IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY

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Management Engineering Section

# ASPECTS OF MATHEMATICAL PROGRAMMING IN FINANCIAL CORPORATE PLANNING

by

J.S.H. Kornbluth

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ABSTRACT

Linear programming has been used as a tool for the investigation of corporate planning and the valuation of resources, the management of bank assets, etc. This thesis uses the LP framework to develop a global corporate model for short to medium term financial corporate planning, and shows the difficulties inherent in both the large scale use of such models and the theoretical application of the dual evaluation process.

Fractional programming is used to analyse corporate planning with respect to objectives which comprise fractional terms. Duality and pricing in linear fractional programming are discussed. Conditions for 'coherent pricing'in linear fractional programming are deduced, and sequential methods for the decentralisation of planning operations with fractional programmes are given.

The use of special methods for fractional programming ( integer programming, upper bound techniques, sensitivity analysis etc.) are also presented.

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J.Kornbluth
Imperial College
London S.W.7.

## NOTATION

### MATHEMATICAL NOTATION

| | |
|---|---|
| $\sum_i \sum_j \sum_k$ | sums over i,j,k, usually i = 1 ... m, j = 1 ... n, k = 1 ... m |
| $\sim$ | varies with, is approximately the same as |
| $\approx$ | any of the symbols $\leq$ , $=$ , $\geq$ ,according to the problem specification. |
| $\hat{\underline{x}}$ , $\underline{x}^*$ | particular values of $\underline{x}$, $\underline{x}^*$ denoting the optimal value of $\underline{x}$ |
| $f(\underline{x}^*)$ | the optimal value of $f(\underline{x})$ |
| $\underline{x}_B$ | the vector of basic $x_i$ |
| $\underline{c}_B$ , $\underline{d}_B$ | the elements of $\underline{c}$ and $\underline{d}$ corresponding to $\underline{x}_B$. ( $\underline{c}^*$ and $\underline{d}^*$ are also used in this context.) |
| $\left[\frac{\partial f}{\partial \underline{x}}\right]_{\underline{x} = \underline{x}^*} \left(\nabla_x f(\underline{x})\right)$ | the value of the partial derivatives of $f(\underline{x})$ with respect to $\underline{x}$ , evaluated at $\underline{x}^*$. |
| $\{ \underline{x} \mid \text{condition A} \}$ | the set of $\underline{x}$ for which condition A holds |
| $\mu_i$, $\overline{\mu}_i$ | sometimes used to denote corresponding variables in the original fractional and Charnes and Cooper forms, $\overline{\mu}_i$ being the transform of the variable $\mu_i$. |

### MODEL NOTATION

| | |
|---|---|
| n / m | a model considering n products over a planning horizon of m time periods. |

## ABBREVIATIONS

| | |
|---|---|
| o.f. | objective function |
| s.t. | such that |
| rhs | right hand side |
| N + S, NS | necessary and sufficient |
| KT, K-T | Kuhn and Tucker |
| BB, B-B | Balinski and Baumol |
| LP | linear programming |
| IP | integer programming |
| FP | fractional programming |
| CCP | chance constrained programming |
| $\exists$ | there exists |

## REFERENCES

| | |
|---|---|
| (1.31) | equation 31 of Chapter 1 |
| Fig. 1.31 | figure 31 of Chapter 1 |
| Table 1.31 | table 31 of Chapter 1 |
| 1.3, 1.3.1 | section 3 and section 3.1 of Chapter 1 |
| Appendix 1.3 | the third appendix for Chapter 1 |
| (31) | reference 31 |

All equations, figures, and tables in the appendices continue the ordering pertaining to the original chapter of the text.

# Chapter 1. Linear Programming and Corporate Modelling: A Review

## 1.1 LP and the Costing of Funds

Linear Programming is the description of problems of the form

$$\max P = \sum_j c_j x_j$$

$$\text{s.t.} \quad \sum_j a_{ij} x_j \approx b_i \qquad i = 1 \ldots m$$

$$x_j \geq 0$$

$$(c_j,\ a_{ij},\ b_i \text{ constants}) \tag{1.1}$$

LP problems are characterised by their attempt to optimise the value of a function of several variables $\{x_j\}$, subject to linear constraints on the levels that may be assigned to the $\{x_j\}$. In the context of Management Science, the variables $\{x_j\}$ may be the activity levels of a system (corporation); the function P then measures an objective of the management of the system concerning the component variables. The constraints $\{b_i\}$ are the 'resources', and the constraint set $\{\underline{x} | \underline{A}.\underline{x} \approx \underline{b}\}$ denotes the allowable (feasible) combinations of $\{x_j\}$.

Theoretical aspects of LP, solution methods, and applications are considered in (16), (33), (55), (83); from these, and many other sources, the advantages of an LP approach to corporate planning may be deduced. We will not discuss this point, but will trace the major advances in the uses of LP in planning and valuation.

LP was initially utilised for solving problems of blending, plant loading, diet calculation, etc. (see e.g. (33), (83)). During the 1950's, more complex problems in Operational Research and Financial Planning were formulated in LP terms. Charnes Cooper and Miller, (24) consider the application of LP to the

Warehouse Problem (10), and the implication of the dual variables for evaluation. Their primal formulation of the problem is the maximisation of undiscounted cummulative profit, for which the dual variables have the dimensions of (compound) interest rates (i.e. "pounds per pound invested per period"). Thus, by attaching financial constraints to production and storage equations, funds are evaluated with respect to optimal corporate behaviour.

Dean (37) proposed a ranking of capital projects on the basis of their internal rates of return (i.e. on the basis of the internal discount rate that would reduce the net present value of the project to zero); this method is criticised by many authors (see e.g. (59)) in that it does not allow for interdependent projects, negative cash flows etc. In this respect, the ranking of projects via the dual evaluation, as suggested in (24) represents a major advance in the field of capital budgeting and resource valuation.*

## 1.2 Dual Interpretations for Capital Budgeting

Lorrie and Savage (59) show that Dean's proposed ranking of projects must fail if:

- a. the projects are interdependent
- b. the total capital expenditure is limited in more that one planning period, or
- c. the stream of returns is not always positive.

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* As noted in (24), there is a similarity between the Warehouse Problem, and the problem of optimal flows through a network; this latter approach is developed by Ford and Fulkerson in (38). Although networks can aid the conceptualisation of the problem, (via the use of flow charts, such as Fig. 2.3 and Fig. 2.4,) the Mathematical Programming approach has decisive economic and computational advantages.

Their formulation of the capital budgeting problem is considered in depth by Weingartner, (88). Although Weingartner's work essentially deals with long term planning, the theory and methodology he develops are also applicable to medium and short planning, and point the way to much of the work in this thesis. His formulation is:

$$\max P = \sum_j b_j x_j$$

$$\text{s.t} \quad \sum_j c_{tj} x_j \le C_t \qquad t = 1 \ldots T$$

$$0 \le x_j \le 1$$

$$(x_j \text{ integers}) \qquad (1.2)$$

where $\{b_j\}$ are the rewards (NPV's) associated with the projects $\{j\}$, $x_j = 0$ or 1 according as the j'th project is rejected or accepted, T is the number of periods to the planning horizon, $c_{ij}$ is the outlay for project j in period i, and $\{C_t\}$ are the maximum possible expenditures for the periods $t = 1 \ldots T$.

(The formulation (1.2) overcomes the last two points raised in (59); the interdependence of projects may also be included in the integer programme (IP), using inequalities of the form $x_j - x_k \le 0$.)

The discussion in (88) contains three important features:

i. the attempt to solve (1.2) using approximate LP techniques,

ii. the use of LP duality to rank the projects $\{x_j\}$,

iii. the use of IP algorithms to give a true optimum for (1.2), and the attempt to associate a dual pricing mechanism with the integer solution, by re-imputation.

Using the LP approximation to (1.2) Weingartner analyses the dual, namely:

$$\min \ \pi = \sum_{t=1}^{T} \rho_t C_t + \sum_j \mu_j$$

$$\text{s.t} \quad \sum_{t=1}^{T} \rho_t c_{tj} + \mu_j \geq b_j \qquad j = 1 \ldots n$$

$$\rho_t, \ \mu_j \geq 0 \qquad (1.3)$$

where: $\{\rho_t\}$ are the shadow prices (or opportunity costs) associated with the budgetary constraints, and $\{\mu_j\}$ are the dual evaluators of the upper bound constraints $\{x_j \leq 1\}$. If (bounded) optimal solutions exist for (1.2) and (1.3), LP duality ensures that $P^* = \pi^*$. Weingartner associates the $\{\mu_j^*\}$ with the goodwill generated by $x_j^*$ (since $\Sigma \mu_j^*$ represents the difference between the value of the firm $P^*$, and the value imputed to resources, $\sum_{t=1}^{T} \rho_t^* C_t$ ). Defining $\gamma_j^* = \sum_{t=1}^{T} \rho_t^* c_{tj} + \mu_j^* - b_j$ , Weingartner shows that the $\{\mu_j^*, \gamma_j^*\}$ provide convenient rankings for the projects.

Baumol and Gomory (43) suggest a method whereby the dual evaluators of the final LP in the method of Integer Forms for IP (42) may be re-imputed to the original constraints to give an efficient[(1)] price allocation. The theoretical difficulties associated with recomputed dual prices are discussed in Chapter 5 of (88). Weingartner suggests an alternative dual approach, namely the use of the LP dual on the restricted (optimal) IP formulation. This formulation does price out resources (an improvement over the prices of Baumol and Gomory) but does not clarify the concept of a free good.

Alcaly and Klevorick (2) have given another variant on the re-imputation process, introducing subsidies to the activities

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(1) see Koopmans (56)

to ensure that LP duality theorems still hold; although their prices are more acceptable economically, the authors note that "the concept of a free good remains disturbing", and admit the "tenuous relationship between dual prices and marginal revenue product in IP".

Balas, (5), has recently formulated a generalised duality theory for discrete programming, which furnishes marginal values for integer programmes. The use of this theory for fractional programmes is discussed in Section 6.6.3.

Although the theoretical application of pricing in IP is still unresolved, Weingartner's work represents the first formalisation of the capital budgeting problem, and forms the basis for many of the later financial planning models. The more realistic estimates of rates of return on capital give a framework in which financing options may be compared.

## 1.3 LP for Accounting and Control

### 1.3.1 Goal Programming and Accounting Models:

Goal Programming (16, Appendix B) is the description applied by Charnes and Cooper to problems of the form:

$$\begin{aligned} \min\ & \sum_i \left(y_i^+ + y_i^-\right) \\ \text{s.t}\ \ & \underline{g}.\underline{x} - \underline{y}^+ + \underline{y}^- = \underline{g}_o \\ & \underline{A}.\underline{x} \leq \underline{b} \\ & \underline{x},\ y_i^+,\ y_i^- \geq 0 \end{aligned} \tag{1.4}$$

in which the variables $\{x_i\}$ are considered as 'sub-goals' to the 'goal' $\underline{g}_0$.

Ijiri (53) shows how the analysis of break-even points may be transferred to a goal programming problem, and how the formulation of (1.4) may be used to analyse the operations of a firm which has multiple goals. Using 'non-archimedean' weightings, Ijiri ranks goals in the order in which they are to be achieved, producing a single objective function for (1.4), and via the generalised inverse for $\underline{A}$, (see e.g. 71), devises methods by which deviations from goal attainment may be controlled, (Appendix A (53)). Ijiri also applies goal programming to the analysis of the spread sheet accounts of a firm, via the incidence matrix of the accounting network. The model presented in (53) uses the changes in accounts as performance indicators; the objective of maximizing net addition to retained earnings is optimised subject to restrictions on the account levels and their inter-actions. (In this model, each account is represented by a model variable.)

1.3.2 Feedback Indicators and Control of Performance

For the set of goals $\underline{v}$, the sub-goals $\underline{x}$, and the relationship $\underline{A}.\underline{x} = \underline{b}$, Ijiri defines an indicator set $\underline{w}$ by:

$$\underline{c}.\underline{x} = \underline{w} \qquad (1.5)$$

(If $\underline{x}$ and $\underline{w}$ are n and k resp., $\underline{c}$ is any kxn matrix). He shows that the necessary and sufficient condition for $\underline{v}$ to a uniquely determined function of $\underline{w}$ is that each row of $\underline{A}$ be expressable as a linear combination of the rows of $\underline{c}$; in this case, $\underline{w}$ is a perfect indicator set. (Where $\underline{v}$ is not uniquely determined by $\underline{w}$, the set $\underline{w}$ is said to be in imperfect indicator set). The case of imperfect indicators is analysed using the generalised inverse of $\underline{A}$, whereby Ijiri demonstrates how the imperfect indicators may be used to determine whether the system is operating within prespecified limits $\underline{A}.\check{\underline{x}}_o$ and $\underline{A}.\hat{\underline{x}}_o$. The development is important where management wish to review a restricted number of statistics (indicators) from which a global (i.e. subgoal) performance may be surmised, and controlled.

1.3.3 Opportunity Costing and Departmental Control

Samuels, (73), investigates a different aspect of control derivable from the LP model. He attempts to formulate a model in which the dual evaluators are used to price resources, and the divisions of the company are charged (controlled) by their deviation from optimal usage of resources.

The calculation of the (opportunity) costs to be charged against erring departments is made using the dual evaluators of resources at the previous optimum: Samuels asserts that, by duality, the accounting procedures will balance up the total optimal budgeted value with the opportunity charges to product accounts and their marginal contributions.

The model discussed is one of three products (X, Y, Z), and three resources, floor space, supervisor time and machine time:

$$\begin{aligned} \max P &= 2X + 3Y + 4Z \\ \text{s.t.} \quad & 5X + Y + Z \leq 8000 \text{ (floor space)} \\ & X + 5Y + Z \leq 8000 \text{ (supervisor time)} \\ & X + Y + 5Z \leq 8000 \text{ (machine time)} \end{aligned} \tag{1.6}$$

The optimal solution is $P^* = £10,284$, $X^* = 1142$, $Y^* = 1143$, $Z^* = 1143$, with dual evaluators $(\frac{5}{28}, \frac{12}{28}, \frac{19}{28})$

Samuels considers three situations:

a. Suppose X overproduces, (say $\hat{X} = 1183$) and this causes Z to produce only 942 units, because of insufficient floor space. (The nature of causality is not stated explicitly: this is discussed in Appendix 3.2). Dept. X has caused a net loss of £722, (overproduction has generated extra profits of 41 x £2, but has caused Dept. Z to lose 201 x £4), and is billed accordingly.

b. He further demonstrates that if Dept. X is more efficient in its use of supervisor time (i.e. it reduces the coefficient of X in the second row of (1.6)), it can be credited with this saving, (although no other section uses this newly freed quantity of supervisor time).

c. The final example, of overproduction by Dept. X with simultaneous underproduction by Dept. Z is presented thus: "Assume that, for one reason or another, the producers of Z would not have produced more than 1,050 units even if Dept. X had not exceeded its allotment. In this instance the lower than optimal profit should be attached to the departments of both products, X and Z.

The opportunity cost charged to Dept. X is profit lost because the inputs used in the production of X prevented Z from achieving its adjusted output figure of 1,050. Against this, Dept. X is credited with the returns from the extra output it produced because Z could not use all of its original budget.

The most severe restriction on X is floor space; 93 units were made available by Z's failing to achieve its target (i.e. 1,143-1,050). As product X requires 5 units of floor space per unit, this enabled production of 19 extra units of X. This resulted in a credit of £38 to Dept. X.

The other 22 units produced by Dept. X above the budgeted output figure were with resources that should have been used by Z, and so a credit of the contribution on these items is given to Z.

| | | | | | |
|---|---|---|---|---|---|
| X's share of "Loss": 4 x (1,050 - 942) | = | 108 x 4 | = | 432 | |
| Less Contribution from extra X profit | = | 19 x 2 | = | 38 | 394 |
| Z's share of "Loss": 4 x (1,143 - 1,050) | = | 93 x 4 | = | 372 | |
| Less Contribution from extra profit | = | 22 x 2 | = | 44 | 328 |
| Total 'Loss' | | | | | 722 " |

The accounting system presented by Samuels, (according to Bernhard (9)), is:

i. Bill to Dept. X the revenue loss of Z.

ii. Credit to Dept. X the revenue it has generated by using resources that Z had not planned to use.

iii. Bill to Z the loss caused by its inability to produce more than the revised figure for its best performance (regardless of the behaviour of X).

iv. Credit to Dept. Z, X's revenue obtained by using resources allocated to Z that Z had planned to use.

Bernhard, (9), reviewing this system, remarks:

"The main point of Samuels' paper is that any decrease in profit should be charged as an opportunity cost to whichever department(s) was responsible for the deviation."

Commenting on case c. Bernhard notices that ii and iii in the accounting system are in conflict. Suppose Z could have made 1142 units and not 1050. i. suggests billing X with £722, iii. suggests billing X with £804 and crediting Z with £82. Although the net result is the same the second process gives Z credit solely because X has infringed upon it!

Bernhard suggests a modified accounting procedure, changing iii to:

iii' Bill to X the revenue loss of Z. Credit to X the revenue gained by utilizing resources assigned to Z (whether Z has planned to use them or not). Bill to Z: (as before) ..

The algebraic sum of the penalties and bonuses remains the same, but the allocation has been rationalised.

Although Samuels notes that the external supply of limiting factors, and the internal relationships of technology may change within the span of the time period, he does not consider the possibility that the behaviour of the erring department(s) may be so far removed from the optimal solution (plan), as to go beyond the range of the optimal solution. This would invalidate the dual evaluators, and the penalty/bonus scheme under which the department considered itself to be operating.

The dual evaluators may also change due to information flows during the period. These difficulties (associated with the choice of time period) are discussed in Appendix 3.2.

#### 1.3.4 LP and Asset Valuation

The use of dual prices for (long term) fixed asset valuation (and depreciation) is discussed by Carsberg (11); the article is based on two papers by Wright (94, 95). Wright proposes a valuation of assets based, not on sunk costs, but on the minimum of replacement cost, realizable value or output value, and suggests that dual prices may be used as measures of opportunity values.

Carsberg notes the following points:

a. due to degeneracy, the marginal values derived at the optimum may be "direction dependent" (see Strum (76)).

Thus they may measure change in the direction of either an increase or decrease of asset holding, but not necessarily in both. He suggests that separate calculation is necessary before the marginal values (of dual prices) and opportunity values may be associated.

b. in situations where there is a limit on the possible holding of an asset, the value associated with that asset by the dual may be in excess of replacement cost; this might prove unacceptable under normal accounting conventions.

c. measuring the value of each asset according to either the dual price (for increasing the asset holding) or the opportunity value (for decreasing the holding) will not give a true valuation of assets that are lost 'jointly'. Under these conditions, the dual prices are liable to overvalue the firm with respect to its net cash flow. Carsberg associates this overvaluation with the accounting problem of 'jointness'.

Unfortunately, the model presented by Carsberg is very simplified. Viewing the production of one item over five years ($X_1 \ldots X_5$), which is produced on two types of machines A and B (each of which has a life of two years), he deduces a model of the type shown in Fig. 1.1. (Equations (1) to (5) are sales constraints; equations (6) to (10) and (11) to (15) are machine requirements for A and B). In this (single-product) case, the optimal basis is very stable; the solution is: "manufacture the product in each period up to capacity"; consequently the range of the solution is large. The use of the dual evaluators to evaluate all the machinery is possible (if not theoretically desirable), since the loss of one machine, Maximize:

$$12x_1 + 15x_2 + 9x_3 + 6x_4 + 3x_5 - 100a_1 - 70a_2 - 50a_3 - 40a_4 - 40a_5 - 80b_1 - 70b_2 - 60b_3 - 50b_4 - 40b_5$$

Subject to:

$$\begin{array}{lll}
x_1 & \leq 100 \cdots & (1) \\
x_2 & \leq 150 \cdots & (2) \\
x_3 & \leq 130 \cdots & (3) \\
x_4 & \leq 120 \cdots & (4) \\
x_5 & \leq 100 \cdots & (5) \\
10x_1 - 100a_1 & \leq 0 \cdots & (6) \\
10x_2 - 100a_1 - 100a_2 & \leq 0 \cdots & (7) \\
10x_3 - 100a_2 - 100a_3 & \leq 0 \cdots & (8) \\
10x_4 - 100a_3 - 100a_4 & \leq 0 \cdots & (9) \\
10x_5 - 100a_4 - 100a_5 & \leq 0 \cdots & (10) \\
5x_1 - 60b_1 & \leq 0 \cdots & (11) \\
5x_2 - 60b_1 - 60b_2 & \leq 0 \cdots & (12) \\
5x_3 - 60b_2 - 60b_3 & \leq 0 \cdots & (13) \\
5x_4 - 60b_3 - 60b_4 & \leq 0 \cdots & (14) \\
5x_5 - 60b_4 - 60b_5 & \leq 0 \cdots & (15)
\end{array}$$

$$x_j,\ a_j,\ b_j \geq 0$$

Fig. 1.1 The Carsberg Model

by parametric methods, will not invalidate the optimal basis.

If the model were to consider multi-product firms, the loss of one machine (i.e. the loss of a predetermined number of capacity units) might alter the production schedules and invalidate the optimal basis; thus machines cannot be valued for depreciation using only the optimal dual prices. The sensitivity of the basis in the multi-product environment might imply very different accounting results for small changes of inputs. Parametric analysis will be required even when considering the loss of just one machine (asset), "all other things being the same". In this sense the dual evaluations incorporate the concept of jointness into their pricing mechanism; the dual price is the marginal change in the objective function per marginal change in a single resource - "all other things remaining the same". The attempt to overcome this constraint in interpretation and ascribe values to individual assets that will be independent of the remainder of the firm must, to some extent, be arbitrary - and hence the 'over-valuation' of assets noted by Carsberg.

In this case, the philosophy of LP and the conventions of accounting are in direct opposition. The difficulties of jointness (c.f. Wright) can be overcome by the use of LP, but, the LP solution cannot be dismembered to give valuations that will accord with accounting conventions. A second instance of this difficulty occurs with "free goods". Machinery, (resources) which are not fully utilised are given zero value. A tangible, useful, asset is written off instantly if it is not utilised to capacity; i.e. if its associated constraint is not binding.

Futhermore, the valuations of an LP model are dependent on the objectives of the firm; multiple values (differing according to objective and utilisation) may be present where a firm has more than one measure of performance, or can be put into more than one operating environment. (Multiple values are considered in Chapter 3). The extent to which the choice of the basic time unit affects the dual evaluators is discussed in Section 3.3.4 and Appendix 3.2.

## 1.4 LP Models for Asset Management and Banking

### 1.4.1 Introduction

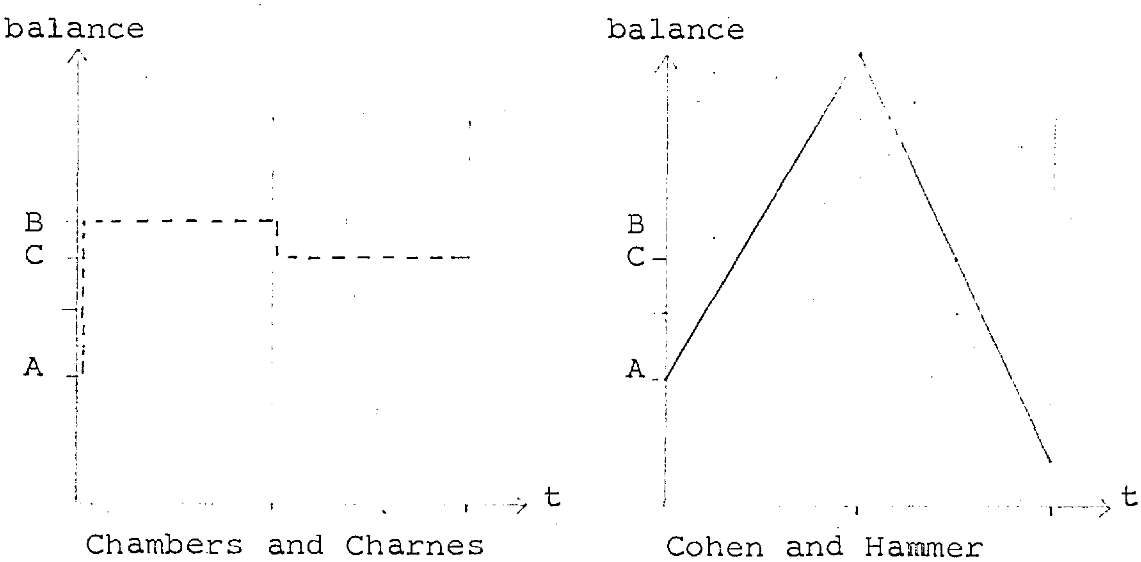
In the applications of LP to problems involving set-up times, batch quantities, non-divisibility of resources, etc. many assumptions have to be made regarding the relevance of the purely linear approach. (See e.g. Sections 2.7, 3.3 and Appendix 3.3). During the last decade, attention has been focused on the uses of Mathematical Programming in Banking, Portfolio Selection, etc. (e.g. (14), (29), (30), and (74)). Much of this work is similar to the models emanating from the simple Warehouse Model, but the detail is more refined. Also, because of the nature of the resources, the assumptions of linearity, divisibility, etc. are easily justified.

### 1.4.2 The Chambers and Charnes Model (14)

This model of the operations of banks uses a multi-period LP formulation. The constraints deal with desired liquidity ratios, security purchase restrictions, etc., the objective function being the undiscounted return on the bank's loans and investments.

### 1.4.3 The Cohen and Hammer Model

Cohen and Hammer (29), develop a more detailed model for asset management, introducing many of the safety regulations that abound in standard banking procedures; policy considerations, loan-related feedback mechanisms, etc. The two models (14) and (29) differ in their treatment of cash flows and the availability of funds. Chambers and Charnes assume instantaneous changes at inter-temporal links - "the desired average balance in a partical category is identically equal to the spot balance of that category at every instant within each period". Cohen and Hammer assume that the rates of cash flow are constant within each period. The two systems are compared in Figure 1.2.




<u>Figure 1.2 Spot and Average Balances</u>

As a result of these assumptions, the average period balance sheets will balance in the Cohen and Hammer Model, but this will not necessarily be the case for any spot balances at the beginning or end of periods.

(This 'unreality' is justified by the authors on the grounds that their model is used for intermediate and long range planning, not for day to day transactions. It is also noted (29) that, "any degree of realism in this respect can be incorporated into the model by appropriately shortening the durations of the planning periods".)

Cohen and Hammer consider three possible objective functions for maximisation:

(1) the value of stockholders' equity during the final period

(2) the present value of the net income stream plus realised capital gains and losses during the planning period

(3) the sum of (2) and the net present value of (1)

Objective (1) avoids the calculation of an internal discount rate, and implies a (true) willingness to postpone current income in favour of an ultimately higher value of stockholders' equity. The second objective allows future gains to be discounted against risk, but the optimal solution is said to be sensitive to discount rates. (3) is justifiable because only the first period decisions are required; the final value of stockholders' equity is included to allow for the horizon in these decisions, (otherwise terminal stocks will be deemed worthless).

1.4.4 The Use of Unequal Time Periods

For the models presented in (14) and (29), the time to the planning horizon was divided into equal periods. Orgler (69), has suggested a model in which unequal periods of time are used, varying from daily considerations near the decision instant to longer (monthly) considerations at the planning horizon.

In models for day to day decision making this approach has the advantage of computational brevity, since detail at the planning horizon is not required. Where such detail is needed (in Chapter 2, etc.) unequal time periods will not be so useful.

## 1.5 Normative Models and the Behavioural Science Approach

### 1.5.1 Assumptions in Normative Planning

In this review of normative models for industry and banking, we have tacitly assumed that in most cases, the objective function is a reasonable expression of management's aims, thereby associating the short term optimisation with the first stages of a global, corporate, long-term plan. The validity of this approach is questioned by Charnes and Stedry (26), (27). In (26), they re-iterate the distinctions between the normative (Operational Research, Economic) approach, and the descriptive attempts of Cyert and March, Simon, etc. (32, 36). The first approach is said to have the following characteristics:

> "*i*. explicit long-run profitability maximization for the firm as a whole,
>
> *ii*. focus on the design of internal systems to achieve this aim,
>
> *iii*. the rigorous use of mathematical tools in the solution of the organizational problems posed."
>
> (see (26), page 147)

(The original abstraction that firms "maximize profit" is linked to a further assumption "that the individuals within the firm are 'rational'".) Although Charnes and Stedry concede that the concept of profit maximization could be broadened to include utility maximization, they suggest that "the assumptions of long-run profit or utility maximization are non-operational, (even) if, logically or tautologically, they can be shown to be valid. Such aims as "good employee morale", "no layoffs unless necessary", etc. are not readily translatable into terms of profit and loss". Charnes and Stedry (26, page 150) find no evidence that firms do construct long term profit functions from which they derive short term statements of purpose. (See also Cyert, Dill and March (31)). In contrast to the normative approach, the descriptions of the behavioural scientist have the advantages of encompassing all the aims and aspirations of the firm, (at various levels of the organizational heirarchy), but there is a dearth of viable mathematical tools that can be used in analyses of the relationships between aims, policies and strategies.

Two models are presented in (26) and (27) which broaden the scope of mathematical programming for modelling the aims and aspirations of management; Goal Programming (introduced in Section 1.3.1) and Chance Constrained Programming.

1.5.2 Chance-Constrained Programming

Initially developed by Charnes and Cooper (18), CCP uses the following formulation:

Replace $\sum_j a_{ij}x_j \le b_i$, $(i = 1 \ldots m)$, with the probabilistic constraint:

$$\Pr\left\{\sum_j a_{ij}x_j \le b_i\right\} \ge 1 - \alpha_i \qquad i = 1 \ldots m \qquad (1.7)$$

Each expression, $(i = 1 \ldots m)$, becomes a statement of policy, with respect to the goal $b_i$. The objective function for the programme can take one of three suggested forms:

the E model: $\max E(\underline{c}.\underline{x})$

the V model: $\max E(\underline{c}.\underline{x} - \underline{c}_o.\underline{x}_o)^2$

the P model: $\max \Pr\left\{\underline{c}.\underline{x} \ge \underline{c}_o.\underline{x}_o\right\}$

(See e.g. (19), (20), (21), (22))

Optimum decision rules for a limited (and severely restricted) class of problems are considered in (20) and (21), but, although the application of this technique to (large scale) business problems is very attractive, the lack of theoretical analysis and computer algorithms makes it impossible.

Some work has been carried out by Charnes and Sten Thore (28) on liquidity levels for financial institutions, and by Charnes, Cooper and Symonds, (25), on problems with very special forms of associated probability laws. As yet, the forms of constraints that can be accommodated are very limited; most of the calculations in the literature are specific to particular problems, and do not furnish general algorithms. Nevertheless, managerial awareness of risk in financial planning may not be ignored. Fractional Programming under conditions of risk and uncertainty is considered in Appendix 6.4 and may well represent a fruitful field for further research.

## 1.6 Ratios, Performance Measures and Fractional Programming

### 1.6.1 Ratios

In recent times, British Industry has witnessed an increasing emphasis on productivity and financial ratios; 'productivity' has become an established yardstick in labour efficiency and wage negotiations, and the use of financial ratios has been much publicised as the result of such takeovers as that of A.E.I. by G.E.C. (Indeed, long term measurements have assumed a short term importance that completely distorts the economic picture.) Both sets of ratios attempt to combine into one factor a series of complex relationships. Cohen and Hammer, (30), note "the fact that the bankers pay attention to such simple and naive rules of thumb as the ratio of loans to deposits, capital to risk assets and mortgages to savings deposits indicates their awareness of the interactions that exist among these various accounts".

Initially, productivity ratios were simple, economic guides to the output potentials of different plots of land, or the conversion efficiency of an engineering process. Even in their simplest form they represent two major forms of comparison; the "input creativity" emphasizes the non-comparability of inputs and outputs, whereas the "conversion efficiency" stresses the reduction of both to common terms (e.g. B.T.U. equivalents), (40). Recently, work has been published that emphasizes the relationship between the productivity measures and the aims of the organization: as Professor Gold states (40):

"in as much as different systems are likely to have different objectives, and each system is likely to have a variety of performance criteria, it follows that each system may be characterized by an array of productivity relationships at a given time, and also that identical measurements may have a widely disparate meanings in different systems".

1.6.2 Value Added and Total Efficiency

Gold's work on the uses of ratios within the company has its parallel in inter-company and inter-industry studies. Professor Ball, (7), also mentions this association between aims and performance measures: "there is a great temptation (here) to embark on the search for the Golden Index, the single statistic that can be taken as a measure of the success and efficiency of the enterprise. A popular candidate for this role is the rate of return on capital." (7, page 6).
Giving reasons why no such "Golden Index" can exist, Professor Ball writes: "The starting point in any discussion of efficiency must be to specify the set of objectives that one is seeking

to attain. It is necessary to measure efficiency in relation to objectives, otherwise it has no meaning". He proceeds to argue that it is necessary to include a measure of "value added" to the battery of statistics that are used to assess performance." He introduces the concept of the 'total efficiency' of a firm - compounded of price and technological efficiency, (analogous to the 'efficient points' in Koopmans (56)). Comparison by 'total efficiency' is suggested as method of inter-firm and inter-industry judgements, but serves little purpose in advancing the normative objectives of management.

1.6.3 <u>Programming with Ratio Requirements</u>

Chambers (13) has considered the allocation of funds between competing projects (over the medium term) where a company wishes to restrict the values that will appear in reported results. His model, similar to that of Weingartner (88), includes constraints on the lower bounds that may be taken by such ratios as current assets to current liabilities, and return on gross assets. He also incorporates policy decisions on the minimum acceptable growth of profits, and shows how these constraints impinge on the optimal schedule of investments.

The inclusion of minimum levels for ratios derived from LP variables poses no new problems. The constraint

$$\frac{\underline{c}.\underline{x} + c_o}{\underline{d}.\underline{x} + d_o} \geq \lambda \tag{1.8}$$

is readily converted into the linear constraint

$$(\underline{c} - \lambda\underline{d}).\underline{x} + c_o - \lambda d_o \geq 0 \tag{1.9}$$

provided that $\underline{d}.\underline{x} + d_o$. is always positive.

For a firm whose aims can be expressed as the attainment of a set of goals $\lambda_i$, by the set of ratios $\left\{\frac{\underline{c}_i \cdot \underline{x} + c_{o_i}}{\underline{d}_i \cdot \underline{x} + d_{o_i}}\right\}$, a linear goal programming formulation may be derived. The programme

$$\min \sum_i y_i$$

$$\text{s.t.} \quad \frac{\underline{c}_i \cdot \underline{x} + c_{o_i}}{\underline{d}_i \cdot \underline{x} + d_{o_i}} + y_i = \lambda_i \qquad i = 1 \ldots m$$

$$\underline{x}, y_i \geq 0 \qquad \underline{d}_i \cdot \underline{x} + d_{o_i} \geq 0 \tag{1.10}$$

can be approximated to by

$$\min \sum_i y_i$$

$$\text{s.t.} \quad (\underline{c}_i - \lambda_i \underline{d}_i) \cdot \underline{x} + (c_{o_i} - \lambda_i d_{o_i}) + y_i = 0 \qquad i = 1 \ldots m$$

$$\underline{x}, y_i \geq 0 \tag{1.11}$$

(The linear nature of the $y_i$ has been lost; $\{y_i\}$ include a heavy weighting on the basis of the i'th denominator).

1.6.3 Fractional Programming

The problem

$$\max \frac{\underline{c} \cdot \underline{x} + c_o}{\underline{d} \cdot \underline{x} + d_o}$$

$$\text{s.t. } \underline{x} \in S$$

where $S = \left\{\underline{x} \mid \underline{A} \cdot \underline{x} \leq \underline{b}, \ \underline{x} \geq 0\right\}$

and $\underline{d} \cdot \underline{x} + d_o \neq 0$ for $\underline{x} \in S$ (1.12)

has been described by Charnes and Cooper (17) as "programming with linear fractional functionals". In (17) they prove that the optimal solution to (1.12) can be obtained by solving at most two linear programmes, either

$$\begin{aligned} \max \quad & \underline{c}.\underline{y} + c_o t \\ \text{s.t.} \quad & \underline{A}.\underline{y} - \underline{b}t \le \underline{0} \\ & \underline{d}.\underline{y} + d_o t = 1 \\ & \underline{y},\ t \ge 0 \end{aligned} \tag{1.13}$$

$$\begin{aligned} \text{or} \max \quad & -\underline{c}.\underline{y} - c_o t \\ \text{s.t.} \quad & \underline{A}.\underline{y} - \underline{b}t \le \underline{0} \\ & -\underline{d}.\underline{y} - d_o t = 1 \\ & \underline{y},\ t \ge 0 \end{aligned} \tag{1.14}$$

If $(\underline{y}^*, t)$ is optimal for (1.13) or (1.14), then $\underline{x}^* = \frac{1}{t^*} . \underline{y}^*$ is optimal for (1.12).†

Martos (65), has shown that the problem (1.12) can be solved by "simplex-like" methods; such a method is given in (64).

Wagner and Yuan (85) have proved an algorithmic equivalence between (17) and (64).

Joksch (54) considers a more general class of objective functions which may be solved by parametric methods. For (1.12) the algorithm finds the value of $\theta$ which maximises $f(\theta)$, where

$$\begin{aligned} f(\theta) = \max \quad & \frac{\underline{c}.\underline{x} + c_o}{\theta} \\ \text{s.t.} \quad & \underline{A}.\underline{x} \le \underline{b} \\ & \underline{d}.\underline{x} + \underline{d}_o = \theta \\ & \underline{x} \ge \underline{0} \end{aligned} \tag{1.15}$$

The development of fractional programming (and the corresponding recent developments in computer technology and LP capabilities) enable such ratios as "return on capital", or "return on assets" to be included in the set of objectives

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† In later theoretical work we assume that (1.13) is "the Charnes and Cooper Form" of (1.12); generalisations to include (1.14) present no added difficulties.

for normative corporate planning. The considerable emphasis placed on these ratios by contemporary management, justifies the inclusion of fractional programming as a useful management tool. Although it cannot be claimed that 'return on assets' is the unitary objective for corporate strategy, the use of fractional programming enhances the normative approach to corporate planning - making it more realistic for both management and the management scientist.

# Chapter 2 The Mathematical Model of the Firm

## 2.1 Introduction

The broad outline of the LP tools available for corporate modelling has been sketched in Chapter 1. A primary intention of the project was to use these techniques, and to develop new methods, for modelling the planning process in a firm.

In the following sections, we describe the test firm, its technology and planning process, develop the mathematical formulation of the model, and show how the data for the model is closely allied to both the structure of the firm, and the structure of the bounded variable algorithm for LP.

### 2.2.1 The Firm

As a basis for the development of the LP model, a study of a particular firm was undertaken; the company studied is part of an international corporation whose operations in the United Kingdom consist of the import, production, marketing and export of a range of electrical appliances. The study was limited to the operations within the United Kingdom, since the individual companies have considerable autonomy.

### 2.2.2 The Product Range

The product range of the firm falls into two major categories: domestic appliances, and industrial appliances, and the second category is further subdivided, according to the particular specification of the product, into three sub-catagories: light duty, medium duty, and heavy duty.

The numerical division of the product range between these categories was:

| domestic | light duty | medium duty | heavy duty |
|---|---|---|---|
| 55 | 13 | 44 | 113 |

Further classifications were electrical wiring specifications, earthing requirements, colour codes, etc. varying between markets. Typically, a domestic product could have up to twelve individual specifications; an industrial product would have at most three or four variants.

2.2.3 The Manufacturing Facilities

The manufacturing facilities of the company were divided between its three factories in England; two of these being 'adjacent' in the London area - the third in the North of England.

Production was organised in batches, according to the pertaining production schedules and estimated requirements. The final stages of production for each batch comprised assembly, testing and packing, these activities being kept strictly separate for the domestic and industrial ranges.

The machinery of the factories was coded into a series of work centre classes. A typical (numerical) breakdown of the basic machining centre is shown in Figure 2.1. Codes between 1000 and 9999 were used. Machinery (and production) was allocated between factories to keep the costs of transporting unfinished parts to a minimum; factories were assumed to specialise in particular ranges of product.

2.2.4 Raw Materials, Storage and Inventory

On each factory floor locations were assigned for raw materials - mainly metal bar, electrical wire, and castings. Materials were released from stores according to production schedules; work-in-progress was stored in bins on the shop floor, or returned (for temporary storage) to specified areas of the factory floor.

After final assembly and testing, finished goods passed into the warehouses located at the factory. Goods were either

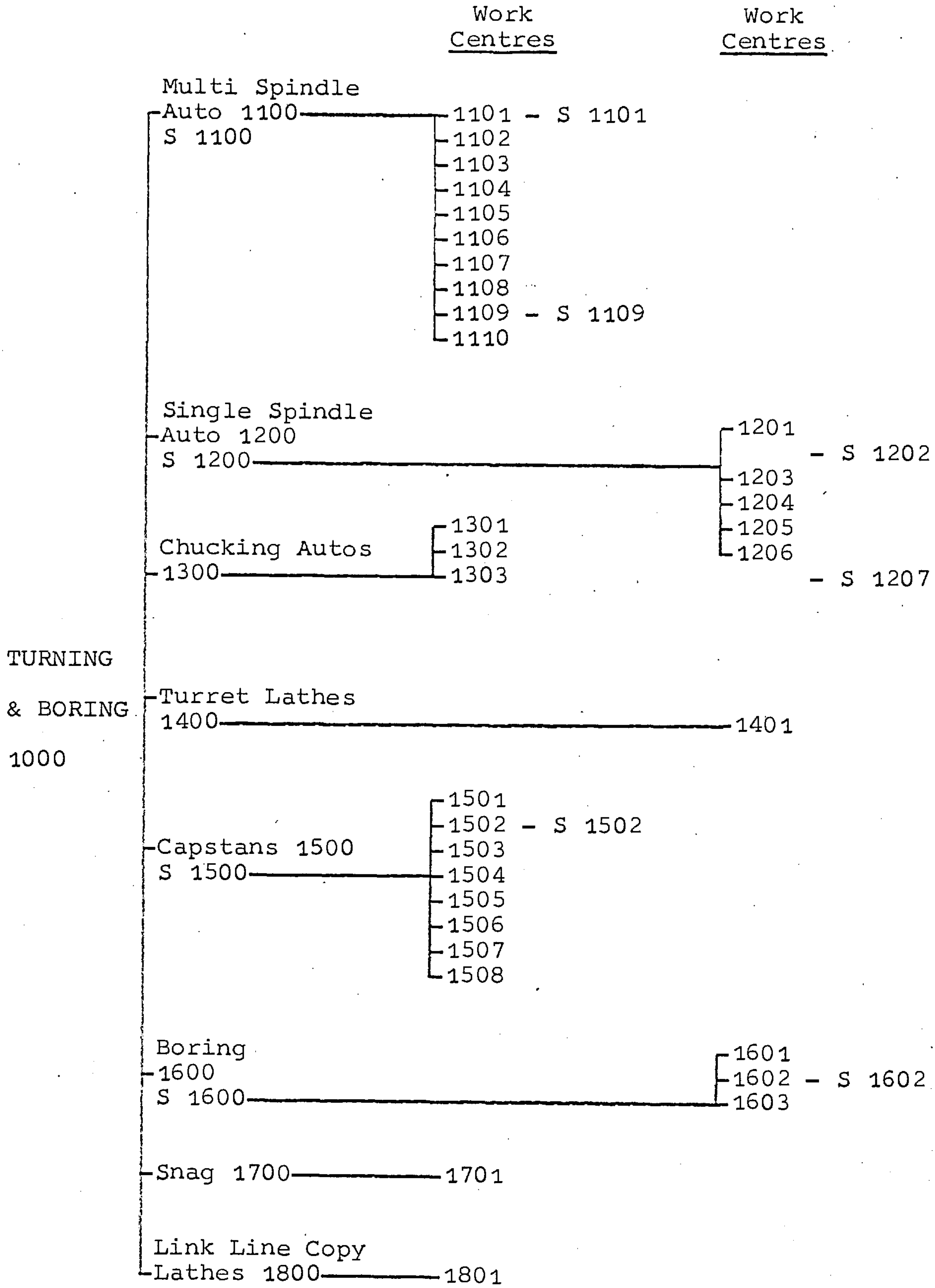



Figure 2.1

Basic Machining Centre - 1000 - (Turning and Boring) despatched immediately to purchasers or company warehouses, or stored at the factory, (see 2.2.7). The company used its own fleet of vehicles and public road hauliers.

2.2.5 The Market

The company divided the outlet for its product into five categories, each market outlet having a different characteristic and associated discount. These are shown below in Fig. 2.2.

| Market Sector | Repayment | Orders | Discount |
|---|---|---|---|
| 1 | slow | large | 30% |
| 2 | fast | large (erratic) | 40% |
| 3 | fast | large (steady) | 40% |
| 4 | medium | large | 40% |
| 5 | fast | medium | intercompany |

Fig. 2.2 Discounts and Repayment Times

2.2.6 Marketing and Promotion

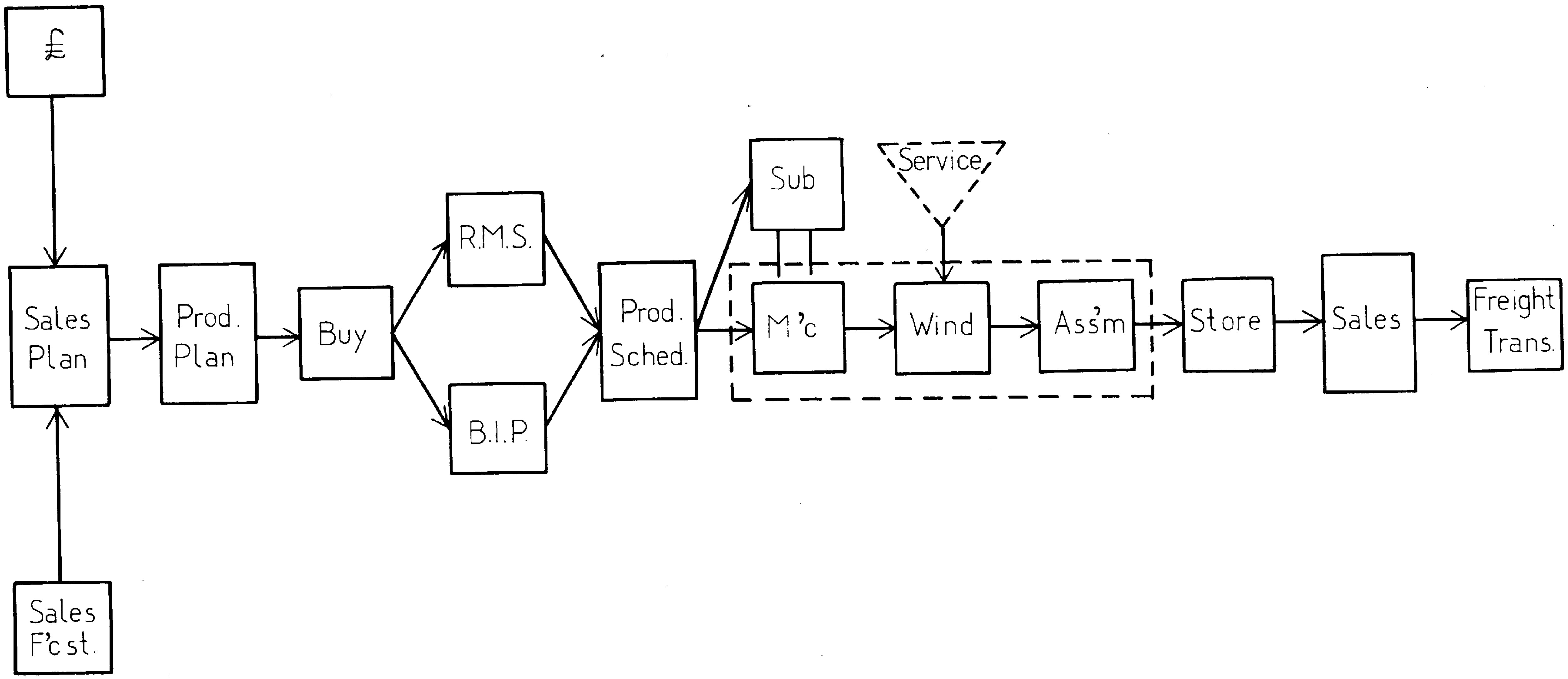
The firm employed a sales force whose major function was marketing products via the dealer/distributor network. Dealers and distributors were contacted regularly in order to ensure that they were fully stocked to meet expected responses from advertising campaigns.

The sales campaigns were organised by the Marketing Division of the company, and used two primary methods of communication: the press, and commercial television.

Much of the advertising in the press was carried out in association with the Mail Order Houses, with whom costs were shared. The television advertising campaigns were directly controlled by the company, and geared towards promoting an early response for seasonal fluctuations, i.e. towards extending the periods of seasonal demand.

The response rates to promotion, i.e. monthly sales figures were derived from an analysis of the returns of the guarantee

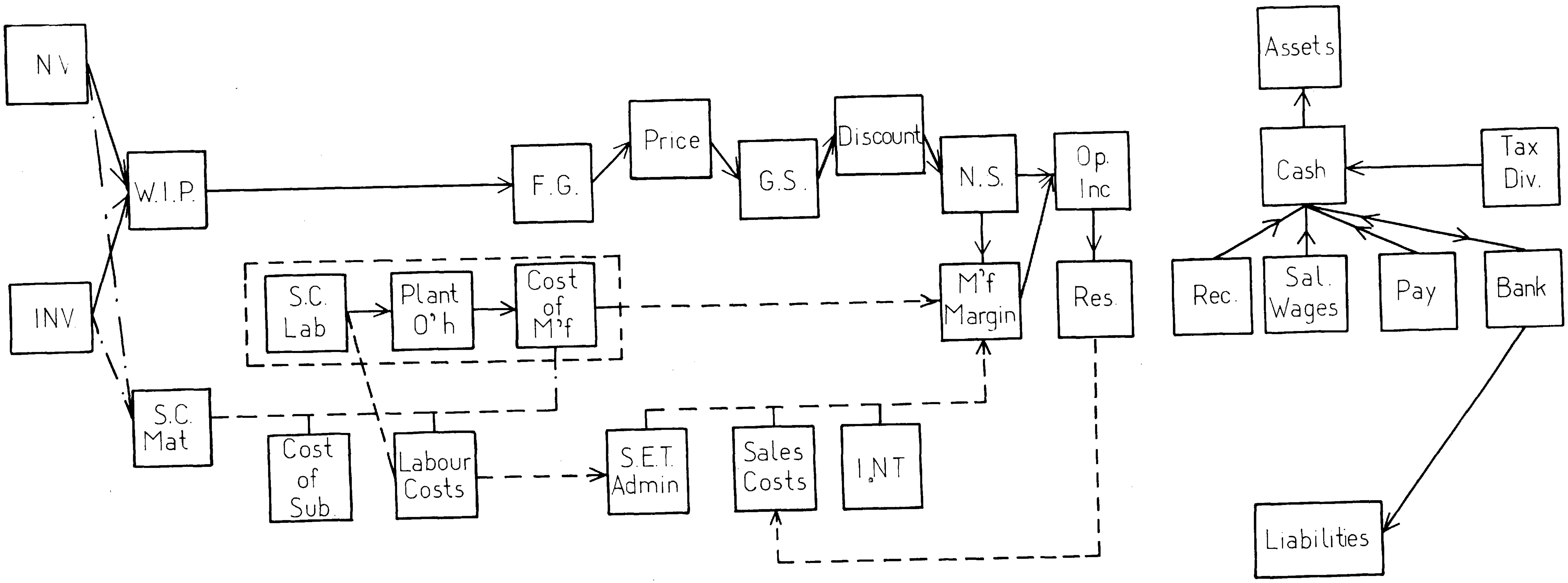
Fig 2.3 The Physical Flows




Key for Fig. 2.3

| | | | |
|---|---|---|---|
| £ | : Financial position before planning | Sub | : Subcontracted work |
| Prod. Plan | : Production plan for the next year (and/or three years) | M'c | : Machine shops of company |
| Buy | : Purchasing of raw materials by the firm | Wind | : Winding shops of company |
| R.M.S. | : Raw materials store | Ass'm | : Assembly lines of company |
| B.I.P. | : Bought in parts | Freight Trans. | : After the activity of 'Sales' the goods are shipped from the warehouses in various ways - combined here under the heading 'freight and transport'. |
| Prod. Sched. | : These stores are depleted according to the production schedules arising out of the annual production plan | | |

Fig. 2.4 The Financial Flows




Key for Fig. 2.4

| | | | |
|---|---|---|---|
| INV. | : Inventories of raw materials and bought parts | S.E.T. Admin | : Debts incurred due to labour force |
| W.I.P. | : Work in progress account | S.C. Lab | : Standard cost of labour hours used in production |
| F.G. | : Finished goods account | Plant O'h | : Plant overheads incurred due to production |
| Price | : Price structure for finished goods | Cost of M'f | : Total cost (at standard) of the manufactured items |
| G.S. | : Gross sales | M'f Margin | : Manufacturing margin |
| N.S. | : Net sales | Res. | : Company reserves |
| Op. Inc. | : Operating income | Rec. | : Receivables |
| S.C. Mat | : Standard cost of materials | Sal. Wages | : Salaries and wages that must be paid in cash |
| Cost of Sub. | : Costs incurred due to subcontracted work | Pay | : Payables |
| | | Bank | : Loans and/or repayments with bank. |

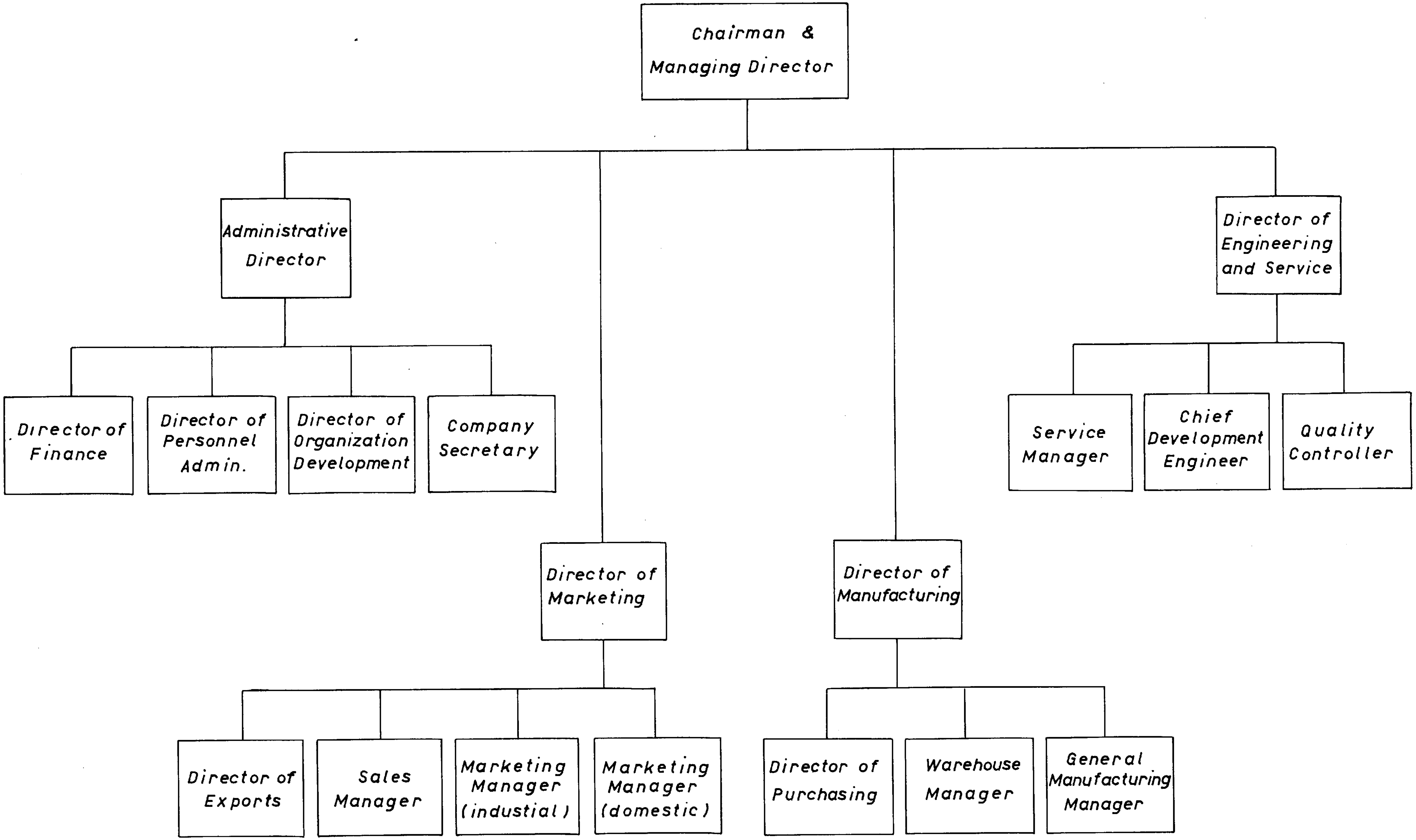



FIG. 2.5 THE MANAGEMENT STRUCTURE

cards supplied with each product.

2.2.7 The Physical Flows through the Firm

A schematic chart of the flow of physical goods through the firm is shown in Fig. 2.3. Inputs to the physical flows for any period of time were determined (initially) by the financial situation at the planning moment, and the projections of sales forecasts. Based on these estimates, materials were purchased; these eventually passed through the manufacturing processes of the firm to be despatched as finished goods.

2.2.8 The Accounting Procedures and the Financial Flows

The company used an "integrated standard costing system" based on the standard costs of some two years standing. Any deviations from these costs were allocated to rate variance accounts according to standard practices.

For the purposes of planning the corporate strategy over the short/medium term, (i.e. 1/3 years), a flow chart for the financial accounts was drawn up - Fig. 2.4. This chart shows the financial flows corresponding to the physical flows of Figure 2.3. The chart is given in two sections. After the derivation of the operating income, the balance could either be transferred to assets (bank or cash) or could be used to generate reserves. The generating of reserves was used to supply extra funds for the marketing of goods - in particular, reserves were used to increase promotional expenditure on advertising.

2.2.9 The Management Heirarchy and the Committee Structure

A study was made of the structure of the management system, the relationships between the management and committee structures,and the information flows. (A chart of the management structure is shown in Fig. 2.5).

A 3-dimensional picture of the firm was composed: the basic physical and financial flows were drawn on to sheets of clear perspex. Other sheets of perspex were used to show the basic management functions, and the inter-relationships between the management and the committee structure. A final sheet was used to identify the information flows between physical and financial centres, and the management and committees concerned.

The sheets were drawn so that any number could be viewed concurrently; a view through the chart of committee structures and physical and financial flows showed the manner in which each committee interacted with these flows, both from the central and informational view-points.

The total 'sandwich' is illustrated below in Fig. 2.6.

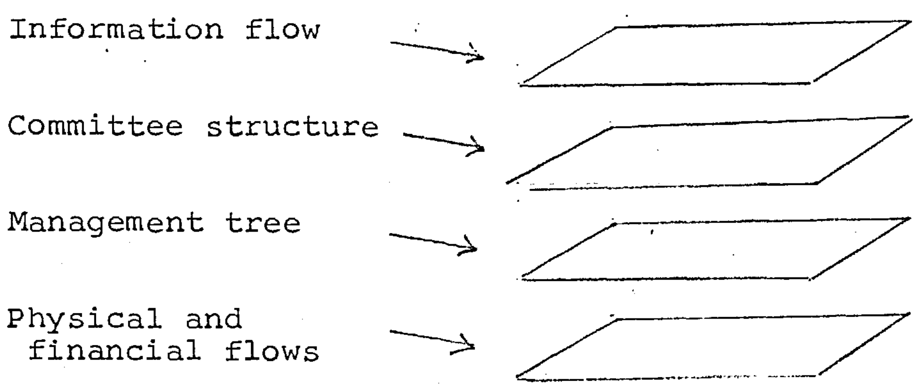



Fig. 2.6. The perspex charts

The committees appearing on the second perspex sheet were:

i. Management Advisory Committee (MAC)
ii. Management Operating Committee (MOC)
iii. Marketing Committee
iv. Manufacturing Committee
v. Finance Committee
vi. Plant Loading Committee
vii. Inventory Committee

These are related, stratigraphically within the firm, in three levels; the Board, the Planning Level, and the Control

Level.

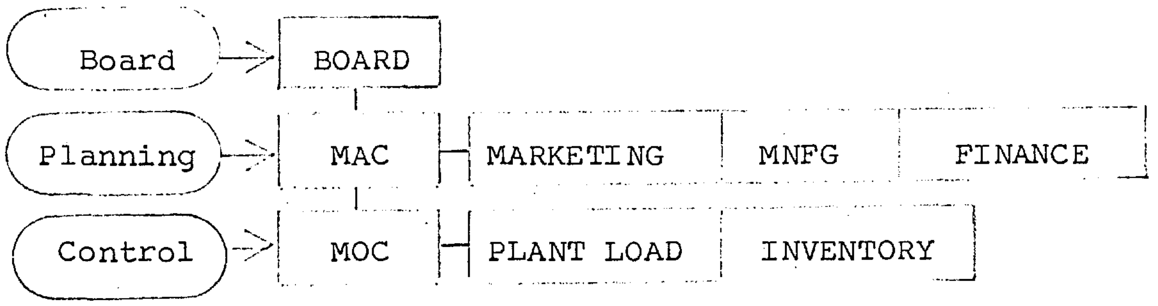



Fig. 2.7 The Committee Structure

The planning committees provided the inputs of policy and objectives for the model; the output was intended for submission at the planning level for approval, then at the control level for application.

## 2.3 The Corporate Aims and Strategic Planning

### 2.3.1 The Planning Processes and Performance Measures

The policy of the corporation was to conduct its forward planning in three stages: the construction of a ten-year plan; the construction of a three-year plan (updated) to correspond with the current ten-year plan; the construction of annual (operating) plans and budgets.

The U.K. company followed similar procedures. Eight measures were listed in the company's report on Financial Planning, by which performance was judged, and concerning which the ten-year plan developed detailed projections. They were: Return on Assets; Return on Fixed and Current Assets; Ratio of Net Sales to Total Assets; Ratio of Income before Taxes to Net Sales; Growth of Total Assets; Growth of Net Sales; Growth of Income before Taxes; Growth of Earnings per Share.

The ten-year plan was an extrapolation of these measures over the coming decade. Once these estimates of performance measures were published, they became the standard performance measures for current evaluation of operations.

The performance measures themselves were of different importance; a trend of the past decade has been towards the reliance on the ratio of Return on Assets. (The concept of Productivity of Assets has been reviewed in Section1.6, and is amplified in Chapter 3).

Considerable emphasis on "Return on Assets" and "Growth" is prevalent in the medium and short term plans. (The medium term plan is a more accurate (and updated) version of the ten-year plan).

2.3.2 The Annual Plan and Operating Budget

As a result of the planning operations i and ii above, the annual budgets were planned in April - June. At the end of the annual planning period, these budgets became operational, i.e. they were the control budgets for the coming year (October to October).

The construction of the annual operating budgets was itself a three stage process.

Stage 1, April: A financial assessment was made of the Income and Surplus, Balance Sheets and Cash Flows for the year ahead.

Stage 2, May-June: Production plans, Market Policies etc. were prepared, in order to achieve the proposals of Stage 1. Forecasts were obtained on all market fronts for use in the planning of operations.

Stage 3, June-July: The forecasts and plans of stage 2 were consolidated into a series of working plans and budgets which became operational.

The general nature of the planning process (for each stage) is summarised in Figure 2.8.

The final stages of acceptance or suggestion of modifications

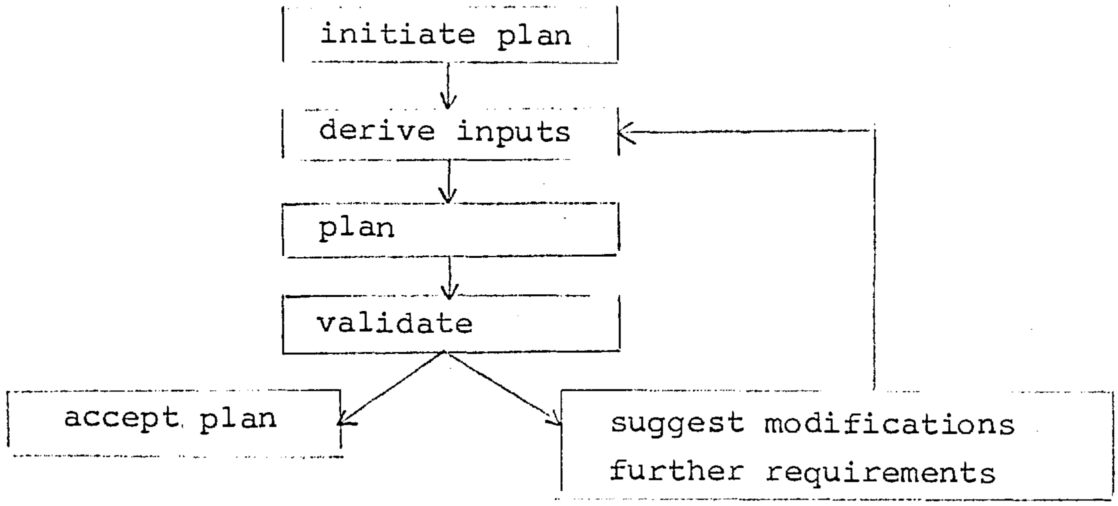



Fig. 2.8 The General Planning Process

or amendments, emphasise the circular nature of the planning process.

The model for short term planning was intended to enter the 'plan' stage for the one year exercises;it was designed to utilise the forecasts of Stage 2 of this process, and produce a more detailed set of production plans and balance sheets for further appraisal by management.

The major advantage in the planning process would be the speed with which Stage 3 could be enacted; this would allow a series of possible budgets to be considered.

## 2.4 The Model

The planning model proposed for the firm was a multistage LP model; the planning horizon (one year) would be split into a series of (equal) periods (months) and the interaction of the variables defining the period activities of the firm would thus model the progress of the firm to the planning horizon.

The advantages of the linear approach to such planning problems have been discussed extensively. See e.g. (16), (83). The addition of accounting systems to such formulations of production planning poses no new problems; the accounting system, and in particular, the standard costing system, is a linear concept of constant returns to scale.

The interactions of the variables of the model take two forms; inter-period and intra-period types. These form two distinct groups within the model, and model different functions within the firm. An outline of these constraints is given in Section 2.4.

For the 'initial model' it was assumed that the factories should not be considered as separate units; the company was assumed to be a homogeneous unit. No transportation costs between factories were included; (these would have been of integer (i.e. non-linear) type).

The model would thus be useful as either a global model of the United Kingdom operations, or as a model of any individual factory, which could be inserted into a decomposition process. These (and further) assumptions are discussed in Section 2.7.

### 2.4.1 Intra-Period Constraints

The intra-period constraints are representations of the accounting procedures. In setting up the accounting network of the firm, we have not used the explicit approach of Ijiri (53). This approach implies the inclusion of many variables as (explicit combinations of existing variables, the calculations being carried out in 'equality type' constraints of the form:

$$x_k = \sum_j x_j a_{jk} \qquad (2.1)$$

This is unnecessary when the row sum $\sum x_j a_{jk}$ will suffice; the inclusion of such equalities is computationally undesirable. Thus most of the accounting activities are modelled by unconstrained row sums of the corresponding multiples of variables. These accounting rows are:

i. Gross Sales: The value of goods sold in each period is the sum of the product of the sales figures and the list prices.

ii. Standard Costs of Sales: These row sums indicate the standard costs incurred during the production of the goods sold in each period.

iii. Overhead Accounts: These accounts are determined by adjustments to the standard costs to account for rate and usage variances (see Appendix 2.2). They are equality constraints of the type outlined above, and are included because of the importance of the overhead accounts (due to the lack of updating of the standard costs).

iv. Discount on Sales: The trade discounts on gross sales are determined from the discount structures of Section 2.2.5.

v. Net Sales: The net sales figures per period are deduced from the gross sales and discount rates.

vi. Manufacturing margin: The estimated manufacturing margin on current sales is calculated from the net sales, standard costs and overhead accounts.

### 2.4.2 Inter-Period Constraints

The inter-period constraints fall into three main sections: accounting sums; capacity constraints; and continuity constraints (balance equations).

a. Accounting sums and equations: The variables for "the amounts stored in each period" were omitted; they are linearly dependent on "the amounts produced and sold". This means that some accounting and storage constraints are of the inter-period type although they are logically of the intra-period type. Due to the method of formulation the following are also inter-period constraints:

i. Work-in-progress: During the periods prior to the completion of a product, it will be accounted as work-in-progress. This row is the sum over 'incomplete' products of their contributions to the work-in-progress account.

ii. Finished Goods: The finished goods row accounts for the change in the level of the finished goods account due to production and sales during the periods of the total planning period. Finished goods are valued at list price.

iii. Payables: The amounts falling due for payment in each period are calculated; payments are staggered according to the lag between receipt and the date for settling accounts.

iv. Receivables: The amounts expected in receipts are similarly summed. Both 'Payables' and 'Receivables' are used in the Cash Continuity Equation of part c. below.

v. Bank Charges: The interest charge for the period is calculated on the difference between bank loans and repayments. Bank charges also appear in the 'Payables' account.

vi. Marketing Expenses: The marketing expenses are calculated on the basis of sales of the present (or succeeding) periods. (See Section 2.7.5.)

b. Capacity Constraints: At most stages of production and storage, physical constraints of capacity are operative; these are:

i. Work centre capacity constraints: For each work centre, the planned usage may not exceed the total capacity available. Allowance may be included for subcontracted work.

ii. Labour force requirements: The labour force requirement (for machine operatives) can be calculated from the proposed work centre usages; the labour requirements may be bounded by the maximum size of the labour force.

iii. Storage capacity: The spot balance of products stored at the end of each period may not exceed the storage facilities;

the increase in stored product may not exhaust the storage space available at the beginning of the planning period. This increase is calculated from the difference between production and sales figures, in which the increase of stored product is implicit.

iv. Materials usage: For each period, it is desired that the raw materials required for production be on hand at the beginning of the period; this requirement ensures a steady flow of materials into the system, corresponding with the back up stocks held on the factory floor.

The materials requirement is calculated from the production plans for the succeeding three periods.

c. Continuity equations: In common with all multistage models, the LP model outlined here requires inter-period constraints to define the manner in which material balances, etc., are carried over between periods. These continuity equations are often implicit (as in the case of storage of completed products). Continuity equations explicit in the model formulation are:

i. Materials balance equation: Raw materials available in a period is equal to the raw materials available in the previous one adjusted for usage and extra purchase.

ii. Cash continuity equation: The cash on hand at the end of a period is calculated from the cash on hand at the end of the previous period, adjusted for payments and receipts. The spot balance adjustment is analogous to making up the monthly cash accounts.

A further set of constraints were added to the inter-period set in order to reflect the time that different products remain in store before sales - in order to include the rate of turnover of stocks into the financial scheme.

iii. Storage requirements: No finished product is available for sale, unless it has been in store for an appropriate length of time. This lag is determined by the storage lag data discussed below in Appendices 2.2.1 and 2.2.4.

2.4.3 The Bounds on Admissible Activity Levels

The levels at which activities may take place are controlled by two sections of the model:

a. the constraints: these define the allowable levels of activities by regulating their inter-actions.
b. management policies: stipulating levels of activities that they consider desirable; these may be minimum sales targets, cash balances, etc.

The management policy decisions are entered into the model by bounding the activity levels of the model variables. Explicit inclusion has been made of the following bounds:

i. Minimum sales: Sales of each product must exceed the given minimum sales pattern.

ii. Cash balances: The cash balance at the end of each period must lie between pre-specified limits.

iii. Bank loan restrictions: Upper and lower limits are placed on the amount that may be borrowed per period.

iv. Raw materials balance: The materials balance at the end of each period must lie within a specified range.

Other bounds may be introduced into the model after the initial tableau has been set up by the matrix generating programme, e.g.

a. Total loan restriction: The total outstanding loan may be restricted by bounding the admissible level of interest payments.
b. Upper sales limits: If planned sales exceed the market forecast, this forecast may be introduced as

an upper limit on sales.

### 2.4.4 The Objectives

The matrix generating programme also formulated the basic set of (unconstrained) rows which could serve as objective functions for the programming model. These were:

1. Change in Current Assets.
2. Change in Current Liabilities.
3. Gross Sales.

From these three rows, and a knowledge of the asset and liability positions of the firm at the beginning of the planning period, the performance measures of Section 2.3.1 can be deduced.

The company emphasised its desire to make operations independent of current taxation policies, hence all measures are calculated "pre-tax". Net sales, although not explicitly included in the objective set, can be deduced from the row sums of period sales.

The formulation of the objective functions was:

1. Change in Current Assets: Change in current assets is accounted for by changes in finished goods, materials, cash and outstanding accounts.
2. Change in Current Liabilities: Additions to current liabilities derive from changes in the outstanding loan, and outstanding debt.
3. Gross Sales: The sum of all monthly gross sales for the total planning period.

## 2.5 The Mathematical Formulation

Corresponding to the logical exposition of the model in Section 2.4 , a mathematical formulation was devised. This is presented in Appendix 2.1.

The arrays used for this formulation are shown in Tables 2.1 to 2.7. The features defined are:

i. the model parameters
ii. the model variables
iii. the production/technology arrays
iv. the accounting data
v. the accounting and storage lags
vi. the technological capacities
vii. the bounds on the acceptable variable levels.

The matrix generating and report programmes are listed in Appendix 2.3.

## 2.6 The Association with the Bounded Variable Algorithm

### 2.6.1 Normative models and planning procedures

Corporate planning may be characterised by the following concepts; given the present organisation of the firm, its 'status-quo' in technological development and resources, and, bearing in mind the objectives of its management, what plans should be envisaged for 'optimal' operations in the coming planning period.

These plans may encompass changes of the organisation itself, advancements in technology, and adjustments to resources, and will include the proposed future use of each of these factors in the manner most suited to management aims. The desirability of any plan will not necessarily be quantifiable in such terms as 'Return on Assets', 'Sales', etc., (as in 2.3). Sociological norms will also be present, as well as factors not directly under the company's control - market share, market value, etc. Any plans which are 'normative' with respect toquantifiable elements such as financial ratios, sales, etc. will also have to be compatible with the aspirations of the firm and its management. In this context, the normative approach to planning is one of validating the mathematically 'optimal' plan with respect to the non-quantifiable demands of the firm, and rejecting (or re-formulating) such plans when they do not satisfy such requirements.

2.6.2 The Elements of Normative Planning

The elements of normative planning outlined above are:

i. The organisation of the firm: This comprises its structure both in the management and technological senses, and the framework of the production processes that it utilises. Under 'organisation', we include the management heirarchy, the committee structure and the information flow, as well as the basic framework of the flow diagrams of the physical and financial resources, Figs. 2.3 and 2.4.

This outline is complemented by the technological factors and resources to give 'the model of the firm'.

ii. The technology and resources: The ways in which the basic framework is utilised depend on the present state of the firm's technological development and the resources it commands.

Its development is characterised by such items as the product range, the use of the technological framework by the product range, the firm's ability to introduce innovations, the productive efficiencies, etc. The resources on hand are those factors which may be disposed of, by management, in pursuit of production and sales.

iii. The objectives of the management: These are divided between 'aims' and 'policies'. The aims of management comprise their desire to optimise behaviour, attain targets, 'perform' well, etc. Any attempts to achieve these aims may be constrained by policies which describe the bounds in which management chooses to operate. These bounds may be on financial holdings (cash, loans, etc.), stock holdings, or a restriction on performance levels.

In this model, the policies form added restrictions to the attainment of objectives.

2.6.3 The Bounded Variable Algorithm:

The programming model, (described in Section 2.4 and Appendix 2.3) was designed for use with standard LP codes. The package used for all solutions was the Mathematical Programming System on an IBM 360/65 computer (the MPS/360 package). This code, in common with most standard LP codes, uses the bounded variable form of the revised simplex algorithm, (16) and (68); it solves the LP:

$$\begin{aligned} &\max\ \underline{c}.\underline{x} \\ &\text{s.t. } \underline{A}.\underline{x} \le \underline{b} \\ &\underline{L} \le \underline{x} \le \underline{U} \end{aligned} \qquad (2.2)$$

where $\underline{L}$ and $\underline{U}$ are lower and upper bounds on the admissible levels of the value of $\underline{x}$, $\underline{c}.\underline{x}$ is the objective function, and $\{\underline{x} | \underline{A}.\underline{x} \le \underline{b}\}$ is the constraint set.

Considering the matrix $\underline{A}$ and the vector $\underline{b}$, we may distinguish two separate features of the constraint set: the structure of $\underline{A}$ and $\underline{b}$, i.e. the positions of non-zero entries; the values of $\underline{A}$ and $\underline{b}$, i.e. the actual matrix entries. 'Three dimensionally' the form of the bounded variable algorithm may be described by Fig. 2.9.

These levels are related to the elements of normative planning in the following way:

i. the underlying framework of constraints and capacities is derived directly from the present organisation of the company, the layout of its production facilities, the current accounting

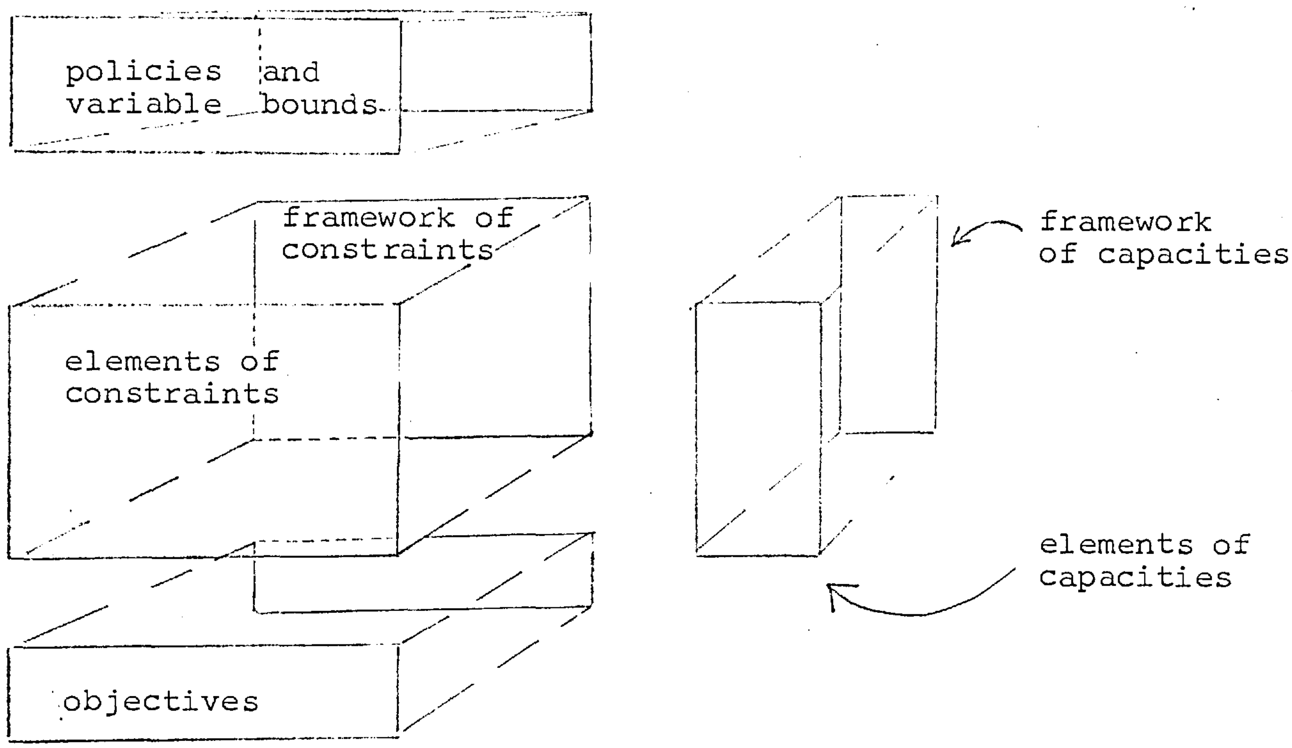



Fig. 2.9 The underlying structure of the bounded variable model

procedures, the information flows, etc.

ii. the present technology and resources determine the entries into the frameworks for A and b (as defined by the present organisation). Efficiencies determine the machine centre usages, input requirements, etc., the resources determine the plant capacities available, the material inputs, etc.

iii. the objectives and bounds are directly related to the quantifiable aims and policies of management. Their aspirations are measured in the set of objectives; the policy levels are included in the bounds on admissible activity levels.

### 2.6.4 The Relationship between the Model and the Committee Structure

The committees directly related with the formulation of the model are those of the planning level: the MAC and the Marketing, Manufacturing and Finance Committees. (see Fig. 2.8). These have an effect on two of the elements of planning, the organisation and the objectives; they directly determine the framework of the constraints (via the organisation) and the objectives and bounds, (via the aims and policies of management). Their effects are:

i. Management Advisory Committee: The MAC provides consultation to the Board on matters concerned with major changes of structure in assets, research, facilities and organisation. In terms of a mathematical model these enter as either the framework for the constraints and capacities, or the proposals for an integer programming (capital budgeting) type of model. Interest here is restricted to linear models, hence the committee has the effect of suggesting the constraint framework.

It also establishes the new aims and objectives of the firm, or modifications thereto, and specifically formulates the profit objectives, i.e. its major role includes the inputs for the bounds and constraints.

These inputs are further modified by:

ii. Marketing Committee: where marketing policy is formulated. For the model this policy is included as sales bounds and suggestions for marketing expenditures.

iii. Manufacturing Committee: where manufacturing policy is proposed for the approval of the MAC.

The technology of the firm is not decided in committee as a short term planning objective; use is made of the technology to determine optimal policies.

## 2.7 The Assumptions

### 2.7.1 Introduction

The model presented in Section 2.4 and Appendix 2.1 is a deterministic, multi-stage LP model, to be used as a planning tool for upper management. Implicit in the formulation are assumptions concerning both the nature of the interactions allowed in the model, and the possible control parameters that can be utilised to govern such interactions. We will discuss the general nature of these assumptions under three headings - linearity, determinism, and time structure and stock control - and append details of any further assumptions made in individual constraints.

2.7.2 <u>Linearity</u>

In assuming that the planning processes can be modelled using an LP format, all possibilities of capital investment on plant and facilities, restrictions of minimum batch quantities, and allowances for machine set-up times have been excluded.

The model was intended for short to medium term planning, and it would be expected that any capital commitments arising out of a scarcity of capacity during the planning period (and demonstrated by the model) would require a lead time longer than the planning period itself. (Capital expenditures undertaken before the beginning of the planning period would assume a deterministic form and any associated changes in capacity could be built into the model with the data outlined above).

The lack of restriction on minimum batch quantities is somewhat more important; a failure to include such quantities could lead to impractical planning. The inclusion of minimum batches for production implies either the association of integer variables with production activities, or nonlinear equations of the form

$$\text{PROD (I,J)} \times (\text{PROD (I,J)} - \text{MIN (I,J)}) \geq 0 \qquad (2.3)$$

where MIN (I,J) = minimum batch quantity for product I in period J. (For definitions of PROD (I,J), etc., see Appendix 2.1).

The non-negativity requirement on PROD (I,J) and the equation (2.3) would ensure that if PROD (I,J) were non-zero, it would be greater than MIN (I,J). Such an inequality can only be handled in the LP environment using Seperable Programming (see e.g. (68), (51)); the inclusion of such constraints for each product and every period would greatly expand the problem size. It was assumed that deficiencies introduced by omitting batch quantities could be adjusted (post-optimally) by manipulation.

Economic batch quantities can only be accommodated using Integer Programming techniques.

Set-up times have also been omitted, these again are non-linear. A true estimate of utilised capacity is obtained by the inclusion of set-up times for each non-zero PROD (I,J) - using Integer Programming. As with batch quantities, set-up times have not been included, due to the size restrictions on computable integer programmes, and the difficulties of dual interpretations, (see Section 3.3.2).

A post-optimal scan can be made, to assess the effect of both these omissions - the process of post-optimal adjustment has been outlined in Section 2.3.2. "Suggesting modifications" in Fig. 2.8 could include making allowances for the proposed set-up times by appropriate reductions of capacities and can take care of minimum batch quantities by the addition of further bounds on the non-zero production variables. (Such considerations have been made in Appendix 3.2).

The assumptions of linearity and constant returns to scale do not contradict the normal accounting procedures of standard costing, indeed the action of the programme in making up accounts at the end of each month, exactly models accounting practice, both with respect to linearity and time segmentation.

A further assumption associated with linearity is the fact that products (variables) are independent - physically and algebraically. Hence, two products will appear totally dissimilar if they differ by only one attribute - say standard wiring. It is thus necessary to remove all trivial differences between product items before defining the product set (PROD (I,J) J = 1, ... NPROD). This implies a loss of detail in the planning, but will reduce the number of variable that must be included in the model.

2.7.3 Determinism

In Chapter 1 we have mentioned the Stochastic Approach to Mathematical Programming proposed by Charnes and Cooper (18). These methods have the advantages of introducing a well defined concept of risk into the planning process, and of being much more akin to management psychology in their treatment of constraints, but this attractiveness is marred by the lack of computational success, and the complexity of any programmes developed. It is still far from practical to attempt to model a large scale operation using Chance Constrained Programming. The only approach to risk, at present, appears to be the use of deterministic models, with a post-hoc risk analysis made by management on the basis of successive optimisations.

As has been shown by Wagner, the optimum of an LP approximation using mean value estimates may vary markedly from the mean value of the stochastic programme. (See e.g. 84). Such will be the case where many of the matrix elements are themselves stochastic.

In this model (and in the short term planning context), this stochastic nature need not present too much of an obstacle to a deterministic approach followed by adequate sensitivity analysis, since the variation is not large.

The stochastic nature of forecasts is a necessary feature of the input data, and must be dealt with using comprehensive post-optimal analysis.

2.7.4 Time segmentation and stock control

The normal practice in multistage models is to define variables as pertaining either to the ends of respective periods, (such as "cash at the end of period I"), or to some indefinite time in period I, for accounting at the end of the period (e.g. "materials purchased during period I"). (Equations such as (2.22) and (2.23) in Appendix 2.1 define the continuity of these variables at intertemporal links.)

We have already noted that there are two approaches to the method of specifying continuity and growth of stock holdings. Cohen and Hammer propose a justification for the average balance approach on the basis that, a the model is a medium/long term planning tool, and b time periods can be shortened (arbitrarily) for greater realism. In the model described above, we have assumed that holdings of cash, raw materials, finished goods and work in progress are all to be considered as spot balances referring to holdings at the end of respective periods. With the variables for cash holding, this assumption is valid on the grounds of regular accounting practice. (The treatment of finished goods and work-in-progress is similarly justified).

The reasons for regarding materials and stored product as spot balances are two-fold:

a. the balance equations are simplified,

b. with both materials and stock, the end of the period values are used in equations modelling the flow of the respective item through the system in addition to providing the continuity equations. Average balances have little meaning in this case. Difficulties arise in measuring the warehouse utilisation, (2.20). The spot balance here is measured against the absolute capacity.

It might be more meaningful to use some average measure to compare with capacity, but both average and (adjusted) spot balances are unrealistic. The quantity to be compared with storage capacity is the maximum amount held during the period. To obtain this figure we would have to make further assumptions on the rates of change of stocks due to production and sales, and might enter into non-linear systems when trying to cope with both increasing and decreasing stock levels. In this case the normal appeal to the correctness of the length of the time period is made.

The final problem associated with the time structure of the models is its interpolation into the real time world; i.e. the adjustments to the initial and final stages of the model to ensure a smooth (and feasible) transition between true operating time and the model's planning period and vice versa.

At the planning horizon, the accounts for work-in-progress need adjustment allowing for work to be completed beyond the scope of the model; use of machine facilities during the terminal periods will be an underestimate of the actual use that will be made; materials on hand at the planning horizon must allow for a reasonable continuity of production.

These end conditions must be satisfied to prevent the model "running the firm into the ground". It is assumed that such definitions can be provided - ab initio, or deduced by the processes of sensitivity analysis and re-optimisation outlined above.

2.7.5 Other assumptions

i. Work Centre Usage: To find the usage of each work centre per period (and to compare this with capacity) the total sum of hours planned for each facility is made, (2.18). Because of the linearity of this sum, no account can be taken of the order in which jobs would be scheduled. The LP model assumes that all feasible plans, (with respect to the constraint set of Appendix 2.1) are also feasible for scheduling through the work shops. This will be the case if work-centre aggregation is meaningful. In our case, either like machines are combined into work centres, or the work centre represents part of a flow line. In both these instances the aggregation will not lead to scheduling conflict.

ii. Marketing Expenses: In addition to the constraints of Section 2.4, additional rows calculating the marketing outlays were included. For the 26/12 model it was assumed that a flat rate of £1 was paid per item marketed; in this case the marketing expense is equal to the monthly sales (in units). No forward lag was allowed - the expense was made to fall due during the period of sale. (This may be altered using the MARK and MRKLAG arrays).

The association between advertising expenditure and sales is necessarily deterministic in this model. Other factors may also include such as effects of substitution and correlation between sales;

a. Substitution: Suppose sales of product i can be satisfied with a sale $\beta_{ij}$ of product j. Then, considering say the maximum possible market for sales of type i we could formulate the rows

$$\text{"sales of i"} \leq \text{maximum for i}$$

$$\text{of } \sum_j (\beta_{ij} \times X_j) \leq \text{maximum for i} \qquad (2.4)$$

$$\text{where } \beta_{ij} = 1$$

b. Association: Conversely if the sale of product i implies a possible sale of $\alpha_{ij}$ of product j, sales of product i may be thought to increase the potential for sales of product j.

i.e. "sales of i" $\leq$ maximum + $\sum_j \alpha_{ij} \times X_j$ (2.5)

where $\alpha_{ij} = 0$

Both of these marketing models could be included in the model, but data was available for neither the expenditure calculations nor the substitution/association effects. It was thus assumed that a constant rate of expenditure would be built into the model; the rate at which unit sales imply unit marketing costs being given by the diagonal elements of the MARK array.

It was further assumed that transportation and storage costs were reflected in the standard costing system.

iii. Homogeneity of facilities: As mentioned in Section 2.4, no account has been taken of the separation between factories. The model is of a single production unit, and can model either the total U.K. operations, ignoring the separation into three factories, or the operations of one factory. Three such models could be combined into a LP decomposition algorithm for global optimisation. The case of fractional objectives in decomposition is considered in Chapter 5.

2.8 Summary

An LP model for short to medium term planning has been proposed. This model is formalised in Appendix 2.1. The discussion of the assumptions underlying this model has concentrated on three aspects: linearity determinism and time-segmentation. Further discussion is presented in Appendices 2.2 and 3.2 and the dual evaluation is considered in Chapter 3.

## Chapter 3 Performance Measures and Multiperiod Models

### 3.1 Introduction

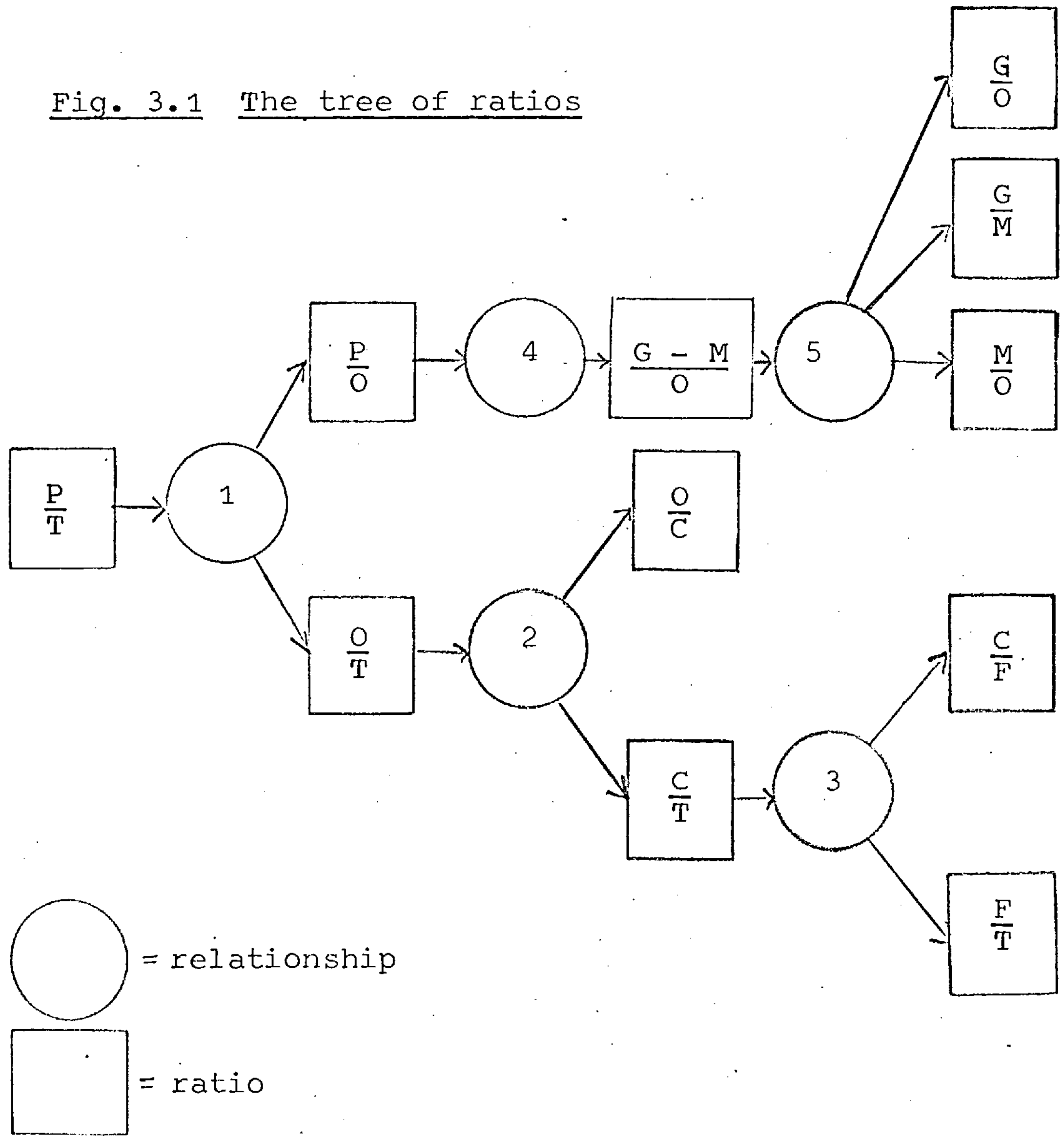
In the previous chapter the LP model for short term planning was introduced. Examples of optimal solutions, sensitivity analyses, and other post-optimal tests are presented in Appendix 3.1.

This chapter concentrates on the assumptions under the interpretation of the dual variables in multistage LP's as prices, and develops the use of LP dual evaluators and reduced costs for resource valuation, ratio analysis and product ranking.

### 3.2 Productivity and Financial Ratios

Interpretations of productivity and financial ratios range from "evaluations of past performance" to "criteria for management control" and "statistics for inter-firm comparisons". In many of these cases different inferences may be drawn from the same ratios, regardless of their primary function, and regardless of the objectives of the firm. It has been suggested by Gold and Kraus (41) that for the purposes of control some of these ratios may be shown to be part of a tree which disaggregates the basic ratio of "profit to total investment" into its constituent parts. (Such a tree is shown in Figure 3.1). In (41) they quote the different emphases placed by various firms on sections of the tree, e.g. Dupont consider (profit:sales), (sales:total investment) and (profit: total investment) as key ratios. Monsanto on the other hand use (profit:investment), (net income:investment), (sales: property), (selling expenses:sales), (operating expenses: sales) and (cost of goods:sales).

Fig. 3.1 The tree of ratios




**Key**

P = profit

T = total investment

O = output

C = capacity

G = gross sales

F = fixed investment

M = costs

**Relationships**

1. $\frac{P}{T} = \frac{P}{O} \cdot \frac{O}{T}$

2. $\frac{O}{T} = \frac{O}{C} \cdot \frac{C}{T}$

3. $\frac{C}{T} = \frac{C}{F} \cdot \frac{F}{T}$

4. $\frac{P}{O} = \frac{G - M}{O}$

5. $\frac{G}{O} = \frac{G}{M} \cdot \frac{M}{O}$

Considering the use of ratios for either inter-firm or intra-firm comparisons, and in both planning and reporting situations, many of these ratios are ill-defined. 'Capacity' is measurable as a unitary statistic , only if the firm produces just one product, with a fixed statement of resources.

In normal industrial conditions, the capacity of a production unit, manufacturing a number of interdependent products with fixed resources, cannot be defined as a single statistic without the inclusion of some management objective regarding the most desirable product mix. There may be many 'efficient'† combinations of production, and the mapping of these combinations into one financial estimate of 'capacity' is meaningless if corporate objectives are ignored. The applications of LP to corporate planning amplify this aspect of business ratios (i.e. their dependence on management objectives). As we have noted in Appendix 2.3, the ratios derived from an LP model vary with the objective function used for optimisation , thus ratios may be expected to differ within an industry because of differences of objectives, as well as differences of productive efficiency. (Amey uses LP to clarify the concepts of economic efficiency and business efficiency - see (3)).

This point becomes more apparent when considering such terms as 'output to capacity'. For planning, 'capacity' may be defined in two ways:

i. the 'capacity' to produce say goods and services, is that value of goods and services that is theoretically attainable whilst the firm pursues some definite objective (with fixed resources).

ii. 'capacity' is the maximum value of goods and services that the firm can produce, regardless of its objectives.

† in the sense of Koopmans (56)

Mathematically, if $\underline{x}$ is the vector of production, storage variables .........., the technological constraint set is $\{\underline{x} \mid \underline{A}.\underline{x} \leq b\}$, and the function $O(\underline{x})$ measures the value of output of goods and services, we may define $C(\underline{x})$, the capacity, as:

$$C(\underline{x}) = \{O(\underline{x}) \mid \max f(\underline{x}),\ \underline{A}.\underline{x} \leq \underline{b}\}$$

according to (i) (3.1)

or

$$C(\underline{x}) = \{\max O(\underline{x}) \mid \underline{A}.\underline{x} \leq \underline{b}\}$$

according to (ii). (3.2)

Alternatively, the two definitions may be thought of as:

$$C(\underline{x}) = \{O(\underline{x}) \mid \underline{x} \text{ maximises } M_1.f(\underline{x}) + M_2.O(\underline{x}),\ \text{s.t } \underline{A}.\underline{x} \leq \underline{b}\}$$

where

<u>i</u>. $M_1$, $M_2$ have a non-archimedian order property $M_1 >> M_2$ (see (16 pp. 756-767)) (3.3)

and <u>ii</u>. where $M_1 = 0$, $M_2 = 1$ (3.4)

In the first case, the planned output and capacity may be identical; the objective function is reflected in the capacity level itself. The ratio 'capacity to fixed investment' is the rate of turnover of the fixed investment - planned with respect to the company's objective. In the second case, the ratio 'output to capacity' is not unity; it represents the extent to which management have sacrificed the attainment of maximum output in order to optimise their objective.

Both approaches emphasize the central role played by the objective function in determining the physical and financial measures of performance.

<u>Example</u>: Consider a firm manufacturing three items on two machines, A and B, each machine having a capacity of 10 hours. Every unit of product one requires 2 hours on machine A and

1 hour on machine B. Units of product two and three each require one hour on machine B. Let the net profit per unit be £4, £2, £2 respectively and the 'output value' per unit be £6, £8, £9 respectively. The LP for maximising net profit is:

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 + 2x_3 \\ \text{s.t.} \quad & 2x_1 \leq 10 \quad \text{(machine A)} \\ & x_1 + x_2 + x_3 \leq 10 \quad \text{(machine B)} \\ & x_1, \; x_2, \; x_3 \geq 0 \end{aligned} \tag{3.3}$$

The optimal solution is

$$x_1 = 5$$

$$x_2 + x_3 = 5$$

i.e. the value of output may range between £70 and £75 without affecting the objective. In this case, capacity (using the first definition) is £75, (with respect to maximising net profit). The maximum value of output is £90, (when $x_3 = 10$, $x_1 = x_2 = 0$). Thus according to the second definition the absolute capacity (regardless of objectives) is £90.

(Similar treatment can be given to other ratios of Figure 3.1 (e.g. 'sales to output') using examples of greater detail.)

## 3.3 Reduced Costs and Dual Evaluators

### 3.3.1 Introduction

Thus far, we have shown how the LP approach may be used to generate optimal plans for company operations, how such plans are interpreted as production schedules, financial accounts and operating ratios, and how these measures vary with the objectives of the firm. But, the major advantage of this approach is not in the attainment of the optimal solution; duality and the pricing of resources are the primary attractions of the method, since each evaluator (or reduced cost) assigns a monetary value to a proposed (or possible) change. Moreover, this evaluation is made in terms of the objective of the firm i.e. with respect to the attainment (or increase) of the planning objective. (For the objectives mentioned in Section 2.4.4 these may be such values as "pounds change in gross sales per extra unit of resource", or "decrease in profit per unit increase of production level").

For the short term corporate planner, the LP approach has two advantages:

a. it gives a guide to 'optimal' policy

b. it evaluates resources with respect to that optimum. (These values may then be utilised in revising capital investment decisions, company policies, etc.)

Such benefits rely heavily on the assumptions of total linearity in the system, the presence of a unitary objective (which in itself is linear) and the accuracy of the correspondence between the 'model' and 'reality'. The cogency with which these assumptions may be justified is the sole guide to the acceptability of the approach.

### 3.3.2 The Linearity of the System

The model described in Section 2.4 was constructed on the assumptions of linear relationships within the firm (of production, storage, etc.) and possible further, linear, relationships between the firm and its external environment, (e.g. marketing, the inclusion of transportation in standard costing, etc.)

As a consequence of this assumption, the dual evaluators of the model are interpreted as the values of the resources of 'the model' and hence as the resources of 'the firm'. In many cases, the linearity is questionable; costs are certainly not linear, and, like overheads, cannot be considered deterministic in the real environment; many of the variables of the model imply solutions which should take account of the non-divisibility of activities by the inclusion of integer specifications. The theoretical and practical implications for duality in these circumstances are important, as are the specific interpretations that are placed on the values themselves.

The assumptions of linearity with respect to costs and overheads have been considered in Section 2.7; they are justifiable if the model is a planning tool, and do not impair the useful interpretation of the dual variables. The assumption concerning the integer values of certain variables is more serious.

In Chapter 1, we have introduced the ideas of Baumol and Gomory, (43), Weingartner (88), etc., concerning pricing (and duality) in Integer Programming. In both works, the implication is clear: the association of dual-variables with prices of resources is tenuous.

In our model we have included neither set-up times, nor batch quantities for production runs. Does this ommission invalidate the pricing of machine capacity (and all other prices)? The exclusion of set-up times was tested with respect to the 3/5 model. The capacities for the various work centres were decreased to allow for the set-up times implied by the basic optimal set and the LP was re-optimised; allowances were made for one set-up per month and one set-up per batch of a hundred items. With both changes, the optimal (basic) set was unaltered, thus the dual evaluators did not change.

In fact, neither of the revised capacity sets fell outside the range for the row dealing with the respective capacity.

(Details are given in Appendix 3.3).

The real problem, including set-up times on machinery, should contain the following rows:

$$\sum_j a_{ij}x_j + \sum_j s_{ij}y_j \le c_i \qquad i = 1\ldots m \qquad (3.4)$$

and

$$x_j - k_j y_j \le 0 \qquad y_j \text{ integers} \qquad (3.5)$$

where $a_{ij}$ is the usage of facility i by activity j, $s_{ij}$ is the set-up time required by $k_j$ units of activity j on facility i, there are m facilities.

We have solved the amended problems with

$$\sum_j a_{ij}x_j \le c_i \qquad i = 1\ldots m \qquad (3.6)$$

and

$$\sum_j a_{ij}x_j \le c_i - \sum_j s_{ij}\bar{y}_j \qquad (3.7)$$

where $\bar{y}_j$ is an estimate of the number of set-ups required for the j'th activity.

If we can assume that the capacity figures $\{c_i\}$ comprise infinitely divisible resources, then the dual evaluators of the LP's containing rows (3.6) and (3.7) will give the marginal values of these resources. The 'range' for which these values hold is given by the minimum of:

i. the variation of $\{c_i\}$ that preserve the present basis (i.e. the LP range of $\{c_i\}$)

ii. the change of $\{c_i\}$ that preserve the estimates $\{\bar{y}_i\}$

This range can only be derived using parametric analysis of the right hand side of (3.7); see Appendix 3.3. With respect to increasing capacity (r.h.s.) values, the objective function will increase (where marginality holds) until the allowances for set-ups become insufficient. When more capacity has been generated (to allow for additional set-up) the objective function will show an initial fall, (productive capacity has been removed for set-up time), and then rise according the dual evaluators during the new extent of set-up allocation. The typical graph (Figure 3.2) is deduced in Appendix 3.3.

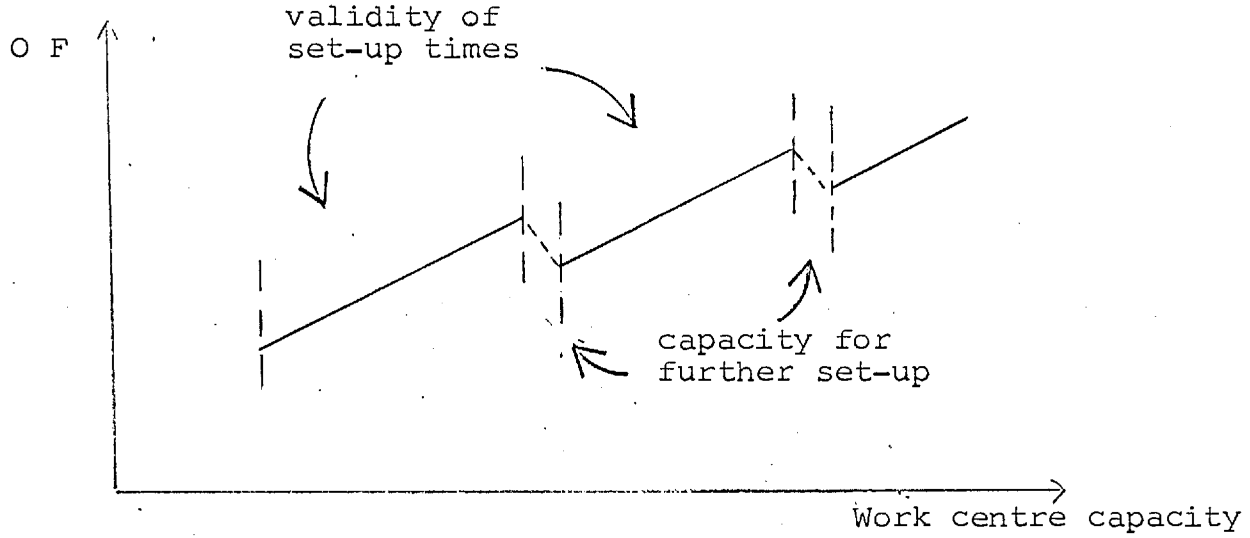



Fig. 3.2

The inclusion of the integer estimates does imply that the dual evaluators will not price out resources (i.e. the right hand side of the initial LP).

Based on the assumption that production and sales activities can take any real values, the concept of marginality is retained, by the adaptation of the mixed-integer problem to a series of linear problems. (The integer problems of Baumol and Gomory, Weingartner, etc. (43), (88) are not so amenable, because of the disparity between the integer and linear forms).

3.3.3 The Reality of 'One Objective'

In many of the models presented in Chapter 1, (Charnes Cooper and Miller (24), Chambers and Charnes (14), Ijiri (53), Samuels (13), Carsberg (11), etc.) the aspirations of management have been summarised in one, linear, objective function. If the optimisation of this expression is accepted as the sole aim of management, the duality of the LP model provides a unitary set of dual prices,[†] which may then be used for the marginal evaluation of resources. As already mentioned, (Section 1.5), this normative approach to Corporate Modelling has been queried by Charnes and Stedry in the light of studies by behavioural scientists. The absence of a unitary long-term objective function (31, also 26 page 147) makes the normative approach seem unrealistic, and throws doubt on the extent to which dual evaluators, derived from one objective function, may be used to price resources for management decisions.

For the two methods suggested in (26) there are difficulties in the use of duality for pricing. Dual variables can be defined for the certainty equivalent of a Chance Constrained Programme, (see e.g. (67)), but for large problems CCP is impractical. The objective in Goal Programming is the minimization of a distance function. Dual variables for the formulation in equations (1.4) represent the marginal change in distance from goals per unit change of resource. (One attractive feature in Goal Programming is the possible use of the optimal canonical form to give the marginal rate of change of the achievement of each goal with respect to changes of each resource.

e.g. Consider a two goal problem; "$2X + 3Y$ to approach 44," "$X + Y$ to approach 20", with constraints $2X + Y \leq 12$, $X, Y \geq 0$.

The formulation is:

$$\begin{aligned} \min \quad & s_1 + s_2 \\ \text{s.t.} \quad & 2X + 3Y + s_1 = 44 \\ & X + Y + s_2 = 20 \\ & 2X + Y + Z = 12 \\ & X, Y, Z, s_1, s_2 \geq 0 \end{aligned}$$

the optimal solution is $Y^* = 12$, $s_1^* = 8$, $s_2^* = 8$.

† in degenerate cases, the prices in this set may not be unique - see e.g. Carsberg (11)

For this optimum, the first line of the canonical form reads

$$-4X^{*} + s_1^{*} - 3Z^{*} = (1, 0, -3).\begin{pmatrix}44\\20\\12\end{pmatrix}$$

The rate of change of $s_1^*$ with respect to changes in the third resource (now at 12 units) is -3. This is the rate at which the distance between performance and goal is being decreased.

In Corporate Planning, a series of objectives must be considered. (Eight such objectives have been listed in Section 2.3.1 for linear or fractional programming). For each objective function, there will be an optimal set of dual evaluators, representing the marginal values of resources with respect to that objective. In these situations, management must review the arrays of prices, and arrive at a subjective evaluation and ranking of all resources. (Alternatively, the (linear) objectives may be weighted to form some utility function. The dual evaluators will then rank resources with respect to utility - but will not generate monetary prices).

The contrast between such dual prices (and their associated optimal policies) may be extremely illuminating for management. The sets of dual prices provide financial evaluations which take account of the firm's strategies, activities and policies; they highlight the fact that resources can only be valued with respect to corporate objectives, operating constraints and the external environment, and that these values may well differ from economic or accounting values.

The existence of multiple sets of prices does vitiate the use of dual evaluators as penalties and bonuses in control models, (such as those of Samuels (73)), and casts doubt on the use of dual prices for assets valuation and accounting, (as suggested by Carsberg (11)). ( See Appendix 3.2.)

### 3.3.4 The Effect of Time Segmentation

The multiperiod models used by Cohen and Hammer (29), Chambers and Charnes (14), etc., were intended as operating models, which would suggest strategies for the first period of the planning horizon. (The models would be run at the beginning of every period to give the strategy for the immediate period). In these cases, the time segmentation (i.e. the length of period considered) is not so serious. The only criterion that need be considered is whether the model has included all future periods that might affect present strategy, via their interactions with the present decision variables.

The model of Section 2.4 has a different purpose. Its function is to view the whole of the company's operations up to the planning horizon; the evaluations thus obtained are intended to give a picture of the values of resources over the whole planning period. Here, two problems must be resolved:

a. the selection of the correct length for the time period

b. the selection of the appropriate number of time periods beyond the horizon.

a. The Time Period: As shown in Appendix 2.4, difficulties were encountered with models of over 1400 rows. Since the row dimension per period is fixed, consideration of time periods of less than one month (even for the short term) would make the model unwieldy.

One assumption that is implicit in 'time-segmented' multiperiod models is that activities that are scheduled for a particular time period must be independent of time within that period.

e.g. if the optimum schedule for period one is: $x_1$ units of product 1, $x_2$ units of product 2; then the manufacturing processes of that period may be organized in any order, provided that the totals $x_1$ and $x_2$ are attained. There is no allowance made for further assumptions e.g. product 1 must be made before product 2.

For planning, this assumption is justifiable. Taking 'a month' as the basic unit of time, we can assume that the production targets for each period can be satisfactorily scheduled on the shop floor. (Production will appear to be instantaneous in the model). The assumption of 'scheduling within the time period' is vital to the interpretation of the model and the dual evaluators. In Appendix 3.2 we show that the failure to define the correct time unit leads to failure in the interpretation of the dual variables. The use of a month as the basic time unit is not tenable for control models, due to the rapidity of change within the period. From Appendix 3.2 it would appear that the size of model required for control operations is large - and consequently expensive.

b. The 'End Effect': In Section 2.7.4 we have discussed the amendments that must be made to the constraints (and resources) of the later periods of the model, to make it correspond with reality. The problem of identifying the effect of the termination of the model at an arbitrary point in time has not been investigated; it can only be solved in conjunction with the implementation of the model by the test firm.

## 3.4 Pricing and Rationalisation with Multistage Models

### 3.4.1 Capacity Evaluations

In Section 2.6.3, we have shown how the bounded variable algorithm may be used to illucidate the structure of the planning process. If $\underline{x}$ is the vector of planning variables for one time period only, the elements $b_i$ in equation (2.2) represent the plant capacities etc. for that time period. If, however, $\underline{x}$ spans two periods, the vector $\underline{b}$ may be split into $\underline{b}_1$ and $\underline{b}_2$ referring to the capacities of the first and second time period respectively.

At the optimum, each capacity in $\underline{b}_1$ and $\underline{b}_2$ will have an associated dual evaluator. Table 3.1 is the set of non-zero evaluators for the 18 work centres of the 26/12 model, using ASSETS as the objective function. These show a marked variation over time, implying that there is no unitary value that can be ascribed to increasing plant capacities etc. Such (marginal) values are time-dependent as well as objective-dependent.

If new plant is installed for a particular work centre at the beginning of a period, its total capacity is increased for that period, and for all subsequent periods throughout the life of the new plant. From the figures of Table 3.1, the value (over the year) of installing an extra unit of capacity for work centre 3, to be operating during January to December is £17.86 , i.e. the sum of the marginal values of extra capacity for work centre 3 over each month.

[The marginal values may be added, provided that the system is linear, and that the proposed change does not invalidate the present basis. (If the change in r.h.s. is $\underline{\delta b}$, and $\hat{\underline{b}} = \underline{b} + \underline{\delta b}$ is within the range of acceptable '$\underline{b}$' for the present basis to be optimal, the change in value of the objective function is "$\underline{c}.\underline{B}^{-1}.\underline{\delta b}$"). Further consideration of the summation of marginal values is given in Appendix 3.3.]

The extent to which capacity may be increased without invalidating the present (optimal) basis must be found using parametric analysis.

| Centre | 3 | 11 | 13 | 14 | 15 | 18 |
|---|---|---|---|---|---|---|
| Jan. | 9.98 | 1.02 | 1.46 | 0.20 | 2.43 | 1.48 |
| Feb. | 0.0 | 1.02 | 1.46 | 0.20 | 2.43 | 4.67 |
| Mar. | 1.32 | 0.43 | 1.65 | 0.0 | 1.68 | 4.49 |
| Apr. | 0.91 | 1.17 | 1.44 | 0.05 | 1.26 | 4.49 |
| May | 0.79 | 1.16 | 1.33 | 0.0 | 1.27 | 4.72 |
| Jun. | 1.17 | 1.14 | 1.36 | 0.02 | 1.28 | 4.44 |
| Jul. | 0.48 | 1.22 | 1.38 | 0.0 | 1.22 | 4.86 |
| Aug. | 1.4 | 1.13 | 1.34 | 0.01 | 1.29 | 4.40 |
| Sep. | 0.22 | 1.25 | 1.41 | 0.0 | 1.20 | 4.97 |
| Oct. | 1.59 | 1.13 | 1.32 | 0.01 | 1.30 | 5.42 |
| Nov. | 0.0 | 1.29 | 1.44 | 0.0 | 1.18 | 0.0 |
| Dec. | 0.0 | 1.5 | 1.36 | 0.0 | 1.03 | 0.0 |

Table 3.1 Monthly Changes of Dual Evaluators

By summation, the monthly dual evaluators provide an estimate of the value for the first year of a unit increase in capacity; individually they demonstrate the distribution of this value over time.

The non-zero marginal values for increased capacity for the 26/12 model are shown in Table 3.2

| Work Centre | 3 | 11 | 13 | 14 | 15 | 18 |
|---|---|---|---|---|---|---|
| Value | 17.86 | 13.46 | 16.95 | .49 | 17.57 | 43.94 |

Table 3.2 Marginal Values of Extra Plant

In Table 3.2, work centre 18 seems to give the greatest rewards for investment. From the theoretical viewpoint, 'limited' funds should be allocated to increasing the capacity of work centre 18, provided that the units purchased do not invalidate the present optimal basis, by increasing the right hand side entries beyond the range of feasibility. (In general, such investments must be judged using a new right hand side.)

3.4.2 Average Reduced Costs

The vectors $\underline{x}$ and $\underline{b}$ of (2.2) may be split into sub-vectors $\underline{x}_1$, $\underline{x}_2$, ...., $\underline{b}_1$, $\underline{b}_2$, ..... where each pair $(\underline{x}_i, \underline{b}_i)$ represent the planning variables and (right hand side) capacities for the i'th period. Each vector $\underline{x}_i$ will have its own associated set of optimal reduced costs, (representing the net losses that would be sustained by deviating from the optimum activity levels).

For the production variables associated with each period, we will obtain a series of reduced costs. (Table 3.3 is the set of reduced costs for the 26 products of the 26/12 model. As with Table 3.2 the objective function was ASSETS.)

| ODUCT | MONTH 1 | MONTH 2 | MONTH 3 | MONTH 4 | MONTH 5 | MONTH 6 | MONTH 7 | MONTH 8 | MONTH 9 | MONTH10 | MONTH11 | MONTH12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 564.85 | 564.85 | 459.73 | 663.82 | 633.98 | 628.58 | 646.60 | 631.77 | 641.07 | 636.32 | 744.85 | 849.92 |
| 2 | 0.00 | 0.00 | 10.19 | 0.00 | 90.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 3 | 133.66 | 133.66 | 284.44 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 4 | 19.94 | 19.94 | 4.77 | 145.00 | 134.15 | 129.61 | 137.48 | 131.00 | 135.07 | 132.99 | 162.56 | 190.80 |
| 5 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 35.63 | 70.98 |
| 6 | 317.86 | 317.86 | 173.97 | 393.02 | 383.28 | 379.21 | 386.27 | 380.46 | 384.11 | 382.24 | 492.57 | 600.81 |
| 7 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 8 | 545.92 | 545.92 | 193.44 | 333.09 | 317.86 | 311.49 | 322.53 | 313.44 | 319.14 | 316.23 | 339.71 | 361.52 |
| 9 | 580.83 | 580.83 | 396.59 | 420.02 | 421.30 | 421.84 | 420.91 | 421.67 | 421.19 | 421.44 | 452.75 | 483.85 |
| 10 | 742.72 | 742.72 | 629.71 | 665.97 | 671.39 | 673.66 | 669.73 | 672.97 | 670.93 | 671.97 | 737.77 | 803.39 |
| 11 | 945.00 | 945.02 | 651.36 | 621.98 | 629.43 | 632.55 | 627.15 | 631.60 | 628.81 | 630.23 | 654.89 | 679.99 |
| 12 | 865.21 | 865.21 | 822.53 | 679.76 | 702.49 | 712.00 | 695.52 | 709.09 | 700.57 | 704.92 | 788.08 | 872.49 |
| 13 | 505.80 | 505.80 | 444.10 | 342.65 | 355.45 | 360.81 | 351.52 | 359.16 | 354.37 | 356.82 | 387.58 | 419.22 |
| 14 | 1293.07 | 1293.07 | 692.17 | 1023.62 | 1007.95 | 1001.39 | 1012.76 | 1003.40 | 1009.27 | 1006.27 | 1128.63 | 1248.23 |
| 15 | 819.84 | 819.84 | 669.14 | 558.56 | 579.22 | 587.87 | 572.88 | 585.21 | 577.48 | 581.43 | 636.14 | 692.20 |
| 16 | 593.85 | 593.85 | 454.51 | 427.81 | 538.83 | 443.45 | 435.45 | 442.03 | 437.91 | 440.02 | 510.37 | 581.02 |
| 17 | 610.30 | 610.30 | 441.53 | 544.07 | 544.53 | 544.72 | 544.39 | 544.67 | 544.49 | 544.58 | 629.97 | 714.49 |
| 18 | 550.04 | 550.04 | 410.53 | 478.48 | 484.27 | 486.69 | 482.49 | 485.95 | 483.78 | 484.89 | 568.21 | 651.18 |
| 19 | 1391.74 | 1391.74 | 1459.33 | 707.05 | 804.15 | 844.80 | 774.37 | 832.33 | 795.97 | 814.55 | 946.92 | 1086.97 |
| 20 | 1039.12 | 1039.12 | 846.35 | 668.26 | 706.91 | 723.10 | 695.06 | 718.13 | 703.66 | 711.06 | 801.27 | 894.14 |
| 21 | 1166.00 | 1166.00 | 917.75 | 739.32 | 757.86 | 765.61 | 752.17 | 763.23 | 756.29 | 759.84 | 845.99 | 932.96 |
| 22 | 0.00 | 0.00 | 0.00 | 92.30 | 27.62 | 0.55 | 47.46 | 8.85 | 33.07 | 206.90 | 60.24 | 93.32 |
| 23 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 24 | 1214.23 | 1214.23 | 1668.10 | 973.33 | 1057.96 | 1093.39 | 1032.00 | 1082.52 | 1050.83 | 1067.02 | 1177.38 | 1294.49 |
| 25 | 1183.04 | 1183.04 | 1776.41 | 939.10 | 1038.38 | 1079.94 | 1007.93 | 1067.19 | 1030.02 | 1049.01 | 1157.80 | 1274.73 |
| 26 | 1787.86 | 1787.86 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Table 3.3 The Reduced Costs for the 26/12 Model

| ODUCT | MONTH 1 | MONTH 2 | MONTH 3 | MONTH 4 | MONTH 5 | MONTH 6 | MONTH 7 | MONTH 8 | MONTH 9 | MONTH10 | MONTH11 | MONTH12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 2 | 6.36 | 6.36 | 0.00 | 17.09 | 17.09 | 16.37 | 18.71 | 13.44 | 23.00 | 9.00 | 23.47 | 10.97 |
| 3 | 0.00 | 0.00 | 0.00 | 23.73 | 23.73 | 23.45 | 24.36 | 22.32 | 26.02 | 20.60 | 26.20 | 26.74 |
| 4 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 5 | 38.96 | 38.96 | 16.69 | 14.53 | 14.53 | 16.17 | 10.84 | 22.85 | 1.06 | 32.96 | 0.00 | 0.00 |
| 6 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 7 | 494.43 | 0.00 | 197.30 | 183.77 | 183.78 | 182.65 | 186.31 | 178.05 | 193.05 | 171.09 | 193.78 | 219.86 |
| 8 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 9 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 11 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 12 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 13 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 14 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 15 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 16 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 17 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 18 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 19 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 20 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 21 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 22 | 9.48 | 9.48 | 11.66 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 23 | 23.80 | 23.80 | 17.61 | 20.30 | 20.30 | 20.60 | 19.62 | 21.84 | 17.82 | 23.70 | 17.62 | 16.18 |
| 24 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 25 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 26 | 0.00 | 0.00 | 4.20 | 2.37 | 2.37 | 2.17 | 2.83 | 1.33 | 0.00 | 0.68 | 4.19 | 5.17 |

Table 3.4 The Activity Levels for the 26/12 Model

| PRODUCT | AVERAGE REDUCED COST |
|---|---|
| 1 | 639.27 |
| 2 | 8.34 |
| 3 | 45.97 |
| 4 | 111.94 |
| 5 | 8.87 |
| 6 | 382.63 |
| 7 | 0.00 |
| 8 | 351.69 |
| 9 | 453.60 |
| 10 | 696.07 |
| 11 | 689.83 |
| 12 | 759.82 |
| 13 | 395.27 |
| 14 | 1059.98 |
| 15 | 639.98 |
| 16 | 491.59 |
| 17 | 568.16 |
| 18 | 509.71 |
| 19 | 987.49 |
| 20 | 795.51 |
| 21 | 860.25 |
| 22 | 47.52 |
| 23 | 0.00 |
| 24 | 1160.45 |
| 25 | 1148.88 |
| 26 | 297.97 |

Table 3.5  Average Reduced Costs

The average reduced cost (over twelve months) is shown in Table 3.5. For product rationalisation these statistics are meaningless, apart from indicating which products are always basic - these have zero average reduced cost. In Section 3.3.4 we consider three other statistics which produce rankings that may be more meaningful to management.

### 3.4.3 The 'Sales to Costs' Ratio

At the aggregate level, the ratio 'sales to costs', of Figure 3.1, estimates the total capability (efficiency) of the firm's production system when converting 'costs' to 'sales'. This interpretation also holds at the disaggregated (product by product) level; in both cases, costs are measured by average (or incurred) values.

The resource evaluation of the optimal solution to the linear programming model gives a set of marginal (and average) values for resources. (From these figures, resources may be valued at their marginal rates).

Consider the normal formulation:

$$\begin{aligned} \max \quad & \underline{c}.\underline{x} \\ \text{s.t.} \quad & \underline{A}.\underline{x} \le \underline{b} \\ & \underline{x} \ge \underline{0} \end{aligned} \qquad (3.4)$$

Let the optimal solution be $\underline{x}^*$, the columns of $\underline{A}$ be $\underline{a}_i$, the dual evaluators be $\underline{v}$, and let $\{z_i\} = \{c_i - \underline{v}.\underline{a}_i\}$ be the reduced costs. Then $\underline{v}.\underline{a}_i$ represents the marginal value (cost) of inputs to activity i (at the optimum), $c_i$ represents the return from (say) the sale of product i, and the ratio $\theta_i = \dfrac{c_i}{\underline{v}.\underline{a}_i}$ is the rate of conversion of input value to output value by activity i, (at the optimum). Ranking activities by the $\theta_i$ statistics in the single-period model we have:

i. an activity is basic if and only if

$\theta_i = 1$ (i.e. $z_i = 0$)

ii. any activity for which $\theta_i < 1$ is rejected; non-basic activities may be ranked by $\theta_i$ $(0 \leq \theta_i \leq 1)$

For programmes in which variables $x_i$ have upper bounds a further modification may be introduced:

The model is now

$$\begin{aligned} \max \quad & \underline{c}.\underline{x} \\ \text{s.t.} \quad & \underline{A}.\underline{x} \leq b \\ & \underline{0} \leq \underline{x} \leq \underline{U} \end{aligned} \qquad (3.9)$$

Weingartner (88, page 54) associates a goodwill value with the dual evaluator of any upper bound constraint that is tight at the optimum. For basic activities $x_i$ not at their upper bound, the ratio $\theta_i = \frac{c_i}{\underline{v}.\underline{a}_i}$ is unity, since $c_i - \underline{v}.\underline{a}_i = 0$.

For a variable at its upper bound (say $x_j$) let $p_j$ be the dual evaluator of the constraint $x_j \leq u_j$

Then, $c_j - \underline{v}.\underline{a}_j - u_j.p_j = 0$ (3.10)

(optimality condition)

i.e. $c_j > \underline{v}.\underline{a}_j$

$$\theta_j = \frac{c_i}{\underline{v}.\underline{a}_j} > 1 \qquad (3.11)$$

In the bounded variable model, the ranking by $(\theta_i)$ is not confined to the range $0 \leq \theta_i \leq 1$. The properties of the ranking with upper bounds are given in Table 3.6.

| Non basic activities | Basic activities | |
|---|---|---|
| | not at bound | at bound |
| $\theta < 1$ | $\theta = 1$ | $\theta > 1$ |

Table 3.6 $\theta$ Ranking for Upper Bounds

Ranking by ($\theta_i$) eliminates possible confusion between columns that are near multiples of one another.

Suppose $c_i \sim kc_j \qquad k >> 1$

$\underline{a}_i \sim k\underline{a}_j$

then $c_i - \underline{v}.\underline{a}_i \sim k(c_j - \underline{v}.\underline{a}_j)$

i.e. the reduced cost of the j'th activity is $\frac{1}{k}$ times that of the i'th activity, yet, in cases where the corresponding $x_i$, $x_j$ are infinitely divisible, the net effect of changing either $x_i$ or $x_j$ is the same. The fact that $\theta_i \sim \theta_j$ reflects this.

Example

Consider the capital budgeting problem posed in (88)

$$\text{max;} \quad 14x_1 + 17x_2 + 17x_3 + 15x_4 + 40x_5 + 12x_6 + 14x_7 + 10x_8 + 12x_9$$

$$\text{s.t.} \quad 12x_1 + 54x_2 + 6x_3 + 6x_4 + 30x_5 + 6x_6 + 48x_7 + 36x_8 + 18x_9 \leq 50$$

$$3x_1 + 7x_2 + 6x_3 + 2x_4 + 35x_5 + 6x_6 + 4x_7 + 3x_8 + 3x_9 \leq 20$$

$$0 \leq x_i \leq 1 \qquad i = 1\ldots9 \qquad (3.12)$$

The solution, reduced costs, $\theta$ rankings (and Weingartner's ranking) are shown below in Table 3.7

| Project | Activity | Input cost | Upper limit | Reduced cost | $\theta$ Ratio | $\theta$ Rank | W Rank |
|---|---|---|---|---|---|---|---|
| 1 | 1.0 | 14.0 | 1.0 | 6.77 | 1.94 | 2 | 2 |
| 2 | 0.0 | 17.0 | 1.0 | -3.41 | 0.83 | 7 | 7 |
| 3 | 1.0 | 17 | 1.0 | 5.00 | 1.41 | 4 | 3 |
| 4 | 1.0 | 15 | 1.0 | 10.45 | 3.29 | 1 | 1 |
| 5 | 0.0 | 40.0 | 1.0 | -29.31 | 0.57 | 8 | 8 |
| 6 | 0.96 | 12.0 | 1.0 | 0.0 | 1.0 | 5 | 5 |
| 7 | 0.04 | 14.0 | 1.0 | 0.0 | 1.0 | 5 | 5 |
| 8 | 0.0 | 10.0 | 1.0 | -0.5 | 0.95 | 6 | 6 |
| 9 | 1.0 | 12.0 | 1.0 | 3.95 | 1.49 | 3 | 4 |

Table 3.7 The Solution to Weingartner's Problem (88)

### 3.4.4 Statistics for Product Ranking

For a multiperiod model, the $\{\theta_i\}$ (corresponding to similar activities in different time periods) will show a time dependence. Data from the 26/12 model is presented in Tables 3.3 and 3.4, the average reduced cost for each product being shown in Table 3.5. The average reduced costs provide little guidance for product ranking. Three further measures are suggested:

Let $x_i$ be the optimal amount of x produced in period i,

$\underline{a}_i$ be the corresponding 'column'

$s_i$ be the entry for '$x_i$' in the objective function

$\underline{v}$ be the dual evaluators

$$P(x) = \frac{\Sigma x_i s_i}{\Sigma x_i (\underline{v} \cdot \underline{a}_i)}$$

$$Q(x) = \frac{\Sigma s_i}{\Sigma(\underline{v} \cdot \underline{a}_i)}$$

and $$R(x) = \frac{1}{N} \sum \frac{s_i}{\underline{v} \cdot \underline{a}_i} \qquad (3.12)$$

where N is the number of time periods being considered. P is a productivity measure, aggregating the sales and cost figures according to monthly production levels. The 'usefulness' of a product is measured in terms of increasing values for P. Q is a similar statistic, omitting the weighting by production level. R is the average of the $\theta_i$ over the total planning periods. Rankings for the 26 production variables of the 26/12 model are shown in Table 3.8; the monthly $\theta$ statistics are given in Table 3.9.

Since the model has no upper bounds on production levels, the P statistics are either 1 or 0, depending on whether the product is produced, or not. (Either $x_i = 0$ and $s_i \neq \underline{v} \cdot \underline{a}_i$,

| PRODUCT | P VALUE | Q VALUE | R VALUE |
|---|---|---|---|
| 1 | 0.00 | 0.36 | 0.37 |
| 2 | 0.98 | 0.98 | 0.98 |
| 3 | 1.00 | 0.93 | 0.94 |
| 4 | 0.00 | 0.74 | 0.76 |
| 5 | 1.00 | 0.98 | 0.98 |
| 6 | 0.00 | 0.56 | 0.57 |
| 7 | 1.00 | 1.00 | 1.00 |
| 8 | 0.00 | 0.57 | 0.58 |
| 9 | 0.00 | 0.56 | 0.57 |
| 10 | 0.00 | 0.49 | 0.49 |
| 11 | 0.00 | 0.52 | 0.52 |
| 12 | 0.00 | 0.56 | 0.56 |
| 13 | 0.00 | 0.63 | 0.64 |
| 14 | 0.00 | 0.46 | 0.47 |
| 15 | 0.00 | 0.58 | 0.58 |
| 16 | 0.00 | 0.65 | 0.65 |
| 17 | 0.00 | 0.64 | 0.64 |
| 18 | 0.00 | 0.69 | 0.69 |
| 19 | 0.00 | 0.69 | 0.70 |
| 20 | 0.00 | 0.71 | 0.71 |
| 21 | 0.00 | 0.69 | 0.69 |
| 22 | 1.00 | 0.97 | 0.97 |
| 23 | 1.00 | 1.00 | 1.00 |
| 24 | 0.00 | 0.62 | 0.62 |
| 25 | 0.00 | 0.62 | 0.62 |
| 26 | 1.00 | 0.91 | 0.94 |

Table 3.8 The P, Q, R Statistics

| PRODUCT | MONTH 1 | MONTH 2 | MONTH 3 | MONTH 4 | MONTH 5 | MONTH 6 | MONTH 7 | MONTH 8 | MONTH 9 | MONTH10 | MONTH11 | MONTH12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.39 | 0.39 | 0.44 | 0.36 | 0.36 | 0.37 | 0.36 | 0.37 | 0.36 | 0.37 | 0.33 | 0.30 |
| 2 | 1.00 | 1.00 | 0.98 | 1.00 | 0.86 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 3 | 0.83 | 0.83 | 0.71 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 4 | 0.94 | 0.94 | 0.98 | 0.69 | 0.71 | 0.71 | 0.70 | 0.71 | 0.70 | 0.71 | 0.66 | 0.63 |
| 5 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 0.95 | 0.9[illegible] |
| 6 | 0.61 | 0.61 | 0.74 | 0.55 | 0.56 | 0.56 | 0.56 | 0.56 | 0.56 | 0.56 | 0.50 | 0.45 |
| 7 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 8 | 0.46 | 0.46 | 0.71 | 0.59 | 0.60 | 0.60 | 0.59 | 0.60 | 0.60 | 0.60 | 0.58 | 0.57 |
| 9 | 0.50 | 0.50 | 0.60 | 0.58 | 0.58 | 0.58 | 0.58 | 0.58 | 0.58 | 0.58 | 0.56 | 0.55 |
| 10 | 0.47 | 0.47 | 0.51 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 | 0.47 | 0.45 |
| 11 | 0.44 | 0.44 | 0.53 | 0.54 | 0.54 | 0.54 | 0.54 | 0.54 | 0.54 | 0.54 | 0.53 | 0.52 |
| 12 | 0.53 | 0.53 | 0.54 | 0.59 | 0.58 | 0.58 | 0.58 | 0.58 | 0.58 | 0.58 | 0.55 | 0.53 |
| 13 | 0.58 | 0.58 | 0.61 | 0.67 | 0.66 | 0.65 | 0.66 | 0.66 | 0.66 | 0.66 | 0.64 | 0.62 |
| 14 | 0.41 | 0.41 | 0.57 | 0.47 | 0.47 | 0.48 | 0.47 | 0.48 | 0.47 | 0.48 | 0.45 | 0.42 |
| 15 | 0.52 | 0.52 | 0.57 | 0.61 | 0.61 | 0.60 | 0.61 | 0.60 | 0.61 | 0.61 | 0.58 | 0.56 |
| 16 | 0.61 | 0.61 | 0.67 | 0.68 | 0.63 | 0.68 | 0.68 | 0.68 | 0.68 | 0.68 | 0.64 | 0.61 |
| 17 | 0.62 | 0.62 | 0.69 | 0.65 | 0.65 | 0.65 | 0.65 | 0.65 | 0.65 | 0.65 | 0.61 | 0.58 |
| 18 | 0.68 | 0.68 | 0.74 | 0.71 | 0.70 | 0.70 | 0.70 | 0.70 | 0.70 | 0.70 | 0.67 | 0.64 |
| 19 | 0.62 | 0.62 | 0.61 | 0.76 | 0.74 | 0.73 | 0.74 | 0.73 | 0.74 | 0.73 | 0.70 | 0.67 |
| 20 | 0.65 | 0.65 | 0.69 | 0.74 | 0.73 | 0.72 | 0.73 | 0.73 | 0.73 | 0.73 | 0.70 | 0.68 |
| 21 | 0.62 | 0.62 | 0.67 | 0.72 | 0.71 | 0.71 | 0.72 | 0.71 | 0.71 | 0.71 | 0.69 | 0.67 |
| 22 | 1.00 | 1.00 | 1.00 | 0.94 | 0.98 | 0.99 | 0.97 | 0.99 | 0.98 | 0.89 | 0.96 | 0.94 |
| 23 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 24 | 0.61 | 0.61 | 0.53 | 0.66 | 0.64 | 0.63 | 0.64 | 0.63 | 0.64 | 0.64 | 0.61 | 0.59 |
| 25 | 0.61 | 0.61 | 0.51 | 0.66 | 0.64 | 0.63 | 0.65 | 0.64 | 0.64 | 0.64 | 0.62 | 0.59 |
| 26 | 0.65 | 0.65 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Table 3.9 The $\theta$ Statistics

or $x_i \neq 0$ and $s_i = \underline{v}.\underline{a}_i$, by simplex optimality criteria). The P statistics show which products might be dropped from the range (i.e. those with zero values). With Q and R rankings there are some small differences in rankings (with products 3, 8, 12, 18, 21 and 26). None of these differences suggest major alterations in ranking. These statistics only give a guide for product rationalisation. A true picture of rationalisation can only be obtained by re-optimising the model, flagging out the products that are to be dropped.

The rankings P, Q, R, (and the dual evaluators and reduced costs) are dependent on the objectives used in optimisations - no single set can be proposed as the unique ranking for the firm's products.

## 3.5 Conclusions

i. Planning criteria and performance measures are objective dependent.

ii. The dual evaluators and valuations are similarly objective dependent.

iii. The underlying assumptions for dual pricing are:

   a. linearity

   b. one objective function

   c. a close correspondence between model and reality.

   Where these are contradicted, (integer values, multiple objectives, long time periods, etc.) dual prices must be treated with caution.

iv. With multiperiod models, dual evaluators may have to be summed to give estimates of the marginal value of capacity.

v. Reduced costs give little guide for product rationalisation. Three statistics have been suggested to aid management in this task.

# Chapter 4 Duality and Pricing in Fractional Programming

## 4.1.1 Introduction

In Chapter 3, ranking of resources with respect to a series of objectives was discussed. As suggested in Section 2.3.1, corporate objectives may include such terms as 'return on assets', which are not linear, but fractional. In this chapter we investigate the nature of the dual prices in fractional programming, for the general, and linear, constraint cases.

## 4.1.2 Duality Theorems for Fractional Programming

Considering the following problem:

$$\max \quad f(\underline{x}) = \frac{\underline{c}\cdot\underline{x} + \alpha}{\underline{d}\cdot\underline{x} + \beta} \qquad (4.1)$$

$$\text{s.t.} \quad g_i(\underline{x}) \le 0 \qquad i = 1 \ldots m$$

Swarup (77, 79, 80) has proved the following theorems:

Theorem 1: Let $f(\underline{x})$, $g_1(\underline{x}) \ldots g_m(\underline{x})$ be differentiable on $E^n$, $f(\underline{x})$ as given in (4.1), each $g_i(\underline{x})$ convex; then the necessary and sufficient conditions for $\underline{x}^* \varepsilon S$ to be a solution to (4.1) is that $\exists\, \underline{u}^* \varepsilon E^m$ s.t.

$$\text{i} \quad \nabla_x f(\underline{x}^*) - \nabla_{x_i} \sum_{i=1}^{m} u_i^* \cdot g_i(\underline{x}^*) = \underline{0}$$

$$\text{ii} \quad \sum_{i=1}^{m} u_i^* \cdot g_i(\underline{x}^*) = 0 \qquad (4.2)$$

$$\text{iii} \quad g_i(\underline{x}^*) \le 0 \qquad i = 1 \ldots m$$

$$\text{iv} \quad u_i^* \ge 0 \qquad i = 1 \ldots m$$

where

a $S = \{\underline{x} \mid g_i(\underline{x}) \le 0,\ i = 1 \ldots m\}$

b $\underline{d}\cdot\underline{x} + \beta \ne 0$ for $\underline{x} \varepsilon S$

According to Wolfe (93), the dual problem to (4.1) is

$$\min \ \chi(\underline{x}, \underline{u})$$

$$\text{s.t. } \nabla_x \chi(\underline{x}, \underline{u}) = \underline{0} \ , \quad \underline{u} \geq \underline{0} \tag{4.3}$$

where $\chi(\underline{x}, \underline{u}) = f(\underline{x}) - \underline{u}'.g(\underline{x})$

Let $D(\underline{x}, \underline{u}) = \left\{(\underline{x}, \underline{u}) \middle| \nabla_x \chi(\underline{x}, \underline{u}) = \underline{0} \ , \ \underline{u} \geq \underline{0}\right\}$ with $\underline{x}$ unconstrained. Swarup also proves:

Theorem 2: If $\underline{x}^*$ is an optimal solution for (4.1),

$\Longrightarrow \exists \ \underline{u}^*$ s.t. $(\underline{x}^*, \underline{u}^*) \varepsilon D$

s.t. $\chi(\underline{x}^*, \underline{u}^*) = f(\underline{x}^*)$

He does not use the non-negativity requirement on $\underline{x}$, i.e. $\underline{x} \geq \underline{0}$ or $-x_j \leq 0$. If we include these constraints in (4.1) and extend $u_i$ to $(\underline{u}^*, \underline{v}^*)$, then 4.2$\underline{i}$ reads

$$\frac{\partial f^*}{\partial x_j} - \sum_k u_k^* \cdot \frac{\partial g_k(\underline{x}^*)}{\partial x_j} + v_j^* = 0$$

$$\text{or} \quad \frac{\partial f^*}{\partial x_j} - \sum_k u_k^* \cdot \frac{\partial g_k(\underline{x}^*)}{\partial x_j} = -v_j^* \tag{4.4}$$

Now if $x_j^* > 0$ we have $v_j^* = 0$ (from $\underline{ii}$ of (4.2))

$$\text{therefore } x_j^* \cdot \left(\frac{\partial f^*}{\partial x_j} - \Sigma u_k^* \cdot \frac{\partial g_k(\underline{x}^*)}{dx_j}\right) = 0$$

$$\text{or} \quad \underline{x}^* \cdot \left[\nabla_x f(\underline{x}^*) - \nabla_x \Sigma u_k^* \cdot g_k(\underline{x}^*)\right] = 0 \tag{4.5}$$

Also from (4.4) and $\underline{iv}$ of (4.2), we have

$$\frac{\partial f^*}{\partial x_j} - \Sigma u_k^* \frac{\partial g_k(\underline{x}^*)}{\partial x_i} = -v_i^* \leq 0$$

$$\text{i.e. } \nabla_x f(\underline{x}^*) - \nabla_x \sum_k u_k^* \cdot g_k(\underline{x}^*) \leq \underline{0} \tag{4.6}$$

Hence, if we include the non-negativity requirement on $\underline{x}$, we must amend the equations (4.2) to:

$\underline{i}$ $\quad \nabla_x f(\underline{x}^*) - \nabla_x \sum_k u_k^* \cdot g_k(\underline{x}^*) \leq \underline{0}$

$\underline{ii}$ $\underline{x}^* \cdot \left[\nabla_x f(\underline{x}^*) - \nabla_x \sum_k u_k^* \cdot g_k(\underline{x}^*)\right] = 0$ (4.7)

$\underline{iii}$ $\sum_k u_k^* \cdot g_k(\underline{x}^*) = 0$

$\underline{iv}$ $g_k(\underline{x}^*) \leq 0 \qquad k = 1 \ldots m$

$\underline{v}$ $u_k^* \geq 0 \qquad k = 1 \ldots m$

This is the more usual form of the Kuhn Tucker Conditions (57) and is the form of KT Conditions used by Balinski and Baumol (6) in their work on the economic interpretation of the dual. It is the form we will assume throughout this chapter.

## 4.2 The Interpretation of the Non-Linear Dual Variables as Marginal Values

4.2.1 The power of the dual programme in LP is well known, and its economic interpretation is in widespread use. The interpretation of the non-linear dual, although lacking some of the desirable features of the linear dual, can still prove a powerful tool in the evaluation of non-linear programming problems. The extent of the interpretation depends on the properties of the objective and constraint functions. In this section the main reference is the work of Balinski and Baumol (6). We will develop the ideas that they have presented for the concave objective function, and show how these cannot be applied to the FP case, where the objective function is only continuous, differentiable and quasi-monotonic.

Define $\pi = \pi(\underline{b}) = \left\{\max f(\underline{x}) \mid \underline{g}(\underline{x}) \leq \underline{b}, \ \underline{x} \geq \underline{0}\right\}$

We will refer to the $u_i^*$ of (4.7) as the dual evaluators. In order to interpret the dual evaluators of the LFP as the marginal values (prices) of the resources $b_k$, we need to show that $u_k^* \sim \partial\pi/\partial b_k$, i.e. where the marginal value of $\pi$ with respect to $b_k$ is defined, its value is given by $u_i^*$.

We show how far the dual analysis can be carried in FP, and why the concept of pricing is not always well defined.

4.2.2 Marginality where f is concave

An outline of the work in (6) is as follows:
In order to interpret the dual variable $u_i^*$ as the 'marginal value' (in terms of an economic price) for an extra unit of the i'th resource, it is necessary to show that:

$$" u_i^* = \frac{\partial \pi}{\partial b_i} " \qquad (4.8)$$

Even in LP, the discontinuities in $\partial\pi/\partial b_i$ do not always allow this result to be proved. However, it is possible to show that

$$\frac{\partial \pi}{\partial b_{i+}} \le u_i^* \le \frac{\partial \pi}{\partial b_{i-}} \qquad (4.9)$$

and for any point where $\partial\pi/\partial b_i$ is defined, its value is given by (4.8).

To lend credence to the 'price' allocation we need diminishing returns to scale. This is also implied by (4.9); for $\frac{\partial \pi}{\partial b_{i+}} \le \frac{\partial \pi}{\partial b_{i-}}$, and for $\delta > 0$, we have:

$$\frac{1}{2\delta}\left(\left[\frac{\partial \pi}{\partial b_i}\right]_{b_i+\delta} - \left[\frac{\partial \pi}{\partial b_i}\right]_{b_i-\delta}\right) \le 0$$

i.e. where the second derivative of $\pi$ with respect to b exists, it is negative.

In order to deduce the inequalities (4.9), BB define

$$\pi(y_k) = \max_{\underline{x}} \left\{ f(\underline{x}) \,\middle|\, \begin{matrix} g_i(\underline{x}) \le b_i, \; i \ne k \\ g_k(\underline{x}) \le y_k \end{matrix} \right\}$$

and show that:

i $\pi(y_k)$ exists in a neighbourhood of $b_k$,

ii $\pi(y_k)$ is continuous, and

iii the partial derivatives of (4.9) exist.

The proofs given in (6) depend heavily on the constraint qualification for $g_i(\underline{x})$ and on the equivalence between the Kuhn Tucker Conditions and the Saddle Point Conditions for the Lagrangian.

Kuhn and Tucker (57) proved that the sufficient conditions for a saddle point are the "Kuhn-Tucker Conditions" (4.7) and the concavity/convexity of f, $\underline{g}$.

We do not have f concave; f is quasi-monotonic. This is insufficient to prove the equivalence between the Saddle Point and Kuhn-Tucker Conditions. Thus we cannot show

$$\frac{\partial \pi}{\partial b_{i+}} \leq u_i^* \leq \frac{\partial \pi}{\partial b_{i-}} .$$

Indeed a quasi-monotonic function need not have left and right derivatives defined at all points. Consider the step function that lies between two rays that pass through the origin:

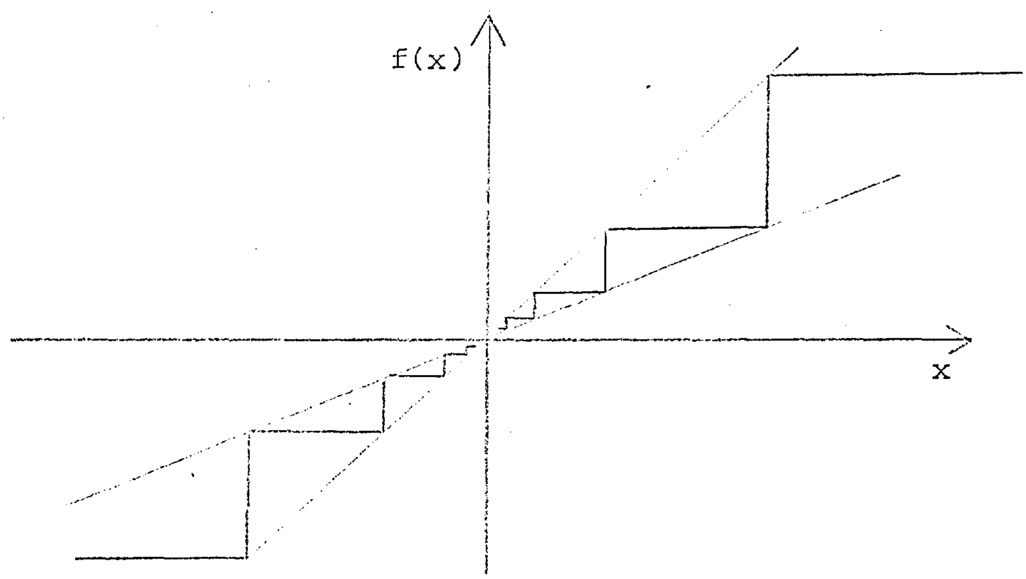



Fig 4.1 A quasi-monotonic function with no derivatives at x = 0

f(x) is monotonic (quasi-monotonic) but there are no one-sided derivatives at x = 0.

It is possible to give sufficient conditions for the existence of the marginal values $\partial\pi/\partial b_k$.

Where the constraints are linear, we have the $x_i^*$ in terms of an optimal inverse basis $\underline{B}^{-1}$, since for basic $x_i^*$, $\underline{x}^* = \underline{B}^{-1}\underline{b}$. Hence $f(\underline{x}^*) = f(\underline{B}^{-1}\underline{b})$ and the partial derivatives can be shown to exist.

In the non-linear case we can give a generalisation of the equation $\underline{x}^* = \underline{B}^{-1}\underline{b}$. For concave f, $\pi$ is also concave:

$$\begin{aligned}\pi(\theta b_{k'} + (1-\theta)b_{k''}) &= \max_{\underline{x}} \left\{ f(\underline{x}) \,\middle|\, g_i(\underline{x}) \le \theta b_{k'} + (1-\theta)b_{k''},\ \underline{x} \ge \underline{0} \right\} \\ &\ge f(\theta \underline{x}' + (1-\theta)\underline{x}'') \\ &\ge \theta f(\underline{x}') + (1-\theta) f(\underline{x}'') \\ &= \theta\pi(b_{k'}) + (1-\theta)\pi(b_{k''}) \qquad (4.10)\end{aligned}$$

proved in (6), where $\underline{x}'$ and $\underline{x}''$ are the optimal solutions for $b_{k'}$ and $b_{k''}$.

For quasi-monotonic f we do not have such a strong result.

Let $S_K = \left\{ \underline{x} \,\middle|\, g_i(\underline{x}) \le b_i,\ i \ne k,\ g_k(\underline{x}) \le y_k \right\}$.

Then $y_{k_1} \ge y_{k_2} \Rightarrow S_{K_1} \supseteq S_{K_2} \Rightarrow \pi(y_{k_1}) \ge \pi(y_{k_2})$, (4.11)

i.e. $\pi$ is monotonic in each argument.

But this does not guarantee the existence of partial derivatives at all points of $E^n$, nor does this give diminishing returns to scale.

4.2.3 Marginality where f is quasi-monotonic

We can state sufficient conditions for the partial derivatives of $\pi(b_k)$ to exist in segments of the total range of $b_k$.

Lemma 1: For $b_{k'} \le b_k \le b_{k''}$ the sufficient condition for the partial derivatives of $\pi(b_k)$ to exist (with appropriate left and right hand derivatives at the ends of the range) is that $\exists\, \Phi_i(b_k)$ s.t.

$$x_i = \Phi_i(b_k) \qquad \text{for all } i$$

$\Phi_i$: continuous, differentiable with respect to $b_k$ in the range $(b_{k'}, b_{k''})$.

<u>Proof</u>: If such $\Phi_i$ exist, then

$$\begin{aligned} \pi(b_k) &= f(\underline{x}(b_k)) \\ &= f(\underline{\Phi}(b_k)) \\ &= \frac{\underline{c}.\underline{\Phi}(b_k) + \alpha}{\underline{d}.\underline{\Phi}(b_k) + \beta} \qquad b_{k'} \le b_k \le b_{k''} \end{aligned}$$

$\underline{d}.\underline{\Phi}(b_k) + \beta > 0$ by assumption, therefore partial derivatives exist as required.

Hence we have:

LFP <u>iii</u>: If $\exists\, \Phi_i(b_k)$ as defined in Lemma 1 for each of the required ranges $b_{k_0} \le b_{k_1} \le b_{k_2} \le \ldots$

then $\frac{\partial\pi}{\partial b_{k+}}$ and $\frac{\partial\pi}{\partial b_{k-}}$ exist,

and we will have $u_i^* = \frac{\partial\pi}{\partial b_k}$ where this is defined.

But, as previously stated, we do not have the inequalities (4.9).

This reflects the general situation in fractional programming that returns to scale need not be diminishing. Since it is a requirement for a coherent pricing system that there exist diminishing returns to scale, the dual evaluators, although equivalent to the marginal values, will not serve as 'economic' prices in all cases.

4.2.4 <u>Linear Constraints</u>

The case where the $g_i(\underline{x})$ are linear can be treated separately.

It has been proved by Martos (65) that the problem

$$\begin{aligned} &\max\ f(\underline{x}) \\ &\text{s.t. } \underline{A}.\underline{x} \le \underline{b} \\ &\qquad \underline{x} \ge \underline{0} \end{aligned} \tag{4.12}$$

has an extremum point solution. This result has a two-fold importance:

a. it allows us to use simplical methods for solving (4.12)

b. it implies that at the optimal vertex the optimal solution $\underline{x}^*$ is given by:

$$\underline{x}^* = \underline{B}^{-1}.\underline{b}$$

where $\underline{B}^{-1}$ is the inverse basis (see e.g. (44)).

The ranges $b_{k_1} \ldots b_{k_n}$ etc. are given by the points where a further iteration is necessary, i.e. where the present basis no longer remains optimal (or feasible).

Between changes of basis the i'th rows of $\underline{B}^{-1}$ provide the $\Phi_i$ of the previous lemma. Thus Lemma 1 of Section 4.2.2 provides proof that the dual variables can be equated with the marginal values of resources, if the situation is one of diminishing returns to scale.

We could, however, use the CC form of (4.12) to prove the existence of $\frac{\partial \pi}{\partial b+}$ and $\frac{\partial \pi}{\partial b-}$ .

In the marginal work of Mills and Williams ((66) and (92) respectively) we have the following conditions for the existence of marginal values. (Once again we present their theorems in order to aid exposition).

Marginal Values of Linear Programmes

(The notation used is that of Williams.) Consider the problems

i. max. $\underline{c}.\underline{x}$, $\quad \underline{x} \geq \underline{0}$, $\quad \underline{A}.\underline{x} \leq \underline{b}$

ii. min. $\underline{\pi}.\underline{b}$, $\quad \underline{\pi} \geq \underline{0}$, $\quad \underline{\pi}.\underline{A} \geq \underline{c}$

Given $\underline{H}$, $\hat{\underline{b}}$, $\hat{\underline{c}}$, define:

i' max. $(\underline{c} + \alpha\hat{\underline{c}}).\underline{x}$, $\quad \underline{x} \geq \underline{0}$, $\quad (\underline{A} + \alpha\underline{H}).\underline{x} \leq \underline{b} + \alpha\hat{\underline{b}}$

ii' min. $\underline{\pi}.(\underline{b} + \alpha\hat{\underline{b}})$, $\quad \underline{\pi} \geq \underline{0}$, $\quad \underline{\pi}.(\underline{A} + \alpha\underline{H}) \geq \underline{c} + \alpha\hat{\underline{c}}$

The 'marginal value' is discussed for small values of $\alpha$ and is defined as:

$$f'(0) = \lim_{\alpha \to 0} \frac{\Phi(\tilde{\underline{A}} + \alpha\check{\underline{H}}) - \Phi(\tilde{\underline{A}})}{\alpha}$$

where: $\tilde{\underline{A}} = \begin{pmatrix} \underline{A} & \underline{b} \\ \underline{c} & 0 \end{pmatrix}$ , $\tilde{\underline{H}} = \begin{pmatrix} \underline{H} & \hat{\underline{b}} \\ \hat{\underline{c}} & 0 \end{pmatrix}$ ,

and $\Phi$ is defined as the value of the LP (if it exists)

i.e. $\Phi(\underline{A}) = \max \underline{c}.\underline{x}...$

$= \min \underline{\pi}.\underline{b}...$

$f'(0)$ is the marginal value of $\tilde{\underline{A}}$ with respect to $\tilde{\underline{H}}$.

Let $S(\tilde{\underline{A}}) = \{\underline{x} \mid \underline{x} \geq \underline{0},\ \underline{A}.\underline{x} \leq \underline{b}\}$ and $T(\tilde{\underline{A}}) = \{\underline{\pi} \mid \underline{\pi} \geq \underline{0},\ \underline{\pi}.\underline{A} \geq \underline{c}\}$.

Theorem I (Williams). For given $\tilde{\underline{A}}$, the N+S conditions that $f'(0)$ exists for every $\tilde{\underline{H}}$ are that both the primal and dual optimal sets of $\tilde{\underline{A}}$ are bounded.

Equivalently, that the regularity conditions

$$\begin{aligned} R_1&:\ \underline{y} \geq 0,\ \underline{A}.\underline{y} \leq 0 \Rightarrow \underline{c}.\underline{y} < 0 \\ R_2&:\ \underline{\rho} \geq 0,\ \underline{\rho}.\underline{A} \geq 0 \Rightarrow \underline{\rho}.\underline{b} > 0 \end{aligned} \qquad (4.13)$$

are satisfied by $\tilde{\underline{A}}$.

Theorem II Let $\tilde{\underline{A}}$ satisfy $(R_1, R_2)$; then $f'(0)$ of $\underline{A}$ is given by

$$f'(0) = \max_{\underline{x}^o \varepsilon S^o(\underline{A})} \ \min_{\pi^o \varepsilon T^o(\underline{A})} \ \psi\,(\tilde{\underline{H}}, \underline{x}^o, \underline{\pi}^o)$$

where:

$$S^o(\tilde{\underline{A}}) = \{\underline{x}^o \mid \underline{x}^o \geq \underline{0},\ \underline{A}.\underline{x}^o \leq \underline{b},\ \underline{c}.\underline{x}^o \geq \underline{c}.\underline{x},\ \text{all } \underline{x}\varepsilon S(\tilde{\underline{A}})\}$$

$$T^o(\tilde{\underline{A}}) = \{\underline{\pi}^o \mid \underline{\pi}^o \geq \underline{0},\ \underline{\pi}^o.\underline{A} \geq \underline{c},\ \underline{\pi}^o.\underline{b} \leq \underline{\pi}.\underline{b},\ \text{all } \underline{\pi}\varepsilon T(\hat{\underline{A}})\}$$

and $\psi(\tilde{\underline{H}}, \underline{x}^o, \underline{\pi}^o) = \underline{c}.\underline{x}^o + \underline{\pi}^o.\underline{b} - \underline{\pi}^o.\underline{H}.\underline{x}^o$

These two theorems apply for the simple LP model only, (i.e. $\underline{A}.\underline{x} \leq \underline{b}$), since only for this type can the regularity conditions be guaranteed (if there exist feasible solutions). For the case where equalities are found in the constraint set, we have further theorems.

Define $\tilde{\underline{A}}^* = \begin{pmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{b}_1 \\ \underline{A}_{21} & \underline{A}_{22} & \underline{b}_2 \\ \underline{c}_1 & \underline{c}_2 & \underline{0} \end{pmatrix}$

with which we associate two LP's

$$I^*: \quad \max \underline{c}_1 \cdot \underline{x}_1 + \underline{c}_2 \cdot \underline{x}_2 \qquad \text{s.t.} \quad \begin{aligned} \underline{A}_{11} \cdot \underline{x}_1 + \underline{A}_{12} \cdot \underline{x}_2 &\le \underline{b}_1 \\ \underline{A}_{21} \cdot \underline{x}_1 + \underline{A}_{22} \cdot \underline{x}_2 &= \underline{b}_2 \\ \underline{x}_1 &\ge \underline{0} \end{aligned}$$

$$II^*: \quad \min \underline{\pi}_1 \cdot \underline{b}_1 + \underline{\pi}_2 \cdot \underline{b}_2 \qquad \text{s.t.} \quad \begin{aligned} \underline{\pi}_1 \cdot \underline{A}_{11} + \underline{\pi}_2 \cdot \underline{A}_{21} &\ge \underline{c}_1 \\ \underline{\pi}_1 \cdot \underline{A}_{21} + \underline{\pi}_2 \cdot \underline{A}_{22} &= \underline{c}_2 \\ \underline{\pi}_1 &\ge \underline{0} \end{aligned}$$

also define $\underline{\tilde{H}}^* = \begin{pmatrix} \underline{H}_{11} & \underline{H}_{12} & \underline{\hat{b}}_1 \\ \underline{H}_{21} & \underline{H}_{22} & \underline{\hat{b}}_2 \\ \underline{\hat{c}}_1 & \underline{\hat{c}}_2 & \underline{0} \end{pmatrix}$

<u>Theorem I*</u> N+S condition that $f'(0)$ exists is that the primal and dual sets of $\underline{\tilde{A}}^*$ are bounded or that the (amended) regularity conditions $(R_1^*, R_2^*)$ be satisfied by $\underline{\tilde{A}}$,

$$R_1^*: (\underline{y}_1, \underline{y}_2) \neq 0; \qquad \left.\begin{aligned} \underline{y}_1 \ge \underline{0}, \ \underline{A}_{11} \cdot \underline{y}_1 + \underline{A}_{12} \cdot \underline{y}_2 &\le \underline{0} \\ \underline{A}_{21} \cdot \underline{y}_1 + \underline{A}_{22} \cdot \underline{y}_2 &= \underline{0} \end{aligned}\right\} \Rightarrow \underline{c}_1 \cdot \underline{y}_1 + \underline{c}_2 \cdot \underline{y}_2 < 0$$

$$R_2^*: (\underline{\rho}_1, \underline{\rho}_2) \neq 0; \qquad \left.\begin{aligned} \underline{\rho}_1 \ge \underline{0}, \ \underline{\rho}_1 \cdot \underline{A}_{11} + \underline{\rho}_2 \cdot \underline{A}_{21} &\ge \underline{0} \\ \underline{\rho}_1 \cdot \underline{A}_{12} + \underline{\rho}_2 \cdot \underline{A}_{22} &= \underline{0} \end{aligned}\right\} \Rightarrow \underline{\rho}_1 \cdot \underline{b}_1 + \underline{\rho}_2 \cdot \underline{b}_2 > 0$$

<u>Theorem II*</u> Let $\underline{\tilde{A}}^*$ satisfy $(R_1^*, R_2^*)$.

The marginal value $f'(0)$ is given by:

$$f'(o) = \max_{\underline{x}^o \in S(\underline{\tilde{A}}^*)} \ \min_{\pi^o \in T(\underline{\tilde{A}}^*)} \ \psi^*(\underline{\tilde{H}}^*, \underline{x}^o, \underline{\pi}^o)$$

Where $\psi^*$ is the Lagrangian form

$$\psi^* = \underline{\hat{c}}_1 \cdot \underline{x}_1^o + \underline{\hat{c}}_2 \cdot \underline{x}_2^o + \underline{\pi}^o \cdot \underline{\hat{b}}_1 + \underline{\pi}_2^o \hat{b}_2 - (\underline{\pi}_1^o, \underline{\pi}_2^o) \begin{pmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{pmatrix} \begin{pmatrix} \underline{x}_1^o \\ \underline{x}_2^o \end{pmatrix}$$

We can use <u>this</u> marginality of the <u>ordinary</u> linear problem to prove the existence of marginal values for the fractional programme with linear constraints.

Using the CC Equivalence given in Chapter 1 we have

$$F1 \equiv F2$$

where

$$F1 = \max \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta} \quad \text{s.t.} \quad \underline{A}.\underline{x} \le \underline{b} \qquad \underline{x} \ge \underline{0}$$

$$F2 = \max \underline{cy} + \alpha t \quad \text{s.t.} \quad \underline{A}.\underline{y} - \underline{b}t \le \underline{0}$$

$$\underline{d}.\underline{y} + \underline{\beta}t = 1 \qquad \underline{y},\ t \ge 0$$

(providing $\underline{d}.\underline{x} + \beta$ is positive for all $\underline{x}$ s.t. $\underline{A}.\underline{x} \le \underline{b}$)

Now F2 is a linear programming problem of the second type, and we can use theorems $I^*$ and $II^*$ to deduce that marginal values exist with respect to changes in $\underline{b}$, i.e. $'\frac{\partial F2}{\partial b_i}'$ exists, and by equivalence $'\frac{\partial F_1}{\partial b_{i+}}' = '\frac{\partial F_2}{\partial b_{i+}}'$. The form exists due to $I^*$ and $II^*$ above, using $\tilde{\underline{H}} = \begin{pmatrix} \underline{0} & \hat{\underline{b}} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} \end{pmatrix}$

Thus for the linear fractional programme we have an existence theorem for the right hand derivatives with respect to each of the resources, the existence of this derivative depending only on the boundedness of the optimal solution set to the problem.

Note: $\frac{\partial F_2}{\partial b_{i+}}$ exists as a marginal value of the LP (in terms enunciated by Williams). $\frac{\partial F_1}{\partial b_{i+}}$ exists by equivalence, and since $b_i$ is a right hand side variable for F1 we have the existence of the dual prices . We are now in a position to discuss the dual evaluators in terms of marginal returns and losses. This we shall do following the strict economic interpretation, but first we must discuss the more immediate implications of non-linearity. (Although the model we will later present is one with linear constraints, (a considerable computational simplification,) we will discuss the general case with convex $g_i(\underline{x})$.)

## 4.3 The Economic Interpretation of the Non-Linear Dual in Fractional Programming

### 4.3.1 The KT Conditions

Under the conditions stated in Section 4.2, the dual evaluators for the fractional programme exist, and are equivalent to the marginal value of resources, where there are diminishing returns to scale.

As in (6) we could give an economic interpretation to the Kuhn Tucker Conditions (shown in Section 4.1) for such cases of diminishing returns.

The KT Conditions are:

i $\nabla_x f(\underline{x}^o) - \nabla_x \sum_k u_k^o \cdot g_k(\underline{x}^o) \leq \underline{0}$

ii $\underline{x}^o \cdot \left[\nabla_x f(\underline{x}^o) - \nabla_x \sum_k u_k^o \cdot g_k(\underline{x}^o)\right] = 0$

iii $\sum_k u_k^o \cdot g_k(\underline{x}^o) = 0$

iv $g_k(\underline{x}^o) \leq 0 \qquad k = 1 \ldots m$

v $u_k^o \geq 0 \qquad k = 1 \ldots m$

Assume $u_i^o$ is the marginal value of the i'th resource, $\frac{\partial f(\underline{x}^o)}{\partial x_i}$ is the marginal profit yield of $x_i$.

$\frac{\partial g_k(\underline{x}^o)}{\partial x_i}$ is the amount of the k'th input required to produce the next unit of $x_i$ (at the optimum) - it is the marginal input requirement for $x_i$.

$\sum u_k^o \cdot \frac{\partial g_k(\underline{x}^o)}{\partial x_i}$ is the total 'value' of resources required to produce the next unit of $x_i$ (at the optimum) hence the set of constraints i imply that the net rate of increase of value of objective is less or equal to zero.

ii implies that, if the net rate of increase of value of the o.f. is negative for any $x_i^o$, that $x_i^o$ is at its lower limit - zero.

If $x_i^o$ is positive, we must have $\frac{\partial f(\underline{x}^o)}{\partial x_i} - \Sigma u_k^o . \frac{\partial g_k(\underline{x}^o)}{\partial x_i} = 0$ implying that a further increase in $x_i^o$ will not increase the value of $f(\underline{x}^o)$

The condition iii provides the concept of free goods: A free good is one whose increase of supply will not increase the possibilities of increasing the objective function. If a particular constraint $g_k(\underline{x}^o)$ is strictly negative,

$$u_k^o \cdot g_k(\underline{x}^o) = 0 \Rightarrow u_k^o = 0,$$

i.e. it is a free good. Thus, if a resource $b_k$ is a free good, it has a zero marginal (accounting) value.

4.3.2 Economic Rent

Economic rent $R^*$ is defined as the rent payable to the owner of scarce facilities without which a company cannot operate (58). This will be the difference between the total yield $f(\underline{x}^*)$ and the marginal value of all inputs, (at the optimum).

i.e. $R^* = f^* - \pi^* = f(\underline{x}^*) - \sum_k u_k^o \cdot g_k(\underline{x}^o)$

This term appears in the non-linear dual problem in (6).

In LFP, where the constraints are linear, we will show how the dual evaluators, the objective function, and the economic rent vary with changes of resource availability.

4.3.3 Summary

We have thus far shown how the application of the price concept to the dual variables of fractional programming is weakened because of two factors:

i. the existence proofs for the partial derivatives $\frac{\partial\pi}{\partial b_{k+}}$, $\frac{\partial\pi}{\partial b_{k-}}$, $\frac{\partial\pi}{\partial b_k}$ are less powerful, and

ii. there is the possibility that returns to scale may not be diminishing.

Where the partial derivatives do exist, and the returns to scale do diminish, we can allow a full economic interpretation of the Kuhn Tucker Conditions. In the remaining sections we will consider the case with linear constraints, and show which cases do, in fact, engender a coherent pricing system with diminishing returns to scale.

## 4.4 Association between the duals of the original form and the CC form

### 4.4.1 An Algebraic Approach

The initial problem is

$$\max \quad f(\underline{x}) = \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta}$$

$$\text{s.t.} \quad \underline{A}.\underline{x} \leq \underline{b} \qquad (4.14)$$

Let $\underline{v}_F$ = dual evaluators of the original fractional programme, (4.14),

$(\underline{v}_{CC}, v)$ = dual evaluators of the CC form, with v referring to the denominator row.

We will show that $\underline{v}_F = t^*.\underline{v}_{CC}$. Similar results for 'reduced costs' are given in Appendix 4.2.

The dual of (4.14) is

$$\min \quad \underline{v}_F.\underline{b}$$

$$\text{s.t.} \quad \underline{v}_F.\underline{A} \geq \left[\frac{\partial f}{\partial \underline{x}}\right]_{\underline{x}=\underline{x}^*} \qquad (4.15)$$

Let $\hat{\underline{c}} = \left[\frac{\partial f}{\partial \underline{x}}\right]_{\underline{x}=\underline{x}^*}$ (with rearrangement where necessary)

The dual of (4.15) is

$$\max \quad \hat{\underline{c}}.\underline{x}$$

$$\text{s.t.} \quad \underline{A}.\underline{x} \leq \underline{b} \qquad (4.16)$$

By the equivalence of (4.14) and (4.16) the dual evaluators of (4.16) will be $\underline{v}_F$.

Wagner and Yuan (85) have shown the association between the optimal inverse basis of the CC form and the 'original' inverse basis.

Let the original basis be $\underline{B}$.

Then the CC basis is $\underline{B}^*$ where

$$\underline{B}^* = \begin{pmatrix} \underline{B} & -\underline{b} \\ \underline{d}^* & \beta \end{pmatrix}$$

Now let $(\underline{B}^*)^{-1} = \begin{pmatrix} \underline{M}_{11} & \underline{M}_{12} \\ \underline{M}_{21} & \underline{M}_{22} \end{pmatrix}$

corresponding to the $\underline{B}$, $-\underline{b}$, $\underline{d}^*$, $\beta$

They show that:

$$\begin{aligned} \underline{M}_{11} &= \underline{B}^{-1} - \underline{B}^{-1}.\underline{b}(\beta + \underline{d}^*.\underline{B}^{-1}.\underline{b})^{-1}\underline{d}^*.\underline{B}^{-1} \\ \underline{M}_{12} &= (\beta + \underline{d}^*.\underline{B}^{-1}.\underline{b})^{-1}.\underline{B}^{-1}.\underline{b} \\ \underline{M}_{21} &= -(\beta + \underline{d}^*.\underline{B}^{-1}.\underline{b})^{-1}\underline{d}^*.\underline{B}^{-1} \\ \underline{M}_{22} &= (\beta + \underline{d}^*.\underline{B}^{-1}.\underline{b})^{-1} \end{aligned} \qquad (4.17)$$

Now $\underline{B}^{-1}.\underline{b} = \underline{x}^*$

and $(\beta + \underline{d}^*.\underline{B}^{-1}.\underline{b})^{-1} = \dfrac{1}{(\underline{d}^*.\underline{x}^* + \beta)} = t^*$

therefore we can write (4.17) as:

$$\begin{aligned} \underline{M}_{11} &= \underline{B}^{-1} - \underline{x}^*.t^*.\underline{d}^*.\underline{B}^{-1} \\ \underline{M}_{12} &= t^*.\underline{x}^* \\ \underline{M}_{21} &= -t^*.(\underline{d}^*.\underline{B}^{-1}) \\ \underline{M}_{22} &= t^* \end{aligned} \qquad (4.18)$$

Now the dual evaluators are given by:

$$\underline{v}_F = \hat{\underline{c}}^*.\underline{B}^{-1}$$

$$(\underline{v}_{CC}, v) = (\underline{c}^*, \alpha).(\underline{B}^{*-1}) \qquad (4.19)$$

Using (4.18) we can write:

$$\underline{v}_{CC} = \underline{c}^* . \underline{M}_{11} + \alpha . \underline{M}_{21}$$

$$f(\underline{x}) = \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta}, \text{ and } t = \frac{1}{\underline{d}.\underline{x} + \beta}$$

therefore, $$\frac{\partial f}{\partial x_i} = \frac{c_i(\underline{d}.\underline{x} + \beta) - d_i(\underline{c}.\underline{x} + \alpha)}{(\underline{d}.\underline{x} + \beta)^2}$$

i.e. $\hat{\underline{c}}_i^* = (c_i - d_i.f^*).t^*$

Now $\underline{c}^*.\underline{M}_{11} = \underline{c}^*.\underline{B}^{-1} - \underline{c}^*.\underline{x}^*.t^*.\underline{d}^*.\underline{B}^{-1}$

therefore $\underline{v}_{CC} = (\underline{c}^* - \underline{c}^*.\underline{x}^*.t^*.\underline{d}^* - \alpha t^*.\underline{d}^*).\underline{B}^{-1}$

$$= (\underline{c}^* - (\underline{c}^*.\underline{x}^* + \alpha)t^*.\underline{d}^*).\underline{B}^{-1}$$

Hence $\underline{v}_{CC} = (\underline{c}^* - \underline{d}^*.f^*).\underline{B}^{-1}$

$\underline{v}_F = (\underline{c}^* - \underline{d}^*.f^*).\underline{B}^{-1}.t^*$

i.e. $\underline{v}_F = t^*.\underline{v}_{CC}$ Q.E.D. (4.20)

### 4.4.2 Variation of Marginal Values (dual evaluators) with Change of Resources: Returns to Scale

At the optimum point we know that the 'fractional' evaluators are 't' times the evaluators of CC formulation.

i.e. $\underline{v}_F = t^*.\underline{v}_{CC}$

Now at this point, $$t^* = \frac{1}{\underline{d}^*.\underline{x}^* + \beta} = \frac{1}{\underline{d}^*.\underline{B}^{-1}.\underline{b} + \beta}$$

where $\underline{B}^{-1}$ is the current inverse basis.

If some of the resources are allowed to vary, i.e. we allow a change $\delta\underline{b}$, this basis may still remain optimal (as in LP), assuming that the problem has no degeneracy.

Assuming that the basis has not changed, the evaluators of the rows of the CC formulation will not have changed; these are piecewise constant.

Thus for such a point $\hat{\underline{b}} = \underline{b} + \delta\underline{b}$,

we have $$\hat{\underline{v}}_F = \frac{1}{(\underline{d}^*.\underline{B}^{-1}.\hat{\underline{b}} + \beta)} \cdot \underline{v}_{CC}$$

We can predict what the marginal values will be up to the next basis change. The next basis change can be deduced by ranging the right hand side of the CC formulation or by parametric analysis on the last column of that tableau.

Thus we have nearly as much knowledge about the marginal values of the fractional programme as we have about the dual variables of the linear programme.

The elements of $(\underline{d}^*.\underline{B}^{-1})$ will determine whether the marginal value increases or decreases with $b_k$:

$$\hat{\underline{v}}_F \sim \underline{v}_{CC} \cdot \frac{1}{(\sum_i \theta_i b_i + \beta)} \quad \text{where } \theta_i = (\underline{d}^*.\underline{B}^{-1})_i \tag{4.21}$$

and if $\theta_i > 0$ $\exists$ diminishing returns to scale.

Thus the $\underline{d}^*$- vector plays a vital role in determining whether prices exist or not.

$$\frac{\partial \underline{v}_F}{\partial b_i} = -\theta_i \frac{1}{(\Sigma\theta_i b_i + \beta)^2} \cdot \underline{v}_{CC}$$

and $$\frac{\partial \underline{v}_F}{\partial b_i} \leq 0 \Leftrightarrow \theta_i \geq 0 \tag{4.22}$$

Similarly we can fully determine the value of the objective function, the total value of input factors, and the economic rent, and their marginal rates of change:

$$\hat{f} = \frac{\underline{c}^*.\hat{\underline{x}} + \alpha}{\underline{d}^*.\hat{\underline{x}} + \beta} = \frac{\underline{c}^*.\underline{B}^{-1}.\hat{\underline{b}} + \alpha}{\underline{d}^*.\underline{B}^{-1}.\hat{\underline{b}} + \beta} \tag{4.23}$$

$$\hat{R} = \hat{f} - \hat{\pi} = \frac{\underline{c}^*.\underline{B}^{-1}.\hat{\underline{b}} + \alpha - \hat{\underline{b}}.\underline{v}_{CC}}{(\underline{d}^*.\underline{B}^{-1}.\hat{\underline{b}} + \beta)} \tag{2.24}$$

Let $\theta_i = (\underline{d}^*.\underline{B}^{-1})_i$

$\Phi_i = (\underline{c}^*.\underline{B}^{-1})_i$

$\mu_i = (\Phi_i - v_{CC_i})$

$$\hat{f} = \frac{\underline{\Phi}.\hat{\underline{b}} + \alpha}{\underline{\theta}.\hat{\underline{b}} + \beta}$$

$$\frac{\partial \hat{f}}{\partial b_i} = \frac{(\underline{\theta}.\hat{\underline{b}} + \beta)\Phi_i - (\underline{\Phi}.\hat{\underline{b}} + \alpha)\theta_i}{(\underline{\theta}.\hat{\underline{b}} + \beta)^2}$$

$$\frac{\partial \hat{f}}{\partial b_i} \geq 0 \Leftrightarrow \frac{\Phi_i}{\theta_i} \geq \frac{\underline{\Phi}.\hat{\underline{b}} + \alpha}{\underline{\theta}.\hat{\underline{b}} + \beta} = \hat{f} \quad (4.25)$$

Similarly

$$\hat{R} = \hat{f} - \hat{\pi} = \frac{\underline{\mu}.\hat{\underline{b}} + \alpha}{\underline{\theta}.\hat{\underline{b}} + \beta}$$

$$\frac{\partial \hat{R}}{\partial b_i} \geq 0 \Leftrightarrow \frac{\mu_i}{\theta_i} \geq \frac{\underline{\mu}.\hat{\underline{b}} + \alpha}{\underline{\theta}.\hat{\underline{b}} + \beta} = \hat{R} \quad (4.26)$$

A knowledge of the present inverse basis is sufficient to determine the marginal rates of increase of objective function and rent with increases in factor input. Thus, for the case of FP with linear constraints the returns to scale are determined by $(\underline{d}^*.\underline{B}^{-1})$ as follows:

If $\theta_i = (\underline{d}^*.\underline{B}^{-1})_i$, then

$\theta_i \geq 0 \; \forall i \Longrightarrow$ diminishing returns to scale (d.r.t.s)

and $\theta_i < 0$ some $i \Longrightarrow$ increasing returns to scale.

(4.27)

### 4.4.3 A Check via the CC Form

We have shown that the conditions for diminishing returns to scale are that:

$$\theta_i = (\underline{d}^*.\underline{B}^{-1})_i \geq 0$$

where $\underline{d}^*$ is the vector of the denominator entries corresponding to the basic variables. According to (4.17) we can derive the optimal inverse basis of the CC form in terms of $\underline{B}^{-1}$. $\underline{M}_{21}$ is given by $\underline{M}_{21} = -t^*.(\underline{d}^*.\underline{B}^{-1})$. Thus the condition that $\theta_i$ be positive is the same as requiring that the components of $\underline{M}_{21}$ be negative.

$t^*$ is positive by assumption, hence

$$\theta_i \geq 0 \Longleftrightarrow (\underline{M}_{21})_i \leq 0 \qquad (4.23)$$

Where the calculation has been made using the CC form, an inspection of the last row of the optimal inverse gives the conditions for diminishing returns to scale.

## 4.5 Conclusions

Although the economic interpretation of pricing cannot easily be applied to the general case of LP with nonlinear (convex) constraints, we have shown conditions under which the dual variables of the FP with linear constraints do have an economic interpretation.

In so doing we have used only the optimal inverse basis and the denominator, thus the coherence of the marginal pricing is easily checked.

Examples in Appendix 4.1 show that even simple FP problems can exhibit increasing returns to scale.

# Chapter 5 Decomposition of Linear Fractional Programmes

## 5.1 Decomposition and Decentralisation

5.1.1 The Linear Decomposition Principle of Dantzig and Wolfe (36), has found use in two applications of mathematical programming:

(i) it is a method of solving large programmes with special structure, namely

$$\max \Sigma \underline{c}_i . \underline{x}_i$$
$$\text{s.t.} \quad \underline{A}_i . \underline{x}_i \leq \underline{b}_i \qquad i = 1 \ldots m$$
$$\Sigma \, \underline{M}_i . \underline{x}_i \leq \underline{b}$$
$$\underline{x}_i \geq \underline{0}$$

(ii) it is also a method of formalising the planning process of a decentralised firm.

The importance of linear decomposition in the analysis of the decentralisation arises from its analogy with a 'transfer-price mechanism' for decentralised planning. In Sections 5.2 onwards we present two approaches to the decomposition of Linear Fractional Programmes, together with an analysis of the transfer prices generated in such applications.

### 5.1.2 Decentralisation and Transfer Pricing with Nonlinear Objectives

The economist's approach to decentralised planning has been characterised by an attempt to apply a "market clearing mechanism" (simple price/quantity relationships) to the decentralised firm, surmising that market adjustments within the firm will enable each division to act in a manner which is optimal both with respect to its own objectives and with respect to the aims of the organization. Thus, from the economic standpoint, the problem is a search for that set of prices - the transfer prices - which will equate supply and demand within the organization for each market. Arrow and

Hurwicz (4) have shown that gradient methods can be used to calculate such prices - but the method is one of infinite iteration. For cases where the objectives of the divisions and the technological constraints are linear, the Decomposition Principle provides a finite mechanism for such calculations. (See (36) and other references quoted in (23)).

Considering the following problem:

$$\min f(\underline{u}) = \phi_1(\underline{u}_1) + \phi_2(\underline{u}_2) + \ldots \phi_n(\underline{u}_n)$$

$$\begin{array}{llll} \text{s.t.} & \underline{B}_1(\underline{u}_1) & & \le \underline{b}_1 \\ & & \underline{B}_2(\underline{u}_2) & \le \underline{b}_2 \\ & & \quad\ddots & \vdots \\ & & \qquad \underline{B}_n(\underline{u}_n) & \le \underline{b}_n \\ \text{and} & \underline{C}_1(\underline{u}_1) + \underline{C}_2(\underline{u}_2) + \ldots & \underline{C}_n(\underline{u}_n) & \ge \underline{0} \end{array}$$

where $f(\underline{u})$ is the objective function for the corporation, $\left\{\underline{B}_i(\underline{u}_i) \le \underline{b}_i\right\}$ are the sets of divisional constraints, and $\Sigma \underline{C}_i(\underline{u}_i) \ge \underline{0}$ are the corporate constraints,

Charnes, Cooper and Kortanek (23) have shown that decentralised planning by price alone, where the objective function is separable, is possible only if each $\phi_k$ is strictly convex. Other models require more information during the planning process than can be provided by a pricing system of penalties and subsidies.

Whinston (90, 91) discusses the problem of transfer prices via the Kuhn Tucker Conditions associated with the optimal allocation of resources in the firm. He considers models of the form:

$$\max \sum_i f_i(\underline{x}_i)$$

$$\text{s.t.} \sum_i g_{ij}(\underline{x}_i) \le k_j$$

$$\underline{x}_i \ge \underline{0}$$

and concludes that, from an interpretation of the KT Conditions, namely:

$$\frac{\partial f_i}{\partial x_i} - \sum_j \lambda_j^o \cdot \frac{\partial g_{ij}}{\partial x_i} \leq 0$$

$$x_i^o \cdot \left[\frac{\partial f_i}{\partial x_i} - \sum_j \lambda_j^o \cdot \frac{\partial g_{ij}}{\partial x_i}\right] = 0$$

$$\sum_i g_{ij}(\underline{x}^o) \leq k_j$$

$$\lambda_j^o \cdot (\sum_i g_{ij}(\underline{x}^o) - k_j) = 0$$

$$x_i^o,\ \lambda_j^o \geq 0 \qquad \text{for all } i,j$$

a pricing-correction mechanism can be derived. These adjustments are:

$$\frac{d\lambda_j}{dt} = \begin{cases} 0 \text{ if } \lambda_j = 0 \text{ and } k_j - \sum_i g_{ij}(x_i) > 0 \\ \delta\left\{\sum_i g_{ij}(x_j) - k_j\right\} \text{ otherwise} \end{cases}$$

This analysis is similar to that of Koopmans (56), using the 'custodian' price setting technique.

Whinston further shows that in the case of externalities in the objective function (indicating a technological or economic dependence between divisions), other information such as lower bounds on production, may be required to promote optimal divisional behaviour, e.g. for the objective function:

$$\max \quad f_1(\underline{x}_1, \underline{x}_2) + f_2(\underline{x}_2) + \ldots \quad f_n(\underline{x}_n)$$

$$\text{subject to } \sum_i g_{ij}(\underline{x}_i) \leq k_j \qquad j = 1 \ldots m$$

$$\underline{x}_i \geq \underline{0}$$

a gaming situation develops between divisions one and two. Price guides are no longer sufficient as a mechanism for motivating optimal behaviour.

Hass (45) considers the decomposition of a quadratic programme:

$$\max \quad \pi(\underline{X}, \underline{Y}) = \underline{P}.\underline{X} + \underline{Q}.\underline{Y} + \underline{Z}.\underline{\emptyset}.\underline{Z}$$

$$\text{s.t.} \quad \underline{C}.\underline{Z} \leq \underline{R}$$

$$f_i(\underline{Y}) \leq s_i \qquad i = 1 \ldots a$$

$$g_i(\underline{X}) \leq t_i \qquad i = 1 \ldots a$$

$$\underline{X}, \underline{Y} \geq \underline{0}$$

where: $\underline{P}$, $\underline{X}$ are m vectors

$\underline{Q}$, $\underline{Y}$ are n vectors

$\underline{Z} = (\underline{X}, \underline{Y})$

$\underline{\emptyset}$ is (m + n) by (m + n), symmetric and negative definite

$\underline{f}$, $\underline{g}$ are convex, etc.

$\underline{C}$, $\underline{R}$ have dimension k by (m + n) and k respectively.

He partitions $\underline{\emptyset}$ and $\underline{C}$ as follows:

$$\underline{\emptyset} = \begin{pmatrix} \underline{\emptyset}_1 & \underline{\emptyset}_3 \\ \underline{\emptyset}_3 & \underline{\emptyset}_2 \end{pmatrix} \begin{matrix} m \\ n \end{matrix} \qquad \underline{C} = \begin{pmatrix} \underline{C}_1 & \underline{C}_2 \end{pmatrix} k$$

$$\qquad\; m \quad\; n \qquad\qquad\qquad\qquad m \quad n$$

and shows that the quadratic decomposition is effected by supplying correction factors to $\underline{P}$, $\underline{Q}$ according to the optimal solution of the present 'executive programme'.

If $\underline{\lambda} = (\lambda_1, \ldots \lambda_k)$ are the (provisional) marginal costs (or revenues) associated with corporate resources, and $\hat{\underline{X}}$, $\hat{\underline{Y}}$ are the present optimal solutions for $\underline{X}$, $\underline{Y}$ in the executive programme, the amendments are:

$$\underline{P} \rightarrow \underline{P} + \underline{\emptyset}_1'.\underline{X} - \underline{C}_1'.\underline{\lambda} + 2\, \underline{\emptyset}_3'.\hat{\underline{Y}} \qquad \text{for div. 1}$$

$$\underline{Q} \rightarrow \underline{Q} + \underline{\emptyset}_2'.\underline{Y} - \underline{C}_2'.\underline{\lambda} + 2\, \underline{\emptyset}_3'.\hat{\underline{X}} \qquad \text{for div. 2}$$

In this case, not only the prices ($\lambda_i$) but also the inter-divisional dependencies $\underline{\emptyset}_3'.\hat{\underline{X}}$ and $\underline{\emptyset}_3'.\hat{\underline{Y}}$ are being given to the divisions.

Hass shows that this is equivalent to a search for 'efficient' functions, rather than 'efficient' prices - these functions are shown to be linear, e.g. demand curves of the form a - bx.

The interest of Hass's work lies in the possible inclusion of price dependence between divisions; the profit for an activity $X_i$ may depend on $\underline{Y}$ - this is reflected in the $\underline{\emptyset}$ of the total objective, and the profit amendments $\underline{\emptyset}_3' . \hat{\underline{X}}$ and $\underline{\emptyset}_3' . \hat{\underline{Y}}$ of the revised divisional programme.

### 5.1.3 Decomposition with Fractional Objective Functions

The decomposition of a linear fractional programme is complicated by the non-separability of the objective function. For the programme:

$$\max \frac{\underline{c}_1\underline{x}_1 + \underline{c}_2\underline{x}_2}{\underline{d}_1\underline{x}_1 + \underline{d}_2\underline{x}_2 + \beta}$$

$$\begin{array}{llcl} \text{s.t.} \quad \text{(i)} & \underline{A}_1\underline{x}_1 & \leq & \underline{b}_1 \\ \text{(ii)} & \qquad\quad \underline{A}_2\underline{x}_2 & \leq & \underline{b}_2 \\ \text{(iii)} & \underline{M}_1\underline{x}_1 + \underline{M}_2\underline{x}_2 & \leq & \underline{b} \\ & \qquad\quad \underline{x}_1, \underline{x}_2 & \geq & \underline{0} \end{array} \qquad (5.1)$$

where $\underline{d}_1\underline{x}_1 + \underline{d}_2\underline{x}_2 + \beta > 0$ and bounded for all feasible $(\underline{x}_1, \underline{x}_2)$, no 'divisional' objective function presents itself. The denominator acts as an externality between the divisions.

We will consider two approaches to the problem:

i. The linear approach: Using the Charnes and Cooper Equivalence, divisions will be given linear objective functions. The form of the master programme will have slight differences from that of the ordinary linear decomposition.

ii. The fractional approach: The objective function of (5.1) will be split into two parts, each division will be asked to optimise a function derived from marginal values of activities at the previous executive optimum, subsidised or penalised with the usage of corporate resources at these marginal values. An equivalence is drawn between the two approaches.

(A decomposition principle has been anunciated by Chadda (12) but this method does not allow for iterative planning processes.)

## 5.2 The Linear Approach

### 5.2.1 The Charnes and Cooper Form

Consider the CC form of (5.1). This is:

$$\begin{aligned}
&\max && \underline{c}_1\underline{y}_1 + \underline{c}_2\underline{y}_2 \\
&\text{s.t.}\ \text{(i)} && \underline{A}_1\underline{y}_1 - \underline{b}_1 t \le \underline{0} \\
&\text{(ii)} && \underline{A}_2\underline{y}_2 - \underline{b}_2 t \le \underline{0} \\
&\text{(iii)} && \underline{M}_1\underline{y}_1 + \underline{M}_2\underline{y}_2 - \underline{b}t \le \underline{0} \\
&\text{(iv)} && \underline{d}_1\underline{y}_1 + \underline{d}_2\underline{y}_2 + \beta t = 1 \\
& && \underline{y}_1, \underline{y}_2 \ge \underline{0}, \ t > 0
\end{aligned} \qquad (5.2)$$

Although the constraint set of (5.1) was in decomposable form, that of (5.2) contains a further dependence between the divisions one and two; the objective function, however, is now separable, and linear. By the initial conditions on $\underline{d}_1\underline{x}_1 + \underline{d}_2\underline{x}_2 + \beta$, t is always positive and non-zero. Rewrite (5.2), introducing two variables $t_1$, $t_2$ as follows.

$$\begin{aligned}
&\max && \underline{c}_1\underline{y}_1 + \underline{c}_2\underline{y}_2 \\
&\text{s.t.}\ \text{(i)} && \underline{A}_1\underline{y}_1 - \underline{b}_1 t_1 \le \underline{0} \\
&\text{(ii)} && \underline{A}_2\underline{y}_2 - \underline{b}_2 t_2 \le \underline{0} \\
&\text{(iii)} && \underline{M}_1\underline{y}_1 + \underline{M}_2\underline{y}_2 - \underline{b}t \le \underline{0} \\
&\text{(iv)} && \underline{d}_1\underline{y}_1 + \underline{d}_2\underline{y}_2 - \beta t = 1 \\
&\text{(v)} && t_1 - t = 0 \\
& && t_2 - t = 0 \\
& && t_1, t_2, \underline{y}_1, \underline{y}_2 \ge 0, \ t > 0
\end{aligned} \qquad (5.3)$$

The coefficient matrix of (5.3) is

$$\begin{pmatrix} \mathbf{A}_1 & -\mathbf{b}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & -\mathbf{b}_2 & \mathbf{0} \\ \mathbf{M}_1 & \mathbf{0} & \mathbf{M}_2 & \mathbf{0} & -\mathbf{b} \\ \mathbf{d}_1 & 0 & \mathbf{d}_2 & 0 & \beta \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{5.4}$$

Clearly this is now in linear decomposition form.

The divisional programmes are:

$$\max f_\alpha = \mathbf{c}_\alpha \cdot \mathbf{y}_\alpha \qquad \text{s.t.} \quad \mathbf{A}_\alpha \cdot \mathbf{y}_\alpha - \mathbf{b}_\alpha \cdot t_\alpha \le \mathbf{0}$$
$$\mathbf{y}_\alpha,\ t_\alpha \ge 0$$
$$\alpha = 1,\ 2 \tag{5.5}$$

These are unbounded in $t_\alpha$, thus according to Dantzig and Wolfe (36) the master program need only consider non-negative combinations of divisional programmes, (and not convex combinations of such programmes).

Assume divisions one and two have submitted programmes $\{\mathbf{y}_\alpha^i,\ t_\alpha^i\}$, $i = 1 \ldots k_\alpha$, $\alpha = 1,\ 2$

At this point the master programme will be:

$$\max \quad \sum_i \mu^i \cdot \mathbf{c}_1 \cdot \mathbf{y}_1^i + \sum_j \nu^j \cdot \mathbf{c}_2 \cdot \mathbf{y}_2^j$$

$$\text{s.t.} \quad \text{(i)} \quad \sum_i \mu^i \cdot \mathbf{M}_1 \cdot \mathbf{y}_1^i + \sum_j \nu^j \cdot \mathbf{M}_2 \cdot \mathbf{y}_2^j - \mathbf{b}t \le \mathbf{0}$$

$$\text{(ii)} \quad \sum_i \mu^i \cdot \mathbf{d}_1 \cdot \mathbf{y}_1^i + \sum_j \nu^j \cdot \mathbf{d}_2 \cdot \mathbf{y}_2^j + \beta t = 1$$

$$\text{(iii)} \quad \sum_i \mu^i \cdot t_1^i - t = 0$$

$$\text{(iv)} \quad \sum_j \nu^j \cdot t_2^j - t = 0$$

$$\mu^i,\ \nu^j \ge 0 \tag{5.6}$$

where sums are for: $i = 1 \ldots k_1$

$j = 1 \ldots k_2$

Let the dual evaluators at the optimum of (5.6) be

$(\underline{\pi}, \pi_d, \pi_1, \pi_2)$ corresponding to rows (i), (ii), (iii) and (iv). According to Baumol and Fabian (8), we may interpret these as 'provisional prices' and use them to form subsidy/penalty revisions for the divisional programmes.

The decomposition process is as follows:

(a) assume $k_\alpha$ solutions from division $\alpha$, $(\alpha = 1, 2)$.

(b) form master programme (5.6) and optimise.

(c) revise objective functions for divisions in exactly the same manner as for linear decomposition, but omit the last two rows. The new divisional objective functions are:

$$\sum_i \left\{ c_{\alpha_i} - \underline{\pi}.\underline{M}_{\alpha_i} - \pi_d . d_{\alpha_i} \right\} y_{\alpha_i} \qquad \alpha = 1, 2$$

The 'denominator row' is considered to be a 'corporate resource'. ($\underline{M}_{\alpha_i}$ is the i'th column of $\underline{M}_\alpha$)

(d) solve the divisional programmes and test for optimality. Let the new divisional optima be $\hat{f}_\alpha$. If $\pi_\alpha \geq \hat{f}_\alpha$ $(\alpha = 1, 2)$, the present solution is optimal. (This condition is proved later). If $\pi_\alpha < \hat{f}_\alpha$ for $\alpha = 1$ or $2$, update $k_\alpha$ and go to step (b).

5.2.2 Bounding the divisional subprogrammes

In order to obtain bounded solutions to (5.5) it may be necessary to put an arbitrary bound on t. (Let this be $t^o$.) (In the original programme (5.2), t is always bounded since $t = \underline{d}_1\underline{x}_1 + \underline{d}_2\underline{x}_2 + \beta$, and $\underline{d}_1\underline{x}_1 + \underline{d}_2\underline{x}_2 + \beta > 0$ for all feasible $(\underline{x}_1\underline{x}_2)$—by assumption).

Lemma: For the LP

$$\begin{aligned} \max \quad & f = \underline{c}.\underline{y} \\ \text{s.t.} \quad & \underline{A}.\underline{y} - \underline{b}.t \leq \underline{0} \\ & t \leq t^o \\ & \underline{b}, \underline{y}, t \geq 0, \; t^o > 0 \end{aligned} \qquad (5.7)$$

if a (bounded) optimum exists, say $f^*$, then:

$$t^* = t^o, \text{ or } f^* = 0 \text{ and } 0 \le t^* \le t^o.$$

Proof: Suppose a (bounded) optimum exists. Let $\pi$ be the dual evaluator of the last row of (5.7).

Then, by LP duality

$$f^* = \pi_t . t^o \qquad \text{with } \pi_t \ge 0 \tag{5.8}$$

But $t^o > 0$, therefore:

either $f^* = 0 \Rightarrow \pi_t = 0$; the last row of (5.7) is slack

or $f^* > 0 \Rightarrow \pi_t > 0$; the last row of (5.7) is tight $\Rightarrow t^* = t^o$.

Q.E.D.

Corollary: Equivalence: For $0 < t < \infty$, and $\underline{y} = \underline{x}.t$, the programmes:

$$\max f = \underline{c}.\underline{y} \quad \text{s.t. } \underline{A}.\underline{y} - \underline{b}t \le \underline{0} \qquad t \le t^o \tag{5.9}$$

$$\max \quad \underline{c}.\underline{x} \quad \text{s.t. } \underline{A}.\underline{x} \le \underline{b} \tag{5.10}$$

are equivalent if bounded solutions exist for (5.10) for which $f^* > 0$.

Proof: $f^* > 0 \Rightarrow t^* = t^o$.

Equivalence by division.

In this chapter we will assume that divisions only tender programmes of strictly positive value. We also assume that 't' is strictly positive (and bounded). Thus we may amend the decomposition method of (5.21) as follows:

(a)-(c) remain the same

(d) solve the equivalent divisional subprogrammes

$$\max \sum_i (c_{\alpha_i} - \underline{\pi}.M_{\alpha_i} - \pi_d . d_{\alpha_i}) x_{\alpha_i}$$

$$\text{s.t. } \underline{A}_\alpha . \underline{x}_\alpha \le \underline{b}_\alpha$$

$$\underline{x}_\alpha \ge \underline{0} \tag{5.11}$$

(e) assume $t^*$ is the optimal value of t in the master programme about to be solved in step (b). This is permissible because any arbitrary $t^*$ may be chosen.

(f) go to (b), i.e. solve (5.6)

N.B. Assumption (e) alters the form of rows (iii) and (iv) These are now:

$$\left.\begin{array}{l}\sum_i \mu^i - 1 = 0 \\ \sum_j \nu^j - 1 = 0\end{array}\right\} \quad \text{i.e. convexity constraints}$$

5.2.3 The Optimality Criterion

The optimality criterion cited in (d) now becomes clear; it is identical in application with that of the ordinary linear decomposition:

$\hat{f}_\alpha$ is the net profit contribution of the new solution from division $\alpha$

$\pi_\alpha$ is the relative marginal profitability of a transfer of some company resources to division $\alpha$.(See (8), page 13.) For optimality $\pi_\alpha \geq \hat{f}_\alpha$ .

## 5.3 The Fractional Approach

### 5.3.1 The Executive Programme

Assume that, in accordance with step (d) of Section 5.2.2, the divisions have tendered the plans

$$\{X^i_\alpha\} \quad i = 1 \dots k_\alpha , \quad \alpha = 1, 2$$

Assume that the corporate management now wish to use these plans to form a global optimum; the method of forming the executive programme corresponds with the linear decomposition approach.

The executive programme is:

$$\max \quad \frac{\Sigma\, \bar{\mu}^i\, \underline{c}_1.\underline{X}_1^i \;+\; \Sigma\, \bar{\nu}^j\, \underline{c}_2.\underline{X}_2^j}{\Sigma\, \bar{\mu}^i\, \underline{d}_1.\underline{X}_1^i \;+\; \Sigma\, \bar{\nu}^j\, \underline{d}_2.\underline{X}_2^j + \beta}$$

$$\begin{aligned} \text{s.t.} \quad & \text{(i)}\ \Sigma\, \bar{\mu}^i\, \underline{M}_1.\underline{X}_1^i + \Sigma\, \bar{\nu}^j\, \underline{M}_2.\underline{X}_2^j \le \underline{b} \\ & \text{(ii)}\ \Sigma\, \bar{\mu}^i = 1 \\ & \text{(iii)}\ \Sigma\, \bar{\nu}^j = 1 \end{aligned} \qquad (5.12)$$

(where the sums over i and j are as (5.6)), $\bar{\mu}^i, \bar{\nu}^j \ge 0$

Equations (ii) and (iii), the convex combinations, are required to maintain feasibility.

The CC form of (5.12) is

$$\begin{aligned} \max \quad & \Sigma\, \alpha^i.\underline{c}_1.\underline{X}_1^i + \Sigma\, \gamma^j.\underline{c}_2.\underline{X}_2^j \\ \text{s.t.} \quad & \text{(i)}\ \Sigma\, \alpha^i.\underline{M}_1.\underline{X}_1^i + \Sigma\, \gamma^j.\underline{M}_2.\underline{X}_2^j - \underline{b}t \le 0 \\ & \text{(ii)}\ \Sigma\, \alpha^i.\underline{d}_1.\underline{X}_1^i + \Sigma\, \gamma^j.\underline{d}_2.\underline{X}_2^j + \beta t = 1 \\ & \text{(iii)}\ \Sigma\, \alpha^i - t = 0 \\ & \text{(iv)}\ \Sigma\, \gamma^j - t = 0 \\ & \alpha^i,\ \gamma^j \ge 0 \end{aligned} \qquad (5.13)$$

where the transformation

$$\left.\begin{aligned} t.\underline{\bar{\mu}} &= \underline{\alpha} \\ t.\underline{\bar{\nu}} &= \underline{\gamma} \end{aligned}\right\} \text{ has been applied, } t > 0.$$

<u>Lemma:</u> The optimal solution vectors to (5.13) and (5.6) differ only by the scale parameter $t^*$ applied to the unbounded solutions to (5.7) in Section 5.2.2.

<u>Proof:</u> Re-write the activities and constraints in (5.13) as:

$$\max \quad \Sigma\left(\frac{\alpha^i}{t^*}\right).\underline{c}_i(\underline{x}^i.t^*) \quad \text{s.t. } \ldots\ldots\ldots$$

The $(\underline{x}^i.t^*)$ are the same as the $(\underline{y}^i)$. Thus the activities $\frac{\alpha^i}{t^*}$ and $\mu^i$ are identical,

$$\text{i.e.} \quad \alpha^i = t^*.\mu^i \qquad (5.14)$$

<u>Corollary:</u> The dual evaluators of both (5.6) and (5.13) are identical.

Proof: Given two problems

$$\max \ \underline{c}.\underline{x} \qquad \text{s.t.} \quad \underline{A}.\underline{x} \leq \underline{b}$$

and

$$\max \ \frac{\underline{c}.k\underline{x}}{k} \qquad \text{s.t.} \quad \frac{\underline{A}}{k}.k\underline{x} \leq \underline{b}$$

the dual evaluations are identical, if k is non-zero and constant.

For $k = t^*$, the above result follows.

5.3.2 The Fractional Algorithm

Let $\overline{\pi}$, $\overline{\pi}_1$, $\overline{\pi}_2$, be the dual evaluators of rows (i), (ii) and (iii) at the optimum, of (5.12).

The proposed algorithm is:

(a) assume $k_\alpha$ solutions from division $\alpha$, ($\alpha = 1, 2$).

(b) form the executive programme (5.12) and optimise.

Let $\underline{x} = (\underline{x}_1, \underline{x}_2)$ be the 'optimal' programme derived.

(c) derive the marginal values of production for each variable $\underline{x}_{1_i}$, $\underline{x}_{2_j}$ at the present solution levels, i.e. form the vector $\left[\frac{\partial f}{\partial x_{\alpha_i}} - \overline{\pi}.\underline{M}_{\alpha_i}\right]$

(d) present each division with these new marginal figures and request optimisation with respect to these new (linear) objectives.

(e) test for optimality with new divisional solutions $\hat{f}_\alpha$.

If $\overline{\pi}_\alpha > \hat{f}_\alpha$ ($\alpha = 1, 2$) the present solution to (b) is optimal.

If $\overline{\pi}_\alpha < \hat{f}_\alpha$ ($\alpha = 1$ or $2$), update $k_\alpha$ and go to step (b).

5.3.3 Comments on Algorithm

As will be shown in Section 5.4, the two approaches to the decomposition of (5.1) are identical apart from constant factors at each level of updating the master programmes.

The linear method stresses the planning approach of treating the denominator as a corporate resource that divisions are 'allowed to use'. This approach also makes it quite clear to divisions that a penalty/subsidy process is being used.

In the second algorithm, the emphasis on the fractional nature of the problem is maintained, by concentrating on the net marginal increase to a fractional objective function.

The optimality condition (e) of Section 5.3.2 follows from the associations derived in Section 5.4 and the optimality conditions for the linear approach.

## 5.4 The Association between the Two Algorithms

### 5.4.1 The dual evaluators of the master programmes

Let the optimal value of the denominator of (5.12) be $\hat{d}$, and let $\hat{t} = \hat{d}^{-1}$.

By fractional programming duality, and by the lemma of Section 5.3.1,

$$\hat{t}(\underline{\pi}, \pi_1, \pi_2) = (\underline{\bar{\pi}}, \bar{\pi}_1, \bar{\pi}_2) \qquad (5.16)$$

### 5.4.2 The association between the revised divisional objective functions

Assume that divisions 1 and 2 have submitted $k_\alpha$ propositions ($\alpha = 1, 2$).

According to the linear algorithm, the objective function for division $\alpha$ is revised to

$$\sum_i \left\{c_{\alpha_i} - \underline{\pi}.\underline{M}_{\alpha_i} - \pi_d . d_{\alpha_i}\right\} y_{\alpha_i} \qquad \alpha = 1, 2$$

$$\text{or} \quad \sum_i \left\{c_{\alpha_i} - \underline{\pi}.\underline{M}_{\alpha_i} - \pi_d . d_{\alpha_i}\right\} x_{\alpha_i} \qquad \alpha = 1, 2 \qquad (5.17)$$

in the revised version.

According to (5.15) of the fractional approach of Section 5.3, the divisional objective is a vector whose i'th component is

$$\left[\frac{\partial f}{\partial x_{\alpha_i}}\right]_{\underline{x} = \hat{\underline{x}}} - \underline{\pi}.\underline{M}_{\alpha_i} \qquad (5.18)$$

Now $$\left[\frac{\partial f}{\partial x_{\alpha_i}}\right]_{\underline{x} = \hat{\underline{x}}} = (c_{\alpha_i} - \hat{f}.d_{\alpha_i}).\hat{t}$$

But, by linear and fractional duality on (5.6) and (5.12)

$$\hat{f} = \pi_d \qquad (5.19)$$

Using (5.19) and (5.16) we can write (5.18) as

$$(c_{\alpha_i} - \pi_d.d_{\alpha_i}).\hat{t} - \hat{t}\ \underline{\pi}.\underline{M}_{\alpha_i}$$

$$= \hat{t}\left\{c_{\alpha_i} - \underline{\pi}.\underline{M}_{\alpha_i} - \pi_d.d_{\alpha_i}\right\} \qquad (5.20)$$

Comparing (5.17) and (5.20) we see that at the $k_\alpha$'th stage the objective functions for divisions $\alpha$ are in effect the same, whichever algorithm is used, the difference being a scalar multiplier.

## 5.5 The Optimal Dual Solution

### 5.5.1 Introduction

Walker (5) has shown that for linear decomposition, the final tableaux of the executive and divisional programmes provide, not only the primal solution vector, but also the full dual evaluation. The final executive programme gives the dual evaluators of the rows of the 'executive' section of the initial tableau, whilst the derived divisional programmes give the dual evaluation of their respective rows.

Thus for the problem:

$$\begin{array}{llll} \max & \underline{c}_1\underline{\lambda}_1 + \underline{c}_2\underline{\lambda}_2 & & \\ \text{s.t.} & \text{(i)}\ \underline{A}_1\underline{\lambda}_1 & \leq & \underline{b}_1 \\ & \text{(ii)}\ \qquad \underline{A}_2\underline{\lambda}_2 & \leq & \underline{b}_2 \\ & \text{(iii)}\ \underline{M}_1\underline{\lambda}_1 + \underline{M}_2\underline{\lambda}_2 & \leq & \underline{b} \\ & \qquad \underline{\lambda}_i & \geq & \underline{0} \end{array} \qquad (5.21)$$

the final executive programme gives the dual evaluators for rows (iii). Let these be $\underline{\pi}_b^*$.

The (dual) solutions to

$$\begin{aligned} \min \quad & \underline{\pi}_\alpha . \underline{b}_\alpha \\ \text{s.t.} \quad & \underline{\pi}_\alpha . \underline{A}_\alpha \geq \underline{c}_\alpha - \underline{\pi}_b^* . \underline{M}_\alpha \\ & \underline{\pi}_\alpha \geq \underline{0} \\ & \alpha = 1, 2 \end{aligned} \tag{5.22}$$

give the dual evaluators for (i) and (ii) according as $\alpha = 1, 2$. ((5.22) are simply the dual forms of the final divisional subprogrammes).

In the linear case, the $\{\underline{\pi}_b^*, \underline{\pi}_\alpha^*\}$ are automatically generated by the final iterations; Walker's proofs rely upon the linear duality theorems equating the optimal primal and dual objective functions, i.e. he relies on the fact that:

$$\underline{c}_1\underline{\lambda}_1^* + \underline{c}_2\underline{\lambda}_2^* = \sum_\alpha \underline{\pi}_\alpha^* \underline{b}_\alpha + \underline{\pi}_b^* . \underline{b} \tag{5.23}$$

Because of the non-linearity of the fractional objective function, (and the presence of economic rent in the dual objective), this equality is not upheld in the fractional case. The value of the primal objective function does not equal the total implied value of all resources, i.e.

$$\frac{\underline{c}_1\underline{\lambda}_1^* + \underline{c}_2\underline{\lambda}_2^*}{\underline{d}_1\underline{\lambda}_1^* + \underline{d}_2\underline{\lambda}_2^* + \beta} \neq \sum_\alpha \underline{\pi}_\alpha^* . \underline{b}_\alpha + \underline{\pi}_b^* \, \underline{b} \tag{5.24}$$

where the $\underline{\lambda}^*$'s and $\underline{\pi}^*$'s refer to the optimal solution to

$$\begin{aligned} \max \quad & \frac{\underline{c}_1\underline{\lambda}_1 + \underline{c}_2\underline{\lambda}_2}{\underline{d}_1\underline{\lambda}_1 + \underline{d}_2\underline{\lambda}_2 + \beta} = f(\underline{\lambda}) \\ \text{s.t.} \quad & \text{(i)} \quad \underline{A}_1\underline{\lambda}_1 \leq \underline{b}_1 \\ & \text{(ii)} \quad \underline{A}_2\underline{\lambda}_2 \leq \underline{b}_2 \\ & \text{(iii)} \quad \underline{M}_1\underline{\lambda}_1 + \underline{M}_2\underline{\lambda}_2 \leq \underline{b} \\ & \underline{\lambda}_\alpha \geq \underline{0} \end{aligned} \tag{5.25}$$

Let the optimal value in (5.25) be $f(\underline{\lambda}^*) = f^*$

We will show that Walker's proof can be adapted to the non-linear case, and will prove that the dual evaluators of the final tableaux for the method outlined in Section 5.3 are the dual evaluators for the total problem.

5.5.2 The dual programmes

The dual to (5.25) is: find $(\underline{\pi}_1^*, \underline{\pi}_2^*, \underline{\pi}^*, \underline{\lambda}_1^*, \underline{\lambda}_2^*) = \underline{\theta}$ s.t. $\underline{\theta}$ is the solution to:

$$\begin{aligned} \min \quad & \underline{\pi}_1\underline{b}_1 + \underline{\pi}_2\underline{b}_2 + \underline{\pi}.\underline{b} \\ \text{s.t.} \quad & \underline{\pi}_1\underline{A}_1 + \underline{\pi}\,\underline{M}_1 \geq \left[\frac{\partial f}{\partial \underline{\lambda}_1}\right]_{\underline{\lambda} = \underline{\lambda}^*} \\ & \underline{\pi}_2\underline{A}_2 + \underline{\pi}\,\underline{M}_2 \geq \left[\frac{\partial f}{\partial \underline{\lambda}_2}\right]_{\underline{\lambda} = \underline{\lambda}^*} \end{aligned} \tag{5.26}$$

Let $\left[\frac{\partial f}{\partial \underline{\lambda}_i}\right]_{\underline{\lambda} = \underline{\lambda}^*} = \underline{f}_i'^* = \overline{\underline{c}}_i$

where $\underline{\lambda}^*$ is optimal for (5.25). Form the primal:

$$\begin{aligned} \max \quad & \underline{f}_1'^*.\underline{x}_1 + \underline{f}_2'^*.\underline{x}_2 = \overline{\underline{c}}_1.\underline{x}_1 + \overline{\underline{c}}_2.\underline{x}_2 \\ \text{s.t.} \quad & \text{(i)}\ \underline{A}_1\underline{x}_1 \leq \underline{b}_1 \\ & \text{(ii)}\ \underline{A}_2\underline{x}_2 \leq \underline{b}_2 \\ & \text{(iii)}\ \underline{M}_1\underline{x}_1 + \underline{M}_2\underline{x}_2 \leq \underline{b} \end{aligned} \tag{5.27}$$

(5.27) is a linear decomposition problem.

Consider the following programmes:

Assume that $k_\alpha$ solutions have been tendered from divisions $\alpha$, ($\alpha = 1, 2$) for the decomposition of (5.27).

The master programme is:

$$\begin{aligned} \max \quad & \sum_{i=1}^{k_1} \mu_1^i.\overline{\underline{c}}_1.\underline{\lambda}_1^i + \sum_{i=1}^{k_2} \mu_2^i.\overline{\underline{c}}_2.\underline{\lambda}_2^i \\ \text{s.t.} \quad & \text{(i)}\ \Sigma\, \mu_1^i.\underline{M}_1\underline{\lambda}_1^i + \Sigma\, \mu_2^i.\underline{M}_2.\underline{\lambda}_2^i \leq \underline{b} \\ & \text{(ii)}\ \Sigma\, \mu_1^i = 1 \\ & \text{(iii)}\ \Sigma\, \mu_2^i = 1 \\ & \mu_\alpha^i \geq 0 \end{aligned} \tag{5.28}$$

Assume that the dual evaluators for rows (i)-(iii) of (5.28) are $(\underline{\pi}^*, \sigma_1, \sigma_2)$

The final divisional programmes for (5.27) are

$$\max \sum_i (\overline{c}_{\alpha_i} - \underline{\pi}^* . \underline{M}_{\alpha_i}) \lambda_i$$

$$\text{s.t. } \underline{A}_\alpha . \underline{\lambda}_\alpha \leq \underline{b}_\alpha \qquad (5.29)$$

$$(\alpha = 1, 2)$$

The dual of (5.29) is:

$$\min \ \pi_\alpha b_\alpha$$

$$\text{s.t. } \underline{\pi}_\alpha . \underline{A}_\alpha \geq \overline{c}_\alpha - \underline{\pi}^* . \underline{M}_\alpha$$

$$(\alpha = 1, 2)$$

<u>Lemma</u>: Let $\underline{x}^*$ be the optimal vector for the problem

$$\max \ f(\underline{x}) = \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta} \qquad \text{s.t. } \underline{A}.\underline{x} \leq \underline{b} \qquad (5.30)$$

$$\underline{x} \geq \underline{0}$$

Let $f(\underline{x}^*) = f^*$, and let the optimum dual evaluators be $\underline{\pi}^*$.

Consider the problem:

$$\max \sum_i x_i . \left[ \frac{\partial f}{\partial x_i} \right]_{\underline{x} = \underline{x}^*} \qquad (5.31)$$

$$\text{s.t. } \underline{A}.\underline{x} \leq \underline{b}$$

$$\underline{x} \geq \underline{0}$$

Then: $\underline{x}^*$ is optimal for (5.31) and $\underline{\pi}^*$ are the dual evaluators.

<u>Proof</u>: By Martos (65), the fractional programme has one unique solution over the constraint set $\{\underline{A}.\underline{x} \leq \underline{b}, \ \underline{x} \geq 0\}$ i.e. (5.30) has a unique solution. From Swarup's work, the KT Conditions for such an optimum are that there exists $\underline{x}^*, \underline{\pi}^* \geq 0$

$$\text{(i)} \quad \left[ \frac{\partial f}{\partial x_i} \right]_{\underline{x} = \underline{x}^*} - \underline{\pi}^* . \underline{A}_i \leq 0 \quad \forall i$$

$\underline{A}_i$ is i'th column of $\underline{A}$

$$\text{(ii)} \quad x_i^* \left[ \frac{\partial f}{\partial x_i} \right]_{\underline{x} = \underline{x}^*} - \underline{\pi}^* . \underline{A}_i = 0 \quad \forall i$$

$$\text{(iii)} \quad \underline{A}.\underline{x}^* \leq \underline{b} \qquad (5.32)$$

$$\text{(iv)} \quad \underline{\pi}^*.(\underline{A}.\underline{x}^* - \underline{b}) = 0$$

But conditions (5.32) are the KT Conditions for the problem (5.31), i.e. ($\underline{x}^*$, $\underline{\pi}^*$) is a sadde point for (5.31). But there is only one solution to (5.31), hence proof .....

Theorem: (Walker (86)). If (5.28) is the final optimum tableau for the executive programme for (5.27), then $\underline{\pi}^*$, and the corresponding $\underline{\pi}_1$, $\underline{\pi}_2$ from (5.30) are the components of the dual evaluators of (5.27).

Proof: see (86)

Lemma: The dual evaluators $\underline{\pi}^*$ from (i) of (5.28) are the dual evaluators of the rows (iii) of (5.25).

Proof: By the theorem just quoted, the $\underline{\pi}^*$ of (5.28) are also the evaluators of rows (iii) of (5.27).

By the lemma just proved, the dual evaluators of (5.27) and (5.25) are identical, hence:

Corollary 2: The Final Fractional Dual, (equivalent to the theorem of Walker). The final dual solutions to steps (b), (c) and (d) of the decomposition of 5.3.2 give the dual evaluators of the programme (5.25).

Proof: Using lemmas already proved.

Example of computation are shown in Appendix 5.1.

### 5.5.3 The final 'prices'

As has been shown, the final solutions to the executive and divisional programmes furnish the dual (marginal) evaluations of the total problem, thus they provide the desired 'transfer prices'. These 'prices' however , are the marginal value of inputs and outputs; they do not equate the value of total supply and demand in each market due to the non-linearity of the objective function.

As can be surmised from the work on strictly convex functions by Charnes, Cooper and Kortanek (23), these marginal prices will be insufficient to promote optimal behaviour from divisions. (Methods of control in decentralised firms are discussed in Section 5.8).

## 5.6 The Optimal Inverse

### 5.6.1 Introduction

According to Chapter 4, the optimal inverse basis is needed in order to test the 'returns to scale' of any fractional programme. Without this definition of 'returns', it is impossible to associate the dual evaluators (marginal values) with economic prices. For any sensitivity analysis to be effected, the optimal inverse basis is also a pre-requisite.

In this section we will consider the methods available for the calculation of the optimal inverse basis. This will be approached indirectly by first considering the problem of finding the range of possible changes in right hand side elements that 'maintain' the present basis. Throughout this work we will assume that the problem is non-degenerate at all vertices of the simplex.

Most techniques of post-optimal analysis in LP use the optimal inverse basis as a starting point.

i.e. for the problem:

$$\max \ \underline{c}.\underline{x} \quad \text{s.t.} \quad \underline{A}.\underline{x} \leq \underline{b}, \quad \underline{x} \geq \underline{0} \qquad (5.33)$$

we have $\underline{x}^* = \underline{B}^{-1}.\underline{b}$ for some $\underline{B}^{-1}$. (5.34)

With a non-degenerate problem (at its optimum):

$$\frac{\partial x_i^*}{\partial b_j} = (\underline{B}^{-1})_{ij}$$

and the range of values over which any $b_k$ can vary whilst the problem remains optimal/feasible is given by the extent to which $\underline{b}$ in (5.34) 'preserve' the condition that $\underline{x}^* \geq \underline{0}$ . ( See e.g. (44).)

We will use the marginal values $\frac{\partial x_i^*}{\partial b_j}$ calculated indirectly to form the optimal inverse basis for the decomposed linear fraction programme. To aid exposition the linear case is presented, and the Baumol and Fabian metaphor of corporate planning is maintained. Clearly the method can be interpreted as an adjunct to decomposition for the solution of large scale problems.

5.6.2 Notation

In the following sections the divisional weights have not been separated. The solutions $\underline{X}_i^*$, $i \in I$, are of the form $\begin{pmatrix} \underline{X}_1 \\ \underline{0} \end{pmatrix}$ (from division 1) or $\begin{pmatrix} \underline{0} \\ \underline{X}_2 \end{pmatrix}$ (from division 2).

$\mu_i^*$ is the optimal weight attached to the i'th plan

$b_k$ refers to the k'th entry in the relevant r.h.s.

5.6.3 The marginal variations of basic $\underline{X}_i$ with changes in resources

We will assume that the information available to the central organization is:

(i) the series of solutions $\{\underline{X}_i^*\}$,

(ii) the optimal weights $\{\mu_i^*\}$, (5.35)

(iii) the optimal inverse basis of the final executive programme $\underline{B}_\mu^{-1}$

(iv) the technology matrix and r.h.s. of the corporate section of (5.21).

a. Change of $b_i$ contained in corporate constraints:

Using (i) and (ii) of (5.35) we have the optimal programme

$$\underline{X}^* = \sum_i \mu_i^* \underline{X}_i^* \quad (5.36)$$

Since the problem is not degenerate, (assumed), a small change in $b_i$ in the corporate section will not induce a change in any of the sets of penalties and subsidies given to divisions,

i.e. the divisional solutions $\{\underline{X}_i^*\}$ are 'independent' of $b_i$ (for corporate resources). Consider changing a particular resource level in the corporate r.h.s., say $b_k$. (5.36) can be written as

$$\underline{X}^* = \sum_i \mu_i^*(b_k).\underline{X}_i^*$$

and $$\frac{\partial \underline{X}^*}{\partial b_k} = \sum_i \frac{\partial \mu_i^*(b_k)}{\partial b_k} . \underline{X}_i^* \quad (5.37)$$

But, from (iii) of (5.35), and the assumption of non-degeneracy:

$$\begin{pmatrix} \mu_i^* \end{pmatrix} = \underline{B}_\mu^{-1} . \begin{pmatrix} \underline{b} \\ 1 \\ 1 \end{pmatrix} \quad \text{for basic } \mu_i^*$$

(We ignore non-basic $\mu_i^*$ and assume that $\underline{\mu}^*$ comprises the non-zero $\mu_i^*$ only).

Hence $$\frac{\partial \mu_i^*}{\partial b_k} = \left(\underline{B}_\mu^{-1}\right)_{ik}$$

Since $\underline{B}_\mu^{-1}$ is known, the terms $\frac{\partial \underline{X}_i^*}{\partial b_k}$ may be calculated directly.

b. Change of $b_k$ contained in a divisional r.h.s.:

For this case, both $\{\mu_i^*\}$ and $\{\underline{X}_i^*\}$ are $b_k$-dependent.

Hence $$\frac{\partial \underline{X}^*}{\partial b_k} = \sum_j \left( \frac{\partial \mu_j^*}{\partial b_k} . X_j + \mu_j^* \frac{\partial \underline{X}_j^*}{\partial b_k} \right) \qquad (5.38)$$

Assume that each division supplies the marginal values of the optimal solutions with respect to its own resources, i.e. assume $\frac{\partial \underline{X}_j^*}{\partial b_k}$ are known. (In the computational approach for large programmes, the $\frac{\partial X_j}{\partial b_k}$ are known from the inverse basis of the 'divisional' programmes.) The $\left\{\frac{\partial \mu_i^*}{\partial b_k}\right\}$ may be calculated as follows:

i. formulate the executive LP, $\varphi$, in terms of the $(B_i^{-1})$ and $\underline{b}$ instead of $X_i^*$, using $\underline{X}_i^* = (\underline{B}_i^{-1}.\underline{b})$. For a small change in $b_k \in \underline{b}$, $\varphi$ and $\{\mu_i^*\}$ vary with $b_k$, i.e. $\varphi = \varphi(b_k)$, and the solutions to the final executive programme are $\mu_i^* = \mu_i^*(b_k)$.

ii. if the marginal values of the LP $\varphi$ exist, we can find $\mu_i^*(b_k + \Delta_k)$, for some small $\Delta_k$.

The N + S conditions for the existence of the marginal values of an LP have been considered in Section 4.2.3; they are due to Williams (92).

Assuming that these hold for the executive programme, the terms $\frac{\partial \mu_i^*}{\partial b_k}$ may be derived from

$$\frac{\partial \mu_i^*}{\partial b_k} = \lim_{\Delta_k \to 0} \left( \frac{\mu_i^*(b_k + \Delta_k) - \mu_i^*(b_k)}{\Delta_k} \right) \qquad (5.39)$$

(By the assumption of non-degeneracy, we can assume that $\exists$ a $\delta_k$ s.t. for $\Delta_k \leq \delta_k$ the $\mu_i^*(b_k + \Delta_k)$ are defined and that the limit exists).

The executive programme $\varphi(b_k)$ will be of the form:

$$\begin{aligned} \max \quad & \underline{c}.\sum_i \mu_i(b_k) \ \ \underline{B}_i^{-1}.(\underline{b} + \underline{\Delta}) \\ \text{s.t.} \quad & \underline{M}.\sum_i \mu_i(b_k) \ \ \underline{B}_i^{-1}.(\underline{b} + \underline{\Delta}) \le \underline{b} \\ & \sum \mu_i^1 = 1 \quad \text{convexity} \\ & \sum \mu_i^2 = 1 \quad \text{constraints} \end{aligned} \tag{5.40}$$

where: $\underline{\Delta} = (0, 0, \ldots\ldots b_k, 0, 0 \ldots\ldots 0)$, $\underline{b}$ is corporate r.h.s., and the $\mu_i^1$, $\mu_i^2$ refer to respective divisions. $\mu_i^\alpha \ge 0$. From the formulations (5.39) and (5.40), the right hand side of (5.38) may be obtained.

Since these calculations have only depended on the linearity of the constraint set, they are applicable to linear fractional programming; the theorems of Williams will not be immediately applicable, but, using methods similar to Section 5.5, they may be used via the Charnes and Cooper Equivalent forms.

A direct method for computing $\frac{\partial \mu_i}{\partial b_k}$ is shown below in Section 5.6.4.

Calculations illustrating the theory of Sections 5.6.3, and 5.6.4 are presented in Appendices 5.2 and 5.3.

### 5.6.4 Direct calculation of the "perturbed inverse basis"

Assume that the columns of the executive programme for basic $\mu_i^*$ are given by $\underline{A}$. ($\underline{A}$ is m by m)

A small change in $b_k$ will change the column values of $\underline{A}$ according to the matrix elements $\frac{\partial \underline{X}_i^*}{\partial b_k}$, (assuming that the change is sufficiently small to retain optimality/feasibility etc.)

Assume that the perturbed matrix for $\underline{A}$ is $\underline{A} + \underline{H}(\delta b_k)$, where $\delta b_k$ represents the small change in $b_k$. The new inverse basis is $[\underline{A} + \underline{H}(\delta b_k)]^{-1}$ which exists, if the conditions enunciated by Williams are satisfied, (and $[\underline{A} + \underline{H}(\delta b_k)]$ is non-singular). Now: $\underline{H}(\delta b_k)$ is linear in $\delta b_k$, since it is the weighted sum of terms which are linear in $\delta b_k$, i.e. it can be written as $\underline{H}.\delta b_k$ where $\underline{H}$ is a matrix of scalar values.

$$\underline{\mu}^*(b_k) = \underline{A}^{-1}.\underline{b}$$

and $\underline{\mu}^*(b_k + \delta b_k) = [\underline{A} + \underline{H}.\delta b_k]^{-1} .\hat{\underline{b}}$

where $\hat{\underline{b}}$ is the r.h.s. of the executive programme, $\hat{b}_k = b_k + \delta b_k$

$$\frac{\partial \underline{\mu}}{\partial b_k}(b_k) = \lim_{\delta b_k \to 0} \left( \frac{[\underline{A} + \underline{H}.\delta b_k]^{-1}.\hat{\underline{b}} - \underline{A}^{-1}.\underline{b}}{\delta b_k} \right) \tag{5.41}$$

(if the r.h.s. exists)

$$[\underline{A} + \underline{H}.\delta b_k]^{-1} = \left(\underline{A}.[\underline{I} + \underline{A}^{-1}.\underline{H}.\delta b_k]\right)^{-1}$$
$$= [\underline{I} + \underline{A}^{-1}.\underline{H}.\delta b_k]^{-1}.\underline{A}^{-1}$$

But we may make $\delta b_k$ as small as we please, i.e. if $\underline{D} = \underline{A}^{-1}.\underline{H}.\delta b_k = (d_{ij})$ we can find $\varepsilon$, $\delta$ s.t.

$$|d_{ij}| < \varepsilon \text{ for } \delta b_k < \delta.$$

Hence we can ensure that $(\underline{A}^{-1}.\underline{H}.\delta b_k)^m \to \underline{0}$, as $m \to \infty$ and can expand $(\underline{I} + \underline{D})^{-1}$ to give:

$$(\underline{I} + \underline{D})^{-1} = \underline{I} - \underline{D} + \underline{D}^2 \ldots\ldots \tag{5.42}$$

Using (5.42) we can re-write (5.41) as

$$\frac{\partial \underline{\mu}(b_k)}{\partial b_k} = \lim_{\delta b_k \to 0} \left( \frac{\underline{A}^{-1} - \underline{D}.\underline{A}^{-1} + \underline{D}^2.\underline{A}^{-1}).\hat{\underline{b}} - \underline{A}^{-1}.\underline{b}}{\delta b_k} \right)$$
$$= \lim_{\delta b_k \to 0} \left( \frac{- \underline{A}^{-1}.\underline{H}.\underline{A}^{-1}.\underline{b}.\delta b_k + 0(\delta b_k)}{\delta b_k} \right)$$
$$= -\underline{A}^{-1}.\underline{H}.\underline{A}^{-1}.\underline{b} \tag{5.43}$$

Typical calculations for $\underline{H}$ are also shown together with the worked example in Appendix 5.3.

N.B. $\underline{A}^{-1}.\underline{b}$ is the present solution (immediately available)

$\underline{A}^{-1}$ is the optimal inverse (immediately available)

only $\underline{H}$ need be determined.

## 5.7 The Provisional Dual Pricing Theorems

### 5.7.1 Provisional pricing in the linear case

For the LP

$$\max \ \underline{c}.\underline{x} \qquad \text{s.t.} \quad \underline{A}.\underline{x} = \underline{b}, \quad \underline{x} \geq \underline{0} \qquad (5.44)$$

Let $\hat{\underline{x}} = \underline{B}^{-1}.\underline{b}$

$\hat{\underline{c}}$ be the $\{c_i\}$ term corresponding to $\hat{x}$

$\pi = \hat{\underline{c}}.\underline{B}^{-1}$

For any solution $\hat{\underline{x}}$, not necessarily optimal, Baumol and Fabian have proved the following theorems:

Theorem 1: For non-basic $x_k$, the marginal change in objective function upon inclusion of $x_k$ is given by

$$\Delta_k = - \underline{\pi}.\underline{A}_k + \hat{c}_k \qquad (5.45)$$

where $\underline{A}_k$ is the column of $\underline{A}$ pertaining to $x_k$.

Theorem 2: For basic $x_j$, $\Delta_j = 0$

$$\text{i.e.} \quad \underline{\pi}.\underline{A}_j = c_j \qquad (5.46)$$

Theorem 3: $\underline{\pi}.\underline{b} = \hat{\underline{c}}.\hat{\underline{x}}$ (5.47)

These theorems are proved in (6), and allow the interpretation of the $\pi$'s of executive programmes in linear decomposition as provisional dual prices.

### 5.7.2 Provisional pricing for fractional decomposition

In steps (d) of 5.2.2 and (c) of 5.3.2, we have amended the objective functions of the divisional programmes using the $\pi$'s of the corporate constraint rows as marginal values of the corporate resources. We formed the expressions:

$$\max \sum_i \left\{ c_{\alpha_i} - \underline{\pi}.\underline{M}_{\alpha_i} - \pi_d . d_{\alpha_i} \right\} x_{\alpha_i}$$

and $$\max \sum_i \left( \frac{\partial f}{\partial x_{\alpha_i}} - \overline{\underline{\pi}}.\underline{M}_{\alpha_i} \right) x_{\alpha_i}$$

In so doing it was assumed that the Provisional Dual Pricing Theorems quoted in 5.7.1 held for the fractional executive programmes. This will now be proved, using the duality relationships derived in Section 4.4. We use the same notation as in Section 4.4, namely:

assume the initial problem is

$$\max \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta} \quad \text{s.t.} \quad \underline{A}.\underline{x} \le \underline{b}, \quad \underline{x} \ge \underline{0} \qquad (5.48)$$

The 'Charnes and Cooper form' of this is

$$\max \underline{c}.\underline{y} + \alpha t \quad \text{s.t.} \quad \underline{A}.\underline{y} - \underline{b}t \le \underline{0}, \quad \underline{d}.\underline{y} + \beta t = 1, \quad \underline{y}, t \ge 0 \qquad (5.49)$$

Let the present solution to (5.48) be described by $\hat{\underline{x}} = \underline{B}^{-1}.\underline{b}$

The corresponding inverse basis of the CC form, $(\underline{B}^*)^{-1}$ is given by

$$(\underline{B}^*)^{-1} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

where:

$$M_{11} = \underline{B}^{-1}. -\hat{\underline{x}}.\hat{t}.\hat{\underline{d}}.\underline{B}^{-1}$$
$$M_{12} = \hat{t}.\hat{\underline{x}}$$
$$M_{21} = -\hat{\underline{t}}(\hat{\underline{d}}.\underline{B}^{-1})$$
$$M_{22} = \hat{t}. \qquad (5.50)$$

In Chapter 6 we show that if $\underline{a}_k = \underline{B}^{-1}.\underline{A}_k$, then $\underline{w}_k = (\underline{B}^*)^{-1}.\begin{pmatrix} \underline{a}_k \\ d_k \end{pmatrix}$

is given by :

$$\underline{w}_k = \begin{pmatrix} \underline{B}^{-1}.\underline{a}_k + \hat{t}.\hat{\underline{x}}(d_k - \hat{\underline{d}}.\underline{B}^{-1}.\underline{a}_k) \\ \hat{t}\,(d_k - \hat{\underline{d}}.\underline{B}^{-1}.\underline{a}_k) \end{pmatrix} \qquad (5.51)$$

Define $\underline{\pi}_F$ by:

$$\underline{\pi}_F = \left[\frac{\partial \hat{f}}{\partial \underline{x}}\right]_{\underline{x} = \hat{\underline{x}}} . \underline{B}^{-1} \qquad (5.52)$$

where $\hat{f}$ is differentiated with respect to basic $x_j$ only.

Define $\underline{\pi}_{cc}$, $\pi_d$ by:

$$(\underline{\pi}_{cc}, \pi_d) = (\hat{\underline{c}}, \alpha).(\underline{B}^*)^{-1} \qquad (5.53)$$

(This is the same definition as in 5.7.1 for the linear case).

Theorem 1F: For non-basic $x_k$ in (5.48), the marginal change upon inclusion of $x_k$ is given by $\Delta_k$, where

$$\Delta_k = -\underline{\pi}_F.\underline{A}_k + \left[\frac{\partial f}{\partial x_k}\right]_{\underline{x} = \hat{\underline{x}}} \qquad (5.54)$$

Proof: Theorem 1 holds for the Charnes and Cooper form of (5.48).

Let the corresponding solution be $(\hat{y}, \hat{t})$

Then:

$$\Delta_{y_k} = -\underline{\pi}_{cc}.\begin{pmatrix}\underline{A}_k \\ d_k\end{pmatrix} + c_k$$

$$= -(\hat{\underline{c}}, \alpha).(\underline{B}^*)^{-1}\begin{pmatrix}\underline{A}_k \\ d_k\end{pmatrix} + c_k$$

$$= -\hat{\underline{c}}.\underline{B}^{-1}.\underline{a}_k - \hat{\underline{c}}.\hat{\underline{x}}.\hat{t}(d_k - \hat{\underline{d}}.\underline{B}^{-1}.\underline{a}_k)$$

$$- \alpha\,\hat{t}(d_k - \hat{\underline{d}}.\underline{B}^{-1}.\underline{a}_k) + c_k$$

Now $(\hat{\underline{c}}.\hat{\underline{x}} + \alpha)\hat{t} = \hat{f}$

$$\therefore \quad \Delta_{y_k} = -\left\{\hat{\underline{c}}.\underline{B}^{-1}.\underline{a}_k - \hat{f}.\hat{\underline{d}}.\underline{B}^{-1}.\underline{a}_k\right\} + (c_k - d_k.\hat{f})$$

$$= \left\{\underline{c} - \hat{f}.\underline{d}\right\} . (\underline{B}^{-1}.\underline{a}_k) + (c_k - \hat{f}.d_k)$$

Now $\hat{\underline{c}} - \hat{f}.\hat{\underline{d}} = \frac{1}{\hat{t}} . \left[\frac{\partial \hat{f}}{\partial \underline{x}}\right]_{\underline{x} = \hat{\underline{x}}}$

and $c_k - \hat{f}.d_k = \frac{1}{t}\left[\frac{\partial f}{\partial x_k}\right]_{\underline{x} = \hat{\underline{x}}}$

$$\therefore \quad \Delta_{y_k} = \frac{1}{t}\left\{-\pi_F.\underline{A}_k + \left[\frac{\partial f}{\partial x_k}\right]_{\underline{x} = \hat{\underline{x}}}\right\}$$

but $\Delta_k = \hat{t}\,\Delta_{y_k}$ by relation $\hat{t}.\hat{\underline{x}} = \hat{\underline{y}}$ (shown in Appendix 4.2)

$$\therefore \quad \Delta_k = -\underline{\pi}_F.\underline{A}_k + \left[\frac{\partial f}{\partial x_k}\right]_{\underline{x} = \hat{\underline{x}}}$$

Theorem 2F: For basic $x_j$, $\Delta_j = 0$

$$\text{i.e. } \underline{\pi}_F.\underline{A}_k = \left[\frac{\partial \hat{f}}{\partial x_j}\right]_{\underline{x} = \hat{\underline{x}}}$$

Proof: The same proof applies; but $\Delta_{y_k} = 0$ because $y_j$ is basic, $\therefore \Delta_j = 0$ ($x_j$ is basic)

$$\text{i.e. } \underline{\pi}_F.\underline{A}_k = \left[\frac{\partial f}{\partial x_j}\right]_{\underline{x} = \hat{\underline{x}}}$$

N.B. Theorem 3 has no fractional equivalent, i.e.

$$\underline{\pi}_F.\underline{b} \neq \sum_{(x_j \text{ basic})} \hat{x}_j . \left[\frac{\partial \hat{f}}{\partial x_j}\right]_{\underline{x} = \hat{\underline{x}}}$$

The normal failure due to non-linearity of the objective function occurs, but the lack of Theorem 3F does not preclude the interpretation of the π's of the fractional executive programme as marginal values.

The full interpretation as economic prices would require diminishing returns to scale, etc. But as has been seen, these are not necessary for the operation of the decomposition algorithms.

## 5.8 Control in Decentralised Organizations

### 5.8.1 Control in Decentralised Organizations

At the termination of a decomposition process, the 'executive' calculates the optimal weights to be attached to the divisional subprogrammes. As has been outlined in (8) these weights form convex combinations of divisional solutions; the optimal divisional programme is an interior point of the divisional constraint set - and cannot be reached by programming methods which have extremum point optima.

Thus pricing alone is insufficient (in the linear case) to ensure that divisions act optimally. At the end of the decomposition the optimal solutions are announced as production fiats. This will also be the case in fractional programming because of the persistence of the divisional extremum point solution.

Charnes Cooper and Kortanek (23) have shown that it is possible to set goals for each division, based on the optimal solution; divisions are then asked to optimise a function containing severe penalties for any deviations from the prescribed goals. These are termed 'pre-emptive goals'.

Such goals are also definable for the divisions in the fractional cases; they will differ little from those of the linear case, due to the similarity of the divisional programmes in both the linear and fractional decompositions.

5.9 Summary

In this chapter we have shown how decomposition methods may be applied to fractional programming problems, using both the original and Charnes and Cooper forms.

We have proved the appropriate duality and pricing theorems (where possible), and have shown how the bases of the decomposition method can be used in the construction of the total optimal inverse basis. Examples are presented in Appendices 5.1, 5.2 and 5.3.

# Chapter 6 Special Methods in Fractional Programming

## 6.1 Introduction

In this chapter we discuss the special methods available in FP, which can be based on the equivalence between the algorithms of Martos (64) and Charnes and Cooper (17). Emphasis is placed on the latter approach since it utilises existing codes; IP with fractional objectives is also considered together with aspects of pricing with integer programmes. Stochastic Programming with fractional objectives is reviewed in Appendix 6.4.

## 6.2 Basis Relationships in FP

Wagner and Yuan (85), have shown the equivalence of the algorithms of Martos (64) and Charnes and Cooper (17). Their work shows that the two methods proceed to the optimum via the same pivot paths. Charnes and Cooper have also shown that for any vector ($\underline{y}$, t), feasible for (1.23), t is strictly positive.

Thus we may assume that any set of pivot operations $\{$remove $x_s$, introduce $x_r\}$ has a corresponding set of operations in the CC form, namely $\{$remove $y_s$, introduce $y_r\}$.

## 6.3 The Bounded Variable Algorithm

### 6.3.1 The CC Form

Consider the problems

$$\max f(\underline{x}) = \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta}$$

$$\text{s.t. } \underline{i} \quad \underline{A}.\underline{x} \le \underline{b}$$

$$\underline{ii} \quad \underline{0} \le \underline{x} \le \underline{U} \qquad (6.1)$$

and

$$\max f(\underline{y}) = \underline{c}.\underline{y} + \alpha t$$

$$\text{s.t.} \quad \underline{i} \quad \underline{A}.\underline{y} - \underline{b}t \le 0$$
$$\underline{d}.\underline{y} + \beta t = 1$$
$$\underline{ii} \quad \underline{y} - \underline{U}t \le 0 \;, \; \underline{y} \ge \underline{0} \qquad (6.2)$$

(6.2) is the CC form of (6.1), but it does not display the upper bounded variable characteristics of (6.1), because of the inclusion of the variable t in the rows ii. For a problem of this form, with many upper bounds (e.g. the capital budgeting problem where projects are bounded by unity), the resulting CC form (6.2) appears cumbersome due to the explicit inclusion of all upper bounds in the rows ii of (6.2).

The CC form cannot be used for a bounded variable algorithm for the solution of fractional programmes.

6.3.2 The Parametric Approach

Using the method of Joksch (54), the problem (6.1) becomes

$$\max \frac{\underline{c}.\underline{x} + \alpha}{\theta} = f(\theta)$$
$$\text{s.t.} \quad \underline{A}.\underline{x} \le \underline{b}$$
$$\underline{d}.\underline{x} + \beta = \theta$$
$$\underline{0} \le \underline{x} \le \underline{U} \qquad (6.3)$$

For any fixed $\theta$, (6.3) is a normal bounded variable LP.

6.3.3 Variations on Martos' Algorithm

In order to solve (6.1) directly, the only variation required for the normal LP bounded variable algorithm is that of the selection of the pivot column; this can be achieved by adaptation of Martos' algorithm, ((64)), according to methods outlined in (35) and (68).

### 6.3.4 Dual evaluators in upper bound formulations

Weingartner (88), uses the dual evaluators associated with the upper bound formulation to rank basic and non-basic projects. Such rankings can be applied in FP; once again the mapping is effected via the variable 't'.

Consider the problem:

$$\max \quad f(\underline{x}) = \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta}$$

$$\text{s.t.} \quad \underline{i} \quad \underline{A}.\underline{x} \leq \underline{b}$$

$$\underline{ii} \quad x_i \leq 1 \qquad i = 1 \ldots n \tag{6.4}$$

Let the dual evaluators for (6.4) be ($\underline{\pi}_F$, $\underline{\mu}_F$), and the optimal value be $f^* = f(\underline{x}^*)$

The CC form of (6.4) is

$$\max \quad f(\underline{y}) = \underline{c}.\underline{y} + \alpha t$$

$$\text{s.t.} \quad \underline{i} \quad \underline{A}.\underline{y} - \underline{b}t \leq \underline{0}$$

$$\underline{d}.\underline{y} + \beta t = 1$$

$$\underline{ii} \quad \underline{y} - \underline{e}t \leq \underline{0} \tag{6.5}$$

where $\underline{e}$ is the vector (1, 1, 1, ... 1). Let the evaluators of (6.5) be ($\underline{\pi}_C$, $\pi_d$ , $\underline{\mu}_C$). By LP duality, $\pi_d = f^*$.

Let $\gamma_j$ = '$z_j - c_j$' defined for the CC form ('y' variables only) as in (88), $\underline{\gamma}_C = (\gamma_j)$

then $\gamma_j = \underline{\pi}_C.\underline{A}_j + d_j.f^* + \mu_{C_j} - c_j$

$= \underline{\pi}_C.\underline{A}_j + \mu_{C_j} - (c_j - d_j.f^*)$

Now $\underline{\pi}_C = \underline{\pi}_F.\frac{1}{t^*}$ and $\underline{\mu}_C = \underline{\mu}_F.\frac{1}{t^*}$ (from Section 4.4.1)

$$\text{therefore } \gamma_j = \frac{1}{t^*}\left\{\underline{\pi}_F.\underline{A}_j + \mu_{F_j} - (c_j - d_j.f^*)t^*\right\} \tag{6.6}$$

But the numerator of (6.6) is the marginal value sum of inputs for the j'th variable minus the marginal return, evaluated at the optimum of (6.4).

i.e. $\gamma_j . t^* = \underline{\pi}_F . \underline{A}_j + \mu_{F_j} - \left[\frac{\partial f}{\partial x_j}\right]_{\underline{x} = \underline{x}^*} = \gamma_{F_j}$

where $\gamma_{F_j}$ is the '$\gamma_j$' defined for the form (6.4)

$$\text{i.e.} \quad \underline{\gamma}_C = \underline{\gamma}_F . \frac{1}{t^*}$$

The natural ranking of ($\underline{\gamma}$, $\underline{\mu}$) is preserved in FP, and can be deduced from the ranking in the CC form.

### 6.3.5 Productivity Ratios

In Chapter 3, we have suggested a second ranking for variables (projects), the ratio of marginal return to the sum of the marginal values of inputs (i.e. for LP the ratios $\theta_j = \frac{c_j}{\underline{v} . \underline{a}_j}$ ). These rankings are not strictly preserved between the CC form and the original fractional form.

Considering the CC form (6.5), the definition of $\theta_j$ would give

$$\theta_j = \frac{c_j}{\left\{\underline{\pi}_C . \underline{A}_j + d_j . f^*\right\}} \quad \text{(ignoring the upper bounds)}$$

$\hat{\theta}_j$, the equivalent $\theta$'s for the original form would be

$$\hat{\theta}_j = \frac{\underline{c}_j}{\underline{\pi}_F . \underline{A}_j}$$

$\theta_j > \theta_k \not\Rightarrow \hat{\theta}_j > \hat{\theta}_k$ because of the term $d_j . f^*$;
defining $\theta_j$ for the CC form as $\frac{\underline{c}_j}{\underline{\pi}_C . \underline{A}_j}$ , the ranking is preserved.

## 6.4 Sensitivity Analysis in FP

In order to describe the optimal solution to a mathematical programming problem, three pieces of information are required: the primal solution, the dual variables, and the 'robustness' of the solution to changes of input data. Sensitivity is required before the solution can be used for decision making.

Sensitivity in FP can be approached using the sensitivity analysis of LP applied to the CC form, as outlined in the following sections.

6.4.1 Changes in the r.h.s elements $\{b_k\}$ that 'preserve' optimality are derived from the optimal inverse basis using the formula

$$\underline{x}^* = \underline{B}^{-1}\underline{b}$$

For changes of $\underline{b}$, the basis is feasible only if the corresponding $\underline{x}^*$ is positive. ( See e.g. (68).)

In the CC form, the elements $\underline{b}$ appear in the matrix of constraints. Nevertheless the range for $\{b_k\}$ can be deduced from the range of the r.h.s. of the CC form.

Assume that the range of the k'th row of the CC form is $\delta_k$ (for an increase in $b_k$): then, a basis change occurs when

$$\underline{A}_k . \hat{\underline{y}} - b_k . \hat{t} = \delta_k$$

where $(\hat{\underline{y}}, \hat{t})$ is the value of $(\underline{y}, t)$ at the end point of the range of the k'th row. Up to this point the present basis is optimal, i.e.

$$\underline{A}_k . \hat{\underline{y}} - b_k . \hat{t} \leq \delta_k$$

$$\text{or} \quad \underline{A}_k . \hat{\underline{x}} \leq b_k + \frac{\delta_k}{\hat{t}} \qquad (6.7)$$

therefore the range of $b_k$ (increasing) is given by $\frac{\delta_k}{\hat{t}}$

where $\delta_k$ is the range of the k'th row of the CC form and $\hat{t}$ is the value of t at the limit of the range. (Similar analysis applies for decreasing $b_k$: worked examples are shown in Appendix 6.1)

6.4.2 <u>Changes in the $\{c_i\}$ terms</u>

Allowable changes in the $\{c_i\}$ terms may be deduced directly from the CC form; this is an LP, for which sensitivity to changes in $\{c_i\}$ is readily available, see e.g. (68).

6.4.3 Changes in the $\{d_i\}$ terms

Let $\sigma_i(d_i)$ be the i'th reduced cost in the CC optimal tableau, let the dual evaluators be $(\underline{\pi}_{CC}, \pi)$. Let the i'th column of the initial tableau be $\begin{pmatrix}\underline{A}_j \\ d_i\end{pmatrix}$ and the optimal solution be

$$f^* = \frac{\underline{c}.\underline{x}^* + \alpha}{\underline{d}.\underline{x}^* + \beta} = \frac{u^*}{v^*}, \text{ with } \pi = f^*$$

Let $\overline{\sigma}_i(d_i)$ be the i'th 'reduced cost' in the original form. Consider changes caused by perturbation of $d_i$ by an amount $\Delta_i$ where $d_i + \Delta_i = d_i'$

From Appendix 4.2 (4.41) we know that

$$\sigma_i = \frac{1}{t^*} . \overline{\sigma}_i$$

$$\text{Now } \overline{\sigma}_i = \left[\frac{\partial f}{\partial x_i}\right]_{\underline{x}=\underline{x}^*} - \left[\frac{\partial f}{\partial \underline{x}_B}\right]_{\underline{x}=\underline{x}^*} . \underline{B}^{-1}.\underline{A}_i \qquad (6.8)$$

where $\{\underline{x}_B\}$ are basic activities, $\underline{B}^{-1}$ is the optimal inverse of the original fractional form,

therefore, $\overline{\sigma}_i = (c_i - d_i.f^*)t^* - \sum_j (c_j - d_j.f^*)t^*.a_{ij}$

where $\underline{B}^{-1}.\underline{A}_i = \underline{a}_i$

$$\text{and } \sigma_i = (c_i - d_i.f^*) - \sum_j (c_j - d_j.f^*).a_{ij} \qquad (6.9)$$

a. Non-basic $\{x_i\}$: $f^*$ and $a_{ij}$ are independent of $d_i$, therefore, for the present basis to be optimal we require that

$$\sigma_i(d_i') = c_i - (d_i + \Delta_i)f^* - \sum_j (c_j - d_j.f^*).a_{ij} \leq 0$$

$\forall$ non basic i

i.e. $\sigma_i(d_i') = \sigma_i - \Delta_i.f^* \leq 0$

therefore, $d_i$ may be reduced by an amount

$$\Delta_i \le \left|\frac{\sigma_i}{f^*}\right| \text{ whilst } x_i \text{ remains non-basic} \tag{6.10}$$

b. Basic $\{x_i\}$: If $x_i$ is basic, $f^*$ and $t^*$ vary with $d_i$; optimality is preserved if $\sigma_k(d_i') \le 0$ for all non-basic $x_k$

$$\text{i.e. if } \sigma_k(d_i') = c_k - d_k.\frac{u^*}{v^* + \Delta_i.x_i^*}$$

$$- \sum_{j\neq i}(c_j - d_j.\frac{u^*}{v^* + \Delta_i.x_i^*}) . a_{kj}$$

$$- \left(c_i - (d_i + \Delta_i).\frac{u^*}{v^* + \Delta_i.x_i^*}\right) .a_{ki}$$

$$\le 0 \qquad \text{non basic } k. \tag{6.11}$$

Let $t^*$ vary with $\Delta_i$ i.e. $t^* = (v^* + \Delta_i.x_i^*)^{-1}$

Rearranging terms we have

$$\sigma_k(d_i') = t^*\Big[c_k(v^* + \Delta_i.x_i^*) - d_k.u^*$$

$$- \sum_{j\neq i}\left\{c_j(v^* + \Delta_i.x_i^*) - d_j.u^*\right\} a_{kj}$$

$$- \left\{c_i(v^* + \Delta_i.x_i^*) - (d_i + \Delta_i)u^*\right\} a_{ki}\Big]$$

$$= t^*\Big[\Delta_i\left\{c_k.x_i^* - \sum_j c_j.x_i^*.a_{kj} + u^* a_{ki}\right\}$$

$$+ c_k.v^* - d_k.u^* - \sum_j(c_j.v^* - d_j.u^*)a_{kj}\Big]$$

Let $\eta_i = c_k.x_i^* - \sum_j c_j.x_i^*.a_{kj} + u^*.a_{ki}$

then $\sigma_k(d_i') = t^*\left\{\Delta_i.\eta_i + v^*.\sigma_k\right\} \le 0$

$$\forall k \tag{6.12}$$

i.e. $\Delta_i \le \dfrac{-v^*\sigma_k}{\eta_i}$

$d_i$ may be decreased by an amount $\rho_k = \dfrac{v^*}{\eta_i}\sigma_k$ before $x_k$ will become a 'profitable non-basic' activity; hence 'range' for $d_i$ is $\min_k\{\rho_k\}$

(Worked examples are given in Appendix 6.1; all components of $\{\rho_k\}$ are obtainable from the CC optimal solutions)

Alternatively, an algebraic formulation may be used, i.e. noting the changes of the inverse basis with changes of $\{d_i\}$. The approach is shown in Appendix 6.1.3: although useful for exposition, it has no computational value.

## 6.5 IP with Fractional Objective Functions

6.5.1 In the exposition of FP methods we have thus far assumed that variables are real valued, but many formulations are only meaningful if model variables are integer valued (e.g. capital budgeting, etc.). Branch and Bound Techniques for (linear) IP may easily be adapted for FP, at the possible expense of computational efficiency in the tree search.

Cutting plane methods (42), (43), may also be applied to FP; Swarup (78) has given one approach via the direct method, formulating his own dual algorithm for FP. The CC form may also be utilised for integer work as follows.

### 6.5.2 An Integer Algorithm for FP

Consider the problem:

$$\max \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta} \quad \text{s.t.} \quad \underline{A}.\underline{x} \leq \underline{b}, \quad \underline{x} \geq \underline{0}, \quad x_i \text{ integers} \tag{6.13}$$

where $\underline{c}$, $\underline{d}$, $\underline{A}$, $\underline{b}$ have integer entries.

(6.13) has the CC form:

$$\max \underline{c}.\underline{y} + \alpha t \quad \text{s.t.} \quad \underline{A}.\underline{y} - \underline{b}t \leq \underline{0}, \quad \underline{d}.\underline{y} + \beta t = 1, \quad \underline{y},\ t \geq 0 \tag{6.14}$$

with the added requirement that $\frac{1}{t^*}.\underline{y}^*$ be integer valued.

The convexification algorithm utilising the CC form may be stated as follows:

<u>i</u> optimise the CC form and test $\frac{1}{t^*}.\underline{y}^*$ for optimality (i.e. for integer values)

<u>ii</u> all $\frac{y_i^*}{t^*}$ integers - stop.

if not:

<u>iii</u> map the final CC tableau back to the original form, giving "the greatest fractional row".

<u>iv</u> form the cutting plane and add it to the constraint set.

<u>v</u> map the new constraint into the CC form.

<u>vi</u> use the LP dual simplex method to restore feasibility, and optimise. Go to <u>ii</u>.

Let $\hat{\underline{A}}_{CC}$ be the present (optimal) canonical form for the CC method, and the solution vector be $\underline{y}^* = t^*.\underline{x}^*$. Let $\{x_k^*\}$ denote the fractional part of $x_k^*$. Using the Method of Integer Forms, (42), we select the row for which $\{x_k^*\}$ is a maximum, i.e. for which $\left\{\frac{y_k^*}{t^*}\right\}$ is a maximum, say $x_i^*$.

Assume that the original (optimal) canonical form is $\underline{\bar{A}} = (\bar{a}_{ij})$ (Mappings for $\hat{\underline{A}}_{CC} \rightarrow \underline{\bar{A}}$ are given in Section 6.5.3)

The cutting plane is

$$\sum_j \{\hat{a}_{ij}\}.\ x_j \geq \{x_i^*\} \qquad (6.15)$$

i.e. in the CC form this constraint is

$$\sum_j \{\hat{a}_{ij}\}.\ y_j - \{x_i^*\}t \geq 0 \qquad (6.16)$$

(6.16) is added to the matrix $\hat{\underline{A}}_{CC}$, and the dual simplex algorithm implemented as in step <u>vi</u>.

(Worked examples are shown in Appendices 6.2 and 6.3).

6.5.3 The mapping between optimal tableaux

Two methods are available for finding the equivalent canonical form mentioned in the previous paragraph.

i. The optimal solution can be deduced from the basic rows in the CC form, using the equivalence between pivoting sequences outlined in 6.5.1.

The optimal inverse of the original form can be deduced from the sequence of pivots in the CC form. The efficiency of inversion routines makes this heuristic method attractive for large scale programming; it also provides an accurate computation of the row elements of the original form.

ii. The second method utilises the Wagner-Yuan Equivalence (85).

Let $\underline{W}_k = \begin{pmatrix} \underline{W}_{k_y} \\ W_{k_t} \end{pmatrix}$ be the k'th column of the 'optimal' tableau of the CC form.

Let $\underline{z}_k$ be the k'th column of the optimal tableau of the original form.

Let $\underline{\overline{B}}^{-1}$ be the CC inverse basis

Let $\underline{y}_B$ be the first m entries of the present r.h.s. of the CC form

$$\text{i.e.} \quad \underline{y}_B = (\underline{\overline{B}})^{-1} . \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Let $\underline{x}_B$, $\underline{d}_B$ .... correspond to $\underline{y}_B$ and let $\underline{a}_k$ denote the k'th column of $\underline{A}$. Then, using the equivalence outlined in Section 4.4.1, we have

$$\underline{W}_k = (\underline{\overline{B}})^{-1} . \begin{pmatrix} \underline{a}_k \\ d_k \end{pmatrix} = \begin{pmatrix} \underline{B}^{-1}.\underline{a}_k + t\underline{x}_B(d_k - \underline{d}_B.\underline{B}^{-1}.\underline{a}_k) \\ t(d_k - \underline{d}_B.\underline{B}^{-1}.\underline{a}_k) \end{pmatrix}$$

Now $\underline{z}_k = \underline{B}^{-1}.\underline{a}_k$

therefore, $\underline{W}_k = \begin{pmatrix} \underline{z}_k + t.\underline{x}_B(d_k - \underline{d}_B.\underline{z}_k) \\ t(d_k - \underline{d}_B.\underline{z}_k) \end{pmatrix}$

The last entry in the column $W_k$ is known (or can be readily calculated if the revised simplex method is used); this is '$W_{k_t}$ - the entry in the denominator row.'

Thus $\underline{W}_k = \begin{pmatrix} \underline{W}_{k_y} \\ W_{k_t} \end{pmatrix} = \begin{pmatrix} \underline{z}_k + \underline{y}_B . W_{k_t} . \frac{1}{t} \\ W_{k_t} \end{pmatrix}$

$$\underline{z}_k = \underline{W}_{k_y} - \underline{y}_B . \left( \frac{W_{k_t}}{t} \right) \qquad (6.17)$$

Thus all the required elements of the optimal tableau of the original form may be calculated (see Appendix 6.2).

## 6.6 Pricing for Integer Programming with Fractional Objective Functions

### 6.6.1 Introduction

With the emergence of algorithms to solve (linear) integer and mixed integer programmes, economists and experts in mathematical programming have been faced with the problem of interpreting the value of resources in the light of such optimisations. Since the dual pricing mechanism for linear programmes is so powerful, duality has provided the major springboard for (such) resource evaluation.

Methods have been devised by Gomory and Baumol (43), and Alcaly and Klevorick (2), for "re-imputing" the dual variables(at the optimal tableau of the cutting plane algorithm)back to the original resources.

A similar method has been used by Weingartner (88) as outlined in Section 1.2.

Dual pricing mechanisms have been seen to fail in some LFP cases , because of the lack of diminishing returns to scale; both marginality and diminishing returns to scale are absent in integer programming. Frank (39a) has proposed defining the marginal value of resources as $\frac{\Delta z}{\Delta b_i}$ where the $\Delta b_i$ represent unit changes in the resources, and $\Delta z$ the concommitant changes in objective function, but the results are not generally applicable.

6.6.2 <u>Pricing via Recomputed Dual Variables</u>

In LP, the recomputation process has the following properties:

<u>i</u> the recomputed prices eliminate the possibility of profitable output - i.e. recomputation preserves the normal linear optimality criteria,

<u>ii</u> a good has a zero price if it is a free good in the economic sense,

<u>iii</u> if there exist 'n' original inequalities such that these alone determine the same integer optimum as the total problem, the dual evaluators of the reduced problem give a unique set of recomputed prices. ( See (88).)

The general deficiencies of non-unique recomputations, the inability to cope with free goods (i.e. a good should be a free good if and only if it has a zero price), etc. all throw doubt on the pricing system of recomputed dual variables.

The fact that the optimal integer solution has been found using 'combinations of resources' as cutting planes indicates that resources can no longer be considered independent. Weingartner notes that the concept of a free good is not one which has a unique interpretation in integer programming.

Consider the programme of Figure 6.1. (The integer points are those on the lattice points.)

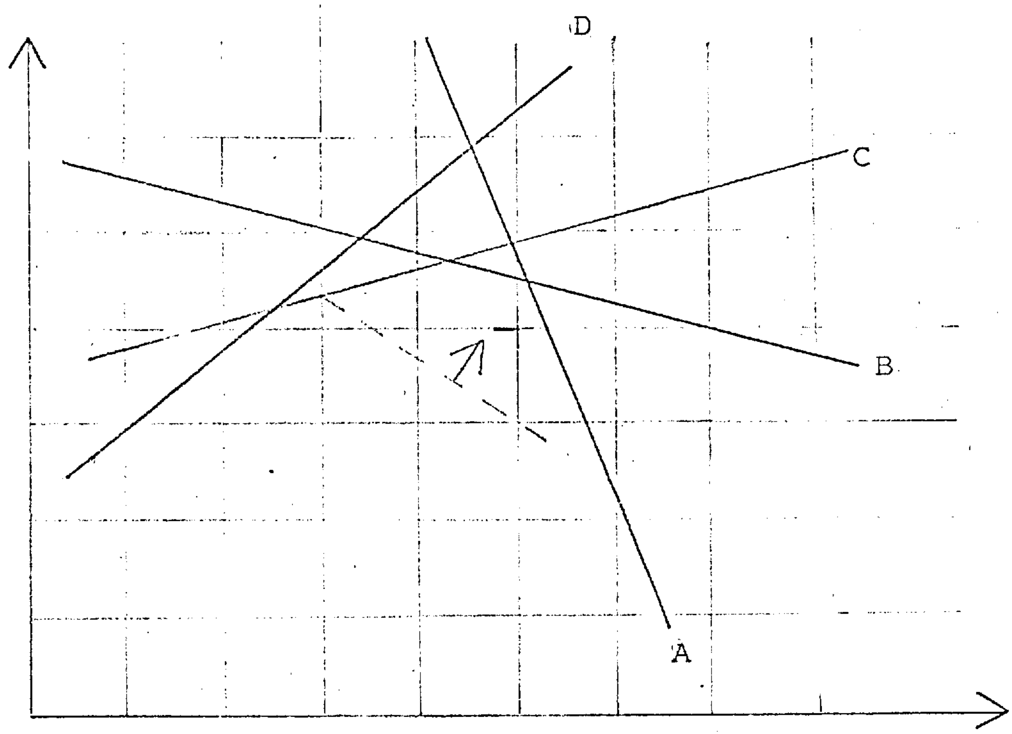



Figure 6.1 A typical integer lattice

Only the resource level corresponding to hyperplane D represents a truly free good. Either B or C may be removed without affecting the optimum; but the removal of both gives a different optimum. Neither B nor C represents a truly free good; they are not independent.

A further criticism of the Baumol/Gomory Prices has been made by Alcaly and Klevorick (2). In the linear case, the recomputed prices do not exhaust input factors; i.e. the pricing does not equate the value of inputs with the value of outputs.

Alcaly and Klevorick suggest two methods to overcome this; the first introduces a constant term to balance input and output. This is a 'subsidy' to the firm to keep it to 'integer production'. The method has all the failings of the Gomory and Baumol Method. The second method is one which redistributes prices amongst goods in such a way as to (artificially) ensure that a free good has a non-zero price.

All these methods are of doubtful practical use, but since they can easily be applied to the fractional programming case, the appropriate recomputations have been considered in Appendix 6.3.

There is an additional problem when recomputing dual evaluators in a non-linear environment; the dual evaluators of the intermediate non-linear programmes are not piecewise constant.

(The dual evaluators of the CC Form, (an LP), are piecewise linear, but those of the original form are not.) An implicit assumption in recomputed dual 'prices' is that the dual evaluators, themselves, are piecewise constant. This does not hold in the fractional case (or any case with a non-linear objective function). The recomputed prices of Appendix 6.3 ignore this non-linearity; like all recomputed prices they can only serve as guides to resource evaluation.

6.6.3 <u>Pricing via Minmax Duality Theory</u>

The difficulties of pricing by recomputed duals highlight the fact that the integer programming problem, as formulated in (6.13), has no dual - hence any interpretation of 'dual' prices is erroneous.

Balas (5), in his work on Duality in Discrete Programming has suggested the following approach to the dual of the integer programming problem. His work amplifies that of Wolfe (93), Mangasarian (61, 62) and Huard (48).

Let $X_1$, $U_1$ be arbitrary sets of vectors.

Let $X_2$, $U_2$ be sets of vectors in real space.

Balas defines two problems:

$$\min_{\underline{u}_1} \max_{\underline{x}, \underline{u}_2} K(\underline{x}, \underline{u}) - \underline{u}_2 \nabla_{\underline{u}_2} K(\underline{x}, \underline{u})$$

$$\text{s.t. } \nabla_{\underline{u}_2} K(\underline{x}, \underline{u}) \geq \underline{0} \qquad (6.18)$$

$$\underline{x}_2, \underline{u}_2 \geq \underline{0}$$

$$\underline{x}_1 \varepsilon X_1 \quad \underline{u}_1 \varepsilon U_1$$

and

$$\max_{\underline{x}_1} \min_{\underline{u}, \underline{x}_2} K(\underline{x}, \underline{u}) - \underline{x}_2 . \nabla_{\underline{x}_2} K(\underline{x}, \underline{u})$$

$$\text{s.t. } \nabla_{\underline{x}_2} K(\underline{x}, \underline{u}) \leq \underline{0} \qquad (6.19)$$

$$\underline{x}_2, \underline{u}_2 \geq \underline{0}$$

$$\underline{x}_1 \varepsilon X_1 \quad \underline{u}_1 \varepsilon U_1$$

where $K(\underline{x}, \underline{u})$ is the Lagrangian function

$$K(\underline{x}, \underline{u}) = f(\underline{x}) - \underline{u}.\underline{F}(\underline{x})$$

Balas proves that (6.18) and (6.19) are symmetric dual to each other. (Assumptions are made concerning the separability of $K(\underline{x}, \underline{u})$ with respect to either $\underline{u}_1$ or $\underline{x}_1$).

Let $U_1$ denote integer valued dual variables

$U_2$ denote real valued dual variables

$X_1$ denote integer valued primal variables

$X_2$ denote real valued primal variables

We are at 'liberty' to assume that the dual variables for the dual to (6.11) are real or integer valued. In the case of the linear objective function, integer programming implies discrete, integer-valued changes in the value of the objective for discrete changes of resources. Hence integer values are logically acceptable. In the case of the fractional objective, as in (6.11), this assumption is less justified due to the non-linearity of the objective function.

In either case, (real or integer-valued dual variables), it is readily seen that for a pure integer problem (6.11)

$$X_1 = \{\text{set of integers}\}$$

$$X_2 = \emptyset$$

The constraint set of (6.19) is empty, and the objective function is optimised for non-negative $\{\underline{u}_2\}$.

The implication of Balas' formulation is that dual 'prices' do not exist in pure integer programming since any reasonable allocation of dual variables in (6.19) will be possible. (If the dual variables are also integers the constraint set of (6.18) is empty. If they are real, (6.18) is the 'normal' integer programming problem with an additional allocation for $\underline{u}_2$ which is unconstrained).

Prices are generated in the mixed integer case. (Such prices are similar to the marginal values derived in Appendix 3.4). Here a penalty can be applied to, say, 'opening a new factory', when the returns to production are known. The penalty/subsidy mechanism in mixed integer programming derives its meaning from the pricing mechanism generated for the real valued variables and resources; in this case the Balas formulation preserves the normal economic criteria for profitable production.

# Chapter 7 Summary and Conclusions

## 7.1 Fractional Programming

In Chapters 4, 5, and 6 we have shown that LP methods have close counterparts in fractional programming, except in the application of duality and marginal pricing. We have given the conditions under which the marginal values of a fractional programme do show diminishing returns to scale. The methods of decomposition in FP, integer programming, post-optimal analysis, etc., have also been covered, and we have noted that a form of goal programming is also possible.

## 7.2 LP and Corporate Planning

The role of LP in corporate planning has not yet been defined. Linear models such as those of Cohen and Hammer (29), Chambers (13), and Chambers and Charnes (14) have been proposed as viable approaches to financial planning; the model developed in this thesis is intended to aid corporate financial planners in their short to medium strategic planning.

As we have seen, some authors demur. Objections are raised against the use of normative programming methods for corporate planning (and in particular against LP) because of the implied use of only one objective function, the disparity between the model and the real system, and the total neglect of sociological factors inherent in planning. In Chapter 3, we have shown that the optimal strategy and valuation of a firm varies according to the objectives (and environment), and that the differences between the model and the real system make the use of dual prices more difficult than LP theory would suggest. However, these difficulties, (an absence of one objective function, a multiplicity of interests, and an abstraction for planning purposes) are inherent in the planning exercise itself. They are not introduced by the LP approach. In this respect, the tool used for the 'solution of a problem' cannot be blamed for the initial intractibility of the problem itself. Multiple objectives, compromises with reality, etc., are part of the difficulty of corporate planning.

The inclusion of fractional programming for corporate modelling considerably broadens the scope of the linear approach. As we have seen, ratios can now be included as both objectives and constraints, without altering the basic linear approach. Fractional programmes can be used to rank alternatives as well as evaluate resources. This availability of a range of mathematical forms for the objective in one model framework, the present advances in integer and mixed-integer programming, and the speed and sophistication of the LP approach to planning (as compared to that of the accountant/ economist) still weigh heavily in favour of the use of linear models for corporate planning, (with the provisos outlined in Chapter 3).

The same justification cannot be applied to LP models used for control, or the valuation of assets, where it is vital to have a close correspondence between the model and the real system. The complexity of such models, and the difficulties associated with their solution (and interpretation) imply that control models based on LP would be impractical and expensive, even if the difficulties raised in Chapter 3 could be overcome. Similarly there are serious doubts attached to the use of LP models for asset valuation because of the presence of multiple corporate objectives.

Further work is necessary in the area of fractional programming in order to increase its 'planning power'. The compatability of the performance ratios of divisions and central management needs further study, as does the possibility of using decomposition in the setting of target performance ratios for a decentralised organisation. A second major area that requires further research is the analysis of risk and uncertainty in FP, using the methods outlined in Appendix 6.4, and the use of goal programming to analyse the importance of performance objectives for the corporate planner.

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# APPENDICES

## Appendix 2.1 The Mathematical Formulation

2.1.1 The size of the model is determined by a series of input parameters to the matrix generating programme, which define the extent of detail in the data. The variables defined in Table 2.2 are used for exposition only; being activity levels for the linear programme they appear as column names. Tables 2.3 and 2.4 contain the basic data on the product ranges, production requirements, "use of technology", and basic accounting data, used in constructing the set of constraints.

In order to model the time dependence of the accounting procedures, and the different rates of turnover for individual accounts, lags are introduced into the system. These provide the basic description of the possible cash flows through the planning period. The lags are derived from a study of the times between the incurring of a debt and the date at which it is settled, and are introduced into the mathematical formulation to ensure that the model will exhibit the same tardiness in settling accounts.

Lags are also introduced into the sales/storage equations to ensure that finished goods remain in the warehouse for some time prior to despatch. Here again the length of the lag has direct bearing on the cash flow, the amount of capital tied up in stocks, and warehouse utilisation.

The technological capacities and variable bounds model the physical and managerial restrictions on the possible operations of the firm during the planning period. Capacities and bounds, built in to the matrix generator are listed in Tables 2.6 and 2.7. These arrays, and the data used in the model are amplified in Appendix 2.2. (In all the following equations, the time subscript I = 0 or I ≤ 0 implies an input to the model, rather than a variable activity level, e.g. RAWM (O,L) is the input to the model of the L'th type of raw material.

| Variable | Interpretation |
|---|---|
| NPROD | number of products considered |
| NWC | number of work centres in the model |
| NSUB | number of work centres that can be 'subcontracted' |
| NLF | types of labour available |
| NRM | types of raw materials considered |
| NSCS | number of standard cost accounts in the model |
| NOH | number of overhead accounts in the model |
| NM | number of periods (months) to be considered, i.e. the planning horizon |

Table 2.1 The model parameters

| Array | Dimensions | Interpretation |
|---|---|---|
| PROD | I = 1, NM | amount of product K completed in |
| | K = 1, NPROD | period I |
| SALE | I = 1, NM | amount of product K sold in |
| | K = 1, NPROD | period I |
| SUB | I = 1, NM | amount of hours of work centre |
| | J ε SUBWC | SUBWC (J) subcontracted during |
| | | period I |
| RAWM | I = 1, NM | amount of raw materials of type J |
| | J = 1, NRM | stored at the end of period I |
| RMIN | I = 1, NM | amount of raw materials of type J |
| | J = 1, NRM | purchased during period I |
| MRKT | I = 1, NM | amount expended on promotion of |
| | J = 1, NPROD | product J in period I |
| STCS | I = 1, NM | J'th standard cost of sales in |
| | J = 1, NSCS | period I |
| OVHD | I = 1, NM | J'th overhead account of period I |
| | J = 1, NOH | |
| CASH | I = 1, NM | cash on hand at the end of period I |
| BNKL | I = 1, NM | amount borrowed during period I |
| BNKR | I = 1, NM | amount repayed during period I |
| BNKC | I = 1, NM | interest charges in period I |
| PAYS | I = 1, NM | total amount payable in period I |
| RECS | I = 1, NM | total amount receivable during |
| | | period I |

Table 2.2 The Model Variables

| Array | Dimensions | Interpretation |
|---|---|---|
| MCREQ | I = 1, 3<br>J = 1, NWC<br>K = 1, NPROD | hours of work centre J, required in period I - 1 before completion, for one unit of product K. |
| WCLF | J = 1, NWC<br>L = 1, NLF | hours of labour of type L required for one hour of production of facility J. |
| RMREQ | I = 1, 3<br>K = 1, NPROD<br>L = 1, NRM | raw materials of type L required in period I - 1 before completion, for one unit of product K. |
| SUBWC | J = 1, NSUB | work centres on which subcontracting is permissible. |

Table 2.3 Production/Technology Arrays

| Array | Dimension | Interpretation |
|---|---|---|
| LIST | K = 1, NPROD | list price of product K |
| SPACE | K = 1, NPROD | volume of product K in storage |
| SCSP | J = 1, NSCS<br>K = 1, NPROD | J'th standard cost of sale of one unit of product K |
| DISCP | K = 1, NPROD | discount allowed on list price of product K |
| SUBP | J = 1, NSUB | cost of subcontracting one hour's work of facility SUBWC (J) |
| WAGES | J = 1, NLF | hourly wage rate for J'th type of labour |
| RMB | J = 1, NRM | cost per unit of raw materials of type J |
| WIPP | I = 1, 2<br>K = 1, NPROD | value (for work-in-progress) of the K'th product, I periods before completion |
| MARK | I = 1, NPROD<br>K = 1, NPROD | rates at which unit sales imply costs of advertising (See Section 2.7.5) |
| OHRATE | J = 1, NOH | rate at which the J'th overhead account is calculated from the standard costs |
| ALPHA | | the rate of interest on loans |

Table 2.4 Accounting Data

| Array | Dimension | Interpretation |
| --- | --- | --- |
| LAG | K = 1, NPROD | minimum storage time for product K |
| RECLAG | K = 1, NPROD | lag between despatch of product K and receipt of payment |
| SUBLAG | J = 1, NSUB | lag on payment for use of J'th subcontracting facility |
| LABLAG | J = 1, NLF | lag on payment of wages for J'th type of labour |
| RMLAG | J = 1, NRM | lag on payment for raw materials of type J |
| MRKLAG | K = 1, NPROD | lag on payment for marketing expenditure for product of type K |
| OHLAG | J = 1, NOH | lag on payment of J'th overhead account |
| ALFLAG | | lag on interest payments |

Table 2.5 Accounting and Storage Lags

| Array | Dimensions | Interpretation |
|---|---|---|
| CAPWC | I = 1, NM<br>J = 1, NWC | capacity of work centre J in period I |
| CAPLF | I = 1, NM<br>J = 1, NLF | capacity of labour force (of type J) in period I |
| CAPST | I = 1, NM | storage capacity in period I |

Table 2.6 Technological capacities

| Bounds | Dimensions | Interpretation |
|---|---|---|
| POLICY | I = 1, NM<br>K = 1, NPROD | minimum sales of product K in period I |
| CASHLO | I = 1, NM | minimum cash balance at the end of period I |
| CASHUP | I = 1, NM | maximum cash balance at the end of period I |
| BANKLO | I = 1, NM | minimum bank loan during period I |
| BANKUP | I = 1, NM | maximum bank loan during period I |
| RMLO | I = 1, NM<br>J = 1, NRM | minimum materials balance of type J at the end of period I |
| RMUP | I = 1, NM<br>J = 1, NRM | maximum materials balance of type J at the end of period I |

Table 2.7 Bounds on acceptable variable levels

### 2.1.2 The Intra-Period Constraints

i. Gross sales:

$$\text{Gross sales (I)} = \sum_{K=1}^{NPROD} \text{SALE (I,K) . LIST (K)} \qquad I = 1...NM \qquad (2.6)$$

ii. Standard costs of sales:

$$\text{STCS (I,J)} = \sum_{K=1}^{NPROD} \text{SALE (I,K) . SCSP (J,K)} \qquad I = 1...NM,\; J = 1...NSCS \qquad (2.7)$$

iii. Overhead accounts:

$$\text{OVHD (I,J)} = \text{STCS (I,J) . OHRATE (J)} \qquad I = 1...NM,\; J = 1...NOH \qquad (2.8)$$

iv. Discount on sales:

$$\text{Discount (I)} = \sum_{K=1}^{NPROD} \text{SALES (I,K) . DISCP (K).LIST (K)} \qquad I = 1...NM \qquad (2.9)$$

v. Net sales:

$$\text{Net sales (I)} = \sum_{K=1}^{NPROD} \text{SALES (I,K) . LIST (K)} \left[\text{1-DISCP (K)}\right] \qquad I = 1...NM \qquad (2.10)$$

Manufacturing margin

$$\text{Manufacturing margin (I)} = \text{net sales (I)} - \sum_{J=1}^{NSCS} \text{STCS (I,J)} \qquad I = 1...NM \qquad (2.11)$$

### 2.1.3 The Inter-Period Constraints

a. Accounting sums and equations

i. Work-in-progress:

$$\text{Work-in-progress (I)} = \sum_{J=1}^{2} \text{PROD (I+J,K) . WIPP (3-J,K)} \qquad I = 1....NM-2 \qquad (2.12)$$

(appropriate adjustment is needed for end of planning horizon to allow for production beyond the NM'th period).

ii. Finished goods:

Change in finished goods account over initial value

$$= \sum_{J=1}^{I} \sum_{K=1}^{NPROD} (PROD\ (I,K) - SALE\ (I,K))\ .\ LIST\ (K) \qquad (2.13)$$

iii. Payables:

Let $\hat{I}$ be an adjustment for I corresponding to the relative accounting lag: e.g. let BNKC $(\hat{I})$ = BNKC (I - ALFLAG)

$$I = 1...NM$$

then:

$$PAYS\ (I) = \sum_{J=1}^{NOH} OVHD\ (\hat{I},J) + \sum_{J=1}^{NRM} RMIN\ (\hat{I},J)\ .\ RMB\ (J)$$

$$+ \sum_{J=1}^{NPROD} MRKT\ (\hat{I},J) + BNKC\ (\hat{I}) + \sum_{J=1}^{NSUB} SUB\ (\hat{I},J)\ .\ SUBP\ (J)$$

$$+ \sum_{M=1}^{NLF} \sum_{L=1}^{NWC} WAGES\ (M)\ .\ WCLF\ (I,M)\ .$$

$$\left\{ \sum_{J=0}^{2} \sum_{K=1}^{NPROD} (MCREQ\ (J+1,L,K)\ .\ PROD\ (I+J,K)) - SUB\ (I,L) \right\}$$

$$I = 1...NM \qquad (2.14)$$

(for consideration of the last term see b.i. below: (Work Centre Capacity)).

iv. Receivables:

$$RECS\ (I) = \sum_{K=1}^{NPROD} SALE\ (\hat{I},K)\ .\ LIST\ (K) \left[ 1 - DISCP\ (K) \right]$$

$$I = 1...NM \qquad (2.15)$$

v. Bank charges:

$$BNKC\ (I) = \sum_{J=1}^{I} \left\{ BNKL\ (J) - BNKR\ (J) \right\}\ ALPHA$$

$$I = 1...NM \qquad (2.16)$$

If BNKC (I) is constrained to be positive

$$\sum_{J=1}^{I} \left\{ BNKL\ (J) - BNKR\ (J) \right\} \geq 0$$

i.e. total repayments cannot exceed total loans.

If BNKC (I) is not constrained, the model is able to invest (as well as borrow) at the interest rate ALPHA.

vi. Marketing Expenses:

$$MRKT\ (I,J) = \sum_{K=1}^{NPROD} SALE\ (I,K) . MARK\ (J,K) \qquad (2.17)$$

b. Capacity constraints

i. Work centre capacity:

$$\sum_{J=0}^{2} \sum_{K=1}^{NPROD} MCREQ(J+1,L,K) . PROD(I+J,K) - SUB(I,L) \leq CAPWC\ (I,L)$$

$$I = 1...NM$$

$$L = 1...NWC \qquad (2.18)$$

ii. Labour force capacity:

$$\sum_{L=1}^{NWC} WCLF(L,M) . \left\{ \sum_{J=0}^{2} \sum_{K=1}^{NPROD} MCREQ(J+1,L,K) . PROD(I+J,K) - SUB(I,L) \right\} \leq CAPLF\ (I,M)$$

$$I = 1...NM$$

$$M = 1...NLF \qquad (2.19)$$

iii. Storage capacity:

$$\sum_{J=1}^{I} \sum_{K=1}^{NPROD} (PROD(J,K) - SALE(J,K)) . SPACE(K) \leq CAPST(I)$$

$$I = 1...NM \qquad (2.20)$$

CAPST is the storage space (over and above that used at the onset of the model) available in period I.

iv. Materials usage:

$$\sum_{J=0}^{2} PROD\ (I+J,K) . RMREQ\ (J+1,K,L) \leq RAWM\ (I-1,L)$$

$$I = 2...NM \qquad (2.21)$$
$$L = 1...NRM$$

c. Continuity Constraints

i. Materials balance:

$$RAWM\ (I,L) = RAWM\ (I-1,L) + RMIN\ (I,L) - \sum_{J=0}^{2} PROD\ (I+J,K) .$$

$$RMREQ\ (J+1,K,L) \qquad (2.22)$$

$$I = 1...NM$$

$$L = 1...NRM$$

For I = 1, the initial input of raw materials is used on the right hand side, i.e. raw materials input = RAWM (0,L)

ii. Cash continuity:

$$CASH(I) = CASH(I-1) - PAYS(I) + RECS(I)$$

$$I = 1...NM \qquad (2.23)$$

iii. Storage requirements:

$$SALE(I,K) \le \sum_{J=0}^{I-LAG(K)} (PROD(J,K) - SALE(J,K))$$

$$I = 1...NM \qquad (2.24)$$

### 2.1.4 The bounds on variable levels

i. Minimum sales policy:

$$SALE(I,K) \ge POLICY(I,K)$$

$$K = 1...NPROD$$

$$I = 1...NM \qquad (2.25)$$

ii. Cash balance:

$$CASHLO(I) \le CASH(I) \le CASHUP(I)$$

$$I = 1...NM \qquad (2.26)$$

iii. Limits on bank loans:

$$BANKLO(I) \le BNKL(I) \le BANKUP(I)$$

$$I = 1...NM \qquad (2.27)$$

iv. Raw materials balance:

$$RMLO(I,J) \le RAWM(I,J) \le RMUP(I,J)$$

$$J = 1...NRM$$

$$I = 1...NM \qquad (2.28)$$

### 2.1.5 The Objective Function

i. 'Change' in current assets:

$$ASSETS = \sum_{I=1}^{NM} \sum_{K=1}^{NPROD} \{PROD(I,K) - SALE(I,K)\} . LIST(K)$$

$$+ \sum_{J=1}^{NRM} RAWM(NM,J) . RMB(J) + \sum_{\hat{I}>NM} SALE(I,K) . LIST(K)$$

$$[1-DISCP(K)] + CASH(NM) \qquad (2.29)$$

where $\hat{I} = I + \text{RECLAG}(K)$

ii. 'Change' in current liabilities:

$$\text{LIABLES} = \sum_{I=1}^{NM} \left\{ \text{BNKL}(I) - \text{BNKR}(I) \right\}$$

$$+ \sum_{\hat{I}>NM} \left\{ \sum_{K=1}^{NSUB} \left[ \text{SUB}(I,K) \cdot \text{SUBP}(K) \right] \right.$$

$$\sum_{K=1}^{NRM} \text{RMIN}(I,K) \cdot \text{RMB}(K) + \sum_{K=1}^{NPROD} \text{MRKT}(I,K)$$

$$\left. \sum_{K=1}^{NOH} \text{OVHD}(I,K) + \text{BNKC}(I) \right\}$$

where $\hat{I} = I$ + appropriate accounting lag (2.30)

iii. Gross Sales:

$$\text{GROSSALE} = \sum_{I=1}^{NM} \sum_{K=1}^{NPROD} \text{SALE}(I,K) \cdot \text{LIST}(K) \qquad (2.31)$$

2.1.6 The size of the model

The size of the model is determined by the input parameters of Table 2.1. For the equations outlined above, these parameters determine the size of the problem as follows: Let HIGH be the row dimension per period and LONG be the row dimension per period. Then

$$\text{HIGH} = 3\text{NTOOL} + \text{NWC} + \text{NLF} + 2\text{NRM} + \text{NSCS} + \text{NOH} + 11$$

$$\text{LONG} = 4\text{NTOOL} + \text{NWC} + \text{NSUB} + \text{NLF} + 2\text{NRM} + \text{NSCS} + \text{NOH} + 12 \qquad (2.32)$$

The total dimension of the initial tableau is NM x HIGH by NM x LONG; any objective functions are added to this.

## Appendix 2.2 Model Data and the Aggregation Programmes

### 2.2.1 Introduction

The data for the models was obtained from the test firm, and processed for use with the LP model described in Section 2.4 and Appendix 2.1. (The processing was carried out on an IBM 1130 machine).

The input data for the model consists of: the technological data; the accounting data; the time lags; and the input parameters. Details of the data preparation for these sections are listed below.

The work presented in this section gives details of the figures used in the 26/12 model; the small models 3/5, etc. are obtained by taking the first 3 items of production or accounting data.

These computations were intended primarily to test the model, and its reactions to subsequent analysis and theoretical applications. It is not in the interest of the test firm to present figures that bear too close a relation to their actual results, therefore, where data was not immediately available at the time of computation, broad assumptions have been made concerning the unknown figures. Thus, the numerical results presented do not conflict with the firm's wish that such items should be confidential. The 'assumed' data is in areas where no processing was necessary; prices, market requirements, etc. The treatment of all processed data, and the allied assumptions are fully documented.

### 2.2.2 The Technological Data

2.2.2.1 Work Centre Aggregation: As mentioned in Section 2.2.4, the company used a coding system for each of its work centres; in total there were 215 such work centre codes.

For each work centre, card data was available specifying the monthly capacity in machine hours. Such cards are shown in Table 2.8 for the months of June to October for centres 1101 to 2702.

From Appendix 2.1 we know that the row dimensions of the LP model vary with the number of work centres considered per month, thus using 215 centres in a twelve month model would immediately involve 2580 rows; (the capacity of standard LP packages is 4095 rows).

Consultation with the production staff at the firm resulted in the conclusion that it would be adequate, for planning purposes, to consider eighteen 'aggregate' work centres for the model. (These 'centres' are listed in Table 2.9). Data such as that in Table 2.8 was then aggregated to give the firm's total monthly capacity for the new work centres, for the twelve month period October to September. This data is shown in Table 2.10 and was used for the work centre capacities of the models, i.e. the WCCAP array.

Management policy insisted that all heat treatment, winding, packing, etc. be done on the firm's machinery. Thus the work centres on which subcontracting was allowed were numbers one to nine, omitting three and four. This is summarised in Table 2.11.

For the data arrays we have NSUB = 7, and SUBWC = $\{1, 2, 5, 6, 7, 8, 9\}$.

2.2.2.2 Production requirements: For each product of the firm's range, data was available showing how much time was required per hundred units of production. A typical set of requirements is shown in Table 2.12. This data was reorganised to give the requirements, per hundred units, on the aggregate

| CENTRE | JUN | JUL | AUG | SEP | OCT |
|---|---|---|---|---|---|
| 1101 | 2035 | 1696 | 848 | 2035 | 1696 |
| 1102 | 0 | 0 | 0 | 0 | 0 |
| 1103 | 0 | 0 | 0 | 0 | 0 |
| 1104 | 1017 | 848 | 424 | 1017 | 848 |
| 1105 | 0 | 0 | 0 | 0 | 0 |
| 1106 | 0 | 0 | 0 | 0 | 0 |
| 1107 | 0 | 0 | 0 | 0 | 0 |
| 1108 | 0 | 0 | 0 | 0 | 0 |
| 1109 | 1526 | 1272 | 636 | 1526 | 1272 |
| 1110 | 0 | 0 | 0 | 0 | 0 |
| 1201 | 0 | 0 | 0 | 0 | 0 |
| 1202 | 508 | 424 | 212 | 508 | 424 |
| 1203 | 0 | 0 | 0 | 0 | 0 |
| 1204 | 0 | 0 | 0 | 0 | 0 |
| 1205 | 0 | 0 | 0 | 0 | 0 |
| 1206 | 0 | 0 | 0 | 0 | 0 |
| 1207 | 0 | 0 | 0 | 0 | 0 |
| 1301 | 0 | 0 | 0 | 0 | 0 |
| 1302 | 0 | 0 | 0 | 0 | 0 |
| 1303 | 0 | 0 | 0 | 0 | 0 |
| 1401 | 0 | 0 | 0 | 0 | 0 |
| 1501 | 0 | 0 | 0 | 0 | 0 |
| 1502 | 652 | 544 | 272 | 652 | 544 |
| 1503 | 0 | 0 | 0 | 0 | 0 |
| 1504 | 0 | 0 | 0 | 0 | 0 |
| 1505 | 0 | 0 | 0 | 0 | 0 |
| 1506 | 0 | 0 | 0 | 0 | 0 |
| 1507 | 0 | 0 | 0 | 0 | 0 |
| 1508 | 0 | 0 | 0 | 0 | 0 |
| 1601 | 0 | 0 | 0 | 0 | 0 |
| 1602 | 652 | 544 | 272 | 652 | 544 |
| 1603 | 0 | 0 | 0 | 0 | 0 |
| 1701 | 326 | 272 | 136 | 326 | 272 |
| 1801 | 0 | 0 | 0 | 0 | 0 |
| 2101 | 0 | 0 | 0 | 0 | 0 |
| 2102 | 691 | 576 | 288 | 691 | 576 |
| 2103 | 0 | 0 | 0 | 0 | 0 |
| 2201 | 691 | 576 | 288 | 691 | 576 |
| 2301 | 345 | 288 | 144 | 345 | 288 |
| 2401 | 0 | 0 | 0 | 0 | 0 |
| 2501 | 691 | 576 | 288 | 691 | 576 |
| 2502 | 691 | 576 | 288 | 691 | 576 |
| 2503 | 345 | 288 | 144 | 345 | 288 |
| 2601 | 1036 | 864 | 432 | 1036 | 864 |
| 2602 | 0 | 0 | 0 | 0 | 0 |
| 2603 | 2764 | 2304 | 1152 | 2764 | 2304 |
| 2604 | 0 | 0 | 0 | 0 | 0 |
| 2701 | 345 | 288 | 144 | 345 | 288 |
| 2702 | 0 | 0 | 0 | 0 | 0 |

TABLE 2.8 TYPICAL DATA OF MONTHLY WORK CENTRE CAPACITIES

| No. | Description |
|---|---|
| 1 | Multi-spindle automatic lathes, single-spindle automatic lathes etc. |
| 2 | Other lathes and boring equipment |
| 3 | Heat treatment |
| 4 | Gear cutting |
| 5 | Grinding |
| 6 | Drilling |
| 7 | Milling |
| 8 | Pressing |
| 9 | Finishing |
| 10 | Field winding |
| 11 | Stator winding |
| 12 | Armature winding |
| 13 | Degreasing and hand spraying |
| 14 | Assembly ( domestic ) |
| 15 | Assembly ( industrial ) |
| 16 | Testing and inspection |
| 17 | Packaging |
| 18 | Final inspection |

Table 2.9 The Aggregated Work Centres

| | OCT | NOV | DEC | JAN | FEB | MAR | APR | MAY | JUN | JUL | AUG | SEP |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 12080 | 12080 | 14494 | 12080 | 12080 | 14494 | 12080 | 12080 | 14494 | 12080 | 12080 | 6040 |
| 2 | 16200 | 16200 | 19438 | 16200 | 16200 | 19438 | 16200 | 16200 | 19438 | 16200 | 16200 | 8100 |
| 3 | 16136 | 16136 | 19358 | 16136 | 16136 | 19358 | 16136 | 16136 | 19358 | 16136 | 16136 | 8068 |
| 4 | 12720 | 12720 | 15262 | 12720 | 12720 | 15262 | 12720 | 12720 | 15262 | 12720 | 12720 | 6500 |
| 5 | 11928 | 11928 | 14311 | 11928 | 11928 | 14311 | 11928 | 11928 | 14311 | 11928 | 11928 | 5964 |
| 6 | 16368 | 16368 | 19639 | 16368 | 16368 | 19639 | 16368 | 16368 | 19639 | 16368 | 16368 | 8184 |
| 7 | 4456 | 4456 | 5346 | 4456 | 4456 | 5346 | 4456 | 4456 | 5346 | 4456 | 4456 | 2228 |
| 8 | 11384 | 11384 | 13659 | 11384 | 11384 | 13659 | 11384 | 11384 | 13659 | 11384 | 11384 | 5692 |
| 9 | 11024 | 11024 | 13226 | 11024 | 11024 | 13226 | 11024 | 11024 | 13226 | 11024 | 11024 | 5512 |
| 10 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |
| 11 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |
| 12 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |
| 13 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |
| 14 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |
| 15 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |
| 16 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |
| 17 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |
| 18 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 |

TABLE 2.10 MONTHLY CAPACITIES FOR AGGRETATED WORK CENTRES

| WORK CENTRE | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 - 18 |
|---|---|---|---|---|---|---|---|---|---|---|
| LABOUR REQUIREMENT | .968 | .823 | .5 | .75 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| SUBCONTRACTING | YES | YES | NO | NO | YES | YES | YES | YES | YES | NO |

TABLE 2.11 LABOUR REQUIREMENTS AND SUBCONTRACTING ON WORK CENTRES

| Centre | Hours per 100 Items |
|---|---|
| 1101 | 57 |
| 1102 | 480 |
| 1103 | 162 |
| 1106 | 143 |
| 1202 | 113 |
| 1206 | 207 |
| 1301 | 350 |
| 1501 | 333 |
| 1502 | 500 |
| 1504 | 550 |
| 1601 | 333 |
| 1602 | 375 |
| 1603 | 167 |
| 1701 | 43 |
| 2101 | 117 |
| 2102 | 35 |
| 2301 | 5 |
| 2502 | 22 |
| 2503 | 2 |
| 2603 | 175 |
| 2701 | 50 |
| 2702 | 19 |
| 2801 | 35 |
| 3101 | 1025 |
| 3202 | 225 |
| 3203 | 742 |
| 3501 | 100 |
| 4101 | 225 |
| 4103 | 735 |
| 4401 | 353 |
| 4601 | 33 |

Table 2.12 Typical Work Centre Requirements

work centres described in Table 2.9. For the twenty six products considered thus far, the machine requirements are shown in Table 2.13.

Estimates of the set-up times required per product on the aggregate work centres were also obtained. This set of estimates is shown in Table 2.14. No account was taken here (or in the firm) of the effect of sequencing of products on the set-up times between production runs. The data of Table 2.15 is used for the MCREQ arrays of the model.

2.2.2.3 <u>The Labour Force Requirements</u>: Many of the work centres (of the firm) did not involve full time operator attention, i.e. the time used on work centres was no direct guide to the labour force requirements. A study was undertaken to determine the operator time required per hour of machine time per aggregate work centre. The results are shown in Table 2.11 as the labour requirement (in hours) of each centre, per hour operating time. The estimated total available per month was 90,000 man-hours. It was assumed, during these computations, that there was only one form of labour; the hourly wage was taken as £0.375.

2.2.2.4 <u>Raw Materials Requirements</u>: For the testing of the model, it was assumed that there would be only one type of raw material input - thus 'raw materials' could be considered as one homogeneous resource. The requirements for each product could be allocated according to the use of work centres and stages of production.

As a starting assumption it was assumed that the materials requirement per month of the production were identical. Thus, for production spread over three months, a third of the raw materials input was required each month. This assumption over-

| CENTRE | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 18 | 0 | 26 | 29 | 20 | 0 | 0 | 5 | 1 | 0 | 473 | 0 | 217 | 0 | 0 | 0 | 41 | 0 |
| 2 | 78 | 12 | 75 | 104 | 49 | 119 | 6 | 5 | 44 | 51 | 114 | 87 | 14 | 426 | 0 | 0 | 0 | 0 |
| 3 | 155 | 78 | 79 | 111 | 33 | 155 | 18 | 6 | 76 | 0 | 50 | 0 | 315 | 0 | 0 | 0 | 55 | 0 |
| 4 | 25 | 0 | 25 | 29 | 20 | 2 | 0 | 3 | 1 | 51 | 114 | 109 | 10 | 207 | 0 | 0 | 0 | 0 |
| 5 | 149 | 1 | 41 | 76 | 33 | 111 | 0 | 9 | 55 | 0 | 238 | 0 | 60 | 187 | 0 | 0 | 65 | 0 |
| 6 | 64 | 20 | 51 | 54 | 58 | 176 | 13 | 26 | 37 | 0 | 540 | 0 | 43 | 0 | 0 | 0 | 0 | 0 |
| 7 | 92 | 52 | 58 | 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 96 | 125 | 127 | 58 | 63 | 69 | 15 | 48 | 142 | 0 | 358 | 0 | 48 | 264 | 0 | 0 | 67 | 0 |
| 9 | 35 | 44 | 19 | 44 | 15 | 95 | 12 | 7 | 66 | 0 | 301 | 0 | 52 | 0 | 282 | 0 | 62 | 0 |
| 10 | 88 | 15 | 52 | 100 | 84 | 144 | 12 | 13 | 175 | 0 | 427 | 0 | 52 | 0 | 292 | 0 | 52 | 0 |
| 11 | 204 | 140 | 89 | 96 | 49 | 183 | 74 | 10 | 72 | 0 | 400 | 0 | 47 | 0 | 373 | 0 | 37 | 0 |
| 12 | 135 | 137 | 71 | 177 | 108 | 134 | 0 | 20 | 197 | 0 | 610 | 0 | 94 | 0 | 284 | 0 | 86 | 0 |
| 13 | 127 | 22 | 74 | 114 | 128 | 163 | 12 | 2 | 132 | 0 | 302 | 0 | 57 | 0 | 211 | 0 | 48 | 0 |
| 14 | 124 | 176 | 70 | 46 | 11 | 157 | 10 | 21 | 106 | 0 | 888 | 0 | 56 | 0 | 384 | 0 | 495 | 0 |
| 15 | 107 | 110 | 38 | 129 | 40 | 168 | 12 | 14 | 151 | 0 | 523 | 0 | 49 | 0 | 394 | 0 | 45 | 0 |
| 16 | 103 | 112 | 40 | 110 | 57 | 168 | 12 | 14 | 151 | 0 | 523 | 0 | 71 | 0 | 300 | 0 | 58 | 0 |
| 17 | 111 | 118 | 41 | 92 | 63 | 168 | 12 | 14 | 136 | 0 | 523 | 0 | 71 | 0 | 344 | 0 | 58 | 0 |
| 18 | 224 | 148 | 95 | 136 | 68 | 229 | 78 | 10 | 135 | 0 | 522 | 0 | 47 | 0 | 328 | 0 | 37 | 0 |
| 19 | 297 | 1134 | 370 | 599 | 343 | 367 | 196 | 77 | 419 | 0 | 1321 | 0 | 83 | 0 | 444 | 0 | 75 | 0 |
| 20 | 281 | 411 | 231 | 306 | 167 | 527 | 135 | 64 | 505 | 0 | 826 | 0 | 0 | 0 | 456 | 0 | 119 | 0 |
| 21 | 353 | 172 | 111 | 202 | 86 | 190 | 103 | 57 | 279 | 0 | 802 | 0 | 205 | 0 | 438 | 0 | 83 | 0 |
| 22 | 107 | 310 | 101 | 15 | 72 | 166 | 0 | 236 | 218 | 0 | 0 | 0 | 650 | 0 | 0 | 0 | 37 | 0 |
| 23 | 248 | 195 | 53 | 77 | 112 | 118 | 8 | 14 | 44 | 0 | 0 | 0 | 59 | 0 | 420 | 0 | 66 | 0 |
| 24 | 403 | 350 | 215 | 503 | 197 | 247 | 16 | 73 | 189 | 0 | 832 | 0 | 115 | 0 | 446 | 0 | 132 | 0 |
| 25 | 422 | 332 | 203 | 546 | 197 | 250 | 16 | 73 | 184 | 0 | 832 | 0 | 115 | 0 | 449 | 0 | 134 | 0 |
| 26 | 407 | 1704 | 651 | 294 | 441 | 675 | 454 | 27 | 475 | 0 | 1433 | 0 | 90 | 0 | 619 | 0 | 108 | 0 |

Table 2.13 Work Centre Requirements ( hours per hundred items )

| CENTRE | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 460 | 0 | 40 | 20 | 100 | 0 | 0 | 30 | 0 | 0 | 6000 | 0 | 3000 | 0 | 0 | 0 | 4000 | 0 |
| 2 | 730 | 0 | 120 | 100 | 150 | 150 | 30 | 50 | 30 | 0 | 20 | 100 | 30 | 120 | 0 | 0 | 0 | 0 |
| 3 | 1590 | 160 | 120 | 130 | 150 | 160 | 30 | 70 | 40 | 0 | 430 | 0 | 5250 | 0 | 0 | 0 | 1000 | 0 |
| 4 | 540 | 0 | 40 | 20 | 100 | 10 | 0 | 110 | 0 | 0 | 20 | 100 | 10 | 50 | 0 | 0 | 0 | 0 |
| 5 | 300 | 0 | 40 | 30 | 70 | 120 | 0 | 70 | 20 | 0 | 60 | 0 | 10 | 30 | 0 | 0 | 10 | 0 |
| 6 | 630 | 40 | 60 | 50 | 130 | 80 | 10 | 30 | 10 | 0 | 4500 | 0 | 500 | 0 | 0 | 0 | 0 | 0 |
| 7 | 610 | 50 | 60 | 30 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 920 | 170 | 120 | 40 | 210 | 60 | 20 | 160 | 40 | 0 | 60 | 0 | 10 | 30 | 0 | 0 | 20 | 0 |
| 9 | 500 | 110 | 50 | 30 | 80 | 90 | 10 | 120 | 10 | 0 | 60 | 0 | 10 | 0 | 40 | 0 | 20 | 0 |
| 10 | 920 | 80 | 70 | 60 | 170 | 190 | 10 | 90 | 30 | 0 | 70 | 0 | 10 | 0 | 70 | 0 | 20 | 0 |
| 11 | 1720 | 250 | 80 | 70 | 220 | 140 | 50 | 60 | 30 | 0 | 60 | 0 | 10 | 0 | 50 | 0 | 10 | 0 |
| 12 | 1110 | 240 | 50 | 80 | 170 | 160 | 0 | 90 | 30 | 0 | 60 | 0 | 10 | 0 | 40 | 0 | 10 | 0 |
| 13 | 990 | 40 | 70 | 70 | 210 | 190 | 10 | 40 | 20 | 0 | 60 | 0 | 10 | 0 | 40 | 0 | 10 | 0 |
| 14 | 810 | 180 | 80 | 30 | 60 | 80 | 10 | 80 | 40 | 0 | 5810 | 0 | 10 | 0 | 60 | 0 | 70 | 0 |
| 15 | 0 | 0 | 20 | 10 | 40 | 0 | 0 | 0 | 0 | 0 | 5750 | 0 | 10 | 0 | 50 | 0 | 10 | 0 |
| 16 | 930 | 150 | 50 | 60 | 110 | 130 | 10 | 40 | 40 | 0 | 5750 | 0 | 10 | 0 | 50 | 0 | 20 | 0 |
| 17 | 0 | 0 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5750 | 0 | 10 | 0 | 50 | 0 | 20 | 0 |
| 18 | 1850 | 250 | 100 | 90 | 240 | 180 | 60 | 50 | 30 | 0 | 5500 | 0 | 10 | 0 | 60 | 0 | 10 | 0 |
| 19 | 1600 | 860 | 110 | 230 | 330 | 220 | 70 | 120 | 50 | 0 | 6000 | 0 | 10 | 0 | 60 | 0 | 10 | 0 |
| 20 | 1430 | 340 | 90 | 130 | 190 | 270 | 90 | 140 | 40 | 0 | 5000 | 0 | 0 | 0 | 50 | 0 | 20 | 0 |
| 21 | 1170 | 310 | 90 | 70 | 110 | 170 | 70 | 150 | 40 | 0 | 50 | 0 | 10 | 0 | 70 | 0 | 10 | 0 |
| 22 | 530 | 380 | 30 | 10 | 40 | 150 | 0 | 160 | 30 | 0 | 0 | 0 | 5250 | 0 | 0 | 0 | 750 | 0 |
| 23 | 890 | 230 | 60 | 60 | 130 | 60 | 10 | 60 | 10 | 0 | 0 | 0 | 10 | 0 | 60 | 0 | 10 | 0 |
| 24 | 0 | 0 | 20 | 40 | 0 | 0 | 0 | 0 | 0 | 0 | 5000 | 0 | 10 | 0 | 50 | 0 | 20 | 0 |
| 25 | 1720 | 320 | 90 | 170 | 190 | 250 | 10 | 180 | 40 | 0 | 5000 | 0 | 10 | 0 | 50 | 0 | 20 | 0 |
| 26 | 1280 | 900 | 120 | 110 | 350 | 310 | 130 | 100 | 30 | 0 | 5500 | 0 | 10 | 0 | 60 | 0 | 10 | 0 |

Table 2.14 Set-up Requirements ( hours per product )

| | LIST | STANDARD COSTS | | | SIZE |
|---|---|---|---|---|---|
| 1 | 375.000 | 135.709 | 20.450 | 23.110 | 0.679 |
| 2 | 570.000 | 142.430 | 38.970 | 48.110 | 0.765 |
| 3 | 700.000 | 166.660 | 44.060 | 55.300 | 0.792 |
| 4 | 330.000 | 124.770 | 19.170 | 21.940 | 0.686 |
| 5 | 700.000 | 153.410 | 33.729 | 40.819 | 1.525 |
| 6 | 500.000 | 144.739 | 42.360 | 51.839 | 0.686 |
| 7 | 479.999 | 205.739 | 37.770 | 44.809 | 2.236 |
| 8 | 479.999 | 158.210 | 37.660 | 45.019 | 1.286 |
| 9 | 600.000 | 138.910 | 47.569 | 56.800 | 0.543 |
| 10 | 675.000 | 163.340 | 58.819 | 71.460 | 0.543 |
| 11 | 760.000 | 198.589 | 76.089 | 93.220 | 0.678 |
| 12 | 000.000 | 259.090 | 83.400 | 99.649 | 1.794 |
| 13 | 700.000 | 186.130 | 63.599 | 78.360 | 0.792 |
| 14 | 930.000 | 173.070 | 68.590 | 83.500 | 1.149 |
| 15 | 909.999 | 236.570 | 67.410 | 82.860 | 0.655 |
| 16 | 945.000 | 238.529 | 66.619 | 81.779 | 0.655 |
| 17 | 020.000 | 230.969 | 67.830 | 83.419 | 0.524 |
| 18 | 175.000 | 199.029 | 61.639 | 75.509 | 0.655 |
| 19 | 300.000 | 460.679 | 230.899 | 291.030 | 1.351 |
| 20 | 950.000 | 562.119 | 161.329 | 203.230 | 3.126 |
| 21 | 940.000 | 526.050 | 139.230 | 167.520 | 4.660 |
| 22 | 715.000 | 512.609 | 115.080 | 139.540 | 3.326 |
| 23 | 500.000 | 351.179 | 140.980 | 165.850 | 1.696 |
| 24 | 900.000 | 516.939 | 157.029 | 195.630 | 3.259 |
| 25 | 900.000 | 453.809 | 153.109 | 190.809 | 3.225 |
| 26 | 350.000 | 641.810 | 329.450 | 421.179 | 2.156 |

TABLE 2.15 LIST PRICES, STANDARD COSTS AND PACKAGE SIZES

stated the requirement, since it was noted that for many products, the materials requirement was greater during the final production month. (Further study of materials requirements will be necessary before implementation. At present the firm uses no planning of raw materials associated with production plans, thus even the crude splitting by three will show a possible saving over present methods). The data for RMREQ is thus the standard cost of materials, of Table 2.15, the price (RMB) is unity.

2.2.3 The Accounting Data

2.2.3.1 Basic Figures: The basic accounting data for the twenty six product model is shown in Table 2.15; the figures are given per hundred items. The standard costs are in the order: materials, labour, overheads. Finished goods were stored in metal bins at the two main warehouses (attached to two of the factories, one near London and one in the North of England). Their capacities were 5130 and 1257 bins respectively, giving a total of approximately 6500 bins (allowing some storage at the third factory). The space figures of Table 2.15 are the number of bins required per hundred items of product.

2.2.3.2 Treatment of Overhead Accounts: In the model, the standard cost of sales is calculated from the sum of the respective standard costs: it was proposed that the actual cost of sales be estimated in a similar way. Considering the accounts for 1966 and 1967, Table 2.17, we can estimate the total variances on each account, based on a summary of these figures.

| 1966 Account | Standard | Rate Var. | % | Usage Var. | % |
|---|---|---|---|---|---|
| Material | 4,024,930 | -141,488 | -3.5 | 115,875 | 2.9 |
| Labour | 791,685 | 42,489 | 5.3 | 91,707 | 11.5 |
| O'head | 956,780 | 568,575 | 59.4 | 99,297 | 10.4 |

| 1967 Account | Standard | Rate Var. | % | Usage Var. | % |
|---|---|---|---|---|---|
| Material | 4,772,552 | -146,517 | -3.0 | -17,617 | -0.03 |
| Labour | 940,175 | 58,311 | 6.2 | 79,244 | 8.4 |
| O'head | 1,140,157 | 444,639 | 39.0 | 86,576 | 7.6 |

Table 2.16 The overhead accounts

Using the total variance over the standard cost as an estimate of the deviation from standards we have:

| Unit Variance | 1966 | 1967 |
|---|---|---|
| Materials | -0.006% | -0.03 % |
| Labour | .169% | .146% |
| O'head | .698% | .465% |

Table 2.17 Estimates of unit variance

From Table 2.17 we can judge the approximate rates for total variance per unit of standard cost.

For the present valculations these were assumed† to be materials: 0.03%, labour: 0.15%, o'heads: 0.7%. These are the values used for the OHRATE array, for calculation of overhead variance accounts from the incurred standard costs.

† the positive value of materials variance ensures an even greater demand for cash.

### 2.2.4 Accounting and Storage Lags

2.2.4.1 Storage lags: The purpose of the storage lag associated with each product was to ensure that the time flow of the product through the firm was correctly modelled. Limited data was available on the storage of each product on a monthly basis. Initially, it was hoped to estimate the shelf life of a product (in storage) by calculating the time to sell all stocks held at the moment of completion of a product batch. This turnover period would estimate the time spent by this product batch, (on a FIFO basis) in the company's warehouse. However, for this exercise, data was required on stock holdings of all products at a fixed time, and all subsequent production and sales figures. These were not available. Records of monthly production, storage and sales were updated at irregular (and different) intervals of time. For some products it was possible to estimate the 'shelf life' from the data available. The results achieved are shown in Table 2.18.

Table 2.18 Estimated Lag Per Product (in Months)

| ITEM | AUG | SEP | OCT | NOV | DEC | JAN | FEB | MAR | APR | MAY | JUN | JUL |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 4 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 3 | 3 | 5 | 7 |
| 10 | 5 | 5 | 6 | 6 | 4 | 3 | 3 | 3 | 3 | 2 | 2 | 1 |
| 11 | 4 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 3 |
| 12 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 4 | 3 | 2 | 2 |
| 15 | 2 | 3 | 3 | 5 | 5 | 6 | 6 | 5 | 4 | 3 | 2 | 5 |
| 16 | 1 | 2 | 2 | 4 | 3 | 4 | 3 | 3 | 2 | 1 | 1 | 2 |
| 17 | 1 | 2 | 2 | 3 | 3 | 3 | 2 | 2 | 1 | 1 | 3 | 3 |
| 18 | 2 | 1 | 2 | 2 | 3 | 3 | 4 | 3 | 2 | 3 | 3 | 2 |
| 19 | 2 | 2 | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 3 | 3 |
| 20 | 6 | 4 | 4 | 3 | 4 | 5 | 5 | 4 | 3 | 3 | 1 | 2 |
| 21 | 5 | 5 | 4 | 3 | 2 | 2 | 4 | 5 | 5 | 5 | 4 | 4 |
| 22 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| 23 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 2 | 1 | 1 |
| 24 | 1 | 1 | 1 | 1 | 2 | 4 | 5 | 4 | 6 | 5 | 5 | 5 |
| 25 | 4 | 3 | 2 | 2 | 2 | 2 | 1 | 1 | 2 | 3 | 3 | 4 |
| 26 | 8 | 9 | 9 | 8 | 7 | 6 | 6 | 5 | 5 | 4 | 3 | 3 |

It must be remembered that these figures are a global estimate of 'shelf life' - covering all possible market outlets; they will thus disguise the Mail Order deliveries which would not appear in storage records due to the rapidity of their despatch after completion. The figures of Table 2.18 are thus an over-estimate of the shelf life.

It was felt necessary to make the lag of storage a variable input. In this way changes of market outlet per product can be judged by corresponding changes of the lags on despatch and payment, (LAG and RECLAG) and the discount allowed (DISCP).

For the test calculations of the 26/12 model products were allowed to be sold immediately - i.e. if possible. The action of the LP model does not conflict with the desired FIFO basis for sales.

2.2.4.2 Other accounting lags: The remaining lags on accounting constraints are divided between the periods in which the company settles its debt, or accounts for costs incurred, and the periods in which it expects to receive payments for sales.

1. The periods over which accounts were stretched were zero for payments of wages, interest charges, and marketing expenses. Subcontracting fees were paid one month in arrears (for the test model).

2. It was assumed that overheads would be accounted for at the end of period in which they were incurred.

3. It was further assumed that payments were made within a month of despatch.

As with the storage lags, these values of input data may be altered at will, to model different marketing situations.

2.2.5 Input parameters

The input parameters for the 26/12 model are detailed below in Tables 2.19 and 2.20.

| parameter | NPROD | NWC | NSUB | NLF | NRM | NSCS | NOH | NM |
|---|---|---|---|---|---|---|---|---|
| value | 26 | 18 | 7 | 1 | 1 | 3 | 3 | 12 |

Table 2.19 Input parameters for 26/12 model

| Item | Input value |
|---|---|
| Raw materials | £ 5,000 |
| Cash | £50,000 |
| Finished product | 10 units of each product (1 unit = 100 items) |

Table 2.20 Input values

Control Variables and policy levels

For the initial test of the 26/12 model, cash and bank loans were bounded. No restrictions were placed on sales, and an upper limit on materials holding was set at £5,000.

| Item | Lower bound | Upper bound |
|---|---|---|
| Cash | £50 | £100,000 |
| Bank loans | – | £150,000 |
| Materials | – | £5,000 |
| Unit sales | – | – |

Table 2.21 Control variables and bounds

## Appendix 2.3 The Programmes

2.3.1 The matrix generating programme for the model is listed below. This programme provided the input data for the LP.

### 2.3.2 The Output of the Model

At the optimum of the LP, the output generated by the procedure SOLUTION comprises three parts; the objective function, the row values (and the dual evaluators) and the column values (and the reduced costs). Typical printout of this solution is shown in Figures 2.10 to 2.12. A report generator was written for the model which would translate the output of SOLUTION (filed onto magnetic disk) into the more useful form of optimal schedules for production, storage and sales. This routine also provided the month by month cash flow statements and income and surplus accounts. Details of this programme,(AKOUNT), are given below, and a sample of the output from AKOUNT for a run of the 3/5 model is shown in Figures 2.13 to 2.15. Given the asset position of the firm at the opening of the first period, the routine could also provide balance sheets, and the set of operating and financial ratios.

Since the optimal solution to the model varies with the objective function, the operating and financial ratios derived from the model, will reflect management objectives. It will thus become clear that management objectives will have direct influence on the firm's optimal strategy, its financial accounts, its operating ratios, and its resource valuation .

```
        EXECUTOR.     MPS/360 V2-M0                                              PAGE

SOLUTION   (OPTIMAL)

TIME =   0.31 MINS. ITERATION NUMBER =       8

        ...NAME...      ...ACTIVITY...      DEFINED AS

        FUNCTIONAL      136522.09181        GROSSALE
        RESTRAINTS                          JSUKRH
        BOUNDS....                          JKBOUND
```

Fig. 2.10 Typical MPS Output

| | NUMBER | ...ROW... | AT | ...ACTIVITY... | SLACK ACTIVITY | ..LOWER LIMIT. | ..UPPER LIMIT. | .DUAL ACTIVITY |
|---|---|---|---|---|---|---|---|---|
| | 203 | MYWCLF15 | BS | . | 10000.00000 | NONE | 10000.00000 | . |
| | 204 | MYWCLF16 | BS | . | 10000.00000 | NONE | 10000.00000 | . |
| | 205 | MYWCLF17 | BS | 1719.76153 | 8280.23847 | NONE | 10000.00000 | . |
| | 206 | MYWCLF18 | BS | . | 10000.00000 | NONE | 10000.00000 | . |
| | 207 | MYLERQ1 | BS | 8068.95522 | 81931.04478 | NONE | 90000.00000 | . |
| | 208 | MYSTCN1 | BS | . | . | NONE | . | . |
| | 209 | MYSTCN2 | BS | . | . | NONE | . | . |
| | 210 | MYSTCN3 | BS | . | . | NONE | . | . |
| | 211 | MYSTCP1 | BS | 22.36000- | 6522.36000 | NONE | 6500.00000 | . |
| | 212 | MYINEQ1 | BS | 5000.00000- | 5000.00000 | NONE | . | . |
| A | 213 | MYINCN1 | EQ | . | . | . | . | . |
| | 214 | MYWPPP1 | BS | . | . | NONE | NONE | . |
| | 215 | MYGRSE1 | BS | 60560.53730 | 60560.53730- | NONE | NONE | . |
| | 216 | MYEGAC1 | BS | . | . | NONE | NONE | . |
| A | 217 | MYMARK1 | UL | . | . | NONE | . | . |
| A | 218 | MYMARK2 | UL | . | . | NONE | . | . |
| A | 219 | MYMARK3 | UL | . | . | NONE | . | . |
| A | 220 | MYSCSR1 | EQ | . | . | . | . | . |
| A | 221 | MYSCSR2 | EQ | . | . | . | . | . |
| A | 222 | MYSCSR3 | EQ | . | . | . | . | . |
| A | 223 | MYOHDR1 | EQ | . | . | . | . | . |
| A | 224 | MYOHDR2 | EQ | . | . | . | . | . |
| A | 225 | MYOHDR3 | EQ | . | . | . | . | . |
| | 226 | MYDSCR1 | BS | 3028.02686 | 3028.02686- | NONE | NONE | . |
| | 227 | MYNSLR1 | BS | . | . | NONE | NONE | . |
| | 228 | MYMEMR1 | BS | . | . | . | . | . |
| A | 229 | MYPAYR1 | EQ | . | . | . | . | . |
| A | 230 | MYRECR1 | EQ | . | . | . | . | . |
| A | 231 | MYCASR1 | EQ | . | . | . | . | . |
| A | 232 | MYBKCR1 | EQ | . | . | . | . | . |
| | 233 | MYSTRQ1 | UL | 10.00000 | . | NONE | 10.00000 | 482.22222- |
| | 234 | MYSTRQ2 | UL | 10.00000 | . | NONE | 10.00000 | 570.00000- |
| | 235 | MYSTRQ3 | UL | 10.00000 | . | NONE | 10.00000 | 700.00000- |
| | 236 | ASSETS | BS | 38550.00000 | 38550.00000- | NONE | NONE | . |
| | 237 | LIABLE | BS | . | . | NONE | NONE | . |
| | 238 | GROSSALE | BS | 186522.09181 | 186522.09181- | NONE | NONE | 1.00000 |

Fig. 2.11 Rows Section of MPS Output

SECTION 2 - COLUMNS

| | NUMBER | .COLUMN. | AT | ...ACTIVITY... | ..INPUT COST.. | ..LOWER LIMIT. | ..UPPER LIMIT. | .REDUCED COST. |
|---|---|---|---|---|---|---|---|---|
| | 239 | JAPROD1 | BS | . | . | . | NONE | . |
| | 240 | JAPROD2 | BS | 23.47418 | . | . | NONE | . |
| | 241 | JAPROD3 | BS | 30.70273 | . | . | NONE | . |
| | 242 | JASALE1 | LL | 10.00000 | 375.00000 | 10.00000 | NONE | 107.22222- |
| A | 243 | JASALE2 | LL | . | 570.00000 | . | NONE | . |
| A | 244 | JASALE3 | LL | 20.00000 | 700.00000 | 20.00000 | NONE | . |
| A | 245 | JASUBW1 | LL | . | . | . | NONE | . |
| | 246 | JASUBW2 | BS | 93.99963 | . | . | NONE | . |
| A | 247 | JASUBW3 | LL | . | . | . | NONE | . |
| A | 248 | JASUBW4 | LL | . | . | . | NONE | . |
| A | 249 | JASUBW5 | LL | . | . | . | NONE | . |
| A | 250 | JASUBW6 | LL | . | . | . | NONE | . |
| A | 251 | JASUBW7 | LL | . | . | . | NONE | . |
| | 252 | JARAWM1 | BS | 3006.37990 | . | 50.00000 | NONE | . |
| A | 253 | JARMIN1 | LL | . | . | . | NONE | . |
| | 254 | JAMARK1 | BS | 700.00000 | . | . | NONE | . |
| | 255 | JAMARK2 | BS | . | . | . | NONE | . |
| | 256 | JAMARK3 | BS | 2800.00000 | . | . | NONE | . |
| | 257 | JASTCS1 | BS | 4590.29000 | . | . | NONE | . |
| | 258 | JASTCS2 | BS | 1085.70000 | . | . | NONE | . |
| | 259 | JASTCS3 | BS | 1337.10000 | . | . | NONE | . |
| | 260 | JAOVHD1 | BS | 145.39899 | . | . | NONE | . |
| | 261 | JAOVHD2 | BS | 162.85500 | . | . | NONE | . |
| | 262 | JAOVHD3 | BS | 935.97000 | . | . | NONE | . |
| A | 263 | JACASH1 | UL | 50000.00000 | . | 50.00000 | 50000.00000 | . |
| A | 264 | JABNKL1 | UL | 50000.00000 | . | . | 50000.00000 | . |
| | 265 | JABNKR1 | BS | 29272.42264 | . | . | NONE | . |
| | 266 | JABNKC1 | BS | 1450.93042 | . | . | NONE | . |
| | 267 | JAPAYS1 | BS | 36139.14694 | . | . | NONE | . |
| | 268 | JARECS1 | BS | 16862.50000 | . | . | NONE | . |

Fig. 2.12 Columns Section of MPS Output

INCOME AND SURPLUS ACCOUNT MONTH 3

| | |
|---|---|
| GROSS SALES | 17750. |
| LESS CASH DISCOUNT | 887. |
| NETT SALES | 16862. |
| LESS STANDARD COST | 7113. |
| OTHER COSTS | |
| TOTAL COST OF SALES | 7113. |
| MANUFACTURING MARGIN | 9749. |
| LESS DIRECT SALES | |
| PROMOTION | |
| DISTRIBUTION | |
| TOTAL SELLING EXPENSES | 3500. |
| ADMIN AND GENERAL | 0. |
| LOAN INTEREST | 0. |
| TOTAL SELLING EXPENSES | 3500. |
| S.E.T. | 0. |
| OPERATING INCOME | 6249. |
| INCOME BEFORE TAX | 6249. |
| LESS TAX | 0. |
| NETT INCOME | 6249. |

Fig. 2.15 Income and Surplus from AKOUNT

PRODUCTION,STORAGE AND SALES SCHEDULES

MONTH 1

| | TOOL1 | TOOL2 | TOOL3 | TOOL4 | TOOL5 | TOOL6 |
|---|---|---|---|---|---|---|
| PRODUCE | 0.0 | 23.47 | 30.70 | | | |
| SELL | 10.00 | 0.0 | 20.00 | | | |
| STORE | 0.0 | 33.47 | 20.70 | | | |

MONTH 2

| | TOOL1 | TOOL2 | TOOL3 | TOOL4 | TOOL5 | TOOL6 |
|---|---|---|---|---|---|---|
| PRODUCE | 13.20 | 23.47 | 21.61 | | | |
| SELL | 10.00 | 56.95 | 20.00 | | | |
| STORE | 3.20 | 0.00 | 22.31 | | | |

MONTH 3

| | TOOL1 | TOOL2 | TOOL3 | TOOL4 | TOOL5 | TOOL6 |
|---|---|---|---|---|---|---|
| PRODUCE | 13.20 | 23.47 | 21.61 | | | |
| SELL | 10.00 | 0.0 | 20.00 | | | |
| STORE | 6.40 | 23.47 | 23.92 | | | |

MONTH 4

| | TOOL1 | TOOL2 | TOOL3 | TOOL4 | TOOL5 | TOOL6 |
|---|---|---|---|---|---|---|
| PRODUCE | 3.60 | 23.47 | 28.22 | | | |
| SELL | 10.00 | 0.0 | 52.14 | | | |
| STORE | -0.00 | 46.95 | -0.00 | | | |

MONTH 5

| | TOOL1 | TOOL2 | TOOL3 | TOOL4 | TOOL5 | TOOL6 |
|---|---|---|---|---|---|---|
| PRODUCE | 10.00 | 23.47 | 23.81 | | | |
| SELL | 10.00 | 70.42 | 23.81 | | | |
| STORE | -0.00 | 0.00 | -0.00 | | | |

Fig. 2.13 Schedules from AKOUNT

| CASH FLOW STATEMENT | MONTH 2 |
|---|---|
| GROSS SALES | 1197. |
| INCOME BEFORE TAX | 17525. |
| | |
| CASH AT BEGINNING | 50000. |
| | |
| RAW MATERIALS | 3119. |
| WAGES | 9190. |
| OVERHEAD ACCOUNTS | 3746. |
| BANK CHARGES | 1001. |
| RECEIPTS | |
| NETT SALES | 47700. |
| | |
| CASH AT END | 50000. |

Fig. 2.14 Cash Flow Statement from AKOUNT

The Matrix Generator - Main Programme

```
      INTEGER*2 IA(4000,2)
      COMMON IA(4000,2),AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50),
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      DIMENSION NAME(50,2)
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
C
 6000 FORMAT(36I2)
 6001 FORMAT(1H1,' DATA INPUT',//' NTOOL   NWC   NSUB   NLF   NRM   NSCS
     1 NOH    NM',//,I4,7I6)
 9991 FORMAT(2A4,2X,5F8.4)
 9992 FORMAT(20X,F5.2)
 9993 FORMAT(3A2)
 9994 FORMAT(20F4.3)
 9995 FORMAT(12F6.4)
 9996 FORMAT(70A1)
 9997 FORMAT(70I1)
 9998 FORMAT(3F6.2)
 9999 FORMAT(12F6.2)
C
C   DATA INPUT
C
      READ(5,6000)NTOOL,NWC,NSUB,NLF,NRM,NSCS,NOH,NM
      READ(5,9995) ALPHA
      WRITE(6,6001) NTOOL,NWC,NSUB,NLF,NRM,NSCS,NOH,NM
      READ(5,9991) ((NAME(I,J),J=1,2),LIST(I),(SCSP(J,I),J=1,NSCS),
     1 SPACE(I),I=1,NTOOL)
      READ(5,9992) (RMREQ(1,J,1),J=1,NTOOL)
      CALL AMCREQ
      READ(5,9994) ((WCLF(J,I),J=1,NWC),I=1,NLF)
      DO 1 I=1,NTOOL
      DO 1 J=1,NTOOL
      MARK(I,J)=0.0
      IF(I.EQ.J)MARK(I,J)=1.0
    1 CONTINUE
      READ(5,9994)  (DISCP(J),J=1,NTOOL)
      READ(5,9999) (SUBP(J),J=1,NSUB)
      READ(5,9995) (WAGES(J),J=1,NLF)
      READ(5,9995) (RMB(J),J=1,NRM)
      READ(5,9997)(LAG(J),J=1,NTOOL)
      READ(5,9997)(SUBWC(J),J=1,NSUB)
      READ(5,9995) (OHRATE(J),J=1,NOH)
      READ(5,9997) (RECLAG(J),J=1,NTOOL)
      READ(5,9997) (SUBLAG(J),J=1,NSUB)
      READ(5,9997) (LABLAG(J),J=1,NLF)
       READ(5,9997) (RMLAG(J),J=1,NRM)
      READ(5,9997) (MRKLAG(J),J=1,NTOOL)
      READ(5,9997) (OHLAG(J),J=1,NOH)
      READ(5,9997) ALFLAG
      HIGH=3*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+11
      LONG=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+12
      NROW=0
      KMONTH=3
      CALL APRE1(KMONTH)
      CALL XLAG(1)
      NPRE2=NROW
      KMONTH=2
      CALL APRE1(KMONTH)
      CALL XLAG(2)
      NPRE1=NROW
      CALL ABLOCK
      CALL XLAG(3)
      NBLOCK=NROW
      CALL APOST1
      CALL XLAG(4)
      NPOST1=NROW
      CALL APOST2
      CALL XLAG(5)
```

```
      NPOST2=NROW
C
      WRITE(6,6002) NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
 6002 FORMAT(///,'  ENTRIES INTO STORAGE ARRAYS  ',5X,5I10)
      WRITE(6,6003) HIGH,LONG
 6003 FORMAT(///,'  SIZE OF EACH BLOCK IS  ',I4,' X ',I4)
C
C   M.P.S. OUTPUT
C
      CALL TITLE(NAMEC,LONG)
      CALL TITLE(NAMER,HIGH)
      KK=HIGH+1
      READ(5,9993) (NAMER(KK,J),J=1,3)
      CALL ASSETS
      CALL MPSOUT
      STOP
      END


      SUBROUTINE AMCREQ
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50),
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
      INTEGER SPRED
C
C   WORK IN PROGRESS CALCULATIONS AND MATERIAL REQUIREMENTS
C
      DO 11 J=1,NTOOL
      RMREQ(1,J,1)=RMREQ(1,J,1)/3.0
      DO 12 I=1,2
   12 WIPP(I,J)=LIST(J)*(3.0-FLOAT(I))/3.0
      DO 13 L=1,NRM
      DO 13 I=1,3
   13 RMREQ(I,J,L)=RMREQ(1,J,1)
   11 CONTINUE
C
C   MACHINE REQUIREMENTS OF PRODUCTS,SPREAD OVER TIME
C  WORK CENTRES 1-9          PREVIOUS TWO OR THREE MONTHS ( READ SPRED)
C  WORK CENTRES 10-18 LAST MONTH ONLY
C
      IZXQ=3
      SPRED =3
      DO 1 J=1,3
      DO 1 K=1,NWC
      DO 1 L=1,NTOOL
    1 MCREQ(J,K,L)=0.
      DO 2 I=1,NTOOL
      READ(5,5002) (B(J),J=1,NWC)
 5002 FORMAT(//,11X,10F6.0,/,11X,10F6.0)
      DO 3 J=1,NWC
      GO TO (4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5),J
    4 CONTINUE
      DO 20 K=1,SPRED
      NN=IZXQ-K
      MCREQ(NN,J,I)=B(J)/FLOAT(SPRED)
   20 CONTINUE
      GO TO 3
    5 CONTINUE
      MCREQ(1,J,I)=B(J)
    3 CONTINUE
    2 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE APRE1(LMONTH)
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,10,50),WCLF(10,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMLC(800,3),LAG(50),
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,MSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
      KMONTH=LMONTH
C
C SETTING UP THE MATRIX BLOCK PRE1.
C FOR THIS MATRIX MONTH=1
C FOR THE OTHER MATRICES,PRE2 ETC. USE MONTH =3,4...
C NOW WE SET UP PRE2 IN THE PRE1 AREA
C
C 1 WORK CENTRE EQUATION
      DO 28 I=1,NWC
      DO 28 J=1,NTOOL
      NROW=NROW+1
      IA(NROW,1)=I
      IA(NROW,2)=J
   28 AA(NROW)=MCREQ(KMONTH,I,J)
C 3 LABOUR FORCE REQUIREMENTS
C
      KROW=NWC
      DO 6 I=1,NLF
      IROW=KROW+I
      DO 7 J=1,NTOOL
      ICOL=J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=0.0
      DO 8 K=1,NWC
    8 AA(NROW)=AA(NROW)+WCLF(K,I)*MCREQ(KMONTH,K,J)
    7 CONTINUE
    6 CONTINUE
C
C 8 INPUT REQUIREMENTS
C
      KROW=NWC+NTOOL+NLF+1
      DO 29 J=1,NTOOL
      DO 29 I=1,NRM
      IROW=KROW+I
      NROW=NROW+1
      IA(NROW,2)=J
      IA(NROW,1)=IROW
   29 AA(NROW)=RMREQ(KMONTH,J,I)
C
C
C
C 9 INPUT CONTINUITY
C
      KROW=KROW+NRM
      DO 30 I=1,NRM
      DO 30 J=1,NTOOL
      IROW=KROW+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=J
      AA(NROW)=RMREQ(KMONTH,J,I)
   30 CONTINUE
C
C 10 WORK IN PROGRESS ACCOUNT
C
      KROW=KROW+NRM
      IROW=KROW+1
      DO 34 J=1,NTOOL
```

```
      L=KMONTH-1
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=J
   34 AA(NROW)=WIPP(L,J)
C
C  11 MARKETING
C
C
C     WAGES
C     ASSUMING THAT WAGES CANNOT BE DEFRAYED FOR ANY LENGTH OF TIME
C
      IROW=3+NOH+NSCS+2*NTOOL+2*NRM+NLF+NWC+5
      DO 900 I=1,NTOOL
      ICOL=I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=0.0
      DO 900 K=1,NWC
      DO 900 J=1,NLF
  900 AA(NROW)=AA(NROW)-MCREQ(KMONTH,K,I)*WCLF(K,J)*WAGES(J)
      RETURN
      END
```

```
C
      SUBROUTINE ABLOCK
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7).ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50),
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
C
C 1 WORK CENTRE REQUIREMENTS
C
      DO 1 IROW=1,NWC
      DO 1 ICOL=1,NTOOL
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
    1 AA(NROW)=MCREQ(1,IROW,ICOL)
      KCOL=3*NTOOL+NWC
      DO 4 I=1,NSUB
      ICOL=KCOL+I
      IROW=SUBWC(I)
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
    4 CONTINUE
C
C 3 LABOUR FORCE REQUIREMENTS
C
      KROW=NWC
      DO 6 I=1,NLF
      IROW=KROW+I
      DO 7 J=1,NTOOL
      ICOL=J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=0.0
      DO 8 K=1,NWC
    8 AA(NROW)=AA(NROW)+WCLF(K,I)*MCREQ(3,K,J)
    7 CONTINUE
      DO 2 J=1,NSUB
      ICOL=3*NTOOL+NWC+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      K=SUBWC(J)
      AA(NROW)=-WCLF(K,I)
    2 CONTINUE
    6 CONTINUE
C
C 6 STORAGE CAPACITY
C
      IROW=NTOOL+NLF+NWC+1
      DO 10 J=1,NTOOL
      ICOL=J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=SPACE(J)
      ICOL=2*NTOOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
   10 AA(NROW)=-SPACE(J)
```

```
C
C  8 INPUT REQUIREMENTS
C
      KROW=NWC+NLF+NTOOL+1
      DO 12 I=1,NRM
      IROW=KROW+I
      DO 12 J=1,NTOOL
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=J
   12 AA(NROW)=RMREQ(1,J,I)
C
C  9 INPUT CONTINUITY
C
      KROW=KROW+NRM
      KCOL=3*NTOOL+NWC+NSUB+NLF
      DO 13 I=1,NRM
      IROW=KROW+I
      DO 14 J=1,NTOOL
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=J
   14 AA(NROW)=RMREQ(1,J,I)
      ICOL=KCOL+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
      ICOL=KCOL+NRM+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
   13 CONTINUE
C
C 11 GROSS SALES
C
      KCOL=2*NTOOL
      IROW=IROW+2
      DO 15 J=1,NTOOL
      ICOL=KCOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
   15 AA(NROW)=LIST(J)
C
C 12 INCREMENT OF FINISHED GOODS ACCOUNT
C
      IROW=IROW+1
      DO 16 J=1,NTOOL
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=J
      AA(NROW)=LIST(J)
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=2*NTOOL+J
      AA(NROW)=-LIST(J)
   16 CONTINUE
C
C 13 MARKETING EFFECTS
C
C    AMENDED PROGRAMME TO TAKE ACCOUNT OF THE MARKETING VARIABLES
C    WHICH VARY WITH GROSS SALES
C
```

```
C
      KROW=4+2*NRM+NTOOL+NLF+NWC
      KCOL=2*NTOOL
      DO 17 I=1,NTOOL
      IROW=KROW+I
      ICOL=KCOL+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=MARK(I,I)*LIST(I)
      ICOL=3*NTOOL+NWC+NSUB+NLF+2*NRM+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
   17 CONTINUE
C
C 14 STANDARD COST OF SALES
C
      KROW=KROW+NTOOL
      DO 19 I=1,NSCS
      IROW=KROW+I
      DO 20 J=1,NTOOL
      ICOL=2*NTOOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
   20 AA(NROW)=SCSP(I,J)
      ICOL=3+4*NTOOL+2*NRM+NLF+NSUB+NWC+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
   19 CONTINUE
C
C 15 OVERHEAD ACCOUNTS
C
      KROW=NWC+NLF+2*NTOOL+4+NSCS+2*NRM
      KCOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+6
      DO 40 I=1,NOH
      IROW=KROW+I
      ICOL=KCOL+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
      ICOL=3+4*NTOOL+2*NRM+NLF+NSUB+NWC+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=OHRATE(I)
   40 CONTINUE
C
C THE REMAINING PART OF THE OVERHEAD ACCOUNT IS SET IN THE MAIN PROGRAMM
C
C  DISCOUNT
C
      KROW=NOH+NSCS+2*NTOOL+2*NRM+NLF+NWC+5
      IROW=KROW
      DO 21 J=1,NTOOL
      ICOL=2*NTOOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
   21 AA(NROW)=DISCP(J)*LIST(J)
C
C
C 18 MANUFACTURING MARGIN
C
      IROW=KROW+2
C
C 19 PAYABLES
C
      IROW=KROW+3
      ICOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+11
      NROW=NROW+1
      IA(NROW,1)=IROW
```

```
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
C   WAGES
C   ASSUMING THAT WAGES CANNOT BE DEFRAYED FOR ANY LENGTH OF TIME
C
C     WAGES ARE CALCULATED ON THE BASIS OF     LFREQ=(MCREQ.PROD-SUB).WCL
C
C
      DO 900 I=1,NTOOL
      ICOL=I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=0.0
      DO 900 K=1,NWC
      DO 900 J=1,NLF
  900 AA(NROW)=AA(NROW)-MCREQ(1,K,I)*WCLF(K,J)*WAGES(J)
      DO 902 II=1,NSUB
      ICOL=3*NTOOL+NWC+II
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=0.0
C
C     IF THERE IS NO LAG ON SUBCONTRACTING PAYMENTS WE HAVE TO ADJUST TH
C     ROW VALUE UNDER SUB BY THE COST OF THAT CONTRACT
C      THIS IS NECESSARY TO AVOID HAVING A DOUBLE ENTRY FOR THE SAME
C     MATRIX POSITION
C
      DO 901 I=1,NLF
  901 AA(NROW)=AA(NROW)+WAGES(I)*WCLF(II,I)
      IF(SUBLAG(II).EQ.0) AA(NROW)=AA(NROW)-SUBP(II)
  902 CONTINUE
C
C  20 RECEIVABLES
C
      IROW=KROW+4
      ICOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+12
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
C
C  21 CASH CONTINUITY EQUATIO
C
      IROW=KROW+5
      KCOL=4*NTOOL+NWC+NSUB+NLF+2*NRM +NSCS+6+NOH
      ICOL=KCOL+1
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
      ICOL=KCOL+2
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
      ICOL=KCOL+3
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
      ICOL=KCOL+4
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
      ICOL=KCOL+5
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
      ICOL=KCOL+6
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
```

```
C
C 22 BANK CHARGES
C
      IROW=KROW+6
      ICOL=KCOL+2
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-ALPHA
      ICOL=KCOL+3
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)= ALPHA
      ICOL=KCOL+4
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
C
C LAG REQUIREMENTS FOR BLOCK
C 23 LAG STORE REQUIREMENT
C
       KCOL=2*NTOOL
      KROW=2*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+11
      DO 30 J=1,NTOOL
      IROW=KROW+J
      ICOL=KCOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
   30 CONTINUE
C
C  TOOL LAG
C
      KROW=2*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+11
      KCOL=NTOOL
      DO 56 J=1,NTOOL
      IROW=KROW+J
      KK=LAG(J)+1
      GO TO (57,58,59),KK
   57 CONTINUE
      ICOL=J
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
   58 CONTINUE
   59 CONTINUE
   56 CONTINUE
C
C  OVERHEAD LAG
C
      RETURN
      END
```

```
      SUBROUTINE APOSTI
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50),
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
C
C SETTING UP POST
C THIS IS FOR THE AREA IMMEDIATELY BELOW THE MONTH'S MATRIX
C
C
C 6 STORAGE CAPACITY
C
      IROW=NTOOL+NLF+NWC+1
      DO 10 J=1,NTOOL
      ICOL=J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=SPACE(J)
      ICOL=2*NTOOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
   10 AA(NROW)=-SPACE(J)
C
C 8 INPUT REQUIREMENTS
C
      KROW=NWC+NLF+NTOOL+1
      KCOL=3*NTOOL+NWC+NSUB+NLF
      DO 31 I=1,NRM
      IROW=KROW+I
      ICOL=KCOL+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
   31 CONTINUE
C
C 9 INPUT CONTINUITY
C
      KROW=KROW+NRM
      DO 32 I=1,NRM
      IROW=KROW+I
      ICOL=KCOL+I
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
   32 CONTINUE
C
C 21 CASH CONTINUITY
C
      KCOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+7
      KROW=NWC+NLF+2*NTOOL+2*NRM+NSCS+NOH+10
      ICOL=KCOL
      IROW=KROW
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
```

```
C 22 BANK CHARGES
C
      IROW=KROW+1
      ICOL=KCOL+1
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-ALPHA
      ICOL=KCOL+2
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=ALPHA
C
C  TOOL LAG
C
      KROW=2*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+11
      KCOL=NTOOL
      DO 56 J=1,NTOOL
      IROW=KROW+J
      KK=LAG(J)+1
      GO TO (57,58,59),KK
   57 CONTINUE
   58 CONTINUE
      ICOL=J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
      ICOL=2*NTOOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
   59 CONTINUE
   56 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE APOST2
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50),
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
C
C  6 STORAGE CAPACITY
C
      IROW=NTOOL+NLF+NWC+1
      DO 10 J=1,NTOOL
      ICOL=J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=SPACE(J)
      ICOL=2*NTOOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
   10 AA(NROW)=-SPACE(J)
C
C   BANK CHARGES
      KCOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+8
      KROW=NWC+NLF+2*NTOOL+2*NRM+NSCS+NOH+11
      IROW=KROW
      ICOL=KCOL
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-ALPHA
      KCOL=KCOL+1
      IROW=KROW
      ICOL=KCOL
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)= ALPHA
C
C   TOOL LAG
C
      KROW=2*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+11
      KCOL=NTOOL
      DO 56 J=1,NTOOL
      IROW=KROW+J
      ICOL=J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=-1.0
      ICOL=2*NTOOL+J
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=1.0
   56 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE TITLE(MAMEC,KONG)
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),MCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50),
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      DIMENSION MAMEC(800,3)
      DIMENSION NUMBER(10)
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
      READ(5,2003) IBLAN
 2003 FORMAT(1A1)
       DO 30 I=1,800
      DO 30 J=1,3
   30 MAMEC(I,J)=IBLAN
      READ(5,2000) (NUMBER(J),J=1,10)
 2000 FORMAT(50A1)
      READ(5,2005) (MONTH(K),K=1,12)
 2005 FORMAT(12A2)
      L=0
      KROW=0
      READ(5,2004) NROW
 2004 FORMAT(I2)
      DO 2 I=1,NROW
      READ(5,2001) NAME,NDIM,JCODE
 2001 FORMAT(A4,I2,A1)
       DO 3 J=1,NDIM
      KROW=KROW+1
      ICODE(KROW)=JCODE
      MAMEC(KROW,1)=NAME
    3 CONTINUE
C
C   DIMENSIONING THE COLUMN NAMES
C
C   KTENS IS THE NUMBER OF TENS AT PRESENT REGISTERED
C
C   KUNITS IS THE NUMBER OF UNITS REGISTERED
C
      KTENS=0
      KUNITS=0
      LI=2
      LJ=3
      DO 6 KK=1,NDIM
      L=L+1
      KUNITS=KK-KTENS*10+1
      IF(KK.GE.10) GO TO 7
      MAMEC(L,LI)=NUMBER(KUNITS)
      GO TO 6
C
C
C
C   UPDATE KTENS EVERY TIME WE COME ACROSS ZERO AS OUR UNIT TERM
C
    7 IF(KUNITS.EQ.11) KTENS=KTENS+1
      IF(KUNITS.EQ.11) KUNITS=1
      LTENS=KTENS+1
      MAMEC(L,LI)=NUMBER(LTENS)
      MAMEC(L,LJ)=NUMBER(KUNITS)
    6 CONTINUE
    2 CONTINUE
C THE COLUMN AND ROW NAMES HAVE BEEN SET UP
      RETURN
      END
```

```
      SUBROUTINE ASSETS
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50),
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
C
      KK=NM*LONG
      DO 1 J=1,KK
      LIABLE(J)=0.0
    1 B(J)=0.0
C
C  RAW MATERIALS
C
      KCOL=3*NTOOL+NWC+NSUB+NLF+(NM-1)*LONG
      DO 20 J=1,NRM
      ICOL=KCOL+J
   20 B(ICOL)=1.0
C
C   CASH
C
      ICOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+7+(NM-1)*LONG
      B(ICOL)=1.0
C
      DO 2 K=1,NM
C
C    WORK IN PROGRESS
C
      DO 11 I=1,2
      DO 11 J=1,NTOOL
      JJ=(NM-I)*LONG+J
   11 B(JJ)=WIPP(I,J)
C
C    FINISHED GOODS
C
      DO 3 J=1,NTOOL
      ICOL=(K-1)*LONG+J
      B(ICOL)=LIST(J)
      ICOL=ICOL+2*NTOOL
    3 B(ICOL)=-LIST(J)
C
C  BANK LOAN OUTSTANDING
C
      KCOL=(K-1)*LONG+4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+7
      ICOL=KCOL+1
      LIABLE(ICOL)=1.0
      ICOL=KCOL+2
      LIABLE(ICOL)=-1.0
    2 CONTINUE
C
C SETTING UP THE LIABILITIES NOT YET PAID DUE TO LAGS
C
      DO 10 K=1,2
      KK=(NM-K)*LONG
      KNM=K-1
C
C RECEIVABLES
C
      DO 4 J=1,NTOOL
      IF(RECLAG(J).LE.KNM) GO TO 4
      ICOL=2*NTOOL+J+KK
      B(ICOL)=B(ICOL)+LIST(J)*(1.0-DISCP(J))
    4 CONTINUE
C
C  LIABILITIES
C
```

```
      DO 5 J=1,NSUB
      IF(SUBLAG(J).LE.KNM) GO TO 5
      KCOL=3*NTOOL+NWC+J+KK
      LIABLE(KCOL)=SUBP(J)
    5 CONTINUE
      DO 6 J=1,NRM
      IF(RMLAG(J).LE.KNM) GO TO 6
      KCOL=3*NTOOL+NWC+NLF+NRM+NSUB+J+KK
      LIABLE(KCOL)=RMB(J)
    6 CONTINUE
      DO 7 J=1,NTOOL
      IF(MRKLAG(J).LE.KNM) GO TO 7
      KCOL=3*NTOOL+NWC+NLF+NSUB+2*NRM+J+KK
      LIABLE(KCOL)=MARK(J,J)
    7 CONTINUE
      DO 8 J=1,NOH
      IF(OHLAG(J).LE.KNM) GO TO 8
      KCOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+6+J+KK
      LIABLE(KCOL)=1.0
    8 CONTINUE
      IF(ALFLAG.LE.KNM) GO TO 9
      KCOL=4*NTOOL+NWC+NLF+NSUB+2*NRM+NSCS+NOH+10+KK
      LIABLE(KCOL)=1.0
    9 CONTINUE
   10 CONTINUE
      RETURN
      END
```

## Output of MPS Data

```
      SUBROUTINE MPSOUT
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50)
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      DIMENSION IRHS(8)
      DIMENSION CAPWC(18),CAPLF(2),POLICY(50)
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
 2000 FORMAT(50A1)
 2001 FORMAT(10A1)
 2002 FORMAT(10F3.1)
 2995 FORMAT(1X,'N',2X,'LIABLE')
 2997 FORMAT(1X,'N',2X,'GROSSALE')
 2998 FORMAT(1X,'N',2X,'ASSETS')
 2999 FORMAT('NAME',10X,'JSHKLP1')
 3000 FORMAT('ROWS')
 3001 FORMAT(1X,A1,2X,A2,A4,2A1)
 3002 FORMAT('COLUMNS')
 3003 FORMAT(4X,A2,A4,2A1,2X,A2,A4,2A1,2X,F11.4)
 3004 FORMAT('RHS')
 3006 FORMAT('BOUNDS')
 5000 FORMAT(10F8.2)
 6000 FORMAT(4X,'JSHKRH  ',2X,A2,A4,2A1,2X,F12.5)
 7000 FORMAT(1X,'LO',1X,'JKBOUND ',2X,A2,A4,2A1,2X,F12.5)
 7001 FORMAT(1X,'UP',1X,'JKBOUND ',2X,A2,A4,2A1,2X,F12.5)
 7002 FORMAT('ENDATA')
 7004 FORMAT(/,'  MINIMUM SALES',(/,27X,10F8.1))
 7005 FORMAT(/,'  MINIMUM STORAGE'.(/,27X,10F8.1))
 7006 FORMAT(/,'  LIMITS ON CASH',(/,27X,10F8.1))
 7007 FORMAT(/,'  LIMITS ON BANK DEALINGS',(/,27X,10F8.1))
 7008 FORMAT(/,'  LIMITS ON RESOURCE STORAGE',(/,27X,10F8.1))
 8000 FORMAT(1H1,//,10X,'CONSTRAINTS ON OPERATION')
 8001 FORMAT(//,5X,'MONTH  ',I2)
 9000 FORMAT(1X,'LO',1X,'BOUND1  ',2X,A2,A4,2A1,2X,F12.5)
 9001 FORMAT(1X,'UP',1X,'BOUND1  ',2X,A2,A4,2A1,2X,F12.5)
 9003 FORMAT(4X,'BASICRHS',2X,A2,A4,2A1,2X,F12.5)
      NN=NM*LONG
      MM=NN*HIGH
      WRITE(9,2999)
C
C   ROWS SECTION
C
      WRITE(9,3000)
      DO 1 K=1,NM
    1 WRITE(9,3001)  (ICODE(J),MONTH(K),(NAMER(J,L),L=1,3),J=1,HIGH)
      WRITE(9,2998)
      WRITE(9,2995)
      WRITE(9,2997)
C
C   COLUMNS SECTION
C
      WRITE(9,3002)
      CALL SETA
C
C   RIGHT HAND SIDE
C
      WRITE(9,3004)
      CALL SETB
      DO 5 K=1,NM
      DO 5 I=1,HIGH
      II=(K-1)*HIGH+I
      WRITE(9,6000) MONTH(K),(NAMER(I,L),L=1,3),B(II)
    5 CONTINUE
C
C   TO OUTPUT THIS RHS IN MPSOUT
C
```

```
      DO 4 K=1,NM
      DO 4 I=1,HIGH
      II=(K-1)*HIGH+I
    4 WRITE(9,9003) MONTH(K),(NAMER(I,L),L=1,3),AA(II)
      WRITE(9,3006)
C
C   CHANGES FOR MPSOUT
C
      DO 19 J=1,NN
      B(J)=0.0
   19 AA(J)=0.0
C
C   INCLUDE BASIC RANGES IN THE AA ARRAY
C
C   STORE THESE IN AA FOR LOWER BOUNDS
C
C    STORE IN B FOR UPPER BOUNDS
C
C   BOUNDS
C
      WRITE(6,8000)
      DO 6 K=1,NM
      WRITE(6,8001) K
      KK=(K-1)*LONG
C
C   MINIMUM SALES FOR EACH PERIOD
C
      READ(5,5000)(POLICY(J),J=1,NTOOL)
      WRITE(6,7004) (POLICY(J),J=1,NTOOL)
      DO 7 J=1,NTOOL
      ICOL=2*NTOOL+J
      KCOL=KK+ICOL
      AA(KCOL)=POLICY(J)
      WRITE(9,7000) MONTH(K),(NAMEC(ICOL,L),L=1,3),POLICY(J)
    7 CONTINUE
C
C   FINAL CASH REQUIRED AT END OF PERIOD
C
      READ(5,5000) CASHLO,CASHUP
      WRITE(6,7006) CASHLO,CASHUP
      ICOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+7
      KCOL=KK+ICOL
      B(KCOL)=CASHUP
      WRITE(9,7000) MONTH(K),(NAMEC(ICOL,L),L=1,3),CASHLO
      WRITE(9,7001) MONTH(K),(NAMEC(ICOL,L),L=1,3),CASHUP
C
C   LIMITS ON BANK LOANS PER PERIOD
C
      READ(5,5000) CASHLO,CASHUP
      WRITE(6,7007) CASHLO,CASHUP
      ICOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+8
      KCOL=KK+ICOL
      AA(KCOL)=CASHLO
      B(KCOL)=CASHUP
      WRITE(9,7000) MONTH(K),(NAMEC(ICOL,L),L=1,3),CASHLO
      WRITE(9,7001) MONTH(K),(NAMEC(ICOL,L),L=1,3),CASHUP
C
C   LIMITS ON RAW MATERIALS AT END OF EACH PERIOD
C   LOWER BOUND   UPPER BOUND
C
      READ(5,5000)  (POLICY(J),J=1,NRM)
      WRITE(6,7008) (POLICY(J),J=1,NRM)
      KCOL=3*NTOOL+NWC+NSUB+NLF
      DO 12 J=1,NRM
      ICOL=KCOL+J
      WRITE(9,7000) MONTH(K),(NAMEC(ICOL,L),L=1,3),POLICY(J)
   12 CONTINUE
      READ(5,5000) (POLICY(J),J=1,NRM)
      DO 13 J=1,NRM
      ICOL=KCOL+J
      WRITE(9,7001) MONTH(K),(NAMEC(ICOL,L),L=1,3),POLICY(J)
      ICOL=KK+KCOL+J
      B(ICOL)=POLICY(J)
   13 CONTINUE
    6 CONTINUE
C
      NB=NM/3
      DO 20 K=1,NB
C
```

```
C     BOUNDING FINAL CASH
C
      KCOL=(K-1)*LONG*3+2*LONG+4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+NOH+7
      AA(KCOL)=LIABLE(4000)
      DO 20 J=1,NRM
C
C
C     BOUNDING FINAL RAW MATERIALS
C
      KCOL=(K-1)*3*LONG+2*LONG+3*NTOOL+NWC+NSUB+NLF+J
      B(KCOL)=RMB(J)
      AA(KCOL)=RMB(J)
   20 CONTINUE
      DO 21 K=1,NM
      DO 21 J=1,LONG
      I=(K-1)*LONG+J
      IF(AA(I).GT.0.0) WRITE(9,9000) MONTH(K),(NAMEC(J,L),L=1,3),AA(I)
      IF(B(I).GT.0.0)  WRITE(9,9001) MONTH(K),(NAMEC(J,L),L=1,3),B(I)
   21 CONTINUE
C
C     RANGES
C
      WRITE(9,7002)
      RETURN
      END
```

```
      SUBROUTINE XLAG(NTYPE)
      INTEGER*2 IA(4000,2)
      COMMON IA(4000,2),AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50)
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
      KROW=NOH+NSCS+2*NTOOL+2*NRM+NLF+NWC+6
      IROW=2*NTOOL+NOH+NSCS+2*NRM+NLF+NWC+9
C
C     RECEIVABLES
C
      DO 20 J=1,NTOOL
      ICOL=2*NTOOL+J
      X=-LIST(J)*(1.0-DISCP(J))
      KK=RECLAG(J)+3
      IF(KK-NTYPE) 20,21,22
   22 IROW=KROW
   21 CONTINUE
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=X
   20 CONTINUE
C     PAYABLES
      KROW=NOH+NSCS+2*NTOOL+2*NRM+NLF+NWC+7
      IROW=NOH+NSCS+2*NTOOL+2*NRM+NLF+NWC+8
C
C     SUBCONTRACTING COSTS
C
      DO 30 J=1,NSUB
      IROW=II
      KK=SUBLAG(J)+3
      IF(KK.EQ.3) GO TO 30
      X=-SUBP(J)
      ICOL=3*NTOOL+NWC+J
      IF(KK-NTYPE) 30,31,32
   32 IROW=KROW
   31 CONTINUE
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=X
   30 CONTINUE
C
C     RAW MATERIALS BOUGHT
C
      DO 50 J=1,NRM
      IROW=II
      ICOL=3*NTOOL+NWC+NLF+NRM+J+NSUB
      KK=RMLAG(J)+3
      X=-RMB(J)
      IF(KK-NTYPE) 50,51,52
   52 IROW=KROW
   51 CONTINUE
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=X
   50 CONTINUE
C
C     MARKETING EXPENSES
C
      DO 60 J=1,NTOOL
      IROW=II
      ICOL=3*NTOOL+NWC+NLF+NSUB+2*NRM+J
      KK=MRKLAG(J)+3
      X=-MARK(J,J)
      IF(KK-NTYPE) 60,61,62
```

```
   62 IROW=KROW
   61 CONTINUE
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=X
   60 CONTINUE
C
C     OVERHEAD EXPENSES
C
      DO 70 J=1,NOH
      IROW=II
      ICOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+NSCS+6+J
      KK=OHLAG(J)+3
      X=-1.0
      IF(KK-NTYPE) 70,71,72
   72 IROW=KROW
   71 CONTINUE
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=X
   70 CONTINUE
C
C     BANK CHARGES
C
      IROW=II
      ICOL=4*NTOOL+NWC+NLF+NSUB+2*NRM+NSCS+NOH+10
      KK=ALFLAG+3
      X=-1.0
      IF(KK-NTYPE)90,91,92
   92 IROW=KROW
   91 CONTINUE
      NROW=NROW+1
      IA(NROW,1)=IROW
      IA(NROW,2)=ICOL
      AA(NROW)=X
   90 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SETB
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50)
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
       DIMENSION CAPWC(20),CAPLF(5)
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
      EQUIVALENCE(LIABLE(4000),CASHIN)
C
C SETTING UP THE RIGHT HAND SIDE OF THE MATRIX
C
      WRITE(6,7000)
 7000 FORMAT(1H1,//,10X,' INPUTS TO THE MODEL')
 5000 FORMAT(10F8.2)
      MM=NM*HIGH
      DO 1 I=1,MM
    1 B(I)=0.
C
C WORK CENTRE CAPACITY
C
      READ(5,5000) (CAPWC(J),J=1,NWC)
      WRITE(6,8002) (CAPWC(J),J=1,NWC)
 8002 FORMAT(/,'  CAPACITY OF WORK CENTRES',(/,27X,10F8.1))
      DO 10 K=1,NM
      DO 10 I=1,NWC
      IROW=(K-1)*HIGH+I
   10 B(IROW)=CAPWC(I)
C
C  CAPACITY OF LABOUR FORCE PER PERIOD
C
      READ(5,5000)(CAPLF(J),J=1,NLF)
      WRITE(6,7003) (CAPLF(J),J=1,NLF)
 7003 FORMAT(/,'  CAPACITY OF LABOUR FORCE',(/,27X,10F8.1))
      DO 9 I=1,NLF
      DO 9 K=1,NM
      IROW=(K-1)*HIGH+I+NWC
      B(IROW)=CAPLF(I)
    9 CONTINUE
C
C  STORAGE CAPACITY
C
      READ(5,5000) CAPST
      WRITE(6,7001) CAPST
 7001 FORMAT(//,'  STORAGE CAPACITY..............',F10.2,' BINS')
      DO 2 K=1,NM
      I=(K-1)*HIGH+NWC+NLF+NTOOL+1
    2 B(I)=CAPST
C
C INPUT OF TOOLS  RHS FOR STORAGE LAG INEQUALITY
C
       K=1
      KROW=(K-1)*HIGH+NWC+NLF+2*NTOOL+2*NRM+NSCS+NOH+11+NTOOL
      IROW=(K-1)*HIGH+NWC+NLF+2*NTOOL+2*NRM+NSCS+NOH+12
      READ(5,5000) (B(J),J=IROW,KROW)
      WRITE(6,7004)(B(J),J=IROW,KROW)
 7004 FORMAT(//,'  INPUT OF FINISHED PRODUCTS....',10F10.2)
      DO 3 I=1,NTOOL
      KROW=NWC+NLF+2*NTOOL+2*NRM+11+NSCS+NOH+I
      DO 3 K=2,NM
      IROW=(K-1)*HIGH+KROW
    3 B(IROW)=B(KROW)
C
C  INPUT OF RAW MATERIALS
C
      READ(5,5000) (RMB(J),J=1,NRM)
      WRITE(6,7002) (RMB(J),J=1,NRM)
```

```
7002 FORMAT(//,'  INPUT OF RAW MATERIALS........',3F10.2)
     DO 7 I=1,NRM
     IROW=NWC+NLF+NTOOL+1+I
     B(IROW)=RMB(J)
     IROW=IROW+NRM
   7 B(IROW)=RMB(I)
C
C  INPUT OF CASH
C
     READ(5,5000) CASHIN
     WRITE(6,7007) CASHIN
7007 FORMAT(//,'  INPUT OF CASH..................',F10.2)
     IROW=2*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+10
     B(IROW)=CASHIN
C
C   INPUT OF SECOND R.H.S FOR USE WITH BASIC
C
C  FIRST SET OF ROWS ARE SPLIT IN THE RATIOS 1,1    .33,.66,  .66,.33
C
     DO 19 J=1,MM
  19 AA(J)=0.0
     KROW=2*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+11
     DO 202 K=1,NM
     DO 202 J=1,NTOOL
     IROW=KROW+J+(K-1)*HIGH
     AA(IROW)=B(IROW)/FLOAT(NM)
 202 CONTINUE
     NB=NM/3
     DO 200 K=1,NB
     KK=3*(K-1)*HIGH
     JJ=NWC+NLF+NTOOL+1+NRM
     DO 201 I=1,3
     DO 201 J=1,JJ
     IROW=KK+J+(I-1)*HIGH
     AA(IROW)=B(J)*(4.0-FLOAT(I))/3.0
 201 CONTINUE
     DO 203 J=1,NRM
     JJ=NWC+NLF+NTOOL+NRM+1+J
     IROW=KK+JJ
     AA(IROW)=B(JJ)
 203 CONTINUE
     JJ=2*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+10
     IROW=KK+JJ
     AA(IROW)=B(JJ)
 200 CONTINUE
     RETURN
     END
```

```
      SUBROUTINE SETA
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50)
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
      KCOL=4*NTOOL+NWC+NSUB+NLF+2*NRM+3
      DO 500 K=1,NM
      DO 500 J=1,LONG
      GO TO (503,502,504,504,504,504,504,504,504,504,504,504),K
  501 CONTINUE
  504 II=1
      JJ=NPRE2
      KMONTH=K-2
      CALL WRITEA
  502 II=NPRE2+1
      JJ=NPRE1
      KMONTH=K-1
      CALL WRITEA
  503 II=NPRE1+1
      JJ=NBLOCK
      KMONTH=K
      CALL WRITEA
      IF(NM-K) 508,508,704
  704 II=NBLOCK+1
      JJ=NPOST1
      KMONTH=K+1
      CALL WRITEA
      IF(NM-K-1) 508,508,505
  505 KMONTH=KMONTH+1
      II=NPOST1+1
      JJ=NPOST2
      CALL WRITEA
      IF(KMONTH-NM) 505,508,508
  508 CONTINUE
      IROW=HIGH+1
      KK=(K-1)*LONG+J
      IF(B(KK).EQ.0.0) GO TO 506
      WRITE(9,3004) MONTH(K),(NAMEC(J,L),L=1,3),(NAMER(IROW,L),L=1,3),
     1 B(KK)
  506 CONTINUE
      IROW=HIGH+2
      IF(LIABLE(KK).EQ.0.0) GO TO 507
      WRITE(9,3004) MONTH(K),(NAMEC(J,L),L=1,3),(NAMER(IROW,L),L=1,3),
     1 LIABLE(KK)
  507 CONTINUE
      L=2*NTOOL+1
      LL=3*NTOOL
      KK=J-2*NTOOL
      IF(L.LE.J.AND.J.LE.LL) WRITE(9,3005) MONTH(K),(NAMEC(J,N),N=1,3)
     1,LIST(KK)
 3005 FORMAT(4X,A2,A4,2A1,2X,'GROSSALE',2X,F11.4)
  500 CONTINUE
 3003 FORMAT(4X,A2,A4,2A1,2X,A2,A4,2A1,2X,F11.4)
 3004 FORMAT(4X,A2,A4,2A1,2X,3A2,2X,2X,F11.4)
      RETURN
      END
```

```
      SUBROUTINE WRITEA
      INTEGER*2 IA(4000,2)
      COMMON IA,AA(4000),B(4000),
     1 MCREQ(3,18,50),WCLF(18,1),SPACE(50),RMREQ(3,50,2),LIST(50),
     2 MARK(50,50),SCSP(3,50),DISCP(50),SUBP(7),WAGES(2),RMB(2),
     3 WIPP(2,50),SUBWC(7),ICODE(800),NAMER(800,3),NAMEC(800,3),LAG(50)
     4OHRATE(6)
      COMMON MONTH(12),RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6),ALFLAG
      COMMON LIABLE(4000)
      COMMON NROW
      COMMON KMONTH
      COMMON I,J,K,N,NT,NM,II,JJ,ICOL,IROW,HIGH,LONG,ALPHA,NWC,NLF,
     1NTOOL,NSUB,NRM,NSCS,NOH
      COMMON NPRE2,NPRE1,NBLOCK,NPOST1,NPOST2
      REAL LIABLE
      REAL MCREQ,LIST,MARK
      INTEGER SUBWC,HIGH,OHLAG,ALFLAG,MRKLAG,RMLAG,SUBLAG,RECLAG
      DO 601 I=II,JJ
      IF(IA(I,2)-J) 601,401,601
  401 IROW=IA(I,1)
      WRITE(9,3003) MONTH(K),(NAMEC(J,L),L=1,3),MONTH(KMONTH),
     1(NAMER(IROW,L),L=1,3),AA(I)
  601 CONTINUE
 3003 FORMAT(4X,A2,A4,2A1,2X,A2,A4,2A1,2X,F11.4)
      RETURN
      END
```

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```
      DIMENSION WAGES(3)
      DIMENSION BETA(3)
      DIMENSION Y(50)
      DIMENSION X(4000),RX(4000)
      DIMENSION RMB(2),SUBP(7),TOTLAB(2),LIST(50),RMIN(2),WCLF(18,2)
      DIMENSION RECLAG(50),SUBLAG(7),LABLAG(2),RMLAG(2),
     1 MRKLAG(50),OHLAG(6)
      REAL LABREQ(2)
      REAL LIABLE
      REAL LIST
C
C   DATA INPUT
C
      READ(5,6001)NTOOL,NWC,NSUB,NLF,NRM,NSCS,NOH,NM
      READ(5,5003)  (LIST(J),J=1,NTOOL)
      READ(5,9994)  ((WCLF(J,I),J=1,NWC),I=1,NLF)
      READ(5,5004)  (SUBP(J),J=1,NSUB)
      READ(5,5002)  (WAGES(J),J=1,NRM)
      READ(5,5002)  (RMB(J),J=1,NRM)
      READ(5,9997)  (RECLAG(J),J=1,NTOOL)
      READ(5,9997)  (SUBLAG(J),J=1,NSUB)
      READ(5,9997)  (LABLAG(J),J=1,NLF)
       READ(5,9997)  (RMLAG(J),J=1,NRM)
      READ(5,9997)  (MRKLAG(J),J=1,NTOOL)
      READ(5,9997)  (OHLAG(J),J=1,NOH)
      READ(5,9997)  ALFLAG
      READ(5,5001)  (Y(J),J=1,NTOOL)
      READ(5,5001)   (RMIN(J),J=1,NRM)
      READ(5,5001)  CASHIN
      READ(5,5005)  ASSETS
      READ(5,5001)   (BETA(I),I=1,3)
C
      WRITE(6,6500)  NTOOL,NWC,NSUB,NLF,NRM,NSCS,NOH,NM
      WRITE(6,6501)
      WRITE(6,6502)(RECLAG(J),J=1,NTOOL)
      WRITE(6,6503)(SUBLAG(J),J=1,NSUB)
      WRITE(6,6504)(LABLAG(J),J=1,NLF)
      WRITE(6,6505)  (RMLAG(J),J=1,NRM)
      WRITE(6,6506)(MRKLAG(J),J=1,NTOOL)
      WRITE(6,6507)(OHLAG(J),J=1,NOH)
      WRITE(6,6508)ALFLAG
      WRITE(6,6509)(RMB(J),J=1,NRM)
      WRITE(6,6510)(WAGES(J),J=1,NRM)
      WRITE(6,6511)CASHIN
      WRITE(6,6512)(Y(J),J=1,NTOOL)
      WRITE(6,6515)  (RMIN(J),J=1,NRM)
      WRITE(6,6514)  ASSETS
      WRITE(6,6513)  (BETA(I),I=1,3)
C
      HIGH=3*NTOOL+NWC+NLF+2*NRM+NSCS+NOH+11
      LONG=3*NTOOL+NSUB+2*NRM+NSCS+NOH+6
C
      DO 1 J=1,NLF
    1 TOTLAB(J)=0.0
C
C    OPINCM IS THE OPERATING INCOME TO DATE
C
      OPINCM=0.0
C
C
      KK=NM*HIGH+3
      WRITE(6,9996) KK
      READ(3,5000,END=800) (RX(J),J=1,KK)
      KK=LONG*NM
      WRITE(6,9996) KK
      READ(3,5000,END=800)   (X(J),J=1,KK)
 4000 FORMAT(1H1)
 4001 FORMAT(1H ,//,'  INCOME AND SURPLUS ACCOUNT               MONTH ',I3,
     1///)
 4002 FORMAT(30H  GROSS SALES                  ,F10.0)
 4003 FORMAT(30H  LESS CASH DISCOUNT           ,F10.0)
 4004 FORMAT(30H  NETT SALES                   ,F10.0)
 4005 FORMAT(30H  LESS STANDARD COST           ,F10.0)
 4006 FORMAT(30H       OTHER COSTS             ,F10.0)
 4007 FORMAT(30H  TOTAL COST OF SALES          ,F10.0)
 4008 FORMAT(30H  MANUFACTURING MARGIN         ,F10.0)
 4009 FORMAT(30H  LESS DIRECT SALES            ,F10.0)
 4010 FORMAT(30H       PROMOTION               ,F10.0)
```

```
4011 FORMAT(30H       DISTRIBUTION                    ,F10.0)
4012 FORMAT(30H  TOTAL SELLING EXPENSES               ,F10.0)
4013 FORMAT(30H  ADMIN AND GENERAL                    ,F10.0)
4014 FORMAT(30H  LOAN INTEREST                        ,F10.0)
4015 FORMAT(30H  TOTAL SELLING EXPENSES               ,F10.0)
4016 FORMAT(30H  S.E.T.                               ,F10.0)
4017 FORMAT(30H  OPERATING INCOME                     ,F10.0)
4018 FORMAT(30H  INCOME BEFORE TAX                    ,F10.0)
4019 FORMAT(30H  LESS TAX                             ,F10.0)
4020 FORMAT(30H  NETT INCOME                          ,F10.0)
4021 FORMAT(30H  PROVISION FOR DEPRECIATION           ,F10.0)
4030  FORMAT(30H  BALANCE SHEET             MONTH ,I5)
4031 FORMAT(30H ASSETS                                 )
4032 FORMAT(30H  CASH                               ,F10.0)
4033 FORMAT(30H  ACCOUNTS RECEIVABLE                ,F10.0)
4034 FORMAT(30H INVENTORIES                         ,F10.0)
4035 FORMAT(30H    FINISHED GOODS                   ,F10.0)
4036 FORMAT(30H    W.I.P.                           ,F10.0)
4037 FORMAT(30H      MATERIALS                      ,F10.0)
4038 FORMAT(30H     IN TRANSIT                      ,F10.0)
4040 FORMAT(30H  TOTAL CURRENT ASSETS               ,F10.0)
4041 FORMAT(30H  TOTAL INVENTORIES                  ,F10.0)
4042 FORMAT(30H P.P.E.                              ,F10.0)
4043 FORMAT(30H AT ORIGINAL COST                    ,F10.0)
4044 FORMAT(30H  LESS ALLOWANCES                    ,F10.0)
4045 FORMAT(30H  TOTAL ASSETS                       ,F10.0)
4049 FORMAT(1H1,//,'    CASH FLOW STATEMENT                MONTH ',I2,///)
4050 FORMAT(30H  CASH AT BEGINNING                ,F10.0)
4051 FORMAT('  EXPENDITURES        ')
4052 FORMAT(30H RAW MATERIALS                     ,F10.0)
4053 FORMAT(30H WAGES                             ,F10.0)
4054 FORMAT(30H OVERHEAD ACCOUNTS                 ,F10.0)
4056 FORMAT(' BANK CHARGES                       ',F10.0)
4057 FORMAT(' RECEIPTS                           ',F10.0)
4058 FORMAT(30H NETT SALES                        ,F10.0)
4059 FORMAT(30H CASH AT END                       ,F10.0)
4061 FORMAT(//,'   SALES TO ASSETS              ',F20.4)
4062 FORMAT(//,'   INCOME TO ASSETS             ',F20.4)
4063 FORMAT(//,'   INCOME PER MAN HOUR          ',F20.4)
4064 FORMAT(//,'   SALES PER MAN HOUR           ',F20.4)
4065 FORMAT(//,'   STOCK TO SALES RATIO         ',F20.4)
4066 FORMAT(//,'   VALUE ADDED PER MAN HR       ',F20.4)
4067 FORMAT(//,'   VALUE ADDED TO ASSETS        ',F20.4)
4068 FORMAT(//,'   VALUE ADDED TO SALES         ',F20.4)
4069 FORMAT(//,'   DEBTORS TO CREDITORS         ',F20.4)
4070 FORMAT(//,'      OPERATING RATIOS    ',//)
4071 FORMAT(//)
4072 FORMAT('      FINISHED GOODS           ',F20.0,//)
4073 FORMAT('      GROSS SALES              ',F20.0,//)
4074 FORMAT('      TOTAL ASSETS             ',F20.0,//)
4075 FORMAT('      LIABILITIES              ',F20.0,//)
4076 FORMAT('      VALUE ADDED BY FIRM      ',F20.0,//)
4077 FORMAT(30X,'                ')
5000 FORMAT(F20.4)
5001 FORMAT(10F8.2)
5002 FORMAT(12F6.4)
5003 FORMAT(10X,F8.4)
5004 FORMAT(12F6.2)
5005 FORMAT(F12.2)
6000 FORMAT(1H1)
6001 FORMAT(36I2)
6002 FORMAT(//,'   MONTH ',I2,/,15X,'TOOL1',5X,'TOOL2',5X,'TOOL3',
    15X,'TOOL4',5X,'TOOL5',5X,'TOOL6',5X,'TOOL7')
6003 FORMAT(/,' PRODUCE   ',(7F10.2))
6004 FORMAT(/,' STORE     ',(7F10.2))
6005 FORMAT(/,' SELL      ',(7F10.2))
6006 FORMAT(1H0,/,20X,' PRODUCTION,STORAGE AND SALES SCHEDULES')
6500 FORMAT(1H ,//,' DATA INPUT',//,' PRODUCTS M/C S',5X,'SUBS',6X,'LAB
    1OUR',4X,'MATERIALS STANDARDS O-HEADS',3X,'PERIODS',//,8(2X,I4,4X))
6501 FORMAT(///,' LAGS')
6502 FORMAT(/,'      RECEIPTS                ',8X,50I1)
6503 FORMAT(/,'      CONTRACT  PAYMENTS',8X,50I1)
6504 FORMAT(/,'      LABOUR    PAYMENTS',8X,50I1)
6505 FORMAT(/,'      MATERIAL  PAYMENTS',8X,50I1)
6506 FORMAT(/,'      ADVERT    PAYMENTS',8X,50I1)
6507 FORMAT(/,'      O-HEADS   PAYMENTS',8X,50I1)
6508 FORMAT(/,'      INTEREST  PAYMENTS',8X,50I1)
6509 FORMAT(//,' RMB                       ',/,20F6.2)
6510 FORMAT(//,' WAGES                     ',/,20F6.2)
```

```
 6511 FORMAT(//,' CASH IN              ',/,F12.3)
 6512 FORMAT(//,' PRODUCT INPUT        ',/,20F6.2)
 6513 FORMAT(//,' BETA RATES           ',/,20F6.2)
 6514 FORMAT(//,' OTHER ASSET INPUT',/,F20.4)
 6515 FORMAT(//,' INPUT OF MATERIALS',/,10F10.2)
 9994 FORMAT(20F4.3)
 9996 FORMAT(///,'  DIMENSION',I10)
 9997 FORMAT(70I1)
C
C  WRITING OUT THE PRODUCTION SCHEDULES ETC.
C  FOR THE REMAINDER WE NEED THE INITIAL CONDITIONS THAT HAVE BEEN
C   BEEN SUPPLIED TO THE MAIN PROGRAMME
C
      WRITE(6,4000)
      WRITE(6,6006)
      DO 7 K=1,NM
      WRITE(6,6002) K
      KK=(K-1)*LONG
      L=KK+1
      LL=KK+NTOOL
      WRITE(6,6003) (X(J),J=L,LL)
      L=LL+1
      LL=LL+NTOOL
      WRITE(6,6005) (X(J),J=L,LL)
      DO 8 J=1,NTOOL
      L=KK+J
      LL=KK+NTOOL+J
      Y(J)=Y(J)+X(L)-X(LL)
    8 CONTINUE
      WRITE(6,6004) (Y(J),J=1,NTOOL)
    7 CONTINUE
C
C    CALCULATING THE VALUE OF FINAL STOCKS
C
      STOCK =0.0
      DO 30 J=1,NTOOL
   30 STOCK=STOCK+Y(J)*LIST(J)
C
C CALCULATION OF THE INCOME AND SURPLUS ACCOUNT
C INPUT FOR THIS WILL BE..... THE FINAL VALUES FROM THE SIMPLEX TABLEAU
C THE LENGTH OF EACH MONTHS MATRIX, THE NUMBER OF TOOLS CONSIDERED, ETC.
C
      KNUM=3*NTOOL+NSUB+2*NRM
C
C  INCOME AND SURPLUS
C
      DO 6 K=1,NM
      KK=(K-1)*LONG
      KH=(K-1)*HIGH
      WRITE(6,4000)
      WRITE(6,4001) K
C
C  GROSS SALES
C
      I=NWC+NLF+NTOOL+3+KH+2*NRM
      WRITE(6,4002) RX(I)
C
C  DISCOUNT
C
      J=I+NTOOL+NSCS+NOH+2
      WRITE(6,4003) RX(J)
      WRITE(6,4077)
C
C  NET SALES
C
      Y(1)=RX(I)-RX(J)
      WRITE(6,4004) Y(1)
C
C  STANDARD COST OF SALES
C
      Y(2)=0.
      DO 2 J=1,NSCS
      I=KK+KNUM+J
    2 Y(2)=Y(2)+X(I)
      WRITE(6,4005) Y(2)
      WRITE(6,4006)
      WRITE(6,4007)          Y(2)
      WRITE(6,4077)
C
```

```
C  MANUFACTURING MARGIN
C
      Y(3)=Y(1)-Y(2)
      WRITE(6,4008) Y(3)
      Y(4)=0.
C
C  TOTAL SELLING EXPENSES
C
      DO 3  J=1,NTOOL
      I=KK+2*NTOOL+NSUB+2*NRM+J
    3 Y(4)=Y(4)+X(I)
      WRITE(6,4009)
      WRITE(6,4010)
      WRITE(6,4011)
      WRITE(6,4012)  Y(4)
C ADMIN AND GENERAL IS GIVEN BY BETA(1) TIMES THE TOTAL LABOUR FORCE
C    CALCULATE THE REQUIRED LABOUR FORCE IN HOURS FROM THE ROW VALUES
C    GIVING THE TIMES REQUIRED AT EACH MACHINE CENTRE
C
      DO 100 I=1,NLF
      LABREQ(I)=0.0
      DO 100 J=1,NWC
      JJ=KH+J
      LABREQ(I)=LABREQ(I)+RX(JJ)*WCLF(J,I)
      TOTLAB(I)=TOTLAB(I)+LABREQ(I)
  100 CONTINUE
      Y(5)=0.
      DO 4  J=1,NLF
    4 Y(5)=Y(5)+LABREQ(J)*BETA(1)
      WRITE(6,4013)  Y(5)
      Y(6)=0.
      WRITE(6,4014)  Y(6)
      Y(7)=Y(4)+Y(5)
      WRITE(6,4015)  Y(7)
C
C RATE AT WHICH S.E.T. IS PAID IS BETA(2)
C
C THIS COULD BE + OR -, AND COULD BE TAKEN TO REFER TO ANY OTHER GENERAL
C
C EXPENSE THAT VARIES WITH LABOUR FORCE
C
      Y(8)=0.
      DO 5  J=1,NLF
    5 Y(8)=Y(8)+LABREQ(J)*BETA(2)
      WRITE(6,4016)  Y(8)
      WRITE(6,4077)
      Y(9)=Y(3)-Y(7)-Y(8)
C
C  INCOME BEFORE TAX
C
      WRITE(6,4017)  Y(9)
      WRITE(6,4018)  Y(9)
      OPINCM=OPINCM+Y(9)
C
C COMPANY TAX IS ASSUMED TO BE PAID AT A FLAT RATE OF BETA(3) PER CENT
C
      Y(10)=Y(9)*BETA(3)
      IF(Y(9).LT.0.0) Y(10)=0.0
      WRITE(6,4019)  Y(10)
      WRITE(6,4077)
      Y(11)=Y(9)-Y(10)
      WRITE(6,4020)  Y(11)
C
C   CASH FLOW STATEMENT
C
      WRITE(6,4049)  K
C
C  GROSS SALES
C
      I=NWC+NLF+NTOOL+3+KK+2*NRM
      WRITE(6,4002)  RX(I)
C
C  INCOME BEFORE TAX
C
      WRITE(6,4018)  Y(9)
      WRITE(6,4071)
C
C  CASH AT BEGINNING
C
```

```
      Z=CASHIN
      I=KK-LONG+KNUM+NSCS+NOH+1
      IF(I.GT.0) Z=X(I)
      WRITE(6,4050) Z
      WRITE(6,4071)
C
C   EXPENDITURES
C
C
C
C   RAW MATERIALS
C
      Y(10)=0.0
      DO 220 J=1,NRM
      ICOL=KK-RMLAG(J)*LONG+2*NTOOL+NSUB+NRM+J
      IF(ICOL.LE.0) GO TO 220
      Y(10)=Y(10)+X(ICOL)*RMB(J)
  220 CONTINUE
      WRITE(6,4052) Y(10)
C
C   WAGES
C
      Y(11)=0.0
      DO 23 J=1,NLF
      IROW=KH-LABLAG(J)*HIGH+NWC+J
      IF(IROW.LT.0) GO TO 23
      Y(11)=Y(11)+RX(IROW)*WAGES(J)
   23 CONTINUE
      WRITE(6,4053) Y(11)
C
C   OVERHEAD ACCOUNTS
C
      Y(12)=0.0
      DO 24 J=1,NOH
      ICOL=KK-OHLAG(J)*LONG+KNUM+NSCS+J
      IF(ICOL.LE.0) GO TO 24
      Y(12)=Y(12)+X(ICOL)
   24 CONTINUE
      WRITE(6,4054) Y(12)
C
C   BANK CHARGES
C
      Y(13)=0.0
      I=KK-ALFLAG*LONG+KNUM+NSCS+NOH+4
      IF(I.GT.0) Y(13)=X(I)
      WRITE(6,4056) Y(13)
C
C   RECEIPTS
C
      WRITE(6,4057)
C
C   NETT SALES
C
      WRITE(6,4004) Y(1)
      WRITE(6,4071)
C
C   CASH AT END
C
      I=(K-1)*LONG+KNUM+NSCS+NOH+1
      WRITE(6,4059) X(I)
    6 CONTINUE
C     PRINTING OUT THE OPERATING RATIOS
      WRITE(6,4000)
      WRITE(6,4070)
C
C     ASSETS,LIABILITIES AND GROSS SALES ARE THE LAST THREE ROW VALUES
C     CALCULATING THE COSTS OF MATERIALS INPUT AND CONTRACT INPUT
C     SUBRACTED FROM GROSS SALES THESE GIVE THE VALUE ADDED BY THE FIRM
C
      COSTS=0.0
      DO 31 K=1,NM
      I=(K-1)*LONG+3*NTOOL+NSUB+2*NRM+1
      COSTS=COSTS+X(I)
      DO 31 J=1,NSUB
      I=(K-1)*LONG+2*NTOOL+J
   31 COSTS=COSTS+X(I)*SUBP(J)
C
      KK=NM*HIGH
      J=KK+1
      TOTASS=RX(J)+ASSETS
```

```
      J=KK+2
      LIABLE=RX(J)
      J=KK+3
      SALES=RX(J)
      VALADD=SALES-COSTS
      WRITE(6,4000)
      WRITE(6,4072) STOCK
      WRITE(6,4073) SALES
      WRITE(6,4074) TOTASS
      WRITE(6,4075) LIABLE
      WRITE(6,4076) VALADD
C
C     OPERATING INCOME HAS ALREADY BEEN CALCULATED
C     ALL THE FIGURES WILL BE GROSSED UP TO AN EQUIVALENT TWLEVE MONTH
C     PERIOD BY USING THE FACTOR 12/NM
C
      AFACT=12.0/FLOAT(NM)
C
C
C     SALES TO ASSETS
C
      Z=SALES*AFACT/TOTASS
      WRITE(6,4061) Z
C
C     INCOME TO ASSETS
C
      Z=OPINCM*AFACT/TOTASS
      WRITE(6,4062) Z
C
C     LABOUR PRODUCTIVITY
C
      TLABHR=0.0
      DO 35 I=1,NLF
   35 TLABHR=TLABHR+TOTLAB(I)
C
C     INCOME PER MAN HOUR
C
      Z=OPINCM/TLABHR
      WRITE(6,4063) Z
C
C     SALES PER MAN HOUR
C
      Z=SALES/TLABHR
      WRITE(6,4064) Z
C
C     VALUE ADDED STATISTICS
C
C     VALUE ADDED PER MAN HOUR
C
      Z=VALADD/TLABHR
      WRITE(6,4066) Z
C
C     VALUE ADDED TO TOTAL ASSETS
C
      Z=VALADD/TOTASS
      WRITE(6,4067) Z
C
C     VALUE ADDED PROPORTION OF SALES
C
      Z=VALADD/SALES
      WRITE(6,4068) Z
C
C     STOCK TURNOVER
C
      Z=STOCK*AFACT/SALES
      WRITE(6,4065) Z
C
C     DEBTORS TO CREDITORS
C
      Z=-1.0
      IF(LIABLE.GT.0.0) Z=ASSETS/LIABLE
      WRITE(6,4069) Z
      GO TO 802
C
  800 WRITE(6,801) J
  801 FORMAT(//,' ERROR OCCURED AT J =',I5)
  802 CONTINUE
      STOP
      END
```

## Appendix 2.4 Computational Difficulties and Solution Strategies

### 2.4.1 LP Models

The model described in Chapter 2 is a second version of the matrix generating programme (MGP). The initial version contained many of the row sums and implied variables as explicit column values, (e.g. work in progress, work centre usage, etc.), and had a larger column dimension than the version described in Chapter 2, (compare examples 6 and 7 in Table 2.23). With small models, this earlier formulation was found to be satisfactory; the explicit formulation enabled management to comprehend the model more readily; larger models soon gave rise to computational difficulties and the revised form of the MGP was used. (Apart from example 6 of Table 2.23, all results are obtained using the formulation of Section 2.5 and Appendix 2.2).

| NO | ROWS | COLS | DATA | TIME | LP/CC | CRASH | COMMENTS |
|---|---|---|---|---|---|---|---|
| 1 | 238 | 150 | 3/5 | 0.66 | LP | YES | |
| 2 | 239 | 151 | 3/5 | 0.5 | CC | NO | |
| 3 | 239 | 151 | 3/5 | 0.4 | CC | NO | |
| 4 | 238 | 150 | 3/5 | 0.49 | LP | YES | |
| 5 | 258 | 151 | 3/5 | $\infty$ | CC | NO | INDIFFERENCE |
| {6 | 565 | 684 | 3/12 | 9.07 | LP | NO | ROW AND DJ CHECKS |
| 7 | 576 | 360 | 3/12 | 1.63 | LP | YES | |
| 8 | 240 | 150 | 3/5 | 0.79 | LP | YES | |
| 9 | 239 | 150 | 3/5 | 4.59 | LP | YES | INDIFFERENCE M66-1274, 4.1 MIN |

Table 2.23 Sample Times for Smaller LP Models

2.4.1.1 Small Models: The computation times for small models (i.e. 3/5 and 3/12 models) are compared in Table 2.23. The use of the CRASH† procedure is noted in the CRASH column;

† for MPS terminology see (51)

the entry LP (or CC) in the LP/CC column denotes the use of the linear or Charnes and Cooper (fractional) programming, algorithm. The time noted is for optimisation only, i.e. from the time of setting up of the problem to its optimisation. Generally a further 3-5 minutes must be added to this time to allow for the matrix generating, compilation and data conversion steps outlined in Appendices 2.2 and 2.3.

Although the use of 'CRASH' appears to have retarded the solution of small problems, later experience with this routine proved beneficial. Even with such small jobs, some difficulties were manifest. Example 9 of Table 2.23, arrived at an indifference plane during computation; for 1208 iterations the objective function remained constant, i.e. the degeneracy due to 'computation' had not been overcome.

All jobs detailed in Table 2.23 were run using the HASP system (50), i.e. with core size restricted to 32K bytes but with no charge for input/output time. The critical level of row dimension between HASP and NON-HASP was found to be between 900-1000 rows. Above 1000 rows, jobs had to use total core (65K bytes) and were run using the on-line, NON-HASP, system with a consequent rise in computation time.

2.4.1.2 <u>Large Models</u>: A 26/8 model was set up and run under the HASP system. (The dimensions for this model, and the 26/12 model are shown in Table 2.24). Computation had to be effected in four stages due to time restrictions on the computer unit. The carry-over of information between stages was effected using the basis preservation techniques. For this model, the solution time was 103 minutes. Part of this large solution time was due to the difficulty of finding the first feasible solution. Since the model was run under the

HASP system, CRASH gave little help. (A discussion of the value of the CRASH routine is presented below in 2.4.1.3). The eta files soon exhausted core, and the routine did not complete its second phase satisfactorily.

The difficulties of the 26/12 model are discussed below in 2.4.1.3. The large number of fixed rows (equalities) in the 26/8 model was due to the explicit inclusion of all

| | | Total | Normal | Free | Fixed | Bounded |
|---|---|---|---|---|---|---|
| 26/8 | Rows (LOG. VAR.) | 930 | 224 | 2 | 704 | 0 |
| | Columns (STR. VAR.) | 1192 | 1016 | 0 | 0 | 176 |
| 26/12 | Rows (LOG. VAR.) | 1395 | 1260 | 75 | 60 | 0 |
| | Columns (STR. VAR.) | 1188 | 1152 | 0 | 0 | 36 |

Table 2.24 The 26/8 and 26/12 model dimensions

accounting variables in equality rows. For the 26/12 model, and all subsequent models, the row sums (normal rows) were used.

2.4.1.3 The 26/12 Model: The normal setting of MPS tolerances was used to attempt the computation of the 26/12 model; computational difficulties arose immediately. These were:

i. row checks; left and right hand sides of row sums differing by more than XTOLERR

ii. non-zero reduced costs (DJ's) for basic variables (i.e. DJ's in excess of XTOLDJ)

iii. singularities in the basis during inversion.

All three signify computational, rather than theoretical errors, (or errors of formulation).

The process of amending the tolerances to facilitate solution is presented in Table 2.25.

| NO. | SCALE | XEPS | CRASH | XFREQINV | XTOLPIV | XTOLV | XTOLDJ | XOBJ | COMMENTS |
|---|---|---|---|---|---|---|---|---|---|
| 1 | no | 0.1 | no | 100 | *10.0 | *50.0 | *10.0 | ASSETS | DUAL effective but singularities occur at iteration 132 |
| 2 | no | not used | yes | 50 | *90.0 | *50.0 | *10.0 | " | Slower convergence to feasible solution. Singularities at iteration 183, after 18 min. |
| 3 | no | 0.1 | yes | 50 | *100.0 | 0.5 | 0.5 | " | Etas ex-core during CRASH. 206 infeasibilities; loss of control. |
| 4 | no | 0.1 | yes | 50 | *100.0 | 0.05 | 0.05 | " | Singularity at first inversion XTOLV or XTOLDJ too high. |
| 5 | no | 0.1 | no | 30 | *1000.0 | 0.5 | 0.5 | " | XTOLPIV not critical above $10^{-6}$ (DUAL) |
| 6 | yes | 0.1 | no | 50 | *100.0 | 0.5 | 0.5 | GROSSALE | Unsatisfactory control of infeasibility after iteration 44. XTOLPIV is not affecting the accuracy |
| 7 | yes | 0.1 | no | 50 | *100.0 | 0.5 | 0.5 | ASSETS | Similar to No.6 XFREQINV too high ? |
| 8 | yes | 0.001 | no | 35 | *100.0 | 0.5 | 0.5 | " | Better than Nos. 6 & 7. XTOLDJ and XTOLV are too large |

able 2.25 The solution strategies for the 26/12 model.

Standard tolerances are given in No.10

Scale factors for tolerances are denoted '*10.0'

| NO. | SCALE | XEPS | CRASH | XFREQINV | XTOLPIV | XTOLV | XTOLDJ | XOBJ | COMMENTS |
|---|---|---|---|---|---|---|---|---|---|
| 9 | yes | 0.1 | no | 50 | •10.0 | •10.0 | •10.0 | ASSETS | Near feasible after 70 iterations. Sum of infeasibilities increases thereafter. |
| 10 | yes | not used | no | 35 | $10^{-8}$ | $10^{-7}$ | $10^{-7}$ | " | Normal controls are too low |
| 11 | yes | 0.001 | twice | 50 | •10.0 | •10.0 | •10.0 | " | 22.9 min. for CRASH-INVERT-CRASH-INVERT 5 infeasibilities at the end. |
| 12 | yes | 0.01 | twice | 50 | •10.0 | •10.0 | •10.0 | " | 12.4 min for CRASH-INVERT-CRASH-INVERT Feasible after 2 min. |
| 13 | yes | 0.001 | – | 50 | •10.0 | •10.0 | •10.0 | " | Basis from No.11 not feasible. XEPS is too small. |
| 14 | no | 0.01 | – | 50 | •10.0 | •10.0 | •10.0 | " | Very slow rise in OF. System error during the use of XDZPCT = 0.1 |
| 15 | yes | 0.001 | – | 50 | •10.0 | •10.0 | •10.0 | " | Continuation of No.14. Large number of singularities. Basis abandoned. |
| 16 | yes | not used | twice | 50 | •10.0 | •10.0 | •10.0 | " | 80 min. to solution,XPRICE = 4 |
| 17 | yes | 0.01 | twice | 50 | •10.0 | •10.0 | •10.0 | " | 60 min. to solution,XPRICE = 4 |
| 18 | yes | 0.01 | three times | 50 | •10.0 | •10.0 | •10.0 | " | Third CRASH ineffective. |
| 19 | yes | 0.01 | twice | 50 | •10.0 | •10.0 | •10.0 | " | 90 min to soln. XDZPCT = 0.25 too high |

able 2.25 continued.

Row Checks: The maximum row error, (even after scaling), was less than $10^{-3}$. In all strategies the row check marker was put to zero until the 'optimum' was reached, (XCHECKSW = 0); row errors thus introduced were removed by inversion.

Use of the Dual Algorithm: The first few strategies (1, 5, 6, 7, 8) attempted to use the dual algorithm, since this should be more effective in removing infeasibilities. For these problems, this was not found to be true.

The dual algorithm operates on major iterations only, and the consequent loss of speed (especially under NON-HASP) was found to be unjustifiable.

Scaling: Automatic scaling was soon utilized; the intrinsic scaling introduced in the data was insufficient and it was deduced (from comparisons between 4,5 and 6,7) that the lack of further scaling was detrimental to the condition of the inverse basis. The condition of the inverse basis was further improved by the use of the slower (but more accurate) form of the inversion routine - i.e. XINVERT was set at 1.

The tolerance levels: After a few initial attempts at raising the tolerances by more than a factor of ten (strategies 1 to 8) it was deduced that such action was not aiding solution; a comparison of the "paths", i.e. a comparison of the incoming and outgoing vectors in 5, 6, 8 and 9 showed that XTOLDJ and XTOLV should not be raised by more than a factor of 10.0.

Raising XTOLPIV to $10^{-7}$ (i.e. multiplication by 10.0) was found to be vital; this, and the need for the accurate form for inversion imply that the inverse basis would soon become unstable again, if the dimensions of the LP were increased any further.

Initial infeasibility and the CRASH routines: Despite the change of form in the MGP (introducing inequalities into the system, and removing equalities) the major computational difficulty was the attainment of the first (good) feasible solution. In small programmes CRASH was found to be of little value since there were few infeasibilities; this value increased with programme size, as long as the eta vectors (the components of the inverse basis) could remain in core. For the 26/8 model under HASP, CRASH was very ineffective. The result of the etas exhausting core during a CRASH procedure is to leave the basis in a worse position for later (PRIMAL) optimisation (see e.g. strategy 3 in Table 2.25).

Under NON-HASP, with the 26/12 model, a 'double crash' procedure was tried, using inversion between the 'crashes' to concentrate the eta files and enable them to come into core again. The limit of the "multiple crash" procedure was found to be CRASH - INVERT - CRASH - INVERT; a further CRASH had little effect, (strategy 18). In Table 2.26, the START, FINISH columns give the number of infeasibilities at the beginning and the end of the CRASH routine; the time taken by CRASH is noted under the TIME column.

| XEPS | START | FINISH | TIME | SCALE | COMMENTS |
|---|---|---|---|---|---|
| .0 | 24 | 5 | 6.1 | NO | |
| .0 | 24<br>6 | 6<br>4 | 7.1<br>4.8 | YES | INVERT CALLED |
| .0 | 24<br>6 | 6<br>4 | 7.2<br>4.8 | YES | INVERT CALLED |
| .001 | 36<br>18 | 18<br>5 | 12.0<br>10.3 | YES | ETAS EX-CORE<br>INVERT CALLED |
| .01 | 60<br>33 | 33<br>5 | 10.1<br>11.1 | YES | ETAS EX-CORE<br>INVERT CALLED |
| .01 | 60<br>33 | 33<br>5 | 11.1<br>11.0 | YES | ETAS EX-CORE<br>INVERT CALLED |
| .1 | 24 | 206 | 0.7 | NO | ETAS EX-CORE |

Table 2.26 Epsilon Perturbation and the CRASH Routine

Epsilon perturbation: As can be deduced from the formulation of the initial tableau in Chapter 2, the right hand side vectors contain a large number of zeros. (For the 26/12 model, approximately 60% of the r.h.s. is zero.) Perturbation methods, (50) are available in MPS - according to user-specified values of ε; the perturbation strategy uses a perturbed r.h.s. to find a 'pseudo optimum', which is assumed to be near to the real optimum (with ε = 0). Two values were tested; ε = 0.01 and ε = 0.0001. Their effect on the CRASH procedure is clearly recognizable - as the results in Table 2.26 show. The eta files fill up more quickly with the higher values of ε (e.g. ε = 0.1). (For the strategies (16, 17 and 19) the movement from 'pseudo-optimal' to optimal solution using the statements

```
XEPS = 0.0
DUAL
PRIMAL
```

was very rapid, requiring, at most, one or two minor iterations).

The effect of epsilon perturbation on the time to solution cannot be deduced so easily from the results obtained. It would seem that the value of epsilon does not affect the nature or quality of the inverse basis produced by CRASH; it only affects the time taken by the procedure itself. Thereafter, the choice between perturbations is governed by the ultimate proximity of the pseudo and real optima - an unknown.

For the 26/12 data the minimum positive right hand side was 10.0, hence at most the perturbation was by 1%.

Multiple pricing: Pricings of 2, 4 and 7 were used. The results are shown in Table 2.27. The average inter-inversion times were 2.9, 2.0 and 2.5 respectively. (Although a lower level of pricing can also extend the time to solution, by increasing matrix reading time as opposed to introducing too many vectors of only little merit, it was assumed that a pricing of 4 would be the most appropriate level for this problem.) The reduction in matrix reading time for each inter-inversion period was judged to be more valuable even allowing for the possible increase in the number of inversions required.

| Pricing | Inter-inversion times | Average |
|---|---|---|
| 7 | 2.62, 2.66, 2.85, 2.14, 2.45 2.50, 2.86 | 2.58 |
| 4 | 3.22, 1.80, 1.79, 1.62, 1.59 | 2.00 |
| 2 | 2.24, 2.15, 2.16, 4.53, 4.11 2.3 | 2.91 |

Table 2.27 Inter-inversion times with multiple pricing

Systems faults: Apart from the computational errors that occurred during the attempts to optimise the 26/12 model, system failures also occurred. With such extensive use of disk files and data transfers the probability of either finding a 'bad track' or of an input/output error is high. Such errors occur in reading the matrix, transferring data between scratch files, updating the eta vectors, etc., and are natural hazards of large-scale LP work. A careful control of the disks was attempted; files were separated across disk drives to minimize reading times using the 'SEP' parameter of the IBM/360 Job Control Language (see (49)).

A controlled method of saving the basis was implemented. The feasible basis was updated on the problem file (PBFILE) every 15 to 20 minutes. This meant that any loss of programme control due to system faults could only waste a maximum of 20 minutes computation; the end effect of operator cancellation of the job was also eliminated by storing the final inverse basis before allowing the job to terminate.

2.4.1.4 <u>Solutions of the 26/12 Model</u>: The solution of the 26/12 model is shown in Tables 3.3 and 3.4, and is discussed in Appendix 3.1.

2.4.2 <u>Fractional Models</u>

2.4.2.1 <u>Introduction</u>: The difficulties in computation of the fractional programme† arise directly from the form of the constraint set itself. Using the Charnes and Cooper form, the constraint set is

$$\begin{aligned} \underline{A}\cdot\underline{y} - \underline{b}t &\leq \underline{0} \\ \underline{d}\cdot\underline{y} + \beta t &= \theta \end{aligned} \qquad (2.33)$$

where $\theta$ is arbitrary for the problem.

In the original form the constraint set is

$$\underline{A}\cdot\underline{x} \leq \underline{b}. \qquad (2.34)$$

We can note immediately that:

<u>i</u>. the right hand side of (2.33) is composed of all, but one, zero terms.

<u>ii</u>. computationally, the level of $\theta$ is important when referencing in incoming vector, and may affect the feasibility of the solution procedure by allowing 'wrong' decisions when pivoting.

The major difficulties of fractional programming are that:

<u>a</u>. due to the appearance of the right hand side vector in the constraint set, the inverse basis may be ill-conditioned.

<u>b</u>. due to the formulation of the right hand side, degeneracy is unavoidable.

<u>c</u>. the initial value of t as it enters the basis must be non-zero in order for the solution to be attained.

† using the Charnes and Cooper method

2.4.2.2 <u>The Inverse Basis</u>: From equations (4.17) we know that the inverse basis for (2.33) is given by $(\underline{B}^*)^{-1}$ where

$$(\underline{B}^*)^{-1} = \begin{pmatrix} \underline{M}_{11} & \underline{M}_{12} \\ \underline{M}_{21} & \underline{M}_{22} \end{pmatrix}$$

$$\underline{M}_{11} = \underline{B}^{-1} - \underline{x}^*.t^*.\underline{d}^*.\underline{B}^{-1}$$

$$\underline{M}_{12} = t^*.\underline{x}^*$$

$$\underline{M}_{21} = -t^* (\underline{d}^*.\underline{B}^{-1})$$

$$\underline{M}_{22} = t^*$$

and $\underline{B}^{-1}$ is the inverse basis of the corresponding basis to (2.34).

The terms of the $\underline{M}_{ij}$ matrices may give an ill-conditioned matrix $(\underline{B}^*)^{-1}$ even though $\underline{B}^{-1}$ is itself well-conditioned. The level of t will be important; this is dependent on $\theta$.

2.4.2.3 <u>The initial difficulties</u>: The difficulties with the Charnes and Cooper method for solving (2.33) with the 3/5 data arose when the computation arrived at a solution in which the programme was feasible, with a zero value for the objective. Further iterations showed no improvement in this level, although there was little evidence of cycling. (See strategy 1 of Table 2.28). At the second attempt, the pivot tolerance was increased; as a result the programme hovered between the feasible regions, with a zero value for the objective.

Inspection of the solution showed that 't' had become basic at the zero level, due to the use of CRASH; once t becomes basic, the problem iterates endlessly. Two strategies were attempted; epsilon perturbation, and lower bounds for admissible t.

2.4.3.4 The Strategies:

i. Epsilon Perturbation: As in 2.4.3.1 above, perturbation proved very useful. It ensures that, initially, no degeneracy occurs, i.e. t cannot enter at the zero level.

ii. Bounding t: An arbitrary lower bound on t was entered into the BOUNDS section of MPS. (This was FEAS: $t \geq 0.00001$).

The programme was optimised twice, with and without this bound, using the first optimal basis as a starting point for the second optimisation, via the SAVE/RESTORE routines. The programme was:

```
SETUP ('MAX', 'SCALE', 'BOUND', 'FEAS')
CRASH
PRIMAL
SAVE
SETUP ('MAX', 'SCALE')
RESTORE
PRIMAL
```

The first SETUP ensures that CRASH does not enter t at zero; the second SETUP (by omitting the vector FEAS), removes the arbitrary bound on t. This method is analogous to the two stage method of perturbation, but has the disadvantage of requiring three extra routines, (SAVE, SETUP AND RESTORE).

Both strategies 7, 8 and 9 (in Table 2.28) used a double CRASH procedure. This has not proved useful in the cases where $\theta = 1$, but has shown some reduction in solution time for the case using perturbation of 0.01 and $\theta =$ 10,000; (this seems to be due to the fact that the epsilon differs markedly from the only non-zero right hand side entry).

| No | $\theta$ | XEPS | BOUNDS | CRASH | TIME |
|---|---|---|---|---|---|
| 1 | 1.0 | 0.0 | NO | YES | $\infty$ |
| 2 | 1.0 | 0.01 | NO | YES | 0.47 |
| 3 | 1.0 | 0.0 | YES | YES | 0.41 |
| 4 | $10^4$ | 0.0 | NO | YES | $\infty$ |
| 5 | $10^4$ | 0.01 | NO | YES | 1.25 |
| 6 | $10^4$ | 0.0 | YES | YES | 0.49 |
| 7 | $10^4$ | 0.01 | NO | TWICE | 0.9 |
| 8 | 1.0 | 0.01 | YES | TWICE | 0.59 |
| 9 | 1.0 | 0.0 | YES | TWICE | 0.51 |

Table 2.28 Strategies for Fractional Programming

| | TOTAL | NORMAL | FREE | FIXED | BOUNDED |
|---|---|---|---|---|---|
| Rows | 254 | 171 | 27 | 56 | 0 |
| Cols | 151 | 151 | 0 | 0 | 0 |
| 1648 Elements - density = 1.60 | | | | | |

Table 2.29 Dimensions of 3/5 model

2.4.2.5 The Parametric Approach: A further approach to the problem of fractional programming is the parametric approach of Joksch; this method uses parametrization of $\theta$ in the problem

$$\begin{aligned} &\max\ (\underline{c}.\underline{x}+\alpha)/\theta \\ &\text{s.t.}\ \ \underline{A}.\underline{x}\le\underline{b} \\ &\qquad \underline{d}.\underline{x}=\theta-\beta \\ &\qquad \underline{x}\ge\underline{0} \end{aligned} \tag{2.36}$$

This method was attempted for the 3/5 model, taking the denominator over a wide range of values. The results were:

i. time to solve (2.36) for $\theta-\beta$ = 20,000 was 0.44 minutes

ii. total time including parametric analysis, 0.97 min.

In fact the optimum for the model occured when $\underline{d} \cdot \underline{x}$ was a minimum, i.e. at $\underline{d} \cdot \underline{x} = 0$.

## Appendix 3.1 General Results and Model Capabilities

The data and results for the first optimisation of the 26/12 model have been described in Chapters 2 and 3, and the computational difficulties with the models have been considered in Appendix 2.4. For tests i and ii of the model, the 26/12 model was used; the remainder were based on 3/5 models. The tests were:

i. Change of minimum sales policies

The minimum sales policy for the non-basic products 1, 12 and 20 (shown in Table 3.10) was imposed on the optimum of the 26/12 model.

| Product | J | F | M | A | M | J | J | A | S | O | N | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 5 | 5 | 10 | 15 | 20 | 25 | 15 | 5 | 5 | 5 | 5 |
| 12 | 10 | 10 | 15 | 20 | 25 | 15 | 10 | 5 | 5 | 5 | 5 | 5 |
| 20 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |

Table 3.10 The Minimum Sales Policy

Without the minimum sales policy the optimal profit (maximum ASSETS) was £2,152,960. The loss due to the policy was £18,951. The 'decision' was further tested by increasing the minimum sales policy for product 1 by x% of the amount shown in Table 3.12. The levels x% at which basis changes occurred are shown below in Table 3.11. The graph of profit (ASSETS) against x is shown in Figure 3.3. The cost of the decision to increase x can be measured directly by the loss of profit.

ii. The evaluation of raw materials

The dual evaluators of the raw materials balance equations were '1.48', implying that if the system could include extra units of raw materials into these balances the net increase in

| x% | ASSETS |
|---|---|
| 0.39 | £2,125,763 |
| 6.94 | £2,120,280 |
| 12.27 | £2,115,755 |
| 13.61 | £2,114,610 |
| 14.43 | £2,113,912 |
| 15.71 | £2,112,996 |
| 16.60 | £2,111,994 |
| 19.97 | £2,108,772 |
| 22.97 | £2,105,840 |
| 23.45 | £2,105,387 |
| 26.36 | £2,102,497 |

Table 3.11 Variation of ASSETS with X%

profit would be £1.148 per unit. Since units were assumed to cost £1 each, this figure represents the maximum price the firm should pay for its raw materials.

The input of raw materials to the model was £5,000. This amount was increased (by parametric analysis); the return of £1.148 per unit was maintained up to the input level £5,440. Thereafter, the row "input of raw materials" was not a binding constraint, and the dual evaluator for increasing input fell to £1 per unit - i.e. the cost price. (This is shown in Figure 3.4). From the formulation of Section 2.4 it can be seen that raw materials and cash are to some extent interchangeable where there are no lags on payments and the input of materials is tight. Hence the initial dual evaluator for the cash continuity constraint was also £1.148. (This was the case for the rows calculating overheads, payables, etc., since a unit change in any of these rows would imply a unit change of cash holding.)

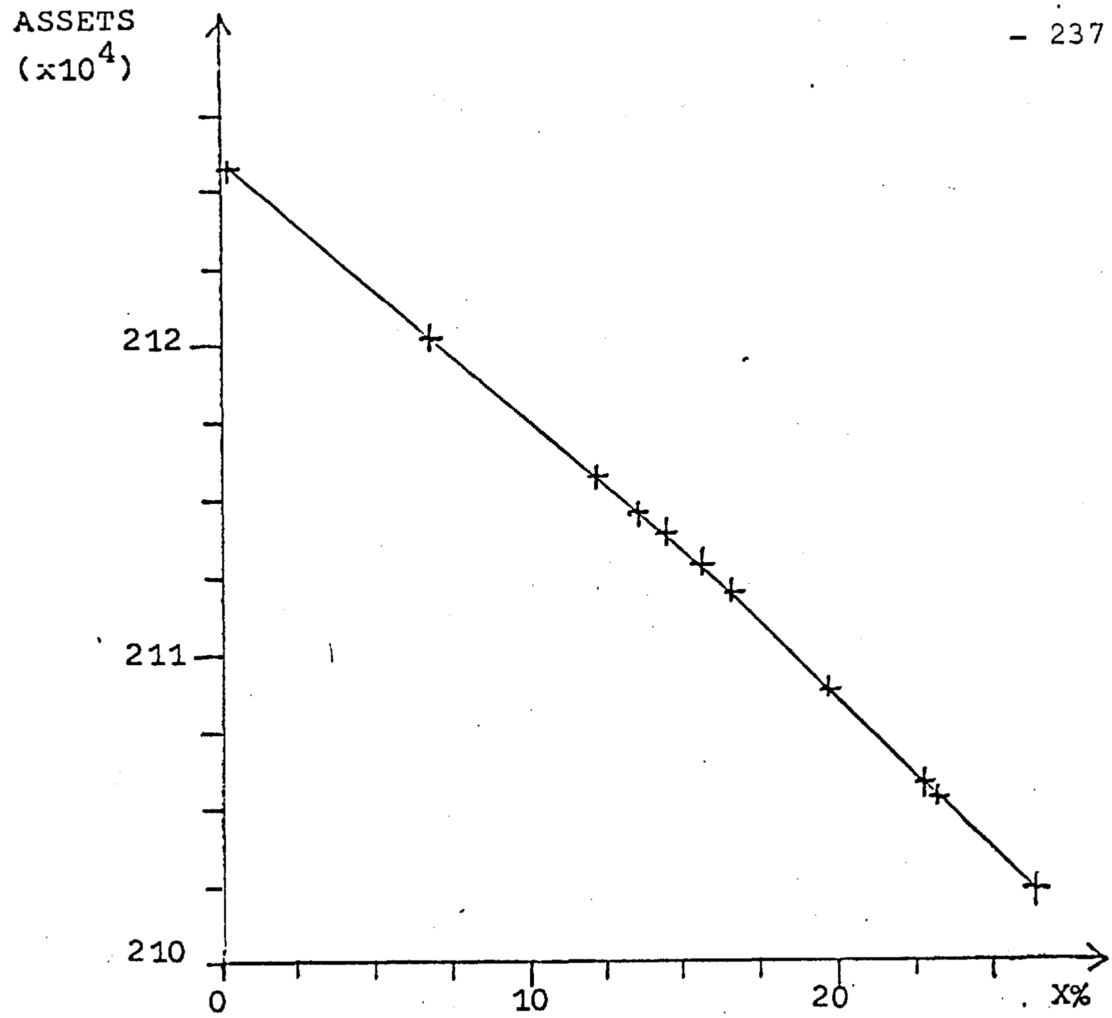


Fig 3.3 Variation of ASSETS with X%

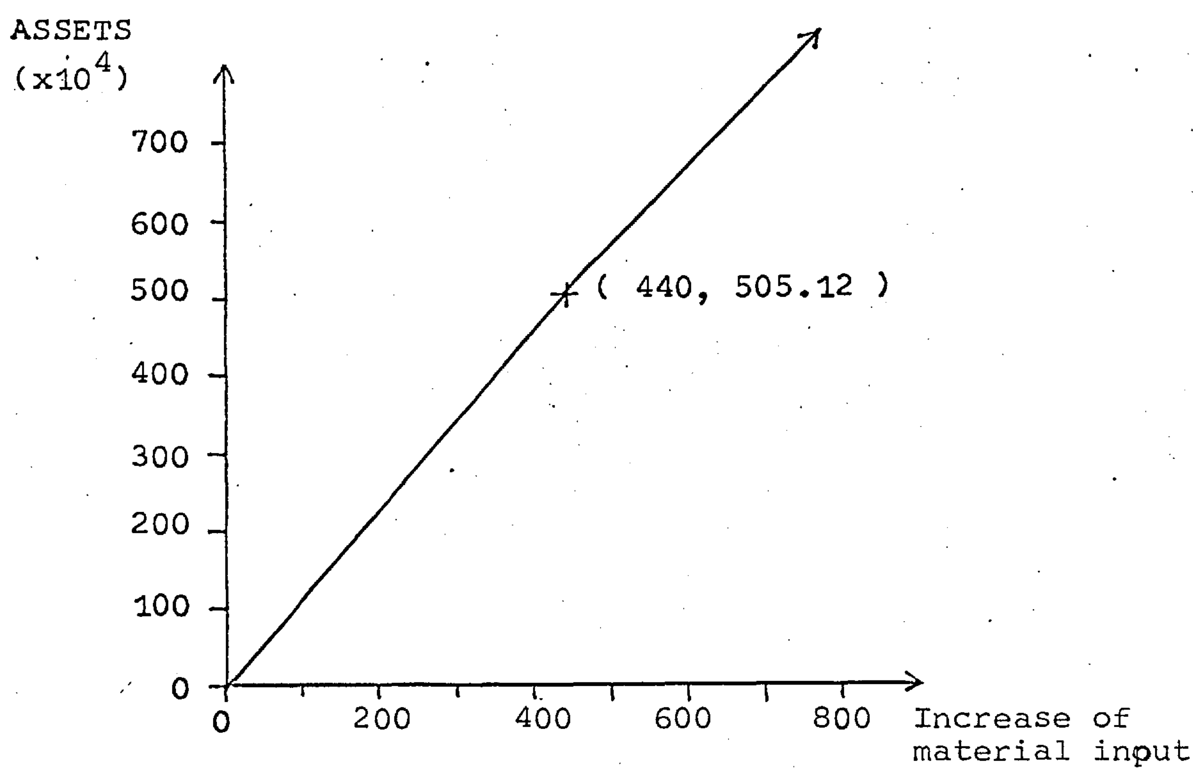


Fig. 3.4 Increase of ASSETS with input of materials

iii. The marginal evaluation of plant capacity

The remarks of Appendix 3.3 and of Section 3.3.4 apply to all cases of marginal evaluation. Thus, although Figure 3.4 represents the linear change of profit with raw material input, the underlying, mixed-integer, structure of the problem must be borne in mind.

iv. The range of the solution

For the 3/5 model used in v. below , the range of the initial solution (i.e. the LP solution with no allowance for set-up times) is shown below in Table 3.12, together with the 'activities' of each of the rows. The range of the optimal solution has less power in the case of the financial planning models for two reasons:

a. the underlying structure is a mixed integer (non-linear) programme (c.f. Section 3.3.2)

b. changes in one right hand side entry may imply alterations to other entries (e.g. increasing capacities in January as in Section 3.4.1). The range of the LP solution is valid for changes in only one r.h.s. entry at a time.

v. Parametrics

Work centre parametrisation will be shown in Appendix 3.3. Other parametrisations were carried out to test the model's adaptability to cash shortage.

A 3/5 model was used which had the following inputs: £5,000 raw materials, £5,000 cash, and 10 units each of product. A minimum sales policy of (10, 0, 20) units per month was imposed for the three products; market expenditure was assumed to be 25% of gross sales value. There were no lags on payments Financial bounds on the model were:

| | | |
|---|---|---|
| Cash: | lower bound £0 | upper bound £5,000 |
| Bank loan: | lower bound £0 | upper bound £5,000 |

The optimal solution (maximising ASSETS) was £52,357. The results of decreasing cash input are shown in Table 3.13. The column 'DUAL' is the associated dual evaluator, and the amount of decrease is given by XPARAMx£1,000, (the parameter value times the amount of change).

| XPARAM | ASSETS | DUAL |
|---|---|---|
| 0.0 | 52357.0 | 1.414 |
| 1.08 | 50821.0 | 1.419 |
| 2.59 | 48683.0 | 1.475 |

Table 3.13 Parametric Analysis of Cash Input

Clearly, the more the input of cash is lowered, the greater becomes the value associated with return on extra cash. Because there were no lags on payments, the model was always able to generate sufficient funds to maintain feasibility, even when the initial input of cash was decreased to zero. (There was no basis change above XPARAM = 2.59)

The uses of parametric analysis to test the sensitivity of the model to changes in the right hand side (or objective function) are straightforward. Testing the sensitivity of the model to input data is more difficult. The normal sensitivity analysis allows for changes of any row or column, but not for changes throughout the matrix. Since the model is composed of a series of similar submatrices, the effect of changing input data is to change many rows (or columns) simultaneously. These changes cannot be fully investigated without setting up entirely new sets of problem data. Methods similar to Section 5.6 and outlined in the theorems I and II of Section 4.3 could be used to test the marginal rate of change of the objective function with respect to a matrix of perturbations. Sections of input data for which the marginal rate of change of the value of the LP is small changes of data, are not sensitive regions. Those areas for which the marginal rates of change are high will be areas of sensitivity; in these cases input data should be verified.

vi. Inclusion of bounds on financial ratios

The ratio 'current assets to current liabilities' was bounded in two 3/5 models. In both cases it was assumed that the initial level of the account was zero, thus the required ratio was ASSETS/LIABLE. Two sets of data were used:

A: A 3/5 model with accounting lags of one period.

Input of cash £50,000

Input of materials £50,000

Bounds on cash £50 to £50,000

Bounds on loans £0 to £50,000

Bounds on materials £50 (lower bound)

Minimum sales (10, 0, 10) per month

B: A 3/5 model with accounting lags of two periods.

Input and bounds as above

Minimum sales (10, 0, 20) per month.

Both sets of data calculated market expenditure as 25% of gross sales.

a. Using Data A and the constraint ASSETS/LIABLE ≥ 1.7, the following results were obtained.

| Row | Objective Function | |
|---|---|---|
| | ASSETS | GROSSALE |
| ASSETS | 140414. | 21195. |
| LIABLE | 74584. | 12526. |
| GROSSALE | 53750. | 215949. |
| A/L ratio | 1.875 | 1.7 |

Table 3.14 Results with A/L ≥ 1.7

b. Using Data B and the constraint ASSETS/LIABLE ≥ 1.5, a similar analysis gave:

| Row | Objective Function | |
|---|---|---|
| | ASSETS | GROSSALE |
| ASSETS | 186909. | 165158 |
| LIABLE | 124606. | 110105 |
| GROSSALE | 88750 | 186522 |
| A/L ratio | 1.5 | 1.5 |

Table 3.15 Results with A/L ≥ 1.5

Comparing Tables 3.14 and 3.15 with ASSETS as the objective function, we see that the ratio A/L has become binding in the second case, because of the increase of minimum sales and the lengthening of the accounting lags.

Using the new row (ASSETS - 1.5 LIABLE) as the row to be parametrised, it is possible to subtract multiples of LIABLE to sweep out the series of solutions for the various levels of the constraint. Initial levels of current assets and liabilities can be included by adjusting the right hand side entry corresponding to the 'ratio' constraint.

## Appendix 3.2 LP Models for Control

In Chapter 1 we introduced Samuels' model for financial control using the dual evaluations of the optimal solution to a linear programming model of the firm, and Bernhard's comments on the accounting procedures. The model, (1.6), was:

$$\begin{aligned} \max P &= 2x + 3y + 4z \\ \text{s.t.} \quad & 5x + y + z \le 8000 \text{ (floor space)} \\ & x + 5y + z \le 8000 \text{ (supervisor time)} \\ & x + y + 5z \le 8000 \text{ (machine time)} \end{aligned} \qquad (3.13)$$

The optimal solution was:

$\left\{P^* = £10,284,\ x^* = 1142,\ y^* = 1143,\ z^* = 1143\right\}$

with dual evaluators $\left\{\lambda_1 = 5/8,\ \lambda_2 = 12/28,\ \lambda_3 = 19/28\right\}$.

The underlying assumptions of both papers require careful examination. If we use a formulation such as (3.13) for planning purposes, we assume that activities of production etc. take place instantaneously (at the beginning or end of a period), or that the order in which these activities (or any fraction of the activities) are carried out is unimportant. (Indeed for planning purposes these assumptions have been justified in Section 3.3.4; they are dependent on the time period chosen, and are implicit in an LP formulation). As we have shown in Chapter 3, if the assumptions of linearity, the existence of one objective, and the reality of the time segmentation do hold, the dual evaluators may be interpreted as the marginal value of resources.

If we use (3.13) as a control tool, there must be some further assumption regarding the information flow - within the model time period. (Samuels has implied a time-structure within his operating period by suggesting that overproduction by department X has <u>caused</u> department Z to produce only 942 units - "because there were not enough units of floor space available after department X used more than its budget"; but he gives no suggestion as to the knowledge of department Z at the time when it was about to start production.) Clearly information and control systems should be closely related. Departments can only be rewarded (or penalised) for their success in achieving company aims at their current state of knowledge. If X overproduces, and Z cannot make more than 942 units, (say it produces 900 units) its penalty should reflect the total failure (243 units) (unmitigated by the chance factor of overproduction by X), and not its relative failure (of 43 units). Moreover, these penalties should be at the rate which department Z believes to be operative. Conversely if there is an information system, which instantaneously recognizes overproduction of department X the controlling mechanism should alter the targets for departments Y and Z <u>and</u> the penalty/bonus scheme, and they should be informed of the new operating situation.

Consider example (3.13) with the following two assumptions:

<u>i</u>. departments use production facilities consecutively,

<u>ii</u>. at the end of a particular run, all departments know the state of the firm's resources, and aims are updated accordingly.

(a) Suppose Y is the first to utilise production facilities and produces the required amount, X overproduces, and Z is forced to underproduce; the accounting procedure should be that of Samuels in Chapter 1.

(b) Suppose X is the first to use facilities, and overproduces, ($\hat{X}$ = 1183). The 'optimal' situation has changed and both planning and control should reflect this. For the remainder of the period the problem is:

$$\max \quad (2X) + 3Y + 4Z = P$$
$$\text{s.t.} \quad (5X) + Y + Z \leq 2085$$
$$(X) + 5Y + Z \leq 6817 \qquad (3.14)$$
$$(X) + Y + 5Z \leq 6817$$

This has an optimal solution

$$\left\{P^* = £7438,\ X^* = 0,\ Y^* = 902,\ Z^* = 1183\right\}$$

(The right hand side entries give the capacities remaining, after department X has utilised all facilities.)

Originally the total profit was £10,284. Now it is £1183 x 2 + £7438 = £9804. Department X has caused a loss of £480 if departments Y and Z are informed about their new targets, and are capable of changing plans, (i.e. there is no ordering of parts, or other time dependence). Given the information structure we have defined, the loss caused by department X is much less than that of Samuels' work. How does this opportunity cost relate to the marginal use of materials? What dual evaluators should be used for accounting purposes if we wish to keep to the original idea of a marginal cost system?

If X produces optimum value $X^* = 1142$, the new programme is given by:

$$\max \quad 3Y + 4Z$$
$$\text{s.t.} \quad Y + Z \leq 2290$$
$$5Y + Z \leq 6858 \qquad (3.15)$$
$$Y + 5Z \leq 6858$$

(by optimality of X we may drop it from (3.15)) with the solution $\left\{P^* = £8001,\ Y^* = 1143,\ Z^* = 1143\right\}$ and dual evaluators

$$\left\{\lambda_1 = 0,\ \lambda_2 = 11/24,\ \lambda_3 = 17/24\right\} \qquad (3.16)$$

profit of department X = sum of resources used = £2 x 1142.
Now we can see precisely what happens when X overproduces by

41 units. Initially it uses up resources at the costs given by the λ's in (3.16), but by parametric analysis we can show that there is a basis change after production of an extra 6/7 units of X.

At the basis change the dual evaluators become

$$\left\{\lambda_1 = 11/4,\ \lambda_2 = 0,\ \lambda_3 = 1/4\right\} \qquad (3.19)$$

The gross opportunity cost to be charged against X in this case is

$$\frac{6}{7}\left(5.0 + 1.\frac{11}{24} + 1.\frac{17}{24}\right) + \left(41 - \frac{6}{7}\right)\left(5.\frac{11}{4} + 1.0 + 1.\frac{1}{4}\right)$$

$$= \frac{6}{7}\cdot\frac{23}{24} + \frac{231}{7}\cdot\frac{56}{4} = 1 + 562 = £563$$

But X has made an extra return of £2 x 41. Allowing for rounding to integers we have: net billing to X = £563 − 82 = £480, the opportunity cost under our assumed information structure.

(c) The cases of 'simultaneous' over- and under-production. These have already been quoted above; a combination of 'overproduction causing underproduction' and 'overproduction recouping losses due to underproduction'. Under our assumptions this is impossible and it seems unlikely that a working situation could be found for which Samuels' assumptions would be valid.

If, according to Samuels, both X and Z act simultaneously, X should bear the penalty for overproducing regardless of Z's failure, and Z should bear the cost of its underproduction, regardless of the fact that its loss was partially recouped by another department. The opportunity cost is <u>the cost that could have been caused</u>, not that which actually was caused due to a fortuitous (and simultaneous) occurence. The imbalance would have to appear in a rectification account; this would be the cost of lack of information.

i.e. bill to X $(-£2 \times 41 + £804) = £722$

bill to Z : $£4 \times (1143 - 942) = £804$

rectification: £804 : the amount about which Z was uninformed.

From these examples it is clear that if the assumption of ordering activities within the time period is violated (as it is in the example presented by Samuels) the duality theorems will not give correct marginal evaluations. Samuels has taken a time period that is too long. If the time period were short enough, the problem of ordering activities would disappear, but the problem would expand to unmanageable (and uneconomic) dimensions.

Further criticism may be made of Samuels' paper and the recent work of Carsberg, because both assume the existence of only one objective function for the firm. For planning or control, this assumption is somewhat difficult to justify, consequently the use of duality for such explicit pricing exercises as financial penalties and depreciation is open to serious questioning.

## Appendix 3.3 The Effect of Set-Up Times

### 3.3.1 The Model

The effect of set-up times for machines was tested on the 3/5 model, ( a model considering the first three products of Table 2.16 (in Appendix 2.2) over a period of five months). The model used was a simplified, yet extreme case; cash was bounded by £50 and £50,000, bankloans by £0 and £50,000; the inputs of raw materials and cash were £5000 and £5000 respectively; all payments were lagged by two months and ASSETS was used as the objective function.

For this model, the optimal solution gave the following results:

(a) ASSETS = £229,360

(b) Production schedules of $\{0, 23.4, 30.7\}$ per period

(3) Work centre capacity constraints 13 and 14 of each period were binding, with dual evaluators £2.222 per unit and £1.265 per unit respectively

(4) No set-up times were allowed, i.e. all of the 10,000 hours per period on centres 13 and 14 were used for production

The set-up times for each product batch are shown in Table 3.16. (We have assumed that these are the set-up requirements for a batch of 10 units in the model solution).

### 3.3.2 The Revised Problem

Assuming that the probable set-up requirements for the model would be 3 'set-ups' per month, per product, the capacities for work centres were changed (in the right hand side vector), and the model was re-optimised. This optimal solution gave the results :

(1) ASSETS = £213,668

(2) Production schedules of $\{0, 21.3, 27.9\}$ for each product

in each period

(3) Work centre capacity constraints 13 and 14 of each period were binding, with dual evaluators £2.222 per unit and £1.265 per unit respectively

(4) The utilised capacity in work centre 14 was 9100 hours. 900 hours were taken by set-up requirement (150 hours each for six batches).

| CENTRE | PRODUCT 1 | PRODUCT 2 | PRODUCT 3 |
|---|---|---|---|
| 1 | 460 | 730 | 160 |
| 2 | 0 | 0 | 120 |
| 3 | 40 | 120 | 130 |
| 4 | 20 | 100 | 150 |
| 5 | 100 | 150 | 160 |
| 6 | 0 | 150 | 130 |
| 7 | 0 | 30 | 50 |
| 8 | 30 | 50 | 40 |
| 9 | 0 | 30 | 0 |
| 10 | 150 | 150 | 150 |
| 11 | " | " | " |
| 12 | " | " | " |
| 13 | " | " | " |
| 14 | " | " | " |
| 15 | " | " | " |
| 16 | " | " | " |
| 17 | " | " | " |
| 18 | " | " | " |

Table 3.16 Work Centre Set-Up Times

The allowances of set-up times for the batches have caused a drop in the monthly production figures from $\{0, 23.4, 30.7\}$ to $\{0, 21.3, 27.9\}$ and a corresponding change in objective function, £229,360 to £213,666, but the optimal basis from the original solution gave an optimal solution to the revised problem without requiring any further iterations, thus the dual evaluators for work centres 13 and 14 show no change - the marginal values of extra capacity are unaltered.

Also, the allowance for three batches per month for products 2 and 3 implies that the mixed integer solution should be sought for the range

$$20 \leq \text{PROD}\ (I,J) \leq 30 \quad \begin{matrix} I = 1...5 \\ J = 2,\ 3 \end{matrix} \qquad (3.18)$$

The optimal schedules for the revised problem adhere to this; the solution may be assumed to be the required 'mixed-integer' optimal solution.

### 3.3.3 Parametrisation of Capacities

Neither products 1 nor 3 utilise work centre 14, (see Table 2.13 of Appendix 2.2). Parametrisation of work centre 14 was carried out, as if new plant were installed at the end of (the previous) December, to be operative through the months January to May, (i.e. the change column added capacity to the right hand entries for work centre 14 for each month, January to May).

#### (a) The Original Model

With parametric analysis applied to the original problem, the first basis change occurred when the input requirement constraint for period three became binding, (i.e. when the input of raw materials became insufficient to allow for production during periods 3, 4 and 5, without purchases in period 3). This basis change (occuring when utilised capacity on centre 14 was 14662 hours and ASSETS were £368,902. (At this basis change capacity on work centre 11 during periods 1, 2, 3 became binding).

(b) The Revised Model

In order to remain within the logical range allowed by the set-up times (i.e. only 3 batches per month for products 2 and 3), the variables PROD (I,J), were bounded above by 30 , for I = 1...5, J = 2, 3, using the REVISE procedure.

The demand XDOFREQ1 was directed to printing out a solution at the rate XFREQ1 = 1, i.e. at every iteration: Parametrisation was used to detect the point at which the new limits PROD (I,J) ≤ 30 became binding - this point corresponded to the following solution:

(1) ASSETS = £236,944

(2) Production schedules of $\{0, 30.0, 27.5\}$ for each product in each period

(3) Work centre capacity constraints 13 of each period were binding, with dual evaluators £2.222 per unit

(4) The utilised capacity on work centre 14 was 12,780 hours (900 hours were taken by set-up requirements). Total capacity was 13680 hours.

(The basis of this solution was punched onto cards.)

The binding constraints

$$\text{PROD (I, 2)} \le 30.0 \qquad I = 1...5 \qquad (3.19)$$

make the capacity constraints for work centre 14 appear slack. To allow the variables PROD (I,2) to take values greater than 30.0 further allowances for set-up times were made on all work centres, apart from work centre 14. Parametrisation of work centre 14 was continued from the 'revised' utilisable figure of 12780, assuming that the 1050 hours required for set-up times (four batches of product 2, three batches of product 3) would be accounted for by the installation of new plant at the end of December.

With the assumption that plant had been installed that would be just sufficient to allow for the required four set-ups for product 2, an optimal solution was obtained, (utilizing the punched basis). This solution was:

(1) ASSETS = £235,277

(2) Production schedules of $\{0,\ 30.0,\ 27.0\}$ per product per period

(3) Work centre capacity constraints 13 were binding in all periods, with dual evaluators £2.222 per unit. Capacity constraints on work centre 14 were binding in period 2, 3 4 and 5, with dual evaluators £1.265 per unit

(4) The utilised capacity on work centre 14 was 12,780. (1050 hours were taken by set-up requirements). Total capacity was 13,830. (The change of production of item three from 27.5 to 27.0 is caused by the set-up time of product 2 on work centre 13. Product 3 uses 315 hours on work centre 13 (per unit of product); a reduction in capacity of 150 hours for the set-up time of product 2 reduces the production of item 3 by approximately 0.5)

The bounds on PROD (I,2) were altered to

$$\text{PROD}\ (I,2) \leq 40.0 \quad I = 1...5 \qquad (3.20)$$

to allow the production of item 2 to utilise the next range $30.0 \leq \text{PROD}\ (2,J) \leq 40.0$: $J = 1...5$, and the parametric analysis was continued.

The constraint on input requirement in period three became tight, at the following point:

(1) ASSETS = £262,222

(2) Production schedules of $\{0,\ 40.0,\ 26.6\}$ per period

(3) Work centre capacity constraints 13 of each period were

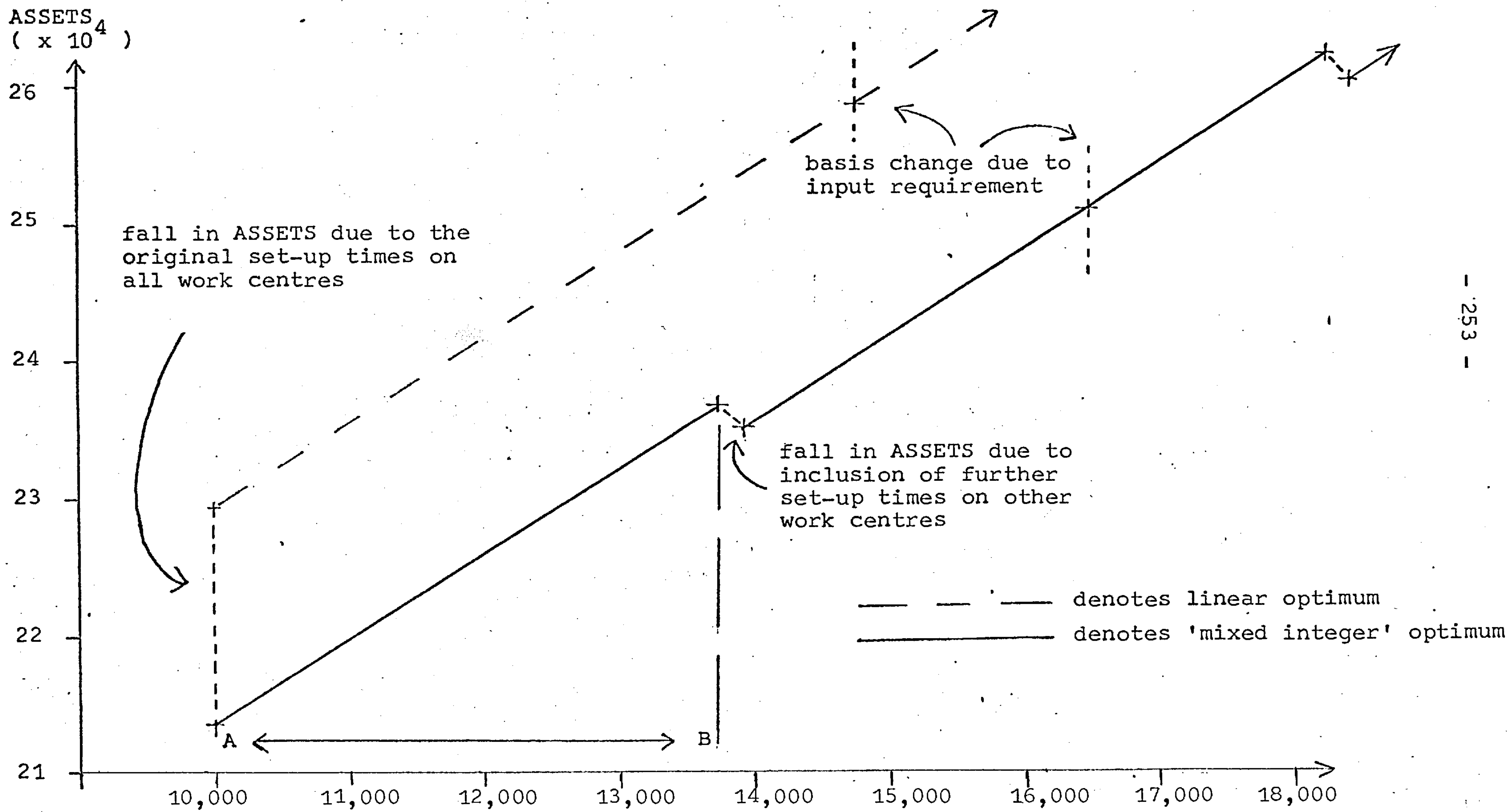


Fig 3.5 Variation of ASSETS with Work Centre Capacity

binding, with evaluators £2.222 per unit

(4) The utilised capacity on work centre 14 was 17040 hours (total capacity being 17040 + 1050 = 18090 hours).

Allowing for one further batch of product 2, the solution $x_2 = 40.0$ ($40.0 \leq x_2 \leq 50.0$) was:

(1) ASSETS = £260,555

(2) Production schedules $\{0, 40.0, 26.15\}$ per period

(3) Utilisation of work centre 14 was 17040 hours (total capacity being 17040 + 1200 = 1840 hours)

The evaluators for work centre 13 and 14 remained unchanged at £2.222 and £1.265 per unit, and did not change with parametrisation until the constraints $x_2 \leq 50.0$ became binding.

These results are summarised in Figure 3.5.

### 3.3.4 Conclusions

From the results of Section 3.3.3, and Figure 3.5 we may conclude that:

(a) the dual evaluators given by the revised solutions, within the logical range of allowable production do give the marginal values of resources. The range of applicability of these values is, however, more severely restricted; this has been noted in Section 3.3.2.

(b) the actual change in objective function due to simultaneous changes in right hand side elements, may be deduced from the sum of the dual evaluators and the respective amounts of change, if the amounts of change do not extend beyond the optimality (feasibility) of the present basis. Considering the range AB on Figure 3. 5, the change in profit is £23,276 (= £236,944 - 213,668). The dual evaluators for each of the five work centres is £1.265 per unit and the number of extra units on each is

$3680 = \{13{,}680 - 10{,}000\}$; for these extra units:

$£23{,}276 = 5 \times \{3680 \times £1.265\}$

(c) with multiple resources and set-up times the general change of profit with resource is a toothed function. Losses are caused when generating capacity for set-up times, due to the reduction of overall production levels. This reduction is caused by the removal of utilised capacity from existing bottlenecks, in order to allow for (non-productive) set-up times.

## Appendix 4.1 Examples: Returns to Scale

a. Problem 1

$$\max f = \frac{100x_1 + 5x_2}{x_1 - 1.5x_2 + 10}$$

$$\left.\begin{array}{rl} \text{s.t.} \quad x_1 - x_2 & \le 0 \\ x_2 & \le 5 \\ x_1,\ x_2 & \ge 0 \end{array}\right\} \quad \underline{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \qquad (4.29)$$

The solution is: $x_1^* = x_2^* = 5 \qquad f = \frac{105.5}{7.5}$

Now consider changes in $b_2$:

For increases in $b_2$ we have, $x_1^* = x_2^* = b_2$

(until the denominator approaches zero);

the problem is, in effect, $\max F = \frac{105x}{10 - 0.5x}$ (4.30)

$$\text{s.t.} \quad x \le b$$

| $x^* = b$ | $105x^*$ | $10 - 0.5x^*$ | $F^*$ |
|---|---|---|---|
| 6 | 630 | 7 | 90 |
| 8 | 840 | 6 | 140 |
| 10 | 1050 | 5 | 210 |
| 12 | 1260 | 4 | 315 |
| 14 | 1470 | 3 | 490 |
| 16 | 1680 | 2 | 840 |

Table 4.1

From Table 4.1, the problem (4.29) clearly exhibits increasing returns to scale.

Now $\underline{d}^* = (1, -1.5)$, and the inverse basis $= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \underline{B}^{-1}$,

$\underline{\theta} = \underline{d}^* . \underline{B}^{-1} = (1, -0.5)$, i.e. $\theta_i < 0$, as proved above.

b. Problem 2

$$\max f = (10x_1 + x_2)/(1.5x_1 - x_2 + 6)$$

$$\begin{array}{rl} \text{s.t.} \quad x_1 - x_2 & \le 0 \\ x_2 & \le 5 \\ x_1,\ x_2 & \ge 0 \end{array} \qquad (4.31)$$

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

The solution is: $x_1^* = x_2^* = 5$.

For changes of $b_2$ the function exhibits diminishing returns to scale. The inverse basis is $\underline{B}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\underline{d} = (1.5, -1)$, and $\underline{\theta} = \underline{d}.\underline{B}^{-1} = (1.5, 1).\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (1.5, 1)$.

All $\theta_i \geq 0 \Rightarrow$ diminishing returns to scale.

The problem becomes: $\max F = \dfrac{11x}{(0.5x + 6)}, \; x \leq b$ (4.32)

| $x^* = b$ | $11\,x^*$ | $0.5x^* + 6$ | $F^*$ |
|---|---|---|---|
| 6 | 66 | 9 | 7.33 |
| 8 | 88 | 10 | 8.8 |
| 10 | 110 | 11 | 10.0 |
| 12 | 132 | 12 | 11.0 |
| 14 | 154 | 13 | 11.846 |

Table 4.2

As can be seen by Table 4.2, the function exhibits diminishing returns to scale, i.e. $\dfrac{\partial F^*}{\partial b}$ is decreasing.

c. Dual evaluators and $\dfrac{\partial \pi}{\partial b_i}$:

At the optimum to problem 1, $f^* = \dfrac{(100,5).(x_1, x_2)'}{(1, -1.5).(x_1, x_2)' + 10}$

Now $(x_1^*, x_2^*) = \underline{B}^{-1}.\underline{b}$

therefore $\pi^*(b_2) = \dfrac{(100, 105).(b_1, b_2)'}{(1, -1.5).\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + 10}$

$= \dfrac{105b_2}{10 - 0.5b_2} \qquad (b_1 = 0)$

and $\left[\dfrac{\partial \pi^*}{\partial b_2}\right]_{b_2 = 5} = \dfrac{105(7.5 + 2.5)}{7.5^2} = 105.\dfrac{10}{7.5}$ (4.33)

Also $\pi^*(b_1) = \dfrac{100b_1 + 525}{b_1 + 7.5}$ $\qquad (b_2 = 5)$

and $\left[\dfrac{\partial \pi^*}{\partial b_1}\right]_{b_1 = 0} = \dfrac{30}{7.5}$ (4.34)

The Charnes and Cooper form of problem 1, (4.29), is:

$$\begin{aligned} \max \quad & 100y_1 + 5y_2 \\ \text{s.t.} \quad & y_1 - y_2 \le 0 \\ & y_2 - 5t \le 0 \\ & y_1 - 1.5y_2 + 10t = 1 \\ & y_i,\ t \ge 0 \end{aligned} \qquad (4.35)$$

for which the optimal solution is: $y_1^* = \frac{2}{3}$, $y_2^* = \frac{5}{7.5}$, $t^* = \frac{4}{30}$

$v_{CC_1} = 30$, $v_{CC_2} = 105.\frac{4}{3}$, i.e. $x_1^* = 5$, $x_2^* = 5$

and from the dual evaluators of the CC form, we have:

$$v_{F_1} = \frac{30}{7.5} = \left[\frac{\partial \pi^*}{\partial b_1}\right]_{b_1 = 0}$$

$$v_{F_2} = \frac{105.10}{(7.5)} = \left[\frac{\partial \pi^*}{\partial b_2}\right]_{b_2 = 5}$$

Although the dual evaluators exist, and can be derived from either the original or the CC form, no concept of pricing can be given, due to increasing returns to scale.

The optimal inverse basis to (4.32), $\underline{B}^{*-1}$, is given by:

$$\underline{B}^{*-1} = \left(\begin{array}{cc|c} \frac{1}{3} & \frac{4}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{4}{3} & \frac{2}{3} \\ \hline \frac{-4}{30} & \frac{2}{30} & \frac{4}{30} \end{array}\right) \qquad (4.36)$$

Now according to (4.28) the signs of the entries in $\underline{M}_{21}$ should be negative for diminishing returns to scale; the second entry is positive showing that for $b_2$, the returns to scale are increasing.

<u>Changes of $d_2$ in problem 1;</u>

Suppose $d_2$ changes by an amount $\Delta d_2$; will there be a change to diminishing returns to scale?

let $\tilde{d}_2 = d_2 + \Delta d_2$, $d_1 = 1$

for $\underline{B}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\underline{\tilde{d}}^* . \underline{B}^{-1} = (1, \tilde{d}_2) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (1, 1 + \tilde{d}_2)$$

i.e. d.r.t.s. $\Leftrightarrow 1 + \tilde{d}_2 > 0 \Leftrightarrow 1 + 1.5 + \Delta d_2 > 0$

$\Leftrightarrow \Delta d_2 > -2.5$

Further implications of changes in $\underline{d}$ are considered in later work.

If $d_1$ were to change, $d_1$ would have to increase beyond 1.5 for the returns to scale to be diminishing.

From the form of $M_{21}$ in (4.18) it is clear that, for each $d_i$, the range of values for $d_i$ is divided into only two disjoint parts, one of diminishing returns to scale, and one of increasing returns to scale.

<u>Problem 3</u>

$$\max \quad \frac{4x_1 + x_2 + 4x_3 + x_4}{x_1 + x_2 + x_3 + x_4 + 1}$$

$$\begin{aligned} \text{s.t.} \quad x_1 + 2x_2 + 2x_3 + x_4 &\leq 40 \\ x_1 + x_2 &\leq 30 \\ 2x_1 + x_2 &\leq 20 \\ x_3 &\leq 10 \\ x_4 &\leq 10 \\ x_3 + x_4 &\leq 15 \end{aligned}$$

The problem is solved using the CC form.

Optimal solution is:

$$y_1^* = \frac{20}{42}\ ,\ y_3^* = \frac{20}{42}\ ,\ t^* = \frac{2}{42}\ ,\ v_3 = \frac{4}{42}\ ,\ v_4 = \frac{4}{21}\ ,$$

i.e. $x_1^* = 10\ ,\ x_3^* = 10\ ,\ u_3 = \frac{8}{42^2} = \frac{2}{21^2}\ ,\ u_4 = \frac{4}{21^2}$.

Optimal inverse is:

$$\begin{pmatrix} s_1 \\ s_2 \\ x_1 \\ x_3 \\ s_5 \\ s_6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\underline{d}^* = (0, 0, 1, 1, 0, 0)$

$\underline{\theta} = \underline{d}^* . \underline{B}^{-1} = (0, 0, \frac{1}{2}, 1, 0, 0)$

$\theta_i > 0$ diminishing returns to scale.

## Appendix 4.2 The Reduced Costs of Fractional Programming

In the normal LP usage, 'reduced costs' $\sigma_i$ are defined as $\sigma_i = c_i - \underline{c}_B.\underline{B}^{-1}.\underline{a}_i$

where $\underline{a}_i$ is the i'th column of the original tableau. Now, if $f(\underline{x}) = \underline{c}.\underline{x}$, $c_i = \frac{\partial f}{\partial x_i}$, and

$$\sigma_i = \left[\frac{\partial f}{\partial x_i}\right]_{\underline{x}=\hat{\underline{x}}} - \left[\frac{\partial f}{\partial \underline{x}_B}\right]_{\underline{x}=\hat{\underline{x}}} .\underline{B}^{-1}.\underline{a}_i \tag{4.37}$$

where $\left[\frac{\partial f}{\partial \underline{x}_B}\right]$ denotes differentiation with respect to basic variables only. This concept of 'reduced cost' may be considered as a marginal return, and may be generalised to the fractional case, f(x) defined as in (1.12).

Thus when Martos (64) uses the terms $\Delta_i$ to rank incoming activities, where

$$\Delta_i = (d_0 + \underline{d}_B.\underline{x}_B).(c_i - \underline{c}_B.\underline{B}^{-1}.\underline{a}_i) - (c_0 + \underline{c}_B.\underline{x}_B).(d_i - \underline{d}_B.\underline{B}^{-1}.\underline{a}_i) \tag{4.38}$$

he is, in effect, using a multiple of the marginal return for each activity, since

$$\frac{\Delta_i}{(d_0 + \underline{d}_B.\underline{x}_B)^2} = (c_i - d_i.f^*).t^* - (\underline{c}_B - \underline{d}_B.f^*).\underline{B}^{-1}.\underline{a}_i.t^* \tag{4.39}$$

where $t^* = (d_0 + \underline{d}_B.\underline{x}_B)^{-1}$ and $f^*$ is the value of the objective, for the present solution $\underline{x}_B = \underline{B}^{-1}.\underline{b}$

$$\text{i.e.} \quad \Delta_i.(t^*)^2 = \left[\frac{\partial f}{\partial x_i}\right]_{\underline{x}=\hat{\underline{x}}} - \left[\frac{\partial f}{\partial \underline{x}_B}\right]_{\underline{x}=\hat{\underline{x}}} .\underline{B}^{-1}.\underline{a}_i \tag{4.40}$$

$\Delta_i.(t^*)^2$ is the marginal return for introducing the i'th activity. (Call this $\bar{\sigma}_i$)

By Wagner and Yuan (85), $\Delta_i = \frac{\hat{\sigma}_i}{t^*}$, where $\hat{\sigma}_i$ is the reduced cost (marginal return) in the CC form,

$$\hat{\sigma}_i = c_i - (\underline{c}_B, \alpha).\underline{B}^{*-1}.\begin{pmatrix} a_i \\ d_i \end{pmatrix}$$

Hence $\bar{\sigma}_i = t\hat{\sigma}_i$ (4.41)

## Appendix 5.1 The Decomposition Process

Example:

$$\max \frac{4x_1 + 2x_2 + 4y_1 + 3y_2}{x_1 + x_2 + y_1 + y_2 + 1}$$

$$\begin{aligned} \text{s.t.}\quad x_1 + 2x_2 + 2y_1 + y_2 &\le 15 \\ x_1 + 3x_2 &\le 30 \\ 2x_1 + x_2 &\le 20 \\ y_1 &\le 10 \\ y_2 &\le 10 \\ y_1 + y_2 &\le 15 \\ x_i, y_i &\ge 0 \end{aligned} \qquad (5.56)$$

Optimal solution is:

$\hat{f} = \frac{100}{27}$, $x_1 = 10$, $x_2 = 0$, $y_1 = 2.5$, $y_2 = 0$

Dual evaluators $\sim$(0.14, 0.0, 0.07, 0.0, 0.0, 0.0) for the CC form.

Assume an initial all-slack basis:

Solution: $f^* = 0$, $\pi_1 = \pi_2 = \underline{\pi} = 0$

Using the first method, of Section 5.22, the divisional programmes are:

Div. 1 $\max 4x_1 + 2x_2$ s.t.

$$\begin{aligned} x_1 + 3x_2 &\le 30 \\ 2x_1 + x_2 &\le 20 \\ x_i &\ge 0 \end{aligned}$$

Solution: $x_1 = 6$, $x_2 = 8$, $\hat{f}_1 = 40$

(We neglect the solution $x_1 = 10$, $x_2 = 0$, in order to force iterations).

Div. 2 $\max 4y_1 + 3y_2$ s.t.

$$\begin{aligned} y_1 &\le 10 \\ y_2 &\le 10 \\ y_1 + y_2 &\le 15 \\ y_i &\ge 0 \end{aligned}$$

Solution: $y_1 = 10$, $y_2 = 5$, $\hat{f}_2 = 55$

Policy: Accept both since $\hat{f}_\alpha \ge \pi_\alpha$, $\alpha = 1,2$

Form Executive Programme:

This is: max $\dfrac{0\mu_1 + 40\mu_2 + 0\,v_1 + 55v_2}{0\mu_1 + 14\mu_2 + 0\,v_2 + 15v_2 + 1}$

$$\begin{aligned} \text{s.t.} \quad 0\mu_1 + 22\mu_2 + 0\,v_1 + 25v_2 &\le 15 \\ \mu_1 + \mu_2 &= 1 \\ v_1 + v_2 &= 1 \\ \mu_i,\ v_i &\ge 0 \end{aligned} \qquad (5.57)$$

The optimal solution to the CC form of (5.57) is

$\bar{\mu}_1 = \frac{1}{10}$, $\bar{v}_1 = \frac{2}{50}$, $\bar{v}_2 = \frac{3}{50}$, $\hat{t} = \frac{1}{10}$

Hence $\mu_1 = .1$, $v_1 = \frac{2}{5}$, $v_2 = \frac{3}{5}$, $\hat{f} = \frac{165}{50}$

$\pi_1 = 0$, $\pi_2 = 0$, $\pi_d = \frac{165}{50}$

$\pi_{cc} = 0$ (duals of CC form equivalent to (5.6) of Section 5.2.1)

Revise divisional objective functions:

Method 1 (Section 5.2.2)

Revise according to $c_i - (d_i.\pi_d - \underline{\pi}_{cc}.\underline{M}_i)$

where $\underline{\pi}_{cc}$ are dual evaluators of executive rows in the CC form of (5.57).

Method 2 (Section 5.3.2)

Revise according to $(c_i - d_i.\hat{f})\hat{t} - \underline{\pi}_F.\underline{M}_i$

where $\underline{\pi}_F$ are dual evaluators of executive rows in (5.57).

Now $\underline{\pi}_F = \hat{t}.\underline{\pi}_{cc}$; we will use the first method throughout.

Optimality test: $\hat{f}_1 > \pi_1$ $\therefore$ not optimal

Revised objectives are:

Div. 1 $c_1 : 4 - 1.\frac{165}{50} - 0(1) = \frac{7}{10}$

$c_2 : 2 - 1.\frac{165}{50} - 0(2) = \frac{-13}{10}$

Div. 2 $c_1 : 4 - 1.\frac{165}{50} - 0(2) = \frac{7}{10}$

$c_2 : 3 - 1.\frac{165}{50} - 0(1) = \frac{-3}{10}$

Now proposals are:

Division 1 : $x_1 = 10,\ x_2 = 0,\ \hat{f}_1 = 40 > \pi_2 = 0$ $\therefore$ accept.

Division 2 : $y_1 = 10,\ y_2 = 0,\ \hat{f}_2 = 40 > \pi_2 = 0$ $\therefore$ accept.

New executive programme has the solution:

$$\mu_1 = 0 \quad \mu_2 = 0 \quad \mu_3 = 1$$

$$v_1 = \tfrac{3}{4} \quad v_2 = 0 \quad v_3 = \tfrac{1}{4}$$

$$\hat{f} = \frac{100}{27}$$

$$\pi_1 = \frac{40}{27} \quad \pi_2 = 0 \quad \pi_d = \frac{100}{27} \quad \pi = \frac{4}{27}$$

(duals of CC form equivalent to (5.6) of Section 5.2.1)

Revised divisional objectives are:

Div. 1 $\quad c_1 : 4 - \frac{100}{27} - \frac{4}{27} = \frac{4}{27}$

$\quad c_2 : 2 - \frac{100}{27} - \frac{8}{27} = -2$

Div. 2 $\quad c_1 : 4 - \frac{100}{27} - \frac{8}{27} = 0$

$\quad c_2 : 3 - \frac{100}{27} - \frac{4}{27} = \frac{-23}{27}$

New solutions are:

Division 1 : $x_1 = 10,\ x_2 = 0 \quad \hat{f}_1 = \frac{40}{27} = \pi_1$

$\therefore$ do not accept

Division 2 : $y_1 = 0,\ y_2 = 0 \quad \hat{f}_2 = 0 \leq \pi_2$

$\therefore$ do not accept

$\therefore$ solution to the previous executive programme is optimal.

i.e. solution is $\quad x_1 = 10 \quad x_2 = 0$

$$y_1 = 2.5 \quad y_2 = 0$$

$$f = \frac{100}{27}$$

The optimal dual evaluators:

From the CC form of the final executive programme $\pi_{cc} = \frac{4}{27}$

$\therefore$ for original fractional form $\pi_F = \frac{4}{27} \cdot \frac{2}{27} = \frac{8}{(27)^2}$

For the divisions we have the final programmes:

<u>Div. 1</u>

$$\max \frac{4}{27} x_1 - 2x_2$$

$$\text{s.t.} \quad x_1 + 3x_2 \le 30$$

$$2x_2 + x_2 \le 20$$

$$x_1, x_2 \ge 0$$

<u>Div. 2</u>

$$\max 0.y_1 - \frac{23}{27} y_2$$

$$\text{s.t.} \quad y_1 \le 10$$

$$y_2 \le 10$$

$$y_1 + y_2 \le 15$$

$$y_1, y_2 \ge 0$$

Dual evaluators are $(0, \frac{2}{27})$ and $(0, 0, 0)$.

Thus for the CC form of (5.56) we have the dual evaluators

$(\frac{4}{27}, 0, \frac{2}{27}, 0, 0, 0)$

Now $\hat{t} = \frac{2}{27}$, $\therefore$ the evaluators for fractional form are

$(.14, 0.0, 0.07, 0.0, 0.0, 0.0).\frac{2}{27}$

## Appendix 5.2 Sensitivity Analysis and the 'Perturbed Inverse Basis'

Example: The problem (5.58) is taken from Baumol and Fabian (8)

$$\begin{aligned}
\max \quad & P = x_1 + x_2 + 2y_1 + 2y_2 \\
\text{s.t.} \quad & x_1 + 2x_2 + 2y_1 + y_2 \le 40 \\
& x_1 + 3x_2 \le 30 \\
& 2x_1 + x_2 \le 20 \\
& y_1 \le 10 \\
& y_2 \le 10 \\
& y_1 + y_2 \le 15 \\
& x_i, y_i \ge 0
\end{aligned} \tag{5.58}$$

Optimal inverse basis is $\underline{B}^{-1}$ where,

$$\underline{B}^{-1} = \begin{pmatrix} \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} & 0 & -\frac{2}{3} \\ -\frac{5}{3} & 1 & \frac{1}{3} & \frac{5}{3} & 0 & \frac{5}{3} \\ -\frac{1}{3} & 0 & \frac{2}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} x_2 \\ \text{slack} \\ x_1 \\ y_1 \\ \text{slack} \\ y_2 \end{pmatrix} \tag{5.59}$$

Final tableau of executive programme is:

$$\begin{aligned}
\max \quad & 10\mu_3 + 14\mu_4 + 0\,v_1 + 25\,v_4 \\
\text{s.t.} \quad & 10\mu_3 + 22\mu_4 + 0\,v_1 + 25\,v_4 \le 40 \\
& \mu_3 + \mu_4 = 1 \\
& v_1 + v_4 = 1
\end{aligned} \tag{5.60}$$

Optimal inverse basis:

$$\begin{pmatrix} v_4 \\ \mu_4 \\ \mu_3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{12} & -\frac{10}{12} & -\frac{25}{12} \\ -\frac{1}{12} & \frac{22}{12} & \frac{25}{12} \end{pmatrix} = \underline{\underline{B}}^{-1} \tag{5.61}$$

Tendered solutions are:

$$\begin{aligned} \text{For } '\mu_3' &: (10,\ 0,\ 0,\ 0) \\ '\mu_4' &: (\ 6,\ 8,\ 0,\ 0) \\ 'v_1' &: (\ 0,\ 0,\ 0,\ 0) \\ 'v_4' &: (\ 0,\ 0,\ 10,\ 5) \end{aligned} \qquad (5.62)$$

Change of corporate resource 'b': (initially at 40 units)

$\frac{\partial \mu_3}{\partial b} = -\frac{1}{12}$, $\frac{\partial \mu_4}{\partial b} = \frac{1}{12}$, $\frac{\partial v_4}{\partial b} = 0$, given by first column of $\underline{\underline{B}}^{-1}$.

From $\underline{B}^{-1}$ we know: $\frac{\partial x_1^*}{\partial b} = -\frac{1}{3}$, $\frac{\partial x_2^*}{\partial b} = \frac{2}{3}$, $\frac{\partial y_1^*}{\partial b} = \frac{\partial y_2^*}{\partial b} = 0$ (5.63)

Using the formula $\frac{\partial x_i^*}{\partial b} = \sum \frac{\partial \mu_i^*}{\partial b} \cdot x_i^*$, (5.37) of 5.6.3

we have:

$$\begin{aligned} \frac{\partial x_1^*}{\partial b} &= 10\left(-\frac{1}{12}\right) + 6\left(\frac{1}{12}\right) = -\frac{1}{3} \\ \frac{\partial x_2^*}{\partial b} &= 0\left(-\frac{1}{12}\right) + 8\left(\frac{1}{12}\right) = \frac{2}{3} \\ \frac{\partial y_1^*}{\partial b} &= 0.(10) = 0 \\ \frac{\partial y_2^*}{\partial b} &= 0(5) = 0 \end{aligned} \qquad (5.64)$$

Q.E.D.

Change in $b_k$:$b_k$ contained in divisional r.h.s.

We now use the formula: (5.38) of Section 5.6.3

i.e. $$\frac{\partial \underline{x}^*}{\partial b_k} = \sum_j \left[\frac{\partial \mu_j^*}{\partial b_k} \cdot \underline{x}_j^* + \mu_j^* \cdot \frac{\partial \underline{x}_j^*}{\partial b_k}\right]$$

The second terms are known from the solutions and the respective optimal inverse bases.

The $\frac{\partial u_i^*}{\partial b_k}$ will be calculated from these and further LP's as follows:

1. Use inverse bases of the various divisional solutions to give $x_i^*(b_k)$

2. derive $\frac{\partial x_i^*}{\partial b_k}$

3. form the final executive programme in terms of a variable $b_k$, i.e. form the $\varphi(b_k)$

4. solve for $\varphi(b_k)$ and $\varphi(b_k + \delta b_k)$ and from analytical expressions derive the $\frac{\partial \mu_i^*}{\partial b_k}$.

1. Divisional subproblems:

Division 1: constraints are
$$x_1 + 3x_2 \le 30$$
$$2x_1 + x_2 \le 20$$

For $\underline{x}_4^* \sim \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$: inverse is $\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$

For $\underline{x}_3^* \sim \begin{pmatrix} \text{slack} \\ x_1 \end{pmatrix}$ : inverse is $\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

Putting $x_i^*$ in terms of $\underline{b}$ we have

$$\underline{x}_3^* : x_{31}^* = \frac{b_2}{2}, \quad x_{32}^* = 0$$

$$\underline{x}_4^* : x_{41}^* = -\frac{b_1}{5} + \frac{3}{5} \cdot b_2, \quad x_{42}^* = \frac{2}{5}b_1 - \frac{1}{5}b_2 \qquad (5.65)$$

Suppose we are considering changes of a resource of division 1. The solutions tendered by division 2 are independent of changes in resource level of division 1. Thus we form the executive programme

$$\max \mu_3 \left[\frac{b_2}{2}\right] + \mu_4\left[\frac{-b_1}{5} + \frac{3b_2}{5} + \frac{2b_1}{5} - \frac{b_2}{5}\right] + 0\,v_1 + 25v_4$$

$$\text{s.t. } \mu_3 \left[\frac{b_2}{2}\right] + \mu_4\left[\frac{-b_1}{5} + \frac{3b_2}{5} + \frac{4b_1}{5} - \frac{2b_2}{5}\right] + 0\,v_1 + 25v_4 \le 40$$

$$\mu_3 + \mu_4 = 1$$

$$v_1 + v_4 = 1$$

$$(5.66)$$

This is the same as

$$\max \quad \mu_3\left[\frac{b_2}{2}\right] + \mu_4\left[\frac{b_1}{5} + \frac{2b_2}{5}\right] + 0\,v_1 + 25v_4$$

$$\text{s.t.} \quad \mu_3\left[\frac{b_2}{2}\right] + \mu_4\left[\frac{3b_1}{5} + \frac{b_2}{5}\right] + 0\,v_1 + 25v_4 \leq 40$$

$$\mu_3 + \mu_4 = 1$$

$$v_1 + v_4 = 1 \qquad (5.67)$$

Consider changes of first resource in division 1, i.e. put $b_1 = 30 + \delta$, $b_2 = 20$. The executive programme becomes:

$$\max \quad 10\mu_3 + \left[14 + \frac{\delta}{5}\right]\mu_4 + 0\,v_1 + 25v_4$$

$$\text{s.t.} \quad 10\mu_3 + \left[22 + \frac{3\delta}{5}\right]\mu_4 + 0\,v_1 + 25v_4 \leq 40$$

$$\mu_3 + \mu_4 = 1$$

$$v_1 + v_4 = 1 \qquad (5.68)$$

Optimal solution is

$$v_4(\delta) = 1$$

$$\mu_4(\delta) = \frac{5}{12 + 3\delta/5}$$

$$\mu_3(\delta) = \frac{7 + 3\delta/5}{12 + 3\delta/5} \qquad (5.69)$$

assuming

$$22 + \frac{3\delta}{5} > 0$$

$$12 + \frac{3\delta}{5} > 0$$

$$18 - \frac{3\delta}{5} > 0$$

and $$7 + \frac{3\delta}{5} > 0$$

Hence:

$$\frac{\partial\mu_3^*}{\partial b_1} = \lim_{\delta \to 0}\left(\frac{1}{\delta}\left[\frac{7 + \frac{3\delta}{5}}{12 + \frac{3\delta}{5}} - \frac{7}{12}\right]\right)$$

$$= \lim_{\delta \to 0}\left[\frac{\frac{36\delta}{5} - \frac{21\delta}{5}}{144\delta}\right] = \frac{1}{48}$$

$$\frac{\partial\mu_4^*}{\partial b_1} = -\frac{1}{48} \qquad (5.70)$$

Now from a knowledge of "total" optimal inverse basis we know

$$\frac{\partial x_1^*}{\partial b_1} = \frac{\partial x_2^*}{\partial b_1} = 0$$

and applying the formula (5.38) of 5.3.6 we have:

$$\frac{\partial x_1^*}{\partial b_1} = \frac{1}{48}(10 - 6) + \frac{7}{12} \cdot 0 + \frac{5}{12} \cdot \left[-\frac{1}{5}\right] = 0$$

$$\frac{\partial x_2^*}{\partial b_1} = \frac{1}{48}(0 - 8) + \frac{7}{12} \cdot 0 + \frac{5}{12} \cdot \left[\frac{1}{5}\right] = 0$$

Q.E.D.

Similarly we may test changes with respect to $b_2$.

Using $b_1 = 30$, $b_2 = 20 + \delta$, a similar analysis leads to:

$$v_4 = 1$$

$$\mu_3(\delta) = \frac{7 + \frac{\delta}{5}}{12 - \frac{3\delta}{10}}$$

$$\mu_4(\delta) = \frac{5 + \frac{\delta}{5}}{12 - \frac{3\delta}{10}}$$

$$\frac{\partial \mu_3}{\partial b_2} = \frac{1}{32} \qquad \frac{\partial \mu_4}{\partial b_2} = -\frac{1}{32}$$

From the 'total' inverse basis we know that:

$$\frac{\partial x_1^*}{\partial b_2} = \frac{2}{3} \qquad \frac{\partial x_2^*}{\partial b_2} = -\frac{1}{3}$$

Apply the formula:

$$\frac{\partial x_1^*}{\partial b_2} = \frac{1}{32}(10 - 6) + \frac{7}{12} \cdot \frac{1}{2} + \frac{5}{12} \cdot \frac{3}{5} = \frac{2}{3}$$

$$\frac{\partial x_1^*}{\partial b_2} = \frac{1}{3}(-8) + \frac{7}{12} \cdot 0 + \frac{5}{12}(-\frac{1}{5}) = -\frac{1}{3}$$

Q.E.D.

Thus by calculating all terms $\frac{\partial x_i}{\partial b_k}$ the total inverse basis may be derived.

The fractional case is no different except that the executive programme is more difficult to compute. The theory remains the same since at all stages the '$\underline{x} = \underline{B}^{-1}\underline{b}$' optimal relationship holds.

## Appendix 5.3 Direct Calculation of the 'Perturbed Inverse Basis'

In the calculations of Appendix 5.2, the basic columns of the final executive tableau are

$$\begin{array}{cc} & \begin{array}{ccc} \mu_3 & \mu_4 & v_4 \end{array} \\ \underline{A} = & \begin{pmatrix} 10 & 22 & 25 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\underline{A}^{-1} = \begin{pmatrix} -\frac{1}{12} & \frac{22}{12} & \frac{25}{12} \\ \frac{1}{12} & -\frac{10}{12} & -\frac{25}{12} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{H} = \begin{pmatrix} 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

from (5.68) of Appendix 5.2.

$$\underline{b} = \begin{pmatrix} 40 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{A}^{-1}.\underline{b} = \left(\frac{7}{12}, \frac{5}{12}, 1\right) = (\mu_3^*, \mu_4^*, v_4^*)$$

Using the formula (5.43) of section 5.6.4 we have:

$$\text{"}\frac{\partial \mu}{\partial b_k}\text{"} = -\underline{A}^{-1}.\underline{H}.\underline{A}^{-1}.\underline{b}$$

$$= -\underline{A}^{-1}.\underline{H}.\begin{pmatrix} \frac{7}{12} \\ \frac{5}{12} \\ 1 \end{pmatrix}$$

$$= - \begin{pmatrix} 0 & -\frac{1}{20} & 0 \\ 0 & \frac{1}{20} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{7}{12} \\ \frac{5}{12} \\ 1 \end{pmatrix}$$

$$= - \begin{pmatrix} -\frac{1}{48} \\ \frac{1}{48} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{48} \\ -\frac{1}{48} \\ 0 \end{pmatrix}$$

These are the same as the marginal figures derived in (5.70) of Appendix 5.2.

## Appendix 6.1 Sensitivity Analysis

### 6.1.1 Changes in r.h.s. elements

Consider the problem:

$$\max \frac{3.1x_1 + 3x_2}{x_1 + x_2 + 1}$$

$$\text{s.t.} \quad \begin{aligned} x_1 \quad\quad &\le 2 \\ x_2 &\le 2 \\ x_1 + x_2 &\le 3 \qquad x_i \ge 0 \end{aligned} \tag{6.20}$$

Direct approach: let the slacks be $S_1$, $S_2$, $S_3$

Optimal basis is $(x_1, x_2, S_2)$

Inverse basis is $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \underline{B}^{-1}$

Consider changes in $b_3$, say $\delta b_3$; optimality (and feasibility) conditions are that:

$$\underline{\theta} = \underline{B}^{-1} . \begin{pmatrix} 2 \\ 2 \\ 3 + \delta b_3 \end{pmatrix} \ge \underline{0}$$

i.e. $\theta_1 = 2 \ge 0$

$\theta_2 = -2 + 3 + \delta b_3 \ge 0$

$\theta_3 = 2 + 2 - 3 - \delta b_3 \ge 0$

$\Longrightarrow$ $\underline{-1 \le \delta b_3 \le 1}$

The CC Form of (6.20) is

$$\max \quad 3.1y_1 + 3y_2$$

$$\text{s.t.} \quad \begin{aligned} y_1 + y_2 + t + S_0 &= 1 \\ y_1 - 2t + S_1 &= 0 \\ y_2 - 2t + S_2 &= 0 \\ y_1 + y_2 - 3t + S_3 &= 0 \qquad \underline{y}_i, S_i, t \ge 0 \end{aligned} \tag{6.21}$$

Optimal basis is $(y_1, t, S_2, y_2)$

Inverse basis is $\begin{pmatrix} \frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ \frac{1}{4} & 1 & 1 & -\frac{5}{4} \\ \frac{1}{4} & -1 & 0 & \frac{3}{4} \end{pmatrix} = \underline{\bar{B}}^{-1}$

$$\underline{\bar{b}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix}$$

Consider changes of $\bar{b}_3$ by say $\delta$ .

$\underline{\bar{\theta}} = \underline{\bar{B}}^{-1}.\underline{\bar{b}} \geq \underline{0}$ is required for feasibility

$$\left.\begin{array}{l} \bar{\theta}_0 = \frac{1}{2}(1-\delta) \\ \bar{\theta}_1 = \frac{1}{4}(1-\delta) \\ \bar{\theta}_2 = \frac{1}{4}(1-5\delta) \\ \bar{\theta}_3 = \frac{1}{4}(1+3\delta) \end{array}\right\} \text{ i.e. } -\frac{1}{3} < \delta < \frac{1}{5}$$

Now at $\delta = 5$
$$\left.\begin{array}{l} \bar{\theta}_0 = y_1 = \frac{2}{5} \\ \bar{\theta}_1 = t = \frac{1}{5} \\ \bar{\theta}_2 = S_2 = 0 \\ \bar{\theta}_3 = y_2 = \frac{2}{5} \end{array}\right\}$$

i.e. $x_1 = x_2 = 2 \qquad t = \frac{1}{5}$

At $\delta = -\frac{1}{3}$
$$\left.\begin{array}{l} \bar{\theta}_0 = y_1 = \frac{2}{3} \\ \bar{\theta}_1 = t = \frac{1}{3} \\ \bar{\theta}_2 = S_2 = \frac{2}{3} \\ \bar{\theta}_3 = y_2 = 0 \end{array}\right\}$$

i.e. $x_1 = 2 \quad S_2 = 2 \quad t = \frac{1}{3}$

thus we have $-\frac{1}{3} \leq \delta \leq \frac{1}{5}$

implying $\underline{-1 \leq \delta b_3 \leq 1}$

Hence limits for range of $\delta b$ are given by the appropriate corrections to the range of $\overline{\delta b}$ as in Section 6.4.1

6.1.2 (a) <u>Changes in $c_j$</u>

Consider a third activity $x_3$, i.e.

$$\max \quad \frac{3.1x_3 + 3x_2 + \gamma x_3}{x_1 + x_2 + x_3 + 1}$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + x_3 \le 2 \\ & x_2 + x_3 \le 2 \\ & x_1 + x_2 \le 3 \qquad x_i \ge 0 \end{aligned} \tag{6.22}$$

Assume $\gamma$ is initially zero. What value must $\gamma$ attain in order for $x_3$ to enter the basis.

Solution is $x_1 = 2$, $x_2 = 1$, $S_2 = 1$, $f^* = \frac{9.2}{4} = 2.3$

$$\underline{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \qquad \underline{A}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Using the CC form of the (6.22)

$$\overline{\underline{B}}^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 0 & \frac{3}{4} & \frac{1}{4} \\ 1 & 1 & -\frac{5}{4} & \frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{6.23}$$

(Rows have been arranged to have denominator last.)

$\underline{c}^* = (3.1, 3, 0, 0) \sim (y_1, y_2, S, t)$

$(\underline{\pi}_{CC}, \pi) = (.1, 0, .7, \frac{9.2}{4})$

$$\begin{pmatrix} \underline{A}_3 \\ \underline{d}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\sigma_3 = \gamma - \frac{9.2}{4} - 0.1 = \gamma - \frac{9.6}{4}$$

therefore, for $\gamma < 2.4$ $x_3$ does not enter the basis

$\gamma \geq 2.4$ $x_3$ replaces $x_1$

(As $x_3$ replaces $x_1$, the constraint on $x_2$ is relaxed; i.e.

$$x_3 \nearrow \Rightarrow x_1 \searrow \Rightarrow x_2 \nearrow)$$

(b) Changes in basic $c_j$

Suppose $\gamma = 2$. At what level of $c_1$ will $x_1$ leave the basis of (6.22).

For the basic set, $(y_1, y_2, S, t)$, the CC inverse basis is of the form

$$\underline{\bar{B}}^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 0 & \frac{3}{4} & \frac{1}{4} \\ 1 & 1 & -\frac{5}{4} & \frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$(\underline{c}^*, \alpha) = (c_1, 3, 0, 0)$

$(\underline{\pi}_{CC}, \pi) = (c_1 - 3, 0, \frac{-c_1}{2} + \frac{9}{4}, \frac{c_1}{2} + \frac{3}{4})$

$A_{s_1}$ is column associated with slack variable $s_1$.

$\underline{A}'_{s_1} = (1, 0, 0, 0)$

$\underline{A}'_{s_3} = (0, 0, 1, 0)$

$\underline{A}'_3 = (1, 1, 0, 1)$

Consider the 'reduced costs' $\sigma_i$ :

$\sigma_{s_1} = 0 - (c_1 - 3) \leq 0$ if $c_1 \geq 3$

$\sigma_3 = 2 - (c_1 - 3) - \frac{c_1}{2} - \frac{3}{4} \leq 0$ if $\frac{3c_1}{2} \geq \frac{17}{4}$

i.e. if $c_1 \geq \frac{17}{6}$

$\sigma_{s_3} = 0 - (\frac{9}{4} - \frac{c_1}{2}) \leq 0$ if $c_1 \leq \frac{9}{2}$

For the present basis to be optimal:

$c_1 \geq 3$ (otherwise $S_1$ will enter the basis)

$c_1 \leq \frac{9}{2}$ (otherwise $S_3$ will enter the basis)

6.1.3 Changes in $d_j$

Suppose $\gamma$ is fixed at 1; by how much must $d_3$ be reduced in order for $x_3$ to enter the basis. Let the change be $\Delta d_3$. By (6.10) of Section 6.4.3 (a),

If $\Delta d_3 \leq \left| \frac{\sigma_3}{f^*} \right|$, $x_3$ will not enter the basis

Now $\sigma_3 = 1 - 2.4 = -1.4$

therefore, for $x_3$ to enter the basis $d_3$ must be reduced by an amount $\Delta d_3$, where

$$\Delta d_3 = \left| -\frac{1.4}{2.3} \right|$$

Changes in basic $d_j$:

Consider changes in $d_2$, in the original form and the CC form.

$$\underline{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \underline{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

We use the equation (4.18) for $M_{11}$ in terms of $d_2$.

$$\underline{B}^{-1}.\underline{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{d}^*.\underline{B}^{-1} = (1-d_2,\ 0,\ d_2)$$

$$t^* = (\beta + \underline{d}^*.\underline{B}^{-1}.\underline{b})^{-1} = \frac{1}{2 + d_2 + 1} = \frac{1}{3 + d_2}$$

Hence:

$$M_{11} = \underline{B}^{-1} - \frac{1}{(3 + d_2)} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} . (1 - d_2,\ 0,\ d_2)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} - \frac{1}{(3+d_2)} \cdot \begin{pmatrix} 2-2d_2 & 0 & 2d_2 \\ 1-d_2 & 0 & d_2 \\ 1-d_2 & 0 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+3d_2}{3+d_2} & 0 & \frac{-2d_2}{3+d_2} \\ \frac{-4}{3+d_2} & 0 & \frac{3}{3+d_2} \\ \frac{2+2d_2}{3+d_2} & 1 & \frac{-3-2d_2}{3+d_2} \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} (\frac{2}{3+d_2}) \\ (\frac{1}{3+d_2}) \\ (\frac{1}{3+d_2}) \end{pmatrix}$$

$$M_{21} = (-\frac{1-d_2}{3+d_2}, 0, -\frac{d_2}{3+d_2})$$

$$M_{22} = \frac{1}{3+d_2}$$

for $d_2 = 1$.

$$M_{11} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ 1 & 1 & -\frac{5}{4} \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

$$M_{21} = (0, 0, -\tfrac{1}{4})$$

$$M_{22} = \tfrac{1}{4}$$

$$(\underline{c}^*, \alpha) = (3.1, 3.0, 0)$$

Consider the changes of $d_2$ on the columns $A_{s_1}$, $A_3$, and $A_{s_3}$.

$$\sigma_{s_1} = 0 - (\underline{c}^*,\alpha)\underline{B}^{-1}(d_2)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -(3.1, 3, 0, 0) \begin{pmatrix} \frac{1+3d_2}{3+d_2} \\ \frac{-4}{3+d_2} \\ \frac{2+2d_2}{3+d_2} \\ \frac{-(1-d_2)}{3+d_2} \end{pmatrix}$$

$$= \frac{-1}{3+d_2}(3.1 + 9.3d_2 - 12)$$

$$\leq 0 \quad \text{if } d_2 \geq \frac{8.9}{9.3} \quad \text{(ignoring the solution } d_2 \leq -3)$$

Similarly:

$$\sigma_{s_3} = 0 - (3.1,\ 3,\ 0,\ 0)\underline{\overline{B}}^{-1}(d_2)\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$(3+d_2).\sigma_{s_3} = (3.1)(2d_2) - 9$$

$$= 6.2d_2 - 9 \leq 0 \qquad \text{if } d_2 \leq \frac{9}{6.2}$$

If $d_2 > \frac{9}{6.2}$ $s_3$ will enter the basis.

Also

$$\sigma_3 = 1 - (3.1,\ 3,\ 0,\ 0)\underline{\overline{B}}^{-1}(d_2)\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$$

$$= 1 - (3.1,\ 3.\ 0,\ 0)\begin{pmatrix}\frac{3+3d_2}{(3+d_2)}\\ \frac{-3}{(3+d_2)}\\ \frac{6+3d_2}{(3+d_2)}\\ \frac{d_2}{(3+d_2)}\end{pmatrix}$$

$$= 1 - \frac{3.1(3+3d_2)}{(3+d_2)} + \frac{9}{(3+d_2)}$$

$$= \frac{2.7 - 8.3d_2}{(3+d_2)}$$

$$\sigma_3 \leq 0 \qquad d_2 \geq \frac{2.7}{8.3}$$

Hence the range for $d_2$ is $\frac{8.9}{9.3} < d_2 < \frac{9}{6.2}$

## Appendix 6.2 Integer Programming

Consider the problem

$$\max \quad \frac{2x_1 + x_2 + 1}{x_1 + x_2 + 1}$$

$$x_1 + 2x_2 \leq 4$$

$$2x_1 + x_2 \leq 3 \tag{6.24}$$

$$x_1,\ x_2 \text{ integers} \geq 0.$$

Optimal tableaux are:

| original form | | | | | CC form | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | | $t$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | |
| | | | | | 1 | 0 | $\frac{1}{5}$ | 0 | $-\frac{1}{5}$ | $\frac{2}{5}$ |
| 0 | $1\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{3}$ | 0 | 0 | 2 | 1 | $-1$ | 1 |
| 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{3}{2}$ | 0 | 1 | $\frac{4}{5}$ | 0 | $\frac{1}{5}$ | $\frac{3}{5}$ |

(N.B. For ease of programming, the denominator has been made the first row of the CC form, and 't' the first column).

Consider the column for $y_2$

$$\underline{y}_B = (1, \tfrac{3}{5}) \quad t = \frac{2}{5} \quad W_{k_2} = \begin{pmatrix} 2 \\ \frac{4}{5} \end{pmatrix} \quad W_{t_2} = \frac{1}{5}$$

Using the formula (6.17) we have

$$z_2 = \begin{pmatrix} 2 \\ \frac{4}{5} \end{pmatrix} - \begin{pmatrix} 1 \\ \frac{3}{5} \end{pmatrix} \cdot \frac{1}{5} \cdot \frac{5}{2} = \begin{pmatrix} 2 \\ \frac{4}{5} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{3}{10} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$$

Similarly, calculations can be made for all the required columns.

N.B. $x_1$ is basic in second constraint row as is $y_1$.

$x_2$ is basic in first constraint row as is $y_2$.

Thus pivoting on (row 2, $x_1$) and (row 1, $x_2$) for the original form will produce the optimal tableau.

The cutting row is derived from the second constraint.

$\{x_i\} = \frac{1}{2}$

$(\{\hat{a}_{ij}\}) = (0, \frac{1}{2}, 0, \frac{1}{2})$

In the CC form the cutting plane is

$$\tfrac{1}{2}t - \tfrac{1}{2}y_2 - \tfrac{1}{2}y_4 \le 0$$

At this point the (infeasible) tableau in the CC form is

| $t$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | |
|---|---|---|---|---|---|---|
| (1) | 0 | $\frac{1}{5}$ | 0 | $-\frac{1}{5}$ | 0 | $\frac{2}{5}$ |
| 0 | 0 | 2 | 1 | -1 | 0 | 1 |
| 0 | 1 | $\frac{4}{5}$ | 0 | $\frac{1}{5}$ | 0 | $\frac{3}{5}$ |
| $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 1 | 0 |
| 0 | 0 | $\frac{4}{5}$ | 0 | $\frac{1}{5}$ | 0 | |

Pivot on t to restore canonical form; thereafter, pivoting according to the dual simplex rules leads to the optimal tableau

| $t$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | |
|---|---|---|---|---|---|---|
| 1 | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 0 | 0 | $\frac{7}{2}$ | 1 | 0 | $\frac{5}{2}$ | $\frac{5}{2}$ |
| 0 | 0 | $\frac{3}{2}$ | 0 | 1 | $\frac{5}{2}$ | $\frac{1}{2}$ |
| 0 | 0 | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | |

Optimal solution is:

$$t = \frac{1}{2},\ y_1 = \frac{1}{3},\ \ y_3 = \frac{3}{2},\ \ y_4 = \frac{1}{2}$$

giving $x_1 = 1, x_2 = 0, x_3 = 3, x_4 = 1$

## Appendix 6.3 Recomputation of Dual Evaluators

Consider the problem:

$$\max \quad \frac{3x_1 + x_2 + 1}{x_1 + x_2 + 1} = Z_I$$

$$\text{s.t.} \quad 10x_1 + 5x_2 \le 11$$

$$x_2 \le 1 \qquad (6.25)$$

$$x_1, x_2 \ge 0 \quad \text{(integers)}$$

The algorithmic approach is shown in figure 6.2.

At the LP optimum the solution is

$$x_1 = \frac{11}{10}$$

$$x_2 = 0$$

$$t = \frac{10}{21} \qquad (6.26)$$

$$z^*_{LP} = \frac{43}{21}$$

The cutting plane is given by

$$\frac{1}{2} x_2 + \frac{1}{10} S_1 \ge \frac{1}{10} \qquad (6.27)$$

In the Charnes and Cooper Form this is

$$\frac{1}{2} y_2 + \frac{1}{10} S_1 \ge \frac{1}{10} . t \qquad (6.28)$$

Inserting this, the optimal tableau (6) is obtained

The integer programming optimum to (6.25) is

$t = \frac{1}{2}$ $x_1 = 1$ $'S'_2 = 1$ $'S'_3 = 1$

$z^*_I = 2$

The dual evaluators of the corresponding form are

$\pi_1 = 0$ $\pi_2' = 0$ $\pi_3 = \frac{1}{2}$

(In the CC form they are

$\pi_{CC_1} = 0$ $\pi_{CC_2} = 0$ $\pi_{CC_3} = 1$ )

Now the cutting plane in terms of $\{x_i^*\}$ was given by (6.27), hence using Baumol and Gomory (48) we have the recomputed duals

$$\pi'_1 = 0 + \frac{1}{10}\cdot\frac{1}{2} = \frac{1}{20}$$

$$\pi'_2 = 0 + 0$$

$$\pi'_3 = 0$$

The LP duals at the optimum were

$$\bar{\pi}_1 = \frac{2}{21}\cdot\frac{10}{21} = \frac{20}{(21)^2}$$

$$\bar{\pi}_2 = 0$$

Because of the structure of the problem the dual evaluations are very similar.

N.B. The recomputation has been made assuming a linear change between the LP optimum and the IP optimum. From Chapter 4 we know that this is not true for fractional programmes. Evaluators are not piecewise constant. However, the added complication of such calculations seems out of all proportion to the associated gain of information.

The associated subsidy (Alcaly and Klevorick (2)) would be the r.h.s. value of the cutting plane constraint, i.e. $\frac{1}{10}$

$\alpha_I$ = value of inputs from recomputed duals

$$= \left(\frac{1}{20}, 0\right)\cdot\begin{pmatrix}11\\0\end{pmatrix} = \frac{11}{20}$$

$S_I$ = subsidy = $\frac{1}{10}$

$$\alpha_I + S_I = \frac{13}{20}$$

But $Z^*_I = 2$

thus value of inputs < value of output, a typical result of integer programming.

Any balance would be of the form of economic rent - but clearly this definition of 'value' is a tenuous one.

| | t | $y_1$ | $y_2$ | $s_1$ | $s_2$ | | rhs | comments |
|---|---|---|---|---|---|---|---|---|
| * | 1 | 1 | 1 | 0 | 0 | | 1 | |
| $s_1$ | -11 | 10 | 5 | 1 | 0 | | 0 | non-basic |
| $s_2$ | -1 | 0 | 1 | 0 | 1 | | 0 | |
| | -1 | -3 | -2 | 0 | 0 | | | |

| | t | $y_1$ | $y_2$ | $s_1$ | $s_2$ | | rhs | comments |
|---|---|---|---|---|---|---|---|---|
| t | 1 | 1 | 1 | 0 | 0 | | 1 | |
| $s_1$ | 0 | 21 | 16 | 1 | 0 | | 11 | basic |
| $s_2$ | 0 | 1 | 2 | 0 | 1 | | 1 | |
| | 0 | -2 | -1 | 0 | 0 | | | |

| | t | $y_1$ | $y_2$ | $s_1$ | $s_2$ | | rhs | comments |
|---|---|---|---|---|---|---|---|---|
| t | 1 | 0 | $\frac{5}{21}$ | $\frac{-1}{21}$ | 0 | | $\frac{10}{21}$ | |
| $y_1$ | 0 | 1 | $\frac{16}{21}$ | $\frac{1}{21}$ | 0 | | $\frac{11}{21}$ | optimal |
| $s_2$ | 0 | 0 | $\frac{26}{21}$ | $\frac{-1}{21}$ | 1 | | $\frac{10}{21}$ | |
| | 0 | 0 | $\frac{11}{21}$ | $\frac{2}{21}$ | 0 | | $\frac{43}{21}$ | |

| | t | $y_1$ | $y_2$ | $s_1$ | $s_2$ | | rhs | comments |
|---|---|---|---|---|---|---|---|---|
| t | 1 | 0 | $\frac{5}{21}$ | $\frac{-1}{21}$ | 0 | 0 | $\frac{10}{21}$ | |
| $y_1$ | 0 | 1 | $\frac{16}{21}$ | $\frac{1}{21}$ | 0 | 0 | $\frac{11}{21}$ | non-basic |
| $s_2$ | 0 | 0 | $\frac{26}{21}$ | $\frac{-1}{21}$ | 1 | 0 | $\frac{10}{21}$ | |
| * | 0.1 | 0 | -0.5 | -0.1 | 0 | 1 | 0 | |
| | 0 | 0 | $\frac{11}{21}$ | $\frac{2}{21}$ | 0 | 0 | | |

Fig. 6.2 Tableaux for Integer Algorithm

| | t | $y_1$ | $y_2$ | $s_1$ | $s_2$ | | rhs | comments |
|---|---|---|---|---|---|---|---|---|
| t | 1 | 0 | $\frac{5}{21}$ | $\frac{-1}{21}$ | 0 | 0 | $\frac{10}{21}$ | |
| $y_1$ | 0 | 1 | $\frac{16}{21}$ | $\frac{1}{21}$ | 0 | 0 | $\frac{11}{21}$ | infeasible |
| $s_2$ | 0 | 0 | $\frac{26}{21}$ | $\frac{-1}{21}$ | 1 | 0 | $\frac{11}{21}$ | |
| $s_3$ | 0 | 0 | $\frac{-11}{21}$ | $\frac{-2}{21}$ | 0 | 1 | $\frac{-1}{21}$ | |
| | 0 | 0 | $\frac{11}{21}$ | $\frac{2}{21}$ | 0 | 0 | | |
| t | 1 | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | |
| $y_1$ | 0 | 1 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | integer optimum |
| $s_2$ | 0 | 0 | $\frac{3}{2}$ | 0 | 1 | $\frac{-1}{2}$ | $\frac{1}{2}$ | |
| $s_3$ | 0 | 0 | $\frac{11}{2}$ | 1 | 0 | $\frac{-21}{2}$ | $\frac{1}{2}$ | |
| | 0 | 0 | 0 | 0 | 0 | 1 | | |

Fig. 6.2 (Continued)

## Appendix 6.4 Risk and Uncertainty in FP

### 6.4.1 Introduction

Much literature has been devoted to the extension of LP for cases in which the programme data are subject to stochastic variation (e.g. (25), (28), (34), (39), (60), (82), (84), (87), and (89).) Such extensions deal with the maximisation of the expected value of a linear objective e.g. (25), maximisation of some merit and penalty function (89), etc. Some formulations do allow extensions to FP, the resultant programmes being quadratic, or convex problems.

### 6.4.2 The Expected Value Approach

Using the assumption that distributions of variables are 'normal', significant simplifications are made in stochastic LP, e.g. (22, 25); in (20) and (25) the resultant programmes are LP's. With FP, such simplifications do not readily occur; the assumption of the 'normal' distribution is not helpful.

Consider $z = \dfrac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta}$, where $\underline{c}$, $\alpha$, $\underline{d}$, $\beta$ are normal variates (with known parameters). z can be written $z = \frac{r}{s}$ where r and s are normal.

Assume r and s have, say, $N(0, \sigma_r)$ and $N(0, \sigma_s)$. z has a Cauchy distribution of the form $f(z) = \dfrac{1}{\beta\pi((\frac{z}{\beta})^2+1)}$

where $\beta = \dfrac{\sigma_r}{\sigma_s}$. However, this assumes that the denominator can take all values. It is possible to expand the function

$$z = \frac{r + r_0}{s + s_0};$$

$$z = (r + r_0)(s + s_0)^{-1} = \frac{r + r_0}{s_0} \quad 1 - \frac{s}{s_0} + 0\ (\frac{s}{s_0})^2$$

$$\simeq \frac{(r + r_0)(s_0 - s)}{s_0^2} \qquad (6.29)$$

If r has the distribution $(\theta, \sigma_r)$ and s has $(\theta, \sigma_s)$, then z can approximately be described by the distribution $(\frac{r_0}{s_0}, \sigma)$ where $\sigma^2 = \frac{\sigma_r^2}{s_0^2} + \sigma_s^2 \cdot \left(\frac{r_0}{s_0^2}\right)^2$

6.4.3 <u>The Utility Theory Approach</u>

R.J. Freund (39) uses a utility approach to risk, maximising the form $\int r.(1 - e^{-ar})dr$ (where r is some measure of return.)

For objectives which have a normal distribution, the maximisation becomes

$$\max \ E(u) = \int_{-\infty}^{\infty} (1 - e^{-ar}) e^{-\left(\frac{r-\mu}{\sigma}\right)^2 \frac{1}{2}} .dr$$

$$= \max \ E(\mu) = \mu - \frac{a}{2} \sigma^2 \qquad (6.30)$$

where r has $(\mu, \sigma)$. If we consider a fraction $z = \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta}$ for deterministic $\underline{d}$, $\alpha$, $\beta$ and normal $c_i$, for any choice of $\underline{x}$, z has a normal distribution

$$\mu = \frac{\overline{\underline{c}}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta}$$

$$\sigma^2 = \frac{\Sigma \underline{x}_i^2 \ \sigma_i^2}{(\underline{d}.\underline{x} + \beta)^2}$$

or

$$\frac{\Sigma \ x_i x_j \sigma_{ij}}{(\underline{d}.\underline{x} + \beta)^2}$$

where $\{\sigma_i\}$ are the standard deviations of each $c_i$ and $\sigma_{ij}$ is the variance/covariance matrix.

The utility approach then has the form

$$\max \ \frac{\overline{\underline{c}}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta} - \frac{a}{2} \frac{\Sigma x_i x_j \sigma_{ij}}{(\underline{d}.\underline{x} + \beta)^2} = f(\underline{x})$$

$$f(\underline{x}) = \frac{(\overline{\underline{c}}.\underline{x} + \alpha)(\underline{d}.\underline{x} + \beta) - \frac{a}{2} \Sigma \ x_i x_j \sigma_{ij}}{(\underline{d}.\underline{x} + \beta)^2} \qquad (6.31)$$

This non-linear fractional programme can be solved using Swarup's Algorithm (81).

6.4.4 Uncertainty

Considering the linear form:

$$\begin{aligned} \min \quad & \underline{c}.\underline{x} + E_{\zeta}(\underline{q}, \underline{y}) = f(\underline{x}) \\ \text{s.t.} \quad & \underline{A}.\underline{x} \leq \underline{b} \\ & \underline{T}.\underline{x} + \underline{M}.\underline{y} = \underline{\zeta} \\ & \underline{x}, \underline{y} \geq 0 \end{aligned} \qquad (6.32)$$

The convex certainty equivalent is of the form min $\underline{c}.\underline{x} + Q(\underline{x})$. Examples are given in (87), but these rely on the linearity of $\underline{c}.\underline{x}$.

This analysis applied to a fractional programme the certainty equivalent would have the form

$$\max \tilde{f}(\underline{x}) = \frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta} - E_{\zeta}(\underline{q}, \underline{y})$$

$$= \max \frac{N(\underline{x})}{D(\underline{x})}$$

which can be optimised using Ritter's method (72), if N is convex, and D is linear.

6.4.5 Chance Constrained Programming

Charnes and Cooper (18) consider three objective functions for the Chance Constrained Programme:

$$\begin{aligned} f(\underline{x}) &= E(\underline{c}.\underline{x}) \\ &= E(\underline{c}.\underline{x} - \underline{c}_0.\underline{x}_0)^2 \\ &= P\{\underline{c}.\underline{x} \geq \underline{c}_0.\underline{x}_0\} \end{aligned} \qquad (6.33)$$

known as the E, V and P models.

If $\underline{c}$ is random, $\underline{d}$ deterministic, the linear decision rules (20) may be useful.

The P form gives a simple formulations since

$$P\left\{\frac{\underline{c}.\underline{x} + \alpha}{\underline{d}.\underline{x} + \beta} \geq \theta\right\} \equiv P\left\{(\underline{c} - \theta\underline{d}).\underline{x} + (\alpha - \theta\beta) \geq 0\right\}$$

The fractional and linear P models are identical. Unfortunately, the P model is not easy to solve; but its use in Corporate Planning (maximising the probability of achieving a given return on assets say) is attractive.

6.4.6 Conclusions

Although the fractional objective function presents certain difficulties in Stochastic Programming, the assumption the <u>d</u> is deterministic offers some simplification. Situations in which <u>c</u> is stochastic, <u>d</u> deterministic might represent stochastic return on known investments, etc., and might find some use in Corporate Planning, as might the use of satisficing ratio demands using chance constrained programming.