SYNTHESIS OF COMMUNICATION NETS

by

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ABSTRACT

It is established that some necessary conditions for the realizability of a matrix T as the terminal matrix of an oriented communication net are equivalent. A general method for the synthesis of a terminal matrix of an oriented communication net is introduced.

Necessary and sufficient conditions are given for the synthesis of a matrix T as the terminal matrix of an oriented communication net having special topological structure, namely: double tree net, double loop net or separable net. A necessary and sufficient condition for the synthesis of a triangular terminal matrix is also obtained.

Some properties of minimum total edge capacity realizations of symmetric terminal matrices are found. Using these properties, the problem of synthesizing a given symmetric terminal matrix with minimum total edge capacity and minimum number of edges, having non-zero capacities, is solved.

Two lemmas, which are useful in reducing the work in the evaluation of the terminal matrix of a non-oriented communication net which has weights on both edges and nodes, are given. Such a net is termed a radio-wire

communication net. A necessary and sufficient condition for the realizability of a symmetric matrix T as the terminal matrix of a radio-wire communication net is presented.

Finally, necessary conditions are obtained for the synthesis of a symmetric matrix as the terminal matrix of a radio-wire communication net containing a given set of edges and nodes which have unlimited capacities.

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CHAPTER I

An Algorithm For Synthesizing An Oriented Communication Net

1.1 Introduction

An oriented communication net, N, is an oriented graph with weights on edges. The weight of an edge is a real non-negative number called the "edge capacity"; that is, the capacity of transferring information in the direction indicated by the orientation of the edge. Without loss of generality we shall assume that there is at most one edge leading from any node to another in N. Also, we shall assume, until Chapter IV, that no capacity constraints on nodes are admitted. Let the net N contain n nodes labelled by 1,2,...,n. An edge from node i to node j will be denoted by e_{ij} and its capacity by c_{ij}. The maximum allowable information from node i to node j in N is called the "terminal capacity" from node i to node j and represented by t_{ij}. The matrix $T = [t_{ij}]$ is called a "terminal matrix"* of a communication net if $t_{ij} (i \neq j)$

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* Sometimes referred to in the literature as "terminal capacity matrix".

is the terminal capacity from node i to node j and t_{ii} is the symbol ⓘ , representing node i; numerically t_{ii} is assumed to be ∞ unless capacity constraints on node i are admitted.

Throughout all the thesis the matrix $T = [t_{ij}]$ is assumed to be a square matrix of order n, where every off-diagonal element t_{ij}, $i \neq j$, is a real non-negative number, and every diagonal element t_{ii} is the node symbol ⓘ .

An important problem related to oriented communication nets is to obtain necessary and sufficient conditions for the realizability of terminal matrices. Gomory and Hu,[6] Mayeda,[10] and Tang and Chien[13] have given necessary conditions. However, these conditions are not sufficient, except for 3 by 3 terminal matrices, as has been shown by Tang and Chien.[13]

In this chapter we shall present a method for realizing a given realizable terminal matrix, T, that is finding an oriented net with associated edge capacities such that its terminal matrix coincides with the given T.

1.2 Some Properties of the Terminal Matrix of an Oriented Communication Net.

We shall first present some definitions and results which will be required in this chapter and the next chapters.

Definition (1.1). A subnet of N consisting of a sequence of distinct nodes $i_1, i_2, \ldots, i_m$ and edges $e_{i_1 i_2}, e_{i_2 i_3}, \ldots, e_{i_{m-1} i_m}$ is called a "directed path from node i_1 to node i_m", or simply "directed path (i_1, i_m)". A path (i_1, i_m)" is a subnet that becomes a directed path (i_1, i_m) by reversing the orientations of some of the edges.

Definition (1.2). A "cut (i,j)" of a connected oriented net N is a minimal set of edges the removal of which destroys all directed paths (i,j). If the removal of the edges of a cut (i,j) partitions the nodes of N into $V_i = \{i_1, \ldots, i_p\}$ and $V_j = \{j_1, \ldots, j_q\}$ such that $i_1, i_2, \ldots, i_p$ are i and all nodes reachable from i after the removal of all edges of the cut (i,j) and $j_1, j_2, \ldots, j_q$ are all other nodes, then this cut (i,j) will be represented by $K_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$. The capacity of a cut (i,j) is the sum of the capacities of all edges of the cut (i,j). The capacity of $K_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$ is denoted by $C(K_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q})$. A "minimum cut (i,j)" is a cut (i,j) whose capacity is not more than the capacity of any other cut (i,j) in N.

A fundamental property of communication nets is contained in a theorem formulated and proved by Ford and Fulkerson,[3] and independently by Elias, Feinstein and Shannon.[2] It is called the "Max-Flow Min-Cut Theorem" and it reads:

Theorem (1.1). "For an oriented communication net the maximum flow from node i to node j is equal to the capacity of a minimum cut (i,j)."

Definition (1.3). A matrix T is "semiprincipal partitionable" if upon rearranging the rows and the corresponding columns (if necessary) T is partitionable into

$$T = \left[\begin{array}{c|c} A_1 & T_1 \\ \hline C_1 & B_1 \end{array}\right], \qquad (1.1)$$

such that T_1 is a uniform matrix with element value t_1 minimal for T (i.e. smallest among all elements of T), A_1 and B_1 are square submatrices whose diagonal elements are the node symbols and both A_1 and B_1 are again partitionable in the same fashion satisfying the same conditions until finally each submatrix becomes a one-by-one matrix. The matrices A_1 and B_1 are called the "resultant main submatrices" by the semiprincipal partitioning process.

Necessary conditions of a realizable terminal matrix are contained in the following theorems which are given by Tang and Chien:[13]

<u>Theorem (1.2)</u>. "A realizable terminal matrix of an oriented communication net is semiprincipal partitionable."

<u>Theorem (1.3)</u>. "Let $t_{ij}(i, j = 1, 2, \ldots, n,\ i \neq j)$ be any element of a realizable terminal matrix T. Then

$$t_{ij} \geq \min \left\{ t_{ik},\ t_{kj} \right\} ,$$

for all $k = 1, 2, \ldots, n$."

Theorem (1.3) is also given by Gomory and Hu.[6]

<u>Definition (1.4)</u>. An "S-submatrix" of a realizable terminal matrix T of an oriented communication net N is a submatrix of T consisting of all elements of T which are at the intersections of the rows representing the nodes in V_r and the columns representing the nodes in V_c, where $V_r = \{r_1, r_2, \ldots, r_\alpha\}$ and $V_c = \{c_1, c_2, \ldots, c_\beta\}$, $\alpha + \beta = n$, are the collections of nodes such that every node of N is either in V_r or V_c but not in both. Such an S-submatrix will be denoted by $s_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$.

It is clear that for every cut $K_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$ of N there corresponds one S-submatrix, $s_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$,

$p + q = n$, in the terminal matrix T of N.

Definition (1.5). Let $V_r = \{r_1, \ldots, r_\alpha\}$ and $V_c = \{c_1, \ldots, c_\beta\}$ be any partitioning of the set of nodes of N such that $V_r \cap V_c = \emptyset$, the empty set. We define* a "seg(V_r, V_c)", denoted by $g_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$, as the set of all edges e_{ij} of N such that $i \in V_r$ and $j \in V_c$. It must be noticed that seg $(V_r, V_c) \neq$ seg(V_c, V_r). The capacity of a seg $g_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$, denoted by $C(g_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta})$, is the sum of the capacities of all edges in the seg (V_r, V_c).

One can easily see that for every S-submatrix, $s_{r_1 r_2, \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$ of T there corresponds one and only one seg $g_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$, and every seg $g_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$ contains a cut (r_x, c_y), where $x = 1, 2, \ldots, \alpha$ and $y = 1, 2, \ldots, \beta$. Therefore by Theorem (1.1) we arrive at the following lemma:

Lemma (1.1). "For every S-submatrix $s_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$ of the terminal matrix T of an oriented communication net N,

$$C(g_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}) \geq t_{ij} ,$$

for all $i \in V_r$ and $j \in V_c$."

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*This definition of a seg is slightly different from that given by Reed.[11]

The following theorem is given by Mayeda:[10]

Theorem (1.4). "Every element in a terminal matrix T, except the diagonal elements, belongs to at least one S-submatrix in which this element is the largest."

Mayeda[10] asserts, without proof, that theorems (1.3) and (1.4) are equivalent. His assertion is not obvious; we shall prove it here.

Theorem (1.5). "The following statements are equivalent:

(a) For every element t_{ij}, $i \neq j$, of T

$$t_{ij} \geq \min \left\{ t_{ik}, t_{kj} \right\} , \tag{1.2}$$

for all $k = 1,2,\ldots,n$.

(b) For every element t_{ij}, $i \neq j$, of T there is an S-submatrix containing t_{ij} as a largest element."

Proof. (1) Assume that matrix T satisfies (a), we shall show that it also satisfies (b). Let t_{ij} be any element of T and let $\left\{ i_1, i_2, \ldots, i_a \right\}$ be all elements in $\left\{ 1,2,\ldots,n \right\}$ such that

$$t_{ij} < t_{ii_r} , \quad r = 1,2,\ldots,a, \tag{1.3}$$

and let $\left\{ j_1, j_2, \ldots, j_b \right\}$ be all elements in $\left\{ 1,2,\ldots,n \right\}$ such that

$$t_{ij} < t_{j_r j} , \quad r = 1,2,\ldots,b. \tag{1.4}$$

The sets $\{i_1, i_2, \ldots, i_a\}$ and $\{j_1, j_2, \ldots, j_b\}$ have no elements in common, because otherwise if p is in both of them then

$$t_{ij} < t_{ip}, t_{pj} ,$$

contradicting (1.2)

Now let

$$Q = \{1, 2, \ldots, n\} - \{i, i_1, i_2, \ldots, i_a\} - \{j, j_1, j_2, \ldots, j_b\} ;$$

and $\{n_1, n_2, \ldots, n_d\}$ be all elements of Q such that

$$t_{i_r n_x} > t_{ij} , \qquad x = 1, 2, \ldots, d, \tag{1.5}$$

for at least one $i_r \in \{i_1, i_2, \ldots, i_a\}$. Finally, let $\{m_1, m_2, \ldots, m_c\}$ be the complement of $\{n_1, n_2, \ldots, n_d\}$ in Q. We shall prove that $s_{ii_1i_2\ldots i_an_1n_2\ldots n_d, jj_1j_2\ldots j_bm_1m_2\ldots m_c}$ is an S-submatrix of T which contains t_{ij} as largest element. Suppose, if possible, that

$$t_{\alpha\beta} > t_{ij} , \tag{1.6}$$

where $\alpha \in \{i, i_1, \ldots, i_a, n_1, \ldots, n_d\}$ and $\beta \in \{j, j_1, \ldots, j_b, m_1, \ldots, m_c\}$

We have three cases:

(A) If $\alpha \in \{i, i_1, \ldots, i_a\}$ and $\beta \in \{j, j_1, \ldots, j_b\}$, then from (1.3), (1.4) and (1.6),

$$\begin{aligned} t_{ij} &< \min\{t_{i\alpha}, t_{\alpha\beta}, t_{\beta j}\} \\ &= \min\{t_{i\alpha}, \min\{t_{\alpha\beta}, t_{\beta j}\}\} \\ &\leq \min\{t_{i\alpha}, t_{\alpha j}\} \quad , \end{aligned}$$

contradicting (1.2)

(B) If $\alpha \in \{i, i_1, \ldots, i_a\}$ and $\beta \in \{m_1, \ldots, m_c\}$ then we get contradiction to the definition of $\{n_1, \ldots, n_d\}$.

(C) If $\alpha \in \{n_1, \ldots, n_d\}$ and $\beta \in \{j, j_1, \ldots, j_b, m_1, \ldots, m_c\}$ then from (1.2), (1.5) and (1.6) we have, for at least one $i_r \in \{i_1, \ldots, i_a\}$,

$$t_{i_r\beta} \geq \min \left\{t_{i_r\alpha}, t_{\alpha\beta}\right\},$$

$$> t_{ij},$$

which has been shown in Cases (A) and (B) above to lead to a contradiction.

Hence in any case

$$t_{\alpha\beta} \leq t_{ij},$$

for all $\alpha \in \{i, i_1, \ldots, i_a, n_1, \ldots, n_d\}$, and $\beta \in \{j, j_1, \ldots, j_b, m_1, \ldots, m_c\}$.

Thus (a) implies (b).

(2) Suppose that T satisfies (b), and let t_{ij} be any element of T and $s_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$ be an S-submatrix of T in which t_{ij} is a largest element. If $k \in \{1, 2, \ldots, n\}$, $k \neq i, j$, then either $k \in \{r_1, r_2, \ldots, r_\alpha\}$ or $k \in \{c_1, c_2, \ldots, c_\beta\}$. Therefore either t_{kj} or t_{ik} is an element of $s_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$. Thus

$$t_{ij} \geq \min \left\{t_{ik}, t_{kj}\right\},$$

for all $k = 1, 2, \ldots, n$.

Hence the theorem.

Now we shall prove that Theorem (1.4) implies Theorem (1.2), that is

<u>Theorem (1.6)</u>. "If for every element t_{ij}, $i \neq j$, of matrix T there is at least one S-submatrix of T in which t_{ij} is a largest element, then T is semiprincipal partitionable matrix."

<u>Proof</u>. Let t_1 be a minimal element of T and let T_1 ($\equiv s_{r_1 r_2 \ldots r_\alpha, c_1 c_2 \ldots c_\beta}$) be an S-submatrix of T in which t_1 is a largest element, then T_1 is a uniform matrix of element value t_1 and by rearranging the rows and columns (if necessary) T can be partitioned into

$$T = \left[\begin{array}{c|c} A_1 & T_1 \\ \hline C_1 & B_1 \end{array}\right],$$

where A_1 and B_1 are square matrices; those are the resultant submatrices of T. Now let t_2 be a minimal element of A_1, and $s_{r'_1 r'_2 \ldots r'_{\alpha'}, c'_1 c'_2 \ldots c'_{\beta'}}$ be an S-submatrix of T in which t_2 is a largest. Let $\{p_1, p_2, \ldots, p_\gamma\}$ and $\{q_1, q_2, \ldots, q_\delta\}$, $\gamma + \delta = \alpha$, be all the elements of $\{r_1, r_2, \ldots, r_\alpha\}$ which are in $\{r'_1, r'_2, \ldots, r'_{\alpha'}\}$ and $\{c'_1, c'_2, \ldots, c'_{\beta'}\}$, respectively, then $s_{p_1 p_2 \ldots p_\gamma, q_1, q_2 \ldots q_\delta}$,

($\equiv T_2$) say, is an S-submatrix of A_1. Since t_2 is a largest element of T_2 and smallest of A_1, and so of T_2, then T_2 is a uniform matrix of element value t_2. Thus A_1 can be partitioned as

$$A_1 = \left[\begin{array}{c|c} A_2 & T_2 \\ \hline C_2 & B_2 \end{array}\right] .$$

The above partitioning can be carried out until each resultant main submatrix becomes a one-by-one matrix. Thus T is semiprincipal partitionable.

Hence the theorem.

It must be noticed that the converse of Theorem (1.6) is not necessarily true. This follows from the matrix in (1.7) which is semiprincipal partitionable but does not contain an S-submatrix in which element 3 is largest.

$$T = \begin{bmatrix} \textcircled{1} & 2 & 1 \\ 5 & \textcircled{2} & 1 \\ 3 & 4 & \textcircled{3} \end{bmatrix} \qquad (1.7)$$

<u>Corollary (1.1)</u>. "If for every element t_{ij}, $i \neq j$, of matrix T,

$$t_{ij} \geq \min \left\{ t_{ik}, t_{kj} \right\} ,$$

for all $k = 1,2,\ldots,n$,

then T is semiprincipal partitionable."

This follows directly from Theorems (1.5) and (1.6).

Mayeda also shows that Theorem (1.4) is not sufficient for the realizability of terminal matrices of oriented communication nets.

Let S_{ij}, $i \neq j$, be the set of all S-submatrices of T each of which contains t_{ij} as a largest element. We shall prove the following theorem which will be required in the next section.

Theorem (1.7). "For every element t_{ij}, $i \neq j$, of a realizable terminal matrix T of an oriented communication net N, there is at least one S-submatrix in S_{ij} whose corresponding seg is a cut (i,j) of N and its capacity is equal to t_{ij}."

Proof. By Theorem (1.1), there is a minimum cut (i,j), say $K_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$, of N such that

$$C(K_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}) = t_{ij} .$$

Consider the S-submatrix $s_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$ of T. By Lemma (1.1)

$$C(g_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}) = t_{ij} \geq t_{pq} ,$$

where t_{pq} is a maximal element of $s_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q}$. But t_{ij} is an element of this S-submatrix. Thus $t_{ij} = t_{pq}$

and $s_{i_1 i_2 \ldots i_p, j_1 j_2 \ldots j_q} \in S_{ij}$.

Hence the theorem.

One can easily see that if t_{pq} is a maximal element of T, then S_{pq} consists of 2^{n-2} S-submatrices; all combinations of rows containing row p but **not containing** row q.

The following lemma is useful in obtaining S_{ij}, $i \neq j$, for some element, t_{ij}, of T.

<u>Lemma (1.2)</u>. "Let M be a submatrix of the terminal matrix T whose elements are all those on the intersections of rows $i, i_1, i_2, \ldots, i_a$ and columns $j, j_1, \ldots, j_b$ where

$$\{i, i_1, \ldots, i_a\} \bigcap \{j, j_1, \ldots, j_b\} = \emptyset ,$$

$$t_{ii_r} > t_{ij} , \quad \text{for all } r = 1, \ldots, a,$$

and

$$t_{j_r j} > t_{ij}, \quad \text{for all } r = 1, \ldots, b.$$

Then M is a submatrix of every S-submatrix in S_{ij}."

<u>Proof</u>. Suppose, if possible, that $s_1 \in S_{ij}$ does not contain M, then either

(1) there is at least one i_r which is a column of s_1, then t_{ii_r} is an element of s_1; thus it must not be more than t_{ij}, contradicting the hypothesis; or

(2) there is at least one j_r which is a row of s_1, then $t_{j_r j}$ is an element of s_1 and it must not be more than

t_{ij}, contradicting the hypothesis also.

Hence the lemma.

Corollary (1.2). "If $t_{ij} < t_{ik}$, for all $k \neq j$, then S_{ij} consists of one S-submatrix only which is column j without ⓙ . Similarly, if $t_{ij} < t_{kj}$ for all $k \neq i$, then S_{ij} consists of one S-submatrix which is row i without ⓘ ."

The proof follows directly from Lemma (1.2).

Lemma (1.3). "Let t_{ij} and t_{pq} be any two elements of a realizable terminal matrix T, such that $t_{ij} \neq t_{pq}$, then $S_{ij} \cap S_{pq} = \emptyset$."

Proof. If $S_{ij} \cap S_{pq}$ is not empty then let s be an S-submatrix in both S_{ij} and S_{pq}. Thus each of t_{ij} and t_{pq} is maximal element of s. Hence $t_{ij} = t_{pq}$, contradicting the hypothesis. Hence the lemma.

1.3 Synthesis Procedure.

Suppose we have a matrix T which satisfies the necessary conditions in Theorem (1.4), that is for every element t_{ij}, $i \neq j$, there is at least one S-submatrix of T in which this element is largest. Then the following method will lead to a realization for T if it is realizable as a terminal matrix of an oriented communication net. The procedure will be described in the following steps:

(1) Obtain S_{ij} for each element t_{ij}, $i \neq j$, of T, and let $\{S_{ij}\}$ be the set whose elements are all S_{ij}, $i,j = 1,2,\ldots,n$, $i \neq j$.

(2) Let $\{S_{ij}^{(1)}\} = \{S_{i_1 j_1}^{(1)}, S_{i_2 j_2}^{(1)}, \ldots, S_{i_\alpha i_\alpha}^{(1)}\}$ be the set of all elements of $\{S_{ij}\}$ each of which consists of one S-submatrix only; then obtain S_1, where

$$S_1 = U_{r=1}^{\alpha} S_{i_r j_r}^{(1)}.$$

(3) Let $\{S_{ij}^{(2)}\} = \{S_{i_1 j_1}^{(2)}, S_{i_2 j_2}^{(2)}, \ldots, S_{i_\beta j_\beta}^{(2)}\}$ be all elements of $\{S_{ij}\}$ such that

$$S_{i_r j_r}^{(2)} \notin \{S_{ij}^{(1)}\} \quad ,$$

and

$$S_{i_r j_r}^{(2)} \cap S_1 \neq \emptyset,$$

for all $r = 1, 2, \ldots, \beta$. The set $\{S_{ij}^{(2)}\}$ can be obtained easily by applying Lemma (1.3). Then obtain S_2 and $\{S_{ij}^{(3)}\}$, where

$$S_2 = S_{i_m j_m} \, U_{r=1}^{\beta} (S_{i_r j_r}^{(2)} - S_1 \cap S_{i_r j_r}^{(2)}) \quad ,$$

$$\{S_{ij}^{(3)}\} = \{S_{ij}\} - \{S_{ij}^{(1)}\} - \{S_{ij}^{(2)}\} - S_{i_m j_m} \, ,$$

$$\equiv \{ S_{i_r j_r}^{(3)} \mid r = 1, 2, \ldots, \gamma, \} \text{ say,}$$

in which $S_{i_m j_m}$ is the set of all S-submatrices each of which contains $t_{i_m j_m}$ as largest element and $t_{i_m j_m}$ is a maximal element of T.

(4) Let $S_3^{(1)}, S_3^{(2)}, \ldots, S_3^{(y)}$ be all different sets such that each contains exactly one element from every $S_{i_r j_r}^{(3)}$, $r = 1, \ldots, \gamma$ and no other elements. Now we have y cases, each corresponds to one of $S_3^{(1)}, \ldots, S_3^{(y)}$. Let us take the first one. Obtain E_1 and I_1 corresponding to $S_3^{(1)}$, where

$$E_1 = S_1 \cup S_3^{(1)},$$

$$I_1 = S_2 \cup [U_{r=1}^{\gamma} S_{i_r j_r}^{(3)} - S_3^{(1)}] \, .$$

It can be easily seen that each S_{ij}, except $S_{i_m j_m}$ has one of its elements in E_1 and the others in I_1. Also, every S-submatrix of T is contained in either E_1 or I_1.

(5) For each S-submatrix in E_1 form an equality by setting the capacity of its corresponding seg in N equal to the largest element of this S-submatrix, where N is the complete oriented communication net[*] of n nodes and unknown edge capacities, c_{ij}, $i \neq j$. Thus we get

$$[A_{11}^{(1)} \; A_{12}^{(1)}] \begin{bmatrix} C_1 \\ \\ C_2 \end{bmatrix} = T_1 \quad , \tag{1.8}$$

where

$$T_1 = \begin{bmatrix} t_{i_1 j_1} \\ \vdots \\ t_{i_a j_a} \end{bmatrix} , \quad t_{i_r j_r} \leq t_{i_{r+1} j_{r+1}}, \quad r = 1,2,\ldots,a-1,$$

in which $t_{i_r j_r}$, $r = 1,2,\ldots,a$, is a largest element in the corresponding S-submatrix in E_1, a is the number of elements in E_1 and C,

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} ,$$

is a column matrix whose elements are the edge capacities of N, which are arranged in an order such that

$$C_1 = \begin{bmatrix} c_{i_1 j_1} \\ \vdots \\ c_{i_a j_a} \end{bmatrix} .$$

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[*]A complete oriented communication net is an oriented net that contains edges e_{ij} and e_{ji} for all $i \neq j$.

Matrices $A_{11}^{(1)}$ and $A_{12}^{(1)}$ are the coefficient matrices (0, + 1 elements only) of the equations. From the ordering of the equations and the variables we can easily see that $A_{11}^{(1)}$ is a lower triangular matrix with +1 along the diagonal; thus it is non-singular.

(6) For each S-submatrix of I_1 form an inequality by making the capacity of its corresponding seg, in N, more than or equal the value of a largest element in this S-submatrix; and let the first 2^{n-2} of the inequalities correspond to those in $S_{i_m j_m}$. Thus we get

$$[A_{21}^{(1)} \; A_{22}^{(1)}] \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \geq T_2 \tag{1.9}$$

Since the edge capacities are non-negatives then

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \geq 0. \tag{1.10}$$

From (1.8) we have

$$C_1 = (A_{11}^{(1)})^{-1} T_1 - (A_{11}^{(1)})^{-1} A_{12}^{(1)} C_2, \tag{1.11}$$

thus (1.9) becomes

$$A_{21}^{(1)}[(A_{11}^{(1)})^{-1}T_1 - (A_{11}^{(1)})^{-1}A_{12}^{(1)}C_2] + A_{22}^{(1)}C_2 \geq T_2 \; . \tag{1.12}$$

Therefore from (1.10), (1.11) and (1.12) the set of constraints in (1.8), (1.9) and (1.10) reduces to

$$\begin{bmatrix} A_{22}^{(1)}-A_{21}^{(1)}(A_{11}^{(1)})^{-1}A_{12}^{(1)} \\ -(A_{11}^{(1)})^{-1}A_{12}^{(1)} \\ U \end{bmatrix} C_2 \geq \begin{bmatrix} T_2-A_{21}^{(1)}(A_{11}^{(1)})^{-1}T_1 \\ -(A_{11}^{(1)})^{-1}T_1 \\ 0 \end{bmatrix} \tag{1.13}$$

where U is a unit matrix of order n(n-1)-a.

If the constraints in (1.13) are consistent then they must have a solution. Any solution of (1.13) which satisfies at least one of the first 2^{n-2} constraints with equality is a realization of T, because: (1) from (1.8), for any pair of nodes, say i and j, there is a cut (i,j) in $\bar{N}$ whose capacity is equal to t_{ij}, and (2) every cut (i,j) of $\bar{N}$ is a seg of an S-submatrix and thus it is contained in either I_1 or E_1 and so its capacity is not less than t_{ij} (Lemma (1.1)), where $\bar{N}$ is the net of the solution.

If: (1) the constraints in (1.13) are inconsistent or (2) they are consistent but become inconsistent if any of the first 2^{n-2} constraints becomes equality, then we must go back to Step 4 and take $S_3^{(2)}$ then obtain I_2 and E_2 and repeat the procedure. The method must be repeated for the other sets $S_3^{(3)},\ldots,S_3^{(y)}$ until we arrive at a solution. If none of these sets leads to a solution, then T is not realizable as a terminal matrix of an oriented communication net (Theorem (1.7)).

The method described above is very laborious, but it can be simplified by making use of the following lemma:

<u>Lemma (1.4)</u>: "Let X be a square lower triangular matrix of order h with nonzero diagonal elements, and let X be partitioned as

$$X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & x_{hh} \end{bmatrix}, \qquad (1.14)$$

where X_{11} is a square lower triangular matrix of order (h-1), and X_{21} is a row matrix, 1 x (h-1). Then the inverse of X is given by

$$X^{-1} = \begin{bmatrix} X_{11}^{-1} & 0 \\ \frac{-X_{21}X_{11}^{-1}}{x_{hh}} & \frac{1}{x_{hh}} \end{bmatrix}. \qquad (1.15)$$

<u>Proof</u>. The lemma is true, because carrying out the multiplication of the right hand sides of (1.14) and (1.15) we get a unit matrix of order h.

Let s_1 be an S-submatrix in $S_{i_a j_a}$ which is also in E_1, then its corresponding equality occupies the last row in (1.8). If there is $s_2 \in S_{i_a j_a}$, $s_2 \neq s_1$, such that $S_3^{(1)} - s_1 \cup s_2$ is a member of $\{S_3^{(2)}, S_3^{(3)}, \ldots, S_3^{(y)}\}$, then take $S_3^{(2)}$ to be $S_3^{(1)} - s_1 \cup s_2$. Thus

$$E_2 = E_1 - s_1 \cup s_2 \quad ,$$

$$I_2 = I_1 \cup s_1 - s_2$$

If the constraint corresponding to s_2 occupies the j^{th} row of (1.9) then $[A_{11}^{(2)}\ A_{12}^{(2)}]$ is $[A_{11}^{(1)}\ A_{12}^{(1)}]$ with the a^{th} row replaced by the j^{th} row of $[A_{21}^{(1)}\ A_{22}^{(1)}]$, and $[A_{21}^{(2)} A_{22}^{(2)}]$ is $[A_{21}^{(1)}\ A_{22}^{(1)}]$ with the j^{th} row replaced by the a^{th} row of $[A_{11}^{(1)}\ A_{12}^{(1)}]$. Thus if we partition $A_{11}^{(1)}$ and $A_{11}^{(2)}$ as in (1.14),

$$A_{11}^{(1)} = \begin{bmatrix} (A_{11}^{(1)})_{11} & 0 \\ (A_{11}^{(1)})_{21} & 1 \end{bmatrix} ,$$

$$A_{11}^{(2)} = \begin{bmatrix} (A_{11}^{(2)})_{11} & 0 \\ (A_{11}^{(2)})_{21} & 1 \end{bmatrix} ,$$

then by Lemma (1.4), $(A_{11}^{(2)})^{-1}$ is $(A_{11}^{(1)})^{-1}$ with the last row replaced by $[-(A_{11}^{(2)})_{21}(A_{11}^{(1)})_{11}^{-1} \quad 1]$, = R say. So it is very easy to find $(A_{11}^{(2)})^{-1}$ from $(A_{11}^{(1)})^{-1}$.

Similarly $(A_{11}^{(2)})^{-1} A_{12}^{(2)}$ is $(A_{11}^{(1)})^{-1} A_{12}^{(1)}$ except the last row which must be $RA_{12}^{(2)}$; and also $(A_{11}^{(2)})^{-1}T_1$ is $(A_{11}^{(1)})^{-1}T_1$ except the last element which must be RT_1. Other matrices required in (1.13) can be found from those by some simple multiplications.

The entire method of synthesis will be illustrated in the following example:

<u>Example (1.1)</u>. Consider the matrix

$$T = \begin{bmatrix} \textcircled{1} & 5 & 4 & 4 \\ 13 & \textcircled{2} & 4 & 4 \\ 9 & 8 & \textcircled{3} & 6 \\ 25 & 13 & 9 & \textcircled{4} \end{bmatrix}.$$

The set of all S_{ij}, $i \neq j$, are obtained:

$$S_{13} = S_{14} = S_{23} = S_{24} = \left\{s_{12,34}\right\} .$$

$$S_{12} = \left\{s_{1,234}\right\} ,$$

$$S_{21} = \left\{s_{2,134}, s_{23,14}\right\} ,$$

$$S_{31} = \left\{s_{3,124}\right\} ,$$

$$S_{32} = \left\{s_{13,24}\right\} ,$$

$$S_{34} = \left\{s_{123,4}\right\} ,$$

$$S_{41} = \left\{s_{4,123}, s_{42,13}, s_{43,12}, s_{423,1}\right\} ,$$

$$S_{42} = \{s_{14,23}, s_{134,2}\} ,$$

$$S_{43} = \{s_{124,3}\} .$$

Therefore

$$S_1 = \{s_{12,34}, s_{1,234}, s_{123,4}, s_{13,24}, s_{124,3}, s_{3,124}\},$$

$$\{S_{ij}^{(2)}\} = \emptyset ,$$

$$S_2 = \{s_{4,123}, s_{42,13}, s_{43,12}, s_{423,1}\}, \quad \text{and}$$

$$\{S_{ij}^{(3)}\} = \{S_{21}, S_{42}\} .$$

Thus

$$S_3^{(1)} = \{s_{2,134}, s_{14,23}\} ,$$

$$S_3^{(2)} = \{s_{2,134}, s_{134,2}\} ,$$

$$S_3^{(3)} = \{s_{23,14}, s_{14,23}\} , \quad \text{and}$$

$$S_3^{(4)} = \{s_{23,14}, s_{134,2}\} .$$

Therefore we have 4 cases, at most.

<u>Case 1</u>.

$$E_1 = \{s_{12,34}, s_{1,234}, s_{123,4}, s_{13,24}, s_{124,3}, s_{3,124}, s_{2,134}, s_{14,23}\},$$

$$I_1 = \{s_{43,12}, s_{4,123}, s_{42,13}, s_{423,1}, s_{23,14}, s_{134,2}\} .$$

The set of constraints in the order mentioned above is given by

$$\overset{\qquad\qquad A_{11}^{(1)} \qquad\qquad\qquad\qquad\qquad\qquad A_{12}^{(1)}}{\left[\begin{array}{cccccccc|cccc}
1 & & & & & & & & 1 & 1 & 1 & 0 \\
1 & 1 & & & & & & & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & & & & & & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & & & & & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & & & & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]}
\left[\begin{array}{c}
C_{13} \\ C_{12} \\ C_{34} \\ C_{32} \\ C_{43} \\ C_{31} \\ C_{21} \\ C_{42} \\ \hline C_{14} \\ C_{23} \\ C_{24} \\ C_{41}
\end{array}\right]
=
\overset{T_1}{\begin{bmatrix} 4 \\ 5 \\ 6 \\ 8 \\ 9 \\ 9 \\ 13 \\ 13 \end{bmatrix}} , \qquad (1.16)$$

and

$$\overset{\qquad\qquad A_{21}^{(1)} \qquad\qquad\qquad\qquad\qquad\qquad A_{22}^{(1)}}{\left[\begin{array}{cccccccc|cccc}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]}
\left[\begin{array}{c}
C_{13} \\ C_{12} \\ C_{34} \\ C_{32} \\ C_{43} \\ C_{31} \\ C_{21} \\ C_{42} \\ \hline C_{14} \\ C_{23} \\ C_{24} \\ C_{41}
\end{array}\right]
\geq
\overset{T_2}{\begin{bmatrix} 25 \\ 25 \\ 25 \\ 25 \\ 13 \\ 13 \end{bmatrix}} . \qquad (1.17)$$

Then we find $(A_{11}^{(1)})^{-1}$ and all other required matrices,

$$(A_{11}^{(1)})^{-1} = \begin{bmatrix} 1 & & & & & & & \\ -1 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 1 & -1 & -1 & 1 & & & & \\ -1 & 0 & 0 & 0 & 1 & & & \\ -1 & 1 & 0 & -1 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, \qquad (1.18)$$

$$(A_{11}^{(1)})^{-1}A_{12}^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad (1.19)$$

$$A_{21}^{(1)}(A_{11}^{(1)})^{-1}A_{12}^{(1)} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \qquad (1.20)$$

$$(A_{11}^{(1)})^{-1}T_1 = \begin{bmatrix} 4 \\ 1 \\ 6 \\ 1 \\ 5 \\ 2 \\ 13 \\ 3 \end{bmatrix} , \qquad (1.21)$$

$$A_{21}^{(1)}(A_{11}^{(1)})^{-1}T_1 = \begin{bmatrix} 6 \\ 8 \\ 18 \\ 15 \\ 21 \\ 5 \end{bmatrix} \qquad (1.22)$$

Hence the set of constraints is given by

$$\left[\begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \begin{bmatrix} c_{14} \\ c_{23} \\ c_{24} \\ c_{41} \end{bmatrix} \geq \left[\begin{array}{r} 19 \\ 17 \\ 7 \\ 10 \\ -8 \\ 8 \\ \hline -4 \\ -1 \\ -6 \\ -1 \\ -5 \\ -2 \\ -13 \\ -3 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] . \qquad (1.23)$$

From the sixth row of (1.23) we have contradiction, that is $0 \geq 8$, thus no solution for (1.16) and (1.17). Hence we must consider the next case.

Case 2.

$$E_2 = \{ s_{12,34}, s_{1,234}, s_{123,4}, s_{13,24}, s_{124,3}$$
$$s_{3,124}, s_{2,134}, s_{134,2} \}.$$

$$I_2 = \{ s_{43,12}, s_{4,123}, s_{42,13}, s_{423,1}, s_{23,14}, s_{14,23} \}.$$

Using the modification mentioned in the discussions following Lemma (1.4), we find that $(A_{11}^{(2)})^{-1}$ is identical with that in (1.18) except the last row which must be [0 0 1 -1 0 0 0 1]. Thus the last row of $(A_{11}^{(2)})^{-1}A_{12}^{(2)}$ is [0 0 1 0]; thus it is identical with that in (1.19). Also, the last element of $(A_{11}^{(2)})^{-1}T_1$ is 11. Hence obtaining $A_{21}^{(2)}(A_{11}^{(2)})^{-1}(A_{12}^{(2)})$ and $A_{21}^{(2)}(A_{11}^{(2)})^{-1}T_1$, we arrive finally at

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} c_{14} \\ c_{23} \\ c_{24} \\ c_{41} \end{bmatrix} \geq \begin{bmatrix} 11 \\ 9 \\ 7 \\ 10 \\ -8 \\ -8 \\ \hline -4 \\ -1 \\ -6 \\ -1 \\ -5 \\ -2 \\ -13 \\ -11 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (1.24)$$

which is consistent and can be reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} c_{14} \\ c_{23} \\ c_{24} \\ c_{41} \end{bmatrix} \geq \begin{bmatrix} 11 \\ -4 \\ -1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{1.25}$$

The set in (1.24) remains consistent even if the first constraint becomes equality. A solution of (1.25) which makes the first constraint of (1.24), which is also the first of (1.25), equality is given by

$$c_2 = \begin{bmatrix} x \\ 0 \\ 0 \\ 11-x \end{bmatrix}, \quad 0 \leq x \leq 4 .$$

Hence substituting in

$$c_1 = (A_{11}^{(2)})^{-1}T_1 - (A_{11}^{(2)})^{-1}A_{12}^{(2)}c_2 ,$$

we get for the transpose of C

$$C' = [4-x,\ 1,\ 6-x,\ 1,\ 5+x,\ 2+x,\ 13,\ 11,\ x,\ 0,\ 0,\ 11-x],$$

which is a family of realizations of T.

CHAPTER II

Realizability Conditions of Special Types of Oriented Communication Nets

2.1 Introduction

We have mentioned in Chapter I that the necessary and sufficient conditions for a matrix T to be realizable as a terminal matrix of an oriented communication net have not been found yet. We have also seen that the general method presented in Chapter I for the realization of a realizable matrix T is long and laborious. Thus it is worthwhile to present some sufficient conditions for the realizability of some types of oriented communication nets. Tang and Chien[13] showed that a matrix T satisfying some conditions (Theorem 2.6 of this Chapter) is realizable. Recently, 1962, Mayeda[10] showed that a completely partitionable matrix is realizable.

In this chapter, we shall present necessary and sufficient conditions for a matrix T to be realizable as a terminal matrix of communication nets having tree or loop structures. We shall also show that Tang and Chien's and Mayeda's results are special cases of ours. A method for realizing a terminal matrix T which is lower or upper triangular will also be given in this chapter.

2.2 Double-Tree Nets.

Definition (2.1). A "double-tree net" is a connected oriented communication net containing no loop consisting of more than two edges. That is a double tree net may contain two parallel edges of opposite orientations (see Fig.2.2).

Definition (2.2). A "double linear tree net" (or double Hamiltonian path net) is a double tree net which contains a path containing all the nodes of the net.

Definition (2.3). In the definition of the semi-principal partitionable matrix (Definition (1.3) the sub-matrix C_1 in (1.1) will be called a "C-submatrix of T". In general for each resultant main submatrix of order more than one there is one C-submatrix obtained by applying the semiprincipal partitioning process to that resultant main submatrix. Thus a semiprincipal partitionable matrix of order n contains exactly (n-1) C-submatrices.

Definition (2.4). A matrix T is said to be "principal partitionable"[9] if it is symmetric and semiprincipal partitionable, that is, T is semiprincipal partitionable with $C_r = T_r'$ for each $r = 1,2,\ldots,n-1$, where T_r' is the transpose of T_r.

The following theorem is given by Mayeda[9]:

<u>Theorem (2.1)</u>. "A matrix T is realizable as a terminal matrix of a non-oriented communication net if and only if T is principal partitionable."

<u>Definition (2.5)</u>. A matrix $T = [t_{ij}]$ is said to be "completely partitionable[7,10]" if its rows and columns can be rearranged to form a matrix $\bar{T} = [\bar{t}_{ij}]$ with the property that matrices $T^{(1)} = [t^{(1)}_{ij}]$ and $T^{(2)} = [t^{(2)}_{ij}]$ are both (without further rearrangement of their rows and columns) principal partitionable, where $t^{(1)}_{ij} = t^{(1)}_{ji} = \bar{t}_{ij}$ for $i < j$ and $t^{(2)}_{ij} = t^{(2)}_{ji} = \bar{t}_{ij}$ for $i > j$.

<u>Lemma (2.1).</u> "The terminal matrix of a double linear tree net is completely partitionable."

<u>Proof</u>. Let N be a given double linear tree net. Label the nodes of N by $1, 2, \ldots, n$ such that $e_{12}, e_{23}, \ldots e_{n-1,n}$ is a path $(1,n)$ of N. Let $T = [t_{ij}]$ be the terminal matrix of N with the diagonal elements in the order $1, 2, \ldots, n$. Any change in the capacity of edge $e_{\ell k}$, $\ell > k$, does not change the terminal capacity t_{ij}, $i < j$, of N. Thus $T^{(1)}$ and $T^{(2)}$ are the terminal matrices of N_1 and N_2, respectively, where $T^{(1)} = [t^{(1)}_{ij}]$, $t^{(1)}_{ij} = t^{(1)}_{ji} = t_{ij}$ for $i < j$, $T^{(2)} = [t^{(2)}_{ij}]$, $t^{(2)}_{ij} = t^{(2)}_{ji} = t_{ij}$ for $i > j$, and $N_1 (N_2)$ is N with all edges $e_{ij} (e_{ji})$, $i > j$, removed and all other edges replaced by non-oriented edges. By Theorem (2.1), $T^{(1)}$ and $T^{(2)}$ are

principal partitionable. Hence T is completely partitionable.

Mayeda shows that a completely partitionable matrix is realizable and its realization is a double linear tree net whose edge capacity matrix, $C = [c_{ij}]$, is given by

$$c_{ij} = \begin{cases} t_{ij}, & \text{if } i - j = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence we arrive at the following theorem:

<u>Theorem (2.2)</u>. "A matrix T is realizable as the terminal matrix of a double linear tree net if and only if it is completely partitionable."

<u>Definition (2.6)</u>. A matrix $T = [t_{ij}]$ is a "tree terminal matrix" if T is semiprincipal partitionable with each C-submatrix, C_r, $r = 1,2,\ldots,n-1$, containing at least one element, say t_{ij}, such that for every other element in C_r, say $t_{\ell k}$, the following relation holds:

$$t_{\ell k} = \min \{t_{\ell i}, t_{ij}, t_{jk}\}. \qquad (2.1)$$

The element t_{ij} will be called a "connective element of C_r".

<u>Lemma (2.2)</u>. "In Definition (2.6), the relation (2.1) is equivalent to

$$t_{\ell k} = \min \left\{ t_{\ell i},\ t_{ik} \right\} , \tag{2.2a}$$

$$= \min \left\{ t_{\ell j},\ t_{jk} \right\} . \tag{2.2b}$$

Proof. Suppose (2.1) holds, then

$$t_{ik} = \min \left\{ t_{ii},\ t_{ij},\ t_{jk} \right\}$$

$$= \min \left\{ t_{ij},\ t_{jk} \right\} , \tag{2.3}$$

because $t_{ii} = \infty$, by definition.

Similarly,

$$t_{\ell j} = \min \left\{ t_{\ell i},\ t_{ij} \right\} . \tag{2.4}$$

Thus substituting (2.3) and (2.4) in (2.1), we obtain (2.2a) and (2.2b), respectively.

Now suppose (2.2) holds, then putting $k = j$ in (2.2a) we get relation (2.4). Substituting $t_{\ell j}$ from (2.4) in (2.2b) we obtain (2.1).

Hence the lemma.

One can easily see that a connective element of C_r is not less than any other element in C_r. Thus we consider only the maximum elements of C_r when we look for a connective element of C_r. Another remark is that the element of a C-submatrix consisting of one element only is the connective element of that C-submatrix.

<u>Example (2.1)</u>. To illustrate Definition (2.6) consider the matrix

$$T = \left[\begin{array}{cc:ccc} \textcircled{1} & 2 & 1 & 1 & 1 \\ 7 & \textcircled{2} & 1 & 1 & 1 \\ \hline 7 & 10 & \textcircled{3} & 4 & 3 \\ 7 & 8 & 8 & \textcircled{4} & 3 \\ 6 & 6 & 6 & 4 & \textcircled{5} \end{array}\right] . \qquad (2.5)$$

The element t_{32} is the only connective element of the C-submatrix, C_1, where

$$C_1 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 7 & 10 \\ 7 & 8 \\ 6 & 6 \end{array}\right] \end{array} ,$$

because for $i = 3$ and $j = 2$, eqn (2.1) holds for each of t_{31}, t_{41}, t_{42}, t_{51}, t_{52} which are the elements of C_1 other than t_{32}. Similarly t_{53} is the connection element of C_2, where

$$C_2 = 5 \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \left[\begin{array}{cc} 6 & 4 \end{array}\right] \end{array} .$$

The connective element of $C_3 = 2\,\overset{1}{[7]}$ and $C_4 = 4\,\overset{3}{[8]}$ are t_{21} and t_{43}, respectively. Thus every C-submatrix of the matrix T given in (2.5) has a connective element; so T is *a* tree-terminal matrix.

<u>Theorem (2.3)</u>. "A matrix T is realizable as the terminal matrix of a double tree net if, and only if, T is a tree terminal matrix."

<u>Proof</u>. Let N be a double tree net and c_{ji} a minimal edge capacity in N. The removal of edges e_{ij} and e_{ji} from N will cut it into two disjoint double tree nets, N_a and N_b, as shown in Fig.(2.1). Since there is one and only one directed path from any node a to any other node b in N, then the terminal capacity t_{ab} is equal to the smallest edge capacity in the directed path (a,b). Thus the terminal matrix T of N is partitionable into

$$T = \left[\begin{array}{c|c} A_1 & T_1 \\ \hline C_1 & B_1 \end{array}\right],$$

where T_1 is a uniform submatrix with element value $t_1 = c_{ji}$ = minimal for T, A_1 is the terminal matrix of N_a and B_1 is the terminal matrix of N_b. If $t_{\ell k}$ is any element in C_1 then ℓ is a node in N_b and k is a node in N_a. The directed path (ℓ,k) consists of the directed path (ℓ,i), edge e_{ij}, and the directed path (j,k). Thus

$$t_{\ell k} = \min \left\{ t_{\ell i},\ t_{ij},\ t_{jk} \right\} .$$

By similar method, the terminal matrices A_1 and B_1 of the double tree nets N_a and N_b, respectively, can be partitioned in the same fashion with the elements of their C-submatrices satisfying the corresponding equalities. We can carry on this procedure until finally each subnet becomes a single node. Thus T is a tree terminal matrix (Definition 2.6).

Now, suppose T is a tree terminal matrix of order n. We shall prove by induction on n that T is realizable as a terminal matrix of a double tree net. The assertion is true for n = 2. Suppose that it is true for $n \leq k$, and consider T of order n = k+1. Since T is a tree terminal matrix then it can be partitioned as in (1.1). Because A_1 and B_1 are tree terminal matrices of orders $\leq k$, then by induction hypothesis the realizations of A_1 and B_1, say N_a and N_b, respectively, are double tree nets. Now consider the net N consists of N_a, N_b and edges e_{ij} and e_{ji} of capacities $c_{ij} = t_{ij}$ and $c_{ji} = t_{ji}$ in which t_{ij} is a connective element of C_1. The terminal capacity $t'_{\ell k}$ of N, where ℓ is a node of N_b and k is a node of N_a, is given by

$$t'_{\ell k} = \min \left\{ t'_{\ell i}, t'_{ij}, t'_{jk} \right\}$$

$$= t_{\ell k} \ .$$

Thus T is the terminal matrix of N.

Hence the theorem.

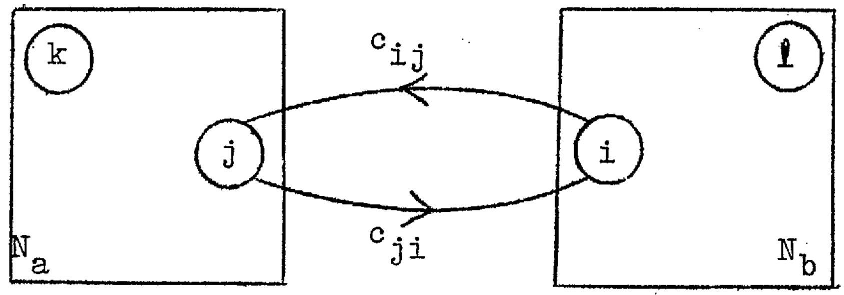


Fig.(2.1). A double tree net, N, for illustrating the proof of Theorem (2.3).

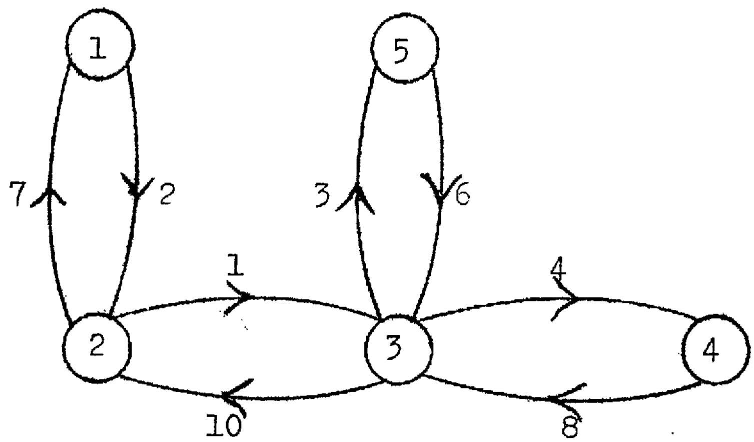


Fig.(2.2). A realization of T in (2.5).

The synthesis procedure of a tree terminal matrix of order n is summarized in the following steps:

(1) Partition T by the semiprincipal partitioning process.

(2) Find out a connective element for each C-submatrix of T. Let $t_{i_r j_r}$ be a connective element of C_r, $r = 1,2,\ldots,n-1$.

(3) A double tree net of T is determined by the edge capacity matrix, $C = [c_{ij}]$, where

$$c_{ii} = \text{node symbol } ⓘ, \quad i = 1,2,\ldots,n,$$

$$c_{i_r j_r} = t_{i_r j_r},$$

$$c_{j_r i_r} = t_{j_r i_r}, \qquad r = 1,2,\ldots,n-1,$$

and every other element is zero.

For example, a realization for the matrix given in (2.5) is shown in Fig. (2.2).

<u>Corollary (2.1)</u>. "A completely partitionable matrix is a tree terminal matrix."

The proof follows from Theorems (2.2) and (2.3). Thus Mayeda's result is a special case of ours.

<u>Corollary (2.2)</u>. 'The number of distinct elements in a tree terminal matrix is not morethan 2(n-1)."

The proof follows from Theorem (2.3) and the fact that the number of edges in a double tree net of n nodes is not more than 2(n-1).

Corollary (2.3). "A matrix of real non-negative numbers given by

$$T = \begin{bmatrix} \textcircled{1} & t_1 & t_1 \cdots\cdots & t_1 & t_1 \\ t_2 & \textcircled{2} & t_2 \cdots\cdots & t_2 & t_2 \\ & & \vdots & & \\ t_{n-1} & t_{n-1} & t_{n-1} \cdots & \textcircled{n-1} & t_{n-1} \\ t_n & t_{n+1} & t_{n+2} \cdots & t_{2(n-1)} & \textcircled{n} \end{bmatrix},$$

where $t_1 \leq t_2 \leq \cdots\cdots \leq t_{n-1} \leq \min \left\{ t_n, t_{n+1}, \ldots, t_{2(n-1)} \right\}$,

is realizable as a terminal matrix of a double star net, that is, double tree net which is star-shaped."

Proof. The C-submatrix, C_r, obtained in the r^{th} $(r \leq n-1)$ semiprincipal partitioning step is given by

$$C_r = \begin{array}{c} \\ r+1 \\ r+2 \\ \vdots \\ n-1 \\ n \end{array} \begin{array}{c} r \\ \begin{bmatrix} t_{r+1} \\ t_{r+2} \\ \vdots \\ t_{n-1} \\ t_{n+r-1} \end{bmatrix} \end{array} .$$

The element t_{nr} (= t_{n+r-1}) is a maximal element of C_r, and it is a connective element of C_r because for each $\ell(n > \ell > r)$,

$$\min \left\{ t_{\ell n}, t_{nr}, t_{rr} \right\} = t_{\ell n} = t_{\ell r} .$$

Thus by Theorem (2.3), T is realizable and the edge capacity matrix of its realization is given by $C = [c_{ij}]$, where

$$c_{ii} = \text{node symbol } (i) , \quad i = 1,\ldots,n,$$

$$c_{ij} = \begin{cases} t_{ij} , & \text{for } i = n, \quad j = 1,2,\ldots,n-1; \\ & \quad i = 1,2,\ldots,n-1, \quad j = n, \\ 0 , & \text{otherwise.} \end{cases}$$

Hence the corollary.

Since each C-submatrix of a tree terminal matrix may have many connective elements, then there may be many double tree net realizations for a given tree terminal matrix. The number of these nets is determined by the following corollary:

<u>Corollary (2.4)</u>. "If a tree terminal matrix T of order n, has a unique semiprincipal partitioning form, then it has $\prod_{r=1}^{n-1} m_r$ distinct double tree net realizations, where m_r is the number of connective elements of the r^{th} C-submatrix."

2.3 Double-Loop Nets.

Definition (2.7). A "double loop net" is a connected oriented communication net which becomes a single loop if each set of parallel edges (not necessarily of the same orientation) is replaced by one edge.

This section will be divided into two subsections.

2.3.1. Analysis of the Terminal matrix of a Double Loop Net.

Consider a double loop net, N, of n nodes labelled by 1,2,...,n, as shown in Fig. (2.3), where

$$\begin{aligned} &c_{1n} \leq c_{i+1,i} \,, \\ &c_{m,m+1} \leq c_{n1} \,, \quad c_{i,i+1} \,, \end{aligned} \tag{2.6}$$

for all i = 1,2,...,n-1, in which all edge capacities are real non-negative numbers. Let S_a be the set of nodes 1,2,...,m and S_b the set of nodes m+1, m+2,...,n. Finally, let N_a and N_b be the double linear tree nets obtained from N by removing edges e_{n1}, e_{1n}, $e_{m,m+1}$, $e_{m+1,m}$. The terminal matrices of N_a and N_b will be denoted by $T^{(a)} = [t_{ij}^{(a)}]$, (i,j = 1,2,...,m) and $T^{(b)} = [t_{ij}^{(b)}]$, (i,j = m+1, ...,n), respectively.

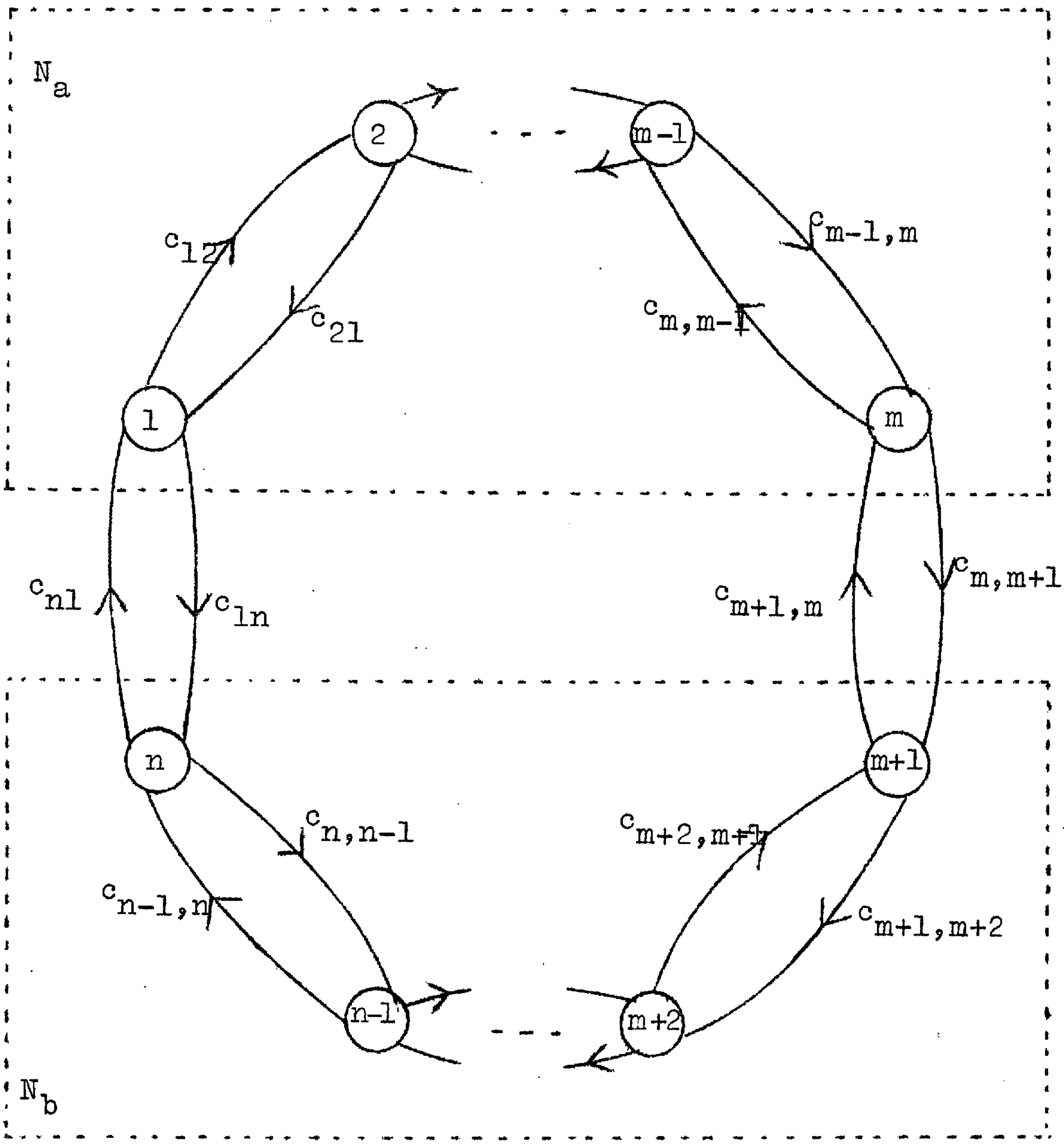


Fig.(2.3). A double loop net, N, of n nodes.

It is clear that in the double loop net N there are exactly two directed paths, which are edge-disjoint, leading from any node to another. Moreover, (1) one of the directed paths from a node in S_a to a node in S_b contains edge e_{1n} and the other contains edge $e_{m,m+1}$, (2) one of the directed paths (i,j), where $i,j \in S_a$, contains e_{1n} ($e_{m,m+1}$) if $i < j$ ($i > j$) and the other contains neither e_{1n} nor $e_{m,m+1}$, (3) one of the directed paths (i,j), where $i,j \in S_b$, contains edge e_{1n}($e_{m,m+1}$) if $i < j$ ($i > j$) and the other contains neither e_{1n} nor $e_{m,m+1}$, and (4) each of the directed paths from a node in S_b to a node in S_a contains neither e_{1n} nor $e_{m,m+1}$. Thus using (2.6) we get

$$t_{ij} = \begin{cases} c_{1n} + c_{m,m+1}, \ (\equiv t_1, \text{ say}), & \text{if } i \in S_a \text{ and } j \in S_b, \\ c_{1n} + t_{ij}^{(a)}, & \text{if } i,j \in S_a \text{ and } i < j, \\ c_{m,m+1} + t_{ij}^{(a)}, & \text{if } i,j \in S_a \text{ and } i > j, \\ c_{1n} + t_{ij}^{(b)}, & \text{if } i,j \in S_b \text{ and } i < j, \\ c_{m,m+1} + t_{ij}^{(b)}, & \text{if } i,j \in S_b \text{ and } i > j. \end{cases} \qquad (2.7)$$

The following lemma follows directly from the fact that $T^{(a)}$ and $T^{(b)}$ are completely partitionable (Lemma (2.1)):

Lemma (2.3). "The terminal matrix T of a double loop net N is partitionable (without rearranging rows and columns) into

$$T = \begin{bmatrix} \overbrace{A}^{m} & \overbrace{T_1}^{n-m} \\ C & B \end{bmatrix} \begin{matrix} \}m \\ \}n-m \end{matrix}$$

such that T_1 is a uniform matrix with element value t_1 minimal for T, and A and B are completely partitionable matrices."

From now on we shall write, for shortness,

$$\min \{t_{m+1,n}, t_{1m}\} = f ,$$

$$\min \{t_{n,m+1}, t_{m1}\} = h.$$

Lemma (2.4). "For the double loop net N,

$$\begin{aligned} &\min \{t_{i,m+1}, t_{mj}\} \geq h, \\ &\min \{t_{in}, t_{1j}\} \geq f, \end{aligned} \qquad (2.8)$$

for all $i \in S_b$ and $j \in S_a$.

Proof. From Fig. (2.3), one can easily see that for $i = m+1,\ldots,n$ and $j = 1,\ldots,m$,

$$t^{(b)}_{i,m+1} \geq t^{(b)}_{n,m+1} ,$$

$$t^{(a)}_{mj} \geq t^{(a)}_{m1} ,$$

$$t^{(b)}_{in} \geq t^{(b)}_{m+1,n} ,$$

and

$$t^{(a)}_{1j} \geq t^{(a)}_{1m} .$$

Thus

$$\min \left\{ t^{(b)}_{i,m+1}, t^{(a)}_{mj} \right\} \geq \min \left\{ t^{(b)}_{n,m+1}, t^{(a)}_{m1} \right\} ,$$

$$\min \left\{ t^{(b)}_{in}, t^{(a)}_{1j} \right\} \geq \min \left\{ t^{(b)}_{m+1,n}, t^{(a)}_{1m} \right\} .$$

Hence (2.8) follows by direct substitution from (2.7).

Now, turning to the case when node $i \in S_b$ and $j \in S_a$, we can easily see that:

$$t_{ij} = \min \left\{ t^{(b)}_{i,m+1}, t^{(a)}_{mj}, c_{m+1,m} \right\} + \min \left\{ t^{(b)}_{in}, t^{(a)}_{1j}, c_{n1} \right\} ,$$

which becomes, by using (2.7),

$$t_{ij} = \min \left\{ t_{i,m+1}, t_{mj}, (c_{m+1,m} + c_{m,m+1}) \right\}$$
$$+ \min \left\{ t_{in}, t_{1j}, (c_{n1} + c_{1n}) \right\} - t_1, \qquad (2.9)$$

where $t_1 = c_{m,m+1} + c_{1n}$.

Consequently,

$$t_{n1} = \min \left\{ h, (c_{m+1,m} + c_{m,m+1}) \right\} + c_{n1} - c_{m,m+1},$$

$$t_{m+1,m} = \min \left\{ f, (c_{n1} + c_{1n}) \right\} + c_{m+1,m} - c_{1n} . \qquad (2.10)$$

Comparing $(c_{m+1,m} + c_{m,m+1})$ and h, and $(c_{n1} + c_{1n})$ and f, we have the following four cases:

Case (A):*

$$c_{m+1,m} + c_{m,m+1} > h$$

and (2.11)

$$c_{n1} + c_{1n} > f.$$

Then from (2.10) we obtain

$$c_{n1} = t_{n1} + c_{m,m+1} - h \ ,$$

(2.12)

$$c_{m+1,m} = t_{m+1,m} + c_{1n} - f \ .$$

Substituting (2.12) in (2.11) and (2.9) we get

$$t_{m+1,m},\ t_{n1} > f + h - t_1 \qquad (2.13)$$

and

$$t_{ij} = \min\left\{t_{i,m+1},\ t_{mj},\ (t_{m+1,m} + t_1 - f)\right\} + \min\left\{t_{in},\ t_{1j},\ (t_{n1} + t_1 - h)\right\} - t_1, \qquad (2.14)$$

for all $i \in S_b$ and $j \in S_a$.

Case (B):**

$$c_{m+1,m} + c_{m,m+1} > h,$$

(2.15)

$$c_{n1} + c_{1n} \leq f \ .$$

---

*This case does not occur if $c_{n1} = c_{m,m+1}$ and/or $c_{m+1,m} = c_{1n}$.

**This case does not occur if $c_{m+1,m} = c_{1n}$.

Then from (2.10) we obtain

$$c_{n1} = t_{n1} + c_{m,m+1} - h ,$$
$$c_{m+1,m} = t_{m+1,m} - t_{n1} - c_{m,m+1} + h. \tag{2.16}$$

From (2.6) we have

$$c_{n1} \geq c_{m,m+1} . \tag{2.17}$$

Substituting (2.16) in (2.17), (2.15) and (2.9) and using Lemma (2.4) in the latter, we get

$$t_{n1} \geq h,$$
$$t_{m+1,m} > t_{n1} \leq f + h - t_1 , \tag{2.18}$$

and

$$t_{ij} = \min \left\{ t_{i,m+1}, t_{mj}, (t_{m+1,m} - t_{n1} + h) \right\} + t_{n1} - h , \tag{2.19}$$

for all $i \in S_b$ and $j \in S_a$.

<u>Case (C)</u>.***

$$c_{m+1,m} + c_{m,m+1} \leq h,$$
$$c_{n1} + c_{1n} > f . \tag{2.20}$$

***This case does not occur if $c_{n1} = c_{m,m+1}$.

Then from (2.10) we obtain

$$c_{n1} = t_{n1} - t_{m+1,m} - c_{1n} + f ,$$
$$c_{m+1,m} = t_{m+1,m} + c_{1n} - f . \tag{2.21}$$

From (2.6) we have

$$c_{m+1,m} \geq c_{1n} . \tag{2.22}$$

Substituting (2.21) in (2.22), (2.20) and (2.9) and using Lemma (2.4) in the latter we get

$$t_{m+1,m} \geq f ,$$
$$t_{n1} > t_{m+1,m} \leq f + h - t_1 , \tag{2.23}$$

and

$$t_{ij} = \min \left\{ t_{in}, t_{1j}, (t_{n1} - t_{m+1,m} + f) \right\}$$
$$+ t_{m+1,m} - f , \tag{2.24}$$

for all $i \in S_b$ and $j \in S_a$.

<u>Case (D)</u>:

$$c_{m+1,m} + c_{m,m+1} \leq h,$$
$$c_{n1} + c_{1n} \leq f . \tag{2.25}$$

Then from (2.9) and using Lemma (2.4), we obtain

$$t_{ij} = c_{n1} + c_{m+1,m} \quad (= t_2, \text{ say}), \tag{2.26}$$

for all $i \in S_b$ and $j \in S_a$.

Thus constraints (2.25) make submatrix C uniform. Adding the inequalities in (2.25) and using (2.6) we get

$$t_1 \leq t_2 \leq f + h - t_1 \tag{2.27}$$

Before summing up the previous results, we shall define a loop-terminal matrix.

Definition (2.8). A square matrix T of order n is called a "loop-terminal matrix" if it can be partitioned, by rearranging its rows and columns (if necessary), into

$$T = \overset{\quad m \qquad n-m}{\left[\begin{array}{c:c} A & T_1 \\ \hdashline C & B \end{array}\right]} \begin{array}{l} \}\, m \\ \}\, n-m \end{array} \tag{2.28}$$

such that

1) T_1 is a uniform matrix with an element value t_1 minimal for T,

2) A and B are completely partitionable (without further rearrangements of rows and columns), and

3) one of the following cases holds:

(A) inequalities (2.13) and equalities (2.14),

(B) " (2.18) " " (2.19),

(C) " (2.23) " " (2.24),

or (D) " (2.27) " " (2.26).

Summing up the results of this subsection we arrive at the fact that the terminal matrix of a double loop net is a loop-terminal matrix.

2.3.2. Synthesis of a loop-terminal Matrix.

In this subsection, we shall prove the second part of the following theorem

Theorem (2.4). "A matrix T is realizable as the terminal matrix of a double loop net if, and only if, T is a loop terminal matrix."

Proof. It has been shown in §2.3.1 that the terminal matrix of a double loop net is a loop terminal matrix. The second part will be proved here.

Let $T = [t_{ij}]$ be a loop-terminal matrix partitioned as in (2.28). Assume that x and y be any non-negative real numbers such that

$$x + y = t_1 \tag{2.29}$$

Moreover, let $T^{(a)} = [t_{ij}^{(a)}]$ and $T^{(b)} = [t_{ij}^{(b)}]$ be A and B, respectively, with x subtracted from each element above the diagonal and y subtracted from each element below the diagonal; that is

$$t_{ij}^{(a)} = \begin{cases} t_{ij} - x, & \text{if } i < j, \\ t_{ij} - y, & \text{if } i > j, \text{ where } i,j = 1,2, \\ & \ldots, m,\ i \neq j, \end{cases}$$

and

$$t_{ij}^{(b)} = \begin{cases} t_{ij} - x\,, & \text{if } i < j, \\ t_{ij} - y\,, & \text{if } i > j, \quad \text{where } i,j = m+1, \\ & \quad \ldots, n,\ i \neq j. \end{cases} \tag{2.30}$$

The elements of $T^{(a)}$ and $T^{(b)}$ are non-negative real numbers. Since A and B are completely partitionable then $T^{(a)}$ and $T^{(b)}$ are completely partitionable without rearranging the rows and columns. Thus $T^{(a)}$ and $T^{(b)}$ are realizable as terminal matrices of double linear tree nets with edge capacities given by

$$\begin{aligned} c_{i,i+1} &= t_{i,i+1} - x\,, \\ c_{i+1,i} &= t_{i+1,i} - y\,, \end{aligned} \tag{2.31}$$

where $i = 1,2,\ldots,m-1$ for $T^{(a)}$ and $i = m+1,\ldots,n-1$ for $T^{(b)}$.

It is clear that the order of the nodes of the double linear tree nets of $T^{(a)}$ and $T^{(b)}$ will be as in A and B; that is, $1,2,\ldots,m$, and $m+1,\ldots,n$, respectively.

Connect these two nets by adding edges $e_{1n}, e_{m,m+1}$, e_{n1} and $e_{m+1,m}$. The edge capacities of e_{1n} and $e_{m,m+1}$ are x and y, respectively; the capacities of e_{n1} and $e_{m+1,m}$ will be determined in each of the four cases corresponding to Definition (2.8). Let this double loop net be denoted by N_k. We shall prove for each case that the terminal matrix, say $K = [k_{ij}]$, of N_k is identical with the given loop-terminal matrix T.

Before considering each case we observe from (2.29) and (2.31), that

$$\begin{aligned} c_{i,i+1} &\geq c_{m,m+1} , \\ c_{i+1,i} &\geq c_{1n} , \end{aligned} \tag{2.32}$$

for $i = 1,2,\ldots,m-1,m+1,\ldots,n-1$.

Case (A): If T satisfies (2.13) and (2.14) then let the values of c_{n1} and $c_{m+1,m}$ in N_k be

$$\begin{aligned} c_{n1} &= t_{n1} - h + c_{m,m+1} , \\ c_{m+1,m} &= t_{m+1,m} - f + c_{1n} . \end{aligned} \tag{2.33}$$

Substituting (2.33) in (2.13) we obtain

$$\begin{aligned} c_{n1} &> c_{m,m+1} , \\ c_{m+1,m} &> c_{1n} . \end{aligned} \tag{2.34}$$

Thus using (2.32) and (2.34) and analysing N_k as in §2.3.1, we get equation (2.7) with k_{ij} in the left hand side instead of t_{ij}; which gives by substituting $t_{ij}^{(a)}$ and $t_{ij}^{(b)}$ from (2.30):

$$k_{ij} = t_{ij}$$

for all i and j except $i = m+1,\ldots,n$ and $j = 1,2,\ldots,m$.

From (2.33) and using (2.13) we obtain

$$c_{m+1,m} + c_{m,m+1} > h \ (= \min \{k_{n,m+1}, k_{m1}\}) ,$$

$$c_{n1} + c_{1n} > f \ (= \min \{k_{m+1,n}, k_{1m}\}).$$

Thus continuing the analysis of N_k as in Case (A) of §2.3.1, and using (2.14) we arrive at

$$k_{ij} = t_{ij}$$

for all $i \in S_b$ and $j \in S_a$.

Hence $K = T$.

<u>Case (B)</u>: If T satisfies (2.18) and (2.19), then let

$$\begin{aligned} c_{n1} &= t_{n1} - h + c_{m,m+1} , \\ c_{m+1,m} &= t_{m+1,m} - t_{n1} + h - c_{m,m+1} . \end{aligned} \qquad (2.35)$$

Thus substituting (2.35) in (2.18), we obtain

$$c_{n1} \geq c_{m,m+1} ,$$

$$c_{m+1,m} > c_{1n} ,$$

and

$$c_{m+1,m} + c_{m,m+1} > h ,$$

$$c_{1n} + c_{n1} \leq f .$$

Hence following the analysis of Case (B) we get, as in the synthesis of Case (A), $K = T$.

<u>Case (C)</u>: If T satisfies (2.23) and (2.24), then let

$$\begin{aligned} c_{n1} &= t_{n1} - t_{m+1,m} - c_{1n} + f , \\ c_{m+1,m} &= t_{m+1,m} + c_{1n} - f . \end{aligned} \qquad (2.36)$$

Substituting (2.36) in (2.23) we obtain

$$c_{n1} > c_{m,m+1} ,$$

$$c_{m+1,m} \geq c_{1n} ,$$

and

$$c_{m+1,m} + c_{m,m+1} \leq h \ ,$$

$$c_{n1} + c_{1n} > f \ .$$

Hence following the analysis of Case (C) we get, as in the synthesis of Case (A), $K = T$.

Case (D): If submatrix C of T is a uniform of element value t_2 and

$$t_1 \leq t_2 \leq f + h - t_1 \ , \tag{2.27}$$

then there are always two positive real numbers α and β such that

$$\alpha + \beta = t_2 \ ,$$

$$c_{m,m+1} \leq \alpha \leq f - c_{1n} \ , \tag{2.37}$$

$$c_{1n} \leq \beta \leq h - c_{m,m+1} .$$

This can be proved by plotting (2.37) (see Fig.2.4) and noticing that the point $(c_{m,m+1}, c_{1n})$ is always inside the region bounded by $\alpha = 0$, $\beta = 0$ and $\alpha + \beta = t_2$ (or on the boundary $\alpha + \beta = t_2$) because $c_{m,m+1} + c_{1n} = t_1 \leq t_2$, and the point $(f - c_{1n},\ h - c_{m,m+1})$ is always outside the same region (or on the boundary) because $f + h - (c_{1n} + c_{m,m+1}) \geq t_2$. Thus any point on the line **ab** in Fig.2.4 satisfies (2.37). Let (α_o, β_o) be any such point.

Now, let the values of c_{n1} and $c_{m+1,m}$ be given by

$$c_{n1} = \alpha_o \ ,$$

$$c_{m+1,m} = \beta_o \ .$$

Then using (2.37) we obtain

$$c_{n1} \geq c_{m,m+1}$$

$$c_{m+1,m} \geq c_{1n} \ ,$$

and

$$c_{m+1,m} + c_{m,m+1} \leq h \ ,$$

$$c_{n1} + c_{1n} \leq f \ .$$

Thus following the analysis of Case (D) we get, as in the synthesis of the previous Cases, K = T .

Hence the theorem.

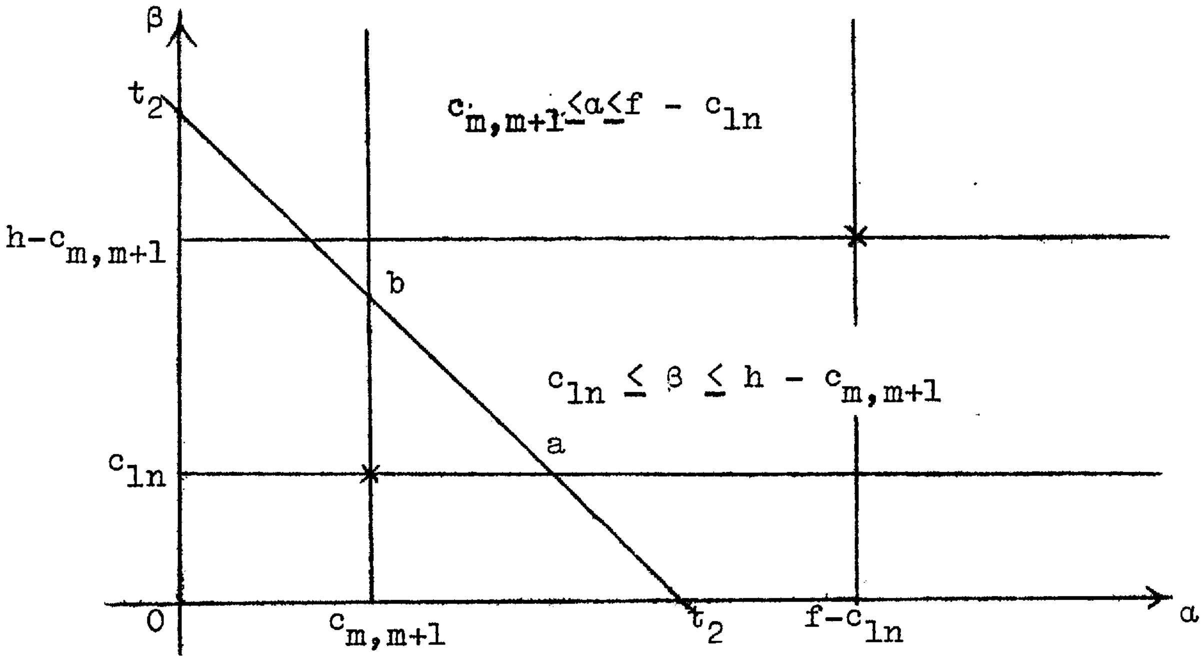


Fig.(2.4). Plots of relations (2.37).

It should be noticed that a loop terminal matrix satisfying Case (D) is not necessarily completely partitionable. This can be seen by considering the loop terminal matrix given in (2.39), *page 60*.

Corollary (2.5). "If t_M is a maximal element of the submatrix C of a loop terminal matrix T, then

$$t_M \begin{cases} \leq t_{m+1,m} + t_{n1} + t_1 - f - h \text{ , if T satisfies (2.14)} \\ = t_{m+1,m} \text{ ,} \qquad \text{if T satisfies (2.19),} \\ = t_{n1} \text{ ,} \qquad \text{if T satisfies (2.24).} \end{cases}$$

The proof is obvious.

The following corollary follows directly from Corollary 2.5 and Definition 2.8:

Corollary (2.6). "Suppose T has a unique partitioning,

$$T = \left[\begin{array}{c|c} A & T_1 \\ \hline C & B \end{array}\right] ,$$

where T_1 is a uniform of element value t_1 minimal for T and A and B are completely partitionable without rearranging rows and columns. If t_M is a maximal element of C and

$$t_M > \max\left\{(t_{m+1,m} + t_{n1} + t_1 - f - h),\ t_{m+1,m}, t_{n1}\right\} ,$$

then T is not a loop terminal matrix."

The synthesis procedure indicated in the proof of Theorem 2.4, of a loop-terminal matrix, will be illustrated in the following two examples.

Example 2.2. Consider the matrix

$$\left[\begin{array}{cc:ccc} \textcircled{1} & 4 & 3 & 3 & 3 \\ 9 & \textcircled{2} & 3 & 3 & 3 \\ \hdashline 7 & 6 & \textcircled{3} & 5 & 5 \\ 8 & 6 & 8 & \textcircled{4} & 6 \\ 9 & 6 & 7 & 7 & \textcircled{5} \end{array}\right] . \qquad (2.38)$$

The resultant main submatrices A and B are completely partitionable, but the C-submatrix C is not uniform, so we exclude Case (D). Since $m = 2$, $n = 5$, then

$$f = \min\{t_{12}, t_{35}\} = 4 ,$$

$$h = \min\{t_{21}, t_{53}\} = 7 .$$

It is easy to see that inequalities (2.13) and (2.18) do not hold, but the inequalities in (2.23) do hold. Thus considering (2.24) we find that for $i = 3,4,5$ and $j = 1,2$, this equality holds in T. Hence T is a loop-terminal matrix satisfying Case (C). Therefore the edge capacities c_{32} and c_{51} are given by

$$c_{32} = 6 + x - 4 = 2 + x,$$

$$c_{51} = 9 - 6 - x + 4 = 7 - x,$$

where $x = c_{15}$ and $c_{23} = y = 3 - x$, in which $0 \leq x \leq 3$. The other edge capacities are given by

$$c_{12} = 4 - x\ , \qquad c_{21} = 6 + x,$$
$$c_{34} = 5 - x\ , \qquad c_{43} = 5 + x,$$
$$c_{45} = 6 - x\ , \qquad c_{54} = 4 + x.$$

The double loop net is shown in Fig. 2.5.

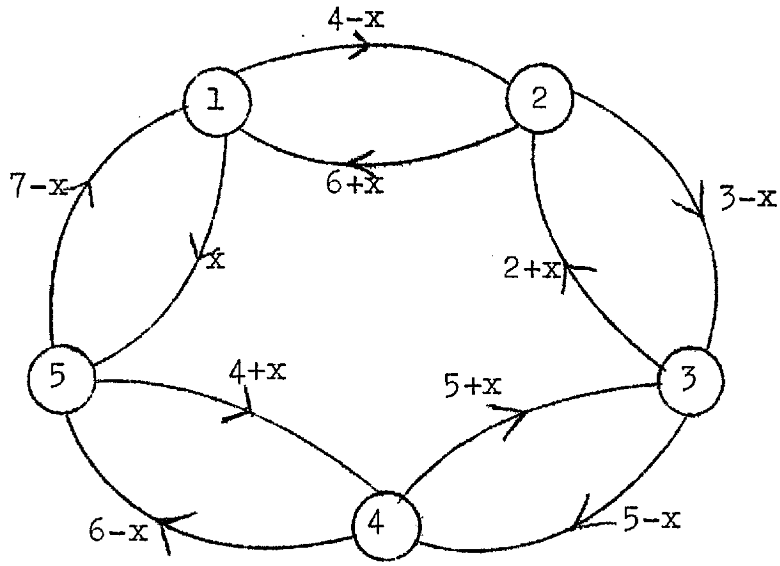


Fig.(2.5). A realization of T in (2.38), where $0 \leq x \leq 3$.

<u>Example 2.3</u>. Consider the matrix

$$T = \left[\begin{array}{cc|cc} ① & 5 & 2 & 2 \\ 6 & ② & 2 & 2 \\ \hline 8 & 8 & ③ & 7 \\ 8 & 8 & 9 & ④ \end{array}\right] \qquad (2.39)$$

As in Example 2.2,

$$f = \min \{t_{12}, t_{34}\} = 5$$
$$h = \min \{t_{21}, t_{43}\} = 6\ ,$$

$t_1 = 2$ and $t_2 = 8$.

Therefore, inequalities (2.27) hold and T is a loop-terminal matrix satisfying Case (D). Let $c_{14} = x$, $0 \leq x \leq 2$, then

$$c_{23} = 2 - x,$$

$$c_{12} = 5 - x, \qquad c_{21} = 4 + x,$$

$$c_{34} = 7 - x, \qquad c_{43} = 7 + x.$$

To determine c_{41} and c_{32}, let $c_{41} = \alpha$, then $c_{32} = 8 - \alpha$, where

$$\begin{aligned} 2 - x \leq \alpha \leq 5 - x, \\ x \leq 8 - \alpha \leq 4 + x. \end{aligned} \qquad (2.40)$$

One can easily combine the inequalities in (2.40) and get

$$4 - x \leq \alpha \leq 5 - x \ . \qquad (2.41)$$

The double loop net of (2.39) is shown in Fig.2.6.

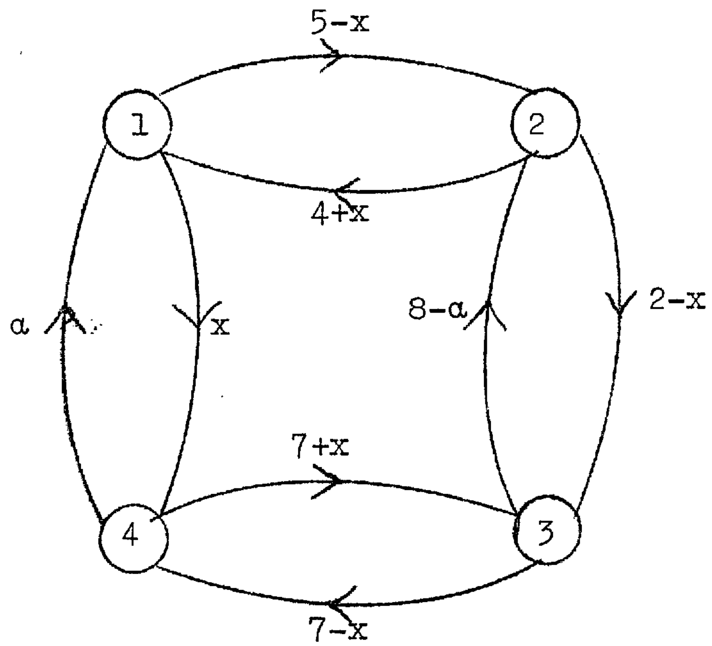


Fig.(2.6). A realization of T in (2.39), where $0 \leq x \leq 2$ and $4 - x \leq \alpha \leq 5 - x$.

2.4 Separable Oriented Nets.

Definition (2.9). A net N, oriented or non-oriented, is "nonseparable"[15] if every subnet of N, consisting of more than one node, has at least two nodes in common with its complement. All other nets are "separable".

Definition (2.10). A matrix T is called a "separable-terminal matrix" if it can be partitioned into

$$T = \begin{bmatrix} A_1 & C_1 & B_1 \\ R_1 & \textcircled{v_c} & R_2 \\ B_2 & C_2 & A_2 \end{bmatrix} , \qquad (2.42)$$

such that

$$T_1 = \begin{bmatrix} A_1 & C_1 \\ R_1 & \textcircled{v_c} \end{bmatrix} \quad \text{and } T_2 = \begin{bmatrix} \textcircled{v_c} & R_2 \\ C_2 & A_2 \end{bmatrix} .$$

are realizable as terminal matrices, and every element of B_1 and B_2, say t_{ij}, satisfies

$$t_{ij} = \min \left\{ t_{iv_c}, \ t_{v_cj} \right\} . \qquad (2.43)$$

Theorem (2.5). "A matrix T is realizable as the terminal matrix of a separable oriented net if and only if it is a separable terminal matrix."

Proof. Let N be a separable oriented net, and N_1 and N_2 be its components (not necessarily non-separable) at the cut-node* v_c; and also let T be the terminal matrix of N. By rearranging the rows and columns (if necessary), T can be partitioned as in (2.42). Since N is separable, then every directed path between two nodes of $N_1(N_2)$ does not contain any edge of $N_2(N_1)$. Thus T_1 and T_2 are the terminal matrices of N_1 and N_2, respectively. Moreover, any directed path (i, j), where i in N_1 and j in N_2, consists of a directed path (i, v_c) in N_1 and a directed path (v_c, j) in N_2. Thus every element of B_1 and B_2, say t_{ij}, must satisfy (2.43). Therefore T is a separable terminal matrix.

Now, let T be any separable terminal matrix partitioned as in (2.42), and let N_1 and N_2 be the realizations of T_1 and T_2, respectively. Form the net N' by identifying node v_c of N_1 with node v_c of N_2. If t'_{ij} is the terminal capacity of N' from node i in N_1 to node j in N_2, then (as in the proof of the first part)

$$\begin{aligned} t'_{ij} &= \min \left\{ t'_{iv_c},\ t'_{v_c j} \right\} \\ &= \min \left\{ t_{iv_c},\ t_{v_c j} \right\} \\ &= t_{ij}\ , \text{ by Definition (2.10).} \end{aligned}$$

---

*A "cut-node"[11] is a single node which is common to a subgraph and its complement.

Thus N' is a realization of T.

Hence the theorem.

Corollary (2.7). "A tree-terminal matrix of order n, $n \geq 3$, is a separable terminal matrix."

The proof follows from Theorems (2.3) and (2.5).

The following theorem is given by Tang and Chien[13]:

Theorem (2.6). "A matrix T is realizable if it can be partitioned, by rearranging the nodes in such a fashion that the following conditions are satisfied:

(1) Each submatrix corresponding to a sub-collection of nodes lying along the diagonal line is square and contains elements with values no smaller than the value of any of the elements in an off-diagonal submatrix.

(2) Each off-diagonal submatrix is a uniform matrix.

(3) Each submatrix along the diagonal line is realizable.

(4) Treating these submatrices along the diagonal line as nodes, the matrix T is realizable."

A realization net, N, of matrix T which satisfies the conditions of Theorem (2.6) is a separable net containing at least m-1 cut-nodes, where m is the number of submatrices along the diagonal line. Thus, by Theorem 2.5 we arrive at

<u>Corollary (2.8)</u>. "Every matrix T satisfying the conditions of Theorem (2.6) is a separable matrix."

Excluding the trivial case when every submatrix along the diagonal line consists of a single node, the converse of Corollary (2.8) is not necessarily true; that is, not every separable matrix satisfies the conditions of Theorem (2.6). This follows from the following example.

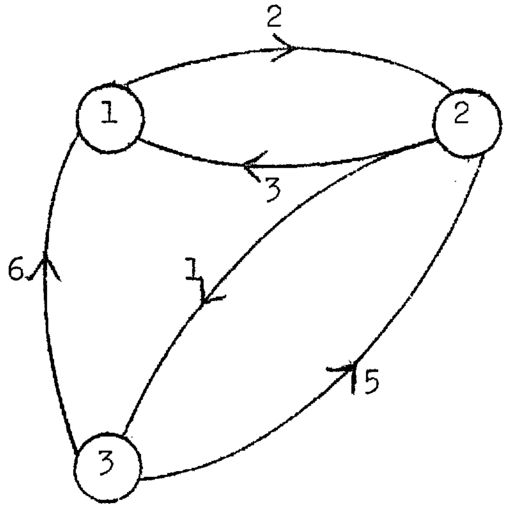
<u>Example (2.4)</u>. Consider the matrix T given in (2.44) which is partitioned corresponding to Definition (2.10), with node 3 as the cut-node, v_c, and submatrices T_1 and T_2 are given in (2.45).

$$T = \left[\begin{array}{cc|c|c} ① & 2 & 1 & 1 \\ 4 & ② & 1 & 1 \\ \hline 9 & 7 & ③ & 4 \\ \hline 8 & 7 & 8 & ④ \end{array}\right] \tag{2.44}$$

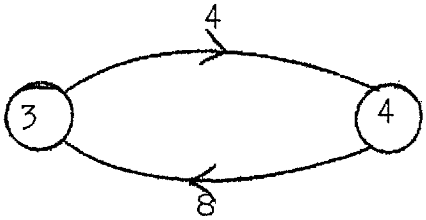
$$T_1 = \begin{bmatrix} ① & 2 & 1 \\ 4 & ② & 1 \\ 9 & 7 & ③ \end{bmatrix}, \qquad T_2 = \begin{bmatrix} ③ & 4 \\ 8 & ④ \end{bmatrix} \tag{2.45}$$

One can easily see that T is separable matrix; its realization is shown in Fig.2.7

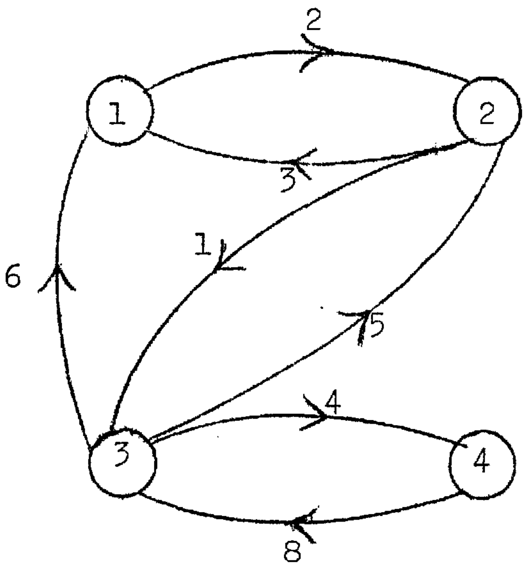
It can be easily seen that there is no partitioning of the matrix in (2.44), except the trivial one, that satisfies condition (1) of Theorem (2.6). Hence our Theorem (2.5) is more general than Theorem (2.6).



N_1: A realization of T_1.



N_2: The realization of T_2.



N: A realization of T in (2.44).

Fig.(2.7).

2.5 A necessary and sufficient Condition for the Realizability of a Triangular Terminal Matrix.

Definition (2.11). "A descending net", N, is an oriented communication net of n nodes labelled 1,2,...,n and edges e_{ij} $(i \neq j)$ of capacities c_{ij} such that

$$c_{ij} = 0 \text{ , if } i < j,$$

$$c_{ij} \geq 0 \text{ , if } i > j.$$

Lemma (2.5). "The terminal matrix, $T = [t_{ij}]$, of a descending net, N, is a lower or upper triangular matrix if the node order along the diagonal line is 1,2,...,n, or n,n-1,...,1, respectively."

Proof. Since there is no directed path (i,j) if $i < j$ in N, then there is no flow from i to j; that is, $t_{ij} = 0$.

Hence the lemma.

It is always possible to transform an upper triangular terminal matrix to a lower triangular terminal matrix by rearranging the rows and columns such that the node order along the diagonal line is reversed. Thus all the following results are applicable to upper triangular terminal matrices as well, after appropriate transformations.

Definition (2.12). An "h-subnet", $N^{(h)}$, of a descending net N is the net obtained from N by deleting every edge e_{ij}, $i - j > h$, where h is a positive integer not more than n-1.

Lemma (2.6). "Let $N^{(h)}$, where h is a positive integer $\leq$ n-1, be an h-subnet of a descending net N, then

$$t_{ij} = t_{ij}^{(h)}, \quad i > j,$$

if

$$i - j \leq h,$$

where $t_{ij}^{(h)}$ is the terminal capacity from i to j in $N^{(h)}$."

Proof. Let $e_{\ell k}$ be any edge of N which is not in $N^{(h)}$, that is, $\ell - k > h$, then

$$\ell - k > i - j.$$

Therefore, if $\ell \leq i$, then $k < j$. Thus $e_{\ell k}$ has one or both of its nodes in the subset of nodes $\{1, 2, \ldots, j-1, i+1, \ldots, n\}$. Therefore, by Definition (2.11), $e_{\ell k}$ is not contained in any directed path (i,j) in N; that is the removal of $e_{\ell k}$ does not change the terminal capacity (i,j).

Hence the lemma.

Let $C = [c_{ij}]$ be the edge capacity matrix of a descending net N whose terminal matrix is $T = [t_{ij}]$, then one can easily see that

$$c_{j+1,j} = t_{j+1,j}, \quad j = 1, 2, \ldots, n-1,$$

and (2.46)

$$c_{j+2,j} = t_{j+2,j} - \min\{t_{j+2,j+1}, t_{j+1,j}\},$$

for all $j = 1, 2, \ldots, n-2$.

<u>Theorem (2.7)</u>. "A lower triangular matrix, $T = [t_{ij}]$, is realizable as a terminal matrix of an oriented communication net, N, if and only if,

$$t_{ij} \geq t_{ij}^{(i-j-1)} , \qquad (2.47)$$

for all i and j such that $i > j + 1$, where $t_{ij}^{(i-j-1)}$ is the terminal capacity (i,j) in the (i-j-1)-subnet, $N^{(i-j-1)}$, of N."

<u>Proof</u>. It must be noticed that any realization of a lower triangular terminal matrix is a descending net. Thus N is a descending net. Since $N^{(i-j-1)}$ is a subnet of N, then condition (2.47) is necessary for the realizability of T. Now, suppose T satisfies (2.47) and assume that we have found $N^{(r)}$ for some r, $1 \leq r \leq n-2$, then we can construct $N^{(r+1)}$ by adding to $N^{(r)}$ all edges e_{ij} such that $i - j = r + 1$, whose capacities are given by

$$c_{ij} = t_{ij} - t_{ij}^{(r)} ,$$

$$\geq 0 , \qquad \text{by} \quad (2.47)$$

But using (2.46), we can easily find $N^{(2)}$. Thus we can obtain $N^{(3)}$, $N^{(4)}$, ..., $N^{(n-1)}$, successively. By Lemma (2.6), $N^{(n-1)}$ is the realization net N of T.

Hence the theorem.

The procedure of the synthesis of a lower triangular matrix T will be illustrated in the following example.

Example (2.5). Consider the following lower triangular matrix,

$$T = \begin{bmatrix} \textcircled{1} & & & & & \\ 3 & \textcircled{2} & & & & \\ 6 & 6 & \textcircled{3} & & & \\ 15 & 8 & 4 & \textcircled{4} & & \\ 14 & 12 & 4 & 2 & \textcircled{5} & \\ 17 & 11 & 8 & 6 & 1 & \textcircled{6} \end{bmatrix} \qquad (2.48)$$

By (2.46), we get

$$c_{21} = 3, \quad c_{32} = 6, \quad c_{43} = 4, \quad c_{54} = 2, \quad c_{65} = 1,$$

$$c_{31} = 3, \quad c_{42} = 4, \quad c_{53} = 2, \quad c_{64} = 5 \ .$$

The 2-subnet, $N^{(2)}$, is shown in Fig.(2.8a), from which we find $t_{41}^{(2)} = 6$, $t_{52}^{(2)} = 4$ and $t_{63}^{(2)} = 5$. Therefore, $c_{41} = 9$, $c_{52} = 8$ and $c_{63} = 3$. The 3-subnet, $N^{(3)}$, is shown in Fig.(2.8b). Similarly from $N^{(3)}$ we find $t_{51}^{(3)} = 7$ and $t_{62}^{(3)} = 9$, thus $c_{51} = 7$ and $c_{62} = 2$; and the 4-subnet, $N^{(4)}$, is shown in Fig.(2.8c). Finally, from $N^{(4)}$ we obtain $t_{61}^{(4)} = 11$; thus $c_{61} = 6$ and the realization of T is shown in Fig.(2.8d).

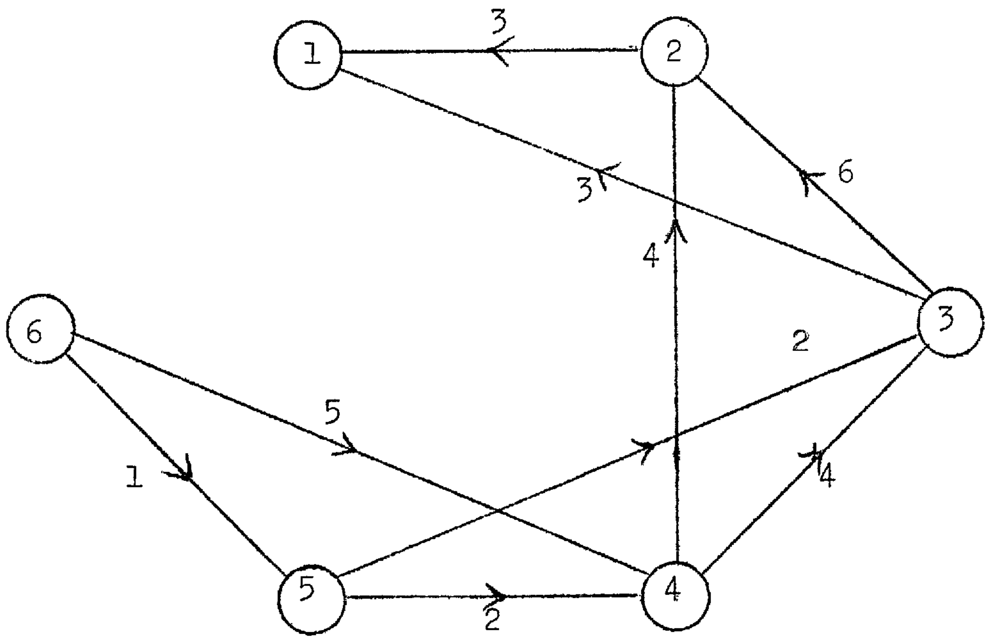


Fig.(2.8a). $N^{(2)}$.

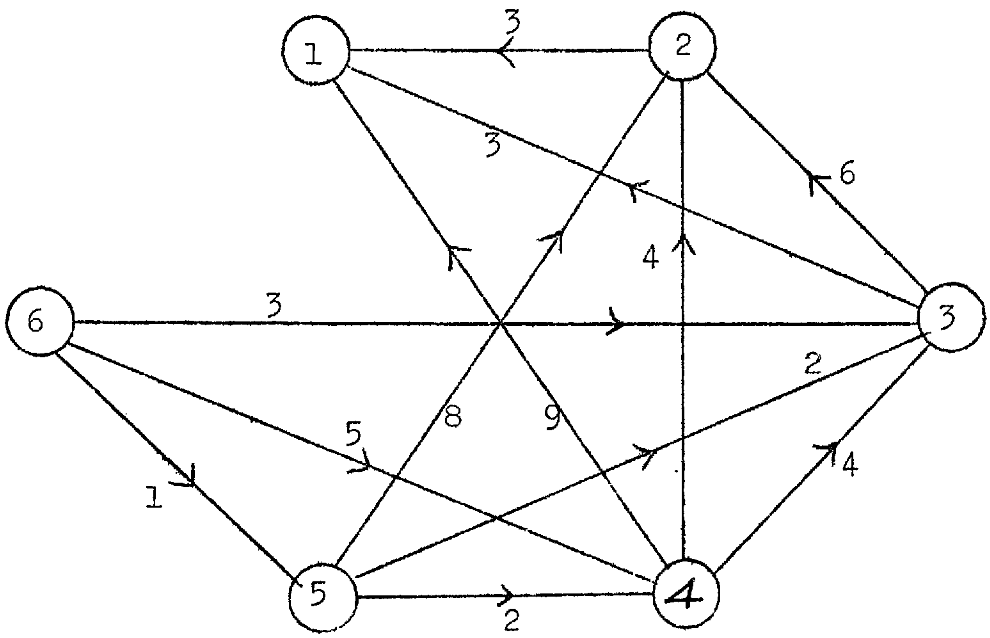


Fig. (2.8b). $N^{(3)}$.

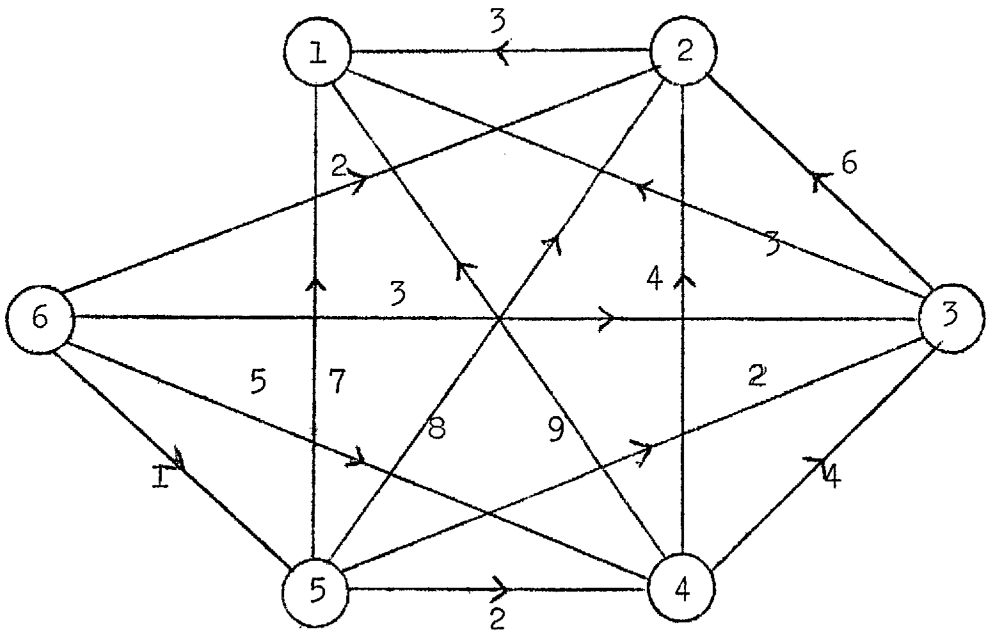


Fig.(2.8c). $N^{(4)}$

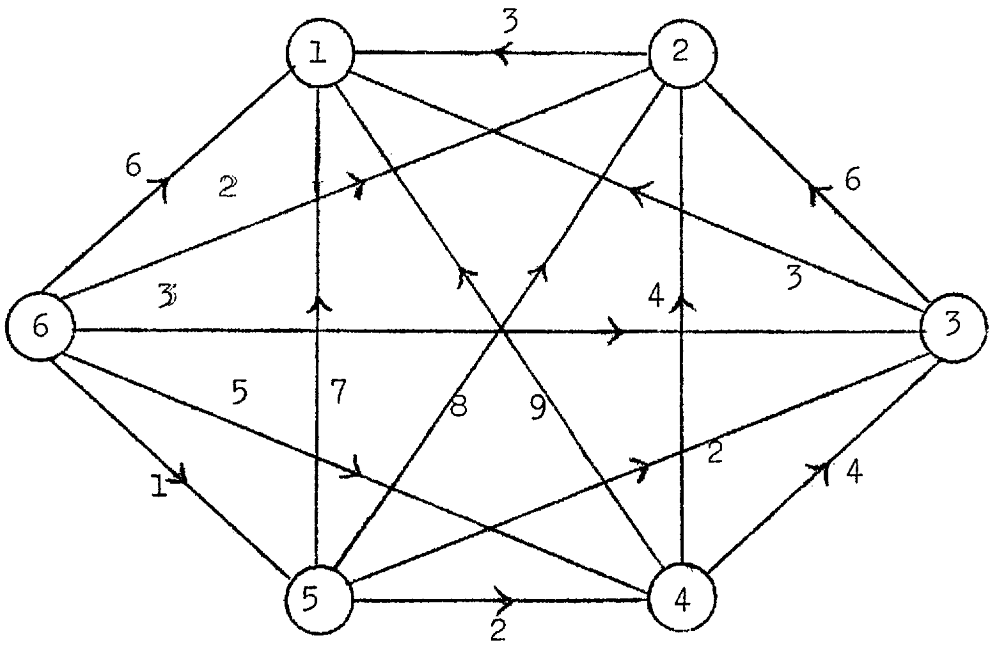


Fig.(2.8d). $N^{(5)} = N$.

CHAPTER III

On the Synthesis of NON-Oriented Communication Nets with Minimum Total Edge-Capacity and Minimum Number of Edges.

3.1 Introduction.

A non-oriented communication net is an undirected graph with capacities as weights. We shall assume that no capacity constraints on nodes are admitted. The flow in any edge may go in either (but not both) direction as long as its magnitude does not exceed the edge capacity. The edge matrix, $E = [e_{ij}]$, edge capacity matrix, $C = [c_{ij}]$, and the terminal matrix, $T = [t_{ij}]$, for a nonoriented communication net, N, are defined in the same way as in the oriented case. It must be noticed that in the non-oriented case E, C and T are symmetrical matrices. We shall assume, as before, that N is connected and contains exactly n nodes labelled by 1,2,...,n.

It is easy to see that a cut (i,j) of a nonoriented communication net N is a cut-set (i,j) of N, that is, a minimal set of edges whose removal separates N into two disconnected nets, each being connected and one containing node i and the other node j. For nonoriented nets, we shall use the term cut-set instead of cut. The capacity of a cut-set (i,j) is the sum of the capacities of its edges; and a minimum cut-set (i,j) of N is a cut-set (i,j) whose capacity is not more than the capacity of any other cut-set (i,j) of N. A minimum cut-set (i,j) of N will be denoted by S_{V_i, V_j} if the removal of all its edges separates N into two subnets N_i and N_j, where V_i and V_j are the sets of all nodes in N_i and N_j respectively.

It has been shown[9] that a symmetric matrix T is realizable as the terminal matrix of a nonoriented net if, and only if, T is principal partitionable (Definition 2.4). Gomoy and Hu[6] have shown that a necessary and sufficient condition for a matrix T to be realizable as the terminal matrix of a nonoriented communication net is that for all i,j,k = 1,2,...,n,

$$t_{ij} = t_{ji} \geq \min \{t_{ik}, t_{kj}\} \quad .$$

The total sum of edge capacities of a nonoriented communication net is called the "total edge capacity". Several methods for the synthesis of nonoriented nets with minimum total edge capacity are known; those are listed below:

(1) Method of equal distribution,[8]

(2) Method of decomposition of matrices,[6]

(3) Method of elementary matrices,[14]

(4) Method of successive expansion.[14]

The last method will be discussed in detail in the next section.

In this chapter, we shall add some properties of the minimum total edge capacity realization (in short, minimum realization) of nonoriented communication nets. Some results on the minimum realization with minimum number of edges, having nonzero capacities, will also be given.

3.2 A minimum Realization of a Symmetric Terminal Matrix.

A symmetric terminal matrix T, i.e. a terminal matrix of a nonoriented communication net, is principal partitionable. Wing and Chien[14] show that T can also be partitioned uniquely into the form:

$$T = \begin{bmatrix} A_1 & T_{12} & \cdots & T_{1k} \\ T'_{12} & A_2 & \cdots & T_{2k} \\ & & \ddots & \\ T'_{1k} & T'_{2k} & \cdots & A_k \end{bmatrix}, \qquad (3.1)$$

for some $k \leq n$, such that:

1) $A_1, A_2, \ldots, A_k$ are square submatrices,

2) Every T_{ij} is a uniform matrix with element value t_1, T'_{ij} is the transpose of T_{ij},

3) Every element in $A_1, A_2, \ldots, A_k$ is greater than t_1,

4) Every submatrix $A_1, A_2, \ldots, A_k$ can be partitioned in the same way and satisfies the same conditions.

This partitioning will be called "Wing-Chien partitioning".

<u>Definition (3.1)</u>. The "index of partitioning", I_p, of a symmetric terminal matrix T is the number of operations necessary to partition T, by Wing-Chien partitioning, into a form in which every diagonal submatrix is either of order 2 x 2 or 1 x 1, with the provision that each operation is to be applied to one diagonal submatrix at a time.

It is well-known that the set of all edges incident at any node i of a nonoriented net N contains a cut-set (i,r), for all $r \neq i$. Thus if N is any realization of a symmetric terminal matrix T, then

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} \geq t_{io} , \quad i = 1,2,\ldots,n, \qquad (3.2)$$

where $t_{io} = \max \left\{ t_{ir} \,\middle|\, r = 1,\ldots,n,\ i \neq r \right\}$.

Thus the total edge capacity, E_t, of N must satisfy

$$E_t = \sum_{i>j} c_{ij} \geq \tfrac{1}{2} \sum_{i=1}^{n} t_{io} , \qquad (3.3)$$

$i,j = 1,2,\ldots,n.$

Hence any realization of T that satisfies (3.3) as an equality, and so (3.2), is a minimum realization, N_m, of T. We have mentioned in §3.1, four such realizations. The one which is of interest here is the Method of Successive Expansion which may be summarized in the following steps:

1) Partition T by Wing-Chien partitioning (eqn 3.1),

2) Treating each diagonal submatrix $A_1, \ldots, A_k$, as a node construct a loop with each edge capacity equal to $t_{1/2}$, i.e. half the capacity of the first partitioning step.

3) To realize each of A_i $(i = 1, \ldots, k)$ which is of order more than 1, a new loop is formed, by repeating steps 1) and 2) on A_i, to take the place of the corresponding node and each edge in the new loop will have a capacity of $t_{2/2}$ (where t_2 is a minimal element of A_i) except for one edge, which has a capacity of $(t_2 - t_1)/2$ and is the one and only one edge which the new loop shares with the original loop.

4) Each submatrix is carried out in the same way until each node in the net obtained represents one node in T and not a diagonal submatrix of order more than 1.

The net obtained by the above method of realization satisfies (3.3) as an equality; and if every set of parallel edges replaced by one edge then this net, N_m, will contain exactly $(I_p + n - 1)$ edges, where I_p is the index of partitioning of T. Due to the manner of construction, Nm is ~~This follows from the fact that,~~ N_m ~~is~~ planar[15] net consisting of I_p meshes.

The following theorem follows from (3.2) and the above discussions:

Theorem (3.1). "A realization N_m of a symmetric terminal matrix T is minimum if, and only if, for every node i of N_m

$$Q_i(N_m) = t_{io}\ , \quad i = 1,\ldots,n, \qquad (3.4)$$

where $Q_i(N_m)$ is the sum of the capacities of all edges incident at node i in N_m, that is

$$Q_i(N_m) = \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} \text{ "}$$

3.3 Properties of the Minimum Realizations of a Symmetric Terminal Matrix.

Without loss of generality, we may assume that T contains no zero elements, that is, $t_{ij} > 0$ for all $i \neq j$.

Theorem (3.2). "A minimum realization N_m of a symmetric terminal matrix T is a non-separable net."

Proof. Suppose, if possible, that N_m is a separable net, then it must contain at least one cut-node[(*)], say node i. Let $N_1, N_2, \ldots, N_\alpha$ ($\alpha \geq 2$) be all the maximally connected subnets[(**)] of the net obtained from N_m by removing node i and all edges incident at it. There is at least one edge in N_m connecting node i with a node in N_r for each $r = 1, \ldots, \alpha$. Thus for every $j = 1, 2, \ldots, n$, $j \neq i$, there is a cut-set (i,j) in N_m which is a proper subset of the set of all edges incident at node i. Therefore

$$t_{io} < Q_i(N_m) ,$$

contradicting Theorem (3.1). Thus N_m is a nonseparable net. Hence the theorem.

Corollary (3.1). "A minimum realization of a symmetric terminal matrix T contains no cut-set consisting of one edge only if the order of T is more than 2."

Proof. Any net which consists of more than 2 nodes and contains a cut-set consisting of a single edge is separable. Hence the assertion (Theorem (3.2)).

(*) Theorem (3.1), Ref.12.

(**) Let S be a non-empty set of nodes of an unconnected net, N, and $\bar{S}$ the complement of S in N such that there exists a path between any two nodes in S and no path between any node of S and any node of $\bar{S}$. The subnet of N which consists of all nodes in S and all edges having their nodes in S is called a "maximally connected subnet" of N.

<u>Definition (3.2)</u>. Let V_ℓ be a set of nodes of a nonoriented communication net N. A"(V_ℓ)-condensed net, denoted by N(V_ℓ)", of N is a net obtained from N by identifying(*) all the nodes in V_ℓ; the new node will be denoted by v_ℓ.

The following lemma is given by Gomoy and Hu[6] (Lemma 1):

<u>Lemma (3.1)</u>(**). "Let $S_{V_\ell;V_k}$ be a minimum cut-set (ℓ,k) of a nonoriented net N, then the terminal capacity (i,j), $i,j \in V_k$, $i \neq j$, is the same in both N and N(V_ℓ), where N(V_ℓ) is the (V_ℓ)-condensed net of N."

<u>Theorem (3.3)</u>. "Let $S_{V_\ell;V_k}$ be a minimum cut-set (ℓ,k) of a minimum realization N_m of a symmetric terminal matrix T, then the (V_ℓ)-condensed net, $N_m(V_\ell)$, is a minimum realization of $\bar{T} = [\bar{t}_{ij}]$ if $t_{\ell k}$ is a minimal element of row ℓ, where $\bar{T}$ is T with: (1) all rows and columns corresponding to the nodes in V_ℓ deleted and (2) row and column corresponding to node v_ℓ added, with element value $t_{\ell k}$."

(*)If the nodes of an edge are identified (shorted), the edge is removed from the net, that is, no self-loop edges are allowed.

(**)A rigorous proof of this assertion is given by Ford and Fulkerson[5] (Lemma 3.1, page 179).

<u>Proof</u>. By Lemma (3.1), the terminal capacity (i,j), $i,j \in V_k$, in $N_m(V_\ell)$ is equal to t_{ij}. Shorting an edge is equivalent to making its capacity ∞, thus it does not decrease the terminal capacity between any pair of nodes. Since $t_{\ell k}$ is a minimal element of row ℓ in T and the set of all edges incident at node v_ℓ in $N_m(V_\ell)$ is a cut-set (v_ℓ, r), $r \in V_k$, and its capacity is equal to $t_{\ell k}$, then the terminal capacity (v_ℓ, r) in $N_m(V_\ell)$ is equal to $t_{\ell k}$. Thus T is the terminal matrix of $N_m(V_\ell)$.

Because N_m is a minimum realization of T, then by Theorem (3.1), eqn (3.4) holds for every node $i \in V_k$ in both N_m and $N_m(V_\ell)$. Thus $N_m(V_\ell)$ is a minimum realization of T. Hence the theorem.

<u>Theorem (3.4)</u>. Let N_m be a minimum realization of a symmetric terminal matrix T, and let N_m contain a node, say r, of second degree, that is only two-edges, say e_{rp} and e_{rq}, are incident at r, where $c_{rq} \geq c_{rp} > 0$. Then $N_m^{(1)}$ is a minimum realization of $T^{(1)}$, in which $N_m^{(1)}$ is N_m with edge e_{rq} short-circuited (see Fig.3.1) and $T^{(1)}$ is T with row and column r deleted."

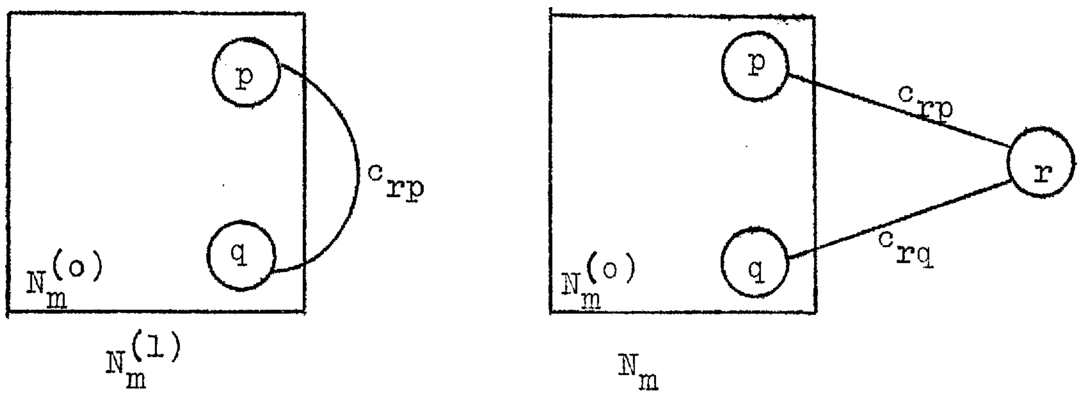


Fig.(3.1). Illustration of Theorem (3.4).

<u>Proof</u>. It is obvious that the terminal capacity (i,j), $i \neq j \neq r$, in $N_m^{(1)}$ is equal to t_{ij}; thus $N_m^{(1)}$ is a realization of $T^{(1)}$.

Let $t_{xo}^{(1)} = \max \left\{ t_{xy} \mid y = 1,2,\ldots,n,\ y \neq x \neq r \right\}$, then $t_{xo} = \max \left\{ t_{xo}^{(1)},\ t_{xr} \right\}$, for all $x = 1,2,\ldots,n$, $x \neq r$. Suppose i ($\neq q,r$) is any node of N_m and $S_{V_i;V_q}$ is a minimum cut-set (i,q) in N_m. Since $c_{rp} \leq c_{rq}$, then $S_{V_i;V_q}$ is a cut-set (i,r). Thus

$$t_{iq} \geq t_{ir}\ , \text{ for all } i \neq r,q.$$

Therefore

$$t_{io}^{(1)} = t_{io}\ , \text{ for all } i \neq r,q.$$

Since N_m is a minimum realization of T and

$$Q_i(N_m) = Q_i(N_m^{(1)})\ , \quad i \neq r,q,$$

then by Theorem (3.1)

$$Q_i(N_m^{(1)}) = t_{io}^{(1)}, \text{ for all } i \neq r,q. \qquad (3.5)$$

To complete the proof of the theorem we must prove that

$$Q_q(N_m^{(1)}) = t_{qo}^{(1)} \ . \qquad (3.6)$$

If

$$t_{qr} < Q_q(N_m) \ ,$$

then

$$t_{qo}^{(1)} = t_{qo} = Q_q(N_m) \geq Q_q(N_m^{(1)}) \ ,$$

and by (3.2)

$$t_{qo}^{(1)} = Q_q(N_m^{(1)}).$$

If

$$t_{qr} = Q_q(N_m),$$

then

$$Q_q(N_m^{(o)}) = \min \left\{ t_{qp}^{(o)}, \ c_{rp} \right\} \ ,$$

where $T^{(o)} = [t_{ij}^{(o)}]$ is the terminal matrix of $N_m^{(o)}$ (see Fig. 3.1).

Thus

$$Q_q(N_m^{(1)}) = c_{rp} + \min \left\{ t_{qp}^{(o)}, \ c_{rp} \right\}$$

$$\leq t_{qp}$$

and by (3.2),

$$Q_q(N_m^{(1)}) = t_{qp} = t_{qo} \ .$$

Therefore node q in $N_m^{(1)}$ satisfies (3.6) in any case.

Hence the theorem.

Let N_1 and N_2 be nonoriented nets which have the same nodes, and let C_1 and C_2 be the edge capacity matrices of N_1 and N_2, respectively, with the same node ordering; then the net obtained by superimposing N_1 and N_2,

$$N = N_1 + N_2 ,$$

is the net whose edge capacity matrix C is given by

$$C = C_1 + C_2,$$

and its nodes are those of N_1 or N_2.

<u>Theorem (3.5)</u>. "Suppose $N_m^{(1)}$ and $N_m^{(2)}$ are minimum realizations of $T^{(1)} = [t_{ij}^{(1)}]$ and $T^{(2)} = [t_{ij}^{(2)}]$, respectively, where both have the same nodes in the same order. Let

$$T = T^{(1)} + T^{(2)},$$

and

$$N_m = N_m^{(1)} + N_m^{(2)} ,$$

then N_m is a minimum realization of T if, and only if:

(1) for each node pair (i,j), there exists a cut-set (i,j) which is minimum for both $N_m^{(1)}$ and $N_m^{(2)}$, and

(2) for each row i there exists a column, say r, such that the entry (i,r) is a maximal element of row i (excluding the node symbol) in both $T^{(1)}$ and $T^{(2)}$,

that is

$$t_{io}^{(1)} = t_{ir}^{(1)} \quad \text{and} \quad t_{io}^{(2)} = t_{ir}^{(2)} \ . \text{"}$$

<u>Proof</u>. (a) Suppose conditions (1) and (2) are satisfied. Let $\bar{t}_{ij}$ be the capacity of a minimum cut-set (i,j), say S_{V_i,V_j}, of N_m. Then the set of all edges $e_{\ell k}^{(1)}(e_{\ell k}^{(2)})$, $\ell \in V_i$ and $k \in V_j$, contains a cut-set (i,j) of $N_m^{(1)}(N_m^{(2)})$. Therefore

$$\bar{t}_{ij} \geq t_{ij}^{(1)} + t_{ij}^{(2)} \ .$$

By condition (1), there is a cut-set (i,j), say $S_{V_i';V_j'}$, which is a minimum cut-set (i,j) for both $N_m^{(1)}$ and $N_m^{(2)}$. But $S_{V_i';V_j'}$ is also a cut-set (i,j) of N_m with capacity $t_{ij}^{(1)} + t_{ij}^{(2)}$, thus

$$\bar{t}_{ij} \leq t_{ij}^{(1)} + t_{ij}^{(2)} \ .$$

Hence $$\bar{t}_{ij} = t_{ij} \ ,$$

that is, T is the terminal matrix of N_m.

From condition (2), for each row i

$$t_{io} = t_{io}^{(1)} + t_{io}^{(2)} \ .$$

But by Theorem (3.1),

$$t_{io}^{(1)} = Q_i(N_m^{(1)}) \ ,$$

and

$$t_{io}^{(2)} = Q_i(N_m^{(2)}) \ .$$

Since

$$Q_i(N_m) = Q_i(N_m^{(1)}) + Q_i(N_m^{(2)}) \ ,$$

then

$$t_{io} = Q_i(N_m) \ ,$$

and N_m is a minimum realization of T.

(b) Suppose that N_m is a minimum realization of T. Any minimum cut-set (i,j), say $S_{V_i;V_j}$, of N_m is a cut-set (i,j) for both $N_m^{(1)}$ and $N_m^{(2)}$. Since

$$t_{ij} = t_{ij}^{(1)} + t_{ij}^{(2)} \ ,$$

then $S_{V_i;V_j}$ is a minimum cut-set (i,j) for both $N_m^{(1)}$ and $N_m^{(2)}$. Hence condition (1).

Now, let

$$t_{ir} = Q_i(N_m) \ ,$$

where i is any node of N_m. Since

$$t_{ir} = t_{ir}^{(1)} + t_{ir}^{(2)} \ ,$$

$$t_{ir}^{(1)} \leq Q_i(N_m^{(1)}) \ ,$$

$$t_{ir}^{(2)} \leq Q_i(N_m^{(2)}) \ ,$$

and

$$Q_i(N_m) = Q_i(N_m^{(1)}) + Q_i(N_m^{(2)}) ,$$

then

$$t_{ir}^{(1)} = Q_i(N_m^{(1)}),$$

$$t_{ir}^{(2)} = Q_i(N_m^{(2)}) ,$$

and condition (2) follows.

Hence the theorem.

Theorem (4.1) of Reference 5, page 190, that is the fact that the " Method of Decomposition of Matrices" gives a minimum realization, follows from our theorem above.

3.4 Synthesis of Non-Oriented Communication Nets With Minimum Total-Edge Capacity and Minimum Number of Edges.

In this section we shall present some results on a minimum realization, with minimum number of edges, of a terminal matrix of a nonoriented communication net. The edges counted are those having non-zero capacities. We shall assume that the minimal element t_1 of T is not zero.

Lemma (3.2). "The minimum number of edges, m, of a minimum realization, N_m, of a symmetric terminal matrix, T, of order 4 is given by

$$m = I_p + 3 , \tag{3.7}$$

where I_p(= 1,2) is the index of partitioning of T."

Proof. If $I_p = 1$, then by Theorem (3.2), eqn (3.7) is true. Now, let $I_p = 2$, then T must have the following form:

$$T = \left[\begin{array}{cc:c:c} \textcircled{1} & t_3 & t_2 & t_1 \\ t_3 & \textcircled{2} & t_2 & t_1 \\ \hdashline t_2 & t_2 & \textcircled{3} & t_1 \\ \hdashline t_1 & t_1 & t_1 & \textcircled{4} \end{array}\right] , \tag{3.8}$$

where $t_1 < t_2 \leq t_3$.

By Theorem (3.1), the minimum total edge capacity, E_t, is given by

$$E_t = \tfrac{1}{2}(t_1 + t_2 + 2t_3) \cdot \qquad (3.9)$$

Suppose, if possible that N_m of (3.8) consists of 4 edges, that is N_m is a loop (Theorem 3.2), then

$$E_t = Q_4(N_m) + Q_x(N_m) , \qquad (3.10)$$

where x is the node opposite to node 4 in the loop; that is, x = 1,2 or 3. Thus (3.10) gives

$$E_t = t_1 + t_2 \text{ (or } t_3) . \qquad (3.11)$$

From (3.9) and (3.11) we get $t_1 \geq t_2$, which contradicts the assumption in (3.8). Thus N_m must consist of at least 5 edges if $I_p = 2$.

Hence the lemma.

<u>Theorem (3.6)</u>. "If T is a symmetric terminal matrix of order n, then there is no minimum realization of T consisting of n edges only if the index of partitioning, I_p, of T is more than 1."

<u>Proof</u>. Since $I_p > 1$, then $n \geq 4$. The case for n = 4 has been shown to be true in Lemma (3.2). The proof will be completed by induction on n. Let the theorem be true for any symmetrical terminal matrix of order

$h > 4$ and $I_p > 1$, and consider a matrix T of order (h+1) and partitioning index $I_p > 1$. Suppose, if possible, that its minimum realization, N_m, consists of (h+1) edges; then by Theorem (3.2), N_m is a loop. If $I_p > 2$, then by Theorem (3.4), $N_m^{(1)}$ is a minimum realization of $T^{(1)}$, where $N_m^{(1)}$ is N_m with the edge of greatest capacity at node r, any node in N_m, short-circuited, and $T^{(1)}$ is T with row and column r deleted. This contradicts the induction hypothesis, because $N_m^{(1)}$ is a loop, $T^{(1)}$ is of order h and the index of partitioning of $T^{(1)}$ is more than 1. Now, suppose $I_p = 2$. Let the first step of the Wing-Chien partitioning of T be as given in (3.1). One submatrix only, say A_1, along the diagonal line is of partitioning index 1, the others are of orders 1 or 2. Since $(h+1) > 4$, then either A_1 is of order more than 3 and/or the total order of $A_2, \ldots, A_k$ is more than 1. Thus there is at least one node, say r, whose deletion from T gives $T^{(1)}$ having partitioning index equal 2 also. A loop minimum realization, $N_m^{(1)}$, of $T^{(1)}$ is obtained from N_m by shorting the edge of greatest capacity incident at node r (Theorem 3.4). This also contradicts the induction hypothesis. Thus N_m cannot be a loop.

Hence the theorem.

The following corollaries follow directly from the above theorem.

Corollary (3.2). "A minimum realization with minimum number of edges of a symmetric terminal matrix is a loop if, and only if, its index of partitioning is 1."

Corollary (3.3). "The minimum number of edges of a minimum realization of a symmetric terminal matrix, whose order is n and index of partitioning is 2, is (n+1)."

The converse of the above assertion is not necessarily true; this can be seen by examining Example (3.1), (page 94). That is because some of the edge capacities become zero if certain relationships between some edge capacities exist. Thus in order to consider the minimum realization with minimum number of edges for any symmetric terminal matrix whose index of partitioning is more than 2, the following definition is needed.

Definition (3.3). A "variable terminal matrix", T, is a terminal matrix of a nonoriented communication net whose elements $0 < t_1, t_2, \ldots, t_\alpha$ ($I_p \leq \alpha \leq n-1$, I_p is the index of partitioning of T) which are obtained by Wing-Chien partitioning of T, are arbitrary variables taking any set of real positive values that do not change the partitioning structure of T. No particular relationships between some of the t's are assumed.

Let N_m be any minimum realization of a variable terminal matrix T of order n, then one can easily observe that every edge capacity of N_m is a function of some t's, and every edge not in N_m has identically zero capacity. Let $A_1, A_2, \ldots, A_k$ be the submatrices along the diagonal line of T corresponding to the first step of Wing-Chien partitioning (eqn 3.1), and let T_o be T with $t_1 = 0$. Then N_{m_o} is a realization of T_o, where N_{m_o} is N_m with t_1 set equal to zero in each edge capacity function. Matrix T_o is a symmetric terminal matrix of an unconnected communication net, that is, N_{m_o} is unconnected. Each A_i $(i = 1,\ldots,k)$ is a symmetric terminal matrix of one maximally connected subnet of N_{m_o}, say $N_{m_o}^{(i)}$. Moreover, $N_{m_o}^{(i)}$ is a minimum realization of A_i. Since $t_1, t_2, \ldots, t_\alpha$ are arbitrary variables and no particular relationships hold between some of them then every edge in $N_{m_o}^{(i)}$, for all $i = 1,2,\ldots,k$, is an edge in N_m, possibly with different capacity value. Now, we can prove the following result:

Theorem (3.7). "The minimum number of edges, m, of a minimum realization, N_m, of a variable terminal matrix," T, of a nonoriented communication net is given by

$$m = n + I_p - 1 \, , \qquad (3.12)$$

where n is the order of T and I_p is its index of partitioning."

Proof. ~~Using~~ Since 3.12 hold for the method of Successive Expansion, we need only to prove that the number of edges of any minimum realization satisfies

$$m \geq n + I_p - 1 \, . \qquad (3.13)$$

This is true for n = 3 or 4. The proof will be completed by induction on n. Let (3.13) be true for any T of order $h \leq n - 1$ and consider T of order n. Suppose N_{m_o}, T_o, $N_{m_o}^{(i)}$ and A_i, $i = 1,2,\ldots,k$, are defined as in the previous discussions. Moreover, let n_i and $I_p^{(i)}$ be the order and the partitioning index, respectively, of A_i. Since $n_i \leq n - 1$ and $N_{m_o}^{(i)}$ is a minimum realization of A_i, then by the induction hypothesis, the number of edges, m_i, of $N_{m_o}^{(i)}$ satisfies

$$m_i \geq n_i + I_p^{(i)} - 1, \text{ for all } i = 1,2,\ldots,k.$$

Since N_m contains no cut-set consisting of one edge only (Corollary 3.1) and every edge of N_{m_o} is an edge of N_m, then

$$m \geq k + \sum_{i=1}^{k} m_i \, ,$$

$$\geq \sum_{i=1}^{k} (n_i + I_p^{(i)}) \, .$$

But

$$\sum_{i=1}^{k} n_i = n ,$$

and

$$\sum_{i=1}^{k} I_p^{(i)} = I_p - 1 ,$$

thus (3.13) is true, which completes the proof of the theorem.

Theorem (3.7) may not be true if there are some relationships between some of the t's. This will be illustrated in the following example:

Example (3.1). Consider the terminal matrix

$$T = \left[\begin{array}{ccc|c|cc} \textcircled{1} & t_4 & t_4 & t_2 & t_1 & t_1 \\ t_4 & \textcircled{2} & t_4 & t_2 & t_1 & t_1 \\ t_4 & t_4 & \textcircled{3} & t_2 & t_1 & t_1 \\ \hline t_2 & t_2 & t_2 & \textcircled{4} & t_1 & t_1 \\ \hline t_1 & t_1 & t_1 & t_1 & \textcircled{5} & t_3 \\ t_1 & t_1 & t_1 & t_1 & t_3 & \textcircled{6} \end{array}\right] , \qquad (3.14)$$

where $0 < t_1 < t_2, t_3,$

$t_2 < t_4,$

and $t_4 = t_1 + t_2$.

The relation (3.12) gives m = 8, but the net shown in Fig. (3.2) consists of 7 edges only and it is a minimum realization of T. Considering the sum of the capacities of the edges incident at node 1 or 3, we observe that this net cannot be minimum if $t_4 \neq t_1 + t_2$.

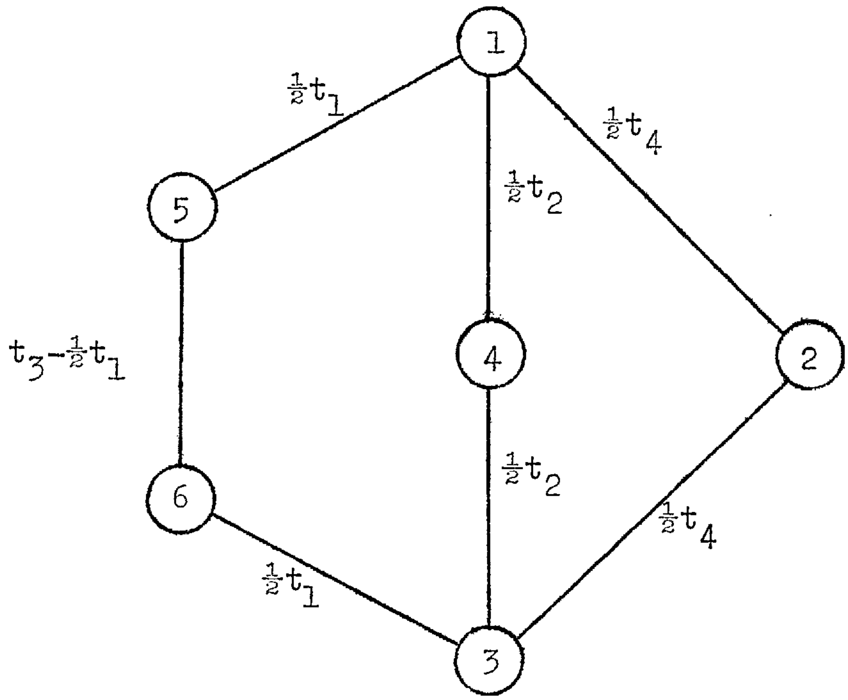


Fig.(3.2) A minimum realization of the matrix given in (3.14).

CHAPTER IV

"SYNTHESIS OF RADIO-WIRE-COMMUNICATION NETS"

4.1 Introduction.

In the previous chapters we have considered the synthesis of communication nets which are assumed to have weights on edges only; the nodes are assumed to have sufficient capacities to handle all information flowing into them.

In a recent paper, 1962, Yau[16] considered the synthesis of a radio-communication net, that is, a net which has weights on nodes only; the edges have unlimited capacities. In this chapter and the next one, we shall investigate the synthesis of more general communication models which ~~we~~ Yau ~~shall~~ calls radio-wire-communication nets. In these nets the nodes and the edges are assumed to have weights which are real non-negative numbers, called capacities. The capacity of an element may be very large (∞). We mean by an element of a radio-wire communication net, N, either an edge or a node.

It is assumed that for every node the maximum amount of information which can be transmitted and the maximum amount which can be received are the same, and each not exceeding the capacity of the node. It is also assumed that the edges are nonoriented, that is, the flow in any edge may go in either direction (but not both) as long as its magnitude does not exceed the edge capacity.

The nodes of a radio-wire-communication net, N, will be denoted by $v_1, v_2, \ldots, v_n$, and the edge between nodes v_i and v_j is denoted by $e_{ij} = e_{ji}$. The capacity of a node v_i will be represented by c_i, and the capacity of e_{ij} by $c_{ij} = c_{ji}$.

The maximum amount of information flow between a pair of nodes (v_i, v_j) of N is called the "terminal capacity (v_i, v_j)". When we find the terminal capacity between two nodes, all other nodes may be considered as relay stations without changing their capacities. Therefore, in the analysis, a multiterminal communication net is essentially treated in the same way as a two-terminal communication net.

The terminal matrix, $T = [t_{ij}]$, of N is a symmetric matrix with entry t_{ij}, $i \neq j$, being the terminal capacity (v_i, v_j), and t_{ii} being the node v_i, represented by symbol $\textcircled{v_i}$, for all $i, j = 1, \ldots, n$.

The purpose of this chapter is to present a necessary and sufficient condition for a symmetric matrix T to be realizable as the terminal matrix of a radio-wire communication net.

4.2 Analysis of Radio-Wire-Communication Nets.

In radio-wire-communication nets the concept of cut-set must be generalized to include nodes as well as edges. This has been done by Yau.[17]

Definition (4.1). A "generalized cut-set (v_i, v_j)" of a radio-wire-communication net, N, is a minimal set of elements whose removal(†) destroys all paths (v_i, v_j) in N. A generalized cut-set (v_i, v_j) will be denoted by S_{ij}. It must be noticed that S_{ij} contains neither v_i nor v_j.

Yau used the term "cut-set" to mean generalized cut-set. Since the term "cut-set" is widely used in the literature to represent an edge-cut-set(*), then we prefer to use the term "generalized cut-set". Some properties of generalized cut-sets (v_i, v_j) together with a method for obtaining all of them are given by Yau.[17] The capacity of a generalized cut-set, S_{ij}, denoted by $C(S_{ij})$, is the

---

(†) If a node is removed from N then every edge incident at that node must be deleted.

(*) Edge-, node- and mixed-cut-sets[17] are generalized cut-sets consisting of edges, nodes and edges and nodes, respectively.

A cut-set (v_i, v_j) is called, by Yau, basic cut-set (v_i, v_j).

sum of the capacities of all elements in S_{ij}. A "minimum generalized cut-set (v_i,v_j) of N is a generalized cut-set (v_i,v_j) whose capacity is not larger than the capacity of any other generalized cut-set (v_i,v_j) of N.

Dantzig and Fulkerson,[1] and Ford and Fulkerson[4] proved that the Max-Flow Min-Cut Theorem is also valid for a communication net with weights on both edges and nodes. In our notations, this theorem states:

Theorem (4.1). "The maximum flow, t_{ij}, between a pair of nodes (v_i,v_j) in a radio-wire communication net N is given by

$$t_{ij} = \min \left\{c_i, c_j, C(S_{ij}^{(m)})\right\} ,$$

where c_i and c_j are the capacities of nodes v_i and v_j, respectively, and $C(S_{ij}^{(m)})$ is the capacity of a minimum generalized cut-set (v_i,v_j) of N."

Now, we shall present some results which simplify the work for finding the terminal matrix for a given radio-wire-communication net.

Lemma (4.1). "In any radio-wire communication net, N, there exists a minimum generalized cut-set (v_i,v_j), for all $i \neq j$, which does not contain node v_r if

$$\tfrac{1}{2}Q_r(N) \leq c_r \tag{4.1}$$

(†) This, of course, excludes edges incident at any node contained in the cut-set.

where $Q_r(N)$ is the sum of the capacities of all edges incident at node v_r in N".

<u>Proof.</u> Let $S_{ij}^{(m)}$ be any minimum generalized cut-set (v_i, v_j) and N_{ij} be N with all the elements of $S_{ij}^{(m)}$ removed. Moreover, let N_i and N_j be maximal connected subnets of N_{ij} which contain nodes v_i and v_j, respectively. If $S_{ij}^{(m)}$ contains node v_r, then each of the sets $(S_{ij}^{(m)} \bigcup E_{r_i} - v_r)$ and $(S_{ij}^{(m)} \bigcup E_{r_j} - v_r)$ is a generalized cut-set (v_i, v_j), where $E_{r_i}(E_{r_j})$ is the set of all edges each of which is incident at node v_r and at a node in $N_i(N_j)$. Since neither E_{r_i} nor E_{r_j} has a common element with $S_{ij}^{(m)}$, and(†)

$$C(S_{ij}^{(m)} \bigcup E_{r_i} - v_r) \geq C(S_{ij}^{(m)}) ,$$

$$C(S_{ij}^{(m)} \bigcup E_{r_j} - v_r) \geq C(S_{ij}^{(m)}) ,$$

then

$$C(E_{r_i}),\ C(E_{r_j}) \geq c_r .$$

But

$$E_{r_i} \bigcap E_{r_j} = \emptyset ,$$

and from (4.1)

$$C(E_{r_i} \bigcup E_{r_j}) \leq 2c_r ,$$

(†) C( ) = sum of the capacities of all elements in the set ( ).

then

$$C(E_{r_i}) = C(E_{r_j}) = c_r \quad .$$

Thus each of $(S_{ij}^{(m)} \bigcup E_{r_i} - v_r)$ and $(S_{ij}^{(m)} \bigcup E_{r_j} - v_r)$ is a minimum generalized cut-set (v_i, v_j) of N.

Hence the lemma.

By the above lemma, if

$$\tfrac{1}{2}Q_r(N) \leq c_r$$

for all $r = 1,2,\ldots,n$, then we can obtain all $S_{ij}^{(m)}$, for $i,j = 1,2,\ldots,n$, $i \neq j$, by treating N as an ordinary non-oriented communication net with weights on edges only. In this case T can be found by evaluating (n-1) flow problems only, by using the Gomory and Hu[6] technique which depends on Lemma (3.1).

The following assertion is similar to Lemma (3.1) for radio-wire communication nets. Indeed, one can easily see that Lemma (3.1) follows directly from our Lemma (4.2).

Let V_r be any set of nodes of a radio-wire communication net, N. We extend Definition (3.2) as follows: A "(V_r)-condensed net", denoted by $N(V_r)$, of N is a net obtained from N by identifying all nodes in V_r; the new node is denoted by v_r' and given infinite capacity.

Let $S_{ij}^{(m)}$ be any minimum generalized cut-set (v_i, v_j) of N, and let N_i', N_j', N_1'', N_2'', ..., N_α'' be the maximal connected subnets of the net N_{ij} obtained from N by removing all the elements of $S_{ij}^{(m)}$, where N_i' and N_j' contain nodes v_i and v_j, respectively. Moreover, let N_i be N_i' and some or all of N_1'', N_2'', ..., N_α'', and let N_j be N_j' and the remainder of N_1'', N_2'', ..., N_α''. We shall call N_i and N_j the parts of N_{ij}. Now, we can state our lemma.

<u>Lemma (4.2)</u>. "If

$$C(S_{ij}^{(m)}) \leq c_i \,, \tag{4.2}$$

then the terminal capacity (v_ℓ, v_k) in N is equal to the terminal capacity (v_ℓ, v_k) in the (V_i)-condensed net, where nodes v_ℓ and v_k are in N_j, and V_i is the set of all nodes in N_i."

<u>Proof</u>. It is sufficient to prove that there exists a minimum generalized cut-set (v_ℓ, v_k) in N which does not contain any element of N_i.

Let $S_{\ell k}^{(m)}$ be a minimum generalized cut-set (v_ℓ, v_k) of N. Suppose $S_{\ell k}^{(m)}$ contains some elements of N_i. Let: N_ℓ and N_k denote the parts of $N_{\ell k}$ corresponding to $S_{\ell k}^{(m)}$, V_{ij}, $V_{\ell k}$, V_i, V_j, V_ℓ and V_k denote the sets of all nodes in $S_{ij}^{(m)}$, $S_{\ell k}^{(m)}$, N_i, N_j, N_ℓ and N_k, respectively,

$$V_1' = V_i \cap V_\ell \;,$$
$$V_2' = V_i \cap V_{\ell k} \;,$$
$$V_3' = V_i \cap V_k \;,$$
$$V_4' = V_j \cap V_\ell \;,$$
$$V_5' = V_j \cap V_{\ell k} \;,$$
$$V_6' = V_j \cap V_k \;,$$
$$V_7' = V_{ij} \cap V_\ell \;,$$
$$V_8' = V_{ij} \cap V_{\ell k} \;,$$
$$V_9' = V_{ij} \cap V_k \;, \quad \text{and}$$

E_{pq} = the set of all edges each of which has one **node** in V_p' and the other in V_q', $p \neq q$, $p, q = 1,2,\ldots,9$. Fig.4.1 illustrates the connection between $S_{ij}^{(m)}$ and $S_{\ell k}^{(m)}$ in N, where each E_{pq} is represented by one edge, for simplicity. In the above notations, the sets $S_{ij}^{(m)}$ and $S_{\ell k}^{(m)}$ are given by

$$S_{ij}^{(m)} = E_{14} \cup E_{15} \cup E_{16} \cup E_{24} \cup E_{25} \cup E_{26} \cup E_{34} \cup E_{35} \cup E_{36} \cup V_7' \cup V_8' \cup V_9' \;,$$

$$S_{\ell k}^{(m)} = E_{13} \cup E_{16} \cup E_{19} \cup E_{34} \cup E_{46} \cup E_{49} \cup E_{37} \cup E_{67} \cup E_{79} \cup V_2' \cup V_5' \cup V_8' \;.$$

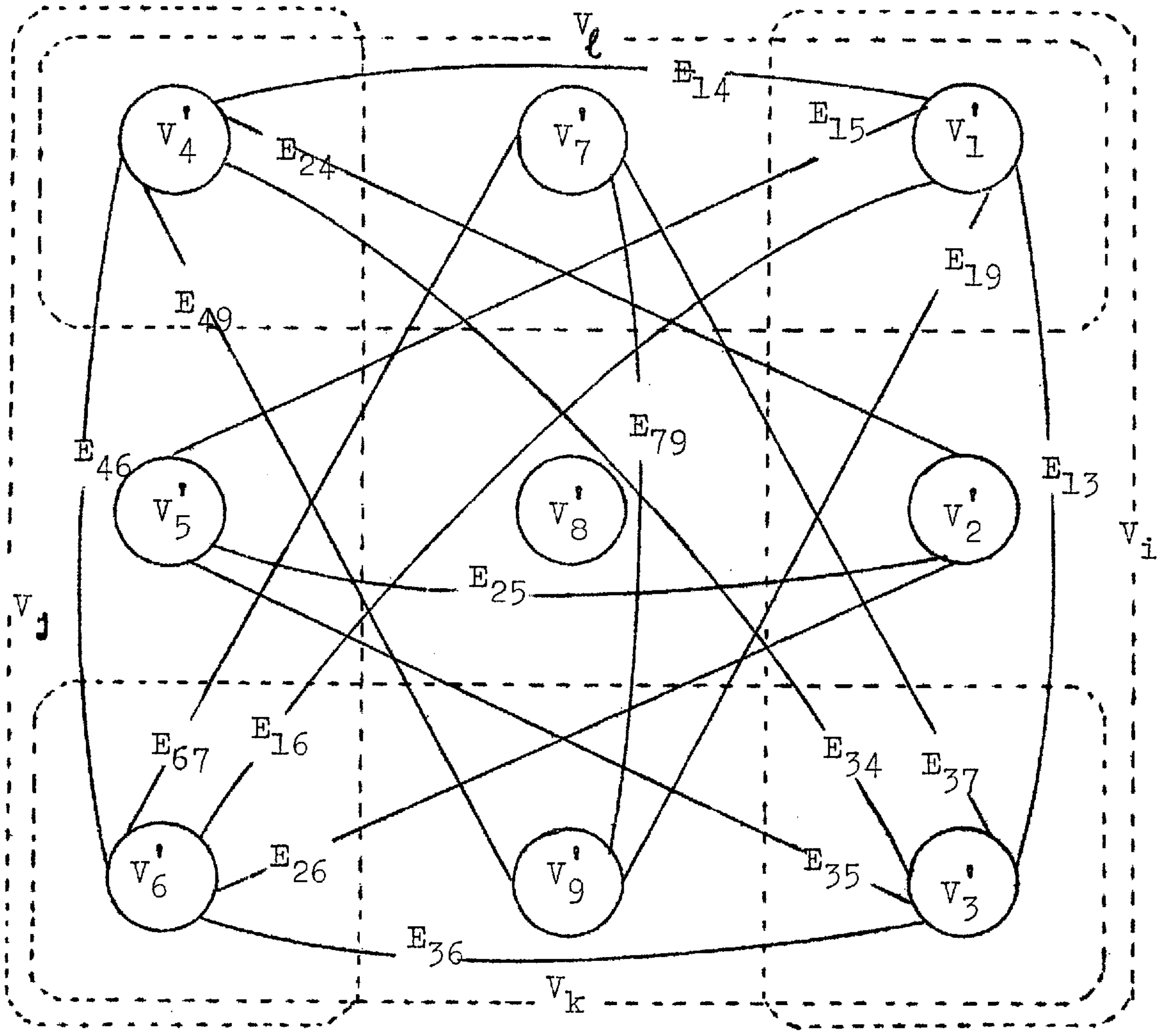


Fig.4.1. An illustration for the proof of Lemma (4.2).

We have to consider two cases only:

<u>Case (1)</u>. Node $v_i \in V_1'$. Then it can easily be seen that the set S_{ij}' contains a generalized cut-set (v_i, v_j) of N, where

$$S'_{ij} = E_{13} \cup E_{14} \cup E_{15} \cup E_{16} \cup E_{19} \cup V'_2 \cup V'_7 \cup V'_8 .$$

Since

$$C(S^{(m)}_{ij}) \leq C(S'_{ij}) ,$$

then

$$C(E_{24} \cup E_{25} \cup E_{26} \cup E_{34} \cup E_{35} \cup E_{36} \cup V'_9) \leq C(E_{13} \cup E_{19} \cup V'_2) \tag{4.3}$$

The set of elements $S'_{\ell k}$ given by

$$S'_{\ell k} = S^{(m)}_{\ell k} \cup (E_{24} \cup E_{25} \cup E_{26} \cup E_{34} \cup E_{35} \cup E_{36} \cup V'_9) - (E_{13} \cup E_{19} \cup V'_2)$$

$$= (E_{16} \cup E_{26} \cup E_{36} \cup E_{46} \cup E_{67} \cup V'_9 \cup V'_5 \cup V'_8) \cup (E_{24} \cup E_{25} \cup E_{34} \cup E_{35} \cup E_{49} \cup E_{37} \cup E_{79}),$$

contains a generalized cut-set (v_ℓ, v_k). This set $S'_{\ell k}$ contains no elements of N_i, and from (4.3)

$$C(S'_{\ell k}) \leq C(S^{(m)}_{\ell k}).$$

Thus $S'_{\ell k}$ must be a minimum generalized cut-set (v_ℓ, v_k).

Case (2). Node $v_i \in V'_2$.

The set of elements $S''_{\ell k}$ given by

$$S''_{\ell k} = S^{(m)}_{\ell k} \cup S^{(m)}_{ij} - (E_{13} \cup V'_2) ,$$

contains a generalized cut-set (v_ℓ, v_k). Since

$$C(S_{ij}^{(m)}) \leq c_i ,$$

then

$$C(S_{\ell k}^{''}) \leq C(S_{\ell k}^{(m)}) .$$

Thus $S_{\ell k}^{''}$ is a minimum generalized cut-set (v_ℓ, v_k).

The case, node $v_i \in V_3^{'}$, is exactly similar to Case (1).

Hence the lemma.

It must be mentioned that Lemma (4.2) may not be true if $C(S_{ij}^{(m)}) > c_i$, as illustrated in the net given in Fig.4.2, from which we observe that

$$S_{14}^{(m)} = \{e_{23}, e_{56}\} ,$$

$$C(S_{14}^{(m)}) = 6 > c_1 = 1 .$$

The set

$$S_{35}^{(m)} = \{e_{45}, e_{26}, v_1\} ,$$

is the only minimum generalized cut-set (v_3, v_5), and it contains two elements, namely, e_{26} and v_1, from N_1.

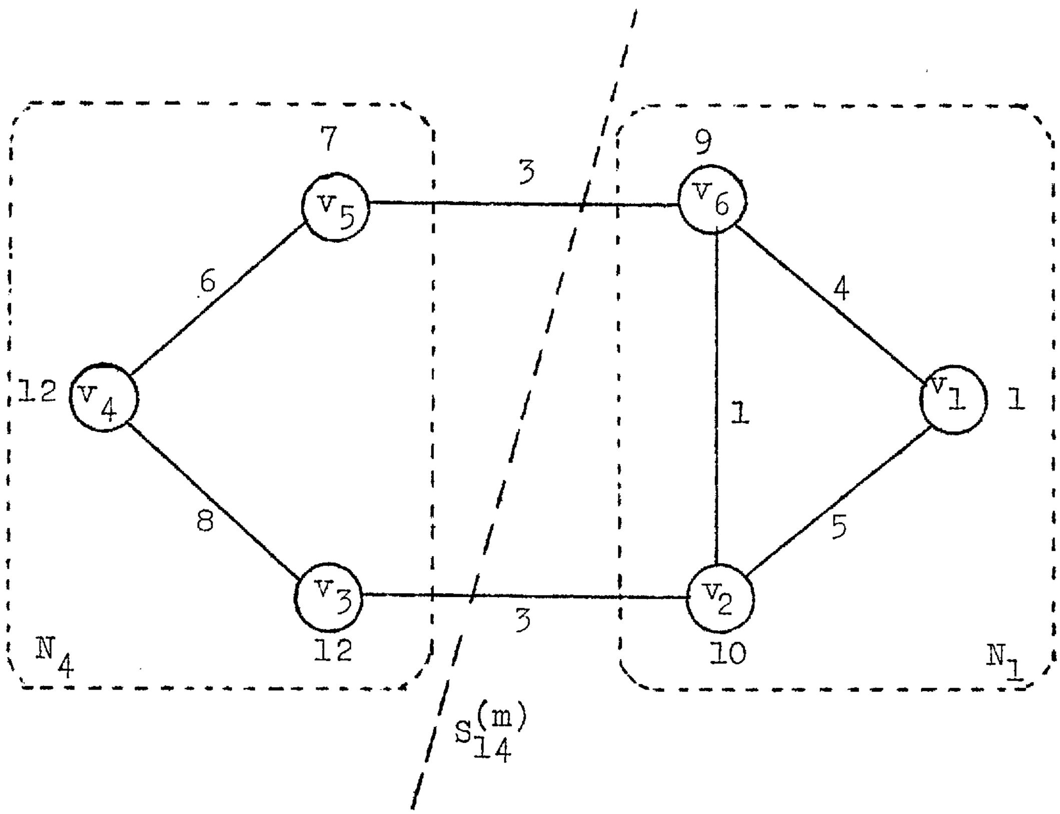


Fig. 4.2

4.3 Synthesis of Radio-Wire-Communication Nets.

Lemma (4.3). "A symmetric matrix, $T = [t_{ij}]$, is principal partitionable if, and only if, for every t_{ij}, $i \neq j$,

$$t_{ij} \geq \min \left\{ t_{ik}, t_{kj} \right\} , \tag{4.4}$$

for all $k \neq i,j$."

The proof follows directly from Theorem (2.1) and the fact that T is realizable as a terminal matrix of a nonoriented communication net if, and only if, T satisfies (4.4) for every $i \neq j \neq k$[6].

Lemma (4.4). "The terminal matrix, $T = [t_{ij}]$, of a radio-wire-communication net, N, satisfies (4.4) for every $i \neq j \neq k$."

Proof. Let $S_{ij}^{(m)}$ be a minimum generalized cut-set (v_i, v_j), then we have two cases:

Case (1).

$$t_{ij} = C(S_{ij}^{(m)}) .$$

If $v_k \in S_{ij}^{(m)}$, then

$$C(S_{ij}^{(m)}) \geq c_k \geq t_{ik},\ t_{kj} .$$

If $v_k \not\in S_{ij}^{(m)}$, then $S_{ij}^{(m)}$ contains a generalized cut-set (v_i, v_k) and/or (v_k, v_j).

Thus inequality (4.4) holds in both cases.

Case (2).

$$t_{ij} = c_i \text{ (or } c_j) < C(S_{ij}^{(m)}) \text{ .}$$

Since

$$t_{ik} \leq c_i \quad \text{and} \quad t_{kj} \leq c_j \text{ ,}$$

then (4.4) holds in this case as well.

Hence the lemma.

Theorem (4.2). "A symmetric matrix T is realizable as the terminal matrix of a nonoriented radio-wire communication net if, and only if, T is a principal partitionable matrix."

Proof. From Lemmas (4.3) and (4.4), the terminal matrix of a nonoriented radio-wire communication net is principal partitionable.

Now, suppose T is principal partitionable, then by Theorem (2.1) it is realizable as the terminal matrix of a nonoriented communication net, and therefore of a radio-wire net. Hence the theorem]Let N' be any such realization. Each node of N' has infinite capacity. Now, obtain a nonoriented radio-wire communication net N from N' by giving each node, say v_i, finite capacity c_i determined by

$$c_i = Q_i(N'),$$

where $Q_i(N')$ is the sum of the capacities of all edges incident at node v_i in N'. By Theorem (4.1) and using Lemma (4.1), it follows that N is a radio-wire realization of T with finite node capacities.

~~Hence the theorem~~.

<u>Corollary (4.1)</u>. "A symmetric matrix $T = [t_{ij}]$ is realizable as the terminal matrix of a nonoriented radio-wire-communication net if, and only if, for every element t_{ij}, $i \neq j$, of T,

$$t_{ij} \geq \min \left\{ t_{ik}, t_{kj} \right\} ,$$

for all $k \neq i,j$."

The proof follows directly from Lemmas (4.3) and (4.4) and Theorem (4.2).

<u>Corollary (4.2)</u>. "The terminal matrix of a radio-wire communication net of n nodes contains at most (n-1) distinct elements."

The proof is obvious.

Let $C_T(N)$ denote the total element capacity of a radio-wire communication net N, that is $C_T(N)$ is the sum of the capacities of all edges and nodes of N. A radio-wire realization of a principal partitionable matrix T with minimum total element capacity is described in the next theorem.

<u>Theorem (4.3)</u>. "A radio-wire-communication net, N, is a realization, with minimum $C_T(N)$, of a symmetric terminal matrix T if, and only if, for every node v_i of N,

$$Q_i(N) = c_i = t_{io} , \qquad (4.5)$$

where

$$t_{io} = \max \left\{ t_{ir} \;\middle|\; r = 1,2,\ldots,i-1,i+1,\ldots,n \right\} ."$$

Proof. From Theorem (4.1), for every node v_i of any radio-wire communication net, N, whose terminal matrix is T, the following relations hold

$$c_i,\ Q_i(N) \geq t_{io} \qquad (4.6)$$

Thus

$$C_T(N) = \sum_{i=1}^{n} (c_i + \tfrac{1}{2}Q_i(N))$$

$$\geq \frac{3}{2} \sum_{i=1}^{n} t_{io} .$$

Therefore, if (4.5) holds for every node v_i of N, then

$$C_T(N) = \frac{3}{2} \sum_{i=1}^{n} t_{io} ,$$

and so N has minimum $C_T(N)$.

In the proof of Theorem (4.2), we can choose a non-oriented communication net N' to be a minimum realization of T, that is, every node of N' satisfies (3.4), Theorem (3.1). Thus it is possible to obtain a radio-wire realization for T satisfying (4.5). Thus by (4.6), every node v_i of a radio-wire realization with minimum total element capacity of T must satisfy (4.5).

Hence the theorem.

It is worthwhile to mention that if N' is obtained by using the Method of Successive Expansion (§3.2), then N will contain a minimum number of edges, corresponding to the results of §3.4, as well as minimum total element capacity.

CHAPTER V

Necessary Conditions For Conditional Synthesis of Radio-Wire-Communication Nets

5.1 Introduction

An interesting problem concerning the synthesis of radio-wire-communication nets is the following: What are the necessary and sufficient conditions for the realizability of a given symmetric matrix T such that a given set of elements, of the realization net, have unlimited capacities. It can easily be verified that the synthesis problems of ordinary nonoriented communication nets, radio-communication nets[16] and radio-wire communication nets presented in Chapter IV are included in this general synthesis problem.

Let S_∞ be the set of the given elements which must have unlimited capacities. A realization of a symmetric principal partitionable matrix T of order n which contains S_∞ will be called a (T, S_∞)-realization and denoted by N. The ∞-subnet of N, denoted by $N^{(\infty)}$, is defined as a radio-wire-communication net consisting of all nodes v_i, $i = 1, \ldots, n$, of N whose capacities c_i are given by

$$c_i = \infty, \quad \text{if } v_i \in S_\infty,$$
$$= t_{io}, \quad \text{if } v_i \notin S_\infty,$$

where

$$t_{io} = \max \left\{ t_{ir} \;\middle|\; r = 1,\ldots,i-1,i+1,\ldots,n \right\} ,$$

and whose edges are all those in S_∞, each having ∞ capacity.

From the results of Chapter IV, matrix T must be principal partitionable. Another necessary condition for the (T, S_∞)-realization will be obtained from $N^{(\infty)}$ in this chapter.

5.2 Necessary conditions For (T,S_∞)-Realizations.

Suppose that (T,S_∞) is realizable and N is its realization. Then T can be partitioned by Wing-Chien partitioning. We shall assume, without loss of generality, that the order of the nodes along the diagonal of T in its Wing-Chien partitioning is $v_1,v_2,\ldots,v_n$.

A square submatrix of T whose diagonal elements are node symbols $v_\alpha,v_{\alpha+1},\ldots,v_\beta$, $1 \leq \alpha \leq \beta \leq n$, will be denoted by $D_{\alpha,\alpha+1,\ldots,\beta}$, and the value of a minimal element of $D_{\alpha,\alpha+1,\ldots,\beta}$ will be represented by $t_{\alpha,\alpha+1,\ldots,\beta}$. Finally, the set of nodes $v_\alpha,v_{\alpha+1},\ldots,v_\beta$ will be represented by $V_{\alpha,\alpha+1,\ldots,\beta}$. For example, $T \equiv D_{1,2,\ldots,n}$, $D_r = [\textcircled{v_r}]$, $r = 1,2,\ldots,n$. If $D_{\alpha,\alpha+1,\ldots,\beta}$ is of order more than one, i.e. $\beta-\alpha \geq 1$, then its resultant main submatrices by Wing-Chien partitioning process, denoted by $D_{\alpha,\alpha+1,\ldots,\alpha_1}$, $D_{\alpha_1+1,\ldots,\alpha_2},\ldots, D_{\alpha_k+1,\ldots,\beta}$, are the submatrices obtained by applying the first operation of Wing-Chien partitioning on $D_{\alpha,\alpha+1,\ldots,\beta}$, where k+1 is the number of these submatrices. It must be noticed that each element of any of these submatrices is greater than $t_{\alpha,\alpha+1,\ldots,\beta}$. For illustration, consider the terminal matrix given in (5.1), which is in Wing-Chien partitioning form.

$$T = \left[\begin{array}{c|c|cc|c|c} \textcircled{v_1} & 2 & 2 & 2 & 2 & 2 \\ \hline 2 & \textcircled{v_2} & 3 & 3 & 2 & 2 \\ \hline 2 & 3 & \textcircled{v_3} & 5 & 2 & 2 \\ 2 & 3 & 5 & \textcircled{v_4} & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & \textcircled{v_5} & 4 \\ \hline 2 & 2 & 2 & 2 & 4 & \textcircled{v_6} \end{array}\right]. \quad (5.1)$$

The resultant main submatrices of $D_{1,2,\ldots,6}$, by Wing-Chien partitioning process, are

$$D_1 = [\textcircled{v_1}],\ D_{2,3,4} = \begin{bmatrix} \textcircled{v_2} & 3 & 3 \\ 3 & \textcircled{v_3} & 5 \\ 3 & 5 & \textcircled{v_4} \end{bmatrix} \text{ and}$$

$$D_{5,6} = \begin{bmatrix} \textcircled{v_5} & 4 \\ 4 & \textcircled{v_6} \end{bmatrix}.$$

The minimal element of $D_{2,3,4}$ has value $t_{2,3,4} = 3$, and its resultant main submatrices are

$$D_2 = [\textcircled{v_2}] \text{ and } D_{3,4} = \begin{bmatrix} \textcircled{v_3} & 5 \\ 5 & \textcircled{v_4} \end{bmatrix}.$$

<u>Lemma (5.1)</u>. "In Wing-Chien partitioning of T, let $D_{\alpha_a,\ldots,\alpha_b}$ and $D_{\alpha_c,\ldots,\alpha_d}$ be any two resultant main sub-matrices, each of order more than one, of $D_{\alpha,\alpha+1,\ldots,\beta}$, where $\alpha \leq \alpha_a < \alpha_b < \alpha_c < \alpha_d \leq \beta$. If (T, S_∞) is realizable then S_∞ contains no edge e_{ij} such that $v_i \in V_{\alpha_a,\ldots,\alpha_b}$ and $v_j \in V_{\alpha_c,\ldots,\alpha_d}$."

<u>Proof</u>. Suppose, if possible, that $e_{ij} \in S_\infty$ and $v_i \in V_{\alpha_a,\ldots,\alpha_b}$ and $v_j \in V_{\alpha_c,\ldots,\alpha_d}$, then $c_{ij} = \infty$.

Thus

$$t_{ij} = \min \left\{c_i, c_j\right\} \tag{5.2}$$

Let $v_\ell \in V_{\alpha_a,\ldots,\alpha_b}$ and $v_k \in V_{\alpha_c,\ldots,\alpha_d}$, then

$$t_{i\ell},\ t_{jk} > t_{\alpha,\alpha+1,\ldots,\beta}.$$

But

$$t_{i\ell} \leq c_i \quad \text{and} \quad t_{jk} \leq c_j\ ,$$

therefore

$$\min \left\{c_i, c_j\right\} > t_{ij}\ ,$$

contradicting (5.2).

Hence the lemma.

The phrase "identifying the nodes of $D_{\alpha,\alpha+1,\ldots,\beta}$ in N" will mean: if $V_{\alpha,\alpha+1,\ldots,\beta}$ consists of one node only no change is made in N, otherwise the nodes $V_{\alpha,\alpha+1,\ldots,\beta}$ are identified in N and the new node, called a combined node and denoted by $v_{\alpha,\alpha+1,\ldots,\beta}$, is given unlimited capacity. If the nodes of an edge are identified that edge must be deleted. Any set of parallel edges resulting from this process is replaced by one having capacity equal to the sum of capacities of all these parallel edges.

Definition (5.1). The "signature of $D_{\alpha,\alpha+1,\ldots,\beta}$ in N with respect to Wing-Chien partitioning of T, $\beta > \alpha$" denoted by $N^{(wc)}_{\alpha,\alpha+1,\ldots,\beta}$, is a radio-wire-communication net obtained from N by identifying the nodes of each of (1) resultant main submatrices of $D_{\alpha,\alpha+1,\ldots,\beta}$, and (2) largest resultant main submatrices each of which does not contain $D_{\alpha,\alpha+1,\ldots,\beta}$. To illustrate this definition, consider the matrix given in (5.1) which is the terminal matrix of the net N shown in Fig.(5.1). The signature of $D_{2,3,4}$ in N with respect to Wing-Chien partitioning is shown in Fig.5.2.

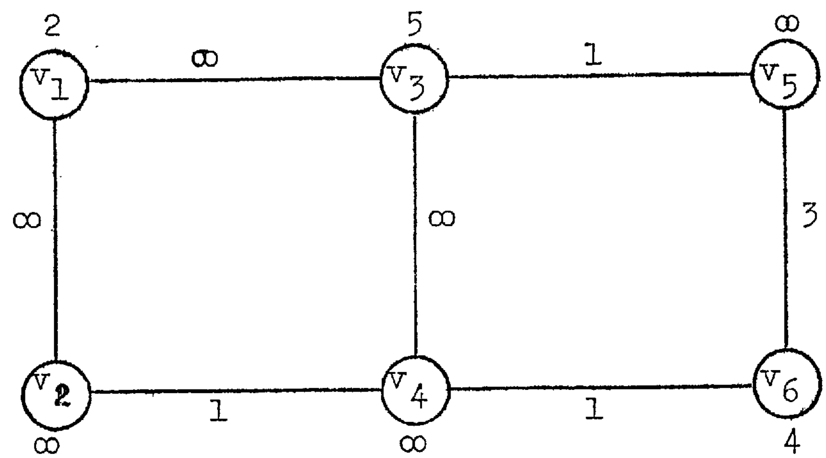


Fig.5.1

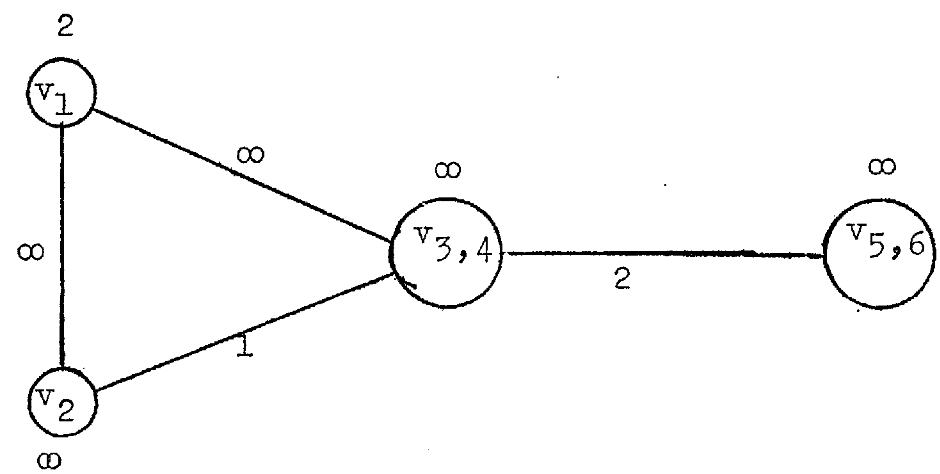


Fig.5.2

Lemma (5.2). "Let T be the terminal matrix of N and $D_{\alpha_a,\alpha_a+1,\ldots,\alpha_b}$ and $D_{\alpha_c,\alpha_c+1,\ldots,\alpha_d}$ any two resultant main submatrices of $D_{\alpha,\alpha+1,\ldots,\beta}$ by Wing-Chien partitioning. The terminal capacity (v_p,v_q), $v_p \equiv v_{\alpha_a,\ldots,\alpha_b}$, $v_q \equiv v_{\alpha_c,\ldots,\alpha_d}$ (*), in $N^{(wc)}_{\alpha,\alpha+1,\ldots,\beta}$, denoted by $t_{pq}(N^{(wc)}_{\alpha,\ldots,\beta})$, is equal to $t_{\alpha,\alpha+1,\ldots,\beta}$."

Proof. If there is at least one resultant main submatrix of T not containing $D_{\alpha,\alpha+1,\ldots,\beta}$ and of order more than one, then there must be a minimum generalized cutset, say S_1, of N such that

$$C(S_1) = t_{1,2,\ldots,n} .$$

If $t_{1,2,\ldots,n} < t_{\alpha,\alpha+1,\ldots,\beta}$, then the removal of S_1 does not separate any two nodes of $V_{\alpha,\alpha+1,\ldots,\beta}$. Let N_1' be the maximal connected subnet of $N-S_1$ (**) which contains

(*) This means that nodes $v_{\alpha_a,\ldots,\alpha_b}$ and $v_{\alpha_c,\ldots,\alpha_d}$ are given, for simplicity, labels v_p and v_q, respectively.

(**) The net $N-S_1$ is the net obtained from N by removing every element in S_1. If a node is removed then every edge incident at it must be deleted.

the set of nodes $V_{\alpha,\alpha+1,\ldots,\beta}$, and let $\overline{N}_1'$ be the complement of N_1' in $N-S_1$. Moreover, let N_1 be N with each set of nodes of a resultant main submatrix of T in $\overline{N}_1'$ identified. Then by Lemma (4.2), if v_x and v_y are any two nodes in N_1', then

$$t_{xy}(N_1) = t_{xy}(N) = t_{xy} \ .$$

If N_1' contains the nodes of a resultant main submatrix of T which is of order more than 1 and not containing $D_{\alpha,\alpha+1,\ldots,\beta}$, then there must be another minimum generalized cut-set, say S_2, of N_1 such that

$$C(S_2) = t_{1,2,\ldots,n} \ .$$

The removal of S_2 from N_1 will not separate any two nodes of $D_{\alpha,\alpha+1,\ldots,\beta}$. As in the case of S_1, let N_2' be the maximal connected subnet of $N_1 - S_2$ which contains the nodes of $V_{\alpha,\alpha+1,\ldots,\beta}$, and let $\overline{N}_2'$ be the complement of N_2' in $N_1 - S_2$. Then

$$t_{xy}(N_2) = t_{xy},$$

where N_2 is N_1 with each set of nodes of a resultant main submatrix of T in $\overline{N}_2'$ identified, and v_x and v_y are any two nodes in N_2' which are also in N_1'.

We repeat the above process until we arrive at the net N_i such that

$$t_{xy}(N_i) = t_{xy} ,$$

where N_i is N with the set of nodes of each resultant main submatrix of T, except the one containing $D_{\alpha,\alpha+1,\ldots,\beta}$, say $D^{(1)}$, identified and v_x and v_y are any two nodes of $D^{(1)}$.

We repeat the above process on the resultant main submatrices of $D^{(1)}$, and so on until we arrive at the net N_j such that

$$t_{xy}(N_j) = t_{xy}$$

where N_j is N with the set of nodes of each largest resultant main submatrix not containing $D_{\alpha,\alpha+1,\ldots,\beta}$ identified, and v_x, $v_y \in V_{\alpha,\alpha+1,\ldots,\beta}$. The net $N^{(wc)}_{\alpha,\alpha+1,\ldots,\beta}$ can be obtained from N_j by identifying the nodes of each resultant main submatrix of $D_{\alpha,\alpha+1,\ldots,\beta}$. Thus if $V_{\alpha_a,\alpha_a+1,\ldots,\alpha_b}$ or $V_{\alpha_c,\alpha_c+1,\ldots,\alpha_d}$ consists of one node whose capacity, in N (and so in N_j), is equal to $t_{\alpha,\alpha+1,\ldots,\beta}$, then

$$t_{pq}(N^{(wc)}_{\alpha,\alpha+1,\ldots,\beta}) = t_{\alpha,\alpha+1,\ldots,\beta} \qquad (5.3)$$

If neither $V_{\alpha_a,\alpha_a+1,\ldots,\alpha_b}$, nor $V_{\alpha_c,\alpha_c+1,\ldots\alpha_d}$ consists of one node whose capacity in N is equal to $t_{\alpha,\alpha+1,\ldots,\beta}$, then there must be a minimum generalized cut-set (v_x,v_y), $v_x \in V_{\alpha_a,\ldots,\alpha_b}$ and $v_y \in V_{\alpha_c,\ldots,\alpha_d}$, say S_{j+1}, in N_j such that

$$C(S_{j+1}) = t_{\alpha,\alpha+1,\ldots,\beta} .$$

The removal of S_{j+1} does not separate any two nodes belonging to the same resultant main submatrix, by Wing-Chien partitioning process, of $D_{\alpha,\ldots,\beta}$. Thus S_{j+1} is also a generalized cut-set (v_p, v_q) in $N^{(wc)}_{\alpha,\alpha+1,\ldots,\beta}$ and hence (5.3) follows.

Hence the lemma.

<u>Corollary (5.1)</u>. "If (T, S_{∞}) is realizable and $N^{(\infty)}$ is its ∞-subnet, then

$$t_{pq}(N^{(\infty)(wc)}_{\alpha,\alpha+1,\ldots,\beta}) \leq t_{\alpha,\alpha+1,\ldots,\beta}, \qquad (5.4)$$

for every resultant main submatrix $D_{\alpha,\alpha+1,\ldots,\beta}$ in Wing-Chien partitioning of T, where $N^{(\infty)(wc)}_{\alpha,\alpha+1,\ldots,\beta}$ is the signature of $D_{\alpha,\alpha+1,\ldots,\beta}$ in $N^{(\infty)}$ with respect to Wing-Chien partitioning of T, and v_p and v_q are as defined in Lemma (5.2)."

The proof follows directly from Lemma (5.2) and the fact that $N^{(\infty)}$ is a subnet of N.

<u>Definition (5.2)</u>. Given (T, S_{∞}) where T is principal partitionable, the net $N^{(\infty)}$ will be called "satisfactory with respect to Wing-Chien partitioning" if, and only if, in Wing-Chien partitioning of T, for every pair of resultant main submatrices of T and of every resultant

main submatrix of order more than one, inequality (5.4) holds.

Corollary (5.1) provides us with another necessary condition for the realizability of (T, S_{∞}), namely, $N^{(\infty)}$ must be satisfactory with respect to Wing-Chien partitioning.

5.3 Satisfactory Principal Partitioning.

All definitions and notations mentioned previously in §5.2 are based on Wing-Chien partitioning of T. We extend them to any principal partitioning of T. To avoid confusion, if $D_{\gamma,\gamma+1,\ldots,\delta}$ is any resultant main submatrix in a principal partitioning of T then its signatures with respect to this partitioning in N and $N^{(\infty)}$ will be represented by $N^{(p)}_{\gamma,\gamma+1,\ldots,\delta}$ and $N^{(\infty)(p)}_{\gamma,\gamma+1,\ldots,\delta}$, respectively, where $\delta > \gamma$. Other notations will be used as before. It must be noticed that T and every resultant main submatrix of order more than one, in a principal partitioning of T, has exactly two resultant main submatrices, and a minimal element of a resultant main submatrix of $D_{\gamma,\gamma+1,\ldots,\delta}$, by a principal partitioning process, may be equal to the minimal element of $D_{\gamma,\gamma+1,\ldots,\delta}$, that is, $t_{\gamma,\gamma+1,\ldots,\delta}$. Thus Lemma (5.2) may not be true for any principal partitioning of T. But it is true for at least one principal partitioning as we shall show in the next statement.

Lemma (5.3). "Let T be the terminal matrix of N, then there exists at least one principal partitioning of T (called satisfactory) such that for every resultant main submatrix $D_{\gamma,\gamma+1,\ldots,\delta}$ of order more than one whose resultant main submatrices, by this satisfactory principal partitioning, are $D_{\gamma,\gamma+1,\ldots,\gamma_e}$ and $D_{\gamma_e+1,\ldots,\delta}$,

$$t_{p',q'}(N^{(sp)}_{\gamma,\gamma+1,\ldots,\delta}) = t_{\gamma,\gamma+1,\ldots,\delta},$$

where,

$v_{p'} \equiv v_{\gamma,\gamma+1,\ldots,\gamma_e}$, $v_{q'} \equiv v_{\gamma_e+1,\gamma_e+2,\ldots,\delta}$,

$1 \leq \gamma \leq \gamma_e < \delta \leq n$, and $N^{(sp)}_{\gamma,\gamma+1,\ldots,\delta}$ is the signature of $D_{\gamma,\gamma+1,\ldots,\delta}$ in N with respect to this satisfactory principal partitioning of T."

Proof. We shall obtain a satisfactory principal partitioning of T by the following steps:

(1) Partition T by Wing-Chien partitioning.

(2) If there are only 2 resultant main submatrices, by Wing-Chien partitioning, in T then these are also the resultant main submatrices of T in the satisfactory principal partitioning. If there are more than 2 resultant main submatrices in T, find $N^{(wc)}_{1,2,\ldots,n}, \equiv N^{(1)}$, say. Then consider the following two cases:

(a) $N^{(1)}$ contains a node, say v_r, such that $c_r = t_{1,2,\ldots,n}$. In this case, T is partitioned as D_r and $D_{1,2,\ldots,r-1,r+1,\ldots,n}$.

(b) $N^{(1)}$ contains no such node, then there is a generalized cut-set (v_x,v_y), say S_1, such that $C(S_1) = t_{1,2,\ldots,n}$. Let N_1' be the maximal connected subnet of $N^{(1)} - S_1$ which contains v_x. In this case T is partitioned as $D_{r_1,r_2,\ldots,r_a}$ and $D_{r_{a+1},r_{a+2},\ldots,r_n}$, where $v_{r_1},v_{r_2},\ldots,v_{r_a}$ are all nodes of the resultant main submatrices of T, in its Wing-Chien partitioning, whose corresponding combined nodes are in N_1', and $v_{r_{a+1}},v_{r_{a+2}},\ldots,v_{r_n}$ are all other nodes of N.

(3) To partition any of the submatrices just obtained in Step (2) which is of order more than 2, we partition it first by Wing-Chien partitioning and obtain its signature corresponding to this partitioning, then repeat Step (2).

(4) Continue this procedure successively. In general, suppose we have arrived at the submatrix D which is of order more than 2. To partition it into two submatrices, first partition it by Wing-Chien process and within the partitioning, already reached, of T find its signature in N, say N(D), and with respect to it repeat Step (2) replacing $t_{1,2,\ldots,n}$ by the value of a minimal element of D and $N^{(1)}$ by N(D).

By Lemma (4.2), the terminal capacity in N(D) between any pair of nodes corresponding to the resultant main submatrices of D, by Wing-Chien partitioning process, is equal to the terminal capacity between the same pair of nodes in the signature of $\bar{D}$ in N with respect to Wing-Chien partitioning of T, and hence equal to the minimal element of D, by Lemma (5.2), where $\bar{D}$ is the largest resultant main submatrix of T, in Wing-Chien partitioning, such that $\bar{D}$ contains D and the value of a minimal element of $\bar{D}$ is equal to that of D. Thus the partitioning obtained by this method is satisfactory.

Hence the lemma.

The procedure will be illustrated later (in page 130) by an example.

Corollary (5.2). "If (T, S_∞) is realizable and $N^{(\infty)}$ is its ∞-subnet, then

$$t_{p'q'}(N^{(\infty)(sp)}_{\gamma,\gamma+1,\ldots,\delta}) \leq t_{\gamma,\gamma+1,\ldots,\delta}, \qquad (5.5)$$

for every resultant main submatrix $D_{\gamma,\gamma+1,\ldots,\delta}$, $\gamma < \delta$, in a satisfactory principal partitioning of T, where $N^{(\infty)(sp)}_{\gamma,\gamma+1,\ldots,\delta}$ is the signature of $D_{\gamma,\gamma+1,\ldots,\delta}$ in $N^{(\infty)}$ with respect to the satisfactory principal partitioning of T, and $v_{p'}$ and $v_{q'}$ are as defined in Lemma (5.3)."

The proof follows directly from Lemma (5.3) and the fact that $N^{(\infty)}$ is a subnet of N.

<u>Definition (5.3)</u>. Given (T, S_∞), where T is principal partitionable, $N^{(\infty)}$ will be called "satisfactory with respect to a principal partitioning", if, and only if, there is at least one principal partitioning of T such that for T and for every resultant main submatrix of order more than one, inequality (5.5) holds.

<u>Theorem (5.1)</u>. "Given (T, S_∞), where T is principal partitionable, then $N^{(\infty)}$ is satisfactory with respect to a principal partitioning if, and only if, it is satisfactory with respect to Wing-Chien partitioning."

<u>Proof</u>. If $N^{(\infty)}$ is satisfactory with respect to Wing-Chien partitioning of T, then by replacing N by $N^{(\infty)}$ and " = " sign in Steps (2b) by " $\leq$ " sign in the procedure described in the proof of Lemma (5.3) we get a procedure for obtaining a satisfactory principal partitioning of T from a given (T, S_∞).

If $N^{(\infty)}$ is satisfactory with respect to a principal partitioning of T, then let D be any resultant main submatrix, of order more than one, in Wing-Chien partitioning of T. Since the signature of D in $N^{(\infty)}$ with respect to a principal partitioning of T is the signature of D in $N^{(\infty)}$ with respect to Wing-Chien partitioning with the capacities of some elements increased to ∞, then by Lemma (5.3) $N^{(\infty)}$ is satisfactory with respect to Wing-Chien partitioning as well.

Hence the theorem.

The following example illustrates the procedure of obtaining a satisfactory principal partitioning of T from its Wing-Chien partitioning if (T, S_∞) is given and $N^{(\infty)}$ is known to be satisfactory with respect to Wing-Chien partitioning.

Example (5.1). Consider the matrix T which has been partitioned by Wing-Chien partitioning.

$$T = \begin{bmatrix} \textcircled{v_1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \textcircled{v_2} & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & \textcircled{v_3} & 4 & 2 & 2 & 2 & 2 \\ 1 & 2 & 4 & \textcircled{v_4} & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & \textcircled{v_5} & 3 & 3 & 3 \\ 1 & 2 & 2 & 2 & 3 & \textcircled{v_6} & 3.5 & 3.5 \\ 1 & 2 & 2 & 2 & 3 & 3.5 & \textcircled{v_7} & 3.5 \\ 1 & 2 & 2 & 2 & 3 & 3.5 & 3.5 & \textcircled{v_8} \end{bmatrix} \quad (5.6a)$$

The set S_∞ is given by

$$S_\infty = \left\{ e_{13}, e_{16}, e_{18}, e_{27}, e_{34}, e_{56}, e_{57}, v_3, v_6, v_7, v_8 \right\} \quad (5.6b)$$

The net $N^{(\infty)}$ of this (T,S_∞) is shown in Fig.5.3. By obtaining each of $N^{(\infty)(wc)}_{1,2,\ldots,8}$, $N^{(\infty)(wc)}_{2,3,\ldots,8}$, $N^{(\infty)(wc)}_{3,4}$, $N^{(\infty)(wc)}_{5,6,7,8}$ and $N^{(\infty)(wc)}_{6,7,8}$, one can easily see that $N^{(\infty)}$ is satisfactory with respect to Wing-Chien partitioning. Thus $N^{(\infty)}$ is satisfactory with respect to a principal partitioning. We shall obtain a satisfactory principal partitioning of T.

The submatrices D_1 and $D_{2,3,\ldots,8}$ are the resultant main submatrices of T in the required satisfactory principal partitioning. To partition $D_{2,3,\ldots,8}$, we partition it first by Wing-Chien partitioning, as in (5.6a), and then obtain its signature in $N^{(\infty)}$ with respect to this partitioning, as shown in Fig.5.4.

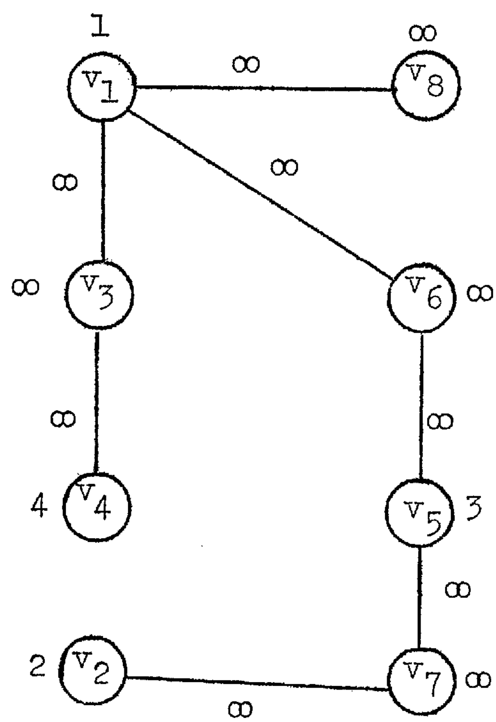


Fig.5.3 The ∞-subnet of (T,S_∞) of Ex.5.1.

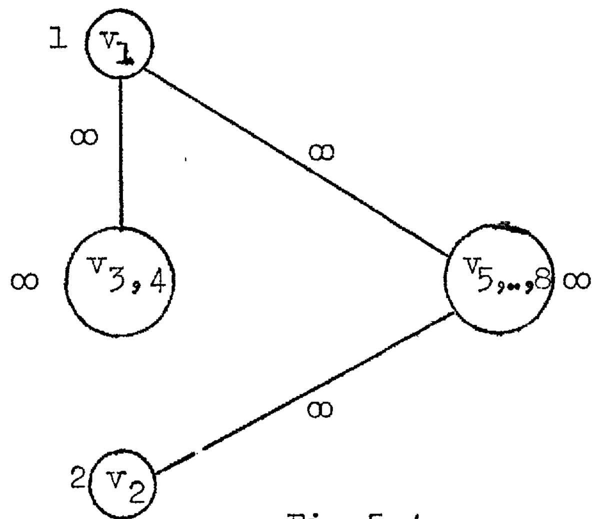


Fig.5.4

Since v_2 is a node corresponding to a resultant main submatrix, by Wing-Chien partitioning, of $D_{2,3,\ldots,8}$, then it is partitioned as D_2 and $D_{3,4,\ldots,8}$. Since $D_{3,4,\ldots,8}$ consists of $D_{3,4}$ and $D_{5,6,7,8}$ only, by Wing-Chien partitioning, then these are its resultant main submatrices in the satisfactory principal partitioning. Similarly, $D_{3,4}$ is partitioned into D_3 and D_4; and $D_{5,\ldots,8}$ is partitioned into D_5 and $D_{6,7,8}$. To partition $D_{6,7,8}$, we partition it by Wing-Chien partitioning and then obtain its signature in $N^{(\infty)}$ with respect to this and the satisfactory partitioning obtained up to this step. This signature, say N', is shown in Fig. (5.5). The minimum generalized cut-set (v_6,v_8) in N' is v_1, whose removal does not separate v_6 from v_7, thus $D_{6,7,8}$ can be partitioned as $D_{6,7}$ and D_8. If we consider the minimum generalized cut-set (v_6,v_7) in N', which is v_5, then $D_{6,7,8}$ can also be partitioned into $D_{6,8}$ and D_7. But we cannot partition it into D_6 and $D_{7,8}$, because for every generalized cut-set (v_x,v_y), $x,y \in \{6,7,8\}$, $x \neq y$, the nodes v_7 and v_8 are separated by its removal. One can see the reason for that by noticing that if D_6 and $D_{7,8}$ are taken as the resultant main submatrices of $D_{6,7,8}$ then its signature in $N^{(\infty)}$ with respect to this partitioning will be as in Fig.5.6. From which we find that the terminal capacity $(v_6,v_{7,8})$ is equal to 4, which is greater than $t_{6,7,8}$ (= 3.5), contradicting the definition of the satisfactory principal partitioning. The final satisfactory principal partitioning of T is shown in (5.7).

$$T = \begin{bmatrix} \textcircled{v_1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \textcircled{v_2} & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & \textcircled{v_3} & 4 & 2 & 2 & 2 & 2 \\ 1 & 2 & 4 & \textcircled{v_4} & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & \textcircled{v_5} & 3 & 3 & 3 \\ 1 & 2 & 2 & 2 & 3 & \textcircled{v_6} & 3.5 & 3.5 \\ 1 & 2 & 2 & 2 & 3 & 3.5 & \textcircled{v_7} & 3.5 \\ 1 & 2 & 2 & 2 & 3 & 3.5 & 3.5 & \textcircled{v_8} \end{bmatrix} . \quad (5.7)$$

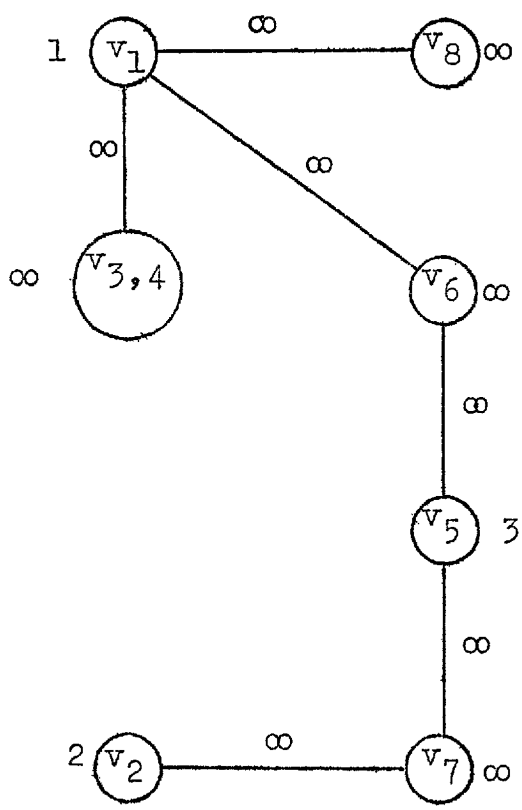


Fig.5.5 N'

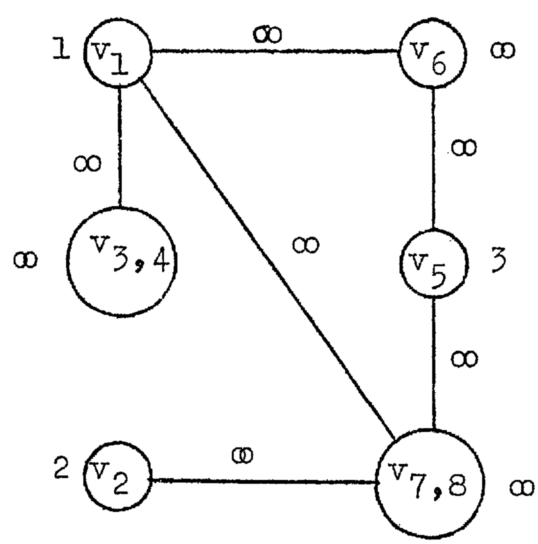


Fig. 5.6

The matrix T, in (5.7), can in fact be realized by the following procedure:

Starting from the combined node $v_{1,2,\ldots,n}$ ($\equiv N_o$, say), obtain nets N_1, N_2, ..., N_f, successively, such that N_f contains no combined nodes and every N_j, $0 \leq j < f$, contains at least one combined node. Suppose that N_{i-1}, $1 \leq i < f$, which contains the combined node $v_{\alpha,\alpha+1,\ldots,\beta}$, has been obtained. To obtain N_i apply the following steps:

(1) Delete from N_{i-1} node $v_{\alpha,\alpha+1,\ldots,\beta}$ together with all edges incident at it.

(2) Add nodes $v_{\alpha,\alpha+1,\ldots,\gamma}$ and $v_{\gamma+1,\gamma+2,\ldots,\beta}$ whose capacities are given in $N^{(\infty)(sp)}_{\alpha,\alpha+1,\ldots,\beta}$, where $D_{\alpha,\alpha+1,\ldots,\gamma}$ and $D_{\gamma+1,\gamma+2,\ldots,\beta}$ are the resultant main submatrices of $D_{\alpha,\alpha+1,\ldots,\beta}$ in the satisfactory principal partitioning of T.

(3) Connect node $v_{\alpha,\alpha+1,\ldots,\gamma}(v_{\gamma+1,\gamma+2,\ldots,\beta})$ with node $v_{i_1,i_2,\ldots,i_m}$ of N_{i-1} by an ∞-capacity edge if $e_{xy} \in S_\infty$, $x \in \{\alpha,\alpha+1,\ldots,\gamma\}$ ($\{\gamma+1,\gamma+2,\ldots,\beta\}$) and $y \in \{i_1,i_2,\ldots,i_m\}$. Denote the net obtained by $\bar{N}_i$.

(4) Find a minimum generalized cut-set (v_{p_i},v_{q_i}), $v_{p_i} \equiv v_{\alpha,\alpha+1,\ldots,\gamma}$, $v_{q_i} \equiv v_{\gamma+1,\gamma+2,\ldots,\beta}$, in $\bar{N}_i$. Denote this cut-set by S_i. Let N_1' and N_2' be the maximal

connected subnets of $\bar{N}_i - S_i$ which contain nodes v_{p_i} and v_{q_i}, respectively.

(5) Obtain N_i from $\bar{N}_i$ by (a) connecting an edge of capacity $[t_{\alpha,\alpha+1,\ldots,\beta} - t_{p_i q_i}(\bar{N}_i)]$ between nodes v_{p_i} and v_{q_i}, and (b) reconnecting all finite-capacity edges which were incident at $v_{\alpha,\alpha+1,\ldots,\beta}$ in N_{i-1} by using the following technique: Let $e_{xp_{i-1}}$, $v_{p_{i-1}} \equiv v_{\alpha,\alpha+1,\ldots,\beta}$, be any finite-capacity edge incident at $v_{p_{i-1}}$ in N_{i-1}. If $C(S_i) \leq t_{\alpha,\alpha+1,\ldots,\beta}$ then reconnect $e_{xp_{i-1}}$ between nodes v_x and v_{p_i} (v_{q_i}) if v_x in N_1' (N_2'). If v_x is in neither N_1' nor N_2' then $e_{xp_{i-1}}$ can be reconnected with either v_{p_i} or v_{q_i} unless one of them is combined, say v_{p_i}, and the other is not combined; in this case $e_{xp_{i-1}}$ is reconnected between v_x and v_{p_i}. If $C(S_i) > t_{\alpha,\alpha+1,\ldots,\beta} = c_{q_i}$ then $e_{xp_{i-1}}$ is reconnected between v_x and v_{p_i}.

A realization of (T, S_∞), which is given in (5.6), is shown in Fig.5.7.

Consideration of a number of examples seems to suggest that the above procedure of the synthesis for (T, S_∞) is always valid, and so the conditions (1) T is principal partitionable and (2) $N^{(\infty)}$ is satisfactory with respect to Wing-Chien partitioning are sufficient for the realization of (T, S_∞), but no proof of this has yet been found.

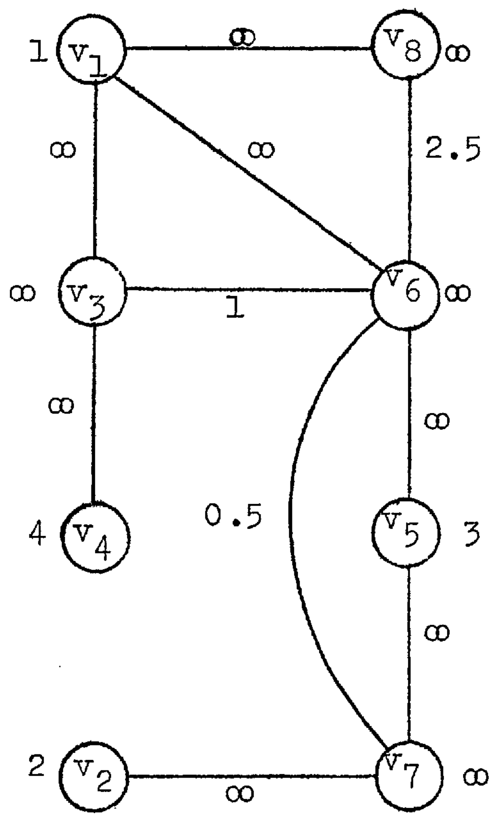


Fig.5.7 A realization of (T, S_∞) given in (5.6).

CONCLUSION

Several problems on the synthesis of communication nets have been solved in this thesis. We hope that the ideas and techniques presented here will prove to be useful in tackling other problems which have not yet been solved. Some of these unsolved problems in this field are:

(1) To obtain necessary and sufficient conditions for realizability of a matrix T as the terminal matrix of an oriented communication net which are general and easy to check on the given T.

(2) To find an optimal synthesis for each of the special terminal matrices introduced in Chapter II, i.e. a tree-terminal matrix, a loop-terminal matrix, a separable terminal matrix and a triangular terminal matrix.

(3) To find a sufficient condition for realizability of a terminal matrix in which there are k different entries for a fixed k.

(4) To obtain, depending on Lemma 4.2, a systematic method for evaluating the terminal matrix of a non-oriented radio-wire-communication net by solving (n-1) flow problems.

(5) To find necessary and sufficient conditions for realizability of a given (T, S_{∞}).

REFERENCES

1. Dantzig, G.B., and Fulkerson, D.R.: "On the Max-Flow Min-Cut Theorem of Networks, Linear Inequalities and Related Systems", Annals of Math. Study 38, pp.215-221, (Princeton, 1956).
2. Elias, P., Feinstein, A., and Shannon, C.E.: "A Note on the Maximal Flow Through a Network", IRE Trans. on Information Theory, Vol.IT-2, pp.117-119; December, 1956.
3. Ford, L.R., and Fulkerson, D.R.: "Maximal Flow Through a Network", Canad.J.Math., Vol.8, pp.399-404; 1956.
4. Ford, L.R., and Fulkerson, D.R.: "A Simple Algorithm For Finding Maximal Network Flows and an Application to the Hitchcock Problem", Canad.J. Math., Vol.9, pp.210-218; 1957.
5. Ford, L.R., and Fulkerson, D.R.: <u>Flows in Networks</u>, Princeton University Press, Princeton, 1962.
6. Gomory, R.E., and Hu, T.C.: "Multi-terminal Network Flows", IBM Report No.RC-318, September, 1960.
7. Jelinek, F., and Mayeda, W.: "On the Maximum Number of Different Entries in the Terminal Capacity Matrix of Oriented Communication Nets", IEEE Trans. on Circuit Theory, Vol.CT-10, No.2, pp.307-308, June, 1963.

8. Kim, W.H., and Chien, R.T.: Topological Analysis and Synthesis of Communication Networks, Columbia University Press, New York, 1962.

9. Mayeda, W.: "Terminal and Branch Capacity Matrices of a Communication Net", IRE Trans.on Circuit Theory, Vol.CT-7, No.3; pp.261-269; September, 1960.

10. Mayeda, W.: "On Oriented Communication Nets", IRE Trans.on Circuit Theory, Vol.CT-9, No.3, pp.261-267; September, 1962.

11. Reed, M.B.: "The Seg: A New Class of Subgraphs", IRE Trans.on Circuit Theory, Vol.CT-8, No.1; pp.17-22; March, 1961.

12. Seshu, S., and Reed, M.B.: Linear Graphs and Electrical Networks, Addison-Wesley, Reading (Mass.), 1961.

13. Tang, D.T., and Chien, R.T.: "Analysis and Synthesis Techniques of Oriented Communication Nets", IRE Trans.on Circuit Theory, Vol.CT-8, No.1; pp.39-43; March, 1961.

14. Wing, O., and Chien, R.T.: "Optimal Synthesis of a Communication Net", IRE Trans.on Circuit Theory, Vol.CT-8, No.1; pp.44-49; March, 1961.

15. Whitney, H.: "Non-separable and Planar Graphs", Trans.Am.Math.Soc., Vol.34; pp.339-362; 1932.

16. Yau, S. S.: "Synthesis of Radio-Communication Nets", IRE Trans.on Circuit Theory, Vol.CT-9, No.1; pp.62-68; March, 1962.

17. Yau, S. S.: "A Generalization of the Cut-Set", J.Franklin Inst., Vol.273, No.1; pp.31-48; January, 1962.

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ON TWO-TREE TRANSFORMATIONS AND THE SEPARATION OF TWO-TREE SETS

BY

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ABSTRACT

The present paper presents a necessary and sufficient condition for a two-tree (ij,lk) to be obtained from another two-tree (ij,lk) of a connected graph, by a finite number of elementary two-tree (ij,lk) transformations. A procedure for separating the common terms between $T_{i,k}$ and $T_{j,l}$ into two groups, one of them $T_{ij,lk}$ and the other $T_{il,jk}$, is given in this paper.

I. INTRODUCTION

In 1959, Fujisawa (1)[2] introduced the concept of the distance between any two trees of a connected graph G. And he had introduced, in the same paper, the concept of an elementary operation (or elementary transformation) on a tree of G. He used this operation to list all the trees of G.

In 1960, Watanabe (7) and more recently Hakimi (2) have used the concept of distance to find other procedures for listing all the trees of G. Watanabe's definition of the distance differs from that given by Fujisawa and Hakimi, and a slightly modified version of it is used in the present paper.

Hakimi and Mayeda (3) introduced the concept of an elementary two-tree (12,0) transformation, and they showed that it is possible to obtain any term of $T_{12,0}$ from another term in $T_{12,0}$ by a finite number of elementary two-tree (12,0) transformations.

In this paper, we shall give a definition for the distance between any two subgraphs of G which have the same number of edges (branches). And we shall introduce the concept of an elementary two-tree (ij, lk) transformation, and find a necessary and sufficient condition for two terms of

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

$T_{ij,lk}$ to be derivable one from the other by a finite number of elementary two-tree (ij, lk) transformations.

Next, we shall use the concept of distance to separate the common terms between $T_{i,k}$ and $T_{lj,}$ into two groups, one of them $T_{ij,lk}$ and the other $T_{il,jk}$. This result is useful in the synthesis of four-terminal networks without mutual inductance by topological considerations (4).

II. ELEMENTARY TWO-TREE (ij,lk) TRANSFORMATIONS

We assume a connected graph G containing e edges and v vertices. A complete discussion of the properties of linear graphs may be found in the references cited in this paper. Only those definitions which are needed in the paper are given here.

A subgraph g of G may be represented by the product of its edges. If b is an edge of G, then $g_1 \cdot b = g_2$, where g_2 is a subgraph of G containing all the edges of g_1 and edge b; if b is an edge of g_1, g_2 contains two edges b in parallel.

The *union of* g_1 *and* g_2 is represented by $g_1 \cup g_2 = g_3$, where g_3 contains all edges and vertices of g_1 and g_2. (The same notation is used for the union of subgraphs as well as for the union of sets.)

The *intersection of two sets* S_1 *and* S_2 is represented by $S_1 \cap S_2 = S_3$, where S_3 is a set containing only those elements of S_1 and S_2 which are in both.

A *two-tree* (ij, lk) of G is a pair of unconnected, circuitless subgraphs, each subgraph being connected, and one of them containing vertices i and j, the other containing vertices l and k, and together including all the vertices of G. Hence the two-tree contains $(v\text{-}2)$ edges.

Let $T_{ij,lk}$ be the set of all two-trees (ij, lk) of the graph G, and let t_f be any member of $T_{ij,lk}$. It will be represented by

$$t_f = b_{f1} \cdot b_{f2} \cdots b_{f_{v-2}},$$

in which b_{fy} $(1 \leq y \leq v - 2)$ is an edge of t_f.

If g contains an edge b, then $\frac{g}{b}$ is g with edge b removed (open-circuited).

Definition 1. Let b_x be any edge of G; the operation

$$t_x = t_f \cdot b_x / b_{fy}, \tag{1}$$

where $1 \leq y \leq v - 2$, is called an *elementary two-tree* (ij, lk) *transformation on* t_f if t_x is a two-tree (ij, lk) of G. b_{fy} does not always exist such that t_x is in $T_{ij,lk}$; that is, this transformation is not necessarily always possible. We shall see that the existence of such a transformation depends on b_x.

Let

$$P_{ij}^{(f)} \equiv b_{fij}^{(1)} \cdot b_{fij}^{(2)} \cdots b_{fij}^{(a)}$$

consist of edges of t_f which form a path between i and j; and suppose that $b_{fij}^{(1)}$ is connected to vertex i and $b_{fij}^{(a)}$ is connected to vertex j, and in general $b_{fij}^{(y)}$ is connected to $b_{fij}^{(y-1)}$, where $2 \leq y \leq a$. Let similar assumptions be made for

$$P_{lk}^{(f)} \equiv b_{flk}^{(1)} \cdot b_{flk}^{(2)} \cdots b_{flk}^{(b)}.$$

Furthermore, let $V_{ij}^{(f)}$ and $V_{lk}^{(f)}$ be the sets of all vertices of G which are contained in $P_{ij}^{(f)}$ and $P_{lk}^{(f)}$, respectively.

Definition 2.[3] We shall call an edge b a *bridged-edge with respect to* t_f, ($t_f \in T_{ij,lk}$), if $t_f \cdot b$ is a tree of G. (The edge b cannot be in t_f.)

Definition 3. The unique path which passes through a bridged-edge b, has one of its terminals on $P_{ij}^{(f)}$ and the other on $P_{lk}^{(f)}$, and contains no other vertices of $V_{ij}^{(f)}$ and $V_{lk}^{(f)}$ is called a *bridged-path* (b) with respect to t_f and denoted by $B(t_f, b)$. In $t_f \cdot b$, the paths $P_{il}^{(f)}$, $P_{ik}^{(f)}$, $P_{jk}^{(f)}$ and $P_{jl}^{(f)}$ contain $B(t_f, b)$ in common.

Definition 4. b is called a *direct bridged-edge* with respect to t_f if $B(t_f, b)$ contains no edges other than b, that is, $b \equiv B(t_f, b)$. That is, b has one of its vertices on $P_{ij}^{(f)}$ and the other on $P_{lk}^{(f)}$.

Let b_x be an edge of G not in t_f.

If b_x is not a bridged-edge with respect to t_f, then $t_f \cdot b_x$ contains a circuit in one of the connected subgraphs of t_f and any edge of this circuit other than b_x may be taken as b_{fy} such that t_x, (see Eq. 1) is a two-tree (ij, lk) of G.

If b_x is a bridged-edged with respect to t_f but is not a direct bridged-edge, then $t_f \cdot b_x$ is a tree and $B(t_f, b_x)$ contains at least one edge (say b_{fy}) other than b_x; and because $B(t_f, b_x)$ has no vertices other than its terminals in common with $P_{ij}^{(f)}$ and $P_{lk}^{(f)}$, then b_{fy} is not an edge of $P_{ij}^{(f)}$ or $P_{lk}^{(f)}$. Thus t_x is a two-tree (ij, lk) of G.

If b_x is a direct bridged-edge with respect to t_f, then $t_f \cdot b_x$ is a tree of G and contains paths $P_{li}^{(f)}$, $P_{ik}^{(f)}$, $P_{jl}^{(f)}$ and $P_{jk}^{(f)}$ all passing through b_x. Thus on removing any b_{fy} (other than b_x) there must remain one path via b_x between i or j and k or l. Hence there is no b_{fy} such that t_x is a two-tree (ij, lk) of G.

Now, we are able to prove the following statement.

Lemma 1. Given a two-tree, say t_f, in $T_{ij,lk}$, every two-tree, say t_h, in $T_{ij,lk}$ can be obtained by a finite number of elementary two-tree (ij, lk) transformations, if

$$V_{ij}^{(f)} \cap V_{lk}^{(h)} = \phi,$$

where ϕ is a null set.

The proof of Lemma 1 is in the Appendix.

[3] The definition of bridged-edge given here is equivalent to Wing & Kim's (7) definition.

Let t_f and t_h be in $T_{ij,lk}$. If we can obtain t_h from t_f by a finite number of elementary two-tree (ij, lk) transformations, then we shall represent this relation between t_f and t_h by

$$t_h = \mathfrak{L}^{(ij,lk)}(t_f).$$

But if we cannot obtain t_h from t_f by a finite number of elementary two-tree (ij, lk) transformations, we write

$$t_h \neq \mathfrak{L}^{(ij,lk)}(t_f).$$

It is easy to see that, if

$$t_h = (\neq)\,\mathfrak{L}^{(ij,lk)}(t_f),$$

then

$$t_f = (\neq)\,\mathfrak{L}^{(ij,lk)}(t_h).$$

Lemma 2 of Hakimi and Mayeda (3) which states: "Given a two-tree in $T_{12,0}$, every two-tree in $T_{12,0}$ can be obtained by a finite number of elementary two-tree (12,0) transformations," is a special case of Lemma 1 given here.

Lemma 2. Let t_f and t_h be in $T_{ij,lk}$. If $t_f \cup t_h$ contains a path $P_{ij}^{(x)}$ such that

$$V_{ij}^{(x)} \cap V_{lk}^{(f)} = \phi$$

and

$$V_{ij}^{(x)} \cap V_{lk}^{(h)} = \phi,$$

then

$$t_h = \mathfrak{L}^{(ij,lk)}(t_f).$$

Lemma 1 is a special case of this lemma when $P_{ij}^{(x)} \equiv P_{ij}^{(f)}$ or $P_{ij}^{(x)} \equiv P_{ij}^{(h)}$.[4]

Proof: Let

$$P_{ij}^{(x)} = b_1^{(x)} \cdot b_2^{(x)} \cdots b_u^{(x)},$$

taking the edges in order starting at vertex i.

By the first condition of the lemma, $b_r^{(x)}$ is not a direct bridged edge with respect to $t_f^{(r-1)}$, where

$$t_f^{(r)} = t_f^{(r-1)} \cdot b_r^{(x)}/b_f^{(r)}, \qquad (1 \leq r \leq u),$$

in which $b_f^{(r)}$ is an edge of t_f. Each $t_f^{(r)}$ is in $T_{ij,\,k}$. Thus starting with t_f and using the edges of $P_{ij}^{(x)}$ in order, we get a two-tree (ij, lk), $t_f^{(u)}$, by u elementary two-tree (ij, lk) transformations. That is,

$$t_f^{(u)} = \mathfrak{L}^{(ij,lk)}(t_f).$$

By the second condition and using Lemma 1,

$$t_h = \mathfrak{L}^{(ij,lk)}(t_f^{(u)}).$$

Hence

$$t_h = \mathfrak{L}^{(ij,lk)}(t_f),$$

[4] Two subgraphs of G identically equal if they have the same edges.

which proves the lemma.

The next statement is the general case of this transformation.

Theorem 1. If t_f and t_h are in $T_{ij,lk}$, and $t_f \cup t_h$ contains a set of $(u + 2)$ paths

$$P_{lk}^{(f)},\ P_{ij}^{(1)},\ P_{lk}^{(2)},\ \cdots,\ P_{lk}^{(u)},\ P_{ij}^{(h)} \qquad \text{(if } u \text{ is even)}$$

or

$$P_{lk}^{(f)},\ P_{ij}^{(1)},\ P_{lk}^{(2)},\cdots,\ P_{ij}^{(u)},\ P_{lk}^{(h)} \qquad \text{(if } u \text{ is odd)},$$

such that no two successive members have common vertices, then

$$t_h = \mathfrak{L}^{(ij,lk)}(t_f).$$

Proof: Starting with t_f and applying the technique used in the first part of the proof of Lemma 2, we can obtain t_1, containing $P_{ij}^{(1)}$,

$$t_1 = \mathfrak{L}^{(ij,lk)}(t_f).$$

In general we can obtain t_{r+1} from t_r, $(1 \leq r \leq u)$,

$$t_{r+1} = \mathfrak{L}^{(ij,lk)}(t_r),$$

where t_r contains $P_{ij}^{(r)}$ and t_{r+1} contains $P_{lk}^{(r+1)}$ if r is odd; or t_r contains $P_{lk}^{(r)}$ and t_{r+1} contains $P_{ij}^{(r+1)}$ if r is even. t_{u+1} contains $P_{ij}^{(h)}$ (or $P_{lk}^{(h)}$).

By Lemma 1,

$$t_h = \mathfrak{L}^{(ij,lk)}(t_{u+1}).$$

Thus

$$t_h = \mathfrak{L}^{(ij,lk)}(t_f).$$

Hence the theorem.

Corollary 1. If $P_{ij}^{(f)} \equiv P_{ij}^{(h)}$, where t_f and $t_h \in T_{ij,lk}$, then

$$t_h = \mathfrak{L}^{(ij,lk)}(t_f).$$

The proof is obvious.

Definition 5. Let g_1 and g_2 be any subgraphs of G which have the same number of edges. The *distance*[5] between g_1 and g_2 is denoted by $d(g_1, g_2)$ = number of edges in g_1 (or g_2) which are not in g_2 (or g_1). $d(g_1, g_2)$ could be written in the alternative form

$$d(g_1, g_2) = \tfrac{1}{2}\ \text{number of edges of}\ g_1 \oplus g_2.^{6}$$

We call g_1 and g_2 *neighboring* if $d(g_1, g_2) = 1$.

Lemma 3. If $d(t_f, t_h) = 1$, where t_f and $t_h \in T_{ij,lk}$, then

$$\begin{aligned} V_{ij}^{(f)} \cap V_{lk}^{(h)} &= \phi \\ V_{lk}^{(f)} \cap V_{ij}^{(h)} &= \phi. \end{aligned} \tag{3}$$

---

[5] Several authors (1,2) have given definitions for the distance between two trees. This definition is a general definition for distance; and it differs from Watanabe's definition which is as follows: The distance between two subgraphs is the total number of edges in them, diminished by the number of common edges.

[6] The ring sum, $g_1 \oplus g_2$, of g_1 and g_2 consists of edges of g_1 and g_2 which are not in both.

Proof: Let b_f be the edge of t_f which is not in t_h, and let b_h be the edge of t_h which is not in t_f. Then

$$t_h = t_f \cdot b_h / b_f.$$

If $t_f \cdot b_h$ contains a circuit, then Eq. 3 follows. If $t_f \cdot b_h$ is a tree of G, then b_f must be contained in $B(t_f, b_h)$. Thus t_f and t_h will have the same paths between i and j and between l and k; and Eq. 3 follows. Hence the lemma.

The following theorem is the converse of Theorem 1.

Theorem 2. If t_f and t_h are in $T_{ij,\,lk}$, and $t_f \cup t_h$ does not contain a set of $(U + 2)$ paths (where U is any finite number):

$$P^{(f)}_{lk},\ P^{(1)}_{ij},\ P^{(2)}_{lk}, \cdots,\ P^{(u)}_{lk},\ P^{(h)}_{ij} \qquad \text{(if } U \text{ is even)}$$

or

$$P^{(f)}_{lk},\ P^{(1)}_{ij},\ P^{(2)}_{lk}, \cdots,\ P^{(u)}_{ij},\ P^{(h)}_{lk} \qquad \text{(if } U \text{ is odd)},$$

such that no two successive members have common vertices, then

$$t_h \neq \mathfrak{L}^{(ij,\,lk)}(t_f).$$

Proof: Suppose, if possible,

$$t_h = \mathfrak{L}^{(ij,\,lk)}(t_f);$$

then there is a set of two-trees (ij, lk) of G

$$t_f,\ t^{(1)},\ t^{(2)}, \cdots,\ t^{(\lambda)},\ t_h$$

in which λ is finite, such that

$$d(t^{(1)}, t_f) = d(t^{(r)}, t^{(r+1)}) = d(t^{(\lambda)}, t_h) = 1,$$

where $1 \leq r < \lambda$. Moreover,

$$t_f \cup t_h \cup \left[\bigcup_{r=1}^{\lambda} t^{(r)} \right] = t_f \cup t_h.$$

Thus, by Lemma 3, $t_f \cup t_h$ contains a set of paths

$$P^{(f)}_{lk},\ P^{(1)}_{ij},\ P^{(2)}_{lk}, \cdots,\ P^{(\lambda)}_{lk},\ P^{(h)}_{ij} \qquad \text{(if } \lambda \text{ is even)}$$

or

$$P^{(f)}_{lk},\ P^{(1)}_{ij},\ P^{(2)}_{lk}, \cdots,\ P^{(\lambda)}_{ij},\ P^{(h)}_{lk} \qquad \text{(if } \lambda \text{ is odd)},$$

such that no two successive members have common vertices. This contradicts the hypothesis of the theorem. Hence the theorem.

Thus we have found a necessary and sufficient condition such that

$$t_h = \mathfrak{L}^{(ij,\,lk)}(t_f).$$

There are many special cases of Theorem 2, but they are long and tedious to state and to prove. We omit these cases since they are not of great interest.

III. THE SEPARATION OF $T_{i,k} \cap T_{j,l}$ INTO TWO GROUPS

We denote by $T_{a,b}$ the set of all two-trees (a, b) of a graph G.
Given $T_{i,k}$ and $T_{j,l}$ only, it is easy to see that

$$T_{i,k} \cap T_{j,l} = T_{ij,lk} \cup T_{il,jk}.$$

Our present problem is to find how to separate $T_{ij,lk}$ from $T_{il,jk}$ (4,5,6,9). It is necessary to notice that it is impossible to say which set of two-trees belongs to $T_{ij,lk}$ and which to $T_{il,jk}$; it is required only to separate the set

$$F \equiv T_{ij,lk} \cup T_{il,jk}$$

into two groups, one of them $T_{ij,lk}$ and the other $T_{il,jk}$.

To find a procedure for separating the set F, we need to prove a number of preliminary results.

Definition 6. A maximal connected set (ij, lk), denoted by $S_{ij,lk}$, is a non-empty set of two-trees (ij, lk) of G, such that

(1) if $t_h = \mathcal{L}^{(ij,lk)}(t_f)$, where $t_f \in S_{ij,lk}$, then $t_h \in S_{ij,lk}$;

(2) if $t_h \neq \mathcal{L}^{(ij,lk)}(t_f)$, where $t_f \in S_{ij,lk}$, then $t_h \notin S_{ij,lk}$.

Definition 7. Let S_1 and S_2 be any two sets of two-trees of G. We say S_1 *and* S_2 *are m-joined sets* (or S_1 is m-joined with S_2), if there exist $t_1^{(m)} \in S_1$ and $t_2^{(m)} \in S_2$ such that

$$d(t_1^{(m)}, t_2^{(m)}) = m,$$

and for all other elements $t_1^{(x)} \in S_1$ and $t_2^{(y)} \in S_2$

$$d(t_1^{(x)}, t_2^{(y)}) \geq m.$$

Now, let

$$D_1 = \{d_1^{(1)}, d_2^{(1)}, \cdots, d_m^{(1)}\}$$

be the set of all edges of $t_1^{(m)}$ which are not in $t_2^{(m)}$; and let

$$D_2 = \{d_1^{(2)}, d_2^{(2)}, \cdots, d_m^{(2)}\}$$

be the set of all edges of $t_2^{(m)}$ which are not in $t_1^{(m)}$. Furthermore, let

$$C = \{c_1, c_2, \cdots, c_{v-m-2}\}$$

be the set of all edges which are in both $t_1^{(m)}$ and $t_2^{(m)}$.

Let $d_s^{(1)}$, $1 \leq s \leq m$, be any element of D_1. If $t_2^{(m)} \cdot d_s^{(1)}$ contains a circuit, then this circuit must contain at least one element of D_2, say $d_s^{(2)}$. Then $t_2^{(m)} \cdot d_s^{(1)}/d_s^{(2)}$ is a member of S_2, and

$$d(t_1^{(m)}, t_2^{(m)} \cdot d_s^{(1)}/d_s^{(2)}) = m - 1.$$

This is impossible, by Definition 7. Thus $t_2^{(m)} \cdot d_s^{(1)}$ is a tree. Hence each edge of D_1 is a bridged-edge with respect to $t_2^{(m)}$; and each edge of D_2 is a bridged-edge with respect to $t_1^{(m)}$.

Let $S^{(1)}_{ij,\,lk}$ and $S^{(2)}_{ij,\,lk}$ be m-joined sets of G, and $d(t_1^{(m)}, t_2^{(m)}) = m$, where

$$t_1^{(m)} \in S^{(1)}_{ij,\,lk} \text{ and } t_2^{(m)} \in S^{(2)}_{ij,\,lk}.$$

Let $P^{(1)}_{ij}$ and $P^{(1)}_{lk}$ be the paths of $t_1^{(m)}$ between i and j and between l and k, respectively. Similarly, let $P^{(2)}_{ij}$ and $P^{(2)}_{lk}$ be the paths of $t_2^{(m)}$.

From Definition 7, we can see that $B(t_1^{(m)}, d_s^{(2)})$, $1 \leq s \leq m$, contains no edges of D_1, since otherwise $S^{(1)}_{ij,\,lk}$ and $S^{(2)}_{ij,\,lk}$ will not be m-joined sets. Thus $B(t_1^{(m)}, d_s^{(2)})$ consists of $d_s^{(2)}$ and edges of C only. Similarly, $B(t_2^{(m)}, d_s^{(1)})$ contains no edges of D_2, and consists of $d_s^{(1)}$ and edges of C only. Moreover, in $t_1^{(m)} \cup t_2^{(m)}$, $B(t_1^{(m)}, d_s^{(2)}$ is unique for each s; and similarly for $B(t_2^{(m)}, d_s^{(1)})$.

Lemma 4. No two bridged paths $B(t_1^{(m)}, d_s^{(2)})$ and $B(t_1^{(m)}, d_q^{(2)})$, $s \neq q$ and $1 \leq s, q \leq m$, have any vertices in common except possibly one only of their terminals.

Proof: $t_1^{(m)} \cdot d_s^{(2)} \cdot d_q^{(2)}$ is a connected subgraph of G containing all the vertices and one circuit only which contains $d_s^{(2)}$, $d_q^{(2)}$ and at least one edge of D_1, say $d_r^{(1)}$, which is either an edge of $P^{(1)}_{ij}$ or of $P^{(1)}_{lk}$. Thus, this circuit does not consist of $B(t_1^{(m)}, d_s^{(2)})$ and $B(t_1^{(m)}, d_q^{(2)})$ only; that is, the terminals of $B(t_1^{(m)}, d_s^{(2)})$ can not be terminals of $B(t_1^{(m)}, d_q^{(2)})$.

If $B(t_1^{(m)}, d_s^{(2)})$ and $B(t_1^{(m)}, d_q^{(2)})$ have common vertices other than their terminals, then they have at least one edge of C, say c_1, in common, otherwise $t_1^{(m)} \cdot d_s^{(2)} \cdot d_q^{(2)}$ would contain two (or more) circuits. Thus

$$t_1^{(\hat{m})} \equiv \frac{t_1^{(m)} \cdot d_s^{(2)} \cdot d_q^{(2)}}{d_r^{(1)} \cdot c_1}$$

is a two-tree (ij, lk). If $d_r^{(1)}$ is an edge of, say $P^{(1)}_{ij}$, then $t_1^{(\hat{m})}$ contains $P^{(1)}_{lk}$, and by Corollary 1, $t_1^{(\hat{m})}$ belongs to $S^{(1)}_{ij,\,lk}$. But

$$d(t_1^{(\hat{m})}, d_2^{(m)}) = m - 1,$$

which contradicts the hypothesis. Hence the lemma.

The next statement follows directly from Lemma 4.

Lemma 5. In $t_1^{(m)} \cup t_2^{(m)}$, there is no path which contains more than one edge of D_1 (or D_2), has its terminals on $P^{(2)}_{ij}$ and $P^{(2)}_{lk}$ (or $P^{(1)}_{ij}$ and $P^{(1)}_{lk}$), and contains no other vertices of $V^{(2)}_{ij}$ and $V^{(2)}_{lk}$ (or $V^{(1)}_{ij}$ and $V^{(1)}_{lk}$).

We are now in a position to prove the following important theorem.

Theorem 3. All the edges of D_1 are contained in $P^{(1)}_{ij} \cup P^{(1)}_{lk}$.

Proof: Let $d_r^{(1)}$ be in D_1 and suppose if possible that $d_r^{(1)}$ is neither in $P^{(1)}_{ij}$ nor in $P^{(1)}_{lk}$. Let v_1 and v_2 be the terminals of $B(t_2^{(m)}, d_r^{(1)})$, where $v_1 \in V^{(2)}_{ij}$ and $v_2 \in V^{(2)}_{lk}$. Furthermore, let $d_r^{(1)}$ be in $g^{(1)}_{ij}$, the part of $t_1^{(m)}$ which contains vertices i and j. Then v_1 and v_2 are in $g^{(1)}_{ij}$, since $B(t_2^{(m)}, d_r^{(1)})$ consists of $d_r^{(1)}$ and edges of C, that is all the edges of $B(t_2^{(m)}, d_r^{(1)})$ are in $g^{(1)}_{ij}$. Now v_1 and v_2 cannot both be in $V^{(1)}_{ij}$, since $g^{(1)}_{ij}$ contains no circuits and $d_r^{(1)}$ is not an edge of $P_{ij}^{(1)}$ by our assumption. If $B(t_2^{(m)}, d_r^{(1)})$ and $P_{ij}^{(1)}$ have no common vertex, then it is easy to see that there is a path in $g_{ij}^{(1)}$, say P_r, which contains $d_r^{(1)}$ and has only one vertex in common with $P_{ij}^{(1)}$, which must be a terminal of P_r, the other terminal being either v_1 or v_2.

Let v_c be the common vertex of P_r and $P_{ij}^{(1)}$ or of $B(t_2^{(m)}, d_r^{(1)})$ and $P_{ij}^{(1)}$.

Now we have to consider two cases: (1) the path between v_c and v_1 in $g_{ij}^{(1)}$ contains $d_r^{(1)}$; or (2) the path between v_c and v_2 in $g_{ij}^{(1)}$ contains $d_r^{(1)}$.

Case 1: On $P_{ij}^{(2)}$, there is at least one edge of D_2 between v_1 and j, since otherwise $g_{ij}^{(1)}$ will contain circuits. Let $d_r^{(2)}$ be the first edge of D_2 after v_1 between v_1 and j on $P_{ij}^{(2)}$. Let v_3 and v_4 be the vertices of $d_r^{(2)}$, v_3 being the one nearer to v_1. Then v_3 must be in $g_{ij}^{(1)}$. Also v_4 cannot be in $g_{ij}^{(1)}$, that is, v_4 must be in $g_{lk}^{(1)}$, since otherwise $t_1^{(m)} \cdot d_r^{(2)}$ would contain a circuit, and this leads to a new term of $S_{ij,\,lk}^{(1)}$ having distance $(m - 1)$ from $t_2^{(m)}$. Thus $t_1^{(m)} \cdot d_r^{(2)}$ is a tree of G. The path between v_4 and v_c in $t_1^{(m)} \cdot d_r^{(2)}$ contains no vertices of $V_{ij}^{(1)}$ except v_c, since otherwise $g_{ij}^{(1)}$ contains a circuit or else $d_r^{(1)}$ is an edge of $P_{ij}^{(1)}$. Hence $B(t_1^{(m)}, d_r^{(2)})$ contains the path between v_c and v_1 in $t_1^{(m)} \cdot d_r^{(2)}$, which is impossible since $B(t_1^{(m)}, d_r^{(2)})$ cannot contain an edge of D_1, as we have shown in the previous discussion. Thus $d_r^{(1)}$ must be an edge of $P_{ij}^{(1)}$.

Case 2: Between v_2 and l on $P_{lk}^{(2)}$, there is at least one edge of D_2, since otherwise there will be a path in $t_1^{(m)}$ between l and a vertex of $P_{ij}^{(1)}$ which is impossible. Thus by the same technique that was applied to Case 1, we can show that $d_r^{(1)}$ must be an edge of $P_{ij}^{(1)}$.

If we had assumed initially that $d_r^{(1)}$ was in $g_{lk}^{(1)}$, then it would follow similarly that $d_r^{(1)}$ must be an edge of $P_{lk}^{(1)}$. Hence the theorem.

Thus any path between a vertex of $V_{ij}^{(2)}$ and a vertex of $V_{lk}^{(2)}$ which contains no other vertices of either, is a bridged-path with respect to $t_2^{(m)}$, and is contained in either $P_{ij}^{(1)}$ or $P_{lk}^{(1)}$.

Moreover, from Lemmas 4 and 5, and Theorem 3, between any two successive edges of D_1 on $P_{ij}^{(1)}$ (or $P_{lk}^{(1)}$) there is at least one vertex of either $V_{ij}^{(2)}$ or $V_{lk}^{(2)}$.

Now, to simplify the discussion in the following subsection, we short-circuit all the edges of C in $t_1^{(m)} \cup t_2^{(m)}$ and we denote the new graph by g_s. If $t_{1s}^{(m)}$ and $t_{2s}^{(m)}$, respectively, represent $t_1^{(m)}$ and $t_2^{(m)}$ with the edges of C short-circuited, then

$$g_s = t_{1s}^{(m)} \cup t_{2s}^{(m)}.$$

Each bridged-path with respect to $t_1^{(m)}$ (or $t_2^{(m)}$) in $t_1^{(m)} \cup t_2^{(m)}$ becomes a direct bridged-edge with respect to $t_{1s}^{(m)}$ (or $t_{2s}^{(m)}$), respectively. Note that $t_{1s}^{(m)}$ and $t_{2s}^{(m)}$ are two-trees $(i'j', l'k')$, but not two-trees (ij, lk) of G unless C is empty; where vertices i', j', l' and k' of g_s correspond to i, j, l, and k, respectively.

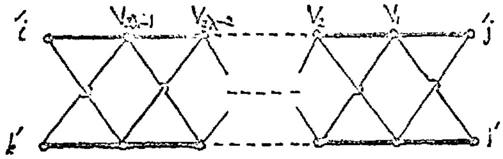

FIG. 1. g_s for $t_1^{(m)} \in S^{(1)}_{ij,\, lk}$ and $t_2^{(m)} \in S^{(2)}_{ij,\, lk}$

The number of edges in g_s is $2m$.

Hence, from Lemmas 4 and 5, and Theorem 3, g_s will be as in Fig. 1, where thick lines denote edges of $t_{1s}^{(m)}$, thin lines denote edges of $t_{2s}^{(m)}$.

Thus the number of edges of D_2 is

$$m = 2(2\lambda - 1) + 2 = 4\lambda.$$

Thus, we have the following theorem:

Theorem 4. If $S^{(1)}_{ij,\, lk}$ and $S^{(2)}_{ij,\, lk}$ are m-joined sets, then

$$m = 4\lambda,$$

where λ is a positive integer.

It is easy to see that all previous discussion (from Definition 7 onwards) except Theorem 4 and Fig. 1 hold if $S^{(2)}_{ij,\, lk}$ is replaced by $S^{(2)}_{il,\, jk}$ while $S^{(1)}_{ij,\, lk}$ remains. In this case g_s will be as in Fig. 2 where thick lines denote edges of $t_{1s}^{(m)}$, thin lines denote edges of $t_{2s}^{(m)}$.

Thus the number of edges of D_2 is

$$m = 2(2\lambda) + 2 = 2(2\lambda + 1).$$

Thus we have the following theorem:

Theorem 5: If $S^{(1)}_{ij,\, lk}$ and $S^{(2)}_{il,\, jk}$ are m-joined sets, then

$$m = 2(2\lambda + 1),$$

where λ is zero or a positive integer.

Corollary 2. Each set $S_{ij,\, lk}$ is two-joined with $T_{il,\, jk}$, and each set $S_{il,\, jk}$ is two-joined with $T_{ij,\, lk}$.

The proof of this statement follows from Fig. 2.

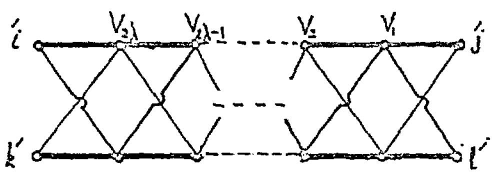

FIG. 2. g_s for $t_1^{(m)} \in S^{(1)}_{ij,\, lk}$ and $t_2^{(m)} \in S^{(2)}_{il,\, jk}$.

Next, we shall prove a simple lemma which will be needed in explaining the separation procedure.

Lemma 6. There is no two-tree (ij, lk) which has a neighboring two-tree (il, jk).

Proof: Suppose, if possible, $t_1 \in T_{ij,\,lk}$ and $t_2 \in T_{il,\,jk}$ are neighboring, that is

$$d(t_1, t_2) = 1;$$

then there is one and only one edge of t_1 which is not in t_2. Let this edge be in the part which contains vertices i and j. Then the edges of the path between l and k in t_1 are also in t_2; that is, there is a path between l and k in t_2, which is impossible. Hence the lemma.

We are now able to solve the problem posed at the beginning of this section: given the sets $T_{i,k}$ and $T_{j,l}$ of two-trees of a graph G, and hence the set

$$F \equiv T_{i,k} \cap T_{j,l} = T_{ij,\,lk} \cup T_{il,\,jk},$$

find a procedure for separating F into two groups f_1 and f_2, one of them $T_{ij,\,lk}$ and the other $T_{il,\,jk}$.

IV. SUMMARY OF PROCEDURE

Our procedure is summarized in the following steps:

(*a*) Find, from F, all the maximal connected sets, $S^{(r)}$, where $r = 1, 2, \ldots, n$, n being finite. This can be done by the following steps:

1. Choose any member (term) of F, say t_0; and let $\{t_1\}$ be the set of all the neighbors of t_0.
2. Find the neighbors of the members of $\{t_1\}$ which are not t_0 or in $\{t_1\}$. Let the set of all these new members be $\{t_2\}$.
3. Repeat this procedure until a set $\{t_q\}$, $1 \leq q$, is found such that the neighbors of all its members are either in $\{t_{q-1}\}$ or in $\{t_q\}$; that is $\{t_{q+1}\} = \phi$.
4. The union of all these sets will form a maximal connected set, say $S^{(1)}$, that is

$$S^{(1)} = \bigcup_{x=0}^{q} \{t_x\},$$

where $\{t_0\} \equiv t_0$. All the members of $S^{(1)}$ are either in $T_{ij,\,lk}$ or in $T_{il,\,jk}$, by Lemma 6.

5. Apply steps 1, 2, 3, and 4 to the complement in F of $\bigcup_{y=1}^{z} S^{(y)}$, where $1 \leq z < n$, to find $S^{(z+1)}$. Since F contains a finite number of terms, then

$$F = \bigcup_{y=1}^{n} S^{(y)}.$$

We denote by $\{S^{(r)}\}$ the set of all sets $S^{(r)}$.

(*b*) Choose a member of $\{S^{(r)}\}$, say $S^{(1)}$, and find from $\{S^{(r)}\}$ all the maximal connected sets which are 4λ-joined with $S^{(1)}$. The union of $S^{(1)}$ and those sets is the first group, f_1. The union of the remaining sets of $\{S^{(r)}\}$ will be the second group, f_2.

We can check the result by:

1. Using Theorems 4 and 5; that is, each maximal set of f_2 should be 4λ-joined with each other such set, and each maximal set of f_1 should be $2(2\lambda+1)$-joined with each maximal set of f_2;
2. Using Corollary 2 for a further check.

We now give a simple method for finding the value of m for two maximal connected sets. When v is large and the number of terms of each of these sets is large, a quick method for finding m is very desirable. Let $S^{(1)}$ and $S^{(2)}$ be m-joined sets, and let

$$S^{(1)} = \{t_1^{(1)},\ t_2^{(1)},\cdots,\ t_a^{(1)}\}$$

and

$$S^{(2)} = \{t_1^{(2)},\ t_2^{(2)},\cdots,\ t_b^{(2)}\}.$$

Construct a matrix $M = [m_{ij}]$ of order $a \times b$, with rows corresponding to the members of $S^{(1)}$ and columns corresponding to the members of $S^{(2)}$, and elements

$$m_{ij} = d(t_i^{(1)},\ t_j^{(2)}).$$

Then m is the minimum element in M.

It is worthwhile to illustrate the whole procedure by an example.

Example

We shall not set down $T_{i,k}$ and $T_{j,l}$ since they contain a large number of terms; but we enumerate their common terms, which are

$$\begin{aligned} F = \{&y_1y_2y_3y_4y_5y_6y_{13}y_{15}, && y_3y_6y_7y_8y_9y_{12}y_{13}y_{15}, && y_2y_5y_7y_{10}y_{11}y_{12}y_{14}y_{15},\\ &y_7y_8y_9y_{10}y_{11}y_{12}y_{13}y_{15}, && y_2y_3y_5y_6y_7y_{12}y_{13}y_{14}, && y_1y_4y_8y_9y_{10}y_{11}y_{13}y_{15},\\ &y_1y_3y_4y_6y_8y_9y_{13}y_{14}, && y_2y_5y_7y_{10}y_{11}y_{12}y_{13}y_{15}, && y_1y_3y_4y_6y_8y_9y_{13}y_{15},\\ &y_1y_2y_4y_5y_{10}y_{11}y_{13}y_{15}, && y_2y_3y_5y_6y_7y_{12}y_{13}y_{15}, && y_3y_6y_7y_8y_9y_{12}y_{13}y_{14},\\ &y_1y_4y_8y_9y_{10}y_{11}y_{13}y_{14}, && y_2y_3y_5y_6y_7y_{12}y_{14}y_{15}, && y_1y_2y_3y_4y_5y_6y_{14}y_{15},\\ &y_1y_2y_4y_5y_{10}y_{11}y_{13}y_{14}, && y_3y_6y_7y_8y_9y_{12}y_{14}y_{15}, && y_1y_2y_3y_4y_5y_6y_{13}y_{14},\\ &y_1y_2y_4y_5y_{10}y_{11}y_{14}y_{15}, && y_7y_8y_9y_{10}y_{11}y_{12}y_{13}y_{14}, && y_2y_5y_7y_{10}y_{11}y_{12}y_{13}y_{14},\\ &y_1y_4y_8y_9y_{10}y_{11}y_{14}y_{15}, && y_1y_3y_4y_6y_8y_9y_{14}y_{15}, && y_7y_8y_9y_{10}y_{11}y_{12}y_{14}y_{15}\}. \end{aligned}$$

Pick up any member of F, say $y_1y_2y_3y_4y_5y_6y_{13}y_{15}$. There are two neighbors of this member, namely

$$y_1y_2y_3y_4y_5y_6y_{14}y_{15} \quad \text{and} \quad y_1y_2y_3y_4y_5y_6y_{13}y_{14};$$

these are the members of $\{t_1\}$. We find that $\{t_2\} = \phi$; thus the first maximal connected set is

$$\{y_1y_2y_3y_4y_5y_6y_{13}y_{15};\ y_1y_2y_3y_4y_5y_6y_{14}y_{15};\ y_1y_2y_3y_4y_5y_6y_{13}y_{14}\}.$$

Similarly finding the other maximal connected sets, we get the following complete set of the maximal connected sets:

$$S^{(1)} = \{y_1y_2y_3y_4y_5y_6y_{13}y_{15},\quad y_1y_2y_3y_4y_5y_6y_{14}y_{15},\quad y_1y_2y_3y_4y_5y_6y_{13}y_{14}\}$$
$$S^{(2)} = \{y_3y_6y_7y_8y_9y_{12}y_{13}y_{15},\quad y_3y_6y_7y_8y_9y_{12}y_{13}y_{14},\quad y_3y_6y_7y_8y_9y_{12}y_{14}y_{15}\}$$
$$S^{(3)} = \{y_2y_5y_7y_{10}y_{11}y_{12}y_{14}y_{15},\quad y_2y_5y_7y_{10}y_{11}y_{12}y_{14}y_{13},\quad y_2y_5y_7y_{10}y_{11}y_{12}y_{13}y_{15}\}$$
$$S^{(4)} = \{y_7y_8y_9y_{10}y_{11}y_{12}y_{13}y_{15},\quad y_7y_8y_9y_{10}y_{11}y_{12}y_{13}y_{14},\quad y_7y_8y_9y_{10}y_{11}y_{12}y_{14}y_{15}\}$$
$$S^{(5)} = \{y_2y_3y_5y_6y_7y_{12}y_{13}y_{14},\quad y_2y_3y_5y_6y_7y_{12}y_{13}y_{15},\quad y_2y_3y_5y_6y_7y_{12}y_{14}y_{15}\}$$
$$S^{(6)} = \{y_1y_4y_8y_9y_{10}y_{11}y_{13}y_{15},\quad y_1y_4y_8y_9y_{10}y_{11}y_{13}y_{14},\quad y_1y_4y_8y_9y_{10}y_{11}y_{14}y_{15}\}$$
$$S^{(7)} = \{y_1y_3y_4y_6y_8y_9y_{13}y_{14},\quad y_1y_3y_4y_6y_8y_9y_{13}y_{15},\quad y_1y_3y_4y_6y_8y_9y_{14}y_{15}\}$$
$$S^{(8)} = \{y_1y_2y_4y_5y_{10}y_{11}y_{13}y_{14},\quad y_1y_2y_4y_5y_{10}y_{11}y_{14}y_{15},\quad y_1y_2y_4y_5y_{10}y_{11}y_{13}y_{15}\}.$$

Consider $S^{(1)}$ and $S^{(2)}$:

$S^{(2)}$ \ $S^{(1)}$	$y_1y_2y_3y_4y_5y_6y_{13}y_{15}$	$y_1y_2y_3y_4y_5y_6y_{14}y_{15}$	$y_1y_2y_3y_4y_5y_6y_{13}y_{14}$
$y_3y_6y_7y_8y_9y_{12}y_{13}y_{15}$	4	5	5
$y_3y_6y_7y_8y_9y_{12}y_{13}y_{14}$	5	5	4
$y_3y_6y_7y_8y_9y_{12}y_{14}y_{15}$	5	4	5

Thus $S^{(1)}$ and $S^{(2)}$ are 4-joined sets. In a similar way or by direct computation we find that $S^{(1)}$ is 4-joined with $S^{(3)}$ and $S^{(6)}$. But $S^{(1)}$ is 6-joined with $S^{(4)}$, and 2-joined with $S^{(5)}$, $S^{(7)}$ and $S^{(8)}$. Thus

$$f_1 = \{S^{(1)},\ S^{(2)},\ S^{(3)},\ S^{(6)}\}$$

and

$$f_2 = \{S^{(4)},\ S^{(5)},\ S^{(7)},\ S^{(8)}\}.$$

It is easy to check this result.

V. CONCLUSION

In the present paper, we have found two main results, namely:

1. A necessary and sufficient condition for a two-tree (ij, lk) to be obtained from another two-tree (ij, lk) of G, by a finite number of elementary two-tree (ij, lk) transformations:

2. A procedure for separating the common terms between $T_{i,k}$ and $T_{j,l}$ into two groups, one of them $T_{ij,\,lk}$ and the other $T_{il,\,jk}$.

In synthesizing two terminal-pair networks without mutual inductance from the open circuit impedance matrix, Z_{oc}, using topological methods, the given functions z_{11}, z_{12} and z_{22}, where

$$Z_{oc} = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix},$$

are expressed in terms of elements admittance functions $y_1,\ y_2, \ldots,\ y_e$;

(see Mayeda and Seshu (4)). The problem of selecting a number of elementary positive real functions[7] $y_1, y_2, \ldots, y_t$ such that the given positive real functions z_{11}, z_{12}, z_{22} are expressible in terms of them, remains unsolved (4,5,6).

However, assuming z_{∞} is given by

$$z_{\infty} = \frac{1}{V(Y)} \begin{bmatrix} W_{1',1}(Y) & W_{12,1'2'}(Y) - W_{12',1'2}(Y) \\ W_{12,1'2'}(Y) - W_{12',1'2}(Y) & W_{2,2'}(Y) \end{bmatrix}$$

where

$V(Y) = \Sigma$ (tree-admittance products),
$W_{1,1'}(Y) = \Sigma$ (two-tree (1, 1')-admittance products),
$W_{2,2'}(Y) = \Sigma$ (two-tree (2, 2')-admittance products),
$W_{12,1'2'}(Y) = \Sigma$ (two-tree (12, 1'2')-admittance products), and
$W_{12',1'2}(Y) = \Sigma$ (two-tree (12', 1'2)-admittance products),
in which 1,1' are the input vertices and 2,2' are the output vertices, Seshu (4,5) gave a procedure for determining the network, (to within a 2-isomorphism), starting with $V(Y)$, $W_{2,2'}(Y)$, $W_{1,1'}(Y)$, and $[W_{12,1'2'}(Y) - W_{12',1'2}(y)]$. It is to be noted that the same network realizes also the negative of the given $[W_{12,1'2'} - W_{12',1'2}]$ if the labels of output vertices are interchanged.

It is clear that, using the procedure given in Section III, we can carry out Seshu's synthesis when only $V(Y)$, $W_{1,1'}(Y)$ and $W_{2,2'}(Y)$ are given.

Acknowledgment

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REFERENCES

(1) T. FUJISAWA, "On a Problem of Network Topology," *IRE Trans. on Circuit Theory*, Vol. CT-6, No. 1, pp. 261–266 (Sept., 1959).

(2) S. L. HAKIMI, "On Trees of a Graph and their Generation," JOUR. FRANKLIN INST., Vol. 272, pp. 347–359 (1961).

(3) S. L. HAKIMI AND W. MAYEDA, "On Coefficients of Polynomials in Network Functions," *IRE Trans. on Circuit Theory*, Vol. CT-7, No. 1, pp. 40–44 (March, 1960).

(4) W. MAYEDA AND S. SESHU, "Topological Formulas for Network Functions," Bull. No. 446, Univ. of Ill. Engineering Experiment Station, 1957.

(5) S. SESHU, "Topological Considerations in the Design of Driving-point Functions," *IRE Trans. on Circuit Theory*, Vol. CT-2, No. 4, pp. 356–367 (December, 1955).

(6) S. SESHU AND M. B. REED, "Linear Graphs and Electrical Networks," Reading (Mass.), Addison-Wesley, 1961.

(7) H. WATANABE, "A Computational Method for Network Topology," *IRE Trans. on Circuit Theory*, Vol. CT-7, No. 3, pp. 296–302 (September, 1960).

(8) O. WING AND W. H. KIM, "The Path Matrix and Its Realisability," *IRE Trans. on Circuit Theory*, Vol. CT-6, No. 3, pp. 267–272 (September, 1959).

(9) H. T. LEE, "Graphs, Network Discriminants and Positive Homogeneous Multilinear Algebraic Forms," D.E.E. Dissertation, Polytechnic Institute of Brooklyn, June, 1960.

[7] An elementary positive real function (5) is a function of one of the three forms $1/R$, $1/pL$, pC, where p is a complex variable and R, L, C are positive real constants.

APPENDIX

Proof[8] *of Lemma 1*

Add $b_{h\,lk}^{(1)}$ to t_f. Since $b_{h\,lk}^{(1)}$ is not a direct-bridged edge with respect to t_f, then there exists an edge, say $b_f^{(1)}$, in t_f such that

$$t_f^{(1)} = t_f \cdot b_{h\,lk}^{(1)}/b_f^{(2)}$$

is a two-tree $(ij,\ lk)$ of G.

Repeat the operation by adding $b_{h\,lk}^{(1)}$ to $t_f^{(1)}$, and so on.

In general we use the recursion formula

$$t_f^{(r)} = t_f^{(r-1)} \cdot b_{h\,lk}^{(r)}/b_f^{(r)}, \qquad 1 \leq r \leq d,$$

where $t_f^{(0)} \equiv t_f$, and d is the number of edges in $P_{lk}^{(h)}$.

In each $t_f^{(r)}$, the path between vertices i and j is $P_{ij}^{(f)}$, since

$$V_{ij}^{(f)} \cap V_{lk}^{(h)} = \phi;$$

hence all $t_f^{(r)}$ are in $T_{ij,\ lk}$.

Thus we obtain a two-tree $(ij,\ lk)$, $t_f^{(d)}$, by a finite number of elementary two-tree $(ij,\ lk)$ transformations; and $t_f^{(d)}$ contains $P_{lk}^{(h)}$.

Now, we repeat the operation on $t_f^{(d)}$ by adding in order

$$b_{hij}^{(1)},\ b_{h\,ij}^{(2)}, \cdots,\ b_{hij}^{(c)}$$

which are the edges of $P_{ij}^{(h)}$ in order (that is, $b_{hij}^{(1)}$ is connected to i and $b_{hij}^{(c)}$ is connected to j, and in general $b_{hij}^{(r)}$ is connected to $b_{hij}^{(r-1)}$, $2 \leq r \leq c$).

Since

$$V_{ij}^{(h)} \cap V_{lk}^{(h)} = \phi,$$

then in each step, $b_{hij}^{(y)}$ $(1 \leq y \leq c)$ is not a direct bridged-edge with respect to $t_f^{(d+y-1)}$. Thus all of

$$t_f^{(d+1)},\ t_f^{(d+2)}, \cdots,\ t_f^{(d+c)}$$

are in $T_{ij,\ lk}$ and contain $P_{lk}^{(h)}$; $t_f^{(d+c)}$ contains both $P_{ij}^{(h)}$ and $P_{lk}^{(h)}$.

To complete the transformation we make use of those edges of t_h which have common vertices with $V_{lk}^{(h)}$. After that we make use of the edges of t_h which have common vertices with the previous set of edges; and so on until we use (in order) all the edges of t_h which are in the part that contains vertices l and k.

Then we turn to the edges of t_h which have common vertices with $V_{ij}^{(h)}$, then those which have common vertices with them, and so on until we use (in order) all the edges of t_h which are in the part that contains vertices i and j. In each of the above steps, the two-tree $(ij,\ lk)$ transformation is possible, since on adding one of the previous edges in the order mentioned above, we get either a circuit which contains an edge of t_f which is not in t_h or a tree which contains paths between the pairs of vertices $(i,\ l)$, (i,k), $(j,\ l)$ and (j,k) that have at least one edge of t_f in common which is not in the paths between (i, j) and (l, k).

Thus the lemma is proved.

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[8] The technique used here is similar to that used by Hakimi and Mayeda (3).

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and quicker results. The use of the above discussions lies in obtaining the zero distribution of any real polynomial in the unit circle and in simplifying the procedures for obtaining the zeros of any form of $F(z)$. Such a general form of $F(z)$ could represent the characteristic equation of a linear discrete feedback system or a pulsed network.

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REFERENCES

[1] M. Marden, "The Geometry of the Zeros of a Polynomial in a Complex Variable," American Mathematical Society, c... X, pp. 155–157; 1949.
[2] M. Marden, "The number of zeros of a polynomial in a circle," *Proc. Nat. Acad. Sci.*, U. S. A., vol. 39, pp. 15–17; 1948.
[3] A. Cohn, "Uber die anzhal der wurzeln einer algebraischen gleichung in einem kreise," *Math. Zeit.*, vol. 14, pp. 110–148; August, 1922.
[4] E. I. Jury, "On the roots of a real polynomial inside the unit-circle and a stability criterion for linear discrete system," presented at the Second International Federation of Automatic Control Congress, Basle, Switzerland, Paper No. 413/7; September 4, 1963.
[5] F. F. Bonsall and M. Marden, "Zeros of self-inversive polynomials," *Proc. Amer. Math Soc.*, vol. 3, pp. 471–475; June, 1952.
[6] Germán Ancochea, "Zeros of self-inversive polynomials," *Proc. Amer. Math. Soc.*, vol. 4, pp. 900–902; December, 1935.
[7] A. T. Kempner, "On the complex roots of algebraic equations," *Bull. Amer. Math. Soc.*, vol. 41, pp. 809–843; December, 1935.
[8] E. I. Jury and J. Blanchard, "A stability test for linear discrete systems in table form," Proc. IRE, vol. 44, pp. 1947–1948; December, 1961.
[9] E. I. Jury, "Theory and Application of the 2-Transform Method," John Wiley and Sons, Inc., New York, N. Y.; 1964.

On the Sign of a Tree Pair

Several methods for determining the sign of a tree pair[1] (complete tree[2,3]) have been described by several authors. The method of interest here is that of Frisch and Kim.[1] For completeness, we shall outline briefly the terminology used by Frisch and Kim.

The graph $\mathcal{G}$ of a general linear network consists of two kinds of edges: ordinary edges and active edge pairs. Each ordinary edge is weighted by its self-admittance and each active edge pair by its mutual admittance. An active edge pair consists of two edges: current edge and voltage edge; they occur in pairs and between different node pairs. The current edge is indicated by double arrows and the voltage edge by a triangular-shaped arrow.

A triode of Fig. 1(a) is characterized as a linear active device by

$$I_p = g_m V_{GK} + g_p V_{PK}$$

where g_m is the transconductance and g_p is the plate conductance. The graph of this triode is shown in Fig. 1(a′).

A transistor, shown in Fig. 1(b), is characterized by the following set of equations:

$$I_e = g_e(V_e - V_n)$$

$$I_c = ag_e(V_e - V_n) + g_c(V_c - V_n)$$

$$I_b = g_b(V_b - V_n)$$

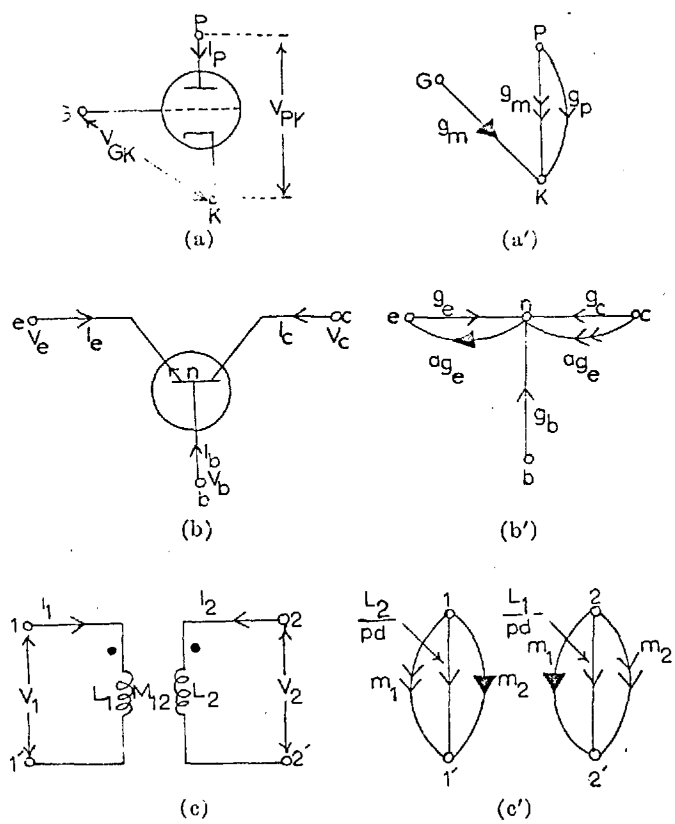

Fig. 1—(a) A triode. (a′) Triode graph. (b) A transistor. (b′) Transistor graph. (c) A transformer. (c′) Transformer graph, where $m_1 = m_2 = -M_{12}/pd$.

and

$$I_e + I_c + I_b = 0$$

where g_e is the emittor conductance, g_b is the base conductance, g_c is the collector conductance and a is the current amplification factor. The graph of this transistor is shown in Fig. 1(b′).

The graph of the transformer [Fig. 1(c)], which is determined by

$$I_1 = \frac{L_2}{pd} V_1 - \frac{M_{12}}{pd} V_2$$

$$I_2 = -\frac{M_{12}}{pd} V_1 + \frac{L_1}{pd} V_2$$

where

$$d = L_1 L_2 - M_{12}^2 \neq 0$$

and p is the complex frequency variable, is shown in Fig. 1(c′).

The graph $\mathcal{G}$ can be decomposed into two subgraphs: current graph $\mathcal{G}_I$, which consists of all the current edges and the ordinary edges, and voltage graph $\mathcal{G}_V$, which consists of all the voltage edges and the ordinary edges.

Fig. 2 shows a network with its graph $\mathcal{G}$, in which

$$y_3 = pC_3, \qquad y_4 = \frac{L_5}{pd}, \qquad y_5 = \frac{L_4}{pd},$$

$$m_6 = m_7 = -\frac{M_{45}}{pd},$$

and

$$d = L_4 L_5 - M_{45}^2 \neq 0.$$

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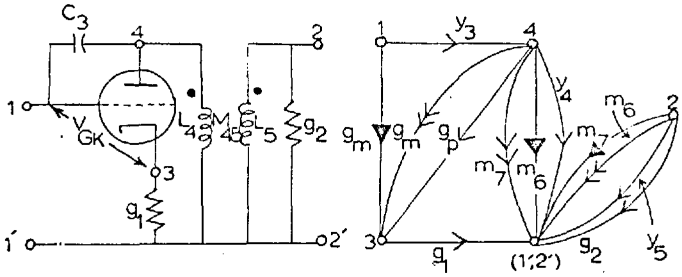
[1] I. T. Frisch and W. H. Kim, "Properties of 2-semi-isomorphic graphs and their applications: active network analysis," *J. Math. Phys.*, vol. 2, pp. 627–635; August, 1961.

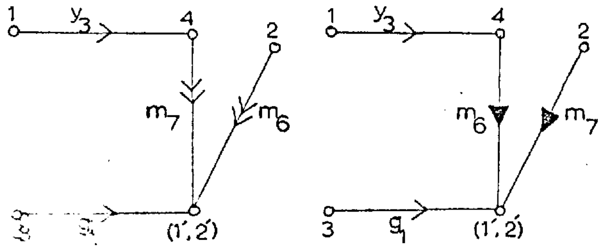
[2] C. L. Coates, "General Topological Formulas for Linear Network Functions," General Electric Res. Lab., Schenectady, N. Y., Rept. No. 57-RL-1746; 1957.

[3] W. Mayeda, "Topological Formulas for Active Networks," University of Illinois, Urbana, Interim Tech. Rept. No. 8, U. S. Army Contract No. DA-11-022-ORD-1983; 1958.



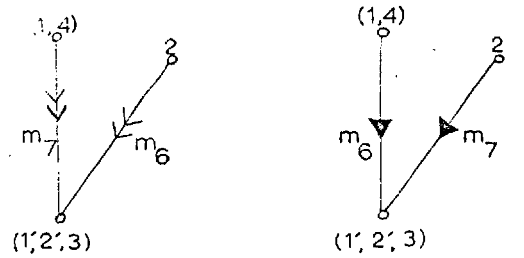
(a) (b)

Fig. 2—(a) A network. (b) Its graph.



(a) (b)

Fig. 3—Tree pair $g_1y_3m_6m_7$ of Fig. 2 (b). (a) Current tree. (b) Voltage tree.



(a) (b)

Fig. 4—Reduced tree pair of the tree pair $g_1y_3m_6m_7$. (a) Reduced current tree. (b) Reduced voltage tree.

A pair of trees of $\mathcal{G}_I$ and $\mathcal{G}_V$ that contain the same edges are called a tree pair of $\mathcal{G}$. Fig. 3 shows the voltage and current trees of the tree pair $g_1y_3m_6m_7$ of the graph in Fig. 2(b).

The principal node of an edge of a given tree is a node of the edge such that the tree path between it and the reference node contains the edge.

The reduced tree pair of a tree pair is a pair of subgraphs derived from the current and voltage trees by short circuiting all ordinary edges in the tree pair.

Let A_I and A_V be the reduced incidence matrices of $\mathcal{G}_I$ and $\mathcal{G}_V$, respectively, with respect to the same reference node. Let A_{I_i} and A_{V_i}, respectively, be square submatrices of A_I and A_V whose columns correspond to the edges of a tree pair τ_i of $\mathcal{G}$. Then the sign of τ_i is given by $|A_{I_i}\cdot A_{V_i}'|$, where A_{V_i}' is the transpose of A_{V_i}.

The following lemma is that of Frisch and Kim.[1]

Lemma 1: The sign of a tree pair is given by $(-1)^\gamma\Pi_k$ (sign of active edge pairs of the reduced tree pair of a tree pair), where γ is the number of interchanges of edges in the current graph or in the voltage graph needed to give the current and voltage edges of each active pair in the reduced tree pair the same principal nodes, and k is the number of active edge pairs in the reduced tree pair.

Example 1: In order to find the sign of the tree pair $g_1y_3m_6m_7$ by the formula of Lemma 1, the ordinary edges g_1 and y_3 are reduced and the reduced tree pair is shown in Fig. 4. Taking node (1′, 2′, 3) as a reference node we find that the principal nodes of the edge pairs m_6 and m_7 in the reduced current tree [Fig. 4(a)] are 2 and (1, 4), and in the reduced voltage tree [Fig. 4(b)] are (1, 4) and 2, respectively. Thus m_6 and m_7 have positive signs, and the number of interchanges of edges needed to give m_6 and m_7 the same principal nodes is one. Therefore, from Lemma 1, the sign of the tree pair is (-1).

Some results which reduce the work in determining the signs of all tree pairs of $\mathcal{G}$ will be given in this communication.

Let t be a 2-tree of a directed graph G, and e_1 and e_2 be edges of G such that $e_1\cdot t$ and $e_2\cdot t$ are trees of G. t consists of two unconnected subgraphs, each being connected and called a *part* (or a maximal connected subgraph[4]) of t. Let t_1 and t_2 be the parts of t. Each of e_1 and e_2 has one of its nodes in t_1 and the other in t_2. If *both* e_1 and e_2 have their directions from t_1 to t_2 or from t_2 to t_1, then we shall say that e_1 and e_2 have the same direction with respect to t_1 and t_2; otherwise, they have opposite directions.

Lemma 2: If g is a subgraph of $\mathcal{G}$ consisting of ordinary edges only and e is an active edge pair such that eg is a tree pair of $\mathcal{G}$, then the sign of eg is determined by e.

In other words, if e_I and e_V are the current and voltage edges of e, respectively, and g_1 and g_2 are the parts of g, then the sign of eg is $(+1)$ if e_I and e_V have the same direction with respect to g_1 and g_2; otherwise, (-1). The proof follows from Lemma 1.

By Lemma 2 and from the triode and transistor graphs [Fig. 1(a), (b)] we have:

Corollary 1: The sign of a tree pair that consists of ordinary edges and the active edge pair of a triode (a transistor) is positive (negative).

Corollary 2: 1) The sign of a tree pair of $\mathcal{G}$ that consists of ordinary edges and one active edge pair of a transformer [Fig. 1(c)] is $(+1)$ if its ordinary edges form a 2-tree (12, 1′2′) of $\mathcal{G}$, where (1, 1′) and (2, 2′) are the node pairs of the transformer. 2) The sign of a tree pair that consists of ordinary edges and the two active edge pairs of a transformer is (-1).

Proof: 1) Let m_1 be an active edge pair of a transformer and g_1 be a subgraph of $\mathcal{G}$ consisting of ordinary edges such that m_1g_1 is a tree pair of $\mathcal{G}$. Then g_1 is a 2-tree, separating nodes 1 and 1′ as well as 2 and 2′; that is, g_1 is either a 2-tree (12, 1′2′) or a 2-tree (12′, 1′2) of $\mathcal{G}$. Applying Lemma 2, we find that if g_1 is a 2-tree (12, 1′2′) then the sign of m_1g_1 is $(+1)$, and if g_1 is a 2-tree (12′, 1′2) then the sign of m_1g_1 is (-1).

2) Let g_2 be a subgraph of $\mathcal{G}$ consisting of ordinary edges such that $m_1m_2g_2$ is a tree pair of $\mathcal{G}$, where m_2 is the other active edge pair of the same transformer. Then g_2 must be a 3 tree of $\mathcal{G}$. It is easy to see that g_2 is a 3 tree (12, 1′, 2′), (12′, 1′, 2), (1′2′, 1, 2) or (1′2, 1, 2′) of $\mathcal{G}$. Applying Lemma 1 we find that the sign of $m_1m_2g_2$ is (-1) for each of the four cases of g_2. Hence the Corollary.

Lemma 3: Let $e_1, e_2, \cdots, e_b$ be the edges of a connected-directed graph G, A the reduced incidence matrix of G with the rth column corresponding to $e_r (r = 1, 2, \cdots, b)$, e_it, and e_jt $(1 \le i < j \le b)$ two trees of G, A_i and A_j square submatrices of A whose columns correspond to the edges of e_it and e_jt, respectively, and n the number of all edges of t which are in the sequence $e_{i+1}, e_{i+2}, \cdots, e_{j-1}$. Then

$$|A_i|\cdot|A_j| = (-1)^n\cdot\alpha$$

where $\alpha = +1$, if e_i and e_j have the same direction with respect to the two parts of t, $\alpha = -1$, otherwise.

Proof: Let $\bar{A}_j$ be A_i with column e_i replaced by column e_j, then looking upon e_it and e_jt as the current and voltage trees of one tree pair, and using Lemma 2 we get

$$|\bar{A}_j|\cdot|A_i| = \alpha.$$

But $\bar{A}_j$ can be obtained from A_j by n interchanges of columns, thus

$$|A_j| = (-1)^n\cdot|\bar{A}_j|.$$

Hence the Lemma.

[4] S. Seshu and M. B. Reed, "Linear Graphs and Electrical Networks," Addison-Wesley Publishing Co., Reading, Mass.; 1961.

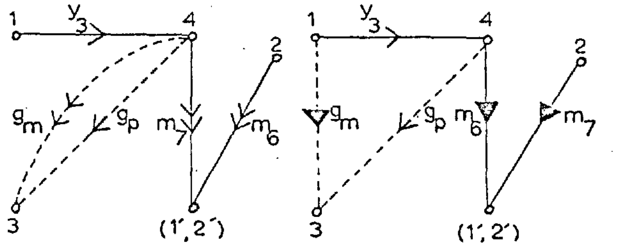


Fig. 5—Current and voltage graphs of $g_m g_p y_3 m_6 m_7$. (a) Current graph. (b) Voltage graph.

Talbot's Lemma[5] and Theorem 5 of Watanabe[6] are special cases of our result above.

Next we present a simple method for determining the relative sign of two tree pairs that differ by one edge only.

Let $e_1, e_2, \cdots, e_b$ be the ordinary and active edge pairs of the graph $\mathcal{G}$, and let the columns of A_I and A_V appear in the order $e_1, e_2, \cdots, e_b$. Let g be a subgraph of $\mathcal{G}$ such that $e_i g$ and $e_j g (i < j)$ are tree pairs of $\mathcal{G}$, where e_i and e_j are edges (ordinary or active edge pairs) of $\mathcal{G}$. Moreover, let g_I, g_V, e_{iI}, e_{iV} and e_{jI}, e_{jV} be the current and voltage graphs (edges) of g, e_i and e_j, respectively. Denote the parts of g_I and g_V by g_{I_1}, g_{I_2} and g_{V_1}, g_{V_2}, respectively, and let A_{I_i} and A_{I_j} be submatrices of A_I whose columns correspond to the edges of $e_i g$ and $e_j g$, respectively. Similarly, let A_{V_i} and A_{V_j} be submatrices of A_V whose columns correspond to the edges of $e_i g$ and $e_j g$, respectively. Then by Lemma 3, we have

$$
\begin{aligned}
|A_{I_i}| \cdot |A_{I_j}| &= (-1)^n, \text{ if } e_{iI} \text{ and } e_{jI} \text{ have the same direction with respect to } g_{I_1} \text{ and } g_{I_2} \\
&= (-1)^{n+1}, \text{ if } e_{iI} \text{ and } e_{jI} \text{ have opposite directions with respect to } g_{I_1} \text{ and } g_{I_2} \\
|A_{V_i}| \cdot |A_{Vj}| &= (-1)^n, \text{ if } e_{iV} \text{ and } e_{jV} \text{ have the same direction with respect to } g_{V_1} \text{ and } g_{V_2} \\
&= (-1)^{n+1}, \text{ if } e_{iV} \text{ and } e_{jV} \text{ have opposite directions with respect to } g_{V_1} \text{ and } g_{V_2},
\end{aligned}
$$

where n is the number of all ordinary and active edge pairs of g which are in the sequence $e_{i+1}, e_{i+2}, \cdots, e_{j-1}$.

The sign of $e_i g$ and $e_j g$ are given by $|A_{I_i}| \cdot |A_{V_i}'|$ and $|A_{I_j}| \cdot |A_{Vj}'|$, respectively where A_{V_i}' indicates the transpose of A_{V_i}. Hence we arrive at the following theorem.

Theorem: The tree pairs $e_i g$ and $e_j g$ of $\mathcal{G}$ are of the same sign if e_i and e_j have (or have not) the same direction with respect to the parts of g in *both* current and voltage graphs; otherwise they are of opposite signs.

This result is useful in reducing the work in finding the signs of all tree pairs, and in evaluating the network functions of a general linear network.[3]

Example 2: The set of all tree pairs of $\mathcal{G}$, shown in Fig. 2, is

$$
\begin{aligned}
&(g_1 y_3 y_4 y_5,\ g_1 g_2 y_3 y_4,\ g_1 y_3 y_5 g_p,\ y_3 y_4 y_5 g_p,\ g_1 g_2 y_3 g_p,\ g_2 y_3 y_4 g_p), \\
&(g_1 y_3 y_5 g_m,\ y_3 y_4 y_5 g_m,\ g_2 y_3 y_4 g_m,\ g_1 g_2 y_3 g_m), \\
&(g_1 y_3 m_6 m_7,\ y_3 g_p m_6 m_7,\ y_3 g_m m_6 m_7).
\end{aligned}
$$

The tree pairs in the first brackets consist of ordinary edges only, so they have positive signs. Every tree pair in the second brackets contains one active edge pair g_m only; therefore by Corollary 1, all the trees in the second brackets have positive signs. The first two tree pairs in the third brackets consist of ordinary edges and the two active edge pairs (m_6 and m_7) of the transformer; therefore they have negative signs (Corollary 2). To find the sign of $y_3 g_m m_6 m_7$, we shall compare it with $y_3 g_p m_6 m_7$. These two tree pairs have edges y_3, m_6 and m_7 in common. The current and voltage graphs of $g_p g_m y_3 m_6 m_7$ are shown in Fig. 5, where the noncommon edges g_p and g_m are drawn in dotted line. The parts of $y_3 m_6 m_7$ in both current and voltage are $y_3 m_6 m_7$ and an isolated vertex 3. Edges g_p and g_m have the same direction with respect to the parts of $y_3 m_6 m_7$ in both current and voltage graphs. Therefore, by the Theorem, $y_3 g_m m_6 m_7$ and $y_3 g_p m_6 m_7$ have the same sign; that is $y_3 g_m m_6 m_7$ has negative sign.

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[5] A. Talbot, "Topological Analysis of General Linear Networks," presented at 6th Midwest Symp. on Circuit Theory, Madison, Wis.; May, 1963.
[6] H. Watanabe, "A computational method for network topology," IRE Trans. on Circuit Theory, vol. CT-7, pp. 296–302; September, 1960.

A Comment on Minimum Feedback Arc Sets

In a recent paper Younger[1] proposed a procedure for finding minimum feedback arc sets for a directed graph, *i.e.*, minimum sets of arcs which when removed leave the resultant graph free of directed loops. It may be worth mentioning that the problem can be formulated and solved by dynamic programming techniques and as a quadratic assignment problem.

Younger points out that the problem of finding a minimum feedback arc set is equivalent to the problem of finding an optimum linear ordering of the nodes of the graph, or as Tucker[2] previously observed, an optimum assignment of node "potentials." An ordering is optimal when as few arcs as possible are directed from nodes in position i of the order to nodes in position j, where $i \leq j$.

An optimum ordering of the nodes can be found from optimum orderings of subsets of nodes. Let S denote an arbitrary subset of nodes, and let $f(S)$ stand for the number of arcs in a minimum arc set for the subgraph consisting of the nodes in S and all the arcs between them. Let $c(k, S')$ denote the number of arcs from node k to nodes in the subset S'. Then, assuming no self-loops, we have the functional equation

$$f(S) = \min \{ f(S') + c(k, S') \},$$

where the minimization is carried out over all subsets S' and nodes k, such that $S = S' \cup \{k\}$.

In carrying out computations with the functional equation, computer memory requirements increase as 2^n and computation time as $n2^n$, where n is the number of nodes. Computations for graphs with up to 15 nodes can be carried out efficiently on an IBM 7090, and various means can be used to obtain solutions for larger problems. (Compare the "suboptimization" procedures of Held and Karp.[3])

The problem can be formulated as a quadratic assignment problem of the Koopmans-Beckmann type[4] as follows. Let C denote the nxn connection matrix of the graph and let U denote an nxn upper triangular matrix, where

$$
\begin{aligned}
U_{ij} &= 1 \quad \text{if} \quad i \leq j \\
&= 0 \text{ otherwise.}
\end{aligned}
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[1] D. H. Younger, "Minimum feedback arc sets for directed graphs," IEEE Trans. on Circuit Theory, vol. CT-10, pp. 238–245; June, 1963.
[2] A. W. Tucker, "On Directed Graphs and Integer Problems," Symp. on Combinatorial Problems, Princeton University, Princeton, N. J.; 1960.
[3] M. Held and R. M. Karp, "A dynamic programming approach to sequencing problems," *J. Soc. Industr. Appl. Math.*, vol. 10, pp. 196–210; March, 1962.
[4] T. C. Koopmans and M. J. Beckmann, "Assignment Problems and the Location of Economic Activities," *Econometrica*, vol. 25, pp. 52–76; January, 1957.