TOPICS IN REGGE POLE THEORY

by

John David Jenkins

A thesis submitted to the University of London for the degree of Ph.D.

July 1970

Imperial College, London.

ABSTRACT

This thesis is presented in two distinct parts.

In part one, a recurrence relation and a generating function for the Feynman amplitude for the scattering of four massive spinless particles, due to the exchange of a massive spin s particle, described by an (s,0) particle-field of Weinberg, are obtained. These are then used, together with the Van Hove model, to calculate the corresponding Reggeized scattering amplitude. The main contributions to this amplitude are: a Regge pole, a fixed branch point, and the background integral. The fixed branch point is found to play an important role with regard to the singularity structure of the amplitude at t=0, when at least one pair of the masses of the incoming and outgoing particles in the t-channel is equal. This work is based on an article in Il Nuovo Cimento 61A, 721 (1969).

In part two, the $U(6) \otimes U(6) \otimes O(3)$ symmetry scheme is used, through the vector dominance model, to calculate pole graphs for pseudoscalar meson photoproductio on nucleon-octet particles. The results of this calculation, together with the Van Hove model, are then used to calculate Regge pole contributions to the s-channel helicity amplitudes. Finally, Regge cut contributions are introduced by applying absorption corrections to the s-channel helicity amplitudes.

The results of a χ^2 fit to the experimental data for the reaction $\gamma p \rightarrow \pi^+ n$, in which evasive π, ρ, and A_2 Regge pole exchanges, together with absorptive Regge cuts, were considered, are presented. The results obtained demonstrate the important role of the Regge cut contributions at t=0; and the agreement with experiment is good.

ACKNOWLEDGEMENTS

I should like to take this opportunity of thanking Professor P.T. Matthews for his patience and encouragement during the preparation of this work, and for numerous helpful discussions with regard to part one of this thesis.

I should also like to thank the University of London for the support of a University of London Postgraduate Studentship.

Finally, thanks are due to Frances for her typing of this thesis.

CONTENTS

PART ONE

INTRODUCTION

In reference (1) Durand calculated the Feynman amplitude for the scattering of four unequal mass, spinless particles with a spin-s particle exchanged. In that calculation, the spin-s particle was described by a particle-field which transforms under the $(\frac{1}{2}s,\frac{1}{2}s)$ representation of the auxiliary Lorentz group. It was shown that such an exchange contributes to the first s+1 partial-waves in the direct channel. In other words, such an exchange effectively involves the spins 0,1,...,s, rather than just pure spin s.

On-shell the lower-spin contributions vanish, and the exchange is pure spin s as expected. However, the role of the lower-spin contributions becomes evident at the off-shell point t=0. At this point any pure spin exchange is singular in Durand's model, and the singularities in the spin 0,1,...,s-1 contributions combine to cancel the singularity in the spin-s contribution, thus giving a finite overall contribution at t=0.

On Reggeizing the above Feynman amplitude by way of the Van Hove model [(2)], Durand showed that the pure spin s contribution gives rise to the usual Regge pole term, whilst the lower-spin contributions give rise to fixed "daughter" Regge pole terms. Before proceeding, it must be remarked that in a more detailed model which takes into account self-energy insertions in the exchanged-particle propagator, the above fixed "daughter" Regge poles are converted into the usual moving "daughter" Regge poles of Freedman and Wang [(3,4)]. Again the role of either the fixed or moving "daughter" Regge pole terms is to combine and cancel the singularity in the parent Regge pole term. Now it may be noted that the $(\frac{1}{2}s,\frac{1}{2}s)$ particle-field used in references (1) and (3) has many redundant components, whose dependence is given by the equations of motion of the field. These redundant components have no effect on-shell, but off-shell it is precisely they which carry the lower-spin contributions mentioned above.

Thus by using the model Feynman amplitude of reference (1) within the context of the Van Hove model, a connection between the existence of redundant components in the propagated particle-field and the existence of the moving "daughter" Regge poles of Freedman and Wang [4] has been traced.

Bearing in mind the remarks of the previous paragraph, part one of this thesis contains an investigation of the results of carrying out the above programme of Durand [1] using, instead of his propagated particle-field, one which has no redundant components. To this end, a spin s particle is here described by an (s,0) particle-field of Weinberg [5] ; since such a field has no redundant components in a spinor basis.

In chapter 1 a resumé of the Feynman rules for high-spin particles, due to Weinberg [5], is given. This is followed in chapter 2 by an explicit construction of the (s,0) particle-field of Weinberg and its propagator, firstly in a spinor basis, and secondly, for s integral, in a tensor basis. The tensor (s,0) particle-field (s integral) is then used, in preference to the spinor (s,0) particle-field throughout the remainder of part one. This is because a discussion of the non-covariant terms, which appear in the (s,0) particle-field propagator [6], is much simpler in terms of the manifestly covariant tensor particle-field. Chapter 2 closes with a discussion of these contact terms.

In chapter 3 the formalism of chapters 1 and 2 is used to construct the interaction Hamiltonian density required for the calculation of the above Feynman amplitude, and the amplitude is calculated. Chapter 4 contains a discussion of the Feynman amplitude calculated in chapter 3; and this is followed by the Reggeization of that amplitude by means of the Van Hove model. The Reggeized amplitude is then discussed with special reference to the point t=0, and the chapter is closed with a summary and the conclusions of the work of part one.

There are two appendices. Appendix A deals with some properties of covariant matrices which are required in chapters 1, 2, 3; whilst appendix B contains a demonstration of the equivalence of using either a tensor (s,0), or a spinor (s,0), particle-field in the calculation of chapter 3.

The following list of notations will be adhered to throughout part one:

ϵ_{ijk} ($\epsilon_{123}=1$) is the totally antisymmetric rotation group tensor of rank three.

$\epsilon_{\mu\nu\lambda\rho}$ ($\epsilon_{0123}=1$) is the totally antisymmetric Lorentz group tensor of rank four.

$\underline{p}$, $\hat{p}$, p denote respectively the vector $\underline{p}$, a unit vector in its direction, and its magnitude.

$a^\dagger$ is the Hermitian conjugate of the operator a.

a^* is the complex conjugate of the complex number a.

The summation convention is used for all repeated indices, be they spinor, vector or tensor indices.

Equations are numbered from (1) within each chapter. Reference to them is made by their numbers within their chapters of origin, and by their numbers, prefixed by the numbers of their chapters of origin, without them.

CHAPTER 1

The purpose of this chapter is to present explicit Feynman rules for a perturbation-theoretic calculation of the elements of the relativistic S-matrix in a formalism which differs as little as possible from one spin to another.

Section 1 is concerned with the main assumptions on which the formalism is to be based, whilst sections 2-5 describe the evolution of the formalism from these assumptions. Finally section 6 contains an explicit statement of the Feynman rules in the desired form.

Section 1. The assumptions of Weinberg's formalism:

The presentation of the above-mentioned Feynman rules is achieved on the basis of the following three main assumptions:

(i) The S-matrix is assumed to be given by Dyson's formula

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T\left(H_I(t_1) \ldots H_I(t_n)\right) dt_1 \ldots dt_n,$$

where the Hamiltonian, H, has been written as the sum of a free-particle part, H_0, and an interaction part, H_I, and $H_I(t)$ is defined to be the interaction Hamiltonian in the interaction representation. That is

$$H_I(t) = e^{iH_0 t} H_I \, e^{-iH_0 t}$$

(ii) The S-matrix is assumed to be invariant under the transformations of the Poincaré group. A sufficient condition for this invariance is

$$H_I(t) = \int_{-\infty}^{\infty} \mathcal{H}_I(t, \underline{x}) \, d^3x,$$

where:

(a) $\mathcal{H}_I(x)$ is a Poincaré scalar. That is, to each Poincaré transformation $x \to \Lambda x + a$, there corresponds a unitary operator $U(\Lambda, a)$ such that

$$U(\Lambda, a)\,\mathcal{H}_I(x)\,U^{-1}(\Lambda, a) = \mathcal{H}_I(\Lambda x + a).$$

(b) For $(x-y)$ spacelike

$$[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0.$$

In terms of the newly defined interaction Hamiltonian density, $\mathcal{H}_I(x)$, the S-matrix now becomes

$$(1)\quad S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T\{\mathcal{H}_I(x_1) \ldots \mathcal{H}_I(x_n)\}\, d^4x_1 \ldots d^4x_n.$$

By a consideration of the form of the S-matrix given by (1) it is easily seen that the conditions (a) and (b) are together sufficient for the Poincaré invariance of the S-matrix. For the expression (1) is evidently a Poincaré invariant provided that the step functions implicit in the definition of the time-ordered product are Poincaré scalars. Now a step-function is a Poincaré scalar if and only if its argument is timelike or lightlike, and condition (b) guarantees that no step-function with spacelike argument appears. Hence the S-matrix, as given by (1), is a Poincaré invariant.

(iii) The interaction Hamiltonian density is to be constructed from the creation and annihilation operators of the free-particles described by H_0. In order to be certain that $\mathcal{H}_I(x)$, so constructed, will satisfy properties (a) and (b) it is formed as a function of one or more fields, $\psi_a(x)$, which are linear combinations of the creation and annihilation operators, and which have the following properties.

(c) If $U(\Lambda,a)$ is the unitary operator defined above then

$$U(\Lambda,a)\,\Psi_a(x)\,U^{-1}(\Lambda,a) = D_a{}^b(\Lambda^{-1})\,\Psi_b(\Lambda x+a),$$

where $D_a{}^b(\Lambda^{-1})$ is some representation of Λ^{-1}.

(d) For (x-y) spacelike

$$[\Psi_a(x), \Psi_b^\dagger(y)]_\pm = 0,$$

where $\pm$ refer to anticommutator or commutator respectively. these fields must then be coupled in an invariant manner, and such that any product of fields appearing contains an even number of fields which anticommute for spacelike separations.

Thus the first step in the construction of a Poincaré invariant S-matrix involves the construction of particle-fields which satisfy properties (c) and (d). Since the construction of these fields rests heavily on the properties of the Poincaré group, it will be necessary to give a review of some of its properties. This is the subject of the next section.

Section 2. The Poincaré group:

Take any two elements x_1^μ, x_2^μ (μ=0, 1, 2, 3) of a four dimensional real vector space, and consider the group of real, inhomogeneous linear transformations of the vector space into itself which leave invariant the quadratic form

$$g_{\mu\nu}(x_1^\mu - x_2^\mu)(x_1^\nu - x_2^\nu) \tag{2}$$

where the metric tensor $g_{\mu\nu}=g^{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$,

and the summation convention is used. Such a transformation will have the general form

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$$

where $\Lambda^\mu{}_\nu$ and a^μ are independent of x . Invariance of (2) requires that

(3) $$g_{\mu\nu}\Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho = g_{\lambda\rho}.$$

If further to the above it is required that

$$\det(\Lambda^\mu{}_\nu) = 1$$

and

$$\Lambda^0{}_0 \geq 1,$$

then the resulting subgroup is called the Poincaré group.

Poincaré transformations for which $\Lambda^\mu{}_\nu$ is the unit matrix evidently form the group of translations of the vector space, whereas Poincaré transformations for which $a^\mu = 0$ form the group of proper orthochronous Lorentz transformations, henceforth called simply the Lorentz group.

The general infinitesimal Poincaré transformation is given by

(4) $$x'^\mu = x^\mu + \epsilon^{\mu\nu} x_\nu + \epsilon^\mu,$$

where the parameters $\epsilon^{\mu\nu}$ and ϵ^μ are infinitesimal, and from (3)

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}.$$

On any representation space for the Poincaré group the operator corresponding to (4) is defined as

(5) $$1 + \tfrac{i}{2}\epsilon^{\mu\nu} J_{\mu\nu} + i\epsilon^\mu P_\mu,$$

where $J_{\mu\nu}$ and P_μ are the infinitesimal generators of the group in that particular representation. The operator corresponding to a finite Poincaré transformation is obtained by exponentiation, and is given by

(6) $$\exp\left(\tfrac{i}{2}\epsilon^{\mu\nu}J_{\mu\nu} + i\epsilon^{\mu}P_{\mu}\right).$$

By a consideration of a representation space of Poincaré-scalar point functions, for example, explicit representations for $J_{\mu\nu}$ and P_{μ} are obtained, and the following commutation relations deduced.

(7) $$[P_{\mu}, P_{\nu}] = 0.$$

(8) $$[J_{\mu\nu}, J_{\lambda\rho}] = -i(g_{\nu\lambda}J_{\mu\rho} + g_{\mu\rho}J_{\nu\lambda} - g_{\mu\lambda}J_{\nu\rho} - g_{\nu\rho}J_{\mu\lambda}).$$

(9) $$[P_{\lambda}, J_{\mu\nu}] = i(g_{\lambda\nu}P_{\mu} - g_{\lambda\mu}P_{\nu}).$$

The above comprises all the properties of the Poincaré group needed for the present, and the remainder of this section is devoted to a discussion of the properties of the Lorentz subgroup.

From the expressions (4) and (5) it is seen that this subgroup of the Poincaré group is generated by the $J_{\mu\nu}$. However, when dealing with the Lorentz group it is often more convenient to introduce new infinitesimal generators J_i, K_i (i=1,2,3), defined in terms of the $J_{\mu\nu}$ by

(10) $$K_i = J_{0i}$$

(11) $$J_i = -\tfrac{1}{2}\epsilon_{ijk}J_{jk}$$

Whence the commu tation relations of J_i , K_i are

(12) $[J_i , J_j] = i\epsilon_{ijk} J_k$

(13) $[K_i , K_j] = -i\epsilon_{ijk} J_k$

(14) $[K_i , J_j] = i\epsilon_{ijk} K_k$

Note that (12) demonstrates explicitly that the J_i are the infinitesimal generators of the rotation subgroup, whereas the operator

$$e^{-i\eta\cdot\underline{K}}$$

where η is defined by

$$\cosh\eta = (1-v^2)^{-\frac{1}{2}} = \gamma$$

$$\sinh\eta = \gamma v$$

$$\hat{\eta} = \hat{v}$$

corresponds to the Lorentz transformation given by the matrix

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma v_j \\ \gamma v_i & \delta_{ij} + v^{-2}(\gamma-1) v_i v_j \end{pmatrix}$$

This is just the matrix of the Lorentz transformation from one inertial frame to another which is moving with velocity $-\underline{v}$ with respect to the former, the frames being such that their space axes are identically oriented and (0 0 0 0) gives the same event in both frames.

Finally an alternative set of generators, which gives a decomposition of the Lorentz group of use in later chapters, is introduced. Define $\underline{L}$ and $\underline{M}$ by

$$\underline{L} = \frac{1}{2}(\underline{J} + i\underline{K})$$
$$\underline{M} = \frac{1}{2}(\underline{J} - i\underline{K}) .$$

The commutation relations of L_i, M_i are then given by

$$[L_i , L_j] = i\epsilon_{ijk} L_k$$
$$[M_i , M_j] = i\epsilon_{ijk} M_k$$
$$[L_i , M_j] = 0 .$$

Note that the components of $\underline{L}$ and $\underline{M}$ both satisfy the algebra of SU(2), the group of 2x2 unitary matrices with unit determinant, and that $\underline{L}$ and $\underline{M}$ commute. Thus the SU(2) $\otimes$ SU(2) decomposition of the Lorentz group is exhibited.

For integral values of $2j_1$ and $2j_2$ the $(2j_1+1)(2j_2+1)$-dimensional irreducible representation (j_1, j_2) is defined in terms of this decomposition as follows

$$\langle m_1 m_2 | \underline{L} | m_1' m_2' \rangle = \underline{J}^{(j_1)}_{m_1 m_1'} \, \delta_{m_2 m_2'} \tag{16}$$
$$\langle m_1 m_2 | \underline{M} | m_1' m_2' \rangle = \delta_{m_1 m_1'} \, \underline{J}^{(j_2)}_{m_2 m_2'} ,$$

where $\underline{J}^{(j)}_{mm'}$ is just the usual $(2j+1)$- dimensional unitary irreducible representation of the rotation group generators:(7)

$$(J_1 + iJ_2)^{(j)}_{mm'} = \left[(j \mp m')(j \pm m' + 1)\right]^{1/2} \delta_{m\, m'\pm 1} \tag{17}$$
$$(J_3)^{(j)}_{mm'} = m\,\delta_{mm'} .$$

The representations (j_1, j_2), for all integral values of $2j_1$ and $2j_2$, exhaust all the finite-dimensional irreducible representations of the Lorentz group. And in each such representation the infinitesimal generators may be written in terms of the unitary representations of the rotation group generators through equations (16).

The apparatus required for the construction of one-particle physical states, in terms of which creation and annihilation operators are defined, has now been set up, and so these topics form the subject of the next section.

Section 3. One-particle states

A system is said to be elementary if the space of all its possible physical states forms a representation space for a unitary irreducible representation of the Poincaré group. Amongst such systems are those which consist of just one particle which is considered to be elementary, or which it is convenient to treat as being such.

Since the only assumption to be made about one-particle states is the above, namely that they transform under unitary irreducible representations of the Poincaré group, then such a state will be wholly specified by the eigenvalues of a complete commuting subset of the group generators in the corresponding representation.

It follows from the commutation relations of the Poincaré group that the Casimir operator

$$P^{\mu} P_{\mu} = P^{2},$$

together with the generators P_{μ} may be taken as forming part of the complete commuting set. For the purpose of completing the set the Pauli-Lubanski spin operator, W^{μ}, is introduced.[8] It is defined by

$$W^{\mu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} P_{\nu} J_{\lambda\rho}. \tag{18}$$

From the antisymmetry of $\epsilon^{\mu\nu\lambda\rho}$, and the fact that the P_{μ} commute, it follows that

$$P_\mu W^\mu = 0 .$$

Thus W^μ has only three independent components. It also satisfies the following commutation relations

$$(19) \quad [W^\mu, W^\nu] = -i\epsilon^{\mu\nu\lambda\rho} P_\lambda W_\rho$$

$$(20) \quad [W^\mu, P_\nu] = 0$$

$$(21) \quad [W_\mu, J_{\lambda\rho}] = i(g_{\mu\rho} W_\lambda - g_{\mu\lambda} W_\rho) .$$

From these commutation relations it is seen that the Casimir operator

$$W^\mu W_\mu = W^2 ,$$

together with an arbitary component of W, W_3 for example, will complete the commuting set required. Thus the possible physical states of a one-particle system are specified by the eigenvalues of the operators P^2, P_μ, W^2, W_3 in the representation corresponding to the system. These eigenvalues will be denoted by m^2, p_μ, w^2, w_3 respectively, and the one-particle states written

$$|m^2 p_\mu w^2 w_3\rangle$$

The quantities m and p_μ are to be interpreted as the mass and momentum of the particle respectively. The following will only be concerned with the case $m > 0$, and this restriction is to be understood henceforth.

Before dealing with w^2 and w_3, note that the subspace of one-particle states defined by keeping m^2, p_μ, w^2 fixed is invariant under transformations generated by W_μ. Also the algebra of the W_μ, when restricted to this subspace, is closed. Thus the transformations of this subspace into itself, which are generated by the W_μ, form a group. This group is called the little group of the momentum p_μ.

It turns out to be most convenient to specify w^2 and w_3 in the rest-frame of the particle, for then the little group takes on a well-known form, and w^2 and w_3 lend themselves to easy interpretation. In this frame

$$p_\mu = m_\mu = (m\ 0\ 0\ 0),$$

and so W_μ is effectively given by

$$W_\mu = (0 \quad mJ_1 \quad mJ_2 \quad mJ_3)$$

when acting on the states $|m^2\ m_\mu\ w^2\ w_3\rangle$. Thus the little group of m_μ is the rotation group, and the operators W^2 and W_3 reduce to $m^2 \underline{J}^2$ and mJ_3 respectively. The corresponding values of w^2 and w_3 are given by

$$w^2 = m^2 s(s+1)$$

and

$$w_3 = m s_3$$

where s is to be interpreted as the spin of the particle, and s_3 the spin-projection along the 3-axis. And so the final form of the labelling of one-particle states is

$$|m^2\ p_\mu\ s\ s_3\rangle,$$

although this will often be abbreviated to

$$|p_\mu\ s_3\rangle.$$

The one-particle states are assumed to be normalized as follows

$$(22)\quad 2\pi\,\theta(p_0)\,\delta(p^2-m^2)\langle p'_\mu s'_3 | p_\mu s_3\rangle = (2\pi)^4\,\delta^{(4)}(p'_\mu - p_\mu)\,\delta_{s'_3 s_3}.$$

The next consideration is to explore the transformation properties of the one-particle states under the Poincaré group. Since the one-particle states were chosen to be eigenstates of the momentum operator they transform simply under translations. Thus for a translation with parameter l^μ

$$(23)\quad e^{il^\mu P_\mu}|p_\mu s_3\rangle = e^{il^\mu p_\mu}|p_\mu s_3\rangle.$$

Preparatory to a discussion of the more complicated transformation properties of the one-particle states under the Lorentz group, the notion of a boost is introduced. A boost, $B(p_\mu)$, is defined as being the Lorentz transformation from the rest-frame of the particle to a frame in which the particle has momentum p_μ. This allows the possibility of defining one-particle states, of arbitrary momentum p_μ, in terms of the corresponding rest-frame states as follows

$$(24)\quad |p_\mu s_3\rangle = B(p_\mu)|m_\mu s_3\rangle = N(p_\mu)B(p_\mu)|m\, s_3\rangle,$$

where $N(p_\mu)$ is a normalization factor, and the new rest-frame states $|ms_3\rangle$ are normalized as follows

$$(25)\quad \langle m s'_3 | m\, s_3\rangle = \delta_{s'_3 s_3}.$$

Since one-particle states transform under unitary irreducible representations of the Poincare group, $B(p_\mu)$ is here to be understood as a unitary irreducible representation of the corresponding boost. The boost convention used here is

$$B(p_\mu) = e^{-i\underline{\epsilon}\cdot\underline{k}},$$

with ϵ given by

$$\cosh\epsilon = \frac{p_0}{m}$$

$$\text{(26)}\quad \sinh\epsilon = \frac{p}{m}$$

$$\hat{\epsilon} = \hat{p}\,.$$

Now, with the concept of a boost, an explicit discussion of the Lorentz transformation properties of the one-particle states may be given.

Let Λ be a Lorentz transformation which takes the particle momentum from p_μ to p'_μ, and let $U(\Lambda)$ be the unitary operator representing Λ in the space of one-particle states $|p_\mu s_3\rangle$. Then the transformed state is $U(\Lambda)|p_\mu s_3\rangle$. In order to obtain an explicit form for this transformation boosts are introduced as follows

$$U(\Lambda)|p_\mu s_3\rangle = N(p_\mu)B(p'_\mu)B^{-1}(p'_\mu)U(\Lambda)B(p_\mu)|m s_3\rangle,$$

$$= \sum_{s'_3}\int_{-\infty}^{\infty} N(p_\mu)B(p'_\mu)|q_\mu s'_3\rangle \langle q_\mu s'_3|B^{-1}(p'_\mu)U(\Lambda)B(p_\mu)|m s_3\rangle\,\theta(q_0)\,\delta(q^2-m^2)\frac{d^4q}{(2\pi)^3}$$

on the introduction of a complete set of one-particle states. At this stage note that the operator $B^{-1}(p'_\mu)U(\Lambda)B(p_\mu)$ corresponds to a sequence of Lorentz transformations which transform the momentum as follows

$$m_\mu \rightarrow p_\mu \rightarrow p'_\mu \longrightarrow m_\mu\,.$$

Thus this sequence of transformations leaves m_μ invariant, and is hence a transformation of the little group of m_μ, i.e. a rotation. Such a rotation is called a Wigner rotation. As a consequence of this remark the above now reduces to

$$U(\Lambda)|p_\mu s_3\rangle = \sum_{s_3'} N(p_\mu) B(p_\mu') |m_\mu s_3'\rangle\langle m_\mu s_3'| B^{-1}(p_\mu') U(\Lambda) B(p_\mu) |m\, s_3\rangle$$

$$= \sum_{s_3'} N(p_\mu) N^*(p_\mu') |p_\mu' s_3'\rangle \mathcal{D}^{(s)}_{s_3' s_3}\left(B^{-1}(p_\mu') U(\Lambda) B(p_\mu)\right), \tag{27}$$

Where, on account of (25) and the fact that $B^{-1}(p_\mu')U(\Lambda)B(p_\mu)$ is a unitary operator, $\mathcal{D}$ is just the usual rotation matrix. (9) From now on the normalization factor $N(p_\mu)\, N^*(p_\mu')$ in (27) will be assumed to have been absorbed into the rotation matrix.

Now, as the next step in the construction of particle-fields, creation and annihilation operators are introduced. They are defined in terms of the one-particle states and an invariant, non-degenerate vacuum state, $|0\rangle$, by the following relations

$$a^\dagger(p_\mu s_3)|0\rangle = |p_\mu s_3\rangle$$

$$2\pi\, \theta(p_0)\, \delta(p^2-m^2)\, a(p_\mu s_3)|p_\mu' s_3'\rangle = (2\pi)^4\, \delta^{(4)}(p_\mu' - p_\mu)\, \delta_{s_3' s_3} |0\rangle \tag{28}$$

$$a(p_\mu s_3)|0\rangle = 0\,.$$

The creation and annihilation operators are required to satisfy either commutation or anticommutation relations, which, for consistency with (22) and (28), must have the form

$$2\pi\, \theta(p_0)\, \delta(p^2-m^2) \left[a(p_\mu s_3), a^\dagger(p_\mu' s_3')\right]_\pm$$

$$(29) \quad = (2\pi)^4 \delta^{(4)}(p'_\mu - p_\mu)\, \delta_{s'_3 s_3}$$

$$[a(p_\mu s_3), a(p'_\mu s'_3)]_\pm = 0$$

$$[a^\dagger(p_\mu s_3), a^\dagger(p'_\mu s'_3)]_\pm = 0 .$$

The Poincaré transformation properties of $a^\dagger(p_\mu s_3)$ follow immediately from equations (23), (27) and (28). They are

$$(30) \quad e^{i l^\mu P_\mu}\, a^\dagger(p_\mu s_3)\, e^{-i l^\mu P_\mu} = e^{i l^\mu p_\mu}\, a^\dagger(p_\mu s_3)$$

$$(31) \quad e^{-i \underline{\eta} \cdot \underline{K}}\, a^\dagger(p_\mu s_3)\, e^{i \underline{\eta} \cdot \underline{K}} = \sum_{s'_3} a^\dagger(p'_\mu s'_3)\, \mathcal{D}^{(s)}_{s'_3 s_3}\left(e^{i \underline{\epsilon}' \cdot \underline{K}}\, e^{-i \underline{\eta} \cdot \underline{K}}\, e^{-i \underline{\epsilon} \cdot \underline{K}}\right) .$$

Since all the representations involved in (30) and (31) are unitary, the Poincaré transformation properties of $a(p_\mu s_3)$ follow very simply by just taking the Hermitian conjugate of these equations. They are

$$(32) \quad e^{i l^\mu P_\mu}\, a(p_\mu s_3)\, e^{-i l^\mu P_\mu} = e^{-i l^\mu p_\mu}\, a(p_\mu s_3)$$

$$(33) \quad e^{-i \underline{\eta} \cdot \underline{K}}\, a(p_\mu s_3)\, e^{i \underline{\eta} \cdot \underline{K}} = \sum_{s'_3} a(p'_\mu s'_3)\, \mathcal{D}^{(s)}_{s_3 s'_3}\left(e^{i \underline{\epsilon} \cdot \underline{K}}\, e^{i \underline{\eta} \cdot \underline{K}}\, e^{-i \underline{\epsilon}' \cdot \underline{K}}\right)$$

Now that the transformation properties of the creation and annihilation operators are known, only one more concept is needed for the final construction of the particle-fields which satisfy properties (c) and (d) of section 1. This concept is the subject of the next section.

Section 4. Auxiliary groups and operators:

The transformation properties of the creation and annihilation operators under translations suggest that some sort of Fourier transform of them will lead to particle-fields satisfying (c). However the usual Fourier transform will not do, as the momentum dependence of the Lorentz transformation properties of the creation and annihilation operators precludes the desired transformation properties of the fields. In order to overcome this difficulty it is necessary to decouple this momentum dependence. This is achieved by taking explicit representations of the operators in the product corresponding to the Wigner rotation. Since this involves taking explicit representations of the operators $\underline{J}$ and $\underline{K}$, representations of a group which has these amongst its generators are required. Such a group is called an auxiliary group. The simplest example of an auxiliary group is that which has $\underline{J}$ and $\underline{K}$ as its generators, i.e. the Lorentz group. This is the only auxiliary group which will be used in the following.

Let the vectors $|a\rangle$, where a is a collective label, form an orthonormal basis for a representation space of the auxiliary Lorentz group. Then formally

(34) $$\langle a | b \rangle = \delta_{ab}$$

(35) $$|a\rangle\langle a| = 1 .$$

The insertion of complete sets of the vectors $|a\rangle$ into (33) leads to

(36) $$e^{-i\eta\cdot\underline{K}}\, a(p_\mu s_3)\, e^{i\eta\cdot\underline{K}} = \sum_{s'_3} \langle m s_3 | a\rangle \langle a| e^{i\varepsilon\cdot\underline{K}} |b\rangle\langle b| e^{i\eta\cdot\underline{K}} |c\rangle\langle c| e^{-i\varepsilon\cdot\underline{K}} |d\rangle \langle d | m s_3\rangle\, a(p'_\mu s'_3) .$$

In this expression the overlap functions $\langle m s_3 | a\rangle$ must be calculated separately for each representation. For example if $|a\rangle$

corresponds to a finite-dimensional representation labelled in the $SU(2) \otimes SU(2)$ decomposition by

$$|a\rangle = |j_1 s_1 ; j_2 s_2\rangle ,$$

then

$$\langle m s_3 | a\rangle = \langle m s_3 | j_1 s_1 ; j_2 s_2\rangle$$

is just a Clebsch- Gordan coefficient. [9] Note also that the appearance of the overlap functions means that in order to be able to construct non-trivial fields only representations, which, when restricted to the rotation subgroup contain the spin s amongst their components, may be used.

Returning to (36), note that it may be rearranged in the following manner

$$(37) \quad e^{-i\eta\cdot\underline{K}}\Big(\sum_{s_3'} \langle a| e^{-i\underline{\epsilon}\cdot\underline{K}} |b\rangle\langle b|m s_3'\rangle a(p_\mu s_3')\Big)$$

$$e^{i\eta\cdot\underline{K}} = \langle a| e^{i\eta\cdot\underline{K}} |b\rangle \sum_{s_3'} \langle b| e^{-i\underline{\epsilon}'\cdot\underline{K}} |c\rangle\langle c| m s_3'\rangle\, a(p_\mu' s_3') .$$

Thus the operator

$$A_a(p_\mu) = \sum_{s_3} u_a(p_\mu s_3)\, a(p_\mu s_3)$$

where

$$(38) \quad u_a(p_\mu s_3) = \langle a| e^{-i\underline{\epsilon}\cdot\underline{K}} |b\rangle\langle b| m s_3\rangle ,$$

has the following momentum-independent Lorentz transformation property

$$(39) \quad e^{-i\eta\cdot\underline{K}} A_a(p_\mu) e^{i\eta\cdot\underline{K}} = \left(e^{i\eta\cdot\underline{K}}\right)_a{}^b A_b(p_\mu') .$$

The operator $A_a(p_\mu)$ is called an auxiliary operator, and $u_a(p_\mu s_3)$ is the corresponding particle-spinor.

The transformation property of $A_a(p_\mu)$ under translations follows immediately from (32), and it is

$$(40) \quad e^{il^\mu P_\mu} A_a(p_\mu) e^{-il^\mu P_\mu} = e^{-il^\mu p_\mu} A_a(p_\mu).$$

A related auxiliary operator, $A^a(p_\mu)$, the operator dual to $A_a(p_\mu)$, is defined as follows

$$(41) \quad A^a(p_\mu) = \sum_{s_3} u^a(p_\mu s_3)\, a^\dagger(p_\mu s_3)$$
$$= \sum_{s_3} \langle m s_3 | b \rangle \langle b | e^{i\underline{\epsilon}\cdot\underline{K}} | a \rangle a^\dagger(p_\mu s_3).$$

The spinor $u^a(p_\mu s_3)$ is said to be the particle-spinor dual to $u_a(p_\mu s_3)$. The transformation properties of $A^a(p_\mu)$ are as follows

$$(42) \quad e^{-i\underline{\eta}\cdot\underline{K}} A^a(p_\mu) e^{i\underline{\eta}\cdot\underline{K}} = A^b(p'_\mu)\left(e^{-i\underline{\eta}\cdot\underline{K}}\right)_b{}^a$$

$$(43) \quad e^{il^\mu P_\mu} A^a(p_\mu) e^{-il^\mu P_\mu} = e^{il^\mu p_\mu} A^a(p_\mu)$$

Finally auxiliary operators, which will later be seen to correspond to antiparticles, are introduced. They are constructed on the basis of the following observation. If $\underline{J}$ is any representation of the rotation group generators, then there exists a matrix C such that [(9)]

$$(44) \quad C^{-1}\,\underline{J}\,C = -\underline{J}^T,$$

and $-\underline{J}^T$ also satisfies the commutation relations of the rotation group generators. Thus $-\underline{J}^T$ defines another representation of the rotation group which is equivalent to the first. With the aid of this matrix C new auxiliary operators are defined as follows

$$(45)\quad \tilde{B}_a(p_\mu) = \sum_{s_3} v_a(p_\mu s_3)\, b^\dagger(p_\mu s_3)$$
$$= \sum_{s_3' s_3} \langle a|e^{-i\epsilon\cdot\underline{K}}|b\rangle\langle b|m s_3'\rangle C_{s_3' s_3}\, b^\dagger(p_\mu s_3),$$

and its dual

$$(46)\quad \tilde{B}^a(p_\mu) = \sum_{s_3} v^a(p_\mu s_3)\, b(p_\mu s_3)$$
$$= \sum_{s_3' s_3} C^{-1\, s_3 s_3'}\langle m s_3'|b\rangle\langle b|e^{i\epsilon\cdot\underline{K}}|a\rangle b(p_\mu s_3),$$

where the antiparticle creation and annihilation operators, $b^\dagger(p_\mu s_3)$ and $b(p_\mu s_3)$, are assumed to satisfy anticommutation or commutation relations analogous to (29), and the quantities $v_a(p_\mu s_3)$ and $v^a(p_\mu s_3)$ are said to be the antiparticle-spinor and its dual antiparticle-spinor, respectively. The transformation properties of these auxiliary operators are

$$(47)\quad e^{-i\eta\cdot\underline{K}}\,\tilde{B}_a(p_\mu)\,e^{i\eta\cdot\underline{K}} = \left(e^{i\eta\cdot\underline{K}}\right)_a{}^b\,\tilde{B}_b(p_\mu')$$

$$(48)\quad e^{il^\mu P_\mu}\,\tilde{B}_a(p_\mu)\,e^{-il^\mu P_\mu} = e^{il^\mu p_\mu}\,\tilde{B}_a(p_\mu),$$

and

$$(49)\quad e^{-i\eta\cdot\underline{K}}\,\tilde{B}^a(p_\mu)\,e^{i\eta\cdot\underline{K}} = \tilde{B}^b(p_\mu')\left(e^{-i\eta\cdot\underline{K}}\right)_b{}^a$$

$$(50)\quad e^{il^\mu P_\mu}\,\tilde{B}^a(p_\mu)\,e^{-il^\mu P_\mu} = e^{-il^\mu p_\mu}\,\tilde{B}^a(p_\mu).$$

Now particle-fields may be constructed by taking appropriate linear combinations of these auxiliary operators. This is done, and the properties of the fields explored, in the next section.

Section 5. Particle-fields:

The first step in the construction of the fields is to find the appropriate combinations of the auxil-

iary operators which satisfy the transformation property (c).
For the lower index case an inspection of equations (39), (40), (47) and (48) reveals that the field must be of the form

$$(51)\quad \psi_a(x) = \int_{-\infty}^{\infty}\left(\xi\, A_a(p_\mu)\,e^{-ip\cdot x} + \eta\,\tilde{B}_a(p_\mu)\,e^{ip\cdot x}\right)(2\pi)^{-3}\theta(p_0)\,\delta(p^2-m^2)\,d^4p\,.$$

Whilst for the upper index case a similar inspection of equations (42), (43), (49) and (50) gives

$$(52)\quad \psi^a(x) = \int_{-\infty}^{\infty}\left(\xi\, A^a(p_\mu)\,e^{ip\cdot x} + \eta\,\tilde{B}^a(p_\mu)\,e^{-ip\cdot x}\right)(2\pi)^{-3}\theta(p_0)\,\delta(p^2-m^2)\,d^4p\,.$$

In both of these expressions ξ and η are invariant functions of p_μ which will be determined by enforcing property (d) on the fields. For the lower index case consider

$$[\psi_a(x),\ \psi_b^\dagger(y)]_\pm\,.$$

Substitution of the explicit form (51) , and use of the commutation or anticommutation relations of the particle and antiparticle creation and annihilation operators leads to

$$\int_{-\infty}^{\infty}\sum_{s_3} u_a(p_\mu s_3)\,u_b^*(p_\mu s_3)\left(|\xi|^2 e^{-ip\cdot(x-y)} \pm |\eta|^2 e^{ip\cdot(x-y)}\right)(2\pi)^{-3}\theta(p_0)\,\delta(p^2-m^2)\,d^4p\,.$$

At this stage it should be noted that

$$\Pi_{ab}(p_\mu) = \sum_{s_3} u_a(p_\mu s_3)\,u_b^*(p_\mu s_3)$$

has the following property, which may be deduced from Koller's explicit form for the spinors,[10]

$$(53) \quad \Pi_{ab}(-p_\mu) = (-1)^{2s}\, \Pi_{ab}(p_\mu).$$

So now the above becomes

$$\Pi_{ab}(i\partial_\mu)\int_{-\infty}^{\infty}\Big(|\xi|^2 e^{-ip\cdot(x-y)} \pm (-1)^{2s} |\eta|^2 e^{ip\cdot(x-y)}\Big)\theta(p_0)(2\pi)^{-3}\delta(p^2-m^2)\,d^4p.$$

Now it is well known that such an integral vanishes outside the light cone if and only if

$$|\xi|^2 = \mp(-1)^{2s}\,|\eta|^2. \quad (11)$$

An immediate consequence is

$$\mp(-1)^{2s} = 1.$$

Thus if s is integral the plus sign, hence commutator, must be taken; whilst if s is half-integral the minus sign, hence anti-commutator, must be taken. And so this formalism leads to the usual connection between spin and statistics. A second consequence is

$$|\xi| = |\eta|.$$

Thus every particle has an antiparticle which enters into interactions with equal coupling strength. A redefinition of the phases of the creation and annihilation operators allows the choice

$$\xi = \eta = 1.$$

The field now has its final form

$$(54) \quad \psi_a(x) = \int_{-\infty}^{\infty}\sum_{s_3}\Big(u_a(p_\mu s_3)\,a(p_\mu s_3)\,e^{-ip\cdot x} + v_a(p_\mu s_3)\,b^\dagger(p_\mu s_3)\,e^{ip\cdot x}\Big)(2\pi)^{-3}\theta(p_0)\,\delta(p^2-m^2)\,d^4p.$$

And its commutator or anticommutator is given by

$$(55)\quad [\psi_a(x), \psi_b^\dagger(y)]_\pm = i\Pi_{ab}(i\partial_\mu)\Delta(x-y),$$

where

$$(56)\quad \Delta(x) = i\int_{-\infty}^{\infty}(e^{-ip\cdot x} - e^{ip\cdot x})\theta(p_0)\delta(p^2-m^2)\frac{d^4p}{(2\pi)^3}.$$

Only one more quantity is now needed in order that an explicit exposition of the Feynman rules may be given; it is the propagator of the field $\psi_a(x)$, and is defined as

$$\langle 0|T\{\psi_a(x)\psi_b^\dagger(y)\}|0\rangle.$$

A substitution of the explicit form (51) quickly leads to

$$\int_{-\infty}^{\infty}\Pi_{ab}(p_\mu)\left(\theta(x_0-y_0)e^{-ip\cdot(x-y)} + (-1)^{2s}\theta(y_0-x_0)\right.$$
$$\left. e^{ip\cdot(x-y)}\right)(2\pi)^{-3}\theta(p_0)\delta(p^2-m^2)\,d^4p.$$

Use of (53) now gives

$$\Pi_{ab}(i\partial_\mu)\int_{-\infty}^{\infty}\left(\theta(x_0-y_0)e^{-ip\cdot(x-y)} + \theta(y_0-x_0)\right.$$
$$\left. e^{ip\cdot(x-y)}\right)(2\pi)^{-3}\theta(p_0)\delta(p^2-m^2)\,d^4p$$

+ non-covariant contact terms.

The non-covariant contact terms arise as a consequence of commuting the derivatives past the θ-functions. Now there is a theorem of Matthews[6] which states that it is possible to cancel these non-covariant terms by the addition of appropriate non-covariant contact interactions to $\mathcal{H}_I(x)$, and that these contact interactions may be neglected in an exposition of the Feynman rules, except in that they cancel the undesirable non-covariant contact terms. So, neglecting these terms, the above becomes[12]

$$i\Pi_{ab}(i\partial_\mu)\int_{-\infty}^{\infty}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\frac{d^4p}{(2\pi)^4}$$

$$=i\int_{-\infty}^{\infty}\frac{\Pi_{ab}(p_\mu)\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\frac{d^4p}{(2\pi)^4}$$

Thus the propagator in momentum space is

$$\frac{i\,\Pi_{ab}(p_\mu)}{p^2-m^2+i\epsilon} \tag{57}$$

A procedure completely analogous to the above few paragraphs may also be carried out for the upper index field. The details are ommitted.

The complete framework required for an exposition of the Feynman rules has now been set up, and their explicit statement is given in the next section.

Section 6. The Feynman rules

The Feynman rules will, by way of illustration, be given for the particular example of a three particle vertex which is described by an interaction Hamiltonian density of the form

$$\mathcal{H}_I(x) = g\,A^{abc}\,\psi_a^{(1)}(x)\,\psi_b^{(2)}(x)\,\psi_c^{(3)}(x)$$

+ Hermitian conjugate.

In this expression g is the coupling constant at the vertex, and A^{abc} is some function of invariant quantities which may, or may not, contain derivatives. The S-matrix is now calculated from (1), using Wick's theorem as usual to derive the Feynman rules.

(i) For each vertex include a factor

$$-igA^{abc},$$

where for every $i\partial_\mu$ which appears in A^{abc} is substituted the momentum of the particle on whose field the derivative acts. Contributions from non-covariant contact interactions are to be ignored.

(ii) For each internal line include the covariant part of the propagator

$$\langle 0 | T(\psi_a(x)\psi_b^\dagger(y)) | 0 \rangle .$$

(iii) For external lines include the following quantities:

$$u_\alpha(p_\mu A)\, e^{-ip\cdot x} \quad - \quad \text{particle destroyed}$$

$$u_\alpha^*(p_\mu A)\, e^{ip\cdot x} \quad - \quad \text{particle created}$$

$$v_\alpha^*(p_\mu A)\, e^{-ip\cdot x} \quad - \quad \text{antiparticle destroyed}$$

$$v_\alpha(p_\mu A)\, e^{ip\cdot x} \quad - \quad \text{antiparticle created.}$$

(iv) Integrate over all vertex positions, and sum over all repeated dummy indices.

(v) Supply a minus sign for each closed fermion loop.

With the above statement of the Feynman rules all the basic material required for the ensuing calculation is now at hand.

CHAPTER 2

The primary aim of this chapter is the explicit construction of particle-fields which describe a particle of spin s, and which transform under either the $(s,0)$ or $(0,s)$ representations of the auxiliary Lorentz group. The spinors corresponding to these fields will be constructed from the basic $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ particle-spinors respectively. Consequently section 1 is concerned with a detailed account of the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the auxiliary Lorentz group.

Section 1. $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations

In the unitary irreducible representation of the rotation group with weight $\frac{1}{2}$ the infinitesimal generators are given by the three Pauli spin-matrices $\frac{1}{2}\underline{\sigma} = \frac{1}{2}(\sigma_1 \ \sigma_2 \ \sigma_3)$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus equations 1(16) lead the lower $(\frac{1}{2},0)$ representation of the Lorentz group to be defined by

$$\tfrac{1}{2}(\underline{J} + i\underline{K})_a{}^b = \tfrac{1}{2}(\underline{\sigma})_a{}^b$$
$$\tfrac{1}{2}(\underline{J} - i\underline{K})_a{}^b = 0.$$

Consequently in this representation

$$(1) \qquad (\underline{J})_a{}^b = \tfrac{1}{2}(\underline{\sigma})_a{}^b \quad , \quad (\underline{K})_a{}^b = -\tfrac{i}{2}(\underline{\sigma})_a{}^b$$

As noted in 1(44), $-\frac{1}{2}(\underline{\sigma}^T)^b{}_a$ gives a representation of the rotation group generators which is equivalent to that given by $\frac{1}{2}(\sigma)_a{}^b$. Consequently a representation of the Lorentz group equivalent to the lower $(\frac{1}{2},0)$ representation is defined as follows

$$\tfrac{1}{2}(\underline{J} + i\underline{K})^b{}_a = -\tfrac{1}{2}(\underline{\sigma}^T)^b{}_a$$

$$\tfrac{1}{2}(\underline{J} - i\underline{K})^b{}_a = 0 .$$

This representation is called the upper $(\frac{1}{2},0)$ representation, and in it the infinitesimal generators are given by

$$(2)\quad (\underline{J})^b{}_a = -\tfrac{1}{2}(\underline{\sigma}^T)^b{}_a \;, \quad (\underline{K})^b{}_a = \tfrac{i}{2}(\underline{\sigma}^T)^b{}_a .$$

The equivalence of the upper and lower $(\frac{1}{2},0)$ representations is expressed by 1(44) for the Pauli matrices, namely

$$C^{-1\,ab}(\underline{\sigma})_b{}^c\, C_{cd} = -(\underline{\sigma}^T)^a{}_d \;,$$

where the form of the Pauli matrices readily gives

$$(3)\quad C_{ab} = (-1)^{1/2-a}\,\delta_{a\,-b}$$

and

$$(4)\quad C^T = -C .$$

Thus it is seen that the matrices C^{-1} and C respectively raise and lower $(\frac{1}{2},0)$ representation indices. In the following such indices, be they upper or lower, will be referred to as undotted indices, and the corresponding representations as undotted representations.

In a similar fashion the lower $(0,\frac{1}{2})$ representation is defined by

$$\tfrac{1}{2}(\underline{J} + i\underline{K})_{\dot a}{}^{\dot b} = 0$$

$$\tfrac{1}{2}(\underline{J} - i\underline{K})_{\dot a}{}^{\dot b} = -\tfrac{1}{2}(\underline{\sigma}^T)_{\dot a}{}^{\dot b} \;,$$

whence

(5) $(\underline{J})_{\dot a}{}^{\dot b} = -\frac{1}{2}(\underline{\sigma}^T)_{\dot a}{}^{\dot b}$ and $(\underline{K})_{\dot a}{}^{\dot b} = -\frac{i}{2}(\underline{\sigma}^T)_{\dot a}{}^{\dot b}$.

The equivalent upper $(0,\frac{1}{2})$ representation is accordingly defined by

$$\frac{1}{2}(\underline{J}+i\underline{K})^{\dot b}{}_{\dot a} = 0$$

$$\frac{1}{2}(\underline{J}-i\underline{K})^{\dot b}{}_{\dot a} = \frac{1}{2}(\underline{\sigma})^{\dot b}{}_{\dot a},$$

whence

(6) $(\underline{J})^{\dot b}{}_{\dot a} = \frac{1}{2}(\underline{\sigma})^{\dot b}{}_{\dot a}$ and $(\underline{K})^{\dot b}{}_{\dot a} = \frac{i}{2}(\underline{\sigma})^{\dot b}{}_{\dot a}$.

In this case the raising and lowering matrices are numerically the same as in the undotted case, though they are said now to raise and lower dotted indices. The $(0,\frac{1}{2})$ representations are accordingly called dotted representations.

If D_a , D^a , $D_{\dot a}$ and $D^{\dot a}$ are four quantities which transform as their indices suggest, then their transforms under the Lorentz transformation Λ are given explicitly by (1), (2), (5) and (6) as follows

(7) $D'_a = (e^{-\frac{1}{2}\eta\cdot\sigma})_a{}^b D_b,$

(8) $D'^a = D^b(e^{\frac{1}{2}\eta\cdot\sigma})_b{}^a,$

(9) $D'_{\dot a} = D_{\dot b}(e^{-\frac{1}{2}\eta\cdot\sigma})^{\dot b}{}_{\dot a},$

(10) $D'^{\dot a} = (e^{\frac{1}{2}\eta\cdot\sigma})^{\dot a}{}_{\dot b} D^{\dot b}.$

<u>Section 2. $(\frac{1}{2},0)$ spinors and fields.</u>

From (7) it is seen that in

the $(\frac{1}{2},0)$ representation the boost $B(p_\mu)$ is given by

$$\left(e^{-\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_a{}^b .$$

Since also, as stated in section 4 of chapter 1, the overlap function $\langle a|ms_3\rangle$ in this representation is just the Clebsch- Gordan coefficient

$$\langle \tfrac{1}{2}a\,0\,0|\tfrac{1}{2}s_3\rangle = \delta_a{}^{s_3},$$

the particle-spinor is given by

$$(11)\quad u_a(p_\mu s_3) = \left(e^{-\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_a{}^b \delta_b{}^{s_3}$$
$$= \left(e^{-\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_a{}^{s_3} .$$

The next few paragraphs will be concerned with showing that once the expression (11) for the particle-spinor is known, then all the other relevant $(\frac{1}{2},0)$ spinors may be constructed from (11) together with the aid of the raising and lowering matrices.

From 1(45) the $(\frac{1}{2},0)$ antiparticle-spinor is given by

$$(12)\quad v_a(p_\mu s_3) = u_a(p_\mu s_3')\,C_{s_3's_3},$$

where the summation convention has now been extended to include spin indices. This practice will be pursued throughout the rest of part one. Now the matrix C, although it acts in a different space from the raising and lowering matrices, is seen from 1(44) to be numerically the same as the lowering matrix for undotted indices. In fact it is the lowering matrix for the spin-projection labels, and C^{-1} is accordingly the raising matrix for such labels. Thus with what is essentially the lowering matrix for undotted indices the lower $(\frac{1}{2},0)$ antiparticle-spinor may be constructed from (11).

In order to construct the dual particle-spinor consider

$$C^{-1\,ab}\,v_b(p_\mu s_3) = C^{-1\,ab}\left(e^{-\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_b{}^{s_3'}C_{s_3's_3}$$
$$= \left(e^{\frac{1}{2}\underline{\epsilon}.\underline{\sigma}^T}\right)^a{}_{s_3} \qquad \text{by } 1(44)$$
$$= \left(e^{\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_{s_3}{}^a$$

(13) i.e. $$u^a(p_\mu s_3) = C^{-1\,ab}\,v_b(p_\mu s_3).$$

Finally the dual antiparticle-spinor is constructed by a consideration of

$$C^{-1\,ab}\,u_b(p_\mu s_3) = C^{-1\,ab}\left(e^{-\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_b{}^{s_3}$$
$$= \left(e^{\frac{1}{2}\underline{\epsilon}.\underline{\sigma}^T}\right)^a{}_{s_3'}\,C^{-1\,s_3's_3} \qquad \text{by } 1(44)$$
$$= -C^{-1\,s_3s_3'}\left(e^{\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_{s_3'}{}^a \qquad \text{by } (4)$$
$$= -v^a(p_\mu s_3)$$

(14) i.e. $$v^a(p_\mu s_3) = -\,C^{-1\,ab}\,u_b(p_\mu s_3).$$

Thus it is seen that with the aid of the raising and lowering matrices the particle-spinor can be used to construct the corresponding antiparticle-spinor, and the spinors dual to both. So from now on only particle spinors will be studied in any detail.

A result that will be of use later is that if $u_\alpha(p_\mu s_3)$ is a $(\frac{1}{2},0)$ particle-spinor then $u_\alpha^*(p_\mu s_3)$ transforms under the lower $(0,\frac{1}{2})$ representation, and

(15) $$u_\alpha^*(p_\mu s_3) = -\,v_{\dot\alpha}(p_\mu s_3).$$

The proof is as follows:

The transform of $u_\alpha^*(p_\mu s_3)$ is

$$\{(e^{-\frac{1}{2}\eta\cdot\sigma})_a{}^b u_b(p_\mu s_3)\}^*$$
$$= (e^{-\frac{1}{2}\eta\cdot\sigma^T})_a{}^b u_b^*(p_\mu s_3),$$

since, as may be seen from their explicit form, the Pauli matrices satisfy $\sigma^* = \sigma^T$,

$$= u_b^*(p_\mu s_3)\,(e^{-\frac{1}{2}\eta\cdot\sigma})^b{}_a .$$

But from (9) this is seen to be just the manner in which a lower $(0,\frac{1}{2})$ quantity transforms. This completes the first part of the assertion, and the second follows from an inspection of the explicit forms of $u_a^*(p_\mu s_3)$ and $v_{\dot{a}}(p_\mu s_3)$.

In a similar fashion it may be shown that

$$(16) \quad v_a^*(p_\mu s_3) = u_{\dot{a}}(p_\mu s_3).$$

Now that the lower $(\frac{1}{2},0)$ spinors have been effectively constructed in equations (11) and (12), 1(54) may be used to write the corresponding field as

$$(17) \quad \psi_a(x) = \int_{-\infty}^{\infty} \Big(u_a(p_\mu s_3)\, a(p_\mu s_3)\, e^{-ip\cdot x} + v_a(p_\mu s_3)\, b^\dagger(p_\mu s_3)\, e^{ip\cdot x} \Big)\theta(p_0)\,\delta(p^2-m^2)\,\frac{d^4p}{(2\pi)^3}.$$

The propagator corresponding to this field is given in momentum space by 1(57) as follows

$$i\,\frac{u_a(p_\mu s_3)\, u_b^*(p_\mu s_3)}{p^2 - m^2 + i\epsilon}\,.$$

The quantity $u_a(p_\mu s_3)u_b^*(p_\mu s_3)$ will be denoted by $\Pi_{a\dot{b}}(p_\mu)$, where the dotted index is a consequence of the transformation property of $u_b^*(p_\mu s_3)$ proved above. It is given explicitly by

$$\Pi_{a\dot{b}}(p_\mu) = \left(e^{-\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_a{}^{s_3}\left(e^{-\frac{1}{2}\underline{\epsilon}.\underline{\sigma}}\right)_{s_3\dot{b}}$$
$$= \left(e^{-\underline{\epsilon}.\underline{\sigma}}\right)_{a\dot{b}}$$
$$= (\cosh\epsilon - \hat{\epsilon}.\underline{\sigma}\sinh\epsilon)_{a\dot{b}}$$

as a consequence of the commutation and anticommutation relations of the Pauli matrices which are given in appendix A,

$$= \frac{1}{m}(p_0 - \underline{p}.\underline{\sigma})_{a\dot{b}}$$
$$= \frac{p^\mu}{m}(\sigma_\mu)_{a\dot{b}},$$

where the covariant matrix $(\sigma_\mu)_{a\dot{b}}$ is defined and discussed in appendix A.

An analysis similar to the above may be carried through for $(0,\frac{1}{2})$ spinors and fields. In this case the particle-field with an upper dotted index is given by

$$(18)\quad \psi^{\dot{a}}(x) = \int_{-\infty}^{\infty}\Big(u^{\dot{a}}(p_\mu s_3)\,a^\dagger(p_\mu s_3)\,e^{ip\cdot x} + v^{\dot{a}}(p_\mu s_3)\,b(p_\mu s_3)\,e^{-ip\cdot x}\Big)\theta(p_0)\,\delta(p^2-m^2)\frac{d^4p}{(2\pi)^3},$$

and the propagator for this field is given by

$$\frac{i\,\tilde{\Pi}^{\dot{a}b}(p_\mu)}{p^2-m^2+i\epsilon}.$$

The explicit form of $\tilde{\Pi}^{\dot{a}b}(p_\mu)$ is given by

$$\tilde{\Pi}^{\dot{a}b}(p_\mu) = u^{\dot{a}}(p_\mu s_3)\,u^{\dot{b}*}(p_\mu s_3)$$

$$= \left(e^{\underline{\epsilon}\cdot\underline{\sigma}}\right)^{\dot{a}b}$$
$$= \left(\cosh\epsilon + \hat{\underline{\epsilon}}\cdot\underline{\sigma}\sinh\epsilon\right)^{\dot{a}b}$$
$$= \frac{1}{m}\left(p_0 + \underline{p}\cdot\underline{\sigma}\right)^{\dot{a}b}$$
$$= \frac{p^\mu}{m}\left(\tilde{\sigma}_\mu\right)^{\dot{a}b} ,$$

where again the covariant matrix $(\tilde{\sigma}_\mu)^{\dot{a}b}$ is defined and discussed in appendix A.

Now that the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ fields and their propagators have been constructed, the basic apparatus for the construction of the corresponding $(s,0)$ and $(0,s)$ quantities is available. This construction is the subject of the next section.

Section 3. (s,0) spinors and fields:

As seen in section 2 for the $(\frac{1}{2},0)$ case, once the particle-spinor has been constructed all the other relevant spinors may be constructed from it with the aid of the raising and lowering matrices. Since the $(s,0)$ particle-spinor is to be constructed as a multispinor from 2s $(\frac{1}{2},0)$ particle-spinors, then the indices may be raised or lowered one at a time to obtain all the other spinors which may be required. So, as in section 2, attention is restricted to the particle spinor.

Henceforth, for notational convenience, the basic $(\frac{1}{2},0)$ particle-spinor $U_\alpha(p_\mu s_3)$ will be written as $U_\alpha(p_\mu s)$.

Note that the quantity

$$\sum_{\substack{\text{all permutations} \\ P(a_i)}} U_{P(a_1)}(p_\mu s_1)\cdots U_{P(a_{2s})}(p_\mu s_{2s})$$

is totally symmetric in the 2s undotted spinor indices $a_1 \ldots a_{2s}$; and thus it transforms under the irreducible representation of the auxiliary Lorentz group labelled by (s,0).[7] In order to connect this quantity with the required (s,0) particle-spinor it is only necessary to introduce the correct coupling of the spin-projection indices $s_1 \ldots s_{2s}$. The coupling coefficient is given by

$$(19) \quad \langle s_1 \cdots s_{2s} \mid s\, A \rangle = \left\{ \frac{(s-A)!\,(s+A)!}{(2s)!} \right\}^{1/2} \delta_{\sum_i s_i\, A} .$$

The proof, by induction, is as follows:

For s=1 (19) is just the Clebsch-Gordan coefficient [9]

$$\langle \tfrac{1}{2} s_1\, \tfrac{1}{2} s_2 \mid 1\, A \rangle = \left\{ \frac{(1-A)!\,(1+A)!}{2!} \right\}^{1/2} \delta_{s_1+s_2\, A} .$$

Now note that 2s spin $\frac{1}{2}$ states couple only to a spin s state if and only if the coupling is totally symmetric in the spin $\frac{1}{2}$ states [7]. Hence

$$\langle s_1 \cdots s_{2s} | s\, A \rangle = \frac{1}{2s} \sum_{r=1}^{2s} \left(\langle s_r | \langle s_1 \ldots s_{r-1} s_{r+1} \ldots s_{2s} | \right) | s\, A \rangle ,$$

where $\langle s_1 \ldots s_{r-1}\; s_{r+1} \ldots s_{2s} |$ is totally symmetric in its (2s-1) labels, and hence only couples to spin $s-\frac{1}{2}$. Thus

$$\langle s_1 \cdots s_{2s} | s A \rangle = \frac{1}{2s} \sum_{r=1}^{2s} \langle s_1 \ldots s_{r-1} s_{r+1} \ldots s_{2s} | s-\tfrac{1}{2}\; u \rangle \langle \tfrac{1}{2}\; s_r\;\; s-\tfrac{1}{2}\;\; u | s\;\; A \rangle ,$$

where the Clebsch-Gordan coefficient $\langle \frac{1}{2}\; s_r\; s-\frac{1}{2}\; U | s\; A \rangle$ is given explicitly by[9]

$$\langle \tfrac{1}{2}\, s_r \;\; s-\tfrac{1}{2} \;\; u | s A\rangle = \left\{ \frac{(s-A)!\,(s+A)!}{(s-\frac{1}{2}-u)!\,(s-\frac{1}{2}+u)!\,(2s)!} \right.$$

$$\left. \frac{(2s-1)!}{(\frac{1}{2}+s_r)!\,(\frac{1}{2}-s_r)!} \right\}^{1/2} \delta_{u+s_r\, A} \, ,$$

and the inductive hypothesis gives

$$\langle s_1 \ldots s_{r-1} s_{r+1} \ldots s_{2s} | s-\tfrac{1}{2} \;\; u\rangle = \left\{ \frac{(s-\frac{1}{2}-u)!\,(s-\frac{1}{2}+u)!}{(2s-1)!} \right\}^{1/2} \delta_{\sum\limits_{i \neq r} s_i\, u} .$$

Substitution of these two explicit forms into the above gives

$$\langle s_1 \ldots s_{2s} | s A\rangle = \frac{1}{2s} \sum_{r=1}^{2s} \left\{ \frac{(s-A)!\,(s+A)!}{(2s)!} \right\}^{1/2} \delta_{\sum\limits_i s_i\; A}$$

$$= \left\{ \frac{(s-A)!\,(s+A)!}{(2s)!} \right\}^{1/2} \delta_{\sum\limits_i s_i\; A} .$$

Q.E.D.

So using the expression (19) for the spin-coupling coefficient the normalized lower $(s,0)$ particle-spinor is given by

$$\frac{1}{(2s)!} \sum_{P(a_i)} u_{P(a_1)}(p_\mu s_1) \ldots u_{P(a_{2s})}(p_\mu s_{2s}) \langle s_1 \ldots s_{2s} | s A\rangle$$

$$= u_{a_1}(p_\mu s_1) \ldots u_{a_{2s}}(p_\mu s_{2s}) \langle s_1 \cdots s_{2s} | s A\rangle ,$$

since the spin coupling coefficient is totally symmetric in the s_i . Thus

$$(20) \quad u_{a_1 \ldots a_{2s}}(p_\mu A) = u_{a_1}(p_\mu s_1) \cdots u_{a_{2s}}(p_\mu s_{2s}) \langle s_1 \cdots s_{2s} | s A\rangle$$

is the required particle-spinor in a basis of the $(s,0)$ representation space labelled by the a_i .

Since the only non-trivial SU(2) representation involved in the (s,0) representation is that with weight s, the spin-coupling coefficient (19) is also the coupling coefficient for 2s $(\frac{1}{2},0)$ labels to a single (s,0) label. This may be easily verified by the following consideration of the transformation properties of $\langle sa|a_1 \ldots a_{2s}\rangle$. Look at

$$(e^{-i\eta\cdot\underline{K}})_a{}^b \langle sb|a_1 \ldots a_{2s}\rangle$$
$$= (e^{-\eta\cdot\underline{J}})_a{}^b \langle sb|a_1 \ldots a_{2s}\rangle$$

since $\underline{J} = i\underline{K}$ in the (s,0) representation. Now note that $(\underline{J})_a{}^b$ is a representation of the rotation group generators, and that $\langle sb|a_1 \ldots a_{2s}\rangle$ is just the transformation matrix from a basis for the corresponding representation space labelled by $|a_1 \ldots a_{2s}\rangle$ to one labelled by $|sb\rangle$. Hence the above becomes

$$= \langle sa|b_1 \ldots b_{2s}\rangle (e^{-\eta\cdot\underline{J}})_{b_1}{}^{a_1} \cdots (e^{-\eta\cdot\underline{J}})_{b_{2s}}{}^{a_{2s}} .$$

After a rearrangment this equality may be written as

$$(e^{-\eta\cdot\underline{J}})_a{}^b \langle sb|b_1 \ldots b_{2s}\rangle (e^{\eta\cdot\underline{J}})_{b_1}{}^{a_1} \cdots$$
$$(e^{\eta\cdot\underline{J}})_{b_{2s}}{}^{a_{2s}} = \langle sa|a_1 \cdots a_{2s}\rangle ,$$

which demonstrates explicitly that $\langle sa|a_1 \ldots a_{2s}\rangle$ is an invariant quantity transforming as shown below

$$\langle sa|a_1 \cdots a_{2s}\rangle = (s;\tfrac{1}{2} \cdots \tfrac{1}{2})_a{}^{a_1 \cdots a_{2s}}$$

Thus in this new basis the (s,0) particle-spinor is written as

$$u_a(p_\mu A) = (s;\tfrac{1}{2} \cdots \tfrac{1}{2})_a{}^{a_1 \cdots a_{2s}} u_{a_1 \ldots a_{2s}}(p_\mu A).$$

By exactly similar arguments it may be shown that

$$\langle sa|a_1 \cdots a_{2s}\rangle = (s;\tfrac{1}{2} \cdots \tfrac{1}{2})^{\dot{a}_1 \cdots \dot{a}_{2s}}{}_{\dot{a}}$$

is an invariant quantity which transforms as its indices suggest. Finally

$$
\begin{aligned}
(s;\tfrac{1}{2}\cdots\tfrac{1}{2})_{\dot a_1\cdots\dot a_{2s}}^{\dot a} &= (s;\tfrac{1}{2}\cdots\tfrac{1}{2})^{a}{}_{a_1\cdots a_{2s}} \\
&= C^{-1ab}C_{a_1b_1}\cdots C_{a_{2s}b_{2s}}(s;\tfrac{1}{2}\cdots\tfrac{1}{2})_{b}{}^{b_1\cdots b_{2s}} \\
&= (-1)^{-s+a}(-1)^{s-\sum_i a_i}(s;\tfrac{1}{2}\cdots\tfrac{1}{2})_{-a}{}^{-a_1\cdots -a_{2s}} \\
&= (s;\tfrac{1}{2}\cdots\tfrac{1}{2})_{-a}{}^{-a_1\cdots -a_{2s}} \\
&= (s;\tfrac{1}{2}\cdots\tfrac{1}{2})_{a}{}^{a_1\cdots a_{2s}}\,,
\end{aligned}
$$

where the last two equalities followed from the explicit form (19).

So now the lower $(s,0)$ particle-field may be written as

$$
\begin{aligned}
(21)\quad \psi_a(x) &= (s;\tfrac{1}{2}\cdots\tfrac{1}{2})_{a}{}^{a_1\cdots a_{2s}}\psi_{a_1\cdots a_{2s}}(x) \\
&= (s;\tfrac{1}{2}\cdots\tfrac{1}{2})_{a}{}^{a_1\cdots a_{2s}}\int_{-\infty}^{\infty}\theta(p_0)\,\delta(p^2-m^2) \\
&\quad \big(u_{a_1\cdots a_{2s}}(p\mu A)\,a(p\mu A)\,e^{-ip\cdot x} + \\
&\quad u_{a_1\cdots a_{2s}}(p\mu A')C_{A'A}\,b^{\dagger}(p\mu A)\,e^{ip\cdot x}\big)\frac{d^4p}{(2\pi)^3},
\end{aligned}
$$

with analogous expressions for all the other relevant $(s,0)$ and $(0,s)$ particle fields. Henceforth all $(s,0)$ and $(0,s)$ quantities will be written in bases labelled by $2s$ undotted and dotted indices respectively. With this notation the propagator corresponding to (21) is given by I(57) as

$$
i\,\frac{u_{a_1\cdots a_{2s}}(p\mu A)\,u^{*}_{b_1\cdots b_{2s}}(p\mu A)}{p^2-m^2+i\epsilon}
$$

This may be written in terms of the corresponding quantity in the $(\frac{1}{2},0)$ case by a consideration of the quantity $-i$ times the numerator. This is given explicitly as

$$u_{a_1}(p_\mu s_1)\cdots u_{a_{2s}}(p_\mu s_{2s})\, u^*_{b_1}(p_\mu t_1)\cdots u^*_{b_{2s}}(p_\mu t_{2s})$$

$$\frac{(s-\sum_i t_i)!\,(s+\sum_i t_i)!}{(2s)!}\,\delta_{\sum_i s_i\, \sum_j t_j}$$

To effect a further reduction the following lemma is required.

<u>lemma</u>:

$$\sum_{t_k}(s-\sum_i t_i)!(s+\sum_i t_i)!\,\delta_{\sum_i s_i, \sum_j t_j}\, u^*_{b_1}(p_\mu t_1)\cdots u^*_{b_{2s}}(p_\mu t_{2s})$$

$$=\sum_{\substack{\text{all permutations}\\ P(b_i)}} u^*_{P(b_1)}(p_\mu s_1)\cdots u^*_{P(b_{2s})}(p_\mu s_{2s}).$$

This identity is verified by comparing both sides for the case when an arbitrary number $r(0\leqslant r\leqslant 2s)$ of the indices s_i take on the value $-\frac{1}{2}$. In this case $\sum_i s_i = \frac{1}{2}(2s-r) -\frac{1}{2}r = s-r$, whence the left hand side is given by

$$\sum r!(2s-r)!\, u^*_{b_1}(p_\mu -\tfrac{1}{2})\cdots u^*_{b_r}(p_\mu -\tfrac{1}{2})\,u^*_{b_{r+1}}(p_\mu \tfrac{1}{2})\cdots u^*_{b_{2s}}(p_\mu \tfrac{1}{2}),$$

where the summation is over all possible combinations of r of the indices b_i which correspond to factors of the form $u_{b_i}{}^*(p_\mu\ -\frac{1}{2})$. That the right hand side is also equal to this is seen by noting that corresponding to each combination of r of the b_i there are $r!$ permutations of these b_i amongst themselves, and $(2s-r)!$ permutations of the remaining b_i amongst themselves, and that each such pair of permutations gives rise to an identical term. Thus by firstly breaking down the sum over all permutations into one over combinations as described above the right hand side is also seen to lead to the above expression. Q.E.D.

Returning now to the propagator numerator it is seen that this lemma gives

$$U_{a_1 \quad a_{2s}}(p_\mu A)\, U_{b_1 \quad b_{2s}}^{*}(p_\mu A)$$

$$= \frac{1}{(2s)!} \sum_{P(b_i)} U_{a_1}(p_\mu S_1) \cdots U_{a_{2s}}(p_\mu S_{2s})$$

$$U_{P(b_1)}^{*}(p_\mu S_1) \cdots U_{P(b_{2s})}^{*}(p_\mu S_{2s})$$

$$= \frac{1}{(2s)!} \sum_{P(b_i)} \Pi_{a_1 P(b_1)}(p_\mu) \cdots \Pi_{a_{2s} P(b_{2s})}(p_\mu)$$

from the definition of the propagator numerator for the lower $(\frac{1}{2},0)$ particle-field given in section 2. Thus the propagator of the lower $(s,0)$ particle-field is

$$(22) \quad \frac{i}{p^2 - m^2 + i\epsilon} \frac{1}{(2s)!} \sum_{P(b_i)} \Pi_{a_1 P(b_1)}(p_\mu) \cdots \Pi_{a_{2s} P(b_{2s})}(p_\mu).$$

In the above discussion s could take on either integral or half-integral values, whereas in chapter 3 the restriction to integral values of s will be made. In the case of integral s it will be found simpler to perform calculations, and to overcome some difficulties involving non-covariant contact terms, if a manifestly covariant tensor labelling of the $(s,0)$ particle-fields is used. The transition from a spinor labelling to a tensor labelling , for the case of integral s, is the subject of the next section.

Section 4. Tensor fields for integral s.

In order to effect the change of description of integral spin particles from that by spinor fields to that by tensor fields, the transformation matrix from a spinor $(s,0)$ to a tensor $(s,0)$ representation is required. The general case is constructed from the case $s=1$, which is now studied in detail.

For the purpose of constructing this transformation matrix for the case $s=1$ the covariant matrix

$$(\sigma_{\mu\nu})_a{}^b$$

is introduced. It is just twice the generators of the Lorentz group in the $(\frac{1}{2},0)$ representation, and its explicit form and properties are discussed in appendix A. By means of the raising matrix the above is used to construct the covariant matrix

$$(23)\quad (S^{(1)}_{\mu\nu})^{ab} = \frac{1}{2\sqrt{2}}\, C^{-1\,ac}\,(\sigma_{\mu\nu})_c{}^b .$$

This is just the required transformation matrix from a lower (1,0) spinor representation to a (1,0) tensor representation. The proof is as follows:

By a direct use of equations A(16), A(17), A(18), and its defining equation (23) it is easily shown that $(S^{(1)}_{\mu\nu})^{ab}$ has the following four properties:

(i) It is symmetric in a and b.

(ii) $$(S^{(1)}_{\mu\nu})^{cd}\,(e^{\frac{1}{2}\eta\cdot\sigma})_c{}^a\,(e^{\frac{1}{2}\eta\cdot\sigma})_d{}^b = \Lambda_\mu{}^\lambda\,\Lambda_\nu{}^\rho\,(S^{(1)}_{\lambda\rho})^{ab}.$$

(iii) $$(S^{(1)}_{\mu\nu})^{ab}\,(S^{(1)\mu\nu})^{cd} = \frac{1}{2}\left(C^{-1\,ac}\,C^{-1\,db} + C^{-1\,ad}\,C^{-1\,cb}\right).$$

(iv) $$(S^{(1)}_{\mu\nu})^{ab}\,(S^{(1)}_{\lambda\rho})_{ba} = \frac{1}{4}\left(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda} + i\,\epsilon_{\mu\nu\lambda\rho}\right).$$

Property (ii) verifies that $(S^{(1)}_{\mu\nu})^{ab}$ actually does transform with two upper undotted indices, and is hence at most a mixture of (1,0) and (0,0) representations. Property (i) guarantees that only the (1,0) representation appears.[7] The orthogonality relations (iii) and (iv) complete the properties required in order that $(S^{(1)}_{\mu\nu})^{ab}$ be the desired transformation matrix.

Another property of $(S^{(1)}_{\mu\nu})^{ab}$, which follows immediately from A(15) and (23), is that it is self-dual. That is

$$(24)\quad \frac{i}{2}\epsilon_{\mu\nu}{}^{\lambda\rho}\left(S^{(1)}_{\lambda\rho}\right)^{ab} = \left(S^{(1)}_{\mu\nu}\right)^{ab}.$$

The generalization to the transformation matrix for any positive integer s is achieved by coupling s of the quantities $(S^{(1)}_{\mu\nu})^{ab}$ in a totally symmetric fashion as follows. Define

$$(25)\quad \left(S^{(s)}_{\mu_1\nu_1\cdots\mu_s\nu_s}\right)^{a_1\cdots a_{2s}} = \frac{1}{(2s)!}\sum_{P(a_i)}\left(S^{(1)}_{\mu_1\nu_1}\right)^{P(a_1)P(a_2)}\cdots\left(S^{(1)}_{\mu_s\nu_s}\right)^{P(a_{2s-1})P(a_{2s})}.$$

$(S^{(s)}_{\mu_1\nu_1\cdots\mu_s\nu_s})^{a_1\cdots a_{2s}}$ has the following properties which are immediate consequences of its definition and properties (i) - (iv) of $(S^{(1)}_{\mu\nu})^{ab}$.

(v) It is totally symmetric in the a_i

$$(vi)\quad \left(S^{(s)}_{\mu_1\nu_1\cdots\mu_s\nu_s}\right)^{b_1\cdots b_{2s}}\left(e^{\frac{1}{2}\eta\cdot\sigma}\right)_{b_1}{}^{a_1}\cdots\left(e^{\frac{1}{2}\eta\cdot\sigma}\right)_{b_{2s}}{}^{a_{2s}} = \Lambda_{\mu_1}{}^{\lambda_1}\cdots\Lambda_{\nu_s}{}^{\rho_s}\left(S^{(s)}_{\lambda_1\rho_1\cdots\lambda_s\rho_s}\right)^{a_1\cdots a_{2s}}.$$

(vii) It satisfies complicated orthogonality relations analogous to (iii) and (iv).

Property (vi) verifies that it transforms with 2s upper undotted indices, whilst the symmetry property (v) ensures that only the (s,0) representation appears.[7] The orthogonality relations complete the properties required in order that it be the desired transformation matrix from a spinor (s,0) representation to a tensor (s,0) representation.

Some further properties which may be noted are that $(S^{(s)}_{\mu_1\nu_1\cdots\mu_s\nu_s})^{a_1\cdots a_{2s}}$ is symmetric under the interchange of any two pairs of indices $(\mu_i\nu_i)$, self-dual in each pair of indices $(\mu_i\nu_i)$ separately, and gives zero when contracted with the operator

$$\tfrac{1}{4}\left(g_{\mu_1\mu_2}g_{\nu_1\nu_2}-g_{\mu_1\nu_2}g_{\nu_1\mu_2}+i\epsilon_{\mu_1\nu_1\mu_2\nu_2}\right).$$

These three properties are simple consequences of the definition of $(S^{(s)}_{\mu_1\nu_1\cdots\mu_s\nu_s})^{a_1\cdots a_{2s}}$ and the properties of $(S^{(1)}_{\mu\nu})^{ab}$.

Now that these transformation matrices have been constructed, the transition to the tensor equivalent of the (s,0) particle-spinor is immediate. It is

$$\begin{aligned} u_{\mu_1\nu_1\cdots\mu_s\nu_s}(p_\mu A) &= (S^{(s)}_{\mu_1\nu_1\cdots\mu_s\nu_s})^{a_1\cdots a_{2s}}\, u_{a_1\cdots a_{2s}}(p_\mu A) \\ &= (S^{(1)}_{\mu_1\nu_1})^{a_1 a_2}\cdots(S^{(1)}_{\mu_s\nu_s})^{a_{2s-1}a_{2s}}\, u_{a_1\cdots a_{2s}}(p_\mu A), \end{aligned}$$

since $u_{a_1\cdots a_{2s}}(p_\mu A)$ is totally symmetric in its indices. Using the explicit form (23) this further becomes

$$u_{\mu_1\nu_1\cdots\mu_s\nu_s}(p_\mu A) = \left(\frac{1}{2\sqrt{2}}\right)^s C^{-1\,a_1a_1'}(\sigma_{\mu_1\nu_1})_{a_1'}{}^{a_2} \cdots C^{-1\,a_{2s-1}a_{2s-1}'}(\sigma_{\mu_s\nu_s})_{a_{2s-1}'}{}^{a_{2s}}\, u_{a_1\cdots a_{2s}}(p_\mu A). \tag{26}$$

Also, the corresponding particle-field is just

$$(27)\quad \psi_{\mu_1\nu_1\cdots\mu_s\nu_s}(x) = \left(S^{(s)}_{\mu_1\nu_1\cdots\mu_s\nu_s}\right)^{a_1\cdots a_{2s}} \psi_{a_1\cdots a_{2s}}(x).$$

Before the propagator of this field may be calculated it is necessary to look in more detail at

$$u_{\mu_1\nu_1\cdots\mu_s\nu_s}{}^*(p_\mu A) = \left(\frac{1}{2\sqrt{2}}\right)^s C^{-1\,a_1a_1'}{}^*(\sigma_{\mu_1\nu_1})_{a_1'}{}^{a_2}{}^*$$

$$\cdots C^{-1\,a_{2s-1}a_{2s-1}'}{}^*(\sigma_{\mu_s\nu_s})_{a_{2s-1}'}{}^{a_{2s}}{}^* \, u_{a_1\cdots a_{2s}}{}^*(p_\mu A).$$

The right hand side may be written more explicitly by noting, as shown in section 2, that the complex conjugate of a quantity with undotted indices transforms as if each undotted index were replaced by the corresponding dotted index. This observation, together with the following identities

$$C^{-1\,ab}{}^* = C^{-1\,\dot a\dot b},$$

$$A(27)\quad (\sigma_{\mu\nu})_a{}^b{}^* = -(\tilde\sigma_{\mu\nu})_{\dot a}{}^{\dot b},$$

gives

$$(28)\quad u_{\mu_1\nu_1\cdots\mu_s\nu_s}{}^*(p_\mu A) = \left(\frac{-1}{2\sqrt{2}}\right)^s C^{-1\,\dot a_1\dot a_1'}(\sigma_{\mu_1\nu_1})_{\dot a_1'}{}^{\dot a_2}$$

$$\cdots C^{-1\,\dot a_{2s-1}\dot a_{2s-1}'}(\tilde\sigma_{\mu_s\nu_s})_{\dot a_{2s-1}'}{}^{\dot a_{2s}} \, u_{a_1\cdots a_{2s}}{}^*(p_\mu A).$$

Using equations (22), (26) and (28) the propagator for the tensor (s,0) field may be written as

$$(29)\quad \left(-\tfrac{1}{2}\right)^{3s} C^{-1\,a_1a_1'}(\sigma_{\mu_1\nu_1})_{a_1'}{}^{a_2} \cdots C^{-1\,a_{2s-1}a_{2s-1}'}$$

$$(\sigma_{\mu_s\nu_s})_{a_{2s-1}'}{}^{a_{2s}} \, C^{-1\,\dot b_1\dot b_1'}(\tilde\sigma_{\lambda_1\rho_1})_{\dot b_1'}{}^{\dot b_2} \cdots$$

$$C^{-1\,\dot{b}_{2s-1}\dot{b}'_{2s-1}}\left(\tilde{\sigma}_{\lambda_s\rho_s}\right)_{\dot{b}'_{2s-1}}{}^{\dot{b}_{2s}}\frac{1}{(2s)!}\frac{i}{p^2-m^2+i\epsilon}$$

$$\sum_{P(\dot{b}_i)}\Pi_{a_1 P(\dot{b}_1)}(p_\mu)\cdots\Pi_{a_{2s}P(\dot{b}_{2s})}(p_\mu)\,.$$

Now that this propagator is given, all the apparatus necessary for the calculation of the scattering amplitude for the scattering of four massive spinless particles with a massive spin s particle exchanged has been ammassed. Before embarking upon this calculation , however, it is necessary to make some remarks concerning non-covariant contact terms which must be borne in mind throughout the calculation. These remarks are the subject of the final section of this chapter.

Section 5. Non-covariant contact terms.

Since the propagator for the (s,0) field will not be calculated explicitly in chapter 3, but just this quantity when fully contracted with momenta, it is necessary to make some remarks concerning its explicit calculation beforehand.

By definition, the propagator for the tensor (s,0) field is given by

$$\langle 0|T(\psi_{\mu_1\nu_1\cdots\mu_s\nu_s}(x)\,\psi_{\lambda_1\rho_1\cdots\lambda_s\rho_s}{}^{\dagger}(y))|0\rangle$$

$$=\int_{-\infty}^{\infty}u_{\mu_1\nu_1\cdots\mu_s\nu_s}(p\ \ A)\,u_{\lambda_1\rho_1\cdots\lambda_s\rho_s}{}^{*}(p\ \ A)$$

$$\left(\theta(x_0-y_0)e^{-ip\cdot(x-y)}+\theta(y_0-x_0)e^{ip\cdot(x-y)}\right)$$

$$\theta(p_0)\,\delta(p^2-m^2)\,(2\pi)^{-3}\,d^4p$$
$$=\Pi_{\mu\nu\lambda\rho}(i\partial\ )\int_{-\infty}^{\infty}\left(\theta(x_0-y_0)\,e^{-ip\cdot(x-y)}\right.$$
$$\left.+\,\theta(y_0-x_0)\,e^{ip\cdot(x-y)}\right)\theta(p_0)\,\delta(p^2-m^2)\,\frac{d^4p}{(2\pi)^3}$$

+ non-covariant contact terms,

where the abbreviation

$$\Pi_{\mu\nu\lambda\rho}(p\cdot) = u_{\mu_1\nu_1\cdots\mu_s\nu_s}(pA)\,u^*_{\lambda_1\rho_1\cdots\lambda_s\rho_s}(pA),$$

and the discussion of propagators in section 5 of chapter 1 have been used.

As also stated in section 5 of chapter 1 the non-covariant contact terms, and only they, may, as a consequence of Matthews' theorem, be discarded. The important consequence of this observation is that $\Pi_{\mu\nu\lambda\rho}$ $(i\partial_\alpha)$ must contain no ∂^2 terms; for if such terms appeared, covariant terms, together with the non-covariant contact terms, would be discarded in obtaining the propagator (29). The proof is as follows:

Consider

$$-\partial^2\int_{-\infty}^{\infty}\left(\theta(x_0-y_0)\,e^{-ip\cdot(x-y)}+\theta(y_0-x_0)\,e^{ip\cdot(x-y)}\right)$$
$$\theta(p_0)\,\delta(p^2-m^2)\,(2\pi)^{-3}\,d^4p$$
$$=\int_{-\infty}^{\infty}\left(\theta(x_0-y_0)\,e^{-ip\cdot(x-y)}+\theta(y_0-x_0)\,e^{ip\cdot(x-y)}\right)$$
$$m^2\,\theta(p_0)\,\delta(p^2-m^2)\,(2\pi)^{-3}\,d^4p+2i\,\delta(x_0-y_0)$$
$$\int_{-\infty}^{\infty}\left(e^{-ip(x-y)}+e^{ip\cdot(x-y)}\right)p_0\,\theta(p_0)\,\delta(p^2-m^2)\,\frac{d^4p}{(2\pi)^3}$$

$$+\delta'(x_0-y_0)\int_{-\infty}^{\infty}\left(e^{ip\cdot(x-y)}-e^{-ip\cdot(x-y)}\right)\theta(p_0)\delta(p^2-m^2)\frac{d^4p}{(2\pi)^3}$$

$$=\int_{-\infty}^{\infty}\left(\theta(x_0-y_0)e^{-ip\cdot(x-y)}+\theta(y_0-x_0)e^{ip\cdot(x-y)}\right)$$

$$m^2\theta(p_0)\delta(p^2-m^2)\frac{d^4p}{(2\pi)^3}+i\delta^{(4)}(x-y)+\partial_0\Big(\delta(x_0-y_0)$$

$$\int_{-\infty}^{\infty}\left(e^{ip\cdot(x-y)}-e^{-ip\cdot(x-y)}\right)\theta(p_0)\delta(p^2-m^2)\frac{d^4p}{(2\pi)^3}\Big).$$

The last term in this sum vanishes because the integrand is antisymmetric. Thus the result of commuting terms involving ∂^2 past the θ-functions is to produce, in addition to the non-covariant contact terms, covariant terms involving $\delta^{(4)}(x-y)$. These latter terms may not be discarded. So whenever a term involving p^2 appears in the numerator of (29) p^2 must be replaced by m^2.

CHAPTER 3

In this chapter the Feynman graph for the scattering of four massive spinless particles with a massive spin s particle exchanged is calculated. The tensor $(s,0)$ field 2(27) is here used to describe the spin s particle, the equivalent procedure in terms of spinor fields being exhibited in appendix B.

The first step in the calculation of this Feynman graph is to look in detail at possible forms of the interaction Hamiltonian density for a spin zero-spin zero-spin s three particle vertex. This is done in section 1.

Section 1. Interactions

The interaction Hamiltonian density for the above-mentioned three particle vertex is constructed using the tensor field 2(27) for the spin s particle, and scalar fields for both the spinless particles. Thus the interaction Hamiltonian density is to be constructed by coupling the three fields

$$\phi(x)\ ,\ \chi(x)\ ,\ \psi_{\mu_1\nu_1\ldots\mu_s\nu_s}{}^{\dagger}(x)$$

in an invariant manner. The only way of doing this non-trivially is by the introduction of derivatives, and the interaction Hamiltonian density chosen is

$$(1)\quad \mathcal{H}_I(x) = \partial^{\mu_1}\ldots\partial^{\mu_s}\phi(x)\,\partial^{\nu_1}\ldots\partial^{\nu_s}\chi(x)\,\psi_{\mu_1\nu_1\ldots\mu_s\nu_s}{}^{\dagger}(x) \text{ + Hermitian conjugate.}$$

Note that any rearrangment of the way in which the derivatives act on the scalar fields gives a density which either differs at most by a sign from (1) , or is identically zero. This follows from the symmetry properties of $(S^{(s)}_{\mu_1\nu_1\ldots\mu_s\nu_s})^{a_1\ldots a_{2s}}$

expounded in section 4 of chapter 2. On the other hand if any of the derivatives were to act on the tensor field they would destroy its property of transforming under the (s,0) representation. Since this is the only representation of interest here the possibility of derivatives acting in this manner is omitted. Finally, as shown below

$$g^{\mu_1\nu_1}\psi_{\mu_1\nu_1\cdots\mu_s\nu_s}(x) = g^{\mu_1\mu_2}\psi_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_s\nu_s}(x)$$
$$= g^{\nu_1\nu_2}\psi_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_s\nu_s}(x)$$
$$= 0\,,$$

and so the extraction of any trace terms from the quantity

(2) $$\partial^{\mu_1}\cdots\partial^{\mu_s}\phi(x)\,\partial^{\nu_1}\cdots\partial^{\nu_s}\chi(x)$$

does not alter the interaction Hamiltonian density.

That the first term in the sequence of equalities is zero is an immediate consequence of the antisymmetry of $(\sigma_{\mu\nu})_a{}^b$, which gives

$$(\sigma_\mu{}^\mu)_a{}^b = 0\,.$$

If the second term can be shown to be zero, then the third is automatically zero because of the same antisymmetry mentioned above. To show that the second term is zero it is sufficient to show that

$$(S^{(1)}_{\mu_1\nu_1})^{ab}\,(S^{(1)\mu_1}{}_{\nu_2})^{cd}$$

is antisymmetric under the interchange of any two, or any two pairs, of the indices abcd; for then the second term is zero as a consequence of the total symmetry of the (s,0) spinor field

$$\psi_{a_1\cdots a_{2s}}(x)\,.$$

By use of the explicit form of $(S^{(1)}_{\mu\nu})^{ab}$ given by 2(23) and A(11), together with A(17), it may be shown by inspection that

$$(S^{(1)}_{\mu o})^{ab}(S^{(1)\mu}{}_{o})^{cd} = \frac{1}{2}\left((C^{-1})^{ac}(C^{-1})^{db}+(C^{-1})^{ad}(C^{-1})^{cb}\right),$$

$$(S^{(1)}_{\mu o})^{ab}(S^{(1)\mu}{}_{i})^{cd} = -\frac{i}{8}\,\epsilon_{ijk}(\sigma_j)^{ab}(\sigma_k)^{cd},$$

$$(S^{(1)}_{\mu i})^{ab}(S^{(1)\mu}{}_{j})^{cd} = \frac{1}{8}\left((\sigma_j)^{ab}(\sigma_i)^{cd}-(\sigma_i)^{ab}(\sigma_j)^{cd}\right),$$

$$(S^{(1)}_{\mu i})^{ab}(S^{(1)\mu}{}_{o})^{cd} = \frac{i}{8}\,\epsilon_{ijk}(\sigma_j)^{ab}(\sigma_k)^{cd}.$$

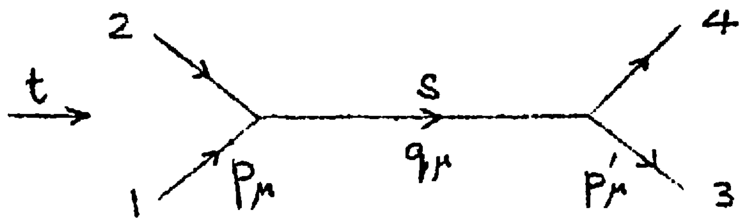
Evidently the first of these quantities is antisymmetric under the interchanges $a\leftrightarrow c$ in the first term and $a\leftrightarrow d$ in the second term, whilst the last three quantities are all antisymmetric under the interchange $(ab)\leftrightarrow(cd)$.

Thus the above equalities are proved, and it is verified that the extraction of trace terms from (3) does not alter the interaction Hamiltonian density. This result has the important consequence that in (1) $\partial^{\mu_1}\ldots\partial^{\mu_s}\phi(x)$ is effectively the same as $\{\partial^{\mu_1}\ldots\partial^{\mu_s}\}\phi(x)$, where the curly brackets indicate that all the traces have been subtracted out in a symmetric manner, which is just an $(\frac{s}{2},\frac{s}{2})$ tensor field describing a spin zero particle. That this field transforms under the $(\frac{s}{2},\frac{s}{2})$ representation is a consequence of its being symmetric and traceless in its s tensor indices.[5] This remark will have an important bearing on appendix B.

Thus it has been shown in the above that the simplest and most logical interaction Hamiltonian density for the description of the above-mentioned three particle vertex is given by (1).

Section 2. The calculation of the amplitude

The contribution to the scattering amplitude for the scattering of four massive spinless particles of the following Feynman graph is to be calculated.



On the basis of the discussion of section 1 the following interaction Hamiltonian density is postulated.

$$(3)\quad \mathcal{H}_I(x) = \sum_{i=1}^{2} g_i\, \partial^{\mu_1}\dots\partial^{\mu_s}\phi_i(x)\,\partial^{\nu_1}\dots\partial^{\nu_s}$$

$$\chi_i(x)\quad \psi_{\mu_1\nu_1\dots\mu_s\nu_s}{}^{\dagger}(x) \text{ + Hermitian conjugate,}$$

where $\phi_1, \chi_1, \phi_2, \chi_2$ are respectively the scalar fields corresponding to the particles 1,2,3 and 4. Also g_1 and g_2 are the coupling constants at the 1-2$\rightarrow$spin s and 3-4$\rightarrow$spin s vertices respectively.

With the above convention the effective wave-functions for the external particles are respectively given by

$$(-i)^s\, p^{\mu_1}\dots p^{\mu_s},$$

$$(-i)^s\, (q-p)^{\nu_1}\dots(q-p)^{\nu_s},$$

$$(i)^s\, p'^{\mu_1}\dots p'^{\mu_s}$$

$$(i)^s\, (q-p')^{\nu_1}\dots(q-p')^{\nu_s}.$$

Also the propagator for the $(s,0)$ tensor field is given by the expressions 2(29). Thus by using these expressions together with the Feynman rules of chapter 1 section 6 the contribution of the above graph is seen to be

$$\left(-\tfrac{1}{2}\right)^{3s} g_1 g_2^* (-i)^2\, p'^{\mu_1} \cdots p'^{\mu_s} (q-p')^{\nu_1} \cdots (q-p')^{\nu_s}$$

$$\frac{i}{q^2-m_s^2}\,\frac{1}{(2s)!}\,(C^{-1})^{a_1 a_1'} (\sigma_{\mu_1\nu_1})_{a_1'}{}^{a_2} \cdots (C^{-1})^{a_{2s-1} a'_{2s-1}}$$

$$(\sigma_{\mu_s\nu_s})_{a'_{2s-1}}{}^{a_{2s}} \sum_{P(\dot b_i)} \Pi_{a_1 P(\dot b_1)}(q_\mu) \cdots \Pi_{a_{2s} P(\dot b_{2s})}(q_\mu)$$

$$(C^{-1})^{\dot b_1 \dot b_1'} (\tilde\sigma_{\lambda_1\rho_1})_{\dot b_1'}{}^{\dot b_2} \cdots (C^{-1})^{\dot b_{2s-1} \dot b'_{2s-1}}$$

$$(\tilde\sigma_{\lambda_s\rho_s})_{\dot b'_{2s-1}}{}^{\dot b_{2s}}\, p^{\lambda_1} \cdots p^{\lambda_s} (q-p)^{\rho_1} \cdots (q-p)^{\rho_s}.$$

As a consequence of the antisymmetry of the quantities $(\sigma_{\mu\nu})_a{}^b$ and $(\tilde\sigma_{\mu\nu})_{\dot c}{}^{\dot d}$ under the interchange $\mu \leftrightarrow \nu$, and a use of the abbreviated notation

$$p^\mu q^\nu (\sigma_{\mu\nu})_a{}^b = (\sigma_{pq})_a{}^b$$

$$p^\mu q^\nu (\tilde\sigma_{\mu\nu})_{\dot a}{}^{\dot b} = (\tilde\sigma_{pq})_{\dot a}{}^{\dot b}$$

$$p^\mu (\sigma_\mu)_{a\dot b} = (\sigma_p)_{a\dot b}$$

$$p^\mu (\tilde\sigma_\mu)^{\dot b a} = (\tilde\sigma_p)^{\dot b a}$$

the above simplifies to

$$(4)\quad \frac{-i g_1 g_2^*}{q^2-m_s^2}\left(-\tfrac{1}{2}\right)^{3s} \frac{1}{(2s)!}\,(C^{-1})^{a_1 a_1'} (\sigma_{p'q})_{a_1'}{}^{a_2}$$

$$\cdots (C^{-1})^{a_{2s-1} a'_{2s-1}} (\sigma_{p'q})_{a'_{2s-1}}{}^{a_{2s}} \sum_{P(\dot b_i)} \Pi_{a_1 P(\dot b_1)}(q_\mu)$$

$$\cdots \Pi_{a_{2s}P(\dot{b}_{2s})}(q_\mu)\,(C^{-1})^{\dot{b}_1\dot{b}_1'}\,(\tilde{\sigma}_{pq})_{\dot{b}_1'}{}^{\dot{b}_2}$$

$$\cdots (C^{-1})^{\dot{b}_{2s-1}\dot{b}_{2s-1}'}\,(\tilde{\sigma}_{pq})_{\dot{b}_{2s-1}'}{}^{\dot{b}_{2s}}$$

$$= \frac{-ig_1g_2^*}{q^2-m_s^2}\left(-\frac{1}{2}\right)^{3s}\frac{1}{(2s)!}A(s)$$

say.

Now it must be remembered from the discussion of chapter 2 section 5 that whenever q^2 appears in the propagator numerator of the (s,0) tensor field it must be replaced by m_s^2. Thus whenever a product of the form

$$\Pi_{a\dot{b}}(q_\mu)\,\tilde{\Pi}^{\dot{b}c}(q_\mu)$$

appears in the calculation of (4) it must, as a consequence of A(23), be written as $\delta_a{}^c$, and not $q^2/m_s^2\,\delta_a{}^c$. However if a contraction occurs between a q_μ originating from the propagator and a q_μ from an external wave-function, then the result is q^2 and not m_s^2.

The next step is to simplify the form of $A(s)$, always bearing the remarks of the above paragraph carefully in mind. For this purpose the following identities, which are proved in appendix A, are required.

A(22) $$(C^{-1})^{ab}\,\Pi_{b\dot{c}}\,(C^{-1})^{\dot{c}\dot{d}} = -(\tilde{\Pi}^T)^{a\dot{d}}$$

A(21) $$(C^{-1})^{\dot{a}\dot{b}}(\tilde{\sigma}_{\mu\nu})_{\dot{b}}{}^{\dot{c}}\,C_{\dot{c}\dot{d}} = -(\tilde{\sigma}_{\mu\nu}{}^T)^{\dot{a}}{}_{\dot{d}}$$

As a consequence of these two identities it follows that

(5) $$A(s) = \sum_{r=1}^{s} 2^{2r-1}\,{}^sP_r\,{}^{s-1}P_{r-1}\,\mathrm{tr}(\sigma_{pq}\Pi\,\tilde{\sigma}_{pq}^T\tilde{\Pi}\ldots)_r\,A(s-r),$$

with

(6) $$A(0) = 1,$$

and where

(7) $$T_r = tr\,(\sigma_{p'q}\,\Pi\,\tilde{\sigma}^T_{pq}\,\tilde{\Pi}\ldots)_r$$

means that the trace is taken over a product of r of the terms one of which is shown in the bracket.

Equation (5) may be seen by firstly noting that the sum over the permutations of the indices $\dot{b}_i$ leads to a sum of terms involving the product of

(8) $$tr\,(C^{-1}\sigma_{p'q}\,\Pi\,C^{-1}\tilde{\sigma}_{pq}\,\Pi^T\ldots)_r$$

with a sum over the permutations of the indices $\dot{b}_i$ remaining after this trace has been taken. Because of the separate complete symmetry of (4) in the terms $(C^{-1}\sigma_{p'q})^{ab}$ and $(C^{-1}\tilde{\sigma}_{pq})^{\dot{a}\dot{b}}$, the latter part of the above-mentioned product is merely A(s-r). Since such terms as the above will appear for all values of r in the range $1 \leq r \leq s$, it only remains to find the number of times each such term occurs. In order to be certain of avoiding double-counting it suffices to take the first factor in (8) to be fixed as $(C^{-1}\sigma_{p'q})^{a_1a_2}$ say. Then there are ${}^sP_r$ ways of choosing the factors $(C^{-1}\tilde{\sigma}_{pq})^{\dot{b}\dot{a}}$, and once this is done there is still the possibility of choosing either $(C^{-1}\tilde{\sigma}_{pq})^{\dot{b}\dot{a}}$ or $(C^{-1}\tilde{\sigma}_{pq})^{\dot{a}\dot{b}}$. Thus in all there are $2^r\,{}^sP_r$ ways of choosing the factors $(C^{-1}\tilde{\sigma}_{pq})^{\dot{b}\dot{a}}$. Similarly, since the first factor $(C^{-1}\sigma_{p'q})^{a_1a_2}$ is fixed, there are $2^{r-1}\,{}^{s-1}P_{r-1}$ ways of choosing the remaining such factors. The proof of (5) is now completed by noting that A(21) and A(22) cause the expression (8) to reduce to (7).

Writing

(9) $$\frac{2^{-2s}}{(s!)^2} A(s) = B(s) ,$$

equation (5) simplifies further to

(10) $$2s\, B(s) = \sum_{r=1}^{s} T_r\, B(s-r) \qquad B(0)=1.$$

So now the calculation of the original amplitude rests on the calculation of the trace T_r. This latter calculation is the subject of the next section.

Section 3. The calculation of T_r

It must be remembered throughout the calculation of the trace

$$T_r = tr(\sigma_{p'q} \Pi \tilde{\sigma}^T_{pq} \tilde{\Pi} \ldots)^r$$

that whenever a product of the form $\Pi \tilde{\Pi}$ or $\tilde{\Pi}\Pi$ occurs it must be replaced by the appropriate Krouecker delta. This follows from the remarks of chapter 2 section 5 and page 58, together with equations A(23) and A(24).

For the first part of this calculation the following identities, which are proved in appendix A, will be required.

A(28) $$\Pi \tilde{\sigma}^T_{pq} \tilde{\Pi} = -\sigma_{pq} + \frac{2i}{m_s}(p\cdot q\,\sigma_q - q^2 \sigma_p)\tilde{\Pi}$$

A(29) $$\tilde{\Pi} \sigma_{pq} \Pi = -\tilde{\sigma}^T_{pq} - \frac{2i}{m_s}(p\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_p)\Pi$$

A(30) $$\sigma_{p'q}(p\cdot q\,\sigma_q - q^2\sigma_p) = (p'\cdot q\,\sigma_q - q^2\sigma_{p'})\tilde{\sigma}^T_{pq}$$

A(31) $$\tilde{\sigma}^T_{pq}(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}) = (p\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_p)\sigma_{p'q}$$

A(32) $$(p\cdot q\,\sigma_q - q^2\sigma_p)(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}) = q^2\sigma_{pq}\sigma_{p'q}.$$

Rather than deal with T_r directly, it is simpler to first consider the matrix product C_r defined by $T_r = \mathrm{tr}(C_r)$. Using A(28) above it may be rewritten as

$$C_r = \left(\sigma_{p'q}\left\{-\sigma_{pq} + \frac{2i}{m_s}(p\cdot q\,\sigma_q - q^2\sigma_p)\tilde{\Pi}\right\}\ldots\right)_r$$

(11) $$= -\sigma_{p'q}\sigma_{pq}C_{r-1} + \frac{2i}{m_s}(p'\cdot q\,\sigma_q - q^2\sigma_{p'})D_r,$$

where

$$D_r = \tilde{\sigma}^T_{pq}\,\tilde{\Pi}\,C_{r-1} \qquad (r \geq 1),$$

C_0 is defined to be 1 , and A(30) has been used. Now look at D_r.

$$D_r = (\tilde{\sigma}^T_{pq}\,\tilde{\Pi}\,\sigma_{p'q}\,\Pi\,\tilde{\sigma}^T_{pq}\,\tilde{\Pi}\ldots)$$

$$= \left(\tilde{\sigma}^T_{pq}\left\{-\tilde{\sigma}^T_{p'q} - \frac{2i}{m_s}(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'})\Pi\right\}\tilde{\sigma}^T_{pq}\tilde{\Pi}\ldots\right)$$

(12) $$= -\tilde{\sigma}^T_{pq}\,\tilde{\sigma}^T_{p'q}\,D_{r-1} - \frac{2i}{m_s}(p\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_p)\,C_{r-1}$$

where A(29) and A(31) have been used respectively. Now D_r is obtained in terms of C_r and C_{r-1} by multiplying (11) by $(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'})$, and noting that as a consequence of A(24) and A(26)

$$(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'})(p'\cdot q\,\sigma_q - q^2\sigma_{p'})$$

$$= q^2(p'\cdot q)^2 + q^4p'^2 - q^2\,p'\cdot q(\tilde{\sigma}_q\sigma_{p'} + \tilde{\sigma}_{p'}\sigma_q)$$

$$= q^2(q^2p'^2 - (p'\cdot q)^2)$$

$$= q^2\beta_2 .$$

Similarly define

$$\beta_1 = p^2 q^2 - (p\cdot q)^2$$
$$\beta = \beta_1 \beta_2$$

and

$$\alpha = q^2 p\cdot p' - p\cdot q\, p'\cdot q .$$

When the above process has been carried out (11) becomes

$$D_r = \frac{m_s}{2iq^2\beta_2}\left(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}\right)\left(C_r + \sigma_{p'q}\sigma_{pq}C_{r-1}\right).$$

Substitution of this expression into (12) gives the following.

$$\frac{m_s}{2iq^2\beta_2}\left(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}\right)\left(C_r + \sigma_{p'q}\sigma_{pq}C_{r-1}\right)$$
$$= \frac{-m_s}{2iq^2\beta_2}\tilde{\sigma}^T_{pq}\,\tilde{\sigma}^T_{p'q}\left(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}\right)\left(C_{r-1} + \right.$$
$$\left.\sigma_{p'q}\sigma_{pq}C_{r-2}\right) - \frac{2i}{m_s}\left(p\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_p\right)C_{r-1},$$
$$(r \geq 2).$$

This expression may be greatly simplified by multiplication throughout by $(p'\cdot q\sigma_q - q^2\sigma_{p'})$, followed by a double application of A(30) to the first term of the right hand side, and an application of A(32) to the second term of the right hand side. The resulting equation is

$$(13)\quad C_r - 2\left(\frac{2q^2}{m_s^2} - 1\right)\sigma_{p'q}\sigma_{pq}C_{r-1}$$
$$+ \sigma_{p'q}\sigma_{pq}\sigma_{p'q}\sigma_{pq}C_{r-2} = 0, \quad (r \geq 2).$$

On writing

$$E_t^{(r)} = \{\sigma_{p'q}\sigma_{pq}\cdots\}_{r-t}C_t, \quad (0 \leq t \leq r-1),$$

(13) becomes

(14) $$E_t^{(r)} - 2\cosh x\, E_{t-1}^{(r)} + E_{t-2}^{(r)} = 0, \quad (r \geqslant t \geqslant 2)$$

with

$$\cosh x = \frac{2q^2}{m_s^2} - 1,$$

and, as a consequence of (13) and the definition of $E_t^{(r)}$,

$$E_r^{(r)} = C_r .$$

Thus the solution of (14) solves (13). The solution of (14) is easily obtained[13], and it is

(15) $$E_t^{(r)} = A e^{tx} + B e^{-tx}, \qquad (t \geqslant 0),$$

where A and B are determined by

$$E_0^{(r)} = \{\sigma_{p'q}\sigma_{pq}\cdots\}_r$$

and

$$E_1^{(r)} = \{\sigma_{p'q}\sigma_{pq}\cdots\}_{r-1}\sigma_{p'q}\Pi\tilde{\sigma}_{pq}^T\tilde{\Pi}.$$

Now as the next step in calculating T_r, $\mathrm{tr}(E_t^{(r)})$ is found explicitly from (15). For this purpose define

(16) $$U_r = \mathrm{tr}(E_0^{(r)}).$$

Then

$$\begin{aligned}\mathrm{tr}(E_1^{(r)}) &= \mathrm{tr}\left(\{\sigma_{p'q}\sigma_{pq}\cdots\}_{r-1}\sigma_{p'q}\Pi\tilde{\sigma}_{pq}^T\tilde{\Pi}\right)\\ &= -\mathrm{tr}\left(\{\sigma_{p'q}\sigma_{pq}\cdots\}_{r-1}\sigma_{p'q}\sigma_{pq}\right)\\ &\quad + \frac{2i}{m_s}\mathrm{tr}\left(\{\sigma_{p'q}\sigma_{pq}\cdots\}_{r-1}\sigma_{p'q}(p\cdot q\sigma_q - q^2\sigma_p)\tilde{\Pi}\right)\end{aligned}$$

by A(28).

$$= \left(\frac{2q^2}{m_s^2} - 1 \right) U_r$$

after use of A(23) and the definition (16). Thus looking at (15) for the values t=0 and t=1 leads to

$$\mathrm{tr}(A) + \mathrm{tr}(B) = U_r$$

and

$$e^x \mathrm{tr}(A) + e^{-x} \mathrm{tr}(B) = U_r \cosh x .$$

These simultaneous equations may be solved to give

$$\mathrm{tr}(A) = \mathrm{tr}(B) = \tfrac{1}{2} U_r .$$

And so finally the following sequence of equalities is valid.

$$(17) \quad T_r = \mathrm{tr}(C_r) = \mathrm{tr}(E_r^{(r)}) = U_r \cosh rx .$$

It only remains to calculate U_r in order that T_r may be determined explicitly. By use of equation A(13)

$$\begin{aligned} U_r &= -\mathrm{tr}(\sigma_{pq}\sigma_{p'q}\{\sigma_{p'q}\sigma_{pq}\cdots\}_{r-1}) \\ &\quad + 2\alpha\, \mathrm{tr}(\{\sigma_{p'q}\sigma_{pq}\cdots\}_{r-1}) \\ &= -\beta U_{r-2} + 2\alpha U_{r-1} \end{aligned}$$

by A(32).

Thus on putting

$$\cosh y = \alpha \beta^{-\frac{1}{2}}$$

it follows that U_r satisfies the reccurrence relation given below.

$$(18) \quad U_r - 2\beta^{1/2} \cosh y\, U_{r-1} + \beta U_{r-2} = 0 , \quad (r \geqslant 2).$$

The solution of (18) is readily obtained in the same manner as that of (14), and it is

(19) $$U_r = \beta^{r/2}\left(A e^{ry} + B e^{-ry}\right),$$

where this time A and B are determined by

$$U_0 = 2$$

and

$$U_1 = \mathrm{tr}\left(\sigma_{p'q}\,\sigma_{pq}\right) = 2\alpha .$$

The first of these equalities follows from the explicit forms of U_1 and U_2, together with (18). The second follows from A(13). Thus looking at (19) for $r=0$ and $r=1$ leads to

$$A + B = 2$$

and

$$A e^{y} + B e^{-y} = 2\cosh y .$$

These simultaneous equations may be solved to give

$$A = B = 1 .$$

Thus

(20) $$U_r = 2\beta^{r/2}\cosh ry ,$$

and so the final form of T_r is given by (17) and (20) as

(21) $$T_r = 2\beta^{r/2}\cosh rx\,\cosh ry$$

with

$$\cosh x = \frac{2q^2}{m_s^2} - 1$$

and

$$\cosh y = \alpha\beta^{-1/2} .$$

Section 4. The amplitude (continued).

Collecting together the results (4), (9), (10), and (21), the Feynman amplitude for the graph of section 2 is given as:

$$(22)\qquad \frac{-ig_1g_2^*}{t^2-m_s^2}\left(-\frac{1}{2}\right)^s \frac{(s!)^2}{(2s)!}\,\beta^{s/2}\,F(s),$$

where F(s) satisfies

$$(23)\qquad sF(s)=\sum_{r=1}^{s}\cosh rx\cosh ry\,F(s-r),$$

with

$$(24)\qquad F(0)=1,$$

$$(25)\qquad \cosh x=\frac{2t}{m_s^2}-1,$$

and

$$(26)\qquad \cosh y=\alpha\beta^{-1/2}=\cos\theta$$

where θ is the centre of momentum frame scattering angle.

Before the amplitude (22) and its Reggeization are discussed in detail in chapter 4, a generating function for F(s), which will facilitate these discussions, is derived.

Multiplying both sides of (23) by z^s, and then summing over all positive integral s gives[14]

$$\sum_{s=1}^{\infty}sF(s)z^s=\sum_{s=1}^{\infty}\sum_{r=1}^{s}\cosh rx\cosh ry\,F(s-r)\,z^s$$

$$=\left(\sum_{r=1}^{\infty}\cosh rx\cosh ry\,z^r\right)\left(\sum_{s=0}^{\infty}F(s)\,z^s\right).$$

On defining

$$(27)\qquad \phi(z)=\sum_{s=0}^{\infty}F(s)\,z^s,$$

and

$$\Psi(z) = \sum_{r=1}^{\infty} \cosh rx \cosh ry \, z^{r-1},$$

this becomes

(28) $$\phi'(z) = \Psi(z)\phi(z)$$

with

$$\Psi(z) = \frac{1}{4z}\sum_{r=1}^{\infty}\left((ze^{x+y})^r + (ze^{x-y})^r + (ze^{y-x})^r + (ze^{-x-y})^r\right)$$

$$= \frac{1}{4}\left(\frac{e^{x+y}}{1-ze^{x+y}} + \frac{e^{x-y}}{1-ze^{x-y}} + \frac{e^{y-x}}{1-ze^{y-x}} + \frac{e^{-x-y}}{1-ze^{-x-y}}\right)$$

$$= \frac{z^3 - \cosh x \cosh y\,(1+3z^2) + (2\cosh^2 x + 2\cosh^2 y - 1)\,z}{z^4 - 4\cosh x \cosh y\,(z+z^3) + 2(2\cosh^2 x + 2\cosh^2 y - 1)z^2 + 1}$$

Now taking this expression for $\Psi(z)$ together with (24) allows (28) to be integrated immediately giving

(29) $$\phi(z) = \left(1 - 4\cosh x \cosh y\,(z+z^3) + 2(2\cosh^2 x + 2\cosh^2 y - 1)z^2 + z^4\right)^{-1/4}.$$

And $\phi(z)$ is just the required generating function for $F(s)$.

CHAPTER 4

A discussion of some of the properties of the Feynman amplitude 3(22) is given in section 1. This is followed in section 2 by the Reggeization of that amplitude; whilst in section 3 the Reggeized scattering amplitude is discussed, and the chapter closed with a discussion and the conclusions of part one.

Section 1. Properties of the Feynman amplitude:

From its explicit form 3(22), together with 3(23) and 3(26), it is evident that the Feynman amplitude is a polynomial in $\cos\theta$, even or odd according as is s. Thus it may be written as a linear sum of Legendre polynomials in the following manner:

$$(1) \quad \sum_{r=0}^{[\frac{1}{2}s]} a_r^{(s)}(t)\, P_{s-2r}(\cos\theta),$$

where $a_r^{(s)}(t)$ is independent of $\cos\theta$, and $[\frac{1}{2}s]$ denotes that the integral part of $\frac{1}{2}s$ is to be taken. Thus in addition to pure spin s, the lower spins $s-2, s-4, \ldots, s-2[\frac{1}{2}s]$ all contribute to the Feynman amplitude. Now it should be noted that the tensor $(s,0)$ particle-field satisfies $\frac{1}{2}s(s-1)$ independent tracelessness conditions. And so it has $\frac{1}{2}s(s-1)$ redundant components. Also, the number of independent field components required to describe one each of particles with spins $s-2, s-4, \ldots, s-2[\frac{1}{2}s]$ is $\frac{1}{2}s(s-1)$. Thus the number of lower-spin contributions to the Feynman amplitude 3(22) is consistent with there being a direct relationship between the existence of redundant components in the propagated tensor particle-field, and the existence of such lower-spin contributions.

As expected, the lower-spin contributions vanish on-shell, and the exchange becomes pure spin s. This may be seen by an inspection of the generating function 3(29), which reduces to the generating function for the Legendre polynomials on-shell.

However, neither the leading contribution, nor any of the lower-spin contributions to the Feynman amplitude 3(22), is singular at the off-shell point $t=0$. This may be seen by noting firstly that all the quantities, other than $F(s)$, which appear in 3(22) are evidently finite at $t=0$; and secondly that, because of the linear independence of the Legendre polynomials, and the fact that the generating function 3(29) reduces to the generating function for these polynomials at $t=0$, the lower-spin contributions to 3(22) separately vanish at that point.

Thus although lower spins still contribute to the Feynman amplitude in the model presented here, they have no role with regard to singularity structure, as in the case of Durand[1]; nor do they have any other obvious special role. They seem to be merely a reflection of the existence of redundant components in the propagated tensor particle-field.

Section 2. Reggeization of the Feynman amplitude:

The Van Hove model[2] is here used to Reggeize the Feynman amplitude 3(22). In this model the mesons are assumed to occur in families of Regge recurrences. Within each such family, a mass-spin relation $m=m(s)$ is satisfied for either all even or all odd non-negative integers s. Further, the members of each such family are assumed to lie on a single corresponding Regge trajectory $s=\alpha(t)$, which satisfies

(2) $$\alpha(m^2(s)) = s \quad \text{and} \quad m^2(\alpha(t)) = t.$$

In order to calculate a Regge pole contribution to the scattering amplitude, the first step is to sum the Feynman amplitudes 3(22) for all the members of the corresponding exchanged family of Regge recurrences. Next the resulting sum is rewritten as a partial-wave series, this being necessary since Regge contributions are assumed to stem from the s-plane singularity structure of the partial-wave amplitudes. Finally a Sommerfeld-Watson transform of the partial-wave series is performed, and the resulting contour integral is written as a sum of Regge contributions and a background integral along the line $\mathrm{Re}\ s = -\frac{1}{2}$.

The first step is immediate, and the result is

$$\sum_{s=0}^{\infty} i\,(2s+1)\,\frac{g_1(s)\,g_2^*(s)}{t-m^2(s)}\left(\frac{1}{4}\beta^{1/2}\right)^s F(s)\,, \tag{3}$$

where $F(s)$ is given by 3(23) or 3(29), and where a factor $\Gamma(2s+2)(2^s\Gamma(s+1)^2)^{-1}$ has been extracted from $g_1(s)\,g_2^*(s)$. It should be noted that the effects of signature have been neglected in (3). This is merely for the sake of notational convenience, and may be remedied by adding to (3) the same expression with $\cos\theta$ replaced by $-\cos\theta$, and dividing the sum by two. In the following signature effects will continue to be neglected, on the understanding that this neglect may be remedied by the above prescription.

The next step is more difficult, and it is achieved by expanding $F(s)$ in partial-wave series. Note that the generating function 3(29) may be rewritten as

$$\phi(z) = \left(1-2e^{y}\cosh x\,z + e^{2y}z^2\right)^{-1/4}\left(1-2e^{-y}\cosh x\,z + e^{-2y}z^2\right)^{-1/4}, \tag{4}$$

and that the generating function for the Gegenbauer polynomials of order $1/4$ is given by [15]

$$(5)\quad \sum_{r=0}^{\infty} C_r^{1/4}(x) z^r = (1-2xz+z^2)^{-1/4}.$$

Hence on the substitution of (5) into (4)

$$\phi(z) = \left(\sum_{r=0}^{\infty} C_r^{1/4}(\cosh x) e^{ry} z^r\right)\left(\sum_{u=0}^{\infty} C_u^{1/4}(\cosh x) e^{-uy} z^u\right),$$

which gives the explicit form of $F(s)$ as

$$(6)\quad F(s) = \sum_{r=0}^{s} C_r^{1/4}(\cosh x)\, C_{s-r}^{1/4}(\cosh x)\, e^{(s-2r)y}$$

$$= \sum_{r=0}^{s} C_r^{1/4}(\cosh x)\, C_{s-r}^{1/4}(\cosh x) \cos(s-2r)\theta$$

by 3(26).

With the expression (6) for $F(s)$, the second step is reduced to expanding $\cos n\theta$ in partial-wave series. This is done by noting that the coefficient of $P_m(\cos\theta)$ in that series is given by

$$a(n,m) = (m+\tfrac{1}{2})\int_{-1}^{1} \cos n\theta\, P_m(\cos\theta)\, d\cos\theta$$

$$= \tfrac{1}{2}(m+\tfrac{1}{2})\int_0^{\pi} \left(\sin(n+1)\theta - \sin(n-1)\theta\right) P_m(\cos\theta)\, d\theta$$

$$= (m+\tfrac{1}{2})\left(\frac{(n-m+2)(n-m+4)\dots(n+m)}{(n-m+1)(n-m+3)\dots(n+m+1)} - \frac{(n-m)(n-m+2)\dots(n+m-2)}{(n-m-1)(n-m+1)\dots(n+m-1)}\right)$$

for $n \geqslant m$ and $n+m$ even, and is zero otherwise. (16) Thus finally

$$(7) \quad a(n,m) = -\frac{n}{8}(2m+1)\frac{\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{n-m-1}{2}\right)}{\Gamma\left(\frac{n-m+2}{2}\right)\Gamma\left(\frac{n+m+3}{2}\right)}$$

for $n \geqslant m$ and $n+m$ even, and is zero otherwise.

Inserting the result (7) into (6), and interchanging the order of summation gives the partial wave expansion of $F(s)$ as

$$(8) \quad F(s) = \sum_{r=0}^{[\frac{1}{2}s]} P_{s-2r}(\cos\theta) \sum_{u=0}^{r} a(s-2u, s-2r)$$
$$C_u^{1/4}(\cosh x)\, C_{s-u}^{1/4}(\cosh x) + \sum_{r=0}^{s-[\frac{1}{2}s]-1} P_{s-2r}(\cos\theta)$$
$$\sum_{u=0}^{r} a(s-2u, s-2r)\, C_u^{1/4}(\cosh x)\, C_{s-u}^{1/4}(\cosh x).$$

Although a more explicit form for the whole of this partial-wave expansion has not been found, an inspection of (8) reveals the exact form of the first two terms. Thus

$$(9) \quad F(s) = {}_2F_1\left(-s, s+\tfrac{1}{2}; \tfrac{3}{4}; 1-\frac{t}{m^2(s)}\right)P_s(\cos\theta) - \frac{1}{3}(2s-3)$$
$$\frac{t}{m^2(s)}\left(\frac{t}{m^2(s)}-1\right) {}_2F_1\left(-s+2, s+\tfrac{1}{2}; \tfrac{7}{4}; 1-\frac{t}{m^2(s)}\right)P_{s-2}(\cos\theta)+\ldots$$

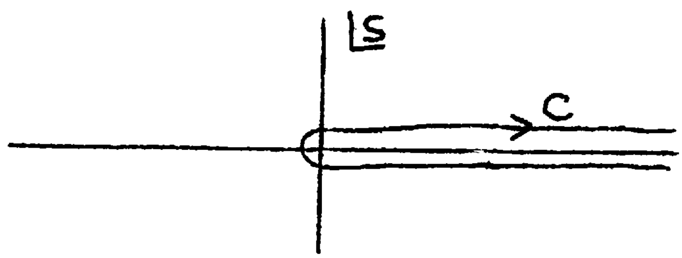
Since the $t=0$ behaviour of the first lower-spin contribution to 3(22) characterises the behaviour of all the lower-spin contributions, in the sense that they all vanish at that point, it is to be expected that the contribution of this lower-spin to the Reggeized scattering amplitude will characterise the corresponding contribution of any other of the lower spins. Thus the expression (9) will be sufficient for the discussions of the ensuing paragraphs.

Now that the partial-wave expansion of $F(s)$, and hence 3(22), has been effectively found, the next step in the calculation of Regge contributions to the scattering amplitude is to effect a Sommerfeld-Watson transformation of the sum over the Feynman amplitudes 3(22) for the corresponding exchanged family of Regge recurrences. The result is[3]

$$(10)\quad \frac{1}{2}\int_C (2s+1)\frac{P_s(-\cos\theta)}{\sin\pi s}\Big\{\frac{g_1(s)g_2(s)}{t-m^2(s)}\left(\tfrac{1}{4}\beta^{1/2}\right)^s {}_2F_1\left(-s,s+\tfrac{1}{2};\tfrac{3}{4};1-\frac{t}{m^2(s)}\right)-\tfrac{1}{3}(2s+5)g_1(s+2)g_2^*(s+2)$$

$$\frac{t}{m^4(s+2)}\left(\tfrac{1}{4}\beta^{1/2}\right)^{s+2} {}_2F_1\left(-s,s+\tfrac{5}{2};\tfrac{7}{4};1-\frac{t}{m^2(s+2)}\right)+\cdots\Big\}ds,$$

where the contour C is as shown below.



In the following $g_1(s)g_2^*(s)$ and $m^2(s)$ are assumed to have no singularities to the right of $\mathrm{Re}\,s = -\frac{1}{2}$ [3]; and the behaviour at infinity of $g_1(s)g_2^*(s)$ is assumed to be such that the contribution of the semicircle at infinity, centre $s = -\frac{1}{2}$, and to the right of $\mathrm{Re}\,s = -\frac{1}{2}$, to the scattering amplitude vanishes. This being so, the expression (10) may be rewritten as the sum of a background integral along $\mathrm{Re}\,s = -\frac{1}{2}$, and the contributions from any singularities of the integrand encountered in the deformation of the contour to $\mathrm{Re}\,s = -\frac{1}{2}$. These singularities are discussed in the next paragraph.

From (2), and the form of the integrand of (10), it follows that only one pole may be encountered in the above deformation of the countour C. This is the Regge pole at $s = \alpha(t)$, and it only appears in the first term of the integrand. On the other hand, each of the first two terms of the integrand possesses a branch point due to the singularity structure of the hypergeometric function. The branch point in the first term is at $s = \alpha(0)$, whilst that in the second is at $s = \alpha(0)-2$. As is customary in the theory of the hypergeometric function, a branch cut extending from the branch point to infinity, along some path in the left half s-plane, is associated with each of the above branch points [17]. These remarks lead to the following expression for the Reggeized scattering amplitude:

$$
\begin{aligned}
(11)\quad & i\pi\left(2\alpha(t)+1\right) g_1(\alpha(t))\, g_2^*(\alpha(t)) \left(\tfrac{1}{4}\beta^{1/2}\right)^{\alpha(t)} \frac{d\alpha(t)}{dt} \frac{1}{\sin\pi\alpha(t)} \\
& P_{\alpha(t)}(-\cos\theta) + \int^{\alpha(0)} \tfrac{1}{2}(2s+1)\, g_1(s)\, g_2^*(s) \left(\tfrac{1}{4}\beta^{1/2}\right)^{s} \\
& \frac{1}{t-m^2} \frac{P_s(-\cos\theta)}{\sin\pi s} \,\mathrm{disc}\, {}_2F_1\left(-s, s+\tfrac{1}{2}; \tfrac{3}{4}; 1-\frac{t}{m^2(s)}\right) ds \\
& -\int^{\alpha(0)-2} \tfrac{1}{6}(2s+1)(2s+5)\, g_1(s+2)\, g_2^*(s+2) \left(\tfrac{1}{4}\beta^{1/2}\right)^{s+2} \\
& \frac{t}{m^4(s+2)} \frac{P_s(-\cos\theta)}{\sin\pi s} \,\mathrm{disc}\, {}_2F_1\left(-s, s+\tfrac{5}{2}; \tfrac{7}{4}; 1-\frac{t}{m^2(s+2)}\right) ds \\
& + \cdots \quad + \text{background integral.}
\end{aligned}
$$

In this expression disc. denotes the discontinuity across the branch cut of the hypergeometric function, and the lower limits of the first and second integrals are respectively the points where the corresponding cuts meet Re $s=-\frac{1}{2}$.

It should be noted that for physical processes Re $\alpha(0) \leq 1$ [18], and that consequently secondary branch point effects would be contained in the background integral. Thus in order to give a strict discussion of these secondary branch point contributions, the Mandelstam form of the Sommerfeld-Watson transform[19] should be used. Even so, the expression (11) is sufficient for the following qualitative discussions.

Section 3. The Reggeized scattering amplitude:

Two classes of contributions to the Reggeized scattering amplitude (11) are considered separately below; firstly with special reference to the point $t=0$. These contributions are:

(i) The secondary cut contribution due to the branch point at $s=\alpha(0)-2$, together with the corresponding part of the background integral. These contributions are those arising from (3) when only the second term in the expression (9) for $F(s)$ is considered.

(ii) The contributions of the Regge pole at $s=\alpha(t)$, the leading branch point at $s=\alpha(0)$, and the corresponding part of the background integral. These contributions are those arising from (3) when only the first term in the expression (9) for $F(s)$ is considered.

In case (i), the contribution to (3) is given by

$$(12) \sum_{s=2}^{\infty} \frac{i}{3}(2s+1)(2s-3)\frac{g_1(s)\,g_2^*(s)}{t-m^2(s)}\left(\frac{1}{4}\beta^{1/2}\right)^s \frac{t}{m^2(s)}\left(\frac{t}{m^2(s)}-1\right)$$

$$_2F_1\left(-s+2,\, s+\frac{1}{2};\, \frac{7}{4};\, 1-\frac{t}{m^2(s)}\right) P_{s-2}(\cos\theta),$$

where it should be noted that[17]

$$ {}_2F_1\left(-s+2, s+\tfrac{1}{2}; \tfrac{7}{4}; 1-\frac{t}{m^2(s)}\right) \underset{t\to 0}{\sim} O(t^{-3/4}). $$

Thus each term in the series (12) vanishes at $t=0$, and hence the corresponding contribution to the Reggeized scattering amplitude also vanishes at that point. This means that the contribution of the secondary branch point at $s=\alpha(0)-2$ gives, together with the corresponding part of the background integral, vanishing contribution to the Reggeized scattering amplitude at $t=0$. So it is seen that the apparent lack of a crucial role for the lower-spin contributions to the Feynman amplitude 3(22) remains, when their contributions to the Reggeized scattering amplitude are considered. This will be discussed further in the next section.

Case (ii) is more interesting, and the corresponding contribution to (3) is

$$ (13) \quad \sum_{s=0}^{\infty} i(2s+1) \frac{g_1(s) g_2^*(s)}{t - m^2(s)} \left(\tfrac{1}{4}\beta^{1/2}\right)^s {}_2F_1\left(s, s+\tfrac{1}{2}; \tfrac{3}{4}; 1-\frac{t}{m^2(s)}\right) P_s(\cos\theta). $$

The contributions of (13) to the Reggeized scattering amplitude are quite different according as either both pairs of masses of the incoming and outgoing particles in the t-channel, are unequal, or at least one pair is equal. These two cases are discussed separately below.

In the former case, $\cos\theta=1$ for $t=0$, and thus the contribution of (13) to the Reggeized scattering amplitude is finite and independent of the crossed channel centre of momentum energy variable at that point. It should be noted that the three Regge contributions in case (ii) are, here, also separately finite at $t=0$. Thus the leading branch point plays no crucial role with regard to the singularity structure of the amplitude at $t=0$, for unequal masses.

However, when at least one of the pairs of masses is equal, t=0 singularities appear in the leading branch point and background integral contributions; and, if $\mathrm{Re}\,\alpha(0)<0$, in the Regge pole term as well. To see how these singularities arise, it should firstly be noted that the small t behaviour of $\beta^{1/2}$ is, for one or both pairs of masses equal, given respectively by $O(t^{1/2})$ or $O(t)$. On taking this remark into account, and noting the explicit appearances of $\beta^{1/2}$ in (11), it is evident that the Regge pole, leading branch point, and background integral contributions are singular as stated above.

Now it follows from the small t behaviour of β, for at least one pair of masses equal, that the only term in (13) to survive at t=0 is the term with s=0. Thus again the t=0 behaviour of the Reggeized scattering amplitude is finite and independent of the crossed channel centre of momentum energy variable. This being so, the singularities in the Regge pole, leading branch point, and background integral contributions must cancel amongst themselves to ensure the finiteness of the Reggeized scattering amplitude at t=0.

Now in the model of Sugar and Sullivan[3], it was shown that the fixed poles, which there cancelled t=0 singularities in the Regge pole term, could only reasonably be discussed at that point. In order to discuss them elsewhere, self-energy corrections to the exchanged-particle propagator, which convert them to t-dependent poles, must be considered. By analogy with their model, it is to be expected that a true description of the above fixed branch point contributions, at points other than t=0, may only be given after the insertion of such self-energy corrections into the exchanged-particle propagator.

To illustrate the qualitative effects, on the fixed branch points, of the introduction of self-energy corrections, the case of the scattering of four equal mass particles, in the approximation that the scattering amplitude satisfies two-particle unitarity, is considered. The argument is that of Blankenbecler and Sugar[20]. This particular case is chosen merely for its simplicity ; and similar considerations for other mass configurations, and with exact self-energy corrections, would be expected to lead to the same qualitative results as presented here.

Using the effective interaction 3(1), and the Born term, $B(s;p',p)$ say, given by 3(22) and (9), the integral equation for the above modified scattering amplitude, $T(s;p',p)$ say, is given by[20]:

$$T(s;p',p) = B(s;p',p) - i\int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{B(s;p',k)\,T(s;k,p)}{\left[(\tfrac{1}{2}q+k)^2-m^2\right]\left[(\tfrac{1}{2}q-k)^2-m^2\right]}, \tag{14}$$

where the kinematics of chapter 3 are used, and m is the common mass of the particles involved. Defining $f(s)$ by

$$T(s;p',p) = f(s)\left(-\tfrac{1}{4}\beta^2\right)^s P_s(\cos\theta) \qquad + \text{ terms}$$

involving lower order Legendre polynomials,

$f(s)$ is easily calculated from (3), (9), and (14), with the result that

$$T(s;p',p) = \frac{i(2s+1)\,g^2(s)\,{}_2F_1\left(-s,s+\tfrac{1}{2};\tfrac{3}{4};1-\frac{t}{m^2(s)}\right)}{m^2(s)-t+ig^2(s)\,I(s;t)\,{}_2F_1\left(-s,s+\tfrac{1}{2};\tfrac{3}{4};1-\frac{t}{m^2(s)}\right)} \tag{15}$$

$$\left(-\tfrac{1}{4}\beta^{1/2}\right)^{s} P_s(\cos\theta) + \cdots,$$

where

$$(16)\quad I(s;t) = i(2s+1)\int_{-\infty}^{\infty} \frac{(q^2k^2 - k\cdot q\; k\cdot q)^s}{[(\tfrac{1}{2}q+k)^2-m^2][(\tfrac{1}{2}q-k)^2-m^2]}\frac{d^4k}{(2\pi)^4}.$$

Strictly speaking, cut-off functions should be introduced into the integral (16) in order to make it convergent[3,20]. However, the only property of the amplitude (15) which is required here, is independent of the cut-off functions. That is the property, evident from the form of (15), that the leading branch point in the modified partial-wave amplitude is again at $s=\alpha(0)$.

Thus the leading branch point is truly a fixed branch point (i.e. its position is independent of t), only the discontinuity across the corresponding branch cut being affected by the insertion of self-energy corrections. As stated above, the same conclusion is expected for a general mass configuration, and for all the secondary branch points.

Hence the final picture of the Reggeized scattering amplitude is one of contributions from:

(i) A moving Regge pole, a fixed leading branch point, and the corresponding part of the background integral.

(ii) Infinitely many fixed secondary branch points, spaced at intervals of two units in s, and the corresponding parts of the background integral.

Section 4. Discussion and conclusions:

Firstly a recursion relation, 3(23), and a generating function, 3(29), were obtained for the Feynman amplitude for the scattering of four massive spinless particles, with a spin s particle exchanged, the spin s particle being described by an (s,0) field of Weinberg. Then an explicit form, (6), for the amplitude, and another for the leading two terms of its partial-wave expansion, given by (9), were calculated.

In the above calculation, in order to cope with some problems concerning the discarding of non-covariant contact terms, which appear in the exchanged-particle propagator, it was found most convenient to write the (s,0) particle-field with respect to a manifestly covariant tensor basis. It was then shown in chapter 2 that such a tensor (s,0) particle-field satisfies certain tracelessness conditions for $s \geqslant 2$. Thus unlike the spinor (s,0) particle-field, the tensor (s,0) particle-field has redundant components for $s \geqslant 2$.

These redundant components, as in the case of Durand[1], manifest themselves by giving rise to lower-spin contributions to the Feynman amplitude 3(22). Nor can these lower-spin contributions be removed by repeating the calculation of chapter 3 using spinor-particle-fields to describe all the particles involved, and constructing the simplest three point interaction involving the spinor fields. For, as shown in appendix B, the simplest such interaction leads to the same results as those obtained in chapter 3.

In order to interpret the remarks of the previous paragraph, it should firstly be noted that a $(\frac{1}{2}s,\frac{1}{2}s)$ spin s particle-field has s^2 redundant components in a spinor basis, whilst in a tensor basis it has $\frac{1}{6}(s+1)(s+2)-(2s+1)$ redundant components. Both these numbers of redundant components are consistent with there being a direct relationship between the content of the lower-spin contributions to the corresponding Feynman amplitude[1], and the existence of these redundant components. However, the relationship is different according as a spinor, or a tensor $(\frac{1}{2}s,\frac{1}{2}s)$ particle-field is used.

Thus the remarks of the previous two paragraphs lend evidence to the assertion that, provided only the simplest three-point interactions are considered, there is a direct relationship between the content of the lower-spin contributions to the Feynman amplitude for the scattering of four massive spinless particles, due to the exchange of a spin s particle, and the number of redundant components in the tensor particle-field of the exchanged particle. And that, excepting accidents, such a relationship does not exist for the corresponding spinor particle-field.

In contrast to the work of Durand[1], the lower-spin contributions found here play no role with regard to the singularity structure of the Feynman amplitudes 3(22) at $t=0$. In fact each such contribution separately vanishes at that point.

In the present chapter, the Van Hove model[2] was employed to Reggeize the Feynman amplitude 3(22). This led to a scattering amplitude with contributions from a moving Regge pole, a leading fixed branch point at $s=\alpha(0)$, an infinite sequence of secondary fixed branch points at $s=\alpha(0)-2, \alpha(0)-4, \ldots$, and a background integral.

The fixed branch point at $s=\alpha(0)-2$, which is assumed to be characteristic of all the other secondary branch points, gives, together with the corresponding part of the background integral (see section 3), vanishing contribution to the scattering amplitude (11) at $t=0$. Also, the secondary branch point contributions are strongly suppressed, relative to the leading branch point contribution, at large values of the crossed channel energy variable; the domain where the Regge model is most important. Thus the secondary branch points, which arise from the redundant components in the exchanged tensor $(s,0)$ particle-field, via the lower-spin contributions to the Feynman amplitude 3(22), play no significant role in the scattering amplitude (11). This,coupled with the lack of any significant role for the lower-spin contributions to 3(22), is suggestive of the possibility that in a more sophisticated model, probably involving more complicated interactions than those of chapter 3 and appendix B, such contributions could be eliminated.

On the other hand, the leading branch point plays an important role with regard to the singularity structure of the scattering amplitude (11) at $t=0$. Two cases arise according as the masses of the incoming and outgoing particles in the t-channel are unequal, or at least one pair of them is equal. In the former case, the Regge pole, leading branch point, and background integral contributions, are separately finite at $t=0$. In the latter case, however, two further cases arise. If $\mathrm{Re}\,\alpha(0)\geqslant 0$, then the Regge pole term is finite at $t=0$, whilst the leading branch point contribution cancels a singularity in the background integral to give a finite scattering amplitude at $t=0$. On the other hand, if $\mathrm{Re}\,\alpha(0)<0$, all three of the above contributions are singular at $t=0$, whilst having a finite sum at that point.

Thus, in the model presented here, the Reggeized scattering amplitude may be looked upon as <u>effectively</u> comprising of a moving Regge pole, and background integral terms, together with a <u>kinematic</u> fixed branch point contribution. The branch point contribution plays no role with regard to the singularity structure of the scattering amplitude at $t=0$ in unequal mass scattering; whilst in equal mass scattering it serves to cancel $t=0$ singularities in either the background integral, or both the background integral and Regge pole terms.

This picture contrasts with the case of Durand[1], in which there is no singularity problem in equal mass scattering; whilst in the unequal mass case a $t=0$ singularity in the Regge pole term is cancelled by the joint effort of an infinite family of "daughter" Regge pole terms, which are often assumed to be of <u>dynamic</u> origin.

One further point of contrast is that, although both models lead to a scattering amplitude which is finite at $t=0$, in the model presented here that amplitude is independent of the crossed channel energy there, whilst in Durand's model it exhibits the characteristic Regge behaviour at that point. In order to see that the loss of Regge behaviour at $t=0$, in the model presented here, is not an inadequacy of the same, three points should be noted.

Firstly, the only moving Regge singularity which contributes to the amplitude (11) is the Regge pole. Secondly, the loss of Regge behaviour at $t=0$ is due to the appearance of a factor t in β, which renders the residue, at this Regge pole, trivially evasive[21]. Finally there is a growing evidence that the high-energy behaviour of scattering amplitudes, in the neighbourhood of $t=0$, is often best explained in terms of moving Regge branch points, together with evasive Regge poles, the branch point effects dominating near $t=0$[22].

Thus it is argued that the Regge model presented here must be complemented by the inclusion of moving Regge branch point effects, in order that a fuller description of the scattering amplitude may be given for all t-values.

This composite picture has the intuitive appeal that any singularities, appearing in the Regge pole and background integral terms at $t=0$, are cancelled in a simple <u>kinematic</u> way by a fixed s-plane branch point contribution; and that the moving Regge branch point contributions, needed to complement the above Regge model, are generated in the Van Hove model by higher order Feynman diagrams[2].

APPENDIX A.

Section 1. Covariant matrices and their properties: In section 1 of chapter 2 the Pauli spin-matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

were introduced. From their definition the following properties are readily verified.

(1) $\mathrm{tr}(\sigma_i) = 0$

(2) $\sigma_i^* = \sigma_i^T$

(3) $[\sigma_i, \sigma_j]_+ = 2\delta_{ij}$

(4) $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$

(5) $C^{-1}\sigma_i C = -\sigma_i^T,$

where the matrix C is given by

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Evidently

(6) $C^T = -C.$

The Pauli matrices may be combined with the unit 2x2 matrix to form the following covariant matrices

(7) $(\sigma_\mu)_{a\dot{b}} = (1 \quad -\underline{\sigma})_{a\dot{b}}$

(8) $$(\tilde{\sigma}_\mu)^{\dot{b}a} = (1 \quad \underline{\sigma})^{\dot{b}a} .$$

By a consideration of the explicit forms of these two matrices, together with the properties (3) and (4) of the Pauli matrices, it is seen that they transform as follows.

(9) $$\left(e^{-\frac{1}{2}\underline{\eta}\cdot\underline{\sigma}}\right)_a{}^b (\sigma_\mu)_{b\dot{c}} \left(e^{-\frac{1}{2}\underline{\eta}\cdot\underline{\sigma}}\right)^{\dot{c}}{}_{\dot{d}} = \Lambda_\mu{}^\nu (\sigma_\nu)_{a\dot{d}}$$

(10) $$\left(e^{\frac{1}{2}\underline{\eta}\cdot\underline{\sigma}}\right)^{\dot{a}}{}_{\dot{b}} (\tilde{\sigma}_\mu)^{\dot{b}c} \left(e^{\frac{1}{2}\underline{\eta}\cdot\underline{\sigma}}\right)_c{}^d = \Lambda_\mu{}^\nu (\tilde{\sigma}_\nu)^{\dot{a}d} .$$

That is they transform as their indices suggest.

It is well known that these two covariant matrices give respectively the transformations from a $(\frac{1}{2},\frac{1}{2})$ spinor representation with two upper labels and a $(\frac{1}{2},\frac{1}{2})$ spinor representation with two lower labels to $(\frac{1}{2},\frac{1}{2})$ tensor representations.[7]

In chapter 2 the covariant matrix which effects the transformation from an $(s,0)$ spinor representation to an $(s,0)$ tensor representation was constructed for $s > 1$ from the case $s=1$, which in turn was constructed from the covariant matrix

(11) $$(\sigma_{\mu\nu})_a{}^b = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_i & -\epsilon_{ijk}\sigma_k \end{pmatrix}_a{}^b .$$

Where $\frac{1}{2}(\sigma_{\mu\nu})_a{}^b$ is just the $(\frac{1}{2},0)$ representation of the Lorentz group generators. The property of being a representation of the Lorentz group generators immediately leads to the commutation relations 1(8), i.e.

(12) $$[\sigma_{\mu\nu}, \sigma_{\lambda\rho}]_a{}^b = 2i\left(g_{\mu\lambda}\sigma_{\nu\rho} + g_{\nu\rho}\sigma_{\mu\lambda} - g_{\mu\rho}\sigma_{\nu\lambda} - g_{\nu\lambda}\sigma_{\mu\rho}\right)_a{}^b .$$

The following identities are also satisfied.

$$(13)\quad [\sigma_{\mu\nu}, \sigma_{\lambda\rho}]_{+\,a}{}^{b} = 2\,(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} + i\,\epsilon_{\mu\nu\lambda\rho})\,\delta_a{}^b.$$

$$(14)\quad \mathrm{tr}\,(\sigma_{\mu\nu}) = (\sigma_{\mu\nu})_a{}^a = 0.$$

$$(15)\quad \tfrac{i}{2}\,\epsilon_{\mu\nu}{}^{\lambda\rho}\,(\sigma_{\lambda\rho})_a{}^b = (\sigma_{\mu\nu})_a{}^b.$$

$$(16)\quad (e^{-\frac{1}{2}\eta\cdot\sigma})_a{}^b\,(\sigma_{\mu\nu})_b{}^c\,(e^{\frac{1}{2}\eta\cdot\sigma})_c{}^d = \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho\,(\sigma_{\lambda\rho})_a{}^d.$$

$$(17)\quad C^{-1\,aa'}(\sigma_{\mu\nu})_{a'}{}^{b}\;C^{-1\,cc'}(\sigma^{\mu\nu})_{c'}{}^{d} = 4\,(C^{-1\,ac}\,C^{-1\,db} + C^{-1\,ad}\,C^{-1\,cb}).$$

$$(18)\quad (\sigma_{\mu\nu})_a{}^b\,(\sigma_{\lambda\rho})_b{}^a = 2\,(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} + i\,\epsilon_{\mu\nu\lambda\rho})$$

Equation (13) is proved by inspection of the explicit form (11) and use of the anticommutation relations of the Pauli matrices given by (3).

Equation (14) is an immediate consequence of the explicit form (11) and the traceless property of the Pauli matrices given by (1)

Equation (15) is an immediate consequence of inspection of the explicit form (11). A quantity which satisfies this property is said to be self-dual, and is automatically antisymmetric under the interchange $\mu \leftrightarrow \nu$.

Equation (16) gives the transformation property of $(\sigma_{\mu\nu})_a{}^b$, which is seen to be just as the indices suggest. The proof is a consequence of the properties (3), (4) and (5) of the Pauli matrices.

Equation (17) follows by inspection of the explicit form of both sides of the equality.

Equation (18) is a consequence of the commutation relations (12) and (13) together with the tracelessness condition (14).

One further property of $(\sigma_{\mu\nu})_a{}^b$ is required. It is that the quantity $(C^{-1})^{ab}(\sigma_{\mu\nu})_b{}^c$ is symmetric under the interchange $a \leftrightarrow c$. The assertion is an immediate consequence of the explicit form (11) and the following result

$$(C^{-1}\underline{\sigma})^{ab} = -(\underline{\sigma}^T C^{-1})^{ab} \qquad \text{by (5)}$$

$$= (\underline{\sigma}^T C^{-1T})^{ab} \qquad \text{by (6)}$$

$$= (C^{-1}\underline{\sigma})^{ba} .$$

In a manner exactly similar to the above the $(0,\frac{1}{2})$ representation of the Lorentz group generators is introduced. It has the explicit form

$$\text{(19)} \qquad \tfrac{1}{2}(\tilde{\sigma}_{\mu\nu}^T)^{\dot a}{}_{\dot b} = \tfrac{1}{2}\begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_i & \epsilon_{ijk}\sigma_k \end{pmatrix}^{\dot a}{}_{\dot b} .$$

Properties akin to the above for $(\sigma_{\mu\nu})_a{}^b$ follow in an exactly similar manner, but since they are not needed here their explicit form and proofs are omitted.

<u>Section 2. Simple identities involving covariant matrices.</u>

$$\text{(20)} \quad C^{-1\,ab}(\sigma_{\mu\nu})_b{}^d C_{dc} = -(\sigma_{\mu\nu}{}^T)^a{}_c .$$

$$\text{(21)} \quad C^{-1\,\dot a\dot b}(\tilde{\sigma}_{\mu\nu})_{\dot b}{}^{\dot d} C_{\dot d\dot c} = -(\tilde{\sigma}_{\mu\nu}{}^T)^{\dot a}{}_{\dot c} .$$

$$\text{(22)} \quad C^{-1\,ab}(\sigma_\mu)_{b\dot d}\, C^{-1\,\dot d\dot c} = -(\tilde{\sigma}_\mu{}^T)^{a\dot c} .$$

$$\text{(23)} \quad (\sigma_\mu)_{a\dot b}(\tilde{\sigma}_\nu)^{\dot b c} = g_{\mu\nu}\,\delta_a{}^c + i(\sigma_{\mu\nu})_a{}^c .$$

(24) $(\tilde{\sigma}_\mu)^{\dot{a}b}(\sigma_\nu)_{b\dot{c}} = g_{\mu\nu}\delta^{\dot{a}}{}_{\dot{c}} - i(\tilde{\sigma}_{\mu\nu}{}^T)^{\dot{a}}{}_{\dot{c}}.$

(25) $(\sigma_\mu)_{a\dot{b}}(\tilde{\sigma}_\nu)^{\dot{b}c} + (\sigma_\nu)_{a\dot{b}}(\tilde{\sigma}_\mu)^{\dot{b}c} = 2g_{\mu\nu}\delta_a^c.$

(26) $(\tilde{\sigma}_\mu)^{\dot{a}b}(\sigma_\nu)_{b\dot{c}} + (\tilde{\sigma}_\nu)^{\dot{a}b}(\sigma_\mu)_{b\dot{c}} = 2g_{\mu\nu}\delta^{\dot{a}}{}_{\dot{c}}.$

(27) $(\sigma_{\mu\nu})_a{}^{b\,*} = -(\tilde{\sigma}_{\mu\nu})_{\dot{a}}{}^{\dot{b}}.$

Eqations (20) and (21) follow immediately from the fact that the components of $(\sigma_{\mu\nu})_a{}^b$ and $(\tilde{\sigma}_{\mu\nu})_{\dot{a}}{}^{\dot{b}}$ are all proportional to some component of $\underline{\sigma}$, and the components of $\underline{\sigma}$ satisfy (5).

Equation (22) follows by inspection of the explicit forms of all the matrices involved.

Equations (23) and (24) follow from the explicit forms of $(\sigma_{\mu\nu})_a{}^b$ and $(\tilde{\sigma}_{\mu\nu}{}^T)^{\dot{a}}{}_{\dot{b}}$, given by (11) and (19) respectively, the explicit forms of $(\sigma_\mu)_{a\dot{b}}$ and $(\tilde{\sigma}_\mu)^{\dot{a}b}$, given respectively by (7) and (8), and finally the commutation relations (3) and (4) of the Pauli matrices.

Equation (25) follows from equation (23) by effecting the interchange $\mu \leftrightarrow \nu$ in this equation, and adding the result to the original equation. Equation (26) follows from equation (24) in an exactly similar manner.

Equation (27) is an immediate consequence of the explicit forms of $(\sigma_{\mu\nu})_a{}^b$ and $(\tilde{\sigma}_{\mu\nu}{}^T)^{\dot{a}}{}_{\dot{b}}$, together with the property (2) of the Pauli matrices.

Section 3. Identities involving contracted covariant matrices:

The results of this section will only be needed in chapter 3, and the notation used here is the same as that which was introduced in section 2 of that chapter. It must also be remembered that whenever a product of the form $\Pi\tilde{\Pi}$ or $\tilde{\Pi}\Pi$ appears it must be replaced by the appropriate Kronecker delta (see page 51).

(28) $\Pi\tilde{\sigma}^T_{pq}\tilde{\Pi} = -\sigma_{pq} + \frac{2i}{m_s}(p\cdot q\,\sigma_q - q^2\sigma_p)\tilde{\Pi}.$

(29) $\tilde{\Pi}\sigma_{p'q}\Pi = -\tilde{\sigma}^T_{p'q} - \frac{2i}{m_s}(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'})\Pi.$

(30) $\sigma_{p'q}(p\cdot q\,\sigma_q - q^2\sigma_p) = (p'\cdot q\,\sigma_q - q^2\sigma_{p'})\tilde{\sigma}^T_{pq}.$

(31) $\tilde{\sigma}^T_{pq}(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}) = (p\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_p)\sigma_{p'q}.$

(32) $(p\cdot q\,\sigma_q - q^2\sigma_p)(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}) = q^2\sigma_{pq}\sigma_{p'q}.$

Proof of (28).

By (24)

$$\Pi\tilde{\sigma}^T_{pq}\tilde{\Pi} = -i\Pi(p\cdot q - \tilde{\sigma}_p\sigma_q)\tilde{\Pi},$$

which, after the application of equations (25) and (26) , respectively, becomes

$$= -i\,p\cdot q + \frac{2i}{m_s}(p\cdot q\,\sigma_q - q^2\sigma_p)\tilde{\Pi} + i\sigma_p\tilde{\sigma}_q$$

$$= -\sigma_{pq} + \frac{2i}{m_s}(p\cdot q\,\sigma_q - q^2\sigma_p)\tilde{\Pi} \qquad \text{by (23).}$$

Q.E.D.

Proof of (29).

By (23)

$$\tilde{\Pi}\sigma_{p'q}\Pi = i\tilde{\Pi}(p'\cdot q - \sigma_{p'}\tilde{\sigma}_q)\Pi,$$

which, after the application of equations (25) and (26) , respectively, becomes

$$= i\,p'\cdot q - \frac{2i}{m_s}\left(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}\right)\Pi - i\tilde{\sigma}_{p'}\sigma_q$$

$$= -\tilde{\sigma}^T_{p'q} - \frac{2i}{m_s}\left(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}\right)\Pi \qquad \text{by (24).}$$

Q.E.D.

Proof of (30).

By (23)

$$\sigma_{p'q}\left(p\cdot q\,\sigma_q - q^2\sigma_p\right) = i\left(p'\cdot q - \sigma_{p'}\tilde{\sigma}_q\right)\left(p\cdot q\,\sigma_q - q^2\sigma_p\right)$$

$$= i\left(p\cdot q\,p'\cdot q\,\sigma_q - q^2 p'\cdot q\,\sigma_p - q^2 p\cdot q\,\sigma_{p'} + q^2\sigma_{p'}\tilde{\sigma}_q\sigma_p\right)$$

after use of (26). Next (25) gives

$$= i\left(p'\cdot q\,\sigma_q - q^2\sigma_{p'}\right)\left(p\cdot q - \tilde{\sigma}_q\sigma_p\right)$$

$$= -\left(p'\cdot q\,\sigma_q - q^2\sigma_{p'}\right)\tilde{\sigma}^T_{qp} \qquad \text{by (24).}$$

$$= \left(p'\cdot q\,\sigma_q - q^2\sigma_{p'}\right)\tilde{\sigma}^T_{pq}.$$

Q.E.D.

Proof of (31).

By (24)

$$\tilde{\sigma}^T_{pq}\left(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}\right) = -i\left(p\cdot q - \tilde{\sigma}_p\sigma_q\right)\left(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'}\right)$$

$$= -i\left(p\cdot q\,p'\cdot q\,\tilde{\sigma}_q - q^2 p\cdot q\,\tilde{\sigma}_{p'} - q^2 p'\cdot q\,\tilde{\sigma}_p + q^2\tilde{\sigma}_p\sigma_q\tilde{\sigma}_{p'}\right)$$

after use of (25). Next (26) gives

$$= -i(p\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_p)(p'\cdot q - \sigma_q\tilde{\sigma}_{p'})$$

$$= -(p\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_p)\,\sigma_{qp'} \qquad \text{by (23).}$$

$$= (p\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_p)\,\sigma_{p'q}.$$

Q.E.D.

<u>Proof of (32).</u>

By (25)

$$(p\cdot q\,\sigma_q - q^2\sigma_p)(p'\cdot q\,\tilde{\sigma}_q - q^2\tilde{\sigma}_{p'})$$

$$= q^2\,p\cdot q\;p'\cdot q + q^4\sigma_p\tilde{\sigma}_{p'} - q^2\,p'\cdot q\,\sigma_p\tilde{\sigma}_q - q^2\,p\cdot q\,\sigma_q\tilde{\sigma}_{p'}$$

$$= q^2\,p\cdot q\;p'\cdot q + q^4(p\cdot p' + i\sigma_{pp'})$$

$$- q^2\,p'\cdot q\,(p\cdot q + i\sigma_{pq}) - q^2\,p\cdot q\,(p'\cdot q + i\sigma_{p'q})$$

after use of (23).

$$= q^4\,p\cdot p' - q^2\,p\cdot q\;p'\cdot q + iq^4\sigma_{pp'}$$

$$+ iq^2\,p\cdot q\,\sigma_{p'q} - iq^2\,p'\cdot q\,\sigma_{pq}.$$

By use of (12) and (13) this then becomes

$$= \tfrac{1}{2}q^2[\sigma_{pq},\sigma_{p'q}]_+ + \tfrac{1}{2}q^2[\sigma_{pq},\sigma_{p'q}]$$

$$= q^2\,\sigma_{pq}\,\sigma_{p'q}.$$

Q.E.D.

APPENDIX B.

In chapter 3 the Feynman graph for the scattering of four massive spinless particles with a spin s particle exchanged was calculated. In this calculation a tensor $(s,0)$ field was used to describe the spin s particle. This appendix will be concerned with showing the equivalence between an approach using a spinor $(s,0)$ field to describe the spin s particle, and the above approach. The first step is to construct the interaction Hamiltonian density, involving spinor fields, which is parallel to 3(1).

Section 1. Interactions.

In order to construct the spin zero-spin zero-spin s three particle vertex it is necessary to find some way of contracting all the spinor indices on the $(s,0)$ particle-field $\psi_{a_1 \cdots a_{2s}}{}^{\dagger}(x)$ in a non-trivial manner. The simplest way of doing this is based on the remarks of page 54, and involves using fields which transform under the $(\frac{s}{2},\frac{s}{2})$ representation to describe all the spinless particles.

The particle-spinor for an $(\frac{s}{2},\frac{s}{2})$ representation field which describes a spin zero particle is given by

$$(1)\quad u_{a_1}(p_\mu s_1)\cdots u_{a_s}(p_\mu s_s)\, u_{\dot b_1}(p_\mu t_1)\cdots u_{\dot b_s}(p_\mu t_s)$$
$$\langle s_1 \cdots s_s | \tfrac{s}{2}\, U\rangle \langle t_1 \cdots t_s | \tfrac{s}{2}\, V\rangle \langle \tfrac{s}{2} U \tfrac{s}{2} V | 0\, 0\rangle .$$

Since the first two spin couplings are totally symmetric in the s_i and t_i respectively, the above is totally symmetric in the a_i and $\dot b_i$ separately. Thus it transforms under the $(\frac{s}{2},\frac{s}{2})$ representation [7]. The spin coupling is evidently correct for a zero spin particle. This expression may be greatly simplified by noting two things. Firstly the Clebsch-Gordan coefficient

$\langle \frac{s}{2} u \frac{s}{2} v | 0 0 \rangle$ is given explicitly by

$$\langle \tfrac{s}{2} u \tfrac{s}{2} v | 0 0 \rangle = \frac{1}{(2s+1)^{1/2}} (-1)^{s-u} \delta_{u\,-v}$$
$$= \frac{1}{(2s+1)^{1/2}} C_{uv}.$$

Secondly, since $\langle s_1 \ldots s_s | \frac{s}{2} u \rangle$ and $\langle t_1 \ldots t_s | \frac{s}{2} v \rangle$ are the transformation matrices from a basis of the rotation group representation with weight $\frac{s}{2}$ and labelled by U or V, to a basis of the same space labelled by s_i or t_i respectively, the following is true.

$$\langle s_1 \cdots s_s | \tfrac{s}{2} u \rangle C_{uv} \langle t_1 \ldots t_s | \tfrac{s}{2} v \rangle$$
$$= C_{s_1 s_1'} \cdots C_{s_s s_s'} \langle s_1' \cdots s_s' | \tfrac{s}{2} v \rangle \langle \tfrac{s}{2} v | t_1 \cdots t_s \rangle,$$

which, by equation 2(19) and the lemma of chapter 2 section 3, becomes

$$= \frac{1}{s!} \sum_{P(t_i)} C_{s_1 P(t_1)} \cdots C_{s_s P(t_s)} .$$

By considering the explicit form of the $v^*_{b_i}(p_\mu t_j)$, and using the results of the above paragraph the expression (1) reduces to the following.

$$(2) \quad \frac{1}{(2s+1)^{1/2}} \frac{1}{s!} \sum_{P(b_i)} \Pi_{a_1 P(b_1)}(p_\mu) \cdots \Pi_{a_s P(b_s)}(p_\mu),$$

where Π is, just as before, proportional to the lower $(\frac{1}{2},0)$ spin $\frac{1}{2}$ particle-field propagator numerator. The field corresponding to the particle-spinor (2) will be denoted by

$$\phi_{a_1\cdots a_s\dot{b}_1\cdots\dot{b}_s}(x).$$

Now that the fields corresponding to the spinless particles have been constructed it only remains to couple the three fields

$$\phi_{a_1\cdots a_s\dot{b}_1\cdots\dot{b}_s}(x),\ \chi_{a_1\cdots a_s\dot{b}_1\cdots\dot{b}_s}(x),\ \psi_{a_1\cdots a_{2s}}^{\dagger}(x),$$

in an invariant manner to obtain the interaction Hamiltonian density for the desired three particle vertex. The density chosen is the following.

$$(3)\quad \mathcal{H}_I(x) = g\,\phi^{a_1\cdots a_s\dot{b}_1\cdots\dot{b}_s}(x)\,\chi_{a_1\cdots a_s}{}^{\dot{b}_{s+1}\cdots\dot{b}_{2s}}(x)$$

$$\psi_{b_1\cdots b_{2s}}{}^{\dagger}(x) + \text{Hermitian conjugate},$$

where the spin zero particle fields appearing are not dual fields of the original fields, but have just had indices raised by means of the raising matrix. This density was chosen because it is the simplest involving the above three fields, and as will be seen later turns out to be exactly parallel with the density 3(1).

<u>Section 2</u>. <u>Equivalence to the tensor approach</u>.

On the basis of the remarks of section 1, the following interaction Hamiltonian density is postulated.

$$(4)\quad \mathcal{H}_I(x) = \sum_{i=1}^{2} h_i\,\phi_i^{a_1\cdots a_s\dot{b}_1\cdots\dot{b}_s}(x)\,\chi_{i\,a_1\cdots a_s}{}^{\dot{b}_{s+1}\cdots\dot{b}_{2s}}(x)$$

$$\psi_{b_1\cdots b_{2s}}{}^{\dagger}(x) + \text{Hermitian conjugate}.$$

Using this expression for the interaction Hamiltonian density, together with (2) for the external particle wave-functions and 2(22) for the propagator of the lower (s,0) particle-field, the Feynman rules give the following as the contribution from the above-mentioned graph.

$$\frac{-ih_1h_2^*}{q^2-m_s^2+i\epsilon}\frac{1}{(2s+1)^2}C^{-1a_1a_1'*}\cdots C^{-1a_sa_s'*}C^{-1\dot b_1\dot b_1'*}\cdots C^{-1\dot b_{2s}\dot b_{2s}'*}\Pi^*_{a_1'\dot b_1'}(p_\mu')\cdots\Pi^*_{a_s'\dot b_s'}(p_\mu')$$
$$\Pi^*_{a_1\dot b_{s+1}'}(q_\mu-p_\mu')\cdots\Pi^*_{a_s\dot b_{2s}'}(q_\mu-p_\mu')\frac{1}{(2s)!}\sum_{P(\dot c_i)}$$
$$\Pi_{b_1P(\dot c_1)}(q_\mu)\cdots\Pi_{b_{2s}P(\dot c_{2s})}(q_\mu)\;C^{-1d_1d_1'}\cdots$$
$$C^{-1d_sd_s'}C^{-1\dot c_1\dot c_1'}\cdots C^{-1\dot c_{2s}\dot c_{2s}'}\Pi_{d_1'\dot c_1'}(p_\mu)\cdots$$
$$\Pi_{d_s'\dot c_s'}(p_\mu)\Pi_{d_1\dot c_{s+1}'}(q_\mu-p_\mu)\cdots\Pi_{d_s\dot c_{2s}'}(q_\mu-p_\mu).$$

This expression may be greatly simplified by use of the following identities.

$$(5)\quad C^{-1aa'*}\Pi^*_{a'\dot b'}C^{-1\dot b\dot b'*}=(\tilde\Pi^T)^{a\dot b*}=\tilde\Pi^{\dot a b}.$$

The first part of this equality follows from A(22), whilst the second part is a consequence of A(2) together with the remarks of page . In a similar manner it follows that

$$(6)\quad C^{-1\dot b\dot b'*}\Pi^*_{a\dot b'}=C^{-1bb'}\Pi_{b'\dot a}.$$

Taking the complex conjugate of equation (5) gives the following equation.

$$(7)\quad C^{-1aa'}\Pi_{a'\dot b'}C^{-1\dot b\dot b'}=\tilde\Pi^{\dot b a}.$$

Finally, the following two equations are immediate consequences of

A(23) and A(24) respectively.

(8) $$\Pi_{b'\dot{a}}(q_\mu - p'_\mu)\,\tilde{\Pi}^{\dot{a}b}(p'_\mu) = p'\cdot(q-p')\delta_{b'}{}^{b} + i(\sigma_{q-p'\,p'})_{b'}{}^{b}$$
$$= p'\cdot(q-p')\delta_{b'}{}^{b} + i(\sigma_{qp'})_{b'}{}^{b}.$$

(9) $$\tilde{\Pi}^{\dot{c}'d}(p_\mu)\,\Pi_{d\dot{c}}(q_\mu - p_\mu) = p\cdot(q-p)\delta^{\dot{c}'}{}_{\dot{c}} - i(\tilde{\sigma}^{T}_{p\,q-p})^{\dot{c}'}{}_{\dot{c}}$$
$$= p\cdot(q-p)\delta^{\dot{c}'}{}_{\dot{c}} - i(\tilde{\sigma}^{T}_{pq})^{\dot{c}'}_{\dot{c}}.$$

Making use of the identities (5), (6), (7), (8) and (9), together with the fact that the propagator for the (s,0) field is totally symmetric in its indices reduces the above expression for the amplitude to the following.

(10) $$\frac{-ih_1h_2^*}{q^2-m_s^2+i\epsilon}\,\frac{(-1)^s}{(2s+1)^2}\,C^{-1\,b_1b_1'}(\sigma_{p'q})_{b_1'}{}^{b_1}\ldots C^{-1\,b_{2s-1}b_{2s-1}'}$$
$$(\sigma_{p'q})_{b_{2s-1}'}{}^{b_{2s}}\,\frac{1}{(2s)!}\sum_{P(\dot{c}_i)}\Pi_{b_1P(\dot{c}_1)}(q_\mu)\ldots\Pi_{b_{2s}P(\dot{c}_{2s})}(q_\mu)$$
$$C^{-1\dot{c}_1\dot{c}_1'}(\tilde{\sigma}_{pq})_{\dot{c}_1'}{}^{\dot{c}_2}\ldots C^{-1\dot{c}_{2s-1}\dot{c}_{2s-1}'}(\tilde{\sigma}_{pq})_{\dot{c}_{2s-1}'}{}^{\dot{c}_{2s}}.$$

But this is just proportional to the expression 3(4) for the corresponding amplitude calculated using a tensor formulation. In fact on writing

$$h_i = \frac{2s+1}{2^{3s/2}}\,g_i \qquad (i=1,2),$$

the two amplitudes are seen to be identically equal.

Thus the equivalence between the tensor and spinor formulations is verified.

REFERENCES

(1) L. Durand III; Physical Review 154, 1537 (1967).

(2) L. Van Hove: . Physics Letters 24B, 183 (1967).

(3) R.L. Sugar and J.D. Sullivan: Physical Review 166, 1515 (1968).

(4) D.Z. Freedman and J.M. Wang: Physical Review 153, 1596 (1967).

(5) S. Weinberg: Physical Review 133, 1318 (1964) and ibid 134, 882 (1964).

(6) P.T. Matthews: Physical Review 76, 684 (1949) and H. Umezawa: Quantum Field Theory (North Holland 1956) Chapter 10.

(7) I.N. Gel'fand, R.A. Minlos and Z.Ya. Shapiro: Representations of the Lorentz and Rotation Groups (Pergamon 1963).

(8) For example S.S. Schweber: Introduction to Relativistic Quantum Field Theory (Harper and Row 1961) Chapter 2.

(9) M.E. Rose: Elementary Theory of Angular Momentum (Wiley 1957).

(10) K. Koller: Nuovo Cimento 54A, 79 (1968).

(11) W. Pauli: Physical Review 58, 716 (1940).

(12) Reference (8) Chapter 13.

(13) J.C. Burkill: A First Course in Mathematical Analysis (Cambridge 1962) Chapter 2.

(14) I am indebted to Dr. John Hawkes for remarks concerning the convolution-like form of 3(23).

(15) A. Erdelyi et al: Higher Transcendental Functions (McGraw-Hill 1953) Volume 2 p.174.

(16) I.S. Gradshteyn and I.M. Ryzhik: Tables of Integrals, Series and Products (Academic Press 1965).

(17) Reference (15) Volume 1.

(18) M. Froissart: Physical Review 123, 1053 (1961).

(19) S.C.Frautschi:Regge Poles and S-matrix Theory (Benjamin 1963).

(20) R. Blankenbecler and R.L. Sugar: Physical Review 168, 1597 (1968).

(21) See for example G.E. Hite: Reviews of Modern Physics 41, 669 (1969) and references therein.

(22) F. Henyey et al: Physical Review 182, 1579 (1969);
J. Froyland and D. Gordon: Physical Review 177, 2500 (1969);
B.J. Hartley, R.W. Moore and K.J.M. Moriarty: Physical Review 187, 339 (1969);
Part two of this thesis.

PART TWO

INTRODUCTION

The work presented in part two of this thesis is the result of a collaboration with the authors named in reference (0), and is based on the work exhibited there. My role in the work was that of setting up the required formalism and doing all tho theoretical calculations.

In charge-exchange pion photoproduction reactions, there is a characteristic peaking of the differential cross-section in the forward direction.[1] This peaking, although suggestive of Reggeized pion exchange, cannot be explained on the basis of that mechanism alone. For in order that angular momentum conservation in the forward direction be satisfied, either some of the pion resdues must vanish there, or there must exist a pion conspirator Regge pole; and the former case leads to a differential cross-section which vanishes in the forward direction.[2]

On the other hand, although the pion together with its conspirator can produce the required forward peaking, the existence of the conspirator leads to discrepancies between the predicted and experimental results in some other reactions.[3]

One possible way out of this rather unsatisfactory situation is to take heed of Mandelstam's demonstration that Regge branch points, as well as Regge poles, should contribute to scattering amplitudes[4]. In this way the forward peaking in charge-exchange pion photoproduction reactions will be produced on the basis of Reggeized pion exchange together with a pion-Pomeron Regge cut.

In the following the Reggeized $U(6) \otimes U(6) \otimes O(3)$ symmetry scheme of Delbourgo and Salam[5] is used to calculate Regge pole contributions to charge-exchange pseudoscalar meson photoproduction amplitudes, via the Vector dominance model[6]; thus providing significant constraints amongst the Regge residues. Regge cut contributions are then introduced by applying absorption corrections to the Regge pole amplitudes. Results of the consequent fit to the experimental data for the reaction $\gamma p \rightarrow \pi^+ n$ are presented.

In chapter 1 a resumé of the formalism of Reggeized $U(6) \otimes U(6) \otimes O(3)$ is given[5], and is extended through the vector dominance model to include photoproduction reactions. Chapter 2 is concerned with a utilization of this formalism to calculate the Regge pole contributions firstly to the invariant amplitudes, and finally to the s-channel helicity amplitudes, for pseudoscalar meson photoproduction, which are required in order that absorption corrections may be applied. Finally chapter 3 contains a discussion of absorption correction Regge cuts, the Regge pole description of the reaction $\gamma p \rightarrow \pi^+ n$, and the results and conclusions of the fitting of the amplitudes, given by this model, to the experimental data.

Except where otherwise stated, the notation set out in the introduction to part one of this thesis will also be used in this second part.

CHAPTER 1

Since the photon is not directly incorporated into the $U(6) \otimes U(6) \otimes O(3)$ symmetry scheme, the first problem in the utilization of this scheme for a description of photoproduction processes is to relate these processes to other processes involving only hadrons. These hadronic processes will then be calculated by a direct use of the $U(6) \otimes U(6) \otimes O(3)$ symmetry. The required relation between photoproduction processes and the corresponding hadronic processes is here assumed to be given through the vector dominance model.[6]

In the vector dominance model the $SU(3)$ U-spin scalar transformation property of the photon[7] gives the covariant T-matrix for the photoproduction of mesons on baryons as follows:

$$(1) \quad T(\gamma B \to MB') = \chi_\rho \left\{ T(\rho^0 B \to MB') + \frac{1}{\sqrt{3}} T(\omega_8 B \to MB') \right\}$$

where the ρ-photon coupling χ_ρ is given by

$$e = \chi_\rho \, g_{\rho\pi\pi}$$

with

$$\frac{e^2}{4\pi} = \frac{1}{137} \quad \text{and} \quad \frac{g^2_{\rho\pi\pi}}{4\pi} = 1 \cdot 8 ,$$

and where ω_8 represents the mixture of the ω and ϕ mesons which transforms as the $SU(2)$ singlet part of the 1^- octet.

Now that the relation (1) between the photoproduction of mesons on baryons and meson-baryon scattering has been given, the remainder of this chapter will be concerned with a resumé of the formalism of Delbourgo and Salam for Reggeization in a $U(6) \otimes U(6) \otimes O(3)$ symmetry scheme. This formalism will then be used in chapter 2 to calculate the high energy contribution to the meson-baryon scattering processes appropriate to the ensuing fit to the experimental data.

The $U(6) \otimes U(6) \otimes O(3)$ symmetry scheme is an orbital excitation model in which the $U(6) \otimes U(6)$ intrinsic spin-unitary spin part of the symmetry treats quark and antiquark spins as distinct and independent, whilst the $O(3)$ orbital part of the symmetry corresponds to the orbital angular momentum of the quark-antiquark system given by $U(6) \otimes U(6)$ part of the symmetry.

All hadrons are assumed to be classified according to representations of the rest symmetry $U(6) \otimes U(6) \otimes O(3)$. A significant empirical feature of the spectroscopy is that only some rather simple representations of this group are realized in nature. They are representations which are characterized by just one quantum number L, where $L(L+1)$ is an eigenvaluo of the Casimir operator of the $O(3)$ subgroup given by $\underline{L}^2$, apart from baryon number.

The generalized helicity subgroup of this rest symmetry is $U_W(6) \otimes O(2)$. If the symmetry is exact for three particle vertices then W-spin is conserved.

The covariant embedding group, which is essential for the construction of relativistically invariant interactions, is in this case $U(6,6) \otimes O(3,1)$.

Section 1. The M-function formalism.

The calculation of scattering amplitudes is effected by means of an M-function approach in a multispinor formalism. This has the merits of making the construction of relativistically invariant interactions straightforward, and of automatically incorporating all mass-dependent kinematic factors into the scattering amplitudes. The latter property of this formalism easily lends itself to an incorporation of symmetry-breaking effects by means of the insertion of physical rather than multiplet masses in the above-mentioned kinematic factors.

In order to construct the multispinor formalism it is necessary to first find the particle-spinors representing all the multiplets involved. These particle-spinors will transform under representations of the covariant embedding group $U(6,6) \otimes O(3,1)$. Note that the covariant embedding group is just an auxiliary group in the sense of part one of this thesis [8].

The intrinsic spin-unitary spin $U(6,6)$ part of the particle-spinor is chosen to be represented by the multispinors of Delbourgo, Salam and Strathdee[9] given by:

$$\bar{\Phi}_A{}^B(p) \qquad A,B = 1, \ldots, 12,$$

for the mesons $(6,\bar{6})$ of the rest symmetry $U(6) \otimes U(6)$, and

$$\Psi_{(ABC)}(p) \qquad A,B,C, = 1, \ldots, 12,$$

which is totally symmetric in A,B,C, for the baryons $(56,1)$ of the same rest symmetry group. These multispinors are also assumed to satisfy the following Bargmann-Wigner equations.

$$(\not{p} - m_1)_A{}^B \bar{\Phi}_B{}^C(p) = \bar{\Phi}_A{}^B(p)(\not{p} + m_1)_B{}^C = 0 \tag{2}$$

and

$$(\not{p} - m_2)_A{}^B \Psi_{(BCD)}(p) = 0, \tag{3}$$

where m_1 and m_2 are respectively the masses associated with the above meson and baryon multiplets.

The orbital $O(3,1)$ part of the particle-spinor, since the orbital angular momentum L of the quark-antiquark system is always integral, is chosen to be a traceless symmetric tensor of rank L [10]. This multispinor must also satisfy the following subsidiary conditions.

(4) $$p^{\mu_1}\,\phi_{\mu_1\cdots\mu_L}(p)=0$$

and

(5) $$(p^2-m^2)\,\phi_{\mu_1\cdots\mu_L}(p)=0.$$

The direct product of the above O(3,1) and U(6,6) multispinors is now taken to give the following multispinors which transform under representations of the covariant embedding group U(6,6) ⊗ O(3,1).

(6) $$\Phi_A{}^B{}_{\mu_1\cdots\mu_L}(p) \qquad (6,\bar{6};L)$$

(7) $$\Psi_{(ABC)\mu_1\cdots\mu_L}(p) \qquad (56,1;L)$$

These particle-spinors both satisfy the subsidiary conditions (4) and (5), and they respectively satisfy the Bargmann-Wigner equations (2) and (3). After reduction with respect to U(3) ⊗ U(2,2) ⊗ O(3,1) the following forms for the particle-spinors (6) and (7) are obtained.

(8) $$\Phi_A{}^B{}_{\mu_1\cdots\mu_L}(p)=\left[\left(\frac{\not p+\mu}{2\mu}\right)\left(\gamma_5\phi_{5\mu_1\cdots\mu_L}+\gamma_\mu\phi^{\mu}{}_{\mu_1\cdots\mu_L}\right)\right]^{\beta b}_{\alpha a}$$

and

(9) $$\Psi_{(ABC)\mu_1\cdots\mu_L}(p)=\frac{\sqrt{3}}{2\sqrt{2}\,m}\left[(\not p+m)\gamma_\mu C\right]_{\alpha\beta}D^{\mu}_{(abc)\gamma\mu_1\cdots\mu_L}+\frac{1}{2\sqrt{6}\,m}\Big(\left[(\not p+m)\gamma_5 C\right]_{\alpha\beta}\epsilon_{abd}\,N_{\gamma c}{}^{d}{}_{\mu_1\cdots\mu_L} + \text{cyclic}$$

permutations of the indices).

In these expressions (8) and (9) the notation of reference (5) is used; and the (L) excitations of the basic spin multiplet particle-spinors, namely the pseudoscalar meson nonet ϕ_5, the vector meson nonet ϕ_μ, the $\frac{1}{2}^+$ baryon octet and the $\frac{3}{2}^+$ baryon decuplet, are exhibited explicitly.

Now that the multispinors (6) and (7) have been constructed, the M-function for a vertex which couples three multiplets of $U(6) \otimes U(6) \otimes O(3)$ may also be constructed. This is done by first noting, as was stated before, that the three-multiplet vertex must be $U_W(6) \otimes O(2)$ invariant; and that the multiplet momenta involved are scalars under the transformations of $U_W(6) \otimes O(2)$. Thus the three-multiplet M-function is constructed by a complete saturation of the indices of the particle-spinors involved amongst themselves, and with momenta. The number of independent ways of doing this gives all the independent couplings that may be constructed.

As an example of such a construction the following effective lagrangians required in the calculations of meson-baryon scattering , are quoted.

(10) $(56,1;0)_{\frac{1}{2}p-q} - (\overline{56},1;0)_{\frac{1}{2}p+q} - (6,\overline{6};L)_p$

$$\mathcal{L}_{\text{effective}} = m^{-L-1}\,\overline{\Psi}^{(ACD)}(\tfrac{1}{2}p+q)\,\Psi_{(BCD)}(-\tfrac{1}{2}p+q)\left[g_0\,\delta_A{}^B + m g_1 \frac{\partial}{\partial q_B{}^A}\right]\overline{\Phi}_L(p,q)$$

and

(11) $(6,\overline{6};0)_{\frac{1}{2}p+q'} - (6,\overline{6};0)_{\frac{1}{2}p-q'} - (6,\overline{6};L)_p$

$$\mathcal{L}_{\text{effective}} = \mu^{-L}\,\overline{\Phi}_A{}^B(\tfrac{1}{2}p+q')\,\overline{\Phi}_C{}^D(\tfrac{1}{2}p-q')$$

$$\Big[h_0^{(-)}\delta_B{}^C\delta_D{}^A + \frac{h_0'^{(-)}}{\mu^2}q'_B{}^A q'_D{}^C + \mu h_1^{(+)}\Big(\delta_B{}^C\frac{\partial}{\partial q'_A{}^D} + \delta_D{}^A\frac{\partial}{\partial q'_C{}^B}\Big) + \mu h_1^{(-)}\Big(\delta_B{}^C\frac{\partial}{\partial q'_A{}^D} - \delta_D{}^A\frac{\partial}{\partial q'_C{}^B}\Big)\Big]\Phi_L(p,q).$$

In both these expressions the quantity $\Phi_L(p,q)$ is the fully contracted $(6,\bar{6};L)$ meson particle-spinor, which is given by

(12) $$\Phi_L(p,q) = q_A{}^B q^{\mu_1}\cdots q^{\mu_L}\,\Phi_B{}^A{}_{\mu_1\cdots\mu_L}(p).$$

The superscripts $(\pm)$ of the couplings h for the meson-meson-meson vertex (11) refer to even and odd values of L respectively. Bose statistics gives that $h^{(+)}=0$ when L is odd, and that $h^{(-)}=0$ when L is even.

With the construction of the effective Lagrangians (10) and (11), the only further apparatus needed, in order that the Feynman graph for the scattering of a $(6,\bar{6};0)$ multiplet by a $(56,1;0)$ multiplet, with a $(6,\bar{6};L)$ multiplet exchanged in the crossed channel, may be calculated, is the propagator for the $(6,\bar{6};L)$ multiplet. This, when fully contracted in the manner of (12), is given by

(13) $$\Delta_L(q,q';p) = i\left(q\cdot q' - \frac{p\cdot q\; p\cdot q'}{M_L^2}\right)\frac{P_L(q,q';p)}{p^2 - M_L^2}$$

where $P_L(q,q';p)$ is the fully contracted propagator numerator corresponding to the $O(3,1)$ part of the particle-spinor. It is discussed in detail by Scadron in reference (10). Asymptotically the expression (13) for the $(6,\bar{6};L)$ meson multiplet propagator becomes

$$(14)\quad \Delta_L(q,q';p) \sim \frac{-i}{p^2-M_L^2}\left(\frac{p\cdot q\; p\cdot q'}{M_L^2} - q\cdot q'\right)^{L+1}.$$

Finally, in order to deal with the effective Lagrangians (10) and (11) some derivatives of (13) are required. They are given asymptotically as follows.

$$(15)\quad \frac{\partial}{\partial q_A{}^B}\Delta_L(q,q';p) \sim \frac{[(M_L+\not p)\not q'(M_L-\not p)]_B{}^A}{8M_L^2}$$

$$\frac{-i}{p^2-M_L^2}\left(\frac{p\cdot q\; p\cdot q'}{M_L^2} - q\cdot q'\right)^{L}$$

and

$$(16)\quad \frac{\partial^2}{\partial q_A{}^B\,\partial q'_C{}^D}\Delta_L(q,q';p) \sim \frac{1}{8M_L^2}(M_L+\not p)_B{}^C$$

$$(M_L-\not p)_D{}^A\;\frac{-i}{p^2-M_L^2}\left(\frac{p\cdot q\; p\cdot q'}{M_L^2} - q\cdot q'\right)^{L}.$$

All the apparatus needed for the calculation of the Feynman graph for the scattering of a $(6,\bar{6};0)$ multiplet by a $(56,1;0)$ multiplet, with a $(6,\bar{6};L)$ multiplet exchanged in the cross channel, has been amassed above. It only remains to give the procedure for the Reggeization of such an amplitude.

<u>Section 2</u>. <u>Reggeization</u>.

In the usual procedure for the Reggeization of scattering amplitudes it is the total angular momentum of the two-particle states in the crossed channel which is allowed to take on complex values. However, here it is not the above quantum number which is complexified, but the quantum number L, which is the excitation number rather than the spin of the exchanged particle.

The Reggeization procedure is carried through by means of a generalization of the Van Hove model.(11) In this model a sum over the relevant Feynman graphs with the exchange of a multiplet of excitation number L is written, for L taking the values L=0,1,2,... . Then this sum is written as an integral, and a Sommerfeld-Watson transform is effected. The contributions of Regge trajectories come from the zeros of the function

$$p^2 - M^2(L) = t - M^2(L) \qquad \text{say.}$$

Thus for t-channel Regge poles the trajectory functions are given by

$$L = \alpha(t) - 1 \qquad \text{say.}$$

In effect the above procedure for Reggeization is equivalent to making the following replacements.

(i) $L \rightarrow \alpha(t) - 1$

and

(ii) $t - M^2(L) \rightarrow \sin\pi(\alpha(t) - 1)$.

It is well known that the replacement (ii) leads to poles at non-sense values of L. This difficulty is overcome by the introduction of the Gell-Mann ghost killing mechanism, which involves multi-plying the Regge amplitude by $-\pi(\Gamma(\alpha))^{-1}$. Or, as is equivalent changing the replacement (ii) to (ii)a below.

(ii)a $t - M^2(L) \rightarrow \Gamma(1 - \alpha(t))$.

Finally, the above simple picture of the procedure of Regg-eization does not take signature into account. So signature is dealt with empirically by the extraction of signature factors

$$\frac{1}{2}\left(1 \pm e^{-i\pi(\alpha(t)-1)}\right) = \frac{1}{2}\left(1 \mp e^{-i\pi\alpha(t)}\right)$$

from the couplings $\bar{h}^{(\pm)}$. It must be remembered that here, unlike the usual case where signature corresponds to even or odd spin, signature corresponds to even or odd excitation number of the exchanged multiplet.

This completes the apparatus required for the calculation of Regge pole contributions to scattering amplitudes, in the context of a $U(6) \otimes U(6) \otimes O(3)$ rest symmetry scheme. The next chapter will be concerned with the application of this formalism to the calculation of the Reggeized scattering amplitudes for the reactions

$$0^- \tfrac{1}{2}^+ \longrightarrow 1^- \tfrac{1}{2}^+ ,$$

and hence, through the vector meson dominance model, together with time-reversal, the calculation of the scattering amplitudes for the reactions

$$\gamma \tfrac{1}{2}^+ \longrightarrow 0^- \tfrac{1}{2}^+ .$$

CHAPTER 2

This chapter, together with the remainder of part two, will only be concerned with charge-exchange processes.

With a view to the calculation of the charge-exchange processes

$$\gamma \tfrac{1}{2}^{+} \longrightarrow 0^{-} \tfrac{1}{2}^{+} ,$$

the $U(6) \otimes U(6) \otimes O(3)$ symmetry scheme is used in section 1 to calculate the Reggeized M-function for the charge-exchange processes

$$0^{-} \; \tfrac{1}{2}^{+} \longrightarrow 1^{-} \; \tfrac{1}{2}^{+}$$

This M-function is then used in section 2 to calculate the Regge pole contribution to the corresponding s-channel helicity amplitudes. Finally, in section 3, the link between these helicity amplitudes and those for the corresponding photoproduction processes is effected by means of time-reversal and the vector dominance model.

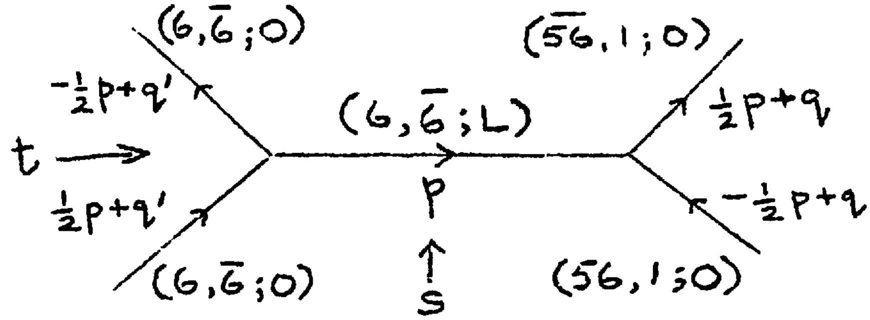
Section 1. The Reggeized M-function.

The effective Lagrangians 1(10) and 1(11) are to be used for the calculation of the Feynman graph for the charge-exchange processes

$$0^{-} \; \tfrac{1}{2}^{+} \longrightarrow 1^{-} \; \tfrac{1}{2}^{+} ,$$

with a multiplet of excitation number L exchanged. Since the couplings h_0 , h_0 and g_0 , appearing in 1(10) and 1(11), do not contribute to charge-exchange scattering, the couplings $h_1^{(\pm)}$ and g_1 are the only ones to be considered.

So, using the effective Lagrangians 1(10) and 1(11) , the particle-spinors 1(6) and 1(7) , and the expression 1(14) for the fully contracted propagator of the exchanged multiplet, the scattering amplitude for the graph



is given by the following.

$$(m\mu)^{-L}\mu\,\overline{\Psi}^{(ACD)}(\tfrac{1}{2}p+q)\,\Psi_{(BCD)}(-\tfrac{1}{2}p+q)\,\overline{\Phi}_{A'}{}^{B'}(\tfrac{1}{2}p+q')$$

$$\overline{\Phi}_{C'}{}^{D'}(\tfrac{1}{2}p-q')\Big[g_1h_1^{(+)}\Big(\delta_{B'}{}^{C'}\frac{\partial}{\partial q'_{A'}{}^{D'}}+\delta_{D'}{}^{A'}\frac{\partial}{\partial q'_{C'}{}^{B'}}\Big)+$$

$$g_1h_1^{(-)}\Big(\delta_{B'}{}^{C'}\frac{\partial}{\partial q'_{A'}{}^{D'}}-\delta_{D'}{}^{A'}\frac{\partial}{\partial q'_{C'}{}^{B'}}\Big)\Big]\frac{\partial}{\partial q_B{}^A}\Delta_L(q,q';p).$$

This readily simplifies to

$$(m\mu)^{-L}\mu\,\overline{\Psi}^{(ACD)}(\tfrac{1}{2}p+q)\,\Psi_{(BCD)}(-\tfrac{1}{2}p+q)\Big(g_1h_1^{(+)}$$

$$[\overline{\Phi}(\tfrac{1}{2}p+q'),\overline{\Phi}(\tfrac{1}{2}p-q')]_+ + g_1h_1^{(-)}[\overline{\Phi}(\tfrac{1}{2}p+q'),$$

$$\overline{\Phi}(\tfrac{1}{2}p-q')]\Big)_{B'}{}^{A'}\frac{\partial^2}{\partial q_B{}^A\,\partial q'_{B'}{}^{A'}}\Delta_L(q,q';p),$$

where the plus suffix on the square bracket denotes the anticommutator. A direct application of 1(16), where from now on the L-dependence of M_L is left understood, simplifies this expression further still to

$$(1)\quad -\frac{i}{2}\left(\frac{p\cdot q\;p\cdot q'\;M^{-2}-q\cdot q'}{m\mu}\right)^L\frac{\mu g_1}{p^2-M^2}\,\overline{\Psi}^{(ACD)}(\tfrac{1}{2}p+q)$$

$$\Psi_{(BCD)}(-\tfrac{1}{2}p+q)\Big(h_1^{(+)}\Gamma_+\big[\bar{\Phi}(\tfrac{1}{2}p+q'),\bar{\Phi}(\tfrac{1}{2}p-q')\big]_+ \Gamma_- + h_1^{(-)}\Gamma_+\big[\bar{\Phi}(\tfrac{1}{2}p+q'),\bar{\Phi}(\tfrac{1}{2}p-q')\big]\Gamma_-\Big)_A^{\ B},$$

where

$$\Gamma_\pm = (2M)^{-1}(M \pm \not{p}) \tag{2}$$

Two properties of $\Gamma_\pm$ which will be used in the following, and which follow immediately from the definition (2), are

$$\Gamma_\pm \not{p} = \not{p}\,\Gamma_\pm = \pm M \Gamma_\pm \tag{3}$$

and

$$\Gamma_+\Gamma_- = 0 . \tag{4}$$

Now that the expression (1) for the contribution of the above graph has been obtained, the next step is to reduce out the contribution to the corresponding graph for

$$0^-(56,1;0) \longrightarrow 1^-(56,1;0) .$$

In order to do this, and the remaining calculations of this section, extensive use of the multiplication and trace properties of the γ-matrices will be made.[12] The notation used is that of reference (9).

Before proceeding, the following relations must be noted.

$$p\cdot q = p\cdot q' = 0 \tag{5}$$

$$q^2 = m^2 - \tfrac{1}{4}M^2 \tag{6}$$

$$q'^2 = \mu^2 - \tfrac{1}{4}M^2 . \tag{7}$$

These follow immediately from the mass-shell relations of the multiplet momenta.

Now, with the aid of 1(8) and 1(9), the above reducing out is achieved by a consideration of the following two quantities.

$$(8)\quad \Gamma_+ \bar{\Phi}(\tfrac{1}{2}p+q')\bar{\Phi}(\tfrac{1}{2}p-q')\Gamma_- = (4\mu^2)^{-1}\phi_5(\tfrac{1}{2}p+q')$$
$$\phi^\mu(\tfrac{1}{2}p-q')\Gamma_+(\tfrac{1}{2}\not{p}+\not{q}'+\mu)\gamma_5(\tfrac{1}{2}\not{p}-\not{q}'+\mu)\gamma_\mu\Gamma_-$$

and

$$(9)\quad \Gamma_+ \bar{\Phi}(\tfrac{1}{2}p-q')\bar{\Phi}(\tfrac{1}{2}p+q')\Gamma_- = (4\mu^2)^{-1}\phi^\mu(\tfrac{1}{2}p-q')$$
$$\phi_5(\tfrac{1}{2}p+q')\Gamma_+(\tfrac{1}{2}\not{p}-\not{q}'+\mu)\gamma_\mu(\tfrac{1}{2}\not{p}+\not{q}'+\mu)\gamma_5\Gamma_-,$$

where only the relevant 0^- and 1^- parts of the meson multiplet contribution have been exhibited.

On making use of the anticommutation relations of the γ-matrices, (8) becomes

$$(4\mu^2)^{-1}\phi_5(\tfrac{1}{2}p+q')\phi^\mu(\tfrac{1}{2}p-q')\Gamma_+(\tfrac{1}{2}\not{p}+\not{q}'+\mu)$$
$$\gamma_\mu(\not{q}'-\tfrac{1}{2}\not{p}-\mu)\gamma_5\Gamma_-,$$

since

$$1(4)\quad (\tfrac{1}{2}p-q')_\mu\phi^\mu(\tfrac{1}{2}p-q')=0,$$
$$=(4\mu^2)^{-1}\phi_5(\tfrac{1}{2}p+q')\phi^\mu(\tfrac{1}{2}p-q')\Gamma_+\Big((\tfrac{1}{2}M+\mu)$$
$$[\gamma_\mu,\not{q}']\gamma_5+\not{q}'\gamma_\mu\not{q}'\gamma_5-(\tfrac{1}{2}M+\mu)^2\gamma_\mu\gamma_5\Big)\Gamma_-$$

by (3).

$$=\frac{(\tfrac{1}{2}M+\mu)}{4\mu^2}\phi_5(\tfrac{1}{2}p+q')\phi^\mu(\tfrac{1}{2}p-q')\Gamma_+\Big([\gamma_\mu,\not{q}']\gamma_5-2\mu\gamma_\mu\gamma_5\Big)\Gamma_-$$

by (7) and since

$$\Gamma_+ \not{q}' \gamma_5 \Gamma_- = \not{q}' \Gamma_- \gamma_5 \Gamma_- = \not{q}' \gamma_5 \Gamma_+ \Gamma_- = 0$$

by (4). Thus the right hand side of (8) is finally given by

$$(10)\quad \phi_5(\tfrac{1}{2}p+q')\phi^\mu(\tfrac{1}{2}p-q')\ \tfrac{1}{2}(1+M/2\mu)\Gamma_+\left(\tfrac{1}{2\mu}\gamma_5[\gamma_\mu, \not{q}']\right.$$
$$\left. - \gamma_\mu\gamma_5\right)\Gamma_- \ .$$

By making the replacement $\not{q}' \to -\not{q}'$ in (8), the final form of the right hand side of (9) is given through (10) as

$$(11)\quad \phi^\mu(\tfrac{1}{2}p-q')\phi_5(\tfrac{1}{2}p+q')\tfrac{1}{2}\left(1+\frac{M}{2\mu}\right)\Gamma_+\left(\frac{1}{2\mu}\gamma_5[\gamma_\mu, \not{q}']+\gamma_\mu\gamma_5\right)\Gamma_- .$$

Collecting the results (10) and (11) together, and temporarily leaving the momentum dependence of the mesons particle-spinors understood, the expression (1) becomes

$$(12)\quad \frac{-i\mu g_1}{p^2-M^2}\left(\frac{\frac{p\cdot q\ p\cdot q'}{M^2}-q\cdot q'}{m\mu}\right)^L \frac{1}{4}\left(1+\frac{M}{2\mu}\right)\overline{\Psi}^{(ACD)}(\tfrac{1}{2}p+q)$$
$$\Psi_{(BCD)}(-\tfrac{1}{2}p+q)\ \Big(\Gamma_+\gamma_\mu\gamma_5\Gamma_-\big(h_1^{(+)}[\phi_5\phi^\mu]_F + h_1^{(-)}$$
$$[\phi_5\phi^\mu]_D\big)+\frac{1}{2\mu}\Gamma_+\gamma_5[\gamma_\mu, \not{q}']\Gamma_-\big(h_1^{(+)}[\phi_5\phi^\mu]_D + h_1^{(-)}[\phi_5\phi^\mu]_F\big)\Big)_A^B ,$$

where D and F represent the usual symmetric and antisymmetric SU(3) couplings respectively.[9]

Expression (12) may be simplified further by a consideration of the expressions

$$(13)\quad \overline{\Psi}(\tfrac{1}{2}p+q)\Gamma_+\gamma_\mu\gamma_5\Gamma_-\Psi(-\tfrac{1}{2}p+q)$$

and

(14) $\overline{\Psi}(\tfrac{1}{2}p+q)\,\Gamma_+\,\gamma_5[\gamma_\mu, \not{q}']\,\Gamma_-\,\Psi(-\tfrac{1}{2}p+q)$

By extensive use of the anticommutation relations of the γ-matrices , the Bargmann-Wigner equations, and the relations (3), (4) and (5), the expressions (13) and (14) respectively reduce to

(15) $\dfrac{p_\mu}{2M}\left(1+\dfrac{2m}{M}\right)\overline{\Psi}(\tfrac{1}{2}p+q)\,\gamma_5\,\Psi(-\tfrac{1}{2}p+q)$

and

(16) $\overline{\Psi}(\tfrac{1}{2}p+q)\left(\dfrac{1}{4M}\left[\not{p},[\not{q}',\gamma_\mu]\gamma_5\right]-iq'^\nu\sigma_{\nu\mu}\gamma_5-\dfrac{2q\cdot q'}{M^2}p_\mu\gamma_5\right)\Psi(-\tfrac{1}{2}p+q).$

The results (15) and (16), together with (12), give the final form of the asymptotic contribution to the above graph for the processes $0^-(56,1;0)\rightarrow 1^-(56,1;0)$ as

(17) $$\frac{-i\mu g_1}{p^2-M^2}\left(\frac{p\cdot q\;p\cdot q'\,M^{-2}-q\cdot q'}{m\mu}\right)^L\frac{1}{8}\left(1+\frac{M}{2\mu}\right)\overline{\Psi}^{(ACD)}(\tfrac{1}{2}p+q)$$

$$\Psi_{(BCD)}(-\tfrac{1}{2}p+q)\Bigg(\frac{p_\mu}{M}\left(1+\frac{2m}{M}\right)\Big(h_1^{(+)}[\phi_5\phi^\mu]_F+h_1^{(-)}$$

$$[\phi_5\phi^\mu]_D\Big)\gamma_5+\Big(\frac{1}{4M\mu}\left[\not{p},[\not{q}',\gamma_\mu]\gamma_5\right]-\frac{i}{\mu}q'^\nu\sigma_{\nu\mu}\gamma_5$$

$$-\frac{2q\cdot q'}{M^2\mu}p_\mu\gamma_5\Big)\Big(h_1^{(+)}[\phi_5\phi^\mu]_D+h_1^{(-)}[\phi_5\phi^\mu]_F\Big)\Bigg)_A^B.$$

Now that the contribution of the above graph to the processes $0^-(56,1;0)\rightarrow 1^-(56,1;0)$ has been reduced out, the next step in the calculation is to reduce out from (17) the contribution to the processes $0^-\ \tfrac{1}{2}^+\longrightarrow 1^-\ \tfrac{1}{2}^+$. It may be shown that the $\tfrac{1}{2}^+$ baryon octet contribution to the quantity:

$$\overline{\Psi}^{ACD}(\tfrac{1}{2}p+q)\,\Psi_{BCD}(-\tfrac{1}{2}p+q)$$

is given by [9]

$$(18)\quad -(24m^2)^{-1}\left(\left(-\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5 C C^{-1}\gamma_5\left(\tfrac{1}{2}\not{p}+\not{q}+m\right)\right)_\beta{}^\alpha \left(\bar{N}^\gamma(\tfrac{1}{2}p+q)N_\gamma(-\tfrac{1}{2}p+q)\right)_{3D+5F} + (12m^2)^{-1}\left(\bar{N}^\gamma(\tfrac{1}{2}p+q) N_\delta(-\tfrac{1}{2}p+q)\left(C^{-1}\gamma_5\left(\tfrac{1}{2}\not{p}+\not{q}+m\right)\right)^{\delta\alpha}\left(\left(-\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5 C\right)_{\beta\gamma}\right)_{3D+2F}$$

where N_γ is usual Dirac spinor, and D,F are the symmetric and antisymmetric SU(3) baryon couplings respectively.

Thus on using the expression (18) it is seen that the following six quantities must be calculated in order that the contribution to the processes $0^- \tfrac{1}{2}^+ \longrightarrow 1^- \tfrac{1}{2}^+$ may be reduced out of (17).

$$(19)\quad \mathrm{tr}\left(C^{-1}\gamma_5\left(\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5\left(-\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5 C\right),$$

$$(20)\quad \mathrm{tr}\left(C^{-1}\gamma_5\left(\tfrac{1}{2}\not{p}+\not{q}+m\right)\left(\frac{1}{4M\mu}[\not{p},[\not{q}',\gamma_\mu]\gamma_5] - \frac{i}{\mu}q'^\nu \sigma_{\nu\mu}\gamma_5 - \frac{2q\cdot q'}{M^2\mu}p_\mu\gamma_5\right)\left(-\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5 C\right),$$

$$(21)\quad \left(C^{-1}\gamma_5\left(\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5\left(-\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5 C\right)^\delta{}_\gamma \bar{N}^\gamma(\tfrac{1}{2}p+q)N_\delta(-\tfrac{1}{2}p+q),$$

$$(22)\quad (4M\mu)^{-1}\left(C^{-1}\gamma_5\left(\tfrac{1}{2}\not{p}+\not{q}+m\right)[\not{p},[\not{q}',\gamma_\mu]\gamma_5]\left(-\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5 C\right)^\delta{}_\gamma \bar{N}^\gamma(\tfrac{1}{2}p+q)N_\delta(-\tfrac{1}{2}p+q),$$

$$(23)\quad -2q\cdot q' M^{-2}p_\mu\left(C^{-1}\gamma_5\left(\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5\left(-\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5 C\right)^\delta{}_\gamma \bar{N}^\gamma(\tfrac{1}{2}p+q)N_\delta(-\tfrac{1}{2}p+q);$$

$$(24)\quad -\frac{i}{\mu} q'^{\nu} \left(C^{-1}\gamma_5 \left(\tfrac{1}{2}\not{p}+\not{q}+m\right)\sigma_{\nu\mu}\gamma_5\left(-\tfrac{1}{2}\not{p}+\not{q}+m\right)\gamma_5 C\right)^{\delta}{}_{\gamma}\, \bar{N}^{\gamma}\left(\tfrac{1}{2}p+q\right) N_{\delta}\left(-\tfrac{1}{2}p+q\right).$$

Of these expressions (19)-(24), only (20) and (21) will be calculated in detail, the remaining results being merely quoted.

In order to calculate (20) firstly note, as may be easily verified, that the trace of a product involving less than four γ-matrices and γ_5 vanishes. Secondly note that

$$(25)\quad \mathrm{tr}\left(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_5\right) = -4\,\epsilon_{\mu\nu\lambda\rho}.$$

Now rewriting (20) as

$$\mathrm{tr}\Big(\left(\tfrac{1}{2}\not{p}-\not{q}-m\right)\left(\tfrac{1}{2}\not{p}+\not{q}+m\right)\Big(\frac{1}{4M\mu}\left[\not{p},[\not{q}',\gamma_\mu]\gamma_5\right] -\frac{i}{\mu} q'^{\nu}\sigma_{\nu\mu}\gamma_5 - \frac{2q\cdot q'}{M^2\mu} p_\mu\gamma_5\Big)\Big),$$

and bearing in mind the above remarks gives

$$\mathrm{tr}\left(\frac{i}{\mu}\left[\not{q},\tfrac{1}{2}\not{p}\right]q'^{\nu}\sigma_{\nu\mu}\gamma_5 - \frac{m}{2M\mu}\not{q}\left[\not{p},[\not{q}',\gamma_\mu]\gamma_5\right]\right)$$

$$(26)\quad = \frac{4}{\mu}\left(1+\frac{2m}{M}\right)\epsilon_{\nu\lambda\rho\mu}\, q^{\nu} p^{\lambda} q'^{\rho}$$

by(25). In an exactly similar fashion it may be shown that (19) vanishes. Now (26) is not the most convenient form of the result of calculating (20). By means of the Bargmann-Wigner equations, together with properties of the γ-matrices, (26) may be cast into the following form more convenient for later usage.

$$(27)\quad \frac{4}{\mu}\left(1+\frac{2m}{M}\right)\left(m^2\left(1-\frac{M^2}{4m^2}\right)[\gamma_\mu,\not{q}']\gamma_5 - q\cdot q' p_\mu\right.$$

$$\gamma_5 + 2m\gamma_5(q_\mu \not{q}' - q\cdot q'\gamma_\mu)\Big).$$

Making use of the property of the γ-matrices that

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T,$$

the expression (21) may be written as

$$(\gamma_5^T(m-\tfrac{1}{2}\not{p}^T-\not{q}^T)\gamma_5^T(m+\tfrac{1}{2}\not{p}^T-\not{q}^T)\gamma_5^T)^\delta{}_\gamma \bar{N}^\gamma(\tfrac{1}{2}p+q)$$

$$N_\delta(-\tfrac{1}{2}p+q)$$

$$=\bar{N}(\tfrac{1}{2}p+q)\gamma_5(\tfrac{1}{2}\not{p}-\not{q}+m)\gamma_5(m-\tfrac{1}{2}\not{p}-\not{q})\gamma_5 N(-\tfrac{1}{2}p+q)$$

$$=\bar{N}(\tfrac{1}{2}p+q)(-\tfrac{1}{2}\not{p}+\not{q}+m)(\tfrac{1}{2}\not{p}+\not{q}-m)\gamma_5 N(-\tfrac{1}{2}p+q)$$

$$=\bar{N}(\tfrac{1}{2}p+q)(-\tfrac{1}{4}M^2+m^2-\tfrac{1}{4}M^2-m^2+m\not{p}-\not{p}\not{q})\gamma_5 N(-\tfrac{1}{2}p+q)$$

$$=\bar{N}(\tfrac{1}{2}p+q)\left(2(m-\not{q})^2-\frac{M^2}{2}\right)\gamma_5 N(-\tfrac{1}{2}p+q)$$

by the Bargmann-Wigner equations.

(28) $$4m^2\left(1-\frac{M^2}{4m^2}\right)\bar{N}(\tfrac{1}{2}p+q)\gamma_5 N(-\tfrac{1}{2}p+q)$$

since

$$\bar{N}(\tfrac{1}{2}p+q)\not{q}\gamma_5 N(-\tfrac{1}{2}p+q) = 0.$$

The simplification of the expressions (22), (23), and (24) is exactly analogous to the above, and the final forms of these expressions are given respectively by

(29) $$\frac{2}{m\mu}\bar{N}(\tfrac{1}{2}p+q)\Big((4m^2+M^2)(q_\mu\gamma_5\not{q}'-q\cdot q'\gamma_5\gamma_\mu)$$

$$-4m\,q\cdot q'\,p_\mu \gamma_5 + 2m\left(m^2 - \frac{M^2}{4}\right)\gamma_5[\gamma_\mu, q\!\!\!/'] \Big) N\left(-\tfrac{1}{2}p+q\right),$$

(30) $$-\frac{2q\cdot q'}{M^2\mu}(4m^2 - M^2)\,p_\mu\,\bar{N}\left(\tfrac{1}{2}p+q\right)\gamma_5 N\left(-\tfrac{1}{2}p+q\right),$$

(31) $$2\mu^{-1}\,\bar{N}\left(\tfrac{1}{2}p+q\right)\Big(\left(m^2 - \tfrac{1}{4}M^2\right)\gamma_5[\gamma_\mu, q\!\!\!/'] - 2q\cdot q'\,p_\mu\gamma_5$$
$$+ 4m\gamma_5\left(q_\mu q\!\!\!/' - q\cdot q'\gamma_\mu\right)\Big) N\left(-\tfrac{1}{2}p+q\right).$$

On collecting all these results together, the expression (17) may be rewritten as

(32) $$\frac{-i\mu}{p^2 - M^2}\left(\frac{\frac{p\cdot q\ p\cdot q'}{M^2} - q\cdot q'}{m\mu}\right)^{L}\frac{1}{8}\left(1+\frac{M}{2\mu}\right)\left(1+\frac{2m}{M}\right)$$
$$\bar{N}\left(\tfrac{1}{2}p+q\right)\Big\{-\frac{2g_F}{\mu}\left(1-\frac{M^2}{4m^2}\right)A^\mu\gamma_5[\gamma_\mu, q\!\!\!/'] +$$
$$\left(\frac{4}{M}\left(\left(1-\frac{M^2}{4m^2}\right)B^\mu - \frac{q\cdot q'}{m\mu}A^\mu\right)g_{D+\frac{2}{3}F} + \frac{2q\cdot q'}{m^2\mu}A^\mu g_F\right)$$
$$p_\mu\gamma_5 + \frac{4}{m\mu}\left(\frac{M}{2m}g_{D+\frac{2}{3}F} - g_F\right)A^\mu\gamma_5\left(q_\mu q\!\!\!/' - q\cdot q'\gamma_\mu\right)$$
$$N\left(-\tfrac{1}{2}p+q\right),$$

where

(33) $$A^\mu = \left(h_D^{(+)} + h_F^{(-)}\right)\phi_5\left(\tfrac{1}{2}p+q'\right)\phi^\mu\left(\tfrac{1}{2}p-q'\right),$$

(34) $$B^\mu = \left(h_F^{(+)} + h_D^{(-)}\right)\phi_5\left(\tfrac{1}{2}p+q'\right)\phi^\mu\left(\tfrac{1}{2}p-q'\right),$$

and the coupling constants now appearing are equal to the corresponding coupling constant, which appeared previously, multiplied by the value of the new suffix at the vertex in question.

The last task of this section is the Reggeization of the scattering amplitude (32) for $0^- \frac{1}{2}^+ \rightarrow 1^- \frac{1}{2}^+$ processes. It is effected by a straightforward application of the procedure given in section 2 of chapter 1, and involves the following replacements:

$$L \rightarrow \alpha_\pm(t) - 1,$$

$$p^2 - M^2 \rightarrow \Gamma(1 - \alpha_\pm(t)),$$

$$M \rightarrow \sqrt{t}$$

and $$h^{(\pm)} \rightarrow \tfrac{1}{2}\beta_\pm\left(1 \mp e^{-i\pi\alpha_\pm(t)}\right)h,$$

where a factor of h, the universal $U(6,6) \otimes O(3,1)$ coupling constant for the relevant three meson multiplet vertex, has been extracted from the residues $\beta_\pm$. These residues have been introduced in order that symmetry breaking may be incorporated into the scheme at a later point.

Now an inspection of (32) after these replacements have been made reveals that the Reggeized scattering amplitude has square root branch points at $t=0$. In order to eliminate these unwanted kinematic singularities, Gribov doubling is introduced.[13] This involves adding to (32) the corresponding quantity with $\sqrt{t} \rightarrow -\sqrt{t}$. The resulting Reggeized scattering amplitude for $0^- \frac{1}{2}^+ \rightarrow 1^- \frac{1}{2}^+$ is then

$$(35) \quad -i\bar{N}(\tfrac{1}{2}p+q)\left(a\gamma_5[\gamma_\mu, \not{q}'] + b p_\mu\gamma_5 + c\gamma_5(q_\mu\not{q}' - q\cdot q'\gamma_\mu)\right)N(-\tfrac{1}{2}p+q)\phi^\mu(\tfrac{1}{2}p-q')$$

where

(36) $$a = -\frac{1}{4}\left(1+\frac{m}{\mu}\right)\left(1-\frac{t}{4m^2}\right)g_F A,$$

(37) $$b = \frac{m\mu}{t}\left(1+\frac{t}{4\mu m}\right)\left(\left(1-\frac{t}{4m^2}\right)B-\frac{q\cdot q'}{m\mu}A\right)g_{D+\frac{2}{3}F}+\left(1+\frac{m}{\mu}\right)\frac{q\cdot q'}{m^2}g_F A,$$

(38) $$C = \frac{1}{2m}\left(\left(1+\frac{t}{4\mu m}\right)g_{D+\frac{2}{3}F}-\left(1+\frac{m}{\mu}\right)g_F\right)A,$$

and

(39) $$A = \left(\tfrac{1}{2}\beta_+\left(1-e^{-i\pi\alpha_+}\right)h_D\,\Gamma(1-\alpha_+)\left(-\frac{q\cdot q'}{m\mu}\right)^{\alpha_+-1}\right.$$
$$\left.+\tfrac{1}{2}\beta_-\left(1+e^{-i\pi\alpha_-}\right)h_F\,\Gamma(1-\alpha_-)\left(-\frac{q\cdot q'}{m\mu}\right)^{\alpha_--1}\right)$$

(40) $$B = \left(\tfrac{1}{2}\beta_+\left(1-e^{-i\pi\alpha_+}\right)h_F\,\Gamma(1-\alpha_+)\left(-\frac{q\cdot q'}{m\mu}\right)^{\alpha_+-1}\right.$$
$$\left.+\tfrac{1}{2}\beta_-\left(1+e^{-i\pi\alpha_-}\right)h_D\,\Gamma(1-\alpha_-)\left(-\frac{q\cdot q'}{m\mu}\right)^{\alpha_--1}\right).$$

The writing of A and B in the above form enables symmetry breaking to be introduced into the trajectory functions $\alpha_\pm$, in addition to the residues. Now that the expressions (35)-(40) for the Reggeized scattering amplitude for the processes $0^-\frac{1}{2}^+\rightarrow 1^-\frac{1}{2}^+$ have been obtained it only remains to calculate the corresponding s-channel helicity amplitudes in terms of this Reggeized contribution. This is dealt with in the next section.

<u>Section 2</u> <u>The s-channel helicity amplitudes:</u>

In connecting the above Reggeized scattering amplitude with the s-channel helicity amplitudes, symmetry breaking is introduced by assuming that the particle-spinors corresponding to the external particles are dependent upon the physical masses of those particles, and not on multiplet masses. This symmetry breaking is necessary to avoid some undesirable kinematic effects.

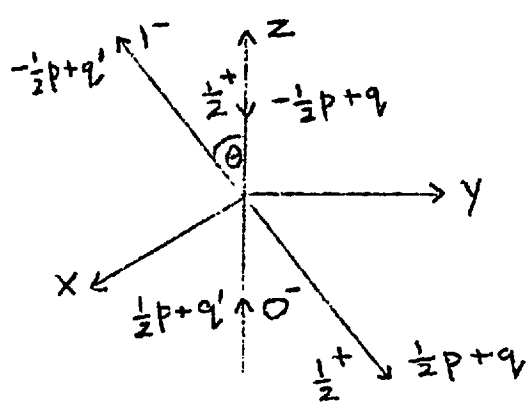
Since the photon can only have helicity ± 1 , the following vector-meson production helicity amplitudes are those considered.

Table 1

$\lambda_1 \lambda_2 \backslash \lambda_3 \lambda_4$	$1 \quad \frac{1}{2}$	$1 \quad -\frac{1}{2}$	$-1 \quad \frac{1}{2}$	$-1 \quad -\frac{1}{2}$
$0 \quad \frac{1}{2}$	Ψ_1	Ψ_3	Ψ_4	$-\Psi_2$
$0 \quad -\frac{1}{2}$	Ψ_2	Ψ_4	$-\Psi_3$	Ψ_1

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the helicities corresponding to the 0^- , incoming $\frac{1}{2}^+$, 1^- , and outgoing $\frac{1}{2}^+$ multiplets respectively. The phase convention is that of Jacob and Wick[14], and parity conservation has been used to reduce the number of independent amplitudes.

The space axes in the centre of momentum frame are oriented as shown in the diagram below, with the scattering taking place in the x-z plane.



On labelling the energies of the four particles by E_1, E_2, E_3, E_4, their masses by m_1, m_2, m_3, m_4, and the magnitudes of the 3-momenta of the incoming and outgoing particles by K and Q respectively, the following explicit expressions ensue.

(41) $$q'^{\mu} = \tfrac{1}{2}\left(E_1+E_3, Q\sin\theta, 0, K+Q\cos\theta\right),$$

(42) $$q^{\mu} = \tfrac{1}{2}\left(E_2+E_4, -Q\sin\theta, 0, -K-Q\cos\theta\right),$$

(43) $$p^{\mu} = \left(E_1-E_3, -Q\sin\theta, 0, K-Q\cos\theta\right).$$

The outgoing meson polarization vector is given by

(44) $$\epsilon_1^{\mu *} = -\tfrac{1}{\sqrt{2}}\left(0, \cos\theta, -i, -\sin\theta\right),$$
$$\epsilon_{-1}^{\mu *} = \tfrac{1}{\sqrt{2}}\left(0, \cos\theta, i, -\sin\theta\right),$$

and the boost convention of Jacob and Wick(14) is such that

(45) $$N(-\tfrac{1}{2}p+q) = \sqrt{E_2+m_2}\left(1 + \frac{i\gamma_5\,\underline{\sigma}\cdot(-\tfrac{1}{2}\underline{p}+\underline{q})}{E_2+m_2}\right)\phi$$

and

(46) $$\bar{N}(\tfrac{1}{2}p+q) = \sqrt{E_4+m_4}\,\phi^{\dagger}\left(1 - \frac{i\gamma_5\,\underline{\sigma}\cdot(\tfrac{1}{2}\underline{p}+\underline{q})}{E_4+m_4}\right)$$

with

(47) $$\phi_{\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix} \qquad \phi_{-\frac{1}{2}} = \begin{pmatrix}-1\\0\end{pmatrix}$$
$$\phi_{\frac{1}{2}}^{\dagger} = \left(\sin\tfrac{1}{2}\theta, -\cos\tfrac{1}{2}\theta\right) \qquad \phi_{-\frac{1}{2}}^{\dagger} = \left(\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\right).$$

The expressions (41)-(47) are all that is needed to calculate the Reggeized contribution to the s-channel helicity amplitudes for the processes $0^- \frac{1}{2}^+ \rightarrow 1^- \frac{1}{2}^+$. As an example, the contribution to the helicity amplitudes of the term

$$\bar{N}(\tfrac{1}{2}p+q)\,\gamma_5\,N(-\tfrac{1}{2}p+q)\quad p\cdot\epsilon_1^* \tag{48}$$

will be calculated in detail. The remaining contributions are calculated in an exactly analogous fashion, and are merely quoted here.

On substitution of the expressions (45) and (46) for the nucleon octet spinors, (48) becomes

$$\phi^\dagger_{\lambda_4}\left(1-\frac{2i\gamma_5 Q\lambda_4}{E_4+m_4}\right)\gamma_5\left(1+\frac{2i\gamma_5 K\lambda_2}{E_2+m_2}\right)\phi_{\lambda_2}\,C^{1/2}\,p\cdot\epsilon_1^*,$$

where

$$C=(E_2+m_2)(E_4+m_4). \tag{49}$$

The following abbreviations, of use later, are also made

$$D_\pm = C \pm QK \tag{50}$$

$$H_\pm = K(E_4+m_4) \pm Q(E_2+m_2). \tag{51}$$

Note that the third and fourth components of the spinors (47), left understood there, vanish, and that from its form γ_5 is easily seen to connect only upper and lower pairs of spinor components. Thus (48) reduces further still to

$$\phi^\dagger_{\lambda_4}\phi_{\lambda_2}\,2i\,p\cdot\epsilon_1^*\,C^{1/2}\left(\frac{Q\lambda_4}{E_4+m_4}-\frac{K\lambda_2}{E_2+m_2}\right).$$

The contribution of (48) to the s-channel helicity amplitudes may be read off directly from this expression. It is as follows

$$
(52)\quad
\begin{aligned}
\Psi_1 &: -i\,(2C)^{-1/2} K H_- \sin\theta \cos\tfrac{1}{2}\theta, \\
\Psi_2 &: \;\; i\,(2C)^{-1/2} K H_+ \sin\theta \sin\tfrac{1}{2}\theta, \\
\Psi_3 &: \;\; i\,(2C)^{-\frac{1}{2}} K H_+ \sin\theta \sin\tfrac{1}{2}\theta, \\
\Psi_4 &: \;\; i\,(2C)^{-\frac{1}{2}} K H_- \sin\theta \cos\tfrac{1}{2}\theta.
\end{aligned}
$$

The contributions to the remaining amplitudes follow from parity conservation as shown in table 1.

In an exactly analogous fashion the contribution of

$$\bar{N}\left(\tfrac{1}{2}p+q\right)\gamma_5[\gamma_\mu, \not{q}']N\left(-\tfrac{1}{2}p+q\right)\epsilon_1^{\mu\,*}$$

to the s-channel helicity amplitudes is shown to be

$$
(53)\quad
\begin{aligned}
\Psi_1 &: i\left(\tfrac{1}{2}C\right)^{-1/2}\left((Q-K)H_- - (E_1+E_3)D_-\right)\sin\tfrac{1}{2}\theta \;+ \\
&\qquad i\,(2C)^{-1/2} K H_- \sin\theta \cos\tfrac{1}{2}\theta, \\
\Psi_2 &: i\left(\tfrac{1}{2}C\right)^{-1/2}\left((Q+K)H_+ + (E_1+E_3)D_+\right)\cos\tfrac{1}{2}\theta \;- \\
&\qquad i\,(2C)^{-1/2} K H_+ \sin\theta \sin\tfrac{1}{2}\theta, \\
\Psi_3 &: -i\,(2C)^{-1/2} K H_+ \sin\theta \sin\tfrac{1}{2}\theta, \\
\Psi_4 &: -i\,(2C)^{-1/2} K H_- \sin\theta \cos\tfrac{1}{2}\theta;
\end{aligned}
$$

the contribution of

$$\bar{N}\left(\tfrac{1}{2}p+q\right)\gamma_5\not{q}'N\left(-\tfrac{1}{2}p+q\right)\; q\cdot\epsilon_1^{*}$$

is given by

$$\Psi_1 : -i(32C)^{-1/2}\left((E_1+E_3)H_+ + (Q+K)D_+\right)K\sin\theta\cos\tfrac{1}{2}\theta,$$

$$\begin{aligned}
&\psi_2:\ i(32C)^{-1/2}\big((E_1+E_3)H_- + (K-Q)D_-\big)K\sin\theta\sin\tfrac{1}{2}\theta,\\
(54)\quad &\psi_3:\ i(32C)^{-1/2}\big((E_1+E_3)H_- + (K-Q)D_-\big)K\sin\theta\sin\tfrac{1}{2}\theta,\\
&\psi_4:\ i(32C)^{-1/2}\big((E_1+E_3)H_+ + (Q+K)D_+\big)K\sin\theta\cos\tfrac{1}{2}\theta;
\end{aligned}$$

and finally the contribution of

$$\bar{N}(\tfrac{1}{2}p+q)\,\gamma_5\,\gamma\cdot\epsilon_1^*\,N(-\tfrac{1}{2}p+q)\quad q\cdot q'$$

is given by

$$\begin{aligned}
&\psi_1:\ -i(\tfrac{1}{2}C)^{-1/2}\ q\cdot q'\ D_+\sin\tfrac{1}{2}\theta,\\
&\psi_2:\ i(\tfrac{1}{2}C)^{-1/2}\ q\cdot q'\ D_-\cos\tfrac{1}{2}\theta,\\
(55)\quad &\psi_3:\ 0,\\
&\psi_4:\ 0.
\end{aligned}$$

Two points must be mentioned before the expressions (52)-(55) are used to write down the s-channel helicity amplitudes. Firstly, in the definition of two-particle states given by Jacob and Wick (14), a factor $(-1)^{s-\lambda}$ is introduced for each of the particles two and four. In the above the nucleon octet members were chosen as particles two and four, and the corresponding phase factors must be introduced into the helicity amplitudes.

Secondly the expressions (35)-(40), when taken together with (52)-(55), give a set of helicity amplitudes which do not obey angular momentum conservation in the forward direction. The offending quantity is the expression (37), and the residues corresponding to any Regge trajectory which contributes to this expression with a non-vanishing value of $g_D+\frac{2}{3}F$ must be made evasive (because of the assumed symmetry there is only one effective residue for each trajectory, and so all the related residues must be evasive if one is). Since $g_{D+\frac{2}{3}F}$ does not vanish for any of the processes of interest, all the residues appearing are assumed to be made evasive by the extraction of a factor $t/4m\mu$.

Bearing these two remarks in mind, and collecting together the results (35)-(40) and (52)-(55), the final form of the s-channel helicity amplitudes for the processes $0^- \frac{1}{2}^+ \rightarrow 1^- \frac{1}{2}^+$ are given as

(56) $$\Psi_1 = (8C)^{-1/2}\Big(a_1\Big(2\sin\tfrac{1}{2}\theta\big((Q-K)H_- - (E_1+E_3)D_-\big) + KH_-\sin\theta\cos\tfrac{1}{2}\theta\Big) - a_2 KH_-\sin\theta\cos\tfrac{1}{2}\theta + a_3\Big(2D_+\sin\tfrac{1}{2}\theta(2s+t-2m^2-2\mu^2) - \big((E_1+E_3)H_+ + (Q+K)D_+\big)K\sin\theta\cos\tfrac{1}{2}\theta\Big)\Big),$$

(57) $$\Psi_2 = (8C)^{-1/2}\Big(a_1\Big(KH_+\sin\theta\sin\tfrac{1}{2}\theta - 2\cos\tfrac{1}{2}\theta\big((Q+K)H_+ + (E_1+E_3)D_+\big)\Big) - a_2 KH_+\sin\theta\sin\tfrac{1}{2}\theta + a_3\Big(2D_-\cos\tfrac{1}{2}\theta(2s+t-2m^2-2\mu^2) - \big((E_1+E_3)H_- + (K-Q)D_-\big)K\sin\theta\sin\tfrac{1}{2}\theta\Big)\Big),$$

(58) $$\Psi_3 = (8C)^{-1/2}\Big((a_1-a_2)H_+ + a_3\big((Q-K)D_- - (E_1+E_3)H_-\big)\Big)K\sin\theta\sin\tfrac{1}{2}\theta,$$

(59) $$\Psi_4 = (8C)^{-1/2}\Big((a_2-a_1)H_- + a_3\big((E_1+E_3)H_+ + (Q+K)D_+\big)\Big)K\sin\theta\cos\tfrac{1}{2}\theta,$$

where

(60) $$a_1 = \frac{-t}{8m\mu}\left(1+\frac{m}{\mu}\right)\left(1-\frac{t}{4m^2}\right)g_F A,$$

(61) $$a_2 = \frac{1}{2}\left(1+\frac{t}{4\mu m}\right)\left(\left(1-\frac{t}{4m^2}\right)B-\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)A\right)g_{D+\frac{2}{3}F} + \frac{t}{8m^2}\left(1+\frac{m}{\mu}\right)\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)g_F A,$$

(62) $$a_3 = \frac{t}{16m^2\mu}\left(\left(1+\frac{t}{4\mu m}\right)g_{D+\frac{2}{3}F}-\left(1+\frac{m}{\mu}\right)g_F\right)A,$$

with

(63) $$A = \frac{1}{2}\beta_+\left(1-e^{-i\pi\alpha_+}\right)h_D\Gamma(1-\alpha_+)\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)^{\alpha_+-1} + \frac{1}{2}\beta_-\left(1+e^{-i\pi\alpha_-}\right)h_F\Gamma(1-\alpha_-)\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)^{\alpha_--1},$$

(64) $$B = \frac{1}{2}\beta_+\left(1-e^{-i\pi\alpha_+}\right)h_F\Gamma(1-\alpha_+)\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)^{\alpha_+-1} + \frac{1}{2}\beta_-\left(1+e^{-i\pi\alpha_-}\right)h_D\Gamma(1-\alpha_-)\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)^{\alpha_--1}.$$

The expressions (56)-(64) form the complete basis for the discussion of Reggeized contributions to the scattering amplitudes for $0^-\,\frac{1}{2}^+ \rightarrow 1^-\,\frac{1}{2}^+$ processes, and hence, through the vector dominance model and time reversal, to the corresponding scattering amplitudes for $\gamma\,\frac{1}{2}^+ \rightarrow 0^-\,\frac{1}{2}^+$ processes. The connection with photoproduction processes will be made in the next section; whilst a discussion of particular processes, and particular Regge pole contributions, is left till chapter 3.

Section 3. s-channel helicity amplitudes for photoproduction.

The s-channel helicity amplitudes for the processes $\gamma\frac{1}{2}^+ \rightarrow 0^-\frac{1}{2}^+$ are labelled as follows

Table 2

$\lambda_3\ \lambda_4 \backslash \lambda_1\ \lambda_2$	$1\ \ \frac{1}{2}$	$1\ \ -\frac{1}{2}$	$-1\ \ \frac{1}{2}$	$-1\ \ -\frac{1}{2}$
$0\ \ \frac{1}{2}$	ϕ_1	ϕ_3	ϕ_4	$-\phi_2$
$0\ \ -\frac{1}{2}$	ϕ_2	ϕ_4	$-\phi_3$	ϕ_1

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the helicities of the photon, and the incoming $\frac{1}{2}^+, 0^-$, outgoing $\frac{1}{2}^+$ multiplets respectively.

Now for any helicity amplitude $M_{\lambda_1\lambda_2\lambda_3\lambda_4}$, time-reversal invariance gives(14)

$$(65)\quad \overline{M}_{\lambda_3\lambda_4\lambda_1\lambda_2} = (-1)^{\lambda_1-\lambda_2-\lambda_3+\lambda_4} M_{\lambda_1\lambda_2\lambda_3\lambda_4},$$

where $\overline{M}_{\lambda_3\lambda_4\lambda_1\lambda_2}$ is a helicity amplitude for the time-reversed process. By an application of the relation (65), the table of helicity amplitudes for the processes $1^-\frac{1}{2}^+ \rightarrow 0^-\frac{1}{2}^+$ is seen to be as in table 3 below, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the helicities of the 1^-, incoming $\frac{1}{2}^+, 0^-$, and outgoing $\frac{1}{2}^+$ multiplets respectively; and the ψ_i are given by the expressions (56)-(59) with the roles of K and Q interchanged.

Table 3.

$\lambda_3\ \lambda_4 \backslash \lambda_1\ \lambda_2$	$1\quad \frac{1}{2}$	$1\quad -\frac{1}{2}$	$-1\quad \frac{1}{2}$	$-1\quad -\frac{1}{2}$
$0\quad \frac{1}{2}$	$-\Psi_1$	Ψ_3	$-\Psi_4$	$-\Psi_2$
$0\quad -\frac{1}{2}$	Ψ_2	$-\Psi_4$	$-\Psi_3$	$-\Psi_1$

Finally an application of the relation 1(1), expressing the vector dominance model, gives the following expressions for the s-channel helicity amplitudes ϕ_i.

(66) $$\phi_1 = -\chi_\rho\left(\Psi_1^{(\rho)} + \frac{1}{\sqrt{3}}\Psi_1^{(\omega_8)}\right),$$

(67) $$\phi_2 = \chi_\rho\left(\Psi_2^{(\rho)} + \frac{1}{\sqrt{3}}\Psi_2^{(\omega_8)}\right),$$

(68) $$\phi_3 = \chi_\rho\left(\Psi_3^{(\rho)} + \frac{1}{\sqrt{3}}\Psi_3^{(\omega_8)}\right),$$

(69) $$\phi_4 = -\chi_\rho\left(\Psi_4^{(\rho)} + \frac{1}{\sqrt{3}}\Psi_4^{(\omega_8)}\right),$$

where the superscripts ρ, w_8 indicate that the contribution from the vector-meson production helicity amplitudes to the particular processes

$$\rho^0\, \tfrac{1}{2}^+ \rightarrow 0^-\, \tfrac{1}{2}^+,$$
$$\omega_8\, \tfrac{1}{2}^+ \rightarrow 0^-\, \tfrac{1}{2}^+$$

is the only one considered; and where the Ψ_i are given by the expressions (56)-(59) with K,Q interpreted respectively as the magnitudes of the outgoing and incoming 3-momenta in the centre of momentum frame.

In order to complete the transition from vector-meson production to photoproduction it is necessary to make some remarks about g uge invariance. Gauge invariance in this case requires that

$$(70) \quad (\tfrac{1}{2}p - q')^{\mu} M_{\mu} = 0,$$

where M_{μ} is defined by rewriting (35) as

$$M_{\mu} \phi^{\mu}(\tfrac{1}{2}p - q').$$

Now it may be verified by direct computation, using (35) - (38), that the only part of M_{μ} which gives non-vanishing contribution to the left hand side of (70) is the pseudoscalar coupling. However this contribution may be rendered gauge invariant by the inclusion of an s-channel contribution proportional to $(\frac{1}{2}p - q')_{\mu}$, which from its very form will give vanishing contribution to the helicity amplitudes in the Coulomb gauge in the centre of momentum frame. Thus the procedure given above for the calculation of the Reggeized s-channel helicity amplitudes for $\gamma\, \frac{1}{2}^{+} \rightarrow 0^{-} \frac{1}{2}^{+}$ gives an effectively gauge invariant result

CHAPTER 3

Section 1. Absorption Model.

The experimental data on $\gamma p \to \pi^+ n$ exhibits a sharp forward peak of width in t of the order of the pion mass squared.[1] Previous work on this photoproduction reaction, using a peripheral model with absorption corrections, has shown that this forward peak can be explained in terms of pion exchange.[7] In this work it can be seen that the elementary pion gives vanishing contribution to the differential cross-section at t=0. However, with the application of absorption corrections a forward peak of the correct width is obtained. In particular consider the s-channel helicity amplitude ϕ_2 (see table 2). Since this helicity amplitude does not involve any net helicity flip, it need not vanish at t=0. However the contribution of an elementary pion to this amplitude has the form

$$(1) \quad \phi_2^{(\pi)} \simeq \frac{t}{t-m_\pi^2} = 1 + \frac{m_\pi^2}{t-m_\pi^2} . \qquad \text{for small } t$$

Now the first term on the right hand side is s-wave, which violates unitarity at high energy. This contribution is mostly removed by the absorption corrections, so that the final result is

$$(2) \quad \phi_2^{(\pi)}(\text{absorbed}) \simeq \frac{m_\pi^2}{t-m_\pi^2} ,$$

which gives the required forward peak. Thus it is seen that as a consequence of the small value of m_π^2, the behaviour of the differential cross-section is dominated by the pion pole for small t. However, in describing the reaction $\gamma p \to \pi^+ n$ it is necessary to consider the exchange of particles with non-zero spin in addition to the pion. As may be seen in references (7,15), the energy and momentum-transfer dependence of the absorption model predictions for the exchange of non-zero spin elementary particles are in disagreement with experiment.

It is in order to overcome this latter disagreement with experiment that in this work a Regge pole model replaces the simple peripheral model of the above work. It remains, however, to discuss the predictions of the Regge pole model near $t=0$, where the absorptive peripheral model gives such good results. As stated in section 2 of chapter 2, the residues of the pion Regge pole are assumed to be evasive in order that angular momentum be conserved in the forward direction. An inspection of the evasive Reggeized pion contribution to the helicity amplitudes reveals that it is zero in the forward direction, exactly as in the peripheral model case. Also, since the pion Regge trajectory must pass through the pion pole, and since m_π^2 is very small the Reggeized pion contribution to ϕ_2 will be of exactly the same form as (1) for small t. Thus if absorption corrections were applied to the Reggeized pion contribution then the result would be of the form (2). All this means that the replacement of the absorptive peripheral model by an absorptive Regge pole model reproduces all the good features of absorptive peripheral π-exchange near $t=0$, whilst overcoming the difficulties involved with the exchange of non-zero spin particles in the former model.

The application of absorption corrections to the Reggeized pion contribution is interpreted as taking into account pion-pomeron Regge cut contributions in addition to those of the pion Regge pole[16]. Thus it is the pion-pomeron Regge cut which dominates the behaviour of the differential cross-section for small t, and gives the forward peak in $\gamma p \rightarrow \pi^+ n$.

In order to introduce the absorption corrections, the ϕ_i are first expanded in partial wave series [14], then modified according to the Watson formula[17]

(3) $$\phi_i'^{J} = \tfrac{1}{2}\left(S^{el\ J}_{initial} + S^{el\ J}_{final}\right)\phi_i ,$$

where $S^{el}_{initial}$ and S^{el}_{final} are the scattering amplitudes for elastic scattering of the initial and final states respectively. The elastic scattering is assumed to be non-spin flip, and given by the Gaussian form

(4) $$S^{el\ J} = 1 - d\exp\left(-J(J+1)/\nu^2p^2\right)$$

with p the magnitude of the three-momentum in the centre of momentum frame, and ν the elastic radius of interaction. Finally, the partial wave series is resummed after the replacement $\phi_i^J \to \phi_i'^J$. In terms of the resulting modified helicity amplitudes ϕ_i', the differential cross-section is given by

(5) $$\frac{d\sigma}{dt} = \frac{\pi}{2K^2}\sum_{i=1}^{4}\left|\frac{\phi_i'}{8\pi S^{1/2}}\right|^2 .$$

Section 2. Regge pole description of $\gamma p \to \pi^+ n$.

Since the vector dominance model is here being used to describe the process $\gamma p \to \pi^+ n$, the scattering amplitudes for this process are given through 1(1) by the scattering amplitudes for the processes

$$\rho^0 p \to \pi^+ n$$

and

$$\omega_8 p \to \pi^+ n .$$

For the former, the possible t-channel exchanges are the π^- and A_2^- mesons, whilst for the latter the possible t-channel exchanges are the ρ^- and B^- mesons.

Little is known of the B meson, though it is generally believed that its Regge trajectory is low-lying. Thus in this work only the π, ρ, and A_2 trajectories will be considered. The relevant SU(3) couplings are given in table 4 below.

Table 4.

Reaction	Baryon vertex		Meson vertex	
	$D+\frac{2}{3}F$	F	D	F
$\rho^0 p \to \pi^+ n$	$\frac{5}{3}\sqrt{2}$	$\sqrt{2}$	0	-2
$\omega_8 p \to \pi^+ n$	$5\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$	$\frac{2}{\sqrt{3}}$	0

It now remains to give a discussion of the role of the π, ρ, and A_2 Regge trajectories within the context of the Reggeized $U(6) \otimes U(6) \otimes O(3)$ symmetry scheme of chapter 2, and their contributions to the quantities A and B defined by 2(63) and 2(64) respectively. In order to do this it must first be noted that the π and ρ mesons belong to the $(6,\bar{6};0)$ representation of $U(6) \otimes U(6) \otimes O(3)$, whilst the A_2 meson belongs to the $(6,\bar{6};1)$ representation. Thus the π and ρ have even N-signature, whilst the A_2 has odd N-signature. Thus the replacement $N \to \alpha - 1$, made in the Reggeization of $U(6) \otimes U(6) \otimes O(3)$, allows the identification of $\alpha_\pm$ in 2(63) and 2(64) with the usual Regge trajectory functions for the ρ and

A_2 mesons respectively. On the other hand the usual pion Regge trajectory function contributes through the replacement $\alpha_+ \longrightarrow \alpha_\pi + 1$. These remarks, together with the SU(3) couplings at the meson vertex (table 4), give the following as the contribution of the π, ρ, and A_2 trajectories to the quantities A and B.

$$(6) \quad A = \tfrac{1}{2}\beta_\rho(1 - e^{-i\pi\alpha_\rho})\Gamma(1-\alpha_\rho)h_D\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)^{\alpha_\rho - 1}$$

$$+\tfrac{1}{2}\beta_{A_2}(1+e^{-i\pi\alpha_{A_2}})\Gamma(1-\alpha_{A_2})h_F\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)^{\alpha_{A_2}-1}$$

and

$$(7) \quad B = \tfrac{1}{2}\beta_\pi(1+e^{-i\pi\alpha_\pi})\Gamma(-\alpha_\pi)\left(\frac{s-m^2-\mu^2+\frac{1}{2}t}{2m\mu}\right)^{\alpha_\pi} .$$

Note that symmetry breaking has been introduced by allowing independent trajectory and residue functions for the π and ρ mesons, even though they belong to the same $U(6) \otimes U(6) \otimes O(3)$ multiplet.

The residues were taken to be constants, and the trajectory functions were parametrized by

$$\alpha(t) = \alpha_0 + \alpha_1 e^{\alpha_2 t} ,$$

consistently with reference (18). This parametization of the trajectory function gives the conventional linear trajectory in the peripheral region:

$$\alpha(t) \simeq \alpha_0 + \alpha_1 + \alpha_1\alpha_2 t .$$

The ρ and A_2 trajectory parameters were taken from reference (18), whilst the number of free parameters in the pion trajectory function was reduced to two by constraining the trajectory (when extrapolated linearly) to pass through the pion pole. In reference (18) the residues corresponding to the ρ and A_2 mesons were not evasive, whereas here they are; thus it cannot be hoped that the residues of reference (18) could be carried over to this work, and β_π, β_ρ, β_{A_2} thus provide three more free parameters.

The differential cross-section for the process $\gamma p \to \pi^+ n$ was then calculated by substituting the values (6) and (7) for A and B into the expressions 2(66)-2(69) for the ϕ_i, expanding the result in a partial-wave series, applying absorption corrections to all the input Regge poles through (3) and (4), resumming the partial wave series, and finally substituting the resulting modified helicity amplitudes into (5). Then the above five free parameters were determined by a χ^2 fit of the differential cross-section to the experimential data using MINUITS (CERN program library no.D506). The results of this minimization procedure are exhibited in table 5, and the absorption coefficients are exhibited in table 6.

Section 3. Discussion and results.

Using the pion trajectory parameters determined from the constraint of passing through the pion pole and from the fit to the experimential data, and the ρ and A_2 trajectory parameters of reference (18), a Chew-Frantschi plot of the π, ρ, and A_2 trajectories is made in figure 1. Note that the π trajectory has a slope of .35 for positive t, in contrast with the ρ and A_2 trajectory slopes of .90 and .86 respectively.

The effect of absorption corrections on elementary and Reggeized pion exchange is shown in figure 2. Unabsorbed elementary and Reggeized pion exchange are taken from reference (7), which used U(6,6) symmetry to relate vertex couplings. Note the similarity between the two cases for small t which was suggested in section 1. The application of absorption corrections to the elementary and Reggeized pion exchanges results in curves 3 and 4 respectively. Again note the similarity between the two cases for small t as they both give a sharp forward peak of width in t of the order of m_π^2. Thus figure 2 demonstrates that the absorptive Reggeized pion exchange does actually reproduce all the good results of absorptive elementary pion exchange at small t. Beyond the region in which the pion to dominate, namely $|t| > m_\pi^2$, a "dip-bump" structure, not exhibited by the data, is present.

As is seen in figure 3, the inclusion of ρ and A_2 Regge poles, together with ρ-pomeron and A_2-pomeron Regge cuts, smooths out the above mentioned "dip-bump" structure, and gives a very good fit to the data for $5.0 \leq P_{Lab} \leq 18.0$ GeV/c and $0 \leq |t| \leq 1$ $(\mathrm{GeV/c})^2$. Data exists for higher momentum transfers, but this was not included in the analysis since it is beyond the peripheral region.

The reaction $\gamma p \rightarrow \pi^+ n$ has been treated in a Regge pole plus Regge cut approach by Froyland and Gordon[19], Henyey et al [20], and Blackmon et al[21]. The model of reference (19) contained evasive π and ρ exchanges, together with Regge cut contributions of adjustable strength. At the expense of 18 free parameters a good representation of the differential cross-section was obtained. However it must be noted that in that model the differential cross -section has, in contrast to the model presented here, a dip at $t \sim m_\pi^2$, the depth of which increases with increasing energy. Reference (20) used π, ρ, and A_2 exchanges, together with adjustable cut contributions, at the expense of 12 free parameters. A dip similar to that appearing in the work of Froyland and Gordon is present, and the agreement with the data is somewhat unsatisfactory for large $|t|$. The model of Blackmon et al[21] used an elementary pion exchange together with Reggeized evasive A_2 exchange. At the expense of five free parameters a reasonable fit to the experimental data was given.

The model presented here, which employs evasive Reggeized π, ρ, and A_2 exchanges together with absorptive correction Regge cuts, is able to reproduce the features of the experimental data very well at the expense of only 5 free parameters. This may be interpreted as further evidence of the role of Regge cuts in the description of strong interactions at high energy.

REFERENCES

(0) P.A. Collins, B.J. Hartley, J.D. Jenkins, R.W. Moore, and K.J. M. Moriarty: Physical Review D.1, 2619 (1970).

(1) A.M. Boyarski, F. Bulos, W. Busza, R. Diebold, S.D. Ecklund, G.E. Fischer, J. Rees, and B. Richter: Physical Review Letters 20, 300 (1968) and contribution to the 1968 Vienna Conference, paper 430;

K. Lubelsmeyer: Pseudoscalar Meson Photoproduction, presented at the Fourth Symposium on Electron and Photon Interactions at High Energies, Liverpool, September 1969.

(2) G.E. Hite: Reviews of Modern Physics 41, 669 (1969) and references therein.

(3) M. LeBellac: Physics Letters 25B, 524 (1967)

(4) S. Mandelstam: Nuovo Cimento 30, 1113 (1963); ibid 30, 1127 (1963); and ibid 30, 1148 (1963).

(5) R. Delbourgo and A. Salam: Physical Review 186, 1516 (1969).

(6) D. Schildknecht: Vector Meson Dominance and Photoproduction, DESY report 69/10, and references therein.

(7) D.G. Fincham, A.P. Hunt, J.H.R. Migueron, and K.J.M. Moriarty: Nuclear Physics B13, 161 (1969).

(8) G. Feldman and P.T. Matthews: Annals of Physics 40, 19 (1966).

(9) R. Delbourgo, M.A. Rashid, A. Salam, and J. Strathdee: Trieste Lectures 1965.

(10) M.D. Scadron: Physical Review 165, 1640 (1968).

(11) L. Van Hove: Physics Letters 24B, 183 (1967).

(12) For example S.S. Schweber: Introduction to Relativistic Quantum Field Theory, (Harper and Row 1961).

(13) V.N. Gribov, L. Okun, and I. Pomeranchuk: Soviet Physico-JETP 18 769 (1967).

(14) M. Jacob and G.C. Wick: Annals of Physics 7, 404 (1959).

(15) F.D. Gault, B.J. Hartley, J.H.R. Migueron, and K.J.M. Moriarty: Nuovo Cimento 62, 269 (1969); and references therein.

(16) R.C. Arnold: Physical Review 140B, 1022 (1965);
G.L. Kane, F. Henyey, D.R. Richards, and M. Ross: Strong Regge Cut Model for Photoproduction of Pions, presented at the Fourth Symposium on Electron and Photon Interactions at High Energies, Liverpool, September 1969.

(17) H.D.D. Watson: Physics Letters 17, 72 (1965).

(18) B.J. Hartley, R.W. Moore, and K.J.M. Moriarty: Physical Review 187, 339 (1969); and ibid to be published.

(19) J. Froyland and D. Gordon: Physical Review 177, 2500 (1969).

(20) F. Henyey, G.L. Kane, J. Pumplin, and M. Ross: Physical Review 182, 1579 (1969).

(21) M.L. Blackmon, G. Kramer, and K. Schilling: Nuclear Physics B12, 495 (1969).

TABLES

Table 5: Absorption coefficients.

p_{lab} (GeV/c)	$c_{initial}$	$\nu_{initial}$ $(GeV/c)^{-1}$	c_{final}	ν_{final} $(GeV/c)^{-1}$
5.0	1	.295	.81	.26
8.0	1	.295	.76	.26
11.0	1	.295	.73	.26
16.0	1	.295	.71	.26
18.0	1	.295	.71	.26

Table 6: Regge parameters.

trajectory	π	ρ	A_2
α_0	-0.520	-0.871 (not varied)	-0.936 (not varied)
α_1	0.513	1.415 (not varied)	1.420 (not varied)
α_2	0.674	0.632 (not varied	0.607 (not varied)
β	21.637	-33.318	35.040

The number of data points used was 99, with a resulting χ^2 of 255.

FIGURE CAPTIONS

(1) Plot of $\alpha(t)$ against t for π, ρ, and A_2 trajectories. Parameters from table 6.

(2) Predictions for elementary pion, Reggeized pion, absorbed elementary pion, and absorbed Reggeized pion exchanges (indicated by curves 1,2,3, and 4 respectively) at 8 GeV /c.

(3) Differential cross-section for $\gamma p \to \pi^+ n$. Data from Boyarski et al (reference (1)).

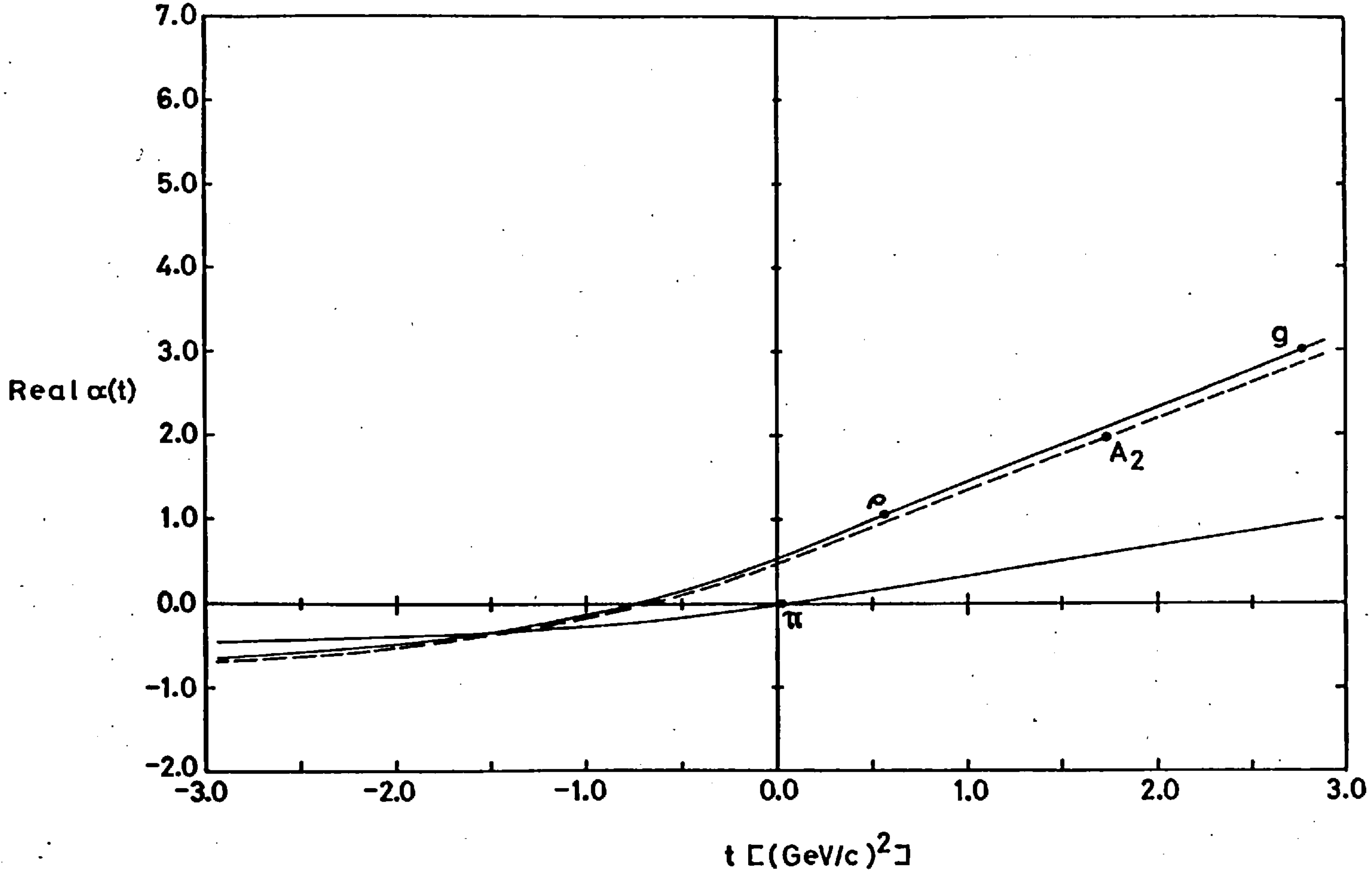

7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
Real α(t)
g
A2
ρ
π
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
t [(GeV/c)2]

Fig.1.

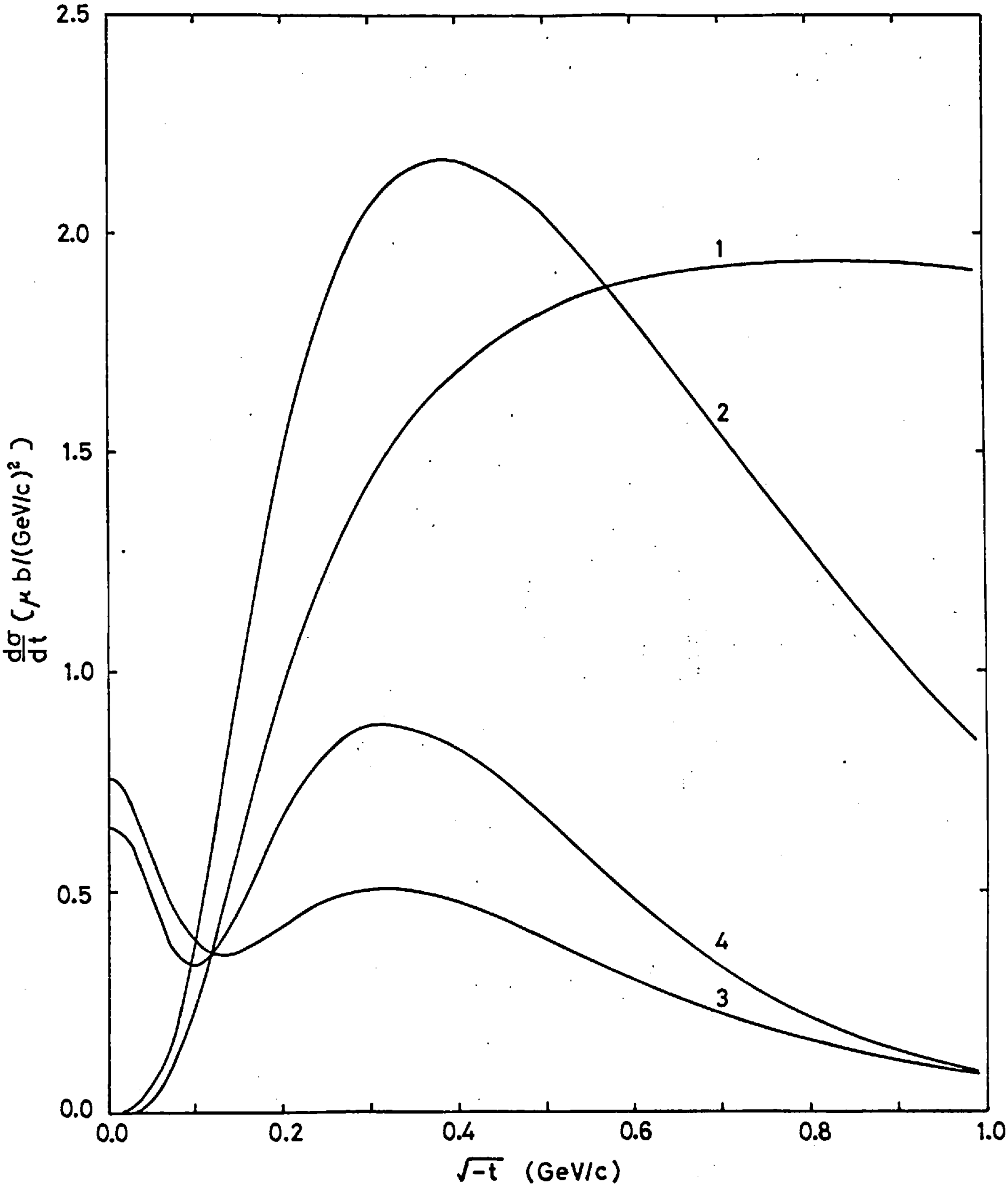
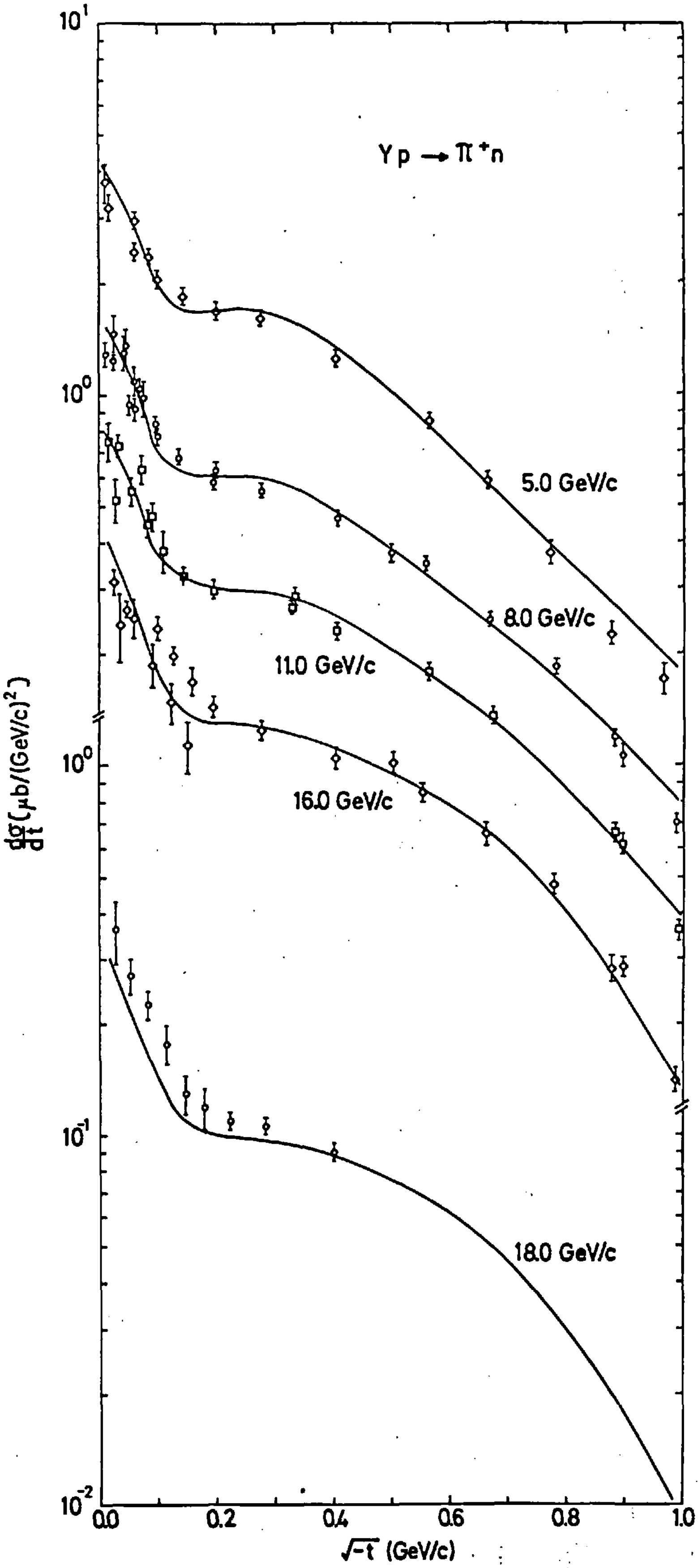

2.5
2.0
1.5
1.0
0.5
0.0
$\frac{d\sigma}{dt}$ (μ b/(GeV/c)^2)
0.0
0.2
0.4
0.6
0.8
1.0
$\sqrt{-t}$ (GeV/c)
1
2
3
4

Fig. 2.



γp → π⁺n
5.0 GeV/c
8.0 GeV/c
11.0 GeV/c
16.0 GeV/c
18.0 GeV/c
10^1
10^0
10^0
10^{-1}
10^{-2}
0.0
0.2
0.4
0.6
0.8
1.0
$\frac{d\sigma}{dt}$ (μb/(GeV/c)^2)
$\sqrt{-t}$ (GeV/c)

Fig. 3.