"THE ANALYSIS AND PROPERTIES OF NONLINEAR RLC NETWORKS CONTAINING COUPLED ELEMENTS"

by

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ABSTRACT

In this thesis, the application of state variable theory to the analysis of nonlinear networks is considered, with the aim of obtaining conditions for the existence of the normal form for a broad class of nonlinear RLC networks containing locally active and/or nonreciprocal coupled elements.

The problem of functional inversion of vector-valued functions, representing one of the key problems in the state variable formulation of nonlinear RLC networks, is examined. A compact and easily applicable criterion for the existence of a unique inverse of an important class of vector valued-functions, reffered to as globally regular functions, is derived.

The analysis is based on the characterization of network elements in terms of the hybrid descriptions. A class of resistive, capacitive and inductive positive network elements, possessing all possible hybrid descriptions and representing a very good model for a large class of locally active elements, is introduced. It is shown that positive network elements possess many interesting properties. A class of positive definite network elements which are strongly locally passive and form a subclass of positive definite network elements is introduced also. (n+1)-terminal network elements representing positive network elements in all different orientations are discussed. Properties of series-parallel interconnections of positive, positive definite and certain other network elements are studied. The existence of a unique solution of one-element-kind networks containing locally active and/or nonreciprocal coupled elements is considered

and a topological approach is taken.

The problem of setting up the normal form equations for nonlinear RLC networks with the normal distribution of independent sources is examined. Sufficient conditions for the existence of a unique network response are given for a broad class of nonlinear RLC networks.

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LIST OF PRINCIPAL SYMBOLS

Much of the work of this thesis involves the use of matrix algebra and it is necessary to distinguish between matrices, vectors and scalars. The general rule used is that matrices are represented by capital letters, vectors by lower case letters which are underlined and scalars by lower case letters. Another convention used is that A^T is the transpose of A.

An attempt has been made to keep the meaning of symbols constant throughout the thesis. Where this has not been possible the meaning is obvious from context, and this is also to be true for symbols not in the list below.

Symbol	Definition	first used on page
B	fundamental loop matrix	48
b	number of branches	49
$\mathcal{B}_C$	set of capacitive branches	52
$\mathcal{B}_E$	set of voltage sources	52
$\mathcal{B}_J$	set of current sources	52
$\mathcal{B}_L$	set of inductive branches	52
$\mathcal{B}_R$	set of resistive branches	52
C	incremental capacitance matrix	43
$\underline{e}$	vector of voltage sources	53
$\underline{E}$	voltage source vector in a network	52
$\underline{f}(\cdot)$	vector-valued function	15

$\underline{f}^{-1}(\cdot)$	inverse of a vector valued function	85
G	incremental conductance matrix	42
G_{in}	indefinite incremental conductance matrix	157
GQLF	generalized quasilinear function	113
$\underline{h}(\cdot)$	hybrid description	37
H	hybrid matrix	41
HNE	Hadamard network element	167
i	current	36
$\underline{i}$	current vector	36
I	identity matrix	50
$\underline{j}$	vector of current sources	53
$\underline{J}$	current source vector in a network	52
L	incremental inductance matrix	42
$N\{\mathcal{B}_a ; \mathcal{B}_b\}$	network derived from a network N by contracting all the branches of $\mathcal{B}_a$ and removing all the branches of $\mathcal{B}_b$	52
PDNE	positive definite network element	161
PNE	positive network element	139
PSDNE	positive semidefinite network element	175
q	charge	19
$\underline{q}$	charge vector	19
Q	fundamental cut-set matrix	48
R	incremental resistance matrix	42
R^n	n-dimensional Euclidean space	17

S	incremental elastance matrix	43
t	time	15
T	tree of a network	49
T_L	L-normal tree	248
T_N	normal tree	68
u. p. d.	uniformly positive definite	109
u. H.	uniformly Hadamard	109
UP	UP (matrix)	139
v	voltage	36
$\underline{v}$	voltage vector	36
Γ	inverse incremental inductance matrix	42
ν	number of nodes in a network	49
φ	flux-linkage	19
$\underline{\varphi}$	flux-linkage vector	19

LIST OF PRINCIPAL SUBSCRIPTS

subscript	subscripts refer to	first used on page
C	capacitive elements	10
E	voltage sources	52
L	inductive elements	39
J	current sources	52
R	resistive elements	36
α	capacitive links	68
β	resistive links	54
γ	inductive links	68
δ	capacitive tree-branches	68
ε	resistive tree-branches	54
ζ	inductive tree-branches	68

LOCATION OF FIGURES

Chapter 1

INTRODUCTION

1.1 MOTIVATION

Early investigations into nonlinear networks were limited to isolated classes of networks. Some special methods were developed to analyse such networks. The usual restrictions placed upon nonlinear networks were either that nonlinear elements possessed characteristics with a small degree of nonlinearity, or that the order of a system was very low. Much work was done on the phase plane analysis of second order systems. The differential equations of Van der Pol's type, describing self-sustained or forced oscillations in nonlinear systems, received special attention. Frequently the analysis, performed on very simple network models, was carried out in order to explain the basis of operation of a nonlinear network, but the proper design of a network, requiring a more complicated network model, was rendered impossible in most cases.

The inherent difficulty, present in any nonlinear network analysis, is that with small exceptions analytic solutions of nonlinear differential equations, which mathematically describe nonlinear dynamic systems, cannot be obtained; it is therefore necessary to resort to numerical methods of some kind to calculate network response. With the advent of digital computers, the numerical solution of differential equations presents no problem in most cases. This stimulated active research in the field of nonlinear networks at large, defined by Liu and Auth[1] as networks with an arbitrary degree of nonlinearity and no limitation on the number of elements.

A fundamental problem in the computer aided analysis of nonlinear networks is to find a suitable characterization. Contrary to the situation in linear network theory, where the analysis can be performed either in the frequency or the time domain, nonlinear networks at large can only be analysed in the time domain. It has now been generally recognized[2-5] that a basic step in the resolution of nonlinear network problems is to reduce the system of algebraic and differential equations, which govern the behaviour of the network to the normal form differential equation, $\dot{\underline{x}} = \underline{f}(\underline{x},t)$, where $\underline{x}$ represents the complete set[5] which uniquely defines all network variables.

The merits of the normal form characterization stem from the fact that such a representation is most amenable to the study of the existence and uniqueness of network response. Similarly some other qualitative properties of network behaviour such as stability and boundedness of network response[6], existence of self-sustained oscillations, etc. can be deduced from this characterization. As a matter of practical concern for computer aided analysis, most numerical methods for solving differential equations assume that differential equations are given in the normal form.

1.2 THE CONCEPT OF STATE

State concepts which are fundamental in modern system theory evolved from the classical theory of dynamics of particles and rigid bodies[7]. Intuitively, the state of a system may be considered to be an independent set of system variables, which must be specified at time $t=t_o$ in order to be able to predict uniquely the future behaviour of the system. More formal definition of the state of a system can be given as follows[8].

The state of a system is a minimum set of numbers (called state variables) which contain sufficient information about the history of the system to allow computation of future behaviour.

To express the definition of state in mathematical form, some notation is needed. Consider a system S and associate with it a set of input variables, denoted by the vector $\underline{u}$, and a set of output variables, denoted by the vector $\underline{y}$. The input and output vectors are assumed to be functions of time t. The set of all possible values which the vector $\underline{u}$ can assume at time t is called the input space. In a similar way, the output space is defined as the set of all possible values which the vector $\underline{y}$ assumes at time t.

Let a set of n state variables be denoted by the vector $\underline{x}$. The state space is then defined as the set of all possible values which the state vector $\underline{x}$ assumes at time t.

We are now in a position to express a definition of state in terms of the state vector $\underline{x}$, the input vector $\underline{u}$, and the output vector $\underline{y}$. The definition of state implies that the state vector $\underline{x}$ can be written in the general form

$$\underline{x}(t) = \underline{F}\left\{\underline{x}(t_o); \underline{u}(t)\right\} \qquad (1.1)$$

where $\underline{F}$ $(\cdot)$ is a single valued function of its arguments. Eqn. (1.1) is called the state equation of the system S and it indicates that the future state of a system is uniquely determined by the state at time t_o and the known input $\underline{u}(t)$ for $t > t_o$, and is independent of values of the state and input before t_o.

The output equation is defined abstractly as

$$\underline{y}(t) = \underline{G}(\underline{x}(t), \underline{u}(t)) \tag{1.2}$$

where again $\underline{G}(\cdot)$ is a single valued function of its arguments. Eqns. (1.1) and (1.2) constitute the state equations of the system S.

Let a system S be described by a set of ordinary differential equations in the normal form

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}(t), t) \tag{1.3}$$

such that eqn. (1.3) possesses a unique solution for given initial conditions $\underline{x}(t_o)$ and the excitation from t_o onward. Then the vector $\underline{x}$ in eqn. (1.3) qualifies as the state vector and eqn. (1.3) is termed the state equation in differential form[9] of system S.

The state of a system at time t is characterized by a point $\underline{x}$ in an n-dimensional state space. For any $\underline{x}(t_o)$ and given $\underline{u}(t)$, the differential equation (1.3) defines trajectory in the state space; the uniqueness of solution of eqn. (1.3) ensures that there is only one trajectory passing through any point $\underline{x}$, i. e. given any initial state $\underline{x}(t_o)$, the future $\underline{x}(t)$ is uniquely determined in the future.

A set of state variables of a given system is not unique. Suppose that an n-vector $\underline{x}$ from an n-dimensional Euclidean space R^n is a state vector. We may take an n-vector $\underline{x}_1$ given by $\underline{x}_1 = \underline{x}_1 = \underline{X}_1(\underline{x})$ as another state vector provided that for every $\underline{x}_1 \in R^n$ there corresponds a unique $\underline{x} \in R^n$ and vice versa.

1.3 HISTORICAL BACKGROUND

Although the field of nonlinear system analysis has a long history, it has been only in relatively recent times that appreciable interest has been shown in the formulation of a general theory for nonlinear electrical networks[10-15].

In 1957 Bashkow[16] introduced a new network characterization in terms of the A matrix. He gave a method for the formulation of a minimal set of first order differential equations for linear time invariant RLC networks. The method was unnecessarily complicated as it required the inclusion of an extra reactive element for each excess* capacitor or inductor in order to simplify the elimination procedure.

In 1959 Bryant[17] presented a systematic treatment of the problem of the state-variable characterization of linear passive RLC networks. He defined the order of complexity of an electrical network as being equal to the dimension of the state vector. The approach is topological and is based on the selection of a particular tree, usually referred to as normal tree[18]. It was shown that the set of capacitor voltages associated with a normal tree and inductor currents associated with the corresponding cotree always defines a state vector for networks containing linear, passive, time invariant RLC elements without coupling. Later the explicit normal form was given for this class of networks[19].

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* Any capacitors and inductors which prevent the formation of a proper tree are called excess[16]; a proper tree[16] of a network contains <u>every</u> capacitive element of the network or every capacitive element plus resistive elements (but no inductive elements).

Subsequently Bryant's approach has been extended to the case of linear, active time invariant networks, linear time-varying networks and nonlinear networks.

In regard to nonlinear network the first paper[2] to consider the problem of the normal form characterization for networks containing two-terminal elements appeared in 1963. Sufficient conditions for the existence of the normal form were given for a class of networks containing nonlinear resistors with strictly monotonic characteristics. Later the same authors treated nonlinear networks with controlled sources[20] subject to certain topological constraints; Palais' theorem[21] on global inverse mapping was applied to obtain a sufficient condition for the existence of the normal form characterization of this class of networks.

Independently and almost simultaneously Stern[5,22], Desoer and Katzenelson[4] and Chua and Rohrer[3] have all proposed methods for the formulation of normal form equations for different classes of nonlinear networks.

In Ref. 4 nonlinear RLC networks containing two-terminal elements were treated and elements with nonmonotonic characteristics were allowed. The dependent variables of the normal form were cut-set charges $\underline{q}$ and loop flux-linkages $\underline{\psi}$ defined with respect to fundamental cut-sets through capacitive branches of the normal tree and fundamental loops of inductive branches in the normal cotree. Sufficient conditions were stated for a network to be determinate, i. e. to possess a unique response for arbitrary initial conditions and given distribution of independent sources. Such conditions are of two kinds:

a) network elements with nonmonotonic characteristics have to satisfy certain topological conditions and

b) network element characteristics have to satisfy suitable Lipschitz conditions.

Chua and Rohrer[23] introduced parametric representation of nonlinear network elements and gave a method for formulation of dynamic equations[3] in the normal form. The dependent variables appearing in the normal form equations are a set of characteristic parameters associated with capacitors in the normal tree and inductors in the normal cotree.

Stern[5,22] studied the normal form description of nonlinear RLC networks containing coupled elements. The concept of the complete set of network variables, defined in this work, is an extention of Bryant's complete set of dynamically independent variables[17] containing only branch voltages and currents as its elements. The class of quasilinear network elements - reciprocal and locally passive elements* - was introduced. With respect to a normal tree a set of cut-set charges $\underline{q}$ and loop flux-linkages $\underline{\varphi}$ was selected as a possible complete set of network variables. In order to determine whether the set $(\underline{q}, \underline{\varphi})$ is complete or not it is necessary to examine the existence and uniqueness of solution of three different one-element-kind networks: resistive, inductive and capacitive. These three networks are described by three sets of algebraic equations referred to as (R), (L) and (C) equations respectively. The question of the existence and uniqueness of solutions for these sets of equations was resolved and some computational schemes were proposed for the following classes of networks:

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* Definitions of reciprocity and local passivity of nonlinear elements will be given in Chapter 2.

(i) networks containing quasilinear elements,

(ii) networks solvable by contraction mapping techniques and

(iii) linearly reducible networks.

A significant result was obtained for networks containing resistive, capacitive and inductive quasilinear elements only; it was proved that this class always possesses the normal form description regardless of network topology.

At approximately the same time Brayton and Moser[24] investigated complete networks - a special class of linearly reducible networks - using the concept of mixed potential function. Some interesting stability results were obtained. In Ref. 25 the link between the parametric approach and the Brayton-Moser approach was provided and in Ref. 26 criteria for the existence of the normal form of complete networks were given.

Besides Stern's work a number of papers treating the normal form characterization of nonlinear RLC networks containing coupled elements have appeared. The tutorial paper by Kuh[27] and the review paper by Kuh and Rohrer[18] were devoted to the problem of setting up the normal form equations generally. Holzmann and Liu[28] extended the parametric representation to coupled elements and combined techiques suggested in Refs. . 3-5 and 17. Two choices of the dependent variables of the normal form were considered:

(i) the cut-set charges $\underline{q}$ and the loop flux-linkages $\underline{\psi}$ [4,5]

(ii) a subset of the characteristic parameters of the inductive and capacitive elements[3]

For either of the two choices of the dependent variables a set of sufficient conditions was proposed. These conditions, stated as main theorems, appear to be given in mathematical terms rather than network terms and are very difficult to apply to practical circuits even with extensive analysis. A corollary of the main theorem expressed in network terms required that subsets of network elements which are not reciprocal and strictly locally passive have to fulfil certain topological conditions.

Varaiya and Liu[29] treated networks assuming the existence of a normal tree such that there was no coupling between resistive link and tree-branches and similarly for inductive and capacitive elements. Another assumption made in this work was that all network elements were locally passive. Sufficient conditions were given for the existence of the normal form of this class of networks. It will be shown in this thesis that their conditions can be relaxed. Two recent contributions in the field of the state variable approach to nonlinear networks were made by MacFarlane[30] and Ohtsuki and Watanabe[31]. Nagrath and Jain[32] studied problems of the state varible description for general classes of nonlinear dynamical systems of lumped multiterminal components.

The difficulties encountered in an attempt to obtain the normal form description of a network arise in two ways. First of all, a given network may be impossible to characterize in the normal form due to a dynamically incomplete specification of the network model; as a consequence either the complete set of network variables cannot be found or the complete set can be selected but the normal form equations cannot be written in terms of this complete set. Secondly, the given network may require an extensive but undesirable a priori computation to test whether or not the network may be characterized in the normal form; this difficulty is present in the class of networks where knowledge of the type of elements and the topology of network is not sufficient to determine whether the normal from description exists or not. Networks possessing this kind of difficulty have been termed irregular networks[33,34]. The class of irregular networks comprises the following two disjoint subclasses:

(i) strictly irregular networks having the property that the set $(\underline{q}, \underline{\varphi})$ is not complete

(ii) potentially irregular networks for which the network topology and the type of elements is not sufficient to determine whether $(\underline{q}, \underline{\varphi})$ is the complete set or not and for which one has to resort to the algebraic relations between branch variables to answer this question.

A topological method was developed to identify irregular networks containing two-terminal elements and a systematic procedure was proposed to modify irregular networks by augmentation with small values of "stray" elements. The augmented network can then be characterized in the normal form. The identification and augmentation of a more general class of irregular networks containing either dependent sources or multiterminal elements was studied in[33].

Only the class of linearly reducible networks may be described by the explicit normal form. For other networks* the normal form cannot be written explicitly and one generally obtains a constrained set of differential equations of the form[35].

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}, \underline{p}) \qquad (1.4a)$$

$$\underline{0} = \underline{g}(t, \underline{x}, \underline{p}) \qquad (1.4b)$$

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* Exception are networks for which the (R), (L) and (C) equations possess an analytic solution

where $\underline{x}$ represents a set of n dynamic variables, $\underline{p}$ a set of k auxiliary variables, eqn. (1.4a) a set of first order differential equations and eqn. (1.4b) a set of k constraint equations. An algorithm was developed[35] to modify a network, which is not necessarily irregular, into a linearly reducible network. The augmented network containing small stray elements is described by a stiff system of differential equations[36]; a computational method based on the approximate solution of differential equations containing certain small parameters was proposed.

Thus it seems that the problem of the formulation of the state equations for a general nonlinear RLC network still presents a difficult problem and no simple, general approach is available to test whether a given network is determinate or not. Nevertheless, before a computer analysis is attempted we must consider the question of existence and uniqueness of the network solution. If the network is not determinate, the computer might yield strange results that depend on the particular algorithm and the particular program being used. For example, if more than one solution is possible then, depending on the algorithm or on round-off errors, etc., the computer might pick one of the solutions without giving the user any indication that another solution exists and that something is basically wrong with his problem or that the chosen model of the physical circuit has been oversimplified.

It is therefore useful to study sufficient conditions ensuring that a given RLC network is determinate. When these conditions are stated in mathematical form (e. g. Theorem 1 of Ref. 28) it is not easy to apply them to practical circuits; such "mathematical" criteria require that certain conditions have to be fulfilled for an infinite number of points in a multidimensional Euclidean space and this presents a serious difficulty for their application in general. Another kind of such conditions for determinateness of a given RLC network represent topological conditions which have to be fulfilled for a given network containing different classes of network elements; on the basis of these conditions a network can then be checked by inspection. As has already been mentioned RLC networks containing quasilinear elements only are determinate regardless of network topology, but network elements which are not strongly locally passive have to satisfy certain topological conditions[4,5]. For example, the usual assumption[4,5] is that in an RLC network each voltage-controlled port of a resistive element (or a voltage-controlled two-terminal resistor) lies in a loop containing capacitors and/or independent voltage sources only and dually the cut-set through each current-controlled port of a resistive element (or a current-controlled two-terminal resistor) contains inductors and/or independent current sources only. The main aim of this thesis is to develop less restrictive topological conditions regarding locally active and nonreciprocal nonlinear coupled elements in an RLC network. To this end new classes of nonlinear network elements will be defined and their properties and interconnections will be examined.

1.4 AIMS AND LAYOUT OF THE THESIS

The main aim of this thesis is to consider state variable theory as a means of analysing nonlinear RLC networks containing coupled elements. The problems of setting up the state equations for a fairly large class of RLC networks will be considered. Since the key problem in the state variable characterization of a given RLC network is the analysis of certain capacitive, inductive and resistive subnetworks, a particular emphasis will be given to the study of one-element-kind networks containing nonreciprocal and/or locally active elements. Sufficient conditions which ensure that a given one-element-kind network possesses a unique solution will be given for networks containing different classes of nonlinear elements. These conditions will then be used in the study of the state variable characterization of nonlinear RLC networks. The thesis will therefore be set out in the following manner.

Chapter 2 contains mostly introductory work. Characterization of nonlinear RLC network elements and fundamental topological concepts are discussed and then different kinds of analyses of resistive networks are presented. An introduction to the formation of the state equations for nonlinear RLC networks is given and the fundamental concepts and results, forming the basis for the later work, are stated.

Chapter 3 represents the necessary mathematical background of the thesis. A simple criterion ensuring that a given vector-valued function possesses an inverse function is derived. This criterion which is based on the Jacobian of a given function is used later in the study of the existence and uniqueness of solutions of one-element-kind networks. The problem of "partial" inversion, important for the transformations from one hybrid description of a network element into another, is treated. Certain classes of invertible functions which appear in the analysis of one-element-kind networks are examined. The concept of a generalized quasilinear function is introduced and the properties of these functions are examined. Finally, some globally asymptotically stable differential equations that may be associated with a given algebraic equation and whose singular points correspond to the solution of this algebraic equation are presented.

In Chapter 4 the concept of positive network elements is introduced and the properties of this class of elements are investigated. Positive network elements may be locally active and many transistor and other active devices can be modelled as positive network elements. Positive definite network elements forming a subclass of positive network elements and representing a generalization of quasilinear elements are introduced and their properties are shown. Two other significant subclasses of positive network elements are treated.

Chapter 5 is devoted to the analysis of one-element-kind networks. The class of positive semidefinite elements - locally passive elements - is introduced. Series-parallel interconnections of network elements are studied and sufficient conditions ensuring that an interconnection of two network elements of one kind results either in a positive network element or in a positive definite network element are given. Different sets of sufficient conditions are presented ensuring that one-element-kind networks containing positive and positive definite (and/or positive semidefinite) network elements possess a unique solution. A number of examples is provided to illustrate the theory developed in Chapter 5.

Chapter 6 is concerned mainly with the problem of uniqueness of solutions of nonlinear RLC networks. An extention to RLC networks containing dependent sources is presented.

Chapter 7 is a discussion of some of the conclusions which can be drawn from the previous 6 chapters and a list of some suggestions of future work.

To make the reading of the thesis easier, each chapter contains a brief introduction and, at the end, a summary discussion. Also to this end, the main results have been stated as theorems and corollaries.

1.5 STATEMENT OF ORIGINALITY

Since the commencement of this work a number of papers have appeared considering similar problems to those of this thesis and a number of similar results have been given. In particular, T. Ohtsuki and H. Watanabe introduced the concept of positive definite network element and studied RLC network containing this class of elements. While extending T. E. Stern's results for networks containing quasilinear elements to the nonreciprocal case the author arrived independently at a very similar concept of network elements which are characterized by generalized quasilinear functions and which he initially termed generalized quasilinear network elements; since this class of elements forms a subclass of positive network elements (to be discussed later) Ohtsuki and Watanabe's term positive definite network element was adopted.

Except where reference is made to the published material, the results and conclusions reported in this thesis were obtained independently by the author, and at the time of writing are believed to be original. A list of original contributions to be presented is given below:

(1) the criterion for the global invertibility of vector-valued functions;

(2) the generalization of local implicit function theorem to consider global behaviour;

(3) the concept of generalized quasilinear function and the properties of generalized quasilinear functions;

(4) the conditions for invertibility of the function

$$\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & F \\ -F^T & 0 \end{bmatrix} \begin{bmatrix} \underline{g}_1(\underline{x}_1) \\ \underline{g}_2(\underline{x}_2) \end{bmatrix}$$

appearing in the analysis of one-element-kind networks which contain locally passive elements;

(5) a globally asymptotically stable differential equation is presented where the Euler method leads to the Newton Raphson method;

(6) the concept of the class of UP matrices and the result that all "partial" inverses of a bounded UP matrix are themselves bounded UP matrices;

(7) the concept of positive network elements - possibly locally active and/or nonreciprocal elements - is introduced and the properties of this class of elements are shown;

(8) the concept of positive definite network elements - strongly locally passive and possibly nonreciprocal elements - is introduced and the properties of positive definite elements are listed;

(9) the conditions ensuring that a three-terminal network represents a positive network element in all three orientations

(10) the conditions that a series-parallel interconnection of two network elements results in a positive network element;

(11) the conditions that a series-parallel interconnection of two network elements results in a positive definite network element;

(12) the result that an (n+m-1)-port network element N formed from an n-port positive network element N_1 and an m-port network element N_2 connecting one of the ports of N_1 to one of the ports of N_2 in parallel (or in series) represents a positive network element;

(13) the result that a resistive network containing positive definite network elements only always possesses a unique solution; similarly RLC networks containing positive definite elements only possess a unique solution regardless of network topology;

(14) the conditions ensuring the existence of a unique solution of one-element-kind networks (Theorems 5.8 - 5.12)

(15) the conditions ensuring the existence of a unique solution for nonlinear RLC networks (Theorems 6.1 - 6.3).

1.6 REFERENCES

1 Liu, R. and Auth, L. V., "Applications of Liapunov's direct method to nonlinear networks at large", Report TREE-642, University of Notre Dame, Notre Dame, Ind., March 1964.

2 Liu, R. and Auth, L. V., "Qualitative synthesis of nonlinear networks", Proc. 1963 First Annual Allerton Conference on Circuit and System Theory, pp. 330-343.

3 Chua, L. O. and Rohrer, R. A., "On the dynamic equations of a class of nonlinear RLC networks", IEEE Trans. on Circuit Theory, vol. CT-12, pp. 475-489, December 1965

4 Desoer, C. A. and Katzenelson, J., "Nonlinear RLC networks", Bell Sys. Tech. J., vol. 44, pp. 161-198, January 1965.

5 Stern, T. E., "On the equations of nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 74-81, March 1966.

6 LaSalle, J. and Lefschetz, S., "Stability by Liapunov's direct method with applications", New York, Academic Press, 1961, pp. 56-67.

7 Timothy, L. K. and Bona, B. E., "State space analysis", McGraw-Hill, New York, 1968, p. 5.

8 DeRusso, P. M., Roy, R. J. and Close, C. M., "State variables for engineers", John Wiley and Sons, New York, 1965, p. 105.

9 Zadeh, L. A. and Desoer, C. A., "Linear system theory", McGraw-Hill, New York, 1963

10 Duffin, R. J., "Nonlinear networks I", Bull. Amer. Math. Soc. vol. 52, pp. 836-838, 1946.

11 Duffin, R. J., "Nonlinear networks II", Bull. Amer. Math. Soc., vol. 53, pp. 963-971, 1947.

12 Duffin, R. J., "Nonlinear networks IIb", Bull. Amer. Math. Soc., vol. 54, pp. 119-127, 1948.

13 Millar, W., "Some general theorems for nonlinear systems possessing resistance", Phil. Mag., vol. 42, pp. 1150-1160, October 1951.

14 Cherry, E. C., "Some general theorems for nonlinear systems possessing reactance", Phil. Mag., vol. 42, pp. 1161-1177, October 1951.

15 Birkhoff, G. and Diaz, J. B., "Nonlinear network problems", Quat. Appl. Math., vol 13, pp. 431-443, January 1956.

16 Bashkow, T. R., "The A matrix, new network description", IRE Trans. on Circuit Theory, vol. CT-4, pp. 117-114, September 1957.

17 Bryant, P. R., "The order of complexity of electrical networks", Proc. IEE, vol 106, Mono 335, Pt. C, pp. 174-188, September 1959.

18 Kuh, E. C. and Rohrer, R. A., "The state variable approach to network analysis", Proc. IEEE, vol. 53, pp. 672-686, July 1965.

19 Bryant, P. R., "The explicit form of Bashkow's A matrix", IRE Trans. on Circuit Theory, vol. CT-9, pp. 303-306, September 1962.

20 Auth, L. V. and Liu, R., "On the existence of the normal form for nonlinear networks at large with controlled sources", Proc. 1964 Second Annual Conference on Circuit and System Theory, pp. 288-292.

21 Palais, R. S., "Natural operations on differential forms", Trans. Amer. Math. Soc., vol. 92 , pp. 125-141, July 1959.

22 Stern, T. E., "Theory of nonlinear networks and systems", Reading, Mass., Addison-Wesley, 1965.

23 Chua, L. O. and Roher, R. A., "On the characterization of nonlinear networks", Proc. of NEC, vol. 21, 1965, Chicago Illinois, pp. 708-713.

24 Brayton, R. K. and Moser, J. K., "A theory of nonlinear networks", Quat. Appl. Math., vol. 22, pp. 1-33, April 1964.

25 Chua, L. O., "On the choice of state variables for the potential functions in nonlinear networks", Proc. IEEE, vol. 53, p. 2110, December 1965.

26 Chua, L. O., "State variable formulation of nonlinear RLC networks in explicit normal form", Proc. IEEE, vol. 53, p. 206, February 1965.

27. Kuh, E. C., "Representation of nonlinear networks", Proc. of NEC, vol. 21, 1965, Chicago, Illinois, pp. 702-707.

28 Holzmann, C. A. and Liu, R., "On the dynamical equations of nonlinear networks with n-coupled elements", Proc. 1965 Third Annual Allerton Conference on Circuit and System Theory, pp. 536-545.

29 Varayia, P. P. and Liu, R., "Normal form and stability of a class of coupled nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 413-418, December 1966.

30 MacFarlane, A. G. J., "Formulation of the state-space equations for non-linear networks", Int. J. Control, vol. 5, pp. 145-161, May 1967.

31 Ohtsuki, T. and Watanabe, H., "State-variable analysis of RLC networks

containing nonlinear coupling elements", IEEE Trans. on Circuit Theory, vol. CT-16, pp. 26-38, February 1969.

32 Nagrath, I. J. and Jain, V. K., "Formulation of state equations for general nonlinear systems of multiterminal components", Proc. IEE, vol. 114, pp. 1647-1651, November 1967.

33 Oh, S. J., "Analysis of nonlinear irregular networks", Tech. Report No. 96, Department of Electrical Engineering, Columbia University, New York, June 1966.

34 Oh, S. J., Stern, T. E. and Meadows, H. E., "On the analysis of nonlinear irregular networks", Proc. of the Symposium on Generalized Networks, Politechnic Press, New York, pp. 653-682, April 1966.

35 Stern, T. E., "Computer-aided analysis of nonlinear networks described by constrained differential equations", Proc. of the International Symposium on Network Theory, Belgrade, pp. 178-190, September 1968.

36 Sandberg, I. W. and Shichman, H., "Numerical integration of stiff nonlinear differential equations", Bell Sys. Tech. J., vol 47, pp. 511-527, April 1968.

Chapter 2

PRELIMINARIES

2.1 INTRODUCTION

Throughout this thesis, use will be made of the basic properties of network elements, topological methods and the conventional methods of analysis of nonlinear networks. In order to put our subsequent discussions on a more rigorous ground, we will define in this chapter some of the frequently used terms and introduce the basic definitions and results on which all the later work is based.

2.2 NETWORK ELEMENTS

2.2.1 Characterization of network elements[1, 2, 3, 4]

The nonlinear networks considered in this thesis will contain resistive, inductive and capacitive elements and voltage and current sources. Each network element can be viewed abstractly as a collection of coupled directed branches, where each branch has associated with it the pair of branch variables, i (current) and v (voltage). Integrals of these quantities [q (charge) and φ (flux)] will also be referred to as branch variables.

Schematically, a network element can be represented either as an n-port - that is n-terminal pair element - or an (n+1)-terminal element having n branches with one node in common. Each branch of an n-port (or (n+1)-

terminal) element will be referred to as a port as far as the characteristics of this element are concerned.

At this point we must delineate the class of network elements which will be considered and for which the analysis will be formulated. Network elements which exhibit hysteretic behaviour are excluded a priori since no satisfactory means of mathematically describing this phenomenon have been advanced[5]. Furthermore it will be assumed that in order to describe a network element, a set of n independent constraint equations, relating branch variables of an n-port (or (n+1)-terminal) element, has to be known. However n-port network elements, characterized by a number of independent equations larger or smaller than n, have been considered [6]. As the simplest n-port elements of this kind, the nullator, the resistive one-port element with $v = 0$ and $i = 0$, and the norator, another resistive one-port with v and i arbitrary, were introduced. Tellegen[7] demonstrated that singular network elements - the nullator and the norator - are mathematical concepts without physical content and have to be regarded as fictitious "mathematical" components. For this reason network elements of this kind will be disregarded.

An n-port (or (n+1)-terminal) network element (possibly time varying) will be termed resistive if its n independent constraint equations can be written implicitly as

$$\underline{f}_R\,(\underline{i}_R,\ \underline{v}_R,\ t) = \underline{0} \tag{2.1}$$

where $\underline{i}_R$, $\underline{v}_R$ are n-vectors, representing resistor currents and voltages respectively, $\underline{f}_R\,(\cdot)$ is a vector-valued function of dimension n and t denotes time. Eqn. (2.1) will be called the <u>implicit description</u> of a resistive element. The function $\underline{f}_R\,(\cdot)$ is assumed to be single valued, continuous and differentiable.

Eqn. (2.1) could define sets of elements as well as individual resistive elements. For example, a set of n one-port resistors is a special case of an n-port in which there is no coupling among any two ports.

Let the n-vectors $\underline{i}_R$ and $\underline{v}_R$ be partitioned as follows

$$\underline{i}_R = \begin{bmatrix} \underline{i}_1 \\ \underline{i}_2 \end{bmatrix} \qquad \underline{v}_R = \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix} \tag{2.2}$$

where $\underline{i}_1$, $\underline{v}_1$ are m-vectors representing currents and voltages of ports 1, 2, ..., m and $\underline{i}_2$, $\underline{v}_2$ are (n-m)-vectors associated with branch variables of ports m + 1, m + 2, ..., n. We can define a set of mixed variables

$$\underline{x}_R = \begin{bmatrix} \underline{v}_1 \\ \underline{i}_2 \end{bmatrix} \qquad \underline{y}_R = \begin{bmatrix} \underline{i}_1 \\ \underline{v}_2 \end{bmatrix} \tag{2.3}$$

The n-vectors $\underline{x}_R$ and $\underline{y}_R$ include one and only one branch variable (voltage or current) from each port.

The implicit description eqn. (2.1) can be rewritten in terms of mixed varibles as

$$\underline{f}_R\ (\underline{x}_R,\ \underline{y}_R,\ t) = \underline{0} \tag{2.4}$$

In certain cases the implicit equation (2.4) can be solved for all $\underline{x}_R \in R^n$, giving

$$\underline{y}_R = \underline{h}_R\ (\underline{x}_R,\ t) \tag{2.5a}$$

or equivalently in component form

$$\left.\begin{aligned} \underline{i}_1 &= \underline{i}_1\ (\underline{v}_1,\ \underline{i}_2,\ t) \\ \underline{v}_2 &= \underline{v}_2\ (\underline{v}_1,\ \underline{i}_2,\ t). \end{aligned}\right. \tag{2.5b}$$

In eqn. (2.5a) $\underline{h}_R\ (\cdot)$ is a single-valued function of $\underline{x}_R$, t. A description such as that given in eqn. (2.5a) is called the hybrid description* of a resistive element. We shall asume that an arbitrary resistive element to be considered can be represented by at least one of the hybrid descriptions, and that every vector-valued function $\underline{h}_R(\cdot)$ belongs to class $C^{(1)}$. **

When the independent variable in the hybrid description (2.5a) is voltage, i. e. $x_i = v_i$, the i-th branch is said to be voltage-controlled[1,2], similarly when in eqn. (2.5a) $x_i = i_i$, the i-th branch is referred to as current-controlled. Two special hybrid descriptions which may or may not exist for a given resistive element are

$$\underline{i}_R = \underline{i}_R(\underline{v}_R, t) \tag{2.5c}$$

$$\underline{v}_R = \underline{v}_R(\underline{i}_R, t) \tag{2.5d}$$

Eqn. (2.5c) describes a resistive element where all branches are voltage-controlled and eqn. (2.5d) characterizes an element with all current-controlled branches.

The implicit description admits very general types of elements. For example, a two-terminal resistive element implicitly described by the relation:

$$i_R^2 + v_R^2 - 1 = 0$$

is neither voltage nor current-controlled; its characteristic in the i_R, v_R plane is a circle and the hybrid description does not exist. Many physical resistive

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* The term hybrid description[3] is equivalent to the term explicit branch relations used in Ref. 4.

** A function $\underline{f}(\underline{x})$ is of class $C^{(1)}$ if it is a continuous function and if its Jacobian matrix $\partial \underline{f}/\partial \underline{x}$ is continuous for all $\underline{x}$.

elements such as vacuum triodes, transistors, ideal transformes, gyrators, dependent and independent voltage and current sources etc. can be characterized in terms of the hybrid description $\underline{h}_R(\cdot)$.

Inductive and capacitive elements are characterized analogously to resistive elements. The implicit description of an n-port (or (n+1)-terminal) inductive element is given as

$$\underline{f}_L(\underline{i}_L, \underline{\varphi}_L, t) = 0 \qquad \underline{v}_L = \dot{\underline{\varphi}}_L \tag{2.6}$$

where $\underline{i}_L, \underline{\varphi}_L, \underline{v}_L$ are n-vectors representing the branch variables of an inductive element and $\underline{f}_L(\cdot)$ has the dimension n. A set of mixed variables $\underline{x}_L, \underline{y}_L$ may be formed in the same way as in the case of a resistive element.

$$\underline{x}_L = \begin{bmatrix} \underline{\varphi}_1 \\ \underline{i}_2 \end{bmatrix} \qquad \underline{y}_L = \begin{bmatrix} \underline{i}_1 \\ \underline{\varphi}_2 \end{bmatrix} \tag{2.7}$$

where $\underline{\varphi}_1, \underline{i}_1$ represent flux-linkages and currents of ports 1, 2 ...., m and $\underline{\varphi}_2, \underline{i}_2$ correspond to flux-linkages and currents of ports m+1, m+2, ..., n. Vectors $\underline{x}_L, \underline{y}_L$ include one and only one branch variable (current or flux-linkage) from each port.

The hybrid description corresponding to the implicit description of an inductive element is given as

$$\underline{y}_L = \underline{h}_L(\underline{x}_L, t) \tag{2.8a}$$

or equivalently in the component form

$$\begin{aligned} \underline{i}_1 &= \underline{i}_1(\underline{\varphi}_1, \underline{i}_2, t) \\ \underline{\varphi}_2 &= \underline{\varphi}_2(\underline{\varphi}_1, \underline{i}_2, t) \end{aligned} \tag{2.8b}$$

Flux-controlled and current-controlled branches of an inductive element are defined as in the case of resistive element. The two particular descriptions of inductive elements are

$$\underline{\varphi}_L = \underline{\varphi}_L(\underline{i}_L, t) \tag{2.8c}$$

$$\underline{i}_L = \underline{i}_L(\underline{\varphi}_L, t) \tag{2.8d}$$

where the first equation describes an inductive element where all branches are current-controlled and the second equation characterizes the case of an inductive element with all flux-controlled branches.

Similary the implicit description of an n-port (or (n+1)-terminal) capacitive element is

$$\underline{f}_C(\underline{v}_C, \underline{q}_C, t) = \underline{0} \qquad \underline{i}_C = \dot{\underline{q}}_C \tag{2.9}$$

where $\underline{v}_C, \underline{q}_C, \underline{i}_C$ are n-vectors associated with the branch variables of the capacitive element and $\underline{f}_C(\cdot)$ is a vector-valued function with n components. A set of mixed variables is formed from $\underline{v}_C, \underline{q}_C$.

$$\underline{x}_C = \begin{bmatrix} \underline{q}_1 \\ \underline{v}_2 \end{bmatrix} \qquad \underline{y}_C = \begin{bmatrix} \underline{v}_1 \\ \underline{q}_2 \end{bmatrix} \tag{2.10}$$

where $\underline{v}_1$, $\underline{q}_1$, describe voltages and charges of ports 1,2,...,m and $\underline{v}_2$, $\underline{q}_2$ correspond to voltages and charges of ports m+1, m+2, ..., n. The hybrid description of a capacitive element is given as

$$\underline{y}_C = \underline{h}_C(\underline{x}_C, t) \tag{2.11 a}$$

or equivalently

$$\underline{v}_1 = \underline{v}_1(\underline{q}_1, \underline{v}_2, t) \qquad (2.11\,b)$$
$$\underline{q}_2 = \underline{q}_2(\underline{q}_1, \underline{v}_2, t)$$

The two special hybrid descriptions

$$\underline{q}_C = \underline{q}_C(\underline{v}_C, t) \qquad (2.11\,c)$$

$$\underline{v}_C = \underline{v}_C(\underline{q}_C, t) \qquad (2.11\,d)$$

are associated with capacitive elements having all voltage-controlled branches or with capacitive elements having all charge-controlled branches.

It will be assumed that functions $\underline{h}_C(\cdot)$ and $\underline{h}_L(\cdot)$ have the entire R^n as their domain and that both are of class $C^{(1)}$. Furthermore, for each capacitive and inductive element to be considered it is assumed that at least one hybrid description exists.

In the analysis of nonlinear networks the incremental parameter matrices, describing the small signal behaviour of network elements, play an important role. Certain classes of network elements will be defined in terms of their incremental parameter matrices. Also some significant concepts such as reciprocity and local passivity of network elements are defined in terms of these matrices.

The incremental parameter matrices of resistive, inductive and capacitive elements are the Jacobian matrices of the hybrid descriptions $\underline{h}_R(\cdot)$, $\underline{h}_L(\cdot)$ and $\underline{h}_C(\cdot)$, associated with these elements. Since functions $\underline{h}_R(\cdot)$, $\underline{h}_L(\cdot)$ and $\underline{h}_C(\cdot)$ are assumed to be of class $C^{(1)}$ the Jacobian matrices of these functions exist.

From the hybrid description of a resistive element (eqn. (2.5a) or (2.5b)) H_R, the hybrid incremental resistance matrix[4] or shortly the <u>hybrid matrix</u>[3], is defined as

$$H_R(\underline{x}_R, t) = \frac{\partial \underline{h}_R}{\partial \underline{x}_R} = \frac{\partial(\underline{i}_1, \underline{v}_2)}{\partial(\underline{v}_1, \underline{i}_2)} = \begin{bmatrix} H_{R11} & H_{R12} \\ H_{R21} & H_{R22} \end{bmatrix} \quad (2.12\,a)$$

Two particular hybrid matrices may be obtained from the two special hybrid descriptions (eqns. (2.5c) and (2.5d)) where all branches of a resistive element are either voltage or current-controlled. The incremental conductance matrix G and the incremental resistance matrix R are defined as

$$G(\underline{v}_R, t) = \frac{\partial \underline{i}_R}{\partial \underline{v}_R} \quad (2.12\,b)$$

and

$$R(\underline{i}_R, t) = \frac{\partial \underline{v}_R}{\partial \underline{i}_R} \quad (2.12\,c)$$

The hybrid matrices of inductive and capacitive elements are defined from the corresponding hybrid descriptions. For inductive elements the hybrid matrix H_L, the incremental inductance matrix L and the inverse incremental inductance matrix Γ are from eqns. (2.8a) - (2.8d)

$$H_L(\underline{x}_L, t) = \frac{\partial \underline{h}_L}{\partial \underline{x}_L} = \frac{\partial(\underline{i}_1, \underline{\varphi}_2)}{\partial(\underline{\varphi}_1, \underline{i}_2)} = \begin{bmatrix} H_{L11} & H_{L12} \\ H_{L21} & H_{L22} \end{bmatrix} \quad (2.13\,a)$$

$$L(\underline{i}_L, t) = \frac{\partial \underline{\varphi}_L}{\partial \underline{x}_L} \quad (2.13\,b)$$

$$\Gamma(\underline{x}_L, t) = \frac{\partial \underline{i}_L}{\partial \underline{\varphi}_L} \quad (2.13\,c)$$

For capacitive elements the hybrid matrix H_C, the incremental capacitance matrix C and the incremental elastance matrix S are from eqns. (2.11 a) - (2.11 d)

$$H_C(\underline{x}_C, t) = \frac{\partial \underline{h}_C}{\partial \underline{x}_C} = \frac{\partial(\underline{v}_1, \underline{q}_2)}{\partial(\underline{q}_1, \underline{v}_2)} = \begin{bmatrix} H_{C11} & H_{C12} \\ H_{C21} & H_{C22} \end{bmatrix} . \tag{2.14a}$$

$$C(\underline{v}_C, t) = \frac{\partial \underline{q}_C}{\partial \underline{v}_C} \tag{2.14b}$$

$$S(\underline{q}_C, t) = \frac{\partial \underline{v}_C}{\partial \underline{q}_C} \tag{2.14c}$$

For the purpose of network analysis it is useful to introduce various kinds of sources as a special class of network elements. In principle, it is possible to include all constant (and time varying) independent voltage (and current) sources into the class of resistive elements[4]. However, an independent voltage source may be viewed as a charge-controlled capacitor with infinite capacitance; similarly an independent current source can be considered as a flux-controlled inductor with infinite inductance. It is for this reason that in this thesis independent sources will be considered as a special class of network elements and will be treated separately.

Another important class of network elements are controlled voltage and current sources. The controlling variable of a controlled voltage (or current) source may be voltage, current, charge or flux and thus RLC networks containing controlled sources are much more general than networks containing RLC elements only. In certain cases a network N of RLC elements and controlled sources can be transformed into an equivalent network N′ consisting of RLC elements only when extra branches are introduced in N. For example, a network containing a capacitor C and voltage-controlled voltage source $E = f(v_c)$ in Fig. 2.1 can be transformed into a network containing the capacitor C and a two-port resistor R when a resistive branch with $i_{R1} = 0$

is introduced across the capacitor C. The hybrid description of R is

$$i_{R1} = 0$$

$$v_{R2} = f(v_{R1})$$

2.2.2 Reciprocity, passivity and local passivity of network elements

In this section RLC network elements will be classified with respect to certain special properties. Reciprocity of a network element is related to the symmetry of certain incremental parameter matrices. Passivity and local passivity are concepts defined with respect to the ability of elements to dissipate energy.

Definition 2.1[4]

A resistive element defined by the hybrid description of the form of eqn. (2.5a), (2.5c) or (2.5d) is said to be reciprocal if the matrix

$$\begin{bmatrix} H_{R11} & H_{R12} \\ -H_{R21} & -H_{R22} \end{bmatrix} \quad \text{or} \quad G(\underline{v}_R, t) \quad \text{or} \quad R(\underline{i}_R, t)$$

is symmetric.

An inductive element defined by the hybrid description of the form of eqn. (2.8a), (2,8c) or (2.8d) is said to be reciprocal if the matrix

$$\begin{bmatrix} H_{L11} & H_{L12} \\ -H_{L21} & -H_{L22} \end{bmatrix} \quad \text{or} \quad L(\underline{i}_L, t) \quad \text{or} \quad \Gamma(\underline{\varphi}_L, t)$$

is symmetric.

A capacitive element defined by the hybrid description of the form of eqn. (2.11a), (2.11c) or (2.11d) is said to be reciprocal if the matrix

$$\begin{bmatrix} H_{C11} & H_{C12} \\ -H_{C21} & -H_{C22} \end{bmatrix} \quad \text{or} \quad C(\underline{v}_C, t) \quad \text{or} \quad S(\underline{q}_C, t)$$

is symmetric.

Similarly to the reciprocity of network elements reciprocity of one-element-kind networks (either resistive or capacitive or inductive) can be defined with respect to a selected set of ports. One-element-kind network or its subnetwork will be said to be reciprocal if it behaves like a reciprocal element when viewed from a selected set of ports.

Reciprocal elements and reciprocal one-element-kind networks have the property that state functions[4] may be obtained for these elements and networks. These functions are energy like functions and are especially important in the stability analysis of networks. Since in this thesis we shall be interested in nonreciprocal networks in particular we shall not make use of the concept of state functions.

Passivity and local passivity are properties associated with the ability of resistive elements to dissipate power. These two concepts may be extended to inductive and capacitive elements[8]. While passivity is defined with respect to the large signal operation of a network element, local passivity is related to the small signal operation.

Definition 2.2

A resistive element, defined by the hybrid description of eqn. (2.5a), is passive if

$$\underline{v}_R^T \underline{i}_R \geq 0 \tag{2.15a}$$

for all $\underline{x}_R \neq \underline{0}$ and all t.

An inductive element, defined by the hybrid description of eqn. (2.8a), is passive if

$$\underline{\varphi}_L^T \underline{i}_L \geq 0 \tag{2.15b}$$

for all $\underline{x}_L \neq \underline{0}$ and all t.

A capacitive element, defined by the hybrid description of eqn. (2.11a), is passive if

$$\underline{q}_C^T \underline{v}_C \geq 0 \tag{2.15c}$$

for all $\underline{x}_C \neq \underline{0}$ and all t. If in eqns. (2.16a) - (2.16c) the quantities on the l. h. s. are positive, then the corresponding network elements are <u>strictly passive.</u> An element is said to be <u>active</u> if it is not passive.

According to this definition ideal transformers and gyrators are passive but semiconductor diodes with the exponential characteristic and transistors are strictly passive. Note that vacuum diodes and triodes are active. There is an interesting class of active resistive elements where it is possible to perform a decomposition into a passive resistive element and a set of independent sources; these network elements will be treated in Section 4.3.2.

In the case of small signal operation of network elements we are interested in their local passivity. For example, transistor is a passive resistive element, but it is its local activity which makes it a useful element in electronic circuits. We shall distinguish three different kinds of local passivity.

Definition 2.3

A resistive (inductive, capacitive) element is said to be locally passive at a point $\underline{x}_R$ ($\underline{x}_L$, $\underline{x}_C$) and time t if its hybrid matrix $H_R(\underline{x}_R, t)$ ($H_L(\underline{x}_L, t)$, $H_C(\underline{x}_C, t)$) is positive semidefinite at that point.

A resistive (inductive, capacitive) element is said to be strictly locally passive at a point $\underline{x}_R$($\underline{x}_L$, $\underline{x}_C$) and time t if its hybrid matrix $H_R(\underline{x}_R, t)$ ($H_L(\underline{x}_L, t)$, $H_C(\underline{x}_C, t)$) is positive definite at that point.

A resistive (inductive, capacitive) element is said to be strongly locally passive if an $\varepsilon > 0$ exists such that $[H_R(\underline{x}_R, t) - \varepsilon I]$($[H_L(\underline{x}_L, t) - \varepsilon I]$, $[H_C(\underline{x}_C, t) - \varepsilon I]$) is positive definite for all $\underline{x}_R$ ($\underline{x}_L$, $\underline{x}_C$) and all t.

A resistive (inductive, capacitive) element is said to be locally active if it is not locally passive.

2.3 NETWORK TOPOLOGY[4,9]

A network can be viewed abstractly as a collection of directed branches. The interconnection of network elements imposes certain constraints on branch voltages and currents, constraints which are embodied in Kirchhoff's voltage and current laws. In order to analyse a given network it is necessary to obtain three sets of independent equations, i. e. Kirchhoff's voltage and current laws and a set of constraint equations characterizing all network elements. In general, the simultaneous use of all three sets of equations yields the required network response for a given set of initial conditions and excitations at the ports of the network. In order to express Kirchhoff's voltage and current laws in a systematic fashion the concepts of the fundamental loop matrix B and the fundamental cut-set matrix Q will be employed.

However, before defining the fundamental loop matrix and the fundamental cut-set matrix, a few additional concepts associated with a graph of a network will be introduced.

A loop is a closed path, along which any node is touched by exactly two braches in the loop.

A network (or subnetwork) is called a self-loop if it consists of a single branch whose end-points are identified: it consists of one branch and one node[1].

A network (or subnetwork) is called an open branch if it consists of a single branch whose end-points are not identified: it consists of one branch and two nodes[1].

A connected graph is a graph which consists of only one separate part. In this thesis it will usually be assumed that the network considered is connected. When a network is not connected each part can be treated separately as far as Kirchhoff's laws are concerned.

A separable graph is a connected graph which can be divided into more than one separate part by the removal of a single node. Otherwise the graph is nonseparable.

A cut-set of a connected graph is a set of branches such that their removal divides the graph into two separate parts and no proper subset of this set of branches has the same property.

A tree of a connected graph is a connected subgraph which contains all the nodes of the graph but does not contain any loops. The branches of a network contained in a particular tree are called tree-branches; those not contained in that tree are called links or chords. When we refer to tree-branches and links it is with respect to a chosen tree. It can be shown that a connected graph having ν nodes and b branches contains $(\nu - 1)$ tree branches and $(b-\nu+1)$ links. Once a tree T of a connected graph is chosen a special class of loops and cut-sets may be defined. If one link is added to a tree T, the resulting graph contains a loop, called a fundamental loop. Each fundamental loop contains exactly one link. Similarly it is possible to form a cut-set containing only one tree branch ofT and some links with respect to the tree. Such a cut-set is called a fundamental cut-set.

The fundamental loops of a connected directed graph with respect to a tree T are the $(b-\nu+1)$ loops formed by each link and the single path in the tree between the nodes of the link. The fundamental loop orientation is chosen to agree with that of the defining link.

If T is a tree of a connected directed graph G, the fundamental system of cut-sets with respect to T is the set of $(\nu - 1)$ cut-sets in which each cut-set includes only one branch of T. The fundamental cut-set orientati-on is to agree with the orientation of the defining branch.

We are now in a position to introduce the fundamental loop matrix and the fundamental cut-set matrix of a connected graph. Consider a connected graph G, having ν nodes and b branches, and choose a tree T. Let the branches of G be numbered consecutively, starting with the links of T. Now we assign numbers $1, 2, \ldots, (b-\nu+1)$ to the fundamental loops so as to coincide with the numbers of the defining links. Similarly, we assign numbers $b-\nu+2, b-\nu+3, \ldots, b$ to the fundamental cut-sets so as to coincide with their defining tree-branches. Using this numbering system we may state the following definition.

Definition 2.4

Given a graph G with tree T, the fundamental loop matrix B is defined as follows: each column of B corresponds to a branch of G (ordered consecutively); each row corresponds to a fundamental loop (ordered consecutively); and

$b_{ij} = 1$ if the j-th branch is in loop i and their orientations coincide;
$b_{ij} = -1$ if the j-th branch is in loop i and their orientations do not coincide;
$b_{ij} = 0$ if the j-th branch is not in loop i.

Because of the way in which the branches are numbered, the matrix B is of the form

$$B = [I, F] \tag{2.16}$$

where I is a $(b-\nu+1) \times (b-\nu+1)$ identity matrix and F a $(b-\nu+1) \times (\nu-1)$ matrix.

Definition 2.5

Given a graph G with tree T, the fundamental cut-set matrix Q is defined as follows: each column of Q corresponds to a branch G (ordered consecutively); each row corresponds to a fundamental cut-set (ordered consecutively); and

$q_{ij} = 1$ if the j-th branch is in the cut-set corresponding to the i-th row of Q, and their orientations coincide;

$q_{ij} = -1$ if the j-th branch is in the cut-set corresponding to the i-th row of Q, and their orientations do not coincide;

$q_{ij} = 0$ if the j-th branch is not in the cut-set corresponding to the i-th row of Q

Because of the way in which the branches are numbered, the matrix Q is of the form.

$$Q = [Q_f, I] \tag{2.17}$$

where Q_f is a $(\nu-1) \times (b-\nu+1)$ matrix and I is a $(\nu-1) \times (\nu-1)$ identity matrix.

When the columns of the matrix B and the matrix Q are arranged in the same order

$$BQ^T = 0 \tag{2.18}$$

where Q^T is the transpose of Q. Thus substituting eqns. (2.16) and (2.17) into eqn. (2.18), Q_f can be expressed in terms of F as

$$Q_f = -F^T \tag{2.19}$$

and $$Q = [-F^T, I] \tag{2.20}$$

Let $\underline{v}$ and $\underline{i}$ be b-vectors composed of the branch voltages and currents. Kirchhoff's voltage and current laws may be expressed as

$$B\underline{v} = \underline{0} \tag{2.21}$$

$$Q\underline{i} = \underline{0} \tag{2.22}$$

Kirchhoff's laws give a total of b independent equations in 2b variables. The remaining equations are supplied by the constraint equations characterizing the network elements.

At this stage it is useful to introduce some notation. Let $\mathcal{B}_E$ and $\mathcal{B}_J$ denote set of branches containing the independent voltage sources $\underline{E}$ and current sources $\underline{J}$ in a network. Similarly let $\mathcal{B}_R$, $\mathcal{B}_L$, $\mathcal{B}_C$ correspond to a set of all resistive, inductive and capacitive branches respectively. In addition, let $N\{\mathcal{B}_a ; \mathcal{B}_b\}$ be the network derived from a network N by contracting* all the branches of $\mathcal{B}_a$ and removing all the branches of $\mathcal{B}_b$.

When a network contains a set of independent voltage sources $\underline{E}$ and a set of independent current sources $\underline{J}$, then $\underline{v}_E$, the set of voltages across branches of $\mathcal{B}_E$, and $\underline{i}_J$, the set of currents flowing through branches of $\mathcal{B}_J$ is known, i. e.

$$\underline{v}_E = \underline{E}$$
$$\underline{i}_J = \underline{J}.$$

In such a case it is convenient to rewrite Kirchhoff's laws (eqns. (2.21), (2.22)) in the form where only unknown voltages and currents appear on the left hand side. Assume that a connected network N has neither voltage-source-only loops nor current-source-only cutsets or equivalently a subnetwork $N\{\mathcal{B}_E ; \mathcal{B}_J\}$ of N is connected and nonseparable. For such a network a tree T exists such that all branches of $\mathcal{B}_E$ are tree-branches and all branches of $\mathcal{B}_J$ are links. Let the total number of branches in $N\{\mathcal{B}_E ; \mathcal{B}_J\}$ be b, the number of nodes ν. The branch voltages and currents in $N\{\mathcal{B}_E ; \mathcal{B}_J\}$ are

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* Contraction of a branch[11] is the process of shrinking the branch to nothing and identifying the two terminal nodes as a single node; this process is often loosely called short-circuiting.

specified by $\underline{v}$ and $\underline{i}$, respectively; for a specific tree they are partitioned as follows:

$$\underline{v} = \begin{bmatrix} \underline{v}_l \\ \underline{v}_t \end{bmatrix} \qquad (2.23a)$$

and

$$\underline{i} = \begin{bmatrix} \underline{i}_l \\ \underline{i}_t \end{bmatrix} \qquad (2.23b)$$

where $\underline{v}_l$ and $\underline{i}_l$ represent link voltages and currents, respectively, while $\underline{v}_t$ and $\underline{i}_t$ represent tree-branch voltages and currents respectively. Kirchhoff's laws equations for the particular tree T of N are given by[10]

$$B\underline{v} = [I, F]\begin{bmatrix} \underline{v}_l \\ \underline{v}_t \end{bmatrix} = \underline{e} \qquad (2.24)$$

$$Q\underline{i} = [-F^T, I]\begin{bmatrix} \underline{i}_l \\ \underline{i}_t \end{bmatrix} = \underline{j} \qquad (2.25)$$

where B is the fundamental loop matrix of $N\{\mathcal{B}_E; \mathcal{B}_J\}$

Q is the fundamental cut-set matrix of $N\{\mathcal{B}_E; \mathcal{B}_J\}$

$\underline{e}$ is an $(b-\nu+1)$-vector, the k-th component of which is the algebraic sum of source voltages which appear in the k-th fundamental loop

$\underline{j}$ is an $(\nu-1)$-vector, the k-th component of which is the algrebraic sum of source currents which appear in the k-th fundamental cut-set.

2.4 INTRODUCTION TO THE ANALYSIS OF ONE-ELEMENT-KIND NETWORKS

As it has already been mentioned, in the process of the normal form characterization of nonlinear RLC networks it is necessary to examine the existence and uniqueness of solution of one-element-kind networks. These one-element-kind networks are subnetworks of a given RLC network and will be discussed in Section 2.5. In this section we shall formulate algebraic equations, governing the behaviour of one-element-kind networks. The analysis of nonlinear resistive networks is almost identical with that of nonlinear capacitive networks or nonlinear inductive networks. Since most of the commonly used nonlinear elements are resistive in nature we shall develop the analysis in terms of resistive networks.

It will be assumed that a resistive network N to be analysed satisfies the following two conditions:

(i) network N is connected

(ii) there are no loops of voltage sources only and no cut-sets of current sources only.

When a network N satisfies these two conditions, $N\{\mathcal{B}_E; \mathcal{B}_J\}$ is nonseparable and connected; a tree T can be chosen such that all voltage sources in N are included in this tree and all current sources lie in its corresponding cotree. Let us choose a tree T and denote the set of all resistive branches of T as $\mathcal{B}_R$, the set of all resistive links as $\mathcal{B}_\beta$ and the set of all resistive tree-branches as $\mathcal{B}_\varepsilon$, where $\mathcal{B}_R = \mathcal{B}_\beta \cup \mathcal{B}_\varepsilon$. Suppose that $\mathcal{B}_R$, $\mathcal{B}_\beta$ and $\mathcal{B}_\varepsilon$ contain b_R, b_β and b_ε branches respectively. With respect to the tree T vectors $\underline{v}_R$ and $\underline{i}_R$, associated with $\mathcal{B}_R$, are partitioned as follows

$$\underline{v}_R = \begin{bmatrix} \underline{v}_\beta \\ \underline{v}_\varepsilon \end{bmatrix} \qquad \underline{i}_R = \begin{bmatrix} \underline{i}_\beta \\ \underline{i}_\varepsilon \end{bmatrix} \tag{2.26}$$

where $\underline{v}_\beta$, $\underline{i}_\beta$ represent link voltages and currents and $\underline{v}_\varepsilon$, $\underline{i}_\varepsilon$ represent tree-branch voltages and currents respectively. In order to deal with a small number of equations we shall not be particularly interested in $\underline{i}_E$, the currents of voltage sources, and $\underline{v}_J$, the voltages across current sources; however, $\underline{i}_E$ and $\underline{v}_E$ can be explicitly obtained as a linear combination of currents and voltages of resistive branches. Thus the unknown quantities are $\underline{v}_R$ and $\underline{i}_R$ and the number of unknown variables is $2b_R$. Hence a set of $2b_R$ independent equations has to be specified in order to compute $\underline{v}_R$ as functions of sources $\underline{E}$ and $\underline{J}$.

The fundamental loop matrix B of the network $N\{\mathcal{B}_E; \mathcal{B}_J\}$ corresponds to the fundamental loops, defined by all resistive links, and is an $b_\beta \times b_R$ matrix. It has the form

$$B = \left[I, F_{\beta\varepsilon}\right] \tag{2.27a}$$

The fundamental cut-set matrix Q of the network $N\{\mathcal{B}_E; \mathcal{B}_J\}$ corresponding to the fundamental cut-sets through all resistive tree-branches, is an $b_\varepsilon \times b_R$ matrix and is of the form

$$Q = \left[-F_{\beta\varepsilon}{}^T, I\right] \tag{2.27b}$$

The most general set of algebraic equations, governing the behaviour of resistive network, can be obtained as follows. Assume that resistive elements are described implicitly by the relation (2.1). Combining eqn. (2.1) and the topological constraints, given by eqns. (2.24) and (2.25), and taking into account eqns. (2.27a) and (2.27b), we arrive at the following set of $2b_R$ implicit equations that determine $\underline{v}_\beta$, $\underline{v}_\varepsilon$, $\underline{i}_\beta$, $\underline{i}_\varepsilon$.

$$B\underline{v}_R = \underline{v}_\beta + F_{\beta\varepsilon}\ \underline{v}_\varepsilon = \underline{e}_\beta \tag{2.28a}$$

$$Q\underline{i}_R = -F_{\beta\varepsilon}{}^T\underline{i}_\beta + \underline{i}_\varepsilon = \underline{j}_\varepsilon \tag{2.28b}$$

$$f_R(\underline{v}_\beta, \underline{v}_\varepsilon, \underline{i}_\beta, \underline{i}_\varepsilon) = \underline{0} \tag{2.28c}$$

In eqns. (2.28) $\underline{e}_\beta$ is an b_β-vector, the k-th component of which is the algebraic sum of source voltages which appear in the k-th fundamental loop and $\underline{j}_\varepsilon$ is an b_ε-vector, the k-th component of which is the algebraic sum of source currents which appear in the k-th fundamental cut-set.

Since eqns. (2.28) represent $2b_R$ equations in $2b_R$ unknowns, it is desirable to reduce these to a simpler form whenever possible. To perform such a reduction, it is necessary to find a smaller set of variables $\underline{x}$, such that all branch variables $\underline{v}_R$, $\underline{i}_R$ are uniquely expressible in terms of $\underline{x}$ and such that a set of implicit equations

$$\underline{f}(\underline{x}) = \underline{0}$$

can be formulated which define the equilibrium conditions of the network. Several alternative forms of the above equation will be considered below.

If all resistive elements are current-controlled, loop analysis is appropriate. The hybrid description of resistive elements has the form

$$\underline{v}_R = \underline{v}_R(\underline{i}_R) \tag{2.29}$$

From eqn. (2.28b)

$$\underline{i}_R = \begin{bmatrix} \underline{i}_\beta \\ \underline{i}_\varepsilon \end{bmatrix} = \begin{bmatrix} I \\ F_{\beta\varepsilon}{}^T \end{bmatrix} \underline{i}_\beta + \begin{bmatrix} \underline{0} \\ \underline{j}_\varepsilon \end{bmatrix} = B^T\underline{i}_\beta + \begin{bmatrix} \underline{0} \\ \underline{j}_\varepsilon \end{bmatrix} \tag{2.30}$$

Eqns. (2.29), (2.30) and (2.28a) can be combined to give[4]

$$B\underline{v}_R(B^T\underline{i}_\beta) + \begin{bmatrix} \underline{0} \\ \underline{j}_\varepsilon \end{bmatrix} = \underline{e}_\beta \tag{2.31}$$

Eqn. (2.31) represents a set of loop equations for the network. Loop equations contain a set of b_β equations and they can be solved for link currents $\underline{i}_\beta$. Once $\underline{i}_\beta$ is known, $\underline{i}_R$ is determined by eqn. (2.30) and then $\underline{v}_R$ is calculated using eqn. (2.29).

Similarly if all resistive elements are voltage-controlled, nodal analysis can be used. The hybrid description of resistive elements is

$$\underline{i}_R = \underline{i}_R(\underline{v}_R) \tag{2.32}$$

From eqn. (2.28a)

$$\underline{v}_R = \begin{bmatrix} \underline{v}_\beta \\ \underline{v}_\varepsilon \end{bmatrix} = \begin{bmatrix} -F_{\beta\varepsilon} \\ 1 \end{bmatrix} \underline{v}_\varepsilon + \begin{bmatrix} \underline{e}_\beta \\ \underline{0} \end{bmatrix} = Q^T\underline{v} + \begin{bmatrix} \underline{e}_\beta \\ \underline{0} \end{bmatrix} \tag{2.33}$$

Eqn. (2.28b) can now be combined with eqns. (2.32) and (2.33) to give[4]

$$Q\underline{i}_R(Q^T\underline{v}_\varepsilon + \begin{bmatrix} \underline{e}_\beta \\ 0 \end{bmatrix}) = \underline{j}_\varepsilon \tag{2.34}$$

which represents a set of node equations for the network. Eqn. (2.34) has b_ε equations and it can be solved for tree voltages $\underline{v}_\varepsilon$, which by eqns. (2.33) and (2.32) determine all other branch variables.

If all resistive elements of N do not possess the hybrid description with $\underline{i}_R$ as independent variable or with $\underline{v}_R$ as independent variable, neither loop nor nodal analysis can be performed. In such a case we shall try to carry out a mixed analysis, resulting in a set of b_R hybrid equations. Hybrid equations are of two forms and one form can be solved for the hybrid set $(\underline{v}_\beta, \underline{i}_\varepsilon)$, the other for the hybrid set $(\underline{i}_\beta, \underline{v}_\varepsilon)$; all other branch variables are then expressible in terms of these two hybrid sets.

Note that hybrid equations can always be obtained from eqns. (2.28).

Namely, calculating $\underline{v}_\beta$ from eqn. (2.28a) and $\underline{i}_\varepsilon$ from eqn. (2.28b) and substituting $\underline{v}_\beta$ and $\underline{i}_\varepsilon$ into eqn. (2.28c), we have the following hybrid equations

$$\underline{f}_R(-F_{\beta\varepsilon}\underline{v}_\varepsilon+\underline{e}_\beta,\underline{v}_\varepsilon,\underline{i}_\beta,F_{\beta\varepsilon}^T\underline{i}_\beta+\underline{j}_\varepsilon)=\underline{0} \tag{2.35}$$

which defines $(\underline{i}_\beta,\underline{v}_\varepsilon)$ as a function of $(\underline{e}_\beta,\underline{j}_\varepsilon)$.

There are another two forms of hybrid equations. The first form exists when a tree T of N can be chosen such that all resistive links are current-controlled and all resistive tree-branches are voltage-controlled. Thus, the hybrid description of resistive elements is

$$\begin{aligned}\underline{v}_\beta &= \underline{v}_\beta(\underline{i}_\beta,\underline{v}_\varepsilon)\\ \underline{i}_\varepsilon &= \underline{i}_\varepsilon(\underline{i}_\beta,\underline{v}_\varepsilon)\end{aligned} \tag{2.36}$$

Combining eqns. (2.28a), (2.28b) and (2.36) a set of hybrid equations is obtained[4]

$$\begin{aligned}\underline{v}_\beta(\underline{i}_\beta,\underline{v}_\varepsilon)+F_{\beta\varepsilon}\underline{v}_\varepsilon &= \underline{e}_\beta\\ -F_{\beta\varepsilon}^T\underline{i}_\beta+\underline{i}_\varepsilon(\underline{i}_\beta,\underline{v}_\varepsilon) &= \underline{j}_\varepsilon\end{aligned} \tag{2.37}$$

The dual case of the above hybrid description (eqn. (2.36) occurs when all resistive links of a tree are voltage-controlled and all resistive tree-branches are current controlled. There

$$\begin{aligned}\underline{i}_\beta &= \underline{i}_\beta(\underline{v}_\beta,\underline{i}_\varepsilon)\\ \underline{v}_\varepsilon &= \underline{v}_\varepsilon(\underline{v}_\beta,\underline{i}_\varepsilon)\end{aligned} \tag{2.38}$$

From eqns. (2.28a), (2.28b) and (2.38) the following hybrid equations result

$$\underline{v}_\beta + F_{\beta\varepsilon}\,\underline{v}_\varepsilon(\underline{v}_\beta, \underline{i}_\varepsilon) = \underline{e}_\beta$$
$$-F_{\beta\varepsilon}{}^T\,\underline{i}_\beta(\underline{v}_\beta, \underline{i}_\varepsilon) + \underline{i}_\varepsilon = \underline{\mathrm{j}}_\varepsilon \qquad (2.39)$$

A special form of the above hybrid equations is obtained when there is no coupling between links and tree-branches of N. Thus

$$\underline{i}_\beta = \underline{i}_\beta(\underline{v}_\beta)$$
$$\underline{v}_\varepsilon = \underline{v}_\varepsilon(\underline{i}_\varepsilon) \qquad (2.40)$$

and the corresponding hybrid equations have the form[8]

$$\underline{v}_\beta + F_{\beta\varepsilon}\,\underline{v}_\varepsilon(\underline{i}_\varepsilon) = \underline{e}_\beta$$
$$-F_{\beta\varepsilon}{}^T\,\underline{i}_\beta(\underline{v}_\beta) + \underline{i}_\varepsilon = \underline{\mathrm{j}}_\varepsilon \qquad (2.41)$$

In many cases the following method of partitioning reduces, still further, the number of implicit equations governing the behaviour of a resistive network N. Branches of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ are of three types: (i) self-loops, (ii) open-branches and (iii) nonseparable connected subnetworks with more than one branch. Assume that the branches are partitioned according to types (i), (ii) and (iii) into sets $\mathcal{B}_{\beta 2}$, $\mathcal{B}_{\varepsilon 2}$, and $\mathcal{B}_{\beta 1 \varepsilon 1} = \mathcal{B}_{\beta 1} \cup \mathcal{B}_{\varepsilon 1}$ with corresponding branch variables $(\underline{v}_{\beta 2}, \underline{i}_{\beta 2})$, $(\underline{v}_{\varepsilon 2}, \underline{i}_{\varepsilon 2})$ and $(\underline{v}_{\beta 1}, \underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{i}_{\varepsilon 1})$. Order the branches so that

$$\underline{i}_\beta = \begin{bmatrix} \underline{i}_{\beta 1} \\ \underline{i}_{\beta 2} \end{bmatrix}, \quad \underline{v}_\beta = \begin{bmatrix} \underline{v}_{\beta 1} \\ \underline{v}_{\beta 2} \end{bmatrix}, \quad \underline{e}_\beta = \begin{bmatrix} \underline{e}_{\beta 1} \\ \underline{e}_{\beta 2} \end{bmatrix}$$
$$\underline{i}_\varepsilon = \begin{bmatrix} \underline{i}_{\varepsilon 1} \\ \underline{i}_{\varepsilon 2} \end{bmatrix}, \quad \underline{v}_\varepsilon = \begin{bmatrix} \underline{v}_{\varepsilon 1} \\ \underline{v}_{\varepsilon 2} \end{bmatrix}, \quad \underline{\mathrm{j}}_\varepsilon = \begin{bmatrix} \underline{\mathrm{j}}_{\varepsilon 1} \\ \underline{\mathrm{j}}_{\varepsilon 2} \end{bmatrix} \qquad (2.42)$$

Then the matrix $F_{\beta\varepsilon}$, appearing in eqns. (2.27a) and (2.27b) has the form[2,12]

$$F_{\beta\varepsilon} = \left[\begin{array}{c|c} F_{\beta 1 \varepsilon 1} & 0 \\ \hline 0 & 0 \end{array}\right] \quad (2.43)$$

It is easy to see[12] that $\mathcal{B}_{\beta 1}$ contains all those resistive links of the tree T associated with fundamental loops containing at least one resistive tree-branch, and $\mathcal{B}_{\beta 2}$ contains all other resistive links. Similarly, $\mathcal{B}_{\varepsilon 1}$ contains all those resistive tree-branches associated with fundamental cut-sets containing at least one resistive link, and $\mathcal{B}_{\varepsilon 2}$ contains all other resistive tree-branches.

By substituting eqn. (2.43) into eqns. (2.28a) and (2.28b) we have

$$\begin{aligned} \underline{v}_{\beta 2} &= \underline{e}_{\beta 2} \\ \underline{i}_{\varepsilon 2} &= \underline{j}_{\varepsilon 2} \end{aligned} \quad (2.44)$$

and thus $\underline{v}_{\beta 2}$ and $\underline{i}_{\varepsilon 2}$ are expressed explicitly as functions of sources $\underline{E}$ and $\underline{J}$; a unique solution in a network can be obtained only when resistive elements in N have voltage-controlled branches belonging to the set $\mathcal{B}_{\beta 2}$ and current-controlled branches belonging to the set $\mathcal{B}_{\varepsilon 2}$.

Hybrid equations, corresponding to eqns. (2.37), can be obtained when $(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{v}_{\beta 2}, \underline{i}_{\varepsilon 2})$ is the independent variable in the hybrid description of resistive elements. Then a set of $(b_{\beta 1} + b_{\varepsilon 1})$ equations has the form

$$\begin{aligned} \underline{v}_{\beta 1}(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{j}_{\varepsilon 2}) + F_{\beta 1 \varepsilon 1} \underline{v}_{\varepsilon 1} &= \underline{e}_{\beta 1} \\ -F_{\beta\varepsilon}^{T} \underline{i}_{\beta 1} + \underline{i}_{\varepsilon 1}(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{j}_{\varepsilon 2}) &= \underline{j}_{\varepsilon 1} \end{aligned} \quad (2.45)$$

Similarly when $(\underline{v}_{\beta 1}, \underline{i}_{\varepsilon 1}, \underline{v}_{\beta 2}, \underline{i}_{\varepsilon 2})$ may be taken as the independent variable, the hybrid equations, corresponding to eqns. (2.41), have the form

$$\begin{aligned} \underline{v}_{\beta 1} + F_{\beta\varepsilon} \underline{v}_{\varepsilon 1}(\underline{v}_{\beta 1}, \underline{i}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{j}_{\varepsilon 2}) &= \underline{e}_{\beta 1} \\ -F_{\beta\varepsilon}^{T} \underline{i}_{\varepsilon 1}(\underline{v}_{\beta 1}, \underline{i}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{j}_{\varepsilon 2}) + \underline{i}_{\varepsilon 1} &= \underline{j}_{\varepsilon 1} \end{aligned} \quad (2.46)$$

In order to examine the question of solvability of the above loop, nodal and hybrid equations, a suitable mathematical criterion is needed. We shall develop such a criterion in Chapter 3 and it will then be applied to these equations in Chapter 5. Since these equations are nonlinear and their number may be very large, we shall not be able to answer the question of the existence and uniqueness of solution for an arbitrary nonlinear resistive network. Therefore, sufficient conditions for the existence and uniqueness of solution of nonlinear one-element-kind networks, containing certain classes of network elements, will be studied, and emphasis will be given to networks containing locally active elements.

2.5 INTRODUCTION TO THE FORMATION OF THE STATE EQUATIONS FOR NONLINEAR RLC NETWORKS

2.5.1 Basic concepts

As an introduction to the concepts involved and a foundation for later work in Chapter 6, this section consists of a thorough presentation of the formation of the state equations for nonlinear RLC networks. Before describing the procedure for the normal form characterization of a given RLC network we have to introduce certain concepts such as: solution of an RLC network, determinate network, complete set etc. Since a number of similar concepts have evolved in the literature we shall review them briefly and emphasize the differences between them.

Definition 2.6[1]

A solution of an RLC network will be called any set of voltages and currents of resistive elements, charges and voltages of capacitive elements and fluxes and currents of inductive elements, which satisfy the Kirchhoff's laws and the branch relations.

Note, that a solution of a nonlinear RLC network does not necessarily determine currents of capacitive elements, $\underline{i}_C$, voltages of inductive elements $\underline{v}_L$, currents of independent voltage sources, $\underline{i}_E$ and voltages of independent current cources, $\underline{v}_J$; if these branch variables are included to form a network solution, the state-variable analysis of a nonlinear network becomes much more involved since in general the number of the necessary equations to be considered increases very much.

Definition 2.7[1]

A network N is said to be determinate if for any value of the initial state $\underline{x}_o$, given at any initial time t_o, and for any value of independent sources $\underline{E}(\cdot)$, $\underline{J}(\cdot)$, there exists one and only one solution for $t \geq t_o$ on some nonvanishing interval $[t_o, t_1)$.

For linear RLC networks the normal form differential equations can be written in terms of a complete set of dynamically indepedent variables which by definition[11] contains branch currents and branch voltages. For non-linear networks the concept of a complete set of dynamically independent variables may be extended to the concept of the complete set that may be defined as follows.

Definition 2.8

A set $\underline{x}$ of network variables will be called complete if to every value of $\underline{x}$, there correspond unique values of voltages and currents of all resistive elements, charges and voltages of all capacitive elements and fluxes and currents of all inductive elements.

Let $\underline{x} \in R^n$, then our definition of the complete set $\underline{x}$ implies that for any value of $\underline{x}$, given at time t_i, the solution of a network is uniquely determined at that time. Our definition of the complete set represents a modified version of the definition given in Ref. 12, where the complete set is defined to determine all branch variables, i. e. currents of capacitive branches, $\underline{i}_C$, and voltages of inductive elements, $\underline{v}_L$, as well. In fact in Refs. 4 and 12 the term complete set is used in accordance with Definition 2.8.

The following example shows that, indeed, networks exist where the complete set does not define uniquely capacitive currents $\underline{i}_C$. Consider the network of two parallel nonlinear capacitors (Fig. 2.2) that are both voltage-controlled and their hybrid descriptions are

$$\begin{aligned} q_1 &= 2v_1^3 - v_1 \\ q_2 &= -v_2^3 + v_2. \end{aligned} \tag{2.47}$$

By definition the complete set for network of Fig. 2.2 determines uniquely q_1, q_2, v_1 and v_2. Suppose that

$$q = q_1 + q_2 \tag{2.48}$$

is a potentially complete set. Since from Fig. 2.2

$$v_1 = v_2 \tag{2.49}$$

q may be expressed as

$$q = v_1^{\,3} = v_2^{\,3} \tag{2.50}$$

Thus

$$\begin{aligned} v_1 &= q^{1/3} \\ v_2 &= q^{1/3} \\ q_1 &= 2q - q^{1/3} \\ q_2 &= -q + q^{1/3} \end{aligned} \tag{2.51}$$

and q forms the complete set. The capacitive currents i_1 and i_2 are

$$\begin{aligned} i_1 &= \dot{q}_1 = C_1\dot{v}_1 \\ i_2 &= \dot{q}_2 = C_2\dot{v}_2 \end{aligned} \tag{2.52}$$

where

$$\begin{aligned} C_1 &= dq_1/dv_1 = 6v_1^{\,2} - 1 \\ C_2 &= dq_2/dv_2 = -3v_2^{\,3} + 1 \end{aligned} \tag{2.53}$$

Differentiating eqn. (2.49) with respect to time t gives

$$\dot{v}_1 = \dot{v}_2 \tag{2.54}$$

From KCL $\quad i_1 + i_2 = 0 \qquad (2.55)$

Substituting eqns. (2.52) and (2.54) into eqn. (2.55) yields

$$(C_1 + C_2)\,\dot{v}_1 = 0 \tag{2.56}$$

If in eqn. (2.48) $q = 0$, it follows that $v_1 = 0$, $v_2 = 0$, $C_1 = -1$, $C_2 = 1$ and from eqn. (2.56) $\dot{v}_1$ is arbitrary and $\dot{v}_2 = \dot{v}_1$ is arbitrary as well; thus by eqn. (2.52) the capacitive currents i_1 and i_2 are not uniquely determined.

Frequently we can obtain the normal form characterization of a network in terms of the complete set $\underline{x}$, $\underline{x} \in R^n$, as

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}, t) \tag{2.57}$$

In general $\underline{f}(\underline{x})$ is not a necessarily continuous function of $\underline{x}$ with the entire R^n as its domain; $\underline{f}(\underline{x})$ may even be a multivalued function of $\underline{x}$. Thus, there are cases when the complete set can be found, but the function appearing in the r. h. s. of the corresponding normal form differential equations is not a continuous function for all $\underline{x} \in R^n$.

The RC network, shown in Fig. 2.3a, has the property that the complete set can be found, but $\underline{f}(\underline{x})$ in the normal form equation is not a continuous function of $\underline{x}$. Let the capacitor (Fig. 2.3a) be linear and let its capacitance be equal to C; let the resistor R in Fig. 2.3a be current-controlled, $v_R = v_R(i_R)$, where the domain of $v_R(\cdot)$ is the entire R^1. The incremental resistance of R is zero at $i_R = i_A$ and $i_R = i_B$ (see Fig. 2.3b). The current i_R is the complete set, since $v_R = v_R(i_R)$, $v_C = v_R$, $q_C = Cv_C$, $(i_C = -i_R)$. The differential equation, written in terms of the complete set i_R, has the form

$$\dot{i}_R = f\,(i_R) = -\frac{1}{C}\left(\frac{dv_R}{di_R}\right)^{-1} i_R \tag{2.58}$$

Since $dv_R/di_R = 0$ at $i_R = i_A$ and $i_R = i_B$, $f(i_R)$ is not a continuous function ar i_A and i_B; thus we cannot expect eqn. (2.58) to have a unique solution $i_R(t)$ for any initial value i_{Ro}.

In the sequel we shall say that the normal form exists when in the differential equation (2.57), written in terms of the complete set $\underline{x}$, $\underline{f}(\underline{x},t)$ is a continuous function of $\underline{x}$ whose domain is the entire R^n. Note that $\underline{f}(\underline{x})$ does not necessarily fulfil the Lipshitz condition and thus even when the normal form exists the solution of a network may not be unique. Therefore the existence of the normal form does not imply that eqn. (2.57) is the state equation in differential form and the complete set is not necessarily a set of state variables. Only when the normal form equations have a unique solution $\underline{x}(t)$, $t_o \leq t < t_1$, for any initial value $\underline{x}_o \in R^n$, given at any initial time t_o, does the complete set become a set of state variables* and eqn. (2.57) represents the state equation.

In Ref. 13 the concept of a set of dynamic variables $\underline{x}_d$ was introduced. This concept is in a certain sense more general than the concept of the complete set. A set of dynamic variables $\underline{x}_d$, $\underline{x}_d \in R^n$, has the property that it defines uniquely all branch variables on a set S, which is a subset of R^n, $S \subset R^n$. Thus it is not necessary for a set of dynamic variables $\underline{x}_d$ to determine all branch variables for any $\underline{x}_d \in R^n$. The space R^n is

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* The term state variable is not used properly in Ref. 12, p. 75; namely the term a set of state variables is more specific than the complete set.

called the dynamic space and a set of first order differential equations, written in terms of $\underline{x}_d$

$$\dot{\underline{x}}_d = \underline{f}_d(\underline{x}_d, t) \tag{2.59}$$

is termed a set of dynamic equations; the domain of the function $\underline{f}_d(\underline{x}_d)$ is a set S', $S' \subset R^n$. When eqn. (2.59) has a unique solution for any initial value $\underline{x}_{do}$, given at any initial time t_o, dynamic variables become state variables a posteriori[13].

2.5.2 Selection of the complete set[1,2,11,13,14]

The normal form equations describing a general RLC network are formulated in two stages:

(i) the complete set $\underline{x}$ is selected for a given network

(ii) the normal form equations are formulated in terms of the complete set $\underline{x}$

We shall assume that a given RLC network N contains resistive, capacitive and inductive elements and in addition independent voltage sources $\underline{E}$ and independent current sources $\underline{J}$. Furthermore N is connected and in N there are no loops of voltage sources only and no cut-sets of current sources only; thus the distribution of independent sources is assumed to be normal[1,2]. Suppose that N has b_R resistive branches, b_C capacitive branches, b_L inductive branches, b_E voltage sources and b_J current cources.

For a broad class of RLC networks the complete set may be obtained on the basis of a normal tree[13,14] T_N. T_N is a complete tree which is chosen according to the set of rules given by Bryant[11]. A normal tree is formed by constructing a complete tree of the graph, derived from a given network, in the following manner. First, all voltage sources are included in the tree, then only capacitive branches are used wherever possible and the resulting subgraph is augmented, first with resistive, and then with inductive branches to form a complete tree. A normal tree contains all the independent voltage sources, the maximum number of capacitive branches, the minimum number of inductive branches, no independent current sources and it is completed with resistive branches. Note that for a network with normal distribution of independent sources a complete tree can always be constructed.

Using the same procedure and subscripts employed by Bryant[11] the branches of the network N, corresponding to RLC elements, can be classified in the following six disjoint subsets with respect to T_N:

S_α : the capacitive links of T_N

S_β : the resistive links of T_N

S_γ : the inductive links of T_N

S_δ : the capacitive tree-branches of T_N

S_ε : the resistive tree-branches of T_N

S_ζ : the inductive tree-branches of T_N

We shall denote the number of branches of S_α, S_β, S_γ, S_δ, S_ε and S_ζ by b_α, b_β, b_γ, b_δ, b_ε and b_ζ respectively.

Kirchhoff's laws of the network N have the form of eqns. (2.24) and

(2.25). According to the above classification the branch voltages and currents of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ can be partitioned as follows

$$\underline{v}_l = \begin{bmatrix} \underline{v}_\alpha \\ \underline{v}_\beta \\ \underline{v}_\gamma \end{bmatrix} \qquad \underline{i}_l = \begin{bmatrix} \underline{i}_\alpha \\ \underline{i}_\beta \\ \underline{i}_\gamma \end{bmatrix} \tag{2.60a}$$

$$\underline{v}_t = \begin{bmatrix} \underline{v}_\delta \\ \underline{v}_\varepsilon \\ \underline{v}_\zeta \end{bmatrix} \qquad \underline{i}_t = \begin{bmatrix} \underline{i}_\delta \\ \underline{i}_\varepsilon \\ \underline{i}_\zeta \end{bmatrix} \tag{2.60b}$$

where subsripts l and t denote links and tree-branches of T_N respectively. Similarly the sources $\underline{e}$ and $\underline{j}$, appearing in eqns. (2.24) and (2.25) can be partitioned conformably as

$$\underline{e} = \begin{bmatrix} \underline{e}_\alpha \\ \underline{e}_\beta \\ \underline{e}_\gamma \end{bmatrix} \qquad \underline{j} = \begin{bmatrix} \underline{j}_\delta \\ \underline{j}_\varepsilon \\ \underline{j}_\zeta \end{bmatrix} \tag{2.61}$$

Because of the way in which a normal tree is defined, the submatrix F, appearing in eqns. (2.24) and (2.25) may be represented in the following form[11]

$$F = \begin{bmatrix} F_{\alpha\delta} & 0 & 0 \\ F_{\beta\delta} & F_{\beta\varepsilon} & 0 \\ F_{\gamma\delta} & F_{\gamma\varepsilon} & F_{\gamma\zeta} \end{bmatrix} \tag{2.62}$$

Combining eqns. (2.24), (2.25), (2.60a,b), (2.61) and (2,62) we obtain Kirchhoff's laws in the form of eqns. (2.63) and (2.64)

$$\underline{v}_\alpha + F_{\alpha\delta}\ \underline{v}_\delta = \underline{e}_\alpha \tag{2.63a}$$

$$\underline{v}_\beta + F_{\beta\delta}\ \underline{v}_\delta + F_{\beta\varepsilon}\ \underline{v}_\varepsilon = \underline{e}_\beta \tag{2.63b}$$

$$-F_{\beta\varepsilon}{}^T\ \underline{i}_\beta - F_{\gamma\varepsilon}{}^T\ \underline{i}_\gamma + \underline{i}_\varepsilon = \underline{i}_\varepsilon \tag{2.63c}$$

$$-F_{\gamma\zeta}{}^T\ \underline{i}_\gamma + \underline{i}_\zeta = \underline{i}_\zeta \tag{2.663d}$$

$$\underline{v}_\gamma + F_{\gamma\delta}\ \underline{v}_\delta + F_{\gamma\varepsilon}\ \underline{v}_\varepsilon + F_{\gamma\zeta}\ \underline{v}_\zeta = \underline{e}_\gamma \tag{2.64a}$$

$$-F_{\alpha\delta}{}^T\ \underline{i}_\alpha - F_{\beta\delta}{}^T\ \underline{i}_\beta - F_{\gamma\delta}{}^T\ \underline{i}_\gamma + \underline{i}_\delta = \underline{i}_\delta \tag{2.64b}$$

where the number of equations is equal to $(b_R+b_L+b_C)$.

The implicit branch relations of RLC elements can be expressed as:

$$\underline{f}_C(\underline{v}_\alpha, \underline{v}_\delta, \underline{q}_\alpha, \underline{q}_\delta, t) = \underline{0} \tag{2.65a}$$

$$\underline{f}_R(\underline{v}_\beta, \underline{v}_\varepsilon, \underline{i}_\beta, \underline{i}_\varepsilon, t) = \underline{0} \tag{2.65b}$$

$$\underline{f}_L(\underline{i}_\gamma, \underline{i}_\zeta, \underline{\varphi}_\gamma, \underline{\varphi}_\zeta, t) = \underline{0} \tag{2.65c}$$

$$\underline{i}_C = \dot{\underline{q}}_C \tag{2.66a}$$

$$\underline{v}_L = \dot{\underline{\varphi}}_L \tag{2.66b}$$

where eqns. (2.65a), (2.65b) and (2.65c) contain b_C, b_R and b_L equations respectively. Since by definition the complete set determines $2(b_R+b_C+b_L)$ network variables $\underline{q}_C, \underline{v}_C, \underline{\varphi}_L, \underline{i}_L, \underline{v}_R, \underline{i}_R$ it is sufficient to consider $(2b_R+b_C+b_\alpha+b_L+b_\zeta)$ eqns. (2.63a-d) and (2.65a-c) which contain only variables $\underline{q}_C, \underline{v}_C, \underline{\varphi}_L, \underline{i}_L, \underline{v}_R, \underline{i}_R$; when all these equations are independent, it is obvious that in principle $(b_\gamma+b_\delta)$ independent variables may be chosen and the complete set $\underline{x}$ has $(b_\gamma+b_\delta)$ components, $\underline{x} \in R^{b_\gamma+b_\delta}$. The remaining equations (2.64a,b) and (2.66a,b) are then used to form the normal form equations $\dot{\underline{x}} = \underline{f}(\underline{x},t)$. However, in order to be able to formulate the normal form, $\underline{x}$ has to be chosen in such a manner that $\dot{\underline{x}}$ may be calculated. In the sequel we shall consider two choices of the complete set:

(i) the cut-set charges $\underline{q}$ and loop flux-linkages $\underline{\varphi}$, determined with respect to the tree T_N [1,2,8,12,14]

(ii) voltages $\underline{v}_\delta$ and currents $\underline{i}_\gamma$ [11] (or characteristic parameters $\underline{x}_\gamma$ and $\underline{x}_\delta$ of capacitive branches and inductive links of T_N [2,13]). As will be demonstrated later, there are examples of networks for which the first choice gives the complete set, but not the second and vice versa. Thus both of the two choices are worth studying.

Let us consider the first choice for the complete set where

$$\underline{x} = \begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix} = \begin{bmatrix} \underline{q}_\delta - F_{\alpha\delta}{}^T \underline{q}_\alpha \\ \underline{\varphi}_\gamma + F_{\gamma\mathfrak{f}}{}^T \underline{\varphi}_\mathfrak{f} \end{bmatrix} \tag{2.67}$$

Combining eqns. (2.63a), (2.65a–c) and (2.67) a set of $2(b_R+b_C+b_L)$ equations is obtained. However, these equations can be partitioned into three sets of equations denoted[12] by (C), (L) and (R):

$$\left.\begin{aligned} &\underline{v}_\alpha + F_{\alpha\delta}\,\underline{v}_\delta = \underline{e}_\alpha \\ &-F_{\alpha\delta}{}^T \underline{q}_\alpha + \underline{q}_\delta = \underline{q} \\ &\underline{f}_C(\underline{v}_\alpha, \underline{v}_\delta, \underline{q}_\alpha, \underline{q}_\delta, t) = \underline{0} \end{aligned}\right\} \quad (C) \tag{2.68a}$$

$$\left.\begin{aligned} &\underline{\varphi} + F_{\gamma\mathfrak{f}}\,\underline{\varphi}_\mathfrak{f} = \underline{\varphi} \\ &-F_{\gamma\mathfrak{f}}{}^T \underline{i}_\gamma + \underline{i}_\mathfrak{f} = \underline{i}_\mathfrak{f} \\ &\underline{f}_L(\underline{i}_\gamma, \underline{i}_\mathfrak{f}, \underline{\varphi}_\gamma, \underline{\varphi}_\mathfrak{f}, t) = \underline{0} \end{aligned}\right\} \quad (L) \tag{2.68b}$$

$$\left.\begin{aligned} &\underline{v}_\beta + F_{\beta\varepsilon}\,\underline{v}_\varepsilon = \underline{e}_\beta - F_{\beta\delta}\,\underline{v}_\delta \\ &-F_{\beta\varepsilon}{}^T \underline{i}_\beta + \underline{i}_\varepsilon = \underline{i}_\varepsilon + F_{\gamma\varepsilon}{}^T \underline{i}_\gamma \\ &\underline{f}_R(\underline{i}_\beta, \underline{i}_\varepsilon, \underline{v}_\beta, \underline{v}_\varepsilon, t) = \underline{0} \end{aligned}\right\} \quad (R) \tag{2.68c}$$

Let $\underline{x}$ be an n-vector. Vector $\underline{x}$ represents the complete set only if the equations (R), (L) and (C) possess a unique solution for any $\underline{x} = \begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix} \in R^n$. Because of the form of the (R), (L), (C) equations it is not necessary to consider the whole set of equations at once but these equations may be solved in three steps. At first the eqns. (C) are solved to obtain $\underline{q}_C(\underline{q})$ and $\underline{v}_C(\underline{q})$; the solution of the eqns. (L) gives $\underline{\varphi}_L(\underline{\varphi})$ and $\underline{i}_L(\underline{\varphi})$. Finally the solutions $\underline{v}_C(\underline{q})$ and $\underline{i}_L(\underline{\varphi})$ are substituted on the r. h. s. of the (R) equations which give $\underline{v}_R(\underline{q}, \underline{\varphi})$ and $\underline{i}_R(\underline{q}, \underline{\varphi})$.

Comparing the (R) equations and the dc equations of resistive networks with sources $\underline{E}$ and $\underline{J}$ (eqns. (2.28a-c)), we see that the l. h.s. of these two sets of equations are identical. Thus the (R) equations correspond to the resistive network which can be obtained from N in the following manner: all capacitors in N are replaced by a set of voltage sources $\underline{E}_C$, where $\underline{E}_C = \underline{v}_C$, and all inductors in N are replaced by a set of current sources $\underline{J}_L$, where $\underline{J}_L = \underline{i}_L$. Similarly the (C) equations represent the governing equations of the capacitive network $N\left\{0; \mathcal{B}_R, \mathcal{B}_L, \mathcal{B}_J\right\}$; namely KCL equations of the network $N\left\{0; \mathcal{B}_R, \mathcal{B}_L, \mathcal{B}_J\right\}$ are obtained from eqn. (2.64b), setting $\underline{i}_\beta = \underline{0}$, $\underline{i}_\gamma = \underline{0}$, $\underline{i}_\delta = \underline{0}$, in the form

$$-F_{\alpha\delta}{}^T \dot{\underline{q}}_\alpha + \dot{\underline{q}}_\delta = \underline{0} \qquad (2.69)$$

Integrating eqn. (2.69) with respect to time t

$$-F_{\alpha\delta}{}^T \underline{q}_\alpha + \underline{q}_\delta = \underline{q}_o \qquad (2.70)$$

where $\underline{q}_o$ is the integration constant. Setting $\underline{q}_o = \underline{q}$ and combining KVL equations of $N\left\{0; \mathcal{B}_R, \mathcal{B}_L, \mathcal{B}_J\right\}$ and implicit branch relations of capacitive elements we obtain the (C) equations.

Analogously the (L) equations are the governing equations of the inductive network $N\left\{\mathcal{B}_E, \mathcal{B}_C, \mathcal{B}_R; 0\right\}$. The second equation of the (L) equations is obtained by integrating KVL equations of $N\left\{\mathcal{B}_R, \mathcal{B}_C, \mathcal{B}_R;0\right\}$.

$$\dot{\underline{\varphi}}_{\gamma} + F_{\gamma\varsigma}{}^T \dot{\underline{\varphi}}_{\varsigma} = \underline{0} \qquad (2.71)$$

and

$$\underline{\varphi}_{\gamma} + F_{\gamma\varsigma}{}^T \underline{\varphi}_{\varsigma} = \underline{\varphi}_o \qquad (2.72)$$

Setting $\underline{\varphi}_o = \underline{\varphi}$ and combining KCL equations of $N\{\mathcal{B}_E, \mathcal{B}_C, \mathcal{B}_R;0\}$ and the implicit branch relations of inductive elements gives the (L) equations.

Thus in order to answer the question whether the set $(\underline{q}, \underline{\varphi})$ is complete or not, we have to study the existence and uniqueness of solution in the capacitive, inductive and resistive subnetworks of N for any value of $\underline{q} \in R^{b_\delta}$ and any value of $\underline{\varphi} \in R^{b_\gamma}$. For a general network N it is extremely difficult to resolve the question of the existence and uniqueness of solution of these three one-element-kind subnetworks of N. Therefore, our main objective will be to find some sufficient conditions ensuring the existence and uniqueness of solution. Such sufficient conditions will be given in Chapter 5.

If eqn. (2.67) is differentiated with respect to t and then substituted in eqns. (2.64a,b) the normal form equations of N are obtained in the following form

$$\begin{bmatrix} \dot{\underline{q}} \\ \dot{\underline{\varphi}} \end{bmatrix} = \begin{bmatrix} F_{\beta\delta}{}^T \underline{i}_{\beta}(\underline{q},\underline{\varphi}) + F_{\gamma\delta}{}^T \underline{i}_{\gamma}(\underline{\varphi}) + \underline{i}_{\delta} \\ -F_{\gamma\delta}\, \underline{v}_{\delta}(\underline{q}) - F_{\gamma\varepsilon}\, \underline{v}_{\varepsilon}(\underline{q},\underline{\varphi}) + \underline{e}_{\gamma} \end{bmatrix} \qquad (2.73)$$

Note that when eqn. (2.73) is integrated numerically the algebraic equations (R), (L), (C) have to be solved at each step of integration to obtain $\underline{i}_{\gamma}(\underline{\varphi}), \underline{v}_{\delta}(\underline{q}),\ \underline{i}_{\beta}(\underline{q},\underline{\varphi})$ and $\underline{v}_{\varepsilon}(\underline{q},\underline{\varphi})$.

Eqn. (2.73) will represent the state equation of N when it possesses a unique solution for any initial value $\underline{x}_o \in R^{b_\gamma + b_\delta}$ and $\underline{x}$ is then the state vector.

The example of the capacitive network given in Fig. 2.2 shows that the complete set $q = q_1+q_2$ does not define capacitive currents $\dot{i}_1$ and i_2. However, in certain practical cases currents of capacitive branches, $\underline{i}_C$, and voltages of inductive branches, $\underline{v}_L$, may be of considerable interest. It is therefore worth studying how to determine $\underline{i}_C$ and $\underline{v}_L$ as functions of the complete set $(\underline{q},\underline{\varphi})$ and under what conditions $\underline{i}_C(\underline{q},\underline{\varphi})$ and $\underline{v}_L(\underline{q},\underline{\varphi})$ are unique.

Taking into account that $\underline{i}_C = \dot{\underline{q}}_C$ and $\underline{v}_L = \dot{\underline{\varphi}}_L$, $\underline{i}_C$ and $\underline{v}_L$ can be obtained by differentiating the (C) and (L) equations respectively. The resulting equations are:

$$\begin{bmatrix} I & F_{\alpha\delta} & 0 & 0 \\ 0 & 0 & -F_{\alpha\delta}^T & I \\ \dfrac{\partial \underline{f}_C}{\partial \underline{v}_\alpha} & \dfrac{\partial \underline{f}_C}{\partial \underline{v}_\delta} & \dfrac{\partial \underline{f}_C}{\partial \underline{q}_\alpha} & \dfrac{\partial \underline{f}_C}{\partial \underline{q}_\delta} \end{bmatrix} \begin{bmatrix} \dot{\underline{v}}_\alpha \\ \dot{\underline{v}}_\delta \\ \underline{i}_\alpha \\ \underline{i}_\delta \end{bmatrix} = \begin{bmatrix} \dot{\underline{e}}_\alpha \\ \dot{\underline{q}} \\ \dfrac{\partial \underline{f}_C}{\partial t} \end{bmatrix} \tag{2.74a}$$

$$\begin{bmatrix} I & F_{\gamma f} & 0 & 0 \\ 0 & 0 & -F_{\gamma f}^T & I \\ \dfrac{\partial \underline{f}_L}{\partial \underline{\mathcal{L}}_\gamma} & \dfrac{\partial \underline{f}_L}{\partial \underline{\mathcal{L}}_f} & \dfrac{\partial \underline{f}_L}{\partial \underline{i}_\gamma} & \dfrac{\partial \underline{f}_L}{\partial \underline{i}_f} \end{bmatrix} \begin{bmatrix} \underline{v}_\gamma \\ \underline{v}_f \\ \dot{\underline{i}}_\gamma \\ \dot{\underline{i}}_f \end{bmatrix} = \begin{bmatrix} \dot{\underline{\varphi}} \\ \dot{\underline{i}}_f \\ \dfrac{\partial \underline{f}_L}{\partial t} \end{bmatrix} \tag{2.74b}$$

where $\dot{\underline{q}}$ and $\dot{\underline{\varphi}}$ are given as functions of $(\underline{q},\underline{\varphi})$ from the normal form equations (eqns. (2.73)). From eqns. (2.74a,b) it follows that $\underline{i}_C$ and $\underline{v}_L$ are uniquely determined for all $(\underline{q},\underline{\varphi})$ when the square matrices in the above equations are nonsingular for all $(\underline{q},\underline{\varphi})$.

It is noted that the set $(\underline{q},\underline{\varphi})$ was obtained through the normal tree T_N. Generally when there are loops of capacitive branches only and/or cut-sets of inductive branches only different normal trees may be formed. Let the set $(\underline{q}^*, \underline{\varphi}^*)$ be associated with another normal tree T_N^*. However, the set $(\underline{q}, \underline{\varphi})$, complete or not, is related to any other set $(\underline{q}^*, \underline{\varphi}^*)$by a nonsingular linear transformation[15]; thus when $(\underline{q}, \underline{\varphi})$ is the complete set any other set $(\underline{q}^*, \underline{\varphi}^*)$ is complete and when $(\underline{q}, \underline{\varphi})$ does not form the complete set no such normal tree T_N^* exists to give the complete set $(\underline{q}^*, \underline{\varphi}^*)$.

As the second choice* of the complete set $\underline{x}$ we shall consider voltages $\underline{v}_\delta$ and currents $\underline{i}_\gamma$. Thus $\underline{x}$

$$\underline{x} = \begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix} \tag{2.75}$$

We shall assume that all capacitive elements are time invariant and voltage-controlled and all inductive elements are time invariant and current-controlled. The hybrid descriptions of capacitive and inductive elements have the form

$$\underline{q}_C = \begin{bmatrix} \underline{q}_\alpha \\ \underline{q}_\delta \end{bmatrix} = \underline{q}_C(\underline{v}_\alpha, \underline{v}_\delta) = \begin{bmatrix} \underline{q}_\alpha(\underline{v}_\alpha, \underline{v}_\delta) \\ \underline{q}_\delta(\underline{v}_\alpha, \underline{v}_\delta) \end{bmatrix} \qquad (2.76$$

$$\underline{\varphi}_L = \begin{bmatrix} \underline{\varphi}_\gamma \\ \underline{\varphi}_f \end{bmatrix} = \underline{\varphi}_L(\underline{i}_\gamma, \underline{i}_f) = \begin{bmatrix} \underline{\varphi}_\gamma(\underline{i}_\gamma, \underline{i}_f) \\ \underline{\varphi}_f(\underline{i}_\gamma, \underline{i}_f) \end{bmatrix} \tag{2.77}$$

It is easy to see that$(\underline{v}_\delta, \underline{i}_\gamma)$ represent the complete set if (R) equations can be solved for all$\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$. Namely, from eqns. (2.63a) and (2.63d) $\underline{v}_\alpha$ and $\underline{i}_f$ are

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* The normal form decription with a more general choice, where the subsets of $\underline{x}$ are the characteristic parameters $\underline{x}_\delta$ and $\underline{x}_\gamma$ of capacitive tree-branches and inductive links, was treated in Refs. 2 and 13.

given in terms of $\underline{v}_\delta$ and $\underline{i}_\gamma$.

$$\begin{aligned} \underline{v}_\alpha &= -F_{\alpha\delta}\,\underline{v}_\delta + \underline{e}_\alpha \\ \underline{i}_f &= F_{\gamma f}{}^T\,\underline{i}_\gamma + \underline{i}_f \end{aligned} \tag{2.78}$$

Thus $\underline{q}_C$ and $\underline{\mathcal{L}}_L$ are expressible explicitly in terms of $(\underline{v}_\delta, \underline{i}_\gamma)$ as

$$\underline{q}_C = \underline{q}_C(-F_{\alpha\delta}\,\underline{v}_\delta + e_\alpha, \underline{v}_\delta) \tag{2.79}$$

$$\underline{\mathcal{L}}_L = \underline{\mathcal{L}}_L(i_\gamma, F_{\gamma f}{}^T\,\underline{i}_\gamma + \underline{i}_f) \tag{2.80}$$

In order to form the normal form equations we have to calculate $\underline{\dot{x}} = \begin{bmatrix} \underline{\dot{v}}_\delta \\ \underline{\dot{i}}_\gamma \end{bmatrix}$. If eqns. (2.79) and (2.80) are differentiated with respect to t we get

$$\underline{i}_C = \begin{bmatrix} \underline{i}_\alpha \\ \underline{i}_\delta \end{bmatrix} = \begin{bmatrix} C_{\alpha\alpha}(-F_{\alpha\delta}\,\underline{\dot{v}}_\delta + \underline{\dot{e}}_\alpha) + C_{\alpha\delta}\,\underline{\dot{v}}_\delta \\ C_{\delta\alpha}(-F_{\alpha\delta}\,\underline{\dot{v}}_\delta + \underline{\dot{e}}_\alpha) + C_{\delta\delta}\,\underline{\dot{v}}_\delta \end{bmatrix} \tag{2.81}$$

$$\underline{v}_L = \begin{bmatrix} \underline{v}_\gamma \\ \underline{v}_f \end{bmatrix} = \begin{bmatrix} L_{\gamma\gamma}\,\underline{\dot{i}}_\gamma + L_{\gamma f}\,(F_{\gamma f}{}^T\,\underline{\dot{i}}_\gamma + \underline{\dot{i}}_f) \\ L_{f\gamma}\,\underline{\dot{i}}_\gamma + L_{ff}\,(F_{\gamma f}{}^T\,\underline{\dot{i}}_\gamma + \underline{\dot{i}}_f) \end{bmatrix} \tag{2.82}$$

where
$$\begin{bmatrix} C_{\alpha\alpha} & C_{\alpha\delta} \\ C_{\delta\alpha} & C_{\delta\delta} \end{bmatrix} = \frac{\partial(\underline{q}_\alpha, \underline{q}_\delta)}{\partial(\underline{v}_\alpha, \underline{v}_\delta)} \tag{2.83}$$

$$\begin{bmatrix} L_{\gamma\gamma} & L_{\gamma f} \\ L_{f\gamma} & L_{ff} \end{bmatrix} = \frac{\partial(\underline{\mathcal{L}}_\gamma, \underline{\mathcal{L}}_f)}{\partial(\underline{i}_\gamma, \underline{i}_f)} \tag{2.84}$$

Substituting eqns. (2.81) and (2.82) into eqns. (2.64a,b)

$$\begin{aligned} M_\delta\,\underline{\dot{v}}_\delta &= K_\delta \\ M_\gamma\,\underline{\dot{i}}_\gamma &= K_\gamma \end{aligned} \tag{2.85}$$

where
$$\begin{aligned} M_\delta &= C_{\delta\delta} + F_{\alpha\delta}{}^T C_{\alpha\alpha} F_{\alpha\delta} - F_{\alpha\delta}{}^T C_{\alpha\delta} - C_{\delta\alpha} F_{\alpha\delta} \\ K_\delta &= \underline{i}_\delta + F_{\beta\delta}{}^T \underline{i}_\beta + F_{\gamma\delta}{}^T \underline{i}_\gamma + (F_{\alpha\delta}{}^T C_{\alpha\alpha} - C_{\delta\alpha})\underline{\dot{e}}_\alpha \end{aligned} \tag{2.86}$$

$$M_{\gamma} = L_{\gamma\gamma} + F_{\gamma f} L_{ff} F_{\gamma f}^{T} + F_{\gamma f} L_{f\gamma} + L_{\gamma f} F_{\gamma f}^{T}$$
$$K_{\gamma} = \underline{e}_{\gamma} - F_{\gamma\sigma} \underline{v}_{\sigma} - F_{\gamma\epsilon} \underline{v}_{\epsilon} - (F_{\gamma f} L_{ff} + L_{\gamma f}) \dot{\underline{i}}_{f} \qquad (2.86)$$

When M_{σ} and M_{γ} are nonsingular for all $\begin{bmatrix} \underline{v}_{\sigma} \\ \underline{i}_{\gamma} \end{bmatrix}$ normal form equations

$$\begin{bmatrix} \dot{\underline{v}}_{\sigma} \\ \dot{\underline{i}}_{\gamma} \end{bmatrix} = \begin{bmatrix} M_{\sigma}^{-1} K_{\sigma} \\ M_{\gamma}^{-1} K_{\gamma} \end{bmatrix} \qquad (2.87)$$

exist. Thus, when $(\underline{v}_{\sigma}, \underline{i}_{\gamma})$ form the complete set the condition for the existence of the normal form in terms of this complete set is the nonsingularity of certain matrices formed from the incremental cpacitance and the incremental inductance matrices.

As mentioned before there are cases where the normal form equations cannot be written in terms of the set $(\underline{q}, \underline{\varphi})$ but they can still be written in terms of the set $(\underline{v}_{\sigma}, \underline{i}_{\gamma})$ and vice versa. Let us demonstrate this fact by a simple example of a nonlinear RC network shown in Fig. 2.4a, where the resistor R is assumed to be voltage-controlled, $i_{\beta} = i_{\beta}(v_{\beta})$. If the capacitor C is charge controlled as shown in Fig. 2.4b its hybrid description is

$$v_{\sigma} = v_{\sigma}(q_{\sigma}) \qquad (2.88)$$

and the cut-set charge $q = q_{\sigma}$ forms the complete set; the corresponding differential equation is

$$\dot{q} = -i_{\beta}(v_{\beta})\Big|_{v_{\beta} = v_{\sigma}(q)} \qquad (2.89)$$

and the normal form exists as in eqn. (2.89) i_{β} is a continuous function of q for all $q \in R^{1}$. If the capacitor in Fig. 2.4a is voltage-controlled and q_{σ} is a strictly monotonic function of v_{σ}, as shown in Fig. 2.4c, the hybrid description is

$$q_{\sigma} = q_{\sigma}(v_{\sigma}) \qquad (2.90)$$

and v_σ forms the complete set. The normal form equation is then

$$\dot{v}_\sigma = -\left(\frac{dq_\sigma}{dv_\sigma}\right)^{-1} i_\beta (v_\beta) \Big|_{v_\beta = v_\sigma} \tag{2.91}$$

Since $dq_\sigma / dv_\sigma \neq 0$ for the strictly monotonic function of Fig. 2.4c the normal form exists.

Note, that when the capacitor in Fig. 2.4a has the characteristic shown in Fig. 2.4b, v_σ is not the complete set as q_σ is not uniquely defined as a function of v_σ. Similarly when the capacitor C is characterized by the strictly monotonic function $q_\sigma(v_\sigma)$ shown in Fig. 2.4c, where the range of q_σ is (Q_1, Q_2), $q = q_\sigma$ is not the complete set. It is therefore worthwhile to consider both $(\underline{q}, \underline{\varphi})$ and $(\underline{v}_\sigma, \underline{i}_\gamma)$ as a potentially complete set for general networks.

Although the sets $(\underline{q}, \underline{\varphi})$ or $(\underline{v}_\sigma, \underline{i}_\gamma)$, which are both based on the concept of a normal tree, form the complete set for a large class of RLC networks, there are cases where neither $(\underline{q}, \underline{\varphi})$ nor $(\underline{v}_\sigma, \underline{i}_\gamma)$ form the complete set, but the complete set can still be found in terms of another set of network variables. As an example consider the RC network, shown in Fig. 2.5, which contains three linear capacitors with capacitances C_1, C_2 and C_3 and two-port resistor, described as

$$i_{R1} = i_{R1}(v_{R1})$$

$$v_{R2} = Kv_{R1}$$

where K is a negative constant. A normal tree of this network is unique and contains branches C_1, C_2, C_3; the potentially complete sets are: $\underline{q} = (q_1, q_2, q_3)^T$ or $\underline{v}_\sigma = (v_1, v_2, v_3)^T$. It is easy to see that neither $\underline{q}$ nor $\underline{v}_\sigma$ forms the complete set. Namely, from Fig. 2.5

$$Kv_1 + v_2(1-K) - v_3 = 0$$

and v_1, v_2, v_3 or q_1, q_2, q_3 are not independent.

However, if the tree containing branches C_1, C_2, R_2 is chosen, then $\underline{v}_{\mathcal{S}} = (v_1, v_2)^T$ forms the complete set, the normal form equations in terms of this set exist and have the form

$$\dot{v}_1 = \frac{1}{C_1} i_{R1}(v_2 - v_1)$$

$$\dot{v}_2 = - C_2 + C_3(1-K)^{-1}(1 + \frac{KC_3}{C_1}) i_{R1}(v_2 - v_1)$$

Note that the branch R_2 is in fact a voltage-controlled voltage source whose controlling voltage is the difference of voltages v_1 and v_2 across capacitors C_1 and C_2. Since there is a loop of controlled-voltage source R_2 and capacitors C_1 and C_2, the KVL for this loop gives a contraint and thus voltages v_1, v_2, v_3 are not independent. Thus the dimension of vector $\underline{x}$ representing the complete set is 2 and not 3.

There are other networks with this kind of difficulty. Especially for networks containing gyrators, ideal transformers and controlled sources of different kinds, the sets $(\underline{q}, \underline{\varphi})$ or $(\underline{v}_{\mathcal{S}}, \underline{i}_{\mathcal{T}})$, derived on the basis of a normal tree, may not form the complete set and the dimension of the complete set $\underline{x}$ for such networks is frequently less than the dimension of vector $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ or $\begin{bmatrix} \underline{v}_{\mathcal{S}} \\ \underline{i}_{\mathcal{T}} \end{bmatrix}$. In general, there is no systematic method, by which a potentially complete set could be selected for such nonlinear networks. Even for linear active networks, containing passive two-terminal RLC elements and controlled sources, it is not easy to find the complete set[16]. There is a class of linear passive time varying networks, containing two-terminal resistors, capacitors, coupled inductors and gyrators, where the normal form equations can be obtained[17] on the basis of a modified normal tree*. However, when for a nonlinear RLC network $(\underline{q}, \underline{\varphi})$ or $(\underline{v}_{\delta}, \underline{i}_{\gamma})$ do not form the complete set the question, whether the complete set exists or not is even more difficult to answer than in the case of a linear network. Thus we shall not try to find the complete set in such cases but our efforts will rather be directed to find sufficient conditions for the existence of the complete set and normal form characterization of a general nonlinear network, containg locally active elements.

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* A modified normal tree T_M contains a maximal number of capacitors and a minimal number of inductors subject to the condition that each gyrator is either in T_M or in its cotree.

2.6 REFERENCES

1 Desoer, C. A. and Katzenelson, J., "Nonlinear RLC networks", Bell Sys. Tech. J., vol. 44, pp. 161-198, January 1965.

2 Holzmann, C.A. and Liu, R., "On nonlinear n-coupled networks", Technical Report No. EE-663, Department of Electrical Engineering, University of Notre Dame, Ind., April 1966.

3 Ohtsuki, T. and Watanabe, H., "State-variable analysis of RLC networks containing nonlinear coupling elements", IEEE Trans. on Circuit Theory, vol. CT-16, pp. 26-38, February 1969.

4 Stern, T.E., "Theory of nonlinear networks and systems", Reading, Mass., Addison-Wesley, 1965.

5 Chua, L. O. and Rohrer, R. A., "On the characterization of nonlinear networks", Proc. of NEC, vol. 21, 1965, Chicago, Illionois, pp. 708-713.

6 Carlin, H. J., "Singular network elements", IEEE Trans. on Circuit Theory, vo. CT-11, pp. 67-72, March 1964.

7 Tellegen, B. H. D., "On nullators and norators", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 466-469, December 1966.

8 Varayia, P. P. and Liu, R., "Normal form and stability of a class of coupled nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 413-418, December 1966.

9 Seshu, S. and Reed, M. B., "Linear graphs and electrical networks", Reading, Mass., Addison-Wesley, 1961

10 Kuh, E., "Representation of nonlinear networks", Proc. of NEC, vol. 21, 1965, Chicago, Illionois, pp. 702-707.

11 Bryant, P. R., "The order of complexity of electrical networks", Proc. IEE, vol. 106, Mono. 355, pp. 174-188, September 1959.

12 Stern, T. E., "On the equations of nonlinear networks", IEEE Trans. on Circuit Theory vol. CT-13, pp. 74-81, March 1966.

13 Chua, L. O. and Rohrer, R. A., "On the dynamic equations of a class of nonlinear RLC networks", IEEE Trans. on Circuit Theory, vol. CT-12, pp. 475-489, December 1965.

14 Kuh, E. C. and Rohrer, R. A., "The state variable approach to network analysis", Proc. IEEE, vol. 53, pp. 672-686, July 1965.

15 Oh, S. J., Stern, T. E. and Meadows, H. E., "On the analysis of nonlinear irregular networks", Proc. of the Symposium on generalized networks, New York, Polytechic Press, pp. 653-682, April 1966.

16 Purslow, E. J. and Spence, R., "On the order of complexity of an active network", Proc. IEE, vol. 114, pp. 195-198, February 1967.

17 Milić, M. M., "The state-variable characterization of a class of linear time-varying nonreciprocal networks", Proc. of the International Symposium on Network Theory, Belgrade, pp. 31-40, September 1968.

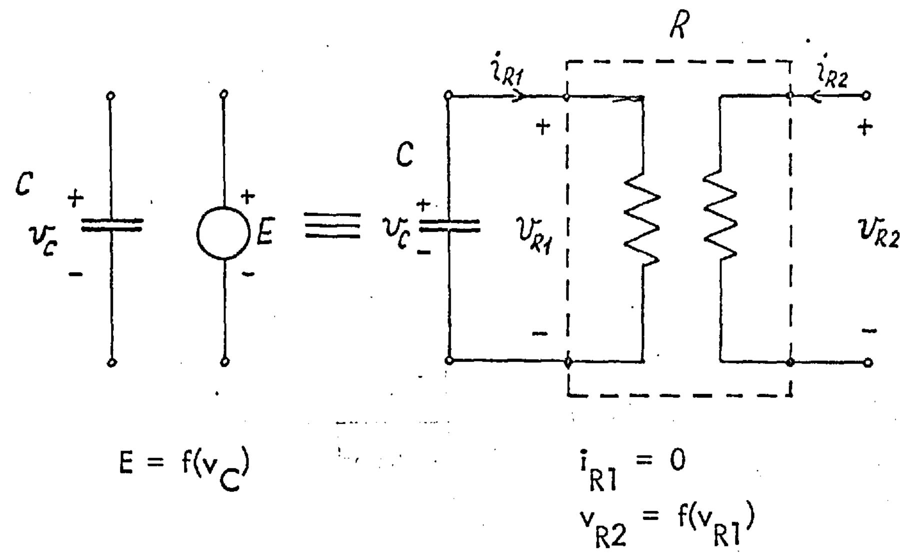


Fig. 2.1. Example of Section 2.2.1.

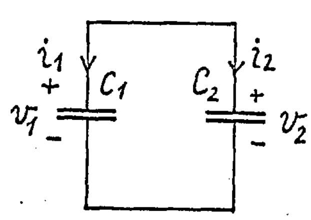


Fig. 2.2. Example of Section 2.5.1

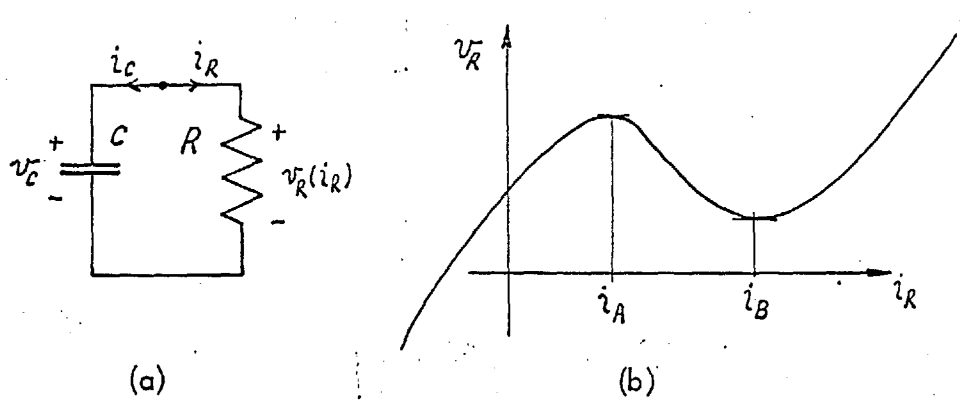
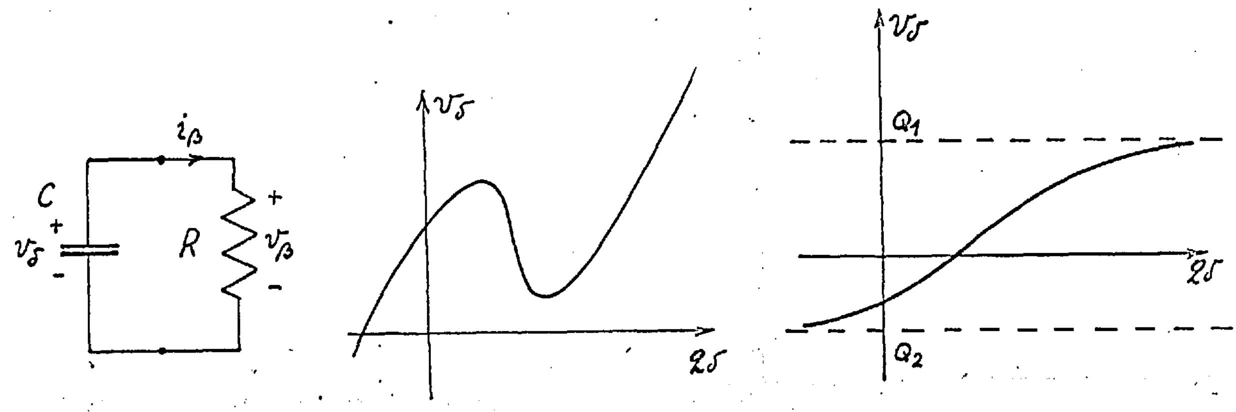


Fig. 2.3. Example of Section 2.5.1.



(a) (b) (c)

Fig. 2.4. Example of Section 2.5.2.

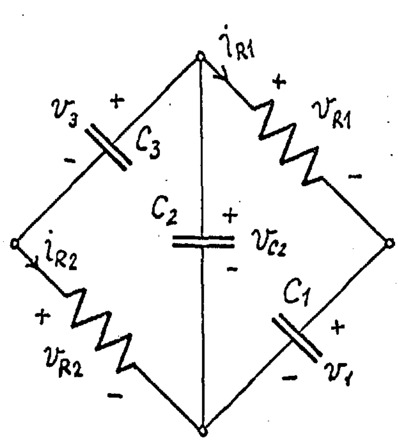


Fig. 2.5. Example of Section 2.5.2.

Chapter 3

FUNCTIONAL INVERSION AND GLOBALLY REGULAR FUNCTIONS

3.1 FUNCTIONAL INVERSION

An important problem in the analysis of nonlinear networks is the question of the existence of the inverse $\underline{x} = \underline{f}^{-1}(\underline{y})$ of a certain vector-valued function $\underline{y} = \underline{f}(\underline{x})$. An example is the coupled n-port resistor where a characterization, say $\underline{v} = \underline{r}(\underline{i})$, is given and the dual description, $\underline{i} = \underline{r}^{-1}(\underline{v})$, is needed. A similar problem arises in the analysis of one-element-kind networks (either resistive or capacitive or inductive) where equations are obtained in a form that requires a functional inversion in order to express the relation between sources and a set of network element variables. The uniqueness of solution for such one-element-kind networks is especially relevant[1,2] in the state variable description of nonlinear networks, where the usual requirement[3,4,5], related to the uniqueness of the network response, is that the inverse function is of class $C^{(1)}$. Bearing this motivation in mind, vector-valued functions $\underline{f}(\underline{x})$ of class $C^{(1)}$ and having a unique inverse of class $C^{(1)}$ will be studied.

Definition 3.1

Given a function $\underline{y} = \underline{f}(\underline{x})$, where $\underline{x}$ and $\underline{y}$ are both n-vectors and the domain of $\underline{f}$ is the entire Euclidean space R^n. A function

$$\underline{x} = \underline{f}^{-1}(\underline{y})$$

will be called an inverse of $\underline{f}$ if its domain is the entire R^n and

$$\underline{f}(\underline{f}^{-1}(\underline{y})) = \underline{y} \text{ for all } \underline{y} \in R^n.$$

This definition implies that the inverse $\underline{f}^{-1}$ is defined in the whole space R^n. If $\underline{f}^{-1}(\underline{y})$ is defined only locally on an open set $U \subset R^n$ around some point $\underline{y}_o$ such an inverse is called a local inverse. A well known inverse function theorem[6] gives conditions which guarantee the existence of a local inverse function $\underline{f}^{-1}$ of class $C^{(1)}$ when $\underline{f}$ is of class $C^{(1)}$.

A transformation of class $C^{(0)}$ which has an inverse of class $C^{(0)}$ is called a homeomorphism of R^n onto itself. A transformation of class $C^{(1)}$ possessing an inverse of class $C^{(1)}$ is called a diffeomorphism of class $C^{(1)}$ of R^n onto itself. The term regular transformation is used in Ref. 7 for a diffeomorphism of class $C^{(1)}$. Throughout this thesis a diffeomorphism of class $C^{(1)}$ of R^n onto itself will be called a globally regular function. In other words, a globally regular function $\underline{f}(\underline{x})$ is of class $C^{(1)}$, its domain is the entire R^n and it possesses an inverse of class $C^{(1)}$.

The existence of an inverse of a function $\underline{f}(\underline{x})$ is related to the question of existence and uniqueness of solution of the equation $\underline{y} = \underline{f}(\underline{x})$. When $\underline{y} = \underline{f}(\underline{x})$ possesses an inverse the corresponding system of n equations, written in component form

$$y_i = f_i(x_1, x_2, \ldots, x_n) \qquad i = 1,2, \ldots, n$$

can be solved for $x_1, x_2, \ldots, x_n$ in terms of $y_1, y_2, \ldots, y_n$ for all $\underline{y} \in R^n$.

3.1.1 Conditions for global regularity

Necessary and sufficient conditions which ensure global regularity of a function have been stated by R. S. Palais (see Cor. 4.3 of Ref. 8). Palais' theorem may be stated in our terminology as follows.

Theorem 3.1[8]

Let $\underline{f}(\underline{x})$ be a vector-valued function of class $C^{(1)}$ where $\underline{x}$ and $\underline{f}$ are n-vectors and the domain of $\underline{f}$ is the entire R^n. Necessary and sufficient conditions that $\underline{f}$ is a globally regular function are:

(i) $\det \partial \underline{f}/\partial \underline{x} \neq 0$ for all $\underline{x}$ and

(ii) $\|\underline{x}\| \to \infty$ implies $\|\underline{f}(\underline{x})\| \to \infty$.

The first condition implies the existence of a local inverse $\underline{f}^{-1}(\underline{y})$ for any $\underline{y} = \underline{f}(\underline{x}_o)$, where $\underline{x}_o$ is an arbitrary point in R^n. The second condition requires that the Euclidean norm $\|\underline{f}(\underline{x})\|$ has to approach infinity for any point on an n-dimensional ball with center at $\underline{x} = \underline{0}$ and the radius approaching infinity. In other words, the distance of the image $\underline{f}(\underline{x})$ in R^n should approach infinity as the distance $\|\underline{x}\|$ approaches infinity. This property can be simply described as radial unboundedness of $\underline{f}$ in all directions.

Palais' theorem is very important since it gives necessary as well as sufficient conditions. For example, a function $\underline{f}(\underline{x})$ of class $C^{(1)}$ whose Jacobian changes sign cannot be a globally regular function. One could object that in our network problems the second condition of Palais' theorem is somewhat unphysical. However, to demonstrate the contrary, let us consider, as an example, the characteristic of a diode described by the exponential function

$$i_D = i_D(v_D) = A(e^{\lambda v_D} - 1).$$

The first condition of Palais' theorem is fulfilled for the function $i_D(v_D)$ but the second condition is not satisfied when $v_D \to -\infty$. Yet the inverse function of $i_D(v_D)$ does not exist for any negative value of current i_D.

Unfortunately it is very rarely that Palais' theorem can be applied directly to resolve the question whether a given function is globally regular or not. One the difficulties in its application is that the radial unboundedness of a given function has to be checked in all directions in R^n. In the case of nonlinear one-element-kind networks the governing equations depend upon topological structure and hybrid descriptions of network elements. For complex one-element-kind networks it is virtually impossible to check whether the second condition of Palais' theorem is fulfilled or not. If the hybrid matrices of all network elements in a network are known it is frequently a simple matter to express the Jacobian matrix of the governing set of equations by the hybrid matrices. In order to satisfy the first condition in Palais' theorem the Jacobian has to be different from zero for all values of the independent variable. It would clearly be very useful, if a sufficient condition, ensuring the fulfillment of condition (ii) in Palais' theorem, can be stated in terms of the Jacobian of $\underline{f}(\underline{x})$ only. We shall show that such a criterion on the Jacobian can indeed be obtained. In order to derive this criterion we shall start with the one-dimensional case and then generalize it to the n-dimensional case.

When $y = f(x)$ and $f(\cdot)$ is of class $C^{(1)}$, a sufficient condition for the existence of the inverse $f^{-1}(y)$ of class $C^{(1)}$ is that

$$|df/dx| \geq \varepsilon > 0 \qquad \text{for all} \quad x \in R^1 \tag{3.1}$$

and ε does not depend upon x. When this condition is extended to the n-dimensional case an ambiguity arises about its meaning. The condition (3.1) has the following possible interpretations when generalized to the n-dimensional case:

(i) $[\partial \underline{f}/\partial \underline{x} - \varepsilon I]$ is positive definite for all $\underline{x} \in R^n$ and $\varepsilon > 0$

(ii) $|\det \partial \underline{f}/\partial \underline{x}| \geq \varepsilon > 0$ for all $\underline{x} \in R^n$.

The explanation (i) is related to quasilinear functions[3] and their extentions to be mentioned later. The condition (ii) can be interpreted geometrically as follows. Let P be an n-dimensional differential cube whose edges are $dx_1 = dx_2 = \ldots = dx_n = \mu$ and let P′ be the image of P under transformation $\underline{f}(\underline{x})$. Then the ratio between vol P′, the volume of P′, and vol P, the volume of P,

$$\frac{\text{vol}P'}{\text{vol}P} \geq \varepsilon > 0$$

Hower, the condition (ii) is not a sufficient condition for a function $\underline{f}(\underline{x})$ to be globally regular. This can be demonstrated with the aid of the following counterexample. Let $\underline{f}$: $R^2 \rightarrow R^2$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underline{f}(\underline{x}) = \begin{bmatrix} e^{x_1} \\ x_2 e^{-x_1} \end{bmatrix} \tag{3.2}$$

then

$$\partial \underline{f}/\partial \underline{x} = \begin{bmatrix} e^{x_1} & 0 \\ -x_2 e^{-x_1}, & e^{-x_1} \end{bmatrix}$$

and $\det \partial \underline{f}/\partial \underline{x} = 1$ everywhere in R^2. Yet, the function $\underline{f}(\underline{x})$, defined by eqn. (3.2) is not globally regular. Clearly, the inverse function does not exist for any $y_1 < 0$ and the function $\underline{f}$ is not radially bounded for $x_2 = 0$, $x_1 \rightarrow -\infty$. An additional condition is necessary to ensure global regularity. We shall show that for functions with bounded Jacobian matrices the condition $|\det \partial \underline{f}/\partial \underline{x}| \geq \varepsilon > 0$ is sufficient to ensure global regularity. This result is stated formally in the form of the following theorem[9].

Theorem 3.2

Given a function

$$\underline{y} = \underline{f}(\underline{x})$$

where $\underline{x}$, $\underline{f}$ and $\underline{y}$ are n-vectors, $\underline{f}$ is of class $C^{(1)}$ and it is defined for all $\underline{x} \in R^n$. Sufficient conditions for $\underline{f}(\underline{x})$ to be globally regular are:

$$\text{(i)} \quad |\det \partial \underline{f}/\partial \underline{x}| \geq \varepsilon > 0 \qquad \text{for all } \underline{x} \in R^n \tag{3.3}$$

and ε does not depend upon $\underline{x}$

(ii) the Jacobian matrix $\partial \underline{f}/\partial \underline{x}$ is bounded, i. e. there exists a value $M > 0$ such that

$$|\partial f_i/\partial x_j| \leq M \qquad \text{for all } \underline{x} \in R^n \tag{3.4}$$

In addition $\partial \underline{f}^{-1}/\partial \underline{y}$, the Jacobian matrix of the inverse of $\underline{f}$, is bounded for all $\underline{y} \in R^n$.

Proof[10]

Before carrying out the proof of Theorem 3.2 some preliminary remarks will be given and then three useful lemmas will be stated.

In the following let R_x^n denote a Euclidean space such that $\underline{x} \in R_x^n$ and let R_y^n be the space of the images $\underline{y} = \underline{f}(\underline{x}) \in R_y^n$. Furthermore, let $\underline{\varphi}(t) = \left[\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)\right]^T$ be a differentiable function of the real variable t, $\alpha \leq t < \beta$, whose values are points in the space R_x^n. Then $\underline{x} = \underline{\varphi}(t)$ represents a rectifiable curve[11] in R_x^n. The length of this curve between two points corresponding to the values $t = \alpha$ and $t = \tau$, is equal to $s(\tau) = \int_\alpha^\tau \| \partial \underline{\varphi}(t)/\partial t \| \, dt$. If $s(\tau) \to \infty$ as $\tau \to \beta$, we shall say that the length of the curve $\underline{x} = \underline{\varphi}(t)$, $\alpha \leq t < \beta$, is infinite. The corresponding curve $\underline{y} = \underline{f}\left[\underline{\varphi}(t)\right] = \underline{\psi}(t)$ in the space R_y^n is also differentiable.

Let $d\underline{x} = \left[dx_1, dx_2, \ldots, dx_n\right]^T$ be a differential of $\underline{x}$. Then the arc length differential ds which is equal to the distance of points $\underline{x}$ and $(\underline{x} + d\underline{x})$ in R_x^n on the curve $\underline{\varphi}(t)$ is

$$ds = (d\underline{x}^T d\underline{x})^{1/2} = \left[(\partial \underline{\varphi}/\partial t)^T \, \partial \underline{\varphi}/\partial t\right] \, dt \tag{3.5}$$

The images of points $\underline{x}$ and $(\underline{x} + d\underline{x})$ are points $\underline{y} = \underline{f}(\underline{x})$ and $\underline{y} + d\underline{y} = \underline{f}(\underline{x} + d\underline{x})$. The arc length differential ds' with respect to the curve $\underline{y} = \underline{\psi}(t) = \underline{f}\left[\underline{\varphi}(t)\right]$ is

$$ds' = (d\underline{y}^T d\underline{y})^{1/2} = \left[(\partial \underline{\varphi}/\partial t)^T (\partial \underline{f}/\partial \underline{x})^T \, \partial \underline{f}/\partial \underline{x} \, \partial \underline{\varphi}/\partial t\right] dt \tag{3.6}$$

The ratio ds'/ds is a function of the vector $\underline{x}$ and the direction of the differential $d\underline{x}$. We shall study such functions where

$$ds'/ds \geq \eta > 0 \tag{3.7}$$

for all $\underline{x}$ and all directions of differentials $d\underline{x}$. When the condition (3.7) is fulfilled, then given an arc in R_x^n the arc length of its image in R_y^n is not smaller than η times the arc length of the original arc. Thus when the arc length of an arc in R_x^n approaches infinity, so does the arc length of its image in R_y^n. In the first lemma we show that a function $\underline{f}(\underline{x})$ satisfying conditions (i) and (ii) of Theorem 3.2 obeys the inequality (3.7).

Lemma 3.1

The ratio ds'/ds for a function $\underline{y} = \underline{f}(\underline{x})$ which satisfies both conditions of Theorem 3.2 has a lower bound

$$ds'/ds \geq \frac{\varepsilon}{(nM)^{n-1}} > 0 \tag{3.8}$$

Proof: From eqn. (3.6) ds'/ds is equal to

$$(ds'ds)^2 = \|d\underline{y}\|^2 / \|d\underline{x}\|^2 = \left[d\underline{x}^T(\partial\underline{f}/\partial\underline{x})^T\,\partial\underline{f}/\partial\underline{x}\;d\underline{x}\right]/d\underline{x}^T d\underline{x}$$

Since $\partial\underline{f}/\partial\underline{x}$ is nonsingular, $(\partial\underline{f}/\partial\underline{x})^T\,\partial\underline{f}/\partial\underline{x}$ is positive definite. Thus using the result given in Ref. 12.

$$d\underline{x}^T(\partial\underline{f}/\partial\underline{x})^T\,\partial\underline{f}/\partial\underline{x}\;\;d\underline{x} \geq \lambda_{min}\,d\underline{x}^T\,d\underline{x} \tag{3.9}$$

where λ_{min} is the smallest eigenvalue of $(\partial\underline{f}/\partial\underline{x})^T\,\partial\underline{f}/\partial\underline{x}$. Writing the determinant as the product of its eigenvalues and taking into account eqn. (3.3)

$$\det\left[(\partial\underline{f}/\partial\underline{x})^T\;\partial\underline{f}/\partial\underline{x}\right] = \prod_{i=1}^{n}\lambda_i \geq \varepsilon^2 \tag{3.10}$$

The application of Gresgorin's theorem[13] about approximate location of the eigenvalues of a matrix and the inequality (3.4) yield the following upper bound for the largest eigenvalue λ_{max} of $\left[(\partial\underline{f}/\partial\underline{x})^T\;\partial\underline{f}/\partial\underline{x}\right]$

$$\lambda_{max} \leq n^2M^2 \tag{3.11}$$

Combining eqns. (3.10) and (3.11)

$$\lambda_{min} \geqslant \frac{\varepsilon^2}{(nM)^{2(n-1)}} \tag{3.12}$$

The inequality (3.8) follows from the expression for $(ds'/ds)^2$, eqn. (3.9) and (3.12).

Q. E. D.

For the proof of Theorem 3.2 we need two additional lemmas where $\underline{f} : R_x^n \longrightarrow R_y^n$ is a $C^{(1)}$ map satisfying both conditions of Theorem 3.2

Lemma 3.2

Let $\underline{x}_o$ be any point of R_x^n, $\underline{y}_o = \underline{f}(\underline{x}_o)$ its image in R_y^n, and $\underline{y} = \underline{\psi}(t)$, $0 \leq t \leq 1$, an arbitrary arc of class $C^{(1)}$ in the space R_y^n with the initial point $\underline{y}_o$: $\underline{\psi}(\underline{0}) = \underline{y}_o$. Then there exists in the space R_x^n one and only one arc $\underline{x} = \underline{\varphi}(t)$, $0 \leq t \leq 1$, of class $C^{(1)}$, with the initial point $\underline{x}_o$ whose map by $\underline{f}$ is the given arc $\underline{y} = \underline{\psi}(t)$, hence

$$\underline{f}\left[\underline{\varphi}(t)\right] = \underline{\psi}(t), \quad \underline{\varphi}(0) = \underline{x}_o \tag{3.13}$$

Proof: It is sufficient to find a continuous function $\underline{x} = \underline{\varphi}(t)$ satisfying eqn. (3.13). The differentiability of $\underline{\varphi}(t)$ follows from the differentiability of the map $\underline{f}$ and the function $\underline{\psi}(t)$, and the uniqueness follows from the inverse function theorem.

Since $\underline{y}_o = \underline{f}(\underline{x}_o)$, there exists a neighbourhood U_o of $\underline{x}_o$ which is mapped by $\underline{f}$ homeomorphically onto a neighbourhood V_o of $\underline{y}_o$. The inverse map $\underline{g}_o(\underline{y})$, defined on V_o, is also of class $C^{(1)}$ and is uniquely determined by the initial condition $\underline{g}_o(\underline{y}_o) = \underline{x}_o$. If $t_1 > 0$ is sufficiently small, the arc $\underline{y} = \underline{\psi}(t)$, $0 \le t \le t_1$, lies in the neighbourhood V_o. Hence we have $\dot{\underline{\varphi}}(t) = \underline{g}_o\left[\underline{\psi}(t)\right]$ for $t \in \left[0, t_1\right]$.

If Lemma 3.2 does not hold, there exists a number t^*, $0 \le t^* \le 1$, such that the function $\underline{\varphi}(t)$ satisfying eqn. (3.13) is defined in the half-open interval $\left[0, t^*\right)$, but cannot be defined in any closed interval $\left[0, \alpha\right]$ with $\alpha \geq t^*$. Now, take any increasing sequence of positive numbers t_n converging to t^*, so that $\lim t_n = t^*$. Write $\underline{x}_n = \underline{\varphi}(t_n)$, so that $\underline{y}_n = \underline{f}(\underline{x}_n)$. The continuity of $\underline{\psi}(t)$ implies $\lim \underline{y}_n = \underline{\psi}(t^*) = \underline{y}^*$. Consider now the sequence $\left\{\underline{x}_n\right\}$. If it is not bounded, then the length of the arc $\underline{x} = \underline{\varphi}(t)$ between the points $\underline{x}_o$ and $\underline{x}_n$ tends to infinity as $n \to \infty$. The image of this arc is the arc $\underline{y} = \underline{\psi}(t)$ between the points $\underline{y}_o$ and $\underline{y}_n$. The length of the latter arc is less than the arc between the points $\underline{y}_o$ and $\underline{y}^*$, which is finite. Hence, the assumption that the sequence $\left\{\underline{x}_n\right\}$ is not bounded contradicts the condition (3.8). Thus $\left\{\underline{x}_n\right\}$ is bounded and it has at least one limit point $\underline{x}^*$. A subsequence converging to $\underline{x}^*$ can be extracted from $\left\{\underline{x}_n\right\}$. Without loss of generality we may assume that this subsequence is $\left\{\underline{x}_n\right\}$. Hence we have $\underline{y}^* = \lim \underline{y}_n = \lim \underline{f}(\underline{x}_n) = \underline{f}(\underline{x}^*)$. Let V^* be a neighbourhood of $\underline{y}^*$, where the inverse map $\underline{x} = \underline{g}^*(\underline{y})$, $\underline{g}^*(\underline{y}^*) = \underline{x}^*$, of the map $\underline{f}$ exists. Since $\underline{g}^*(\underline{V}^*) = \underline{U}^*$ is a neighbourhood of $\underline{x}^*$, and since $\underline{x}^* = \lim \underline{x}_n$, we have $\underline{x}_n \in U^*$ for sufficiently large $\underline{n}$, say for $n \geq n_o$. Then $\underline{g}^*(\underline{y}_n) = \underline{x}_n$ if $n \geq n_o$. The function $\underline{g}^*\left[\underline{\psi}(t)\right]$ is defined in a neighbourhood of t^*. It follows from the uniqueness of the inverse map $\underline{g}^*(\underline{y})$ in the neighbourhood V^* that $\underline{\varphi}(t) = \underline{g}^*\left[\underline{\psi}(t)\right]$ for all $t < t^*$, where both functions are defined. If we put $\underline{\varphi}^*(t) = \underline{\varphi}(t)$ for $0 \le t < t^*$ and $\underline{\varphi}^*(t) = \underline{g}^*\left[\underline{\psi}(t)\right]$, $t \geq t^*$, we get a continuous function $\underline{\varphi}^*(t)$ satisfying eqn.

(3.13) and defined in an interval $[0,\alpha]$ with $\alpha \geq t^*$. This contradicts the choice of t^*.

Q. E. D.

The point $\underline{y}_1 = \underline{\psi}(1)$ is the image of the point $\underline{x}_1 = \underline{\varphi}(1) \in R^n_x$. Since $\underline{y}_o$ can be joined to any other point $\underline{y}_1$ of R^n_y by a differentiable arc, it follows that a map $\underline{f}$: $R^n_x \rightarrow R^n_y$ satisfying both conditions of Theorem 3.2 is onto.

The compactness of the interval $[0, 1]$ implies the existence of a finite sequence of real numbers $t_o = 0 < t_1 < t_2 \ \ldots < t_{n-1} < t_n = 1$ with the following property: if $\underline{y}_k = \underline{\psi}(t_k)$, there exists an open neighbourhood V_k of $\underline{y}_k$ where the inverse map $\underline{x} = \underline{g}_k(\underline{y})$, $\underline{g}_k(\underline{y}_k) = \underline{x}_k$, of the map $\underline{f}$ is defined. The arc $\underline{y} = \underline{\psi}(t)$, $t_k \leq t \leq t_{k+1}$, belongs to V_k and we have $\underline{\varphi}(t) = \underline{g}_k[\underline{\psi}(t)]$ for $t \in [t_k, t_{k+1}]$ and $k = 0, 1, \ldots n-1$.

Lemma 3.3

Let $\underline{x}_o \in R^n_x$ be arbitrary and $\underline{y}_o = \underline{f}(\underline{x}_o)$. If the function $\underline{y} = \underline{\psi}(t,u)$, $0 \leq t, u \leq 1$, whose values are points of R^n_y, is of class $C^{(1)}$ and $\underline{\psi}(0,0) = \underline{y}_o$, then there exists a continuous function $\underline{x} = \underline{\varphi}(t,u)$, $0 \leq t, u \leq 1$, $\underline{\varphi}(t,u) \in R^n_x$, such that

$$\underline{f}\left[\underline{\varphi}(t,u)\right] = \underline{\psi}(t,u), \quad \underline{\varphi}(0,0) = \underline{x}_o \tag{3.14}$$

Proof: According to Lemma 3.2 there exists a unique arc $\underline{x} = \underline{\theta}(u)$, $0 \leq u \leq 1$, $\underline{\theta}(u) \in R^n_x$, $\underline{\theta}(0) = \underline{x}_o$, whose map by $\underline{f}$ is the arc $\underline{y} = \underline{\psi}(0,u)$, $0 \leq u \leq 1$, in the space R^n_y.

We shall now define the function $\underline{\varphi}(t,u)$ as follows. The equation $\underline{y} = \underline{\psi}(t,u)$, $0 \leq t \leq 1$, represents for a fixed $u \in [0, 1]$, an arc of class $C^{(1)}$ in R^n_y. Since $\underline{f}[\underline{\theta}(u)] = \underline{\psi}(0,u)$, there exists a unique arc $\underline{x} = \underline{\varphi}_u(t)$ of class $C^{(1)}$, depending upon u, such that $\underline{f}[\underline{\varphi}_u(t)] = \underline{\psi}(t,u)$ and $\underline{\varphi}_u(0) = \underline{\theta}(u)$. If we put $\underline{\varphi}(t,u) = \underline{\varphi}_u(t)$, $0 \leq t,u \leq 1$, then $\underline{\varphi}(t,u)$ evidently satisfies eqn. (3.14). It must be verified that it is a continuous function of the variables t, u. For this purpose take any number $\alpha \in [0, 1]$ and write $\underline{x}^* = \underline{\theta}(\alpha)$, $y^* = \underline{\psi}(0,\alpha) = \underline{f}(\underline{x}^*)$. The curve $\underline{x} = \underline{\varphi}(t,\alpha)$ has the initial point $\underline{\varphi}(0,\alpha) = = \underline{\theta}(\alpha) = \underline{x}^*$ and is mapped onto $\underline{y} = \underline{\psi}(t,\alpha)$. We can now find, as described above, a sequence of numbers $t_o = 0 < t_1 \ldots < t_n = 1$, and a corresponding sequence of open balls $V_o, V_1, \ldots, V_{n-1}$ of R^n_y such that the arc $\underline{y} = \underline{\psi}(t,\alpha)$, $t_k \leq t \leq t_{k+1}$, belongs to V_k, and such that the inverse map $\underline{x} = \underline{g}_k(\underline{y})$ of $\underline{f}$ exists for $\underline{y} \in V_k$. The branch $\underline{g}_k(\underline{y})$ is determined by $\underline{g}_k(\underline{x}_k)$, where $\underline{x}_k = \underline{\varphi}(t_k,\alpha)$, $\underline{y}_k = \underline{\psi}(t_k,\alpha)$. Since V_k is open and $\underline{\psi}(t,u)$ continuous, there exists a $\delta_k > 0$ so that $\underline{\psi}(t,u) \in V_k$ for $t \in [t_k, t_{k+1}]$, $u \in [\alpha - \delta_k, \alpha + \delta_k] \cap [0, 1]$. Let $\delta = \min(\delta_o, \delta_1, \delta_2, \ldots, \delta_{n-1})$. If we write $\underline{\varphi}_o(t,u) = \underline{g}_k[\underline{\psi}(t,u)]$ for $t \in [t_k, t_{k+1}]$, $u \in [\alpha - \delta, \alpha + \delta] \cap [0, 1] = S$, then the function $\underline{\varphi}_o(t,u)$ is defined and continuous on the strip $0 \leq t \leq 1$, $u \in S$. We have $\underline{\varphi}_o(0,\alpha) = \underline{x}^*$. It follows that $\underline{\varphi}_o(0,u) = \underline{\theta}(u) = \underline{\varphi}(0,u)$ for $u \in S$. Since $\underline{f}[\underline{\varphi}_o(t,u)] = \underline{f}[\underline{\varphi}(t,u)]$, we conclude by Lemma 3,2, that $\underline{\varphi}_o(t,u) = \underline{\varphi}(t,u)$ in the rectangle $t \in [0, 1]$, $u \in S$. Hence the function $\underline{\varphi}(t,u)$ is continuous there. The number $\alpha \in [0, 1]$ being arbitrary, $\underline{\varphi}(t,u)$ is continuous everywhere in the square $0 \leq t,u \leq 1$.

Q. E. D.

Remark: Since $\underline{\psi}(t,u)$ is of class $C^{(1)}$, it is obvious that $\underline{\varphi}(t,u)$ is also of class $C^{(1)}$.

Let us return to the proof of the theorem. Assume that a $C^{(1)}$ map satisfies conditions (i) and (ii) of Theorem 3.2. Then we already know that $\underline{f}$ is onto R^n_y. Suppose that $\underline{x}_o$, $\underline{x}_1 \in R^n_x$ are mapped into the same point $\underline{y}_o \in R^n_y$, hence $\underline{f}(\underline{x}_o) = \underline{f}(\underline{x}_1) = \underline{y}_o$. The equation $\underline{x} = \underline{x}_o + t(\underline{x}_1 - \underline{x}_o) = \underline{\varphi}(t)$, $0 \leq t \leq 1$, represents the line segment betwen the points $\underline{x}_o$ and $\underline{x}_1$. The image of this segment is the closed curve $\underline{y} = \underline{f}\left[\underline{\varphi}(t)\right] = \underline{\psi}(t)$, since $\underline{\psi}(0) = \underline{f}(\underline{x}_o)$ $= \underline{f}(\underline{x}_1) = \underline{\psi}(1)$. Let

$$\underline{\psi}(t,u) = \underline{y}_o + (1-u)\left[\underline{\psi}(t) - \underline{y}_o\right] \tag{3.15}$$

By Lemma 3.3 there exists a continuous function $\underline{\varphi}(t,u)$, $0 \leq t,\ u \leq 1$, such that $\underline{f}\left[\underline{\varphi}(t,u)\right] = \underline{\psi}(t,u)$ and $\underline{\varphi}(0,0) = \underline{x}_o$. For $u = 0$ we have $\underline{\psi}(t,0) = \underline{\psi}(t)$. Since $\underline{y} = \underline{\psi}(t)$ is the image of $\underline{x} = \underline{\varphi}(t) = \underline{x}_o + t(\underline{x}_1 - \underline{x}_o)$ and of $\underline{x} = \underline{\varphi}(t,0)$, and since $\underline{\varphi}(0) = \underline{\varphi}(0,0) = \underline{x}_o$, we conclude that

$$\underline{\varphi}(t,0) = \underline{x}_o + t(\underline{x}_1 - \underline{x}_o) \tag{3.16}$$

Furthermore, $\underline{\psi}(0,u) = \underline{y}_o$, so that the arc $\underline{y} = \underline{\psi}(0,u)$ is reduced to the point $\underline{y}_o$. Since $\underline{\varphi}(0,0) = \underline{x}_o$ and since the inverse image is uniquely determined by the initial point, we have $\underline{\varphi}(0,u) = \underline{x}_o$. Hence $\underline{\varphi}(0,1) = \underline{x}_o$. We obtain from eqn. (3.15) that $\underline{\psi}(t,1) = \underline{y}_o$. This implies $\underline{\varphi}(t,1) = \underline{x}_o$. On the other hand, we have $\underline{\psi}(1,u) = \underline{y}_o$ from eqn. (3.15), and $\underline{\varphi}(1,0) = \underline{x}_1$ from eqn. (3.16). It follows that $\underline{\varphi}(1,u) = \underline{x}_1$. Now we get $\underline{\varphi}(1,1) = \underline{x}_o$ from $\underline{\varphi}(t,1) = \underline{x}_o$ and $\underline{\varphi}(1,1) = \underline{x}_1$ from $\underline{\varphi}(1,u) = \underline{x}_1$. Hence $\underline{x}_1 = \underline{x}_o$. Therefore, the map $\underline{f}$ has the property that any $\underline{y}_o \in R^n_y$ is the image of one point $\underline{x}_o$ only. We know already that $\underline{f}$ is onto R^n_y. Thus $\underline{f}^{-1}$ exists, is everywhere defined, and as $\underline{f}$ is a $C^{(1)}$ map, the same holds for $\underline{f}^{-1}$.

It remains to show that $\partial \underline{f}^{-1}/\partial \underline{y}$ is bounded. The Jacobian matrix $\partial \underline{f}^{-1}/\underline{y}$ is

$$\partial \underline{f}^{-1}/\partial \underline{y} = (\partial \underline{f}/\partial \underline{x})^{-1}\Big|_{\underline{x} = \underline{f}^{-1}(\underline{y})} \tag{3.17}$$

The boundedness of $\partial \underline{f}/\partial \underline{x}$ ensures the boundedness of adj $\partial \underline{f}/\partial \underline{x}$, the adjoint matrix of $\partial \underline{f}/\partial \underline{x}$, and taking into account the inequality (3.3) it follows that the Jacobian matrix $\partial \underline{f}^{-1}/\partial \underline{y}$ is bounded for all $\underline{y} \in R^n_y$. This completes the proof of Theorem 3.2

The following corollary concerning the functional inversion of a function $\underline{y} = \underline{f}(\underline{x},t)$ where t is an additional scalar parameter has useful applications in network analysis.

<u>Corollary 3.1</u>

Given a function $\underline{y} = \underline{f}(\underline{x},t)$ where $\underline{f}$, $\underline{x}$ and $\underline{y}$ are n-vectors, t is a scalar and $\underline{f}$ is defined for all $\begin{bmatrix}\underline{x}\\ t\end{bmatrix} \in R^{n+1}$. Suppose that

(i) $\underline{f} \in C^{(1)}$ in $\underline{x}$ and $\underline{f} \in C^{(0)}$ in t

(ii) $|\det \partial \underline{f}/\partial \underline{x}| \geq \varepsilon > 0$ for all $\begin{bmatrix}\underline{x}\\ t\end{bmatrix}$

(iii) $\partial \underline{f}/\partial \underline{x}$ is bounded for all $\begin{bmatrix}\underline{x}\\ t\end{bmatrix}$.

Then the inverse function $\underline{x} = \underline{f}^{-1}(\underline{y},t) = \underline{g}(\underline{y},t)$, defined for all $\begin{bmatrix}\underline{y}\\ t\end{bmatrix}$, exists; moreover $\underline{f}^{-1}$ is of class $C^{(1)}$ in $\underline{y}$ and it is of class $C^{(0)}$ in t. In addition the Jacobian matrix $\partial \underline{f}^{-1}/\partial \underline{y}$ is bounded for all $\begin{bmatrix}\underline{y}\\ t\end{bmatrix}$.

<u>Proof:</u> Since the function $\underline{f}(\underline{x},t)$ satisfies the conditions of Theorem 3.2

for all t, clearly, the inverse function $\underline{x} = \underline{f}^{-1}(\underline{y},t)$ exists for all t, it is of class $C^{(1)}$ in $\underline{y}$ and $\partial \underline{f}^{-1}/\partial \underline{y} = \partial \underline{g}/\partial \underline{y}$ is bounded for all $\begin{bmatrix} \underline{y} \\ t \end{bmatrix}$. It remains to show that $\underline{f}^{-1}(\underline{y},t)$ is continuous in t.

We want to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|\underline{x} - \underline{x}_o\| = \|\underline{g}(\underline{y}_o, t_o + \Delta t) - \underline{g}(\underline{y}_o, t_o)\| < \varepsilon \qquad (3.18a)$$

for all points $\begin{bmatrix} \underline{y}_o \\ t \end{bmatrix}$ for which

$$\left\| \begin{bmatrix} \underline{y}_o \\ t \end{bmatrix} - \begin{bmatrix} \underline{y}_o \\ t_o + \Delta t \end{bmatrix} \right\| < \delta \qquad (3.18b)$$

Let $\underline{f}(\underline{x}_o, t_o) = \underline{y}_o$ and $\underline{f}(\underline{x}_o, t_o + \Delta t) - \underline{y}_o = \Delta \underline{y}$. Then $\underline{x}_o = g(\underline{y}_o + \Delta \underline{y}, t_o + \Delta t)$. Since $\underline{f}(\underline{x},t)$ is continuous in t then for every $\varepsilon_1 > 0$ there exists a $\delta_1 > 0$ such that

$$\|\underline{y}\| = \|\underline{y} - \underline{y}_o\| = \|\underline{f}(\underline{x}_o, t_o + \Delta t) - \underline{f}(\underline{x}_o, t_o)\| < \varepsilon_1 \qquad (3.19a)$$

for all points $\begin{bmatrix} \underline{x}_o \\ t \end{bmatrix}$ for which

$$\left\| \begin{bmatrix} \underline{y}_o \\ t_o + \Delta t \end{bmatrix} - \begin{bmatrix} \underline{y}_o \\ t_o \end{bmatrix} \right\| < \delta_1 \qquad (3.19b)$$

The difference $[\underline{x} - \underline{x}_o]$ can be recast in the form

$$\underline{x} - \underline{x}_o = \underline{g}(\underline{y}_o, t_o + \Delta t) - \underline{g}(\underline{y}_o + \Delta \underline{y}, t_o + \Delta t)$$

Using the mean-value theorem[14] for scalar functions of vector variables and denoting the i-th component of $\underline{g}$ by g_i we have

$$\underline{x} - \underline{x}_o = \begin{bmatrix} \partial g_1/\partial \underline{y}|_{\underline{y}=\underline{y}_1} \\ \partial g_2/\partial \underline{y}|_{\underline{y}=\underline{y}_2} \\ \cdot \\ \cdot \\ \cdot \\ \partial g_n/\partial \underline{y}|_{\underline{y}=\underline{y}_n} \end{bmatrix} \Delta \underline{y}$$

where $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n$ are points lying on the line segment with endpoints $\underline{y}_o$ and $\underline{y}_o + \Delta\underline{y}$. Applying inequality (3.18a) and taking into account that $\partial \underline{g}/\partial \underline{y} = \partial \underline{f}^{-1}/\partial \underline{y}$ is bounded the following relation is obtained

$$A \, \|\Delta \underline{y}\| < \|\underline{x} - \underline{x}_o\| < \varepsilon \qquad (3.20)$$

where A is a positive value. Therefore the condition $\|\underline{x} - \underline{x}_o\| < \varepsilon$ implies that

$$\|\Delta \underline{y}\| < \varepsilon/A .$$

If in eqn. (3.19a) ε_1 is chosen as $\varepsilon_1 = \varepsilon/A$ then by conditions (3.19a) and (3.19b) for any $|(t_o + \Delta t) - t_o| < \delta_1$ the norm $\|\Delta\underline{y}\| < \varepsilon_1 = \varepsilon/A$ and by eqn. (3.20) $\|\underline{x} - \underline{x}_o\| < \varepsilon$. Thus δ in eqn. (3.18b) is equal to δ_1 defined in eqn. (3.19b).

Q. E. D.

3.1.2 Functions possessing "partial" inverses

In network analysis it is often desirable to transform a given hybrid description of, say, a resistive n-port into another hybrid description with a different set of independent variables. When a set of independent variables $\underline{x}$ in $\underline{y} = \underline{f}(\underline{x})$ is replaced by a set of dependent variables $\underline{y}$ in the new hybrid description it is necessary to perform a functional inversion. Similarly when in the new description only a subset of dependent variables is replaced by the corresponding subset of independent variables the necessary operation to be performed will be called "partial" functional inversion. We would like to find criteria which ensure the existence of a "partial" inverse. The problem may be formulated mathematically as follows.

Given a function $\underline{y} = \underline{f}(\underline{x})$, partition n-vectors $\underline{x}, \underline{y}$ and $\underline{f}$ conformably in the following manner,

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \quad \underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} \quad f = \begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \end{bmatrix} \tag{3.21a}$$

where $\underline{x}_1$, $\underline{y}_1$ and $\underline{f}_1$ are m-vectors, $\underline{x}_2, \underline{y}_2$ and $\underline{f}_2$ are (n-m)-vectors. The function $\underline{y} = \underline{f}(\underline{x})$ can then be written as

$$\begin{aligned} \underline{y}_1 &= \underline{f}_1(\underline{x}_1, \underline{x}_2) \\ \underline{y}_2 &= \underline{f}_2(\underline{x}_1, \underline{x}_2) \end{aligned} \tag{3.21b}$$

Define n-vectors $\underline{w}$ and $\underline{z}$,

$$\underline{w} = \begin{bmatrix} \underline{y}_1 \\ \underline{x}_2 \end{bmatrix} \cdot \quad \underline{z} = \begin{bmatrix} \underline{x}_1 \\ \underline{y}_2 \end{bmatrix} \tag{3.22}$$

The function $\quad \underline{z} = \underline{g}(\underline{w})$, $\hfill$ (3.23a)

which may be written as

$$\begin{aligned} \underline{x}_1 &= \underline{g}_1(\underline{y}_1, \underline{x}_2) \\ \underline{y}_2 &= \underline{g}_2(\underline{y}_1, \underline{x}_2) \end{aligned} \tag{3.23b}$$

and whose domain is the entire R^n, is then defined as a "partial" inverse of $\underline{f}(\underline{x})$. Sufficient conditions for the existence of a "partial" inverse are given in the following theorem.

Theorem 3.3

Suppose a function $\underline{y} = \underline{f}(\underline{x})$ of class $C^{(1)}$ is written in the form

$$\begin{aligned} \underline{y}_1 &= \underline{f}_1(\underline{x}_1, \underline{x}_2) \\ \underline{y}_2 &= \underline{f}_2(\underline{x}_1, \underline{x}_2) \end{aligned} \tag{3.24}$$

where $\underline{x}_1$, $\underline{y}_1$ and $\underline{f}_1$ are m-vectors and $\underline{x}_2$, $\underline{y}_2$ and $\underline{f}_2$ are (n-m)-vectors. Suppose that

$$\text{(i)} \quad \left| \det \partial \underline{f}_1 / \partial \underline{x}_1 \right| \geq \varepsilon_1 > 0 \tag{3.25}$$

is fulfilled for all $\underline{x} \in R^n$ and

(ii) the Jacobian matrix $\partial(\underline{f}_1, \underline{f}_2)/\partial(\underline{x}_1, \underline{x}_2)$ is bounded. Then

(a) the "partial" inverse

$$\begin{bmatrix} \underline{x}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} \underline{g}_1(\underline{y}_1, \underline{x}_2) \\ \underline{g}_2(\underline{y}_1, \underline{x}_2) \end{bmatrix} \tag{3.26}$$

exists and is of class $C^{(1)}$ in $\underline{y}_1$ and $\underline{x}_2$; in addition

(b) the Jacobian matrix $\partial(\underline{g}_1, \underline{g}_2)/\partial(\underline{y}_1, \underline{x}_2)$ may be expressed in terms of $\partial(\underline{f}_1, \underline{f}_2)/\partial(\underline{x}_1, \underline{x}_2)$ as

$$\frac{\partial(\underline{g}_1, \underline{g}_2)}{\partial(\underline{y}_1, \underline{x}_2)} = \left[\begin{array}{c|c} \left[\dfrac{\partial \underline{f}_1}{\partial \underline{x}_1}\right]^{-1} & -\left[\dfrac{\partial \underline{f}_1}{\partial \underline{x}_1}\right]^{-1} \dfrac{\partial \underline{f}_1}{\partial \underline{x}_2} \\ \hdashline \dfrac{\partial \underline{f}_2}{\partial \underline{x}_1}\left[\dfrac{\partial \underline{f}_1}{\partial \underline{x}_1}\right]^{-1} & \dfrac{\partial \underline{f}_2}{\partial \underline{x}_2} - \dfrac{\partial \underline{f}_2}{\partial \underline{x}_1}\left[\dfrac{\partial \underline{f}_1}{\partial \underline{x}_1}\right]^{-1} \dfrac{\partial \underline{f}_1}{\partial \underline{x}_2} \end{array}\right]$$

(3.27)

Proof

Part (a) follows from Theorem 3.2. Regarding $\underline{x}_2$ as a parameter it is evident that all conditions of Theorem 3.2 are fulfilled for function $\underline{y}_1 = \underline{f}_1(\underline{x}_1, \underline{x}_2)$. Thus $\underline{x}_1 = \underline{f}_1^{-1}(\underline{y}_1, \underline{x}_2)$ exists, is of class $C^{(1)}$ and is defined for all $\underline{w} = \begin{bmatrix} \underline{y}_1 \\ \underline{x}_2 \end{bmatrix} \in R^n$.

Let $\underline{X}_1$, $\underline{X}_2$, $\underline{Y}_1$ and $\underline{Y}_2$ represent vectors related by linear approximation for $\underline{f}(\underline{x}_1, \underline{x}_2)$ and $\underline{g}(\underline{y}_1, \underline{x}_2)$ around the points $(\underline{x}_1, \underline{x}_2)$ and $(\underline{y}_1, \underline{x}_2)$ respectively. Then from eqn. (3.21b) and (3.23b) respectively

$$\begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} = \left[\begin{array}{c|c} \dfrac{\partial \underline{f}_1}{\partial \underline{x}_1} & \dfrac{\partial \underline{f}_1}{\partial \underline{x}_2} \\ \hline \dfrac{\partial \underline{f}_2}{\partial \underline{x}_1} & \dfrac{\partial \underline{f}_2}{\partial \underline{x}_2} \end{array}\right] \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} \tag{3.28a}$$

$$\begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix} = \left[\begin{array}{c|c} \dfrac{\partial \underline{g}_1}{\partial \underline{y}_1} & \dfrac{\partial \underline{g}_1}{\partial \underline{x}_2} \\ \hline \dfrac{\partial \underline{g}_2}{\partial \underline{y}_1} & \dfrac{\partial \underline{g}_2}{\partial \underline{x}_2} \end{array}\right] \begin{bmatrix} \underline{Y}_1 \\ \underline{X}_2 \end{bmatrix} \tag{3.28b}$$

When the set of eqns. (3.28b) is solved for $\underline{X}_1$, $\underline{Y}_2$ in terms of $\underline{Y}_1$, $\underline{X}_2$ the result of part (b) follows.

Q. E. D.

When $\underline{x}$ and $\underline{y}$ are n-vectors related by $\underline{y} = \underline{f}(\underline{x})$, there are 2^n possible different selections of independent variables. Theorem 3.3 can easily be extended to give conditions under which any of $(2^n - 1)$ different "partial" inverses exist.

Corollary 3.2

Given a vector valued function $\underline{y} = \underline{f}(\underline{x})$, where $\underline{x} \in R^n$; $\underline{y} \in R^n$ and $\underline{f} \in C^{(1)}$. Suppose that

(i) the Jacobian matrix $\partial \underline{f}/\partial \underline{x}$ is bounded and

(ii) for all principal minors of $\partial \underline{f}/\partial \underline{x}$ of order m,

$m = 1,2,\ldots, n$

$$\left| \det\, (\partial \underline{f}/\partial \underline{x})_{(p)(p)} \right| \geq \epsilon^m > 0 \qquad (3.29)$$

where (p) denotes a subset of m rows and columns not deleted from $\partial \underline{f}/\partial \underline{x}$. Then all $(2^n - 1)$ possible different "partial" inverses of class $C^{(1)}$ can be obtained from $\underline{f}(\underline{x})$.

3.1.3 Implicit functions

Frequently the relation between an n-vector $\underline{x}$ and an m-vector $\underline{y}$ is given implicitly in the form

$$\underline{f}(\underline{x},\underline{y}) = \underline{0} \qquad (3.30)$$

where $\underline{f}$ is an n-vector. Let $\underline{f}(\underline{x}_o,\underline{y}_o) = \underline{0}$. The implicit function theorem[15] yields conditions for the existence of a unique function $\underline{x} = \underline{g}(\underline{y})$, defined locally in a neighbourhood of $\underline{y}_o$. In the analysis of one-element-kind networks the resulting algebraic equations may have the form of eqn. (3.30) where the components of $\underline{y}$ are independent sources and the components of $\underline{x}$ are the unknown network element variables. In order to solve such a network it is necessary to express $\underline{x} = \underline{g}(\underline{y})$ where $\underline{y}$ is not restricted to a small region in R^m. Thus it appears to be fruitful if the conditions for the existence of a unique function $\underline{x} = \underline{g}(\underline{y})$ for all $\underline{y} \in R^m$ can be obtained. A direct application of Theorem 3.2 yields the following result.

Theorem 3.4

Suppose a function $\underline{f}(\underline{x},\underline{y})$ is of class $C^{(1)}$, $\underline{x}$ and $\underline{f}$ are n-vectors, $\underline{y}$ is an m-vector and $\underline{f}$ is defined for all $\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \in R^{n+m}$. Suppose that $\underline{f}(\underline{x},\underline{y}) = \underline{0}$ and

(i) $\partial \underline{f}/\partial(\underline{x},\underline{y})$ is bounded and

(ii) $|\det \partial \underline{f}/\partial \underline{x}| \geq \varepsilon > 0$ for all $\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \in R^{n+m}$. (3.31)

Then there exists a unique function $\underline{x} = \underline{g}(\underline{y})$ of class $C^{(1)}$ with values in R^n and defined for all $\underline{y} \in R^m$ such that

$$\underline{f}(\underline{g}(\underline{y}),\underline{y}) = \underline{0} \qquad \text{for all } \underline{y} \in R^m \tag{3.32}$$

and moreover $\partial \underline{g}/\partial \underline{y}$ is bounded for all $\underline{y} \in R^m$.

Proof

Using Theorem 3.2 the proof can be given along the lines of the proof of the implicit function theorem[15].

Define the function $\underline{F}(\underline{x},\underline{y})$

$$\begin{bmatrix} \underline{z} \\ \underline{w} \end{bmatrix} = \underline{F}(\underline{x},\underline{y}) = \begin{bmatrix} \underline{f}(\underline{x},\underline{y}) \\ \underline{y} \end{bmatrix} \tag{3.33}$$

The Jacobian matrix of $\underline{F}$ is

$$\frac{\partial \underline{F}}{\partial(\underline{x},\underline{y})} = \begin{bmatrix} \frac{\partial \underline{f}}{\partial \underline{x}} & \frac{\partial \underline{f}}{\partial \underline{y}} \\ 0 & I \end{bmatrix} \tag{3.34}$$

From eqn. (3.34) and the inequality (3.31)

$$|\det \partial \underline{F}/\partial(\underline{x},\underline{y})| = |\det \partial\underline{f}/\partial\underline{x}| \geq \varepsilon > 0 \qquad (3.35)$$

and $\underline{F}(\underline{x},\underline{y})$ satisfies the first condition of Theorem 3.2. Since $\partial\underline{f}/\partial(\underline{x},\underline{y})$ is bounded, the Jacobian matrix of $\underline{F}$ is bounded. Thus, $\underline{F}(\underline{x},\underline{y})$ satisfies all conditions of Theorem 3.2 and hence, $\underline{F}$ is a globally regular function. Therefore the inverse of $\underline{F}$

$$\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = \begin{bmatrix} \underline{\phi}(\underline{z},\underline{w}) \\ \underline{w} \end{bmatrix} \qquad (3.36)$$

exists and is of class $C^{(1)}$.

From the definition of the inverse

$$\underline{f}\,[\underline{\phi}(\underline{z},\underline{w}),\ \underline{w}] = \underline{z} \qquad \text{for all } \begin{bmatrix} \underline{z} \\ \underline{w} \end{bmatrix} \in R^{n+m} \qquad (3.37)$$

If we now define

$$\underline{g}(\underline{y}) = \underline{\phi}(\underline{0},\underline{y}) \qquad \text{for all } \underline{y} \in R^m \qquad (3.38)$$

then setting $\underline{z} = \underline{0}$ in eqn. (3.37) and taking into account $\underline{y} = \underline{w}$ (eqn. (3.33)) we obtain eqn. (3.32). The uniqueness of $\underline{g}(\underline{y})$ follows from the fact that $\begin{bmatrix} \underline{\phi}(\underline{z},\underline{w}) \\ \underline{w} \end{bmatrix}$ is the inverse of $\underline{F}(\underline{x},\underline{y})$. The Jacobian matrix $\partial\underline{g}/\partial\underline{y}$ can be obtained from eqn. (3.32) as

$$\frac{\partial \underline{g}}{\partial \underline{y}} = -\left(\frac{\partial \underline{f}}{\partial \underline{x}}\right)^{-1} \frac{\partial \underline{f}}{\partial \underline{y}}$$

Applying conditions (i) and (ii) to the above expression it is easy to see that $(\partial\underline{f}/\partial\underline{x})^{-1}$ is bounded and thus $\partial\underline{g}/\partial\underline{y}$ is bounded.

Q. E. D.

We shall state a useful corollary of Theorem 3.4; it yields conditions that guarantee the global regularity of the function $\underline{x} = \underline{g}(\underline{y})$ which is the solution of implicit equation $\underline{f}(\underline{x},\underline{y}) = \underline{0}$.

Corollary 3.3

Suppose $\underline{f}(\underline{x},\underline{y}) = \underline{0}$ where $\underline{f}$, $\underline{x}$ and $\underline{y}$ are n-vectors, $\underline{f} \in C^{(1)}$ and it is defined for all $\begin{bmatrix} \underline{x} \\ - \\ \underline{y} \end{bmatrix} \in R^{2n}$. Suppose $\partial \underline{f}/\partial(\underline{x},\underline{y})$ is bounded, $|\det \partial \underline{f}/\partial \underline{x}| \geq \varepsilon_1 > 0$ and $|\det \partial \underline{f}/\partial \underline{y}| \geq \varepsilon_2 > 0$. Then there exists a globally regular function $\underline{x} = \underline{g}(\underline{y})$ such that

$$\underline{f}(\underline{g}(\underline{y}),\underline{y}) = \underline{0} \qquad \text{for all } \underline{y} \in R^n. \tag{3.39}$$

Proof: In order to prove the global regularity of $\underline{g}(\underline{y})$ then using Theorem 3.2 it is necessary to show that $\partial \underline{g}/\partial \underline{y}$ is bounded and that $\varepsilon_o > 0$ exists such that $|\det \partial \underline{g}/\partial \underline{y}| \geq \varepsilon_o$. From eqn. (3.39)

$$\partial \underline{g}/\partial \underline{y} = -(\partial \underline{f}/\partial \underline{x})^{-1} \; \partial \underline{f}/\partial \underline{y} \tag{3.40}$$

Since $\partial \underline{f}/\partial(\underline{x},\underline{y})$ is bounded, $|\det \partial \underline{f}/\partial \underline{x}| \geq \varepsilon_1 > 0$, it follows that $(\partial \underline{f}/\partial \underline{x})^{-1}$ is bounded and it is clear from eqn. (3.40) that $\partial \underline{g}/\partial \underline{y}$ is bounded. Similarly, the boundedness of $\partial \underline{f}/\partial \underline{x}$ implies the existence of a value $M_1 > 0$ such that $|\det (\partial \underline{f}/\partial \underline{x})^{-1}| \geq M_1$. Therefore $|\det \partial \underline{g}/\partial \underline{y}| =$ $= |\det(\partial \underline{f}/\partial \underline{x})^{-1}| \; |\det \partial \underline{f}/\partial \underline{y}| \geq M_1 \varepsilon_2$ and $\varepsilon_o \geq M_1 \; \varepsilon_2$.

Q. E. D.

3.2 SOME CLASSES OF GLOBALLY REGULAR FUNCTIONS

In this section we shall study some classes of globally regular functions that occur frequently in network problems. When the Jacobian matrix associated with a given function is bounded Theorem 3.2 may be applied. A subclass of globally regular functions containing functions with unbounded Jacobian matrices will be mentioned as well. Before proceeding further three useful definitions will be stated. The terminology introduced by Minty[16] will be used in Definition 3.2.

Definition 3.2

A function $\underline{f}$: $R^n \to R^n$ is called monotonic provided, for any $\underline{x}_1, \underline{x}_2 \in R^n$, we have

$$[\underline{f}(\underline{x}_2) - \underline{f}(\underline{x}_1)]^T [\underline{x}_2 - \underline{x}_1] \geq 0. \tag{3.41a}$$

If
$$[\underline{f}(\underline{x}_2) - \underline{f}(\underline{x}_1)]^T [\underline{x}_2 - \underline{x}_1] > 0 \tag{3.41b}$$

$\underline{f}$ is strictly monotonic.

If
$$[\underline{f}(\underline{x}_2) - \underline{f}(\underline{x}_1)]^T [\underline{x}_2 - \underline{x}_1] > \mu \|\underline{x}_1 - \underline{x}_1\|^2, \quad \underline{x}_1 \neq \underline{x}_2 \tag{3.41c}$$

with $\mu > 0$, $\underline{f}$ is strongly monotonic.

Let us now extend the definition of a uniformly positive definite matrix[2] to the nonsymmetric case.

Definition 3.3*

A square n x n matrix $A(\underline{x})$ is said to be uniformly positive definite

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* The same definition appeared in Ref. 17.

(u.p.d.) in $\underline{x}$ if there exists a $\mu > 0$ such that

$$\underline{z}^T \left[A(\underline{x}) - \mu I \right] \underline{z} > 0 \qquad \underline{z} \neq \underline{0} \tag{3.42}$$

for all $\underline{x}$ and μ does not depend upon $\underline{x}$.

Definition 3.4[3]

A square n x n matrix $A(\underline{x})$ is said to be uniformly Hadamard (u. H.) in $\underline{x}$ if it is continuous and bounded and if there exists a $\mu > 0$ such that

$$a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| > \mu \qquad \text{for all } \underline{x} \text{ and all } i \tag{3.43}$$

Lemma 3.4

Suppose $A(\underline{x})$ is u. p. d. Then

$$\det A > \mu^n > 0 \qquad \text{for all } \underline{x} \tag{3.44}$$

Proof[9]: Let

$$B = \left[A(\underline{x}) - \mu I \right].$$

The determinant det A can be written in the form

$$\det A = \det \left[\mu I + B \right] = \det \left[\mu I \right] \left[\det \;\; I + (\mu^{-1} I) B \right] \tag{3.45}$$

Using the diagonal expansion of a matrix[18] and taking into account that B is positive definite

$$\det \left[\; I + (\mu^{-1} I) B \right] > 1. \tag{3.46}$$

The inequality (3.44) follows from eqns. (3.45) and (3.46).

Q. E. D.

Lemma 3.5

Suppose $A(\underline{x})$ is a u. H. matrix. Then

$$|\det A| > \mu^n > 0 \qquad \text{for all } \underline{x} \tag{3.47}$$

Proof: Using Gresgorin's theorem[13] it follows from eqn. (3.47) that any eigenvalue λ_i of A is bounded away from zero,

$$|\lambda_i| > \mu > 0 \qquad i=1,2,\ldots,n \tag{3.48}$$

Since $|\det A| = \prod_{i=1}^{n} |\lambda_i|$, $|\det A| > \mu^n > 0.$

Q. E. D.

Lemma 3.6

A necessary and sufficient condition for a function $\underline{f}(\underline{x}) \in C^{(1)}$ to be strongly monotonic is that the Jacobian matrix $\partial \underline{f}/\partial \underline{x}$ of $\underline{f}$ is u.p.d.

Proof of sufficiency: Consider a one-dimensional arc $\underline{x}(\theta)$, $0 \le \theta \le 1$, given by

$$\underline{x}(\theta) = \underline{x}_1 + (\underline{x}_2 - \underline{x}_1)\theta \tag{3.49}$$

Then

$$\partial \underline{x}/\partial \theta = (\underline{x}_2 - \underline{x}_1) \tag{3.50}$$

and

$$(\underline{x}_2 - \underline{x}_1)^T [\underline{f}(\underline{x}_2) - \underline{f}(\underline{x}_1)] =$$

$$= (\underline{x}_2 - \underline{x}_1)^T \int_{\theta=0}^{1} \left.\frac{\partial f}{\partial \underline{x}}\right|_{\underline{x} = \underline{x}(\theta)} (\underline{x}_2 - \underline{x}_1) d\theta > \mu \|\underline{x}_2 - \underline{x}_1\|^2$$

where the last inequality follows from the fact that $\partial \underline{f}/\partial \underline{x}$ is u. p. d.

Thus $\underline{f}$ is strongly monotonic*.

Proof of necessity: From the definition of a strongly monotonic function

$$(\underline{x}_2 - \underline{x}_1)^T \left[\underline{f}(\underline{x}_2) - \underline{f}(\underline{x}_1)\right] > \mu \left\| \underline{x}_2 - \underline{x}_1 \right\|^2, \ \mu > 0 \qquad (3.51)$$

Let $\underline{X} = \underline{x}_2 - \underline{x}_1$ and $\| \underline{x}_2 - \underline{x}_1 \| \to 0$. Then from eqn. (3.51)

$$\underline{X}^T \partial \underline{f}/\partial \underline{x} \ \underline{X} > \mu \underline{X}^T \underline{X}$$

and thus $\partial \underline{f}/\partial \underline{x}$ has to be u. p. d.

Q. E. D.

The first result will be concerned with strongly monotonic functions of class $C^{(1)}$.

3.2.1 Class of strongly monotonic functions

Theorem 3.5**

Suppose a function $\underline{f}$: $R^n \to R^n$ is a strongly monotonic function of class $C^{(1)}$. Then $\underline{f}$ is a globally regular function.

Proof

By Lemma 3.6 a strongly monotonic function has a u. p. d. Jacobian

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* Similarly for $\underline{f}(\underline{x}) \in C^{(1)}$ positive definiteness of $\partial \underline{f}/\partial \underline{x}$ for all $\underline{x}$ ensures strict monotonicity of $\underline{f}$ and positive semidefiniteness of $\partial \underline{f}/\partial \underline{x}$ implies monotonicity.

** The same result appeared in Ref. 17

matrix $\partial \underline{f}/\partial \underline{x}$. According to Lemma 3.4 $\det \partial \underline{f}/\partial \underline{x} > \mu^n > 0$ and the first condition in Palais' theorem is fulfilled.

Applying the Schwartz inequality to the criterion for strong monotonicity (eqn. (3.41 c)) yields

$$\| \underline{f}(\underline{x}_2) - \underline{f}(\underline{x}_1) \| > \mu \| \underline{x}_2 - \underline{x}_1 \| \tag{3.52}$$

Let $\underline{x}_1$ = const. and $\underline{x}_2 = \underline{x}$. Then $\|x\| \to \infty$ in eqn. (3.52) implies $\|\underline{f}(\underline{x})\| \to \infty$ and the second condition in Palais' theorem is fulfilled. Hence $\underline{f}$ is a globally regular function.

Note that in Theorem 3.5 there is no requirement for $\partial \underline{f}/\partial \underline{x}$ to be bounded. Since by Lemma 3.6 strong monotonicity is equivalent to a u. p. d. Jacobian matrix $\partial \underline{f}/\partial \underline{x}$, a function $\underline{f}(\underline{x})$ where $\underline{f} \in C^{(1)}$ and its Jacobian matrix is u. p. d. is a globally regular function.

The inverse $\underline{f}^{-1}(\underline{y})$ of a strongly monotonic function $\underline{f}(\underline{x})$ of class $C^{(1)}$ is a strictly monotonic function of class $C^{(1)}$. Namely, u. p. d. $\partial \underline{f}/\partial \underline{x}$ implies that $\partial \underline{f}/\partial \underline{x}$ is positive definite and therefore $(\partial \underline{f}/\partial \underline{x})^{-1} = \partial \underline{f}^{-1}/\partial \underline{y}$ is positive definite. However, the function $\underline{f}^{-1}(\underline{y})$ is not necessarily strongly monotonic or equivalently its Jacobian matrix is not necessarily u. p. d. This can be demonstrated by the following one-dimensional example.

Let $y = f(x)$ be defined in the following manner:

$$\begin{aligned} y &= x + 1 \qquad && x \leq 0 \\ y &= e^x && x > 0 \end{aligned} \tag{3.53}$$

As the derivative dy/dx is continuous at $x = 0$ this function is of class $C^{(1)}$ and furthermore it is evident that df/dx is u. p. d. The inverse function

$x = f^{-1}(y)$ is from eqn. (3.53)

$$\begin{aligned} x &= y - 1 \qquad & y &\leq 1 \\ x &= \ln y \qquad & y &> 1 \,. \end{aligned} \tag{3.54}$$

Since $dx/dy = 1/y$, $y > 1$ and $dx/dy \rightarrow 0$ as $y \rightarrow \infty$, the Jacobian matrix of $f^{-1}(y)$ is not u.p.d. and $f^{-1}(y)$ is strictly monotonic only. The function given in eqn. (3.53) is a counterexample[19] for the second part of Theorem 3 in Ref. 17. In order to ensure a strongly monotonic inverse of a strongly monotonic function of class $C^{(1)}$ an additional restriction has to be placed upon $\underline{f}$. Strongly monotonic functions possessing strongly monotonic inverse will be termed generalized quasilinear functions.

3.2.2 Generalized quasilinear functions

Definition 3.5

A strongly monotonic function $\underline{f}(\underline{x})$: $R^n \rightarrow R^n$, where $\underline{f} \in C^{(1)}$, will be said to be a generalized quasilinear function (GQLF) if its Jacobian matrix is bounded for all $\underline{x}$. Equivalently a GQLF is a function of class $C^{(1)}$ and has a u. p. d. and bounded Jacobian matrix.

GQLFs are an extention of quasilinear functions[2]. If a GQLF $\underline{f}(\underline{x})$ has a symmetric Jacobian matrix for all $\underline{x}$, it is a quasilinear function. GQLFs that serve as a basis for the definition of positive definite network elements, to be introduced in next chapter, have the following properties:

Property 1

The sum of two GQLFs is a GQLF.

Proof The proof given in Ref. 3, p. 576 for the sum of two quasilinear functions is applicable.

Property 2

If $\underline{f}(\underline{x})$ is a GQLF and given function $\underline{g}(\underline{v}) = A^T\underline{f}(A\underline{v} + \underline{b})$ where $\underline{f}$, $\underline{x}$ and $\underline{b}$ are n-vectors, $\underline{v}$ is an m-vector ($m \leq n$) and A is a constant $n \times m$ matrix of rank m, then $\underline{g}$ is a GQLF.

Proof The proof given in Ref. 3, pp. 576-577 for an equivalent property of quasilinear functions is appropriate.

Note that GQLFs have all the properties[3] of quasilinear functions except that a state function[3] associated with a GQLF may not exist if a GQLF has a nonsymmetric Jacobian matrix. However, the following property of GQLFs does not hold for quasilinear functions.

Property 3

Suppose a function $\underline{f}$: $R^n \to R^n$ is a GQLF. Write $\underline{f}$ in the form

$$\begin{aligned} \underline{y}_1 &= \underline{f}_1(\underline{x}_1, \underline{x}_2) \\ \underline{y}_2 &= \underline{f}_2(\underline{x}_1, \underline{x}_2) \end{aligned} \tag{3.55}$$

where $\underline{x}$ and $\underline{y}$ are partitioned arbitrarily but conformably and $\underline{x}_1$, $\underline{y}_1$ are m-vectors and $\underline{x}_2$, $\underline{y}_2$ are (n-m)-vectors, m = 1, 2, ..., n. Then a "partial" inverse $\underline{g}(\underline{y}_1, \underline{x}_2)$, written in component form

$$\begin{bmatrix} \underline{x}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} \underline{g}_1(\underline{y}_1, \underline{x}_2) \\ \underline{g}_2(\underline{y}_1, \underline{x}_2) \end{bmatrix} \tag{3.56}$$

exists and moreover $\underline{g}(\underline{y}_1, \underline{x}_2)$ is a GQLF. Consequently, all $(2^n - 1)$ different "partial" inverses of a GQLF $\underline{f}$ exist and are themselves GQLFs.

<u>Proof:</u> The existence of a "partial" inverse $\underline{g}(\underline{y}_1, \underline{x}_2)$ follows from the fact that $\underline{f}$ has a u. p. d. and bounded Jacobian matrix $\partial \underline{f}/\partial \underline{x}$. Any principal submatrix of order m (m = 1,2...,n) of $\partial \underline{f}/\partial \underline{x}$ is itself u.p.d. and its determinant has by Lemma 3.4 a lower bound $\mu^m > 0$. Therefore all conditions of Corollary 3.2 are fulfilled and $\underline{g}(\underline{y}_1, \underline{x}_2)$ exists.

To prove that $\underline{g}(\underline{y}_1, \underline{x}_2)$ is a GQLF we have to show that a $\mu_2 > 0$ exists such that $[\partial \underline{g}/\partial(\underline{y}_1, \underline{x}_2) - \mu_2 I]$ is positive definite for all $\begin{bmatrix} \underline{y}_1 \\ \underline{x}_2 \end{bmatrix} \in R^n$. Let $\underline{X}_1$, $\underline{X}_2$, $\underline{Y}_1$, $\underline{Y}_2$ represent vectors related by linear approximation for $\underline{f}(\underline{x})$ or $\underline{g}(\underline{y}_1, \underline{x}_2)$ around points $\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$ and $\begin{bmatrix} \underline{y}_1 \\ \underline{x}_2 \end{bmatrix}$. Then from eqn. (3.55) and (3.56)

$$\begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} = \partial(\underline{f}_1, \underline{f}_2)/\partial(\underline{x}_1, \underline{x}_2) \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} \tag{3.57}$$

$$\begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix} = \partial(\underline{g}_1, \underline{g}_2)/\partial(\underline{y}_1, \underline{x}_2) \begin{bmatrix} \underline{Y}_1 \\ \underline{X}_2 \end{bmatrix} \tag{3.58}$$

As $\partial \underline{f}/\partial \underline{x}$ is u. p. d. it follows from eqn. (3.57)

$$\begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix}^T \begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} > \mu \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix}^T \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} \quad \text{for} \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} \neq \underline{0} \tag{3.59}$$

The scalar product on the left of eqn. (3.59) can be recast in the form

$$\begin{bmatrix} \underline{Y}_1 \\ \underline{X}_2 \end{bmatrix}^T \begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix} > \mu \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix}^T \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} \qquad . \quad (3.60)$$

Substituting eqn. (3.58) into eqn. (3.60)

$$\begin{bmatrix} \underline{Y}_1 \\ \underline{X}_2 \end{bmatrix}^T \frac{\partial(\underline{g}_1, \underline{g}_2)}{\partial(\underline{y}_1, \underline{x}_2)} \begin{bmatrix} \underline{Y}_1 \\ \underline{X}_2 \end{bmatrix} > \begin{bmatrix} \frac{\partial \underline{g}_1}{\partial \underline{y}_1} \underline{Y}_1 + \frac{\partial \underline{g}_1}{\partial \underline{x}_2} \underline{X}_2 \\ \underline{X}_2 \end{bmatrix}^T \begin{bmatrix} \frac{\partial \underline{g}_1}{\partial \underline{y}_1} \underline{Y}_1 + \frac{\partial \underline{g}_1}{\partial \underline{x}_2} \underline{X}_2 \\ \underline{X}_2 \end{bmatrix} \mu$$

(3.61)

The r. h. s. of eqn. (3.61) may be written as a quadratic form $\begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix}^T D^T D \begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix}$ where

$$D = \begin{bmatrix} \frac{\partial \underline{g}_1}{\partial \underline{y}_1} & \frac{\partial \underline{g}_1}{\partial \underline{x}_2} \\ 0 & I \end{bmatrix} \qquad (3.62)$$

From Theorem 3.3 (eqn. (3.27))

$$\frac{\partial \underline{g}_1}{\partial \underline{y}_1} = \left[\frac{\partial \underline{f}_1}{\partial \underline{x}_1} \right]^{-1}$$

and

$$\frac{\partial \underline{g}_1}{\partial \underline{x}_2} = - \left[\frac{\partial \underline{f}_1}{\partial \underline{x}_1} \right]^{-1} \frac{\partial \underline{f}_1}{\partial \underline{x}_2} \ .$$

The boundedness of $\partial \underline{f}_1 / \partial \underline{x}_1$ implies the existence of a lower bound $\varepsilon_1 > 0$ such that det $D > \varepsilon_1$. Since $\underline{f}$ is a GQLF, the matrix D as well as $\partial \underline{g} / \partial(\underline{y}_1, \underline{x}_2)$ is bounded. Thus an $\varepsilon_2 > 0$ exists such that for λ_{min}, the smallest eigenvalue of the positive definite matrix $D^T D$,

$$\lambda_{min} > \varepsilon_2 > 0$$

and using the result in Ref. 12.

$$\begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix}^T D^T D \begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix} > \varepsilon_2 \begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix}^T \begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix} \text{ for all } \begin{bmatrix} \underline{X}_1 \\ \underline{Y}_2 \end{bmatrix} \neq \underline{0} \tag{3.63}$$

Combining eqns. (3.61), (3.62) and (3.63) it follows that $\left[\partial \underline{g}/\partial(\underline{y}_1, \underline{x}_2) - \mu\varepsilon_2 I\right]$ is positive definite and $\mu_2 \geq \mu\varepsilon_2$.

Q. E. D.

3.2.3 Functions with uniformly Hadamard Jacobian matrices

The next result is concerned with functions $\underline{f}$: $R^n \rightarrow R^n$ where $\partial \underline{f}/\partial \underline{x}$ is u. H. It is shown in Ref. 3 that the equation $\underline{f}(\underline{x}) = \underline{0}$ has a unique solution when $\partial \underline{f}/\partial \underline{x}$ is u. H. In fact $\underline{y} = \underline{f}(\underline{x})$ is a globally regular function under this condition.

Theorem 3.6

Suppose the Jacobian matrix $\partial \underline{f}/\partial \underline{x}$ (or its transpose) of a function $\underline{f}$: $R^n \rightarrow R^n$ is u. H. Then $\underline{f}$ is globally regular.

Proof

Since $\partial \underline{f}/\partial \underline{x}$ is u. H., it is bounded, $\underline{f} \in C^{(1)}$ and by Lemma 3.5 $|\det \partial \underline{f}/\partial \underline{x}| > \mu^n > 0$. Hence all conditions of Theorem 3.2 are fulfilled and therefore $\underline{f}$ is globally regular.

Q. E. D.

Some properties of functions with u. H. Jacobian matrices are listed below.

Property 1

Suppose that functions $\underline{f}_1: R^n \to R^n$ and $\underline{f}_2: R^n \to R^n$ have u. H. Jacobian matrices. Then the function $\underline{f} = \underline{f}_1 + \underline{f}_2$ is a globally regular function.

Proof: Since the sum of two u. H. matrices is itself u. H., the Jacobian matrix of $\underline{f}$ is u. H. and Property 1 follows.

Q. E. D.

Property 2

Suppose a function $\underline{f}: R^n \to R^n$ has a u. H. Jacobian matrix. Then all posible $(2^n - 1)$ different "partial" inverses of class $C^{(1)}$ exist.

Proof: For a u. H. matrix all principal submatrices are themselves u. H. and thus by Lemma 3.5 all principal minors of $\partial \underline{f}/\partial \underline{x}$ have a positive lower bound. Therefore all conditions of Corollary 3.2 are satisfied and Property 2 follows.

Q. E. D.

Property 3

Suppose the Jacobian matrix $\partial \underline{f}/\partial \underline{x}$ and its transpose $[\partial \underline{f}/\partial \underline{x}]^T$ of a function $\underline{f}: R^n \to R^n$ are both u. H. Then $\underline{f}$ is a GQLF.

Proof: If $\partial \underline{f}/\partial \underline{x}$ and $[\partial \underline{f}/\partial \underline{x}]^T$ are both u. H. then $\partial \underline{f}/\partial \underline{x}$ is u. p. d. and bounded. Namely, the quadratic form of $\partial \underline{f}/\partial \underline{x}$ is equal to the quadratic form of its symmetric part, i. e.

$$\underline{z}^T \, \partial \underline{f}/\partial \underline{x} \, \underline{z} = \frac{1}{2}\underline{z}^T \left[\partial \underline{f}/\partial \underline{x} + (\partial \underline{f}/\partial \underline{x})^T\right] \underline{z}.$$

The matrix $\left[\partial \underline{f}/\partial \underline{x} + (\partial \underline{f}/\partial \underline{x})^T\right]$ is u. H. and the matrix P

$$P = \left[\partial \underline{f}/\partial \underline{x} + (\partial \underline{f}/\partial \underline{x})^T - \varepsilon I\right] \text{ is u. H. for some } \varepsilon > 0.$$

Hence by Lemma 3.5 all principal minors of P are positive, therefore P is positive definite and $\partial \underline{f}/\partial \underline{x}$ is u. p. d.

Q. E. D.

A special case of Property 3 are functions with u. H. and symmetric Jacobian matrices. These functions belong to the class of quasilinear functions.

3.2.4 Two other classes of globally regular functions

In this section two special classes of functions that appear frequently in the analysis of one-element-kind networks will be treated. Our interest is to find sufficient conditions that ensure global regularity of these functions. In order to discuss these conditions the following lemma and definition will be introduced first.

Lemma 3.7

Let Q and R be two $n \times n$ real constant positive semidefinite (not necessarily symmetric) matrices. Then

$$\det\left[\, I + QR \,\right] \geq 1$$

if (a) R is positive definite (not necessarily symmetric) or

(b) R is symmetric positive semidefinite.

<u>Proof:</u> If R is positive definite we can write

$$\det \left[I + QR \right] = \det \left[R^{-1} + Q \right] \det R$$

The matrix $\left[R^{-1} + Q \right]$ is positive definite. From the inequality[20] that relates the determinant of a positive definite nonsymmetric matrix to the determinant of its symmetric part it follows that

$$\det \left[R^{-1} + Q \right] > \det \left[(R^{-1})_s + Q_s \right]$$

and $$\det R > \det R_s$$

where subscript s denotes the symmetric part of a matrix. Hence

$$\det \left[I + QR \right] > \det \left[I + Q_s (R^{-1})_s \right] \geq 1$$

The last inequality follows from Assertion 2 in the Appendix of Ref. 5.

Let R be symmetric. Since every real symmetric matrix R is orthogonally similar to a diagonal matrix[21] (whose diagonal elements are necessarily the characteristic roots of R), R may be written in the form $R = P \Lambda P^T$, where Λ is a positive semidefinite diagonal matrix and $PP^T = I$. Therefore

$$\det \left[I + QR \right] = \det \left[I + QP\Lambda P^T \right] = \det \left[I + P^T QP\Lambda \right].$$

The matrix $P^T QP$ is positive semidefinite and applying Assertion 1 of the Appendix of Ref. 5 yields: $\det \left[I + QR \right] \geq 1$.

Q. E. D.

Definition 3.6[22]

Let A be a real constant square matrix. A belongs to the class of matrices denoted by P if all principal minors of A are positive.

If all principal minors of A are nonnegative, A belongs to the class of matrices denoted by P_o.

Among other matrices the class P contains positive definite and row (or columns)dominant matrices. The class P_o contains positive semidefinite matrices.

The first function $\underline{y} = \underline{f}(\underline{x})$ to be considered in this section is of the form

$$\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & F \\ -F^T & 0 \end{bmatrix} \begin{bmatrix} \underline{g}(\underline{x}_1) \\ \underline{h}(\underline{x}_2) \end{bmatrix} \qquad (3.64)$$

where $\underline{x}_1$, $\underline{y}_1$ and $\underline{g}$ are m-vectors, $\underline{x}_2$, $\underline{y}_2$ and $\underline{h}$ are (n-m)-vectors and F is a constant m x (n-m) matrix. This function has the form of eqn. (2.41) appearing in mixed analysis of one-element-kind networks and has been studied by Varayia and Liu[5]. Sufficient conditions for $\underline{f}$, defined by eqn. (3.64), to be a globally regular function may be stated as follows.

Theorem 3.7

If the functions $\underline{g}$ and $\underline{h}$ satisfy conditions C1 and C2 or they satisfy conditions C1 and C3, then $\underline{f}$ defined by eqn. (3.64) is a globally regular function.

(C1) $\underline{g}$ and $\underline{h}$ are of class $C^{(1)}$ and the Jacobian matrices $G(\underline{x}_1) = \partial \underline{g}/\partial \underline{x}_1$ and $H(\underline{x}_2) = \partial \underline{h}/\partial \underline{x}_2$ are positive semidefinite and bounded for all $\underline{x}_1$ and $\underline{x}_2$ respectively.

(C2) Either $G(\underline{x}_1)$ is a positive definite matrix (not necessarily symmetric) for all $\underline{x}_1$ or $H(\underline{x}_2)$ is a positive definite matrix (not necessarily symmetric) for all $\underline{x}_2$.

(C3) Either $G(\underline{x}_1)$ is symmetric for all $\underline{x}_1$ or $H(\underline{x}_2)$ is symmetric for all $\underline{x}_2$.

Proof

The Jacobian matrix of $\underline{f}$ (eqn. (3.64))

$$\partial \underline{f} / \partial \underline{x} = \begin{bmatrix} I & FH(\underline{x}_2) \\ -F^T G(\underline{x}_1) & I \end{bmatrix} \tag{3.65}$$

is bounded as G and H are bounded. Applying Theorem 3.2 it is necessary to show that the inequality (3.3) is fulfilled.

Using Lemma 15 of Ref. 23

$$\begin{aligned} \det \ \partial \underline{f} / \partial \underline{x} &= \det \left[I + FH(\underline{x}_2) F^T G(\underline{x}_1) \right] = \\ &= \det \left[I + F^T G(\underline{x}_1) FH(\underline{x}_2) \right] \end{aligned} \tag{3.66}$$

Since the matrices $FH(\underline{x}_2)\ F^T$ and $F^T G(\underline{x}_1) F$ are positive semidefinite, a direct application of Lemma 3.7 to eqn. (3.66) shows that $\det \left[\partial \underline{f} / \partial \underline{x} \right] \geq 1$ if the conditions (C1) and (C2) or (C1) and (C3) are satisfied.

Q. E. D.

Note. The condition (C1) only is not sufficient for $\underline{f}$ (eqn. (3.64)) to be globally regular. When both G and H are positive semidefinite and nonsymmetric, det $\partial \underline{f} / \partial \underline{x}$ is not necessarily different from zero. This may be demonstrated by the following example. Let

$$G(\underline{x}_1) = \begin{bmatrix} \frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix}, \quad H(\underline{x}_2) = \begin{bmatrix} 0 & -1 \\ 1 & \frac{1}{2} \end{bmatrix}, \quad F = I.$$

Then $\det \partial \underline{f}/\partial \underline{x} = \det \left[\, I + GH \,\right] = 0$

Varayia and Liu[5] gave sufficient conditions for $\underline{f}$, defined by eqn. (3.64), to be globally regular. The conditions on $G(\underline{x}_1)$ and $H(\underline{x}_2)$ were more stringent than C2 and C3 but boundedness of G and H was not required. However, their proof is not very convincing*.

The second function that has received considerable attention[24-26] is of the form

$$\underline{y} = \underline{f}(\underline{x}) = A\underline{x} + \underline{g}(\underline{x}) \tag{3.67}$$

where A is a constant n x n matrix, $\underline{g}: R^n \rightarrow R^n$ and the Jacobian matrix $\partial \underline{g}/\partial \underline{x}$ is diagonal for all $\underline{x}$. Eqn. (3.67) describes, say, a linear resistive n-port with conductance matrix A where two-terminal resistive voltage controlled elements are connected in parallel to each port. Eqn. (3.67) plays a central role in the dc analysis of transistor networks[25] when the hybrid description of the transistor has the form

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha_{12} \\ -\alpha_{21} & 1 \end{bmatrix} \begin{bmatrix} g_1(v_1) \\ g_2(v_2) \end{bmatrix} \tag{3.68}$$

---

* At one point they state in the proof of Theorem 1.1 (in our notation): $\det \partial \underline{f}/\partial \underline{x} \geq 1$ implies that $\| \underline{f}(\underline{x}) - \underline{y} \| \rightarrow \infty$ as $\| \underline{x} \| \rightarrow \infty$. The counterexample given in Section 3.1.1 shows that this conjecture is generally not correct. However, it was not shown in Ref. 5, that this is correct in the specific case of function eqn. (3.64).

In eqn. (3.68) i_1 and i_2 are the port currents, v_1 and v_2 are the port voltages; it is assumed, as is the case for the usual large signal model of a physical transistor, that $0 < \alpha_{12} < 1$, $0 < \alpha_{21} < 1$, and that both of the functions $\underline{g}_1$ and $\underline{g}_2$ are continuous and monotonic. The equivalent circuit of a transistor described by the hybrid description (3.68) is shown in Fig. 3.1. The conditions for global regularity of the function $\underline{y} = A\underline{x} + \underline{g}(\underline{x})$ are given in the following theorem.

Theorem 3.8

If $\underline{f}(\underline{x})$ defined by eqn. (3.67) satisfies condition (C1) or (C2), then $\underline{f}$ is globally regular.

(C1) A is a matrix of class P and $\partial \underline{g}/\partial \underline{x}$ is continuous, bounded and diagonal positive semidefinite matrix.

(C2) A is a matrix of class P_o and $\partial \underline{g}/\partial \underline{x}$ is continuous, bounded and diagonal u. p. d. matrix.

Proof

From eqn. (3.67)

$$\det \partial \underline{f}/\partial \underline{x} = \det \left[A + \partial \underline{g}/\partial \underline{x} \right] \qquad (3.69)$$

Suppose that condition (C1) is satisfied. Since $A \in P$, $\partial \underline{g}/\partial \underline{x}$ is diagonal positive semidefinite, the diagonal expansion[18] of $\det \partial \underline{f}/\partial \underline{x}$ yields finally: $\det \partial \underline{f}/\partial \underline{x} \geq \det A > 0$. Hence by Theorem 3.2, $\underline{f}$ is globally regular.

Let condition (C2) be fulfilled. Since $A \in P_o$, $\partial \underline{g}/\partial \underline{x}$ is diagonal and u. p. d., i. e. $\left[\partial \underline{g}/\partial \underline{x} - \mu I\right]$ is positive definite, it follows that

$$\det \partial \underline{f}/\partial \underline{x} \geq \mu^n > 0$$

and $\underline{f}$ is globally regular.

Q. E. D.

In Ref. 25 the uniqueness of solution of eqn. (3.67) was considered under less stringent conditions where the boundedness of $\partial \underline{g}/\partial \underline{x}$ was not required. However, Theorem 3 in Ref. 25 does not guarantee that $\underline{f}$ is a globally regular function.

The classes of functions, treated in Section 3.2, embrace the most important functions occuring in dc analysis of nonlinear networks for which the existence and uniqueness of solution can be guaranteed. There is another point that is worth mentioning. Global regularity of different classes of functions, that were treated before each by a different method, has been established in a rather simple and unified manner, employing both conditions of Theorem 3.2.

3.3 SIMULATION OF ALGEBRAIC EQUATIONS BY DIFFERENTIAL EQUATIONS

Once it is established that a given function $\underline{y} = \underline{f}(\underline{x})$ is globally regular a question arises how to calculate its inverse. It is not our purpose to consider the problem of finding algorithms to compute inverses of globally regular functions in too much detail. The aims of this section will be to show that many existing algorithms are based on or may be deduced from an appropriate differential equation

$$\underline{\dot{x}} = \underline{F}(\underline{x}, \underline{y}) \tag{3.70}$$

associated with a given function $\underline{y} = \underline{f}(\underline{x})$. Differential eqn. (3.70) has the property that for a given $\underline{y}$ its singular point $\underline{x} = \underline{x}_o$ is the solution of the equation $\underline{y} = \underline{f}(\underline{x})$ and hence the problem of computing the inverse $\underline{f}^{-1}(\underline{y})$ is transformed to the problem of finding a singular point of eqn. (3.70). Even when $\underline{F}(\underline{x}, \underline{y}) = \underline{0}$ admits one solution only for any $\underline{y} \in R^n$, there is no guarantee that travelling along the trajectory defined by eqn. (3.70) the singular point will be reached. Namely, the singular point $\underline{x}_o$ may be unstable or differential eqn. (3.70) may have a limit cycle. It is therefore important to search for functions $\underline{F}(\underline{x}, \underline{y})$, associated with a given function $\underline{y} = \underline{f}(\underline{x})$, such that differential eqn. (3.70) represents a globally asymptotically stable[3,5] differential equation. The necessary criterion for global aymptotic stability of a given differential equation is given in Ref. 27 (see also Theorem 8.5 of Ref. 3).

There are two globally asymptotically stable differential equations that can be associated with any globally regular function $\underline{y} = \underline{f}(\underline{x})$ and they are given in the form of the next two theorems.

Theorem 3.9

Differential equation

$$\underline{\dot{x}} = -(\partial \underline{f}/\partial \underline{x})^T \left[\underline{f}(\underline{x}) - \underline{y} \right] \tag{3.71}$$

where $\underline{f}: R^n \rightarrow R^n$ is a globally regular function, is a globally asymptotically stable differential equation and for given $\underline{y}$ its singular point corresponds to the solution of equation $\underline{y} = \underline{f}(\underline{x})$.

Proof[3]

Since $\underline{f}$ is globally regular, by Palais' theorem: (i) $\|\underline{f}\| \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$ and (ii) det $\partial \underline{f}/\partial \underline{x} \neq 0$.

Choosing Liapunov function W

$$W = \left[\underline{f}(\underline{x}) - \underline{y}\right]^T \left[\underline{f}(\underline{x}) - \underline{y}\right] \qquad (3.72)$$

we have

$$\dot{W} = 2\left[\underline{f}(\underline{x}) - \underline{y}\right]^T \partial \underline{f}/\partial \underline{x} \quad \dot{\underline{x}} =$$

$$= -2\left[\underline{f}(\underline{x}) - \underline{y}\right]^T \partial \underline{f}/\partial \underline{x} \quad (\partial \underline{f}/\partial \underline{x})^T \left[\underline{f}(\underline{x}) - \underline{y}\right].$$

By condition (i) $W \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$. By condition (ii) $\dot{W}$ must be negative except at a singular point. Hence the conditions for global asymptotic stability[27] are fulfilled and all trajectories approach a singular point of the differential eqn. (3.71). But by condition (ii) a singular point must be a solution of the equation $\underline{y} = \underline{f}(\underline{x})$.

Q. E. D.

Theorem 3.10

Differential equation

$$\dot{\underline{x}} = -(\partial \underline{f}/\partial \underline{x})^{-1} \left[\underline{f}(\underline{x}) - \underline{y}\right] \qquad (3.73)$$

where $\underline{f}$: $R^n \rightarrow R^n$ is a globally regular function with bounded Jacobian matrix, is a globally asymptotically stable differential equation and for a given $\underline{y}$ its singular point corresponds to the solution of equation $\underline{y} = \underline{f}(\underline{x})$.

Proof

Let Liapunov function W be the same as in the proof of Theorem 3.9

(eqn. (3.72)). Then from eqn. (3.73).

$$\dot{W} = -2 \left[\underline{f}(\underline{x}) - \underline{y}\right]^T \left[\underline{f}(\underline{x}) - \underline{y}\right] = -2\,W.$$

$\dot{W}$ is negative everywhere except at a singular point. As in the proof of Theorem 3.9 $W \to \infty$ as $\|x\| \to \infty$. Thus all conditions for global asymptotic stability[27] are fulfilled and differential eqn. (3.73) is globally asymptotically stable. Since $\underline{f}$ is globally regular and $\partial \underline{f}/\partial \underline{x}$ is bounded it follows that $(\partial \underline{f}/\partial \underline{x})^{-1}$ is nonsingular and a singular point of eqn. (3.73) must be the solution of the equation $\underline{y} = \underline{f}(\underline{x})$.

Q. E. D

Simpler differential equations, leading to the solution of algebraic equations, may be found for specific globally regular functions. The following theorem is valid for strongly monotonic functions of class $C^{(1)}$.

Theorem 3.11

Differential equation

$$\dot{\underline{x}} = -\left[\underline{f}(\underline{x}) - \underline{y}\right] \tag{3.74}$$

where $\underline{f}: R^n \to R^n$ is a strongly monotonic function of class $C^{(1)}$, is globally asymptotically stable and for given $\underline{y}$ its singular point corresponds to the solution of equation $\underline{y} = \underline{f}(\underline{x})$.

Proof[3]

Since $\underline{f}$ is strongly monotonic, $\partial \underline{f}/\partial \underline{x}$ is u. p. d. Choosing Liapunov function $W = \left[\underline{f}(\underline{x}) - \underline{y}\right]^T \left[\underline{f}(\underline{x}) - \underline{y}\right]$ the proof is straightforward.

Q. E. D.

Application of Eulers' method to differential eqn. (3.74) gives

$$\underline{x}_{k+1} = \underline{x}_k - h\left[\underline{f}(\underline{x}_k) - \underline{y}\right] \qquad (3.75)$$

and the singular point of eqn. (3.74) which is the solution of the equation $\underline{y} = \underline{f}(\underline{x})$ is given as

$$\underline{x} = \lim_{k\to\infty} \underline{x}_k \qquad (3.76)$$

Eqn. (3.75) is identical to the algorithm[3] proposed for the computation of the inverse of a quasilinear function. Note, that the algorithm given in eqn. (3.75) is appropriate for calculation of the inverse of GQLFs. When Sandberg's Theorem 1 in Ref. 28 is applied to functions $\underline{f}$: $R^n \rightarrow R^n$ an interesting relation can be found between the iteration process, described in his theorem, and the algorithm of eqn. (3.75). It is shown in[28] that for a strongly monotonic function $\underline{y}=\underline{f}(\underline{x})$, satisfying eqn. (3.41) by definittion, and in addition possessing the property

$$\left\| \underline{f}(\underline{x}_2) - \underline{f}(\underline{x}_1) \right\|^2 \leq \varepsilon \left\| \underline{x}_2 - \underline{x}_1 \right\|^2 \qquad (3.77)$$

for any $\underline{x}_1, \underline{x}_2 \in R^n$ a unique inverse function exists and it may be calculated by the algorithm of eqn. (3.75), which is convergent for step size $h \leq \frac{\mu}{\varepsilon}$.

The inequality (3.77) represents the Lipschitz condition. It is interesting to note that GQLFs satisfy the inequality (3.77). Namely, GQLFs have bounded Jacobian matrices and thus fulfil the condition (3.4). Let $\underline{x}_1, \underline{x}_2 \in R^n$ and consider one-dimensional arc defined by eqn. (3.49). Then

$$\left| f_i(\underline{x}_2) - f_i(\underline{x}_1) \right| = \int_0^1 \left| \frac{\partial f_i}{\partial \underline{x}} \left[\underline{x}_2 - \underline{x}_1\right] \right| d\Theta \leq$$
$$\int_0^1 \left\| \frac{\partial f_i}{\partial \underline{x}}^T \right\| \left\| \underline{x}_2 - \underline{x}_1 \right\| d\Theta$$

Applying the condition (3.4)

$$\left| f_i(\underline{x}_2) - f_i(\underline{x}_1) \right| \leq M n^{1/2} \left\| \underline{x}_2 - \underline{x}_1 \right\|$$

and finally

$$\left\| \underline{f}(\underline{x}_2) - \underline{f}(\underline{x}_1) \right\| \leq M n \left\| \underline{x}_2 - \underline{x}_1 \right\| \tag{3.78}$$

Comparing eqns. (3.77) and (3.78) gives $\epsilon = Mn$. Thus the algorithm of eqn. (3.75) is convergent for a GQLF when $h \leq \mu/Mn$.

Generally different methods of numerical integration are expected to be convergent when (i) a differential equation associated with a given algebraic equation is globally asymptotically stable, (ii) it satisfies the Lipschitz condition, which guarantees the uniqueness of solution and (iii) step size h is small enough. Thus, globally asymptotically stable differential equations may be used to generate a large number of algorithms just by choosing different methods of numerical integration. Further discussion about this problem would exceed the scope of this thesis.

There is another point that is worth mentioning. When the Euler method is applied to differential eqn. (3.73) we have

$$\underline{x}_{k+1} = \underline{x}_k - h\left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]^{-1}_{\underline{x} = \underline{x}_k} \left[\underline{f}(\underline{x}_k) - \underline{y} \right] \tag{3.79}$$

It is interesting to observe that by setting $h = 1$, eqn. (3.79) represents the Newton-Raphson formula for computing the solution of algebraic equation $\underline{y}=\underline{f}(\underline{x})$. The step size in the Newton-Raphson method can be too large for some globally regular functions; thus, the iteration process will not always be convergent. However, the reduction of the step size may result in a convergent procedure.

3.4 SUMMARY

This chapter presents the necessary mathematical background of the thesis. In order to study the properties of one-element-kind and RLC networks the problem of functional inversion of vector-valued functions has been treated. Theorem 3.2, giving a new criterion for global regularity of functions, has been derived. The globally regular property of certain classes of vector-valued functions, corresponding either to hybrid descriptions of network elements or to dc equations of nonlinear networks, has been established on the basis of Theorem 3.2 rather than using a different approach for each separate class as was done previously. Hence, it has been demonstrated that this theorem has useful applications in the field of nonlinear networks. However, being very general, it may succesfully be used in solving other problems where the existence and uniqueness of solutions for a set of algebraic equations is in question.

The question of the existence of "partial" inverses of vector-valued functions, related to the transformations of one hybrid descriptions to another, has been examined. The conditions ensuring the existence of a unique solution of a set of implicit equations, which may correspond to the governing equation of an one-element-kind network, have been given in Theorem 3.4. Some other results, related to Theorem 3.2 and useful in the subsequent treatment, have been stated.

Finally, it has been demonstated that certain existing algorithms for computing the inverse of a globally regular function can be considered as a numerical method for solving a differential equation which is associated with a given function; the question of convergence of an algorithm for computing the inverse is related to the properties of such a differential equation. A globally asymptotically stable differential equation has been found where the Euler method gives the Newton-Raphson formula.

The results of Chapter 3 will be used throughout the rest of this thesis. In order to study the properties of RLC networks containing locally active elements, the concept of positive network elements will be introduced in the next chapter; properties of positive network elements and certain interesting subclasses of positive network elements will be examined.

3.5 REFERENCES

1 Desoer, C. A. and Katzenelson, J., "Nonlinear RLC networks", Bell Sys. Tech. J., vol. 44, pp. 161-198, January 1965.

2 Stern, T. E., "On the equations of nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 74-81, March 1966.

3 Stern, T. E., "Theory of nonlinear networks and systems", Reading, Mass., Addison-Wesley, 1965.

4 Holzmann, C. A. and Liu, R., "On nonlinear n-coupled networks", Technical Report No. EE-663, Department of Electrical Engineering, University of Notre Dame, Ind., April 1966.

5 Varayia, P. P. and Liu, R., "Normal form and stability of a class of coupled nonlinear networks", IEEE Trans. on Circuit Theory, CT-13, pp. 413-418, December 1966.

6 Rudin, W., "Principles of mathematical analysis", McGraw-Hill, New York, 1964, p. 193.

7 Fleming. W. H., "Functions of several variables", Reading, Mass., Addison-Wesley, 1965, p. 115.

8 Palais, R. S., "Natural operations on differential forms", Trans. Amer. Math. Soc., vol. 92, pp. 125-141, 1959.

9 Vehovec, M., "Simple criterion for the global regularity of vector-valued functions", Electronics Letters, vol.5, No. 26, pp. 680-681, December 1969.

10 Vidav, I., "Diffeomorphisms of Euclidean spaces", to appear in Glasnik Matematički.

11 Ref. 7, p. 125.

12 Belmann, R., "Introduction to matrix analysis", McGraw-Hill, New York, 1960, p. 110.

13 Ref. 3, p. 558.

14 Ref. 3, pp. 570-571.

15 Ref. 6. p. 195.

16 Minty, G. J., "Two theorems on nonlinear functional equations in Hilbert space", Bull. Amer. Math. Soc., vol. 69, No. 5, pp. 691-692, 1963.

17 Ohtsuki, T. and Watanabe, H., "State-variable analysis of RLC networks containing nonlinear coupling elements", IEEE Trans. on Circuit Theory, vol. CT-16, pp. 26-38, February 1969.

18 Aitken, A. C., "Determinants and matrices", Oliver and Boyd, Edinburgh, 1962, p. 99.

19 Vehovec, M., "Comment on State-variable analysis of RLC networks containing nonlinear coupling elements", to be published in IEEE Trans. on Circuit Theory.

20 Schneider, H. (Editor), "Recent advances in matrix theory", Proc. of an Advanced Seminar conducted by the Mathematics Research Center, Madison and Milwakee, 1964, p. 138.

21 Perlis, S., "Theory of matrices", Reading, Mass., Addison-Wesley, 1952, p. 186.

22 Fiedler, M. and Pták, V., "Some generalizations of positive definiteness and monotonicity", Numer. Math., vol. 9, No. 2, pp. 163-172, 1966.

23 Bryant, P. R., "The order of complexity of electrical networks", Proc. IEE, Mono. No. 335, vol 106, Pt. C, pp. 174-188, September 1959.

24 Willson, Jr., A. N., "On the solutions of equations for nonlinear resistive networks", Bell Sys. Tech. J., vol. 47, pp. 1755-1773, August 1968.

25 Sandberg, I. W. and Willson, Jr., A.N., "Some theorems on properties of dc equations of nonlinear networks", Bell Sys. Tech. J., vol. 48, pp. 1034, January 1969.

26 Sandberg, I. W., "Theorems on the analysis of nonlinear transistor networks", Bell Sys. Techn. J., vol. 49, pp. 95-114, January 1970.

27 LaSalle, J. and Lefschetz, S., "Stability by Liapunov's direct method with applications", New York, Academic Press, 1961, pp. 56-67.

28 Sandberg, I. W., "On the properties of systems that distort signals - I", Bell Sys. Tech. J., vol. 42, pp. 2033-2046, May 1963.

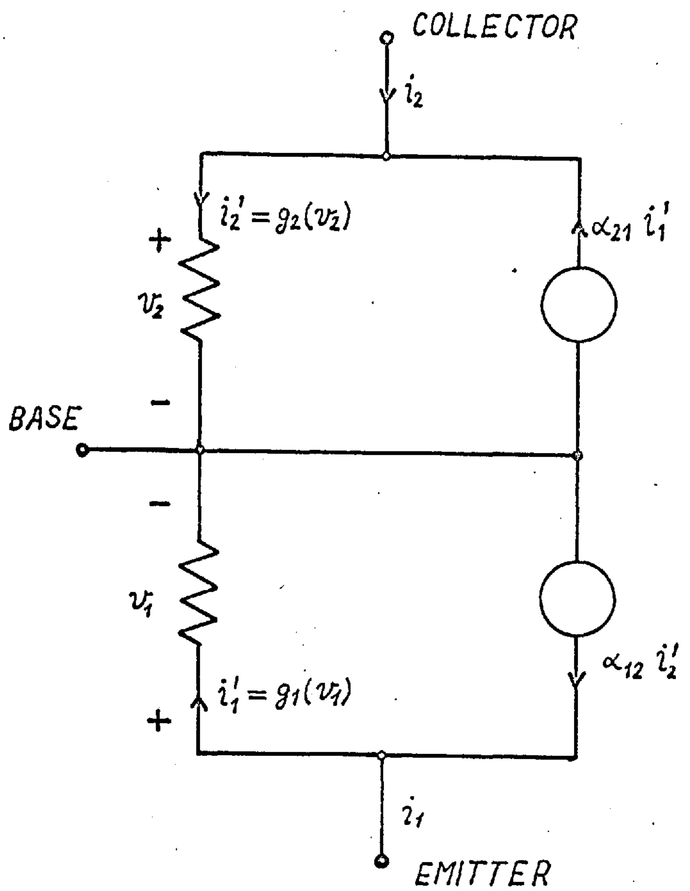

Fig. 3.1. The equivalent circuit of a transistor.

Chapter 4

POSITIVE NETWORK ELEMENTS

When, say, resistive networks are analysed it is significant that the resulting equations may be written in one of the forms, described in Section 2.3, where the number of equations is much smaller than for the basic set of 2 $(b_\beta + b_\varepsilon)$ equations. Suppose, as an example, that loop analysis is desirable. If a resistive element N_R in a network is described by a hybrid description $\underline{h}_R(\underline{x})$ where the independent variable is not the current vector, it is necessary to transform a given hybrid description $\underline{h}_R(\underline{x})$ into another hybrid description $\underline{h}_R(\underline{i})$ with the current vector as the independent variable; this can be done only if the resistive element is current-controlled. When N_R is nonlinear, in general $\underline{h}_R(\underline{i})$ cannot be expressed explicitly in terms of $\underline{h}_R(\underline{x})$, but it is still important to know whether $\underline{h}_R(\underline{i})$ exists or not if for example, numerical method is used to calculate $\underline{h}_R(\underline{i})$ for a given value of $\underline{x}$. Since $\underline{h}_R(\underline{i})$ is generally a "partial" inverse of $\underline{h}_R(\underline{x})$, criteria given in Section 3.1.2 may be applied to establish the existence of a unique $\underline{h}_R(\underline{i})$ in every particular case. Nevertheless, it may happen that for a given network it is impossible to perform either loop or nodal or mixed analysis. It is therefore useful to delineate a class of network elements that possess all hybrid descriptions and thus either of the three analyses may be carried out for a network consisting solely of these elements. A set of network elements possessing the above mentioned property will be termed as positive network elements[1].

As will be shown later, positive network elements are a very general class of network elements and they embrace many locally active as well as locally passive network elements. Many practical nonlinear devices such as transistors, vacuum tubes and some other locally active devices may be modelled as positive network elements. There is another significant point that is worth mentioning; by introducing positive network elements we shall be able to analyse a large class of one-element-kind and RLC networks containing locally active elements.

In this chapter positive network elements will be defined and their properties will be studied and then some important subclasses of positive network elements, having certain special properties, will be discussed. Series-parallel interconnections of positive network elements and some other network elements together with the existence of a unique solution of one-element-kind networks, containing positive network elements, will be investigated in the next chapter.

4.1 POSITIVE NETWORK ELEMENTS AND THEIR PROPERTIES[1]

Before stating a formal definition of a positive network element we need the following definition, representing a generalization of matrices of class P. (See Definition 3.6).

Definition 4.1

Let $A(\underline{x})$ be a real $n \times n$ matrix, depending upon the vector variable $\underline{x} \in R^k$. Denote an arbitrary principal minor of $A(\underline{x})$ by $\det A(\underline{x})_{(p)(p)}$ where (p) denotes a set of m rows and columns not deleted from $A(\underline{x})$ and $m = 1,2,\ldots, n$. Then, $A(\underline{x})$ belongs to the class of matrices denoted by UP if there exists an $\varepsilon > 0$, independent of $\underline{x}$, such that for all principal minors of $A(\underline{x})$

$$\det A(\underline{x})_{(p)(p)} \geq \varepsilon^m > 0 \qquad \text{for all } \underline{x} \in R^k \tag{4.1}$$

Among other matrices the class of u.p.d. matrices belongs to the class of UP matrices. Since any principal submatrix of a u.p.d. matrix is itself u.p.d., then by Lemma 3.4 for any principal minor of order m, $m = 1,2 \ldots, n$

$$\det \ A(x)_{(p)(p)} > \mu^m > 0 \tag{4.2}$$

and eqn. (4.1) is satisfied. A similar reasoning shows that u. H. matrices are a subclass of UP matrices.

Definition 4.2

A resistive (or capacitive or inductive) n-port (or (n+1)-terminal) network element, possessing a hybrid description

$$\underline{y} = \underline{h}(\underline{x})$$

is defined to be positive network element (PNE) if

(i) the hybrid matrix $H = \partial \underline{h} / \partial \underline{x}$ is bounded for all $\underline{x} \in R^n$ and

(ii) H belongs to the class of UP matrices.

Note that in the definition of a PNE there is no requirement for local passivity since matrices of class UP are not necessarily positive semidefinite. The boundedness of the hybrid matrix H is not a severe restriction; namely, when $\underline{h}(\underline{x})$ is a hybrid description, it is of class $C^{(1)}$ and H is bounded everywhere except possibly for $\|\underline{x}\| \to \infty$. Thus the condition (i) in the definition of a PNE is concerned only with the behaviour of H at $\|\underline{x}\| \to \infty$, where the characterization of a network element has no physical significance. Therefore many practical network elements may be modelled as PNEs as will be demonstrated in Section 4.2.

PNEs have many interesting properties that are listed below.

Property 1

All possible hybrid descriptions of an n-port (or (n+1)-terminal) PNE exist.

Proof: Property 1 follows from the definition of a PNE and Corollary 3.2.

Q. E. D.

The following two lemmas will be useful in the proof of Property 2.

Lemma 4.1[2]

Suppose that A is an (n+m) x (n+m) matrix

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

where P is n x n and nonsingular, Q is n x m, R is m x n and S is m x m and nonsingular. Then

$$\det A = \det\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \det P \det\left[S - RP^{-1}Q\right] = $$
$$= \det S \det\left[P - QS^{-1}R\right] \qquad (4.3)$$

Proof: Since P and S are nonsingular A can be rewritten in the form

$$A = \begin{bmatrix} P & 0 \\ 0 & I_{mm} \end{bmatrix} \begin{bmatrix} I_{nn} & P^{-1}QS^{-1} \\ R & I_{mm} \end{bmatrix} \begin{bmatrix} I_{nn} & 0 \\ 0 & S \end{bmatrix}. \qquad (4.4)$$

Applying Lemma 15 of Ref. 3 to eqn. (4.4) yields

$$\det A = \det P \det\left[I_{nn} - P^{-1}QS^{-1}R\right] \det S =$$
$$= \det P \det\left[I_{mm} - RP^{-1}QS^{-1}\right] \det S =$$
$$= \det\left[P - QS^{-1}R\right] \det S = \det P \det\left[S - RP^{-1}Q\right] \qquad (4.5)$$

Q. E. D.

Lemma 4.2

Suppose that A is an $(n+m) \times (n+m)$ matrix

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

where P is $n \times n$ and nonsingular, Q is $n \times m$, R is $m \times n$ and S is $m \times m$ and nonsingular. Then

$$A^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}^{-1} = \begin{bmatrix} (P - QS^{-1}R)^{-1} & -(P-QS^{-1}R)^{-1}QS^{-1} \\ -(S-RP^{-1}Q)^{-1}RP^{-1} & (S-RP^{-1}Q)^{-1} \end{bmatrix} \qquad (4.6)$$

where A^{-1} is partitioned conformably to A.

Proof: Lemma 4.2 may be proved by carrying out multiplication $A^{-1}A$.

Property 2

Any of the 2^n different hybrid matrices of an n-port (or (n+1)-terminal) PNE is UP and bounded.

Proof: Suppose that $\underline{y} = h(\underline{x})$ is a given description of a PNE. Then $\underline{h}^*(\cdot) = \begin{bmatrix} \underline{h}_1^*(\cdot) \\ \underline{h}_2^*(\cdot) \end{bmatrix}$ defined as

$$\begin{bmatrix} \underline{x}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} \underline{h}_1^*(\underline{y}_1, \underline{x}_2) \\ \underline{h}_2^*(\underline{y}_1, \underline{x}_2) \end{bmatrix} \tag{4.7}$$

may represent an arbitrary hybrid description if some rows of $\underline{x}$ and $\underline{y}$ are at first interchanged conformably and then the necessary partitioning is performed. In eqn. (4.7) $\underline{x}_1$, $\underline{y}_1$ are m-vectors and $\underline{x}_2$, $\underline{y}_2$ are r-vectors where $r = n-m$. Denote by H and H* the hybrid matrices associated with $\underline{h}(\cdot)$ and $\underline{h}^*(\cdot)$ respectively. H and H* can be partitioned conformably in the following manner.

$$H = \left[\begin{array}{c|c} \partial \underline{h}_1/\partial \underline{x}_1 & \partial \underline{h}_1/\partial \underline{x}_2 \\ \hline \partial \underline{h}_2/\partial \underline{x}_1 & \partial \underline{h}_2/\partial \underline{x}_2 \end{array}\right] = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{4.8}$$

$$H^* = \left[\begin{array}{c|c} \partial \underline{h}_1^*/\partial \underline{y}_1 & \partial \underline{h}_1^*/\partial \underline{x}_2 \\ \hline \partial \underline{h}_2^*/\partial \underline{y}_1 & \partial \underline{h}^*{}_2/\partial \underline{x}_2 \end{array}\right] = \begin{bmatrix} H_{11}^* & H_{12}^* \\ H_{21}^* & H_{22}^* \end{bmatrix} \tag{4.9}$$

Then Property 2 implies that

$$M_o \geq \det H^*_{(p)(p)} \geq \varepsilon_o > 0 \tag{4.10}$$

where p is an arbitrary subset of s rows and columns and $s = 1, 2 \ldots n$. It can be shown that any principal minor in H^* is expressible as a ratio of two principal minors in H. Using the relation of eqn. (3.27), H^* can be expressed as

$$H^* = \begin{bmatrix} H_{11}^{-1} & -H_{11}^{-1}H_{12} \\ H_{21}H_{11}^{-1} & H_{22} - H_{21}H_{11}^{-1}H_{12} \end{bmatrix} \tag{4.11}$$

H^* may be recast further in the form

$$H^* = \begin{bmatrix} I_{nn} & 0 \\ H_{21} & I_{rr} \end{bmatrix} \begin{bmatrix} H_{11}^{-1} & 0 \\ 0 & H_{22} \end{bmatrix} \begin{bmatrix} I_{nn} & -H_{12} \\ 0 & I_{rr} \end{bmatrix} \tag{4.12}$$

Let the following notation be introduced. Divide the set of the first m rows (and columns) of H and H^* (eqns. (4.8) and (4.9)) into two disjoint subsets S_a and S_b where S_a contains i rows and S_b contains j rows, $i + j = m$; similarly let the set of the remaining $r = n-m$ rows (and columns) of the same matrices be divided into two disjoint subsets S_c and S_d where S_c contains k rows, S_d contains l rows where $k + l = r$. An arbitrary principal minor of H can be written as

$$\det H^*_{(a,c)(a,c)}$$

where $H^*_{(a,c)(a,c)}$ is a principal submatrix obtained from H^* by deleting the rows corresponding to S_b and S_d. Since the rows and columns of $\det H^*_{(a,c)(a,c)}$ may be interchanged, there is no loss of generality when taking S_a as the first i rows of H^*, S_b as the next j rows of H^*, then the next k rows of H^* as S_c and the last l rows as S_d. Thus

$$H^* = \left[\begin{array}{c|c} H^*_{11} & H^*_{12} \\ \hline H^*_{21} & H^*_{22} \end{array}\right] = \left[\begin{array}{cc|cc} H^*_{aa} & H^*_{ab} & H^*_{ac} & H^*_{ad} \\ H^*_{ba} & H^*_{bb} & H^*_{bc} & H^*_{bd} \\ \hline H^*_{ca} & H^*_{cb} & H^*_{cc} & H^*_{cd} \\ H^*_{da} & H^*_{db} & H^*_{dc} & H^*_{dd} \end{array}\right]. \tag{4.13}$$

and when H is partitioned conformably

$$H = \left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] = \left[\begin{array}{cc|cc} H_{aa} & H_{ab} & H_{ac} & H_{ad} \\ H_{ba} & H_{bb} & H_{bc} & H_{bd} \\ \hline H_{ca} & H_{cb} & H_{cc} & H_{cd} \\ H_{da} & H_{db} & H_{dc} & H_{dd} \end{array}\right] \tag{4.14}$$

Denote

$$H^*_{11} = H_{11}^{-1} = K = \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \tag{4.15}$$

where K is partitioned conformably to H_{11}. Using eqn. (4.12) the principal submatrix $H^*_{(a,c)(a,c)}$ is

$$H^*_{(a,c)(a,c)} = \left[\begin{array}{cc|cc} I_{aa} & 0 & 0 & 0 \\ H_{ca} & H_{cb} & I_{cc} & 0 \end{array}\right] \left[\begin{array}{cc|cc} K_{aa} & K_{ab} & 0 & 0 \\ K_{ba} & K_{bb} & 0 & 0 \\ \hline 0 & 0 & H_{cc} & H_{cd} \\ 0 & 0 & H_{dc} & H_{dd} \end{array}\right] \left[\begin{array}{cc} I_{aa} & -H_{ac} \\ 0 & -H_{bc} \\ \hline 0 & I_{cc} \\ 0 & 0 \end{array}\right] \tag{4.16}$$

After the multiplication of the r. h. s. of eqn. (4.16) we get

$$H^*_{(a,c)(a,c)} = \left[\begin{array}{c|c} K_{aa} & -\begin{bmatrix} K_{aa} & K_{ab} \end{bmatrix} \begin{bmatrix} H_{ac} \\ H_{bc} \end{bmatrix} \\ \hline \begin{bmatrix} H_{ca} & H_{cb} \end{bmatrix} \begin{bmatrix} K_{aa} \\ K_{ba} \end{bmatrix} & H_{cc} - \begin{bmatrix} H_{ca} & H_{cb} \end{bmatrix} K \begin{bmatrix} H_{ac} \\ H_{bc} \end{bmatrix} \end{array}\right] \tag{4.17}$$

By Lemma 4.1

$$\det H^*_{(a,c)(a,c)} = \det K_{aa} \det \left[H_{cc} - H_{cb}(K_{bb} - K_{ba}K_{aa}^{-1}K_{ab})H_{bc} \right] \tag{4.18}$$

Applying Lemma 4.2 to eqn. (4.15)

$$K^{-1} = H_{11} = \begin{bmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{bmatrix} =$$

$$= \left[\begin{array}{c:c} (K_{aa}-K_{ab}K_{bb}^{-1}K_{ba})^{-1} & -(K_{aa}-K_{ab}K_{bb}^{-1}K_{ba})^{-1}K_{ab}K_{bb}^{-1} \\ \hdashline -(K_{bb}-K_{ba}K_{aa}^{-1}K_{ab})^{-1}K_{ba}K_{aa}^{-1} & (K_{bb}-K_{ba}K_{aa}^{-1}K_{ab})^{-1} \end{array}\right] \tag{4.19}$$

and from eqn. (4.19)

$$(K_{bb} - K_{ba}K_{aa}^{-1}K_{ab}) = H_{bb}^{-1} \tag{4.20}$$

Substituting eqn. (4.20) in eqn. (4.18)

$$\det H^*_{(a,c)(a,c)} = \det K_{aa} \det \left[H_{cc} - H_{cb}H_{bb}^{-1}H_{bc} \right] \tag{4.21}$$

By Lemma 4.1

$$\det \left[H_{cc} - H_{cb}H_{bb}^{-1}H_{bc} \right] = \frac{1}{\det H_{bb}} \det \begin{bmatrix} H_{bb} & H_{bc} \\ H_{cb} & H_{cc} \end{bmatrix} = \frac{\det H_{(b,c)(b,c)}}{\det \ H_{bb}} \tag{4.22}$$

Applying Lemma 4.1 to eqn. (4.15)

$$\det H_{11} = \det K_{aa} \det\left[K_{bb} - K_{ba} K_{aa}^{-1} K_{ab} \right] \qquad (4.23)$$

and from eqn. (4.23) and using eqn. (4.20)

$$\det K_{aa} = \frac{\det H_{bb}}{\det H_{11}} \qquad (4.24)$$

Eqn. (4.24) is equivalent to the result on minors of the inverse matrix[4]. Substituting eqns. (4.22) and (4.24) into eqn. (4.21) gives finally

$$\det H^*_{(a,c)(a,c)} = \frac{\det H_{(b,c)(b,c)}}{\det H_{11}} \qquad (4.25)$$

Since $H_{(b,c)(b,c)}$ and H_{11} are principal submatrices of a bounded UP matrix H

$$M_1 \geq \det H_{11} \geq \varepsilon^m > 0 \qquad (4.26)$$

$$M_2 \geq \det H_{(b,c)(b,c)} \geq \varepsilon^{j+k} > 0 \qquad (4.27)$$

and from eqns. (4.25), (4.26) and (4.27)

$$\frac{M_2}{\varepsilon^m} \geq \det H^*_{(a,c)(a,c)} \geq \frac{\varepsilon^{j+k}}{M_1} > 0 \qquad (4.28)$$

Therefore eqn. (4.28) satisfies the condition (4.10). Since subsets S_a and S_c are arbitrary and H* may represent arbitrary hybrid matrix of a PNE, it follows from eqn. (4.28) that all hybrid matrices of a PNE are UP and bounded.

Q. E. D.

Since eqn. (4.28) is valid for principal minors of order one, all diagonal entries of any arbitrary hybrid matrix of a PNE are positive and bounded below and above. Hence all incremental driving point conductances (or capacitances or inductances) of a positive resistive (or capacitive or inductive) element are positive.

Property 3 which is a consequence of Property 2 will be stated for resistive elements, although a similar property holds for capacitive and inductive elements.

Property 3

Choose an arbitrary set of m ports of an n-port positive resistive element, N_n, and divide the set of remaining (n-m) ports into two disjoint subsets P_E and P_J. Connect constant voltage sources to ports P_E and constant current sources to ports P_J. Let the chosen ports of N_n define a new m-port N_m. The m-port N_m, defined in such a manner, is then a positive resistive element.

A special case of N_m is the case when m = 1 and the network N_1 is a one-port positive resistive element, i. e. the relation between v_i and i_i, the voltage and current of port P_i, is a quasilinear function for arbitrary values of constant sources at all other (n-1) ports.

Property 4

Any reciprocal PNE is strongly locally passive.

Proof: For any reciprocal* PNE the hybrid matrix H_n corresponding to the one of of the two "nonmixed branch relationships"[5] is

---

* Reciprocal PNEs form a subclass of positive definite network elements defined in Section 4.3.2

symmetric**. In order to show strong local passivity it is necessary to prove that H_n is u.p.d.

Since H_n is a symmetric UP matrix, all of its principal minors are positive and thus H_n is positive definite. From the UP property

$$\det H_n = \prod_{i=1}^{n} \lambda_i > \varepsilon^n > 0 \tag{4.29}$$

where λ_i, $i = 1,2,\ldots,n$, are eigenvalues of H_n and all λ_i are real and positive. The boundedness of H_n implies that any entry of H_n $|(h_n)_{i,j}| \leq M$ for all i and j and hence by Gresgorin's theorem[5] λ_{max}, the largest eigenvalue of H_n,

$$\lambda_{max} \leq nM \tag{4.30}$$

From eqns. (4.29) and (4.30) λ_{min}, the smallest eigenvalue of H_n

$$\lambda_{min} > \frac{\varepsilon^n}{(nM)^{n-1}} > 0 \tag{4.31}$$

Using the result of Ref. 6

$$\underline{z}^T H_n \underline{z} > \lambda_{min} \underline{z}^T \underline{z} > 0, \ \underline{z} \neq \underline{0} \tag{4.32}$$

From eqn. (4.32)

$$\underline{z}^T (H_n - \lambda_{min}) \underline{z} > 0$$

and H_n is a u. p. d. matrix.

Q. E. D.

** For, say a resistive element these two "nonmixed branch relationships" are given by eqns. (2.5c) and (2.5d)

An important conclusion may be obtained from Property 4. Namely, any reciprocal PNE is strongly locally passive and thus a necessary condition for a PNE to be locally active is that it is nonreciprocal. However, nonreciprocity is not sufficient for local activity and among nonreciprocal PNEs there are network elements where the symmetric part of the hybrid matrix is u. p. d.; these PNEs, as well as all reciprocal PNEs, are strongly locally passive and will be treated in Section 4.3.2.

Property 5 will be stated for resistive elements, although analogous property may be proved for capacitive and inductive elements.

Property 5

Let an n-port (or (n+1)-terminal) positive resistive element N_R with the hybrid description

$$\underline{i} = \underline{h}_R(\underline{v}) \tag{4.33}$$

be locally active at some point $\underline{v} = \underline{v}_o$. Then a set of linear (or nonlinear) gyrators may be found such that not all hybrid descriptions of the composite element, consisting of N_R and gyrators connected in parallel to the ports of N_R, will exist.

Proof: Denote the hybrid matrix $H = \partial \underline{h}_R / \partial \underline{v}$ at $\underline{v} = \underline{v}_o$ as H_o, the symmetric part of H_o as H_{os}, $H_{os} = \frac{1}{2}(H_o + H_o^T)$, and the skew-symmetric part of H_o as H_{oss}, $H_{oss} = \frac{1}{2}(H_o - H_o^T)$. Since N_R is locally active at $\underline{v} = \underline{v}_o$, at least one principal minor of H_{os} is negative.

Let $(H_{os})_{(p)(p)}$ be the negative principal minor of H_{os} of order m,

$$\det (H_{os})_{(p)(p)} < 0 \qquad (4.34)$$

where (p) denotes a subset of m rows and columns not deleted from H_{os} and let $\underline{i}_p$ and $\underline{v}_p$ be m-vectors, formed from $\underline{i}$ and $\underline{v}$ by deleting all components except those that correspond to the subset (p). If the set of gyrators possessing skew-symmetric incremental conductance matrix $G_1 = -H_{oss}$ is connected in parallel with N_R, then G, the incremental conductance matrix of the composite element, is

$$G = H_o + G_1 = H_{os} \qquad (4.35)$$

and $$\det G_{(p)(p)} = \det(H_{os})_{(p)(p)} < 0$$

Thus for $G_1 = -H_{oss}$, the principal minor $\det G_{(p)(p)}$ is negative, but for $G_1 = 0$, $\det G_{(p)(p)} > 0$. Since $\det G_{(p)(p)}$ is a continuous function of entries of G_1, a skew-symmetric matrix $G_1 = G_o$ exists such that

$$\det G_{(p)(p)} = \det \left[H_{(p)(p)} + (G_o)_{(p)(p)} \right] = 0 \qquad (4.36)$$

By Palais' theorem, applied to the "partial" inversion, the hybrid description with $\underline{i}_p$ as independent variable does not exist for the composite element since $G_{(p)(p)}$ is singular for $G_1 = G_o$.

Q. E. D.

In a similar manner it is possible to create negative incremental conductances by connecting a set of appropriate gyrators to an n-port positive resistive element which is locally active. For example, suppose that a positive resistive element N_R^* satisfies conditions of Property 5 and in addition $\det H_{os} < 0$ and $\det (H_{os})_{(r)(r)} > 0$, where $\det (H_{os})_{(r)(r)}$ is a principal minor of order (n-1); if a set of gyrators with the incremental conductance matrix $G_1 = -H_{oss}$ is connected in parallel to the ports of H_R^* a negative incremental conductance is obtained at some port of the composite element.

4.2 EXTRAPOLATION OF NETWORK ELEMENT CHARACTERISTICS

The concept of a PNE can be successfully applied in the analysis of practical circuits if the existing devices can be modelled reasonably well as PNEs. Since, say, resistive PNEs have the property that an arbitrary incremental driving point conductance is positive and they may be locally active, it is reasonable to expect that many practical resistive devices, such as transistor and vacuum triode, can be modelled in this manner. We shall try to establish the relation between PNEs and some important representatives of three-terminal resistive devices such as the transistor and vacuum triode. Before going int details certain useful observations, concerning the device models on one side and the analysis and design of practical circuits on the other, will be presented.

(i) Let $\underline{y} = \underline{h}(\underline{x})$ be a hybrid description determined from a chosen physical model of an n-port resistive device. Models are not valid outside some bounded domain $S_1 \subset R^n$; however, by definition a hybrid description $\underline{h}(\cdot)$ has to be defined for all $\underline{x}$.

(ii) In practice devices cannot be used outside some bounded domain $S_2 \subset R^n$; namely, for every device there is a limitation on maximal voltage and/or current or power dissipated in the device etc. These values must not be exceeded in a properly designed circuit.

(iii) Device characteristics cannot be measured outside some bounded domain $S_3 \subset R^n$, otherwise maximum permissible values for voltage, current or power are exceeded.

Regardless of these facts it is still convenient in network analysis or design to have a hybrid description of an n-port network element so that the domain of $\underline{h}(\cdot)$ is the entire R^n. Let us demonstrate the usefulness of the network element characterization in terms of a hybrid description through few examples.

(i) Assume that a resistive network is analysed where for given values of independent sources a unique solution is guaranteed but, say, a resistive element N_R in the network is not characterized for all values of independent variable. If an iteration method (e. g. Newton-Raphson) is used to solve the governing equation, $\underline{y} = \underline{f}(\underline{x})$, of the network, a proper initial value $\underline{x}_o$ has to be chosen. However, even when a proper starting value $\underline{x}_o$ is chosen, it may happen that an intermediate result falls outside the domain of the characterization of N_R and the computation of the solution cannot be continued.

(ii) Assume that it is necessary to design a resistive nework N_R where values of certain elements are adjusted in such manner that the network has the required performance. Furthermore, suppose that in the initial stage of design a chosen network N_R^* does not possess the required performance. When all network elements in N_R^* are not characterized for all values of independent variable it may easily happen that N_R^* possesses no solution for the branch variables. If we can find a solution for N_R^*, a decision can be made, at least in principle, how to alter some parameters of N_R^* in order to obtain the required performance. However, if the network N_R^* possesses no solution, no such decision is possible.

(iii) When characteristics of network elements are not specified for all values of the independent variable it is very difficult to check the existence and uniqueness of solution in a nonlinear RLC network at large for given values of independent sources. However, as mentioned before, we have to be sure that there is a unique solution of a network before its analysis is left to the computer.

We can conclude from the above discussion that it is very helpful to have a network element - representing a device model - which is characterized for all values of independent variable. Thus the idea of a hybrid description of a network element is very useful. However, it is important to point out that outside the region S_1 where the model is valid (or outside the region S_2 where the device can be used or outside the region S_3 where the device can be measured) the device can be modelled arbitrarily. Accepting this point of view it is easy to see that for PNEs the boundedness of the hybrid matrix H is not a severe restriction; since for any hybrid description the function $\underline{h}(\cdot)$ is of class $C^{(1)}$, the hybrid matrix H may become unbounded only for $\|\underline{x}\| \to \infty$, where the characterization of a network element has no physical significance.

Let us consider the frequently used Ebers-Moll static model of the transistor[7] possessing the hybrid description of the form

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \underline{h}(v_1, v_2) = \begin{bmatrix} 1 & -\alpha_{12} \\ -\alpha_{21} & 1 \end{bmatrix} \begin{bmatrix} a_1(e^{\lambda v_1} - 1) \\ a_2(e^{\lambda v_2} - 1) \end{bmatrix} \tag{4.37}$$

The above hybrid description can be obtained from eqn. (3.68) by substituting $g_1(v_1) = a_1(e^{\lambda v_1} - 1)$ and $g_2(v_2) = a_2(e^{\lambda v_2} - 1)$. The function $\underline{h}(v_1, v_2)$ satisfies the first condition of Palais' theorem, namely, det $\partial \underline{h}(v_1, v_2) / \partial(v_1, v_2) > 0$ for all $[v_1, v_2]^T$. Since $[i_1, i_2]^T$ remains bounded for $v_1 \to -\infty$, $v_2 \to -\infty$, the second condition of the same theorem is not satisfied and the function

$\underline{h}(v_1,v_2)$ is not globally regular. As det $\partial\underline{h}(v_1,v_2)/\partial(v_1,v_2) \to 0$ when $v_1 \to -\infty$, $v_2 \to -\infty$ the hybrid matrix $H = \partial\underline{h}(v_1,v_2)/\partial(v_1,v_2)$ does not belong to the class of UP matrices and in addition H is not bounded. Thus the Ebers-Moll model does not satisfy the definition of a PNE. However, the Ebers-Moll model can be transformed into a PNE[1] by performing a slight modification for large $|v_1|$ and $|v_2|$. Namely, when the exponential functions $g_1(v_1) = a_1(e^{\lambda v_1}-1)$ and $g_2(v_2) = a_2(e^{\lambda v_2}-1)$, appearing in eqn. (4.37) are extrapolated linearly outside an interval $[-N,N]$, where N is an arbitrarily large positive value, and in addition dg_1/dv_1 and dg_2/dv_2 are continuous at points $-N$ and N, $g_1(v_1)$ and $g_2(v_2)$ become quasilinear functions. The modified Ebers-Moll model then corresponds to a PNE. Since N is arbitrary, the Ebers-Moll and the modified Ebers-Moll model agree on an arbitrarily large square defined by $-N \le v_1 \le N$, $-N \le v_2 \le N$.

A frequently used static model of a vacuum triode, with grid current equal to zero and a three-halves-power law for plate current[8], is described by eqn. (4.38)

$$\begin{aligned} i_1 &= 0 \\ i_2 &= K(\mu v_1+v_2)^{3/2} \end{aligned} \qquad (4.38)$$

where K is a constant and μ is the voltage amplification factor of the triode. The inverse of the function defined by eqn. (4.38) does not exist. Thus, this model does not correspond to a PNE, but it may be transformed into a PNE in the following manner: an arbitrarily small parasitic conductance $G_{\varepsilon 1}$ is inserted between grid and cathode, another arbitrarily small parasitic conductance $G_{\varepsilon 2}$ is inserted between anode and cathode and in addition the function $(\mu v_1 + v_2)^{3/2}$

is extrapolated linearly outside the region $0 \le \mu v_1 + v_2 \le M$, where M is an arbitrarily large positive value. The hybrid description corresponding to the triode model, modified in this manner, can be expressed as

$$
\begin{aligned}
i_1 &= G_{\varepsilon 1} v_1 \\
i_2 &= G_{\varepsilon 2} v_2 && \mu v_1 + v_2 < 0 \\
i_2 &= K(\mu v_1 + v_2)^{3/2} + G_{\varepsilon 2} v_2 && 0 \le \mu v_1 + v_2 \le M \\
i_2 &= KM^{1/2}(\mu v_1 + v_2) + G_{\varepsilon 2} v_2 && \mu v_1 + v_2 > M
\end{aligned}
\tag{4.39}
$$

As $G_{\varepsilon 1}$ and $G_{\varepsilon 2}$ are arbitrarily small, the difference between the original and the modified model can be made arbitrarily small in a large region $0 \le \mu v_1 + v_2 \le M$.

In a similar manner some other n-port resistive device models that are characterized by the hybrid description $\underline{y}_R = \underline{h}_R(\underline{x}_R)$ and whose hybrid matrix H_R is of class P_o or P for all values of $\underline{x}_R$, can be transformed into PNEs by slight modifications. There are other n-port resistive elements that are characterized by a function $\underline{y}_R = \underline{h}_o(\underline{x}_R)$ but where the domain of $\underline{h}_o(\cdot)$ is a subset $S \subset R^n$ and the "hybrid matrix" H_R is of class P_o or P for all $\underline{x}_R \in S$. It is frequently possible to transform such elements into positive resistive elements by extending the domain of $\underline{h}_o(\underline{x}_R)$ to the entire R^n and then augmenting them with arbitrarily small parasitic resistances or conductances.

4.3 SPECIAL CLASSES OF POSITIVE NETWORK ELEMENTS

It will be shown that special classes of PNEs, possessing additional properties, can be defined. Using these properties it is then possible to study interconnections of different classes of PNEs and also to establish some sufficient conditions for the existence of a unique solution for one-element-kind networks.

4.3.1 (n+1)-terminal elements representing positive network elements in all orientations[9]

(n+1)-terminal elements may be characterized in terms of any of n different orientations. Let us start with the three-terminal resistive network element, shown in Fig. 4.1 and assume that a given resistive three-terminal element corresponds to a PNE when the terminal 3 is common. The choice of the common terminal is arbitrary and the orientation with the common terminal 1 or 2 might be useful in network applications. Generally, the hybrid description with the common terminal 1 or 2 will not satisfy the requirement for a PNE. A question arises; what are sufficient conditions that a three-terminal network element, representing a PNE in orientation with common terminal 3, corresponds to a PNE in the other two orientations? This problem can be easily resolved with the aid of the concept of the indefinite admittance matrix[10].

Suppose that a given three-terminal resistor is characterized in the orientation with the common terminal 3 and it is a PNE in this orientation. Then the following description exists

$$\begin{aligned} i_1 &= i_1(v_1,v_2) \\ i_2 &= i_2(v_1,v_2) \end{aligned} \tag{4.40}$$

with the incremental conductance matrix G_3

$$G_3 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \tag{4.41}$$

where the subscript 3 indicates the common terminal 3. The matrix G_3 is bounded and it belongs to the class UP. Thus an $\varepsilon > 0$ exists such that

$$g_{11} > \varepsilon \quad , \qquad g_{22} > \varepsilon \quad , \qquad \det G_3 > \varepsilon^2 \tag{4.42}$$

The indefinite conductance matrix G_{in} is obtained from G_3 and has the form

$$G_{in} = \begin{bmatrix} g_{11} & g_{12} & -(g_{11}+g_{12}) \\ g_{21} & g_{22} & -(g_{21}+g_{22}) \\ -(g_{11}+g_{21}) & -(g_{12}+g_{22}) & \Sigma g_{ik} \end{bmatrix} \tag{4.43a}$$

where $$\Sigma g_{ik} = g_{11}+g_{12}+g_{21}+g_{22} \tag{4.43b}$$

Denote the incremental conductance matrix for the orientation with the common terminal 1 by G_1 and the incremental conductance matrix for the orientation with the common terminal 2 by G_2. The matrix G_1 (or G_2) is a principal submatrix of the indefinite conductance matrix and is obtained from G_{in} by deleting the first (second) row and the first (second) column.

$$G_1 = \begin{bmatrix} g_{22} & -(g_{21}+g_{22}) \\ -(g_{12}+g_{22}) & \Sigma g_{ik} \end{bmatrix} \tag{4.44a}$$

$$G_2 = \begin{bmatrix} g_{11} & -(g_{11}+g_{12}) \\ -(g_{11}+g_{21}) & \Sigma g_{ik} \end{bmatrix} \tag{4.44b}$$

It is interesting to observe that in G_{in} all principal minors of order 2 have the same value, thus

$$\det G_3 = \det G_1 = \det G_2 \tag{4.45}$$

From the expressions for G_1 and G_2 and taking into account eqns. (4.42) and (4.45) it follows that G_1 and G_2 will be of class UP when

$$\Sigma g_{ik} \geq \varepsilon_1 > 0 \tag{4.46}$$

This result is stated for all three kinds of PNEs - resistive, capacitive and inductive - in the following corollary where c_{ik} denotes an entry of the incremental capacitance matrix $\partial \underline{q}_C / \partial \underline{v}_C$ and γ_{ik} denotes an entry of the incremental inverse inductance matrix $\partial \underline{i}_L / \partial \underline{\varphi}_L$.

Corollary 4.1

If a three-terminal resistive (or capacitive or inductive) network element is a PNE in a given orientation and in addition $\Sigma g_{ik} > \varepsilon_1 > 0$ (or $\Sigma c_{ik} > \varepsilon_1 > 0$ or $\Sigma \gamma_{ik} > \varepsilon_1 > 0$) then it is a PNE in the other two orientations.

One of the properties* of PNEs satisfying the conditions of Corollary 4.1 is the following.

Property 1

When a two-terminal network element N_2 is formed from a three-terminal network element either by connecting the two ports of N_3 in parallel (see Fig.

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* Some other properties of this class of elements will be mentioned in Section 5.2.1.

4.2a) or by connecting the two ports of N_3 in series (see Fig. 4.2b) then the hybrid description of N_2 is a quasilinear function.

Proof: When both ports of a three-terminal resistor N_3 (Fig. 4.2a) are connected in parallel the incremental conductance g of the resulting two-terminal resistor N_2 is equal to

$$g = di/dv = \Sigma g_{ik} > \varepsilon_1 > 0$$

and g is bounded.

The incremental resistance r of N_2 (Fig. 4.2b) is equal to

$$r = dv/di = \Sigma g_{ik}/\det G_3.$$

From eqn. (4.46) and the boundedness of G_3 it follows that an $\varepsilon_2 > 0$ can be found such that $r > \varepsilon_2 > 0$ and r is bounded.

Q. E. D.

Note that the transistor model defined by eqn. (3.68) corresponds to a three-terminal resistor that is a PNE in all orientations if $g_1(v_1)$ and $g_2(v_2)$ are quasilinear functions and $1 - \alpha_{12} > 0$ and $1 - \alpha_{21} > 0$. Thus the modified Ebers-Moll model, described in Section 4.2 possesses the same property. Similarly the modified triode model, described by eqn. (4.39) is a PNE in all three orientations.

Let us now consider the (n+1)-terminal PNE. The resistive case will be discussed although analogous results can be obtained for capacitive and inductive elements. Denote by G_i the incremental conductance matrix with the common terminal i and assume that the hybrid description with the common terminal (n+1) is given. The indefinite conductance matrix G_{in} can be expressed in terms of

$G_{(n+1)}$ as

$$G_{in} = P^T G_{(n+1)} P \tag{4.47}$$

where P is an $n \times (n+1)$ matrix having all entries in the (n+1)-th column equal to -1.

$$P = \left[\begin{array}{c:c} & -1 \\ & -1 \\ I_{nn} & \cdot \\ & \cdot \\ & \cdot \\ & -1 \end{array}\right] \tag{4.48}$$

Eqn. (4.47) can be proved by carrying out the multiplication. Since the sum of all elements in a row (or column) i equal to zero, G_{in} is singular. The incremental conductance matrix with common terminal i can be obtained from G_{in} as

$$G_i = P_i^T G_{(n+1)} P_i \qquad i = 1,2,\ldots,n \tag{4.49}$$

where P_i is given from P by deleting the i-th column

$$P_i = \left[\begin{array}{c:c:c} I_{(i-1)(i-1)} & 0 & \begin{array}{c} -1 \\ \cdot \\ -1 \end{array} \\ \hdashline 0 & 0 & -1 \\ \hdashline 0 & I_{(n-i)\,(n-i)} & \begin{array}{c} -1 \\ \cdot \\ -1 \end{array} \end{array}\right] \tag{4.50}$$

As $\det P_i = -1$ for $i = 1,2,\ldots,n$ it follows from eqn. (4.49) that

$$\det G_i = \det G_{(n+1)} \qquad i = 1,2,\ldots,n \tag{4.51}$$

A resistive (n+1)-terminal PNE represents a PNE in all orientations if all (n+1) different principal submatrices of order n of the incremental indefinite conductance matrix G_{in}' (eqn. (4.47)) belong to the class UP. Since by eqn. (4.51) the determinants of incremental conductance matrices are equal to each other the additional condition $\Sigma g_{ik} > \epsilon_1 > 0$ is sufficient for a three-terminal resistor to represent a PNE in all orientations. In the case of (n+1)-terminal PNE certain other conditions, ensuring the UP property of all (n+1) different principal submatrices of G_{in}, are necessary.

4.3.2 Positive definite network elements

Strongly locally passive network elements with bounded hybrid matrices form an important class of PNEs with many interesting properties.

Definition 4.3

A resistive (or capacitive or inductive) n-port (or (n+1)-terminal) network element with hybrid description $\underline{y} = \underline{h}(\underline{x})$ is defined to be a positive definite network element (PDNE) if the hybrid matrix H, associated with a given description, is continuous bounded and u.p.d. for all $\underline{x} \in R^n$.

It follows from the definition of a PDNE that its hybrid description corresponds to a GQLF. The above definition of a PDNE differs from the definition of a PDNE as given in Ref. 11, where boundedness of the hybrid matrix H is not required. The reasons for the more specific definition of a PDNE, used throughout this thesis, are the following:

(i) PDNEs as defined in this thesis form a subclass of PNEs

(ii) all hybrid matrices that are associated with 2^n different hybrid descriptions of an n-port PDNE are u.p.d. and

(iii) all 2^n different hybrid descriptions satisfy the Lipshitz condition.

PDNEs represent a generalization of quasilinear network elements[5] to the nonreciprocal case. Their special property is that they are strongly locally passive. In the resistive case a fairly complex network containing not only quasilinear resistors but gyrators (linear or nonlinear), ideal transformers and diodes may appear as a resistive PDNE when viewed from an appropriate set of ports. Gyrators and ideal transformers must fulfil some topological restrictions in such resistive network.

Some properties of PDNEs are stated below, others, concerning the interconnections of PDNEs, are given in Chapter 5.

Property 1

All 2^n different hybrid descriptions of an n-port (or (n+1)-terminal) PDNE are GQLFs and all 2^n hybrid matrices are bounded and u.p.d.

Proof: Property 1 is a direct consequence of Property 3 of GQLFs.

Q. E. D.

This property can be given the following physical interpetation. Let an n-port resistive PDNE, N_R, be described by the hybrid description

$$\begin{aligned} \underline{v}_1 &= \underline{h}_1(\underline{i}_1, \underline{v}_2) \\ \underline{i}_2 &= \underline{h}_2(\underline{i}_1, \underline{v}_2) \end{aligned} \tag{4.52}$$

where $\underline{i}_1, \underline{v}_1$ correspond to ports $P_1, P_2, \dots, P_m$ and $\underline{i}_2, \underline{v}_2$ correspond to ports $P_{m+1}, \dots, P_n$. Since the hybrid matrix $H = \partial(\underline{h}_1, \underline{h}_2) / \partial(\underline{i}_1, \underline{v}_2)$ is u.p.d. the network must remain strictly locally passive, when series resistances of value $(-\varepsilon)$, $\varepsilon > 0$, are connected to the ports $P_1, P_2, \dots, P_m$ and parallel conductances of value $(-\varepsilon)$ are connected to the ports $P_{m+1}, \dots, P_n$. According to Property 1 any hybrid matrix is u.p.d.; hence the network, obtained by connecting series resistances $(-\varepsilon_1)$ to an arbitrary set of ports of N_R and parallel conductances $(-\varepsilon_1)$ to the remaining ports of N_R, is locally passive for some $\varepsilon_1 > 0$. Note that not all hybrid matrices of an n-port PDNE, satisfying the definition of Ref. 11, are u.p.d.; namely, the second part of Theorem 3 in Ref. 11 is not correct[12].

Consider an n-port resistive PDNE which is generally not passive. Since the hybrid description of a PDNE is a GQLF, which is strongly increasing, it is possible to extract a series voltage source or a parallel current source at each port of a PDNE in such a way that the remaining n-port is strictly passive. Similarly for a capacitive PDNE a passive capacitor and a set of voltage sources connected to each port of the passive capacitor is obtained. Dually for an inductive PDNE the extraction of a set of current sources, that are parallel to each port, leads to a passive inductive PDNE. The following property will be stated formally for the n-port resistive PDNE.

Property 2

Let an active n-port resistive PDNE, N_R, be described in the form

$$\begin{aligned} \underline{i}_1 &= \underline{h}_1(\underline{v}_1, \underline{i}_2) \\ \underline{v}_2 &= \underline{h}_2(\underline{v}_1, \underline{i}_2) \end{aligned} \tag{4.53}$$

where $\underline{v}_1, \underline{i}_1$ are associated with ports $P_1, P_2, \ldots, P_m$ and $\underline{v}_2, \underline{i}_2$ are associated with ports $P_{m+1}, \ldots, P_n$. A passive resistor N_R' (see Fig. 4.3) can be obtained from N_R by extraction of a set of parallel current sources $\underline{J}_1 = \underline{h}_1(\underline{0}, \underline{0})$ at ports $P_1, P_2, \ldots, P_m$ and a set of series voltage sources $\underline{E}_2 = \underline{h}_2(\underline{0}, \underline{0})$ at ports P_{m+1},

$P_{m+2}, \ldots, P_n$. N_R' is a PDNE and its hybrid description has the form

$$\begin{aligned} \underline{i}_1' &= \underline{h}_1(\underline{v}_1, \underline{i}_2) - \underline{h}_1(\underline{0}, \underline{0}) \\ \underline{v}_2' &= \underline{h}_2(\underline{v}_1, \underline{i}_2) - \underline{h}_2(\underline{0}, \underline{0}) \end{aligned} \tag{4.54}$$

Proof: Choose an m-vector $\underline{\sigma}_1$ and an n-vector $\underline{\sigma}_2$. Since $\underline{h}(\underline{v}_1, \underline{i}_2)$ in eqn. (4.53) is a strongly monotonic function

$$(\underline{v}_1^T - \underline{\sigma}_1^T)(\underline{i}_1 - \underline{h}_1(\underline{\sigma}_1, \underline{\sigma}_2)) + (\underline{i}_2^T - \underline{\sigma}_2^T)(\underline{v}_2 - \underline{h}_2(\underline{\sigma}_1, \underline{\sigma}_2)) > 0 \tag{4.55}$$

for all $(\underline{v}_1^T, \underline{i}_2^T)^T \neq (\underline{\sigma}_1^T, \underline{\sigma}_2^T)^T$. Setting $\underline{\sigma}_1 = \underline{0}$, $\underline{\sigma}_2 = 0$ and taking into account eqn. (4.54) it follows from eqn. (4.55) that

$$\underline{v}_1^T \underline{i}_1' + \underline{i}_2^T \underline{v}_2' > 0 \quad \text{for all } (\underline{v}_1^T, \underline{v}_2^T)^T \neq \underline{0} \tag{4.56}$$

Thus N_R' is strictly passive.

Q. E. D.

Note that the hybrid description (eqn. (4.53)) can be any of 2^n different hybrid descriptions of an n-port resistive PDNE; thus the set of all ports can be divided arbitrarily into two disjoint subsets P_E and P_J and then the proper extraction of series voltage sources at ports P_E and parallel current sources at ports P_J leads to a strictly passive n-port resistor.

Property 3 is analogous to Property 3 of PNEs.

Property 3

Choose an arbitrary set of m ports of an n-port resistive PDNE, N_n, and divide the set of remaining (n−m) ports of N_n into two disjoint subsets P_E and

P_J. Let the chosen ports of N_n define a new m-port N_m. The m-port N_m, defined in such a manner, is then a resistive PDNE.

Property 4

Given an n-port PDNE, N_n, form a (n-1)-port N_{n-1} by connecting ports P_i and P_j of N_n in parallel (or in series). The (n-1)-port N_{n-1} is then a PDNE.

Proof: Consider the resistive case. Assume that ports P_1 and P_2 are connected in parallel as shown in Fig. 4.4. The following description exists for N_n,

$$\underline{i} = \underline{h}(\underline{v}) \tag{4.57}$$

Let $\underline{e}$ and $\underline{j}$ be (n-1)-vectors of port-voltages and port-currents of N_{n-1}. The relation between $\underline{e}$ and $\underline{v}$ and $\underline{j}$ and $\underline{i}$ is from Fig. 4.4

$$\begin{aligned} \underline{v} &= A\underline{e} \\ \underline{j} &= A^T\underline{i} \end{aligned} \tag{4.58}$$

where

$$\underline{A} = \left[\begin{array}{c|c} \begin{matrix} 1 \\ 1 \end{matrix} & 0 \\ \hline 0 & I_{n-2} \end{array}\right] \tag{4.59}$$

Hence
$$\underline{j} = A^T\underline{h}(A\underline{e}) \tag{4.60}$$

Since A is of rank (n-1) by Property 2 of GQLFs, $\underline{j}(\underline{e})$ is a GQLF and thus N_{n-1} is a PDNE. A similar proof, based on the hybrid description of N_n with $\underline{i}$ as independent variable, may be given for the case of the series connection of two ports.

Q.E.D.

This procedure can be extended to arbitrary series-parallel connection of ports of N_n. From a (n-1)-port N_{n-1}, an (n-2)-port N_{n-2}, representing a PDNE, can be formed by connecting two ports of N_{n-1} in series or in parallel; the process may be then continued further.

Property 5

Any (n+1)-terminal PDNE represents a PDNE in all different orientations.

Proof: Consider an (n+1)-terminal resistive PDNE and assume that the orientation with common terminal (n+1), N_{n+1}, is transformed into the orientation with common terminal i, N_i (Fig. 4.5). Denote port-voltages and port-currents of N_{n+1} by $\underline{v}$ and $\underline{i}$, and port-voltages and port-currents of N_i by $\underline{e}$ and $\underline{j}$. Let

$$\underline{i} = \underline{h}_R(\underline{v}) \tag{4.61}$$

be a given hybrid description of N_{n+1} and let

$$\underline{j} = \underline{h}_R^*(\underline{e}) \tag{4.62}$$

be the unknown hybrid description of N_i. The relation between port-voltages $\underline{e}$ and $\underline{v}$ and port-currents $\underline{i}$ and $\underline{j}$ is from Fig. 4.5

$$\begin{aligned} \underline{v} &= P_i \underline{e} \\ \underline{j} &= P_i^T \underline{i} \end{aligned} \tag{4.63}$$

where P_i is defined in eqn. (4.50). Thus

$$\underline{j} = P_i^T h_R(P_i \underline{e}) \tag{4.64}$$

Since P_i is of rank n, by Property 2 of GQLFs $\underline{j}(\underline{e})$ is a GQLF.

The capacitive and inductive case can be proved analogously.

Q. E. D.

4.3.3 Hadamard network elements

Since a u. H. matrix belongs to the class UP, it is apparent that a network element where one of the hybrid matrices is u. H. is a PNE.

Definition 4.4

A resistive (or capacitive or inductive) network element is defined to be a Hadamard network element (HNE) if at least one of its hybrid matrices is u. H.

Note that nonreciprocal HNEs may be locally active. Reciprocal HNEs are strongly locally passive and represent a subclass of PDNEs. It is significant to point out that the inverse matrix of a u. H. matrix is not itself u. H. Thus, contrary to the property of an n-port PDNE, where any of 2^n different hybrid matrices is u. p. d., not all hybrid matrices of a HNE are u. H.

The concept of a HNE will be useful in the study of the interconnections of network elements, treated in the next chapter.

4.4 SUMMARY

In this chapter the concept of PNE has been introduced and the properties of this class of elements have been examined. One of the significant results is that a PNE can be locally active only if it is nonreciprocal. It has been shown that many locally active practical resistive devices where all possible incremental driving point resistances are nonnegative can be modelled as PNEs. This justifies the introduction of the concept of PNE in the nonlinear network theory from the point of practical application.

PDNEs, HNEs and (n+1)-terminal network elements where all different orientations correspond to a PNE form three special classes of PNEs. The properties of these three classes of PNEs, which are not disjoint, have been investigated. The relations between various classes of PNEs can easily be summarized with the aid of the diagram of Fig. 4.6, where the set of all PNEs is divided into 5 disjoint subsets A, B, C, D and E. A contains all PNEs that are locally active at least at one point, E contains all reciprocal PNEs, the set $D \cup E$ contains all strongly locally passive PNEs, the set $C \cup D \cup E$ embraces all strictly locally passive PNEs, the set $B \cup C \cup D \cup E$ contains all locally passive PNEs. All PDNEs are contained in the set $D \cup E$ and all HNEs form a subset of the set $A \cup B \cup C \cup D \cup E$.

Note that the terminology, introduced for different classes of PNEs, is in close relationship with the properties of the hybrid matrices, associated with these classes of PNEs. Namely, PNEs possess UP hybrid matrices, PDNEs have u. p. d. hybrid matrices and one of the hybrid matrices of a HNE is u. H.

The properties of interconnections of different classes of PNEs of one kind (either resistive or capacitive or inductive) will be studied in the next chapter.

4.5 REFERENCES

1 Vehovec, M., "Positive network elements, a new class of elements possessing arbitrary hybrid description", Electronics Letters, vol. 5, No. 26, pp. 681-683, December 1969.

2 Hohn, F. E., "Elementary matrix algebra", New York, McMillan, 1958, p. 127.

3 Bryant, P. R., "The order of complexity of electrical networks", Proc. IEE, Mono. No. 335, vol. 106, Pt. C, pp. 174-188, September 1959.

4 Aitken, A. C., "Determinants and matrices", Oliver and Boyd, Edinburgh, 1962, p. 99.

5 Stern, T. E., "Theory of nonlinear networks and systems", Reading, Mass., Addison-Wesley, 1965.

6 Belmann, R., "Introduction to matrix analysis", McGraw-Hill, New York, 1960, p. 110.

7 Ebers, J. J. and Moll J. L., "Large signal behaviour of junction transistors", Proc. IRE, vol. 42, pp. 1761-1772, December 1954.

8 Angelo, Jr., E. J., "Electronics circuits", McGraw-Hill, New York, 1958, p. 103.

9 Vehovec, M., "Some properties of nonlinear networks containing positive resistive elements", to be published in Proc. of the 13th Midwest Symposium on Circuit Theory, Minneapolis, 1970.

10 Shekel, J., "Matrix representation of transistor circuits", Proc. IRE, vol. 40, pp. 1493-1497, November 1952.

11 Ohtsuki, T. and Watanabe, H., "State-variable analysis of RLC networks containing nonlinear coupling elements", IEEE Trans. on Circuit Theory, vol. CT-16, pp. 26-38, February 1969.

12 Vehovec, M., "Comment on State-variable analysis of RLC networks containing nonlinear coupling elements", to be published in IEEE Trans. on Circuit Theory.

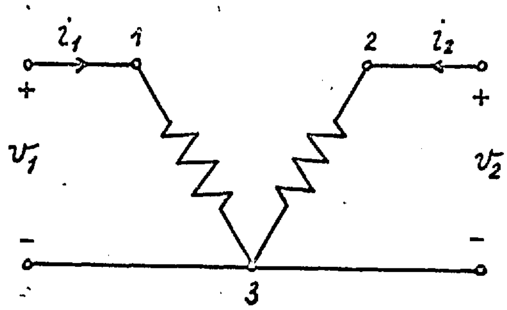

Fig. 4.1. Representation of a resistive three-terminal element

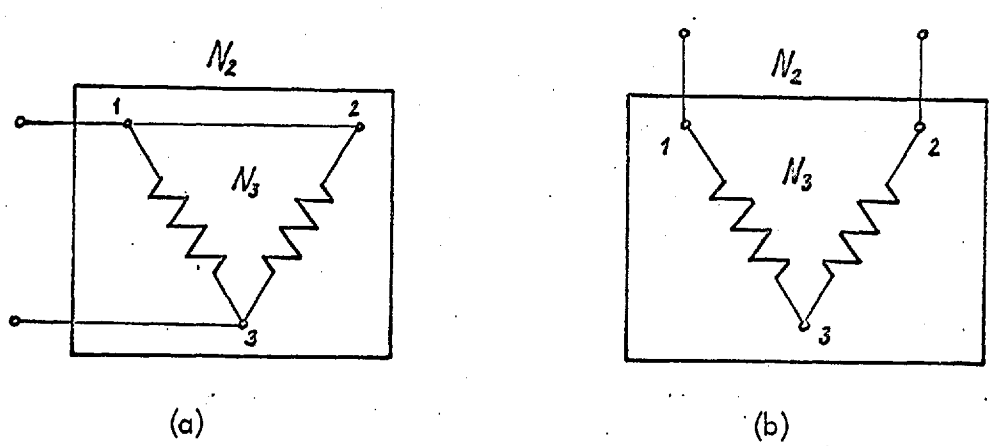

Fig. 4.2. Illustration of Property 1 in Section 4.3.1.

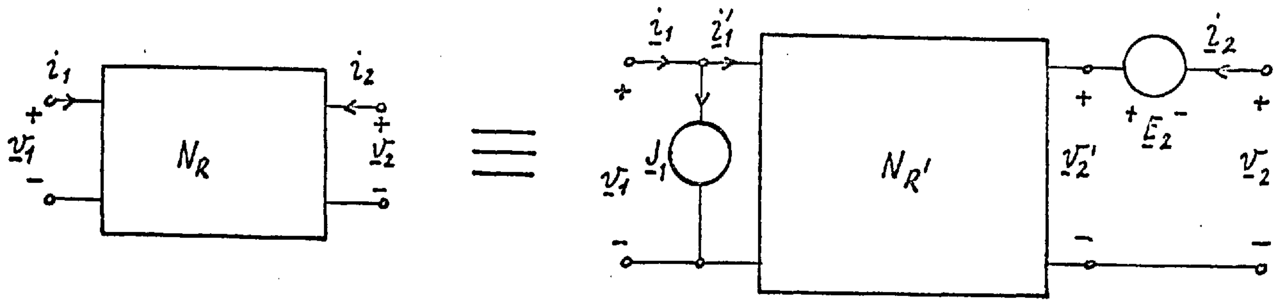

Fig. 4.3. Illustration of Property 2 in Section 4.3.2.

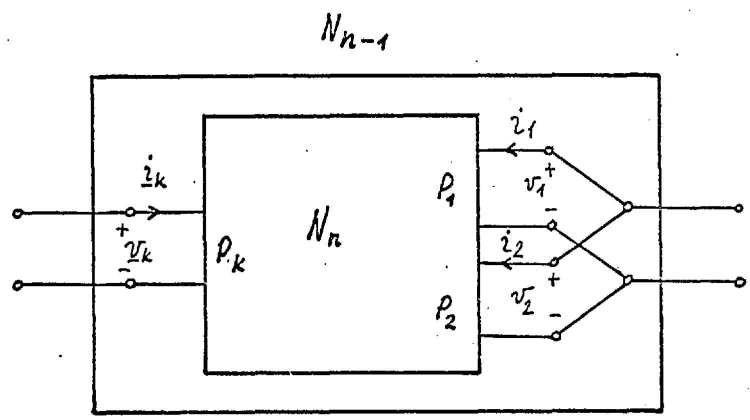

Fig. 4.4. Illustration of Property 4 in Section 4.3.2.

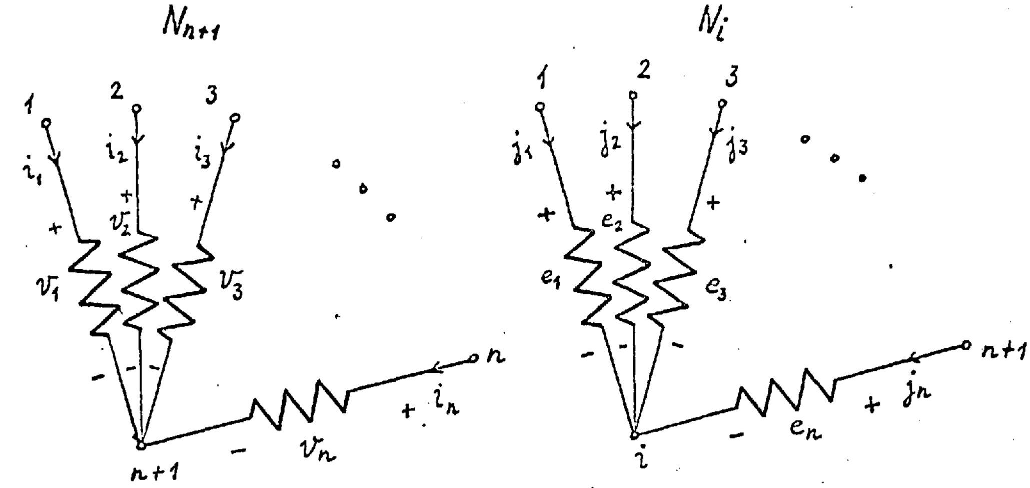

Fig. 4.5. Illustration of Property 5 in Section 4.3.2.

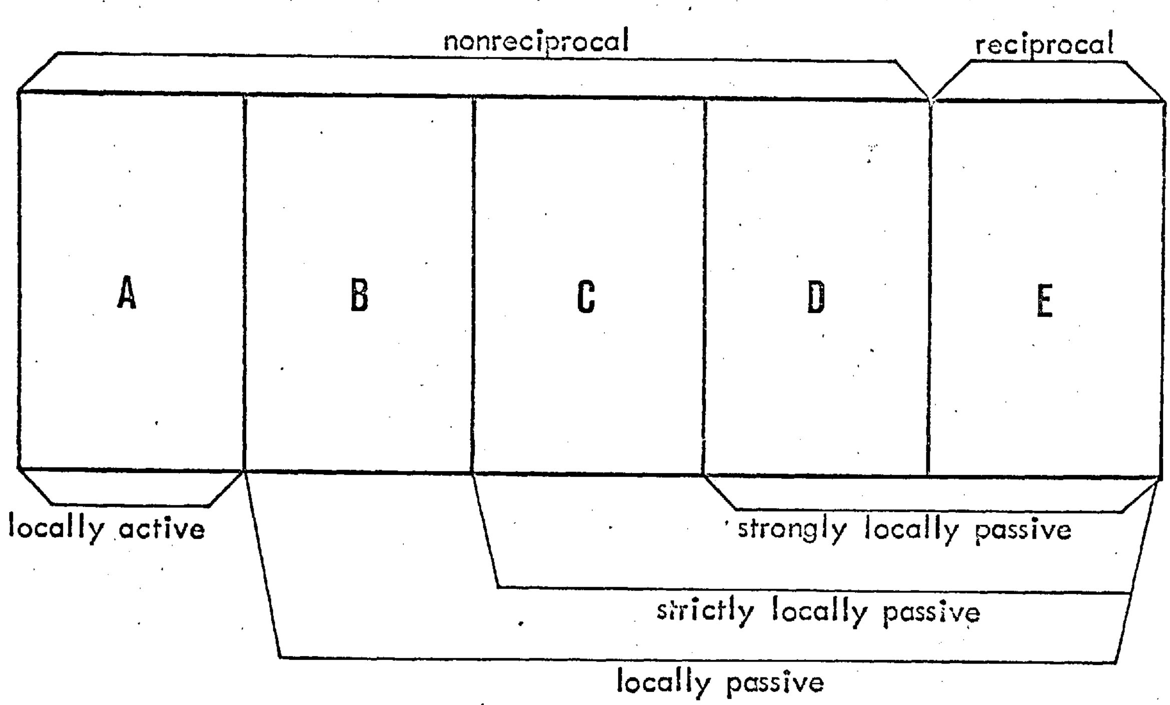

Fig. 4.6. Classification of positive network elements.

Chapter 5

THE ANALYSIS OF ONE-ELEMENT-KIND NETWORKS

5.1 INTRODUCTION

As has already been mentioned, in the state-variable analysis of nonlinear RLC networks, it is necessary to perform the analysis of three one-element-kind networks: resistive, capacitive and inductive. In this chapter one-element-kind networks will be studied. At first series-parallel interconnections of two n-ports will be discussed. In this context a problem of special interest is to determine the classes of elements of one kind that, when interconnected, result in a PNE or PDNE. The reason for studying interconnections resulting in a PNE or PDNE lies in the fact that one may replace an interconnection of two or more network elements of one kind by a PNE or a PDNE. In this way complex one-element-kind networks may frequently be reduced to a single element and the existence and uniqueness of solutions can be more easily studied.

The idea of finding an equivalent network element for an interconnection of two or more network elements is quite old. However, it is important to point out the difference between the linear and nonlinear case. For this purpose consider an example of a parallel-parallel connection of two-port resistive elements N_1 and N_2 shown in Fig. 5.1. In the linear case it is relatively easy to express <u>explicitly</u> the conductance matrix G of the equivalent network element N from a given description for N_1 and N_2, provided that the given matrices of N_1 and N_2 can be transformed into corresponding conductance matrices G_1 and G_2. In the nonlinear case, when N_1 and N_2 are nonlinear, it is generally not possible to obtain an explicit hybrid description for N from given hybrid descriptions of N_1 and N_2, unless N_1 and N_2 are both voltage-controlled. Nevertheless in the study of one-element-kind networks it is helpful to know

i) whether the hybrid description for the equivalent network element, obtained by an interconnection of two or more network elements of one kind, can be obtained in principle and, more specifically,

ii) whether the equivalent network element is a PNE or PDNE.

We shall study series-parallel interconnections of an m-port and an n-port network element; thus a series-parallel interconnection of two n-port network elements is but a special case of such interconnection. For the purpose of studying interconnections of network elements it is necessary to introduce the concept of a positive semidefinite network element that may be defined as follows.

Definition 5.1

A resistive (or capacitive or inductive) n-port (or (n+1)-terminal) network element with the hybrid description $\underline{y} = \underline{h}(\underline{x})$ is defined to be a positive semidefinite network element (PSDNE) if the hybrid matrix $H(\underline{x})$ associated with a given hybrid description is continuous, bounded and positive semidefinite for all $\underline{x} \in R^n$.

When the hybrid matrix of an n-port PSDNE, N, is diagonal, there is no coupling between any pair of ports of N and N is called an uncoupled PSDNE. PSDNE may be considered as an extention of a monotonically increasing two-terminal resistor[1] (or capacitor or inductor) to an n-port network element that may be nonreciprocal in general. It is important to stress that PSDNEs do not necessarily possess all hybrid descriptions and that PSDNEs do not belong to the class of PNEs.

There are many network elements that belong to the class of PSDNEs. For example the diode model with exponential characteristic, the ideal transformer, where both existing hybrid matrices are skew-symmetric, and the gyrator, where the conductance and resistance matrices are skew-symmetric, all belong to the class of PSDNEs. Some properties of PSDNEs are listed below.

Property 1

Assume that $\underline{y} = \underline{h}(\underline{x})$ is the hybrid description of a PSDNE. Then $\underline{h}(\underline{x})$ is a monotonic function.

Proof: Consider a one dimensional arc $\underline{x}(\Theta)$, $0 \le \Theta \le 1$, given by

$$\underline{x}(\Theta) = \underline{x}_1 + (\underline{x}_2 - \underline{x}_1)\Theta \tag{5.1}$$

Then
$$d\underline{x}/d\Theta = \underline{x}_2 - \underline{x}_1 \tag{5.2}$$

and
$$(\underline{x}_2-\underline{x}_1)^T \left[\underline{h}(\underline{x}_2) - \underline{h}(\underline{x}_1)\right] = \left[\underline{x}_2-\underline{x}_1\right]^T \int_0^1 \frac{\partial \underline{h}}{\partial \underline{x}}\bigg|_{\underline{x}=\underline{x}(\theta)} (\underline{x}_2-\underline{x}_1)\, d\Theta =$$

$$= \int_0^1 \left[\underline{x}_2 - \underline{x}_1\right]^T \frac{\partial \underline{h}}{\partial \underline{x}}\bigg|_{\underline{x}=\underline{x}(\Theta)} (\underline{x}_2 - \underline{x}_1)\, d\Theta \tag{5.3}$$

As $\partial\underline{h}/\partial\underline{x}$ is positive semidefinite, $(\underline{x}_2 - \underline{x}_1)^T \frac{\partial \underline{h}}{\partial \underline{x}} (\underline{x}_2 - \underline{x}_1)$ is nonnegative for all $0 \le \Theta \le 1$ and

$$(\underline{x}_2 - \underline{x}_1)^T \left[\underline{h}(\underline{x}_2) - \underline{h}(\underline{x}_1)\right] \geq 0 \tag{5.4}$$

Q. E. D.

The following three properties of PSDNEs are similar to Properties 2,3 and 4 of PDNEs and can be proved analogously.

Property 2

Let an active n-port resistive PSDNE, N_R, be described in the form

$$\begin{aligned} \underline{i}_1 &= \underline{h}_1(\underline{v}_1, \underline{i}_2) \\ \underline{v}_2 &= \underline{h}_2(\underline{v}_1, \underline{i}_2) \end{aligned} \tag{5.5}$$

where $\underline{v}_1$, $\underline{i}_1$ are associated with ports $P_1, P_2, \ldots, P_m$ and $\underline{v}_2$, $\underline{i}_2$ are associated with ports $P_{m+1}, P_{m+2}, \ldots, P_n$. A passive resistive PSDNE N_R' (see Fig. 5.2) can be obtained from N_R by the extraction of a set of parallel current cources $\underline{J}_1 = \underline{h}_1(\underline{0}, \underline{0})$ at ports $P_1, P_2, \ldots, P_m$ and a set of series voltage sources $\underline{E}_2 = \underline{h}_2(\underline{0}, \underline{0})$ at ports $P_{m+1}, P_{m+2}, \ldots, P_n$. N_R' is a PSDNE and its hybrid description has the form

$$\begin{aligned} \underline{i}_1' &= \underline{h}_1(\underline{v}_1, \underline{i}_2) - \underline{h}_1(\underline{0}, \underline{0}) \\ \underline{v}_2' &= \underline{h}_2(\underline{v}_1, \underline{i}_2) - \underline{h}_2(\underline{0}, \underline{0}) \end{aligned} \tag{5.6}$$

Property 3

Choose an arbitrary set of m ports of an n-port resistive PSDNE, N_n, and divide the set of remaining (n-m) ports of N_n into two disjoint subsets P_E and P_J. Connect constant voltage sources to ports P_E and constant current sources to ports P_J. Let the chosen ports of N_n define a new m-port N_m. The m-port N_m, defined in such a manner is then a resistive PSDNE.

Property 4

Given an n-port PSDNE, N_n, form the (n-1)-port N_{n-1} by connecting ports P_i and P_j of N_n in parallel (or in series). The (n-1)-port N_{n-1} is a PSDNE.

Property 5

Suppose that s different hybrid descriptions of an n-port PSDNE exist. Then all hybrid matrices that are associated with the existing hybrid descriptions belong to the class P_o. Thus, for example, all incremental driving point conductances, if they can be defined, are nonnegative for a resistive PSDNE.

Proof For a PSDNE the hybrid matrix belongs to class P_o, since H is positive definite. From proof of Property 2 of PNEs Property 5 follows.

Q. E. D.

5.2 SERIES-PARALLEL INTERCONNECTIONS OF NETWORK ELEMENTS

A series-parallel interconnection of two network elements is shown in Fig. 5.3, where N_1 is an n-port and N_2 is an m-port. The resistive case will be treated throughout this chapter, although the capacitive and inductive case can be dealt with analogously. A set of i ports, P_i, of N_1 and N_2 is connected in parallel and a set of j ports, denoted as P_j, of the same network elements is connected in series; a set of k ports, P_k, of N_1 and a set of l ports of N_2, P_l, are identical for constituent and interconnected network. In this way a (i+j+k+l)-port network element N is formed. Without loss of generality we may assume that the directions of the components of $\underline{v}_1$ are the same as for the components of $\underline{e}_1$ and similarly the components of $\underline{i}_2$ have the same directions as the components of $\underline{j}_2$ (see Fig. 5.3). Note that the interconnection shown in Fig. 5.3 is fairly general; when N_1 and N_2 are two-ports, then the interconnection in Fig. 5.3 may represent either the parallel-parallel, series-series, series-parallel or cascade interconnection of two-ports.

The following theorem gives sufficient conditions for the existence of the hybrid description of network element N obtained by the series-parallel interconnection.

Theorem 5.1

Assume that resistive elements N_1 and N_2 in Fig. 5.3 have the following hybrid descriptions:

$$\underline{y} = \underline{h}_1(\underline{x}) \tag{5.7}$$

$$\underline{z} = \underline{h}_2(\underline{w}) \tag{5.8}$$

where

$$\underline{x} = \begin{bmatrix} \underline{v}_1 \\ \underline{i}_2 \\ \underline{x}_3 \end{bmatrix} \qquad \underline{y} = \begin{bmatrix} \underline{i}_1 \\ \underline{v}_2 \\ \underline{y}_3 \end{bmatrix} \tag{5.9}$$

$$\underline{w} = \begin{bmatrix} \underline{e}_1 \\ \underline{i}_2 \\ \underline{x}_4 \end{bmatrix} \qquad \underline{z} = \begin{bmatrix} \underline{i}_1 \\ \underline{e}_2 \\ \underline{y}_4 \end{bmatrix} \tag{5.10}$$

and $\underline{i}_1$, $\underline{v}_1$, $\underline{e}_1$, $\underline{i}_1$ are i-vectors, $\underline{v}_2$, $\underline{i}_2$, $\underline{e}_2$, $\underline{i}_2$ are j-vectors; k-vectors $\underline{x}_3$, $\underline{y}_3$ and l-vectors $\underline{x}_4$, $\underline{y}_4$ contain a combination of port voltages and port currents of P_k and P_l respectively. Then the hybrid description of the network element N obtained by the series-parallel interconnection of N_1 and N_2, as shown in Fig. 5.3, exists.

Proof

Define vectors $\underline{s}$ and $\underline{u}$, corresponding to the interconnected network, as

$$\underline{s} = \begin{bmatrix} \underline{I}_1 \\ \underline{V}_2 \\ \underline{Y}_3 \\ \underline{Y}_4 \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} \underline{V}_1 \\ \underline{I}_2 \\ \underline{X}_3 \\ \underline{X}_4 \end{bmatrix} \tag{5.11}$$

The relation between $\underline{s}$ and $\underline{u}$ is the unknown hybrid description of N

$$\underline{s} = \underline{h}(\underline{u}) \tag{5.12}$$

From Fig. 5.3 the following relations are obtained

$$\underline{s} = A_1^T \underline{y} + A_2^T \underline{z} \tag{5.13}$$

$$\underline{x} = A_1 \underline{u} \tag{5.14}$$

$$\underline{w} = A_2 \underline{u} \tag{5.15}$$

where A_1 is an (i+j+k+l) x (i+j+k) matrix and A_2 is an (i+j+k+l) x (i+j+p) matrix:

$$A_1 = \begin{bmatrix} I_{ii} & 0 & 0 & 0 \\ 0 & I_{ii} & 0 & 0 \\ 0 & 0 & I_{kk} & 0 \end{bmatrix} \tag{5.16}$$

$$A_2 = \begin{bmatrix} I_{ii} & 0 & 0 & 0 \\ 0 & I_{ii} & 0 & 0 \\ 0 & 0 & 0 & I_{ll} \end{bmatrix} \tag{5.17}$$

Combining eqns. (5.7)-(5.15) the hybrid description of N is

$$\underline{s} = \underline{h}(\underline{u}) = A_1{}^T \underline{h}_1(A_1\underline{u}) + A_2{}^T \underline{h}_2(A_2\underline{u}) \tag{5.18}$$

Q. E. D.

Note that according to Theorem 5.1 the hybrid description of N (eqn. (5.18)) exists when N_1 and N_2 are voltage-controlled with respect to ports P_i which are connected in parallel and current-controlled with respect to ports P_i which are connected in series. When the hybrid descriptions $\underline{h}_1(\cdot)$ and $\underline{h}_2(\cdot)$ are arbitrary, the hybrid descriptions, other than $\underline{h}(\underline{u})$ (eqn. (5.18)) may not exist for N. The following three corollaries may easily be derived from Theorem 5.1.

Corollary 5.1

Assume that N_1 and N_2 are n-port voltage-controlled resistive elements. Then the parallel interconnection of all corresponding ports of N_1 and N_2 results in a voltage-controlled n-port N.

Corollary 5.2

Assume that N_1 and N_2 are n-port current-controlled resistive elements. Then the series interconnection of all corresponding ports results in a current-controlled n-port N.

Corollary 5.3

Assume that N_1 is an n-port PNE and N_2 is an m-port PNE. Then the hybrid description of a network element N obtained by any series-parallel interconnection, as shown in Fig. 5.3, exists.

5.2.1. The series-parallel interconnections resulting in a PNE

It is easy to show that the series-parallel interconnection of two PNEs does not necessarily result in a PNE. Consider, as an example, a parallel-parallel interconnection of 2 linear two-port resistive PNEs N_1 and N_2 with conductance matrices G_1 and G_2:

$$G_1 = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

The conductance matrix G of the interconnected network N

$$G = G_1 + G_2 = \begin{bmatrix} 2 & 10 \\ 10 & 2 \end{bmatrix}$$

and det $G < 0$. Thus the equivalent network element N is not a PNE. Even when N_2 is a reciprocal PDNE and N_1 is a PNE the series-parallel interconnection of N_1 and N_2 does not necessarily result in a PNE. As an example consider the case where N_1 is a linear resistive two-port representing a PNE, and its conductance matrix G_1 is

$$G_1 = \begin{bmatrix} 1 & 0 \\ -20 & 1 \end{bmatrix}$$

and N_2 is a linear reciprocal resistive two-port representing a PDNE, and its conductance matrix G_2 is

$$G_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

The parallel-parallel interconnection of N_1 and N_2 results in a network element N with conductance matrix G

$$G = G_1 + G_2 = \begin{bmatrix} 3 & -1 \\ -21 & 3 \end{bmatrix}$$

and det $G < 0$. Since the parallel-parallel interconnection is but a special case of the series-parallel interconnection, shown in Fig. 5.3, we have proved that the series-parallel interconnection of an n-port PNE and an m-port reciprocal PDNE does not necessarily result in a PNE.

The reason that the resulting network N may not be a PNE when N_1 is a PNE and N_2 is a reciprocal PDNE can be explained by the fact that N_1 may be locally active and N_2 represents a nonlinear feedback network with respect to N_1; thus even when N_2 is locally passive and reciprocal it is still possible for N to have multivalued characteristics with respect to a certain selection of independent variables at the ports of N. We can expect that a PNE will be obtained by the series-parallel interconnection when N_2 does not present any feedback for N_1. The following two theorems give sufficient conditions that a special series-parallel interconnection of two resistive elements results in a PNE.

Theorem 5.2

Suppose N_1 is an n-port resistive PNE and N_2 is an m-port uncoupled resistive PSDNE, where $m \leq n$. Divide the set of m ports of N_2 into two disjoint subsets P_i and P_j containing i and j ports respectively and suppose that N_2 is voltage-controlled with respect to ports P_i and current-controlled with respect to ports P_j. The series-parallel interconnection (Fig. 5.4) where the ports P_i of N_2 are connected in parallel and the ports P_j of N_2 are connected in series with the corresponding ports of N_1 results in an n-port PNE.

Proof

From eqns. (5.7) and (5.8) the corresponding hybrid matrices of N_1 and N_2 are

$$H_1 = \partial \underline{h}_1 / \partial \underline{x} \tag{5.19}$$

$$H_2 = \partial \underline{h}_2 / \partial \underline{w} \tag{5.20}$$

where H_1 is of class UP and H_2 is positive semidefinite and diagonal. It is necessary to show that the hybrid matrix H of the network element N in Fig. 5.4 is UP and bounded. From eqn. (5.18)

$$H = \partial \underline{h} / \partial \underline{u} = A_1^T H_1 A_1 + A_2^T H_2 A_2 \tag{5.21}$$

Since all ports of N_2 are connected either in series or in parallel to the corresponding ports of N_1, P_l is an empty set and from eqns. (5.16), (5.17) and (5.21)

$$H = H_1 + \begin{bmatrix} H_2 & 0 \\ 0 & 0_{kk} \end{bmatrix} \tag{5.22}$$

Application of the diagonal expansion of determinant to all principal minors of H shows that H is a matrix of class UP. Since H_1 and H_2 are both bounded, H is bounded and N is a PNE.

Q. E. D.

It follows from Theorem 5.2 that when a current-controlled monotonically increasing resistor with bounded incremental resistance is added in series to a port of a resistive PNE the interconnected network is a PNE; similarly when a voltage-controlled monotonically increasing resistor with bounded incremental conductance is added in parallel to a port of a resistive PNE the interconnected network is a PNE.

The following theorem[2] regarding the properties of three-terminal resistive elements which represent a PNE in all 3 different orientations has useful application in the dc analysis of nonlinear transistor networks.

Theorem 5.3*

If a two-terminal voltage-controlled resistive PSDNE N_2 is connected to any two terminals of a three-terminal resistive element N_1 representing a PNE in all orientations, the resulting resistive three-terminal element N (Fig. 5.5a) is a PNE in all 3 orientations.

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* It is conjectured that Theorem 5.3 is valid for (n+1)-terminal resistive element which is a PNE in all (n+1) orientations.

Similarly, if a two-terminal current-controlled resistive PSDNE N_2' is connected in series with any port of a three-terminal resistive element N_1 representing a PNE in all 3 orientations, the resulting resistive network element N' (Fig. 5.5b) is a PNE in all three orientations.

Proof

It is sufficient to show that N in Fig. 5.5a and N' in Fig. 5.5b represent a three-terminal resistive element which is a PNE in all 3 orientations. Suppose that the incremental indefinite conductance matrix of a three-terminal element N_1 is given by eqn. (4.43a) and denote the incremental conductance of N_2 (Fig. 5.5a) by g and the incremental resistance of N_2' (Fig. 5.5b) by r; g and r are bounded and nonnegative. Then G_{in}, the incremental indefinite conductance matrix of N (Fig. 5.5a), is

$$G_{in} = \begin{bmatrix} g_{11}+g & g_{12} & -(g_{11}+g_{12}+g) \\ g_{21} & g_{22} & -(g_{21}+g_{22}) \\ -(g_{11}+g_{21}+g) & -(g_{12}+g_{22}) & \Sigma g_{ik}+g \end{bmatrix} \quad (5.23)$$

Since N_1 is a PNE and g is bounded and nonnegative it is easy to see that all principal submatrices of order 2 in G_{in} are UP and bounded; thus N in Fig. 5.5a is a PNE in all 3 orientations.

Similarly, G_{in}', the incremental indefinite conductance matrix of N' in Fig. 5.5b is

$$G_{in}' = \frac{1}{1+rg_{11}} \begin{bmatrix} g_{11} & g_{12} & -(g_{11}+g_{12}) \\ g_{21} & g_{22}+r \det G_3 & -(g_{21}+g_{22}+r \det G_3) \\ -(g_{11}+g_{21}) & -(g_{12}+g_{22}+r \det G_3) & \Sigma g_{ik}+r \det G_3 \end{bmatrix}$$

(5.24)

where G_3 is the incremental conductance matrix of N_1 in the orientation with the common terminal 3. As N_1 is a PNE and r is bounded and nonnegative, clearly, all principal submatrices of order 2 in G'_{in} are UP and bounded. Therefore N′ in Fig. 5.5b is a PNE in all 3 orientations.

Q. E. D.

A PNE can be obtained by the series-parallel interconnection of two HNEs provided certain hybrid matrices of HNEs are u. H.

Theorem 5.4

Assume that resistive elements N_1 and N_2 in Fig. 5.3 are HNEs with the hybrid description given by eqns. (5.7) and (5.8) respectively. If the hybrid matrices $H_1 = \partial \underline{h}_1 / \partial \underline{x}$ and $H_2 = \partial \underline{h}_2 / \partial \underline{w}$ of N_1 and N_2 are u. H. the series-parallel interconnection of N_1 and N_2 shown in Fig. 5.3 results in a resistive element N which is a PNE.

Proof

Partition the $(i+j+k) \times (i+j+k)$ matrix H_1 as

$$H_1 = \begin{bmatrix} H_{1a} & H_{1b} \\ H_{1c} & H_{1d} \end{bmatrix} \tag{5.25}$$

and similarly partition the $(i+j+l) \times (i+j+l)$ matrix H_2 as

$$H_2 = \begin{bmatrix} H_{2a} & H_{2b} \\ H_{2c} & H_{2d} \end{bmatrix} \tag{5.26}$$

where H_{1a} and H_{2a} are $(i+j) \times (i+j)$ matrices. Substituting eqns. (5.16), (5.17), (5.25) and (5.26) into eqn. (5.21) the hybrid matrix H of N is obtained as

$$H = \begin{bmatrix} H_{1a}+H_{2a} & H_{1b} & H_{2b} \\ H_{1c} & H_{1d} & 0 \\ H_{2c} & 0 & H_{2d} \end{bmatrix} \tag{5.27}$$

It is easy to see that H is u. H. since both H_1 and H_2 are u. H.

Q. E. D.

5.2.2 The series-parallel interconnections resulting in a PDNE

As the class of PDNEs has many important properties it is helpful to study interconnections resulting in PDNE. It will be shown that the series-parallel interconnection of 2 PDNEs or of a PDNE and a PSDNE results in a PDNE. These two results are stated formally in the following two theorems.

Theorem 5.5[3]

Let N_1 and N_2 in Fig. 5.3 be resistive PDNEs. Then any series-parallel interconnection, as shown in Fig. 5.3, results in a resistive PDNE N.

Proof

Since N_1 and N_2 are PDNEs the hybrid descriptions given by eqns. (5.7) and (5.8) exist. From eqns. (5.7) and (5.8) the hybrid matrices of N_1 and N_2, $H_1 = \partial \underline{h}_1/\partial \underline{x}$ and $H_2 = \partial \underline{h}_2/\partial \underline{w}$, are u. p. d. and bounded. From eqn. (5.18) the hybrid matrix H of N is equal to

$$H = \partial \underline{h}/\partial \underline{u} = A_1{}^T H_1 A_1 + A_2{}^T H_2 A_2 \tag{5.28}$$

and we want to show that H is bounded and u. p. d.; thus a $\mu > 0$ exists such that the quadratic form

$$\underline{z}^T H \underline{z} > \mu \, \underline{z}^T \underline{z} \qquad \text{for all } \underline{z} \neq \underline{0} \qquad (5.29)$$

Since H_1 and H_2 are u. p. d., $(H_1 - \mu_1 I)$ and $(H_2 - \mu_2 I)$ are positive definite with some $\mu_1 > 0$ and $\mu_2 > 0$ and

$$\underline{z}^T A_1{}^T H_1 A_1 \underline{z} > \mu_1 \underline{z}^T A_1{}^T A_1 \underline{z} \qquad (5.30)$$

$$\underline{z}^T A_2 H_2 A_2 \underline{z} > \mu_2 \underline{z}^T A_2{}^T A_2 \underline{z} \qquad (5.31)$$

From eqn. (5.28)

$$\underline{z}^T H \underline{z} = \underline{z}^T A_1{}^T H_1 A_2 \underline{z} + \underline{z}^T A_2{}^T H_2 A_2 \underline{z} \qquad (5.32)$$

Combining eqns. (5.30) - (5.32) and calculatin $A_1{}^T A_1$ and $A_2{}^T A_2$ yields

$$\underline{z}^T H \underline{z} \geq \underline{z}^T \begin{bmatrix} (\mu_1 + \mu_2) I_{ii} & 0 & 0 & 0 \\ 0 & (\mu_1 + \mu_2) I_{jj} & 0 & 0 \\ 0 & 0 & \mu_1 I_{kk} & 0 \\ 0 & 0 & 0 & \mu_2 I_{pp} \end{bmatrix} \underline{z} \qquad (5.33)$$

The diagonal matrix on the r. h. s. of eqn. (5.33) is positive definite and therefore for all $\underline{z} \neq \underline{0}$

$$\underline{z}^T H \underline{z} > \mu \, \underline{z}^T \underline{z} \qquad \text{if } 0 < \mu < \text{Min}(\mu_1, \mu_2) \qquad (5.34)$$

Thus H is u. p. d. and since H_1 and H_2 are bounded H is bounded as well.

Q. E. D.

Theorem 5.6

Suppose that in Fig 5.4 N_1 is an (i+j+k)-port resistive PDNE and N_2 is an (i+j)-port resistive PSDNE. Let ports P_i of N_2 be voltage-controlled and let ports P_j of N_2 be current-controlled. Then the series-parallel interconnection of N_1 and N_2 shown in Fig. 5.4 results in an (i+j+k)-port resistive PDNE N.

Proof

Let H_1 and H_2 be the hybrid matrices of N_1 and N_2, obtained from the hybrid descriptions of eqns. (5.7) and (5.8); the matrix H_1 is u. p. d. and bounded and H_2 is positive semidefinite and bounded. From eqn. (5.22)

$$H = H_1 + \begin{bmatrix} H_2 & 0 \\ 0 & 0_{kk} \end{bmatrix} \tag{5.35}$$

where the second matrix on the r. h. s. of eqn. (5.35) is positive semidefinite. Since a sum of a u. p. d. and a positive semidefinite matrix is u. p. d. the matrix H is u.p.d. As H_1 and H_2 are bounded H is bounded and N is a PDNE.

Q. E. D.

5.2.3 Special cases of the series-parallel interconnection

As will be demonstrated in this section a special parallel or series interconnection of two resistive PNEs N_1 and N_2 results in a PNE when one of the ports of N_1, P_c, is connected in parallel with one of the ports of N_2, P_d, as shown in Fig. 5.6a; the same result is obtained when P_c and P_d are connected in series as shown in Fig. 5.6b. This property is very useful in the dc analysis of transistor networks.

Theorem 5.7[2]

Suppose N_1 and N_2 are resistive PNEs where N_1 is an n-port and N_2

is an m-port. Let P_c be an arbitrary port of N_1 and let P_d be an arbitrary port of N_2. Then a parallel connection of ports P_c and P_d (Fig. 5.6a) (or a series connection of ports P_c and P_d (Fig. 5.6b)) results in an (n+m-1)-port network element N (or N′) which is a PNE.

Proof

We shall prove Theorem 5.7 for the case of the parallel connection of the ports P_c and P_d; the case of series connection of the same ports can be proved analogously.

Without loss of generality we can assume that P_c is the port number n of N_1 and P_d is the port number 1 of N_2. The following hybrid descriptions of N_1 and N_2 exist.

$$\begin{aligned} \underline{i}_k &= \underline{i}_k(\underline{v}_k, v_n) \\ \underline{i}_n &= \underline{i}_n(\underline{v}_k, v_n) \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} i_1 &= i_1(e_1, \underline{e}_l) \\ \underline{i}_l &= \underline{i}_l(\underline{e}_1, \underline{e}_l) \end{aligned} \tag{5.37}$$

where $\underline{v}_k = (v_1, v_2, \ldots, v_{n-1})^T$, $\underline{i}_k = (i_1, i_2, \ldots, i_{n-1})^T$, $\underline{e}_l = (e_2, e_3, \ldots, e_l)^T$ and $\underline{i}_l = (i_2, i_3, \ldots, i_l)^T$. The incremental conductance matrices $G_1 = \partial(\underline{i}_k, i_n)/\partial(\underline{v}_k, v_n)$ and $G_2 = \partial(i_1, \underline{i}_l)/\partial(e_1, \underline{e}_l)$ of N_1 and N_2 can be partitioned as follows.

$$G_1 = \begin{bmatrix} G_{aa} & G_{ab} & G_{ac} \\ G_{ba} & G_{bb} & G_{bc} \\ G_{ca} & G_{cb} & G_{cc} \end{bmatrix} \tag{5.38}$$

$$G_2 = \begin{bmatrix} G_{dd} & G_{de} & G_{df} \\ G_{ed} & G_{ee} & G_{ef} \\ G_{fd} & G_{fe} & G_{ff} \end{bmatrix} \tag{5.39}$$

where G_{aa} is an $a \times a$ matrix, G_{bb} is a $b \times b$ matrix, G_{cc} is a 1×1 matrix and $(a+b+1) = n$; G_{dd} is a $d \times d$ matrix, G_{ee} is an $e \times e$ matrix, G_{ff} is a 1×1 matrix and $(d+e+1) = m$.

From Fig. 5.6a the incremental conductance matrix G of the resulting network element N is obtained in the form

$$G = \begin{bmatrix} G_{aa} & G_{ab} & G_{ac} & 0 & 0 \\ G_{ba} & G_{bb} & G_{bc} & 0 & 0 \\ G_{ca} & G_{cb} & G_{cc}+G_{dd} & G_{de} & G_{df} \\ 0 & 0 & G_{ed} & G_{ee} & G_{ef} \\ 0 & 0 & G_{fd} & G_{fe} & G_{ff} \end{bmatrix} \tag{5.40}$$

Since G_1 and G_2 are bounded matrices G is bounded. Thus in order to prove that N is a PNE it is necessary to show that G is UP. Let $G_{(b,c,e)(b,c,e)}$ denote the following principal submatrix of G

$$G_{(b,c,e)(b,c,e)} = \begin{bmatrix} G_{bb} & G_{bc} & 0 \\ G_{cb} & G_{cc}+G_{dd} & G_{de} \\ 0 & G_{ed} & G_{ee} \end{bmatrix} \tag{5.41}$$

Since the partitioning of G_1 (eqn. (5.38))and G_2 (eqn. (5.39)) is arbitrary except that G_{cc} and G_{dd} are 1×1 matrices, G is a UP matrix if $\det G_{(b,c,e)(b,c,e)} \geq \varepsilon^{b+e+1} > 0$. By Lemma 4.1

$$\det G_{(b,c,e)(b,c,e)} = \det G_{ee} \det \left[\begin{array}{c|c} G_{bb} & G_{bc} \\ \hline G_{cb} & G_{cc}+G_{dd}-G_{ed}G_{ee}^{-1}G_{de} \end{array}\right] \tag{5.42}$$

where $(G_{cc}+G_{dd}-G_{ed}G_{ee}^{-1}G_{de})$ is a 1×1 matrix. Thus

$$\det G_{(b,c,e)(b,c,e)} = \det G_{ee} \det \begin{bmatrix} G_{bb} & G_{bc} \\ G_{cb} & G_{cc} \end{bmatrix} + (G_{dd}-G_{ed}G_{ee}^{-1}G_{de}) \det G_{bb} \tag{5.43}$$

Applying Lemma 4.1 once more

$$\det \begin{bmatrix} G_{dd} & G_{de} \\ G_{ed} & G_{ee} \end{bmatrix} = (G_{dd}-G_{ed}G_{ee}^{-1}G_{de}) \det G_{ee} \tag{5.44}$$

Substitution of eqn. (5.44) into eqn. (5.43) yields finally

$$\det G_{(b,c,e)(b,c,e)} = \det G_{bb} \det \begin{bmatrix} G_{dd} & G_{de} \\ G_{ed} & G_{ee} \end{bmatrix} + \det G_{ee} \det \begin{bmatrix} G_{bb} & G_{bc} \\ G_{cb} & G_{cc} \end{bmatrix} \tag{5.45}$$

Since G_1 and G_2 are UP matrices, it is obvious from eqn. (5.45) that G is a UP matrix.

Q. E. D.

5.3 THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF ONE-ELEMENT-KIND NETWORKS

The most important reason for studying one-element-kind networks lies in the role they play in the analysis of dynamic nonlinear networks. Very often the solution of a dynamic network N can be obtained only by first analysing related networks: the capacitive network $N\{0; \mathcal{B}_R, \mathcal{B}_L, \mathcal{B}_J\}$, the inductive network $N\{\mathcal{B}_E, \mathcal{B}_C, \mathcal{B}_R; 0\}$ and the resistive network obtained by replacing all capacitive and inductive elements in N by a set of voltage sources and current sources respectively. Another practical reason for studying resistive networks lies in the fact that the steady-state behaviour of many dynamic nonlinear networks is determined by analysing a resistive subnetwork obtained by replacing all capacitive and inductive elements in the dynamic network by open circuits and short circuits, respectively. Indeed, the set of equilibrium states of any dynamic network is simply the solution of this resistive network. Such a resistive network is termed a multivalued memoryless network[4] if it admits more than one solution.

There is no general method for determining all solutions of a multivalued memoryless network. The problem of finding solutions is especially difficult if three-terminal nonlinear resistors are present in a resistive network. If, for example, we use the Newton-Raphson algorithm for computing the solution of a multivalued resistive network it may happen that the procedure is not convergent since the differential eqn. (3.73) is not globally asymptotically stable in such a case. Nevertheless, when a multivalued resistive network N_R contains two-terminal resistors with <u>piecewise linear characteristics</u>, independent and controlled sources, and linear two-port elements (such as gyrators, ideal transformers, etc.) all solutions of N_R can be obtained by the iterative piecewise linear method proposed by Chua[4].

For the purpose of the state-variable analysis of an RLC network N it is necessary to check the existence and uniqueness of solutions of the resistive, capacitive and inductive subnetworks of N. The resistive case will be studied in this section; analogous results can be obtained for the capacitive and inductive case as the (R), (C) and (L) eqns. (eqns. (2.68a)-(2.68c)) have a similar form. If a one-element-kind network possesses a unique solution for any value of independent sources it will be termed a single valued one-element-kind network.

Several forms of the governing equations of resistive networks have been developed in Section 2.3. In principle, Palais' theorem may be used to establish the existence and uniqueness of solutions for these equations. However, for a nonlinear resistive network at large the number of algebraic equations, n, is very large and even with the use of a computer it is virtually impossible to test whether the two conditions in Palais' theorem are fulfilled in R^n. When the hybrid matrices of all resistive elements are bounded and continuous, Theorem 3.2 is applicable; in such a case there is only one condition on the Jacobian of the governing equations that has to be fulfilled. In this manner we obtain a set of very general conditions for a resistive network to be single valued. These conditions are summarized in Theorem 5.8 for the different kinds of analyses treated in Section 2.3. In the following we shall denote

$$F_R = \begin{bmatrix} 0 & F_{\beta\varepsilon} \\ -F_{\beta\varepsilon}^T & 0 \end{bmatrix} \tag{5.46}$$

Theorem 5.8

(a) A resistive network described by a set of eqns. (2.28) is single valued if the Jacobian matrix $\partial \underline{f}_R / \partial(\underline{v}_\beta, \underline{v}_\varepsilon, \underline{i}_\beta, \underline{i}_\varepsilon)$ is bounded and continuous and

$$\left|\det\begin{bmatrix} 1 & F_{\beta\varepsilon} & 0 & 0 \\ 0 & 0 & -F_{\beta\varepsilon}{}^T & 1 \\ \dfrac{\partial \underline{f}_R}{\partial \underline{v}_\beta} & \dfrac{\partial \underline{f}_R}{\partial \underline{v}_\varepsilon} & \dfrac{\partial \underline{f}_R}{\partial \underline{i}_\beta} & \dfrac{\partial \underline{f}_R}{\partial \underline{i}_\varepsilon} \end{bmatrix}\right| \geq \varepsilon > 0 \qquad (5.47a)$$

for all $\underline{v}_\beta$, $\underline{v}_\varepsilon$, $\underline{i}_\beta$, $\underline{i}_\varepsilon$.

(b) A resistive network described by the loop equations (eqns. (2.31)) is single valued if the incremental resistance matrix $R = \partial \underline{v}_R / \partial \underline{i}_R$ is bounded and continuous and

$$|\det [BRB^T]| \geq \varepsilon > 0 \qquad \text{for all } \underline{i}_R. \qquad (5.47b)$$

(c) A resistive network described by the node equations (eqns. (2.34)) is single valued if the incremental conductance matrix $G = \partial \underline{i}_R / \partial \underline{v}_R$ is bounded and continuous and

$$|\det [QGQ^T]| \geq \varepsilon > 0 \qquad \text{for all } \underline{v}_R. \qquad (5.47c)$$

(d) A resistive network described by the hybrid equations (2.37) is single valued if the hybrid matrix $H_1 = \partial(\underline{v}_\beta, \underline{i}_\varepsilon)/\partial(\underline{i}_\beta, \underline{v}_\varepsilon)$ is bounded and continuous and

$$|\det (H_1 + F_R)| \geq \varepsilon > 0 \qquad \text{for all } \underline{i}_\beta, \underline{v}_\varepsilon \qquad (5.47d)$$

(e) A resistive network described by the hybrid equations (2.39) is single valued if the hybrid matrix $H_2 = \partial(\underline{i}_\beta, \underline{v}_\varepsilon)/\partial(\underline{v}_\beta, \underline{i}_\varepsilon)$ is bounded and continuous and

$$|\det (I + F_R H_2)| \geq \varepsilon > 0 \qquad \text{for all } \underline{v}_\beta, \underline{i}_\varepsilon \qquad (5.47e)$$

<u>Proof</u>

A direct application of Theorem 3.2 to the corresponding dc equations of a resistive network gives Theorem 5.8

Q. E. D.

An interesting point related to the condition of eqn. (5.47e) is worth mentioning. Namely, if $\underline{v}_\beta$ and $\underline{i}_\varepsilon$ are calculated from eqns. (2.28a) and (2.28b) and substituted into eqn. (2.38) the hybrid equations are obtained in the following implicit form

$$\begin{aligned} \underline{i}_\beta - \underline{i}_\beta(-F_{\beta\varepsilon}\,\underline{v}_\varepsilon + \underline{e}_\beta, F_{\beta\varepsilon}^T\,\underline{i}_\beta + \underline{j}_\varepsilon) &= \underline{0} \\ \underline{v}_\varepsilon - \underline{v}_\varepsilon(-F_{\beta\varepsilon}\,\underline{v}_\varepsilon + \underline{e}_\beta, F_{\beta\varepsilon}^T\,\underline{i}_\beta + \underline{j}_\varepsilon) &= \underline{0} \end{aligned} \tag{5.48}$$

From eqn. (5.48) $\underline{i}_\beta$ and $\underline{v}_\varepsilon$ are given as functions of $\underline{e}_\beta$ and $\underline{j}_\varepsilon$, and eqn. (5.48) is equivalent to eqn. (2.39). Using Theorem 3.4 sufficient conditions for a resistive network to be single valued is boundedness and continuity of the hybrid matrix $H_2 = \partial(\underline{i}_\beta, \underline{v}_\varepsilon)/\partial(\underline{v}_\beta, \underline{i}_\varepsilon)$ and

$$|\det (I + H_2F_R)| \geq \varepsilon > 0 \qquad \text{for all } \underline{v}_\beta, \underline{i}_\varepsilon \tag{5.49}$$

Note that the conditions (5.47e) and (5.49) have different forms and hence a question arises whether these two conditions are identical or not. In fact it can be shown that

$$\det (I + H_2F_R) = \det (I + F_RH_2) \tag{5.50}$$

since F_R is skew-symmetric. As skew-symmetric matrices are normal * and any normal matrix is unitarily similar to a diagonal matrix[5], F_R can be expressed as

$$F_R = \bar{P}^T \Lambda P \tag{5.51}$$

where Λ is a diagonal matrix containing characteristic roots of F_R and

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* A matrix A is normal[5] if $\bar{A}^T A = A\bar{A}^T$, where $\bar{A}$ is the matrix obtained from A by replacing each element by its conjugate.

$$\bar{P}^T P = I \tag{5.52}$$

Using eqns. (5.51) and (5.52) we have

$$\det (I + H_2F_R) = \det [P(I + H_2\bar{P}^T\Lambda P) \bar{P}^T] =$$
$$\det [I + PH_2\bar{P}^T\Lambda] \tag{5.53}$$

and

$$\det (I + F_RH_2) = \det [P(+\bar{P}^T\Lambda PH_2) \bar{P}^T] =$$
$$= \det[I + \Lambda PH_2\bar{P}^T] \tag{5.54}$$

Applying the diagonal expansion of determinant and taking into account the fact that any principal minor of ($\Lambda PH_2\bar{P}^T$) is equal to the corresponding principal minor of ($PH_2\bar{P}^T\Lambda$) we arrive at eqn. (5.50). Therefore the condition (5.49) is identical to the condition (5.47e).

The conditions imposed in Theorem 5.8 have a rather "mathematical" form and thus it is not easy to test them. We shall aim at developing a set of more specialized criteria that ensure the existence and uniqueness of solution in a resistive network containing certain classes of elements and/or satisfying certain topological restrictions.

The following theorem can be stated for resistive networks containing PDNEs and independent sources only.

Theorem 5.9

If a resistive network consists of PDNEs only it is a single valued resistive network regardless of the network topology.

Proof

Since all resistive elements in a network are PDNEs the hybrid equations can always be expressed in the form of eqn. (2.37) and the hybrid matrix $H = \partial(\underline{v}_\beta, \underline{i}_\varepsilon)/\partial(\underline{i}_\beta, \underline{v}_\varepsilon)$, associated with the hybrid description of eqn. (2.36), is u. p. d. and bounded. The Jacobian matrix of eqn. (2.37) $J_N = \partial(\underline{e}_\beta, \underline{i}_\varepsilon)/\partial(\underline{i}_\beta, \underline{v}_\varepsilon)$ is equal to

$$J_N = (H + F_R) \qquad (5.55)$$

J_N is u. p. d. because F_R is skew-symmetric, and it is bounded because H is bounded. Thus in eqn. (2.37) $\begin{bmatrix} \underline{e}_\beta \\ \underline{i}_\varepsilon \end{bmatrix}$ is a GQLF of $\begin{bmatrix} \underline{i}_\beta \\ \underline{v}_\varepsilon \end{bmatrix}$ and by Property 3 of GQLFs it possesses a unique inverse function.

Q. E. D.

Theorem 5.9 is a generalization of the result in Ref. 6 to the nonreciprocal case; a similar result appeared as Theorem 5 in Ref. 7. The conditions imposed in Theorem 5.9 are very stringent as only PDNEs are allowed to be contained in a resistive network, but the topology is arbitrary. When not all elements in a network belong to the class of PDNEs, the elements which are not PDNEs have to satisfy certain topological conditions. To express the conditions ensuring that a given resistive network N is single valued all branches of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ are separated into three disjoint subsets[7] as follows:

Subset S_1: set of all branches of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ forming nonseparable connected subnetworks with more than one branch. S_1 is further partitioned into two disjoint subsets $S_{\beta 1}$ and $S_{\varepsilon 1}$ with respect to a chosen tree of N,

Subset S_2: $S_2 = S_{\beta 2} \cup S_{\varepsilon 2}$ where

$S_{\beta 2}$ is a set of branches which are self-loops of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ and whose voltages appear as independent variables in the hybrid description,

$S_{\varepsilon 2}$ is a set of branches which are open branches of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ and whose currents appear as independent variables in the hybrid description,

Subset S_3: $S_3 = S_{\beta 3} \cup S_{\varepsilon 3}$ where

$S_{\beta 3}$ is a set of branches which are self-loops of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ and whose currents appear as independent variables in the hybrid description,

$S_{\varepsilon 3}$ is a set of branches which are open branches of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ and whose voltages appear as independent variables in the hybrid description.

According to the above separation the hybrid description corresponding to all resistive elements can be written in the following partitioned form

$$\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \end{bmatrix} = \begin{bmatrix} \underline{h}_1(\underline{x}_1, \underline{x}_2, \underline{x}_3) \\ \underline{h}_2(\underline{x}_1, \underline{x}_2, \underline{x}_3) \\ \underline{h}_3(\underline{x}_1, \underline{x}_2, \underline{x}_3) \end{bmatrix} \tag{5.56}$$

where

$$\underline{x}_2 = \begin{bmatrix} \underline{v}_{\beta 2} \\ \underline{i}_{\varepsilon 2} \end{bmatrix} \qquad \underline{y}_2 = \begin{bmatrix} \underline{i}_{\beta 2} \\ \underline{v}_{\varepsilon 2} \end{bmatrix} \tag{5.57a}$$

$$\underline{x}_3 = \begin{bmatrix} \underline{i}_{\beta 3} \\ \underline{v}_{\varepsilon 3} \end{bmatrix} \qquad \underline{y}_3 = \begin{bmatrix} \underline{v}_{\beta 3} \\ \underline{i}_{\varepsilon 3} \end{bmatrix} \tag{5.57b}$$

and $\underline{x}_1$, $\underline{y}_1$ contain $\underline{v}_{\beta 1}, \underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{i}_{\varepsilon 1}$. The hybrid matrix corresponding to the above hybrid description is partitioned in accordance with the above separation in the form

$$H = \frac{\partial(\underline{y}_1,\underline{y}_2,\underline{y}_3)}{\partial(\underline{x}_1,\underline{x}_2,\underline{x}_3)} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \qquad (5.58)$$

The following theorem ensuring that a given resistive network is single valued can be stated in terms of the hybrid matrix given in eqn. (5.58).

Theorem 5.10

A resistive network containing resistive elements with the hybrid description of eqn. (5.56) is single valued if

(i) the corresponding hybrid matrix H (eqn. (5.58)) is bounded

(ii) $|\det H_{33}| \geq \varepsilon > 0$ for all $\underline{x}_1,\underline{x}_2,\underline{x}_3$ and

(iii) $(H_{11} - H_{13}H_{33}^{-1}H_{31})$ is u. p. d.

Proof

By Theorem 3.3 the following hybrid description of resistive elements exists

$$\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{x}_3 \end{bmatrix} = \begin{bmatrix} \underline{h}_1'(\underline{x}_1,\underline{x}_2,\underline{y}_3) \\ \underline{h}_2'(\underline{x}_1,\underline{x}_2,\underline{y}_3) \\ \underline{h}_3'(\underline{x}_1,\underline{x}_2,\underline{y}_3) \end{bmatrix} \qquad (5.59)$$

The hybrid matrix $H' = \partial(\underline{h}_1', \underline{h}_2', \underline{h}_3')/\partial(\underline{x}_1,\underline{x}_2,\underline{y}_3)$ is bounded and its submatrix $H_{11}' = \partial\underline{h}_1'/\partial\underline{x}_1$ is expressible by H as

$$H_{11}' = H_{11} - H_{13}H_{33}^{-1}H_{31} \qquad (5.60)$$

Partition vectors $\underline{e}_\beta$ and $\underline{i}_\varepsilon$ in accordance with the partitioning of resistive branches

$$\underline{e}_\beta = \begin{bmatrix} \underline{e}_{\beta 1} \\ \underline{e}_{\beta 2} \\ \underline{e}_{\beta 3} \end{bmatrix} \qquad \underline{i}_\varepsilon = \begin{bmatrix} \underline{i}_{\varepsilon 1} \\ \underline{i}_{\varepsilon 2} \\ \underline{i}_{\varepsilon 3} \end{bmatrix}$$

Because of the partitioning of all resistive branches into subsets S_1, S_2 and S_3 the matrix $F_{\beta\varepsilon}$, appearing in eqns. (2.27a) and (2.27b) has the form

$$F_{\beta\varepsilon} = \begin{bmatrix} F_{\beta 1 \varepsilon 1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.61}$$

and by substituting eqn. (5.61) into eqns. (2.28a) and (2.28b) we have

$$\begin{aligned} \underline{v}_{\beta 2} &= \underline{e}_{\beta 2} & \underline{i}_{\varepsilon 2} &= \underline{i}_{\varepsilon 2} \\ \underline{v}_{\beta 3} &= \underline{e}_{\beta 3} & \underline{i}_{\varepsilon 3} &= \underline{i}_{\varepsilon 3} \end{aligned} \tag{5.62}$$

and using eqns. (5.57a) and (5.57b)

$$\underline{x}_2 = \begin{bmatrix} \underline{e}_{\beta 2} \\ \underline{i}_{\varepsilon 2} \end{bmatrix} \qquad \underline{y}_3 = \begin{bmatrix} \underline{e}_{\beta 3} \\ \underline{i}_{\varepsilon 3} \end{bmatrix} \tag{5.63}$$

Therefore

$$\underline{y}_1 = \underline{h}_1'(\underline{x}_1, \underline{e}_{\beta 2}, \underline{i}_{\varepsilon 2}, \underline{e}_{\beta 3}, \underline{i}_{\varepsilon 3}) \tag{5.64}$$

and the relation between $\underline{x}_1$ and $\underline{y}_1$ is a GQLF for all $\underline{e}_{\beta 2}, \underline{i}_{\varepsilon 2}, \underline{e}_{\beta 3}, \underline{i}_{\varepsilon 3}$. By Property 3 of GQLFs eqn. (5.64) can be transformed into the following form

$$\begin{aligned} \underline{v}_{\beta 1} &= \underline{v}_{\beta 1}(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{i}_{\varepsilon 2}, \underline{e}_{\beta 3}, \underline{i}_{\varepsilon 3}) \\ \underline{i}_{\varepsilon 1} &= \underline{i}_{\varepsilon 1}(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{i}_{\varepsilon 2}, \underline{e}_{\beta 3}, \underline{i}_{\varepsilon 3}) \end{aligned} \tag{5.65}$$

where $H'_{11} = \partial(\underline{v}_{\beta 1}, \underline{i}_{\varepsilon 1})/\partial(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1})$ is u. p. d. and bounded. The set of hybrid equations corresponding to eqn. (2.45) has the form

$$\begin{aligned} \underline{v}_{\beta 1}(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{i}_{\varepsilon 2}, \underline{e}_{\beta 3}, \underline{i}_{\varepsilon 3}) + F_{\beta 1 \varepsilon 1}\, \underline{v}_{\varepsilon 1} &= \underline{e}_{\beta 1} \\ -F_{\beta 1 \varepsilon 1}^{T}\, \underline{i}_{\beta 1} + \underline{i}_{\varepsilon 1}(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{i}_{\varepsilon 2}, \underline{e}_{\beta 3}, \underline{i}_{\varepsilon 3}) &= \underline{i}_{\varepsilon 1} \end{aligned} \tag{5.66a}$$

The Jacobian matrix $J_N = \partial(\underline{e}_{\beta 1}, \underline{i}_{\varepsilon 1})/\partial(\underline{i}_{\beta 1}, \underline{v}_{\varepsilon 1})$ of eqn. (5.66a) is

$$J_N = H''_{11} + \begin{bmatrix} 0 & F_{\beta 1 \varepsilon 1} \\ -F_{\beta 1 \varepsilon 1}^{T} & 0 \end{bmatrix} \tag{5.66b}$$

As H''_{11} is u. p. d. and bounded and the second matrix on the r. h.s. of eqn. (5.66b) is skew-symmetric $\begin{bmatrix} \underline{e}_{\beta 1} \\ \underline{i}_{\varepsilon 1} \end{bmatrix}$ is a GQLF of $\begin{bmatrix} \underline{i}_{\beta 1} \\ \underline{v}_{\varepsilon 1} \end{bmatrix}$ for all $\underline{e}_{\beta 2}$, $\underline{i}_{\varepsilon 2}$, $\underline{e}_{\beta 3}$, $\underline{i}_{\varepsilon 3}$ and thus the resistive network in single valued.

Q. E. D.

The result of Theorem 5.10 can be given the following interpretation in network terms. If there are b_R resistive branches in a network N, the set of all resistive elements represents a b_R-port N_R. Let all ports of N_R be classified into six disjoint subsets $P_{\beta 1}, P_{\beta 2}, P_{\beta 3}, P_{\varepsilon 1}, P_{\varepsilon 2}, P_{\varepsilon 3}$ in accordance with the classification of all resistive branches into subsets S_1, S_2 and S_3 (Fig. 5.7). Connect independent voltage sources to ports $P_{\beta 2}$ and $P_{\beta 3}$ and independent current sources to ports $P_{\varepsilon 2}$ and $P_{\varepsilon 3}$ in N_R; then choose ports $P_{\beta 1}, P_{\varepsilon 1}$ as ports of an $(b_{\beta 1}+b_{\varepsilon 1})$-port resistive element N'_R (Fig. 5.7). When N_R is voltage-controlled with respect to ports $P_{\beta 2}$ and $P_{\beta 3}$ and is current-controlled with respect to ports $P_{\varepsilon 2}$ and $P_{\varepsilon 3}$, then the resistive network N is single valued if in addition N'_R represents a PDNE.

A useful corollary of Theorem 5.10 is concerned with a resistive network which contains PDNEs and PNEs.

Corollary 5.4

Suppose that a resistive network N contains PDNEs, PNEs and independent sources. Then, N is single valued if there is not more than one branch of each PNE in any connected nonseparable subnetwork of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ which contains more than one branch*.

Let us assume that all branches which are self-loops in $N\{\mathcal{B}_E; \mathcal{B}_J\}$ are voltage-controlled and all branches that are open loops in $N\{\mathcal{B}_E; \mathcal{B}_J\}$ are current-controlled and thus the subset S_3 is empty. If there is no coupling between branches $S_{\beta 1}$ and $S_{\varepsilon 1}$ the conditions of Theorem 5.10 may be relaxed in the sense that the subnetwork N_R' in Fig. 5.7, derived from a given resistive network, is not necessarily a PDNE.

Theorem 5.11

Assume that in a resistive network N the resistive branches are coupled and

(i) the branches of set $S_{\beta 2}$ are voltage-controlled

(ii) the branches of $S_{\varepsilon 2}$ are current-controlled

(iii) $\underline{i}_{\beta 1}$, $\underline{v}_{\varepsilon 1}$ are expressed as:

$$\begin{aligned} \underline{i}_{\beta 1} &= \underline{h}_{\beta 1}(\underline{v}_{\beta 1}, \underline{v}_{\beta 2}, \underline{i}_{\varepsilon 2}) \\ \underline{v}_{\varepsilon 1} &= \underline{h}_{\varepsilon 1}(\underline{i}_{\varepsilon 1}, \underline{v}_{\beta 2}, \underline{i}_{\varepsilon 2}) \end{aligned} \tag{5.67}$$

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* In other words the subset S_1 in N must not contain more than one branch of each PNE.

where $\underline{h}_{\beta 1}$ and $\underline{h}_{\varepsilon 1}$ are of class $C^{(1)}$.

Then the resistive network N is single valued if in addition conditions (C1) and (C2) or (C1) and (C3) are satisfied simultaneously:

(C1) $\partial \underline{h}_{\beta 1}/\partial \underline{v}_{\beta 1}$ and $\partial \underline{h}_{\varepsilon 1}/\partial \underline{i}_{\varepsilon 1}$ are positive semidefinite (not necessarily symmetric) and bounded

(C2) $\partial \underline{h}_{\beta 1}/\partial \underline{v}_{\beta 1}$ or $\partial \underline{h}_{\varepsilon 1}/\partial \underline{i}_{\varepsilon 1}$ are positive definite (not necessarily symmetric)

(C3) $\partial \underline{h}_{\beta 1}/\partial \underline{v}_{\beta 1}$ or $\partial \underline{h}_{\varepsilon 1}/\partial \underline{i}_{\varepsilon 1}$ are positive semidefinite and symmetric.

Proof

Using eqn. (5.67) the hybrid equations of the resistive network N are obtained from eqn. (2.46) in the form

$$\begin{aligned} &\underline{v}_{\beta 1} + F_{\beta 1 \varepsilon 1}\underline{h}_{\varepsilon 1}(\underline{i}_{\varepsilon 1}, \underline{e}_{\beta 2}, \underline{i}_{\varepsilon 2}) = \underline{e}_{\beta 1} \\ &-F_{\beta 1 \varepsilon 1}^{T}\underline{h}_{\beta 1}(\underline{v}_{\beta 1}, \underline{e}_{\beta 2}, \underline{i}_{\varepsilon 2}) + \underline{i}_{\varepsilon 1} = \underline{i}_{\varepsilon 1} \end{aligned} \tag{5.68}$$

where $\underline{e}_{\beta 2}$, $\underline{i}_{\varepsilon 2}$ may be considered as parameters, Eqn. (5.68) has the same form as eqn. (3.64) and application of Theorem 3.7 gives the result of Theorem 5.11.

Q. E. D.

Theorem 5.11 may be viewed as an extention of Theorem 1.1 in Ref. 8 with respect to conditions (C2) and (C3). In Ref. 8, instead of condition (C2), $\partial \underline{h}_{\beta 1}/\partial \underline{v}_{\beta 1}$ or $\partial \underline{h}_{\varepsilon 1}/\partial \underline{i}_{\varepsilon 1}$ are required to be positive definite symmetric matrices, and instead of condition (C3) these Jacobian matrices are required to be diagonal positive semidefinite matrices; yet in Ref. 8 there is no requirement for $\partial \underline{h}_{\beta 1}/\partial \underline{v}_{\beta 1}$ and $\partial \underline{h}_{\varepsilon 1}/\partial \underline{i}_{\varepsilon 1}$ to be bounded.

In order to interpret the result of Theorem 5.11 in network terms two resistive subnetworks of N will be identified. Let the resistive network when viewed from ports $P_{\beta 1}$ with ports $P_{\beta 2}$ connected to independent voltage sources and ports $P_{\varepsilon 2}$ connected to independent current sources be termed $N_{\beta 1}$ (Fig. 5.8a); the conditions at ports $P_{\varepsilon 1}$ may be arbitrary as branches $P_{\beta 1}$ are not coupled to branches $P_{\varepsilon 1}$. Similarly, let the resistive network when viewed from ports $P_{\varepsilon 1}$ with ports $P_{\beta 2}$ connected to independent voltage sources and ports $P_{\varepsilon 2}$ connected to independent current sources be termed $N_{\varepsilon 1}$ (Fig. 5.8b). By the condition (C1) imposed in Theorem 5.11 $N_{\beta 1}$ and $N_{\varepsilon 1}$ are required to be PSDNEs. When both networks $N_{\beta 1}$ and $N_{\varepsilon 1}$ are nonreciprocal the resistive network N is single valued if at least one of them is strictly locally passive. When at least one of the networks $N_{\beta 1}$ and $N_{\varepsilon 1}$ is reciprocal N is single valued even when $N_{\beta 1}$ and $N_{\varepsilon 1}$ are both locally passive.

An interesting conclusion can be derived from Theorem 5.11 for resistive networks containing gyrators. Such networks may not be single valued even when both branches of each gyrator lie in a tree or in its cotree. An example of a singular linear network is shown in Fig. 5.9, where a gyrator N_R' with the conductance matrix $G_1 = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$ is connected in parallel to a gyrator N_R'' with the conductance matrix $G_2 = -G_1$. Thus the conductance matrix of the interconnected network is $G = G_1 + G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the network in Fig. 5.9 is singular.

Our next result is concerned with a resistive network N containing PDNEs and different kinds of PSDNEs where PSDNEs satisfy certain topological conditions. PNEs can be included as well if there is not more than one branch of each PNE in any connected nonseparable subnetwork of $N\{\mathcal{B}_E; \mathcal{B}_J\}$ containing more than one branch. It is assumed that a tree T_R exists in N such that all voltage controlled PSDNEs, denoted as $N_{\beta b}$, lie in the cotree of T_R and all current-controlled PSDNEs, denoted as $N_{\varepsilon b}$, lie in T_R. Thus the hybrid descriptions of $N_{\beta b}$ and $N_{\varepsilon b}$ are

$$\underline{i}_{\beta b} = \underline{h}_{\beta b}(\underline{v}_{\beta b}) \tag{5.69}$$

$$\underline{v}_{\varepsilon b} = \underline{h}_{\varepsilon b}(\underline{i}_{\varepsilon b}) \tag{5.70}$$

For the third kind of PSDNE, denoted as N_a, we assume that the following three conditions are satisfied:

(i) $b_{\beta a}$, the number of branches of N_a lying in the cotree of T_R, is equal to $b_{\varepsilon a}$, the number of branches of N_a lying in T_R, $b_{\beta a} = b_{\varepsilon a}$,

(ii) N_a is voltage-controlled with respect to its tree branches and is current-controlled with respect to its links,

$$\begin{aligned} \underline{v}_{\beta a} &= \underline{h}_{\beta a}(\underline{v}_{\varepsilon a}) \\ \underline{i}_{\varepsilon a} &= \underline{h}_{\varepsilon a}(\underline{i}_{\beta a}) \end{aligned} \tag{5.71}$$

(iii) H_a, the hybrid matrix of N_a, is skew-symmetric

$$H_a = \partial(\underline{h}_{\beta a}, \underline{h}_{\varepsilon a}) / \partial(\underline{i}_{\beta a}, \underline{v}_{\varepsilon a}) = \begin{bmatrix} 0 & K_a \\ -K_a^T & 0 \end{bmatrix} \tag{5.72}$$

where $$K_a = \partial \underline{h}_{\beta a} / \partial \underline{v}_{\varepsilon a} = -(\partial \underline{h}_{\varepsilon a} / \partial \underline{i}_{\beta a})^T \tag{5.73}$$

and K_a is a square matrix.

Note that when N_a is linear it corresponds to a bank of ideal transformers.

There are no topological restrictions in N for PDNEs, denoted as N_c; with respect to the tree T_R links of N_c are denoted by $N_{\beta c}$ and tree branches of N_c are denoted by $N_{\varepsilon c}$.

If the resistive branches in N are numbered consecutively, starting with links $\mathcal{B}_{\beta a}$ and following with $\mathcal{B}_{\beta b}$, $\mathcal{B}_{\beta c}$ and then tree branches $\mathcal{B}_{\varepsilon a}$, $\mathcal{B}_{\varepsilon b}$, $\mathcal{B}_{\varepsilon c}$ then $F_{\beta\varepsilon}$, the submatrix of the fundamental loop matrix B, can be partitioned into the following 9 submatrices

$$F_{\beta\varepsilon} = \begin{bmatrix} F_{\beta a\varepsilon a} & F_{\beta a\varepsilon b} & F_{\beta a\varepsilon c} \\ F_{\beta b\varepsilon a} & F_{\beta b\varepsilon b} & F_{\beta b\varepsilon c} \\ F_{\beta c\varepsilon a} & F_{\beta c\varepsilon b} & F_{\beta c\varepsilon c} \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & F_3 \\ F_4 & F_5 & F_6 \\ F_7 & F_8 & F_9 \end{bmatrix} \tag{5.74}$$

In eqn. (5.74) both matrices are partitioned conformably; the last form will be used in the sequel for the sake of simplicity in notation.

The following theorem gives sufficient conditions for a resistive network, described above, to be single valued.

<u>Theorem 5.12</u>

Suppose that a resistive network N contains PDNEs, N_c, and PSDNEs, $N_{\beta b}$, $N_{\varepsilon b}$, and N_a with the hybrid descriptions given by eqns. (5.69), (5.70) and (5.71), respectively and that the hybrid matrix H_a of N_a (eqn. (5.72)) is skew-symmetric. Furthermore, assume that a tree T_R exists in N such that all branches of $N_{\varepsilon b}$ and all voltage-controlled branches of N_a lie in T_R but all branches of $N_{\beta b}$ and all current-controlled branches of N_a lie in its cotree. Then, N is single valued if

(i) the hybrid matrices of all resistive elements are bounded

$$\text{(ii)} \quad \det\left[H_a + \begin{bmatrix} 0 & F_1 \\ -F_1^T & 0 \end{bmatrix}\right] \geq \varepsilon > 0 \tag{5.75}$$

(iii) either $N_{\beta b}$ or $N_{\varepsilon b}$ is reciprocal or $N_{\beta b}$ or $N_{\varepsilon b}$ is strictly locally passive.

Proof

As N_c is a PDNE, the hybrid description

$$\begin{aligned} \underline{v}_{\beta c} &= \underline{v}_{\beta c}(\underline{i}_{\beta c}, \underline{v}_{\varepsilon c}) \\ \underline{i}_{\varepsilon c} &= \underline{i}_{\varepsilon c}(\underline{i}_{\beta c}, \underline{v}_{\varepsilon c}) \end{aligned} \tag{5.76}$$

exists. Substituting eqn. (5.74) and the hybrid descriptions of N_a, $N_{\beta b}$, $N_{\varepsilon b}$ and N_c (eqns. (5.69)-(5.71), (5.76) into eqns. (2.24) and (2.25) the following set of hybrid equations of N is obtained

$$\begin{bmatrix} \underline{h}_{\beta a}(\underline{v}_{\varepsilon a}) \\ \underline{v}_{\beta b} \\ \underline{v}_{\beta c}(\underline{i}_{\beta c}, \underline{v}_{\varepsilon c}) \end{bmatrix} + \begin{bmatrix} F_1 & F_2 & F_3 \\ F_4 & F_5 & F_6 \\ F_7 & F_8 & F_9 \end{bmatrix} \begin{bmatrix} \underline{v}_{\varepsilon a} \\ \underline{h}_{\varepsilon b}(\underline{i}_{\varepsilon b}) \\ \underline{v}_{\varepsilon c} \end{bmatrix} = \begin{bmatrix} \underline{e}_{\beta a} \\ \underline{e}_{\beta b} \\ \underline{e}_{\beta c} \end{bmatrix} \tag{5.77}$$

$$-\begin{bmatrix} F_1^T & F_4^T & F_7^T \\ F_2^T & F_5^T & F_8^T \\ F_3^T & F_6^T & F_9^T \end{bmatrix} \begin{bmatrix} \underline{i}_{\beta a} \\ \underline{h}_{\beta b}(\underline{v}_{\beta b}) \\ \underline{i}_{\beta c} \end{bmatrix} + \begin{bmatrix} \underline{h}_{\varepsilon a}(\underline{i}_{\beta a}) \\ \underline{i}_{\varepsilon b} \\ \underline{i}_{\varepsilon c}(\underline{i}_{\beta c}, \underline{v}_{\varepsilon c}) \end{bmatrix} = \begin{bmatrix} \underline{i}_{\varepsilon a} \\ \underline{i}_{\varepsilon b} \\ \underline{i}_{\varepsilon c} \end{bmatrix}$$

where $\underline{i}_\varepsilon$ and $\underline{e}_\beta$ are partitioned conformably to $\underline{i}_\varepsilon$ and $\underline{v}_\beta$. Denote the incremental conductance matrix of $N_{\beta b}$ by $G_{\beta b} = \partial \underline{h}_{\beta b}/\partial \underline{v}_{\beta b}$ and the incremental resistance matrix of $N_{\varepsilon b}$ by $R_{\varepsilon b} = \partial \underline{h}_{\varepsilon b}/\partial \underline{i}_{\varepsilon b}$; similarly let H_c be the hybrid matrix associated with the description of eqn. (5.76)

$$H_c = \partial(\underline{v}_{\beta c}, \underline{i}_{\varepsilon c})/\partial(\underline{i}_{\beta c}, \underline{v}_{\varepsilon c}) = \begin{bmatrix} K_{\beta\beta} & K_{\beta\varepsilon} \\ K_{\varepsilon\beta} & K_{\varepsilon\varepsilon} \end{bmatrix} \qquad (5.78)$$

Using the notation for the hybrid matrices of N_a, $N_{\beta b}$, $N_{\varepsilon b}$ and N_c, the Jacobian matrix J_N associated with eqn. (5.77) may be recast in the form

$$J_N = \left[\begin{array}{cc|cc|cc} 0 & K_a + F_1 & 0 & F_2 R_{\varepsilon b} & 0 & F_3 \\ -K_a^T - F_1^T & 0 & -F_4^T G_{\beta b} & 0 & -F_7^T & 0 \\ \hline 0 & F_4 & 1 & F_5 R_{\varepsilon b} & 0 & F_6 \\ -F_2^T & 0 & -F_5^T G_{\beta b} & 1 & -F_8^T & 0 \\ 0 & F_7 & 0 & F_8 R_{\varepsilon b} & K_{\beta\beta} & K_{\beta\varepsilon} + F_9 \\ -F_3^T & 0 & -F_6^T G_{\beta b} & 0 & K_{\varepsilon\beta} - F_9^T & K_{\varepsilon\varepsilon} \end{array}\right]$$

(5.79)

As all hybrid matrices appearing in J_N are bounded, J_N is bounded and, applying Theorem 3.2, N is single valued if det $J_N \geq \varepsilon_o > 0$. Since by condition (ii) in Theorem 5.12

$$\det\left[H_a + \begin{bmatrix} 0 & F_1 \\ -F_1^T & 0 \end{bmatrix} \right] = \det \left[\begin{array}{c|c} 0 & K_a + F_1 \\ \hline -K_a^T - F_1^T & 0 \end{array}\right] \geq \varepsilon > 0$$

(5.80)

Lemma 4.1 can be applied to det J_N. Hence we have

$$\det J_N = \det(K_a+F_1)^2 \det \left[\begin{array}{c|c|c|c} I & F_5'R_{\varepsilon b} & 0 & F_6' \\ \hline -F_5'^T G_{\beta b} & I & -F_8'^T & 0 \\ \hline 0 & F_8'R_{\varepsilon b} & K_{\beta\beta} & K_{\beta\varepsilon}+F_9' \\ \hline -F_6'^T G_{\beta b} & 0 & K_{\varepsilon\beta}-F_9'^T & K_{\varepsilon\varepsilon} \end{array}\right]$$

(5.81)

where
$$F_5' = F_5 - F_4(K_a^T+F_1^T)^{-1}F_2$$
$$F_6' = F_6 - F_4(K_a^T+F_1^T)^{-1}F_3$$
$$F_8' = F_8 - F_7(K_a^T+F_1^T)^{-1}F_2$$
$$F_9' = F_9 - F_7(K_a^T+F_1^T)^{-1}F_3$$

(5.82)

Using condition (iii) it follows from the proof of Theorem 3.7 that

$$\det \begin{bmatrix} I & F_5'R_{\varepsilon b} \\ -F_5'^T G_{\beta b} & 0 \end{bmatrix} > 1 \qquad (5.83)$$

Applying Lemma 4.1 to the determinant on the extreme right of eqn. (5.81) we have

$$\det J_N = \det(K_a+F_1)^2 \det \begin{bmatrix} I & F_5'R_{\varepsilon b} \\ -F_5'^T G_{\beta b} & I \end{bmatrix} \det M$$

(5.84)

where M is

$$M = H_c + \left[\begin{array}{c|c} 0 & F_9' \\ \hline -F_9'^T & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & -F_8' \\ \hline F_6'^T & 0 \end{array}\right] \left[\begin{array}{c|c} G_{\beta b} & 0 \\ \hline 0 & R_{\varepsilon b} \end{array}\right] \left[\begin{array}{c|c} I & F_5'R_{\varepsilon b} \\ \hline -F_5'^T G_{\beta b} & I \end{array}\right]^{-1} \left[\begin{array}{c|c} 0 & F_6' \\ \hline -F_8'^T & 0 \end{array}\right]$$

(5.85)

A close inspection of eqn. (5.85) reveals that M is u. p. d. since H_c is u. p. d. and $(M - H_c)$ is positive semidefinite. Thus det $M > \dot{\varepsilon}_1 > 0$ and from eqn. (5.84) and using the inequalities (5.80) and (5.83)

$$\det J_N > \varepsilon^2 \varepsilon_1 > 0$$

Therefore the condition (i) in Theorem 3.2 is fulfilled and eqn. (5.77) has a unique solution. Hence N is single valued.

Q. E. D.

<u>Remark:</u> Similarly to Theorems 5.10 and 5.11, Theorem 5.12 can be extended to be applicable to a resistive network N′ containing network elements that are not PDNEs or PSDNEs; assume that $N'\{\mathcal{B}_E; \mathcal{B}_J\}$ consists of subsets $S_1 = S_{\beta 1} \cup S_{\varepsilon 1}$ and $S_2 = S_{\beta 2} \cup S_{\varepsilon 2}$ whereas subset S_3 is empty. As link voltages $\underline{v}_{\beta 2}$ and tree branch currents $\underline{i}_{\varepsilon 2}$ are determined directly by independent voltage and current sources respectively, only $N'' = N'\{\mathcal{B}_E, \mathcal{B}_{\beta 2}; \mathcal{B}_J, \mathcal{B}_{\varepsilon 2}\}$, which is a subnetwork of N′ and where $\underline{v}_{\beta 2}$ and $\underline{i}_{\varepsilon 2}$ are parameters, has to be solved. The set of all resistive elements in N′forms an n′-port N'_R. The set of all resistive elements in N′′ forms an n′′-port N''_R which can be obtained from N'_R as shown in Fig. 5.10. Only N′′ has to obey conditions imposed in Theorem 5.12 for N′ to be single valued. PNEs can obviously be included in N′ provided not more than one branch of each PNE belongs to the set S_1 formed in N′.

According to Theorem 5.12 locally passive nonreciprocal PSDNEs such as gyrators may not lie both in the tree and the cotree. However by an additional topological constraint locally passive nonreciprocal PSDNEs may be placed both in the tree and the cotree.

Corollary 5.5

Assume that in Theorem 5.12 the condition (iii) is replaced by the following condition (iv):

(iv) $F_4 = 0$ and $F_5 = 0$ or $F_2 = 0$ and $F_5 = 0$.

Then the result of Theorem 5.12 holds.

Note that the condition (iv) implies that in the fundamental loop associated with any branch of $\mathcal{B}_{\beta b}$ there is no branch of $\mathcal{B}_{\varepsilon a}$ and $\mathcal{B}_{\varepsilon b}$, or that in the fundamental cut-set associated with any branch of $\mathcal{B}_{\varepsilon b}$ there is no branch of $\mathcal{B}_{\beta a}$ and $\mathcal{B}_{\beta b}$.

So far some results concerning networks of PDNEs, PSDNEs and PNEs have been stated. However, Theorems 5.9 - 5.12 can be combined with the results on interconnections of network elements, as given in Section 5.2; in such a way it is frequently possible first to reduce a subnetwork N_i in a given resistive network N into a PNE or PDNE or PSDNE and then apply one of the theorems or corollaries of Section 5.3. Another approach which is often useful is to reduce a given resistive network N into a PNE or PDNE with respect to the ports corresponding to voltage and current sources of N, and then prove that N is single valued going in the reverse direction from the reduced network to the original one.

5.4 EXAMPLES

In this section we shall present some examples to illustrate the usefulness of the theory developed in this chapter. Resistive networks containing PNEs that may be locally active will be of special interest. Throughout this section we shall assume that each transistor represents a locally active three-terminal resistive element which corresponds to a PNE in all three orientations.

Example 1:

Consider the direct current regulator[9], N, shown in Fig. 5.11a. Assume that Zener diode D is a two-terminal voltage-controlled resistor with monotonic characteristic and R_1, R_2 and R_L are two-terminal linear positive resistors. By Theorem 5.3 R_1 and the transistor T can be substituted by a three-terminal resistive element N′ which is connected to nodes 1,2 and 5 and it is a PNE in all three orientations. Then N′ and D are reduced to a three-terminal resistive element N′′ which is positive in all three orientations. R_2 and N′′ form a three-terminal positive resistive element N′′′. When R_L is added to N′′′ a three-terminal resistive element $\mathcal{N}$ (Fig. 5.11b) which is positive in all three orientations is obtained.

Since a hybrid description with v_1, i_2 as independent variables exists for $\mathcal{N}$, i_1, v_2 (Fig. 5.11b) are defined for $\mathcal{N}$. Retracing the steps from $\mathcal{N}$ to N we see that voltages and currents at both ports of N′′′, N′′ and N′ are uniquely determined and N in Fig. 5.11a is single valued for all values of E.

Example 2:

Consider the multistage dc coupled transistor amplifier N consisting of n identical stages as shown in Fig. 5.12a. We can show that N has u unique solution for any value of sources e, E_1, E_2,...,E_n. Applying Theorem 5.3

several times the i-th stage N_i of N, $i = 1, 2, \ldots, n$, can be reduced to a three-terminal resistive element $\mathcal{N}_i$ which is positive in all three orientations and is connected to nodes $i, i+1$ and $n+2$. In this way the network shown in Fig. 5.12b is obtained. By Theorem 5.7 $\mathcal{N}_1$ and $\mathcal{N}_2$ form a 4-terminal PNE $\mathcal{N}_{1,2}$; $\mathcal{N}_{1,2}$ and $\mathcal{N}_3$ represent a 5-terminal PNE and continuing this procedure we finally arrive at (n+2)-terminal positive resistive element $\mathcal{N}_{1,2,\ldots,n+1} = \mathcal{N}$ shown in Fig. 5.12c. Following the same argument as in example 1 we conclude that the network in Fig. 5.12a is single valued for all values of independent sources; this conclusion represents an interesting network theoretic result.

Example 3:

The resistive network N in Fig. 5.13a is a dc differential amplifier. Let $R_1, R_2, \ldots, R_7$ be two-terminal linear positive resistors. Using Theorem 5.3 R_1, R_2, R_3 and T_1 can be replaced by a three-terminal resistive element $\mathcal{N}_1$ which is connected to nodes 1,4 and 9 and which is positive in all three orientations; R_4, R_5, R_6 and T_2 can be reduced to a three-terminal resistive element $\mathcal{N}_2$ that is connected to nodes 4,8 and 9 and which is positive in all three orientations. Thus N can be replaced by N′ shown in Fig. 13b. Theorem 5.9 can be applied to N′ to prove that N′ is single valued. Note that all resistive elements in N′ belong to the set S_1. In $N'\{\mathcal{B}_E; \mathcal{B}_J\}$ both ports of $\mathcal{N}_1$ are connected in parallel and this parallel interconnection can be replaced by a two-terminal resistor $\mathcal{N}_1'$; using Property 1 of three-terminal resistive elements which represent a PNE in all three orientations we conclude that $\mathcal{N}_1'$ is quasilinear. Similarly $\mathcal{N}_2$ can be replaced in $N'\{\mathcal{B}_E; \mathcal{B}_J\}$ by a quasilinear resistor $\mathcal{N}_2'$. In this way $N'\{\mathcal{B}_E;\mathcal{B}_J\}$ consists of 3 quasilinear resistors $\mathcal{N}_1'$, $\mathcal{N}_2'$ and R_7 and by Theorem 5.9 N' is single valued.

Example 3 is interesting from the following point of view: frequently theorems of Sections 5.3, ensuring that a network N is single valued, cannot be applied directly; but the network has to be reduced at first on the basis of the properties of the series-parallel interconnections until one of Theorems 5.9 – 5.12 is applicable.

Example 4:

The resistive network N, shown in Fig. 5.14, appears in the dynamic analysis of the flip-flop obtained if voltage sources e_1 and e_2 are replaced by two capacitors. Let $R_1, R_2, \ldots, R_6$ be two-terminal linear positive resistors. Since one branch of each transistor is a self-loop of $N\{\mathcal{B}_E;\mathcal{B}_J\}$ N is single valued by Corollary 5.4.

Example 5:

Consider resistive network N, shown in Fig. 5.15a, where N_a is a linear ideal transformer, $N_{\varepsilon b}$ is a linear gyrator, D is a diode with voltage-controlled monotonic characteristic of class $C^{(1)}$ and of bounded slope, R_1 and R_2 are two-terminal linear positive resistors. By Property 3 of PNEs subnetwork N_1 containing T_1 and E_1 represents a quasilinear two-terminal resistor $N_{\varepsilon c}$ (Fig. 5.15b) between nodes 1 and 6. Applying Theorem 5.3 T_2, R_1 and R_2 can be reduced to a three-terminal resistive network N_2' connected to nodes 5,6 and 9 and it is a PNE in all three orientations; N_2' and J can then be reduced to a quasilinear resistor $N_{\beta c}$ between terminals 5 and 6 (Fig. 5.15b). Resistive network N′shown in Fig. 5.15b corresponds to N in Fig. 5.15a. A tree containing branches $\mathcal{B}_{E2}$, $\mathcal{B}_{\varepsilon a}$, $\mathcal{B}_{\varepsilon b1}$, $\mathcal{B}_{\varepsilon b2}$, $\mathcal{B}_{\varepsilon c}$ can be chosen in N′ and thus Theorem 5.12 is applicable for N′. Analysing the subnetwork $N'\{\mathcal{B}_{\varepsilon b}, \mathcal{B}_{\varepsilon c}; \mathcal{B}_{\beta b}, \mathcal{B}_{\beta c}\}$ it is easy to see that the inequality (5.75) is fulfilled. Since in addition $N_{\beta b}$ is reciprocal, N′ is single valued and thus N in Fig. 5.15a is single valued.

5.5 SUMMARY

In this chapter the problem of the existence and uniqueness of solutions in one-element-kind networks has been considered. The emphasis has been laid on resistive networks containing locally active multiterminal elements.

In order to analyse such networks it is helpful to study the properties of the series-parallel interconnections of nonlinear resistive elements such as PNEs, PDNEs and PSDNEs. Particularly interesting results have been obtained for series (or parallel) interconnections of a three-terminal resistive element which is a PNE in all three orientations and certain two-terminal resistors with monotonic characteristics. In this way transistor networks can be reduced to a much simpler form where the existence and uniqueness is more easily studied.

Sufficient conditions ensuring that a given resistive network is single valued have been stated for networks containing different classes of elements and/or satisfying certain topological constraints; thus such conditions are given in network terms rather than in "mathematical" terms and are applicable to many practical networks. A number of transistor networks is presented to illustrate the applicability of the theory developed in this chapter.

5.6 REFERENCES

1 Desoer, C. A. and Katzenelson, J., "Nonlinear RLC networks", Bell Sys. Tech. J., vol. 44, pp. 161-198, January 1965

2 Vehovec M., "Some properties of nonlinear networks containing positive resistive elements", to be published in Proc. of the 13^{th} Midwest Symposium on Circuit Theory, Minneapolis, 1970.

3 Vehovec M., "Normal form of a class of n-coupled nonreciprocal nonlinear networks", to be published in Automatika, Theoretical Supplement.

4 Chua, L. O., "Analysis and synthesis of multivalued memoryless nonlinear networks", IEEE Trans, on Circuit Theory, vol. CT-14, pp. 192-209, June 1967.

5 Perlis, S., "Theory of matrices", Reading, Mass., Addison-Wesley, 1952, pp. 191-195.

6 Stern, T. E., "On the equations of nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 74-81, March 1966.

7 Ohtsuki, T. and Watanabe, H., "State-variable analysis of RLC networks containing nonlinear coupling elements", IEEE Trans. on Circuit Theory, vol. CT-16, pp. 26-38, February 1969.

8 Varayia, P. P. and Liu, R., "Normal form and stability of a class of coupled nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 413-418, December 1966

9 Hurley, R. B., "Junction transistor electronics", John Wiley and Sons, New York, 1958, p. 229.

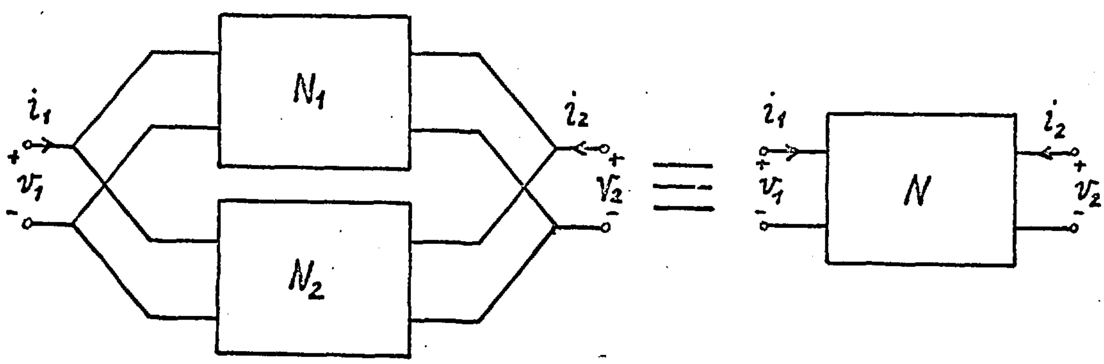

Fig. 5.1. A parallel-parallel interconnection of two-port resistive elements.

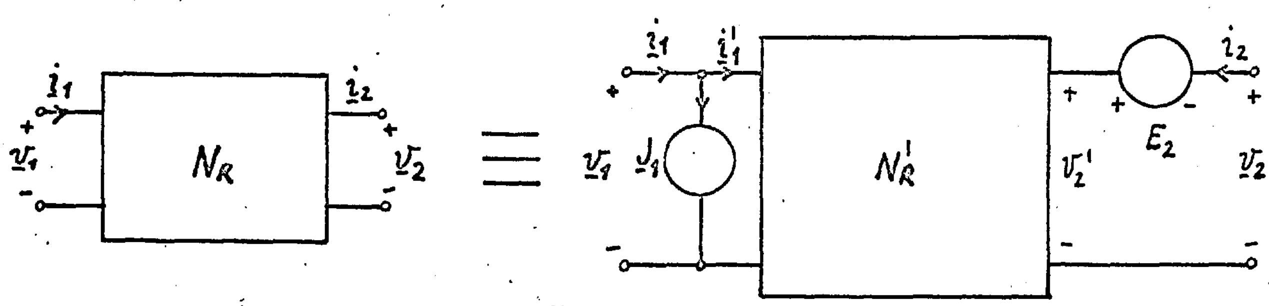

Fig. 5.2. Illustration of Property 2 in Section 5.1.

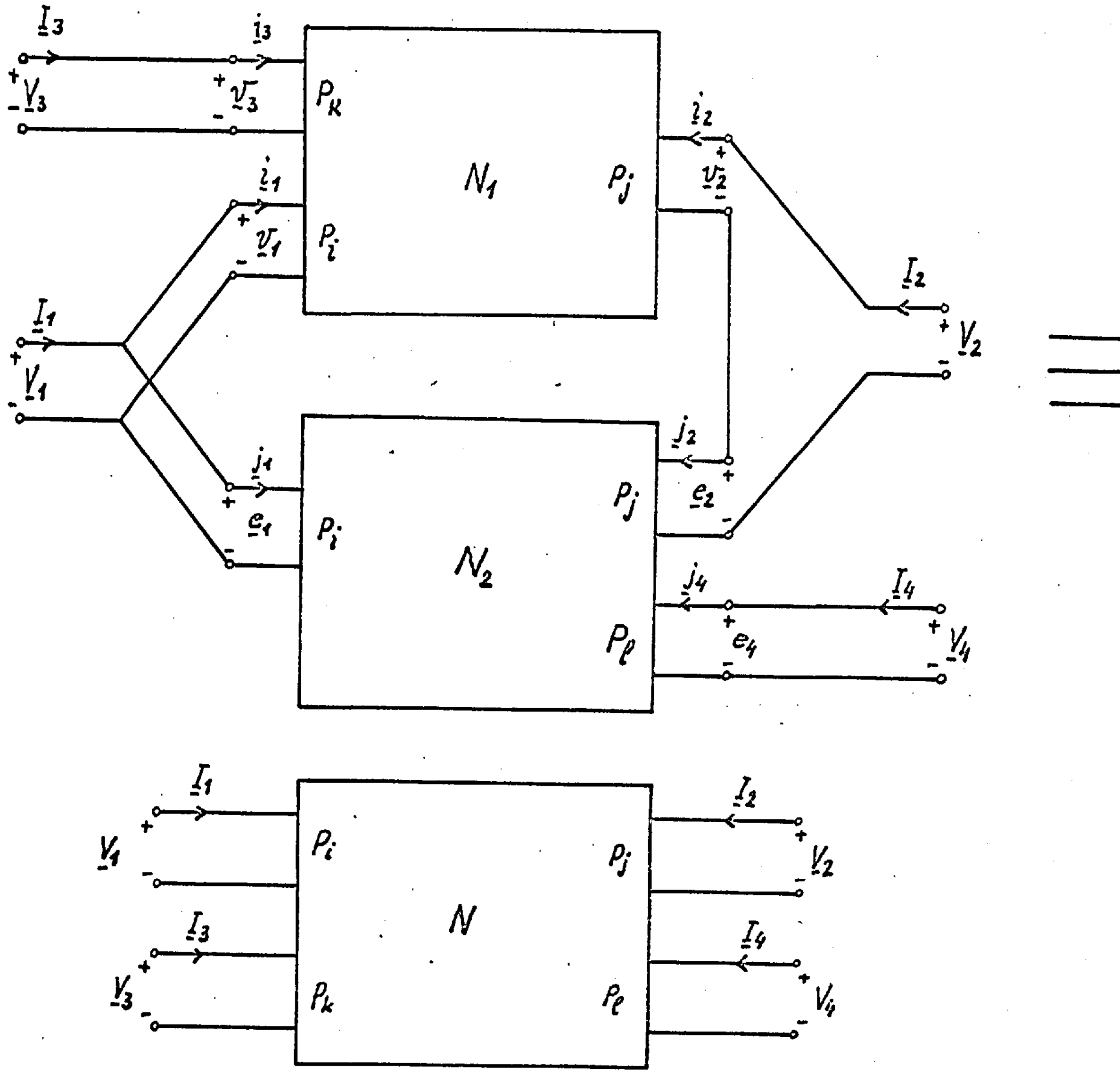

Fig. 5.3. Series parallel interconnection of an n-port N_1 and an m-port N_2.

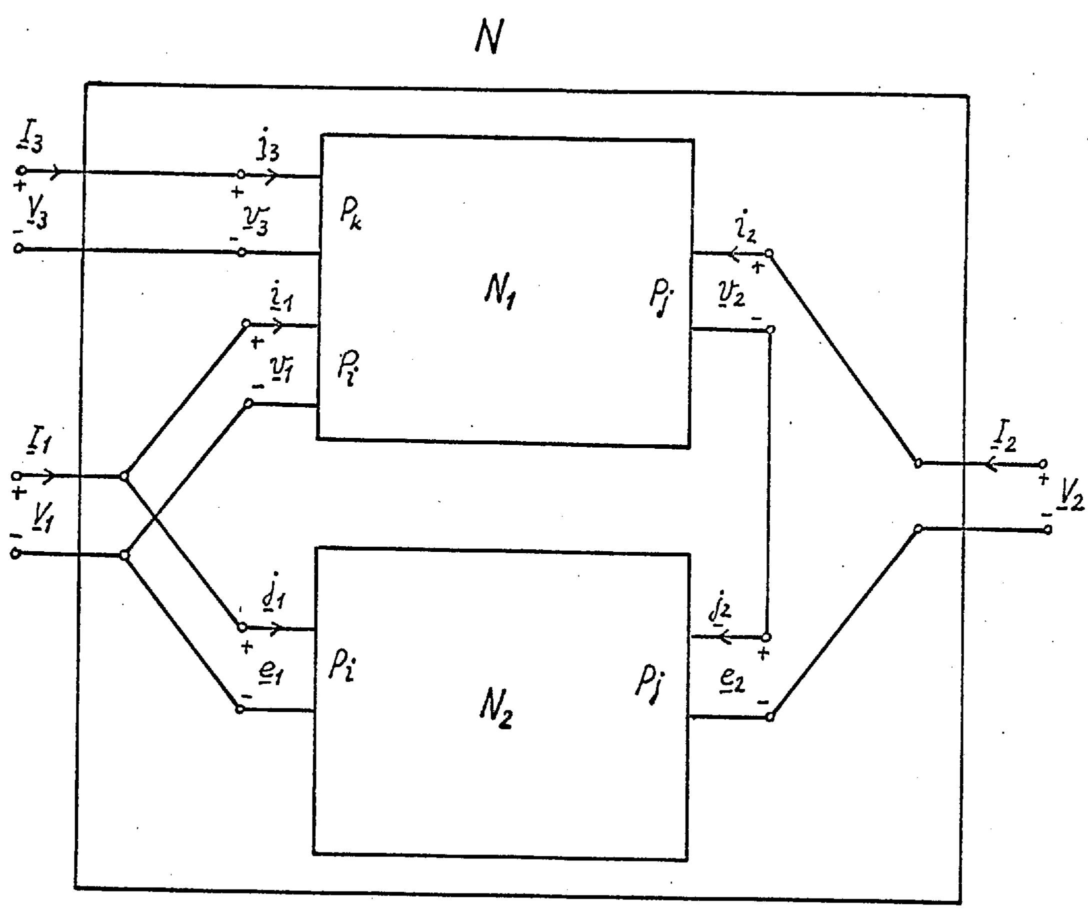

Fig. 5.4. Illustration of Theorem 5.2.

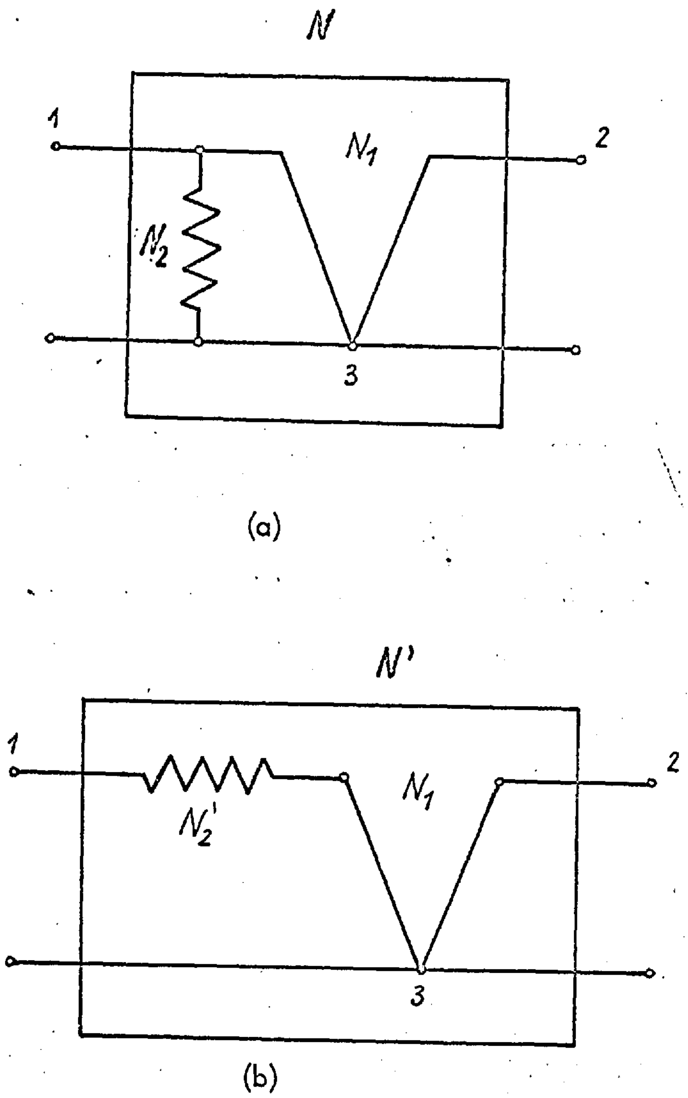

Fig. 5.5. Illustration of Theorem 5.3.

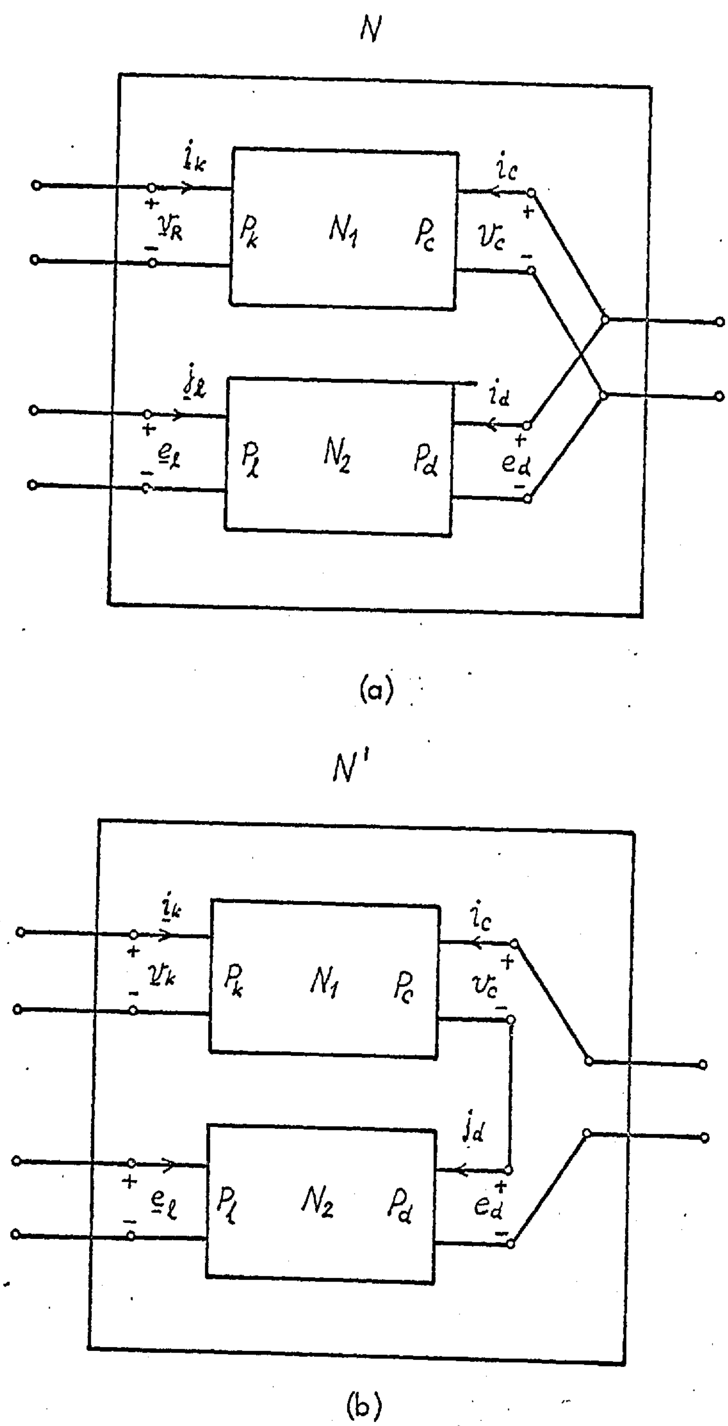

Fig. 5.6. Illustration of Theorem 5.7.

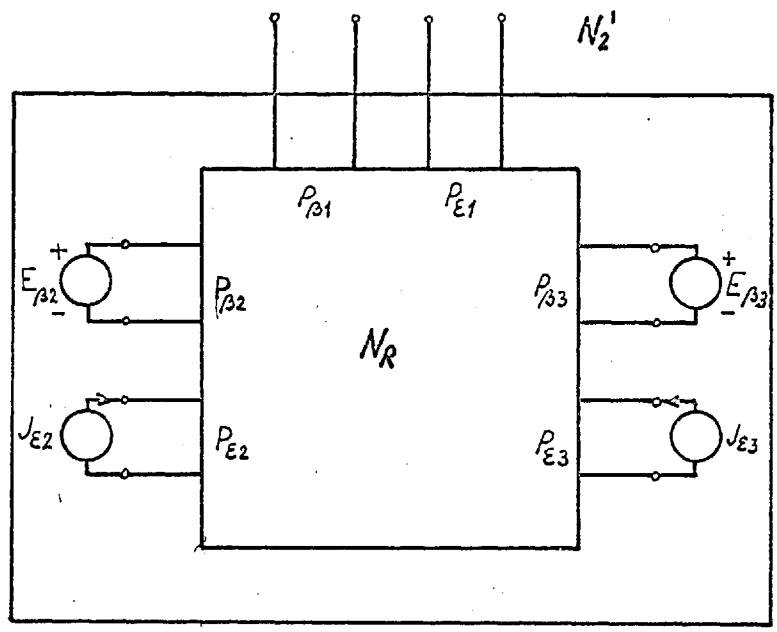

Fig. 5.7. Illustration of Theorem 5.10.

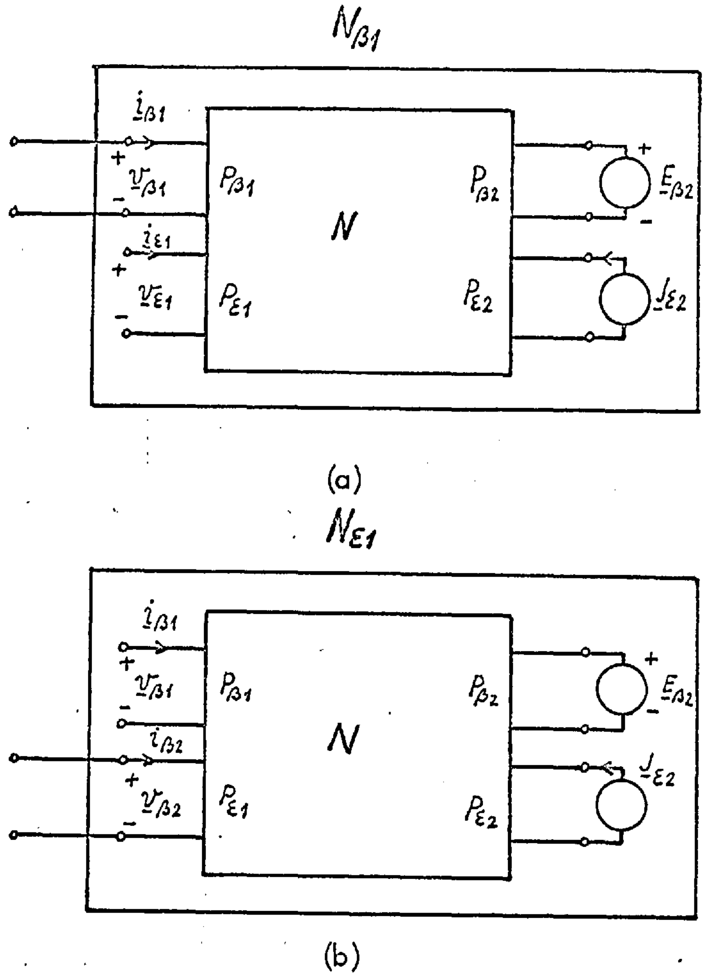

Fig. 5.8. Illustration of Theorem 5.11.

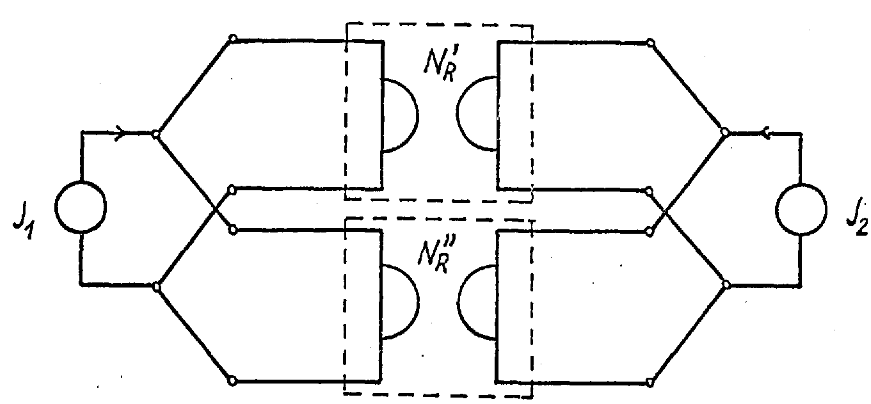

Fig. 5.9. Example of Section 5.3.

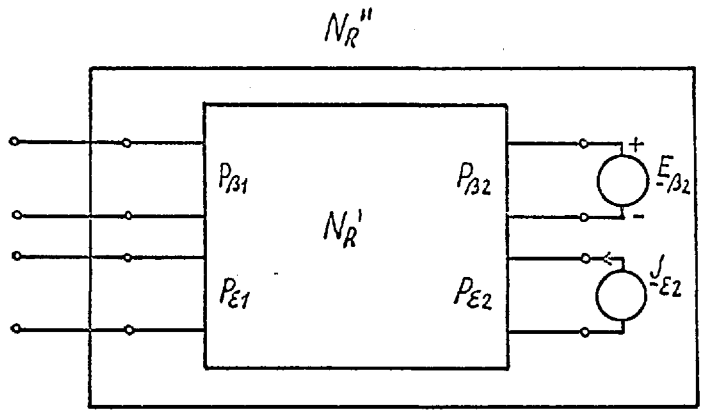

Fig. 5.10. Illustration of Theorem 5.12.

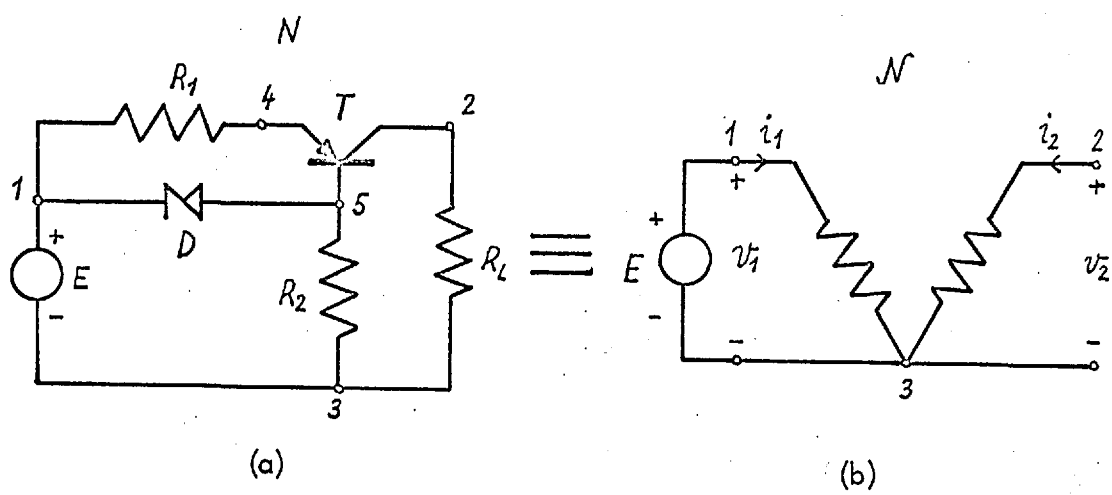

Fig. 5.11. Example 1 of Section 5.4.

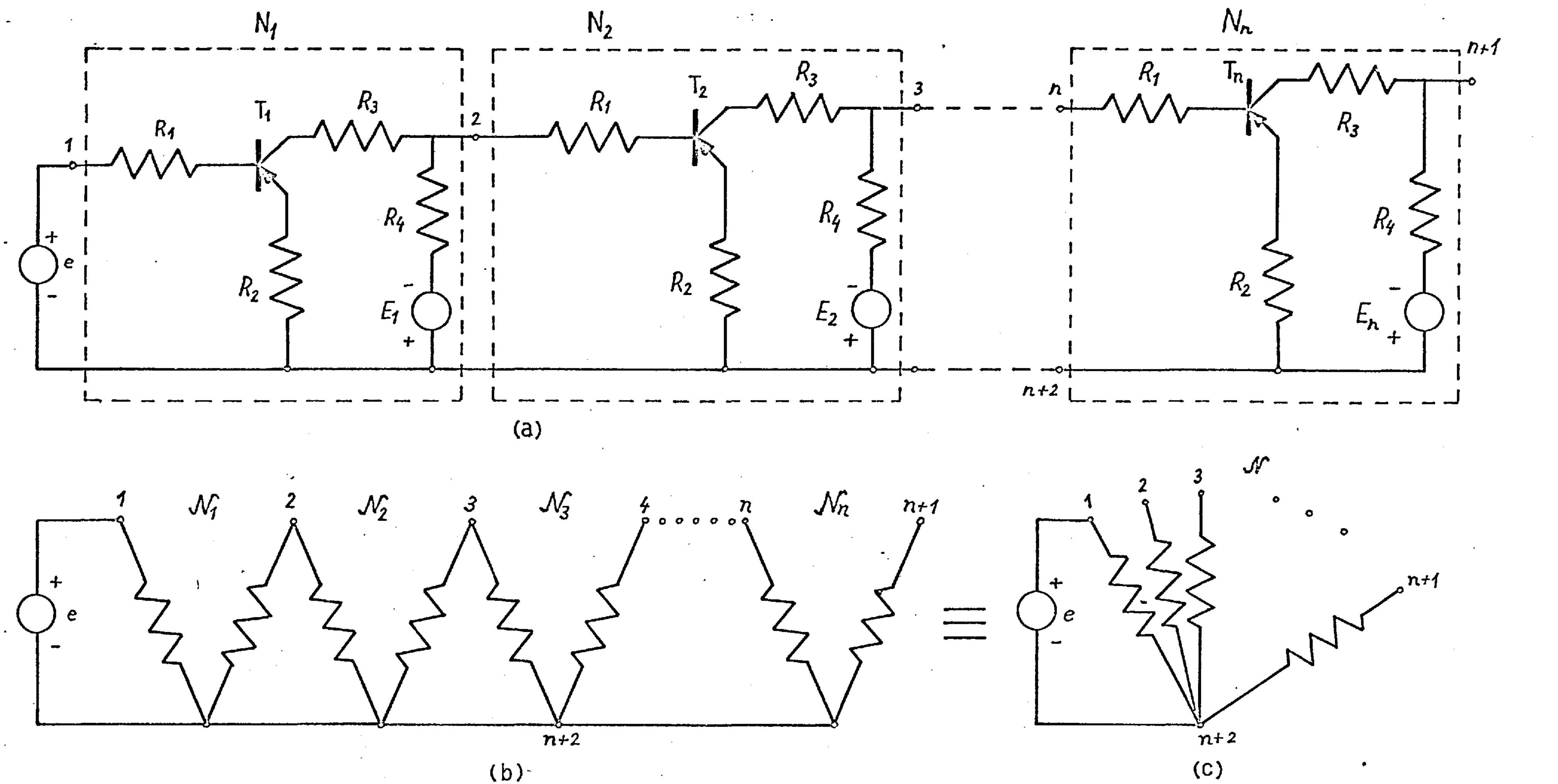

Fig. 5.12. Example 2 of Section 5.4.

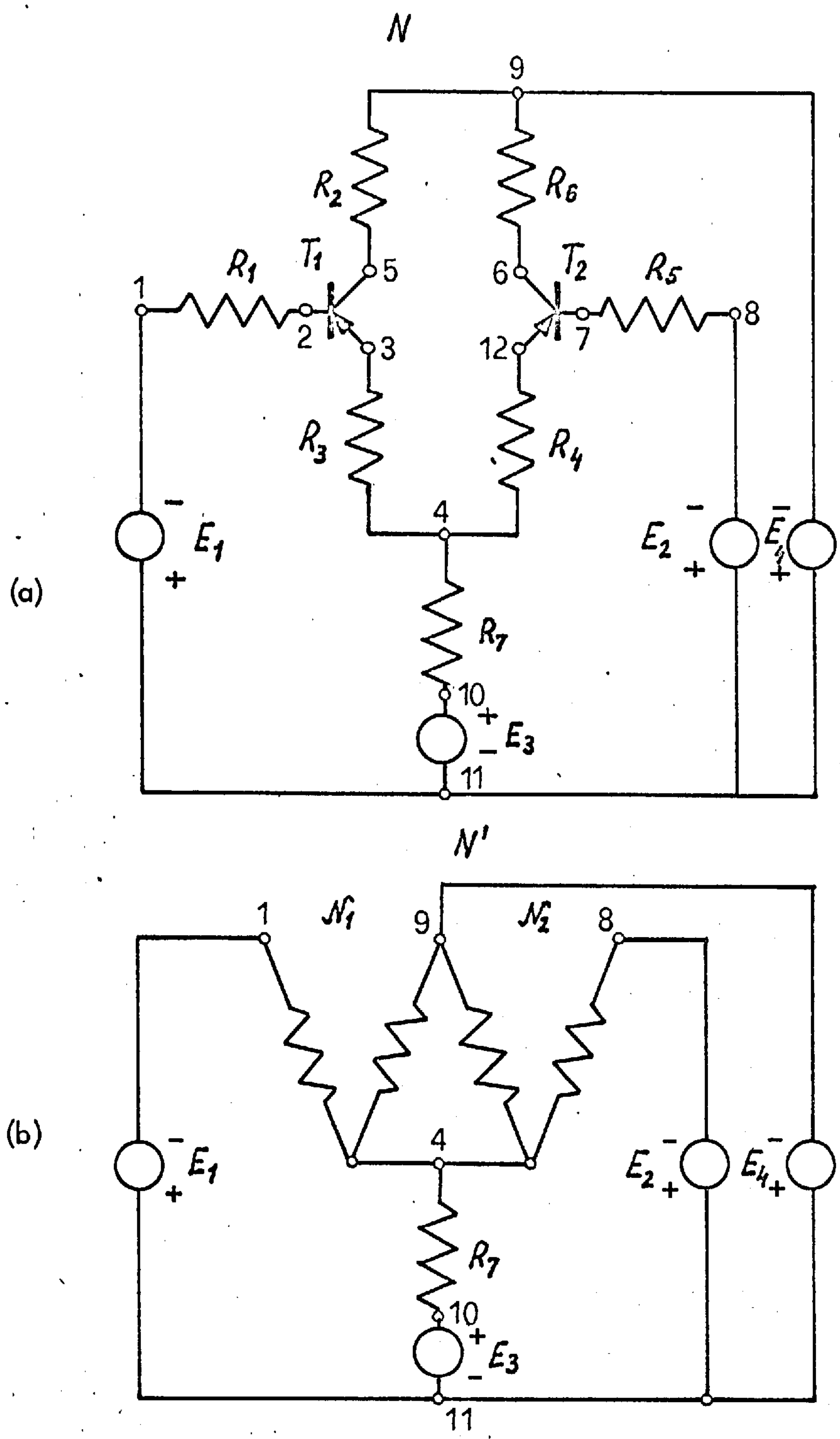

Fig. 5.13. Example 3 of Section 5.4.

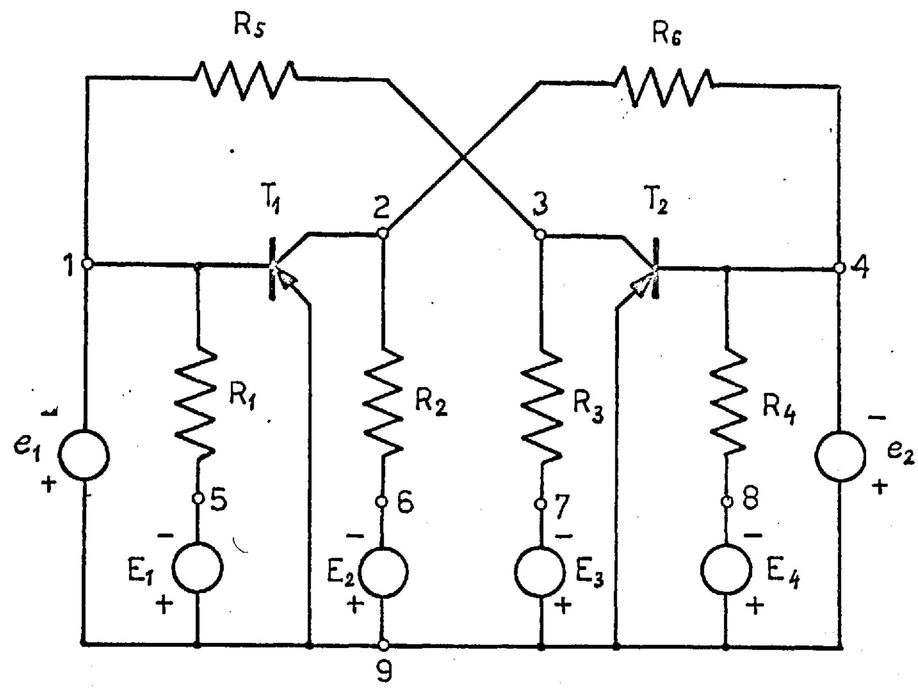

Fig. 5.14. Example 4 of Section 5.4.

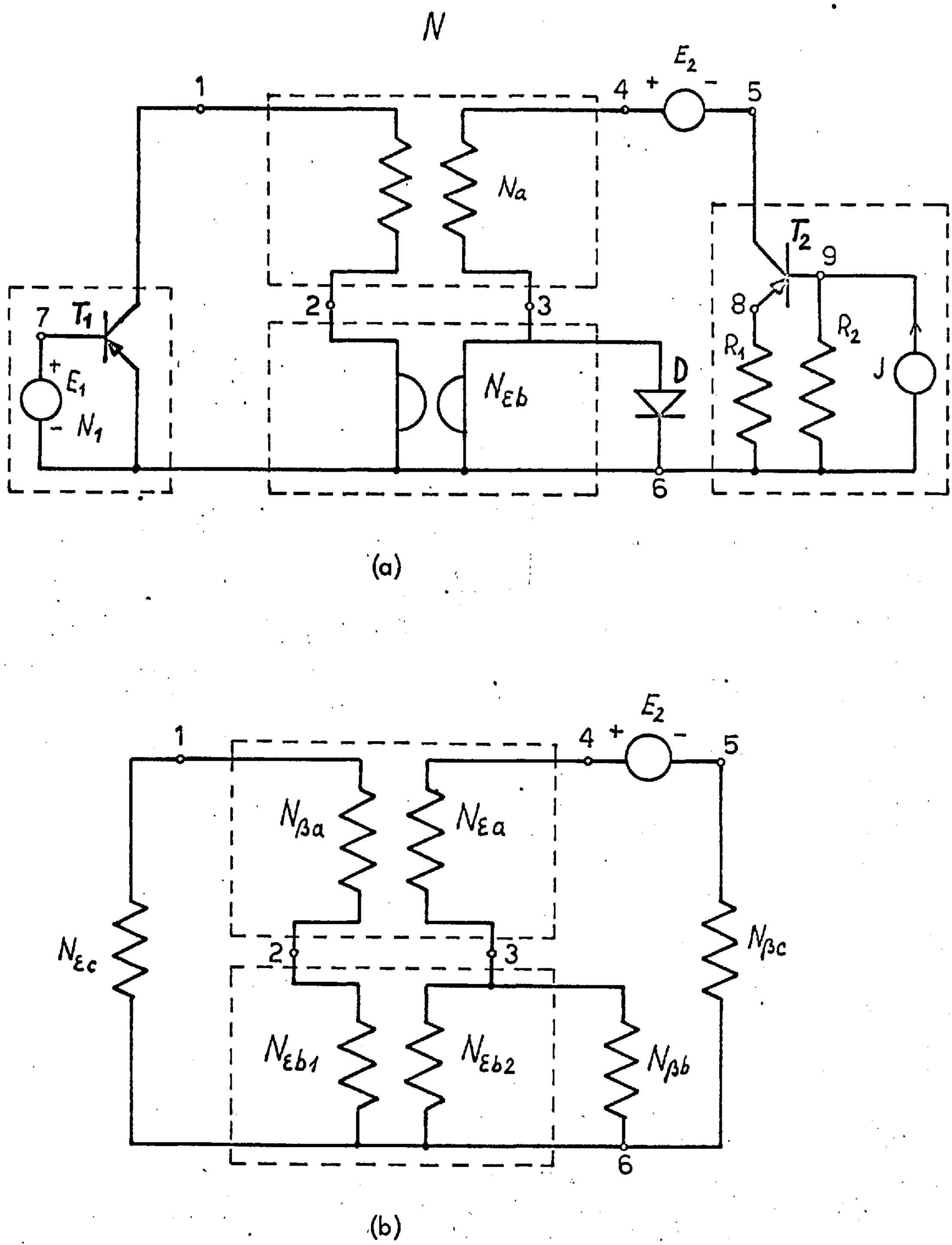

Fig. 5.15. Example 5 of Section 5.4.

Chapter 6

STATE VARIABLE DESCRIPTION OF NONLINEAR RLC NETWORKS

6.1 INTRODUCTION

The formulation of the normal form equations for a general nonlinear RLC network is discussed in Section 2.5 and is based on the concept of a normal tree; two choices of potentially complete sets - $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ and $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\delta \end{bmatrix}$ - are considered. In this chapter we shall aim at developing criteria which ensure the existence of a unique solution of a nonlinear RLC network.

Recently, a considerable attention has been paid to the problem of the reduced state variable formulation for nonlinear RLC networks. The idea is to reduce the order of the normal form differential equations which have to be integrated and in such a way the time necessary for the computation of the network response may be shortened. We shall discuss the reduced state variable formulation at the end of this chapter.

6.2 THE EXISTENCE OF A UNIQUE NETWORK SOLUTION

In the study of RLC network we shall consider $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ and $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\delta \end{bmatrix}$ as a potentially complete set in order to be able to use the criteria developed in Section 5.3. A number of workers have studied the problem of the state equations for nonlinear RLC networks. Some of them[1-6] investigated nonlinear RLC networks containing two-terminal elements only. Others[7-9] placed severe restrictions upon the positioning of network elements which are not locally passive; namely, it is assumed for resistive elements that a normal tree exists such that (i) each voltage-controlled port of a resistive element forms a link and the fundamental loop formed with respect to this link contains independent voltage sources and/or capacitive elements only and (ii) each current-controlled port of a resistive element forms a tree-branch and the fundamental cut-set defined by this tree-branch contains independent current sources and/or inductive elements only. Applying the results of Chapter 5 to the study of RLC networks we shall be able to remove the restrictions of this kind for a large class of nonlinear networks containing locally active and/or nonreciprocal elements.

6.2.1 Nonlinear RLC networks without dependent sources

The first result concerning the existence of a unique solution will be stated in the mathematical form and Corollary 3.1 will be applied to the (C), (L) and (R) equations. It is interesting to observe that Theorem 6.1 gives certain conditions which ensure not only the uniqueness of the network solution but the existence of a unique value of $\underline{i}_C$, currents through all capacitive elements, and $\underline{v}_L$, voltages across all inductive elements, in a network.

Theorem 6.1

Given an RLC (possibly time-varying) network $\mathcal{N}$ with the normal distribution of independent sources, a normal tree T_N and the implicit description of RLC elements in the form of eqns. (2.65a) - (2.65c), if the following conditions are satisfied:

(i) the functions $\underline{f}_R(\underline{i}_\beta, \underline{i}_\varepsilon, \underline{v}_\beta, \underline{v}_\varepsilon, t)$, $\underline{f}_L(\underline{i}_\gamma, \underline{i}_f, \underline{\varphi}_\gamma, \underline{\varphi}_f, t)$ and $\underline{f}_C(\underline{v}_\alpha, \underline{v}_\delta, \underline{q}_\alpha, \underline{q}_\delta, t)$ are of class $C^{(o)}$ in t and of class $C^{(1)}$ in all the remaining independent variables,

(ii) the Jacobian matrices $\partial\underline{f}_R/\partial(\underline{i}_\beta, \underline{i}_\varepsilon, \underline{v}_\beta, \underline{v}_\varepsilon)$, $\partial\underline{f}_L/\partial(\underline{i}_\gamma, \underline{i}_f, \underline{\varphi}_\gamma, \underline{\varphi}_f)$ and $\partial\underline{f}_C/\partial(\underline{v}_\alpha, \underline{v}_\delta, \underline{q}_\alpha, \underline{q}_\delta)$, associated with the functions $\underline{f}_R(\cdot)$, $\underline{f}_L(\cdot)$ and $\underline{f}_C(\cdot)$ respectively, are bounded for all values of independent variables,

(iii) the Jacobian det J_C, det J_L, det J_R associated with the (C), (L) and (R) equations (eqns. (2.68a) - (2.68c)) respectively satisfy the following inequalities for all values of the corresponding independent variables:

$$\det J_C = \det \begin{bmatrix} 1 & F_{\alpha\delta} & 0 & 0 \\ 0 & 0 & 1 & -F_{\alpha\delta}^T \\ \dfrac{\partial\underline{f}_C}{\partial\underline{v}_\alpha} & \dfrac{\partial\underline{f}_C}{\partial\underline{v}_\delta} & \dfrac{\partial\underline{f}_C}{\partial\underline{q}_\delta} & \dfrac{\partial\underline{f}_C}{\partial\underline{q}_\alpha} \end{bmatrix} > \varepsilon_1 > 0 \qquad (6.1)$$

$$\det J_L = \det \begin{bmatrix} 1 & F_{\gamma f} & 0 & 0 \\ 0 & 0 & 1 & -F_{\gamma f}^T \\ \dfrac{\partial\underline{f}_L}{\partial\underline{\varphi}_\gamma} & \dfrac{\partial\underline{f}_L}{\partial\underline{\varphi}_f} & \dfrac{\partial\underline{f}_L}{\partial\underline{i}_f} & \dfrac{\partial\underline{f}_L}{\partial\underline{i}_\gamma} \end{bmatrix} > \varepsilon_2 > 0 \qquad (6.2)$$

$$\det J_R = \det \begin{bmatrix} 1 & F_{\beta\varepsilon} & 0 & 0 \\ 0 & 0 & 1 & -F_{\beta\varepsilon}^T \\ \frac{\partial \underline{f}_R}{\partial \underline{v}_\beta} & \frac{\partial \underline{f}_R}{\partial \underline{v}_\varepsilon} & \frac{\partial \underline{f}_R}{\partial \underline{i}_\varepsilon} & \frac{\partial \underline{f}_R}{\partial \underline{i}_\beta} \end{bmatrix} > \varepsilon_3 > 0 \qquad (6.3)$$

(iv) all independent sources are continuous functions of t.

Then, (a) $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ is the complete set

(b) the normal form equation (2.73) written in terms of the complete set $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$, exists and it possesses a unique solution for all t and all initial conditions.

If conditions (ii) and (iv) are replaced by conditions (v) and (vi) respectively,

(v) $\underline{f}_R(\cdot)$, $\underline{f}_L(\cdot)$ and $\underline{f}_C(\cdot)$ are of class $C^{(1)}$ in all independent variables,

(vi) all independent sources are continuously differentiable functions of t,

then, in addition, $\underline{i}_C$, currents through all capacitive branches, and $\underline{v}_L$, voltages across all inductive branches of the network $\mathcal{N}$ are uniquely determined for all t.

Proof

From conditions (i) and (ii) it follows that the Jacobian matrices J_C, J_L and J_R associated with the eqns. (C), (L) and (R) are continuous and bounded; by condition (iii) $\det J_C > \varepsilon_1 > 0$, $\det J_L > \varepsilon_2 > 0$ and $\det J_R > \varepsilon_3 > 0$. Thus all conditions of Corollary 3.1 are fulfilled and the equations (C), (L) and (R) possess unique solutions for all values of vector $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$, and all values of independent sources; moreover by Corollary 3.1 and condition (iv) the solutions of the (C), (L) and (R) equations are continuous functions of t and they have bounded partial derivatives with respect to $\underline{q}$ and $\underline{\varphi}$ for all ($\underline{q}$, $\underline{\varphi}$, t). Therefore all conditions of Theorem 1 of Ref. 7 are fulfilled and the differential equation (2.73) possesses unique solutions for all t and all initial conditions; these solutions may be continued indefinitely forward and backward in time, i. e. solutions with "finite escape time" are ruled out.

When conditions (v) and (vi) are satisfied, clearly, $\dot{\underline{e}}_{\alpha}$, $\partial \underline{f}_C/\partial t$, $\dot{\underline{i}}_f$, $\partial \underline{f}_L/\partial t$ are defined in eqns. (2.74) for all t. Since by condition (iii) J_C and J_L are both nonsingular, $\underline{i}_C = \begin{bmatrix} \underline{i}_{\alpha} \\ \underline{i}_{\delta} \end{bmatrix}$ is determined from eqn. (2.74a) and $\underline{v}_L = \begin{bmatrix} \underline{v}_{\delta} \\ \underline{v}_f \end{bmatrix}$ is defined from eqn. (2.74b).

(Q. E. D.

Although Theorem 6.1 admits a very general class of nonlinear RLC networks it is stated in an not easily applicable - mathematical - form. From the point of view of the network theorist, it is desirable to express constraints on the existence of the network response in terms of the classes of network elements and the network topology. Using the results of Chapter 5 we arrive at the next two theorems.

Theorem 6.2

An RLC network $\mathcal{N}$ containing (i) resistive, inductive and capacitive PDNEs and (ii) independent sources which are continuous functions of time and are normally distributed is always determinate regardless of the topological structure. The set $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ is the state vector in $\mathcal{N}$ and given any initial state the network solution is uniquely determined in the future for all t.

Proof

Let us show that $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ forms the complete set in $\mathcal{N}$. The (R) equations can be written in the form similar to eqn. (2.37)

$$\begin{aligned} \underline{v}_\beta (\underline{i}_\beta, \underline{v}_\varepsilon) + F_{\beta\varepsilon} \underline{v}_\varepsilon &= \underline{e}_\beta(t) - F_{\beta\delta} \underline{v}_\delta \\ -F_{\beta\varepsilon}^T \underline{i}_\beta + \underline{i}_\varepsilon (\underline{i}_\beta, \underline{v}_\varepsilon) &= \underline{j}_\varepsilon(t) + F_{\gamma\varepsilon}^T \underline{i}_\gamma \end{aligned} \tag{6.4}$$

where the resistive elements in $\mathcal{N}$ are characterized by the hybrid description (2.36). From Theorem 5.9 it follows that eqn. (6.4) has a unique solution

$$\begin{aligned} \underline{i}_\beta &= \underline{i}_\beta (\underline{e}_\beta(t) - F_{\beta\delta} \underline{v}_\delta, \underline{j}_\varepsilon(t) + F_{\gamma\varepsilon}^T \underline{i}_\gamma) \\ \underline{v}_\varepsilon &= \underline{v}_\varepsilon (\underline{e}_\beta(t) - F_{\beta\delta} \underline{v}_\delta, \underline{j}_\varepsilon(t) + F_{\gamma\varepsilon}^T \underline{i}_\gamma) \end{aligned} \tag{6.5}$$

and $\begin{bmatrix} \underline{i}_\beta \\ \underline{v}_\varepsilon \end{bmatrix}$ is a GQLF of $\begin{bmatrix} \underline{e}_\beta(t) - F_{\beta\delta} \underline{v}_\delta \\ \underline{j}_\varepsilon(t) - F_{\gamma\varepsilon}^T \underline{i}_\gamma \end{bmatrix}$. Therefore the Jacobian matrix, associated with the solution of the (R) equations,

$$\partial (\underline{i}_\beta, \underline{v}_\varepsilon) / \partial (\underline{e}_\beta(t) - F_{\beta\delta} \underline{v}_\delta, \underline{j}_\varepsilon(t) + F_{\gamma\varepsilon}^T \underline{i}_\gamma)$$

is u. p. d. and bounded.

Since the capacitive and inductive elements are PDNEs, the following hybrid descriptions exist

$$\begin{aligned} \underline{v}_\alpha &= \underline{v}_\alpha(\underline{q}_\alpha, \underline{v}_\delta) \\ \underline{q}_\delta &= \underline{q}_\delta(\underline{q}_\alpha, \underline{v}_\delta) \end{aligned} \qquad . \quad (6.6)$$

$$\begin{aligned} \underline{\varphi}_\gamma &= \underline{\varphi}_\gamma(\underline{i}_\gamma, \underline{\varphi}_f) \\ \underline{i}_f &= \underline{i}_f(\underline{i}_\gamma, \underline{\varphi}_f) \end{aligned} \qquad (6.7)$$

where $\begin{bmatrix} \underline{v}_\alpha \\ \underline{q}_\delta \end{bmatrix}$ is a GQLF of $\begin{bmatrix} \underline{q}_\alpha \\ \underline{v}_\delta \end{bmatrix}$ and $\begin{bmatrix} \underline{\varphi}_\gamma \\ \underline{i}_f \end{bmatrix}$ is a GQLF of $\begin{bmatrix} \underline{i}_\gamma \\ \underline{\varphi}_f \end{bmatrix}$.

The (C) and (L) eqns. can be recast in the form

$$\begin{aligned} \underline{v}_\alpha(\underline{q}_\alpha, \underline{v}_\delta) + F_{\alpha\delta}\,\underline{v}_\delta &= \underline{e}_\alpha(t) \\ -F_{\alpha\delta}^T\,\underline{q}_\alpha + \underline{q}_\delta(\underline{q}_\alpha, \underline{v}_\delta) &= \underline{q} \end{aligned} \qquad (6.8)$$

$$\begin{aligned} \underline{\varphi}_\gamma(\underline{i}_\gamma, \underline{\varphi}_f) + F_{\gamma f}\,\underline{\varphi}_f &= \underline{\varphi} \\ -F_{\gamma f}^T\,\underline{i}_\gamma + \underline{i}_f(\underline{i}_\gamma, \underline{\varphi}_f) &= \underline{i}_f(t) \end{aligned} \qquad (6.9)$$

where analogously as in eqn. (6.4) $\begin{bmatrix} \underline{e}_\alpha(t) \\ \underline{q} \end{bmatrix}$ is a GQLF of $\begin{bmatrix} \underline{q}_\alpha \\ \underline{v}_\delta \end{bmatrix}$ and $\begin{bmatrix} \underline{\varphi} \\ \underline{i}_f(t) \end{bmatrix}$ is a GQLF of $\begin{bmatrix} \underline{i}_\gamma \\ \underline{\varphi}_f \end{bmatrix}$. Using the arguments of the proof of Theorem 5.9 it follows that $\begin{bmatrix} \underline{q}_\alpha \\ \underline{v}_\delta \end{bmatrix}$ can be obtained from eqn. (6.8) as a GQLF of $\begin{bmatrix} \underline{e}_\alpha(t) \\ \underline{q} \end{bmatrix}$ and similarly $\begin{bmatrix} \underline{i}_\gamma \\ \underline{\varphi}_f \end{bmatrix}$ can be obtained from eqn. (6.9) as a GQLF of $\begin{bmatrix} \underline{\varphi} \\ \underline{i}_f(t) \end{bmatrix}$; hence the functions $\underline{i}_\beta$, $\underline{v}_\varepsilon$, $\underline{i}_\gamma$, $\underline{v}_\delta$, appearing in the normal form equation (2.73), have bounded first partial derivatives with respect to $\underline{q}$ and $\underline{\varphi}$ are continuous in t for all $(\underline{q}, \underline{\varphi}, t)$. Therefore all conditions of Theorem 1 of Ref. 7 are fulfilled and given any initial state $\begin{bmatrix} \underline{q}(t_o) \\ \underline{\varphi}(t_o) \end{bmatrix}$ the state $\begin{bmatrix} \underline{q}(t) \\ \underline{\varphi}(t) \end{bmatrix}$ is determined uniquely in the future for all t.

Q. E. D.

Bryant showed that an RLC network containing linear positive two-terminal RLC elements is always determinate with $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ as the state vector. Theorem 6.2 may be viewed as a fairly general extention of Bryant's result to nonlinear RLC networks containing multiterminal (and possibly nonreciprocal) network elements. At the same time it is a generalization of Stern's result concerning RLC networks of quasilinear elements to the nonreciprocal case.

An RLC network $\mathcal{N}$ which satisfies the conditions imposed in Theorem 6.2 has the following additional properties which are worth mentioning:

(i) the vector $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ represents the complete set,

(ii) if independent sources in $\mathcal{N}$ are differentiable functions of t, the normal form equation (2.87) exists, i. e. its r. h. s. is a continuous function of $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ for all $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ and

(iii) i_C, representing the currents through all capacitive branches, and v_L, representing the voltages across all inductive elements, are uniquely determined if in addition independent sources in $\mathcal{N}$ are continuously differentiable functions of t.

These properties can be shown in the following manner. Since the capacitive and inductive elements in $\mathcal{N}$ are PDNEs, the hybrid descriptions given in eqns. (2.76) and (2.77) exist and $\underline{q}_C$ and $\underline{\varphi}_L$ are defined explicitly in terms of $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ (eqns. (2.79) and (2.80)). From eqn. (6.5) $\underline{i}_\beta$ and $\underline{v}_\varepsilon$ are uniquely determined as functions of $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ and thus $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ represents the complete set.

In order to show that the normal from equation (2.87) exists we have to prove that the functions $M_\gamma{}^{-1}K_\gamma$ and $M_\delta{}^{-1}K_\delta$, appearing in eqn.

(2.87), are continuous functions in $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$. Analysing the expressions for K_γ and K_δ which are given in eqn. (2.86) we conclude that K_γ and K_δ are continuous functions of $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ and are continuous functions of t because $\underline{e}_\alpha(t)$ and $\underline{i}_\xi(t)$ are continuously differentiable functions of t. M_γ is the Jacobian matrix of the function

$$\underline{\Phi}(\underline{i}_\gamma) = \begin{bmatrix} 1 & F_{\gamma\xi} \end{bmatrix} \underline{\varphi}_L\left(\begin{bmatrix} 1 \\ F_{\gamma\xi}^T \end{bmatrix} \underline{i}_\gamma\right) \qquad (6.10)$$

which is formed from $\underline{\varphi}_L(\underline{i}_\gamma, \underline{i}_\xi)$ (eqn. (2.77)). Since $\underline{\varphi}_L(\underline{i}_\gamma, \underline{i}_\xi)$ is a GQLF and $\begin{bmatrix} 1 \\ F_{\gamma\xi}^T \end{bmatrix}$ is of rank b_γ, $\underline{\Phi}(\underline{i}_\gamma)$ is a GQLF of the vector $\underline{i}_\gamma$ by Property 2 of GQLFs and

$$M_\gamma = \partial\underline{\Phi}/\partial(\underline{i}_\gamma) \qquad (6.11)$$

is u. p. d. and bounded; M_γ^{-1} is u. p. d. and bounded and thus $M_\gamma^{-1}(\underline{i}_\gamma)$ is a continuous function of $\underline{i}_\gamma$. Similarly M_δ is the Jacobian matrix of the function

$$\underline{Q}(\underline{v}_\delta) = \begin{bmatrix} -F_{\alpha\delta}^T & 1 \end{bmatrix} \underline{q}_C\left(\begin{bmatrix} -F_{\alpha\delta} \\ 1 \end{bmatrix} \underline{v}_\delta\right) \qquad (6.12)$$

which is formed from $\underline{q}_C(\underline{v}_\alpha, \underline{v}_\delta)$ (eqn. (2.76) and

$$M_\delta = \partial\underline{Q}/\partial(\underline{v}_\delta) \qquad (6.13)$$

is u. p. d. and bounded; M_δ^{-1} is u. p. d. and bounded and thus it is continuous in $\underline{v}_\delta$. Therefore we may conclude that the functions $M_\gamma^{-1}K_\gamma$ and $M_\delta^{-1}K_\delta$ are continuous in $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ for all $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ and thus the normal form equation (2.87) exists. The existence of $\underline{i}_C$ and $\underline{v}_L$ can be deduced from eqns. (2.81) and (2.82).

However, there is another interesting point concerning RLC networks containing PDNEs only. The normal form equation (2.87) does not necessarily fulfil the Lipschitz condition. This can be demonstrated by the following example. Consider an RC network, shown in Fig. 6.1, where the resistor R is linear and positive and the capacitor C is a nonlinear PDNE and its incremental capacitance C_δ is given as:

$$C_\delta(v_\delta) = 1 + v_\delta^{1/3} \qquad -1/2 \le v_\delta \le 1/2$$

$$C_\delta(v_\delta) = 1 + (1/2)^{1/3} \qquad v_\delta > 1/2$$

$$C_\delta(v_\delta) = 1 + (-1/2)^{1/3} \qquad v < -1/2$$

For the network in Fig. 6.1 M_δ^{-1} is equal to

$$M_\delta^{-1} = C_\delta(v_\delta)^{-1} = (1 + v_\delta^{1/3})^{-1} \qquad -1/2 \le v \le 1/2 .$$

Differentiating M_δ^{-1} with respect to v_δ we have

$$dM_\delta^{-1}/dv_\delta = \frac{v_\delta^{-2/3}}{3(1 + v_\delta^{1/3})^2}$$

and $dM_\delta^{-1}/dv_\delta \to \infty$ as $v_\delta \xrightarrow{-} 0$. Thus the normal form equation (2.87) written for the network in Fig. 6.1 does not fulfil the Lipschitz condition.

When an RLC network $\mathcal{N}$ contains network elements which are not PDNEs we can frequently use the results of Chapter 5 to test whether $\mathcal{N}$ is determinate. Since the (C) and (L) equations for the capacitive subnetwork $\mathcal{N}\{0; \mathcal{B}_R, \mathcal{B}_L, \mathcal{B}_J\}$ and the inductive subnetwork $\mathcal{N}\{\mathcal{B}_E, \mathcal{B}_C, \mathcal{B}_R; 0\}$ are analogous to the governing equations of resistive networks the criteria of Chapter 5 can be modified to be applicable for the capacitive and inductive case. However, for most networks of practical interest capacitive and inductive elements are PDNEs or even linear positive two-terminal capacitive and inductive elements; for this class of networks the only problem is the uniqueness of solutions of the resistive network $\mathcal{N}_R$ which is formed from $\mathcal{N}$ (i) by replacing all its capacitive elements by a set of independent voltage sources $\underline{E}_C$ and (ii) by replacing all its inductive elements by a set of independent current sources $\underline{J}_L$. As mentioned before the hybrid equations of $\mathcal{N}_R$ are the (R) equations. We shall therefore state the following theorem for RLC networks where capacitive and inductive elements are restricted to PDNEs for the sake of simplicity.

<u>Theorem 6.3</u>

Given an RLC network $\mathcal{N}$ if

(i) capacitive and inductive elements in $\mathcal{N}$ are PDNEs,

(ii) independent sources are distributed normally and they are continuous functions of time t and

(iii) the resistive network $\mathcal{N}_R$ of $\mathcal{N}$ satisfies the conditions of one of Theorems 5.10 - 5.12 or Corollary 5.4

then (a) $\mathcal{N}$ is determinate and its solutions are defined in the future for all t and

(b) $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ represents the state vector.

<u>Proof</u>

Condition (iii) implies that the (R) eqns. have a unique solution described by eqn. (6.5); $\underline{i}_\beta$, $\underline{v}_\varepsilon$ are functions of class $C^{(1)}$ in $\underline{e}_\beta$, $\underline{v}_\delta$, $\underline{i}_\varepsilon$, $\underline{i}_\gamma$ and have bounded partial derivatives with respect to $\underline{e}_\beta, \underline{v}_\delta, \underline{i}_\varepsilon, \underline{i}_\gamma$.

Since, in addition, the capacitive subnetwork $\mathcal{N}\{0; \mathcal{B}_R, \mathcal{B}_L, \mathcal{B}_J\}$ and inductive subnetwork $\mathcal{N}\{\mathcal{B}_E, \mathcal{B}_C, \mathcal{B}_R; 0\}$ satisfy the conditions of Theorem 6.2, $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ forms the complete set and the normal form equations (2.73) exist. Using the arguments of the proof of Theorem 6.2 we conclude that the solutions of eqn. (2.73) are uniquely defined in the future for all t.

Q. E. D.

Note, that the condition (iii) of Theorem 6.3 can be replaced by the following condition: (i) $\mathcal{N}_R$ is single valued and

(ii) $\underline{i}_\beta$, $\underline{v}_\varepsilon$ given in eqn. (6.5) satisfy the Lipschitz condition with respect to $\underline{e}_\beta$, $\underline{v}_\sigma$, $\underline{i}_\varepsilon$, $\underline{i}_\gamma$. Thus, in the study of the existence of a unique network response of nonlinear RLC networks we may combine the properties of interconnections of resistive networks, described in Section 5.2, together with the results of Section 5.3. This method, although not systematic, could often be successfully applied.

6.2.2 Nonlinear RLC networks containing dependent sources

The problem of the state variable description of nonlinear RLC networks containing dependent sources is more general and more difficult than the state variable characterization of nonlinear RLC networks without dependent sources. The difficulty present in the state variable formulation of nonlinear RLC networks containing dependent sources may be explained as follows. When $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ is selected as a potentially complete set then in the case of networks without dependent sources the set of the (C), (L) and (R) equations has the property that the (C) equations represent an independent set of equations which can be solved separately; similarly the (L) equations form an independent set of equations. The r. h. s. of the (R) equations depends upon $\underline{v}_\sigma$ and $\underline{i}_\gamma$, which are the solutions of the (C) and (L) equations respectively. Thus, speaking in physical terms, the analysis of an RLC network without dependent sources is reduced to the analysis of its capacitive, inductive and resistive subnetworks. However, if controlled sources are present and $\begin{bmatrix} \underline{q} \\ \underline{\mathcal{L}} \end{bmatrix}$ is chosen as a potentially complete set then, in general, the (R), (L) and (C) equations have to be considered together as one set of $2(b_R+b_L+b_C)$ equations. For example, if there is a controlled voltage source $E(x_R)$ and a controlled current cource $J(x_R)$, where x_R is a voltage or current associated with a resistive branch, and $E(x_R)$ lies in a loop defined by a capacitive link and $J(x_R)$ lies in a cut-set defined by an inductive tree-branch, then the (C) equations and the (L) equations become "coupled" to the (R) equations and these two sets of equations are not independent any more. The analysis of sets of $2(b_R+b_L+b_C)$ equations becomes complicated and for nonlinear RLC networks with depedent sources it is more difficult to find topological conditions which ensure the determinateness of a given network than for nonlinear RLC networks without depedent sources.

Nevertheless, conditions obtained for RLC networks without dependent sources can be applied to RLC networks with dependent sources provided the positioning of dependent sources is restricted in such a manner that the independence of the (C) and (L) equations is preserved[10,11]. Networks containing RLC coupled elements and dependent sources will be considered in this section. It will be shown that Katzenelson's results[11] obtained for nonlinear RLC networks containing two-terminal elements and dependent sources can be extended to RLC networks containing coupled elements and dependent sources.

It is reasonable to assume[3,10,12] that dependent sources are not controlled by capacitive currents, $\underline{i}_C$, and inductive voltages, $\underline{v}_L$. Furthermore, the set of all voltage sources $\mathcal{B}_E$ is partitioned into three disjoint subsets $\mathcal{B}_{E1}$, $\mathcal{B}_{E2}$, $\mathcal{B}_{E3}$ and similarly the set of all current sources $\mathcal{B}_J$ is partitioned into three disjoint subsets $\mathcal{B}_{J1}$, $\mathcal{B}_{J2}$, $\mathcal{B}_{J3}$ as follows*:

$\mathcal{B}_{E1}$ ($\mathcal{B}_{J1}$): the set of independent voltage (current) sources

$\mathcal{B}_{E2}$ ($\mathcal{B}_{J2}$): the set of dependent voltage (current) sources which are controlled by the branches of the sets $\mathcal{B}_C$, $\mathcal{B}_L$ and $\mathcal{B}_{E1}$ only

$\mathcal{B}_{E3}$ ($\mathcal{B}_{J3}$): the set of dependent voltage (current) sources which are controlled by the branches of the sets $\mathcal{B}_C$, $\mathcal{B}_L$, $\mathcal{B}_{E1}$ and $\mathcal{B}_{E2}$.

The following theorem gives sufficient conditions for a nonlinear RLC network containing dependent sources to be determinate.

Theorem 6.4

Let $\mathcal{N}$ be a nonlinear RLC network containing independent and dependent sources where all sources are normally distributed. The network is determinate if the following conditions hold:

(i) the network $\mathcal{N}^*$ derived from $\mathcal{N}$ by replacing each dependent source by an independent one satisfies the conditions of one of Theorems 6.1 - 6.3,

(ii) with respect to a normal tree T_N of $\mathcal{N}$ dependent sources satisfy the following:

(a) the fundamental cut-set defined by any tree branch of the set $\mathcal{B}_{E2}$ contains no capacitors and the fundamental cut-set defined by any tree-branch of the set $\mathcal{B}_{E3}$ contains inductors and current sources only,

(b) the fundamental loop defined by any link of the set $\mathcal{B}_{J2}$

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* This method of partitioning is very similar to the partitioning of sources used in Ref. 11.

contains no inductors and the fundamental loop defined by any link of the set $\mathcal{B}_{J3}$ contains capacitors and voltage sources only,

(iii) the dependent sources $\underline{E}_2$, $\underline{J}_2$, $\underline{E}_3$, $\underline{J}_3$ are functions of class $C^{(1)}$ with respect to the controlling variables, they have bounded first partial derivatives with respect to the controlling variables and are continuous functions of t; the independent sources $\underline{E}_1$ and $\underline{J}_1$ are continuous functions of t.

Proof

Since the sources are normally distributed a normal tree T_N exists such that all voltage sources lie in T_N and all current sources form the links with respect to T_N. We can now consider that the vectors $\underline{e}_\alpha$, $\underline{e}_\beta$, $\underline{i}_\varepsilon$, $\underline{i}_f$ appearing on the r. h. s. of the (C), (L) and (R) equations and $\underline{e}_\gamma$, $\underline{i}_\delta$, appearing in the normal form equations(2.73), represent the effect of the independent as well as dependent sources. Using the topological condition (a) it is easy to see that $\underline{e}_\alpha$ depends upon the independent voltage sources $\underline{E}_1$ only and $\underline{e}_\beta$ dependes upon sources $\underline{E}_1$ and $\underline{E}_2$. From the condition (b) it follows that $\underline{i}_f$ depends upon the independent current sources $\underline{J}_1$ and $\underline{i}_\varepsilon$ depends upon sources $\underline{J}_1$ and $\underline{J}_2$. However, the vector $\underline{e}_\gamma$ may depend upon $\underline{E}_1$, $\underline{E}_2$ and $\underline{E}_3$ and similarly the vector $\underline{i}_\delta$ may depend upon $\underline{J}_1$, $\underline{J}_2$ and $\underline{J}_3$. Therefore $\underline{e}_\alpha$, $\underline{e}_\beta$, $\underline{e}_\gamma$, $\underline{i}_\delta$, $\underline{i}_\varepsilon$ and $\underline{i}_f$ can be expressed in the form

$$
\begin{aligned}
\underline{e}_\alpha &= \underline{e}_\alpha\,(\underline{E}_1) & \underline{i}_f &= \underline{i}_f\,(\underline{J}_1) \\
\underline{e}_\beta &= \underline{e}_\beta\,(\underline{E}_1,\underline{E}_2) & \underline{i}_\varepsilon &= \underline{i}_\varepsilon\,(\underline{J}_1,\underline{J}_2) \qquad (6.14)\\
\underline{e}_\gamma &= \underline{e}_\gamma\,(\underline{E}_1,\underline{E}_2,\underline{E}_3) & \underline{i}_\delta &= \underline{i}_\delta\,(\underline{J}_1,\underline{J}_2,\underline{J}_3)
\end{aligned}
$$

Since only the independent sources appear in $\underline{e}_\alpha$ and $\underline{i}_\xi$, the (C) and (L) equations do not change if controlled sources satisfying the conditions (a) and (b) are introduced in $\mathcal{N}\{\mathcal{B}_{E2}, \mathcal{B}_{E3}; \mathcal{B}_{J2}, \mathcal{B}_{J3}\}$; by condition (i) the (C) and (L) equations possess a unique solution $\begin{bmatrix}\underline{v}_C\\ \underline{q}_C\end{bmatrix}$ and $\begin{bmatrix}\underline{i}_L\\ \underline{\varphi}_L\end{bmatrix}$ respectively. When this solution is substituted into the expressions for $\underline{E}_2$ and $\underline{J}_2$, these two sets of controlled sources are defined and thus the r. h. s. of the (R) equations is determined. By condition (i) the (R) equations have a unique solution $\begin{bmatrix}\underline{v}_R\\ \underline{i}_R\end{bmatrix}$. If the values for $\underline{v}_C$, $\underline{q}_C$, $\underline{i}_L$, $\underline{\varphi}_L$, $\underline{v}_R$, $\underline{i}_R$ are substituted into the expressions for $\underline{E}_3$ and $\underline{J}_3$, $\underline{e}_\gamma$ and $\underline{i}_\delta$ are defined and the r. h. s. of the normal form equation (2.73) is determined. Since the differential equation (2.73) satisfies the Lipschitz condition, $\mathcal{N}$ is determinate.

Q. E. D.

6.3 REDUCED STATE VARIABLE FORMULATION

Let us assume that an RLC network possesses the normal form characterization

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) \qquad (6.15)$$

where $\underline{x}$ is the state vector containing n components. From the viewpoint of practical computation, however, it is much desirable for a set of differential equations, which has to be integrated, to have lower order than n if the same solution can be obtained. Such a set of lower order differential equations can indeed be obtained provided the function $\underline{f}(\underline{x}, t)$ has certain properties to be specified in the sequel.

Partition vectors $\underline{x}$ and $\underline{f}$ conformably as follows

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \\ \underline{x}_2 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} \underline{f}_1 \\ \\ \underline{f}_2 \end{bmatrix} \tag{6.16}$$

where $\underline{x}_1$, $\underline{f}_1$ are m-vectors and $\underline{x}_2$, $\underline{f}_2$ are (n-m)-vectors. Then eqn. (6.15) can be written in the partitioned form

$$\underline{x}_1 = \underline{f}_1(\underline{x}_1, \underline{x}_2, t) \tag{6.17}$$

$$\underline{x}_2 = \underline{f}_2(\underline{x}_1, \underline{x}_2, t) \tag{6.18}$$

If the function $\underline{f}_2$ does not depend upon $\underline{x}_1$ and $\underline{x}_2$, i. e.

$$\underline{x}_2 = \underline{f}_2(t) \tag{6.19}$$

$\underline{x}_2(t)$ is obtained directly by the integration of eqn. (6.19) and

$$\underline{x}_2(t) = \underline{x}_2(0) + \int_0^t \underline{f}_2(t)dt \tag{6.20}$$

where $\underline{x}_2(0)$ is the initial value. In general this integration is simple and would not require much computing time; frequently even analytic expressions available for $\int_0^t \underline{f}_2(t)dt$. When the function $\underline{x}_2(t)$ is substituted into eqn. (6.17) we have the following normal form equations

$$\underline{x}_1 = \underline{f}_1(\underline{x}_1, \underline{x}_2(t), t) \tag{6.21}$$

and eqn. (6.21) represents the reduced state variable formulation[13]. It is now necessary to integrate the set of m first order differential equations. Thus, when $\underline{f}$ is of a special form, such that $\underline{f}_2$ depends upon t only, we can expect that less computer time is required to integrate the normal form equations (6.21) of order m than the original state equations (6.15) of order n. However, it is worth pointing out that $\underline{x}_1$ is not the state vector and eqn. (6.21) is not the state equation; $\underline{x}_2(0)$ has to be specified together with $\underline{x}_1(\dot{0})$ to determine the network solution uniquely.

It turns out that the reduced state variable formulation is possible for nonlinear networks containing capacitor and current-source-only cut-sets and/or inductors and voltage-source-only loops. The problem of the reduced state variable formulation for nonlinear RLC networks was considered by Ohtsuki and Watanabe[9] and in more generality by Cahill[14]. Namely, in Ref. 9 it was assumed that there was no cut-set containing capacitors and current sources only and no loop of inductors and voltage sources only, but capacitor-only-cut-sets and inductor-only-loops were allowed. It is interesting to observe that in both papers[9,14] the initial value for the vector $\underline{x}_2$ in our notation was taken $\underline{x}_2(0) = \underline{0}$ and vector $\underline{x}_1$ was termed the state vector.

The reduced state variable formulation of an RLC network $\mathcal{N}$ is easily derived using the concept of the L-normal tree[9] T_L where all voltage sources plus as many inductors as possible are branches of T_L and all current sources plus as many capacitors as possible are links of T_L. For a given RLC network $\mathcal{N}$ vectors $\underline{x}_1$ and $\underline{x}_2$ appearing in eqn. (6.21) can be obtained in the following manner[9,14].

First the normal tree T_N (termed C-normal tree in Ref. 9) is selected in $\mathcal{N}$. Then a particular L-normal tree T_L is chosen[9] such that all inductive tree-branches of T_N are tree-branches of T_L and all capacitive links of T_N are links of T_L. With respect to T_N and T_L all capacitive branches in $\mathcal{N}$ can be partitioned into three disjoint subsets:

S_α : the capacitive links of T_N

$S_{\gamma 1}$: capacitive branches forming tree-branches in T_N and capacitive links in T_L

$S_{\sigma 2}$: the capacitive tree-branches both in T_N and T_L. Similarly, all inductive branches in $\mathcal{N}$ can be partitioned with respect to T_N and T_L into the following three disjoint subsets:

$S_{\gamma 1}$: the inductive branches forming links of T_N and tree-branches of T_L

$S_{\gamma 2}$: the inductive links both of T_N and T_L

$S_{\int}$: the inductive tree-branches in T_N.

Denote by b_α , $b_{\sigma 1}$, $b_{\sigma 2}$, $b_{\gamma 1}$, $b_{\gamma 2}$, $b_{\int}$ the number of branches in S_α , $S_{\sigma 1}$, $S_{\sigma 2}$, $S_{\gamma 1}$, $S_{\gamma 2}$ and $S_{\int}$ respectively. Then[9] $b_{\int}$ is equal to the number of independent inductor and current sources only cut-sets ρ_L, $b_{\gamma 2}$ is equal to the number of independent inductor and voltage sources only loops μ_L, $b_{\sigma 2}$ is equal to the number of independent capacitor and current sources only cut-sets ρ_C and b_α is equal to the number of independent capacitor and voltage sources only loops μ_C in $\mathcal{N}$.

Kirchhoff's current law based on the fundamental cut-sets through capacitive branches $S_{\sigma 2}$ in T_L can be written in the form

$$\begin{bmatrix} 1 & A_{\sigma 1} & A_\alpha \end{bmatrix} \begin{bmatrix} \underline{\dot{q}}_{\sigma 2} \\ \underline{\dot{q}}_{\sigma 1} \\ \underline{\dot{q}}_{\alpha} \end{bmatrix} = \underline{i}^*_{\sigma 2}(t) \tag{6.22}$$

where $A_{\sigma 1}$, A_α are the submatrices in the fundamental cut-set matrix determined with respect to T_L and $\underline{i}^*_{\sigma 2}$ is an $b_{\sigma 2}$-vector, the k-th component of which is the algebraic sum of source currents which appear in the k-th fundamental cut-set through $S_{\sigma 2}$ in T_L. Integrating eqn. (6.22) and introducing the variable $\underline{q}^{**}(t)$ we have

$$\underline{q}^{**}(t) = \underline{q}_{\sigma 2} + A_{\sigma 1}\underline{q}_{\sigma 1} + A_{\alpha}\underline{q}_{\alpha} = \int_0^t \underline{i}^*_{\sigma 2}(t)\, dt + \underline{q}^{**}(0) \qquad (6.23)$$

Where $\underline{q}^{**}(0)$ is the constant of integration. It follows from eqn. (6.23) that $\underline{q}^{**}(t)$ is a known function of time t.

Dually, Kirchhoff's voltage law based on the fundamental loops of inductive links $S_{\gamma 2}$ in T_L can be expressed in the form

$$\begin{bmatrix} 1 & A_{\gamma 1} & A_{f} \end{bmatrix} \begin{bmatrix} \dot{\underline{\psi}}_{\gamma 2} \\ \dot{\underline{\psi}}_{\gamma 1} \\ \dot{\underline{\psi}}_{f} \end{bmatrix} = \underline{e}^*_{\gamma 2}(t) \qquad (6.24)$$

where $A_{\gamma 1}$, A_{f} are the submatrices in the fundamental loop matrix determined with respect to T_L and $e^*_{\gamma 2}$ is an $b_{\gamma 2}$-vector, the k-th component of which is the algebraic sum of sources voltages which appear in the k-th fundamental loop of $S_{\gamma 2}$ in T_L. Integrating eqn. (6.24) and introducing the variable $\underline{\psi}^{**}(t)$ we have

$$\underline{\psi}^{**}(t) = \underline{\psi}_{\gamma 2} + A_{\gamma 1}\underline{\psi}_{\gamma 1} + A_{f}\underline{\psi}_{f} = \int_0^t \underline{e}^*_{\gamma 2}(t)dt + \underline{\psi}^{**}(0) \qquad (6.25)$$

where $\underline{\psi}^{**}(0)$ is the constant of integration and $\underline{\psi}^{**}(t)$ is defined by eqn. (6.25) for any value of t. Note, that $\underline{q}^{**}$ is the vector of cut-set charges and $\underline{\psi}^{**}$ is the vector of loop flux-linkages determined with respect to T_L.

According to the partitioning of sets $S_{\sigma} = S_{\sigma 1} \cup S_{\sigma 2}$ and $S_{\gamma} = S_{\gamma 1} \cup S_{\gamma 2}$, $F_{\alpha\sigma}$ and $F_{\gamma f}$, representing the submatrices of the fundamental loop matrix which is determined with respect to T_N, can be partitioned as follows:

$$F_{\alpha\delta} = \begin{bmatrix} F_{\alpha\delta 1} & F_{\alpha\delta 2} \end{bmatrix} \tag{6.26a}$$

$$F_{\gamma\zeta} = \begin{bmatrix} F_{\gamma 1\zeta} \\ F_{\gamma 2\zeta} \end{bmatrix} \tag{6.26b}$$

Potentially complete sets $\begin{bmatrix} \underline{q} \\ \underline{\varphi} \end{bmatrix}$ and $\begin{bmatrix} \underline{v}_\delta \\ \underline{i}_\gamma \end{bmatrix}$ contain $(b_C + b_L - b_\alpha - b_\zeta)$ components. In order to derive the reduced state variable formulation of an RLC network the following set is selected as a possible state vector

$$\underline{x} = \begin{bmatrix} \underline{q}^* \\ \underline{\varphi}^* \\ \underline{q}^{**} \\ \underline{\varphi}^{**} \end{bmatrix} \tag{6.27}$$

where

$$\underline{q}^* = \underline{q}_{\delta 1} - F^T_{\alpha\delta 1}\underline{q}_\alpha \tag{6.28a}$$

$$\underline{\varphi}^* = \underline{\varphi}_{\gamma 1} + F_{\gamma 1\zeta}\,\underline{\varphi}_\zeta \tag{6.28b}$$

and $\underline{q}^{**}$ and $\underline{\varphi}^{**}$ are given by eqns. (6.23) and (6.25) respectively. Thus a potentially complete set $\underline{x}$ contains:

(i) cut-set charges $\underline{q}^*$ which are based on the fundamental cut-sets through branches of $S_{\delta 1}$ and where the fundamental cut-sets are determined with respect to T_N,

(ii) loop flux-linkages $\underline{\varphi}^*$ which are based on the fundamental loops formed by branches of $S_{\gamma 1}$ and where the fundamental loops are determined with respect to T_N,

(iii) cut-set charges $\underline{q}^{**}$ determined with respect to all capacitive tree-branches in T_L

(iv) loop flux-linkages $\underline{\psi}^{**}$ determined with respect to all inductive links in T_L.

Note, that all components of the vectors $\underline{q}^*$ and $\underline{\psi}^*$ are contained in the vector

$$\begin{bmatrix} \underline{q} \\ \underline{\psi} \end{bmatrix} = \begin{bmatrix} \underline{q}_\delta - F_{\alpha\delta}^T \underline{q}_\alpha \\ \underline{\psi}_\gamma + F_{\gamma\beta} \underline{\psi}_\beta \end{bmatrix}$$

Because $\underline{q}^{**}(t)$ and $\underline{\psi}^{**}(t)$ can be calculated in advance from eqns. (5.23) and (6.25) we can consider that the vector $\underline{x}_2$ appearing in eqn. (6.21) is

$$\underline{x}_2 = \begin{bmatrix} \underline{q}^{**} \\ \underline{\psi}^{**} \end{bmatrix} \qquad (6.29)$$

In order to obtain the reduced state variable formulation it is necessary to express all network variables in terms of the vector $\underline{x}$ given in eqn. (6.27) and then calculate

$$\dot{\underline{x}}_1 = \begin{bmatrix} \dot{\underline{q}}^* \\ \dot{\underline{\psi}}^* \end{bmatrix}$$

in terms of $\underline{q}^*$, $\underline{\psi}^*$, $\underline{q}^{**}$, $\underline{\psi}^{**}$ and t. Combining Kirchhoff's laws given in eqns. (2.63), the implicit description of network elements given in eqns. (2.65) and (2.66), eqns. (6.23), (6.25), (6.28a) and (6.28b) and suitably partitioning the submatrices F_{ij} appearing in eqns. (2.63), the following set of (C*), (L*) and (R*) equations which correspond to the (C), (L) and (R) equations is obtained

$$\begin{aligned}
&\underline{q}_{\delta 1} - F^T_{\alpha\delta 1}\underline{q}_\alpha = \underline{q}^* \\
&\underline{q}_{\delta 2} + A_{\delta 1}\underline{q}_{\delta 1} + A_\alpha \underline{q}_\alpha = \underline{q}^{**} \qquad (C^*) \qquad (6.30a) \\
&\underline{v}_\alpha + F_{\alpha\delta 1}\underline{v}_{\delta 1} + F_{\alpha\delta 2}\underline{v}_{\delta 2} = \underline{e}_\alpha \\
&f_C(\underline{v}_\alpha, \underline{v}_{\delta 1}, \underline{v}_{\delta 2}, \underline{q}_\alpha, \underline{q}_{\delta 1}, \underline{q}_{\delta 2}, t) = \underline{0}
\end{aligned}$$

$$\begin{aligned}
&\underline{\mathcal{L}}_{\gamma 1} + F_{\gamma 1 \zeta}\,\underline{\mathcal{L}}_{\zeta} = \underline{\varphi}^{*} \\
&A_{\gamma 1}\underline{\mathcal{L}}_{\gamma 1} + \underline{\mathcal{L}}_{\gamma 2} + A_{\delta}\underline{\mathcal{L}}_{\zeta} = \underline{\varphi}^{**} \qquad (L^{*}) \\
&-F^{T}_{\gamma 1\zeta}\,\underline{i}_{\gamma 1} - F^{T}_{\gamma 2\zeta}\,\underline{i}_{\gamma 2} + \underline{i}_{\zeta} = \underline{i}_{\zeta} \\
&f_{L}(\underline{i}_{\gamma 1}, \underline{i}_{\gamma 2}, \underline{i}_{\zeta}, \underline{\mathcal{L}}_{\gamma 1}, \underline{\mathcal{L}}_{\gamma 2}, \underline{\mathcal{L}}_{\zeta}, t) = \underline{0}
\end{aligned} \tag{6.30b}$$

$$\begin{aligned}
&\underline{v}_{\beta} - F_{\beta\varepsilon}\,\underline{v}_{\varepsilon} = \underline{e}_{\beta} - F_{\beta\delta}\,\underline{v}_{\delta} \\
&-F^{T}_{\beta\varepsilon}\,\underline{i}_{\beta} + \underline{i}_{\varepsilon} = \underline{i}_{\varepsilon} + F^{T}_{\gamma\varepsilon}\,\underline{i}_{\gamma} \qquad (R)^{*} \\
&f_{R}(\underline{v}_{\beta}, \underline{i}_{\beta}, \underline{v}_{\varepsilon}, \underline{i}_{\varepsilon}, t) = \underline{0}
\end{aligned} \tag{6.30c}$$

Note, that the (R^*) eqns. are identical to the (R) eqns. Once (C^*), (L^*) and (R^*) eqns. are solved $\underline{i}_{\gamma}(\underline{\varphi}^*, \underline{\varphi}^{**})$, $\underline{v}_{\delta}(\underline{q}^*, \underline{q}^{**})$, $\underline{i}_{\beta}(\underline{\varphi}^*, \underline{\varphi}^{**}, \underline{q}^*, \underline{q}^{**})$ and $\underline{v}_{\varepsilon}(\underline{\varphi}^*, \underline{\varphi}^{**}, \underline{q}^*, \underline{q}^{**})$ are obtained; differentiating eqns. (6.28a) and (6.28b) with respecto to t and using eqns. (2.64a) and (2.64b) we finally get the reduced state variable formulation

$$\begin{aligned}
&\dot{\underline{q}}^{*} = \underline{i}_{\delta 1} + F^{T}_{\beta\delta 1}\,\underline{i}_{\beta} + F^{T}_{\gamma\delta 1}\,\underline{i}_{\gamma} \\
&\dot{\underline{\mathcal{L}}}^{*} = -F_{\gamma 1\delta}\,\underline{v}_{\delta} - F_{\gamma 1\varepsilon}\,\underline{v}_{\varepsilon} + \underline{e}_{\gamma 1}
\end{aligned} \tag{6.31}$$

Note that the normal form equations (6.31) are of order

$$d = (b_C + b_L - b_\alpha - b_\zeta - b_{\gamma 2} - b_{\delta 2})$$

which corresponds to the <u>order of state equations</u> as defined in Ref. 9. The reduced state variable formulation is useful when $(b_{\gamma 2} + b_{\delta 2})$ is appreciable compared to the number of components of the state vector $\begin{bmatrix} \underline{q} \\ \underline{\mathcal{L}} \end{bmatrix}$.

6.4 SUMMARY

In this chapter the existence of a unique network response has been considered for a large class of nonlinear RLC networks containing coupled (and possibly locally active and/or nonreciprocal) elements. The results have been extended to a class of RLC networks containing dependent sources. Finally, the reduced state variable formulation has been discussed for a very broad class of nonlinear RLC networks.

6.5 REFERENCES

1 Liu, R. and Auth, L. V., "Qualitative synthesis of nonlinear networks", Proc. 1963 First Annual Allerton Conference on Circuit and System Theory, pp. 330-343.

2 Desoer, C. A. and Katzenelson, J., "Nonlinear RLC networks", Bell Sys. Tech. J., vol. 44, pp. 161-198, January 1965.

3 Chua, L. O. and Rohrer, R. A., "On the dynamic equations of a class of nonlinear RLC networks", IEEE Trans. on Circuit Theory, vol. CT-12, pp. 475-489, December 1965.

4 Brayton, R. K. and Moser, J. K., "A theory of nonlinear networks", Quat. Appl. Math., vol. 22, pp. 1-33, April 1964.

5 MacFarlane, A. G. J., "Formulation of the state-space equations for non-linear networks", Int. J. Control, vol. 5, pp. 145-161, May 1967.

6 Oh, S. J., Stern, T. E. and Meadows, H. E., "On the analysis of nonlinear irregular networks", Proc. of the Symposium on Generalized Networks, Politechnic Press, New York, pp. 653-682, April 1966.

7 Stern, T. E. "On the equations of nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 74-81, March 1966.

8 Holzmann, C. A. and Liu, R., "On the dynamical equations of nonlinear networks with n-coupled elements", Proc. 1965 Third Annual Allerton Conference on Circuit and System Theory, pp. 536-545.

9 Ohtsuki, T. and Watanabe, H., "State-variable analysis of RLC networks containing nonlinear coupling elements", IEEE Trans. on Circuit Theory, vol. CT-16, pp. 26-38, February 1969.

10 Auth, L. V. and Liu, R., "On the existence of the normal form for nonlinear networks at large with controlled sources", Proc. 1964 Second Annual Allerton Conference on Circuit and System Theory, pp. 288-292.

11 Katzenelson, J., "AEDNET: A simulator for nonlinear networks", Proc. IEEE, vol. 54, pp. 1536-1552, November 1966.

12 Oh, S. J. "Analysis of nonlinear irregular networks", Tech. Report No. 96, Department of Electrical Engineering, Columbia University, New York, June 1966.

13 Hakimi, S. L. and Kuo, N., "The rank of a Bashkow-Bryant A-matrix and a network characterization of the interdependence of the state variables", IEEE Trans. on Circuit Theory, vol. CT-13, pp. 331-332, September 1966.

14 Cahill, L. W., "On the selection of state variables for nonlinear RLC networks", IEEE trans. on Circuit Theory, vol. CT-16, pp. 553-555, November 1969.

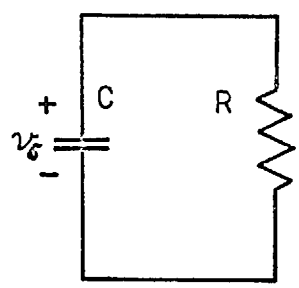

Fig. 6.1. Example of Section 6.2.1.

Chapter 7

CONCLUSIONS

7.1 GENERAL CONCLUSIONS

In this thesis, the formulation of the state equations for nonlinear RLC networks containing coupled elements has been considered from the point of view of analysis. The aim of the research has been to develop the criteria for the existence of a unique network response requiring only the knowledge of the network topology and the distribution of network element types. A particular emphasis has been given to the study of nonlinear networks containing coupled locally active and nonreciprocal elements.

Since the crucial problem in the state variable formulation of non-linear RLC networks is the existence of a unique inverse function of certain vector-valued functions the problem of functional inversion was considered in a great detail. Palais' theorem gives necessary and sufficient conditions for the existence of the global inverse of continuously differentiable vector-valued functions. Unfortunately, it is not easy to apply Palais' theorem in nonlinear network problems. An easily applicable criterion ensuring global invertibility of a large class of vector-valued functions and based on the Jacobian of a given vector-valued function was proposed. This criterion was then used to establish global regularity of few classes of vector-valued functions, important in nonlinear network theory, in a very simple manner. The concept of a generalized quasilinear function possessing bounded and possibly nonsymmetric u. p. d. Jacobian matrix and leading to the concept of a positive definite network element was introduced.

It was assumed that network elements are characterized by the hybrid descriptions. The problem of transformations from one hybrid description into another was examined. The concept of a positive network element possessing all possible hybrid descriptions was introduced. It was shown that positive network elements, which may be locally active, represent a very good model for many practical active devices such as transistors and vacuum triodes. One of the significant properties of positive network elements is that they can be locally active only if they are nonreciprocal and another is that all possible hybrid matrices of a positive network element belong to the bounded matrices of class UP.

Positive definite network elements, representing an extention of the concept of quasilinear elements to the nonreciprocal case and forming the most general passive counterpart of a one-to-one two terminal element, were introduced. These elements, representing a subclass of positive network elements, are strongly locally passive and all possible hybrid matrices of a positive definite network element are u. p. d. Series-parallel interconnections of network elements were studied and sufficient conditions were stated ensuring that the series-parallel interconnection of two network elements results either in a positive or positive definite network element. It was shown that the cascade interconnection of two two-port positive network elements results in a positive network element. The results obtained for interconnections of network elements have useful applications in the dc analysis of transistor networks.

Since the analysis of resistive, capacitive and inductive networks plays an imporant role in the analysis of RLC networks, the problem of the existence of a unique solution for one-element-kind networks was thoroughly studied. Several theorems regarding the conditions for the existence of unique solutions of resistive networks which contain different classes of elements have been presented. It was shown that a resistive network with the normal distribution of independent sources and containing positive definite resistors only possesses a unique solution regardless of the network topology. The applicability of the results obtained in the study of one-element-kind networks was illustrated on a number of examples of transistor networks.

Bryant's method of writing Kirchhoff's laws has been adopted. A potentially complete set of network variables was selected on the basis of a normal tree. Several criteria concerning the existence of a determinate response of nonlinear RLC networks and based on the network topology have been derived. It was shown that these criterial are applicable for a class of nonlinear RLC networks with dependent sources provided dependent sources satisfy certain topological restrictions. The problem of the reduced state variable formulation was considered for a general class of nonlinear RLC networks.

The work carried out in this thesis has led to better understanding of resistive as well as RLC networks containing nonlinear locally active elements and the network theoretic results obtained have justified the introduction of positive and positive definite network elements.

7.2 FUTURE RESEARCH

A number of points in this work could be extended for future research.

(a) Theorem 5.3 in Chapter 5 is concerned with interconnections of a three-terminal resistive element representing a positive network element in all 3 orientations and a two-terminal voltage (or current) controlled resistor with a monotonic characteristic. It is conjectured that the result of Theorem 5.3 can be extended to (n+1)-terminal resistive elements which are positive network elements in all (n+1) different orientations.

(b) Algorithms for dc analysis of nonlinear resistive networks are usually restricted to networks containing quasilinear elements only. For the purpose of the state-variable analysis of nonlinear RLC networks it appears to be useful to develop efficient algorithms for dc analysis of resistive networks allowing different types of network elements.

(c) Several topological criteria for the existence of a unique solution for one-element-kind and RLC networks have been presented in this thesis. By combining the results concerning the properties of interconnections of network elements and the topological criteria a very large class of nonlinear networks can be tested by inspection regarding a unique network solution. However, it appears to be useful to develop a systematic method for checking uniqueness of network solution using computer.

(d) In this thesis it was proved that an RLC network containing positive definite elements is always determinate regardless of the network topology. It may be useful to introduce the class of positive definite RLC n-port networks possessing the property that an arbitrary interconnection of the networks of this class results in a determinate RLC network.

(e) The author has not succeeded in proving Theorem 3.2 using Palais' theorem. It should be possible to prove that the conditions of Theorem 3.2 imply the radial unboundedness of a function $\underline{f}(\underline{x})$. This could lead to a simpler proof of Theorem 3.2 than the proof reported in this thesis.

(f) Study of stability of nonlinear RLC networks containing positive network elements could be of a great interest.