$\bar{3}$]	[21$\bar{3}$]$_2$	35	Obtained from (2) by the operation A[BC] → [BC]A
		[20$\bar{2}$]$_3$		
		[11$\bar{2}$]$_2$		
		[10$\bar{1}$]$_3$		

Tensor	Corr-esponding product	Decomposition		Vectors corresponding to the largest S and the largest I_3 for that value of S.
		irr. rep.	dimension	
4) $T^{[\lambda\mu\nu]}_{(\alpha\beta\gamma)}$	$[111][00\bar{3}]$	$[11\bar{2}]_3$	10	$[pn\Lambda](\bar{\Lambda}\bar{\Lambda}\bar{\Lambda})$
5) $T^{(\lambda\mu\nu)}_{\alpha[\beta\gamma]}$	$[300][0\bar{1}\bar{2}]$	$[3\bar{1}\bar{2}]_1$	35	$(ppp)\bar{\Lambda}[\bar{n}\bar{\Lambda}]$
		$[20\bar{2}]_4$	27	$(ppp)\bar{\Lambda}[\bar{p}\bar{\Lambda}]+(npp)\bar{\Lambda}[\bar{n}\bar{\Lambda}]$
		$[2\bar{1}\bar{1}]_1$	10	$(ppp)\bar{n}[\bar{p}\bar{\Lambda}]+(npp)\bar{n}[\bar{n}\bar{\Lambda}]$ $-(ppp)\bar{\Lambda}[\bar{p}\bar{n}]-(\Lambda pp)\bar{\Lambda}[\bar{\Lambda}\bar{n}]$
		$[10\bar{1}]_4$	8	$(ppp)\bar{p}[\bar{p}\bar{\Lambda}]+(pnp)\bar{p}[\bar{n}\bar{\Lambda}]$ $+(npp)\bar{n}[\bar{p}\bar{\Lambda}]+(nnp)\bar{n}[\bar{n}\bar{\Lambda}]$ $+(\Lambda pp)\bar{\Lambda}[\bar{p}\bar{\Lambda}]+(\Lambda np)\bar{\Lambda}[\bar{n}\bar{\Lambda}]$
6) $T^{\lambda[\mu\nu]}_{\alpha[\beta\gamma]}$	$[210][0\bar{1}\bar{2}]$	$[20\bar{2}]_5$	27	$p[np]\bar{\Lambda}[\bar{n}\bar{\Lambda}]$
		$[2\bar{1}\bar{1}]_2$	10	$p[np]\bar{n}[\bar{n}\bar{\Lambda}]-p[\Lambda p]\bar{\Lambda}[\bar{\Lambda}\bar{n}]$
		$[11\bar{2}]_4$	10	$p[pn]\bar{\Lambda}[\bar{p}\bar{\Lambda}]-n[np]\bar{\Lambda}[\bar{n}\bar{\Lambda}]$
		$[10\bar{1}]_5$	8	$p[np]\bar{p}[\bar{n}\bar{\Lambda}]+n[np]\bar{n}[\bar{n}\bar{\Lambda}]$ $+\Lambda[np]\bar{\Lambda}[\bar{n}\bar{\Lambda}]$
		$[10\bar{1}]_6$	8	$p[pn]\bar{\Lambda}[\bar{p}\bar{n}]+p[p\Lambda]\bar{\Lambda}[\bar{p}\bar{\Lambda}]$ $+p[n\Lambda]\bar{\Lambda}[\bar{n}\bar{\Lambda}]$
		$[000]_2$	1	$p[pn]\bar{p}[\bar{p}\bar{n}]+p[p\Lambda]\bar{p}[\bar{p}\bar{\Lambda}]$ $+p[n\Lambda]\bar{p}[\bar{n}\bar{\Lambda}]+n[pn]\bar{n}[\bar{p}\bar{n}]$ $+n[p\Lambda]\bar{n}[\bar{p}\bar{\Lambda}]+n[n\Lambda]\bar{n}[\bar{n}\bar{\Lambda}]$ $+\Lambda[pn]\bar{\Lambda}[\bar{p}\bar{n}]+\Lambda[p\Lambda]\bar{\Lambda}[\bar{p}\bar{\Lambda}]$ $+\Lambda[n\Lambda]\bar{\Lambda}[\bar{n}\bar{\Lambda}]$

Tensor	Corr-esponding product	Decomposition		Vectors corresponding to the largest S and the largest I_3 for that value of S.
		irr. rep.	dimension	
7) $T^{[\lambda\mu]\nu}_{\alpha[\beta\gamma]}$	$[210][0\bar{1}\bar{2}]$	$[20\bar{2}]_6$	27	Obtained from (6) by the operation $A[BC] \to [BC]A$
		$[2\bar{1}\bar{1}]_3$	10	
		$[11\bar{2}]_5$	10	
		$[10\bar{1}]_7$	8	
		$[00\bar{1}]_8$	8	
		$[000]_3$	1	
8) $T^{[\lambda\mu\nu]}_{\alpha[\beta\gamma]}$	$[111][0\bar{1}\bar{2}]$	$[10\bar{1}]_9$	8	$[pn\Lambda]\,\bar{\Lambda}[\bar{n}\,\bar{\Lambda}]$
9) $T^{(\lambda\mu\nu)}_{[\alpha\beta]\gamma}$	$[300][0\bar{1}\bar{2}]$	$[3\bar{1}\bar{2}]_2$	35	Obtained from (5) by the operation $\bar{A}[\bar{B}\bar{C}] \to [\bar{B}\bar{C}]\bar{A}$
		$[20\bar{2}]_7$	27	
		$[2\bar{1}\bar{1}]_4$	10	
		$[10\bar{1}]_{10}$	8	
10) $T^{\lambda[\mu\nu]}_{[\alpha\beta]\gamma}$	$[210][0\bar{1}\bar{2}]$	$[20\bar{2}]_8$	27	Obtained from (6) by the operation $\bar{A}[\bar{B}\bar{C}] \to [\bar{B}\bar{C}]\bar{A}$
		$[2\bar{1}\bar{1}]_5$	10	
		$[11\bar{2}]_6$	10	
		$[10\bar{1}]_{11}$	8	
		$[10\bar{1}]_{12}$	8	
		$[000]_4$	1	

Tensor	Corr-esponding product	Decomposition		Vectors corresponding to the largest S and the largest I_3 for that value of S.
		irr. rep.	dimen-sion	
11) $T^{[\lambda\mu]\nu}_{[\alpha\beta]\gamma}$	$[210][0\bar{1}\bar{2}]$	$[20\bar{2}]_9$	27	Obtained from (7) by the operation $\bar{A}[\bar{B}\bar{C}] \to [\bar{B}\bar{C}]\bar{A}$
		$[2\bar{1}\bar{1}]_6$	10	
		$[11\bar{2}]_7$	10	
		$[10\bar{1}]_{13}$	8	
		$[10\bar{1}]_{14}$	8	
		$[000]_5$	1	
12) $T^{[\lambda\mu\nu]}_{[\alpha\beta]\gamma}$	$[111][0\bar{1}\bar{2}]$	$[10\bar{1}]_{15}$	8	$[pn\Lambda]\,[\bar{n}\,\bar{\Lambda}]$
13) $T^{(\lambda\mu\nu)}_{[\alpha\beta\gamma]}$	$[300][\bar{1}\bar{1}\bar{1}]$	$[2\bar{1}\bar{1}]_7$	10	$(ppp)\,[\bar{p}\bar{n}\,\bar{\Lambda}]$
14) $T^{\lambda[\mu\nu]}_{[\alpha\beta\gamma]}$	$[210][\bar{1}\bar{1}\bar{1}]$	$[10\bar{1}]_{16}$	8	$p[pn][\bar{p}\bar{n}\,\bar{\Lambda}]$
15) $T^{[\lambda\mu]\nu}_{[\alpha\beta\gamma]}$	$[210][\bar{1}\bar{1}\bar{1}]$	$[10\bar{1}]_{17}$	8	$[pn]p[\bar{p}\bar{n}\,\bar{\Lambda}]$
16) $T^{[\lambda\mu\nu]}_{[\alpha\beta\gamma]}$	$[111][\bar{1}\bar{1}\bar{1}]$	$[000]_6$	1	$[pn\,\Lambda][\bar{p}\bar{n}\,\bar{\Lambda}]$

We have to obtain the results for the product of 3 octets. This means that the tensors T^{λ}_{α}, T^{μ}_{β}, T^{γ}_{ν} we started with are in fact traceless, i.e.

$$T^{\lambda}_{\lambda} = T^{\mu}_{\mu} = T^{\nu}_{\nu} = 0$$

These conditions give rise to linear relations in the vectors belonging to the irreducible spaces corresponding to the same Young's tableau (equivalent representations). Application of these equations reduce the linearly independent ones to precisely the ones required by the decomposition.

$$[10\bar{1}] \otimes [10\bar{1}] \otimes [10\bar{1}]$$

$$= [20\bar{2}] \oplus [2\bar{1}\bar{1}] \oplus [11\bar{2}] \oplus [10\bar{1}]_s \oplus [10\bar{1}]_a \oplus [000]$$

The linear combinations that we selected are given in table VII.

Table VII.

1) $[20\bar{2}] \times [10\bar{1}] = [30\bar{3}] + [3\bar{1}\bar{2}] + [21\bar{3}] + [20\bar{2}]' + [20\bar{2}]'' + [2\bar{1}\bar{1}] + [11\bar{2}] + [10\bar{1}]$

$[3\bar{1}\bar{2}] = 2[3\bar{1}\bar{2}]_1 + [3\bar{1}\bar{2}]_2$

$[21\bar{3}] = -2[21\bar{3}]_1 - [21\bar{3}]_2$

$[20\bar{2}]' = [20\bar{2}]_3 - 2[20\bar{2}]_4 + 2[20\bar{2}]_5 + 3[20\bar{2}]_6 + 3[20\bar{2}]_8 + [20\bar{2}]_9$

$[20\bar{2}]'' = 10[20\bar{2}]_5 + 5[20\bar{2}]_6 + 5[20\bar{2}]_8 + 3[20\bar{2}]_9$

$[2\bar{1}\bar{1}] = 5[2\bar{1}\bar{1}]_1 + \frac{5}{2}[2\bar{1}\bar{1}]_2 - 6[2\bar{1}\bar{1}]_6 + 5[2\bar{1}\bar{1}]_7$

$[11\bar{2}] = -5[11\bar{2}]_1 - \frac{5}{2}[11\bar{2}]_2 - 5[11\bar{2}]_3 + 6[11\bar{2}]_7$

$[10\bar{1}] = 5[10\bar{1}]_1 + 33[10\bar{1}]_{14} - 12[10\bar{1}]_{15} - 12[10\bar{1}]_{17}$

2) $[2\bar{1}\bar{1}] \times [10\bar{1}] = [3\bar{1}\bar{2}] + [20\bar{2}] + [2\bar{1}\bar{1}] + [10\bar{1}]$

$[3\bar{1}\bar{2}] = [3\bar{1}\bar{2}]_2$

$[20\bar{2}] = \frac{1}{2}[20\bar{2}]_3 + [20\bar{2}]_6 + 9[20\bar{2}]_8 + 5[20\bar{2}]_9$

$[2\bar{1}\bar{1}] = \frac{1}{2}[2\bar{1}\bar{1}]_4 + \frac{2}{3}[2\bar{1}\bar{1}]_7$

$[10\bar{1}] = -[10\bar{1}]_{10} + 6[10\bar{1}]_{11} + 3[10\bar{1}]_{15} - 4[10\bar{1}]_{16} - 2[10\bar{1}]_{17}$

3) $[11\bar{2}] \times [10\bar{1}] = [21\bar{3}] + [20\bar{2}] + [11\bar{2}] + [10\bar{1}]$

$[21\bar{3}] = -[21\bar{3}]_2$

$[20\bar{2}] = -\frac{1}{2}[20\bar{2}]_3 + 8[20\bar{2}]_6 + 4[20\bar{2}]_9$

$[11\bar{2}] = -\frac{1}{2}[11\bar{2}]_2 - \frac{2}{3}[11\bar{2}]_3$

$[10\bar{1}] = 12[10\bar{1}]_7 - [10\bar{1}]_{10} + 6[10\bar{1}]_{11} + 3[10\bar{1}]_{15}$
$\quad + 8[10\bar{1}]_{16} + 10[10\bar{1}]_{17}$

4) $[10\bar{1}]_s \times [10\bar{1}] = [20\bar{2}] + [2\bar{1}\bar{1}] + [11\bar{2}] + [10\bar{1}]_{ss} +$
$\quad + [10\bar{1}]_{sa} + [000]$

$[20\bar{2}] = [20\bar{2}]_9$

$[2\bar{1}\bar{1}] = -[2\bar{1}\bar{1}]_6$

$[11\bar{2}] = +[11\bar{2}]_7$

$[10\bar{1}]_{ss} = -2[10\bar{1}]_{14} + 3[10\bar{1}]_{15} + 3[10\bar{1}]_{17}$

$[10\bar{1}]_{sa} = -[10\bar{1}]_{15} + [10\bar{1}]_{17}$

$[000] = [000]_6$

5) $[10\bar{1}]_a \times [10\bar{1}] = [20\bar{2}] + [2\bar{1}\bar{1}] + [11\bar{2}] + [10\bar{1}]_{as}$
$\quad + [10\bar{1}]_{aa} + [000]$

$[20\bar{2}] = -[20\bar{2}]_3 - 2[20\bar{2}]_6 - [20\bar{2}]_9$

$[2\bar{1}\bar{1}] = \frac{1}{2}[2\bar{1}\bar{1}]_4 + \frac{1}{6}[2\bar{1}\bar{1}]_7$

$[11\bar{2}] = -\frac{1}{2}[11\bar{2}]_2 - \frac{1}{6}[11\bar{2}]_3$

$[10\bar{1}]_{as} = 2[10\bar{1}]_7 + 2[10\bar{1}]_8 - 2[10\bar{1}]_{11} - 2[10\bar{1}]_{12}$
$\quad + [10\bar{1}]_3 - [10\bar{1}]_{10}$

$[10\bar{1}]_{aa} = -[10\bar{1}]_3 + 2[10\bar{1}]_9 + [10\bar{1}]_{15}$

$[000] = [000]_3 - [000]_4$

6) $[000] \times [10\bar{1}] = [10\bar{1}]$

$[10\bar{1}] = [10\bar{1}]_{14}$

The linear combinations appear fairly complicated. However, they have been obtained by very simple considerations as orthogonality etc.

In table VII we give the reduction coefficients in the form of isoscalar factors defined earlier. Tables are only given for the +ve strangenesses as the ones for the -ve strangenesses can be inferred by the application of the R-operation defined earlier. It introduces some further changes in the "isoscalar factors":

$$(\pi\pi)_{2(0)} \rightarrow (\pi\pi)_{2(0)} \qquad (1,0,-1)_S \rightarrow (1,0,-1)_S$$

$$(\pi\pi)_1 \rightarrow -(\pi\pi)_1 \qquad (1,0,-1)_A \rightarrow -(1,0,-1)_A$$

$$(\pi K)_{3/2} \rightarrow (\pi\bar{K})_{3/2} \qquad (2,0,-2)' \rightarrow (2,0,-2)'$$

$$(\pi K)_{\frac{1}{2}} \rightarrow -(\pi\bar{K})_{\frac{1}{2}} \qquad (2,0,-2)'' \rightarrow -(2,0,-2)''$$

$$(2,-1,-1) \rightarrow (1,1,-2)$$

$$(3,-1,-2) \rightarrow (2,1,-3)$$

In the product $27 \otimes 8$, two 27 dimensional representations occur. These equivalent ones can again be distinguished by means of this R-operation, as is clear from the different associated phases.

Table VIII.a. Isoscalar Factors.

$$[10\bar{1}] \otimes [10\bar{1}] = [20\bar{2}] \oplus [2\bar{1}\bar{1}] \oplus [11\bar{2}] \oplus [10\bar{1}]_s \oplus [10\bar{1}]_a \oplus [000]$$

$$[11] \otimes [11] = [22] \oplus [30] \oplus [03] \oplus [11]_s \oplus [11]_a \oplus [00]$$

S = 2 (Y = 2)

	$\frac{1}{2}1;\frac{1}{2}1$	
	I = 1	I = 0
$[11\bar{2}]$, [03]	0	−1
$[20\bar{2}]$, [22]	1	0

S = 1 (Y = 1)

	$10;\frac{1}{2}1$		$\frac{1}{2}1;10$		$\frac{1}{2}1;00$	$00;\frac{1}{2}1$
	I=3/2	I=$\frac{1}{2}$	I=3/2	I=$\frac{1}{2}$	I=$\frac{1}{2}$	I=$\frac{1}{2}$
$[10\bar{1}]_s$, $[11]_s$		$\frac{\sqrt{3}}{\sqrt{20}}$		$-\frac{\sqrt{9}}{\sqrt{20}}$	$-\frac{1}{\sqrt{20}}$	$-\frac{1}{\sqrt{20}}$
$[10\bar{1}]_a$, $[11]_a$		$-\frac{1}{2}$		$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$[11\bar{2}]$, [03]		$-\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$[21\bar{1}]$, [30]	$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$			
$[20\bar{2}]$, [22]	$\sqrt{\frac{1}{2}}$	$\frac{1}{\sqrt{20}}$	$\sqrt{\frac{1}{2}}$	$-\frac{1}{\sqrt{20}}$	$\frac{\sqrt{9}}{\sqrt{20}}$	$\frac{\sqrt{9}}{\sqrt{20}}$

$S = 0,\ Y = 0$

	10;10		10;00	00;10	$\frac{1}{2}1;\frac{1}{2}-1$	$\frac{1}{2}-1;\frac{1}{2}1$
	I=2	I=1	I=1	I=1	I=1	I=1
$[000],\quad [00]$						
$[10\bar{1}]_s,\quad [11]_s$		–	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	$-\frac{\sqrt{3}}{\sqrt{10}}$	$-\frac{\sqrt{3}}{\sqrt{10}}$
$[10\bar{1}]_a,\quad [11]_a$		$-\frac{\sqrt{2}}{\sqrt{3}}$			$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$
$[11\bar{2}],\quad [03]$		$-\frac{1}{\sqrt{6}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$
$[2\bar{1}\bar{1}],\quad [30]$		$\frac{1}{\sqrt{6}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$
$[20\bar{2}],\quad [22]$	1		$\frac{\sqrt{3}}{\sqrt{10}}$	$\frac{\sqrt{3}}{\sqrt{10}}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$

	10;10	$\frac{1}{2}1;\frac{1}{2}-1$	$\frac{1}{2}-1;\frac{1}{2}1$	00;00
	I = 0	I = 0	I = 0	I = 0
$[000],\quad [00]$	$\frac{\sqrt{3}}{\sqrt{8}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{8}}$
$[10\bar{1}]s,\quad [11]_s$	$-\frac{\sqrt{3}}{\sqrt{5}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{5}}$
$[10\bar{1}]_a,\quad [11]_a$		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
$[11\bar{2}],\quad [03]$				
$[2\bar{1}\bar{1}],\quad [30]$				
$[20\bar{2}],\quad [22]$	$-\frac{1}{\sqrt{40}}$	$-\frac{\sqrt{3}}{\sqrt{20}}$	$\frac{\sqrt{3}}{\sqrt{20}}$	$\frac{\sqrt{27}}{\sqrt{40}}$

$$([10\bar{1}]I',S';[10\bar{1}]I'',S''|[\lambda_1,\lambda_2,\lambda_3]I,S)$$

$$= ([11]I',Y';[11]I'',Y''|[\mu_1,\mu_2]IY).$$

Table VIII.b. Isoscalar Factors.

$$[11\bar{2}] \otimes [10\bar{1}] = [21\bar{3}] \oplus [20\bar{2}] \oplus [11\bar{2}] \oplus [10\bar{1}]$$

$$[03] \otimes [11] = [14] \oplus [22] \oplus [03] \oplus [11]$$

$S = 3$ $(Y = 3)$

	$02;\frac{1}{2}1$
	$I = \frac{1}{2}$
$[21\bar{3}]$, $[14]$	-1

$S = 2$ $(Y = 2)$

	$\frac{1}{2}1;\frac{1}{2}1$		$02;10$	$02;00$
	$I=1$	$I=0$	$I=1$	$I=0$
$[11\bar{2}]$, $[03]$		$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$
$[20\bar{2}]$, $[22]$	$-\frac{1}{2}$		$\sqrt{\frac{3}{4}}$	
$[213]$, $[14]$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

S = 1 (Y = 1)

	10;½1		½1;10		½1;00	02;½−1
	I = $^3/_2$	I = ½	I = $^3/_2$	I = ½	I = ½	I = ½
[10$\bar{1}$], [11]		$\frac{1}{\sqrt{5}}$		$-\frac{1}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$
[11$\bar{2}$], [03]		$-\sqrt{\frac{1}{2}}$		$-\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$
[20$\bar{2}$], [22]	$-\sqrt{\frac{1}{2}}$	$-\frac{1}{\sqrt{20}}$	$\sqrt{\frac{1}{2}}$	$\frac{\sqrt{49}}{\sqrt{80}}$	$-\frac{\sqrt{9}}{\sqrt{80}}$	$\frac{3}{2}\frac{1}{\sqrt{10}}$
[21$\bar{3}$], [14]	$-\sqrt{\frac{1}{2}}$	$-\frac{1}{2}$	$-\sqrt{\frac{1}{2}}$	$-\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{1}{\sqrt{8}}$

S = 0 (Y = 0)

	$-\frac{3}{2}$−1;½1		10;1(			10;00	½1;½−1	
	I=2	I=1	I=2	I=1	I=0	I=1	I=1	I=0
[10$\bar{1}$], [11]		$\frac{\sqrt{8}}{\sqrt{15}}$		$-\frac{\sqrt{2}}{\sqrt{15}}$	$-\frac{\sqrt{[illegible]}}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	$\frac{\sqrt{2}}{\sqrt{15}}$	$\frac{\sqrt{2}}{\sqrt{5}}$
[11$\bar{2}$], [03]		$-\frac{1}{\sqrt{3}}$	− [illegible]	$-\frac{1}{\sqrt{3}}$			$\frac{1}{\sqrt{3}}$	
[20$\bar{2}$], [22]	$-\sqrt{\frac{3}{4}}$	$-\frac{1}{\sqrt{20}}$	$\frac{1}{2}$	$\frac{\sqrt{9}}{\sqrt{20}}$	$\frac{\sqrt{2}}{\sqrt{5}}$	$-\frac{\sqrt{3}}{\sqrt{10}}$	$\frac{1}{\sqrt{5}}$	$\frac{\sqrt{3}}{\sqrt{5}}$
[21$\bar{3}$], [14]	$\frac{1}{2}$	$-\frac{1}{\sqrt{12}}$	$\sqrt{\frac{3}{4}}$	$\frac{1}{\sqrt{12}}$		$\sqrt{\frac{1}{2}}$	$\frac{1}{\sqrt{3}}$	

$$([11\bar{2}]I',S';[10\bar{1}]\,I'',S''|[\lambda_1,\lambda_2,\lambda_3]IS)$$
$$= ([03]I',Y';[11]\,I'',Y''|[\mu_1,\mu_2]IY)$$

Table VIII.c.

$$[2\bar{1}\bar{1}] \otimes [10\bar{1}] = [3\bar{1}\bar{2}] \oplus [20\bar{2}] \oplus [2\bar{1}\bar{1}] \oplus [10\bar{1}]$$

$$[30] \otimes [11] = [41] \oplus [22] \oplus [30] \oplus [11]$$

	$\frac{3}{2}1;\frac{1}{2}1$	
	I = 2	I = 1
$[20\bar{2}]$, [22]	0	−1
$[31\bar{2}]$, [41]	1	0

S = 1 (Y = 1)

	$\frac{3}{2}1;10$			$\frac{3}{2}1;00$	$10;\frac{1}{2}1$	
	$I=\frac{5}{2}$	$I=\frac{3}{2}$	$I=\frac{1}{2}$	$I=\frac{3}{2}$	$I=\frac{3}{2}$	$I=\frac{1}{2}$
$[10\bar{1}]$, [11]			$-\frac{\sqrt{4}}{\sqrt{5}}$			$-\frac{1}{\sqrt{5}}$
$[2\bar{1}\bar{1}]$, [30]		$\frac{\sqrt{5}}{\sqrt{8}}$		$-\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	
$[20\bar{2}]$, [22]		$-\frac{\sqrt{5}}{\sqrt{16}}$	$\frac{1}{\sqrt{5}}$	$-\frac{3}{4}$	$\frac{1}{\sqrt{8}}$	$-\frac{\sqrt{4}}{\sqrt{5}}$
$[3\bar{1}\bar{2}]$, [41]	1	$-\frac{1}{4}$		$\frac{\sqrt{5}}{\sqrt{16}}$	$\frac{\sqrt{5}}{\sqrt{8}}$	

S = 0 (Y = 0)

	$\frac{3}{2}1;\frac{1}{2}-1$		$10;10$			$10;00$	$\frac{1}{2}-1;\frac{1}{2}1$	
	I=2	I=1	I=2	I=1	I=0	I=1	I=1	I=0
$[10\bar{1}]$, $[11]$		$-\frac{\sqrt{8}}{\sqrt{15}}$		$\frac{\sqrt{2}}{\sqrt{15}}$	$-\frac{\sqrt{3}}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	$\frac{\sqrt{2}}{\sqrt{15}}$	$-\frac{\sqrt{2}}{\sqrt{5}}$
$[2\bar{1}\bar{1}]$, $[30]$		$\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$			$\frac{1}{\sqrt{3}}$	
$[20\bar{2}]$, $[22]$	$-\frac{\sqrt{3}}{\sqrt{4}}$	$\frac{1}{\sqrt{20}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{\sqrt{20}}$	$\frac{\sqrt{2}}{\sqrt{5}}$	$-\frac{\sqrt{3}}{\sqrt{10}}$	$\frac{1}{\sqrt{5}}$	$-\frac{\sqrt{3}}{\sqrt{5}}$
$[3\bar{1}\bar{2}]$, $[41]$	$\frac{1}{2}$	$-\frac{1}{\sqrt{12}}$	$\sqrt{\frac{3}{4}}$	$-\frac{1}{\sqrt{12}}$		$\sqrt{\frac{1}{2}}$	$\frac{1}{\sqrt{3}}$	

$$([2\bar{1}\bar{1}]I',S';\,[10\bar{1}]I'',S''|[\lambda_1,\lambda_2,\lambda_3]IS)$$
$$= ([30]I',Y';\,[11]I'',Y''|[\mu_1,\mu_2]IY)$$

Table VIII.d.

$$[20\bar{2}]\otimes[10\bar{1}] = [30\bar{3}]\oplus[3\bar{1}\bar{2}]\oplus[21\bar{3}]\oplus[20\bar{2}]'\oplus[20\bar{2}]''\oplus[21\bar{1}]\oplus[11\bar{2}]\oplus[10\bar{1}]$$

$$[22]\otimes[11] = [33]\oplus[41]\oplus[12]\oplus[22]'\oplus[22]''\oplus[31]\oplus[13]\oplus[11]$$

S = 3 (Y = 3)

	$12;\frac{1}{2}1$	
	$I=\frac{3}{2}$	$I=\frac{1}{2}$
$[21\bar{3}]$, 14	0	-1
$[30\bar{3}]$, 33	1	0

S = 2 (Y = 2)

	$\frac{3}{2}1;\frac{1}{2}1$		$12;10$			$12;00$	$\frac{1}{2}1;\frac{1}{2}1$	
	I=2	I=1	I=2	I=1	I=0	I=1	I=1	I=0
$[11\bar{2}]$, $[03]$					$-\frac{\sqrt{5}}{\sqrt{6}}$			$-\frac{1}{\sqrt{6}}$
$[20\bar{2}]'$, $[22]'$		$\frac{1}{\sqrt{6}}$		$\frac{1}{2}$		$-\frac{\sqrt{3}}{\sqrt{8}}$	$\frac{\sqrt{5}}{\sqrt{24}}$	
$[20\bar{2}]''$, $[22]''$		$\frac{\sqrt{5}}{\sqrt{14}}$		$-\frac{\sqrt{15}}{\sqrt{28}}$		$-\frac{\sqrt{5}}{\sqrt{56}}$	$-\frac{\sqrt{1}}{\sqrt{56}}$	
$[3\bar{1}\bar{2}]$, $[41]$	$-\frac{1}{\sqrt{3}}$		$\frac{\sqrt{2}}{\sqrt{3}}$					
$[21\bar{3}]$, $[14]$		$-\frac{2}{3}$		$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{2}$	$\frac{\sqrt{5}}{\sqrt{36}}$	$-\frac{\sqrt{5}}{\sqrt{6}}$
$[30\bar{3}]$, $[33]$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{63}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{21}}$		$\frac{\sqrt{2}}{\sqrt{7}}$	$\frac{\sqrt{40}}{\sqrt{63}}$	

The above reduction finally leads to the isoscalar factors for the products $10\otimes 8$, $\overline{10}\otimes 8$, $27\otimes 8$. We have already computed these for $8\otimes 8$. We might like to obtain the same for $10\otimes 10$, $10\otimes\overline{10}$ and $\overline{10}\otimes\overline{10}$. With this end in view, we try to reverse the problem and see whether we could get the isoscalar factors directly from the basis of the two representations. We already know how to compute the matrix elements of the operators $A^3{}_1$, $A^3{}_2$, I_- for any given representation. Thus if we can determine the highest state vectors for the representations occurring in the reduction of the direct product, we shall be able, by means of these operators, to complete the reduction. The quantum numbers i, i_3, s of these highest eigenstates can be determined from the branching law. We order these sets of states corresponding to inequivalent representations by the definition

$$|i,s> \; > |i',s'>$$

if either

$$s > s'$$

or if $s = s'$,

$$i > i'.$$

This definition is possible as two representations in the direct product having the same I,S eigenvalues for their highest states are necessarily equivalent. For let us suppose that (f_1, f_2, f_3), (f_1', f_2', f_3') are two representations occurring in the direct product which have the same (i,s) values for their highest states. Then from equations (3.21)

$$f_1 - f_2 = f_1' - f_2'$$

$$- f_3 = - f_3'$$

Again these occur in the reduction of a direct product, we therefore have

$$f_1 + f_2 + f_3 = f_1' + f_2' + f_3'$$

$$\text{i.e.} \quad f_i = f_i'$$

Now the highest eigenstate in the first representation in the above ordering is just the direct product of the highest states in the basis of the two representations. This representation, which fortunately has multiplicity one, can now be completely determined by means of the operators $A^3{}_1$, $A^3{}_2$, I_-. The next set of equivalent representations will have their highest states having either the same or different quantum numbers than of any state occuring in the first one. In the second case, we have no orthogonality to worry about and we can start directly from suitable total I,S eigenstates. In the second case, however, we have to consider only those linear combinations which are orthogonal to the state in the first representation having the same sub-quantum numbers as the highest states under consideration. For the set of the equivalent representations, we have no method of choice (within the group) and might be required to determine some other criterion to choose suitable sets of linear combinations. After completing this set of states, we can proceed further in an obvious manner. In the above procedure we have implicitly assumed that the number of representations in the decomposition, each of which contains isomultiplets with the same I,S eigenvalues, is just the same as the number of different ways of constructing these eigenvalues. This result is obviously true as is clear from dimensional considerations and is even independent of the particular group under consideration.

vi. Application of the Clebsch-Gordan Coefficients.

1. Calculation of the Scattering Amplitudes.

This is a familar exercise in the context of SU(2) where we express the scattering amplitudes as a sum of a number of invariant ones by applying charge independence. By means of this higher symmetry, believing that it is exact, we can calculate a large number of scattering amplitudes in terms of a few describing various symmetry channels. By way of technique, there is nothing new to be said. One minor point, however, is worth mentioning. In the decomposition of a direct product, some representations have multiplicities higher than one. From the point of view of the group, there is no method of distinguishing between them. So the symmetry allows transitions between equivalent representations in contrast to the case of inequivalent representations where the existence of the symmetry demands that no transition takes place between them.

Let us, for example, consider $M + M \rightarrow M + M$, scattering of two pseudoscalar mesons. We assume that the bosons form an octet as in both the Sakata and the Gell-Mann-Neeman models. Now the decomposition

$$8 \otimes 8$$

leads to $$1 \oplus 8_S \oplus 8_A \oplus 10 \oplus \overline{10} \oplus 27$$

Thus there will be six diagonal invariant amplitudes, namely the ones for

$$1 \rightarrow 1$$
$$8_s \rightarrow 8_s$$
$$8_a \rightarrow 8_a$$
$$10 \rightarrow 10$$
$$\overline{10} \rightarrow \overline{10}$$
$$27 \rightarrow 27$$

and two off-diagonal ones* i.e.

$$8_a \rightarrow 8_s$$
$$8_s \rightarrow 8_a$$

In strong-interaction, time-reversal invariance holds. Thus the two off-diagonal ones are actually equal. This reduces the total number to seven.

Tables expressing the various scattering amplitudes have already appeared.[30] It can be seen from these that it is impossible to verify the symmetry by looking at the cross-sections. Use of R-invariance, which is not a good symmetry, reduces the number of independent amplitudes to 5. This is still inadequate for our purpose.

Though we could not obtain anything useful from the scattering of pseudo-scalar bosons in the two models, we shall see that the Sakata model makes very definite

* The total number (diagonal + off-diagonal) is equal to $\sum_i n_i^2$ where n_i are the various multiplicities of inequivalent representations. Thus in the above case it is $1 + 2^2 + 1 + 1 + 1 = 8$.

predictions in the case of proton - anti-proton annihilation. This reaction in terms of the representations is now

$$3 \otimes \bar{3} \longrightarrow 8 \otimes 8$$

or $$1 \oplus 8 \longrightarrow 1 \oplus 8_s \oplus 8_a \oplus 10 \oplus \overline{10} \oplus 27$$

In this case there are only three amplitudes, namely the ones for $1 \longrightarrow 1$, $8 \longrightarrow 8_s$, $8 \longrightarrow 8_a$. We tabulate below the various annihilation amplitudes.

Table IX.

Process	A_1	A_8^a	A_8^s
$p+\bar{p} \to \pi^+ + \pi^-$	$\frac{1}{\sqrt{24}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{30}}$
$\pi^- + \pi^+$	$\frac{1}{\sqrt{24}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{30}}$
$\pi^o \pi^o$	$\frac{1}{\sqrt{24}}$	0	$-\frac{1}{\sqrt{30}}$
$\pi^{o\prime} \pi^o$	0	0	$-\frac{1}{\sqrt{10}}$
$\pi^o \pi^{o\prime}$	0	0	$-\frac{1}{\sqrt{10}}$
$K^+ K^-$	$\frac{1}{\sqrt{24}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{30}}$
$K^- K^+$	$\frac{1}{\sqrt{24}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{30}}$
$K^o \bar{K}^o$	$\frac{1}{\sqrt{24}}$	0	$\frac{\sqrt{2}}{\sqrt{15}}$
$\bar{K}^o K^o$	$\frac{1}{\sqrt{24}}$	0	$\frac{\sqrt{2}}{\sqrt{15}}$
$\pi^{o\prime} \pi^{o\prime}$	$\frac{1}{\sqrt{24}}$	0	$\frac{1}{\sqrt{30}}$

In this case, much can be said if we also fix the symmetry of the final 2 boson-state. Believing in the existence of the symmetry means taking the true boson multiplets as just two sets of identical particles, and so the symmetry or anti-symmetry of the spatial part of the wave-function would demand symmetry and anti-symmetry of the unitary spin parts of the wave-function. We also know that from the point of view of the unitary spin group, 1 and 8_S are symmetric and 8_A is anti-symmetric. Thus knowing precisely the spatial symmetry of the wave-function will help a great deal in verifying the Sakata model.

For example $p\bar{p}$ annihilation at rest is believed to proceed via s-wave. This means that the spatial wave-function of the two boson-system is anti-symmetric. Thus we must take only the anti-symmetric amplitude 8_A for the various annihilations. This is extremely fortunate as now the annihilation rates are just proportional to the square of the anti-symmetric amplitudes.

Thus we see that if the Sakata model holds, $p+\bar{p} \rightarrow K^0+\bar{K}^0$, annihilation at rest cannot occur. This result, however, depends on the annihilation occurring via s-wave. Can we improve upon this situation?

We see in table IX that the amplitudes for the reactions

$$p + \bar{p} \rightarrow K^0 + \bar{K}^0$$

and $$p + \bar{p} \rightarrow \bar{K}^0 + K^0$$

are equal. Thus

$$\langle p + \bar{p} \mid K_1^0 + K_2^0 \rangle = \langle p + \bar{p} \mid K^0K^0 - \bar{K}^0\bar{K}^0 \rangle$$
$$- \langle p + \bar{p} \mid K^0\bar{K}^0 - \bar{K}^0K^0 \rangle$$

The first matrix element on the right hand side is zero as strong interactions conserve strangeness. The second one vanishes as the matrix elements

$$\langle p + \bar{p} \mid K^0\bar{K}^0 \rangle, \langle p + \bar{p} \mid \bar{K}^0K^0 \rangle$$

are equal.

Thus the Sakata model completely disallows the existence of this decay mode. This result was first noticed by Lipkin, Levinson, Meshkov, Salam and the author.[31]

Experimentally the annihilation $p + \bar{p} \rightarrow K_1^0 + K_2^0$ occurs with the same abundance as $p + \bar{p} \rightarrow K^+ + K^-$.

This is clearly against the Sakata model. Actually this was the first clear-cut prediction of the model which contributed to its downfall. The Gell-Mann-Neeman model, however, does not make any positive prediction.

2. Existence of the Resonances.

The job of assigning particles to various representations has now become classical. We look for the sub-quantum numbers that a particular representation contains and try to find a set of known particles which have those quantum numbers. The rule for the assignment is that these particles must have the same spatial properties, i.e. spin, parity, baryon number, etc. Again the existence of a symmetry demands perfect equality of masses between the various members of the same super-multiplet, or that they are indistinguishable. However the group has operators which allow one to allocate different symmetry quantum numbers to the various members. These quantum numbers and just sufficient to allow complete identification. Evidently these quantum numbers are different from the extra-symmetry ones which must necessarily be the same.

Let us now go back to the Sakata model. It has as its starting point a 3 dimensional representation of the group U(3) or SU(3) and we assume that the particles p, n and Λ belong to this representation. This is valid as this representation contains an isospinor with zero strangeness and an isosinglet with minus one strangeness. The philosophy now is to build all the states from these particles and their anti-particles. The anti-particles form an inequivalent $\bar{3}$ representation. We have already obtained the decomposition $3 \otimes \bar{3}$ which contains an octet and a singlet. Both these are states with baryon number zero.

For the octet, we have two sets of bosons, namely the set of pseudoscalar bosons consisting of π, K, $\bar{K}$, η and the set of vector bosons ρ, ω, K^*, $\bar{K}^*$. Of course we have one more state $K^*(725)$ with the same quantum numbers as the K^* (888) in addition to another vector meson φ. The Sakata model allows an octet and a singlet of vector mesons only.

We emphasize here that the symmetry requires equality of masses of members of a supermultiplet and π, K, $\bar{K}$, η have not that equality. However, we shall consider it to be a violent breakdown of the symmetry.

Before going on to see what the octet model has to say, let us try to allocate positions to the remaining baryons, i.e. to Σ and Ξ. We have to look for these in the decomposition

$$3 \otimes 8$$

which leads to $15 \oplus \bar{6} \oplus 3$,

corresponding to

$$(1,0,0) \otimes (1,0,-1) = (2,0,-1) \oplus (1,1,-1) \oplus (1,0,0)$$

The new representations that appear here are (2,0,-1) and (1,1,-1) of dimensions 15 and $\bar{6}$ respectively. Both $\bar{6}$ and 15 have places for Σ but only 15 has a place for Ξ. These masses are nearly equal, so we may tentatively like to put them in the same 15-dimensional multiplet. This contains a non-strange quartet also which we can take as the 3-3 resonance. This then requires the spins of both Σ and Ξ to be $\frac{3}{2}$. We do not know anything so far about the spin of Ξ. However both these are probably $\frac{1}{2}$ and the choice might not be reasonable. Again putting them in any one of the places requires the existence of a strangeness + 1 baryon while KN system does not resonate. This may be taken as another argument against the model. Finally Λ and Σ have very nearly the same mass. Putting them in different super-multiplets must have some strong reason behind it. These spins are equal and masses nearly equal. If the relative ΛΣ parity came out to be odd, the two were to be placed in different supermultiplets. However, recent experiments strongly favour even ΣΛ relative purity. This therefore is against the existence of a symmetry as demanded by the Sakata model.

In the octet model, the group that is considered is $SU(3)/C_3$ and the lowest non-trivial representation is the octet representation. We have seen that the pseudo-scalar mesons and a set of vector bosons do have exactly the same quantum numbers as required by this representation. In the case of bosons, the nucleon number is zero, so there is no distinction between strangenesses and hypercharge. Also we know that hypercharge is related to an operation (reflection) in the isotopic spin space, while strangess is not. If we look at the eight baryons we see that the hyper-charge and the isotopic spin values are again the same as for the octet representation. This shows that there is a possibility of taking the baryons as also forming an octet. This is very nice as all these have very nearly the same mass and the only relevant parity, namely the $(\Sigma\Lambda)$ parity, appears now experimentally to be even. The spins of N, Σ, Λ are known to be $\frac{1}{2}$. If the Ξ spin also comes out to be $\frac{1}{2}$, this will strongly favour the unitary symmetry in its octet version.

The other argument which we gave against the Sakata model, namely the existence of the annihilation

$$p + \bar{p} \rightarrow K_1^{*0} + K_2^0$$

does not go against this model as now the amplitudes for the reactions

$$\begin{aligned} p + \bar{p} &\rightarrow K^0 + \bar{K}^0 \\ &\rightarrow \bar{K}^0 + K^0 \end{aligned}$$

are unequal. Even if we consider only the assymetric amplitudes, it is clear that the annihilation can still occur on account of the existence of the off-diagonal matrix elements (the symmetric ones are equal).

We have seen that the vector bosons fit quite nicely into an octet and a singlet except for another K*. We can start looking at the boson-baryon resonances. These are to be in the representations that occur in the direct product of 8 and 8. This direct product contains 1,8,8, 10,$\overline{10}$,27. The places for the $\frac{3}{2},\frac{3}{2}$ resonance amongst these could be found in the 10 and 27 dimensional representations. Let us first of all look at the 10-dimensional one. Relative to SU(2) it decomposes into a Y = 1 quartet, a Y = 0 triplet, a Y = -1 doublet and a Y = -2 singlet. All these states must have spin $\frac{3}{2}$ and even parity as the 3-3 resonance at 1240 has these quantum numbers. We have also got two hyperon resonances namely the $Y_1^*(1385)$ and the $\Xi^*_1(1535)$. These are isotopic triplet and doublet respectively with correct value of the hypercharges and spin parity $(\frac{3}{2}^+)$ so as to be accommodated along with the N_1^* in the 10-dimensional multiplet.

To complete the multiplet we require the existence of another resonance, the so-called Ω. The predicted mass of it on the basis of the lowest order symmetry-breaking (see next chapter) should be around (1620). If a resonance is found at this mass value, this will not only enhance our belief in the symmetry, but also in the validity of the lowest order calculations to give fairly reliable estimates.

3. Interactions.

We shall only try to write Yukawa type couplings, i.e. the ones which involve 3 particles. In particular we obtain the technique for writing $\bar{B}BM$ vertex where B and M stand for a baryon (with baryonic number not necessarily equal to one) and a meson respectively. B and $\bar{B}$ belong to contragradient representations. The representations to which B and M belong may or may not be the same. The general Yukawa type coupling is of the form $M_1M_2M_3$ where these three M's belong to possibly different representations.

Firstly we mention a few general remarks. Baryonic number conservation definitely holds and the octet model has no place for it. Thus this is to be assured at the beginning. This can be done by hand by just counting the total baryonic number which must come out to be zero in order to conserve the baryonic number. Similarly the spin and parity considerations which belong to the spatial group will be assumed to have been taken care of. We shall even omit these factors which in spinor cases are matrices. In short, we shall confine our attention to only the symmetry-dependent part of the interactions.

To demonstrate the procedure we go back to the now classic $\bar{N}N\pi$ Yukawa-type coupling in relation to the isotopic spin symmetry. As the pion is supposed to be a vector in the isotopic spin space, we must construct a vector from $\bar{N}$ and N. Such a vector is $\bar{N}\,\underline{\tau}\,N$ where $\underline{\tau}$ are the familiar τ matrices. Thus the interaction will be

$$i\bar{N}\underline{\tau}N \cdot \underline{\pi}$$

where i has been added to make the interaction hermitian.

The property of the above interaction that results in the isotopic spin conservation is that it is a scalar in the isotopic spin space. Again we note that, in this case, there is only one method of constructing a scalar in the same way as there is only one method of constructing a vector from two spinors.

We wish to obtain an expansion of the above expression by a simple procedure. As $\bar{N}\underline{\tau}N$ is a vector, this is the i-spin one that we form from the spinors N and $\bar{N}$, namely

$$\begin{pmatrix} -\bar{n}p \\ \frac{\bar{p}p - \bar{n}n}{\sqrt{2}} \\ \bar{p}n \end{pmatrix}$$

We note that the above has been written in the form of non-hermitian components. In the same sense, the vector $\underline{\pi}$ is

$$\begin{pmatrix} -\pi^+ \\ \pi^o \\ \pi^- \end{pmatrix}$$

and the scalar we obtain will be

$$\bar{p}n\,\pi^+ + \bar{n}p\,\pi^- + \frac{(\bar{p}p - \bar{n}n)}{\sqrt{2}}\,\pi^o$$

In the above we have used the method of constructing a scalar from two vectors.

In the case of SU(3) the procedure is now obvious. For the interaction $M_1M_2M_3$ we have to construct the representation contragradient to M_3 from the direct product of M_1, M_2. This is obvious from the fact that only contragradient representations have a scalar representation in the decomposition of their direct product. (There is no distinction from the point of view of the rotation group between contragradient representations. Thus, in this context the above restriction means that we have to construct the representation to which M_3 belongs out of the direct product of M_1 and M_2). Having obtained the contragradient representation from the direct product of M_1 and M_2, we have simply to determine the scalar that is formed out of this determined set of vectors and the states of M_3. With some phase conventions, this scalar is not just the sum of terms each having the same sign. (The scalar expression obtained in table V is the sum of terms with the same sign only on account of the special choice of phases in table II for the association of particles to the octet representation). For this purpose we need not do the complete reduction to obtain just the scalar. In fact we have only to multiply the states with their charge conjugates (and not R-conjugates). However basis vectors in contragradient representations are related by R-conjugation (up to a common phase). To see where the charge-conjugate state has a phase opposite to that of the R-conjugate, we go back to the basis in terms of the 3-dimensional $\begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$ as for example the ones in table II. The operation of charge-conjugation differs from R-conjugation in the fact that $n \to \bar{n}$ in the former and $n \to -\bar{n}$ in the latter. We show that the number of n's (or p's) in the basis of any representation depend only on y and i_3. Each p or n contributes 1 to y but $\frac{1}{2}$ or $-\frac{1}{2}$ to i_3. Λ contributes nothing to either y or i_3. Thus

$$y = \text{No. of p's} + \text{No. of n's}$$

$$2i_3 = \text{No. of p's} - \text{No. of n's}$$

$$\therefore \text{No. of p's (n's)} = \tfrac{1}{2}y + i_3 \quad (\tfrac{1}{2}y - i_3)$$

Thus the phase associated with the products of vectors in the scalar is just $(-1)^{\frac{1}{2}y-i_3}$. By way of illustration, we quote the scalars in the reduction of the basis of the products

$$8 \otimes 8, \quad 10 \otimes \overline{10}, \quad 27 \otimes 27$$

i) $\underline{8 \otimes 8}$

$$|\tfrac{1}{2},\tfrac{1}{2},1\rangle|\tfrac{1}{2},-\tfrac{1}{2},-1\rangle - |\tfrac{1}{2},-\tfrac{1}{2},1\rangle|\tfrac{1}{2},\tfrac{1}{2},-1\rangle - |1,1,0\rangle|1,-1,0\rangle$$
$$+ |1,0,0\rangle|1,0,0\rangle - |1,-1,0\rangle|1,1,0\rangle + |0,0,0\rangle|0,0,0\rangle$$
$$-|\tfrac{1}{2},\tfrac{1}{2},-1\rangle|\tfrac{1}{2},-\tfrac{1}{2},1\rangle + |\tfrac{1}{2},-\tfrac{1}{2},-1\rangle|\tfrac{1}{2},\tfrac{1}{2},1\rangle$$

ii) $\underline{10 \otimes \overline{10}}$

$$-|\tfrac{3}{2},\tfrac{3}{2},1\rangle|\tfrac{3}{2},-\tfrac{3}{2},-1\rangle + |\tfrac{3}{2},\tfrac{1}{2},1\rangle|\tfrac{3}{2},-\tfrac{1}{2},1\rangle - |\tfrac{3}{2},-\tfrac{1}{2},1\rangle|\tfrac{3}{2},\tfrac{1}{2},-1\rangle$$
$$+|\tfrac{3}{2},-\tfrac{3}{2},1\rangle|\tfrac{3}{2},\tfrac{3}{2},1\rangle - |1,1,0\rangle|1,-1,0\rangle + |1,0,0\rangle|1,0,0\rangle$$
$$-|1,-1,0\rangle|1,1,0\rangle - |\tfrac{1}{2},\tfrac{1}{2},-1\rangle|\tfrac{1}{2},-\tfrac{1}{2},1\rangle + |\tfrac{1}{2},-\tfrac{1}{2},-1\rangle|\tfrac{1}{2},\tfrac{1}{2},1\rangle$$
$$-|0,0,-2\rangle|0,0,2\rangle$$

iii) $\underline{27 \otimes \overline{27}}$

$$
\begin{aligned}
&|1,1,2\rangle|1,-1,-2\rangle - |1,0,2\rangle|1,0,-2\rangle + |1,-1,2\rangle|1,1,-2\rangle \\
&- |\tfrac{3}{2},\tfrac{1}{2},1\rangle|\tfrac{3}{2},-\tfrac{1}{2},-1\rangle - |\tfrac{3}{2},-\tfrac{1}{2},1\rangle|\tfrac{3}{2},\tfrac{1}{2},-1\rangle + |\tfrac{3}{2},-\tfrac{3}{2},1\rangle|\tfrac{3}{2},\tfrac{3}{2},-1\rangle \\
&+ |2,2,0\rangle|2,-2,0\rangle + |2,1,0\rangle|2,-1,0\rangle + |2,0,0\rangle|2,0,0\rangle \\
&- |2,-1,0\rangle|2,1,0\rangle + |2,-2,0\rangle|2,2,0\rangle + |\tfrac{1}{2},\tfrac{1}{2},1\rangle|\tfrac{1}{2},-\tfrac{1}{2},-1\rangle \\
&- |\tfrac{1}{2},-\tfrac{1}{2},1\rangle|\tfrac{1}{2},\tfrac{1}{2},-1\rangle - |1,1,0\rangle|1,-1,0\rangle + |1,0,0\rangle|1,0,0\rangle \\
&- |1,-1,0\rangle|1,1,0\rangle + |\tfrac{3}{2},\tfrac{3}{2},-1\rangle|\tfrac{3}{2},-\tfrac{3}{2},1\rangle - |\tfrac{3}{2},\tfrac{1}{2},-1\rangle|\tfrac{3}{2},-\tfrac{1}{2},1\rangle \\
&+ |\tfrac{3}{2},-\tfrac{1}{2},-1\rangle|\tfrac{3}{2},\tfrac{1}{2},1\rangle - |\tfrac{3}{2},-\tfrac{3}{2},-1\rangle|\tfrac{3}{2},\tfrac{3}{2},1\rangle + |0,0,0\rangle|0,0,0\rangle \\
&- |\tfrac{1}{2},\tfrac{1}{2},-1\rangle|\tfrac{1}{2},-\tfrac{1}{2},1\rangle + |\tfrac{1}{2},-\tfrac{1}{2},-1\rangle|\tfrac{1}{2},\tfrac{1}{2},1\rangle + |1,1,-2\rangle|1,-1,2\rangle \\
&- |1,0,-2\rangle|1,0,2\rangle + |1,-1,-2\rangle|1,1,2\rangle
\end{aligned}
$$

We also know that in the decomposition of the direct product of M_1 and M_2 the representation contragradient to M_3 might occur more than once. In such a case, the symmetry would allow all the interactions that can be obtained this way. In general, from the point of view of the symmetry, the interaction will be an arbitrary linear combination of these.

Let us, for illustration purposes, try to find the Yukawa type interactions when all the 3 M's belong to the 8-dimensional representation. In this case we know that $8 \otimes 8$ contains the octet representation twice (octet representation is self-contragradient). Thus there are two different interactions that we can write, each of which is a scalar from the point of view of the group. We have only to look for the two octets in table V and to form a scalar with the help of the third octet in the manner given again in the same table. The two interactions are given below in terms of the following correspondences for the baryon and the anti-baryon octets:

K^+	p	$\bar{\Xi}^+$
K^0	n	$\bar{\Xi}^0$
$-\pi^+$	$-\Sigma^+$	$-\bar{\Sigma}^+$
π^0	Σ^0	$\bar{\Sigma}^0$
π^-	Σ^-	$\bar{\Sigma}^-$
$-\pi^{0'}$	$-\Lambda$	$-\bar{\Lambda}$
$-\bar{K}^0$	$-\Xi^0$	$-\bar{n}$
K^-	Ξ^-	$\bar{p}$

$\bar{B}BM$ interactions in the Octet Model.

Table X

D type:

$$\frac{\sqrt{3}}{\sqrt{10}}[\bar{\Sigma}^+ n + \bar{\Xi}^0\Sigma^+ + \bar{\Xi}^+(\frac{1}{\sqrt{2}}\Sigma^0 - \frac{1}{\sqrt{6}}\Lambda) + (\frac{1}{\sqrt{2}}\bar{\Sigma}^0 - \frac{1}{\sqrt{6}}\bar{\Lambda})p]K^-$$

$$+\frac{\sqrt{3}}{\sqrt{10}}[-(\frac{1}{\sqrt{2}}\bar{\Sigma}^0 + \frac{1}{\sqrt{6}}\bar{\Lambda})n - \bar{\Xi}^0(\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda) + \bar{\Xi}^+\Sigma^- + \bar{\Sigma}^- p]\bar{K}^0$$

$$+\frac{\sqrt{3}}{\sqrt{10}}[\frac{\sqrt{2}}{\sqrt{3}}\bar{\Sigma}^+\Lambda + \frac{\sqrt{2}}{\sqrt{3}}\bar{\Lambda}\Sigma^+ + \bar{\Xi}^+\Xi^0 + \bar{n}p]\pi^-$$

$$+\frac{\sqrt{3}}{\sqrt{20}}[\frac{2}{\sqrt{3}}\bar{\Sigma}^0\Lambda + \frac{2}{\sqrt{3}}\bar{\Lambda}\Sigma^0 - \bar{\Xi}^0\Xi^0 - \bar{n}n + \bar{\Xi}^+\Xi^- + \bar{p}p]\pi^0$$

$$+\frac{\sqrt{3}}{\sqrt{10}}[\frac{\sqrt{2}}{\sqrt{3}}\bar{\Sigma}^-\Lambda + \frac{\sqrt{2}}{\sqrt{3}}\bar{\Lambda}\Sigma^- + \bar{\Xi}^0\Xi^- + \bar{p}n]\pi^+$$

$$+\frac{1}{2\sqrt{5}}[2\bar{\Sigma}^0\Sigma^0 + 2\bar{\Sigma}^+\Sigma^- + 2\bar{\Sigma}^-\Sigma^+ - 2\bar{\Lambda}\Lambda - \bar{p}p - \bar{\Xi}^+\Xi^- - \bar{\Xi}^0\Xi^0 - \bar{n}n]\eta$$

$$+\frac{\sqrt{3}}{\sqrt{10}}[-(\frac{1}{\sqrt{2}}\bar{\Sigma}^0 + \frac{1}{\sqrt{6}}\bar{\Lambda})\Xi^0 - \bar{n}(\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\bar{\Lambda}) + \bar{\Sigma}^+\Xi^- + \bar{p}\Sigma^+]K^0$$

$$+\frac{\sqrt{3}}{\sqrt{10}}[\bar{\Sigma}^-\Xi^0 + \bar{n}\Sigma^- + (\frac{1}{\sqrt{2}}\bar{\Sigma}^0 + \frac{1}{\sqrt{6}}\bar{\Lambda})\Xi^- + \bar{p}(\frac{1}{\sqrt{2}}\Sigma^0 - \frac{1}{\sqrt{6}}\Lambda)]K^+$$

F type:

$$\frac{1}{\sqrt{6}}\left[-\left(\frac{1}{\sqrt{2}}\bar{\Sigma}^o + \frac{\sqrt{3}}{\sqrt{2}}\bar{\Lambda}\right)p + \bar{\Xi}^+\left(\frac{1}{\sqrt{2}}\Sigma^o + \frac{\sqrt{3}}{\sqrt{2}}\Lambda\right) - \bar{\Sigma}^+ n + \bar{\Xi}^o\Sigma^+\right]K^-$$

$$+ \frac{1}{\sqrt{6}}\left[\left(\frac{1}{\sqrt{2}}\bar{\Sigma}^o - \frac{\sqrt{3}}{\sqrt{2}}\bar{\Lambda}\right)n - \bar{\Xi}^o\left(\frac{1}{\sqrt{2}}\Sigma^o - \frac{\sqrt{3}}{\sqrt{2}}\Lambda\right) - \bar{\Xi}^+\Sigma^- + \bar{\Sigma}^- p\right]\bar{K}^o$$

$$+ \frac{1}{\sqrt{6}}\left(-\sqrt{2}\bar{\Sigma}^o\Sigma^+ + \sqrt{2}\bar{\Sigma}^+\Sigma^o + \bar{n}p - \bar{\Xi}^+\Xi^o\right)\pi^-$$

$$+ \frac{1}{\sqrt{12}}\left(2\bar{\Sigma}^-\Sigma^+ - 2\bar{\Sigma}^+\Sigma^- + \bar{p}p - \bar{\Xi}^+\Xi^- - \bar{n}n + \bar{\Xi}^o\Xi^o\right)\pi^o$$

$$+ \frac{1}{\sqrt{6}}\left(\sqrt{2}\bar{\Sigma}^o\Sigma^- - \sqrt{2}\bar{\Sigma}^-\Sigma^o - \bar{\Xi}^o\Xi^- + \bar{p}n\right)\pi^+$$

$$+ \tfrac{1}{2}\left(\bar{p}p - \bar{\Xi}^+\Xi^- + \bar{n}n - \bar{\Xi}^o\Xi^o\right)\eta$$

$$+ \frac{1}{\sqrt{6}}\left[-\left(\frac{1}{\sqrt{2}}\bar{\Sigma}^o - \frac{\sqrt{3}}{\sqrt{2}}\bar{\Lambda}\right)\Xi^o + \bar{n}\left(\frac{1}{\sqrt{2}}\Sigma^o - \frac{\sqrt{3}}{\sqrt{2}}\Lambda\right) - \bar{p}\Sigma^+ + \bar{\Sigma}^+\Xi^-\right]K^o$$

$$+ \frac{1}{\sqrt{6}}\left[\left(\frac{1}{\sqrt{2}}\bar{\Sigma}^o + \frac{\sqrt{3}}{\sqrt{2}}\bar{\Lambda}\right)\Xi^- + \bar{p}\left(\frac{1}{\sqrt{2}}\Sigma^o + \frac{\sqrt{3}}{\sqrt{2}}\Lambda\right) - \bar{\Sigma}^-\Xi^o + \bar{n}\Sigma^-\right]K^+$$

These interactions were first written by Gell-Mann in his "Eight-fold Way" and given the names D and F types. D type is symmetric and F skew-symmetric in the interchange of any two octets occurring in the interaction.

For this simple case, Salam and Matthews[32] gave a simpler procedure, which corresponds to obtaining a scalar out of three vectors. We know that such a scalar is

$$\underline{a}.(\underline{b} \times \underline{c})$$

Similarly in the product of three 3x3 matrices, we can construct scalars by taking traces. Evidently there are two traces that we can construct, namely

$$\mathrm{tr}(M_1M_2M_3) \text{ and } \mathrm{tr}(M_1M_3M_2)$$

The combinations

$$\mathrm{tr}(M_1M_2M_3 \pm M_1M_3M_2)$$

are just the D and F types considered above.

Thus the symmetry allows an interaction of the type

$$aD + bF$$

i.e. an arbitrary linear combination of the symmetric and the anti-symmetric interactions. In particular, this allows us to write the 12 coupling constants for the $\bar{B}BM$ vertex in terms of just two. In the old days of Global and restricted symmetries, η did not exist, and to try arbitrary choices of equations between the other eight was a job for the theoretician. The final result used to be: "some equal and some vanishing".

The SU(3) symmetry, however, is not that ambitious, but expresses all the twelve in terms of just two parameters which characterize the two types. We give these below in terms of $g_{N\pi\pi} = g_p$ and $\frac{g_{\Xi}}{g_p} = -r$.

Table XI

Yukawa coupling constants in Unitary Symmetry.

$g_{NN\pi} = g_p$

$g_{\Xi\Xi\pi} = - r g_p$

$g_{\Sigma\Lambda\pi} = \frac{1}{\sqrt{3}}(1+r) g_p$

$g_{\Sigma\Sigma\pi} = (1-r) g_p$

$g_{N\Lambda K} = -\frac{1}{\sqrt{3}}(2-r) g_p$

$g_{N\Sigma K} = - r g_p$

$g_{\Xi\Sigma K} = - g_p$

$g_{\Xi\Lambda K} = \frac{1}{\sqrt{3}}(1-2r) g_p$

$g_{NN\eta} = \frac{1}{\sqrt{3}}(1-2r) g_p$

$g_{\Xi\Xi\eta} = -\frac{1}{\sqrt{3}}(2-r) g_p$

$g_{\Lambda\Lambda\eta} = -\frac{1}{\sqrt{3}}(1+r)$

$g_{\Sigma\Sigma\eta} = \frac{1}{\sqrt{3}}(1+r)$

As the vector mesons are also assumed to form an octet, we can write their corresponding interaction in the same manner as above. The question finally arises whether we can write these in terms of just one. This can be dealt with in two manners. Theoretically we would just like to eliminate one on the basis of some other consideration or another postulated invariance. Thus Neeman in his thesis obtained the F type interaction only when he considered gauge transformations which also led him to the existence of eight vector bosons as fundamental intermediating particles. However from gauge invariance we cannot write any Yukawa type Lagragian for ordinary pseudo-scalar mesons. Application of an analogy will rule out D couplings all together. Gell-Mann, on the other hand, tried to introduce R-invariance. As D and F couplings are respectively symmetric and anti-symmetric under the R operator, this invariance would demand the existence of D alone.

The question of whether one or both types exist in nature can also be solved phenonmenonlogically. It appears that experiments cannot be made to agree with only one type of interaction. The situation is even worse. As Gell-Mann pointed out in the "Eightfold Way", the photoproduction data is inconsistent with any value of r. This may be accounted for by a large breakdown of the symmetry in the pseudoscalar boson octet where π and K have a very large mass difference.

Lipkin[33] has shown that the existence of two types of couplings of 3 octets can be considered as a strong point in favour of the SU(3) scheme. In fact, in the case of 3 boson couplings, one of the two types is automatically excluded on account of charge conjugation invariance and the fact that the particles and the anti-particles appear simultaneously in the boson octets. Thus the 3 vector meson or one vector and 2 pseudo-scalar meson vertices must be F type while 3 pseudo-scalar meson or one pseudoscalar meson and two vector meson vertices must necessarily be D type. On the other hand, if we had only one completely symmetric (or anti-symmetric) coupling, we would not have been able to explain the existence of many reactions. Lipkin has further argued that this point goes against G_2, interest in which was again being revived.[34] In this group, only one coupling of 3 septets exists and this happens to be anti-symmetric. G_2 therefore does not allow a coupling of the form MVV or MMM. The same argument applies to the groups C_2 (or B_2)[25] where we consider the mesons to belong to the 10-dimensional representation and in the decomposition

$$10 \otimes 10 = 35 \oplus 35' \oplus 14 \oplus 10 \oplus 5 \oplus 1$$

the 45 asymmetrical components can only be placed in one of the 35 and the 10-dimensional representation.

CHAPTER 3

Symmetry Breaking

i) Symmetry Breaking Interactions.

The most unfortunate aspect of postulating symmetries in elementary particles is the fact that one is expected to think of breaking of the symmetry immediately. This is obvious from the fact that the masses of the particles said to form a supermultiplet are equal only approximately. The breaking of the symmetry then allows the mass degeneracy to be removed and we are expected to obtain the correct mass spectrum from the breaking. Thus if we go back to the history of charge independence, it was supposed that particles with very nearly the same mass form isotopic multiplets and that the correct mass spectrum would be obtained if we did include the electromagnetic effects which do not observe charge independence. In unitary symmetries, likewise, we assume that so far as the very strong interactions are concerned, particles may be grouped into supermultiplets, these particles having nearly the same mass, and that when we consider symmetry breaking we shall again be able to remove the mass degeneracy. Here, however, the situation is slightly different. Neglecting weak interactions altogether, our hierarchy of interactions consists of very strong, strong and electromagnetic interactions. Thus the complete removal of mass degeneracy will now be supposed to take effect in two stages. In the first stage, we turn on the strong interactions. As strong interactions do not have the symmetry of the very strong ones, the supermultiplets will decompose into various isotopic multiplets. In the second stage, when we switch on the electromagnetic interactions, the submultiplets will not be left with any degeneracy. In the above, we were describing only a special case of Pais'[35] hierarchy of interactions. This postulates the existence of a series of interactions with progressively weaker symmetries, i.e. the summetries of an interaction contain the ones for those which are weaker in comparison to it. In the language of group theory, the symmetry groups of the stronger contains the ones for the weaker. In the above context, therefore, we can write the interaction Lagrangian in the form

$$I_{vs} + I_{ms} + I_{em} + \ldots.$$

where I_{vs}, the very strong part, is invariant under the full symmetry group (U(3) or SU(3) in the unitary symmetry models), the medium strong I_{ms} under a subgroup of the full group which in turn includes the subgroup that leaves the electromagnetic interaction I_{em} invariant.

In this chapter we shall be concerned only with the first stage of symmetry breaking, i.e. the breaking of a unitary super-multiplet into isotopic multiplets as a result of the switching on of the strong interaction. If no restriction is imposed on the form of this interaction, obviously no progress can be made. As our goal at this stage is to break supermultiplets into isotopic multiplets, we suppose that this interaction is an operator T that commutes with the isotopic spin, strangeness and the nucleon number operators $\underline{I}$, S, N (Assumption I). This restriction is highly reasonable as we are still in the realm of strong interactions where strangeness and nucleon number are conserved, and any non-commutation with $\underline{I}$ will result in mass-splittings between different members of the isotopic multiplets.

The above restriction alone is still not sufficient for our purpose. We therefore make the further assumption (in analogy with the electromagnetism) that this operator, to lowest order, transforms as the adjoint representation of the group (Assumption II).[36] These restrictions then fix the operator (to lowest order) as the T_3^3 component of a tensor T_ν^μ. This tensor T_ν^μ in $SU(3)/C_3$ is irreducible. However, in U(3), as the adjoint representation is reducible, we can write it as

$$T_\nu^\mu = \delta_\nu^\mu + M_\nu^\mu. \qquad \ldots.(3.1)$$

where M_ν^μ is irreducible and transforms as the 8-dimensional representation.

Henceforth we shall confine ourselves to the U(3) scheme. The results, however, apply to both the schemes.

<u>To nth order</u>, we take this operator to be

$$T_n = T_3^3 + T_3^3T_3^3 + \ldots + T_3^3T_3^3\ldots T_3^3 \quad \text{(n factors)}$$

$$= \sum_{i=1}^{n} \pi_i T_3^3 \qquad \ldots.(3.2)$$

where

$$\pi_i T_3^3 = T_3^3T_3^3\ldots T_3^3 \qquad \text{(i factors)} \qquad \ldots.(3.3)$$

Since every product of tensors T is reducible (under U(3)), we can express (3.3) as

$$\pi_i T_3^3 = \sum_{r=0}^{i} a_r(\delta_3^3)^r M_{33..3}^{33..3} \qquad \ldots.(3.4)$$

(i-r times)

where $M_{33..3}^{33..3}$ is a component of an irreducible tensor.

From our first assumption,

$$[T_3^3, \underline{I}] = [T_3^3, S] = [T_3^3, N] = 0 \qquad \ldots.(3.5)$$

and equation (3.1) above, it follows that M_3^3 also commutes with $\underline{I}$, N, S. By induction now

$$[M_{3..3}^{3..3}, \underline{I}] = [M_{3..3}^{3..3}, S] = [M_{3..3}^{3..3}, N] = 0 \qquad \ldots.(3.6)$$

ii) Okubo's mass formula and its generalization.

According to our assumptions, the lowest order mass splittings are given by the matrix elements of the operator T^3_3 between the same state. Okubo was able to write down these matrix elements in terms of the matrix elements of the operators constructed from the generators. This is in fact generally true that the matrix elements of any operator within the same representation can be expressed as a linear combination of the matrix elements of operators constructed from the generators. This follows from the fact that the matrix algebra A gnerated by the infinitesimal generators within the same representation is the whole matrix algebra.[23] What is remarkable in Okubo's work is the fact that explicit expressions of these operators are obtained. In fact he proves that[19a]

$$< D, \Psi | T^\mu_\nu | D, \Psi>$$

$$= <D,\Psi | a\,\delta^\mu_\nu + bA^\mu_\nu + c(A.A)^\mu_\nu | D, \Psi> \qquad ....(3.7)$$

where D is an arbitrary irreducible representation and Ψ any vector in its basis. Again

$$(A.A)^\mu_\nu = A^{\mu}{}_{\alpha} A^{\alpha}{}_{\nu}$$

where A^μ_ν are the nine generators of the group U(3).

The lowest order mass formula follows from the above if we use the expressions for $A^3{}_3$, $(A.A)^3_3$ in terms of I and S. The final result is that

$$M = a + bS + c(I(I+1) - \tfrac{1}{4}S^2) \qquad \ldots .(3.8)$$

Ginibre and Diu[20] gave a simpler argument to show that there will be only three terms in the above and the simplest linearly independent ones are precisely

$$\delta^\mu_\nu, \; A^\mu_\nu, \; (A.A)^\mu_\nu$$

Thus they obtained this formula rather more directly.

We shall, in the following, try to generalize the above ideas and obtain a mass formula to any particular order.

Lemma I: In any irreducible representation,

$$\underbrace{(A.A\ldots A)}_{n \text{ factors}}{}^{33\ldots 3}_{33\ldots 3} \text{ (m times)}$$

$$= \sum_{\substack{r,s,t\geq 0 \\ r+s+t=m}} a_{rst}(\delta^3_3)^r(A^3_3)^s((A.A)^3_3)^t \qquad \ldots .(3.9)$$

Proof: We have

$$[A^3_3, A^3_3] = [A^3_3, (A.A)^3_3] = [(A.A)^3_3, (A.A)^3_3] = 0 \qquad \ldots .(3.10)$$

Since the Casimic operators $\langle A.A \rangle$ and $\langle A.A.A \rangle$ commute with the generators A^μ_ν, the lemma follows from equation (A.10) in Okubo's paper on replacing T^μ_ν by A^μ_ν.

<u>Theorem I</u>: The mass formula to order n for every representation is a sum of $\frac{(n+1)(n+2)}{2}$ terms, and can be written as[37]

$$M^{(n)} = \sum_{i=0}^{n} \sum_{j=0}^{i} a_{ij}(I(I+1) - \tfrac{1}{4}S^2)^j S^{i-j} \qquad \ldots .(3.11)$$

where a_{ij} are parameters depending upon the representation but independent of the sub-quantum numbers I and S.

Proof: From equation (3.2) we have

$$M^{(n)} = \langle D,\Psi | T_n | D,\Psi \rangle$$

As was argued before, the matrix element of T_n in an irreducible representation must be expressible as linear combination of the matrix elements of suitable operatros constructed from the generators. However, in lemma I, we have seen that the number n of '3' indices on T_n is important. In fact, the lemma shows that in expressing the above matrix element we have only to include the matrix elements of operators like

$$(\delta^3_3)^r (A^3_3)^s ((A.A)^3_3)^t \qquad \begin{pmatrix} r+s+t = n \\ r,s,t \geqslant 0 \end{pmatrix}$$

Omitting δ^3_3 which is just one, we may write

$$M^{(n)} = \sum_{i=0}^{n} \sum_{j=0}^{i} \langle D,\Psi | b_{ij} (A^3_3)^{i-j} ((A.A)^3_3)^j | D,\Psi \rangle \qquad \ldots.(3.12)$$

The formula given in the theorm now follows on writing

$$A^3_3 = - S$$
$$(A.A)^3_3 = (I(I+1) - \tfrac{1}{4}S^2) + aS + b \qquad \ldots.(3.13)$$

where a and b are independent of I and S.

iii) Specialization to Particular Representations.

In the case of the 10-dimensional representation, Gell-Mann[38] remarks that the first order formula

$$M_{10}^{(1)} = a + bS + c(I(I+1) - \tfrac{1}{4}S^2)$$

reduces to

$$M_{10}^{(1)} = a' + b'S$$

on account of the relation

$$I = 1 + \frac{S}{2}$$

valid in this representation.

We wish to point out that the second order formula obtained by Okubo

$$M^2 = a + bS + c(I(I+1) - \tfrac{1}{4}S^2) + dS^2 + eS(I(I+1) - \tfrac{1}{4}S^2) + f(I(I+1) - \tfrac{1}{4}S^2)^2$$

when applied to the 8-dimensional representation becomes

$$M_8^{(2)} = a' + b'S + c'(I(I+1) - \tfrac{1}{4}S^2) + d'S^2$$

as a result of the relations

$$SI(I+1) = \tfrac{3}{4}S$$

$$S^3 = S$$

$$I(I+1)(I(I+2) - 2) = -\frac{15}{16}S^2$$

valid for this representation.

In order to see when and why this happens, we shall look at the formula from a different point of view. Using equations (3.2), (3.4), we can express T_n, the strong symmetry breaking interaction to nth order, as a sum of components of irreducible tensors. Each one of these components commutes with the operators N,S,I (see equation (3.6)). Therefore these appear only in the irreducible tensors which correspond to the representations (with N = 0) containing in their bases an isotpic multiplet with I = S = 0. These representations can only be of the form (f,0,-f) as we prove below.

Lemma II: In representations (f_1,f_2,f_3) with N = 0, the isotopic multiplet I = S = 0 occurs only when $f_3 = - f_1$, $f_2 = 0$.

Proof: From equations (1.28d', 1.28f', 1.28g') we obtain on setting I = S = N = 0

$$f_1' = f_2',\ f_1' = - f_2',\ \text{i.e. } f_1' = f_2' = 0$$

Now using (2.22), $f_2 = 0$.

Finally from (1.28f'), $f_3 = - f_1$

Remark: The group $\frac{SU(3)}{C_3}$ in the Gell-Mann Neeman model has the representations (f_1,f_2,f_3) with a restriction which may be taken as

$$f_1 + f_2 + f_3 = 0$$

So the lemma holds equally well though N is outside the symmetry group.

Next we prove another theorm which gives the number of times these representations occur in the direct product of a representation and its contragradient. This theorm will be of an immense use in the derivation of the final form of the mass formula.

<u>Theorem 2</u>: In the reduction of the direct product of a representation $D \equiv (f_1, f_2, f_3)$ with its contragradient $\bar{D} \equiv (-f_3, -f_2, -f_1)$ the representation $(f, 0, -f)$ occurs d_f times where

$$d_f = \begin{cases} 0 & \text{when } \mu + \nu < f \\ \mu + \nu - f + 1 & \text{when } \mu < f,\ \nu > f \text{ but } \mu+\nu \geq f \\ \mu + 1 & \text{when } \mu < f,\ \nu \geq f \\ \nu + 1 & \text{when } \mu \geq f,\ \nu < f \\ f + 1 & \text{when } \mu \geq f,\ \nu \geq f \end{cases}$$

and

$$\mu = f_1 - f_2, \quad \nu = f_2 - f_3$$

Proof: As some of the integers labelling the representations D and $\bar{D}$ are negative, we first of all consider the representations

$$D_1 \equiv (f_1 - f_3, f_2 - f_3, 0) \equiv (\mu+\nu, \nu, 0)$$

$$\tilde{D}_1 \equiv (f_1 - f_3, f_1 - f_2, 0) \equiv (\mu+\nu, \nu, 0)$$

Corresponding Young's tableaux for $D_1(\tilde{D}_1)$ have $\mu + \nu$ squares in the first and $\nu(\mu)$ squares in the second. We are interested in the representation $(f, 0, -f)$ in the product $D \otimes \bar{D}$. As $D_1(\tilde{D}_1)$ has been obtained from $D(\bar{D})$ by subtracting $f_3(-f_1)$ from each of the three integers labelling the representation, we should look for the representation $(f,0,-f)$ in the product $D_1 \otimes \bar{D}_1$ as associated with the Young's tableau

$$(f+f_1-f_3, f_1-f_3, -f+f_1-f_3) \equiv (\mu+\nu+f, \mu+\nu, \mu+\nu-f)$$

To obtain the product diagrams, we follow the technique already given on page 41. The diagrams we are interested in should have $\mu+\nu+f$, $\mu+\nu$, $\mu+\nu-f$ squares in the first, second and third rows respectively. This is obtained by adding f squares containing α's to the first row of D_1 followed by μ and $\mu+\nu-f$ squares containing some α's and some β's to the second and third rows in a manner that satisfies the three conditions stated on page . for the product diagrams. The condition (ii) requires β additions in the second and third rows to be always on the right of all α's. The condition (iii) of lattice order says that the number of β's to be added to the second rwo must be $\leq f$. Thus the number of diagrams of the above type in the product can be at most $f + 1$ (corresponding to $0,1,2..f$ number of β's added to the second row).

However all these cases are not always possible. To examine this casefully, let us consider μ. If $\mu \geq f$, all the $f + 1$ cases might be possible. But when $\mu < f$ only $\mu + 1$ of these (which correspond to $0,1,2... \mu$ additions of β's to the second row) are possible. All these cases will definitely be possible if we can fill all the squares in the second two with the rest of the α's. As there are only $\mu+\nu-f$ squares to be adjoined to the third row, this requires $\mu+\nu-f > \mu$ or $\nu > f$. On the other hand, when $\mu+\nu-f < \mu$ or equivalent $\nu < f$, then $\mu-(\mu+\nu-f) = f-\nu$ β's (at least) will have to be added to the second row. This will reduce the number of possibilities in each of the above cases by exactly $f-\nu$ to $(f+1)-(f-\nu) = \nu+1$ and $(\mu+1)-(f-\nu) = \mu+\nu-f+1$ respectively. Since the condition (iii) is also satisfied by each of these cases, the theorm follows.

Now in equations (3.2, 3.4) we decomposed T_n, the symmetry breaking interaction to order n, into its irreducible parts. These irreducible parts were represented by $M^{33..3}_{33..3}$ (r indices, $1 \leq r \leq n$). Evidently this can only be a component of a tensor which is completely symmetric in all the upper and lower indices. Also we have seen that it must be of the form (f,0,-f). Thus it is necessarily a component of a tensor which transforms as the (r,0,-r) irreducible representation. In theorem II, we proved that it can occur at most r + 1 times in the reduction of the direct product $D \otimes \bar{D}$. Also we know that T_n differs from T_{n-1} by the addition of another component of an irreducible tensor which belongs to (n,0,-n). Thus we have to add at most n + 1 more operators constructed from the generators such that they are linearly independent of the ones added before. These are precisely what we have already been able to obtain in theorem I. For those representations D which have (f,0,-f) exactly f + 1 times in $D \otimes \bar{D}$ (such representations always exist as is clear from theorem II), we shall have to take all these in the formula to f'th order.

However for a particular representation D, the representation (f,0,-f) may not occur f + 1 times in the reduction of $D \otimes \bar{D}$ (this is evident from theorem II). In such cases there must exist linear relations which will reduce the number of terms to be added at the fth stage to exactly d_f, the number of times the representation (f,0,-f) appears in the reduction of $D \otimes \bar{D}$. In the next few lemmas, we carry it out explicitly and we are able to select the set of linearly independent operators such that the mass formula to order 'n' for a particular representation D takes the form

$$M_D^{(n)} = \sum_{f=0}^{n} \sum_{j=0}^{d_f - 1} a_{fj} (I(I+1) - \tfrac{1}{4}S^2)^j S^{f-j} \qquad \ldots.(3.11)$$

<u>Lemma III</u>: In any irreducible representation

$D \equiv (f_1, f_2, f_3)$ of U(3).

i) I takes the $\mu + \nu + 1$ distinct values $0, \frac{1}{2}, 1, \frac{\mu+\nu}{2}$ with multiplicities $1, 2, \ldots \nu, \nu+1, \nu+1, \ldots \nu+1, \nu, \nu-1, \ldots 1$ respectively.

ii) S takes the $\mu+\nu+1$ distinct values f_1+f_2-n, f_1+f_2-1-n, ..., f_2+f_3-n with multiplicities 1, 2, ..., ν, $\nu+1$, ... $\nu+1$, ν, ..., 1 respectively ($n = f_1+f_2+f_3$).

The proof follows from equations (1.28d',f',g').

<u>Lemma IV</u>: The points (I,S) corresponding to the isotopic multiplets in D form a lattice consisting of $\mu+1$ ($\nu+1$) equally spaced parallel lines with equations of the form

$$I = \tfrac{1}{2}S + C \qquad (I = -\tfrac{1}{2}S' + C')$$

Proof: Eliminating f_1' and f_2' in turn from equations (1.28d') and (1.28g') we obtain

$$I = \tfrac{1}{2}S - f_2' + \tfrac{1}{2}(f_1+f_2+f_3) \qquad \text{....(3.15)}$$

$$I = -\tfrac{1}{2}S + f_1' - \tfrac{1}{2}(f_1+f_2+f_3) \qquad \text{....(3.16)}$$

Corresponding to $\mu+1$ ($\nu+1$) different fixed values of f_2' (f_1'), (3.15) and (3.16) are the equations referred to in the lemma.

From lemmas III and IV we can construct the following lattice of points (I,S) for the representation D.

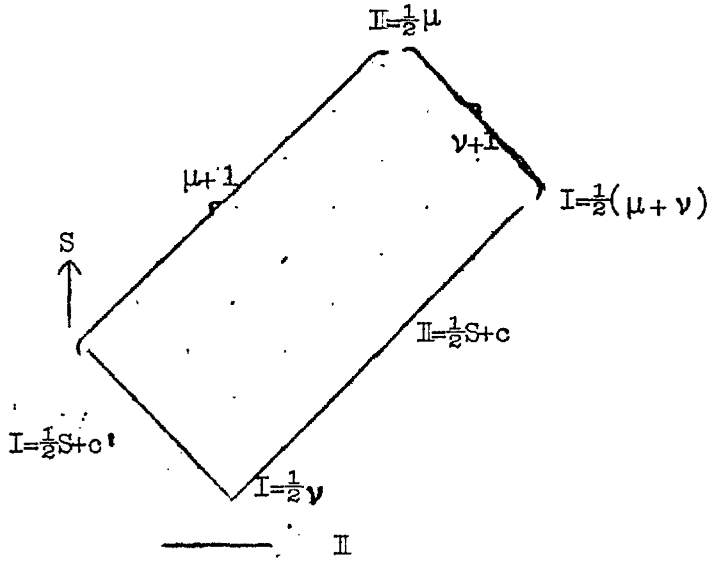


Figure 1': The (I,S) plot for a representation $D \equiv (f_1,f_2,f_3)$. $(\mu = f_1-f_2 \geq \nu = f_2-f_3)$

Lemma V: If A and B are any functions satisfying

i) $$A^{\nu+1} = \sum_{i=1}^{\nu+1} \alpha_i A^{\nu-i+1} B^i + \sum_{\substack{i+j<\nu+1\\ i,j\geq 0}} \alpha_{ij} A^i B^j \quad \ldots.(3.17)$$

and

ii) $$A^i B^{\mu+\nu+1-2i} = \sum_{j=0}^{\nu-1} \sum_{k=0}^{j} \alpha_{ijk} A^k B^{\mu+\nu-j-k} + \sum_{j=0}^{\mu-\nu-1} \sum_{k=0}^{\nu} \beta_{ijk} A^k B^{\nu+1+j-k} + \sum_{\substack{j,k\geq 0\\ j+k<\nu+1}} \gamma_{ijk} A^j B^k \quad \ldots.(3.18)$$

for $i = 0,1,2, \ldots \nu$,

then all other expressions of the form $A^\alpha B^\beta (\alpha,\beta\geq 0)$ not included in the above equations are also expressible in terms of quantities on the right in equation (3.18).

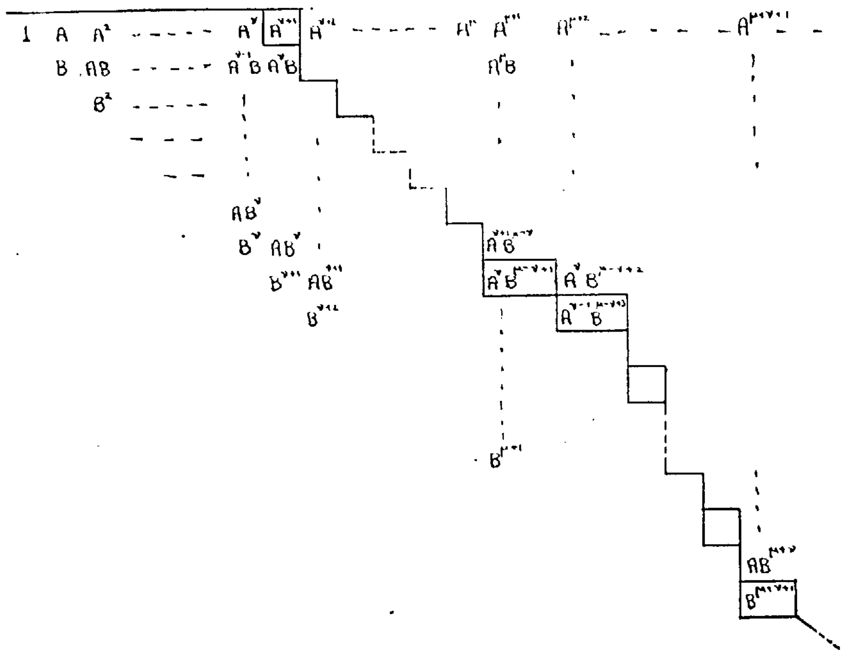


FIG. 3. Schematic representation of terms appearing in the mass formula.

The quantities $A^{\nu+1}$, $A^i\ B^{\mu+\nu+1-2i}$ appearing on the left-hand side of the above equations are enclosed in squares in (Figure 2). Equation (3.17) has on its right-hand side quantities in the $\nu+1$st column except $A^{\nu+1}$ and those on the left of this column. The quantities on the right-hand side of the equations (3.18) are all the quantities below the zig-zag line. The content of the lemma is that all the quantities in the figure above the zig-zag line and not in the squares are expressible in terms of the ones below this line provided that the quantities in the squares satisfy (3.17) and (3.18).

The proof of the above assertion is trivial: we start with the expression (3.17) for $A^{\nu+1}$ and multiply it first by B and then by A, obtaining expressions for $A^{\nu+1}$ B and $A^{\nu+2}$ in terms of the quantities on the right-band side of (3.17) and the ones in $\nu+2$nd column below the zig-zag line. This process is repeated. Slight modification is needed when we approach the stage where we want to express the quantities in the column headed by $A^{\nu+1}$.

Lemma VI.

The conditions (3.17) and (3.18) in Lemma V are in fact satisfied by the functions

$$A = I(I+1) - \tfrac{1}{4}S^2$$

$$B = S$$

Proof.

(i) From Lemma IV we see that all the points of Figure 1' satisfy the relation

$$\prod_{i=1}^{\nu+1} (I(I+1) - \tfrac{1}{4}S^2 + b_i S + c_i) = 0$$

This is condition (3.17).

(ii) To prove equation (3.18), we divide the set of points in Figure 1' into two sets S_i, S_i' $(i = 0,1,\dots,\nu)$ as follows:

Let S take the distinct values $s_1, s_2, \dots s_{\mu+\nu+1}$ expressed as a monotonically increasing sequence. For $i \neq 0$, suppose the set S_i consists of all the points having S as any one of

$$s_1, s_2, \dots, s_i, s_{\mu+\nu-i+2}, \dots, s_{\mu+\nu}, s_{\mu+\nu+1}$$

and let S_o be the null set.

The set S_i' consists of the remaining points in the figure.

Consider first $i \neq 0$.

It is clear from lemma III that the set S_i consists of $i(i+1)$ points. S_i can therefore determine a set of values of the $i(i+1)$ ratios of the $(i(i+1)+1)$ constants α_{rs} such that

$$d(I(I+1) - \tfrac{1}{4}S^2)^i + \sum_{\substack{2r+s<2i \\ r,s>0}} \alpha_{rs}(I(I+1) - \tfrac{1}{4}S^2)^r S^s = 0 \qquad \dots.(3.19)$$

is satisfied by all the points of the set. Here d is necessarily non-zero; for, if it were zero, the equation (3.19) which is now of at most i-1 degree in I(I+1), cannot satisfy all the i distinct points with $S = s_i$ because the corresponding I(I+1) are necessarily distinct and positive-definite.

Thus all the points of the figure satisfy the set of equations

$$[(I(I+1) - \tfrac{1}{4}S^2)^i + \sum_{\substack{2r+S \leq 2i \\ r,S \geq 0}} \frac{a_{rs}}{d} (I(I+1) - \tfrac{1}{4}S^2)^r \, S^s]$$

$$\times \quad (S-s_{i+1}) \ldots . (S-s_{\mu+\nu-i+1}) = 0 \quad \ldots .(3.20)$$

When i = 0, we have instead

$$(S - s_1)(S - s_2) \ldots . (S - s_{\mu+\nu+1}) = 0 \qquad \ldots .(3.21)$$

These give condition (3.18).

(iv) Exact formula for a particular representation.

It is clear from theorem II that the representations (f,0,-f) with $f > f_1 - f_3$ do not occur at all in the reduction of the product $D \otimes \bar{D}$. From this it follows that an exact formula for D (i.e. one true to all orders) is

$$M_D = M_D^{\mu+\nu} \quad (\mu = f_1 - f_2, \; \nu = f_2 - f_3) \qquad \ldots .(3.22)$$

The total number of terms in the formula is equal to

$$\sum_{f=0}^{\mu+\nu} d_f$$

which by theorem II is

$$(\mu+1)(\nu+1)$$

Now as each one of these representations (f,0,-f) contains I = S = 0 multiplet just once, $(\mu+1)(\nu+1)$ is also equal to the number of times I = S = 0 multiplets occur in the direct product of D and $\bar{D}$. We now prove

<u>Theorem III</u>: The number of times I = S = 0 multiplet occurs in the direct product $D \otimes \bar{D}$ is equal to the number of isotopic multiplets in the representation D.

<u>Proof</u>. The representations D and $\bar{D}$ consist of isotopic multiplets of the form (I,S), (I',S') where the set of I', S' is the same as that of I(-S). From (I,S') and (I',S') we obtain

$$\sum_{I=|I-I'|}^{I+I'} (I,\ S+S')$$

This series contains I = S = 0 if and only if

$$S + S' = 0 = I - I'$$

i.e. when

$$S' = -S,\ I' = I.$$

Therefore there exists a unique solution for (I', S') for each (I, S) which satisfies the theorem.

As now each of the multiplets (I, S) of D gives rise to one and only one method of constructing $I = S = 0$ in the direct product $D \otimes \bar{D}$, the theorem follows.

As a final remark, we see that the mass formula M_D has the same number of parameters as the number of isotopic multiplets in D. The arguments above can be carried through even for other symmetry groups proposed for strong interactions with no change in the above conclusion.

Thus the use of a formula which is true to all orders can lead to no predictions at all.[39] In fact such a formula does not make any approximation and hence will not be useful. What is useful above is that we can consider mass formula with some of the last few terms omitted. This will lead to some relations which can then be checked. The whole philosophy of going to higher orders would make sense only when the contributions of the higher order terms is smaller in comparison to those of lower order ones, in order to be sure that only a few (depending upon the representation) terms are essential to obtain the correct mass spectrum. There is however a big distinction in the above higher order formulae and the contribution (finite) of the higher order graphs in perturbation theory. There the expansion is usually carried out in powers of a coupling constant which, being small, automatically leads to the belief that the series might converge. In fact, right from the beginning, a hope dominates the theoretician that this is so. However, the above formulae deal only with the structures of the matrix elements and say nothing about the possible relationships of the various parameters that occur in the theory. Thus no a priori belief exists on which one can claim and obtain progressively weaker contributions to the parameters. Thus a very real problem exists of determining how these parameters can be worked out for various representations knowing only their quantum numbers. Of course these parameters will depend upon the spatial properties of the particles which formed these (we know that these are the same for all the members of a supermultiplet). For this purpose, we shall have to consider the dynamics of not only the particles but also of the symmetries. We shall have to know how from some elementary particles we can obtain others and whether the choice of a set of elementary particles can be uniquely made. The current belief is that the choice is not unique and the "Bootstrap" can be reversed. However, though these questions and their answers will be highly interesting, we shall not be able to go into their details.

(v) Application of the mass formula.

For the eight dimensional representation, we have seen that the general formula is

$$M = M^{(2)}_{8} = a + bS + c(I(I+1) - \tfrac{1}{4}S^2) + dS^2$$

As this representation has four isotopic multiplets, we must consider only the first order formula.

Now from

$$M = a + bS + c(I(I+1) - \tfrac{1}{4}S^2)$$

writing $S = N - Y$,

$$M = a' + b'Y + c'(I(I+1) - \tfrac{1}{4}Y^2)$$

$$\therefore \quad M(Y=1,\ I=\tfrac{1}{2}) = a' + b' + \tfrac{1}{2}c'$$

$$M(Y=-1, I=\tfrac{1}{2}) = a' - b' + \tfrac{1}{2}c'$$

$$M(Y=0,\ I=0) = a'$$

$$M(Y=0,\ I=1) = a' + 2c'$$

The relation that exists can be written as

$$M(Y=1, I=\tfrac{1}{2}) + M(Y=-1, I=\tfrac{1}{2})$$

$$= 2a' + c'$$

$$= \tfrac{1}{2}(3M(Y=0, I=0) + M(Y=0, I=1) \quad \ldots.(3.23)$$

This relation is satisfied to within $\frac{1}{2}\%$ for the baryon octet, and to within 5% for the meson octet. However, for the vector meson octet, it predicts the mass of the isotopic singlet to be 928.MeV. However the known ω and φ mesons both do not have a mass near to this value.

Should we consider it as a failure of the mass formula? Without hurrying to such a conclusion, let us see what dynamical effect the existence of two isotopic singlet vector mesons will have. Evidently from the point of view of the very strong interactions, they must be put in different supermultiplets. We can take one to be a member of an octet (say ω) and the other (say φ) a unitary singlet. Thus if the unitary symmetry were exact, no transition can take place from ω to φ and vice versa. However, this symmetry is not exact, and the very existence of medium strong interactions, which conserve only I and S and bring about the split in the masses of the various istopic multiplets within a supermultiplet, is a manifestation of it. From the point of view of these interactions, $\omega \longleftrightarrow \varphi$ transitions are allowed. Thus the extent of the breakage of the unitary symmetry will have a measure in the transition that occurs between them. Evidently as a result of these transitions, the masses of ω and φ will not be as required by the symmetry. In other words, both ω and φ are not pure states and therefore φ (or ω) may not have the mass as predicted by the lowest order mass formula. After Sakuari,[40] we consider them to be linear combinations of φ_0 and ω_0 which we consider as the actual members of the vector unitary octet and singlet respectively. As these have the same quantum numbers, transitions will occur between them. Thus their mass can be represented as a 2x2 matrix in the form

$$\begin{pmatrix} m(\varphi_o) & a \\ a & m(\omega_o) \end{pmatrix} \qquad \ldots.(3.24)$$

where $a = \varphi_o \leftrightarrow \omega_o$ transition matrix. This matrix is not diagonal and can be diagonalized by means of a unitary transformation. The two eigenvalues are given by

$$\lambda = \tfrac{1}{2}(m(\varphi_o) + m(\omega_o)) \pm \sqrt{\tfrac{1}{4}(m(\varphi_o)+m(\omega_o))^2 + a^2} \qquad \ldots.(3.25)$$

These two eigenvalues are to be 1020 and 781 mev respectively. Also the prediction of the lowest order mass formula for the $m(\varphi_o)$ based on $m_\rho = 750$ mev and $m_K^* = 888$ mev is

$$m(\varphi_o) = 927.5$$

Now from equation (3.25)

$$\lambda_1 + \lambda_2 = m(\varphi_o) + m(\omega_o)$$

$$(\lambda_1 - \lambda_2)^2 = (m(\varphi_o) - m(\omega_o))^2 + 4a^2$$

Using $\lambda_1 = 1020$, $\lambda_2 = 781$, $m(\varphi_o) = 927.5$, we can solve these to obtain

$$m(\omega_o) = 873.5 \text{ mev}$$

$$a = 116 \text{ mev}$$

Again the true eigenvectors, namely φ and ω, may be written as linear combinations of φ_o and ω_o in the form

$$\varphi = \varphi_o \cos\alpha + \omega_o \sin\alpha \qquad \text{....(3.26a)}$$

$$= -\varphi_o \sin\alpha + \omega_o \cos\alpha \qquad \text{....(3.26b)}$$

where

$$\tan\alpha = \frac{92.5}{116} = .80$$

i.e. $\alpha = 39^o$ $(\cos\alpha = .78, \sin\alpha = .63)$ ...(3.26b)

This value of α shows that there is a large mixing going on between φ_o and ω_o which as a consequence gives rise to the mass formula very badly broken.

Let us now try to see the consequences of the above phenomenonlogical considerations. First of all let us look at the $\omega,\varphi \to \gamma$ transitions. As ω_o and φ_o formed essentially members of a unitary singlet and octet and γ can be treated as a component of an octet, we see that

$$\omega_o \not\to \gamma$$

while $\varphi_o \to \gamma$

Thus $$\frac{\omega \to \gamma}{\varphi \to \gamma} \equiv - \frac{\sin\alpha}{\cos\alpha}$$

or $$\frac{\omega \to e^+ + e^-}{\varphi \to e^+ + e^-} = \frac{\sin^2\vartheta}{\cos^2\vartheta} \quad x \quad \text{(phase space part)}$$

$$= .64 \quad x \quad \text{(phase space part)}$$

As a second consequence, we look at the $K\bar{K}$ decay mode of the φ meson (for ω-meson, this decay cannot occur).

$K\bar{K}$ state must be a p-state, i.e. its spatial wave-function is anti-symmetric. To make it satisfy Bose Statistics, we must take the unitary symmetry wave-function also anti-symmetric. Thus $\omega_o \not\to K\bar{K}$ as this is a unitary singlet, and the singlet state constructed out of $K\bar{K}$ is symmetric. Only $\varphi_o \to K\bar{K}$. This φ_o must be taken in the anti-symmetric octet. Now from the Clebsch-Gordan coefficients table, we obtain

$$\left.\begin{array}{lll} \rho \to \pi^+\pi & = & -\frac{1}{\sqrt{3}} \\ \varphi_o \to K^+ + K^- & = & -\frac{1}{2} \\ \varphi_o \to K^o + \bar{K}^o & = & -\frac{1}{2} \\ K^{*+} \to K^+ + \pi^o & = & -\frac{1}{2\sqrt{3}} \\ \quad\;\; \to K^o\pi^+ & = & \frac{1}{2\sqrt{3}} \end{array}\right. \quad X \quad \left\{\begin{array}{l} \text{corresponding} \\ \text{phase-space} \\ \text{factors} \end{array}\right.$$

Using the width of ρ as 100 Mev, we can work out the φ_o and K^{*+} widths as $\sim$30 and 5 mev respectively. Thus $\varphi \to K\bar{K}$ has a width which is about $5 \cos^2\alpha \sim 3$ mev.

CHAPTER 4

Electromagnetic Properties and Unitary Symmetry

We know that the electromagnetic interaction conserves charge, but not every component of the isotopic spin. In fact, it is symmetrical about T_3 axis of the isospin space only, and conserves strangeness as a consequence of Q and T_3 conservation. Thus the electromagnetic interaction is not an isotopic scalar, but a combination of an isotopic scalar and vector. The only such operators which also belong to the adjoint representation (and satisfy T_3 and S conservation laws) are T^1_1 and T^2_2. The choice between these two can be made by looking at the basis for the 3-dimensional representation, in terms of which the electromagnetic interaction may be taken to $\sim p\bar{p}$, i.e. $\sim T^1{}_1$.

Here we remark that we are about to look at two types of effects of this operator. One concerns the second stage referred to in the last chapter, i.e. the electromagnetic mass splittings between various members of an isotopic multiplet. As a second attempt, we shall try to obtain the hyperon magnetic moments and the boson form factors. For the first problem, i.e. the problem of electromagnetic masses, considerations entirely identical to the ones given in the last chapter can be applied. We are only to exploit the symmetry between $T^1{}_1$ and $T^3{}_3$. To understand this symmetry we consider the various irreducible representations decomposed not relative to the isospin group, but with respect to another SU(2) subgroup which is orthogonal to the charge rather than to the strangeness. This is the U-spin approach of Lipkin et al.[21] For example, in this decomposition, we treat p as a singlet and $\binom{n}{\Lambda}$ as a doublet. To obtain the consequences, we have just to apply the transformations

$$\begin{array}{ll} p \to n, & \bar{p} \to \bar{n} \\ n \to \Lambda, & \bar{n} \to \bar{\Lambda} \\ \Lambda \to p, & \bar{\Lambda} \to \bar{p} \end{array} \qquad \ldots.(4.1)$$

Thus for example, the basis for the octet representation takes the form

Table XI

Q	U	U_3		
−1	$\frac{1}{2}$	$\frac{1}{2}$ $-\frac{1}{2}$	$n\bar{p}$ $\Lambda\bar{p}$	π^- K^-
0	1	1 0 −1	$-n\bar{\Lambda}$ $\frac{n\bar{n} - \Lambda\bar{\Lambda}}{\sqrt{2}}$ $\Lambda\bar{n}$	$-K^o$ $-\frac{1}{2}\pi^o + \frac{\sqrt{3}}{2}\pi^{o\prime}$ $\bar{K}^o$
0	0	0	$\frac{-n\bar{n} - \Lambda\bar{\Lambda} + 2p\bar{p}}{\sqrt{6}}$	$\frac{\sqrt{3}}{2}\pi^o + \frac{1}{2}\pi^{o\prime}$
1	$\frac{1}{2}$	$\frac{1}{2}$ $-\frac{1}{2}$	$-p\bar{\Lambda}$ $p\bar{n}$	$-K^+$ π^+

From the analysis presented in Chapter 2, it is clear that the set of (Q,U) for any irreducible representation is the same as of (-S,I). In terms of the weight diagrams, we see that a rotation of 120^o about the centre does not change the diagram. Q and S are measured from two axes which are at 120^o to each other. The symmetry of the diagram now leads to the result. U and I similarly take the same set of values.

Now on completely the same lines as were followed to write the mass formula, we shall obtain the effect of the electromagnetic operator to order n as

$$E_D^{(n)} = \sum_{f=0}^{n} \sum_{j=0}^{d_f^{-1}} a_{fj}(U(U+1) - \tfrac{1}{4}Q^2)^j Q^{f-j} \qquad \ldots\ldots(4.2)$$

This equation is the same as (3.11) where we have just replaced I and S by U and Q respectively as a consequence of the obvious difference between $T^1{}_1$ and $T^3{}_3$. In fact, under this operation, n and Λ of the Sakata model are not split (this is analogous to the fact that under transformations in the isotopic spin space p and n are not split). In more sophisticated language, we shall say that the electromagnetic interactions are U-spin scalars and thus will not create any split between various members of any U-multiplet. When applied to the boson U-octet given in table XI, we obtain the following equations between some matrix elements (we do not write $T^1{}_1$ as these equations are true to all orders in electromagnetism as we are using only the fact that the electromagnetic interactions are U-spin scalars and no additional assumption that, to lowest order, these are just a component $T^1{}_1$ of the adjoint representation).

$$< \pi^- | E | \pi^- > = < K^- | E | K^- > \qquad \ldots.(4.3.i)$$

$$< \pi^+ | E | \pi^+ = < K^+ | E | K^+ > \qquad \ldots.(4.3.ii)$$

$$< K^0 | E | K^0 > = < \bar{K}^0 | E | \bar{K}^0 > \qquad \ldots.(4.3.iiia)$$

$$= < -\tfrac{1}{2}\pi^0 + \tfrac{\sqrt{3}}{2}\pi^{0\prime} | E | -\tfrac{1}{2}\pi^0 + \tfrac{\sqrt{3}}{2}\pi^{0\prime} > \qquad \ldots.)4.3.iiib)$$

$$< -\tfrac{1}{2}\pi^0 + \tfrac{\sqrt{3}}{2}\pi^{0\prime} | E | \tfrac{\sqrt{3}}{2}\pi^0 + \tfrac{1}{2}\pi^{0\prime} > = 0 = < \tfrac{\sqrt{3}}{2}\pi^0 + \tfrac{1}{2}\pi^{0\prime} | E | -\tfrac{1}{2}\pi^0 + \tfrac{\sqrt{3}}{2}\pi^{0\prime} > \qquad \ldots.(4.3.iv)$$

The euation (4.3.iv) follows from the U-spin scalarity of the electromagnetic interaction which forbids any electromagnetic transition between a U-spin scalar and a U-spin vector.

Now from (4.3.iv),

$$< \pi^0 | E | \pi^{0\prime} > = < \pi^{0\prime} | E | \pi^0 > \qquad \ldots.(4.4)$$

$\therefore$ (4.3.iiib) leads to

$$< K^0 | E | K^0 > = \tfrac{1}{4}(< \pi^0 | E | \pi^0 > + 3 < \pi^{0\prime} | E | \pi^{0\prime} > - 2\sqrt{3} < \pi^{0\prime} | E | \pi^0 >) \qquad \ldots.(4.5)$$

Also we can rewrite (4.3.iv) using (4.4) in the form

$$- < \pi^0 | E | \pi^0 > + < \pi^{0\prime} | E | \pi^{0\prime} > + \frac{2}{\sqrt{3}} < \pi^{0\prime} | E | \pi^0 > = 0$$

Eliminating $<\pi^{o}|E|\pi^{o}>$ and $<\pi^{o\prime}|E|\pi^{o\prime}>$ in turn from (4.5), we obtain

$$<K^{o}|E|K^{o}> = <\pi^{o}|E|\pi^{o}> - \sqrt{3}<\pi^{o\prime}|E|\pi^{o}> \qquad \text{....(4.6a)}$$

$$= <\pi^{o\prime}|E|\pi^{o\prime}> - \frac{1}{\sqrt{3}}<\pi^{o\prime}|E|\pi^{o}> \qquad \text{....(4.6b)}$$

In the above we considered the matrix elements of E between two octets. We can also consider these between an octet and the vacuum. The only interesting matrix element amongst these is

$$<0|E|-\tfrac{1}{2}\pi^{o} + \tfrac{\sqrt{3}}{2}\pi^{o\prime}> = 0 \qquad \text{....(4.7)}$$

The above equation is equivalent to

$$<0|E|\pi^{o}> = \sqrt{3}<0|E|\pi^{o\prime}> \qquad \text{....(4.8)}$$

In the above we have been writing those matrix elements of the electromagnetism operator which are true to all orders. In fact the equations (4.3-7) are five relations between 8 diagonal and one off-diagonal matrix element of E. However, if we treat E to the lowest order only, we can obtain two more relations. This is because the formula (4.2), to first order, has only two parameters. This time we shall try to connect diagonal matrix element of E_1, electromagnetism to lowest order, for different U-spin multiplets. In fact, formula (4.2) gives

$$<\pi^{-}|E_1|\pi^{-}> = a - b + \tfrac{1}{2}c \qquad \text{....(4.9a)}$$

$$< K^0 | E_1 | K^0 > = a + 2c \qquad \ldots.(4.9b)$$

$$\frac{\sqrt{3}}{2}\pi^0 + \tfrac{1}{2}\pi^{0\prime} | E_1 | \frac{\sqrt{3}}{2}\pi^0 + \tfrac{1}{2}\pi^{0\prime} > = a \qquad \ldots.(4.9c)$$

$$< K^+ | E_1 | K^+ > = a + b + \tfrac{1}{2}c \qquad \ldots.(4.9d)$$

where on account of E_1 belonging to the octet representation

$$a = -c \qquad \ldots.(4.10)$$

Thus the above equations give rise to two more relations

$$-< K^0 | E_1 | K^0 > = \tfrac{1}{4}(3 < \pi^0 | E_1 | \pi^0 > + < \pi^{0\prime} | E_1 | \pi^{0\prime} >$$

$$+ 2\sqrt{3} < \pi^0 | E_1 | \pi^{0\prime} >) \qquad \ldots.(4.11a)$$

$$= < \pi^- | E_1 | \pi^- > + < K^+ | E_1 | K^+ > \qquad \ldots.(4.11b)$$

In the following we give a few of the interesting physical consequences of the equations obtained above. Some of these were first obtained by Coleman and Glashow.[41] They were later derived by Cabibbo and Gatto.[22] Okubo[19a] used another method to arrive at these. We have tried to obtain them in a more consistent manner by using the U-spin approach.

(a) <u>Form factors of the bosons.</u>

Writing E_1 in place of E in equations (4.3.i-iii) and using charge conjugation ($CE_1C^{-1} = - E_1$) we find

(i) form-factor of π^+ (π^-) is equal to that of K^+(K^-),

(ii) form-factors of K^0 and $\bar{K}^0$ vanish.

(b) Compton Scattering amplitudes.

Replacing E by $E_1 E_1$ we obtain

(i) Compton scattering amplitudes for $K^{\pm}$ are the same as for $\pi^{\pm}$

(ii) the ones for the neutral particles are related as in equations (4.6a,b).

(c) The amplitude for

$$\pi^{o} \to 2\gamma$$

is $\frac{1}{\sqrt{3}}$ times the one for

$$\pi^{o\prime} \to 2\gamma$$

(d) Electromagnetic contributions to the masses of the baryons.

We rewrite equations (4.3.i-iiia), (4.6a,b) for the baryon octet:

$$<\Sigma^- |E| \Sigma^-> = <\Xi^- |E| \Xi^- > \qquad \ldots.(4.12.a)$$

$$<\Sigma^+ |E| \Sigma^+> = <p|E|p> \qquad \ldots.(4.12.b)$$

$$<n|E| n> = <\Xi^o |E| \Xi^o > \qquad \ldots.(4.12.c)$$

$$= <\Sigma^o| E| \Sigma^o> - \sqrt{3} <\Lambda| E| \Sigma^o > \qquad \ldots.(4.12.d)$$

$$= <\Lambda|E|\Lambda> - \frac{1}{\sqrt{3}} <\Lambda|E|\Sigma^o > \qquad \ldots.(4.12.e)$$

The first three of these lead to

$$\delta m_{\Xi^-} - \delta m_{\Xi^o} = \delta m_p - \delta m_n + \delta m_{\Sigma^-} - \delta m_{\Sigma^+} \qquad \ldots.(4.13)$$

This relation is quite well satisfied by the masses of these particles. We emphasize that this relation is <u>true to all orders in electromagnetic and very strong interactions</u>. The medium strong interactions will, however, not allow this relation to be satisfied exactly. In fact if we treat medium strong interactions quite arbitrarily, we will not obtain this relation.

(e) <u>Magnetic moments of the Hyperons</u>.

Considering equations (4.12) and (4.11a,b) with K^0, π^0 $\pi^{0\prime}$, π^-, K^+ replaced by $n, \Sigma^0, \Lambda, \Sigma^-$, p respectively, we can write the anamolous magnetic moments of the hyperons and the $\Lambda - \Sigma$ transition moment in terms of $\mu(\Sigma^-) = \alpha$, $\mu(p) = \beta$ as

$$\mu(\Sigma^-) = \mu(\Xi^-) = \alpha \qquad \ldots.(4.14a)$$

$$\mu(p) = \mu(\Sigma^+) = \beta \qquad \ldots.(4.14b)$$

$$\mu(n) = \mu(\Xi^0) = -(\alpha+\beta) \qquad \ldots.(4.14c)$$

$$\mu(\Sigma^0) = -\mu(\Lambda) = \tfrac{1}{2}(\alpha+\beta) \qquad \ldots.(4.14d)$$

$$\langle \Lambda^0 | E_1 | \Sigma^0 \rangle = \langle \Sigma^0 | E_1 | \Lambda \rangle = \frac{\sqrt{3}}{2}(\alpha+\beta) \qquad \ldots.(4.14e)$$

An important consequence is

$$\mu(\Lambda) = \tfrac{1}{2}\mu(n) \qquad \ldots.(4.15)$$

In the Sakata model, however, we have

$$\mu(\Lambda) = \mu(n) \qquad \ldots.(4.15')$$

as follows from $\begin{pmatrix} n \\ \Lambda \end{pmatrix}$ forming a U-spin spinor. We note, however, that though equation (4.15') is true in the Sakata model to all orders of electromagnetism, equation (4.15) for the octet model is only true to lowest order. Believing the above analysis, the above values should have been another method of prefering one model over the other. Unfortunately the present experimental inaccuracy in $\mu(\Lambda)$ is so large that it fits both.

Clearly relations of the above type will also hold between form-factors of the vector bosons.

CHAPTER 5.

Conclusion

Our attention has been focused on the unitary symmetry schemes of Sakata and Gell-Mann-Neeman and we have been able to see clearly that, if the choice were between these theories only, the octet version would definitely have to be adopted, the main virtue of this model being the placing of all the 8 baryons in the same multiplet which is forced upon us on account of even $\Lambda\Sigma$ parity. In the Sakata model, firstly we shall have to find different places for Λ and Σ, and with a reasonable choice for Σ, we shall have to explain the non-existence of a strangeness one baryon. Again the Sakata model completely forbids

$$p + \bar{p} \rightarrow K^o_1 + K^o_2$$

the occurrence of which has been seen experimentally with the same abundance as

$$p + \bar{p} \rightarrow K^+ + K^-$$

Another distinguishing feature will be the anomalous magnetic moment of Λ which up to now is in agreement with both the schemes.

We have also seen that we can discuss the symmetry breaking with results that are quite good for the pseudo-scalar meson and baryon octets. The vector meson octet, however, presents many difficulties. Firstly we have 10 and we can accommodate only 9. Secondly no choice of 8 fits with the lowest order mass breaking formulae. Attempts have been made to explain both these features. Firstly, according to Nambu and Sakurai,[42] the existence of another K* resonance is in fact the manifestation of symmetry breaking, while the existence of two isosinglet vector mesons results in disagreement with the lowest order mass formula. The second case has also been discussed and consequences derived by many authors. It appears that we shall have to understand the situation better. Simple calculations probably need the existence of another quantum number as has been pointed out by Low and emphasized by Heisenberg.[43]

There is a mathematical-cum-philosophical question which arises. Symmetries based on U(3) or SU(3) are indeed the simplest generalizations of the isotopic spin scheme. But why is it that nature respects the full SU(2) and only a subgroup of SU(3)? Why is it that we have not found any use for the 3-dimensional representation which is, indeed, the fundamental representation for these groups while the corresponding one was necessary for the isotopic spin group SU(2)?[44]

We have seen that Y_1^* can fit quite nicely into the 10-dimensional representation. This being anti-symmetrical allows both $\Sigma\pi$ and $\Lambda\pi$ decay modes. A calculation based on the symmetry and taking into account the phase space gives the branching ratio

$$\frac{Y_1^* \to (\Sigma\pi)_{I=1}}{Y_1^* \to (\Lambda\pi)_{I=1}}$$

as 14%, though the reaction $Y_1^* \to \Sigma\pi$ appears to be highly suppressed. In fact, so far, no event of this type has been seen. Placing Y_1^* in the 27-dimensional representation will forbid this decay mode , but we shall then have to look for its partners to complete the multiplet.

Again the photoproduction data cannot be made to agree with any value of the mixing ratio. This might be expected as the symmetry is very badly broken. The results of the study of Λ-hyperfragments can be reasonably accommodated in the theory.[45] The unfortunate aspect of this problem is that the corresponding value $\alpha \sim \frac{3}{4}$ does not agree with the ones obtained on the basis of dynamical calculations.

This might lead one to the consideration of other symmetry groups like G_2 [34] and C_2 (or B_2). We mentioned in the introduction that these groups could not accommodate all the 8 baryons in the same multiplet. We might, nevertheless, like to consider the consequences of considering some of the 8 baryons to form a representation of these groups. The only candidate which does have some hope is G_2.

This can accommodate the 7 pseudoscalar mesons π, K, $\bar{K}$ in the septet representation. However Λ and Σ have again to be separated. Also Lipkin has shown that this group does not allow couplings between 2 vector mesons and a pseudoscalar meson. In fact, the same argument applies to C_2 (or B_2) where we have to allocate the 10-dimensional representation to the pseudoscalar as well as the vector mesons. In C_2 we need 3 isotopic singlets with different strangenesses in the same multiplet. In these groups, as in G_2, we cannot accommodate Λ and Σ in the same supermultiplet.

Again in the analysis of the basis vector for various irreducible representations of G_2 (B_2 or C_2) we shall require 3 (1) more non-linear operators in addition to $\underline{I}^2$ for complete analysis. These operators will not be expressible as functions of $\underline{I}^2$, I_3, S (or Y). In other words, the first stage of the symmetry breaking will not result in non-degenerate isotopic multiplets for some supermultiplets. Thus the existence of G_2(B_2 or C_2) as a higher symmetry will result in a revision of our ideas about the hierarchy of interactions.

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