I

THEORY OF ELEMENTARY PARTICLES

by

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PREFACE.

The work described in this thesis has been carried out under the supervision of Professor A. Salam between October 1961 and December 1963 in the Department of Theoretical Physics, Imperial College, University of London. The material contained herein is original (except where stated in the text) and has not been previously presented for a degree in this or any other University.

The thesis is based on a paper entitled "Covariant Polarization Analysis of Spin 1 Particles" published in the Proceedings of the Royal Society, A, volume 268, 1962 and the extension of this work to the case of arbitrary spins.

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ABSTRACT

A covariant theory of polarization analysis of particles of arbitrary spin s is developed. This theory is based on a Bhabha type equation which admits multiple mass values for the particles. We begin by describing Stapp's method for obtaining the covariant scattering and density matrices for particles obeying the Dirac equation. In order to apply the method of Stapp to the present case, a complete momentum space expansion of the wave function satisfying the Bhabha type equation is performed. This expansion contains a summation over the various mass states and for each mass state the particle, antiparticle projection operators, the invariant spin projection operators and the helicity projection operators have been obtained. Working in the fusion theory representation of the β matrices it has been found possible to establish for each mass value an orthonormal basis in the 4^{2S} simensional space and to derive certain useful orthonormality relations. For a collision process between particles of spin S and scalar particles the forms of the covariant density and scattering matrices have been derived by expanding them in sums of 2S-fold Kronecker products of Dirac matrices and restricting

them to operate only in the subspace characterized by the particle projection operator and the invariant projection operator for the highest spin S. The covariant scattering equation is then evaluated in the centre of mass frame and after a reduction process it it expressed in terms of (2S+1) dimensional matrices of the rotation group. The relativistic corrections are the same as found by Stapp - the only difference being that the rotational corrections are to be applied to each index of the polarization tensors.

In the last chapter the quantization of the free field is considered and a closed expression for the commutation relations for the free field operators is derived.

CONTENTS

INTRODUCTION

A covariant theory of the polarization analysis of Wolfenstein and Ashkin type has been developed by H. Stapp[1]. This theory is based on the Dirac equation and the covariant forms of the density matrices $\rho(f)$, $\rho'(f')$ and the scattering matrix $S(f', t, f)$ have been obtained by the use of the hole theory condition. f and f' denote the initial and final 4-momenta of the spin $\frac{1}{2}$ particle and t is the momentum of the centre of mass. The density matrices are the same as given by Michel and Wightman[2] and involve 4-vectors p and p' which are orthogonal to f and f' respectively and therefore reduce the 3-vectors $\bar{\underline{p}}$ and $\bar{\underline{p}}'$ in the respective rest frames $\underline{f} = o$, $\underline{f}' = o$. $\bar{\underline{p}}$ and $\bar{\underline{p}}'$ are called the proper polarization vectors. Stapp[1] has also developed a technique for reducing the covariant scattering equation giving the total differential cross-section I.

$$I \quad \rho'(f') = S\,(f',t,f) \quad \rho(f) \quad S^{+}\,(f',t,f)$$

into a form in which the S-matrix is given in the centre-of-mass frame and $\rho(f)'$, $\rho'(f')$ are given in their respective rest frames with certain rotations applied to the proper polarization vectors $\bar{p}_i$ and $\bar{p}_i'$. In this form the scattering equation is expressed in terms of the Pauli matrices and is of the usual form obtained from considerations of space rotation invariance.

The Lorentz transformations involved in the reduction give rise to certain kinematical and rotational corrections when one applies the usual non covariant theory to the analysis of high energy multiple scattering experiments.

In the present work scattering of particles of arbitrary spin s off spinless target particles has been considered and an attempt is made to develop a covariant theory of polarization analysis on the lines laid down by Stapp. For this purpose the wave function $\varphi(x)$ for the free particles of spin s has been supposed to obey the Bhabba type equation

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + ms\right) \varphi(x) = 0$$

There are two types of theories associated with this equation. In the first type β_μ's obey the characteristic equation[3]

$$\beta_\mu^{2s-1} \left(\beta_\mu^2 - 1\right) = 0$$

and $\varphi_\alpha(x)$ satisfies the Klein-Gordon equation. However for $s > 1$, β_μ's have no hermitian representation and for this reason this theory will not be used here.

In the second type of the theory β_μ's obey the characteristic equation

$$(\beta_\mu - s)\,(\beta_\mu - (s-1)) \;\ldots\ldots\; (\beta_\mu + s) = 0$$

In this case β_μ's have hermitian representations but unfortunately instead of the Klein-Gordon equation $\varphi(x)$ satisfies the multiple mass equation[3]

$$\left(\Box - \left(m\,\frac{s}{s}\right)^2\right)\left(\Box - \left(m\,\frac{s}{s-1}\right)^2\right) \;\ldots\ldots\; = 0$$

the last factor being $(\Box - (ms)^2)$ for $n = 2s$ even and $\Box - (ms/\tfrac{1}{2})^2$ for $n = 2s$ odd. This introduces certain difficulties but in view of the fact that we can use the well known fusion theory representation of the β matrices the present work will be based on this second type of the theory.

In order to define a hole theory condition and covariant S and ρ matrices a complete momentum space expansion of $\varphi(x)$ has been performed and positive and negative energy projection operators have been defined. This expansion of $\varphi(x)$ contains a summation over the various mass states $m\frac{s}{s}$, $m\frac{s}{s-1}$ ...... and for each mass state there are besides energy projection operators the invariant spin projection operators belonging to the spin values $s, s-1$ ...., in particular the one which selects the highest spin value s. Working in the fusion theory representation of the β matrices it has been found possible to establish for each mass value an orthonormal basis in the 4^n dimensional space and to derive certain useful orthonormality relations. Free 'particles' or 'anti-particles' of mass 'm' and spin s are described by the momentum space wave functions $U(f)$ with f lying on the lowest mass-shell $f^2 = -m^2$. $U(f)$ belongs to a $(2s+1)$ dimensional sub-space $\eta^{\pm}(f)\ O^{(s)}(f)$. $\eta^{\pm}(f)$ are the 'particle' and the 'anti-particle' projection operators satisfying

$$(\pm i\ \beta_\mu f_\mu + ms)\ \eta^{\pm}(f) = 0$$

$O^{(s)}(f)$ is the spin projector operator which characterises the spin s subspace as it belongs to the eigen value $s(s+1)$ of the invariant spin projection operator $O(f)$

$$O(f)\ O^{(s)}(f) = s(s+1)\ O^{(s)}(f).$$

The $(2s+1)$ independent vectors within the subspaces $\Lambda^{\pm}(f)\ O^{(s)}(f)$ have been chosen as the eigenstates of the helicity operator $\Sigma(\underline{f})$

$$\Sigma(\underline{f})\ Z^{(s_i)} = s_i\ Z^{(s_i)}\ (f)$$

If a non quantised version of scattering theory is employed the effect of the higher mass states can be ignored, covariant S and ρ matrices for particles of the lowest mass m can be defined for 'particles' and 'anti-particles' of half integral spins but only for particles of integral spins. The S and ρ matrices can be shown to obey the hole theory condition

$$S(f',t,f) = \Lambda^{\pm}(f)\ O^{(s)}(f)\ S(f',t,f) O^{(s)}(f) \Lambda^{\pm}(f)$$

The algebraic relations obeyed by the β matrices even for s=1 (The Duffin-Kemmer relation) are so complicated that the only hope of carrying out the analysis lay in using a method which makes the β algebra to depend on the properties of the γ matrices of Dirac algebra. For this reason the well known fusion theory representation of the β matrices has been used and $\phi(x)$ is considered as a spinor of rank . n=2s , each index of which transforms as a Dirac spinor index. The S and ρ matrices are then expanded as a finite sum of n fold Kronecker product of Dirac matrices. The covariant forms of S and ρ matrices are then obtained by using the hole theory condition and the form of the covariant equation $\rho' = S\rho S^{+}$ is written down. These are so complicated that hardly anything concerning the states of polarization can be got out of them. But fortunately Stapps method of reducing the equation mentioned earlier can be applied with some simple modification and an equation is obtained in which the s-matrix is given in the centre of mass frame and the $\rho(s)$ and $\rho'(f')$ matrices are given in their respective rest frames with certain rotations applied them. From this another equation which involves n fold Kronecker product of Pauli's matrices is obtained. This too is highly reducible and a $(2s+1)$ dimensional irreducible equation is extracted from it. This means that we have an equation in terms of the matrices $\vartheta_i^{(s)}$ of the irreducible spin's representation of the rotation group which is the usual non covariant form of the scattering equation.

The rotational and kinematical corrections which would have to be applied if one used the usual scattering theory for the analysis of the high energy multiple scattering experiments are shown to be the same as given by Stapp provided the rotational corrections are applied to each index of the polarization tensors.

The last chapter is devoted to the quantization of the free field $\varphi(x)$. Umezawa and Visconti (17) have obtained the commutation (or anticommutation) relation $\left[\varphi_\alpha(x), \varphi^{\dagger}_\beta(x')\right]_{\mp}$ by rather formal methods but have mentioned no details of the quantization except that the energy is not positive definite. By using the orthonormality relations of the rank n spinors it is shown that free field corresponds to an assembly of Pais-Uhlenbeck (18 ·) oscillators and Sudarshan's (18) method of quantizing such oscillators has been utilised. This involved the introduction of an indefinite matric and consequently the norms of states are not positive definite. The states which contain no particles of masses higher than m have positive norms and we invoke a subsidiary condition which restricts the physical states to have particles of the lowest mass m only.

The classical probability density for antiparticles of integral spins and of mass m has negative values. By calculating the expectation values of the electromagnetic current $j_\mu(x)$ for the state with one particle of mass m or an antiparticle of mass m, it is shown that one obtains the usual values for particles and current densities for arbitrary spins. The expression for $\left[\varphi_\alpha(x), \varphi^{\dagger}_\beta(x')\right]_{\mp}$ given in ref (17) has its own advantages but it involves certain recurrence relations. We have been able to derive the same relation in a closed form.

CHAPTER I.

As the present work leans heavily on Stapps theory for spin $\frac{1}{2}$ particles we begin by describing his method for obtaining the covariant forms of the S and $\mathcal{R}$ matrices. Also the parity non conserving terms which were not given in the original work have been obtained. This is necessary for our purpose as will be seen later on.

Consider a collision process involving spin $\frac{1}{2}$ and spinless particles. Let f and f' be the 4-momenta of the initial and final spin $\frac{1}{2}$ particle and 't' be the momentum of the centre of mass. The S-matrix element in momentum space depends on these three independent vectors and is denoted by $S(f'; t, f)$. For any two 4-vectors $u = (\underline{u} , u_4 = iu_0)$ and ω and Stapp has defined the following quantities

$$\gamma(u) \equiv \frac{\gamma_\mu . u_\mu}{(u_\nu u_\nu)^{\frac{1}{2}}} = \frac{\gamma . u}{(u.u)^{\frac{1}{2}}} \tag{1.1a}$$

$$\gamma(u,w) \equiv \gamma \left(\frac{u}{|u.u|^{\frac{1}{2}}} + \frac{\omega}{|\omega\,\omega|^{\frac{1}{2}}} \right) \tag{1.1b}$$

where the square root in $(u.u)^{\frac{1}{2}}$ is to be taken as positive or positive imaginary. Using the equation

$$\gamma(u)\gamma(u) = \gamma(u,\omega)\gamma(u,\omega) = 1 \tag{12.a}$$

one can prove that

$$\gamma(u)\gamma(u,\omega) = \gamma(u,\omega)\gamma(\omega) \tag{12.b}$$

The particle and anti-particle solutions of Dirac equation

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\psi(x) = 0 \qquad (1,3.a)$$

in momentum space obey the equation

$$(\pm i\gamma_\mu f_\mu + m)\, U_\pm(f) = 0 \qquad (1,4a)$$

or

$$\gamma(f)\, U_\pm\, (f) = \pm\, U_\pm\, (f) \qquad (1,4b)$$

$$U_\pm^\dagger(f)\,\gamma(f) = \pm\, U_\pm^\dagger(f) \qquad (1,4c)$$

where $U^\dagger(f)$ is the adjoint of $U\,(f)$ defined by

$$U^\dagger(f) = U^x(f)\,\gamma_4 \qquad (1,5a)$$

The adjoint of a matrix operator is defined by

$$\gamma^\dagger = \gamma_4\,\gamma^x\,\gamma_4 \qquad (1,5b)$$

For time like **vectors** u and w, $\gamma\,(u)$ and $\gamma\,(u,w)$ are self adjoint

$$\gamma^\dagger(u) = \gamma(u) \qquad (1,6a)$$

$$\gamma^\dagger(u,w) = \gamma(u,w) \qquad (1,6b)$$

The particle and antiparticle projection operators are given by

$$\Lambda^{\pm}(f) = \tfrac{1}{2}\ (1 \pm \gamma'(f)) \tag{1.7a}$$

Suppose now that the initial and final spin $\frac{1}{2}$ particles are both either 'particles' or 'antiparticles'. In these cases Stapp has imposed the hole theory condition on the S matrix

$$\gamma(f')\ S(f',t,f)\gamma(f) = S(f',t,f) \tag{1.8}$$

This equation can be obtained by noticing that in the case being considered the idempotent expression of the s matrix is

$$S = \Lambda^{+}(f')\ S\ \Lambda^{+}(f) \qquad \text{for particles only} \tag{1.9a}$$

or

$$S = \Lambda^{-}(f')\ S\ \Lambda^{-}(f) \qquad \text{for antiparticles only} \tag{1,9b}$$

Since

$$\gamma(f)\Lambda^{\pm}(f) = \Lambda^{\pm}(f)\gamma(f) = \pm\Lambda^{\pm}(f) \tag{1,10a}$$

equation (1.8) holds for both of these cases. It should be mentioned that the case considered here is less general than that considered by Stapp who has put only the condition 1.8 on the s-matrix. This corresponds to the fact that if the incident beam consists of a mixture of particles and antiparticles, by conservation of Baryon number or the lepton number the particles remain particles and the antiparticles remain antiparticles after the scattering. The idempotent expansion of the

S matrix is then

$$S = \Lambda^{+}(f')S\,\Lambda^{+}(f) + \Lambda^{-}(f')\;\;S\,\Lambda^{-}(f)$$

which again gives (1.8). For the sake of simplicity we have chosen the incident particles to be either 'particles' or 'antiparticles' throughout this work. Stapp has further defined a quantity $S'(k',t,k)$ given by

$$S\,(f',t,f) = \gamma(f',t)\;S'\;(k',t,k)\;\gamma(t,f) \qquad (1.11a)$$

where k, and k' are the initial and final relative momenta. The advantage of $S'(k',t,k)$ is that the hole theory condition for it becomes

$$S'(k',t,k)=\gamma(t)\;S'(k',t,k)\gamma(t) \qquad (1.11b)$$

This follows immediately by using (1.2a) and (1.2b). $S'(k',t,k)$ is now expanded in terms of the bases of Dirac algebra.

$$S'(k',t,k)=A+B_{\mu}\gamma_{\mu}+\tfrac{1}{2}\,C_{\mu\nu}\sigma_{\mu\nu}+D_{\mu}\,i\,\gamma_5\gamma_{\mu}+W\gamma_5$$

Application of (1.11b) gives no condition on the invariant A and

$$W = 0 \qquad (1.12a)$$

$$B_{\mu} \propto t_{\mu} \qquad (1.12b)$$

$$C_{\mu\nu}\;t_{\nu} = C_{\nu\mu}t_{\nu} = 0 \qquad (1.12c)$$

$$D_{\mu}t_{\mu} = 0 \qquad (1.12d)$$

In determining the form of vectors $C_{\mu\nu}$ and $D_{\mu\nu}$, Stapp has discarded all terms which are not invariant with respect to parity operation. For example D_μ is determined by the condition (1.12d) and Stapp[1] has selected only the first of the following three independent vectors with this property

$$D_\mu^{(1)} = -iN^{(1)} \quad D^{(1)} \epsilon_{\mu\nu\lambda\sigma} k_\nu k'_\lambda t_\sigma = D^{(1)} n'_\mu \tag{1.13a}$$

$$D_\mu^{(2)} = -iN^{(2)} \quad D^{(2)} \epsilon_{\mu\nu\lambda\sigma} k_\nu n'_\lambda t_\sigma = D^{(2)} n''_\mu \tag{1.13b}$$

$$D_\mu^{(3)} = -iN^{(3)} \quad D^{(3)} \epsilon_{\mu\nu\lambda\sigma} k'_\nu n'_\lambda t_\sigma = D^{(3)} n'''_\mu \tag{1.13c}$$

Normalised in such a way that

$$D_\mu^{(i)} \quad D_\mu^{(i)} = D^{(i)} \quad D^{(i)} \qquad i = 1,2, \text{ or } 3 \tag{1.13d}$$

$D_\mu^{(i)}$ for $i = 1$, is a pseudovector and for $i=1,2$ is a polar vector and consequently $i\gamma_5\gamma_\mu D_\mu^{(i)}$ for $i = 1$ is a scalar and is a pseudoscalar otherwise.

As we shall see later on it is necessary for our purpose to determine the S matrix obeying (1.11a) completely i.e. we must include the parity non conserving terms as well. Writing D_μ as a linear combination of n'_μ, n''_μ, n'''_μ we have

$$D_\mu = D^{(1)} n'_\mu + D^{(2)} n''_\mu + D^{(3)} m'''_\mu \tag{1.14a}$$

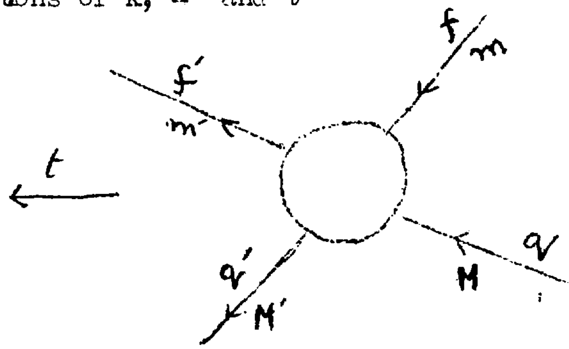
$C_{\nu\mu} = -C_{\mu\nu}$ satisfying (1.12c) is expanded in terms of 4 independent vectors t_μ, k_μ, k'_μ and n'_μ instead of t_μ, k_μ, k_μ' only as has been done for parity conserving case.

$$C_{\mu\nu} = a \Big\{ k_\mu k'_\nu - k_\nu k'_\mu + b\,(t_\mu k_\nu - t_\nu k_\mu)$$

$$+ \; c \;(t_\mu k'_\nu - k'_\mu t_\nu) \; + \; d\,(n'_\mu k_\nu - n'_\nu k_\mu) \qquad (1.15)$$

$$+ \; e \;(n'_\mu k'_\nu - n'_\nu k'_\mu) + G\,(n'_\mu t_\nu - n'_\nu t_\mu) \Big\}$$

Let the masses of initial and final spin zero particles be M and M' and their momenta be q and q' respectively. The masses of initial and final spin $\frac{1}{2}$ particles are denoted by m and m' respectively. Then by definitions of k, k' and t



$$t = f + q = f' + q' \qquad (1.16a)$$

$$k = f - q \qquad (1.16b)$$

$$k' = f' - q' \qquad (1.16c)$$

These equations give at once

$$k.t = M^2 - m^2 \qquad (1.17a)$$

$$k'.t = M'^2 - m'^2 \qquad (1.17b)$$

Multiplying (1.15) by t_ν and equating to zero the coefficients of k_μ, k'_μ and t_μ we obtain the values of the invariants b,c, etc.

$$b = \frac{k'.t}{t.t} = \frac{M'^2 - m'^2}{t.t} \qquad (1.18a)$$

$$c = -\frac{k.t}{t^2} = -\frac{M^2 - m^2}{t.t} \qquad (1.18b)$$

and

$$g = -\frac{e\,k.t + d\,k'.t}{t.t} \qquad (1.18c)$$

B_μ and $C_{\mu\nu}$ are finally given by

$$B_\mu = \frac{1}{(t.t)^{\frac{1}{2}}} B\, t_\mu \qquad (1.19a)$$

$$\begin{aligned} C_{\mu\nu} = {} & a\Big\{ k_\mu k'_\nu - k_\nu k'_\mu + \frac{M'^2 - m'^2}{t.t} (t_\mu k_\nu - t_\nu k_\mu) \\ & - \frac{M^2 - m^2}{t.t} (t_\mu k'_\nu - t_\nu k'_\mu) + d(n'_\mu k_\nu - n'_\nu k_\mu) \\ & + e(n'_\mu k'_\nu - n'_\nu k'_\mu) - \frac{d\,k.t + e\,k'.t}{t.t} (n'_\mu t_\nu - n'_\nu t_\mu) \Big\} \end{aligned} \qquad (1.19b)$$

To separate the parts refering to positive and negative energy states Stapp[1] has proved the following relations (1.22a and 1.22b)

$$\tfrac{1}{2} C_{\mu\nu} \sigma_{\mu\nu} \equiv \tfrac{1}{2} C_{\mu\nu} \left(-\tfrac{i}{2}\right) (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$$

$$= -\tfrac{1}{2}\, c_{\mu\nu}\, \gamma_\mu \gamma_\nu$$

$$= -\tfrac{i}{4}\; c_{\mu\nu}\, \gamma_\sigma \gamma_\varrho\, \epsilon_{\varrho\sigma\mu\nu}\, \gamma_5 \qquad (1.20a)$$

If for $\gamma_\sigma \gamma_\varrho$ in the last equation we write

$$\gamma_\sigma \gamma_\varrho = \frac{\gamma.t\ \gamma.t}{t.t}\ \gamma_\sigma \gamma_\varrho$$

$$= \frac{\gamma.t}{t.t}\ (-\gamma_\sigma \gamma.t + 2\, t_\sigma)\ \gamma_\varrho \qquad (1.20b)$$

$$= \frac{\gamma.t}{t.t}\ (\gamma_\sigma \gamma_\varrho \gamma.t - 2t_\varrho \gamma_\sigma + 2 t_\sigma t_\varrho\ \ )$$

and utilise the condition

$$\gamma(t)\tfrac{1}{2}\, c_{\mu\nu}\sigma_{\mu\nu}\, \gamma(t) = \tfrac{1}{2}\, c_{\mu\nu}\sigma_{\mu\nu} \qquad (1.21)$$

$\frac{1}{2}\, c_{\mu\nu}\sigma_{\mu\nu}$ can be shown to be given by

$$\tfrac{1}{2}\, c_{\mu\nu}\, \sigma_{\mu\nu} = i\gamma(t)\, \gamma_5\, \gamma_\varrho\, c_\varrho \qquad (1.22a)$$

with

$$c_\varrho = \tfrac{1}{2}\ (-it_\sigma)\ \epsilon_{\varrho\sigma\mu\nu}\, c_{\mu\nu}\ \left|t.t\right|^{-\frac{1}{2}} \qquad (1.22b)$$

On substituting for $c_{\mu\nu}$ from (1.19b), c_ϱ is at once seen to be

$$c_\varrho = c^{(1)}\, n'_\varrho + c^{(2)}\, n''_\varrho + c^{(3)}\, n'''_\varrho \qquad (1.23c)$$

and hence $s(k',t,k)$ is of the form

$$S'(k',t,k) = A + B\gamma(t) + i\gamma(t)\gamma_5\,\gamma.(C^{(1)}n' + C^{(2)}n'' + C^{(2)}n''')$$
$$+ i\gamma_5\,\gamma.(D^{(1)}n' + D^{(2)}n'' + D^{(3)}n''') \qquad (1.24a)$$

This can be put in the form in which particle and antiparticle parts are separated by writing

$$1 = \Lambda^+(t) + \Lambda^-(t) \qquad (1.25a)$$

in the first and the last term and

$$\gamma(t) = \Lambda^+(t) - \Lambda^-(t) \qquad (1.25b)$$

in the 2nd and 3rd term on the R.H.S. in (1.24a)

$$S' = \Lambda^+(t)\left\{ F^+ + i\gamma_5 \;\cdot\; (H^+n' + K^+n'' + E^+n''') \right\}$$
$$+ \Lambda^-(t)\left\{ F^- + i\gamma_5 \;\cdot\; (H^-n' + K^-n'' + E^-n''') \right\} \qquad (1.24b)$$

It is to be noted that on account of

$$n'.t = n''.t = n'''.t = o \qquad (1.26)$$

$\Lambda^{\pm}(t)$ commute with $i\gamma_5\,\gamma.n'$ etc., and n', n'', n''' reduce to 3 vectors in the centre of mass frame. For the sake of conciseness let us introduce the notation

$$G^{\pm} n_\mu = H^{\pm} n'_\mu + K^{\pm} n''_\mu + E^{\pm} n'''_\mu \qquad (1.27)$$

The S matrix is related to S' by

$$S\,(f't,f) = \gamma(f',t)\; S'\;(k',t,k)\,\gamma(f,t)$$

$$= \gamma(f',t)\,\gamma(t) \quad S'\;(k',t,k)\,\gamma\,(t)\,\gamma\,(f,t)$$

$$= \Lambda^{+}(f')\;\gamma(f't)\;\;\gamma(t)\quad\left\{\;F^{+}+i\gamma_5\,\gamma.nG^{\pm}\right\}\gamma(t)\gamma(f,t)\Lambda^{+}(f)$$

$$+\;\Lambda^{-}(f')\;\;\gamma(f',t)\gamma(t)\quad\left\{\quad F^{-}+i\gamma_5\;\gamma.nG^{-}\right\}\gamma(t)\gamma(f,t)\cdot\Lambda^{-}(f)$$

$$(1.28)$$

Stapp has let this form of S(f',t,f) remain as it is, but equations (1.9a) and (1.9b) for particles only or antiparticles only show that for 'particle' to 'particle' scattering F^{-} and G^{-} vanish and for antiparticle to anti-particle scattering F^{+} and G^{+} vanish. Denoting the S matrix for these two cases by $S^{\pm}$

$$S^{\pm}\;(f',t,f) = \Lambda^{\pm}(f')\,\gamma\,(f',t)\gamma(t)\;\;(F^{\pm}+G^{\pm}\;i\gamma_5\gamma.n)\gamma(t)\gamma(f,t)\;\Lambda^{\pm}(f)$$

$$= \;\;\gamma(f',t)\gamma(t)\,\Lambda^{\pm}(t)\quad(F^{\pm}+G^{\pm}\;i\gamma_5\gamma.n)\;\gamma\,(t)\gamma(f,t)$$

$$= \;\gamma\,(f',t)\;\gamma\,(t)\quad S'^{\pm}(\;k',t,k)\;\;\gamma\,(t)\,\gamma\,(f,t) \qquad (1.29)$$

The operator $\gamma(t)\;\;\gamma\,(f,t)$ is closely related to the Lorentz transformation (1) between the centre of mass frame and the rest frame of the incident particle. This can be seen as follows.

Let $\underline{f}$ be along the x axis. The Lorentz operator corresponding to the Lorentz transformation which brings the particle to rest is given by

$$L(f) = e^{\frac{1}{2}i\,\gamma_1\gamma_4\,\vartheta} = e^{\frac{1}{2}i(-i\gamma_4\alpha_1)\gamma_4\,\vartheta} \tag{1.30a}$$

$$= e^{-\frac{1}{2}\alpha_1\vartheta} = \cosh\frac{\vartheta}{2} - \alpha_1 \sinh\frac{\vartheta}{2} \tag{1.30b}$$

where ϑ is given by

$$\sinh\vartheta = \frac{U}{\sqrt{1-U^2}} \;, \qquad f = m(U)U = \frac{mU}{\sqrt{1-U^2}} \tag{1.30c}$$

A little calculation shows that

$$L(f) = \frac{1}{\sqrt{2}}\left\{\left(1+\frac{1}{\sqrt{1-U^2}}\right)^{\frac{1}{2}} - \alpha_1 \frac{\frac{U}{\sqrt{1-U^2}}}{\left(1+\frac{1}{\sqrt{1-U^2}}\right)^{\frac{1}{2}}}\right\}$$

$$= \gamma_4 \frac{-i\,(\gamma_1 f_1+\gamma_4 f_4) + \gamma_4 m}{\sqrt{2m\,(m+f_o)}} \tag{1.31a}$$

Writing $\gamma_1 f_1+\gamma_4 f_4 = \gamma_\mu f_\mu$ we see that for arbitrary direction of $\underline{f}$, $L(f)$ is given by

$$L(f) = \gamma_4 \frac{-i\gamma.f + m\gamma_4}{\sqrt{2m\,(f_o+m)}} \tag{1.31b}$$

In the centre of mass frame $\gamma(t) = \gamma_4$ and the calculation of L(f) (1) and $\gamma(t)\ \gamma(f,t)$ in this frame shows that

$$\gamma(t_1)\ \gamma(t_1 f_1) = L(f_1) \tag{1.32a}$$

and hence

$$\gamma(f_1',t_1)\ \gamma(t_1) = L^+(f_1') = L^{-1}(f_1') \tag{1.32b}$$

where the subscript 1 denotes centre of mass values. Thus

$$S\ (f_1',t_1,f_1) = L^{-1}\ (f_1')\ \ S'(k_1',t_1,k_1)\ L(f_1) \tag{1.33}$$

The first factors $L(f_1)$ is the Lorentz operator which reduces the incoming spinors from their centre of mass values $u(f_1)$ to their values in the rest frame of the incident Dirac particles. The unitary operator S' then gives the effect of scattering upon these spinors and finally $L(f_1)$ converts these spinors (in the rest frames) to their values as seen in the centre of mass frames.(1)

The vectors n', n'', n''' are all normal to t, k, k' and in the centre of mass frame reduce to

$$n_1' = (\underline{N}',o) \qquad \underline{N}' = \frac{\underline{k}\wedge\underline{k}'}{|\underline{k}\wedge\underline{k}'|} \tag{1.34a}$$

$$n_2' = (\underline{N}'',o) \qquad \underline{N}'' = \frac{\underline{k}\wedge\underline{N}'}{|\underline{k}\wedge\underline{N}'|} \tag{1.34b}$$

$$n_3' = (\underline{N}''',o) \qquad \underline{N}''' = \frac{\underline{k}'\wedge\underline{N}'}{|\underline{k}'\wedge\underline{N}'|} \tag{1.34c}$$

Since

$$\Lambda^{+}(t_1) = \tfrac{1}{2}(1+\gamma_4) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.35a}$$

$$\Lambda^{-}(t_1) = \tfrac{1}{2}(1-\gamma_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.35b}$$

$$i\gamma_5 \underline{\gamma}\cdot\underline{N} = \begin{pmatrix} \underline{\sigma}\cdot\underline{N} & 0 \\ 0 & -\underline{\sigma}\cdot\underline{N} \end{pmatrix} \tag{1.35c}$$

$S'^{\pm}(k^1, t, k)$ in the centre of mass frame becomes

$$S'^{+}(k_1', t_1, k_1) = \begin{pmatrix} F^{+} + \underline{\sigma}\cdot(H^{+}\underline{N}' + K^{+}\underline{N}'' + E^{+}\underline{N}''') & 0 \\ 0 & 0 \end{pmatrix} \tag{1.36a}$$

$$S'^{-}(k_1', t_1, k_1) = \begin{pmatrix} 0 & 0 \\ 0 & F^{-} + \underline{\sigma}\cdot(H^{-}\underline{N} + K^{+}\underline{N}'' + E^{+}\underline{N}''') \end{pmatrix} \tag{1.36b}$$

<u>Density matrices</u>:-

Stapp has obtained the forms of the covariant density matrices $\rho(f)$ and $\rho'(f')$ also by the application of the hole theory condition.

$$\rho(f) = \Lambda^{\pm}(f)\rho(f)\Lambda^{\pm}(f)$$

which gives

$$\rho(f) = \gamma(f)\ \rho(f)\ \gamma(f)$$

Again expanding $\rho(f)$ in terms of the 16 bases of Dirac algebra it can be shown that

$$\rho(f) = \left[\tfrac{1}{2}\mathrm{tr}\,\rho(f)\right] \Lambda^{\pm}(f) \quad (1 + i\gamma_5 \gamma . p^{\pm})$$

$p^{\pm}$ are 4-vectors orthogonal to f

$$p^{\pm}_{\mu} \; f_{\mu} = o$$

and in the rest frame $\underline{f} = o$, $p^{\pm}_{\mu}$ are 3-dimensional vectors . $p^{\pm}$ are relativistic generalization of the polarization vectors of the non-relativistic theory.

CHAPTER II

The generalization of Stapps theory for spin $\frac{1}{2}$ particles to particles of higher spins will be based on the Bhabha type equation

$$(\beta_\mu \frac{\partial}{\partial x_\mu} + ms)\ \varphi(x) = o \qquad (2.1)$$

With β_μ' s satisfying a certain algebraic relation of degree 2s+1 , s is the highest spin contained in (2.1). There is considerable literature associated with this equation[4]. We give below some of the relevant facts connected with it and then by writing $\varphi(x)$ as a momentum space integral 'particle' and 'antiparticle' projection operators are derived. Later on spin projection operators are introduced and an orthonormal basis in the momentum space spinors is established in the next chapter.

The requirement of Lorentz covariance of (2.1) demands that under a Lorentz transformation [3]

$$x_\mu \rightarrow x'_\mu = a_{\mu\nu}\, x_\nu \qquad (2.2a)$$

$\varphi(x)$ should transform as

$$\varphi(x) \rightarrow \varphi'(x') = \Lambda\ \varphi(x) \qquad (2.2b)$$

with

$$\Lambda\ a_{\nu\mu}\, \beta_\mu\ \Lambda^{-1} = \beta_\nu \qquad (2.2c)$$

or $$\Lambda^{-1}\ \beta_\nu \Lambda = a_{\nu\mu}\beta_\mu \tag{2.2d}$$

Let $I_{\rho\sigma}$ denote the infinitesimal generators of the Lorentz group

$$\Lambda = 1 + \tfrac{1}{2}\epsilon_{\rho\sigma}\ I_{\rho\sigma} \tag{2.3a}$$

$$a_{\rho\sigma} = \delta_{\rho\sigma} + \epsilon_{\rho\sigma}, \quad \epsilon_{\rho\sigma} = -\epsilon_{\rho\sigma} \tag{2.7b}$$

Substituting for Λ and $a_{\rho\sigma}$ in (2.2d) and keeping only the terms up to first order in $\epsilon_{\nu\mu}$ we get

$$\tfrac{1}{2}\epsilon_{\rho\sigma}\ (I_{\rho\sigma}B_\nu - \beta_\nu I_{\rho\sigma}) = -\epsilon_{\nu\mu}\beta_\mu \tag{2.4a}$$

Writing $\epsilon_{\nu\mu}$ in the form

$$\begin{aligned} \epsilon_{\nu\mu}B_\mu &= \tfrac{1}{2}\ (\epsilon_{\rho\mu}\delta_{\rho\nu} - \epsilon_{\mu\sigma}\ \delta_{\sigma\nu})\beta_\mu \\ &= \tfrac{1}{2}\epsilon_{\rho\sigma}(\delta_{\rho\nu}\beta_\sigma - \delta_{\sigma\nu}\ \beta_\rho) \end{aligned} \tag{2.4b}$$

$I_{\rho\nu}$ are seen to satisfy the commutation relation

$$I_{\rho\sigma}\ \beta_\nu - \beta_\nu\ I_{\rho\sigma} = \delta_{\sigma\nu}\beta_\rho - \delta_{\rho\nu}\ \beta_\sigma \tag{2.5}$$

There are two types of theories connected with the equation (2.1).[3] The first is one in which $\varphi(x)$ satisfies Kelin-Gordon equation

(2.6a)

and β_μ' satisfy the characteristic equation

$$\beta_\mu^{2s-1} \quad (\beta_\mu^2 - 1) = 0 \qquad \mu = 1,2,3, \text{ or } 4. \tag{2.6b}$$

and $I_{\rho\sigma}$ is given by

$$I_{\rho\sigma} = \beta_\rho \ \beta_\sigma - \beta_\sigma \ \beta_\rho \tag{2.6c}$$

For $s > 1$, β_μ cannot be hermitian. For if β_μ were hermitian its eigen values would be $\pm$ 1 and 0 and β_μ would satisfy

$$\beta_\mu \quad (\beta_\mu^2 - 1) = 0 \tag{2.7}$$

which is not the characteric equation (2.6b). Also Harish-Chandra[3] has shown that for $s > 1$ the algebra generated by (2.6a) is not finite.

The theory which we shall adopt in this work is the one in which β_μ satisfy the characteristic equation

$$(\beta_\mu+s)\ (\beta_\mu+s-1)\ \ldots\ (\beta_\mu-s) = \prod_{s_i=-s}^{s_i=+s} (\beta_\mu - s_i) = 0 \tag{2.8a}$$

and $$I_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu = [\beta_\mu \beta_\nu] \tag{2.8b}$$

That is $[\beta_\mu\beta_\nu]$ satisfy

$$[\beta_\rho, \beta_\sigma]\beta_\nu - \beta_\nu[\beta_\rho, \beta_\sigma] = \delta_{\sigma\nu} \ \beta_\rho \ - \delta_{\rho\nu} \ \beta_\sigma \tag{2.8c}$$

The eigenvalues of β_μ are $s, s-1, \ldots -s+1, -s$ and all β_μ can be taken to be hermitian. The trouble with this theory is that $\phi(x)$ does not satisfy Klein-Gordon equation but an equation (to be written down later) which shows that for $s > 1$ there are several values of the mass of the particle. For $s = \frac{1}{2}$ and $s = 1$ both of these theories become equivalent. This is easily seen by noting that in these cases the characteristic equation (2.6b) and (2.8a) become the same.

For $S = \frac{3}{2}$, the algebra generated by (2.8a) and (2.8c) has been investigated by Madhavarao[5]. For s=1 we have the Duffin-Kemmer-Petiau theory[6]

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + m\right) \phi(x) = 0 \tag{2.9a}$$

$$[\beta_\rho, \beta_\sigma]\beta_\nu - \beta_\nu[\beta_\rho, \beta_\sigma] = \delta_{\sigma\nu}\beta_\rho - \delta_{\rho\nu}\beta_\sigma \tag{2.9b}$$

$$\beta_\mu(\beta_\mu^2 - 1) = 0 \tag{2.9c}$$

From the last two equations one can obtain the Duffin-Kemmer relation

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\nu}\beta_\mu \tag{2.9d}$$

Defining η_μ by

$$\eta_\mu = 2\beta_\mu^2 - 1 \tag{2.10a}$$

we have for η_μ's the relations[6]

$$\eta_\mu^2 = 1 \qquad \text{no summation over } \mu \tag{2.10b}$$

$$\eta_\mu \eta_\nu = \eta_\nu \eta_\mu \tag{2.10c}$$

$$\beta_\mu \eta_\nu = - \eta_\nu \beta_\mu \qquad \mu \neq \nu \tag{2.10d}$$

$$\beta_\mu \eta_\mu = \beta_\mu = \eta_\mu \beta_\mu \qquad \text{no summation over } \mu \tag{2.10e}$$

With the help of η_μ's and β_μ's one can form 126 basis of the semisimple algebra generated by (2.9d) or equivalently by (2.9a) and (2.9b). Moreover there are three elements which commute with all the base elements. These are (6)

$$1, \quad \sum_{\mu=1}^{4} \eta_\mu - \sum_{\mu<\nu} \eta_\mu \eta_\nu \quad , \eta_1 \eta_2 \eta_3 \eta_4 \, (1 - \sum_\mu \eta_\mu)$$

and thus (6) there are 3 irreducible representations of this algebra of dimensions 10, 5 and 1

$$10^2 + 5^2 + 1^2 = 126$$

There are many ways of proving that the 1o dimensional representation belongs to spin one and the 5 dimensional representation belongs to spin zero(; 3)

It can be easily verified that matrices β_μ given by (7)

$$\beta_\mu = \tfrac{1}{2} \quad (1 \times \gamma_\mu + \gamma_\mu \times 1) \tag{2.11}$$

satisfy the relation (2.9b,c) and hence (2.9d).

X stands for Kronecker product and γ_μ are hermitian Dirac matrices. 1 is unit four dimensional matrix. In this 16 dimensional representation φ is a 16 dimensional vector. φ can also be looked upon as a 4 x 4 matrix $\varphi_{\alpha\beta}$ with each index transforming as a Dirac spinor under Lorentz transformation. Klein[8] has developed a theory of Duffin-Kemmer formalism by utilising the fact that the quantities

$$u = (\gamma_5 \; C^{-1})_{\alpha\beta} \; \varphi_{\alpha\beta}(x) \tag{2.12a}$$

$$u_\mu = (i\gamma_5\gamma_\mu C^{-1})_{\alpha\beta} \; \varphi_{\alpha\beta}(x) \tag{2.12b}$$

to which only the antisymmetric part of $\varphi_{\alpha\beta}$ contributes, constitute the wave functions of the 5 dimensional spin zero equation. The symmetric part contributes to

$$F_{\mu\nu} = (\sigma_{\mu\nu} \; C^{-1})_{\alpha\beta} \, \varphi_{\alpha\beta}(x) \tag{2.12c}$$

$$A_\mu = (\gamma_\mu C^{-1})_{\alpha\beta} \, \varphi_{\alpha\beta}(x) \tag{2.12d}$$

$F_{\mu\nu}$ and A_μ satisfy the usual spin one wave equations. C is the charge conjutation matrix and $\varphi(x)$ satisfies Duffin-Kemmer equation. Thus the 5 independent components of the anti-symmetric $\varphi_{\alpha\beta}$ belongs to spin zero and the 1o components of the symmetric $\varphi_{\alpha\beta}$ belong to the spin 1 sub - spaces. Klein has also shown the $m \; C^{-1}_{\alpha\beta} \; \varphi_{\alpha\beta}$ is the trivial component and is identically zero.

It is very difficult to generalise this method to the case of higher spins but a heuristic way of identifying the symmetric part of the spinor space with the highest spin is the following one.[9]

The symmetric part $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}$ defines an invariant subspace of the Duffin-Kemmer algebra. This is a 1o dimensional subspace. The Dirac equation belongs to $D^{\frac{1}{2},0} + D^{0,\frac{1}{2}}$ representation of the Lorentz group. The Duffin-Kemmer equation with the representation 2.11 of the β matrices belongs to the representation

$$(D^{\frac{1}{2},0} \oplus D^{0,\frac{1}{2}}) \times (D^{\frac{1}{2},0} \oplus D^{0,\frac{1}{2}}) \tag{2.13a}$$

By the generalised Clebsch Gordon theorem the spin one part of the above product representation is (11)

$$D^{1,0} \oplus D^{\frac{1}{2},\frac{1}{2}} \oplus D^{0,1} \tag{2.13b}$$

where each irreducible spin one representation has been taken only once in (2.13b). $D^{\frac{1}{2},\frac{1}{2}}$ occurs twice in (2.13a) but in (2.13b) only one $D^{\frac{1}{2},\frac{1}{2}}$ is included. The spin one representation (2.13b) is (3 + 4+ 3 =)10 dimensional and identifying the symmetric spinor space $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}$ with the representation (2.13b) it follows that this subspace belongs to a spin one representation of the Lorentz group.

Coming now to the case of arbitrary spins, we take the following representation of the β_μ's. (7)

$$\beta_\mu = \tfrac{1}{2} \sum_{\gamma=1}^{n} \Gamma_\mu^{(\gamma)} \tag{2.14a}$$

where

$$\Gamma_\mu^{(\gamma)} = 1 \times 1 \times 1 \; \ldots\ldots \; 1 \times \gamma_\mu \times 1 \; \ldots \times 1 \tag{2.14b}$$

contains $n = 2s$ factors and γ_μ occurs just once in the r . th factor. $\varphi(x)$ has $4^n = 4^{2s}$ components and may be considered as a spinor of r anh n

$$\varphi = \varphi_{\alpha_1 \alpha_2 \dots \alpha_n}$$

which transforms under Lorentz transformations as an 'n' fold Kronecker product of Dirac spinors. This 4^n dimensional representation belongs to the representation

$$\underbrace{(D^{\frac{1}{2},0} \oplus D^{0\frac{1}{2}}) \times (D^{\frac{1}{2},0} \oplus D^{0,\frac{1}{2}}) \;\dots\; \times (D^{\frac{1}{2},0} \oplus D^{0\frac{1}{2}})}_{n \text{ factor}} \qquad (2.15a)$$

Again the spin s part in the product representation (2.15) keeping each irreducible representation only once is

$$D^{\frac{n}{2},0} \oplus D^{\frac{n-1}{2},\frac{1}{2}} \oplus D^{\frac{n-2}{2},\frac{2}{2}} \dots \oplus D^{0,\frac{n}{2}} \qquad (2.15b)$$

The dimensions of this representation are

$$(n+1).\,1 + n.\,2 + (n-1)3 + (n-2).n \dots\dots [n - (n-2)]\, n$$

$$+ \; [n - (n-1)] \; (n+1)$$

$$= \; l.\,1 + (l-1).2 + (l-2)\,3 \dots\dots \; [l - (l-1)]\, l$$

with $l = n+1$

The sum of the series is

$$l\,(1+2+3 \dots l) - (1.2 + 2.3 + 3.4 \dots + (l-1)l)$$

$$= 1 . \frac{l(l+1)}{2} - \frac{l(l-1)(l+1)}{3} = \frac{l(l+1)(l+2)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6} \tag{2.16}$$

But this is exactly the number of independent symmetric components of the spinor $\varphi_{\alpha_1\alpha_2\ldots\alpha_n}$. Hence the invariant subspace defined by the completely symmetric spinors

$$\varphi_{\alpha_1\alpha_2\ldots\alpha_i\alpha_{i+1}\ldots\alpha_j\ldots\alpha_n} = \varphi_{\alpha_1\alpha_2\ldots\alpha_j\alpha_{i+1}\ldots\alpha_i\ldots\alpha_n} \tag{2.17}$$

represents the highest spin value $S = \frac{n}{2}$. Invariance of the subspace means that if $\varphi_{\alpha_1\alpha_2\ldots\alpha_n}$ is completely symmetric then

$$(\beta_\mu \varphi)_{\alpha_1\alpha_2\ldots\alpha_n}$$

is also symmetric. This can easily be proved by using the form (2.14) of β_μ. If a Dirac spinor transforms as

$$\psi_\alpha \longrightarrow L_{\alpha\beta}\psi_\beta \tag{2.18a}$$

under a Lorentz transformation, $\varphi_{\alpha\beta\ldots\ldots\gamma}$ transforms as (7)

$$\varphi_{\alpha\beta\ldots\ldots\gamma} \longrightarrow (L \times L \times L \ldots \times L\ \varphi)_{\alpha\beta\ldots\gamma} \tag{2.18b}$$

$$= L_{\alpha\alpha'}L_{\beta\beta'}\ldots L_{\gamma\gamma'}\ \varphi_{\alpha'\beta'\ldots\gamma'} \tag{2.18c}$$

$$= (L^{(n)} \varphi)_{\alpha\beta\ldots\gamma} \tag{2.18d}$$

(7)
Writing

$$\eta_\mu = \gamma_\mu \times \gamma_\mu \times \gamma_\mu \;\ldots\ldots \times \gamma_\mu \times \ldots \gamma_\mu \qquad (2\text{-}19)$$

(all factors are γ_μ and ~~in~~ no summation over μ)

The adjoint of φ is defined by

$$\varphi^\dagger = \varphi^x \eta_4 \qquad (2.20a)$$

and the adjoint of an operator is defined by

$$A^\dagger = \eta_4 \; A^x \eta_4 \qquad (2.20b)$$

The adjoint of $L^{(n)}$ is

$$L^{(n)\dagger} = \eta_4 L^{(n)x} \eta_4 = \gamma_4 x \gamma_4 \ldots x \gamma_4 \;\; L^x x L^x \ldots x L^x \gamma_4 x \gamma_4 \ldots x \gamma_4$$

$$= \gamma_4 \; L^x \gamma_4 \; x \gamma_4 \; L^x \gamma_4 a \ldots a \;_4 \; L^x \gamma_4$$

$$= L^{-1} \; x \, L^{-1} \ldots x \, L^{-1}$$

$$= (L^{(n)})^{-1} \qquad (2.20c)$$

Defining $\beta(u)$ by

$$\beta(u) = \frac{\beta_\circ u}{(u.u)^{\frac{1}{2}}}, \qquad (u.u)^{\frac{1}{2}} \text{ positive or positive imaginary} \qquad (2.21)$$

For time like u, $\beta(u)$ is self adjoint

(2.21)

$$\beta(f)^{+} = \gamma_4 \, \beta(u)^{\times}(u) \, \eta_4 = \beta(u) \tag{2.22}$$

This is so because β_μ anticomutes with η_4 for $\mu \neq 4$ and commutes with η_4 for $\mu=4$.

The Lorentz transformation defined by

$$x_\mu \mathcal{L}_{\mu\nu}(t) = x_{1\nu} \equiv x_\nu(\underline{t}=o) \tag{2.23a}$$

$$x_\mu = \mathcal{L}_{\mu\nu}(t) \, x_{1\nu} \tag{2.23b}$$

(where $x_{1\nu} = x_\nu(\underline{t}=o)$, is the value of the 4-vector x_ν in the Lorentz frame in which the space part $\underline{t}$ of t vanishes) is represented in the Dirac space by L(t) with the properties

$$L(t)\gamma_\mu L^{-1}(t) = \mathcal{L}_{\mu\nu} \, \gamma_\nu \tag{2.23c}$$

or

$$L^{-1}(t) \, \gamma_\nu \, L(t) = \gamma_\mu \mathcal{L}_{\mu\nu}(t) \tag{2.23d}$$

As shown by Stapp[1] the Lorentz operator which brings the Dirac spinor $u(f_1)$ to its values in the rest frame $\underline{f} = o$ is given by

$$\gamma(t_1)\,\gamma(t_1, f_1) = L(f_1) \tag{2.24a}$$

$$\gamma(f_1', t_1)\,\gamma(t_1) = L^{-1}(f_1') \tag{2.24b}$$

If x is either f or f'

$$L(t)\,\gamma(t)\,\gamma(x,t)\quad L^{-1}(t) = \gamma(t_1)\,\gamma(x_1,t_1) \tag{2.25a}$$

$$L^{-1}(t)\,\gamma(t_1)\,\gamma(x_1,t_1)\;L(t) = \gamma(t)\,\gamma(x,t) \tag{2.25b}$$

In the space of spinors of rank 'n'. The Lorentz operators are given by

$$L^{(n)}(f) = (L(f) \times L(f) \ \ldots\ L(f) \tag{2.26a}$$

$$L^{(n)-1}(f) = L^{-1}(f) \times L^{-1}(f) \ \ldots\ L^{-1}(f)\,. \tag{2.26b}$$

Particle and antiparticle projection operators.

It is profitable to consider first the simpler case of spin 1. One can verify by using the Duffin-Kemmer relation that $\beta(f) = \dfrac{\beta.f}{(f.f)^{\frac{1}{2}}}$ satisfies the characteristic equation.

$$B(f)\left\{\beta^2(f) - 1\right\} = o \tag{2.27}$$

We now make use of the following well known theorem (10)

If A is a linear hermitian operator which satisfies the characteristic equation

$$\prod_{i=1}^{n} (A-A_i) = o \tag{2.28a}$$

A_i are real c members and all A_i are distinct than there are n projection operators

$$\eta^{(i)} = \prod_{j\neq i} \frac{(A-A_j)}{A_i - A_j} \qquad i = 1,2,\ \ldots.n \tag{2.28b}$$

with the properties

$$A\,\eta^{(i)} = A_i\,\eta^{(i)} \qquad (2.28c)$$

$$\eta^{(i)}\eta^{(j)} = \delta_{ij}\quad \eta^{(i)} \qquad (2.28d)$$

$$\sum_{i=1}^{\mu} \eta^{(i)} = 1 \qquad (2.28e)$$

This theorem shows that these are 3 projection operators

$$\eta^{+}(f) = \tfrac{1}{2}\left\{ \beta^2(f)+ 1 \right\} \qquad (2.29a)$$

$$\eta^{-}(f) = \tfrac{1}{2}\left\{ \beta^2(f) - 1 \right\} \qquad (2.29b)$$

$$\eta^{o}(f) = \left\{ 1-\beta^2(f) \right\} \qquad (2.29c)$$

If we take f on the mass shell $f^2 = -m^2$, then $\eta^{\pm}(f)$ satisfy the particle and anti-particle equations respectively

$$(i\beta.f \pm m)\,\eta^{\pm}(f) = o \qquad (2.30a)$$

while $\eta^{o}(f)$ satisfies

$$\beta.f\quad \eta^{o}(f) = o \qquad (2.30b)$$

Let us now perform the fourier analysis of $\varphi(x)$. Writing

$$\varphi(x) = \int e^{if.x}\,\tilde{\varphi}(f)\quad d^4f \qquad (2.31)$$

and substituting in the Duffin Kemmer equation we get

$$(i\ \beta.f + m)\tilde{\varphi}(f) = 0 \tag{2.31a}$$

or $$\beta.f\tilde{\varphi}(f) = im\tilde{\varphi}(f) \tag{2.31b}$$

Multiplying (2.27) by $\tilde{\varphi}(f)$ and using the last equation

$$\frac{im}{(f.f)^{\frac{1}{2}}}\left[-m^2 - f.f\right] = 0 \tag{2.32}$$

This shows that f^2 has a single value equal to $-m^2$. For arbitrary f we define

$$\eta^{\mp}(f) = \frac{1}{-2im^2}\left[i\,(\beta.f)^2 \pm m\beta.f)\right] = \tag{2.33a}$$

$$\eta^{o}(f) = \frac{1}{-2im^2}\left[-2im^2 - 2i\,(\beta.f)^2\right] \tag{2.33b}$$

so that

$$\eta^{+}(f) + \eta^{-}(f) + \eta^{o}(f) = 1 \tag{2.33c}$$

By using the relation $(\beta.f)^2 = f^2\beta.f$ one can show that $\eta^{-}(f)\eta^{+}(f)$ and $\eta^{-}(f)\,\eta^{o}(f)$ contain a factor (f^2+m^2). More precisely

$$\eta^{-}(f)\eta^{+}(f) = \frac{-1}{(-2im)^2}(\beta.f)^2\ (f^2+m^2) \tag{2.34a}$$

$$\eta^{-}(f)\,\eta^{o}(f) = \frac{-2i}{(2im^2)^2}\left\{i(\beta.f)^2 + m\,\beta.f\right\}(f^2+m^2) \tag{2.34b}$$

In view of the fact that

$$\eta^{-}(f) = \left(i\,\beta\cdot f + m\right)\frac{\beta\cdot f}{-2i\,m^2} \qquad (2.35)$$

The solution of (2.31a) can be written

$$\tilde{\varphi}(f) = \delta(f^2+m^2)\,\beta\cdot f\left\{\eta^{+}(f)\,\chi(f) + \eta^{0}(f)\,\chi_0(f)\right\} \qquad (2.36)$$

Where $\chi(f)$ and $\chi_0(f)$ are arbitrary spinors in momentum space. Substituting for $\tilde{\varphi}(f)$ in (2.31) and integrating over f_0 with the help of the δ function one obtains easily

$$\varphi(x) = \int\frac{d^3f}{2\omega}\Big[e^{if\cdot x}\,\beta\cdot f\left\{\eta^{+}(f)\,\chi(f) + \eta^{0}(f)\,\chi_0(f)\right\} + e^{-if\cdot x}(-1)\,\beta\cdot f\left\{+\,\eta^{+}(-f)\,\chi(-f) + \eta^{0}(-f)\,\chi_0(-f)\right\} \qquad (2.37)$$

The 4-momentum f occuring in this equation lies on the mass shell

$$f_0 = w = +\sqrt{f^2 + m^2} \qquad (2.38a)$$

Hence

$$\frac{\beta\cdot f}{im} = \frac{\beta\cdot f}{(f\cdot f)^{\frac{1}{2}}} = \beta(f) \qquad (2.38b)$$

and $\eta^{\pm}(f), \eta^{0}(f)$ occurring in this equation can be written

$$\eta^{\pm}(f) = \frac{1}{2}\left\{\beta^2(f) \pm \beta(f)\right\} \qquad (2.39a)$$

$$\eta^{0}(f) = 1 - \beta^2(f) \qquad (2.39b)$$

These satisfy all the properties of projection operators. Also we have from their definitions

$$\eta^{+}(-f) = \eta^{-}(f) \tag{2.40a}$$

$$\eta^{0}(-f) = \eta^{0}(f) \tag{2.40b}$$

and

$$\beta.f\,\eta^{\pm}(f) = \pm\, i\eta^{\pm}(f)\, m = \pm\,(f\cdot f)^{\frac{1}{2}}\eta^{\pm}(f) \tag{2.41a}$$

$$\beta\cdot f\;\eta^{0}(f) = 0 \tag{2.41b}$$

By virtue of these equations the fourier decomposition of $\varphi(x)$, equation (2.37) can be written

$$\varphi(x) = \int \frac{d^3f}{2\omega}\left\{\eta^{+}(f)\chi(f)e^{if x} + \eta^{-}(f)\chi(-f)e^{-if\cdot x}\right\} \tag{2.42}$$

It is seen that in the momentum space representation the second rank spinor space splits up into three subspaces characterised by the projection operators $\eta^{\pm}(f), \eta^{0}(f)$. The fourier decomposition of $\varphi(\chi)$ satisfying Duffin-Kemmer equation contains only the spinors satisfying particle or anti-particle equations (2.41a), $\eta^{0}(f)$ having been annihilated by the operator $\beta.f$. Moreover $\varphi(x)$ satisfies the Klein-Gordon equation.

Our next task is to generalise this procedure to the case of arbitrary spin s.

Let f be an arbitrary timelike vector. In the Lorentz frame in which

$\underline{f} = o$,

$$\frac{\beta\cdot f}{(f\cdot f)^{\frac{1}{2}}} = \frac{\beta_4 \; f_4}{f_4} = \beta_4 \qquad (2.43a)$$

since β_4 satisfies the characteristic equation (2.8a), $\beta.f$ satisfies

$$\prod_{s_i=-s}^{+s} \left\{ \beta\cdot f - s_i\,(f\cdot f)^{\frac{1}{2}} \right\} = 0 \qquad (2.43b)$$

For even n = 2s, this can be written

$$\left[(\beta\cdot f)^2 - s^2 f.f\right]\left[(\beta\cdot f)^2 - (s-1)^2 f.f\right] \qquad \beta\cdot f = 0 \qquad (2.44a)$$

For n = 2s an odd integer

$$\left[(\beta\cdot f)^2 - s^2 f\cdot f\right]\left[(\beta\cdot f)^2 - (s-1)^2 f.f\right] \cdots \left[(\beta\cdot f)^2 - \tfrac{1}{2} f\cdot f\right], \qquad (2.44b)$$

Substituting for $\varphi(x)$ from (2.31) in (2.1) we get

$$\left(i\,\beta\cdot f + m s\right)\,\tilde{\varphi}(f) = 0 \qquad (2.45a)$$

or $$\beta\cdot f\;\tilde{\varphi}(f) = i m s\;\tilde{\varphi}(f) \qquad (2.45b)$$

Multiplying equations (2.44a) and (2.44b) by $\tilde{\varphi}(f)$ and using (2.45b) one finds that th se are $S = \frac{n}{2}$ possible values of $f^2 = ff$ for even n [8]

$$f^2 = -\left(\frac{s}{s}\right)^2 m^2,\; -\left(\frac{s}{s-2}\right)^2 m^2, \;\ldots\; \left(\frac{s}{1}\right)^2 m^2 \qquad (2.46a)$$

and $\frac{n+1}{2} = \frac{2s+1}{2}$ values for odd n

$$f^2 = -\left(\frac{s}{s}\right)^2 m^2,\ -\left(\frac{s}{s-1}\right)^2 m^2, \cdots\cdots -\left(\frac{s}{1/2}\right)^2 m^2 \tag{2.46b}$$

Let α_λ denote the set

$$\begin{aligned}\{\alpha_\lambda\} &= \frac{s}{s},\ \frac{s}{s-1},\ \frac{s}{s-2}, \cdots\cdots \frac{s}{1}\ \left(\text{or}, \frac{s}{1/2}\right) \\ &= \left\{\frac{s}{s_\lambda}\right\}, \qquad s_\lambda > 0\end{aligned} \tag{2.47}$$

In this case it is much better to put in the momentum integral f^2 on the various mass shells given by (2.46a and b)

$$\varphi(x) = \sum_\lambda \int e^{if\cdot x}\, \delta(f^2 + m^2\alpha_\lambda^2)\, \tilde{\varphi}^{(\lambda)}(f)\, d^4f \tag{2.48a}$$

Integrating this with the help of the δ functions

$$\varphi(x) = \sum_\lambda \int \left\{ \frac{e^{if^{(\lambda)}\cdot x}}{2\omega_\lambda} \tilde{\varphi}^{(\lambda)}(f^{(\lambda)}) + \frac{e^{-if^{(\lambda)}\cdot x}}{2\omega_\lambda} \tilde{\varphi}^{(\lambda)}(-f^{(\lambda)}) \right\} d^3f \tag{2.48b}$$

The time component of $f^{(\lambda)}$ has the mass shell value

$$f_0^{(\lambda)} = +\sqrt{\underline{f}^2 + m^2\alpha_\lambda^2} = \omega_\lambda \tag{2.49a}$$

$$f^{(\lambda)}\cdot f^{(\lambda)} = \underline{f}^2 - f_0^{2(\lambda)} = -m^2\alpha_\lambda^2 \tag{2.49b}$$

Substituting for $\varphi(x)$ from (2.48b) in

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + ms\right)\varphi(x) = 0$$

we obtain

$$\begin{aligned}\sum \int \Big\{ &\left(i\beta\cdot f^{(\lambda)} + ms\right)\tilde{\varphi}^{(\lambda)}(f^{(\lambda)})\, e^{if^{(\lambda)}\cdot x} \\ &+ \left(-i\beta\cdot f^{(\lambda)} + ms\right)\varphi^{(\lambda)}(-f^{(\lambda)})\, e^{-if^{(\lambda)}\cdot x}\Big\} \frac{d^3f}{2\omega_\lambda} \\ = 0 &\end{aligned} \tag{2.50}$$

$\tilde{\varphi}^{(\lambda)}(f^{(\lambda)})$ is thus seen to satisfy

$$\left(i\beta f + ms\right)\tilde{\varphi}^{(\lambda)}(f^{(\lambda)}) = 0 \tag{2.51a}$$

or

$$\left(\beta\cdot f^{(\lambda)} - ims\right)\tilde{\varphi}^{(\lambda)}(f^{(\lambda)}) = 0 \tag{2.51b}$$

For each value of α_λ, $B.f^{(\lambda)}$ satisfies the characteristic equation

$$\prod_{s_i=-s}^{+s}\left(\beta\cdot f^{(\lambda)} - s_i im\alpha_\lambda\right) = 0 \tag{2.52a}$$

For each of these α_λ there is a set of unique projection operators

$$\eta^{s_i}(f^{(\lambda)}) = \prod_{s_j\neq s_i}\left\{\frac{\beta\cdot f^{(\lambda)} - s_j im\alpha_\lambda}{(s_i - s_j)\, im\alpha_\lambda}\right\} \tag{2.52b}$$

Satisfying

$$\left\{\beta\cdot f^{(\lambda)} - s_i im\alpha_\lambda\right\}\eta^{s_i}(f^{(\lambda)}) = 0 \tag{2.53a}$$

$$\sum_{s_i=-s}^{s}\eta^{s_i}(f^{(\lambda)}) = 1 \tag{2.53b}$$

$$\eta^{s_i}(f^{(\lambda)})\,\eta^{s_j}(f^{(\lambda)}) = \delta_{ij}\,\eta^{s_i}(f^{(\lambda)}) \tag{2.53c}$$

Comparing equations (2.51a) and (2.53a) satisfied by $\tilde{\varphi}^{(\lambda)}(f^{(\lambda)})$ and $\eta^{s_i}(f^{(\lambda)})$ one finds that there is one $\eta(f^{(\lambda)})$ which satisfies the same equation as $\tilde{\varphi}^{(\lambda)}(f^{(\lambda)})$; this is the one for which $s_i = s_\lambda$, $s_\lambda > 0$.

$$\left(\beta\cdot f^{(\lambda)} - s_\lambda\alpha_\lambda im\right)\eta^{(s_\lambda)}(f^{(\lambda)}) = 0 \tag{2.54a}$$

gives since

$$s_\lambda \alpha_\lambda = s_\lambda \frac{s}{s_\lambda} = s, \qquad s_\lambda \neq 0 \tag{2.54b}$$

$$(\beta \cdot f^{(\lambda)} - ims)\, \eta^{(s_\lambda)}(f^{(\lambda)}) = 0 \tag{2.54c}$$

There is no other $\eta^{s_i}(f^{(\lambda)})$ which satisfies this equation and hence we can write

$$\tilde{\varphi}^{(\lambda)}(f^{(\lambda)}) = \eta^{(s_\lambda)}(f^{(\lambda)})\, \chi(f^{(\lambda)}) \tag{2.55}$$

where $\chi(f^{(\lambda)})$ is some arbitrary spinor of rank 'n'. Substituting this form of $\tilde{\varphi}^{(\lambda)}(f^{(\lambda)})$ in (2.50)

$$\varphi(x) = \sum_\lambda \int \frac{d^3 f}{2\omega_\lambda} \left\{ e^{if^{(\lambda)} \cdot x} \eta^{(s_\lambda)}(f^{(\lambda)})\, \chi(f^{(\lambda)}) + e^{-if^{(\lambda)} \cdot x} \eta^{(s_\lambda)}(-f^{(\lambda)})\, \chi(-f^{(\lambda)}) \right\} \tag{2.56}$$

$\eta^{(s_\lambda)}(-f^{(\lambda)})\chi(-f^{(\lambda)})$ satisfies the same equation as $\eta^{(-s_\lambda)}(f^{(\lambda)})$

$$(\beta \cdot f^{(\lambda)} + ims)\, \eta^{(-s_\lambda)}(f^{(\lambda)}) = 0 \tag{2.57}$$

Finally putting

$$\eta^{(\pm s_\lambda)}(f^{(\lambda)})\, \chi(\pm f^{(\lambda)}) = u^{(\pm)}(f^{(\lambda)}) \tag{2.58}$$

$$\varphi(x) = \sum_\lambda \int \frac{d^3 f}{2\omega_\lambda} \left\{ e^{if^{(\lambda)} \cdot x} u^{(+)}(f^{(\lambda)}) + e^{-if^{(\lambda)} \cdot x} u^{(-)}(f^{(\lambda)}) \right\} \tag{2.59}$$

where $U^{(\pm)}(f^{(\lambda)})$ satisfy the 'particle' and 'antiparticle' equation

$$\left(\pm i\ \beta\cdot f^{(\lambda)} + m s\right) U^{(\pm)}\left(f^{(\lambda)}\right) = 0 \tag{2.60a}$$

$$f^{(\lambda)} = \left\{ \underline{f}\ ,\ f_0^{(\lambda)} = +\sqrt{\underline{f}^2 + m^2\alpha_\lambda^2} \right\} \tag{2.60b}$$

From (2.59) it is readily seen that $\varphi(x)$ obeys the multiple mass equation[3]

$$\left\{\left(\square - m^2\left(\frac{s}{s}\right)^2\right)\left(\square - m^2\left(\frac{s}{s-1}\right)^2\right) - - - -\right\}\varphi(x) = 0 \tag{2.61}$$

The last factor being $\left(\square - m^2\,\frac{s^2}{1}\right)$ for even $n = 2s$ and $\square - m^2\left(\frac{s}{1/2}\right)^2$ for odd n.

Invariant Spin Projection Operator.

There are two invariants associated with the extended Lorentz group defined as the sum of proper Lorentz group and the group of translations in the 4-space. One of them is the rest mass operator and the other the operator which gives the intrinsic spin of the particle. This is given by[7]

$$O(f^{(\lambda)}) = \frac{i^2}{f^{(\lambda)}\cdot f^{(\lambda)}}\left\{\frac{f^{(\lambda)}\cdot f^{(\lambda)}}{2} I_{\mu\nu} I_{\mu\nu} - I_{\lambda\mu} I_{\lambda\nu} f_\mu^{(\lambda)} f_\nu^{(\lambda)}\right\} \tag{2.62a}$$

The infinitesimal generators $I_{\mu\nu}$ for the equation (2.1) are given by (2.8b). Therefore O(f) is given by

$$O(f^{(\lambda)}) = \frac{i^2}{f^{(\lambda)}\cdot f^{(\lambda)}}\left\{\frac{f^{(\lambda)}\cdot f^{(\lambda)}}{2}[\beta_\lambda,\beta_\mu][\beta_\lambda,\beta_\mu] - [\beta_\lambda,\beta_\mu][\beta_\lambda,\beta_\nu] f_\mu^{(\lambda)} f_\nu^{(\lambda)}\right\} \tag{2.62b}$$

$O(f^{(\lambda)})$ commutes with all the generators of the extended Lorentz group.[7] In the rest frame of the particle $\underline{f} = 0$, $O(f^{(\lambda)})$ reduces to

$$O(\underline{f}=0) \equiv O(0) = \frac{i^2}{2}[\beta_i,\beta_j][\beta_i,\beta_j]$$
$$= \Sigma_i \Sigma_i = \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 \tag{2.63}$$

where Σ_i is defined by

$$\Sigma_i = -i\,\epsilon_{ijk}\,[\beta_j,\beta_k] \tag{2.64}$$

The properties of Σ_i are easily obtained by going over to the representation (2.14) in which Σ_i is given by

$$\Sigma_i = \frac{1}{2}\Big(\bar{\sigma}_i \times 1 \cdots \times 1 + 1 \times \bar{\sigma}_i \times 1 \cdots \times 1$$
$$+ \cdots\cdots + 1 \times 1 \times \cdots 1 \times \bar{\sigma}_i\Big), \text{ n terms} \tag{2.65}$$

with $\bar{\sigma}_i$

$$\bar{\sigma}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{2.66}$$

Σ_i satisfy the commutation relation of angular momentum matrices

$$[\Sigma_i , \Sigma_j] = i\,\epsilon_{ijk}\,\Sigma_k \tag{2.67a}$$

and the characteristic equation

$$\prod_{s_i=-s}^{+s} (\Sigma_j - s_i) = 0 \tag{2.67b}$$

Showing that the eigenvalues of Σ_i are $-s, (s-1), \ldots +s$.

The spin operator (o) = is now easily seen to be given by

$$O(o) = \Sigma^2 = \frac{1}{4}\Big[3n + 2\big(\bar{\sigma}_i \times \bar{\sigma}_i \times 1 \cdots \times 1 + \bar{\sigma}_i \times 1 \times \bar{\sigma}_i \times 1 \cdots \times 1 + \cdots\big) \tag{2.68}$$

The sum involving $\bar{\sigma}_i$ contains ${}^nC_2$ terms each term has $\bar{\sigma}_i$ occurring twice. Denoting such a symmetrical sum of $\bar{\sigma}_i$ by

$$[\bar{\sigma}]^{(n)}_{(2)} \tag{2.69a}$$

$$\Sigma^2 = \frac{1}{4}\left[3n + 2[\bar{\sigma}]^{(n)}_{(2)}\right] \tag{2.69b}$$

For s=1, n=2

$$[\bar{\sigma}]^{(2)}_{(2)} = \bar{\sigma}_i \times \bar{\sigma}_i \equiv a_2 \tag{2.70a}$$

For $s = \frac{3}{2}$, $n = 3$

$$[\bar{\sigma}]^{(3)}_{(2)} = \bar{\sigma}_i \times \bar{\sigma}_i \times 1 + \bar{\sigma}_i \times 1 \times \bar{\sigma}_i + 1 \times \bar{\sigma}_i \times \bar{\sigma}_i \equiv a_3 \tag{2.70b}$$

For s= 1 and $\frac{3}{2}$, the eigenvalues of Σ^2 can easily be calculated by finding the eigenvalues of a_2 and a_3 and utilising the properties of $\bar{a}_i$ matrices

$$a_2^2 = 3 - 2a \tag{2.71a}$$

$$a_3^2 = 9 \tag{2.71b}$$

This shows that the characteristic equation obeyed by Σ^2 is

$$\left(\Sigma^2 - s(s+1)\right)\left(\Sigma^2 - (s-1)s\right) = 0 \quad , \quad s = 1 \text{ or } \tfrac{3}{2} \tag{2.72}$$

Calculation of the eigenvalues of Σ^2 by this method becomes extremely complicated for higher spins and we must fall back on some general considerations. The algebra obeyed by Σ_i defined by (2.65) is the same as obeyed by the spin operators

$$\Sigma_i^p = \frac{1}{2}\left(\sigma_i \times 1 \times 1 \cdots 1 \times 1 + 1 \times \sigma_i \times 1 \cdots 1 \times 1 + \cdots \right.$$
$$\left. \cdots + 1 \times 1 \times \cdots 1 \times \sigma_i\right) \quad , \; n \text{ terms} \tag{2.73}$$

This is the direct product representation

$$D^{\frac{1}{2}} \times D^{\frac{1}{2}} \times \cdots \times D^{\frac{1}{2}} \qquad n \text{ factors} \tag{2.74}$$

where $D^{\frac{1}{2}}$ belongs to the spin $\frac{1}{2}$ representation of the 3-rotative group.

By Clebsch-Gordon theorem a similarity transformation $\mathcal{S}$ reduces (2.74) into the form

$$D^{s} \oplus \alpha_{s-1} D^{s-1} \oplus \alpha_{s-2} D^{s-2} \oplus \cdots \tag{2.75}$$

The representation belonging to the highest spin value s occurs just once while the lower spin representations occur α times in general. $D^{(s)}, D^{(s-1)}$ belong to the eigen spaces corresponding to the eigen values $s(s+1), (s-1)s \ldots$ of the transformed operator

$$\tilde{\Sigma}^{p^2} = \tilde{\Sigma}_i^p \, \tilde{\Sigma}_i^p \tag{2.76}$$

where

$$\tilde{\Sigma}_i^{\ P} = \mathcal{H} \, \Sigma_i^{\ P} \, \mathcal{H}^{-1} \tag{2.77}$$

The eigenvalues of $\tilde{\Sigma}^2$ and hence of Σ^2 are s(s-1), (s-1)s, ...

The characteristic equation of Σ^2 is therefore

$$\left(\Sigma^2 - s(s+1) \right) \left(\Sigma^2 - (s-1)s \right) \cdots\cdots\cdots = 0 \tag{2.78a}$$

By covariance considerations $O(f^{(\lambda)})$ also satisfies this characteristic equation

$$\left\{ O(f^{(\lambda)}) - s(s+1) \right\} \left\{ O(f^{(\lambda)}) - (s-1)s \right\} \cdots\cdots = 0 \tag{2.78b}$$

The last term being $O(f^{(\lambda)})$ for integral s and $O(f^{(\lambda)}) - \frac{1}{2} \, \frac{3}{2}$ for half integral s.

$O\,(f^{(\lambda)})$ is hermitian in the sense

$$O^{\dagger}(f^{(\lambda)}) = \eta_4 \, O^{\times}(f^{(\lambda)}) \, \eta_4 = O(f^{(\lambda)}) \tag{2.79}$$

and by the theorem (2.28) the projection operators are given by

$$O^{(s_j)}(f^{(\lambda)}) = \prod_{\bar{s}_j \neq \bar{s}_i} \frac{O(f^{(\lambda)}) - \bar{s}_j}{\bar{s}_i - \bar{s}_j} \tag{2.80}$$

where s_j denotes the set

$$\{ \bar{s}_j \} = s(s+1), \, (s-1)s, \cdots\cdots , \, 0 \text{ or } \tfrac{1}{2} \cdot \tfrac{3}{2} \\ = \left\{ s_j \, (s_j + 1) \right\} \tag{2.80a}$$

$O^{(s_i)}(f^{(\lambda)})$ satisfy the usual properties of projection operators

$$O^{(s_i)}(f^{(\lambda)}) \, O^{(s_j)}(f^{(\lambda)}) = \delta_{ij} \, O^{(s_i)}(f^{(\lambda)}) \tag{2.81a}$$

$$\sum_{s_i = 0, \frac{1}{2}}^{s} O^{(s_i)}(f^{(\lambda)}) = 1 \qquad (2.81b)$$

$O^{(s_i)}(f^{(\lambda)})$ defines an eigen space of the invariant spin operator $O(f^{(\lambda)})$

$$O(f^{(\lambda)})\; O^{(s_i)}(f^{(\lambda)}) = s_i(s_i+1)\; O^{(s_i)}(f^{(\lambda)}) \qquad (2.81c)$$

The form of $O(f^{(\lambda)})$ for $\underline{f}$=o is obtained by substituting from (2.8) in (2.62b)

$$O(f^{(\lambda)}) = \frac{1}{4}\left[n^2 + 2n - [\gamma_\lambda \gamma_\mu]^{(n)}_{(2)} + 2[\gamma_\lambda \gamma(f^{(\lambda)})]^{(n)}_{(2)} \right] \qquad (2.82)$$

We shall be interested in the following properties of $O^{(s_i)}(f^{(\lambda)})$

(1) $O^{(s_i)}(f^{(\lambda)})$ commutes with $\eta^{s_j}(f^{(\lambda)})$

This can easily be proved by going over to the frame $\underline{f}$ = o. In this frame $O(f^{(\lambda)})$ is given by (2.63), $O(o) = \frac{i^2}{2}\left[\beta_i,\beta_j\right]\left[\beta_i\beta_j\right]$ and $\eta^{s_i}(f^{(\lambda)})$ contains only β_4. By (2.8c) β_4 and $\left[\beta_i,\beta_j\right]$ commute Covariance consideration show that $O(f^{(\lambda)})$ and hence $O^{(s_i)}(f^{(\lambda)})$ and $\eta^{s_j}(f^{(\lambda)})$ commute in any Lorentz frame

(2) Since $O(f^{(\lambda)})$ and hence $O^{(s)}(f^{(\lambda)})$ commutes with all the generators of Lorentz group, in any irreducible representation of the Lorentz group $O^{(s)}(f^{(\lambda)})$ will be given by a scalar matrix. Since $O^{(s)}(f^{(\lambda)})\; O^{(s)}(f^{(\lambda)}) = O(f^{(\lambda)})$ this scalar matrix is just the unit matrix.

(3) $\mathcal{O}\,(f^{(\lambda)})$ is closely related to a base element of the β algebra which commutes with the whole β algebra. This relation will now be investigated and the following result (which might some times be useful) will be proved for $f^2 = -m^2\alpha_0^2 = -m^2$

$$O^{(s)}(f)\,\eta^{\pm}(f)\,K(f) = K(f)\,O^{(s)}(f)\,\eta^{\pm}(f) \tag{2.83}$$

if $\eta^{\pm}(f)$ commutes with $K(f)$.

For s = 1, n = 2 i.e. for the Duffin-Kemmer algebra, the element[10]

$$\sum_{\mu} \eta_\mu - \sum_{\mu<\nu} \eta_\mu \eta_\nu$$

$$= \gamma_\mu \times \gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma_\nu \times \gamma_\mu\gamma_\nu + 2 \tag{2.84}$$

$$= \sum_{\mu} \eta_\mu - \sum_{\mu,\nu} \eta_\mu \eta_\nu + 2$$

Commutes with the whole Duffin-Kemmer algebra. This element is closely connected with the spin operator $O(f)$. Infact since

$$\gamma(f)\,\Lambda^{\pm}(f) = \pm\,\Lambda^{\pm}(f)$$

$$\eta^{\pm}(f) = \Lambda^{\pm}(f) \times \Lambda^{\pm}(f)$$

$$\begin{aligned} O(f)\,\eta^{\pm}(f) &= \tfrac{1}{2}\left\{\gamma_\lambda\gamma(f)\times\gamma_\lambda\gamma(f) - \tfrac{1}{2}\gamma_\lambda\gamma_\mu\times\gamma_\lambda\gamma_\mu + 4\right\} \\ &\qquad \Lambda^{\pm}(f)\times\Lambda^{\pm}(f) \\ &= \tfrac{1}{2}\left\{\gamma_\lambda\times\gamma_\lambda - \tfrac{1}{2}\gamma_\lambda\gamma_\mu\times\gamma_\lambda\gamma_\mu + 4\right\}\Lambda^{\pm}(f)\times\Lambda^{\pm}(f) \\ &\equiv G\,\eta^{\pm}(f) \end{aligned} \tag{2.85}$$

where except for the difference of multiples of unity G is the same element as given in (2.84).

In the general case one defines

$$\eta_\mu^{ij} = 1\times 1\times\cdots 1\times\gamma_\mu\times 1 \quad\cdots\quad 1\times\gamma_\mu\times 1 \quad\cdots\times 1, \quad \text{n factors} \tag{2.86}$$

γ_μ occuring in the ith and jth positions. The element [7]

$$G^{ij} = \sum_\mu \eta_\mu^{ij} - \frac{1}{2}\sum_{\mu,\nu} \eta_\mu^{ij}\eta_\nu^{ij} \qquad \text{for each } i,j \leq n \tag{2.87}$$

commutes with all the elements of β algebra[8]. Again since

$$\eta^\pm(f) = \Lambda^\pm(f)\times\Lambda^\pm(f)\times\cdots\cdots\Lambda^\pm(f)$$

$$O(f)\,\eta^\pm(f) = \frac{1}{2}\left[\frac{n(n+2)}{2} + \frac{1}{2}[\gamma_\lambda\gamma_\mu]^{(n)}_{(2)} + [\gamma_\lambda]^{(n)}_{(2)}\right]\eta^\pm \tag{2.88a}$$

$$= \frac{1}{2}\left[\frac{n(n+2)}{2} + \sum_{i<j}^{n} G^{ij}\right]\eta^\pm(f)$$

Denoting by $\bar{G}$ the factor within the square brackets in (2.88)

$$O(f)\,\eta^\pm(f) = \frac{1}{2}\bar{G}\,\eta^\pm(f) = \frac{1}{2}\eta^\pm(f)\,\bar{G} \tag{2.88b}$$

G has the same commutation property as G^{ij} and it follows that if $\eta^\pm(f)$ commutes with $K(f)$

$$O(f)\,\eta^\pm(f)\,K(f) = K(f)\,\bar{G}\,\eta^\pm(f)\cdot\frac{1}{2}$$

$$= K(f)\,O(f)\,\eta^\pm(f) \tag{2.89}$$

The same result obviously holds flr $O^{(s)}(f)$.

Projection operators for Helicity components.

The operator $\Sigma(f)$ defined by

$$\Sigma(\underline{f}) \equiv \frac{\Sigma_i f_i}{(\underline{f}\cdot\underline{f})^{\frac{1}{2}}} \tag{2.90}$$
$$= \frac{1}{2}\left[\bar{\sigma}(\underline{f})\times 1\times 1 \cdots 1\times 1 + 1\times\bar{\sigma}(\underline{f})\times 1 \cdots \times 1 + \cdots + 1\times 1 \cdots 1\times\bar{\sigma}(\underline{f})\right]$$

where

$$\bar{\sigma}(\underline{f}) = \frac{\bar{\underline{\sigma}}\cdot\underline{f}}{(\underline{f}\cdot\underline{f})^{\frac{1}{2}}} \tag{2.91}$$

reduces to $\bar{\sigma}_i$ for $\underline{f}$ along the ith space axis. $\Sigma(\underline{f})$ is 3-space rotation invariant and so $\Sigma(\underline{f})$ satisfies the characteristic equation

$$\prod_{s_i=-s}^{s}\left(\Sigma(\underline{f}) - s_i\right) = 0 \tag{2.92}$$

The projection operators for different helicity components are given by

$$Z^{s_i}(\underline{f}) = \prod_{s_j\neq s_i}\frac{\Sigma(\underline{f}) - s_j}{s_i - s_j} \tag{2.93}$$
$$s_i = -s, -s+1, \cdots\cdots, s-1, s$$

They obey the usual properties of projection operators.

$$\sum_{s_i=-s}^{s} Z^{s_i}(\underline{f}) = 1 \tag{2.94a}$$

$$Z^{s_i}(\underline{f})\, Z^{s_j}(\underline{f}) = \delta_{ij}\, Z^{s_i}(\underline{f}) \tag{2.94b}$$

$$\Sigma(\underline{f})\, \mathcal{X}^{s_i}(\underline{f}) = s_i\, \mathcal{X}^{s_i}(\underline{f}) \tag{2.94c}$$

Since $\mathcal{O}^{(s_i)}(f)^{(\lambda)}$ commute the generators Σ_i of 3-space rotations (a sub group of the extended Lorentz group), $\mathcal{O}^{(s_i)}(f)^{(\lambda)}$ commute with $\Sigma(\underline{f})$ and hence with $\mathcal{X}^{s_i}(\underline{f})$.

The commuting set of hermitian operators $\beta(f^{(\lambda)})$, $O(f)^{(\lambda)}$ and $\Sigma(\underline{f})$ have simultaneous eigenvectors. Unfortunately this set is not complete but utilising the representation (2.14) of the β matrices it is possible to build up for each mass value m_{α_λ} an orthogonal basis in the 4^n dimensional spinor space. This will be done in the next Chapter and some useful orthonormality relations will be derived.

CHAPTER III

AN ORTHONORMAL BASIS IN THE SPINOR SPACE

We have seen that the projection operators $\eta^{s_\lambda}(f^{(\lambda)})$ satisfy the "particle equation

$$\left(\beta\cdot f^{(\lambda)} - i m s_\lambda \alpha_\lambda\right)\eta^{s_\lambda}(f^{(\lambda)}) = 0 \tag{3.1a}$$

$$\{\alpha_\lambda\} = \frac{s}{s}, \frac{s}{s-1}, \cdots\cdots, \frac{s}{1} \text{ or } \frac{s}{1/2} = \left\{\frac{s}{s_\lambda}\right\} \tag{3.1b}$$

$$= \alpha_0, \alpha_1, \cdots\cdots, \alpha_{s-1} \text{ or } \alpha_{\frac{2s-1}{2}} \tag{3.1c}$$

takes s values 0, 1, 2, ... S for integral S and $\frac{2S+1}{2}$ values 0, 1, 2, ... $\frac{2S-1}{2}$ for half integral S. S_λ takes values

$$S_\lambda = S_0, S_1, S_2, \ldots S_{S-1} \text{ or } S_{\frac{2S-1}{2}} \tag{3.1d}$$

$$= S, S-1, S-2, \ldots 1 \text{ or } \tfrac{1}{2} \tag{3.1e}$$

The "particle" equation can be written in two ways

$$\left(i\beta\cdot f^{(\lambda)} + ms\right)\eta^{s_\lambda}(f^{(\lambda)}) = 0 \tag{3.2a}$$

$$\left(\frac{\beta\cdot f^{(\lambda)}}{i m \alpha_\lambda} - s_\lambda\right)\eta^{s_\lambda}(f^{(\lambda)}) = 0$$

or

$$\left(\beta\cdot(f^{(\lambda)}) - s_\lambda\right)\eta^{s_\lambda}(f^{(\lambda)}) = 0 \tag{3.2b}$$

The antiparticle equation is

$$\left(i\beta\cdot f^{(\lambda)} - ms\right)\eta^{-s_\lambda}(f^{(\lambda)}) = 0 \tag{3.3a}$$

or

$$\left(\beta(f^{(\lambda)}) + s_\lambda\right)\eta^{-s_\lambda}(f^{(\lambda)}) = 0 \tag{3.3b}$$

In the representation (2.14) of the β matrices

$$\beta(f^{(\lambda)}) = \frac{1}{2}\left\{ \gamma(f^{(\lambda)}) \times 1 \times 1 \cdots \times 1 + \cdots \right.$$

$$\left. \cdots + 1 \times 1 \cdots 1 \times \gamma(f^{(\lambda)}) \right\} \tag{3.4}$$

Further

$$\gamma(f^{(\lambda)}) = \frac{\gamma_\mu f_\mu^{(\lambda)}}{(f \cdot f)^{1/2}} = \frac{\gamma_\mu f_\mu^{(\lambda)}}{i m \alpha_\lambda} \tag{3.5a}$$

The projection operators in the Dirac space are

$$\Lambda^{\pm}(f^{(\lambda)}) = \tfrac{1}{2}\left(1 \pm \gamma (f^{(\lambda)})\right) \tag{3.5b}$$

$$\Lambda^{+}(f^{(\lambda)}) + \Lambda^{-}(f^{(\lambda)}) = 1 \tag{3.5c}$$

$$\Lambda^{+}(f^{(\lambda)})\, \Lambda^{-}(f^{(\lambda)}) = \Lambda^{-}(f^{(\lambda)})\, \Lambda^{+}(f^{(\lambda)}) = 0 \tag{3.5d}$$

$$\gamma(f^{(\lambda)})\Lambda^{\pm}(f^{(\lambda)}) = \pm \Lambda^{\pm}(f^{(\lambda)}) \tag{3.5e}$$

$f^{(\lambda)}$ is timelike and $\gamma(f^{(\lambda)})$ and $\Lambda^{\pm}(f^{(\lambda)})$ are self adjoint

$$\gamma^{\dagger}(f^{(\lambda)}) = \gamma_4\, \gamma^{x}(f^{(\lambda)})\, \gamma_4 = \gamma(f^{(\lambda)}) \tag{3.6a}$$

$$\Lambda^{\pm\dagger}(f^{(\lambda)}) = \Lambda^{\pm}(f^{(\lambda)}) \tag{3.6b}$$

In the rest frame $\underline{f} = 0$, $f_0^{(\lambda)} = \alpha_\lambda m$, $\gamma(f^{(\lambda)}) \longrightarrow \gamma(0) = \gamma_4$ and $\Lambda^{\pm}(f^{(\lambda)}) \longrightarrow \Lambda^{\pm}(0) = \frac{1}{2}(1 \pm \gamma_4)$, (3.5e) reduces to

$$\gamma_4 \Lambda^{\pm}(0) = \Lambda^{\pm}(0)\, \gamma_4 = \pm \Lambda^{\pm}(0)$$

Now consider the quantity

$$\Lambda_{r}(f^{(\lambda)}) = \Lambda^{+}(f^{(\lambda)}) \times \Lambda^{-}(f^{(\lambda)}) \; \ldots \times \Lambda^{+}(f^{(\lambda)}), \qquad n = 2S \text{ factors} \tag{3.7a}$$

in which $\Lambda^{-}(f^{(\lambda)})$ occurs r times. This satisfies

$$\beta\,(f^{(\lambda)}) \;\; \Lambda_{r}(f^{(\lambda)}) = \tfrac{1}{2}\,(1 - 1 + \ldots \qquad + 1)\Lambda_{r}(f^{(\lambda)}) = \frac{(n-r)-r}{2}\Lambda_{r}(f^{(\lambda)})$$

$$= (S - r)\;\Lambda_{r}(f^{(\lambda)}) \tag{3.7b}$$

Moreover there are ${}^{n}C_{r}$ quantities of the type (3.7a) whoss square is the same and which are orthogonal to each other but each of them belongs to the same eigenvalue $(S - r)$ of $\beta\,(f^{(\lambda)})$.

It is advantageous to write these down in a table form

TABLE I

For each $f^{(\lambda)}$

$${}^{n}C_{0},\; \left\{ \Lambda^{+} \times \Lambda^{+} \times \qquad \times \Lambda^{+} \times \Lambda^{+} \qquad = \Lambda_{0}^{(1)}(f^{(\lambda)}) \right.$$

$${}^{n}C_{1},\; \left\{ \begin{array}{ll} \Lambda^{-} \times \Lambda^{+} \times - - - - - - \times \Lambda^{+} \times \Lambda^{+} & = \Lambda_{1}^{(1)}(f^{(\lambda)}) \\ \Lambda^{+} \times \Lambda^{-} \times - - - - - - \times \Lambda^{+} \times \Lambda^{+} & = \Lambda_{1}^{(2)}(f^{(\lambda)}) \\ - - - - - - - - - - - - & \\ \Lambda^{+} \times \Lambda^{+} \times - - - - - \times \Lambda^{+} \times \Lambda^{-} & = \Lambda_{1}^{{}^{n}C_{1}}(f^{(\lambda)}) \end{array} \right.$$

$${}^{n}C_{2},\; \left\{ \begin{array}{ll} \Lambda^{-} \times \Lambda^{-} - - - - - - - \times \Lambda^{+} \times \Lambda^{+} & = \Lambda_{2}^{(1)}(f^{(\lambda)}) \\ - - - - - - - - - - - - - - - & \\ - - - - - - - - - - - - - - & \\ \Lambda^{+} \times \Lambda^{+} \times - - - - - - \times \Lambda^{-} \times \Lambda^{-} & = \Lambda_{2}^{{}^{n}C_{2}}(f^{(\lambda)}) \end{array} \right.$$

- - - - - - - - - - - - - - -

$$
{}^{n}C_{r}\,,\quad \left\{\begin{array}{ll}
\Lambda^{-}\times\Lambda^{+}\times\cdots\cdots\times\Lambda^{+}\times\Lambda^{-} & =\Lambda_{r}^{(1)}(f^{(\lambda)})\\
\cdots\cdots\cdots & \\
\cdots\cdots\cdots & \\
\cdots\cdots\cdots & =\Lambda_{r}^{{}^{n}C_{r}}(f^{(\lambda)})
\end{array}\right.
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$
{}^{n}C_{n}\,,\quad \left\{\Lambda^{-}\times\Lambda^{-}\times\cdots\cdots\quad\Lambda^{-}\times\Lambda^{-} \;=\Lambda_{n}^{(1)}(f^{(\lambda)})\right.
$$

The total number of these projection operators is

$$ {}^{n}C_{0} + {}^{n}C_{1} + \ldots {}^{n}C_{r} + \ldots {}^{n}C_{n} = (1+1)^{n} = 2^{n} \tag{3.8}$$

and they are orthogonal to each other in the sense

$$\Lambda_{r}^{(j)}\,(f^{(\lambda)})\;\Lambda_{r'}^{(j')}\,(f^{(\lambda)}) = \Lambda^{(j)}\,(f^{(\lambda)})\,\delta_{jj'}\,\delta_{rr'} \tag{3.9a}$$

The sum of all these projection operators is the identity operator

$$\sum_{r=0}\;\sum_{j=1}^{{}^{n}C_{r}} \Lambda_{r}^{(j)}\,(f^{(\lambda)}) = 1 \tag{3.9b}$$

For $\lambda = o$, ie. $S_{\lambda} = S_{o} = S$, $\alpha_{o} = 1$ there is only one projection operator $\Lambda_{o}^{(1)}\,(f^{(\lambda)})$ which satisfies the particle equation (3.2) and only which satisfies the antiparticle equation (3.3)

$$(\beta\,(f^{(o)}) - s)\,\Lambda_o^{(1)}(f^{(o)}) = o \tag{3.10a}$$

$$(\beta\,(f^{(o)}) + s)\,\Lambda_n^{(1)}(f^{(o)}) = o \tag{3.10b}$$

while any member of the set $\Lambda_r, r \leq \frac{n}{2}$ satisfies

$$\left[\beta\,(f^{(o)}) - (s - r)\right]\Lambda_r(f^{(o)}) = o \qquad r \leq \frac{n}{2} \tag{3.11a}$$

and for $r > \frac{n}{2}$

$$\left[\beta\,(f^{(o)}) + (s - r)\right]\Lambda_r(f^{(o)}) = o \qquad r > \frac{n}{2} \tag{3.11b}$$

In general for any λ the members of the set $\Lambda_\lambda^{(j)}(f^{(\lambda)})$ satisfy the particle equation

$$(i\beta f^{(\lambda)} + ms)\,\Lambda_\lambda^{(j)}(f^{(\lambda)}) = \left[\beta\,(f^{(\lambda)}) - s_\lambda\right]\Lambda_\lambda^{j}(f^{(\lambda)}) = o \tag{3.12a}$$

and the ${}^nC_{n-\lambda} = {}^nC_\lambda$ members of the group $\Lambda_{n-\lambda}^{(j)}(f^{(\lambda)})$ satisfy the antiparticle equation

$$(i\ \beta \cdot f^{(\lambda)} - ms)\,\Lambda_{n-\lambda}^{(j)}(f^{(\lambda)}) = \left[\beta\,(f^{(\lambda)}) + s_\lambda\right]\Lambda_{n-\lambda}^{(j)}(f^{(\lambda)}) = 0 \tag{3.12b}$$

The sum of all the operators $\Lambda_\lambda^{(j)}(f^{(\lambda)})$ is equal to $\eta^+(f^{(\lambda)}) = \eta^{s_\lambda}(f^{(\lambda)})$

That is

$$\eta^+(f^{(\lambda)}) = \sum_{j=1}^{{}^nC_\lambda} \Lambda_\lambda^{(j)}(f^{(\lambda)}) \tag{3.13a}$$

This is so because both the sides satisfy the same particle equation and their squares are the same and both are hermitian. Similarly for the antiparticle projection operators

$$\eta^-(f^{(\lambda)}) = \eta^{-s_\lambda}(f^{(\lambda)}) = \sum_{j=1}^{{}^nC_\lambda} \Lambda_{n-\lambda}^{(j)}(f^{(\lambda)}) \tag{3.13b}$$

$\Lambda^{(j)}_{r}(f^{(\lambda)})$ for $r \neq \lambda$ and $r \neq n-\lambda$ do not satisfy the particle or antiparticle equations but rather those satisfied by

$\eta^{s_i \neq \pm s_\lambda}(f^{(\lambda)})$.

The helicity operator $\Sigma(\underline{f}^{(\lambda)}) = \Sigma(\underline{f})$ given by

$$\Sigma(\underline{f}) = \tfrac{1}{2}\,(\bar{\sigma}(\underline{f}) \times 1 \times 1 \,\ldots\ldots\, \times 1 \;+ 1 \times \bar{\sigma}(\underline{f}) \times 1 \times \ldots\ldots \times 1 + \ldots$$
$$+ \ldots\ldots + 1 \times 1 \times 1 \times \ldots \times 1 \times \bar{\sigma}(\underline{f}) \;), \quad n \text{ terms} \qquad (3.14)$$

has helicity eigenvectors of the type

$$\xi^{\sigma}_{r}(\underline{f}) = \xi^{+}(\underline{f}) \times \xi^{-}(\underline{f}) \times \ldots\ldots \times \xi^{+}(\underline{f}) \qquad (3.15)$$

where

$$\xi^{\pm}(\underline{f}) = \tfrac{1}{2}\,(1 + \bar{\sigma}(\underline{f})) \quad , \quad \bar{\sigma}(\underline{f}) = \frac{\bar{\sigma}_i f_i}{(\underline{f}\cdot\underline{f})^{\frac{1}{2}}} \qquad (3.16a)$$

and satisfies

$$\xi^{\pm}(\underline{f})\,\xi^{\pm}(\underline{f}) = \xi^{\pm}(\underline{f}) \qquad (3.16b)$$

$$\xi^{\mp}(\underline{f})\,\xi^{\pm}(\underline{f}) = 0 \qquad (3.16c)$$

$$\xi^{+}(\underline{f}) + \xi^{-}(\underline{f}) = 1 \qquad (3.16d)$$

$$\bar{\sigma}(\underline{f})\;\xi^{\pm}(\underline{f}) = \pm\,\xi^{\pm}(\underline{f}) \qquad (3.16e)$$

If in (3.15) $\xi^{-}(\underline{f})$ occurs r times then these are ${}^{n}C_{r}$ orthogonal projection operators of this type, they have the same helicity eigen value $s - r$

$$\Sigma(\underline{f}) \quad \xi_{r}^{\sigma}(\underline{f}) = (s-r)\,\xi_{r}^{\sigma} \qquad \sigma = 1, 2, \ldots\ldots, {}^{n}C_{r} \tag{3.17}$$

Again there are ${}^{n}C_{0} + {}^{n}C_{1} + \ldots\ldots {}^{n}C_{n} = 2^{n}$ helicity projection operators and they are orthogonal to each other

$$\xi_{r'}^{\sigma'}(\underline{f}) \quad \xi_{r}^{\sigma}(\underline{f}) = \delta_{r'r} \quad \delta_{\sigma'\sigma}\, \xi_{r}^{\sigma}(\underline{f}) \tag{3.18}$$

It is better to arrange these helicity projection operators in such a way that the first $2S + 1$ belong to the $O^{(s)}(f^{(\lambda)})$ space

$$O^{s}(f^{(\lambda)}) \;\; \chi^{(i)}(\underline{f}) \qquad i = 1, 2, \ldots\ldots 2S + 1 \tag{3.19}$$

and the rest attached to lower spin operators $O^{s_j < s}(f^{(\lambda)})$. There will be several $O^{s_j<s}(f^{(\lambda)})$ subspaces but an orthogonal basis may be constructed within each $s_j < s$ spin space also. Let these be denoted by

$$\xi^{\sigma}(f^{(\lambda)}) \qquad \sigma = 1, 2, \ldots\ldots 2S + 1, \;\; 2S + 2 \ldots\ldots 2^{n} \tag{3.20}$$

For $\sigma = i \leq 2S + 1$

$$\xi^{i}(f^{(\lambda)}) = O^{(s)}(f^{(\lambda)})\,\Sigma^{(i)}(\underline{f}) \qquad i = 1, 2, \ldots 2S + 1 \tag{3.21}$$

For higher values of σ, ξ^{σ} are formed by $O^{s_j<s}(f^{(\lambda)})$ and an appropriate sum of $\xi_{r}^{\sigma}(\underline{f})$. The orthogonality condition reads

$$\xi^{\sigma}(f^{(\lambda)}) \quad \xi^{\sigma'}(f^{(\lambda)}) = \xi^{\sigma}(f^{(\lambda)})\, \delta_{\sigma'\sigma} \qquad = 1, 2, \ldots 2^{n} \tag{3.22}$$

It should be noted that $\Sigma^{(i)}(\underline{f})$ can also be written in terms of $\xi_{r}^{\sigma}(\underline{f})$.

In general $\xi^{\sigma}(f^{(\lambda)})$ is of the form

$$\xi^{\sigma}(f^{(\lambda)}) = O^{(Si)}(f^{(\lambda)}) \left\{ \sum \xi_{r}^{(\alpha)}(\underline{f}) \right\} \tag{3.23}$$

By multiplying $\Lambda_{r}^{(j)}(f^{(\lambda)})$ and $\xi^{\sigma}(f^{(\lambda)})$ one obtains $2^{n}\,2^{n} = 4^{n}$ orthogonal primitive projection operators. Let $U^{\sigma}_{r,j}(f^{(\lambda)})$ denote

$$U^{\sigma}_{r,j}(f^{(\lambda)}) = \Lambda_{r}^{(j)}(f^{(\lambda)})\, \xi^{\sigma}(f^{(\lambda)})\, \mathcal{X}(f^{(\lambda)}) \tag{3.24}$$

Then $U^{\sigma}_{\lambda,j}(f^{(\lambda)})$ satisfies the particle equation (3.2) and $U_{n-\lambda,j}(f^{(\lambda)})$ satisfies the antiparticle equation (3.3). The operator

$$\eta(f^{(\lambda)}) = \gamma(f^{(\lambda)}) \times \gamma(f^{(\lambda)}) \times \ldots\ldots\ldots \gamma(f^{(\lambda)})$$

has the property

$$\eta(f^{(\lambda)})\quad U^{\sigma}_{r,j}(f^{(\lambda)}) = (-1)^{r}\quad U^{\sigma}_{r,j}\quad(f^{(\lambda)}). \tag{3.25}$$

Finally $U^{\sigma}_{r,j}(f^{(\lambda)})$ are orthogonal to each other in the sense

$$U^{+\sigma'}_{r',j'}(f^{(\lambda)})\quad U^{\sigma}_{r,j}\quad(f^{(\lambda)}) = U^{x\sigma'}_{r',j'}(f^{(\lambda)})\,\eta_{4}\,U^{\sigma}_{r,j}\quad(f^{(\lambda)}) \tag{3.26}$$

$$= 0$$

$$\text{if}\quad \sigma \neq \sigma',\ \text{or}\ j \neq j'\ \text{or}\ r \neq r'$$

To normalise these orthogonal vectors their transformation properties under Lorentz transformations along $\underline{f}^{(\lambda)} = \underline{f}$ should be examined.

The Lorentz transformation $\mathcal{L}_{\mu\nu}(f^{(\lambda)})$ which brings $f^{(\lambda)}$ to its rest frame $\underline{f}^{(\lambda)} = \underline{f} = 0$, $f_0^{(\lambda)} = m\alpha_\lambda$ is represented in the Dirac space by $L\,(f^{(\lambda)})$ and is given by

$$L\,(f^{(\lambda)}) = \frac{\gamma_4(-i\gamma_\mu f_\mu^{(\lambda)} + m\,\alpha_\lambda \gamma_4)}{\left[2\,\alpha_\lambda m \;(f_0^{(\lambda)} + m\,\alpha_\lambda)\right]^{\frac{1}{2}}} \tag{3.27a}$$

Its properties are

$$L\,(f^{(\lambda)})\,\mathcal{X}\,(f^{(\lambda)}) = \mathcal{X}\,(\underline{f} = 0)\;, \quad \mathcal{X}\,(f^{(\lambda)}) \text{ a Dirac spinor} \tag{3.27b}$$

$$L\,(f^{(\lambda)})\;\gamma_\mu\;L\;(f^{(\lambda)}) = \mathcal{L}_{\mu\nu}\,(f^{(\lambda)})\,\gamma_\nu\;, \tag{3.27c}$$

$$f_\mu^{(\lambda)}\;\mathcal{L}_{\mu i}\,(f^{(\lambda)}) = 0\;,\quad f_\mu^{(\lambda)}\;\mathcal{L}_{\mu 4}\,(f^{(\lambda)}) = i\,m\,\alpha_\lambda \tag{3.27d}$$

Before we apply

$$L^{(n)}\,(f^{(\lambda)}) = L\,(f^{(\lambda)}) \times L\,(f^{(\lambda)}) \times \ldots\ldots\; L\,(f^{(\lambda)})$$

to the rank n spinors $U^{\sigma}_{r\,j}\,(f^{(\lambda)})$ it is convenient to apply first the rotation operator $R^{(n)}(\tau)$ which corresponds to a space rotation that brings $\underline{f}$ along the ζ-axis

$$f_i\;\tau_{ij} = \tilde{f}_j \qquad\qquad \tilde{f}_1 = \tilde{f}_2 = 0,\;\; \tilde{f}_3 = |\,\underline{f}\,| \tag{3.28a}$$

$$R\,(\tau)\;\;\gamma_i\;\;R^{+}(\tau) = \tau_{ij}\;\gamma_j \tag{3.28b}$$

$$R\,(\tau)\;\;\bar{\sigma}_i\;\;R^{+}(\tau) = \tau_{ij}\;\bar{\sigma}_j \tag{3.28c}$$

Thus

$$R(\pi)\ \frac{\gamma_\mu f_\mu^{(\lambda)}}{(f^{(\ )}\ f^{(\ )})}\quad R^{+}(\pi)\ =\ \frac{\gamma_3\ \tilde{f}_3 + \gamma_4\ f_4^{(\lambda)}}{(f\cdot f)^{\frac{1}{2}}}\ \doteq\ \gamma\,(\tilde{f}^{\ (\lambda)}) \tag{3.29a}$$

$$R(\pi)\ \Lambda^{\pm}\,(f^{(\lambda)})\quad R^{+}(\pi)\ =\ \Lambda^{\pm}\,(\tilde{f}^{\ (\lambda)}) \tag{3.29b}$$

since

$$(\tilde{f}^{(\lambda)}\tilde{f}^{(\lambda)})^{\frac{1}{2}} = (f^{(\lambda)}\ f^{(\lambda)})^{\frac{1}{2}}\ =\ i\ \alpha_\lambda\ m \tag{3.30a}$$

$$\tilde{f}_4^{(\lambda)} = f_4^{(\lambda)} \tag{3.30b}$$

Hence

$$R^{(n)}(\pi)\ \Lambda_r^{(j)}\quad (f^{(\lambda)})\quad R^{\dagger(n)}(\pi) = \Lambda_r^{(j)}\quad (\tilde{f}^{(\lambda)}) \tag{3.31}$$

The effect of the transformation by $R\ (\pi)$ on $\xi^{\pm}\ (\underline{f})$ is

$$R\ \xi^{\pm}\ (\underline{f})\ R^{\dagger}\ =\ \tfrac{1}{2}\,R\ (1 \pm \frac{\bar{\sigma}\cdot\underline{f}}{(\underline{f}\cdot\underline{f})^{\frac{1}{2}}})\ R^{+}$$

$$= \tfrac{1}{2}(1 \pm \frac{\bar{\sigma}_3\ \tilde{f}_3}{\tilde{f}_3})$$

$$= \tfrac{1}{2}\ (1 \pm \bar{\sigma}_3)\ \equiv\ \xi^{\pm}\ (3) \tag{3.32}$$

Also

$$R^{(n)}\ O^{s_i}\ (f^{(\lambda)})\ R^{(n)}\ =\ O^{(s_i)}\ (\tilde{f}^{(\lambda)}) \tag{3.33}$$

Combining these results

$$R^{(n)}\cdot\ \mathcal{U}_{\gamma,j}^{\sigma}\ (f^{(\lambda)}) = R^{(n)}\qquad \Lambda_r^{(j)}(f^{(\lambda)})\ O^{(Si)}(f^{(\lambda)})\left\{\Sigma\ \xi_r^{(a)}(\underline{f})\right\}\mathcal{X}(f^{(\lambda)})$$

$$= \Lambda_{r}^{(j)}(\tilde{f}^{(\lambda)})\ 0^{(Si)}(\tilde{f}^{(\lambda)}) \left\{ \zeta_{r}^{(a)}(3) \right\} \chi(\tilde{f}^{(\lambda)}) \tag{3.34}$$

Operating now by L $(\tilde{f})$ yields the value of $U^{\sigma}_{r,j}(f)$ in the rest frame with spin components along the 3-axis. Since

$$L(\tilde{f}^{(\lambda)}) \Lambda^{\pm}(\tilde{f}^{(\lambda)})\, L^{\dagger}(\tilde{f}^{(\lambda)}) = \Lambda^{\pm}(0) = \tfrac{1}{2}(1 \pm \gamma_4) \tag{3.35}$$

and L (f) boing given by

$$L(\tilde{f}^{(\lambda)}) = \gamma_4 \frac{-i\gamma_3 f_3 - i\gamma_4 f_4^{(\lambda)} + m\alpha_\lambda \gamma_4}{\left\{ 2\alpha m_\alpha (f_0^{(\lambda)} + m\alpha_\lambda) \right\}} \tag{3.36}$$

commutes with $\sigma_3 = i\gamma_1\gamma_2$

$$L^{(n)}(\tilde{f})\, R^{(n)} \cdots U^{\sigma}_{r,j}(f^{(\lambda)}) = \Lambda_{r}^{(j)}(0)\ 0^{(Si)}(o) \left\{ \Sigma\, \zeta_{r}^{(a)}(3) \right\} \chi(o)$$

$$\equiv U^{\sigma}_{r,j}(3,0) \tag{3.37}$$

Writing this equation for $U^{\sigma'}_{r,j'}(f^{(\lambda)})$ taking the hermitian conjugate and multiplying by $\eta_4 = \gamma_4 \times \gamma_4 \cdots\cdots\cdots \times \gamma_4$ from the right

$$U^{*\sigma'}_{r,j'}(f^{(\lambda)})\ R^{\dagger(n)} \cdots L^{*}(\tilde{f}^{(\lambda)})\,\eta_4 = U^{\dagger\sigma'}_{r',j'}(f^{(\lambda)})\ R^{\dagger(n)} \cdots L^{\dagger(n)}(\tilde{f}^{(\lambda)})$$

$$= \chi^{*}(o) \left\{ \Sigma\, \zeta_{r}^{(a')}(3) \right\} 0^{(Si')}(o)\, \Lambda_{r'}^{(j)}(o)\, \eta_4 \tag{3.38}$$

Since $\left\{ \zeta_{r}^{\sigma'}(3) \right\}$, $0^{(Si)}(o)$ and $\Lambda^{(j)}(o)$ are hermitian self adjoint.

Now $\Lambda_{r}^{(j)}(o)$ contains $\Lambda^{-}(o)$, r times therefore

$$\Lambda_{r}^{(j)}(0)\, \eta_4 = (-1)^{r}\, \Lambda_{r}^{(j)}(o) \tag{3.39}$$

giving

$$\mathcal{U}^{\dagger\sigma'}_{r',j'}(f^{(\lambda)})\ \overset{\dagger}{R}^{(n)}(\pi)\ L^{\dagger}\ (\tilde{f}^{(\lambda)}) = (-1)^{r}\mathcal{K}\ (o)\Big\{\Sigma\ \xi_{i}^{(a)}(3)\quad o^{(S'i)}(o)\Lambda_{r'}^{(j')}(o)$$

$$= (-1)^{r}\mathcal{U}^{*\sigma'}_{r',j'}\quad(3,o) \qquad (3.40)$$

From these results one obtains at once

$$\mathcal{U}^{\dagger\sigma'}_{r'j'}(f^{(\lambda)})\ \ \mathcal{U}^{\sigma}_{r,j}\ (f^{(\lambda)}) = (-1)^{r}\ \ \mathcal{U}^{*\sigma'}_{r',j'}(3,o)\ \ \mathcal{U}^{\sigma}_{r,j}\ (3,o)$$

$$= o \quad \text{if}\ \ r',\ j',\ \sigma' \quad \neq r,\ j,\ \sigma \qquad (3.41a)$$

Normalisation of $\mathcal{U}^{\sigma}_{r,j}\ (f^{(\lambda)})$ is affected by setting the positive definite quantity

$$\mathcal{U}^{*\sigma}_{r,j}\ (3,o)\ \mathcal{U}^{\sigma}_{r,j}\ (3,o) = 1 \qquad 3.41b)$$

Hence writing $\epsilon_{r} = (-1)^{r}$

$$\mathcal{U}^{\dagger\sigma'}_{r',j'}(f^{(\lambda)})\ \mathcal{U}^{\sigma}_{r,j}\ (f^{(\lambda)}) = \epsilon_{r}\ \delta_{rr'}\ \ \delta_{j\,j'}\ \delta_{\sigma\sigma'} \qquad (3.42)$$

This is the generalisation of the similar result for the Dirac equation. For the derivation of this result the introduction of $R^{(n)}(\pi)$ was not necessary and one could have directly operated by $L^{(n)}(f^{(\lambda)})$, but then it would have taken some time and space to show that L(f) commutes with

$$\xi^{\pm}(\underline{f}) = \tfrac{1}{2}(1 + \frac{\sigma_{i}f_{i}}{(f\cdot f)^{\frac{1}{2}}})$$

The identity operator is given by

$$1 = \sum_{r=o}^{n}\ \ \sum_{j=1}^{nC_{r}}\ \ \sum_{\sigma=1}^{2^{n}}\ \mathcal{U}^{\sigma}_{r,j}(f^{(\lambda)})\ \ \mathcal{U}^{\dagger\sigma}_{r,j}(f^{(\lambda)})\ \ \epsilon_{r} \qquad (3.43a)$$

or

$$\delta_{\alpha\beta} = \sum_{r,j,\sigma} \mathcal{U}^{\sigma}_{r,j}(f^{(\lambda)})_{\alpha}\, \varepsilon_r\, \mathcal{U}^{\dagger\sigma}_{r,j}(f^{(\lambda)})_{\beta} \quad , \; 1 \leq \alpha, \beta \leq 4^n \tag{3.43b}$$

In this last equation $\mathcal{U}^{\sigma}_{r,j}(f^{(\lambda)})$ is considered a 4^n dimensional vector rather than as a spinor of rank n. The trace of an operator in λ space is defined as

$$\mathrm{tr}_{\lambda}\, Q = \sum_{r,j,\sigma} \mathcal{U}^{\dagger}_{r,j}(f^{(\lambda)})_{\alpha}\; Q_{\alpha\beta}\; \mathcal{U}_{r,j}(f^{(\lambda)})_{\beta}$$

$$= \delta_{\alpha\beta} Q_{\alpha\beta} = Q_{\alpha,\alpha} \tag{3.43c}$$

The spinors belonging to two different mass values λ and λ' are <u>not</u> orthogonal to each other in the sense $U^{\dagger}(f^{(\lambda)})\, U(f^{(\lambda')})$. However there are some useful orthogonality relations with respect to $U^{\dagger}(f^{(\lambda)})\beta_4\, U(f^{(\lambda')})$ type of scalar product. From the free field equation

$$(\beta_i \partial_i + \beta_4 \partial_4 + ms)\, \varphi(x) = 0$$

one can prove in the usual way that the 4-divergence of

$\varphi^{\dagger}(x)\beta_{\mu}\varphi(x)$ vanishes(6)

$$\partial_{\mu}\, \varphi^{\dagger}(x)\, \beta_{\mu}\, \varphi(x) = 0 \tag{3.44}$$

In the non-quantized theory $\varphi^{\dagger}\beta_{\mu}\varphi$ is defined as the probability current density and its 4th component $\varphi^{\dagger}\beta_4\varphi$ as the probability density. $\varphi^{+}\beta_4\varphi$, is conserved in time. In the L. S. Z formulation of the Quantum field theory the free field wave functions are normalised with respect to the type of scalar product which is conserved in time. Hence we must look for the orthonormality properties of $U^{+\sigma'}_{\gamma',j'}(f^{(\lambda')})\ \beta_4\ U^{\sigma}_{\gamma,j}\ (f^{(\lambda)})$. Also the Fourier expansion (2.59) of $\varphi\ (x)$ contains only the "particle" and "antiparticle" spinors,

$U^{(+)}(f^{(\lambda)}) = U^{\sigma}_{\lambda j}\ (f^{(\lambda)})$ and $U^{(-)} = U^{\sigma}_{n-\lambda,j}\ (f^{(\lambda)})$.

These satisfy the particle and the antiparticle equation

$$(\pm\ i\ \underline{\beta}\,\underline{f}\ \pm i\,\beta_4\ f_o^{(\lambda')} + ms)\ \ U^{\sigma'}_{\substack{\lambda' \\ n-\lambda'}\ j'}\ (f^{(\lambda')}) = o \tag{3.45}$$

Taking its Hermitian conjugate and multiplying by $\gamma_4\ U^{\sigma}_{\substack{\lambda \\ n-\lambda}\ j}\ (f^{(\lambda)})$ from the right

$$U^{\sigma'\dagger}_{\substack{\lambda' \\ n-\lambda',\ j'}}(f^{(\lambda')})\ \ (\pm\ i\ \underline{\beta}\,\underline{f} \pm i\,\beta_4 f_o^{(\lambda')} + ms)\ U^{\sigma}_{\substack{\lambda \\ n-\lambda}\ j}\ (f^{(\lambda)}) = o \tag{3.46a}$$

On the other hand

$$U^{\dagger\sigma'}_{\substack{\lambda' \\ n-\lambda',\ j'}}\ (f^{(\lambda)})\ \ (\pm\ i\ \underline{\beta}\,\underline{f} \pm i\beta_4 f_o^{(\lambda)} + ms)\ U^{\sigma}_{\substack{\lambda \\ n-\lambda,\ j}}\ \ (f^{(\lambda)}) = o \tag{3.46b}$$

Subtracting these two equations we get

$$U^{\dagger\sigma'}_{\substack{\lambda' \\ n-\lambda',\ j'}}\ \ (f^{(\lambda')})\,\beta_4\ \ U^{\sigma}_{\substack{\lambda \\ n-\lambda\ j}}\ \ (f^{(\lambda)}) = o \text{ for } \lambda' \neq \lambda \tag{3.47a}$$

We cannot prove that $U^{\sigma}_{n-\lambda',j}(f^{'(\lambda')})\ \beta_4\ U_{\lambda,j}(f^{(\lambda)})$ vanishes but if

$$f^{'(\lambda')} = \left\{ -\underline{f},\ f_{,}^{(\lambda')} \right\} = \left\{ -\underline{f},\ +\sqrt{\underline{f}^2 + m^2\ \alpha_\lambda^{\ 2}} \right\}$$

$$f^{(\lambda)} = \left\{ \underline{f},\ f_o^{(\lambda)} \right\} = \left\{ \underline{f},\ +\sqrt{\underline{f}^2 + m^2 \alpha_\lambda^{\ 2}} \right\}$$

we have

$$(i\,\underline{\beta}\,\underline{f} - i\beta_4\ f_4^{(\lambda')} + ms\ )\ U^{\sigma'}_{n-\lambda',j}(f^{'(\lambda')}) = o$$

or

$$U^{+\sigma'}_{n-\lambda',j'}(f^{'(\lambda')})\ \left[\ i\,\underline{\beta}.\underline{f} - i\beta_4\ f_4^{(\lambda')} + ms\ \right] U^{\sigma}_{\lambda,j}\ (f^{(\lambda)}) = o$$

Also

$$U^{+\sigma'}_{n-\lambda',j'}(f^{'(\lambda')})\ \left[\ i\underline{\beta}\ \underline{f} + i\beta_4\ f^{(\lambda)} + ms\right] U^{\sigma}_{\lambda,j}(f^{(\lambda)}) = o$$

The difference of the last two equations gives

$$U^{+\sigma'}_{n-\lambda',j}(f^{'(\lambda')})\ \beta_4\ U_{\lambda,j}(f^{(\lambda)}) = o \qquad \text{for } \underline{f} = -\ \underline{f} \tag{3.47b}$$

since $f_o^{(\lambda)} + f_o^{(\lambda')} \neq o$.

Thus the β_4 scalar product of mixed particle and antiparticle spinors vanishes for $\underline{f} = -\ \underline{f}$, so that for $\lambda' = \lambda$ it is only necessary to consider products of the type

$$U^{+\sigma'}_{\underset{n-\lambda}{\lambda}j}(f^{(\lambda)})\ \beta_4\ U^{\sigma}_{\underset{n-\lambda}{\lambda}j}(f^{(\lambda)})$$

$$= U^{+\sigma'}_{\underset{n-\lambda}{\lambda}}(f^{(\lambda)})\ \xi^{\sigma'}(f^{(\lambda)})\Lambda^{(j')}_{\underset{n-\lambda}{\lambda}}(f^{(\lambda)})\,\beta_4\ \Lambda^{(j)}_{\underset{n-\lambda}{\lambda}}(f^{(\lambda)})\ \xi^{\sigma}(f^{(\lambda)})\ \ U_{\underset{n-\lambda}{\lambda}}(f^{(\lambda)}) \tag{3.48a}$$

It is easily seen that

$$\Lambda^{(j)}_{\stackrel{\lambda}{n-\lambda}}(f^{(\lambda)})\,\beta_4\,\Lambda^{(j)}_{\stackrel{\lambda}{n-\lambda}}(f^{(\lambda)}) = \Lambda^{(j)}_{\stackrel{\lambda}{n-\lambda}}(f^{(\lambda)})\,\tfrac{1}{2}\Big\{\gamma_4 \times 1 \times \cdots \times 1 + 1 \times \gamma_4 \quad \cdots 1 \tag{3.48}$$

$$+ \cdots \quad + 1 \times 1 \cdots 1 \times \gamma_4 \Big\}\Lambda^{(j)}_{\stackrel{\lambda}{n-\lambda}}(f^{(\lambda)})$$

vanishes if $j' \neq j$ since $\Lambda^{j}_{\stackrel{\lambda}{n-\lambda}}(f^{(\lambda)})$ and $\Lambda^{j}_{\stackrel{\lambda}{n-\lambda}}(f^{(\lambda)})$ contain an equal number (λ or $n-\lambda$) of $\Lambda^{-}(f^{(\lambda)})$ factors and if $j \neq j'$ at least one $\Lambda^{\pm}(f^{(\lambda)})\ \Lambda^{\mp}(f^{(\lambda)})$ will occur in each of the n term on the R.H.S. of the last equation. Now from (3.37) and 3.40)

$$U^{\dagger\sigma'}_{\stackrel{\lambda}{n-\lambda},j'}(f^{(\lambda)})\,\beta_4\,U^{\sigma}_{\stackrel{\lambda}{n-\lambda},j}(f^{(\lambda)}) = (-1)^{\stackrel{\lambda}{n-\lambda}}\,U^{\dagger\sigma'}_{\stackrel{\lambda}{n-\lambda},j'}(3,o)\,L^{(n)}(\tilde{f}^{(\lambda)})\,\beta_4$$

$$L^{+(n)}(\tilde{f}^{(\lambda)})\quad U^{\sigma}_{\stackrel{\lambda}{n-\lambda},j}(3,o)\ \delta_{j\,j'} \tag{3.49}$$

To calculate the R.H.S. consider

$$L^{(n)}(\tilde{f}^{(\lambda)})\,\beta_4\,L^{+(n)}(\tilde{f}^{(\lambda)}) = \tfrac{1}{2}\Big\{L(\tilde{f}^{(\lambda)})\,\gamma_4\ \ L^{+}(f^{(\lambda)}) \times 1 \times 1 \times \cdots 1 \times 1$$

$$+ 1 \times L\,(f^{(\lambda)})\,\gamma_4\ \ L^{+}(\tilde{f}^{(\lambda)}) \times 1 \times 1 \cdots 1 \times 1$$

$$+ \quad 1 \times 1 \times \cdots 1 \times L\,(f^{(\lambda)})\,\gamma_4\,L^{+}\,(f^{(\lambda)}) \Big\} \tag{3.50}$$

But $L\,(\tilde{f}^{(\lambda)})\ \gamma_4\,L^{+}\,(\tilde{f}^{(\lambda)})$

$$= \frac{\gamma_4}{2m\,\alpha_\lambda (f_o^{(\lambda)} + m\,\alpha_\lambda)} \Big\{ \underline{\gamma}\,\underline{f}\,\underline{\gamma}\,\underline{f} + (m\,\alpha_\lambda + f_o^{(\lambda)}) \, (\gamma_4\, i\, \underline{\gamma}\,\tilde{\underline{f}} - i\,\underline{\gamma}\,\tilde{f}\,\gamma_4) + (m\,\alpha_\lambda + f_o^{(\lambda)})^2 \Big\} \qquad (3.51)$$

Writing the spinors in (3.49) in the form

$$U^{\sigma}_{\substack{\lambda \\ n-\lambda,\, j}} (3,o) = \Lambda^{(j)}_{\substack{\lambda \\ n-\lambda}} (o) \; U^{\sigma}_{\substack{\lambda \\ n-\lambda}} (3,o) \qquad (3.52a)$$

and

$$U^{*\sigma'}_{\substack{\lambda \\ n-\lambda}} (3,o) = U^{*\sigma'}_{\substack{\lambda \\ n-\lambda}} (3,o) \; \Lambda^{(j)}_{\substack{\lambda \\ n-\lambda}} (o) \qquad (3.52b)$$

and substituting for $L\, \gamma_4\, L^+$ from (3.51) in (3.49) we wee that $(\gamma_4\, i\, \underline{\gamma}\,\tilde{\underline{f}} - i\,\underline{\gamma}\,\tilde{\underline{f}}\,\gamma_4)$ term vanishes since this will get multiplied with $\Lambda^{\pm}(o)$ from the right and left. Also the γ_4 on the extreme left in (3.51) will give λ or $n-\lambda$ times -1 and $n-\lambda$ or λ times $+1$. The coefficient arising from this is $\frac{-\lambda + 2S - \lambda}{2} = S_\lambda$ or $\frac{-(2S - \lambda) + \lambda}{2} = -S_\lambda$

Thus

$$U^{+\sigma'}_{\substack{\lambda \\ n-\lambda,\, j'}} (f^{(\lambda)})\, \beta_4 \; U^{\sigma}_{\substack{\lambda \\ n-\lambda,\, j}} (f^{(\lambda)}) = \frac{f_o^{(\lambda)} S_\lambda}{m\,\alpha_\lambda}$$

$$\cdot\; (-1)^{\overset{\lambda}{n-\lambda}+1} \; U^{*\sigma'}_{\substack{\lambda \\ n-\lambda,\, j}} (3,o) \; U^{\sigma}_{\substack{\lambda \\ n-\lambda}} (3,o) \qquad (3.53)$$

$$= (-1)^{\overset{\lambda}{n-\lambda}-1} \frac{f_o^{(\lambda)} S_\lambda}{m\,\alpha_\lambda} \; \delta_{j j'}\, \delta_{\sigma \sigma'}$$

It is convenient to rewrite the antiparticle spinors $U_{n-\lambda,j}f^{(\lambda)})$ in the form

$$U^{\sigma}_{n-\lambda,j}(f^{(\lambda)}) = V_{\lambda,j}(f^{(\lambda)}) \tag{3.54}$$

and then collect the β_4 type of orthonormality relations together

$$U^{+\sigma'}_{\lambda',j'}(f^{(\lambda')})\,\beta_4\,U^{\sigma}_{\lambda,j}(f^{(\lambda)}) = (-1)^{\lambda}\,\frac{f_o^{(\lambda)}S_{\lambda}}{m\,\alpha_{\lambda}}\,\delta_{\lambda\lambda'}\,\delta_{\sigma\sigma'}\,\delta_{j'j} \tag{3.55a}$$

$$V^{+\sigma'}_{\lambda',j'}(f^{(\lambda')})\,\beta_4\,V^{\sigma}_{\lambda,j}(f^{(\lambda)}) = (-1)^{2S-\lambda-1}\,\frac{f_o^{(\lambda)}S_{\lambda}}{m\,\alpha_{\lambda}}\,\delta_{\lambda\lambda'}\,\delta_{j'j}\,\delta_{\sigma\sigma'} \tag{3.55b}$$

$$V^{+\sigma'}_{\lambda',j'}(f^{(\lambda')})\,\beta_4\,U^{\sigma}_{\lambda,j}(f^{(\lambda)}) = 0 \tag{3.55c}$$

for $\underline{f}' = -\underline{f}$

POLARIZATION FORMULAE

By taking $\lambda = 0$, $\alpha_\lambda = 1$, the wave function in momentum space of a free particle or an antiparticle of mass m, spin s, momentum $\underline{f}$ and helicity σ can be taken to be $U^{\sigma}_{0,1}(\underline{f}) = U_+(\underline{f})$ or $V^{\sigma}_{0,1}(f) = U^{\sigma}_-(\underline{f})$. Such vectors belong to the subspaces $\eta^{\pm\, s_0}(f)\, 0^{(s)}(f) \equiv \eta^{\pm}(f)\, 0^{(s)}(f)$. In the configuration space the wave functions are

$$\psi_{\pm}(x,f) = U_{\pm}(f)e^{if.x} \tag{4.1}$$

Equations 3.55 in this case become

$$U^{\dagger\sigma'}_{+}(f)\beta_4 U^{\sigma}_{+}(f) = \frac{f_o\, s}{m}\,\delta_{\sigma'\sigma} \tag{4.2a}$$

$$U^{\dagger\sigma'}_{-}(f)\beta_4 U^{\sigma}_{-}(f) = (-1)^{2s-1}\,\frac{f_o\, s}{m}\,\delta_{\sigma'\sigma} \tag{4.2b}$$

$$U^{\dagger}_{-}(-\underline{f},f_o)\beta_4 U_{+}(\underline{f},f_o) = 0 \tag{4.2c}$$

For "antiparticles" of integral spin the particle density is negative, for "particles" or "antiparticles" of half integral spin the particle density is positive as well as for "particles" of integral spin[8]. First consider the case of "particles".

Enclosing the system in a box of volume $\frac{m}{sf_o}$,

$$\int_{\frac{m}{sf_o}} \psi_{\pm}(x,f)\,\beta_4\psi_{\pm}(x,f)\,d^3x = 1 \tag{4.3}$$

the density of states in momentum space is $\frac{m\,d^3f}{sf_o}$. The spinors $\sqrt{\frac{m}{sf_o}}\,U_+(f)$

are normalised with respect to β_4 type of scalar product as shown by (4.2). If the probability of finding a particle in the helicity state σ is W_σ the probability of finding it in a region d^3f at $\underline{f}$ is

$$\omega(d\underline{f}) = \frac{md^3f}{sf_o} \sum_{\sigma'\sigma}^{2s+1} \left| \sqrt{\frac{m}{sf_o}}\; U_+^{\dagger\sigma'}(f)\beta_4 U_+^{\sigma}(f) \sqrt{\frac{m}{sf_o}} \right|^2 W_\sigma \quad (4.4a)$$

$$= \frac{md^3f}{sf_o} \sum_{\sigma'\sigma}^{2s+1} \left(\frac{m}{sf_o}\right)^2 U_+^{\dagger\sigma'}(f)\beta_4 U_+^{\sigma}(f) U_+^{\dagger\sigma}(f)\beta_4 U_+^{\sigma'}(f)\, W_\sigma \quad (4.4b)$$

$$= \frac{md^3f}{sf_o} \sum_{\sigma'\sigma}^{2s+1} U_+^{\dagger\sigma'}(f) U_+^{\sigma}(f)\, W_\sigma U_+^{\dagger\sigma}(f)\, U_+^{\sigma'}(f) \quad (4.4c)$$

where we have used the fact that

$$U_+^{\dagger\sigma'}(f)\; \beta_4 U_+^{\sigma}(f) = \frac{sf_o}{m}\, \delta_{\sigma'\sigma} = U_+^{\dagger\sigma'}(f)\; U_+^{\sigma}(f) \quad (4.5)$$

In equation 4.4c $U_+(f)$ can now be replaced by $U_{r,j}^{\sigma'}(f)\,(-1)^r$ and the summation over σ' extended to 2^n and a summation over r, j performed without altering the value of $\omega(df)$. Using (3.44) and (3.43) this gives

$$\omega(df) = \frac{md^3f}{sf_o} \sum_{\sigma'=1}^{2^n} \sum_{r=1}^{n} \sum_{j=1}^{nC_r} \sum_{\sigma=1}^{(2s+1)} U_{r,j}^{\dagger\sigma'}(f)\; U_{o,1}^{\sigma}(f) W U_{\sigma o,1}^{\dagger}(f) U_{r,j}^{\sigma'}(f)$$

$$= \frac{md^3f}{sf_o}\; \mathrm{tr}\,\rho(f) = \frac{md^3f}{sf_o} \sum_{\sigma=1}^{2s+1} W_\sigma \quad (4.6a)$$

$\rho(f)$ is defined by

$$\rho(f) \equiv \sum_{\sigma}^{2s+1} U_{01}^{\sigma}(f)\, W_{\sigma} U_{01}^{+\sigma}(f) \tag{4.6b}$$

$$= \sum_{\sigma}^{2s+1} U_{+}^{\sigma}(f)\, W_{\sigma} U_{+}^{+\sigma}(f)$$

$\rho(f)$ is a covariant matrix, and $\mathrm{tr}\,\rho(f) = \sum_{\sigma}^{2s+1} W_{\sigma} \cdot \omega(df)$ is invariant since the R.H.S. is so. For antiparticles of half integral spin, (f) can be defined in exactly the same way. The result is

$$\omega(d\underline{f}) = \frac{md^3\underline{f}}{sf_o} \;\mathrm{tr}\,\rho(f) \tag{4.7a}$$

$$= \frac{md^3 f}{sf_o} \sum_{\sigma}^{2s+1} W_{\sigma}$$

$$\rho(f) \equiv \sum_{\sigma}^{2s+1} - U_{n,1}^{\sigma}(f) \quad W_{\sigma} \quad U_{n,1}^{\sigma} \tag{4.7b}$$

The probability density for antiparticles of integral spins is negative and so we cannot define a ρ matrix for them. This difficulty can be removed in the quantised version of the theory as will be shown later on.

For collision process in which the initial and final spin s particles are either 'particles' or 'antiparticles' a non quantised potential scattering theory can be built up. The fact that equation 2.1 with equation (2.8) admits various mass states will have no effect on the scattering since the interaction potentials acting on a state in which the momenta lie on the mass shell $-m^2$ will give a state on the same mass shell.

What we are interested in is what the total differential cross-section for the final spin s particle with momentum $\underline{f'}$ lying in a solid angle $d\Omega$ is given by

$$d\sigma = \sum_{\sigma'=1}^{2s+1} \sum_{\sigma=1}^{2s+1} \left| U_+^{+\sigma'}(f')\, \bar{S}(f',t,f)\; U_+^{\sigma}(f) \right|^2 W_\sigma \tag{4.8}$$

t is the centre of mass momentum given in terms of initial and final momenta by

$$t = f + q = f' + q' \tag{4.9}$$

q and q' being the momenta of the initial and final spin zero particles. $\bar{S}(f',t,f)$ should be a covariant matrix since $d\sigma$ is an invariant. $U_+(f)$ satisfy

$$U_+^{\sigma'}(f') = U_{o,1}^{\sigma'}(f') = \eta^+(f')\, \overset{(s)}{O}(f')\, U_{o,1}^{\sigma'}(f') \qquad \text{for all } \sigma' \leq 2s+1$$

and so $d\sigma$ can be written

$$d\sigma = \sum_{\sigma'=1}^{2^n} \sum_{r,j} \sum_{\sigma}^{2s+1} U_{r,j}^{+\sigma'}(f')\, O^{(s)}(f')\, \bar{S}\; U_{o,1}(f)\, W_\sigma \tag{4.11a}$$

$$U_{o,1}^{+\sigma}(f)\, \bar{S}^+ \eta^+(f')\, O^{(s)}(f')\, U_{r,j}^{\sigma}(f)\, (-1)^r \tag{4.11b}$$

$$= \mathrm{tr}\, \eta^+(f')O^{(s)}\bar{S}(f',t,f)\rho(f)\, \bar{S}^+(f',t,f)\eta^+(f')\, O^{(s)}(f') \tag{4.11c}$$

$\rho(f)$ given by (4.6) satisfies

$$\rho(f) = \eta^+(f)\, O^{(s)}(f)\rho(f)\eta^+(f)\, O^{(s)}(f) \tag{4.12}$$

Substituting 4.12 in 4.11

$$d\sigma = \operatorname{tr} \; S(f',t,f)\rho(f) \; S^{+}(f',t,f) = \operatorname{tr} \; \rho'(f') \tag{4.13}$$

$S(f',t,f)$ and $\bar{S}\,(f',t,f)$ are connected by

$$S\,(f',t,f) = \gamma^{+}(f')\; O^{(s)}(f') \;\; \bar{S}(f',t,f)\,\gamma^{+}(f)\; O^{(s)}(f) \tag{4.14}$$

The final ρ matrix $\rho'(f')$ is defined by

$$\rho'(f') \;=\; S(f',t,f)\,\rho(f) \;\; S^{+}(f',t,f) \tag{4.15}$$

In case tr (f) $= \sum_{\sigma}^{2s+1} W_{\sigma}$ is not equal to unity, equation (4.15) is replaced by

$$d\sigma = \frac{\operatorname{tr}\rho'(f')}{\operatorname{tr}\rho(f)}$$

The average values of a matrix operator A in the state of a particle characterised by the matrix (f) is given by

$$\langle A \rangle = \sum_{\sigma}^{2s+1} U^{+\sigma}(f) \;\; A\, U \;\; (f) \;\; W_{\sigma} \quad \frac{1}{\operatorname{tr}\rho(f)} \tag{4.16}$$

This is easily shown to be given by

$$\langle A \rangle = \frac{\operatorname{tr} A\rho(f)}{\operatorname{tr}\rho\,(f)} \tag{4.17}$$

CHAPTER V

THE S – MATRIX

The spinors $U_{\alpha' \alpha'' \dots \alpha^{(n)}}$ describing the states of particles of the highest spin S are completely symmetric in the spinor indeces $\alpha', \alpha'', \dots, \alpha^{(n)}$. The S-Matrix operating in the space defined by such spinors may therefore be assumed to be completely symmetric in its rows as well as the columns.

$$S(f',t,f)_{\alpha^{(1)}\alpha^{(2)}\dots\alpha^{(n)},\beta^{(1)}\beta^{(2)}\dots\beta^{(n)}} = S(f',t,f)_{p(\alpha^{(1)},\alpha^{(2)}\dots\alpha^{(n)}),p'(\beta^{(1)}\beta^{(2)}\dots\beta^{(n)})} \tag{5.1}$$

p and p' denote any two permutations of 'n' objects acting on $\alpha^{(1)},\alpha^{(2)}\dots\alpha^{(n)}$ and $\beta^{(1)},\beta^{(2)}\dots\beta^{(n)}$ respectively.

In order to utilise the representation (2.14) of β_μ's fully we must write the S-matrix as a Kronecker product

$$S(f',t,f) = \sum_{\ell=1}^{\ell_0} \mathcal{S}_\ell^{(1)} \times \mathcal{S}_\ell^{(2)} \times \mathcal{S}_\ell^{(3)} \times \dots \mathcal{S}_\ell^{(n)} \tag{5.2}$$

where ℓ_0 is some finite integer. It is not possible to write down a sum in Kronecker products which satisfies (5.1) also. However the form

$$S(f',t,f) = \sum_{P,\ell} \mathcal{S}_\ell^{(1)}(f',t,f) \times \mathcal{S}_\ell^{(2)} \times \dots\dots \mathcal{S}_\ell^{(n)} \tag{5.3}$$

where $\sum_P$ denotes the sum of all the $\lfloor n$ permutations of $\mathcal{S}_\ell^{(1)}, \mathcal{S}_\ell^{(2)} \dots \mathcal{S}_\ell^{(n)}$, satisfies (5.1) for $p = p'$ i.e. for permutations on $\alpha^{(1)}, \alpha^{(2)} \dots \alpha^{(n)}$ and $\beta^{(1)}, \beta^{(2)} \dots \beta^{(n)}$ being the same.

This can easily be proved in the following way. Dropping ℓ for the moment

$$S_{\alpha^{(1)}\alpha^{(2)}\ldots\alpha^{(n)},\,\beta^{(1)}\beta^{(2)}\ldots\beta^{(n)}} = \mathcal{S}^{(1)}_{\alpha^{(1)}\beta^{(1)}}\,\mathcal{S}^{(2)}_{\alpha^{(2)}\beta^{(2)}}\cdots\mathcal{S}^{(n)}_{\alpha^{(n)}\beta^{(n)}} \tag{5.4}$$

If the same permutation is applied to $\alpha^{(1)}, \alpha^{(2)} \ldots \alpha^{(n)}$ and $\beta^{(1)}, \beta^{(2)} \ldots \beta^{(n)}$ then the same permutation P is applied to the subscripts $\alpha^{(1)}\beta^{(1)}$, $\alpha^{(2)}\beta^{(2)}, \ldots \alpha^{(n)}\beta^{(n)}$ on the R.H.S. But this is equivalent to the same permutation P being applied to the super scripts $(1), (2) \ldots (n)$ and since all such permutations are being summed up, the result follows.

$S(f', t, f)$ is a invariant matrix but the matrices $\mathcal{S}^{(i)}(f', t, f)$ are not necessarily so. In general $\mathcal{S}^{(i)}(f', t, f)$ will contain tensor indices in the form f'_μ, f_μ, t_μ and γ'_μs. These are all contracted with the ones occurring in other $\mathcal{S}^{(i)}$ so as to leave $S(f', t, f)$ invariant. As far as the tensors formed by f'_μ, t_ν, f_λ are concerned we note that these can be taken over the Kronecker product signs, and contracted with the other tensor indices. So the remaining tensor indices are formed out of γ matrices. As an example $S(f', t, f)$ may be

$$\begin{aligned} S(f', t, f) &= \sum_{P,\ell} \gamma_\nu \mathcal{S}^{(1)}_\ell \gamma_\mu \times \mathcal{S}^{(2)}_\ell \gamma_\mu \times \gamma_\nu \mathcal{S}^{(3)}_\ell \times \ldots\ldots \\ &= \sum_{P\,\ell} \mathcal{S}^{(1)}_{\ell,\nu\mu} \times \mathcal{S}^{(2)}_{\ell,\mu} \times \mathcal{S}^{(3)}_{\ell,\nu} \times \ldots\ldots\ldots \end{aligned} \tag{5.5}$$

To indicate this S is rewritten in the form

$$S(f',t,f) = \sum_L S_L^{(1)} \times S_L^{(2)} \cdots \times S_L^{(n)} \tag{5.6}$$

where $\sum_L$ includes the summations over tensor indices also. Now each $S_L^{(i)}(f',t,f)$ can be expanded in the form

$$S_L^{(i)}(f',t,f) = \Lambda^+(f')S_L^{(i)}\Lambda^+(f) + \Lambda^+(f')S_L^{(i)}\Lambda^-(f)$$
$$+ \Lambda^-(f')S_L^{(i)}\Lambda^+(f) + \Lambda^-(f')S_L^{(i)}\Lambda^-(f) \tag{5.7}$$

Using the notation

$$\Lambda^{\pm}(f')S_L^{(i)}(f',t,f)\Lambda^{\pm}(f) = S_{L\pm\pm}^{(i)}(f',t,f) \tag{5.8a}$$

(5.7) can be written

$$S_L^{(i)}(f',t,f) = S_{L++}^{(i)} + S_{L+-}^{(i)} + S_{L-+}^{(i)} + S_{L--}^{(i)} \tag{5.8b}$$

The matrices on the R.H.S. obey the conditions

$$\Lambda^+(f')S_{L++}^{(i)}\Lambda^+(f) = S_{L++}^{(i)} \tag{5.9a}$$

$$\Lambda^+(f')S_{++}^{(i)}\Lambda^-(f) = 0 \tag{5.9b}$$

etc. Since the initial and final spin s particles are either particles or antiparticles, $S(f',t,f)$ obeys the hole theory condition

$$\eta^{\pm}(f')S(f',t,f)\eta^{\pm}(f) = S(f',t,f) \tag{5.10a}$$

where $$\eta^{\pm}(f) = \Lambda^{\pm}(f) \times \Lambda^{\pm}(f)\cdots \times \Lambda^{\pm}(f) \quad \text{n factors} \tag{5.10b}$$

Substituting the form (5.8) of $\mathcal{S}_L^{(i)}\,(f',t,f)$ in (5.6) and applying the hole theory condition we get at once

$$S = \sum_L \mathcal{S}^{(1)}_{L\underline{++}} \times \mathcal{S}^{(2)}_{L\underline{++}} \times \mathcal{S}^{(3)}_{L\underline{++}} \cdots\cdots \mathcal{S}^{(n)}_{L\underline{++}} \qquad (5.11)$$

From the definitions (5.8a) of $\mathcal{S}^{(i)}_{L\underline{++}}$ it follows that

$$\gamma(f')\,\mathcal{S}^{(i)}_{L\underline{++}}\,(f',t,f)\;\gamma\,(f) = \mathcal{S}^{(i)}_{L\underline{++}}\,(f',t,f) \qquad (5.12)$$

As in Stapps work a matrix $\mathcal{S}'^{(i)}(k',t,k)$ is defined by

$$\mathcal{S}^{(i)}_L\,(f',t,f) = \gamma(f',t)\,\mathcal{S}'^{(i)}_L(k',t,k)\quad \gamma\,(f\,,t) \qquad (5.13)$$

and as before the condition (5.12) is transformed into a commutation relation.

$$\gamma(t)\,\mathcal{S}'^{(i)}_{L\underline{++}}\,(k',t,k)\;\gamma\,(t) = \mathcal{S}'^{(i)}_{L\underline{++}}(k',t,k) \qquad (5.14)$$

A sufficiently general form of $\mathcal{S}'^{(i)}_{L\underline{++}}$ is

$$\mathcal{S}'^{(i)}_{L\underline{++}}\,(k',t,k) = \gamma_\alpha \cdots \gamma_\beta\;\;\bar{\mathcal{S}}^{(i)}_{\ell\underline{++}}(k',t,k)\;\;\gamma_\rho \cdots \gamma_\sigma \qquad (5.15)$$

N and N$'$ are the number of γ matrices on the left and the right of $\bar{\mathcal{S}}^{(i)}(k',\,t,k)$. $\bar{\mathcal{S}}^{(i)}(k',t,k)$ itself is an invariant matrix. Substituting this form (5.15) of $\mathcal{S}'^{(i)}_{L\underline{++}}$ in (5.14)

$$\gamma(t)\,\gamma_\alpha\cdots\gamma_\beta\,\bar{\mathcal{S}}^{(i)}_{\ell\,\underline{\pm\pm}}\,\gamma_\rho\cdots\gamma_\sigma\,\gamma(t) = \gamma_\alpha\cdots\gamma_\beta\,\bar{\mathcal{S}}^{(i)}_{\ell\,\underline{\pm\pm}}\,\gamma_\rho\cdots\gamma_\sigma \tag{5.16}$$

The next obvious step is to multiply the equation by $\gamma_\rho \ldots \gamma_\alpha$ from the left and $\gamma_\sigma \ldots \gamma_\rho$ from the right and use the results

$$\gamma_\alpha \quad \gamma(t) \quad \gamma_\alpha \quad = -2 \quad \gamma\ (t) \tag{5.17a}$$

$$\gamma_\alpha\,\gamma_\alpha \quad = \quad 4 \tag{5.17b}$$

This operation gives

$$\gamma(t)\ \bar{\mathcal{S}}^{(i)}_{\ell\,\underline{\pm\pm}}(k',t,k)\ \gamma\ (t) = \ (-2)^{N+N'}\,\bar{\mathcal{S}}^{(i)}_{\ell\,\underline{\pm\pm}}\ (k',t,k) \tag{5.18}$$

By expanding $\bar{\mathcal{S}}^{(i)}$ in terms of the γ matrices as previously, it can be proved that (5.18) is satisfied by non vanishing $\bar{\mathcal{S}}^{i}$ only for $N + N' = o$. This is easily seen without going into detailed calculations by multiplying (5.18) by $\gamma(t)$ from the left and from the right. This gives

$$(-2)^{\ N\ +\ N'}\ \bar{\mathcal{S}}^{(i)}_{\ell\,\underline{\pm}\ \underline{\pm}} = (-2)^{-(N\ +N')}\,\bar{\mathcal{S}}^{\ i}_{\underline{\pm}\ \underline{\pm}} \tag{5.19}$$

showing that $N + N' = o$ for non trivial $\bar{\mathcal{S}}^{(i)}$. N and N$'$ being positive integers are separately zero. Hence $\mathcal{S}'^{(i)}_{L\underline{\pm\pm}}$ is just the invariant matrix $\bar{\mathcal{S}}^{(i)}_{L\underline{\pm\pm}}$ and we have the rather unexpected result that in the Kronecker product expansion of $S(f',t,f)$ the Dirac matrices $\mathcal{S}^{(i)}(f',t,f)$ are separately ~~co~~invariant. In terms of these matrices $S\ (f',t,f)$ is given by

$$S\,(f',t,f) = \left[\gamma(f',t)\right]^{(n)} \sum_{P,\ell} \left\{ \bar{\delta}^{(1)}_{\ell\,\pm\pm} \times \bar{\delta}^{(2)}_{\ell\,\pm\pm} \cdots \bar{\delta}^{(n)}_{\ell\,\pm\pm} \right\} \left[\gamma(f,t)\right]^{(n)} \tag{5.20}$$

$\left[\gamma(f,t)\right]^{(n)}$ is the n fold Kronecker product of $\gamma(f,t)$

The matrices $\bar{\delta}^{(i)}_{\ell\,\pm\pm}$ (k',t,k) obeying (5.18) with $N + N' = 0$ have already been determined in Chapter I.

$$\bar{\delta}^{(i)}_{\ell\,\pm\pm}(k',\ t,k) = \Lambda^{\pm}(t)\ (F_\ell^{\pm(i)} + G^{\pm i}\ i\gamma_5\ \gamma\cdot n)$$
$$= \gamma(t)\,\Lambda^{\pm}(t)\ (F^{\pm(i)} + G^{\pm i}\ i\gamma_5\gamma\cdot n)\,\gamma(t) \tag{5.21}$$

$S(f',t,f)$ is not completely symmetric in its rows and columns and must be multimplied by $O^{(s)}(f')$ and $O^{s}(f)$ from the left and from the right respectively to describe scattering of particles of spin 's' only.

$$S(f',t,f) = O^{(s)}(f')\left[\gamma(f',t)\gamma(t)\right]^{(n)} \sum_{P,\ell}\ \prod_{i=1}^{n}{}_{\times}$$
$$\left\{\Lambda^{\pm}(t)\ (F_\ell^{\pm(i)} + G_\ell^{\pm(i)}\ i\gamma_5\ \gamma\cdot n)\right\}\left[\gamma(t)\ \gamma(f,t)\right]^{(n)}\ O^{(s)}(f) \tag{5.22}$$

$\prod_x$ is the product symbol for Kronecker products. For further reduction of the S-matrix in the next Chapter it is necessary to show that

$$\left[\gamma(t)\,\gamma(f,t)\right]^{(n)} O^{(s)}(f) = O^{(s)}(t)\left[\gamma(t)\ \gamma(f,t)\right]^{(n)} \tag{5.23}$$

This can be proved in the following way

$$0^{(s)}(t)\quad \left[\gamma(t)\ \gamma(f,t)\right]^{(n)} = L^{\dagger(n)}(t)\ L^{(n)}(t)\ 0^{(s)}(t)\ L^{\dagger(n)}(t)\ L^{(n)}(t)$$

$$\left[\gamma(t)\ \gamma(f,t)\right]^{(n)}\ L^{\dagger(n)}(t)\ L^{(n)}(t) \tag{5.24a}$$

$L^{(n)}(t)$ is the Lorentz operator $L(t) \times 1(t) \ldots\ldots \times L(t)$ which transforms $0^{(s)}(t)$ to its value in the centre of mass frame $\underline{t} = o$.

$$L^{(n)}(t)\ 0^{(s)}(t)\quad L^{\dagger(n)}(t) = 0^{(s)}(o) \tag{5.24b}$$

Further using (2.24) (2.25) it is seen that

$$L^{(n)}(t)\left[\gamma(t)\ \gamma(f,t)\right]^{(n)}\ L^{\dagger(n)}(t) = \left[\gamma(t_1)\ \gamma(f_1,\ t_1)\right]^{(n)}$$

$$= L^{(n)}(f_1) \tag{5.24c}$$

the subscript$_1$ denotes centre of mass values. Hence

$$\begin{aligned}
0^{(s)}(t)\left[\gamma(t)\ \gamma(f,t)\right]^{(n)} &= L^{\dagger(n)}(t)\quad 0^{(s)}(o)\ L^{(n)}(f_1)\ L^{(n)}(t) \\
&= L^{\dagger(n)}(t)\ L^{(n)}(f_1)\ L^{\dagger(n)}(f_1)\ 0^{(s)}(o)\ L^{(n)}(f_1)\ L^{(n)}(t) \\
&= L^{\dagger(n)}(t)\ L^{(n)}(f_1)\ 0^{(s)}(f_1)\ L^{(n)}(t) \\
&= L^{\dagger(n)}(t)\ L^{(n)}(f_1)\ L^{(n)}(t)\ L^{\dagger(n)}(t)\quad 0^{(s)}(f_1)\ L^{(n)}(t) \\
&= L^{\dagger(n)}(t)\ \left[\gamma(t_1)\ \gamma(f_1,t_1)\right]^{(n)}\ L^{(n)}(t)\ L^{\dagger(n)}(t)\ 0^{(s)}(f_1) L^{(n)}(t)
\end{aligned}$$

$$\left[\gamma(t)\ \gamma(f,t)\right]^{(n)}\quad 0^{(s)}(f) \qquad \text{Q.E.D.}$$

It can be proved directly from its definition that $0(f)$ and hence $0^{(s)}(f)$ is self adjoint. Taking the adjoint of (5.23) and replacing f by f

$$O^{(s)}(f') \quad \left[\gamma(f',t)\,\gamma(t)\right]^{(n)} = \left[\gamma(f',t)\,\gamma(t)\right]^{(n)} \quad O^{(s)}(t) \qquad (5.25)$$

Equations (2.81c), (2,83) permit us to write S (f',t,f) in the form

$$S\,(f',t,f) = \left[\gamma(f',t)\,\gamma(t)\right]^{(n)}\, O^{(s)}(t) \sum_{P,\ell} \prod_{i=1}^{n}{}_x \left\{ \Lambda^{\pm}(t)\left(F_{\ell}^{\pm(i)} + G^{\pm i}\, i\,\gamma_5 \gamma \cdot n\right) \right\}$$

$$\left[\gamma(t)\,\gamma(f,t)\right]^{(n)} \qquad (5.26)$$

ρ-matrices

The initial and final density matrices $\rho(f)$ and $\rho(f')$ can be determined in the same manner as the S-matrix.

$$\rho(f) = \frac{\mathrm{tr}\,\rho(f)}{T}\; O^{(s)}(f) \sum_{P,r} C_r^{\pm} \prod_{i=1}^{n}{}_x \left\{ \Lambda^{\pm}(f)\, (1 + i\gamma_5\, \gamma \cdot p^{\pm(i)}) \right\} \qquad (5.27a)$$

$$\rho(f') = \frac{\mathrm{tr}\,\rho(f')}{T'}\; O^{(s)}(f') \sum_{P,r} C_r^{\pm} \prod_{i=1}^{n}{}_x \left\{ \Lambda^{\pm}(f')\, (1 + i\gamma_5\, \gamma \cdot p_r'^{\pm(i)}) \right\} \qquad (5.27b)$$

The 4-vectors $p_{r,\mu}^{\pm(i)}$ and $p_{r,\mu}'^{\pm(i)}$ are orthogonal to f and f$'$ respectively

$$f \cdot p_r^{\pm(i)} = f' \cdot p_r'^{\pm(i)} = 0 \qquad (5.28)$$

and in the respective rest frames $\underline{f} = 0$, $\underline{f}' = 0$ reduce to 3-vectors.

T and T$'$ are the traces of the whole expressions on the right of

$\dfrac{\mathrm{tr}\,(f)}{T}$ and $\dfrac{\mathrm{tr}\,(f')}{T'}$ respectively. For S = 1, T can be easily calculated by going over to the rest frame $\underline{f} = 0$, since the trace remains invariant under unitary transformation

$$T = \sum_r C_r^{\pm}\; (6 + 2\underline{p}_r^{(1)} \cdot \underline{p}_r^{(2)}) \qquad \text{for} \qquad S = \frac{n}{2} = 1 \qquad (5.29a)$$

The form of ρ(f) for s = 1 is

$$\rho(f) = \frac{\mathrm{tr}\,\rho(f)}{T}\, o^{(1)}(f) \sum_{p,r} C_r^{\pm} \cdot \Lambda^{\pm}(f)(1 + i\gamma_5\,\gamma\, p_r^{(\pm)(1)}) \times \Lambda^{\pm}(f)(1 + i\gamma_5\,\gamma\, p^{(\pm)(2)})$$

$$= \frac{\mathrm{tr}\,\rho(f)}{T}\, o^{(1)}(f)\Lambda^{\pm}(f) \times \Lambda^{\pm}(f) \sum_r C_r^{\pm} \Big\{ 2 + (i\gamma_5\gamma_\mu \times 1 + 1 \times i\gamma_5\gamma_\mu)$$

$$\cdot\, (p_{r,\mu}^{\pm(1)} + p_{r,\mu}^{\pm(2)}) + (i\gamma_5\gamma_\mu \times i\gamma_5\gamma_\nu)\, (p_{r,\mu}^{\pm(1)} p_{r,\nu}^{\pm(2)} + p_{r,\mu}^{\pm(2)} p_{r,\nu}^{\pm(1)}) \tag{5.29b}$$

Defining now a 4-vector $P_\mu^{\pm}$ and a 4-tensor $P_{\mu\nu}^{\pm}$ by

$$P_\mu^{\pm} = \sum_r C_r^{\pm} \,(p_{r,\mu}^{\pm(1)} + p_{r,\mu}^{\pm(2)}) \tag{5.29c}$$

$$P_{\mu\nu}^{\pm} = \sum_r C_r^{\pm} \,(p_{r,\mu}^{\pm(1)} p_{r,\nu}^{\pm(2)} + p_{r,\mu}^{(\pm)(2)}\, p_{r,\nu}^{(\pm)(1)}) \tag{5.29d}$$

and
$$\sum 2 C_r^{\pm} = a^{\pm} \tag{5.29e}$$

$$\rho(f) = \frac{\mathrm{tr}\,\rho(f)}{T}\, o^{(1)}(f)\, \eta^{\pm}(f) \Big\{ a^{\pm} + (i\gamma_5\gamma_\mu \times 1 + 1 \times i\gamma_5\gamma_\mu)\, P_\mu^{\pm}$$

$$+\, i\gamma_5\gamma_\mu \times i\gamma_5\gamma_\nu \; P_{\mu\nu}^{\pm} \tag{5.30}$$

$P_{\mu\nu}^{\pm}$ is symmetric in indices μ and ν. $P_\mu^{\pm}$ and $P_{\mu\nu}^{\pm}$ are the relativistic generalisations of the polarization vector and the polarization tensor in terms of which the noncovariant ρ matrix is expressed. In the rest frame $\underline{f}$ = o, the 4-vector P_μ and the 4-tensor $P_{\mu\nu}$ reduce to 3-vector $\bar{p}_j$, and 3-tensor F_{ij}. These are called the proper polarization quantities.

CHAPTER VI

COVARIANT POLARISATION FORMALISM

Covariant scattering equation is obtained by inserting the expressions (5.26), (5.27a) and (5.27b) in the equation

$$\rho'(f') = S(f',t,f)\,\rho(f)\quad S^{\dagger}(f',t,f)$$

$S^{\dagger}(f',t,f)$ is easily formed by remembering that $\gamma(f,t)$, $\gamma(t)$ etc are self adjoint and so

$$\frac{1}{T'}\,\frac{tr\,\rho'(f)}{tr\,\rho(f)}\;O^{(s)}(f')\sum_{P,r} C_r'^{\pm}\prod_{i=1}^{n}{}_{\times}\left\{\Lambda^{\pm}(f')\,(1+i\gamma_5\gamma\cdot p_r'^{\pm(i)})\right\}$$

$$=\left[\gamma(f',t)\,\gamma(t)\right]^{(n)}O^{(s)}(t)\sum_{P,\ell}\prod_{i=1}^{n}{}_{\times}\left\{\Lambda^{\pm}(t)\,(F^{\pm(i)}+G^{\pm(i)}\,i\gamma_5\gamma\cdot n)\right\}$$

$$\left[\gamma(t)\,\gamma(f,t)\right]^{(n)}\;\frac{1}{T}\,O^{(s)}(f)\sum_{P,r}C_r^{\pm}\left\{\prod_{i=1}^{n}{}_{\times}\Lambda^{\pm}(f)\,(1+i\gamma_5\gamma\cdot p_r^{\pm(i)})\right\}$$

$$\left[\gamma(f,t)\,\gamma(f)\right]^{(n)}\;O^{(s)}(t)\sum_{P,\ell'}\prod_{i=1}^{n}{}_{\times}\left\{\Lambda^{\pm}(t)\,(F_{\ell'}^{*\pm(i)}+G_{\ell'}^{*\pm(i)}\,i\gamma_5\gamma\cdot n)\right\}$$

$$\left[\gamma(t)\,\gamma(f',t)\right]^{(n)}\tag{6.1}$$

This equation is so complicated that hardly anything can be got out of it concerning the state of polarisation of the particle after the scattering. But except for the presence of 'n' fold Kronecker products and of spin projection operations $O^{(s)}$, (6.1) is very similar to the equation obtained for spin $\frac{1}{2}$ by Stapp and, by employing the same method, it canbe reduced into a form in which S and $S^\dagger$ are given in the centre of mass frame, $\underline{t} = o$ and $\rho(f)$, $\rho'(f')$ are given in their respective rest frames with a certain rotation applied to each index of the proper polarization tensors. It will be seen that all matrices occurring in this equation are of the form

$$\begin{pmatrix} a_1 & o \\ o & o \end{pmatrix} \times \begin{pmatrix} a_2 & o \\ o & o \end{pmatrix} \times \cdots\cdots\cdots \qquad \times \begin{pmatrix} a_n & o \\ o & o \end{pmatrix}$$

and it would be possible to write it in terms of the Kronecker product of the Pauli matrices a_i. Our next task would be to obtain an equation in terms of the rotation matrices $\theta_i^{(s)}$ of the spin s representation of the 3 dimensional rotation group. Finally the reduced ρ and ρ' matrices will be expressed in terms of the traceless symmetric tensors $T_{ij\ldots\ldots n\ \ldots\ldots}$ formed from $\theta_i^{(s)}$ and the symmetric traceless polarization tensors $\theta_{ij\ldots\ldots\ldots n}$. The S matrix will involve $T_{ij\ldots n\ldots}$ and tensors formed from initial and final momenta k, $\mathbf{k}'$ and $\underline{k} \wedge \underline{\mathbf{k}}'$.

To carry out these reductions the first step is to bring $[\gamma(f',t)\,\gamma(t)]^{(n)}$ and $[\gamma(t)\,\gamma(f,t)]^{(n)}$ on the left hand side of (6.1). This is easily done since

$$\begin{aligned} &[\gamma(f',t)\,\gamma(t)]^{(n)}\ [\gamma(t)\ \gamma(f',t)]^{(n)} \\ &= [\gamma(f',t)\ \gamma(t)\ \ \gamma(t)\ \gamma(f',t)]^{(n)} \\ &= \quad [1]^{(n)} \ \equiv\ 1 \end{aligned} \qquad (6.2)$$

Also $\dfrac{tr\,\rho'(f')}{tr\,\rho(f)}$ is replaced by the total differential cross-section I and the Lorentz transformation $L^{(n)}(t)$ is used to give

$$\frac{1}{T'}\ I\ \ L^{(n)}(t)\left[\gamma\ (t)\,\gamma\ (f',t)\right]^{(n)}\ \overset{\dagger}{L}^{(n)}\ (t)\ L^{(n)}(t)\ \ O^{(s)}(f')$$

$$\sum_{P,r} C'^{\pm}_{r}\ \prod_{i=1}^{n}{}_{x}\left\{\Lambda^{\pm}(f')\ (1 + i\gamma_5\,\gamma\cdot p_r'^{\pm(i)})\right\}\ \ \overset{\dagger}{L}^{(n)}\ (t)\ L^{(n)}\ (t)$$

$$\left[\gamma\ (f',t)\ \gamma(t)\right]^{(n)}\qquad \overset{\dagger}{L}^{(n)}(t) \tag{6.3}$$

$$= L^{(n)}(t)\ O^{(s)}(t)\sum_{P,\ell}\ \prod_{i=1}^{n}{}_{x}\left\{\Lambda^{\pm}(t)\ (F_\ell^{\pm(i)} + G_\ell^{\pm(i)}\ i\,\gamma_5\,\gamma\cdot n)\right\}\ \ \overset{\dagger}{L}^{(n)}\ \ (t)$$

$$L^{(n)}(t)\qquad \left[\gamma(t)\ \ \gamma(f,t)\right]^{(n)}\ \overset{\dagger}{L}^{(n)}(t)\ \ L^{(n)}(t)\ \ \frac{1}{T}\ \ O^{(s)}(f)$$

$$\sum_{P,r} C^{\pm}_{r}\ \ \prod_{i=1}^{n}{}_{x}\left\{\Lambda^{\pm}(f)\ \ (1 + i\,\gamma_5\ \gamma\cdot p_r^{\pm(i)})\right\}\ \ \overset{\dagger}{L}^{(n)}(t)\ L^{(n)}(t)\ \left[\gamma(f,t)\,\gamma(t)\right]^{(n)}$$

$$\overset{\dagger}{L}^{(n)}(t)\ L^{(n)}(t)\ \ O^{(s)}(t)\sum_{P,\ell'}\ \prod_{i=1}^{n}{}_{x}\left\{\Lambda^{\pm}(t)\ (\overset{\times}{F}{}_{\ell'}^{\pm(i)} + \overset{\times}{G}{}_{\ell'}^{\pm(i)}\,i\,\gamma_5\,\gamma\cdot\ n)\right\}$$

$$\overset{\dagger}{L}^{(n)}\ \ (t) \tag{6.4}$$

Let us consider different factors of this equation one by one,

$$L^{(n)}(t)\ \left[\gamma\ (t)\ \ \gamma\ (f',t)\right]^{(n)}\ \ \overset{\dagger}{L}^{(n)}(t) = \left[\gamma(t_1)\ \gamma\ (f',\ t_1)\right]^{(n)}$$

$$= \overset{\dagger}{L}^{(n)}\ \ (f_1') \tag{6.5a}$$

$$L^{(n)}(t)\left[\ \gamma\ (f',\ t)\ \gamma\ (t)\right]^{(n)}\ \overset{\dagger}{L}^{(n)}(t) = \left[\gamma\ (f',t\ )\ \gamma(t\ )\right]$$

$$= \overset{\dagger}{L}^{(n)}\ \ \ (f_1') \tag{6.5b}$$

These two equations and the corresponding ones for f are used on the extreme ends of the $\rho(f)$ and $\rho'(f')$ of equations (6.4). Further we note that

$$L^{(n)}(\chi_1)\, L^{(n)}(t)\, L^{\dagger(n)}(\chi) = R(\chi_1) \qquad , \quad \chi = f \quad \text{or } f' \tag{6.6}$$

is a Lorentz operator corresponding to pure space rotation.(1) This is easily seen in the following way. To fix our ideas let $\chi = f'$

If $U(\underline{f}' = o)$ is the value of a spinor in the frame of rest $\underline{f}' = o$, $L^{\dagger(n)}(f')$ acting on $U(\underline{f}' = o)$ takes it to a frame in which $f' = f'$, i.e. $t = t$

$$L^{\dagger(n)}(f') \quad U(\underline{f}' = o) = U(f') \tag{6.7a}$$

$L^{(n)}(t)$ brings $U(f')$ to the frame $t = t_1$, $\underline{t} = o$, $f' = f'_1$ .... i.e. the centre of mass frame

$$L^{(n)}(t) \quad U(f') = U(f'_1) \tag{6.27b}$$

$L^{(n)}(f'_1)$ brings $U(f'_1)$ back to the frame in which $\underline{f}' = o$. Thus $R(f')$ can only be a pure space rotation.

From (6.6) we derive

$$L^{(n)}(f'_1)\; L^{(n)}(t) = R(f'_1)\; L^{(n)}(f') \tag{6.8a}$$

$$L^{\dagger(n)}(t)\; L^{(n)\dagger}(f'_1) = L^{\dagger(n)}(f')\; R^{\dagger}(f'_1) \tag{6.8b}$$

Using these equations (6.4) becomes

$$\frac{1}{T}\cdot \; I \quad R(f_1') \, L^{(n)}(f') \, O^{(s)}(f') \sum_{P,r} \acute{C}^{\pm} \prod_{i=1}^{n} \times \left\{ \Lambda^{\pm}(f')(1 + i\gamma_5 \gamma \cdot p_r'^{\pm(i)}) \right\}$$

$$\overset{\dagger}{L}{}^{(n)} \; (f') \; R^{\dagger} \; (f_1')$$

$$= L^{(n)}(t) \; O^{(s)}(t) \sum_{P,\ell} \prod_{i=1}^{n} \times \left\{ \Lambda^{\pm}(t) \; (F_\ell^{\pm(i)} + G_\ell^{\pm(i)} \; i\gamma_5 \gamma \cdot n) \right\} \; \overset{\dagger}{L}{}^{(n)}(t)$$

$$R(f_1) \, L^{(n)}(f) \quad \frac{1}{T} \; O^{(s)}(f) \sum_{P,r} C_r^{\pm} \prod_{i=1}^{n} \times \left\{ \Lambda^{\pm}(f)(1 \mp i \gamma_5 \gamma \cdot p_r^{\pm(i)}) \right\}$$

$$\overset{\dagger}{L}{}^{(n)} \; (f) \; R^{\dagger} (f) \; L^{(n)}(t) \; O^{(s)}(t) \sum_{P,\ell'} \prod_{i=1}^{n} \times \left\{ \Lambda^{\pm}(t) \; (F_{\ell'}^{*\pm(i)} + G_{\ell'}^{*\pm(i)} \; i\gamma_5 \gamma \cdot n) \right\}$$

$$\overset{\dagger}{L}{}^{(n)} \quad (t) \tag{6.9}$$

Consider the L.H.S. of this equation. The expression in between $L^{(n)}(f')$ and $\overset{\dagger}{L}{}^{(n)}(f')$ is transformed to a Lorentz frame in which $\underline{f}' = 0$.

Thus $O^{(s)}(f) \longrightarrow O^{(s)}(o)$

$\Lambda^{\pm}(f') \longrightarrow \Lambda^{\pm}(o) = \frac{1}{2}(1 \pm \gamma_4)$ and dropping the suffices on p'

$$\gamma \cdot p' \longrightarrow \gamma \cdot \bar{p}' \tag{6.10a}$$

$$\text{with} \quad \bar{p}_\nu' = p_\nu' \; \mathcal{L}_{\nu\mu}(f') \tag{6.10b}$$

Since $f' \cdot p' = 0$, $\bar{p}'$ is a 3 vector.

Now $R(f')$ commutes with $O^{(s)}(o)$ and

$$R(f_1')\gamma \cdot \bar{p}' \quad R^{\dagger}(f_1) = R(f_1')\, \gamma_i \, R^{\dagger}(f_1') \; \bar{p}_i'$$

$$= r_{ij}(f_1') \, \gamma_j \, \bar{p}_i' \tag{6.11}$$

$\mathcal{R}_{ij}(f_1')$ is the space rotation to which R (f_1') corresponds.

Defining the rotated vector

$$P_j' = \bar{p}_i' \; \mathcal{R}_{ij}(f_1') \tag{6.12a}$$

$$R(f')\, \gamma \cdot \bar{p}' \; R^{\dagger}(f_1) = \underline{\gamma} \cdot \underline{P}' \tag{6.12b}$$

Similar considerations apply to the R.H.S. of (6.9) and defining

$$\bar{p}_\mu = p_\nu \mathcal{L}_{\nu\mu}(f) \tag{6.13a}$$

$$P_j = \bar{p}_i \, \mathcal{R}_{ij}(f_1) \tag{6.13b}$$

$$\bar{n}_\mu = n_\nu \mathcal{L}_{\nu\mu}(t) = (\underline{N}, o) \tag{6.13c}$$

Equation (6.9) reduces to

$$\frac{1}{T'} \; I \; O^{(s)}(o) \sum_{P,r} C_r'^{\pm} \prod_{i=1}^{n}{}_{x} \left\{ \Lambda^{\pm}(o) \; (1 + i\gamma_5 \gamma_j \bar{p}_{r,k}'^{\pm(i)} \mathcal{R}_{kj}(f_1')\right\}$$

$$= O^{(s)}(o) \sum_{P,\ell} \prod_{i=1}^{n}{}_{x} \left\{ \Lambda^{\pm}(o) \; (F^{\pm(i)} + G^{\pm(i)} \; i\gamma_5 \underline{\gamma} \cdot \underline{N})\right\}$$

$$\frac{1}{T} O^{(s)}(o) \sum_{P,r} C_r^{\pm} \prod_{i=1}^{n}{}_{x} \left\{ \Lambda^{\pm}(o)(1 + i\gamma_5 \gamma_j \bar{p}_{r,k}^{\pm(i)} \; \mathcal{R}_{kj}(f_1)\,)\right\}$$

$$O^{s}(o) \sum_{P\,\ell'} \prod_{i=1}^{n}{}_{x} \left\{ \Lambda^{\pm}(o) \; (F^{x\pm(i)} + G^{x\pm(i)} \; i\gamma_5 \underline{\gamma} \cdot \underline{N} \right\} \tag{6.14}$$

It will be noticed that in (6.14) the ρ and ρ' matrix parts are not exactly the same as $\rho(\underline{f}=o)$, $\rho(\underline{f}'=o)$. A rotation $\mathcal{R}_{ij}$ is applied to the proper polarization indices.

From now on we restrict ourselves to the case of 'particles' only and so the superscript $\pm$ over P, P', G, F will be omitted. In the representation of γ matrices we are using

$$\Lambda^{+}(o) = \tfrac{1}{2}(1+\gamma_4) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.15a}$$

$$i\gamma_5\gamma.P_\gamma = \gamma_4\,\underline{\sigma}.\underline{P}^{(i)} = \begin{pmatrix} \underline{\sigma}.\underline{P}_\gamma & 0 \\ 0 & -\underline{\sigma}.\underline{P}^{(1)}_\gamma \end{pmatrix} \tag{6.15b}$$

$$\Lambda^{+}(o)\,(1+i\gamma_5\underline{\gamma}.P^{(i)}_\gamma) = \begin{pmatrix} 1+\underline{\sigma}.P^{(i)}_\gamma & 0 \\ 0 & 0 \end{pmatrix} \tag{6.15c}$$

$$\Lambda^{+}(o)\,(F_1^{(i)}+G_1^{(i)}\,i\gamma_5\underline{\gamma}.\underline{N}) = \begin{pmatrix} (F_1^{(i)}+G_1^{(i)}\,\underline{\sigma}.\underline{N}) & 0 \\ 0 & 0 \end{pmatrix} \tag{6.15d}$$

Keeping in mind the form of $O^{(s)}_{(o)}$ and equation 6.15 we see that the left and right hand sides of (6.14) are of the form

$$\begin{pmatrix} a^{(1)}_\gamma & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} a^{(2)}_\gamma & 0 \\ 0 & 0 \end{pmatrix} \cdots \times \begin{pmatrix} a^{(n)}_\gamma & 0 \\ 0 & 0 \end{pmatrix}$$

$$\equiv A^{(1)}_\gamma \times A^{(2)}_\gamma \;\cdot\;\cdot\;\cdot\; \times A^{(n)}_\gamma \equiv A \tag{6.16a}$$

Acting in the space of spinors of ~~zanh~~ rank n

$$U_{\alpha^{(1)}\,\alpha^{(2)}\ldots\alpha^{(n)}} \qquad \alpha^{(i)} = 1,2,3,4,$$

we have for the matrix element

$$V^{+}A\,U = \sum_{\alpha,\beta=1}^{4} V^{+}_{\alpha^{(1)}\alpha^{(2)}\ldots\alpha^{(n)}} \quad A_{\alpha^{(1)}\alpha^{(2)}\ldots\alpha^{(n)};\beta^{(1)}\beta^{(2)}\ldots\beta^{(n)}}$$

$$U_{\beta^{(1)}\beta^{(2)}\ldots\beta^{(n)}}$$

$$= \sum_{\gamma}\sum_{\alpha\beta}^{4} V^{+}_{\alpha^{(1)}\ldots\alpha^{(n)}} \quad A^{(1)}_{\gamma,\,\alpha^{(1)}\,\beta^{(1)}} \quad A^{(2)}_{\gamma,\,\alpha^{(2)}\beta^{(2)}} \cdots A^{(n)}_{\alpha^{(n)}\beta^{(n)}}$$

$$U_{\beta^{(1)}\beta^{(2)}\ldots\beta^{(n)}} \tag{6.16b}$$

Since from (6.16) $A_{\gamma,\,\alpha^{(m)}\beta^{(m)}} = 0$ if $\alpha^{(m)} > 2$ or for $\beta^{(m)} > 2$ and for $\alpha^{(m)} \leqslant 2$, $\beta^{(m)} \leqslant 2$, $A^{(i)}_{\gamma,\,\alpha^{(m)}\beta^{(m)}} = a^{(i)}_{\gamma\,\alpha^{(m)}\beta^{(m)}}$

Hence

$$V^{+}AU = \sum_{\gamma}\sum_{\alpha,\beta=1}^{2} V^{+}_{\alpha^{(1)}\ldots\alpha^{(n)}} \; a^{(1)}_{\gamma,\alpha,\,(1)\beta^{(1)}} \cdots \; a^{(n)}_{\gamma,\alpha^{(n)}\beta^{(n)}}$$

$$U_{\beta^{(1)}\ldots\ldots\beta^{(n)}}$$

$$= \quad v^{\dagger} \cdot a^{(1)} \times a^{(2)} \ldots\ldots \times a^{(n)}\, u\,.$$

v and u are Pauli spinors of rank n where as V and U were Dirac spinors of rank n. Equation (6.14) is thus equivalent to

$$\frac{I}{T'}\, O_p^{(s)}(o) \quad \sum_{P,\gamma} C'_\gamma \left\{ \prod_{i=1}^{n} x \left(1 + \underline{\sigma} \cdot P'^{(i)}_\gamma \right) \right\}$$

$$= \quad O_p^{(s)}(o) \quad \sum_{P,l} \left\{ \prod_{i=1}^{n} x \left(F_l^{(i)} + G_l^{(i)} \underline{\sigma}.\underline{N} \right) \right\}$$

$$\frac{1}{T}\, O_p^{(s)}(o) \quad \sum_{P,\gamma} \left\{ C_\gamma \prod_{j=1}^{n} x \left(1 + \underline{\sigma}.P^{(i)}_\gamma \right) \right\}$$

$$O_p^{(s)}(o) \quad \sum_{P,l'} \left\{ \prod_{k=1}^{n} x \left(F_{l'}^{x(k)} + G_{l'}^{x(h)} \underline{\sigma}.\underline{N} \right) \right\} \tag{6.17}$$

$O_p^{(s)}(o)$ is now given in terms of the Pauli matrices

$$O_p^{(s)}(o) = \prod_{\bar{s}_i \neq s(s+1)} \frac{(O_p(o) - \bar{s}_i)}{s\,(s+1)-\bar{s}_i} \tag{6.18a}$$

with

$$O_p(o) = \frac{1}{4}\left(3m + 2\left[\sigma_i\right]_{(2)}\right) \tag{6.18b}$$

The S and ρ matrices in 6.17 are given as n-fold Kronecker products of the corresponding quantities for spin $\frac{1}{2}$ particles. The projection operator $O_p^{(s)}(o)$ selects the spin s part. But this equation is given in a highly reducible form and one of the methods to obtain from it the usual $(2s+1)$ dimensional irreducible scattering equation is the following one.

One defines the spin operator

$$J_i = \frac{1}{2}\left(\sigma_i \times 1 \times 1 \ldots \times 1 + 1 \times \sigma_i \times 1 \ldots + 1 \times 1 \ldots \times 1 \times \sigma_i\right) \tag{6.19}$$

It is at once seen that

$$J_i J_i = J^2 = \frac{1}{4}\left(3n + 2\left[\sigma_i\right]_{(2)}^{(m)}\right) = O_p(o) \tag{6.20}$$

and so

$$O_p^{(s)}(o) = \prod_{\bar{s}_i \neq s\,(s+1)} \frac{(J^2 - \bar{s}_i)}{s(s+1)-\bar{s}_i} \qquad (6.21)$$

Equation (6.17) can be completely expressed in terms of J_i's . This is facilatated by the presence of the **permutation** symbol Σ_p. However further analysis is so complicated for the general case of arbitrary spin that it is better to carry out the calculations for s = 1 first.

For S = 1, n = 2

$$\rho_p^{(1)} = \frac{1}{T} O_p^{(1)}(o) \quad \sum_{P,r} c_r \prod_{=1}^{2} \times \left(1 + \sigma \cdot P_r^{(j)}\right)$$

$$= \frac{1}{T} O_p^{(1)}(o) \quad \sum_r c_r \Big(1 + \sigma \cdot \underline{P}_r^{(1)}\Big) \times \Big(1 + \sigma \cdot \underline{P}_r^{(2)}\Big) + \Big(1 + \sigma \cdot P_r^{(2)}\Big) \times \Big(1 + \sigma \cdot \underline{P}_r^{(1)}\Big)$$

$$= \frac{1}{T} O_p^{(1)}(o) \quad \sum_r \Big(2 \cdot 1 \times 1 + 1 \times \sigma \cdot P_r^{(2)} + 1 \times \sigma \cdot \underline{P}_r^{(1)} + \sigma \cdot P_r^{(1)} \times 1 + \sigma \cdot P_r^{(2)} \times 1 + \sigma \cdot \underline{P}_r^{(2)} \times \underline{P}_r^{(1)} + \sigma \cdot P_r^{(1)} \times \sigma \cdot P_r^{(2)}\Big) c_r$$

$$= \frac{1}{T} O_p^{(1)}(o) \quad \sum_r c_r \Big\{ 2 \; 1 \times 1 + (1 \times \sigma + \sigma \times 1) \cdot \big(\underline{P}_r^{(1)} + \underline{P}_r^{(2)}\big) + \big(\sigma \cdot P_r^{(1)} \times \sigma \cdot P_r^{(2)} + \sigma \cdot P_r^{(2)} \times \sigma \cdot P_r^{(1)}\big) \qquad (6.22)$$

The second term on the right hand side is just

$$2\,\underline{J}\cdot(\underline{P}_r^{(1)} + \underline{P}_r^{(2)}) = 2\,\underline{J}\cdot\sum_p \underline{P}_r^{(\alpha)} \tag{6.23}$$

The third term is calculated by considering

$$\sum_p \underline{J}\cdot\underline{P}_r^{(\alpha_1)}\,\underline{J}\cdot\underline{P}_r^{(\alpha_2)} = \underline{J}\cdot\underline{P}_r^{(1)}\,\underline{J}\cdot\underline{P}_r^{(2)} + \underline{J}\cdot\underline{P}_r^{(2)}\,\underline{J}\cdot\underline{P}_r^{(1)}$$

$$= \tfrac{1}{4}\,(\underline{\sigma}\cdot\underline{P}_r^{(1)} \times 1 + 1 \times \sigma\cdot\underline{P}_r^{(1)})(\underline{\sigma}\cdot\underline{P}_r^{(2)} \times 1 + 1 \times \underline{\sigma}\cdot\underline{P}_r^{(2)})$$

$$+ \quad 2 \leftrightarrow 1$$

$$= \tfrac{1}{4}\Big\{\,\underline{\sigma}\cdot\underline{P}_r^{(1)}\,\underline{\sigma}\cdot\underline{P}_r^{(2)} \times 1 + \underline{\sigma}\cdot\underline{P}_r^{(2)}\,\underline{\sigma}\cdot\underline{P}_r^{(1)} \times 1$$

$$1 \times \underline{\sigma}\cdot\underline{P}_r^{(1)}\,\underline{\sigma}\cdot\underline{P}_r^{(2)} + 1 \times \underline{\sigma}\cdot\underline{P}_r^{(2)}\,\underline{\sigma}\cdot\underline{P}_r^{(1)}$$

$$+ 2(\,\underline{\sigma}\cdot\underline{P}_r^{(1)} \times \underline{\sigma}\cdot\underline{P}_r^{(2)} + \underline{\sigma}\cdot\underline{P}_r^{(2)} \times \underline{\sigma}\cdot\underline{P}_r^{(1)}\,)\Big\} \tag{6.24}$$

The first two terms on the R.H.S. can be written

$$\sum_p \underline{\sigma}\cdot\underline{P}_r^{(\alpha_1)}\,\underline{\sigma}\cdot\underline{P}_r^{(\alpha_2)} \times 1 = (\,\underline{\sigma}\cdot\underline{P}_r^{(1)}\,\underline{\sigma}\cdot\underline{P}_r^{(2)} + \underline{\sigma}\cdot\underline{P}_r^{(2)}\,\underline{\sigma}\cdot\underline{P}_r^{(1)}\,) \times 1$$

$$= \underline{P}_r^{(1)}\cdot\underline{P}_r^{(2)} + \underline{P}_r^{(2)}\cdot\underline{P}_r^{(1)} + i\,\underline{\sigma}\cdot(\underline{P}_r^{(1)} \wedge \underline{P}_r^{(2)} + \underline{P}_r^{(2)} \wedge \underline{P}_r^{(1)}) \times 1$$

$$= 2\,\underline{P}_r^{(1)}\cdot\underline{P}_r^{(2)} + 0 \tag{6.25}$$

Thus from (6.24) one obtains

$$\underline{\sigma}\cdot\underline{P}_r^{(1)} \times \underline{\sigma}\cdot\underline{P}_r^{(2)} + \underline{\sigma}\cdot\underline{P}_r^{(2)} \times \underline{\sigma}\cdot\underline{P}_r^{(1)} = \sum_p \underline{\sigma}\cdot\underline{P}_r^{(\alpha_1)} \times \underline{\sigma}\cdot\underline{P}_r^{(\alpha_2)}$$

$$= 2\,(\,\sum_p \underline{J}\cdot\underline{P}^{(\alpha_1)}\,\underline{J}\cdot\underline{P}^{(\alpha_2)} - \underline{P}_r^{(1)}\cdot\underline{P}_r^{(2)}\,) \tag{6.26}$$

Hence $\rho_p^{(1)}$ can be written in terms of J_i's as follows

$$\rho_p^{(1)} = \frac{1}{T} O_p^{(1)}(o) \sum_r Cr \Big\{ 2 - 2\, \underline{P}_r^{(1)} \cdot \underline{P}_r^{(2)} + 2\, \underline{J}\cdot(\underline{P}_r^{(1)} + \underline{P}_r^{(2)})$$

$$+ 2\, J_i J_j \,(P_{r,i}^{(1)} P_{r,j}^{(2)} + P_{r,i}^{(2)} P_{r,j}^{(1)}) \Big\}$$

$$= \frac{1}{T} O_p^{(1)}(o) \sum_r C_r \Big\{ 2 - 2\, \underline{P}_r^{(1)} \cdot \underline{P}_r^{(2)} + 2\, \underline{J}\cdot(P_r^{(1)} + \underline{P}_r^{(2)})$$

$$+ (J_i J_j + J_j J_i)(P_{r,i}^{(1)} P_{r,j}^{(2)} + P_{r,i}^{(2)} P_{r,j}^{(1)}) \Big\} \qquad (6.27a)$$

It is convenient to introduce $N_e^{(i)}$ through

$$G_e^{(i)} \underline{N} = \dot{H}_e^{(i)} \underline{N}^{1} + K_e^{(i)} \underline{N}^{11} + F_e \underline{N}^{111}$$

$$= \underline{N}_e^{(i)} \qquad (6.27b)$$

The reduced S matrix is obtained in terms of $\underline{N}_e^{(i)}$ and J_i's in the same way as above

$$S_p = O_p^{(1)}(o) \sum_{p,e} \prod_{i=1}^{2} (F_e^{(i)} + \underline{J} \cdot \underline{N}_e^{(i)})$$

$$= O_p^{(1)}(o) \sum_e 2\, F_e^{(1)} F_e^{(2)} - 2\, \underline{N}_e^{(1)} \cdot \underline{N}_e^{(2)} + 2\, \underline{J}\cdot(F_e^{(2)} \underline{N}_e^{(1)}$$

$$+ F_e^{(1)} \underline{N}_e^{(2)}) + (J_i J_j + J_j J_i)(N_{e,i}^{(1)} N_{e,j}^{(2)} + N_{e,i}^{(2)} N_{e,j}^{(1)})$$

$$= O_p^{(1)}(o) \sum_e 2\, F_e^{(1)} F_e^{(2)} - 2\, \underline{N}_e^{(1)} \cdot \underline{N}_e^{(2)} + 2\, \underline{J}\cdot \sum_p F_e^{(\alpha_1)} \underline{N}_e^{(\alpha_2)}$$

$$+ (J_i J_j + J_j J_i) \sum_p N_{e,i}^{(\alpha_1)} N_{e,j}^{(\alpha_2)} \qquad (6.27c)$$

To express (6.17) for arbitrary s in terms of the J_i's consider

$$
\begin{aligned}
&\sum_p (1 + \sigma.P_r^{(1)}) \times (1 + \sigma.P_r^{(2)}) \times \ldots \quad (1 + \sigma.P_r^{(n)}) \\
&= \lfloor n + \sum_\alpha \Big\{ \sigma.P_r^{(\alpha)} \times 1 \times 1 \ldots \times 1 + 1 \times \sigma.P_r^{(\alpha)} \times 1 \ldots \times 1 + \\
&\qquad \ldots 1 \times 1 \times \ldots \quad 1 \times \sigma.P_r^{(\alpha)} \Big\} \\
&\quad + \sum_p \Big\{ \sigma.P_r^{(\alpha_1)} \times \sigma.P_r^{(\alpha_2)} \times 1 \times 1 \ldots \times 1 \\
&\qquad + \sigma.P_r^{(\alpha_1)} \times 1 \times \sigma.P_r^{(\alpha_3)} \times 1 \ldots \\
&\qquad + \ldots\ldots \Big\} \\
&\quad + \sum_p \Big\{ \sigma.P_r^{(\alpha_1)} \times \sigma.P^{(\alpha_2)} \times \sigma.P^{(\alpha_3)} \times 1 \times 1 \ldots \times 1 \\
&\qquad + \ldots\ldots \Big\} \\
&\quad - - - - - - - - - \\
&\quad + \sum_p \sigma.P_r^{(\alpha_1)} \times \sigma.P_r^{(\alpha_2)} \times \sigma.P_r^{(\alpha_3)} \times \ldots \sigma.P_r^{(\alpha_n)}
\end{aligned}
\tag{6.28}
$$

The second term within the brackets contains ${}^nC_1$ terms the third term contains ${}^nC_2$ and so on. $\sum_p$ stands for the sum of all possible permutations for example

$$\sum_p \sigma.P_r^{(\alpha_1)} \times \sigma.P_r^{(\alpha_2)} \times 1 \times \ldots \quad \times 1$$

means the sum over all the ${}^nP_2$ permutation of (α_1, α_2).

The second term is just $2\,\underline{J}.\sum_{\alpha}\underline{P}_r^{\alpha}$. To calculate the 3rd term we form

$$\sum_p \underline{J}.\underline{P}_r^{(\alpha_1)}\,\underline{J}.\underline{P}_r^{(\alpha_2)} = \tfrac{1}{4}\sum_p(\underline{\sigma}.\underline{P}_r^{(\alpha_1)} \times 1 \times \ldots \times 1 + 1 \times .\underline{P}_r^{(\alpha_1)}$$

$$\times 1 \times 1 \ldots \times 1 + \ldots\ )$$

$$(\underline{\sigma}.\underline{P}_r^{(\alpha_2)} \times 1 \times \ldots \times 1 + 1 \times .\underline{P}_r^{(\alpha_2)}$$

$$\times 1 \times \ldots \times 1 + \ldots\ )$$

$$= \tfrac{1}{4}\sum_p\Big\{\underline{\sigma}.\underline{P}_r^{(\alpha_1)}\,\underline{\sigma}.\underline{P}_r^{(\alpha_2)} \times 1 \times 1 \ldots \times 1 + 1 \times \underline{\sigma}.\underline{P}_r^{(\alpha_1)}\,.$$

$$\underline{\sigma}.P_r^{(\alpha_2)} \times 1 \ldots \times 1 + \ldots + 2(\underline{\sigma}.P^{(\alpha_1)} \times \underline{\sigma}.\underline{P}^{(\alpha_2)} \times 1 \times 1$$

$$\ldots \times 1 + \underline{\sigma}.\underline{P}^{(\alpha_1)} \times 1 \times \underline{\sigma}.\underline{P}^{(\alpha_2)} + \ldots\ )\Big\} \qquad (6.29a)$$

now

$$\sum_p \underline{\sigma}.\underline{P}_r^{(\alpha_1)}\,\underline{\sigma}.P_r^{(\alpha_2)} = \sum_p \underline{P}_r^{(\alpha_1)}.\underline{P}_r^{(\alpha_2)} + \sum_p i\,\underline{\sigma}.\underline{P}_r^{(\alpha_1)} \wedge \underline{P}_r^{(\alpha_2)}$$

$$= \sum_p \underline{P}_r^{(\alpha_1)}.\underline{P}_r^{(\alpha_2)} + \tfrac{1}{2}\sum_p \underline{\sigma}.(\underline{P}_r^{(\alpha_1)} \wedge \underline{P}_r^{(\alpha_2)} + \underline{P}_r^{(\alpha_2)} \wedge \underline{P}_r^{(\alpha_1)})$$

$$= \sum_p P_r^{(\alpha_1)}.P_r^{(\alpha_2)} + 0 \qquad (6.29b)$$

Therefore

$$\sum_p \underline{\sigma}.\underline{P}_r^{(\alpha_1)} \times \underline{\sigma}.\underline{P}_r^{(\alpha_2)} \times 1 \times 1 \ldots \times 1 + \underline{\sigma}.\underline{P}_r^{(\alpha_1)} \times 1 \times$$

$$\underline{\sigma}.\underline{P}_r^{(\ 2)} \times 1 \ldots \times 1 + \ldots .$$

$$= 2\sum_p \underline{J}\,.\,P_r^{(\alpha_1)}\,\underline{J}.\underline{P}^{(\alpha_2)} - \frac{n}{4}\sum_p \underline{P}_r^{(\alpha_1)}.\underline{P}_r^{(\alpha_2)} \qquad (6.30)$$

When we form $\sum_p \underline{J}.\underline{P}_r^{(\alpha_1)} \; \underline{J}.\underline{P}_r^{(\alpha_2)} \; \underline{J}.P_r^{(\alpha_3)}$ equation (6.29b) shows that terms like

$$\sum_p \underline{\sigma}.\underline{P}_r^{(\alpha_1)} \; \underline{\sigma}.\underline{P}^{(\alpha_2)} \times \underline{\sigma}.\underline{P}^{(\alpha_3)} \times 1 \times 1 \ldots \times 1$$

reduce to

$$\sum_p \underline{P}_r^{(\alpha_1)}.\underline{P}_r^{(\alpha_2)} \quad 1 \times \sigma.P^{(\alpha_3)} \times 1 \times 1 \ldots \times 1 \tag{6.31a}$$

and terms like

$$\sum_p \underline{\sigma}.\underline{P}_r^{(\alpha_1)} \; \underline{\sigma}.\underline{P}_r^{(\alpha_2)} \; \underline{\sigma}.\underline{P}^{(\alpha_3)} \times 1 \times 1 \times \ldots \times 1$$

reduce to

$$\sum_p \underline{P}_r^{(\alpha_1)}.\underline{P}^{(\alpha_2)} \quad \underline{\sigma}.\underline{P}^{(\alpha_3)} \times 1 \times 1 \times \ldots \tag{6.31b}$$

and therefore the third term in (6.28)

$$\sum_p \underline{\sigma}.\underline{P}_r^{(\alpha_1)} \times \underline{\sigma}.\underline{P}_r^{(\alpha_2)} \times \underline{\sigma}.P_r^{(\alpha_3)} \times 1 \times \ldots \times 1$$
$$+ \; \underline{\sigma}.\underline{P}^{(\alpha_1)} \times 1 \times \underline{\sigma}.P_r^{(\alpha_2)} \times \underline{\sigma}.\underline{P}_r^{(\alpha_3)}$$
$$+ \; \ldots\ldots\ldots \qquad {}^{n}C_3 \text{ terms}$$

can be expressed in terms of $\sum_p \underline{J}.P_r^{(\alpha_1)} \; \underline{J}.\underline{P}_r^{(\alpha_2)} \; \underline{J}.P^{(\alpha_3)}$ and $\sum_p \underline{J}.\underline{P}^{(\alpha_1)} \; \underline{P}^{(\alpha_2)}.\underline{P}^{(\alpha_3)}$.

Similar reductions occur for higher terms and we can write

$$\rho_{\mathrm{p}} = \frac{1}{T} O_p^{(s)}(o) \sum_{P,r} C_r \prod_{j=1}^{n} (1 + \sigma \cdot \underline{P}_r^{(j)})$$

$$= \frac{1}{T} O^{(s)}(o) \Big\{ a_o + \underline{J} \cdot \sum_{r,p} a_r(\alpha_1) \underline{P}^{(\alpha_1)}$$

$$+ J_i J_j \sum_{P,r} a_r(\alpha_1 \alpha_2) P_{r,i}^{(\alpha_1)} P_{r,j}^{(\alpha_2)}$$

$$+ J_i J_j J_h \sum_{P,r} a_r(\alpha_1 \alpha_2 \alpha_3) P_{r,i}^{(\alpha_1)} P_{r,j}^{(\alpha_2)} P_{r,h}^{(\alpha_3)}$$

$$\cdots\cdots\cdots\cdots$$

$$+ J_i J_j \ldots J_\ell \sum_{P,r} a_r(\alpha_1 \alpha_2 \ldots \alpha_n) P_{r,i}^{(\alpha_1)} P_{r,j}^{(\alpha_2)} \ldots P_{r,\ell}^{(\alpha_n)} \Big\} \qquad (6.32)$$

The coefficients $a_r(\alpha_1 \alpha_2 \ldots \alpha_n)$ are all completely symetric in $\alpha_1, \alpha_2 \ldots \alpha_n$ and are rotation invarient. This symetry of $a_r(\alpha_1 \alpha_2 \ldots \alpha_n)$ means that the coeficients of $J_i J_j \ldots J_m$ i.e.

$$\sum_{P,r} a_r(\alpha_1 \alpha_2 \ldots \alpha_m) P_{r,i}^{(\alpha_1)} P_{r,j}^{(\alpha_2)} \ldots P_h^{(\alpha_m)}$$

are completely symetric in i, j ... h.

In the same way the s matrix is

$$S_{\mathrm{P}} = O_p^s(o) \sum_{P,\ell} \prod_{i=1}^{n} (F_\ell^{(i)} + \underline{\sigma} \cdot \underline{N}_\ell^{(i)}) \qquad (6.33)$$

$$= O^s(o) \sum_{\ell} \Big\{ b_\ell + \underline{J} \cdot \sum_p b_\ell \; (\alpha_1) \underline{N}_\ell^{(\alpha_1)}$$

$$+ J_i J_j \sum_p b_\ell (\alpha_1 \alpha_2) N_{\ell,i}^{(\alpha_1)} N_{\ell,j}^{(\alpha_2)}$$

$$+ J_i J_j J_h \sum_p b_\ell (\alpha_1 \alpha_2 \alpha_3) N_{\ell,i}^{(\alpha_1)} N_{\ell,j}^{(\alpha_2)} N_{\ell,h}^{(\alpha_3)}$$

. . . .

$$+ J_i J_j \ldots J_h \sum_p b(\alpha_1 \alpha_2 \ldots \alpha_n) N_{\ell,i}^{(\alpha_1)} \ldots N_{\ell,h}^{(\alpha_n)}$$

The invariants $b(\alpha_1 \alpha_2 \ldots \alpha_r)$ are again symetric in $\alpha_1, \alpha_2, \ldots \alpha_r$.

Equation (6.17 can now be completely expressed in terms of J_i's.

J_i's defined by (6.19) satisfy the commutation ~~selections~~ relations

$$[J_i, J_j] = i\,\epsilon_{ijk} J_k \qquad i,j,k = 1,2,3, \qquad (6.34a)$$

and the characteristic equation

$$(J_i - s)(J_i - (s-1)) \ldots (J_i + s) = 0 \qquad (6.34b)$$

These relations can be seen to hold without going into details by noting that β_i satisfy these relations and β_i and J_i are given in the same way in terms of γ_i and σ_i respectively. γ_i and σ_i obey the same algebraic relations.

The matrices $\theta_i^{(s)}$ belonging to the spin s representation of the 3 dimensional rotation group also satisfy the relations (6.34a,b). The matrices for lower spins also satisfy (6.34a,b) but the latter is not the characteristic equation for them.

Consider now equation (6.17) expressed completely in terms of the J_i's. The representation (6.19) of the J_i's is the n fold Kronecker product of spin ½ representation of the rotation group. By Clebsch Gordon decomposition theorem there exists a similarity transformation which transforms the representation

$$D^{1/2} \times D^{1/2} \quad . \; . \quad \times D^{1/2}$$

into the direct form

$$D^{(s)} \oplus \alpha_{s-1} \; D^{(s-1)} + \; . \; . \; . \tag{6.35a}$$

Since J_i's and $\theta_i^{(s)}$'s obey the same algebraic relation this similarity transformation will reduce the J_i's into

$$\theta_i^{(s)} \oplus \alpha_{s-1} \, \theta_i^{(s-1)} \; . \; . \; . \; . \tag{6.35b}$$

As mentioned already the irreducible representation of $O^{(s)}(o)$ is just the (2s + 1) dimension unit matrix $\theta_o^{(s)}$ and therefore this projection operator is reduced by the similarity transformation to the form

$$\theta_o^{(s)} \oplus o \oplus o \; . \; . \; .$$

The effect of the transformed $O^{(s)}(o)$ is that $D^{(s)}$ part of the operators is multiplied by unity while $\alpha_{s-1} D^{(s-1)}$ + $\alpha_{s-2} D^{(s-1)}$ + . . . part just gets annihilated. Hence (6.17) reduces to

$$I \rho' = S \rho S^{+} \tag{6.36a}$$

in which

$$\rho = \frac{1}{T}\Big\{\sum_r a_r + \underline{\theta}^{(s)} \sum_r \sum_p a_r(\alpha_1)\, \underline{P}_r^{(\alpha_1)} +$$

$$\theta_i^{(s)}\, \theta_j^{(s)} \sum_r \sum_p a_r(\alpha_1\alpha_2) P_{r,i}^{(\alpha_1)}\, P_{r,i}^{(\alpha_2)} + \ldots\ldots\ldots +$$

$$\theta_i^{(s)}\, \theta_j^{(s)} \ldots \theta_h^{(s)} \sum_r \sum_p a_r(\alpha_1\alpha_2 \ldots \alpha_n)\, P_{r,i}^{(\alpha_1)}\, P_{r,j}^{(\alpha_2)}$$

$$\ldots P_{r,h}^{(\alpha_n)} \qquad (6.36b)$$

$$S = \sum_\ell b_\ell + \underline{\theta}^{(s)} \sum_\ell \sum_p b_\ell\, (\alpha_1)\, \underline{N}_\ell^{(\alpha_1)} + \theta_i^{(s)}\theta_j^{(s)} \sum_\ell \sum_p$$

$$b_\ell\, (\alpha_1\alpha_2)\, N_{\ell,i}^{(\alpha_1)}\, N_{\ell,j}^{(\alpha_2)} + \ldots\ldots$$

$$+\, \theta_i^{(s)}\, \theta_j^{(s)} \ldots \theta_h^{(s)} \sum_\ell \sum_p b_\ell(\alpha_1\alpha_2 \ldots \alpha_n)\, N_{\ell,i}^{(\alpha_1)}\, N_{\ell,j}^{(\alpha_2)}$$

$$\ldots N_{\ell,k}^{(\alpha_n)} \qquad (6.36c)$$

ρ' is obtained from ρ by replacing a_r's by a_r''s and P_r's by P''s.

If we sum up over r and ℓ we obtain the usual non relativistic cartician forms of the S and ρ matrices. This is done first for the s = 1 case. From (6.27a) we get for ρ

$$\rho = \frac{1}{T} \sum_r c_r \quad (2 - 2\, \underline{P}_r^{(1)} . \underline{P}_r^{(2)})\, \theta_o^{(1)} + 2\underline{\theta}^{(1)} . (\underline{P}_r^{(1)} + \underline{P}_r^{(2)})$$

$$+ (\theta_i^{(1)} \theta_j^{(1)} + \theta_j^{(1)} \theta_i^{(1)}) (P_{r,i}^{(1)} P_{r,j}^{(2)} + P_{r,i}^{(2)} P_{r,j}^{(1)}) \tag{6.37}$$

It is usual to express in terms of the traceless tensor operator $T_{ij}^{(1)}$ and the traceless polarization tensor $Q_{ij}^{(12)}$. This can be done by defining

$$P_{r,i}^{(1)} P_{r,j}^{(2)} + P_{r,i}^{(2)} P_{r,j}^{(1)} = P_{r,ij} \tag{6.38a}$$

$$Q_{r,ij} = P_{r,ij} - \frac{1}{3} \delta_{ij} P_{r,kk} = Q_{r,ji} \tag{6.38b}$$

$Q_{r,ij}$ is traceless, i.e. $Q_{r,ii} = o$, $T_{ij}^{(1)}$ is defined by

$$T_{ij}^{(1)} = \tfrac{1}{2} (\theta_i^{(1)} \theta_j^{(1)} + \theta_j^{(1)} \theta_i^{(1)}) - \frac{2}{3} \delta_{ij} . \theta_o^{(1)} \tag{6.38c}$$

$$T_{ii}^{(1)} = \tfrac{1}{2} (2\theta_i^{(1)} \theta_i^{(1)}) - \frac{2}{3} \delta_{ii}\, \theta_o^{(1)} \tag{6.38d}$$

$$= o$$

Since $\theta_i^{(1)} \theta_i^{(1)} = 2\, \theta_o^{(1)}$, by using the known forms of $\theta_i^{(1)}$ one can prove easily that $\mathrm{tr}\, T_{ij}^{(1)} = o$ for all i,j. Using these equations

$$(\theta_i^{(1)} \theta_j^{(1)} + \theta_j^{(1)} \theta_i^{(1)})\, P_{r,ij} = 2\, T_{ij}^{(1)} Q_{r,ij} + \frac{4.3}{9} P_{r,hh}$$

$$= 2\, T_{ij}^{(1)} Q_{r,ij} + \frac{4}{3}\, 2\, \underline{P}_r^{(1)} \underline{P}_r^{(2)} \tag{6.39}$$

and ρ becomes

$$\rho = \frac{1}{T} \sum_r C_r(6 + 2\underline{P}_r^{(1)}.P_r^{(2)}). \frac{\theta_o^{(1)}}{3} + \underline{\theta}^{(1)} . 2(\underline{P}_r^{(1)} + \underline{P}_r^{(2)})C_r$$

$$+ T_{i\,j}^{(1)} 2Q_{r,i\,j} C_r \tag{6.40}$$

Remembering the value of T for s = 1, equation (5.29a), the first term is just $\frac{1}{3}\,\theta_o^{(1)}$. This is not by chance and the reason for this is that T was originally defined as the trace of the expression on the right of $\frac{tr\cdot\rho(f)}{T}$ in (5.27). This is equivalent to saying that T is the trace of the expression on the right of $\frac{1}{T}$ in (6.40), the trace having remained the same throughout the similarity transformations. Now as $\underline{\theta}^{(1)}$ and $T_{i\,j}^{(1)}$ are traceless and trace of $\theta_o^{(1)}$ is 3, T must be equal to

$$C_r\ (6 + 2\ \underline{P}_r^{(1)}.\underline{P}_r^{(2)})$$

The summation over r can now be performed by defining

$$\sum_r C_r \frac{2(\underline{P}_r^{(1)} + \underline{P}_r^{(2)})}{T} = \frac{1}{3}\,\underline{P} \tag{6.41a}$$

$$\sum_r C_r \frac{2}{T} Q_{r,i\,j} = \frac{1}{3} Q_{i\,j} = \frac{1}{3} Q_{j\,i} \tag{6.41b}$$

Then ρ is simply given by

$$\rho = \frac{1}{3}\,(\theta_o^{(1)} + \underline{\theta}^{(1)}.\underline{P} + T_{i\,j}^{(1)}\ Q_{i\,j}) \tag{6.42a}$$

Similarly for ρ'

$$\rho' = \frac{1}{3}\,(\theta_o^{(1)} + \underline{\theta}^{(1)}.\underline{P}' + T_{i\,j}^{(1)}\ Q'_{i\,j}) \tag{6.42b}$$

These are exactly of the same form as given by Biedenharn.

For S we have from (6.27c)

$$S = \sum_{\ell} \left(2 F_{\ell}^{(1)} F_{\ell}^{(2)} - 2\, \underline{N}_{\ell}^{(1)} \underline{N}_{\ell}^{(1)} \right) \Theta_o^{(1)} + 2\, \underline{\Theta}^{(1)} . \left(F_{\ell}^{(2)} \underline{N}_{\ell}^{(1)} + F_{\ell}^{(1)} \underline{N}_{\ell}^{(2)} \right) + \left(\Theta_i^{(1)} \Theta_j^{(1)} + \Theta_j^{(1)} \Theta_i^{(1)} \right) \left(N_{\ell,i}^{(1)} N_{\ell,j}^{(2)} + N_{\ell,i}^{(2)} N_{\ell,j}^{(1)} \right) \tag{6.43}$$

$N^{(\alpha)}$ is given by

$$\underline{N}^{(\alpha)} = H^{(\alpha)} \underline{N}^{1} + K^{(\alpha)} \underline{N}^{11} + E^{(\alpha)} \underline{N}^{111} \qquad \alpha = 1,2 \tag{6.44a}$$

where

$$\underline{N}^{1} = \frac{\underline{k} \wedge \underline{k}'}{|\underline{k} \wedge \underline{k}'|} \tag{6.44b}$$

$$\underline{N}^{11} = \frac{\underline{k} \wedge \underline{N}'}{|\underline{k} \wedge \underline{N}'|} \tag{6.44c}$$

$$\underline{N}^{111} = \frac{\underline{k}' \wedge \underline{N}'}{|\underline{k}' \wedge N'|} \tag{6.44d}$$

$\underline{N}^{(\alpha)}$ can be expressed in terms of $\underline{k}$ and $\underline{k}'$ and N' as

$$\underline{N}_{\ell}^{(\alpha)} = H^{(\alpha)} \underline{N}^{1} + \underline{k} \left\{ K_{\ell}^{(\alpha)} \frac{\underline{k}.\underline{k}'}{|\underline{k}|} + E_{\ell}^{(\alpha)} \frac{\underline{k}'.\underline{k}'}{|\underline{k}'|} \right\} \frac{1}{|\underline{K} \wedge \underline{K}'|}$$

$$+ \underline{k}' \left\{ - K^{(\alpha)} \frac{\underline{k}.\underline{K}}{|k|} - E^{(\alpha)} \frac{\underline{k}.\underline{K}'}{|k'|} \right\} \frac{1}{|\underline{K} \wedge \underline{K}'|} \tag{6.45a}$$

$$= H^{(\alpha)} \underline{N}^{1} + X^{(\alpha)} \underline{k} + X'^{(\alpha)} \underline{k}' \tag{6.45b}$$

Using the definition of $T^{(1)}_{i\,j}$, S can also be written in terms of $T^{(1)}_{i\,j}$ and $\theta^{(1)}_i$.

$$S = a_o \theta_o^{(1)} + \underline{\theta}^{(1)}.(a_1\underline{N}^1 + a_2\underline{k} + a_3\underline{K}')$$

$$+ 2T^{(1)}_{i\,j}\Big\{ a_4\, N_i' N_j' + a_5(N_i' k_j + N_j' k_i)$$

$$+ a_6(N_i k_j' + N_j k_i') + a_7\, k_i K_j + a_8 K_i' K_j' + a_9(K_i K_j + M_i' K_j')$$

(6.46)

The invariants a_o, a_1, a_2 . . . a_8 are given in terms of the original invariants by

$$a_o = \sum_{\ell} 2\, F_{\ell}^{(1)}\, F_{\ell}^{(2)} + \frac{2}{3}\, \underline{N}_{\ell}^{(1)}.N_{\ell}^{(2)}$$

$$a_1 = \sum_{\ell} \sum_{p} F_{\ell}^{(2)}\, H_{\ell}^{(1)} = \sum F_{\ell}^{(2)}\, H_{\ell}^{(1)} + H_{\ell}^{(2)}\, F_{\ell}^{(1)}$$

$$a_3 = \sum_{\ell} \sum_{p} F_{\ell}^{(2)}\, X_{\ell}^{(1)} \qquad \text{etc.} \qquad (6.47)$$

In the centre of mass frame (6.46) is the most general rotation invariant S matrix that can be formed with the halp of the $\theta^{(1)}_i$ matrices and the initial and final momenta ½$\underline{K}$ and ½$\underline{K}'$ of the particles.

Under time reversal (13)

$$\theta_i^{(1)} \longrightarrow -\,\theta_i^{(1)} \qquad (6.48a)$$

$$\left.\begin{aligned} k &\longrightarrow -\,k' \\ k' &\longrightarrow -\,k \end{aligned}\right\}\quad \underline{N}' \longrightarrow -\,\underline{N}' \qquad \begin{aligned}(6.48b)\\(6.48c)\end{aligned}$$

Hence if time reversal invariance holds

$$a_2 = a_3 \tag{6.49a}$$

$$a_5 = a_6 \tag{6.49b}$$

$$a_7 = a_8 \tag{6.49c}$$

Under space reflection[13]

$$\theta_i^{(1)} \longrightarrow \theta_i^{(1)} \tag{6.50a}$$

$$\left.\begin{array}{l} \underline{k} \longrightarrow -\underline{k} \\ k' \longrightarrow -k' \end{array}\right\} \quad \underline{N}' \longrightarrow \underline{N}' \qquad (6.50b),\ (6.50c)$$

Hence if only parity conserving terms are present

$$a_2 = a_3 = 0 \tag{6.51a}$$

$$a_5 = a_6 = 0 \tag{6.51b}$$

The form 6.46 with 6.49 and 6.51 has been used by Stapp.[14]

In writing the S matrix for spin ½ particles, i.e. $S(k',t,k)$ we had included the parity non conserving term also. When this form is used for deriving the S matrices of higher spins such terms give rise to parity conserving as well as the non conserving terms. Also if only parity conserving term had been kept in $S\ (k',t,k)$ we would have missed a large number of terms in S matrix for higher spins.

Coming back now to the general case, the first thing we can do is to replace in (6.36b) and (6.36c)

$$\theta_i^{(s)}\ \theta_j^{(s)}\ \ldots\ \theta_h^{(s)} \qquad \text{m factors}$$

by $\frac{1}{|\underline{m}}\ \sum_p \theta_i^{(s)}\ \theta_j^{(s)}\ \ldots\ \theta_h^{(s)}$

This is possible on account of the complete symetry of $a(\alpha_1\alpha_2\ \ldots\ \alpha_m)$ and $b(\alpha_1\alpha_2\ \ldots\alpha_m)$ in $\alpha_1\alpha_2\ \ldots\ \alpha_m$. For the P matrix we define traceless tensors

$$\begin{aligned} Q_{r,ijk\ \ldots\ell} = \sum_r \sum_p \frac{a(\alpha_1\alpha_2\ \cdots\alpha_m)}{T}\ P_{r,i}^{(\alpha_1)}\ P_{r,j}^{(\alpha_2)}\ P_{r,k}^{(\alpha_3)} \ldots\ P_\ell^{(\alpha_m)} \\ - \frac{1}{\frac{1}{2}(m^2 - m + 4)}\Big(\underline{P}_r^{(\alpha_1)}\cdot\underline{P}_r^{(\alpha_2)}\,\delta_{ij}\ P_{r,k}^{(\alpha_3)}\ P_{r,t}^{(\alpha_4)} \ldots\ P_\ell^{(\alpha_m)} \\ + \underline{P}_r^{(\alpha_1)}\cdot\underline{P}_r^{(\alpha_3)}\,\delta_{ih}\ P_{r,j}^{(\alpha_2)}\ P_{r,t}^{(\alpha_4)} \ldots\ P_\ell^{(\alpha_m)} \\ + \underline{P}_r^{(\alpha_2)}\cdot\underline{P}_r^{(\alpha_3)}\,\delta_{jk}\ P_{r,i}^{(\alpha_1)}\ P_{r,t}^{(\alpha_3)} \ldots\ P_\ell^{(\alpha_m)} \\ \ldots\ldots\ldots\ldots\ldots\Big) \end{aligned} \qquad (6.52)$$

There are ${}^mC_2$ terms within the ( ) brackets on the R.H.S. as all combinations of tensor indices are taken in the form of Kroneckor $\hat{\delta}$'s. It can easily be proved that if any two of the tensor indices of $Q_{i\,j\,..}$ are contracted the result is zero. This can easily be proved; suppose i and k are contracted, i.e. we calculate $Q_{i\,j\,i\ \ldots\ell}$.

$$Q_{ijit\ldots} = \sum_{r}\sum_{p} \frac{a(\alpha_1\alpha_2 \ldots \alpha_m)}{T} \underline{P}_r^{(\alpha_1)} \cdot \underline{P}_r^{(\alpha_3)} P_{r,j}^{(\alpha_2)} P_{r,t}^{(\alpha_4)} \ldots P_{\ell}^{(\alpha_m)}$$

$$- \frac{1}{\frac{1}{2}(m^2 - m + 4)} \Big(\underline{P}_r^{(\alpha_1)} \cdot \underline{P}_r^{(\alpha_2)} P_{r,j}^{(\alpha_3)} P_{r,t}^{(\alpha_4)} \ldots\ldots P_{r,\ell}^{(\alpha_m)}$$

$$+ 3\underline{P}_r^{(\alpha_1)} \cdot \underline{P}_r^{(\alpha_3)} P_{r,j}^{(\alpha_2)} P_{r,t}^{(\alpha_4)} \ldots\ldots P_{r,\ell}^{(\alpha_m)}$$

$$+ \underline{P}_r^{(\alpha_2)} \cdot \underline{P}_r^{(\alpha_3)} P_{r,j}^{(\alpha_1)} P_{r,t}^{(\alpha_4)} \ldots\ldots P_{r,\ell}^{(\alpha_m)}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big) \qquad (6.53)$$

The presence of the permutation symbol makes it possible to write all terms within the ( ) brackets in the form

$$\underline{P}_r^{(\alpha_1)} \cdot \underline{P}_r^{(\alpha_3)} P_{r,j}^{(\alpha_2)} \ldots P_{\ell}^{(\alpha_m)}$$

The number of such terms within these brackets is

$${}^mC_2 - 1 + 3 = \tfrac{1}{2}(m^2 - m + 4)$$

Hence the R.H.S. of (6.53) vanishes.

From the tensor operator $\frac{1}{2}(\Theta_i^{(s)}\Theta_j^{(s)} + \Theta_j^{(s)}\Theta_i^{(s)})$ we define

$$T_{ij}^{(s)} = \tfrac{1}{2}(\Theta_i^{(s)}\Theta_j^{(s)} + \Theta_j^{(s)}\Theta_i^{(s)}) - \frac{s(s+1)}{3}\,\delta_{ij}\,\Theta_o^{(s)} = T_{ji}^{(s)} \qquad (6.54)$$

$T_{ii}^{(s)}$ vanishes since

$$\Theta_i^{(s)}\,\Theta_i^{(s)} = s(s+1)\,\Theta_o \qquad (6.55a)$$

For the tensor operator of 3rd rank $T^{(s)}_{ijh}$ is defined by

$$T^{(s)}_{ijh} = \frac{1}{\underline{|3}} \sum_{p} \Theta^{(s)}_i \Theta^{(s)}_j \Theta^{(s)}_k - \frac{1}{\underline{|3}} \frac{6\, s(s+1)}{5} \left\{ \delta_{ij} \quad \Theta^{(s)}_k \right.$$

$$\left. + \delta_{ik} \Theta^{(s)}_i + \delta_{ki}\, \Theta_j \right\} \tag{6.56}$$

$T^{(s)}_{ijk}$ is symmetric in $i\,j\,k$ and $T_{i\,i\,k}$ vanishes. This can easily be proved by using (6.55) and

$$\Theta^{(s)}_i \Theta^{(s)}_k - \Theta^{(s)}_h \Theta^{(s)}_i = i\, \delta_{i\,k\,\ell}\ \Theta^{(s)}_\ell \tag{6.55b}$$

It is very difficult to write down the forms of symetric tensor operation of higher orders (up to order $n = 2s$), but in principle it can be done. Summing up over r we define

$$(2s+1) \sum_{r} Q_{r,\,i\,j\,..\,\ell} = Q_{i\,j\,...\,\ell} \tag{6.56}$$

The P matrix then takes the form

$$\rho = \frac{1}{2s+1} \left\{ \Theta^{(s)}_o + \Theta^{(s)}_i Q_i + T^{(s)}_{ij}\, Q_{ij} \right.$$

$$\left. + T^{(s)}_{ijk}\, Q_{ijh} \cdots + T^{(s)}_{ij..\ell}\, Q_{ij\,..\ell} \right. \tag{6.57}$$

The reason for the coefficient of the unit matrix Θ_o being unity has already been given in connection with the spin 1 case.

If $\Theta^{(s)}_i \Theta^{(s)}_j \ldots \Theta^{(s)}_h$ in the s-matrix are replaced by $T^{(s)}_{ij...h}$ then (6.36c) is perhaps the most concise form of writing the complete rotation invarient S matrix. However if the expression (6.55b) for $N^{(\alpha)}_{\ell,i}$ is substituted and a summation is performed over 'ℓ', s can also be written as

$$S = a_o + \Theta_i^{(s)} \left\{ N', \mathbf{K}, k' \right\}_{i} + \ldots\ldots \; T^{(s)}_{ij\ldots k} \left\{ N', k, k' \right\}_{ij\ldots \mathbf{K}}$$

(6.58a)

In this equation $\left\{ N', \mathbf{K}, \mathbf{K}' \right\}_{ij\ldots}$ is a tensor of rank $m \leq 2\ell$ defined as a linear sum of all tensors formed out of the vectors $\underline{N}'$, $\underline{\mathbf{K}}$, $\underline{k}'$. As an example

$$T^{(3)}_{ij\mathbf{K}} \left\{ N', \mathbf{K}, \mathbf{K}' \right\}_{ijK} = T^{(3)}_{ijk} \Big[a_1^3 \, N'_i N'_j N'_{\mathbf{K}} + a_2^3 \, N'_i N'_j k_{\mathbf{K}}$$

$$+ a_3^3 \, N'_i N'_j k'_{\mathbf{K}} + a_4^3 \, N'_i k_j \mathbf{K}_{\mathbf{K}} + a_5^3 \, N'_i K'_j \mathbf{K}'_{\mathbf{K}} + a_6^3 \, N'_i k_j \mathbf{K}_{\mathbf{K}}$$

$$+ a_7^3 \, K_i h_j k_{\mathbf{K}} + a_8^3 \, k'_i k'_j \mathbf{K}'_{\mathbf{K}} + a_9^3 \, K_i k_j k_j + a_{10}^3 \, k_i \mathbf{K}'_j k'_k \Big] \tag{6.58b}$$

The restrictions of time reversal and space reflection invariances on the s matrix can be put in the same way as for the case of spin one particles.

The non covariant forms of the S and ρ matrices obtained here are given in the Cartesian tensor forms. Many authors(15) have used the tensor moment forms of these matrices and have obtained them from the considerations of rotation invariance. For spin s a tensor moment $T^{(s)}{}^J_M$ is a $(2s + 1)$ dimensional matrix which transforms under rotation as the spherical harmonic Y^J_M and has the property(13)

$$\mathrm{tr}\; {}^{(s)}T^J_M = \delta_{Jo} (2s + 1) \tag{6.59a}$$

$$\mathrm{tr}\; {}^{(s)}T^J_M \, {}^{(s)}T^{J'}_{M'} = \delta_{JJ'} \delta_{MM'} (2s + 1) \tag{6.59b}$$

${}^{(s)}T^J_M$ can be written in terms of $\Theta_i^{(5)}$.

If the equation

$$I \rho' = s \boldsymbol{\rho} s^{x} \tag{6.60}$$

with $\boldsymbol{\rho}$, and s given by (6.57) and (6.58) is used to analyse multiple scattering experiments there are certain corrections to be made. The situation is very much similar to the spin ½ case discussed by Stapp[1] and we closely follow his arguments.

In equation 6.14 instead of the proper vectors $\bar{\underline{p}}_r^{(i)}$ and $\bar{\underline{p}}_r^{(i)}$ the rotated vectors $P_{r,j}'^{(i)} = \bar{p}_{r,k}'^{(i)} \boldsymbol{\tau}_{Kj}(f_1')$ and $P_{r,j}^{(i)} = P_{r,h}^{(i)} \boldsymbol{\tau}_{hj}(f_1)$ occur. If these rotations were not present the $\boldsymbol{\rho}$ matrix would have been

$$\bar{\boldsymbol{\rho}} = \frac{1}{2s+1}\Big\{ \theta_o^{(s)} + \theta_i^{(2)}\,\bar{q}_i + T_{ij}^{(s)}\,\bar{q}_{ij} + \dots + T^{s}_{ij..h...}\,\bar{q}_{ij..h...} + \dots + T^{(s)}_{ij..\ell}\,\bar{q}_{ij..\ell} \tag{6.61}$$

and $\bar{\boldsymbol{\rho}}'$ of the same form with the dashed quantities $\bar{q}_{ij...}$ replacing $q_{ij...}$ in (6.61). The tensors $\bar{q}_{ij..k}$ are given in terms of $\bar{\underline{p}}_r^{(i)}$ in exactly the same way as $Q_{ig..h}$ are given in terms of $\underline{P}_r^{(i)}$. 6.52 and 6.56 show that Q and q are related to each other by

$$Q_{j_1 j_2 \dots j_\ell} = \bar{q}_{j_1' j_2' \dots j_\ell'}\, \boldsymbol{\tau}_{j_1' j_1}(f_1)\, \boldsymbol{\tau}_{j_2' j_2}(f_1) \dots \boldsymbol{\tau}_{j_\ell' j_\ell}(f_1) \tag{6.62}$$

The highest rank of each tensor is 2s, $o \leq \ell \leq 2s$. The last equation can be written concisely in the form

$$Q(\ell) = \bar{q}(\ell)\, \mathcal{R}^{(\ell)}(f_1) \tag{6.63a}$$

$$\bar{q}(\ell) = Q(\ell)\, \mathcal{R}^{(\ell)-1}(f_1) \tag{6.63b}$$

The final quantities are related by

$$Q'(\ell) = \bar{q}'(\ell)\, \mathcal{R}^{(\ell)}(f_1') \tag{6.64a}$$

$$\bar{q}'(\ell) = Q'(\ell)\, \mathcal{R}^{(\ell)-1}(f_1') \tag{6.64b}$$

Since $\bar{p}^{(i)}_{r,h}$ and $\bar{p}'^{(i)}_{r,h}$ are the values of $p^{(i)}_{r,\mu}$ and $p'^{(i)}_{r,\mu}$ in the rest frames $\underline{f} = o$ and $\underline{f}' = o$ respectively it is the quantities $\bar{q}_{ij\ldots h}$ rather than $P_{ij\ldots h}$ which are the same in the outgoing beam of one scattering and the incoming beam of the next scattering. Let the superscript (n) denote the quantities referring to the n^{th} scattering and the subscript n on the 4-momenta denote their centre of mass values.

Then

$$\bar{q}'^{(n-1)}_{(\ell)} = \bar{q}^{(n)}_{(\ell)} \tag{6.65}$$

or from (6.63, 6.64)

$$Q'^{(n-1)}(\ell)\, \mathcal{R}^{(\ell)^{-1}}(f'_{n-1}) = Q^{(n)}(\ell)\, \mathcal{R}^{(\ell)^{-1}}(f_n) \tag{6.66a}$$

Thus

$$Q^{(n)}(\ell) = Q'^{(n-1)}(\ell)\, \mathcal{R}^{(\ell)^{-1}}(f'_{n-1})\, \mathcal{R}^{(\ell)}(f_n) \tag{6.66b}$$

The rotations which convert the out going tensors $Q(\ell)$ of the (n-1)th scattering into the incoming tensors of the n^{th} scattering introduce certain differences between the relativistic and the nonrelativistic treatments (in which there are no rotations involved $Q^{(n)}(\ell) = Q^{(n-1)}_{(\ell)}$). In analogy with Stapps work these will be called the rotational corrections. The only difference between the case of spin ½ particles (discussed by Stapp) and the case of arbitrary spins is that these rotational corrections are to be applied to each index of the polarization tensors $Q_{ij\ldots h}$.

There is yet another correction which has to be applied to the nonrelativistic treatment of multiple scattering experiments. This arises out of the use of the relativistic transformation of momenta between the successive Lorentz frames rather than the Galileen transformations. Since the incoming momentum of the n^{th} scattering is the same as the outgoing momentum of the preceeding scattering

$$f^{(n)} = f'^{(n-1)} \tag{6.67}$$

However

$$(f_n)_\mu = f^{(n)}_\mu \mathcal{L}_{\mu\nu}(t^n) \tag{6.68a}$$

$$(f_{n-1})_\nu = f_\mu^{(n-1)} \mathcal{L}_{\mu\nu}(t^{(n-1)}) \tag{6.68b}$$

Thus the relation between the incoming momentum for the n^{th} scattering and the outgoing momentum for the preceeding scattering as measured in their respective centre-of-mass frames is

$$(f_n)_\mu = (f'_{n-1})_\lambda \; \mathcal{L}^{-1}_{\lambda\nu} (f^{(n-1)}) \mathcal{L}_{\nu\mu} (f^{(n)}) \tag{6.69}$$

The major portion of the transformations appearing here will except for the extreme relativistic cases, be given by the Galilean transformations. The remainder has been called the kinematical corrections by Stapp[1].

It is convenient to choose the laboratory frame as the basic reference frame and to assume that the target particles for all the scatterings are at rest in it. From (6.6) and (6.11) the rotation $\mathcal{R}_{\mu\nu}(f_n)$ is given by

$$\mathcal{R}_{\mu\nu}(f_n) = \mathcal{L}^{-1}_{\mu\eta} (f^{(n)}) \mathcal{L}_{\eta\lambda} (f^{(n)}) \mathcal{L}_{\lambda\nu} (f_n) \tag{6.70}$$
$$= \delta_{\mu\nu}$$

Since the Lorentz transformations appearing in this are colinear and their product will be unity. This gives

$$q^{(n)}(\ell) = Q^{(n)}(\ell) \tag{6.71}$$

For the scattered particle $\bar{q}'(\ell)$ and $Q'(\ell)$ will be different. Equation (6.66b) in view of (6.70) becomes

$$Q^{(n)}(\ell) = Q'^{(n-1)}(\ell) \, \mathcal{R}^{(\ell)^{-1}} (f'_{n-1}) \tag{6.72}$$

$\mathcal{R}^{(\ell)}(f'_{n-1})$ is given in terms of $\mathcal{R}(f_{n-1})$ by the Kronecker product

$$\mathcal{R}^{(\ell)}(f'_{n-1}) = \mathcal{R}(f'_{n-1}) \times \mathcal{R}(f'_{n-1}) \times \ldots \times \mathcal{R}(f'_{n-1}) \tag{6.73a}$$

$$\mathcal{R}^{(\ell)^{-1}}(f'_{n-1}) = \mathcal{R}^{-1}(f'_{n-1}) \times \mathcal{R}^{-1}(f'_{n-1}) \ldots \times \mathcal{R}^{-1}(f'_{n-1}) \tag{6.73b}$$

There are two modifications to be made in the non-relativistic theory. The first is that the relation between the momenta in the succession centre of mass frames is given by (6.69). The second is a rotation $R^{-1}(f'_{n-1})$ applied to each tensor index in the outgoing beam before it is interpreted as the incident polarization tensor of the next scattering.

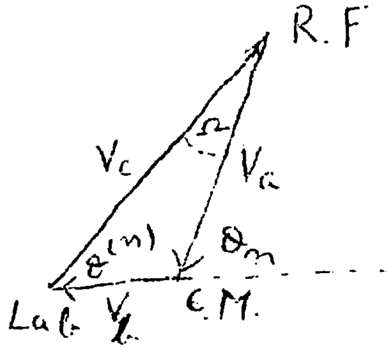
The rotation

$$R^{-1}_{\mu\nu}(f'_1) = \mathcal{L}^{-1}_{\mu\lambda}(f')\,\mathcal{L}_{\lambda\tau}(t)\,\mathcal{L}_{\tau\nu}(f'_1) \qquad (6.74)$$

is the result of three successive transformations. $\mathcal{L}(f_1)$ takes a vector from its value in the rest frame of the scattered particle to the centre of mass frame. $\mathcal{L}(t)$ then takes it from the centre of mass frame to the laboratory frame and finally $\mathcal{L}^{-1}(f')$ takes it from the laboratory frame back to a new rest frame of the scattered particle. The magnitude of this rotation specified by air axial vector $\underline{\Omega}$ has been given by Stapp

$$\operatorname{Sin}|\underline{\Omega}| = \underline{V}_a \wedge \underline{V}_b \; \frac{1 + \gamma^{(a)} + \gamma^{(b)} + \gamma^{(c)}}{(1 + \gamma^{(a)})(1 + \gamma^{(b)})(1 + \gamma^{(c)})} \qquad (6.75)$$

where $\gamma^{(a)}$, $\gamma^{(b)}$ and $\gamma^{(c)}$ are the three Lorentz contraction factors associated with the three transformation listed above and $\underline{V}_a$, $\underline{V}_b$ and $\underline{V}_c$ are the space parts of the three relative relativistic velocities respectively



$\mathcal{R}^{-1}(f)$ is the same rotation as given by G.C. Wick(16) in connection with the Lorentz transformation properties of helicity states.

Chapter VII

The quanitzation of the theory is not directly connected with this work but we shall briefly discuss this question for the following reasons.

(1) We can show that the particle density (for even n values) in the quantised version of the free field theory is positive definite.

(2) We can obtain the commutation reactions for free fields in a closed form whereas the corresponding result of Umezawa and Visconti (17) involved recurrence relations.

It has been mentioned in reference (17) that the Hamiltonian is not positive definite. This means that the quantization involved the introduction of an indefinite matric operator and consequently a subsidiary condition. In this section we discuss the quantization of the free field only and this is sufficient to obtain the results mentioned in the previous paragraph.

Equation 2.59 shows that the momentum space expansion of $\varphi(x)$ satisfying the free field equation (2.1) contains only the "particle" and "antiparticle" spinors and none from the rest of the spinor space. Enclosing the free field $\varphi(x)$ in a box of volume V with periodic boundary conditions $\varphi(x)$ and $\varphi^{\dagger}(x) = \varphi^{\times}(x') \eta_4$ are expanded in the form

$$\varphi_{\alpha}(x) = \sum_{\lambda} \sum_{\underline{f}} \sum_{j=1}^{n_{c_\lambda}} \sum_{\sigma} \sqrt{\frac{m\,\alpha_{\lambda}}{s_{\lambda}\, f_0^{(\lambda)} V}} \Big\{ a^{\sigma}_{\lambda,j\,\underline{f}}\; U^{\sigma}_{\lambda,j}(f^{(\lambda)})_{\alpha}\, e^{i f^{(\lambda)} x}$$

$$+ b^{\times\sigma}_{\lambda,j,\underline{f}}\; V^{\sigma}_{\lambda,j}(f^{(\lambda)})_{\alpha}\, e^{-i f^{(\lambda)} x} \Big\} \qquad (7.1a)$$

$$\varphi^{\dagger}_{\beta}(x') = \sum_{\lambda'} \sum_{\underline{f}'} \sum_{j'=1}^{n_{c_{\lambda'}}} \sum_{\sigma'} \sqrt{\frac{m\,\alpha_{\lambda'}}{s_{\lambda'}\, f_0^{(\lambda')} V}} \Big\{ a^{\times\sigma'}_{\lambda' j' \underline{f}'}\; U^{\dagger\sigma'}_{\lambda',j'}(f'^{(\lambda')})_{\beta}\, e^{-i f'^{(\lambda')} x'}$$

$$+ b^{\sigma'}_{\lambda',j'\underline{f}'}\; V^{\dagger\sigma'}_{\lambda' j'}(f'^{(\lambda')})_{\beta}\, e^{i f'^{(\lambda')} x'} \qquad (7.1b)$$

In these expressions for the first (2s + 1) values of σ, $U^{\sigma}_{\lambda,j}(f^{(\lambda)})$ and $V^{\sigma}_{\lambda j}(f^{(\lambda)})$ belong to $C^{(s)}(f^{(\lambda)})$ subspace and the rest to the subspaces with lower spin values.

The energy momentum vector P_{μ} derived from the free Lagrangian density $-\varphi^{\dagger}(x)\,(\beta_{\mu} \frac{\partial}{\partial x_{\mu}} + ms)\varphi(x)$ is given by

$$P_{\mu} = \frac{1}{i} \int_V d^3x\, \varphi^{\dagger}(x) \beta_4 \frac{\partial}{\partial x_{\mu}} \varphi(x) \qquad (7.2)$$

Substituting the expansions (7.1) and using orthonormality relations (3.55) we obtain for P_μ

$$P_\mu = \sum_{\lambda, j, \sigma, \underline{f}} f_\mu^{(\lambda)} \left\{ (-1)^\lambda \, a^{\times\sigma}_{\lambda, j \underline{f}} \, a^{\sigma}_{\lambda j \underline{f}} - (-1)^{2s+\lambda-1} \, b_{\lambda j \underline{f}} \, b^{\times}_{\lambda j \underline{f}} \right\} \tag{7.3}$$

Let us for the moment neglect the second term on the R.H.S. of the above equation. The Hamiltonian $H = P_o$ is then

$$H = \sum_{\lambda, j, \sigma, \underline{f}} f_o^{(\lambda)} \, (-1)^\lambda \, a^{\times\sigma}_{\lambda, j, \underline{f}} \, a^{\sigma}_{\lambda, j, \underline{f}} \tag{7.4}$$

Owing to the presence of $(-1)^\lambda$ the Hamiltonian is not positive definite. Such a Hamiltonian represents an assembly of Pais-Uhlenbeck oscillators(18). Sudarshan(18) has shown that the quantisation of a Pais-Uhlenbeck oscillator involves the introduction of an indefinite metric operator and his method can be applied to the present case without any difficulty. First the commutation (or anti-commutation) relations are written.

$$\left[a^{\sigma}_{\lambda, j \underline{f}} \, a^{\times\sigma'}_{\lambda' j' \underline{f}'} \right]_{\mp} = \delta_{\sigma\sigma'} \, \delta_{\lambda\lambda'} \, \delta_{jj'} \, \delta_{\underline{f}, \underline{f}'} \, (-1)^\lambda \tag{7.5a}$$

$$\left[b^{\sigma}_{\lambda j \underline{f}}, \, b^{\times\sigma'}_{\lambda' j' \underline{f}'} \right]_{\mp} = \delta_{\sigma\sigma'} \, \delta_{\lambda\lambda'} \, \delta_{jj'} \, \delta_{\underline{f}, \underline{f}'} \, (-1)^\lambda \tag{7.5b}$$

$$[a, b]_{\mp} = [a, b^{\times}]_{\mp} = [a^{\times}, a^{\times}]_{\mp} = [b^{\times}, b^{\times}]_{\mp} = 0 \tag{7.5c}$$

where in accordance with Pauli's principle the upper sign (commutation) is taken for bosons (n = 2s even) and the lower sign (anticommutation) is taken for fermions (2s = n odd).

Using these relations to bring b on the left in (7.3) and neglecting the zero point energy the Hamiltonian becomes

$$H = \sum_{\lambda j \sigma \underline{f}} f_0^{(\lambda)} (-1)^{\lambda} \left\{ a^{\sigma \times}_{\lambda j \underline{f}} a^{\sigma}_{\lambda j \underline{f}} + b^{\sigma \times}_{\lambda j \underline{f}} b^{\sigma}_{\lambda j \underline{f}} \right\} \tag{7.6}$$

Now we introduce the indefinite metric operator

$$\xi = e^{i\pi \sum_{\lambda j \underline{f} \sigma} \lambda (-1)^{\lambda} a^{\times\sigma}_{\lambda j \underline{f}} a^{\sigma}_{\lambda j \underline{f}} + \lambda (-1)^{\lambda} b^{\sigma\times}_{\lambda j \underline{f}} b^{\sigma}_{\lambda j \underline{f}}} \tag{7.7}$$

has the following properties which can easily be proved

$$\xi^2 = 1 \tag{7.7a}$$

$$\xi^{\times} = \xi \tag{7.7b}$$

$$a^{\sigma}_{\lambda j f} \xi = (-1)^{\lambda} \xi \, a_{\lambda j \underline{f}} \tag{7.7c}$$

$$b^{\sigma}_{\lambda j f} \xi = (-1)^{\lambda} \xi \, b^{\sigma}_{\lambda j f} \tag{7.7d}$$

This means that ξ commutes with a_{λ}, $a^{\times}_{\lambda}$, b_{λ}, and $b^{\times}_{\lambda}$ for even λ and anticommutes for odd values of λ. Following Sudarshan( ) we now define the new adjoint operators $a^{\dagger}$ and $b^{\dagger}$ by

$$a^{\dagger\sigma}_{\lambda, j \underline{f}} = \xi \, a^{\times\sigma}_{\lambda j \underline{f}} \, \xi = (-1)^{\lambda} a^{\sigma\times}_{\lambda j f} \tag{7.8a}$$

$$b^{\dagger\sigma}_{\lambda, j \underline{f}} = \xi \, b^{\sigma\times}_{\lambda j \underline{f}} \, \xi = (-1)^{\lambda} b^{\times}_{\lambda j \underline{f}} \tag{7.8b}$$

In terms of $a^{\dagger\sigma}_{\lambda j \underline{f}}$ and $b^{\dagger\sigma}_{\lambda j \underline{f}}$ the Hamiltonian is

$$H = \sum_{\lambda \sigma j \underline{f}} f_0^{(\lambda)} (a^{\dagger\sigma}_{\lambda j \underline{f}} a^{\sigma}_{\lambda j \underline{f}} + b^{\dagger\sigma}_{\lambda j \underline{f}} b^{\sigma}_{\lambda j \underline{f}}) \tag{7.9}$$

with the commutation relations

$$\left[a^{\sigma}_{\lambda j \underline{f}} , a^{\dagger \sigma'}_{\lambda' j' \underline{f}'} \right]_{\mp} = \delta_{\lambda\lambda'} \delta_{\sigma\sigma'} \delta_{jj'} \delta_{\underline{f}\underline{f}'} \qquad (7.10a)$$

$$\left[b^{\sigma}_{\lambda j f} , b^{\dagger \sigma'}_{\lambda' j' \underline{f}'} \right]_{\mp} = \delta_{\lambda\lambda'} \delta_{\sigma\sigma'} \delta_{jj'} \delta_{\underline{f}\underline{f}'} \qquad (7.10b)$$

All other commutators or anticommutators vanish. $a^{\dagger\sigma}_{\lambda j \underline{f}}$ and $b^{\dagger\sigma}_{\lambda j \underline{f}}$ can now be interpreted as the creation operators for particles and antiparticles respectively and $a^{\sigma}_{\lambda j \underline{f}}$, $b^{\sigma}_{\lambda j \underline{f}}$ as the destruction operators for the corresponding quantities. Particle states can be constructed in the usual way by applying $a^{\dagger}$ and $b^{\dagger}$ operators on the vacuum state defined by

$$a^{\sigma}_{\lambda j f} \,|\, o \rangle = b^{\sigma}_{\lambda j f} \,|\, o \rangle = o \qquad (7.11)$$

An N' particle state is

$$|\, N' \rangle = a^{\dagger\sigma_1}_{\lambda_1 j_1 \underline{f}_1} a^{\dagger\sigma_2}_{\lambda_2 j_2 \underline{f}_2} \cdots a^{\dagger\sigma_{N'}}_{\lambda_{N'} j_{N'} \underline{f}_{N'}} \,|\, o \rangle \qquad (7.12)$$

In view of the equations (7.7) and (7.8) the adjoint of the state $|\, N' \rangle$ is given by

$$\langle N' \,| = \langle o \,|\, a^{\sigma_N}_{\lambda_{N'} j_{N'} \underline{f}_{N'}} \cdots\cdots a^{\sigma_2}_{\lambda_2 j_2 \underline{f}_2} a^{\sigma_1}_{\lambda_1 j_1 \underline{f}_1} \, \xi \qquad (7.13a)$$

$$= (-1)^{\lambda_1 + \lambda_2 + \cdots + \lambda_{N'}} \langle o \,|\, a^{\sigma_{N'}}_{N' j_{N'} \underline{f}_{N'}} \cdots a^{\sigma_2}_{\lambda_2 j_2 \underline{f}_2} a^{\sigma_1}_{\lambda_1 j_1 \underline{f}_1}$$

(7.13b)

This shows that the norms of the states are not positive definite. If the state $|N'\rangle$ is properly normalised its norm is given by

$$\langle N' | N' \rangle = (-1)^{\lambda_1+\lambda_2+\cdots\lambda_{N'}} \tag{7.14a}$$

Similarly the expectation value of H in the state N is

$$\langle N' | H | N' \rangle = (-1)^{\lambda_1+\lambda_2+\cdots\lambda_{N'}} \left(f_{1,o}^{\lambda_1} + f_{2,o}^{\lambda_2} \cdots f_{N',o}^{\lambda_{N'}}\right) \tag{7.14b}$$

Thus all the states in which there are an odd number of particles with odd values of λ have negative norms. In the case of electrodynamics the supplementary condition restricted the physical states to have only positive norms and we can try the same trick in the present case also. But before we pick up a supplementary condition and try to obtain a consistent theory, it is better to derive the commutation relations for the field operators.

In the limit

$$V \longrightarrow \infty$$

$$\sum_{\underline{f}} \longrightarrow \frac{V}{(2\pi)^{\frac{3}{2}}} \int d^3f \tag{7.15a}$$

$$V\,\delta_{\underline{f},\underline{f}'} \longrightarrow (2\pi)^3\,\delta(\underline{f} - \underline{f}') \tag{7.15b}$$

and defining

$$a^{\sigma}_{\lambda j}(\underline{f}) = \sqrt{V}\,\frac{1}{(2\pi)^{\frac{3}{2}}}\, a^{\sigma}_{\lambda j \underline{f}} \tag{7.15c}$$

etc., the commutation relation can be written

$$\left[a^{\sigma}_{\lambda j}(\underline{f}),\ a^{\dagger\sigma'}_{\lambda' j'}(\underline{f}') \right]_{\mp} = \delta_{\lambda\lambda'}\, \delta_{jj'}\, \delta(\underline{f} - \underline{f}')\, \delta_{\sigma\sigma'} \qquad (7.16a)$$

$$\left[b^{\sigma}_{\lambda j}(\underline{f}),\ b^{\dagger\sigma'}_{\lambda' j'}(\underline{f}') \right]_{\mp} = \delta_{\lambda\lambda'}\, \delta_{jj'}\, \delta(\underline{f} - \underline{f}')\, \delta_{\sigma\sigma'} \qquad (7.16b)$$

with all other commutators or anticommutators vanishing.

In terms of $a^{\dagger\sigma}_{\lambda j}(f)$ and $b^{\dagger\sigma}_{\lambda j}(\underline{f})$ the expansions of the field are given by

$$\varphi_{\alpha}(x) = \sum_{\lambda j \sigma} \frac{1}{(2\pi)^{\frac{3}{2}}} \int \sqrt{\frac{m\,\alpha_\lambda}{s_\lambda f_0^{(\lambda)}}}\, d^3\underline{f} \left\{ a^{\sigma}_{\lambda j}(\underline{f})\, U^{\sigma}_{\lambda j}(f^{(\lambda)})_{\alpha}\, e^{i f^{(\lambda)}\cdot x} \right.$$

$$\left. + (-1)^{\lambda}\, b^{\dagger\sigma}_{\lambda j}(\underline{f})\, V^{\sigma}_{\lambda j}(f^{(\lambda)})_{\alpha}\, e^{-i f^{(\lambda)}\cdot x} \right\} \qquad (7.17a)$$

$$\varphi^{\dagger}_{\beta}(x') = \sum_{\lambda' j' \sigma'} \frac{1}{(2\pi)^{\frac{3}{2}}} \int \sqrt{\frac{m\,\alpha}{s_\lambda f_0^{(\lambda)}}}\, d^3\underline{f} \left\{ (-1)^{\lambda'}\, a^{\dagger\sigma'}_{\lambda' j'}(f') \right. \cdot$$

$$\left. U^{\dagger\sigma'}_{\lambda' j'}(f'^{(\lambda')})_{\beta}\, e^{-f'^{(\lambda')}x'} + b^{\sigma'}_{\lambda' j'}(\underline{f}')\, V_{\lambda' j'}(f'^{(\lambda')})_{\beta}\, e^{i f^{(\lambda')}\cdot x'} \right\}$$

(7.17b)

Using the commutation relation (7.16) we get

$$\left[\varphi_{\alpha}(x),\ \varphi_{\beta}(x') \right]_{\mp} = \sum_{\lambda j \sigma} \frac{(-1)^{\lambda}}{(2\pi)^3} \int \frac{m\,\alpha_\lambda}{s_\lambda f_0^{(\lambda)}}\, d^3 f$$

$$\left\{ U^{\sigma}_{\lambda j}(f^{(\lambda)})_{\alpha}\, U^{\dagger}_{\lambda j}(f^{(\lambda)})_{\beta}\, e^{i f^{(\lambda)}(x - x')} \right.$$

$$\left. \mp\ V^{\sigma}_{\lambda j}(f^{(\lambda)})\, V^{\dagger\sigma}_{\lambda j}(f^{(\lambda)})\, e^{-i f^{(\lambda)}(x - x')} \right\} \qquad (7.18)$$

The two terms on the righthand side above can be expressed in terms of the particle and antiparticle projection operators $\eta^{\pm}_{\alpha\beta}(f^{(\lambda)}) = \eta^{\pm s_\lambda}_{\alpha\beta}(f^{(\lambda)})$ by using 3.43.

$$\sum_{j\sigma} U^{\sigma}_{\lambda j}(f^{(\lambda)})_{\alpha} \; U^{\dagger\sigma}_{\lambda j}(f^{(\lambda)})_{\beta} = \sum_{j\sigma} \eta^{+s_\lambda}_{\alpha\alpha'}(f^{(\lambda)}) U^{\sigma}_{\lambda j}(f^{(\lambda)}) \; U^{\dagger\sigma}_{\lambda j}(f^{(\lambda)})(-1)^{\lambda+\lambda}$$

$$= \sum_{r=0}^{n} \sum_{j\sigma} (-1)^{\lambda} \; \eta^{+s_\lambda}_{\alpha\alpha'}(f^{(\lambda)}) \; U^{\sigma}_{r,j}(f^{(\lambda)})_{\alpha'} \; U^{\dagger\sigma}_{r,j}(f^{(\lambda)})_{\beta} \; (-1)^{r}$$

$$= (-1)^{\lambda} \; \eta^{+s_\lambda}_{\alpha\alpha'}(f^{(\lambda)}) \; \delta_{\alpha'\beta}$$

$$= (-1)^{\lambda} \; \eta^{+s_\lambda}_{\alpha\beta}(f^{(\lambda)}) \qquad (7.19)$$

Remembering that

$$V^{\sigma}_{\lambda,j}(f^{(\lambda)}) = U^{\sigma}_{n-\lambda\,,j}(f^{(\lambda)})$$

we obtain in the same way as before

$$\mp \sum_{j\sigma} V^{\sigma}_{\lambda,j}(f^{(\lambda)})_{\alpha} \; V^{\dagger\sigma}_{\lambda j}(f^{(\lambda)})_{\beta} = \mp (-1)^{n-\lambda} \eta^{-s_\lambda}_{\alpha\beta}(f^{(\lambda)}) \qquad (7.20)$$

For bosons the upper sign is taken but $n = 2's$ is even and for the fermions the lower sign is taken but n is odd in this case. Hence

$$\left[\varphi_{\alpha}(x), \varphi_{\beta}(x') \right]_{\mp} = \frac{1}{(2\pi)^3} \frac{m\,\alpha_\lambda}{s_\lambda} \int \frac{d^3f}{f_0^{(\lambda)}}$$

$$\left\{ \eta^{s_\lambda}_{\alpha\beta}(f^{(\lambda)}) \, e^{if^{(\lambda)}(x-x')} - \eta^{-s_\lambda}_{\alpha\beta}(f^{(\lambda)}) \, e^{-if^{(\lambda)}(x-x')} \right\}$$

$$(7.21)$$

Since from the definitions of $\eta^{\pm s_\lambda}(f^{(\lambda)})$

$$\eta^{-s_\lambda}(f^{(\lambda)}) = \eta^{+s_\lambda}(-f^{(\lambda)}) \tag{7.22}$$

alternative forms of (7.21) are

$$\left[\varphi_\alpha(x), \varphi_\beta(x')\right]_\mp = \sum_\lambda \frac{m\alpha_\lambda}{s_\lambda} 2 \int d^4f\, \eta^{+s_\lambda}_{\alpha\beta}(f)\,\epsilon(f_o)\,\delta(f^2 + m^2\alpha_\lambda^2)\, e^{if(x-x')} \tag{7.23b}$$

$$= \sum_\lambda \frac{2m\alpha_\lambda}{s_\lambda} \eta^{+s_\lambda}\left(-i\frac{\partial}{\partial x}\right) \frac{1}{(2\pi)^3} \int d^4f\, \epsilon(f_o)\,\delta(f^2 + m^2\alpha_\lambda^2) e^{if(x-x')} \tag{7.23b}$$

The commutation relations given by Umezawa and Visconti (17) are of the following form in our notation

$$\left[\varphi_\alpha(x), \varphi_\beta(x')\right]_\mp = i\, D_{\alpha\beta}(\partial)\, \Delta(x-x') \tag{7.24a}$$

$\Delta(x-x')$ is given by (19)

$$\Delta(x-x') = \frac{-i}{(2\pi)^3} \sum_\lambda \frac{1}{\prod_{\lambda'\neq\lambda} m^2(\alpha_{\lambda'}^2 - \alpha_\lambda^2)} \int d^4f\, e^{if(x-x')}\epsilon(f_o)\,\delta(f^2 + m^2\alpha_\lambda^2) \tag{7.24b}$$

$D(\partial)$ is a covariant operator formed from β_μ and $\frac{\partial}{\partial x_\mu} \equiv \partial_\mu$ with the property

$$(\beta\cdot\partial + ms)_{\alpha\gamma}\, D(\partial)_{\gamma\beta} = \prod_\lambda (\Box - m^2\alpha_\lambda^2)\quad \delta_{\alpha\beta} \tag{7.25}$$

$D(\partial)$ can be expressed in the form (3)

$$D(\partial) = \alpha_o + \alpha_{\mu_1}\partial_{\mu_1} + \alpha_{\mu_1\mu_2}\partial_{\mu_1}\partial_{\mu_2} + \dots + \alpha_{\mu_1\mu_2\cdots\mu_n}\,\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_n} \tag{7.26}$$

where $\alpha_{\mu_1 \ldots \mu_r}$ with $r \leq n = 2s$ are tensors formed from β_μ's. $D(\partial)$ can be determined by substituting (7.26) in (7.25) and comparing coefficients. In this way recurrence relations are obtained for $\alpha_{\mu_1 \ldots \mu_r}$. It is very difficult to prove that the commutation relations (7.23) obtained here are the same as those given by Umezawa and Viscosity for arbitrary's, but in the particular case of $s = \frac{3}{2}$, $D(\partial)$ is easily calculated

$$D(\partial) = 6m^3 - \frac{20}{3} m\partial^2 + \beta\cdot\partial\left(\frac{40}{9}\partial^2 - 4m^2\right) + \frac{8}{3} m(\beta\cdot\partial)^2$$
$$- \frac{16}{9} (\beta\cdot\partial)^3 \qquad (7.27)$$

and there are only two mass states $\alpha_0 m = m$, $\alpha_1 m = 3m$. In this case we can verify by using the definition of $\eta^{s_\lambda}(f^{(\lambda)})$ that the result (7.23) is the same as (7.24) given by Umezawa and Visconti. The form (7.24) has the advantage that the propagator $G_{\alpha\beta}(x - x') = \langle 0 | T\, \varphi_\alpha(x) \;\; \varphi_\beta(x') | 0 \rangle$ can be obtained simply by replacing

$$\int d^4f\, \epsilon(f_0)\, \delta(f^2 + m^2\alpha_\lambda^2)\, e^{if(x-x')}$$

in (7.24b) by

$$\frac{1}{2\pi i} \int \frac{d^4f\, e^{if(x-x')}}{f^2 + m^2\alpha_\lambda^2 - i\epsilon}$$

$G_{\alpha\beta}(x - x')$ can easily be shown to satisfy[19]

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + ms\right) G_{\alpha\beta}(x - x') = -\delta^4(x - x') \qquad (7.28)$$

On the other hand the expression (7.23) is given in a closed form.

Coming back to the question of selecting a supplementary condition which restricts the physical states to have positive norms we note that the single particle states with masses $m\alpha_\lambda$, $\lambda = 1, 3, 5 \ldots$ have negative norms since

$$\langle 0 | \; a^{\sigma}_{\lambda i \underline{f}} \; \zeta \; a^{\dagger}_{\lambda i \underline{f}} \; | 0 \rangle = (-1)^{\lambda} \tag{7.29}$$

Thus we define the physical states to be those which have no particles with odd λ values. Let $|P\rangle$ denote a physical state then it must satisfy

$$\left.\begin{array}{ll} a^{\sigma}_{\lambda j \underline{f}} \, |P\rangle = 0 & (7.30a) \\ b^{\sigma}_{\lambda j \underline{f}} \, |P\rangle = 0 & (7.30b) \end{array}\right\} \text{ for } \lambda = 1, 3, 5 \ldots \text{ all } \sigma, j, \text{ and } \underline{f}$$

We can go a step further and demand that the physical states are those which have only particles with the lowest mass value m. Such state will satisfy

$$\left.\begin{array}{ll} a^{\sigma}_{\lambda j \underline{f}} \, |P\rangle = 0 & (7.31a) \\ b_{\lambda j \underline{f}} \, |P\rangle = 0 & (7.31b) \end{array}\right\} \text{ for } \lambda > 0 \text{ and for all } \sigma, j, \underline{f}$$

This last condition is equivalent to

$$(\Box - m^2)\varphi_j^{(+)}(x) \, |P\rangle = 0 \tag{7.32a}$$

$$(\Box - m^2)\varphi^{\dagger}_j{}^{(+)}(x) \, |P\rangle = 0 \tag{7.32b}$$

$\varphi^{(+)}(x)$) and $\varphi^{\dagger(+)}(x)$ denote the positive frequency parts of $\varphi(x)$ and $\varphi^{\dagger}(x)$ respectively. Equations (7.32) may be taken to be the supplementary conditions for the free fields. As in electrodynamics the advantage of putting a condition on the states rather than on the field operators is that the commutation relation, the propagator and other Greens functions remain unaltered. The physical states satisfying (7.32) have positive norms while the unphysical states have positive as well as negative norms. The expectation values of physical quantities such as the Hamiltonian in physical states are also positive definite. The difficulty connected with particle densities mentioned earlier in 4 disappears in the quantised theory. The electromagnetic current density is given by

$$j_\mu(x) = \frac{ie}{2}\left[\varphi_\alpha^\dagger(x),\ \varphi_\beta(x)\right]_\pm \ \beta_{\mu,\ \alpha\beta} \qquad (7.33)$$

The symmetrised expression (upper sign) is taken for bosons and the antisymmetrised expression (lower sign) is taken for fermions. The vacuum expectation values of $j_\mu(x)$ vanishes. For using the expressions (7.1) and the commutation relations (7.10) we get

$$\langle 0 | \ j_\mu(x) \ | 0 \rangle = \frac{ie}{2} \sum_\lambda \sum_{\underline{f}} \frac{m\ \alpha_\lambda}{s_\lambda\ f_o^{(\lambda)} V}$$

$$\left\{ \pm U_\lambda^\dagger (f^{(\lambda)}) \beta_\mu U_\lambda (f^{(\lambda)}) + V_\lambda^\dagger (f^{(\lambda)}) \beta_\lambda V_\lambda (f^{(\lambda)}) \right. \qquad (7.34)$$

The indices σ and j have been absorbed in $\underline{f}$. From (3.55) one at once derives the relations

$$\overset{\dagger}{U}{}^{\sigma}_{\lambda,j}(f^{(\lambda)})\,\beta_\mu\, U^{\sigma}_{\lambda,j}(f^{(\lambda)}) = (-1)^\lambda \frac{(-i)f_\mu^{(\lambda)} s_\lambda}{m\,\alpha_\lambda} \tag{7.35a}$$

$$\Big\{\overset{\dagger}{V}{}^{\sigma}_{\lambda j}(f^{(\lambda)})\,\beta_\mu\, V^{\sigma}_{\lambda j}(f^{(\lambda)}) = (-1)^{2s-1-\lambda} \frac{(-i)f_\mu^{(\lambda)} s_\lambda}{m\,\alpha_\lambda} \tag{7.35b}$$

These equations show that the quantity within the brackets in (7.34)

$$\frac{-i\, s_\lambda f_\mu^{(\lambda)}}{m\,\alpha_\lambda}\left\{ \pm (-1)^\lambda + (-1)^{2s-1-\lambda}\right\} = 0, \tag{7.36}$$

since the upper sign is taken for 2s even and the lower sign for 2s odd.

Let us now calculate the expectation values of $j_\mu(x)$ for a free "particle" state $a^\dagger_{\lambda\underline{f}}\,|0\rangle$ and for a free "antiparticle" state $b^\dagger_{\lambda\underline{f}}\,|0\rangle$. The result of a straightforward calculation is

$$\langle 0|\; a_{\lambda\underline{f}} \,\{\, j_\mu(x)\; a^\dagger_{\lambda\underline{f}}\; |0\rangle = (-1)^\lambda \frac{e}{V} \frac{f_\mu^{(\lambda)}}{f_0^{(\lambda)}} \tag{7.37a}$$

$$\langle 0|\; b_{\lambda\underline{f}} \,\{\, j_\mu(x)\; b^\dagger_{\lambda\underline{f}}\; |0\rangle = \pm (-1)^{2s-1-\lambda} \frac{e}{V} \frac{f_\mu^{(\lambda)}}{f_0^{(\lambda)}}$$

$$= -\frac{f_\mu^{(\lambda)}}{f_0^{(\lambda)}} \frac{e}{V} (-1)^\lambda \tag{7.37b}$$

Therefore for fermions as well as for bosons the expectation value of $j_\mu(x)$ for a free particle state with even λ is

$$\frac{e}{V}\frac{f_{\mu}^{(\lambda)}}{f_{0}^{(\lambda)}} = \left\{ \frac{e}{V}\underline{v} , \frac{e}{V} \right\} \tag{3.38a}$$

where $\underline{v}$ is the velocity of the particle. For a state with free antiparticle of mass m_{λ}, λ even, the expectation value of j_{μ} in both of the cases is

$$-\frac{e}{V}\frac{f^{(\lambda)}}{f_{0}^{(\lambda)}} = \left(-\frac{e}{V}\underline{v} , -\frac{e}{V} \right) \tag{e.38b}$$

Thus the particle current density for the states $a^{\dagger}_{\lambda,\underline{f}}|0\rangle$ and $b^{\dagger}_{\lambda,\underline{f}}|0\rangle$ for even λ values is $\frac{1}{V}\underline{v}$ and the particle density is $\frac{1}{V}$ both for bosons and fermions. Taking $\lambda = 0$, covariant density matrices can easily be derived in the quantised theory both for particles and antiparticles of integral or half integral spins.

So far we have considered the free field case. For the comparitively simple cases of $s = \frac{3}{2}$ and $s = 2$ there are only two mass states and one might try to develop a quantised theory of interacting fields for particles with these spins but there are several questions which have to be answered satisfactorily. One of these is how should the supplementary condition be modified under the presence of interactions. Another is the question of the unitarity of the s matrix. For whenever we insert a complete set of states not only all the physical but all the unphysical states should be included. The third question we mention is whether such a theory with the supplementary condition (7.32) is a realistic one. The investigation of these problems is outside the scope of this work.

Several authors have employed the Duffin-Kemmer formalism in discussing the quantum electrodynamics of spinless and spin one particles.[20,21] Sometimes it might be useful to have the rules for contracting the Duffin-Kemmer particles in a matrix element according to the method of Lehmann, Symanzik and Zimmerman. These rules can be easily found by using the orthonormality relations (3.42), (3.55) and the expansion (7.1) of $\varphi(x)$ taking $s = 1$, $\lambda = 1$, $j = 1$ and proceeding in the same way as one does in the case of Dirac particles.[22] The 'in' and 'out' destruction operators for particles and antiparticles are given in the sense of weak convergence by

$$a^{i}_{\underline{f}}\binom{in}{out} = \underset{x_0 \to \mp\infty}{\mathrm{Lim}} \sqrt{\frac{m}{f_0 V}} \int_V e^{-if\cdot x}\, u^{\dagger i}(f)\, \beta_4\, \varphi(x)\, d^3x$$

$$b^{i}_{\underline{f}}\binom{in}{out} = -\underset{x_0 \to \mp\infty}{\mathrm{Lim}} \sqrt{\frac{m}{f_0 V}} \int_V e^{-if\cdot x}\, \varphi^{\dagger}(x)\, \beta_4\, V^{i}(f)\, d^3x$$

The in and out fields obey the commutation relations 7.18, 7.21 with $s = 1$, $\lambda = 1$, $j = 1$ and it can easily be shown that the destruction and creation operators obey the commutation relations

$$[a^{i}_{\underline{f}}, a^{j*}_{\underline{f}'}]_- = \delta_{ij}\, \delta_{\underline{f},\underline{f}'}$$

$$[b^{i}_{\underline{f}}, b^{j*}_{\underline{f}'}]_- = \delta_{ij}\, \delta_{\underline{f},\underline{f}'}$$

All other commutators vanish. The indices i and j denote the three helicity states of the spin one particles. As an example we write down the transition matrix element for Compton Scattering of vector bosons

$$\langle f'j; q'e' | S-1 | fi; qe \rangle = (-i)^4 \left(\frac{m\ m}{Vf_0\ Vf'_0\ 2q_0V\, 2q'_0V} \right)^{\frac{1}{2}}$$

$$\int dx\, dy\, dr\, dz\, e^{i(f\cdot x + q\cdot r - f'\cdot y - q'\cdot z)}$$

$$e'_{\nu}\, u^{\dagger j}_{\alpha}(f')\, (-\Box_z) \left(\beta\cdot\frac{\partial}{\partial x} + m\right)_{\alpha\alpha'} \langle 0 | T\, \varphi_{\alpha'}(x)\, \varphi^{\dagger}_{\beta'}(y) A_{\nu}(z)$$

$$j_{\mu}(r) | 0 \rangle \left(\beta\cdot\frac{\partial}{\partial y} - m\right)_{\beta'\beta} u^{i}_{\beta}(f)\, e_{\mu}$$

f, i and f', j are the momenta and the helicities of the initial and final vector bosons, q, e and q' e' are the corresponding quantities for the initial and final photons respectively. In these formulae the 10 dimensional representation of the β matrices may be used. Peaslee[20] has given the traces of these matrices. The contraction formulae are very similar to those for Dirac particles and many results will be similar in form in the two cases.

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