THE MATHEMATICAL FOUNDATIONS

of

QUANTUM FIELD THEORY

by

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ABSTRACT

Two approaches to the mathematical foundations of quantum field theory are considered in detail, the recent constructive attack whose van is represented by J.Glimm and A.Jaffe, and the abstractly axiomatic formulation of I.E.Segal. In the spirit of the work of Glimm and Jaffe, the $\lambda\phi^{2n}$ field theory in two dimensional space-time is shown to exist without cutoffs in the sense that the renormalized Hamiltonian is self adjoint and has a vacuum, which is locally Fock. The structure of I.E.Segal is then developed and it is shown that a natural extension of the definition of renormalized powers of fields of Glimm and Jaffe is the unique one guaranteed by Segal's theory.

PREFACE

This thesis results from work carried out as a postgraduate research student in the Department of Theoretical Physics of Imperial College from October 1966 to September 1969, under the supervision of Professor P.T.Matthews,F.R.S. .

Except as stated in the text, the work in this thesis is original and has not been submitted in this or any other university for any other degree.

I am beholden to Professor Matthews for his patience and encouragement, and to the staff of the Theoretical Physics Department and my fellow students, especially C.J.Isham, for providing a pleasant atmosphere of thriving activity in physics during my time there.

My greatest debt is to Professor R.F.Streater, who introduced me to the axiomatics of quantum field theory and taught me of them, and of mathematical physics in the large. Discussions with him have been repeated sources of stimulation.

In addition I am grateful to Professor I.E.Segal for the hospitality of the Mathematics Department of the Massachusetts Institute of Technology between October 1968 and July 1969. I must thank Professor A.Jaffe, Dr.P.M.Bongaarts, Dr.J.T.Cannon, and Dr. N. Skovhus Poulsen for conversations relevant to this material, and professors and instructors of both M.I.T. and Harvard for making the year so mathematically rewarding.

I acknowledge with gratitude the support of M.I.T. as a Research Associate during my time there, and a Postgraduate Research Studentship of the S.R.C. held from October 1966 to October 1968 and from July to September 1969.

I wish finally to thank Professor R.F.Streater and I.F.Wilde for pertinent comments during the final stages of preparation, and my wife Heather both for her support, her interest in my thesis, and her excellent typing of it.

Parturiunt montes, nascitur ridiculus mus.[1]

P.D.F.Ion

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INTRODUCTION

I have been fascinated since my introduction to the theory of elementary particles by the specious statement that these are but mathematical figments. This is assuredly not the case though often suggested in popular accounts. In fact the quantum theory of fields, which in quantum electrodynamics has been so quantitatively predictive, and in nuclear force theory qualitatively highly successful, leaves much to be desired mathematically. Manipulations that are without the pale of rigorous mathematics have led to these excellent theories. This should be paradoxical and is at least discomfiting.

The study of the mathematical underpinnings of field theory was started many years ago by von Neumann, Friedrichs, Haag, Segal, Wightman, van Hove and others, but progress had, a few years ago, seemed at an end.[2] Recently, however, successes have been attained, notably by J.Glimm, A.Jaffe[3] and I.E.Segal.[4] The first two are following lines set out by A.S.Wightman[5] and are preparing explicit models. Perforce only in two dimensional space-time do they have fairly full results as yet. Their method is that of approximation of an interacting theory which by the well known theorem of Haag, cannot be realized in Fock space, by butchered theories and of taking care of the separate problem of ultra-violet divergences[5] by use of explicit sequences of dressing transformations suggested by perturbation theory.[6] The vacuum is treated by local algebraic field theory's methods.[7] They show that their limits converge to give meaningful operators on a definite new Hilbert space and verify as many of the desirable properties of field theory, as set down in the Wightman[8] and Haag-Kastler[9] axiom systems, as they can. For $(\lambda\phi^4)_2$ they have the fullest results, a well defined Hamiltonian,[10] a vacuum state known to be locally Fock,[11] and correct covariance of the physical and local fields with respect to space and time translations.[12]

Segal has recently announced achievement of similar results of existence of a local field theory for $(\phi^4)_2$ and $(\bar{\psi}\psi\phi)_2$ to those of Glimm and Jaffe.[13] His theorems are based on an extended building up of a general theory,[14] which besides not being so explicitly dimension dependent as the ingenious estimations of Glimm and Jaffe, subsumes much theoretical development of stochastic process theory and of Lie group representation theory.[15] His methods are in addition based on an idiosyncratic development of methods in functional analysis.[16] This makes his work both lengthy and somewhat obscure. Glimm and Jaffe and associates have in the last three years also produced a voluminous amount of material.[17] In spite of this it seemed very much worthwhile to try to understand something of the work of both parties to what is a fairly heated controversy.[18]

In the first part then, I follow Glimm and Jaffe and show that their $(\phi^4)_2$ results may be extended to $(\phi^{2n})_2$ [19] by the use of a stronger theorem of theirs on singular perturbations than they used in $(\phi^4)_2$; in addition a short local Fockness proof is given.[20] In part II an unfortunately sketchy exposition of Segal's foundational theories is given and the relation of it to the treatment of Part I explored. A general theorem of Segal on normal ordering[21] is reproved, both to avoid an error in a published proof and to demonstrate the possibility of handling both Bose and Fermi commutation relations simultaneously.[22]

Unfortunately the need for a fuller treatment than it has been possible to give herein of both theories and their relations one with another and each with diverse realms of mathematics and physics remains.

PART I

THE CONSTRUCTIVE APPROACH OF J.GLIMM AND A.JAFFE

A Model for

$\lambda\phi^{2n}$

Field Theory in Two Dimensional Space-Time.

§1 FOCK SPACE AND THE FREE EVOLUTION

We again define Fock space

$$\underline{\underline{F}} = \underline{\oplus}\,\{\underline{\underline{F}}_n : n \text{ in } \mathbb{N}\}$$

as a Hilbert sum $\underline{\oplus}$ of its n-particle constituents:

$$\underline{\underline{F}}_n = SL_2(R^n) \quad .$$

The vectors of $\underline{\underline{F}}$ will be written f and their components f_n; thus

$$f = (f_0, f_1, f_2, \ldots) \quad .$$

A function f_n in $\underline{\underline{F}}_n$ represents a state with exactly n particles, and $|f_n|^2$ is to be interpreted as the distribution function for their momenta. For vectors of R^n we shall use roman capitals so that for instance

$$K = (k_1, \ldots, k_n) \qquad \text{where } k_1, \ldots, k_n \text{ are}$$

each in R. A permutation π on n ciphers, element of the group P_n of such, will act on such a K in R^n as follows

$$K_\pi = (k_{\pi 1}, \ldots, k_{\pi n}) \quad .$$

We adopt the following notion of, and notation for, the symmetrizing operation

$$\underline{\text{Sym}}_n : L_2(R^n) \to SL_2(R^n)$$
$$: f_n(K) \to (n!)^{-1} \sum\{f_n(K_\pi) : \pi \text{ in } P_n\} \quad .$$

We use the notation $\underline{\text{Sym}}$ for the extension by summing

$$\underline{\text{Sym}} = \oplus\,\{\underline{\text{Sym}}_n : n \text{ in } \mathbb{N}\}$$

so that

$$\underline{\text{Sym}} : \underline{\oplus}\,\{L_2(R^n) : n \text{ in } \mathbb{N}\} \to \underline{\underline{F}} \ ,$$

and we note that both $\underline{\text{Sym}}_n$ and $\underline{\text{Sym}}$ are contractions i.e.

$$||\underline{\mathrm{Sym}}_n f_n|| \leq ||f_n||$$

and $$||\underline{\mathrm{Sym}} f|| \leq ||f|| \, .$$

We proceed now to defining the free motion of particles in terms of the above wave functions; we assume the non-interacting particles should have their configuration evolution defined by the positive energy part $i\partial_t - \sqrt{(-\Delta+m^2)}$

of the Klein-Gordon operator giving the motion of each

$$\Box + m^2 = (i\partial_t - \sqrt{(-\Delta+m^2)})(i\partial_t + \sqrt{(-\Delta+m^2)}).$$

The momentum space evolution is then governed by

$$\partial_t f_1(t,k) = -i\sqrt{(-(ik)^2 + m^2)}f_1(0,k)$$

so

$$f_1(t,k) = e^{-i\sqrt{(k^2+m^2)}}f_1(0,k)$$

for one variable k in R and where we have denoted the time dependent function with the same letter f_1. We note that $\sqrt{(k^2+m^2)}$ is the symbol of the pseudo-differential operator $\sqrt{(-\Delta+m^2)}$.[23]

Note: A partial analysis of quantization and its relation to pseudo-differential operators and their symbols is given in A.Grossmann, G.Loupias, & E.Stein: An algebra of pseudo-differential operators and quantum mechanics in phase space. Ann.Inst.Fourier (Grenoble) 43, (1969), 343-368

For the general case of n non-interacting particles, we write the free evolution as follows:

$$f_n(t,K) = e^{itH_0}f_n(0,K)$$

where $H_0 f_n(t,K) = \mu(K)\ f_n(K)$

where $\mu(K) = \sum\{\mu(k_j) : 1 \leq j \leq n \text{ and } K=(k_1,\dots,k_n)\}$

where $\mu(k) = \surd(k^2+m^2)$

for k in R, with k^2 being the square of the norm of k. Noting that for π in P_n

$$\mu(K_\pi) = \mu(K)$$

we see that H_0 is an operator on $\underline{\underline{F}}$. H_0 is called the free Hamiltonian operator. Its domain is certainly dense for it includes

$$\underline{\underline{F}}' = \oplus\{\underline{\underline{F}}_n : n \text{ in } \mathbb{N}\}$$

the vector space direct sum of the n particle spaces. $\underline{\underline{F}}'$ is then the vector subspace of $\underline{\underline{F}}$ consisting in those state vectors f with only a finite number of particles, or

$$\underline{\underline{F}}' = \{f \text{ in } \underline{\underline{F}} : \text{there exists N in } \mathbb{N} \text{ such that } n>N \text{ implies } f_n=0\} .$$

A putative total Hamiltonian operator H, describing the time evolution of the interacting system in a manner similar to the free evolution should have the form

$$H_0 = H_o + H_I$$

where H_I gives the energy of interaction. We wish to concern ourselves with polynomial self interactions of the boson particles and particularly with those usually written '$\lambda : \phi^4(x):$' or a little more generally '$\lambda : \phi^{2p}(x):$' with $p \geq 2$ an integer. What is meant is of course that the translation invariant interaction H_I is given by the above Hamiltonian densities

$$H_I = \lambda \int_R : \phi^{2p}(x) : dx .$$

where $:\phi^{2p}(x):$ is the Wick ordered operator. This matter is by now a commonplace. But it is not manifestly well defined. In fact, attempts at formulation with observance of the mathematical proprieties show that in general, the so-called operators above are not such at all and that even observing the convention of smearing powers of the field are bilinear forms at best if the dimension of space R is more than 1.

We shall therefore continue the pedantic way and set up with care the formalism. First, we give some comments on the Fourier transform. R, the space of our space time, which is thus $R \times \mathbb{R}^1$, will in the following be taken to be $\mathbb{R}^s$ for some dimension s, or possibly $\mathbb{T}^s = \mathbb{R}^s/\mathbb{Z}^s$. $\mathbb{R}^s$ is isomorphic to its group dual and this isomorphism is taken to be given by $\mathbb{R}^s \to \mathbb{R}^s ; x \to e^{2\pi ix.} = \underline{e}(x.)$

where $\underline{e}(x.) : \mathbb{R}^s \to \mathbb{T} ; k \to \underline{e}(x.k) = e^{2\pi ix.k}$

where x.k is the natural scalar product in $\mathbb{R}^s$. The character associated to x is taken to be $\underline{e}(x.)$ in order that restriction to the subgroup $\mathbb{Z}^s$ provides the correct duality with $\mathbb{T}^s$. In addition one avoids many of the irritating powers of the square root of 2π in many formulas.[24] We shall adopt the usual convention of associating the variable x with configuration space, and k with its dual momentum space. Thus also shall we associate multivectors X with R^n and K with its dual, again R^n; we have the scalar product

$$X.K = \sum\{x_j.k_j : 1 \leq j \leq n \ \& \ X=(x_1,\ldots,x_n) \ \& \ K=(k_1,\ldots,k_n)\}$$

The Fourier transform in $L_2(R^n)$ and thus in $SL_2(R^n)$ is then explicitly given by

$$L_2(R^n) \to L_2(R^n) \; : \; f_n \to \tilde{f}_n$$

where

$$\tilde{f}_n(K) = \int \underline{e}(X.K) f_n(X)\,dX = \int_{R^n} e^{2\pi i (X.K)} f_n(X)\,dX$$

where of course

$dX = dx_1\, dx_2 \;\ldots\; dx_n$ is the product measure on R^n. The inverse transform is then simply

$$f_n(X) = \int \underline{e}(-X.K) \tilde{f}_n(K)\,dK$$

The Fourier transform, as above given, clearly is an isomorphism of $\underline{\underline{F}}_n$ and extends to an isomorphism of $\underline{\underline{F}}$ (with a slight abuse of the notation $\tilde{f}$).

§2 EXISTENTIAL OPERATORS AND BIQUANTIZATION

We adopt a natural domain of 'good' vectors in $\underline{\underline{F}}$, a subspace of $\underline{\underline{F}}'$ and in fact the domain of C^∞ -vectors for the free Hamiltonian as will later be shown. We call this domain $\underline{\underline{D}}_0$ (following Glimm-Jaffe slavishly) and define it by

$$\underline{\underline{D}}_0 = \{f \text{ in } \underline{\underline{F}}' : \text{for every } n \; f_n \text{ is in } \underline{\underline{S}}(R^n)\}$$

where $\underline{\underline{S}}(R^n)$ is the Schwartz space of rapidly decreasing C^∞ functions on R^n. Continuing the interpretation of $\underline{\underline{F}}$ in terms of momenta $\underline{\underline{D}}_0$ is the space of states with a finite number of particles whose momenta tend not to be very high. Since the Fourier transform maps $\underline{\underline{S}}$ into itself (and in fact $\underline{\underline{S}}$ may be characterized as such a space),[25] we know that the Fourier transform maps $\underline{\underline{D}}_0$

into itself and so there is a similar configuration space interpretation.

On this 'natural domain' we shall define the standard annihilation operator $a(k) : \underline{\underline{F}}_n \to \underline{\underline{F}}_{n-1}$.
More precisely perhaps

$$a(k) : \underline{\underline{D}}_0 \cap \underline{\underline{F}}_n \nrightarrow \underline{\underline{D}}_0 \cap \underline{\underline{F}}_{n-1}$$

where $\nrightarrow$ is used to point out that a well defined map is being displayed where the source is the domain, and we do not just have an operator which being in general unbounded may well have as domain a proper subset of the target. For an element f of $\underline{\underline{D}}_0$

$$(a(k)f)_{n-1}(K) = n^{\frac{1}{2}}f_n(k,K)$$

where of course $K = (k_1,\ldots,k_{n-1})$ and so on.

When emphasizing pedantic clarity the following notations will be used for operators (understood to be maps from a domain in the source set to a range in the target set) and maps:-[26]

$f : A \to B$ means f is an operator from A to B so that with the abbreviation $\underline{\text{Domf}}$ and $\underline{\text{Rgef}}$ for the domain and range of f

$f \restriction \underline{\text{Domf}} : \underline{\text{Domf}} \mapsto B$

meaning f (restricted to $\underline{\text{Domf}}$) is a map from $\underline{\text{Domf}}$ to B, in fact $f \restriction \underline{\text{Dom}}\, f : \underline{\text{Dom}}\, f \twoheadrightarrow \underline{\text{Rge}}\, f$
meaning f (restricted to $\underline{\text{Dom}}$.f) is a map from $\underline{\text{Dom}}$.f onto $\underline{\text{Rge}}$ f. We may further use

$f : A \leftrightarrow B$

to mean f is one-to-one from part of A to part of B, and combine this with the previous notation to get the following map notations

$f : A \nrightarrow B$	f maps A to B
$f : A \nleftrightarrow B$	f maps A into B (injection)
$f : A \twoheadrightarrow B$	f maps A onto B (surjection)

$f : A \twoheadrightarrow B$ f maps A into & onto B (bijection)
It should be noted that these are just set theoretic niceties, and as such will only be brought up in delicate situations. The simple arrow $\to$ will normally suffice and will in any case almost always be used for the display transformation of a generic element as in

$f : A \to B : a \to f(a)$

An alternative is $a := f(a)$ (read a becomes $f(a)$), a notation borrowed from the computer language Algol.

On their common domain $\underline{\underline{D}}_0$, any $a(k)$ and $a(k')$ clearly commute i.e.

$$[a(k), a(k')] = 0$$

So any product $a(k_1)\ldots a(k_\beta)$ is well defined on $\underline{\underline{D}}_0$ and will be denoted

$$a(K) \text{ for } K = (k_1, \ldots, k_\beta).$$

The adjoint $a^*(k)$ of $a(k)$ has domain $\{0\}$, but the expression usually written for it makes it plain that it is a densely defined bilinear form on $\underline{\underline{F}}^2$; in fact its domain is $\underline{\underline{D}}_0 \times \underline{\underline{D}}_0$. The usual expression referred to is

$$(a^*(k)f)_{n+1}(K) = (n+1)^{-\frac{1}{2}} \sum_{1\leq j\leq n+1} \{\partial(k-k_j) f_n(K-k_j)\}$$

where if

$$K = (k_1, \ldots, k_n)$$

then

$$K-k_j = (k_1, \ldots, \hat{k}_j, \ldots, k_n)$$

with ^ as deletion operation. The expression which includes a delta function multiplication is obviously no nontrivial operator and is to be interpreted as defining the bilinear form on $\underline{\underline{D}}_0 \times \underline{\underline{D}}_0$ given by

$$a^*(k)(g,f) = \sum_{n=0}^{\infty} n^{-\frac{1}{2}} \sum_{j=1}^{n} \int \delta(k-k_j) g_{n-1}(K-k_j) \times \overline{f_n(K)}\, dK$$

$$= \sum_{n=0}^{\infty} n^{-\frac{1}{2}} \sum_{\pi \text{ in } P_n} \int g_{n-1}(K)\, \overline{f_n((k,K)_\pi)}\, dK$$

so that $a^*(k)$ is an $\underline{\underline{S}}(R)$ valued bilinear form. We check that it is in fact the adjoint of $a(k)$:-

$$\begin{aligned} a(k)(f,g) &= \sum_{n=0}^{\infty} n^{\frac{1}{2}} \int \overline{g_{n-1}(K)}\, f_n(k,K)\, dK \\ &= \sum_{n=0}^{\infty} n^{-\frac{1}{2}} \sum_{\pi \text{ in } P_n} \int \overline{g_{n-1}(K)}\, f_n((k,K)_\pi)\, dK \;. \end{aligned}$$

Thus

$$a^*(k)(g,f) = \overline{a(k)(f,g)}$$

as should be.

Despite the essential incorrectness of this practice, we shall nonetheless call $a^*(k)$ the creation operator. We are led thus to define the 'monomial'

$$a^*(K) = a^*(k_1)\ldots a^*(k_\alpha)$$

for K in R^α and generally the bilinear form for K in R^α, J in R^β

$$\begin{aligned} a^*(K)a(J) &: \underline{\underline{D}}_0 \times \underline{\underline{D}}_0 \to \underline{\underline{S}}\,(R^{\alpha+\beta}) \\ &: (f,g) \to \langle f |\, a^*(K)a(J)g \rangle \end{aligned}$$

We may now define for a kernel

$$c_{\alpha\beta}\,(K,J) \text{ in } \underline{\underline{S}}'\,(R^{\alpha+\beta})$$

the true ($\mathbb{C}$-valued) bilinear form on $\underline{\underline{D}}_0^2$

$$C_{\alpha\beta} = \int c_{\alpha\beta}(K,J)\ a^*(K)a(J)\,dK dJ$$

using the standard integral notation for a distribution.

If $c_{\alpha\beta}(K,J)$ is the kernel of a bounded operator from $\underline{\underline{S}}(R^\beta)$ to $L_2(R^\alpha)$

so that for every f_β in $\underline{\underline{S}}(R^\beta)$

$$\int \left| \int c_{\alpha\beta}(K,J)\ f_\beta(J)\ dJ \right|^2 dK < \infty$$

then $C_{\alpha\beta}$ the biquantization[27] of $c_{\alpha\beta}$ determines an operator on $\underline{\underline{F}}$ whose domain includes $\underline{\underline{D}}_0$. This is built up in the following steps:

$$c_{\alpha\beta} \otimes I_{\gamma\gamma} : \underline{\underline{S}}(R^{\beta}) \otimes \underline{\underline{S}}(R)^{\gamma} \to L_2(R^{\alpha+\gamma})$$

is a bounded operator, and extends to a bounded operator from $\underline{\underline{S}}(R^{\beta+\gamma})$ to $L_2(R^{\alpha+\gamma})$. Then

$\underline{\mathrm{Sym}}_{\beta+\gamma}\ \underline{\mathrm{Sym}}_{\alpha+\gamma}\ (c_{\alpha\beta} \otimes I_{\gamma\gamma})$ is still a bounded operator from $S\underline{\underline{S}}(R^{\beta+\gamma})$ to $SL_2(R^{\alpha+\gamma})$, and so defines an operator (unbounded) from $\underline{\underline{F}}_{\beta+\gamma}$ to $\underline{\underline{F}}_{\alpha+\gamma}$. To finish, sum over all γ in $\mathbb{N}$ these operators to an operator defined on $\underline{\underline{D}}_0$, for in $\underline{\underline{F}}_{\beta+\gamma}$ it is defined on $\underline{\underline{S}}(R^{\beta+\gamma}) \cap \underline{\underline{F}}_{\beta+\gamma}$. Thus

$$C_{\alpha\beta} = \sum_{\gamma=0}^{\infty} \underline{\mathrm{Sym}}_{\alpha+\gamma}\ \underline{\mathrm{Sym}}_{\beta+\gamma}\ c_{\alpha\beta} \otimes I_{\gamma\gamma}$$

It follows from the construction that $C_{\alpha\beta}$ is closable in $\underline{\underline{F}}$ if $c_{\alpha\beta}$ were closable as an operator from $L_2(R^{\beta})$ to $L_2(R^{\alpha})$ Further good properties hold if $c_{\alpha\beta}$ is bounded, but in order to display them we first define the number operator N by

$$(Nf)_n = nf_n$$

on the domain $\{f : \sum \|nf_n\|^2 < \infty\}$ which certainly includes $\underline{\underline{D}}_0$. We have then

PROPOSITION 28

Let $c_{\alpha\beta}$ be a bounded operator from $L_2(R^{\beta})$ to $L_2(R^{\alpha})$ with norm $\|c_{\alpha\beta}\|$. Then its biquantization $C_{\alpha\beta}$ has a closure with domain containing

$\underline{\mathrm{Dom}}\ (N^{\frac{1}{2}(\alpha+\beta)})$ and satisfies the estimate

$$\|(N+I)^{-\alpha/2}\ C_{\alpha\beta}\ (N+I)^{-\beta/2}\| \leq \|c_{\alpha\beta}\|$$

PROOF

$$|(f_{\alpha+\gamma}, C_{\alpha\beta}\ g_{\beta+\gamma})| \leq \|c_{\alpha\beta}\| \{(\alpha+\gamma)!/\gamma!\}^{\frac{1}{2}} \|f_{\alpha+\gamma}\| \times \{(\beta+\gamma)!/\gamma!\}^{\frac{1}{2}} \|g_{\beta+\gamma}\|$$

$$\leq \|C_{\alpha\beta}\| (\alpha+\gamma)^{\alpha/2} \|f_{\alpha+\gamma}\| \ (\beta+\gamma)^{\beta/2} \|g_{\beta+\gamma}\|$$

by a simple application of the Schwarz inequality and a generous estimate for

$$\{(\alpha+\gamma)!/\gamma!\}^{\frac{1}{2}} \quad \text{and} \quad \{(\beta+\gamma)!/\gamma!\}^{\frac{1}{2}}.$$

It is interesting to note that the configuration space annihilation operator $A(x)$, destroying a particle at point x, may be defined by the biquantization of

$$C_{01}(k) = (2\pi)^{-\frac{1}{2}} \quad e^{-ik\cdot x} \quad ,$$

so $$A(x) = (2\pi)^{-\frac{1}{2}} \int e^{-ik\cdot x} \, a(k)dk$$

or $$A(x) = \int \underline{e}(-k\cdot x) \ a(k)dk$$

with the change of variables

$$x,k := (2\pi)^{-\frac{1}{2}}x, \ (2\pi)^{-\frac{1}{2}}k$$

(read x and k become respectively etc.).

Thus we have a simple Fourier transform, and in the configuration space Fock representation the operator

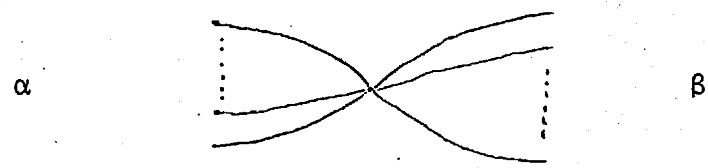
$$(A(x)f)_{n-1}(X) = n^{\frac{1}{2}}f_n(x,X) \ \text{ for } f \text{ in } \underline{\underline{D}}_0$$

with our previous multivariable notations. Again similarly to the above construction, one has $A^*(x)$ and biquantizations of $C_{\alpha\beta}$ in $\underline{\underline{S}}'(R^{\alpha+\beta})$ to

$$C_{\alpha\beta} = \int c_{\alpha\beta}(X,Y) \ A^*(X)A(Y)dXdY$$

a bilinear form on $\underline{\underline{D}}_0 \times \underline{\underline{D}}_0$. The properties of these $C_{\alpha\beta}$ are entirely similar to those of the previous ones as is obvious by the construction.

When Friedrichs' diagrammatic representation[29] is used for these forms, as is necessary for more complicated interactions such as $(Y)_2$, $(\phi^4)_3$ etc. the operator structure of creation and annihilation is given by a diagram



consisting in a vertex, and α legs to the left denoting creations and β legs to the right denoting annihilations. The distribution part $C_{\alpha\beta}(X,Y)$ is termed the numerical kernel of $C_{\alpha\beta}$. These diagrams (especially their generalization to include for instance fermion lines) resemble Feynman's diagrams but distinguish creators and annihilators and are so not 'relativistically covariant'.

We shall, as previously hinted, at times consider the case when R is a rectangular box B in some $\mathbb{R}^s$, That is to say we shall be considering the fields and observables within a bounded region of space B; this is a fairly natural thing to do and it is clear that a box does not have a very pathological form of boundary. By making the box rectangular we allow ourselves to put on simple periodic boundary conditions and B may then be thought of as a torus of the appropriate dimension s. In one dimension we have of course only intervals to consider and no possible ambiguity; in more than one dimension, say three for argument's sake, the boundary conditions one might impose on the (4) non-rectangular Platonic regular solids would produce weird topologies for these compact approximations.[30] It is clear that one may embed $\underline{\underline{F}}(B)$ in $\underline{\underline{F}}(R)$ for one has an obvious embedding in first degree of $L_2(B)$ in $L_2(R)$. Further properties are apparent. If $B_1 \subset B$ (as subsets of R) then $\underline{\underline{F}}(B_1) \subset \underline{\underline{F}}(B)$ in the obvious manner. Further if

$$B = B_1 \cup B_2 \text{ and } B_1 \cap B_2 = \emptyset$$

then

$$\underline{\underline{F}}(B) = \underline{\underline{F}}(B_1) \underline{\odot} \underline{\underline{F}}(B_2)$$

where $\underline{\odot}$ is the Hilbert space completion of the symmetric tensor product.

When we go from the configuration space representation of $\underline{\underline{F}}(B)$ discussed in the last paragraph, the importance of the possible compactification of B to a torus becomes manifest. The torus associated with the box B, is

$$b(\mathbb{R}^s/\mathbb{Z}^s)$$

where we have taken the identification of R with $\mathbb{R}^s$ so that the box B has one vertex at the origin and lies in the wholly positive '2^{+s}-tant' and b is the vector defining B. In other words b is the vector defining the lattice of which B is the fundamental unit cell. Multiplication of b by the coefficients in $\mathbb{Z}^s$ is component-wise. Then $B^{\sim}$ the dual of B, or its momentum space, is

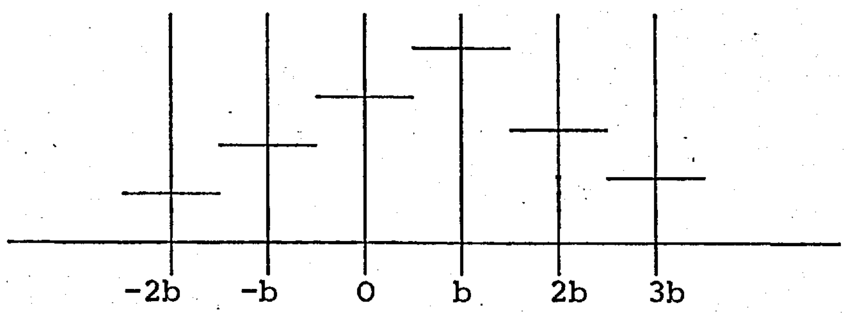
$$\Gamma_B = b'\mathbb{Z}^s,$$

the reciprocal lattice of $b\mathbb{Z}^s$ (as in crystallography) and b' is the vector of $\mathbb{R}^s$ whose components are the reciprocals of those of b. The momentum representation of

Fock space is then $\underline{\underline{F}}(\Gamma_B)$ formed from a space of functions over a lattice. This may also be embedded in

$$\underline{\underline{F}}(R^{\sim}) \simeq \underline{\underline{F}}(R)$$

in a natural way. We consider the extensions of the functions, from definition at the points of the lattice Γ_B, to functions defined on all of R as having the appropriate constant value in a box of size "1/B" symmetrically placed about the lattice point. For example in 1 dimension the following diagram suggests what is to happen.



Analytically we write this prescription assigning an f in $\underline{\underline{F}}(R)$ to an f_Γ in $\underline{\underline{F}}(\Gamma_B)$, in each degree as

$$f_n(K) = f_{\Gamma,n}(\{K\})$$

where $\{K\}$ is the point in the lattice Γ_B^n closest to K in R^n. This is a definition only up to the boundaries of the boxes about each lattice point, but since the union of the boundaries is a set of measure zero and we are interested only in defining L_2 'functions' f_n, this is of no consequence.

The momentum space existential 'operators' are given by $$a^{(*)}{}_B(k) = V^{\frac{1}{2}}\int \chi_{b'}\ (k-\ell)\ a^{(*)}(\ell)\,d\,\ell$$

for k in Γ_B, where $\chi_{b'}$ is the characteristic function of the reciprocal unit cell recentred about the origin

$$\chi_{b'}(\ell) = \begin{cases} 1 & \text{for } -\tfrac{1}{2}b' \leq \ell \leq \tfrac{1}{2}b' \\ 0 & \text{otherwise} \end{cases}$$

(using the simple convention of writing a set of inequalities on components of vector as inequalities on the vectors). Further V is the volume of the box B so we have divided by the square root of the volume of the reciprocal lattice unit cell which is V^{-1}; this is done so that the boxed free Hamiltonian does not have to have an explicit factor of V^{-1} in front of it. Here $a^{(*)}$ stands for a, or a* as the case may be for this typewriter does not have the conventional ≠≠. So these boxed existential operators are again obtained by quantizing a distribution, namely convolution with a characteristic function.

We note that the momentum space representation for the free Hamiltonian $H_{O,B}$ for the system in the box B is

$$H_{O,V} = \sum_{k \text{ in } \Gamma_B} \mu(k)\ a^*{}_B\ (k)\ a_B(k) \quad .$$

§ 3 The Formal Interacting Hamiltonian H_f; $H(g)$; $H(g)_V$; & $H(g)_{V,\kappa}$ Its Form Factor, Volume and Ultra-Violet Cut-off Avatars.

We shall now introduce the interaction we wish to study, $\lambda :\phi^{2p}:$ for $p \geq 2$, in its formal form. Now we must restrict to one space dimension, $R = \mathbb{R}^1$ so that we have even in the simple approximation we shall take well defined operators. This is not in itself sufficient for boson-fermion interactions, and the dressing procedures of Glimm (after Friedrichs) are required then even for $(Y)_2 = (\overline{\psi}\psi\phi)_2$.[31] Such complications we cannot handle and so we continue mimicing after a fashion the Glimm-Jaffe papers.

The interaction in full formal form we wish to study is taking $\lambda = 1$ to lose a further factor to be carried:-

$$H_{I,f} = \int : \phi(x)^{2p} : dx$$

We take as example for formal calculations $2p = 4$.

$$H_{I,f} = \int :\phi(x)^4 : dx$$

where the Wick ordering is done in the usual prescription (creators to the left of annihilators). The field is, formally at least,

$$\begin{aligned}\phi(x) &= 2^{-\frac{1}{2}}(-\Delta+m^2)^{-\frac{1}{4}}\{a^*(x) + a(x)\}\\ &= 2^{-\frac{1}{2}}\{A^*(x)+A(x)\}\end{aligned}$$

Raising it to the power 2p and using the formal commutation relation (it is between bilinear forms)

$$[A^*(x), A(y)] = \delta(x-y)$$

one computes

$$:\phi(x)^4: = 2^{-\frac{1}{2}} \sum_{j=0}^{4} \binom{4}{j} A^*(x)^{4-j} A^j(x).$$

One may do a similar calculation with the expression of the field as a Fourier transform

$$\phi(x) = \int e^{2\pi ikx} \{a^*(k) + a(-k)\}\mu(k)^{-1}dk.$$

Then we have the following general form for $H_{I,f}$ in momentum space which we shall adopt as its <u>definition</u>. It should be noted we have dropped (or absorbed) the overall 2^{-p}.

$$H_{I,f} = \sum_{j=0}^{2p} \binom{2p}{j} V_j,$$

where

$$V_j = \int \delta(|K'| - |K''|)\ \nu(K',K'')^{-\frac{1}{2}} \times a^*(K')\ a(K'')\ d(K',K'')$$

an expression in which we continue the use of our form of Schwartz' multi-index notation. Thus K'' is in $\mathbb{R}^j$ and K' in $\mathbb{R}^{2p-j}$ and $d(K',K'') = dK'dK''$ etc. The numerical kernel of V_j is thus to be

$$V_j(K',K'') = \delta(|K'| - |K''|)\ \nu(K',K'')^{-\frac{1}{2}}$$

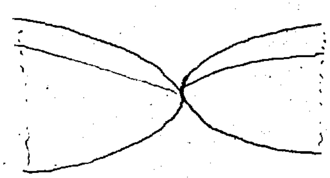
So the action of V_j as a bilinear form on say $\underline{\underline{D}}_0^2$ will be

$$\langle f|\rangle V_j h = \sum_{n=0}^{\infty} (n+2p-j!n)^{\frac{1}{2}} (n+j!n)^{\frac{1}{2}} \int \bar{f}_{n+2p-j}(K,K') \times \delta(|K'| - |K''|\ \nu(K',K'')^{-\frac{1}{2}} \times h_{n+j}(K'',K)\ d(K',K'',K)$$

where (K',K'',K) is in $\mathbb{R}^{2p-j}\times\mathbb{R}^j\times\mathbb{R}^n$ and

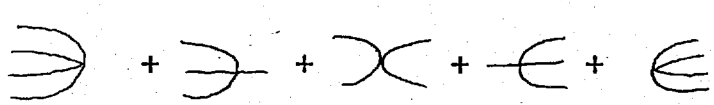
$$(n+2p-j!n)^{\frac{1}{2}} = ((n+2p-j)!/n!)^{\frac{1}{2}}$$

is the characteristic numerical factor coming from the $(2p-j)$ creation operations in $a^*(K')$ and $\{n+j!n\}^{\frac{1}{2}}$ is the similar factor from the annihilations. These factors are both in the numerator, i.e. greater than one, because the symmetrizations are implicit in the use of f and h in $\underline{\underline{D}}_0$. V_j is thus the part of the interaction that annihilates j particles and creates $(2p-j)$ and so would have the Friedrichs diagram:

2p-j  j

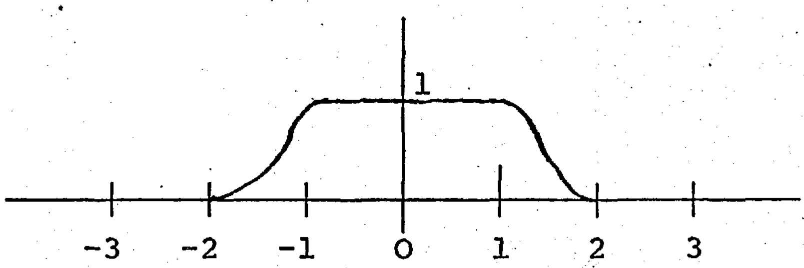
For $:\phi^4:$ then $H_{I,f} = V_0+4V_1+6V_2+4V_3+V_4$

or diagrammatically



Now we have the problem noted above that V_j will in general (if $j \leq 2p$ and it has creators in fact) only define a bilinear form. We are thus led, in order to calculate with operators on the Fock space we know without transgressing Haag's theorem, to introduce a cut-off in the interaction range.

We take a form factor $g(x)$, a real positive smooth (C^∞) function of compact support in $\mathbb{R}$; or g is in $C^\infty_{c,+}(\mathbb{R})$. We shall assume it an even function with support $[-2,+2]$ and to be equal to 1 on $[-1,+1]$, We further assume g monotonic on $[-2,-1]$ and $[1,2]$. So g has a 'bump' function appearance.



These exacting specifications beyond smoothness and compact connected support of g and $(1-g)$ for g are not really necessary, but they are convenient, do not cause a loss of generality, and fix a basic bump function in terms of which we may define other similar functions.

We define then the form factor cut-off interaction $H_I(g)$, by

$$H_I(g) = \int g(x) \; :\phi(x)^{2p}: dx \quad .$$

We take then a sequence of such cut-offs of increasing range given by

$$g_n(x) = g(x/n) \quad \text{(then } g=g_1)$$

so that

$$H_{I,n} = H_I(g_n) = \int g(x/n) :\phi(x)^{2p}: dx \quad .$$

These restricted Hamiltonians of interaction $H_{I,n}$ provide the desired interaction on the spatial region $|x|<n$. Assuming, as one would physically expect, that

disturbance propagates at the speed of light, that is as if free over a small region this restriction should not affect the field. Explicitly, as was pointed out by Guenin[32] (a proof was then given by Segal)[33] in the region

$$\{(x,t) \text{ in } \mathbb{R}^2 : |x|+|t| < n\}$$

the time evolution of the field would be correctly given by H_n as

$$\phi(x,t) = e^{itH_n}\,\phi(x,0)\,e^{-itH_n} ,$$

where

$$H_n = H_o + H_{I,n} .$$

One may then hope to piece together these locally correct dynamics.[32]

We shall discuss this point, and prove Segal's result[34] when we are sure that the total restricted range Hamiltonian is self-adjoint, as will be shown in section 6. First we have to show $H_I(g)$ self-adjoint which will be done in the next section. Further we would wish $H(g)$ to be bounded below and to have a lowest eigenvalue. We shall show the semiboundedness of $H(g)$ first following Glimm[35] in using a technique invented by Nelson.[36] We do this because by carrying out a more general estimation procedure, involving functional integration or path space methods,[37] we obtain the theorem on singular perturbations that will allow us to show $H(g)$ self-adjoint. We here have to avail ourselves of a stronger form of theorem than is required for $:\phi^4:$ interactions.[38]

To prepare the way of this route we shall set out in the remainder of this section the two further increasingly restricted forms of Hamiltonian and their increasingly simple properties. Then we shall be in a position to take these restriction off again as expected.

We commence with $H(g)_V$ the deranged (?!) Hamiltonian in a box with periodic boundary conditions. Boxes are familiar from section 1 and experience therein would suggest the simplification is greatest in the momentum representation. We exhibit $H_I(g)$ there first:

$$H_I(g) = \sum_{j=0}^{2p} \binom{2p}{j} V_j(g)$$

where the numerical kernels are

$$V_j(g)(K',K") = \tilde{g}(|K'|-|K"|)\ \nu(K',K")^{-\frac{1}{2}}$$

for

$$\delta(|K'|-|K"|) = \int \underline{e}(x.|K'|-x.|K"|)dx$$

so

$$\delta(g(x)) = \tilde{g}(|K'|-|K"|).$$

The full action on f in <u>Dom</u> $H_I(g)$, which certainly is dense for it includes $\underline{D}_0$, is

$$H_I(g)f = \sum_{j=0}^{2p} \binom{2p}{j} \quad \sum_{n+j=0}^{\infty} (n+2p-j,n+j!n,n)^{\frac{1}{2}}$$

$$\underline{\mathrm{Sym}} \int \tilde{g}(|K'|-|K"|)\ \nu(K',K")^{-\frac{1}{2}}\ f_{j+n}(K",K)dk"$$

where we avail ourselves of a sort of multinomial factorial notation $(\alpha!\beta)$

$$= (\alpha_1,\ldots,\alpha_m!\beta_1,\ldots,\beta_n)$$

$$= \alpha_1!\alpha_2!\ldots\alpha_m!/\beta_1!\beta_2!\ldots\beta_n!$$

for α in N^m and β in N^n, So the operator is

$$H_I(g) = \sum_{j=0}^{2p} \binom{2p}{j} \int \tilde{g}(|K'|-|K''|)\nu(K',K'')^{-\frac{1}{2}} \times a^*(K')a(K'')dK'dK''$$

$$= \sum_{j=0}^{2p} \binom{2p}{j} \int \tilde{g}(|(K',K'')|)\nu(K'.K'')^{-\frac{1}{2}} \times a^*(-K')a(K'')d(K',K).$$

Recalling the definition of the lattice $\Gamma_B = \Gamma_v$ (for 1 dimension of space) and of the Fock space upon it we may write the free field thereon

$$\phi_v(x) = (2v)^{-\frac{1}{2}} \sum_{k \text{ in } \Gamma_v} \underline{e}(kx)\mu(k)^{-\frac{1}{2}}\{a_v(k)+a_v^*(-k)\}.$$

Then

$$H_I(g)_v = \sum_{j=0}^{2p} \binom{2p}{j} (2v)^{-p} \sum_{(K',K'') \text{ in } \Gamma_v^{2p}} \tilde{g}_v(|K'|-|K''|) \times \nu(K',K'')^{-\frac{1}{2}} a^*(K')a(K'')$$

with $\tilde{g}_v(k)$ a restriction of the Fourier transform

$$\tilde{g}_v(k) = \int_{-v/2}^{+v/2} e^{2\pi ikx} g(x)dx .$$

If the box be so large that it properly contains the support of g then

$$\tilde{g}_v(k) = \tilde{g}(k) .$$

<u>We shall assume from now on that</u>

$$\underline{\text{supp } g \not\subseteq \left(-v/2, +v/2\right)}$$

and we drop the subscript 'v' on g.

We continue and define

$$H(g)_v = H_{o,v} + H_{I,v}(g)$$

the free part plus its interaction. (Note we commuted the subscript 'v' with '(g)', but will from habit unfortunately use both notations $H_{I,v}(g) = H_I(g)_v$.)

Continuing toward simplicity, we cut off the sum over the momentum lattice at some positive κ, the resulting finite lattice is denoted

$$\Gamma_{v,\kappa} = \{k \text{ in } \Gamma_B : |k| < \kappa\} \quad .$$

The physical content of this resection, is a finite number of modes for the field; but then we are back in the familiar territory of the representations of the canonical commutation relations of a finite number of degrees of freedom. We effect a transformation that expresses the system as a set of coupled Schrödinger oscillators and diagonalizes the interaction $H_{I,v}(g)_\kappa$. In fact we move from the exponential basis to the basis of trigonometric functions.[39] For the fields this reads

$$\phi_{v,\kappa}(x) = (2v)^{-\frac{1}{2}}\{\sum_{0\neq k \text{ in } \Gamma_{v,\kappa}} (q_k \underline{\cos} kx + q_{\hat{k}} \underline{\sin} kx) + 2q_0\}$$

and

$$\pi_{v,\kappa}(x) = (2v)^{-\frac{1}{2}}\{\sum_{0\neq k \text{ in } \Gamma_{v,\kappa}} (p_k \underline{\cos} kx + p_{\hat{k}} \underline{\sin} kx) + 2p_0\}$$

where

$$q_k = \tfrac{1}{2}\mu(k)^{-\frac{1}{2}}\{a^*_v(-k) + a_v(k) + a^*_v(k) + a_v(-k)\}$$

$$q_{\hat{k}} = i\tfrac{1}{2}\mu(k)^{-\frac{1}{2}}\{a^*_v(k) + a_v(-k) - a^*_v(-k) - a_v(-k)\}$$

$$p_k = \tfrac{1}{2}\mu(k)^{\frac{1}{2}}\{a^*_v(-k) - a_v(k) - a^*_v(k) + a_v(-k)\}$$

$$p_{\hat{k}} = i\tfrac{1}{2}\mu(k)^{\frac{1}{2}}\{a^*_v(-k) - a_v(k) + a^*_v(k) - a_v(-k)\}$$

and

$$q_0 = \tfrac{1}{2}m^{-\frac{1}{2}}\,\{a^*_v(0) + a_v(0)\}$$

$$p_0 = i\tfrac{1}{2}m^{\frac{1}{2}}\,\{a^*_v(0) - a_v(0)\}$$

and

$$\underline{\cos}\ kx = \cos\ 2\pi kx \quad \text{and} \quad \underline{\sin}\ kx = \underline{\sin}\ 2\pi kx$$

so

$$\underline{e}\ (kx) = \underline{\cos}\ (kx) + i\ \underline{\sin}\ kx \quad .$$

All the operators are defined on the restriction of $\underline{\underline{D}}_0$ to $\Gamma_{V,\kappa}$ which is certainly still dense. On the appropriate domains follow the commutation relations for k and k' in $\Gamma_{V,\kappa}$

$$[q_k, p_{k'}] = i\delta_{kk'}$$
$$[q_{\hat{k}}, p_{\hat{k}'}] = i\delta_{kk'}$$

with the general Kronecker delta symbol on the right. Rewriting the operators of interest we apply a further convention to avoid double counting due to both {q,p} and {q^,p^} being canonical variable sets- we set

$$q(k) = \begin{cases} q_k & \text{for } k > 0 \\ q_{\hat{k}} & \text{for } k < 0 \end{cases}$$

$$p(k) = \begin{cases} p_k & \text{for } k > 0 \\ p_{\hat{k}} & \text{for } k < 0 \end{cases} \quad .$$

Then we have

$$H_{0,V,\kappa} = \tfrac{1}{2} \sum_{0 \neq k \text{ in } \Gamma_{V,\kappa}} \{p(k)^2 + \mu(k)^2 q(k)^2 - \mu(k)\} \quad .$$

This follows from the elementary computations

$$\left\{ \begin{matrix} \mu(k)^2 q(k)^2 \\ p(k)^2 \end{matrix} \right\} = \pm\tfrac{1}{4}\mu(k) \Big[a^*(-k)^2 + a(k)^2 + a^*(k)^2 + a(-k)^2$$
$$\{^+_-\} 2\{a^*(-k)a(k) + a^*(k)a(-k) + a^*(-k)a^*(k)$$
$$\times \quad + a(k)a(-k)\}$$
$$\pm\{a^*(-k)a(-k) + a^*(k)a(k) + a(-k)a^*(k)$$
$$\times \quad + a(k)a^*(-k)\}\Big]$$

where $\{^{+}_{-}\}$ gives the only sign difference between $p(k)^2$ and $\mu(k)^2 q(k)^2$ and the upper or lower sign of $\pm$ is to be taken in each case according as $k > 0$ or $k < 0$. Then using the commutation $[a(k), a^*(k)] = 1$ it is readily verified that $H_{o,v,\kappa}$ has the alleged form.

Taking a further look at our new variables we see that

$$\mu(k)^{-\frac{1}{2}} \{a_v(k) + a_v^*(-k)\} = \begin{cases} q(k)+iq(-k) & \text{for } k > 0 \\ 2q & \text{for } k = 0 \\ q(-k)-iq(k) & \text{for } k < 0. \end{cases}$$

If we denote the left hand side by $\phi(k)$, then the field has the form

$$(2V)^{\frac{1}{2}} \phi(x) = \sum_{k \text{ in } \Gamma_{v,\kappa}} \underline{e}(kx)\, \phi(k)$$

and the $\phi(k)$ may be viewed as independent Gaussian isotropic complex random variables of mean 0 and variance $\mu(k)^{-1}$ (except for $\phi(0)$ which is real). We have as required

$$\phi(-k) = \overline{\phi(k)} \quad .$$

The $\phi(k)$ are random variables as maps from $L^2(\Gamma_{v,\kappa})$ to a Hilbert space.[40]

We continue by diagonalizing $H_{I,v,\kappa}$. We use the formal identity[41] that characterizes Wick products

$$\phi^{2p}_{v,\kappa}(x) = \sum_{j=0}^{2p} (2p!2p-2j,j)(C_\kappa/2)^j :\phi_{v,\kappa}(x)^{2p-2j}:$$

where

$$C_\kappa = v^{-1} \sum_{k \text{ in } \Gamma_{v,\kappa}} \mu(k)^{-1} .$$

This is to be taken as a bit of combinatorial algebra and not as an operator identity; it is the standard contraction formula.[42] By reexpressing $:\phi_{V,\kappa}(g)^{2p}:$ in terms of $\phi_{V,\kappa}(g)^{2p-j}$ for $j \geq 0$ from the above identity it may be seen that it certainly is a polynomial in the above variables $q(k)$. In fact this holds for any interaction term formally

$$P(\phi) = \sum_{\ell=0}^{d} b_k :\phi(g)^{\ell}:$$

for the above formula holds generally in the form

$$\phi^p_{V,\kappa}(x) = \sum_{j=0}^{r} (p!p-2j,j)(C_\kappa/2)^j :\phi_{V,\kappa}(x)^{p-2j}:$$

where r is the greatest integer less than or equal to p.

We have then canonical variables $q(k), p(k)$ and a full Hamiltonian expressed as a polynomial in terms of them. Obviously therefore we should set up the Schrödinger representation of them in terms of differentiation and multiplication operators. Further, this representation is essentially unique by the theorem of Stone & von Neumann for the number of variables k in $\Gamma_{V,\kappa}$ is finite. We will use the trivially renormalized Schrödinger representation on a space with a Gaussian measure, which is dimension independent (the ground state energy is absorbed) and relates to the random variable viewpoint. Let M be the number of modes, the cardinality of $\Gamma_{V,\kappa}$. We shall realize the system on $L_2(\mathbb{R}^M)$, with $q(k)$ being equivalent to multiplication by a coordinate q_k and $p(k)_n$ equivalent to $-i(\frac{\partial}{\partial q_k})$. We set it up as a (direct) tensor product of one mode spaces

$$\underline{\underline{H}}_k = L_2(\mathbb{R}, (\mu(k)/\pi)^{\frac{1}{2}} \cdot \exp(-\mu(k) q_k^2 dq_k).$$

Let

$$\underline{\underline{H}}_{v,\kappa} = \otimes_{k \text{ in } \Gamma_{v,\kappa}} \underline{\underline{H}}_k \ ;$$

this is then

$$L_2(\mathbb{R}^M, \rho_{v,\kappa} dq) \quad \text{with} \quad \rho_{v,\kappa} dq = \Pi_{k \text{ in } \Gamma_{v,\kappa}} \rho_k(q_k) dq_k \ .$$

The representation in $L_2(\underline{\underline{H}}_{v,\kappa})$

$$q(k) : f(q) \to q_k f(q)$$
$$p(k) : f(q) \to -i\rho_k^{-\frac{1}{2}} \left(\frac{\partial}{\partial q_k}\right) (\rho_k^{\frac{1}{2}} f(q))$$

is then irreducible. We shall want to let $M \to \infty$ later and this arrangement is designed to ease the transition.[43]

In one dimension a harmonic oscillator Hamiltonian may be represented on $L_2(\mathbb{R}, \rho(q)dq)$ (dropping the subscript k for a particular mode) as

$$H = -\left(\tfrac{1}{2} \frac{d^2}{dq^2} - \mu q \frac{d}{dq}\right).$$

This representation holds good for each k and on $\underline{\underline{H}}_{v,\kappa}$ our interaction will be

$$H_{v,\kappa}(g) = \sum_{k \text{ in } \Gamma_{v,\kappa}} -(\tfrac{1}{2}\partial_k^2 - \mu(k) q_k \partial_k) + V(q)$$

where $\partial_k = \dfrac{\partial}{\partial q_k}$ and $V(q)$ is the polynomial in the q_k 's that $H_{I,v,\kappa}$ becomes.

We discontinue work on this representation now until section 5 where semi-boundedness of the limit $H(g)$ as $\kappa \to \infty$ and $v \to \infty$ is shown.

§4 SEMIBOUNDEDNESS OF $H_I(g)$ and AN ESTIMATION OF IT WITH RESPECT N from MARKOV PROCESS METHODS

We under take the estimation of $H_I(g)$ in terms of F_τ, an operator interpolation between H_0 and N. Formally

$$F_\tau = \int \mu(k)^\tau \, a^*(k)a(k)dk \quad .$$

Thus

$$F_1 = H_0 \text{ and } F_0 = N \text{ the number operator.}$$

Using the notation

$$\mu^\tau(K) = \sum_{i=1}^n \mu(k_i)^\tau \quad ,$$

so that if K is not in $\mathbb{R}$ $\mu^\tau(K)$ must be distinguished from $\mu(K)^\tau$, the action on ξ in Dom H_0 is

$$(F_\tau \xi)_n (K) = \mu^\tau(K)\xi_n \ .$$

At the end of section 3 we had built up a formalism for description of $H_{v,\kappa}(g)$. We shall prove the required estimates hold in a suitably uniform manner with respect to v and κ and then take the limits $v,\kappa \to \infty$. The representation is in

$$\underline{\underline{H}}_{v,\kappa} = \otimes_k \text{ in } \Gamma_{v,\kappa} \underline{\underline{H}}_k$$

where

$$\underline{\underline{H}}_k \simeq L_2(\mathbb{R}, \rho_k(q_k)dq_k)$$

and

$$\rho_k(q_k) = (\mu(k)/\pi)^{\frac{1}{2}} \exp(-\mu(k)q_k{}^2).$$

For one mode we had the free Hamiltonian as

$$H_{0,k} = -(\tfrac{1}{2}\partial_k^2 - \mu(k)q_k\partial_k),$$

and the interaction as a polynomial perturbation. This representation is well suited to the Markov process point of view and the associated semigroup and path integral methods.

A Markov process[44] on a phase space S is an assignment to every quadruple consisting in, a starting time t in $\mathbb{R}$, an initial point x in S, a finishing time s in $\mathbb{R}$, and a set E of S of a probability $P(t,x;s,E)$ that the system which was in state x at time t will be at time s in a state of the set E; this probability should be such that the future of the system at time t is entirely independent of its past and this is expressed by the Chapman-Kolmogorov equation

$$P(t,x;s,E) = \int_S P(u,y;s,E)P(t,x;u,dy)$$

for $t < u < s$, that is the probability that from x at t the system evolves to be in E at s is the integral over y in S of the probabilities that at some intermediate u it be at y and then evolve to be in E at s, for every intermediate time u. We have a temporally homogeneous Markov process where $P(t,x;s,E)$ depends only on the time interval $s - t$ and not independently on t and s, and further S is a measure space with an invariant measure m. Making this explicit we have a map

$$P: S \times \mathbb{R} \times \underline{\text{Mbl}}S \to \mathbb{R}_+$$

such that

i) $P(t,x,E) \geq 0$; $P(t,x,S) = 1$

ii) $P(t,x,\cdot)$ is countably additive on $\underline{\text{Mbl}}$ S

iii) $P(t,\cdot,E)$ is measurable with respect to m

iv) $P(t+s,x,E) = \int_S P(t,x,E)m(dx)$ (or dmx),

where $\underline{\text{Mbl}}$ S are the sets of S measurable with resepct to m.

Conditions i) & ii) say that for fixed initial conditions we have a probability measure on S; iii) says that the values of this measure vary measurably with space; iv) is the Chapman-Kolmogorov equation saying that the probability of x getting into E in time (t+s) is the integral of the probability of its getting to some intermediate point y in time s times the probability of its getting therefrom to E in time t; v) says that the measure m(E) is the integral over S with weight m of the probabilities of x getting to E in time t, for any specified t. In a suitable space X of functions over S the process gives rise to a linear transformation of elements ψ of X by

$$(T_t\psi)x = \int \psi(y)\ P(t,x,dy)$$

and by iv) this forms a semigroup i.e.

$$T_{t+s} = T_t T_s \quad .$$

We have exactly the above situation here; $\exp(-tH_{o,k})$ is a semigroup of transformations on

$$L_2(\mathbb{E},\rho_k(q_k)dq_k$$

and in fact on

$$L_p(\mathbb{E},\rho_k(q_k)dq_k)$$

for

$$1 \leq p \leq \infty;$$

it has a well known kernel $p^t(q_k,q_k')$ [45] such that

$$(\exp(-tH_{o,k})\psi)q = \int p^t(q,q')\psi(q')\rho_k(q')dq'$$

where

$$\begin{aligned} p^t(q,q') &= (1-e^{-2\mu(k)t})^{-\frac{1}{2}} \\ &\quad\times \exp\left\{-\frac{\mu(k)(q'-e^{-\mu(k)t}q)^2}{1-e^{-2\mu(k)t}}+\mu(k)q'^2\right\} \\ &= \{e^{\mu(k)t/2}/(e^{\mu(k)t}-e^{-\mu(k)t})\} \\ &\quad\times \exp\{-\mu(k)(qe^{\mu(k)t}-q'e^{-\mu(k)t})^2 \\ &\qquad\div (e^{\mu(k)t}-e^{-\mu(k)t})\} \end{aligned}$$

Thus the probability measure is

$$\begin{aligned} P(t,q,dq') = {} & \{\mu(k)/\pi(1-e^{-2\mu(k)t})\}^{\frac{1}{2}} \\ & \times \exp\{-\mu(k)(q'-e^{-\mu(k)t}q)^2 \div (1-e^{-2\mu(k)t})\}dq'. \end{aligned}$$

One may check that the kernel $p^t(q,q')$ has the desired effect by working with it on the orthonormal basis (ψ_j) for $L_2(\mathbb{E},\rho_k(q)dq)$ consisting in the Hermite functions of variable

$$x = \mu(k)^{\frac{1}{2}}q \quad \text{divided by } \pi^{-\frac{1}{4}}e^{-\frac{1}{2}\mu(k)q^2}$$

i.e. $$\psi_j(x) = 2^{-j/2}\,(j!)^{-\frac{1}{2}}\,h_j(x)$$

where

$$h_j(x) = (-1)^j\,e^{x^2}\left(\frac{d}{dx}\right)^j e^{-x^2} \qquad 46$$

By carrying through for F_τ a calculation similar to that for F_0 we find

$$F_{\tau,v,\kappa} = \sum_{k \text{ in } \Gamma_{v,\kappa}} \left(\mu(k)^{\tau-1}p(k)^2 + \mu(k)^{\tau+1}q(k)^2 - \mu(k)^\tau\right)$$

or in differential operator form

$$F_{\tau,k} = -\left(\tfrac{1}{2}\mu(k)^{\tau-1}\partial_k^2 - \mu(k)^\tau q\partial_k\right)$$

with associated integral kernel then

$$p^t_\tau(q,q') = \left(1-e^{-2\mu(k)t}\right)^{-\frac{1}{2}}$$
$$\times\ \exp\{-\mu(k)^\tau\left(qe^{\mu(k)t}-q'e^{-\mu(k)t}\right)^2/\left(e^{\mu(k)t}-e^{-\mu(k)t}\right)\}$$

We pass now to the path space point of view, to make explicit the motive for the introduction of the Feynman-Kac integral[47] (that is motive other than that it provides recognized useful technique). Going back to an abstract semigroup of contractions on each L_2 space of a mode, we consider the space $\underline{\underline{C}}$ of all continuous paths $q=q(s)$ where s ranges over $\Gamma_{V,(\kappa)}$ and $q(s)$ takes its value in

$$L_2(\mathbb{R}_s, {}_s(q_s)dq_s).$$

Had we a different and more common scheme of interpretation s might be a time parameter. On this space $\underline{\underline{C}}$ of paths there is a (Wiener type) measure associated with the semigroup P_t, for in each mode P_t causes evolution of the coordinate $q(s)$ and thus acts on $\underline{\underline{C}}$. We have already for each mode a measure

$$P(t,q,dq') = p^t(q,q')\rho(q')dq'$$

the probability that if $q(0) = q$ then $q(t)$ will lie in

$$\left(q',q' + dq'\right) .$$

We take the σ field of measurable sets generated by those defined by finite families of Borel sets $E_1,\ldots, E_j$ and consisting in

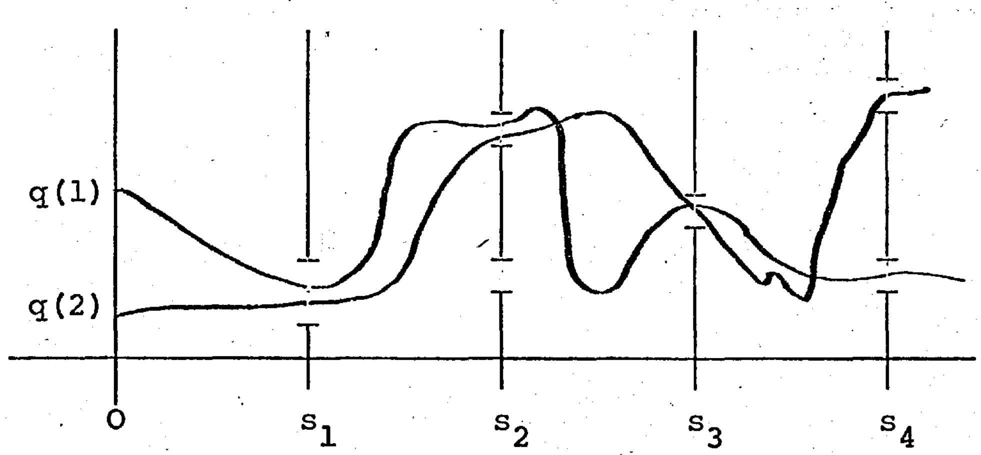
$$\Xi(E_1,\ldots, E_j) = \{q \text{ in } C \text{ such that } q(s_i) \text{ is in } E_i$$
$$\times \qquad \text{for } 1 \leq i \leq j\}$$

and

$$0 = s_1 < s_2 < \cdots < s_j.$$ We then define a measure

$$T(\Xi(E_1,\ldots,E_j))$$
$$= \int_{E_1 x \ldots x E_j} P(s_i - s_{i-1}, q(s_{i-1}), dq(s_i))\rho(q(0))dq(0)$$

and this is the probability that a path which is associated with evolution according to the law P_t will pass through 'specified gates' at several finite times if the starting points are Gaussian distributed.



The paths q (1) and q(2) represent elements of Ξ, and the gaps the Borel set 'gates' E_i . This definition is in a sense forced by the Markov character of the process. We remark that the integral may be extended over further sets E'_ℓ each equal to all of $\mathbb{R}$ from the Chapman-Kolmogorov equation so the sets Ξ are cylinder or tame sets, that is sets defined only by a finite number of conditions on a space of infinite dimension. This measure T permits then the integration of functions on

$$\underline{\underline{F}}_{V,\kappa} = \mathfrak{G}_k \text{ in } \Gamma_{V,\kappa}$$

and in fact on $\underline{\underline{F}}_V$. Further $\underline{\underline{F}}$ is in fact an inductive limit[48] of the spaces $\underline{E}_{v,\kappa}$ and we have a corresponding limit for the path spaces.[49] Doing an integral of a product of j functions

$$V_i \text{ in } L_j \ (\mathbb{E}, \rho_{k_i}(q_{k_i})dq_{k_i})$$

where

$$\rho = \Pi_{k \text{ in } \Gamma_{v,\kappa}} \rho_k(q_k) \quad ,$$

we get

$$\int \Pi_{i=1}^{j} V_i(q(s_i))dT$$
$$= \int dq(0)\rho(q(0))V_1(q(0))$$
$$\times \quad \{\exp(\sigma_1 H_{o,v,\kappa})\Pi_{i=2}^{j}\exp(\sigma_i H_{o,v,\kappa})V_i\}(q(0))$$

where

$$\sigma_i = s_{i-1} - s_i$$

so that

$$\exp(\sigma_i H_{o,v,\kappa})$$

is a contraction from

$$L_j(\mathbb{E},\rho(q(s_i))dq(s_i))$$

to

$$L_j(\mathbb{E},\rho(q(s_{i-1}))dq(s_{i-1}))$$

Thus this devolves to an integral over $q(0)$ and we get by an elementary Hölder inequality

$$\left|\int \Pi V_i(q(s_i))dT\right| \leq \Pi_{i=1}^{j}||V_i||_j \quad .$$

The contractive nature of $\exp(-tH_{o,v,\kappa})$ is a consequence of the following theorem which we quote

<u>THEOREM</u>

If $1 \leq r < \infty$ $\quad \exp(-tH_{o,v,\kappa})$ is a contraction operator on L_r of one mode. For

$T \leq t$, $1 < p$ and $r < \infty$ it is a contraction for

some T independent of v, κ. If p is bounded away

from 1 and r from ∞ then T is independent of p

and r.

<u>PROOF</u>: (<u>SKETCH</u>)[50]

The proof proceeds by showing

$$P_t = \exp(-tH_{o,v,\kappa})$$

a contraction on L_∞ by direct estimate and on L_2 since H_o is positive. One uses the Riesz Thorin Convexity theorem to show it a contraction on L_r for

$$2 \leq r \leq \infty.$$

One notes then that the kernel

$$p^t(q,q')$$

is symmetric, so P_t is symmetric in L_2, thus by duality one has contraction on L_r for

$$1 < r \leq 2$$

and so on L_1 in the limit. One builds the contractiveness from L_p to L_r from that of the map $L_2 \to L_4$ (verified by estimation) and on L_∞ by another application of Riesz-Thorin convexity.[51]

We have then a promising integral on $\underline{\underline{C}}$ and by following Feynman, a use for it. We have in fact the Feynman-Kac formula[52] for the full Hamiltonian semigroup

$$< \Phi|\ \exp(-tH_{v,\kappa})\Psi>$$
$$= \int \overline{\Phi(q(0))}\ \exp(\ -\int_o^t H_{I,v,\kappa}(q(s))ds)$$
$$\times \quad \Psi(q(s))\ dT$$

where Φ, and Ψ are in $\underline{\underline{H}}_{v,\kappa}$ or equivalently in $\underline{\underline{F}}_{v,\kappa}$; this is straightforwardly a statement of a transition probability in terms of the integral over all admissible developments connecting the states with weight according to their probability of evolution. We shall wish to estimate this or its replacement

$$< \Phi|\ \exp(-tF_{v,\kappa}^{(\tau)})\Psi >$$

where
$$F_{v,\kappa}^{(\tau)} = F_{\tau,v,\kappa} + H_{I,v,\kappa} \quad .$$

Now

$$|\ <\Phi|\exp(-tF_{v,\kappa}^{(\tau)}\ )\Psi >\ |$$
$$\leq ||\Phi(q(0))^{-}\Psi(q(t))||_{p'}||\exp(\int_o^t H_{I,v,\kappa}(q(s))ds||_p$$

where

$(1/p') + (1/p) = 1$, for $p > 2$ and large t. The formulation with F_τ changes the kernel to

$$p_\tau^t(q,q')$$

and so modifies the measure to a dT_τ ; otherwise all is the same as for H_o. We require then estimates on these parts and use those of Glimm and Rosen. We look first at the integral. The estimates for

$$I_\kappa = \int_o^t V_{v,\kappa}q(s)dr$$

are such that both the limits $v,\kappa \to \infty$ are uniform and provide both a uniform lower bound for $H(g)$ and resolvent convergence for $H_{v,\kappa}(g)$ to $H(g)$.[53] There are required for the existence of a cutoff vacuum Ω_g the ground state of $H(g)$.[54] Our interaction is as previously reckoned a polynomial V. By examination of the Riemann sums approximating the integral it may be shown that

i) for all $p < \infty$ $\int_0^t V(q(s)ds$ is in $L_p(\underline{\underline{C}}, dT)$

and

ii) for j an even positive integer

$$\|\int_0^t V(q(s))ds\|_j \leq t\|V\|_j$$

We have for the other part in the bound in the inner product the following property:-

There is a T independent of v and κ such that for $t \geq T$ and $1 \leq r < 2$, and $\xi$and η in $L_2(\rho dq)$,

(i') $\xi(q(0))\eta(q(t))$ in $L_r(\underline{\underline{C}}, dT)$

and

(ii') $\|\ \xi(q(0))\eta(q(t))\|_r \leq \|\xi\|_2\|\eta\|_2$

(iii') for r bounded away from 2, T is independent of r.

Thus we have a right hand bound

$$|\langle\Phi|\exp(-tF_{v,\kappa}^{(\tau)})\Psi\rangle|$$
$$\leq \|\Phi\|_2\|\Psi\|_2 \quad \exp(-\int_0^t V_{v,\kappa}(q(s))ds)\|_p$$

So we may infer that

$$\|\exp(-tV_{v,\kappa})\| \leq \|\exp(-\int_0^t V_{v,\kappa}(q(s)).ds\|_p$$

and then

$$-t^{-1}\ln\|\exp(-I_\kappa)\|_p \leq F_{v,\kappa}^{(\tau)} \quad .$$

It should be noted that the only change in using $F^{(\tau)}_{V,\kappa}$ for $\tau < 1$ i.e. not $H_{V,\kappa}$ is in the measure dT_τ and this still gives (i')-(iii'). Using again this further estimate from Glimm, with $C > 0$

$$- C(\ln|\kappa|)^p \leq ||V_{V,\kappa}||$$

so

$$- C\ t(\ln \kappa)^p \leq I_\lambda$$

we estimate I_κ in terms of I_λ. The above estimate derives from the fact that V is a polynomial, so $V_{V,\kappa}$ is a cut off convolution and one has a form of cut off Hausdorff Young theorem and Young's inequality.[49,50] We shall estimate in terms of our probability measure pr defined by dT

$$\begin{aligned} pr\{I_r &\leq -Ct\ln(|\lambda|)^p - 1\} \\ &\leq pr\{|I_\kappa - I_\lambda| \geq 1\} \\ &\leq \int |I_\kappa - I_\lambda|^{2j} dQ \\ &\leq t^{2j} ||V_{V,\kappa} - V_{V,\lambda}||^{2j}_{2j} \end{aligned}$$

by the previous property (ii). Fortunately estimation of the last norm reveals

$$||V_{V,\kappa} - V_{V,\lambda}||^{2j}_{2j} \leq (2pj)!(C_1/\lambda)^j \ (\ln(\lambda))^{2p-1}$$

where j is an integer, and C_1 is independent of both κ and λ. This can be obtained by explicit calculation in Fock space. Application of Stirling's formula for large j gives

$$\begin{aligned} pr\{I_\kappa &\leq - Ct(\ln|\lambda|)^p - 1\} \\ &\leq (2pj)^{2pj} e^{-2p(j+1)} C_3 \lambda^{-j} (\ln|\lambda|)^{2p-1} \end{aligned}$$

where the C_i are always to be taken as constants evolving with the estimates. If one chooses j so that

$$j \leq (2p)^{-1} C_3^{-1/2p} \lambda^{1/2p} \leq j + 1$$

then

$$e^{-2p(j+1)} \leq \exp(- C_3^{-1/2p} \lambda^{\frac{1}{2}p} (\ell n|\lambda|)^{-\frac{1}{2}p} (\ell n\lambda)^{-\frac{1}{2}p}$$

is a bound for the probability. Taking

$$X \propto \ell n|\lambda| + \ell n\ell n|\lambda|$$

this rends

$$\mathrm{pr}\ \{I_\kappa \leq -X-1\} \leq \exp(-C_4 e^{C_5 X^{1/p}}) .$$

Finally then we have

$$\int |e^{-I_\kappa}|^p\, dT = \int e^{-pI_\kappa}\, dT$$
$$\leq e^{-2p} + \textstyle\sum_{n\geq 1} \exp(p(n+2) - C_4 e^{C_5 n^{1/p}})$$

which is a bound that depends on neither v or κ which on being applied to

$$H_{v,\kappa} \geq -t^{-1} \ell n(||\exp(-I_\kappa)||_p)$$

yields a minorant for $H_{v,\kappa}$ independent of both v and κ. This is in fact sufficient to show H(g) bounded below.

Another reckoning, due to Rosen, this time for the differences as the volume varies is

$$||V_{v,\kappa} - V_{w,\kappa}||_{2j}^{2j} \leq (2pj)!(C_6/\lambda)^{2j}$$

of the same form as above, yields better convergence of the approximations, and in fact on

$$\underline{\mathrm{Dom}}\ (N^p) \cap \underline{\mathrm{Dom}}\ H_0$$

$$H_{v,\kappa} \to H \quad \text{strongly as} \quad v,\kappa \to \infty.$$

One also obtains resolvent convergence, the result crucial to proving the existence of a vacuum for the renormalized theory and continuing as far as has been done for $(\phi^4)_2$.

Since H is lower semibounded we take $+E_g$ to be its greatest lower bound, which will be the infimum of the spectrum of the self adjoint H(g). We shall often use the simply renormalized form of Hamiltonian

$$\hat{H}(g) = H(g) - E_g$$

which is then non-negative. It is $\hat{H}(g)$ that has a vacuum Ω_g if E_g is a simple eigenvalue.

§5 Self Adjointness of $H_I(g)$

That $H_I(g)$ in our case is self adjoint is a consequence of Segal's elaborate theory of quantum fields and as is remarked by Rosen, is implicit in the proof of selfadjointness for $:\phi^4(g):$ by Glimm and Jaffe, for there is nothing in the proof apparently peculiar to the degree 4. However, we shall go through some of the details of their approach in the following, for there is an annoying habit in the literature of moving to the closure of an essentially self adjoint operator without saying so.[5]

We shall show that

THEOREM

If g is a real function in $\underline{S}(\mathbb{R})$ then if $P(\phi)$ is a polynomial in

$$H_I(g) = \int P(\phi)\ g(x)\ dx$$

is essentially self adjoint on $\underline{D}_0$.

COROLLARY

$H_I(g)$ on its natural domain is self adjoint.

PROOF

We shall use the fact that for real f, $\phi(f)$ is essentially self adjoint; again $\phi(f)$ on its natural domain will be self adjoint. We take then the maximal abelian W*-algebra generated by the fields and show $H_I(g)$ commutes with it on a large domain and so is essentially self adjoint. Proceeding to details; we let $\underline{\underline{D}}_1$ be the domain generated by the applications of polynomials in the time zero fields to the Fock vacuum Ω_0 , i.e.

$$\underline{\underline{D}}_1 = \mathbb{C}\left[\phi(f) \ : \ f \text{ in } \underline{\underline{S}}(\mathbb{R})\right]\Omega_0$$

$\underline{\underline{D}}_1$ is clearly dense in $\underline{\underline{F}}$; further every Ω in $\underline{\underline{D}}_1$ is an analytic vector for, if $\phi(f)$ has f in $\underline{\underline{S}}(\mathbb{R})$ then

$$\sum_{m=0}^{\infty} (m!)^{-1}||\phi(f)^m\Omega||z^m$$

is an entire function for $\phi(f) < N$. Thus $\phi(f)$ has a dense set of analytic vectors and so by Nelson's theorem[55] is essentially self adjoint; it is symmetric on its natural (maximal) domain if f is real so $\phi(f)$ is a symmetric extension of $\phi(f) \restriction \underline{\underline{D}}_1$ and thus is self adjoint.

Next let

$$\underline{\underline{M}} = W^*\text{-alg}\ \{\phi(f) \ : \ f \text{ in } \underline{\underline{S}}(\mathbb{R})\} \quad ;$$

then $\underline{\underline{M}}$ is maximal abelian i.e. $\underline{\underline{M}} = \underline{\underline{M}}'$ (its commutant). Consider

$$H_I(g) \restriction \underline{\underline{D}}_0 \ ;$$ this restriction of $H_I(g)$ commutes on $\underline{\underline{D}}_0$ with $\underline{\underline{M}}$, for $H_I(g)$ is a bounded function of $\phi(f)$. Thus as an operator with dense domain commuting with a maximal abelian algebra it is essentially self adjoint. This follows from the strong form of the spectral theorem which says any maximal abelian W*-algebra

may be represented as the multiplication algebra of L_∞ functions on L_2 on some measure space M.

It now follows that $H_I(g)$ if formally self adjoint, as it is qua polynomial in ϕ smeared with a real function (this is obvious in momentum space), as a symmetric extension of its essentially self adjoint restriction to $\underline{\underline{D}}_0$ is self adjoint on its natural domain.

Having established the selfadjointness of $H_I(g)$, we are led to wonder about its localization when the function g, that is the smearing, is compactly supported as in the form factor cutoff. We find that the induced operator semigroup

$$\exp\ (iH_I(g)t)$$

is within the local algebra of the support of g. Recapitulating the definition of local algebra of an open region O of space R

$$\underline{\underline{A}}(O) = W^*\text{-alg}\ (\phi(f),\pi(f)\ :\ \text{the support of } f \text{ is within } O);$$

again this means the weak * closed self adjoint algebra of operators generated by the spectral projections of the fields and canonical conjugates based in the region O. We show:-

THEOREM

If g(x) is a real function of $C_c^\infty(O)$, then for our interaction and in fact for any polynomial interaction

$$U(g,t) = \exp(i\ H_I(g)\ t) \text{ is in } \underline{\underline{A}}(O) \cap \underline{\underline{M}}$$

PROOF:-

We shall make us of the result of Araki[56] that the commutant of a local algebra is the algebra of the complement of the closure of the region or

$\underline{\underline{A}}(O)' = \underline{\underline{A}}(O')$ where $O' = R - \bar{O}$.

First remark that $H_I(g)$ as shown in the proof previous commutes with the maximal abelian $\underline{\underline{M}}$ and thus is in $\underline{\underline{M}}$. Now, if f is restricted to have support in O', then U(g,t) commutes with any bounded function of $\phi(f)$ and $\pi(f)$, or U(g,t) is a unitary operator commuting with $\underline{\underline{A}}(O')$ or in $\underline{\underline{A}}(O')'$. Thus U(g,t) is in both $\underline{\underline{A}}(O)$ and $\underline{\underline{M}}$.

Thus we have that our interaction is local in the sense of physics; we shall find that together with the free H_O it provides a correct local dynamics but first we shall have to show H(g) self adjoint and thus a suitable generator for the one parameter group of time translations.

§6 SELF ADJOINTNESS OF H(g)

We prove self adjointness for the $(\phi^{2P})_2$ total Hamiltonian

$$H(g) = H_0 + H_I(g)$$

on its domain $\underline{\text{Dom}}\ H_0 \cap \underline{\text{Dom}}\ H_I(g)$, a furtherance of the result of Rosen that it is essentially self adjoint on this domain; it amounts to showing H(g) closed. Since directly showing an operator closed is notoriously difficult, we have been forced to detour via the strongest known result in the singular perturbation theory of positive self-adjoint. We quote from the Glimm-Jaffe paper on this theory.[57]

Suppose on a Hilbert space $\underline{\underline{H}}$ we have a self adjoint operator $N \geq I$; we define then the scale of Hilbert spaces $\underline{\underline{H}}$ with scalar products

$$\langle\xi|\eta\rangle_\lambda = \langle N^{\lambda/2}\xi|\ N^{\lambda/2}\eta\rangle \quad .$$

We have the standard identifications for non-negative N

$$\underline{\underline{H}}_\lambda \subset \underline{\underline{H}}_0 \subset \underline{\underline{H}}_{-\lambda}$$

where $\underline{\underline{H}}_0 = \underline{\underline{H}}$ and $\underline{\underline{H}}_{-\lambda}$ may be taken as the dual of $\underline{\underline{H}}_\lambda$ which is a set $\underline{\text{Dom}}\ N^{\lambda/2}$. If T is a densely defined bounded operator from $\underline{\underline{H}}_\alpha$ to $\underline{\underline{H}}_\beta$, let $||T||_{\alpha,\beta}$ denote its norm. We set

$$||T|| = ||T||_{0,0}$$

and compute in general

$$||T||_{\alpha,\beta} = ||N^{\beta/2}T\ N^{-\alpha/2}||$$

Suppose now that we have a further operator

$A \geq N$ that commutes with N. Let

$$\underline{\underline{D}} = \bigcap \{\underline{\text{Dom}}\ A^n : n \text{ in } \mathbb{N}\}$$

which is the set of C^∞ (smooth) vectors for A sometimes denoted $C^\infty A$. We assume $\underline{\underline{D}}$ is a core for a second self adjoint operator B. We assume B to be a bounded operator from $\underline{\underline{H}}_\nu$ to $\underline{\underline{H}}_{-\nu}$ and from $\underline{\underline{H}}_\alpha$ to $\underline{\underline{H}}_\beta$ for some α, β, and ν with $\beta > 0$. We assume the following inequalities on bilinear forms on $\underline{\underline{D}} \times \underline{\underline{D}}$

a) $0 \leq aN + B + \text{cst. with } 0 \leq a < \frac{1}{2}$

b) $0 \leq \varepsilon A^2 + \text{cst. } B + (\text{Ad}A^{\frac{1}{2}})^2 B + \text{cst.}$

with $2a + \varepsilon < 1$. If $\nu \geq 2$ we assume additionally that for some $\mu > \nu - 2$

c) $0 \leq \varepsilon N^{\mu+2} + (\text{Ad } N^{(\mu+1)/2})^2 B + \text{cst}$

then we may assert:-

<u>THEOREM</u>

Under the above two paragraphs of hypotheses

A + B is self adjoint.

We wish to apply this to the case of N being the number operator, $A = H_0$ and $B = H_I(g)$ on Fock space $\underline{\underline{F}}$ with $\underline{\underline{D}} = \underline{\underline{D}}_0$, for indeed $\underline{\underline{D}}_0 = C^\infty H_0$.

We list the properties that must be verified:-

(i) $\underline{\underline{D}}_0$ is a core for $H_I(g)$

(ii) there is a $\beta > 0$ and an α such that

$$||N^{\beta/2} H_I(g) N^{-\alpha/2}|| < \infty$$

(iii) there is a ν such that

$$||N^{-\nu/2} H_I(g) N^{-\nu/2}|| < \infty$$

(iv) there is an a, $0 \leq a < \frac{1}{2}$ such that

$$0 \leq aN + H_I + \text{cst.}$$

(v) there is an ε , with $2a + \varepsilon < 1$ such that
$$0 \leq \varepsilon H_0^2 + \text{cst.}H_I + (\mathrm{Ad}H_0^{\frac{1}{2}})^2 H_I + \text{cst.}$$

and maybe if we have to take $\nu \geq 2$

(vi) there is a $\mu > \nu - 2$ such that
$$0 \leq \varepsilon N^{\mu+2} + (\mathrm{Ad}N^{(\mu+1)/2})^2 H_I + \text{cst.} \quad .$$

Condition (i) was shown in proving $H_I(g)$ self adjoint. Condition (iv) with restriction $a > 0$ came as a by product of the proof that $H(g)$ was semibounded by using F_τ with $0 \leq \tau \leq 1$.

We must next seek α, β and ν for (ii) and (iii), so we remind ourselves of the form of $H_I(g)$ for ξ and η in suitable domains

$$\langle\xi|H_I(g)\eta\rangle = \sum_{j=0}^{2p} \binom{2p}{j} \sum_{n+j=0}^{\infty} \{n+2p-j, n+j!n,n\}^{\frac{1}{2}}$$
$$\times \int \overline{\xi}_{n+2p-j}(K,K')\tilde{g}(|K'| - |K''|)\nu(K'K'')^{\frac{1}{2}}$$
$$\times \quad \eta_{n+j}(K'',K)\,dK\,dK'\,dK'' \quad .$$

Looking at the following inequality for Wick monomials W of degree m in G - J I (2.11)[58]:-

For every j such that $|j| \leq m$
$$||(N + I)^{-j/2}W(N + I)^{-(m-j)/2}|| \leq \text{cst.}||W||_{L^2}$$

Each monomial part of our interaction is of degree 2p so that if we take $\nu = +p$ we have
$$||(N + I)^{-p/2}W(N + I)^{-p/2}|| \leq \text{cst.}||W||_{L^2} < \infty$$

for
$$\tilde{g}(|K| - |K''|)\nu(K',K'')^{-\frac{1}{2}}$$ is in L^2. Further if we take $\beta = 1 = -j$ in the above we get
$$||(N + I)^{\frac{1}{2}}W(N + I)^{-p-\frac{1}{2}}|| \leq \infty$$

so with $\beta = 1 > 0$, $\alpha = (2p - 1)$ and $\nu = p$ one has two estimates equivalent to (ii) and (iii), for the addition of I to N only serves to make the operators $(N + I)$ invertible and the estimate easier to prove. These are true for each monomial component of $H_I(g)$ so (ii) and (iii) may be seen to hold for the whole.

We must now tackle the commutator estimates and start with (v) where the presence of the term cst. B turns the trick. We examine matrix elements between elements of $\underline{\underline{D}}_0$. We do the commutator combinatorics first:-

$$(\mathrm{Ad}\ R)(ST) = (\mathrm{Ad}\ R)(S)T + S(\mathrm{Ad}R)T$$

so by induction we will have

$$(\mathrm{Ad}R)(S_1 \ldots S_{2p}) = \sum_{\ell=1}^{2p} S_1 \ldots S_{\ell-1}(\mathrm{Ad}R)(S_\ell)\, S_{\ell+1} \ldots S_{2p}$$

$$(\mathrm{Ad}R)^2(S_1 \ldots S_{2p}) = (\mathrm{Ad}R)((\mathrm{Ad}R)(S_1 \ldots S_{2p}))$$

$$= \sum_{\ell=1}^{2p} S_1 \ldots S_\ell\ (\mathrm{Ad}R)^2(S_\ell) S_{\ell+1} \ldots S_{2p}$$

$$+\ 2 \sum_{m<\ell=2}^{2p} S_1 \ldots (\mathrm{Ad}R)(S_m) \ldots (\mathrm{Ad}R)(S_\ell) \ldots S_{2p}$$

We next adopt the notations

$${}_\ell K = (k_\ell, k_{\ell+1}, \ldots, k_n)$$

and

$$K_j = (k_1, k_2, \ldots, k_j)$$

for shortened rows derived from $K = (k_1, \ldots, k_n)$.

With this set up we note basic relations of commutation:-

$$((\mathrm{Ad}H_0^{\frac12})(a(k))\theta)_n(K)$$
$$= \{\mu(k,K) - \mu(K)\}^{\frac12}(a(k)\theta)_n(K)$$
$$= \lambda_1(k,K)(a(k)\theta)_n(K) \text{ by definition of } \lambda_1.$$
$$= (a_1(k)\theta)_n(K)$$

Repeating this

$$((\mathrm{Ad}H_0^{\frac12})^2(a(k))\theta)_n(K)$$
$$= \lambda_1(k,K)((\mathrm{Ad}H_0^{\frac12})(a(k))\theta)_n(K)$$
$$= \lambda_1(k,K)^2(a(k)\theta)_n(K) = (a_2(k)\theta)_n(K) .$$

Taking adjoints (note this is perfectly good algebra and no analytic claims are made)

$$(\mathrm{Ad}H_0^{\frac12})(a^*(k)) = -a_1^*(k)$$
$$(\mathrm{Ad}H_0^{\frac12})^2(a^*k) = a_2^*(k)$$

Applying all this to an expectation with respect to ξ of $\underline{D}_0$

$$\langle\xi|(\mathrm{Ad}H_0^{\frac12})^2(H_I(g))\xi\rangle = \sum_{j=0}^{2p}\binom{2p}{j}\sum_{n+j=0}^{\infty}\{n+2p-j,n+j!n,n\}^{\frac12}$$
$$\times \int\bar{\xi}_{n+2p-j}(K,K')\tilde{g}(|K'|-|K''|)\nu(K',K'')^{-\frac12}\xi_{n+j}(K'',K)$$
$$\times \{\sum_{\ell=1}^{j}\lambda_2({}_\ell K'',K) + \sum_{\ell=j+1}^{2p}\bar{\lambda}_2(K,K'_{\ell-j})$$
$$+ 2(\sum_{m>\ell=1}^{j}\lambda_1({}_mK'',K)\lambda_1({}_\ell K'',K)$$
$$- \sum_{\ell=1}^{j}\sum_{m=j+1}^{2p}\bar{\lambda}_1(K,K'_{m-j})\lambda_1(K'',K)$$
$$+ \sum_{m>\ell=j+1}^{2p}\bar{\lambda}_1(K,K'_{m-j})\bar{\lambda}_1(K,K'_{\ell-j}))\}$$
$$\times \quad dK'dK''dK$$

where

$$\lambda_1({}_mK'',K) = \mu({}_mK'',K)^{\frac12} - \mu({}_{m-1}K'',K)^{\frac12}$$
$$\bar{\lambda}_1(K,K'_{m-j}) = \mu(K,K'_{m-j})^{\frac12}-\mu(K,K'_{m-j-1})^{\frac12}$$

$$\lambda_2({}_{\ell}K'',K) = \lambda_1({}_{\ell}K'',K)\lambda_1({}_{\ell-1}K'',K)$$
$$\overline{\lambda}_2(K,K'_{m-j}) = \overline{\lambda}_1(K,K'_{m-j-1})\overline{\lambda}(K,K'_{m-j}) \quad .$$

Now we have for the functions λ_1 and λ_2 inequalities

$$|\lambda_1(k,K)| \leq \tfrac{1}{2}\mu(k)\mu(K)^{-\frac{1}{2}} \leq \tfrac{1}{2}\mu(k)(2p)^{-\frac{1}{2}}(n)^{-\frac{1}{2}}$$

and

$$|\lambda_2(k,K)| \leq \mu(k)$$

or

$$|\lambda_2(k,K)| \leq \text{cst. } \mu^2(k)(n+1)^{-\frac{1}{2}} \quad ,$$

which follow from the elementary identity for x positive

$$(1 + x)^{\frac{1}{2}} - 1 \leq x^{\frac{1}{2}} \leq \tfrac{1}{2}x \qquad \text{for } x \leq 4$$

or

$$(1 + x)^{\frac{1}{2}} - 1 \leq \tfrac{1}{2}x \leq x^{\frac{1}{2}} \qquad \text{for } x \geq 4 \ .$$

Now we shall examine

$$\begin{aligned}
&\langle\xi|(\varepsilon H_0^2 + bH_I + c)\xi\rangle \\
&\quad = \textstyle\sum_{n=0}^{\infty}\Big(\varepsilon\int\mu(K)^2\overline{\xi}_n(K)\xi_n(K)\,dK \\
&\qquad\qquad + c\int\overline{\xi}_n(K)\xi_n(K)\,dK\Big) \\
&+b\textstyle\sum_{j=0}^{2p}\Big(\binom{2p}{j}\sum_{n+j=0}^{\infty}\Big[\{n+2p-j,n+j!n,n\}^{\frac{1}{2}} \\
&\quad\int\overline{\xi}_{n+2p-j}(K,K')\tilde{g}(|K'|-|K''|)\nu(K',K'')^{-\frac{1}{2}} \\
&\qquad\qquad \times \quad \xi_{n+j}(K'',K)\,dK'\,dK''\,dK\Big]\Big)
\end{aligned}$$

It is clear from the expression for

$$< \xi \mid (\mathrm{Ad}\, H_0^{\frac{1}{2}})^{\,2} H_I(g)\ \xi >$$

that it bears a great resemblance to H_I. We shall show that

$$X + c_2 \geq c_1 H_I$$

where

$$X = (\mathrm{Ad} H_0^{\frac{1}{2}})^2\ H_I(g) \quad ,$$

we have dropped the (g) and the constants c_1 and c_2 are suitable. To do this we analyse the part of the kernel of

$$< \xi \mid X\ \xi >$$

in parentheses; we shall add the first and second sums each to half of the fourth and remark that these groupings and the remaining term are positive for sufficiently large K. Further the functions are bounded above. We infer that up to a constant, for the low K, we have domination of a multiple of H_I. Now to these rearrangements:-

$$\sum_{\ell=1}^{j} \lambda_2\ ({}_\ell K'',K) + \sum_{\ell=j+1}^{2p} \overline{\lambda}_2 (K,K'_{\ell-j})$$
$$+ 2\big(\sum_{m>\ell=1}^{j} \lambda_1 ({}_m K'',K)\lambda_1 ({}_\ell K'',K)$$
$$- \sum_{\ell=1}^{j} \sum_{m=j+1}^{2p} \overline{\lambda}_1 (K,K'_{m-j})\lambda_1 ({}_\ell K'',K)$$
$$+ \sum_{m>\ell=j+1}^{2p} \overline{\lambda}_1 (K,K'_{m-j})\lambda_1 (K,K'_{\ell-j})\big) =$$

$$= \sum_{\ell=1}^{j} \lambda_1({}_\ell K'',K)(\lambda_1({}_{\ell-1}K'',K) - \sum_{m=j+1}^{2p} \overline{\lambda}_1(K,K'_{m-j}))$$

$$+ \sum_{\ell=j+1}^{2p} \overline{\lambda}_1(K,K'_{\ell-j})(\overline{\lambda}_1(K,K'_{\ell-j-1} - \sum_{m=1}^{j} \lambda_1({}_m K'',K))$$

$$+ 2(\sum_{m>\lambda=1}^{j} \lambda_1({}_m K'',K)\lambda_1({}_\ell K'',K)$$

$$+ \sum_{m>\ell=j}^{2p} \overline{\lambda}_1(K,K''_{m-j})\overline{\lambda}_1(K,K''_{\ell-j}))$$

The λ are certainly positive so the last term "2(...)" is. We examine the subtractions in the first sum:-

$$\sum_{m=j+1}^{2p} \overline{\lambda}_1(K,K'_{m-j}) \leq \tfrac{1}{2}\mu\ (k'_{m-j}) \ /\ \mu(K)^{\frac{1}{2}}$$

a fairly generous estimate achieved by iteration, but showing that for large K (in length or modulus) this is small. The second term may be similarly treated. Overall then for large K this expression is positive.

We write then estimates for constants c_1 and c_2, of bilinear forms

$$X + c_1 \geq c_2 H_I \quad .$$

So then we have

$$-X \leq - c_2 H_I + c_1$$

$$-X - b H_I \leq -(b + c_2) H_I + c_1 \quad .$$

Now we know that for all $\delta > 0$ there is a constant c_3 such that

$$\delta H_0 + H_I + c_3 \geq 0$$

thus

$$\begin{aligned} -X - bH_I &\leq + c_4 (\delta H_0 + c_3) \\ &\leq \varepsilon H_0^2 + c_5 \quad . \end{aligned}$$

Finally then, we achieve

$$0 \leq \varepsilon H_0^2 + X + bH_I + c_5$$

where $c_1, \ldots, c_5$ were suitable constants, and the restriction

$$\varepsilon < 1 - 2a$$

is certainly verifiable for both a and ε may be made small.

Since we are treating the general case $p \geq 2$ and we have taken in condition (iii) $\nu = p$ we are obliged to check (vi). We take $\mu = p - 1 > \nu - 2$ and must verify

$$0 \leq \varepsilon N^{p+1} + (\mathrm{Ad}N^{p/2})^2 H_I + \mathrm{cst.} \quad .$$

Considering $(\mathrm{Ad}N^{p/2})^2(V)$ where V is a Wick monomial part of the interaction of which there are 2p, we see

$$(\mathrm{Ad}N^{p/2})V = N^pV - 2N^{p/2}VN^{p/2} + VN^p \quad .$$

We wish then that

$$0 \leq N^p\ (\varepsilon N+H_I - 2N^{-p/2}H_IN^{p/2} + N^{-p}H_IN^p) + \text{cst.} \quad .$$

The estimate (iv)

$$0 \leq aN+H_I + C_1$$

held true for any $a > 0$ and suitable constant C_1, in particular

$$\varepsilon N + H_I + C_1 \geq 0.$$

We shall add three inequalities to obtain the desired result; from this last follow by pre- and post-multiplication by N^p & 1, $N^{p/2}$ & $N^{p/2}$, and 1 & N^p, respectively

$$0 \leq \varepsilon'\ N^{p+1} + N^pH_I + C_1N^p = A$$

$$0 \leq aN^{p+1} + N^{p/2}H_IN^{p/2} + C_2N^p = B$$

$$0 \leq bN^{p+1} + H_IN^p + C_3N^p = C \quad .$$

Estimating $A - 2B + C$ by using the extreme case of B

$$\varepsilon'\ N^{p+1} + N^pH_I + C_1 - 2aN^{p+1} - 2C_2N^p$$

$$+ bN^{p+1} + bH_IN^p + C_3N^p$$

$$= N^{p+1}(\varepsilon' - 2a + b) + N^pH_I + bH_IN^p + (C_1-2C_2+C_3)N^p$$

If one arranges

$$(\varepsilon' - 2a + b) = \varepsilon$$

as one may and then notes $N^p H_I$ and $H_I N^p$ are both bounded (as adjoints from

$$||N^{-j/2}\ WN^{-(2p-j)/2}|| \leq cst||w||_{L_2} \text{ (with } j = -2p) ;$$

and then arranges C_1, C_2, C_3 which as large enough to be positive, one has with the addition of a constant to dominate $H_I N^p + N^p H_I$

$$\varepsilon N^{p+1} + (adN^{p/2})^2\ H_I + cst \geq 0$$

as required.

Thus we have verified conditions (i) to (vi) of the theorem on singular perturbations and may infer its conclusion

$H_0 + H_I(g)$ is self adjoint on $\underline{\text{Dom}}\ H_0 \cap \underline{\text{Dom}}\ H_I(g)$.

We have, therefore, a good local Hamiltonian as in the ϕ^4 in two dimensions theory.

§7 THE SEQUEL OF THE PROGRAMME

With a Hamiltonian $H(g)$ self adjoint on its natural domain and uniform convergence of the resolvents of $H(g)_{V,\kappa}$ to that of $H(g)$ as the box and ultraviolet cutoffs are removed as has been shown, there are no further obstructions to following the programme of Glimm and Jaffe to its published end through their last two preprints. We shall only sketch the path here.[59]

First, from the semiboundedness of $H(g)$ one may renormalize it to

$$H(g) := H(g) - E_g$$

where E_g is the lowest bound on the spectrum of $H(g)$. The existence of a vacuum vector follows from the compactness of the resolvents of the approximations $H(g)_{V,\kappa}$, which thus each have discrete spectra.[60] Uniform convergence provides a unique vacuum Ω_g up to a phase that may be specified. Uniqueness uses the properties of positive operators, a technique that should be useful in the continuation.

Still in the Fock representation one constructs approximate Heisenberg fields

$$\phi(x,t) = e^{iH(g)t}\,\phi(x,0)\,e^{-iH(g)t}$$

and $H(g)$ by the theorem of Segal, proposed by Guenin provides a correct local dynamics within the causal shadow of the 1-support of g (closure of the set where g takes the value 1).[61] Sufficient (and necessary) for the application of this theorem is self adjointness of H_0, $H_I(g)$ and $H(g)$ which we have. For these Heisenberg fields the properties of locality follow directly from the hyperbolicity of the homogeneous form of the equation of motion for all ultraviolet divergences have been disposed of. Furthermore space time covariance may be set up; if $\alpha = (a,t)$

$$\sigma_\alpha \phi(x,s) = \phi(x + a, s + t)$$

and σ_t is as seen above, unitarily implemented in bounded regions (by taking the 1 support of g large enough). The space translation is also unitarily implemented

$$\sigma_\alpha \phi(x,t) = U(\alpha)\phi(x,0)\; U(\alpha)^{-1}$$

We may then set up a set of local algebras, if $\underline{O}$ is a bounded open region of space-time we have the C*- and W*- algebras associated with it generated by the fields smeared with functions of support within $\underline{O}$

$$\underline{\underline{A}}(\underline{O}) = C^*\mathrm{alg}\{\phi(f),\ \pi(f)\ :\ \mathrm{supp}\ f\ \mathrm{is\ in}\ \underline{O}\}$$

$$\underline{\underline{R}}(\underline{O}) = W^*\mathrm{alg}\{\phi(f),\ \pi(f)\ :\ \mathrm{supp}\ f\ \mathrm{is\ in}\ \underline{O}\}$$
$$= \underline{\underline{A}}(\underline{O})''$$

So we have an algebra with quasi-local structure

$$\underline{\underline{A}} = C^*\mathrm{alg}\ \{\underline{\underline{R}}(\underline{O})\ :\ \underline{O}\ \mathrm{is\ a\ relatively\ compact\ open\ set\ of}\ \mathbb{R}^2\}$$

for it seems that $\underline{\underline{R}}(\underline{O})$ is the most natural algebra to associate with a region;[60] we are thus back with Araki.[61] The Haag-Kastler axioms with the exception of Lorentz covariance are verified.

To obtain finally the physical Hamiltonian and vacuum is a physical Hilbert space one has recourse to this abstract approach from the C*algebra $\underline{\underline{A}}$ above. Every approximate vacuum Ω_g define a state of $\underline{\underline{A}}$ (i.e. a positive linear functional of norm 1 on $\underline{\underline{A}}$, the set of which will be called $\underline{\underline{E}}$) its vacuum expectation

$$\omega_g : A \to (\Omega_g, A\Omega_g) : \underline{\underline{A}} \to \mathbb{C} \quad .$$

The set of ω_g is contained in $\underline{\underline{E}}$ a set which is compact in the natural W* topology on functionals on $\underline{\underline{A}}$, of pointwise convergence on $\underline{\underline{A}}$.[62] Thus the set of ω_g has a convergent subsequence ω_{g_m} and its limit ω is a candidate for the vacuum state. Unfortunately it is not invariant under space translation, though obviously temporally invariant by construction. One therefore takes a sequence of states

$$\omega_n(A) = \int n^{-1}(\Omega_g, \sigma_\alpha(A)\Omega_g)\; h(\alpha/n)\, d\alpha$$

where h is a bump function so that ω_n spreads to become more and more translation invariant. A limit then of a subsequence of these ω_n will be a translationally invariant ω. With then an invariant state ω on a C*-algebra $\underline{\underline{A}}$ one may construct by the method of Gel'fand and Segal a Hilbert space $\underline{\underline{H}}$, a representation π of $\underline{\underline{A}}$ in $\underline{\underline{H}}$, a vector

Ω in $\underline{\underline{H}}$ cyclic for $\pi(\underline{\underline{A}})$ and a unitary representation of the space time group that leaves Ω invariant:-

$$\underline{\underline{H}} = \text{Hilb.Sp.}\ \{\underline{\underline{A}}/\ker\,\omega\}$$

$$\pi : \underline{\underline{A}} \to \underline{\text{BOpr}}\ \underline{\underline{H}} : A \to \pi(A)(\cdot) = (A\cdot)$$

$$||\Omega|| = 1;\quad \pi(\underline{\underline{A}})\Omega \text{ is dense in } \underline{\underline{H}}$$

$$\omega(R) = (\Omega,\pi(A)\Omega)$$

$$U : \mathbb{R}^2 \to \underline{\text{UOpr}}\ \underline{\underline{H}} : \alpha \to U(\alpha)$$

$$U(\alpha)\Omega = \Omega.$$

Two of the desirable properties of field theories blatantly missing above are covariance of the fields and any form of Lorentz representations. But the ω_g are vector states and so the ω_n are normal states on every $\underline{\underline{A}}(\underline{O})$ (equivalently density matrix states or completely additive states). Glimm and Jaffe by showing that the number and energy densities are both bounded and that the vacuum energy E_g is extrinsic in the volume (or support area of g) are finally able to conclude that the physical vacuum ω is locally Fock; this takes them about 100 pages of paper III on $\lambda\phi^4$.

There is however a simpler way of obtaining the result of ω being a locally Fock state, due essentially to Guénin.[63] The states ω_n above are normal states on every $\underline{\underline{A}}(\underline{O})$; by a theorem of Dell'Antonio[64] and Sakai[65] the limit in the W* topology of a sequence of normal states is also normal. Thus the limiting state ω is normal on every $\underline{\underline{A}}(\underline{O})$ and as such carries this von Neumann algebra into an image von Neumann algebra.[66] But Araki has shown the $\underline{\underline{A}}(\underline{O})$ to be type III factors[67] and a result of Griffin[68] then implies that the isomorphism afforded by

$$\pi \restriction \underline{\underline{A}}(\underline{O})$$

since $\underline{\underline{A}}(\underline{O})$ is simple, is unitarily implemented. Thus we have for each local algebra in the physical representation a unitary intertwining $U_{\underline{O}}$ with the local algebra of the same region in the Fock represntation:-

$$U_{\underline{O}} \text{ in } \underline{\text{UOpr}}\ (\underline{\underline{F}},\underline{\underline{H}})$$

and for every

$$A \text{ in } \underline{\underline{A}}(\underline{O}) \qquad \pi(A) = U_{\underline{O}} A U_{\underline{O}}^{-1}$$

Further there is a local vacuum for $\underline{O}$ namely $U_B{}^*\Omega$ but there is a phase ambiguity. In connection with the question of Hepp as to why every one should have to smear with translations to make the approximate vacua, the result that any normal state on a W*-algebra, whose commutant is infinite (e.g. a Type III algebra) is a vector state is of interest in suggesting it should not be necessary.[70] One may note that $\underline{\underline{H}}$ is separable as a unitary image of $\underline{\underline{F}}$.

§8 CONCLUDING REMARKS ON PROBLEMS AND OUTLOOKS

The most obvious deficiency of the above construction is that even though restricted to two dimensional space-time, wherein the Lorentz group is one dimensional and so commutative (since locally compact), Lorentz covariance has not been incorporated. Jaffe and Cannon have stated that they have nearly attained proof of the existence of a self adjoint generator for the Lorentz group and covariance.[69] They are trying to piece together locally correct boosts

$$M = \varepsilon H_0 + H_0(g_0) + H_I(g)$$

where the form factor must give the required function x over some interval for formally

$$M = \int xH(x)\,dx \quad .$$

Then they apparently use similar methods to the ϕ^4 methods. They would then have Lorentz scalar two point functions.

However even with Lorentz covariance one still has no scattering theory of a rigorous sort. To be able to establish Haag-Ruelle scattering theory one requires three properties of the spectrum of space-time translations:[71]

(i) the vacuum Ω is unique

(ii) there is a mass gap, that is 0 is an isolated point of the spectrum

(iii) there is a one particle structure, so that the renormalized mass hyperboloid should support an irreducible representation of the Poincaré group with the renormalized mass m_1.

Toward an answer to the problem (i) one may try to invoke the Alaoglu-Birkhoff theorem,[72] if the Lorentz group is represented, or if not (i.e. for the abelian unitary group of translation) the Dunford theorem,[72] that provides a unique invariant vector by convergence of a sequence of means. In examination of (ii) the only method of attack seems to be the use of compact resolvents which are positive and of the eigenvalue spreading of such operators.[73] Jaffe asserts on general compactness grounds that there should be at most a finite number of discrete eigenvalues and no continuum for $H(g)$ between 0 and the bare mass m_0, but does not seem to be able to keep the limit of such away from 0.

There is one further piece of hope for scattering theory available already. This derives from Ruelle's recent work on integral representations of C*-algebra states.[74] A theorem here applicable states that if a C*-algebra have a quasi-local structure defined by a commutable family of sub-C*-algebras then if for each region the algebra there has a separable closed bilateral ideal such that the restriction of the state ρ under consideration has norm 1 there then ρ has a decomposition into states with trivial algebra at infinity; the algebra at infinity $\underline{\underline{A}}_\infty$ is

$$\bigcap_{\underline{O}} \pi\rho(\underline{\underline{A}}(\underline{O}'))''$$

given by the Gel'fand-Segal construction. However a further theorem states that a state with trivial algebra at infinity has a cluster decomposition property in that for every positive ε and A in $\underline{\underline{A}}$ there is an $\underline{O}$ such that if A' is in $\underline{\underline{A}}(\underline{O}')$ then

$$|\rho(AA') - \rho(A)\rho(A')| \leq \varepsilon||A|| .$$

For other theories in two dimensions, less has been achieved for the presence of non-trivial ultraviolet divergences (those not removed by Wick ordering) necessitates the very complex machinery of dressing transformations and a much more drastic form of changing Hilbert space. Glimm and Jaffe have just announced for the Yukawa theory $\bar{\psi}\psi\phi$ after renormalization, a proof of self-adjointness of H(g) and of resolvent and graph convergence of cut off versions, and locality of them; but as yet they have no vacuum or renormalized H.[75] However in paper III they surmise that the energy estimates they make for ϕ^4 ensuring local Fockness, will remain good for $(\bar{\psi}\psi\phi)_2$ and $(P(\phi) + Q(\phi)\bar{\psi}\psi)$, where P and Q are polynomials in the boson fields. In the light of the use of the theorem on normality of weak * limits of sequences of normal states, this seems plausible. The extrinsicness of the local energy density is violated for $(\phi^4)_3$. Hepp has shown $(P(\phi) + Q(\phi)\overline{\psi}\psi)_2(g)$ is symmetric and densely defined, where the polynomial $P(\phi)$ is positive and dominates the polynomial $Q(\phi)$.[76]

It is hoped by the devotees that there will exist a full non-trivial theory within a few years.

PART II

THE ABSTRACT DEVELOPMENT OF I.E.SEGAL

and

THE RELATION OF PART I TO IT.

§1 QUANTUM FIELDS, FOCK SPACES, & CANONICAL COMMUTATION

Since the abstract approach is here being taken in order to make plain the mathematical completeness of the treatment of but a single example, we shall begin by definitions following those of I.E. Segal.

The usual start would be to define Fock Space (or occupation number space, or the space for the particle representation of the canonical relations) explicitly as follows.

Choosing a standard Hilbert space $L^2(\mathbb{R}^s)$ to which of course any separable Hilbert space is noncanonically isomorphic, one interprets it as the collection of wave functions of single noninteracting particles in space of dimension s (sometimes space-time of dimension s). For the representation of the CCR one constructs from the single particle space, the Hilbert space on which the CCR will be represented as the symmetric, or antisymmetric tensor algebra over $L^2(\mathbb{R}^s)$, with the existential (a neutral adjective to stand for either creation or annihilation) operators, acting as linear maps of degrees ±1, on the graded vector space.

(cf.e.g. for this terminology Chevalley, The Construction And Study of Certain Important Algebras, Publ.Math.Soc. Japan; or H.Nickerson,N.E.Steenrod,D.C.Spencer, Advanced Calculus, Van Nostrand Co.)

Explicitly for the symmetric case as illustration, Fock space $\underline{\underline{F}}$ is the Hilbert space completion of $\underline{\underline{\bar{F}}} = \oplus \underline{\underline{F}}^n = \odot L^2(\mathbb{R}^d)$ ($\odot$ is a standard notation for symmetric tensor algebra or product, cf. Sternberg, Lectures on Differential Geometry, Prentice Hall, 1964) and the homogeneous components $\underline{\underline{F}}^{(n)}$ are given by $\underline{\underline{F}}^{(0)} = \mathbb{C}$ (in general the ground field), and for $n \geq 1$

$$\begin{aligned} \underline{\underline{F}}^{(n)} &= L^2(\mathbb{R}^s) \odot \underline{\underline{F}}^{(n-1)} \\ &= \{f \odot \psi : \psi \in \underline{\underline{F}}^{(n-1)} \ \& \ f \in L^2(\mathbb{R}^s)\} \end{aligned}$$

where by definition $\odot$ is the result of a symmetrizing operation, i.e. an average over the action of the permutation group on the appropriate number of ciphers. P_m, the group of permutations on m ciphers, acts naturally on $V^{\otimes n}$, the n'th tensor power of a vector space V, by linear extension from its action on decomposable tensors, which is

$$\pi(x_1 \otimes \ldots\ldots \otimes x_m) = x_{\pi 1} \otimes \ldots\ldots \otimes x_{\pi m}$$

where $x_i \in V$, and $\pi \in P_m$ is taken to be $\left(\begin{smallmatrix} 1 & \cdots & m \\ \pi 1 & \cdots & \pi m \end{smallmatrix}\right)$. The obvious averager is then

$$x_1 \odot, \ldots, \odot x_m = (m!)^{-1} \sum_{\pi \in P_m} \pi(x_{\pi 1} \otimes, \ldots, \otimes x_{\pi m})$$

In this case then

$$\begin{aligned} L^2(\mathbb{R}^s)^{\odot n} &\cong L^2_{\underline{nsym}}((\mathbb{R}^s)^n) \\ &= \{f(k_1, \ldots, k_n) \in L^2((\mathbb{R}^s)^n) \mid \forall \pi \in P_n \\ &\qquad f(k_1, \ldots, k_n) = f(k_{\pi 1}, \ldots, k_{\pi n})\} \end{aligned}$$

Simply said for the symmetric case, Fock space $\underline{\underline{F}}$ is the Hilbert sum of the homogeneous components $\underline{\underline{F}}^{(n)}$ of the graded symmetric algebra over $\underline{\underline{F}}^{(1)} = L^2(\mathbb{R}^s)$. The existential operators are then:

$$\begin{aligned} &\text{creation} \quad a^* : L^2(\mathbb{R}^s) \to \mathrm{Opr}_{+1}\underline{\underline{F}} : f \mapsto a^*(f) \\ &\qquad\qquad a^*(f) : \underline{\underline{F}}^{(n)} \to \underline{\underline{F}}^{(n+1)} : \psi \to (n+1)^{\frac{1}{2}} f \odot \psi \\ &\text{annihilation} \quad a : L^2(\mathbb{R}^s) \to \mathrm{Opr}_{-1}\underline{\underline{F}} : f \mapsto a(f) \\ &\qquad\qquad a(f) : \underline{\underline{F}}^{(n)} \to \underline{\underline{F}}^{(n-1)} : \psi \to n^{\frac{1}{2}} f \rfloor \psi \end{aligned}$$

where Opr_r for $r \in Z$ denotes the linear operators of degree r on a graded space and $f \lrcorner \psi$ denotes the left interior product of differential geometry, or just contraction. Of course in this case as functions

$$(f \lrcorner \psi)(k_1,\ldots,k_{n-1}) = \int f(k_n)\psi(k_1,\ldots,k_n)dk_n$$

which is the extension of the natural

$$f \lrcorner (g_1 \odot,\ldots,\odot g_n) = (f \lrcorner g_n) g_1 \odot,\ldots,\odot g_{n-1}$$

where in the degree 1 component $\underline{\underline{F}}^{(1)}$,

$$f \lrcorner g = \langle \bar{f} | g \rangle = \langle \bar{g} | f \rangle$$

a symmetric product associated with the Hilbert inner product. Under these conditions $a^*(f)$ and $a(\bar{f})$ are adjoints satisfying the standard commutation relations:

$$a(f)a^*(g) - a^*(g)a(f) = \langle \bar{f} | g \rangle$$

The domain of $a^*(f)$ clearly varies with f for it is

$$\underline{\text{Dom.}}\ a^*(f) = \{\psi \text{ in } \underline{\underline{F}} : || f\odot\psi||^2 < \infty\}$$

A core for a^* i.e. a domain on which the above CCR will hold for all f is

$$\underline{\underline{D}}_\infty = \{\psi = \sum \psi_n \text{ in } \underline{\underline{F}} : \text{ wh } \psi_n \text{ in } \underline{\underline{F}}^{(n)}$$

$$\& \exists N \quad n > N \Rightarrow \psi_n = 0$$

$$\& \forall n \quad \text{supp}\psi_n \text{ cpct} \subset \mathbb{E}^{dn}\}$$

The operator $\Phi(f) = 2^{-\frac{1}{2}}[a(\bar{f}) + a^*(f)]$ is self adjoint and generates a one parameter group

$$W(f) = e^{i\Phi(f)}$$

for which the following (Weyl) relations hold by virtue of the CCR

$$W(f)W(g) = e^{i/2 \mathrm{Im}\langle f|g\rangle}\, W(f+g)$$

Then

$$W: L^2(\mathbb{R}^s) \longrightarrow \text{Un Opr } \underline{\underline{F}}$$

is a Weyl system, the so called

Fock representation of the CCR. Since $L^2(\mathbb{E}^d)$ is infinite dimensional there are many other inequivalent (i.e. there exist no unitary intertwinings) Weyl systems based on it. Lately the exponential representations have been studied by

> J. Fabrey, Exponential Representations of the Canonical Commutation Relations, MIT preprint; and K. Hepp, Renormalized Hamiltonian Dynamics and Representations of the Canonical (Anti) Commutation Relations, Colloque surs les systèmes à un nombre infini de degrés de liberté-CNRS, Paris, 7 Mai 1969.

These are those whose creation operator is given by tensoring with a function w in $\underline{\underline{F}}^{(m)}$ for m>1

$$a^{(m)*}(w) : \underline{\underline{F}}^{(n)} \rightarrow \underline{\underline{F}}^{(n+m)} : \psi \rightarrow [(n+m)!/n!]^{\frac{1}{2}} \, w \otimes \psi$$

The Weyl system associated is obtained by a 'renormalizing' redefinition of scalar product in a way which will come up later.

Thus we had $L^2(\mathbb{E}^s)$ playing the rôle of both parameter space for the degrees of freedom and as generator for the Hilbert space on which the Weyl System was represented. It is further usual to look at the $\mathbb{E}^s$ in question as position (x) space in connection with the first rôle and as momentum (k) space in connection with the second, thus the common formula

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int \tfrac{1}{2}[e^{-2\pi ixk} \, a(k) + e^{2\pi ixk} \, a^*(k)]dk$$

which 'expresses the Fourier transform'.

I.E. Segal has for some time inveighed[77] against the practice of adopting specifics before necessary in this regard, and one might, and is certainly tempted to, look at the algebraic nature of the graded algebra with 'ladder' operators (in fact the anticommutation relation may be taken as stating that the elliptic complex associated with the Fermi existential operators is acyclic, cf. Spanier).[78] We shall follow Segal in not just taking $\widehat{\odot}\underline{\underline{H}}$ for some Hilbert space ($\bar{\odot}$ to denote completion of $\odot$) but by setting out his definitions of the basic kinematics. The different forms of statistics shall not be ignored. The treatment follows Segal

(Quantization & Dispersion for NonLinear Relativistic Equations, Local Non Linear Functions of Quantum Fields, Non Linear Functions of Weak Processes I,II.) [79]

his definitions will be given, the main theorems sketched, and the familiar examples mentioned.

To start, one defines a quantum process which all quantum fields or similar constructs will have to be. The stochastic processes, intriguing though they are in relation to their possible provision of a nexus with Nelson's mechanistic 'brownian' quantum mechanics,[80] will be passed by.

One defines the strong algebraic operations[81] on the set of closed densely defined linear operators on a Hilbert space H (abbreviated ClDsOprH or sometimes Opr H). They are the closures of the usual operations and are defined when they can be

$\dot{+} : \{(A,B)$ in $\underline{\text{Opr}}\ H^2 : \exists (A+B)^-$ in DsOprH$\} \to \underline{\text{Opr}}$H

$: (A,B) \to A\dot{+}B = (A+B)^-$

$\cdot : \{(A,B)$ in $\underline{\text{Opr}}H^2 : \exists (AB)^-$ in DsOprH$\} \to \underline{\text{Opr}}$H

$: (A,B) \to A\cdot B = (AB)^-$

This being so $\forall A$ in $\underline{\text{Opr}}$H $\quad 0\cdot A = 0$

<u>Definition</u>

An operational process with probe space L is a linear map ϕ from L to $\underline{\text{Opr}}$H for some Hilbert space H. If L is a space of functions on a set M ϕ may be said to be an operational process in M. (abbrev. : ϕ in Opr Proc L, H or ϕ in Lin.L,$\underline{\text{Opr}}$H). Equivalence for operational processes is given as follows

if ϕ_i in Opr Proc L, H_i $\quad i=1,2$

the $\phi_1 \simeq \phi_2$

$\iff \exists$ U in UOpr (H_1, H_2)

$\forall$ x in L

$U\ \phi_1(x)\ U^{-1} = \phi_2(x)$

A quantum process will be a special kind of operational process, with a distinguished vector.

<u>Definition</u>

A <u>quantum</u> <u>process</u> with probe space <u>L</u>, is an operational process ϕ in $\mathbb{R}$ Lin.(L, $\underline{\text{Opr}}$H), together with a distinguished unit vector v in H. It is called cyclic if v is cyclic for the von Neumann algebra generated by the image of ϕ, (this will be denoted W* alg.(ϕ(L)) ,

the von Neumann algebra W* alg{$\underline{\underline{A}}$} generated by a family of in general unbounded operators. $\underline{\underline{A}}$ is defined as the double commutant, (or weak closure under usually set, verified conditions, satisfied if there is a cyclic vector for the family or if the family contains a constant,) of the bounded operators determined by $\underline{\underline{A}}$; i.e. the partial isometries and projections of the spectral decompositions of the self adjoint parts of the polar decompositions of elements of $\underline{\underline{A}}$.

Equivalence of quantum processes is the simple restriction of equivalence of the operational processes involved, by requiring that the distinguished vector be carried over

$$\exists\, U \text{ in } \mathrm{Untry}\,(H_1, H_2)$$
$$\forall\, x \text{ in } L \quad U\phi_1(x)\, U^{-1} = \phi_2(x)$$
$$U\, v_1 = v_2$$

The linear functional on W* alg{ϕ(L)} given by taking the expectation with respect to the vector v

$$W^*\mathrm{alg}\{\phi(L)\} \to \mathbb{C} : T \to \langle v|Tv\rangle$$

is called the vacuum state or more shortly, vacuum of ϕ.

At this stage there is a manifest lack of structure. The standard Fock space example has $L = L^2(\mathbb{R}^d)$ $\phi(f) = (a^*f + a(\bar{f}))/2$ and v the Fock vacuum whose only component is in degree zero e.g. (1,0,0.....). What is now required is the building in of the fundamental commutation relations of mechanics, and taking as a possibility both Bose & Fermi statistics one is led to the further assumptive definition:

<u>Definition</u>

An operational process is called canonical if

(i) $\phi(L) \subset$ SAOpr H

(ii) $\exists L_a, L_s \subset$ vsp L $\qquad L = L_a \oplus L_s$

& $\exists$ A in NonDegen A Sym.Bil.Form L_a

& $\exists$ S in NonDegen Sym.Bil. Form L_s

such that

$\forall\ x,y \in L_a$ $\quad e^{i\phi(x)}e^{i\phi(y)} = e^{iA(x,y)}e^{i\phi(y)}e^{i\phi(x)}$

$\forall\ x,y \in L_s$ $\quad \phi(x)\phi(y)+\phi(y)\phi(x)=2S(x,y)I$

$\&\ x\ L_a \& y \in L_s$

$\Rightarrow \phi(x) \leftrightarrow \phi(y)$

where $\leftrightarrow$ denotes strong commutation i.e. in terms of all spectral projections commuting. Verbally an operational process is canonical if it assigns to elements of L self adjoint operators, such that on one subspace L_a of a direct sum decomposition of L they obey Bose commutation relations; on the other L_s they obey Fermi commutation relations, and Bose & Fermi parts do not interfere i.e. they commute. A quantum process will be called canonical if it is such qua operational process. Clearly canonical quantum processes are what was desired. The canonical non quantum process has no vacuum vector and this has often been suggested to be the case for particular models. A very convenient property of canonical processes is that they are in a sense unique if given in that no other decomposition into symmetric and antisymmetric parts is possible, under the extra assumption that L_s is not of finite odd dimension (Scholium 2 in NFWP I § 1).

It is now cogent to set out the theorems on uniqueness of the Bose and Fermi parts of the canonical process, so we define and explore the respective Weyl and Clifford algebras and systems. The attempt will be made to treat the two simultaneously. We proceed to the topologico-algebraic definition.

<u>Definition</u>

Let L be a vector space over $\mathbb{R}$

Let F be a nondegen. bilinear form on L

Let $\underline{\underline{L}}$ be the free noncommutative associative algebra generated by L and the adjoined neutral element e.

Let $\underline{\underline{L}}$ be endowed with the topology which is the inductive limit of the topology of convergence up to finite degrees over finite dimensional subspaces of $\underline{\underline{L}}$.

> Note that this is in fact a double inductive limit, firstly in each degree over the finite dimensional subspaces of L and secondly over finite sets of degrees of the polynomials.

The F algebra over L is defined to be the quotient of $\underline{\underline{L}}$ modulo the closed ideal generated by the relation $\forall x,y \in L \quad xyF(y,x)+yxF(x,y)=F(x,y)F(y,x)e$

If F is antisymmetric this F algebra over L is called the <u>Weyl</u> algebra over (L,F).

If F is symmetric this F algebra over L is called the <u>Clifford</u> algebra over (L,F).

We then remark the following specified properties:

Scholium

If F is symmetric (resp. antisymmetric), the ideal above is algebraically generated by the relations

$$xy + yx = F(x,y)e$$

$$(\text{resp. } xy - yx = F(x,y)e\ )$$

It should be noted that the definition of Clifford algebra is the same as that given by Chevalley[82] in terms of the associated quadratic form $\frac{1}{2}F(x,x)$.

For $M \subset \mathrm{vsp}L$ we shall denote by F-Weyl Alg(M) and F-Cliff Alg (M) the Weyl and Clifford algebras generated by M (with e of course adjoined). Almost always the prefix F specifying the form will be dropped for what form it is will be obvious from the context.

It is well known that the commutation relations defining a Weyl algebra imply that any faithful representation of it as say self adjoint, operators in a Hilbert space (e going onto the unit operator) means that at least some of the operators be unbounded and so not everywhere defined. The trick of Weyl was to unitarize the commutation relations to bounded operators and so we shall define a Weyl <u>system</u> (a special form of antisymmetric canonical process with continuity).

<u>Definition</u>

Let L be a topological vector space

Let A be a nondegenerate antisymmetric bilinear form on L.

A Weyl system over (L,A) is a mapping W to unitary operators in a Hilbert space H such that

(i) $x \to W(x) : L \to U\,\mathrm{Opr}\,H$

is continuous on every finite dimensional subspace of L with respect to the weak topology on OprH.

(ii) $\forall x,y \in L$

$$W(x)W(y) = e^{i/2\, A(x,y)}\, W(x+y)$$

Two Weyl systems will be said to be unitarily equivalent if there is a unitary operator intertwining them in the sense $U \in U\mathrm{Opr}(H_1,H_2)$

such that $UW_1 = W_2U$

In the case that L is finite dimensional, the theorem of Stone and von Neumann says that any Weyl system is unitarily equivalent to the direct sum of copies of a standard Weyl system, that given most frequently in the Schrödinger representation.

In the case of infinite dimensional L the situation is completely different.[83] In fact a Weyl system need not exist for although most of the following construction can in general be carried out, the extra assumption is put in specifically to be sufficient to ensure continuity. It is probably not necessary. We have then the following existence theorem :

THEOREM

Let L be a Hilbert space with inner product $\langle x|y\rangle$. Let $A(x,y) = \mathrm{Im}\langle x|y\rangle$. Then there exists a Weyl system over L,A.

Proof (Sketch)

Every vector $x \in L$ lies in a finite dimensional subspace of L on which A is nondegenerate. On such a subspace by the Stone von Neumann theories there may be built an essentially (i.e. up to equivalence for the category) unique irreducible Weyl system. With this Weyl system associated with the subspace there is associated the C* algebra it generates. The set of finite dimensional subspaces of L is directed with respect to inclusion, i.e. given any two such subspaces there is a third which includes them ($N_1+N_2 \supset N_1, N_2$). The inclusion of subspaces induces inclusive injections of the C* algebras over them. The inclusions of C* algebras satisfy the usual cocycle forms of compatibility condition and so one may form the inductive limit of this directed system of C* algebras and maps satisfying the Weyl relations. The result is a C* algebra $\underline{\underline{A}}$ with a map W from L into it satisfying the Weyl relations.

However, it is by no means clear that

$$x \rightarrow W(x) : L \rightarrow \underline{\underline{A}}$$

is continuous with respect to the weak operator topology of one of the Hilbert spaces on which $\underline{\underline{A}}$ may be faithfully represented.

In finishing this section it might be useful to state that there are many known realizations of these canonical relations. In the case of those associated to a 'lack of interaction' in addition to the Fock or particle number representation, we have the Real wave or renormalized Schrodinger representation which was used in Part I to diagonalize the interaction, and the lesser known complex wave representation that diagonalizes the field operators. All these have their uses. For fermi systems there is also a similar existence theorem for Clifford algebras, but not much is known of special representations.

§2 A PROBLEM OF INTERACTIONS

We have been considering the canonical commutation relations that we would wish our fields to have; that is we have been looking at the quantum basis for a field theory. However as yet there has not been any mention of dynamics or a field equation. We restrict ourselves now to boson (Weyl) systems for this section although the 'abstract Wick ordering' theorem of the next will be demonstrated for both extremes of statistics at onece.

Suppose then that we have a field equation for a relativistic self-interacting boson system of the simplest type

$$(\Box + m^2)\phi = \lambda\phi^p \quad .$$

As a classical equation this nonlinear partial differential equation has only recently been satisfactorily treated,[84] showing the existence of a global weak solution of the Cauchy problem, for fairly general initial data, with p an odd integer for positivity of the energy, (and $\lambda, m > 0$). Letting p become 2p - 1 with p a positive integer, we have our problem of Part I, if only it were clear what was the meaning of ϕ^p. It should be the power of a field ϕ, which if not an operator valued distribution is trivial, and it is well known that multiplication of distributions is not admissible.

We have then to ascribe a meaning to a power of a Weyl system in such a way that the above equation is satisfied.

The usual Wick ordering provides a meaning for powers of free fields and is expressed in Fock space terms; by Haag's theorem use of this representation implies that any translation invariant interaction Hamiltonian leaves the vacuum invariant so the interaction is trivial. However, if the field equation is to be understood as holding for fields with non-trivial interaction (such as $(p + 1)^{-1}\int \phi^{p+1}(x)dx$) then we must have a meaning for such powers independent of the form of the representation of the Weyl relations. It is to produce such that the abstract Wick ordering following is developed. Given a functional on the abstract Weyl algebra (perhaps defining a representation) a method is given, (or at least its existence shown) for changing this functional to one with more desirable properties. These properties are abstractions of those of the change in vacuum expectation values (which are only the moments of a functional) associated with normal ordering such as vanishing of the expectation values of powers. The free field functional has moments only up to order two (n-point functions vanish for $n > 2$) but the general theorem permits the definition of a normal order relative to any functional (e.g. a non-free one). It may be seen then that such an abstract formulation (if not necessary for any given problem might be solvable by sophisticated approximation by perturbations of free field systems) is very satisfying to have, and required if a general theory of quantized fields is to include interactions.

The power of the field which is attained has the desirable properties

(i) the binomial expansion should be applicable to $(\phi + f)^m$ and should be the translate by f of ϕ^m

(ii) as a normalization vacuum expectation should vanish

and is constructed expressly to be so. The interaction formed from this is what is called by Segal a quantized differential form[85] (this in fact is just a transcription of Q-differential form as opposed to C-differential forms in the conventional terminology of Dirac). It is determined by and determines the time evolution given by the commutation form of the Heisenberg equation of motion,

$$-i \frac{d}{dt} \phi(x,t) = |H, \phi(x,t)| \quad ,$$

where

$$H = H_0 + H_I$$

This is in differential form, a condition on the time evolution automorphism; as a differential, a generator of an automorphism group, ad H is a derivation of the global field algebra.[86]

Segal has continued his systematics far and given much more indication of his general methods; but it is lengthy and incomplete and we shall not go further here than to present an alternative proof of the generalized Wick renormalization theorem, which is given in his article in Topology with minor error, separately for Weyl and Clifford systems. After that we shall show the relation of Part I's definition to that in Segal's theory.

§3 THE GENERAL NORMAL ORDERING THEOREM[85]

This approach to field theory involves vacuum expectation values and other linear functionals on certain F-algebras $\underline{E}$ over a vector space L. The F-algebras $\underline{E}$ are graded and have defined on them a degree in which the element 0 has conventionally the degree $-\infty$, and a general element has the degree of the homogeneous component of highest degree. There are then the properties

(i) $\deg(u + v) \leq \max(\deg u, \deg v)$

(ii) $\deg(u\, v) \leq \deg u + \deg v$

(iii) $\deg u = 0$ if and only if there is a λ in $\mathbb{C}^*$ such that $u = \lambda e$

The first problem of renormalization is in a sense that there are natural linear functionals ω defined on parts of $\underline{E}$ but not extendable to all of it. Further ω may give undesirable answers which are finite; it should be noted that the statement above about extendability is only another way of saying that the values on some elements would be infinite contrary to the definition of a functional. It is often desirable to produce a linear functional ω_0 such that

$$\omega_0(e) = 1 \quad .$$

This is the problem of Wick ordering to get rid of trivial vacuum divergences.

We shall deal in the following theorem with the possibility of taking one functional into another by a so called renormalization map, for the two simple cases of a Weyl and a Clifford (skew-symmetric and symmetric) algebra. First, in order to carry both types at once, we make the appropriate definitions.

DEFINITION:

If u in $\underline{\underline{E}}$ is homogeneous of a given degree, then the parity[82] of u, denoted $(-1)^u$, will be ± 1 according as the involution

$z \to -z$ on L induces $u \to \pm u$.

We defined then a bracket operation for elements u of a definite parity by

$$\{u,z\} = uz - (-1)^u zu$$

and extending by linearity to an operation on all of $\underline{\underline{E}}$

$$\{\cdot,z\} : \underline{\underline{E}} \to \underline{\underline{E}}$$

In general one defines

$$\{u,v\} = uv + (-1)^{(\deg u)(\deg v)} vu \quad .$$

In order to combine calculation for $\underline{\underline{W}}(L)$ and C(L) we consider the Weyl algebra as concentrated in even degrees only, i.e.

$$\underline{\underline{W}}_0(L) = \mathbb{C}, \underline{\underline{W}}_1(L) = 0, \quad \underline{\underline{W}}_2(L) = L, \ \underline{\underline{W}}_4(L) = L \odot L$$

and $\quad \underline{\underline{C}}_0(L) = \mathbb{C}, \quad \underline{\underline{C}}_1(L) = L, \quad \underline{\underline{C}}_2(L) = L \wedge L$

Then we have

PROPOSITION

$$\deg \{u,z\} < \deg u$$

PROOF

Considering, without loss of generality, a decomposable element u

$$u = z_1 \ldots z_n$$

$$\{u,z\} = z_1 \ldots z_n z \pm (-1)^u z z_1 \ldots z_n$$

A u induction process would settle the result by repeated commutation if one could shew

$$\deg\{z_1,z\} < \deg z_1$$

But

$$\{z_1,z\} = F(z_1,z)e$$

so the result is obvious.

We seek now a map carrying ω to ω_0 which is a linear map $R : \underline{\underline{E}} \to \underline{\underline{E}}$

such that

(i) $\{R\omega,z\} = R\{u,z\}$ for every u in $\underline{\underline{E}}$ and for every z in L

(ii) $\omega(Ru) = \omega_0(u)$ for every u in $\underline{\underline{E}}$

We shall now show that if L, the phase space, is infinite dimensional then such a map renormalizing a state to a nice vacuum state, that is removing trivial vacuum-vacuum divergences, is defined by only the above properties:-

(ii) that it provides a carrying over of the first state such that

(i) it commutes with the right adjoint action $\{\cdot,z\}$ of L. We have the following

THEOREM

Let L be infinite dimensional and $\underline{\underline{E}}$ either a Weyl or a Clifford algebra, $\underline{\underline{W}}(L)$ or $\underline{\underline{C}}(L)$, over L. With the above conventions on their constructions and on the definition of the bracket {,}, given a state ω, on $\underline{\underline{E}}$ such that

$$\omega(e) \neq 0$$

and a state ω_0 on $\underline{\underline{E}}$ such that

$$\omega_0(e) = 1$$

then there exists a unique map renormalizing ω to ω_0 so that

(i) for all u in $\underline{\underline{E}}$ and for every z in L

$$\{Ru,z\} = R\{u,z\}$$

& (ii) for all u in $\underline{\underline{E}}$. of even degree

$$\omega_0(u) = \omega(Ru)$$

PROOF

To prove this we shall show firstly that if such a map does exist then it is unique, and then set about constructing one. The construction, since the algebra $\underline{\underline{E}}$ is in general infinite dimensional, pieced together from finite dimensional parts as an inductive limit, will involve restriction to finite dimensional subspaces and the associated notion of tame function. But first the uniqueness result as a simple algebraic lemma.

LEMMA

If such a renormalization map as in the

theorem exists, then it is unique.

Proof

The obvious proof is by showing the difference D of two maps R and R' of the kind to vanish. So, letting $D = R-R'$ suppose u to be an element of minimal degree such that $Du \neq 0$. Then, for every 2 in L

$$\{Du,z\} = \{(R-R')u,z\}$$
$$= D\{u,z\}$$

But

$$\deg\{u,z\} < \deg u \ , \quad \text{so} \quad D\{u,z\} = 0.$$

Then

$$\{Du,z\} = 0 \quad \text{for every 2 in L}$$

But

$$\{Du,z_1z_2\} = \{Du,z_1\}z_2 + (-1)^{(Duz_1)}z_1\{Du,z_2\}$$

Where the expansion of the brackets holds since

$$\deg z_1z_2 = \deg z_1 + \deg z_2$$

by the following sublemma:-

SUBLEMMA

Let u,v,w in $\underline{\underline{E}}$ such that

$$\deg vw = \deg v + \deg w$$

then

$$\{u,vw\} = \{u,v\}\ w + (-1)^{(\deg u)(\deg v)}v\{u,w\}$$

Proof

$$\{u,vw\} = uvw - (-1)^{(\deg vw)(\deg u)}vwu$$
$$= uvw - (-1)^{(\deg u)(\deg v)}vuw$$
$$+ (-1)^{(\deg u)(\deg v)}vuw - (-1)^{(\deg v + \deg w)\deg u}vwu$$
$$= \{u,v\}w + (-1)^{(\deg u)(\deg v)}v\{u,w\}$$

So we have

$$\{Du,v\} = 0 \quad \text{for all } v \text{ in } \underline{\underline{E}} .$$

Thus $Du = \lambda e$.

So
$$\begin{aligned}\omega(Du) &= \omega(\lambda e) = \lambda\omega(e)\\ &= \omega(R-R')u\\ &= \omega Ru - \omega R'u\\ &= \omega_0 u - \omega_0 u\\ &= 0 .\end{aligned}$$

Hence we have $\lambda = 0$, but this shows as required that D vanishes always, i.e.

$$R \equiv R'.$$

Continuing the proof of the theorem, we must now construct a map R, renormalizing ω to ω_0 We introduce the notion of a tame function on L which is a function that depends only on a finite number of variables or in otherwords is equal to its restriction to a finite dimensional subspace; the smallest such subspaces is called its support and is essentially given. The definition amounts to this but is technically more directly useful. This notion of tameness is much used in the theory of integration over infinite dimensional spaces.[87]

<u>DEFINITION</u>

A function ϕ on L, which has a fundamental nondegenerate form F, will be called tame if there exists a finite dimensional subspace N of L such that

$$\text{for every } w \text{ in } N \qquad F(z,w) + F(z',w)$$
$$\Rightarrow \phi(z) = \phi(z') .$$

The N of minimal dimension such that this is so is called the support of ϕ.

LEMMA

Let K be a tame map from L to $\underline{E}$. Then there exists u in L such that

$$K(z) = \{u,z\} \qquad \text{if and only if}$$

$$\text{for every } z,z' \text{ in } L \quad \{K(z),z'\} + \{K(z'),z\} = 0 .$$

Proof

We shall first tackle the easy part:- necessity. That the identity holds is an immediate consequence of the (generalized) Jacobi identity

$$\{\{u,z\},z'\} + \{\{z,z'\},u\} + \{\{z',u\},z\} = 0$$

for $\{z,z'\}$ is a scalar and $\{u,e\} = 0$ always. That the mapping be tame follows from the expansion identity for brackets of monomials. For letting u be in $\underline{E}(G)$ where G is a finite dimensional subspace of L, to be a monomial $w_1w_2.....w_r$ with w_i in G. Applying the expansion formula we have

$$\begin{aligned}\{u,z\} &= \{w_1...w_r, z\} \\ &= \{w_1,z\}w_2...w_r + (-1)^{r-1}w_1\{w_2...w_r,z\} \\ &= \textstyle\sum_{k=1}^{r}(-1)^{(r-1)(r-2)...(r-k)}w\ ...w_{k-1}\{w_k,z\} \\ &\qquad \times \quad w_{k+1}...w_r \ .\end{aligned}$$

But then

$$\{w,z\} = \{w,z'\}$$

for all w in G implies

$$\{u,z\} = \{u,z'\}$$

However the condition

$$\{w,z\} = \{w,z'\}$$

is equivalent to

$$F(w,z) = F(w,z') \quad ,$$

Therefore

$$\{u,\cdot\} = K(\cdot)$$

is tame.

To prove sufficiency we must use some of the identities of the general commutator bracket calculus, and a property of nondegenerate bilinear forms on finite dimensional subspaces. First, however, we must be sure that any finite dimensional subspace M of L (in particular the supporting subspace for K) is extendable to a finite dimensional subspace M' on which F is nondegenerate. A demonstration given by Segal in the skew case does not hold. Therefore we include a full elementary proof.

Lemma on Bilinear Forms

Definition: A bilinear form F on a vector space V is termed left nondegenerate if

$$F(\cdot,y) : \underline{V} \to \mathbb{R} : x \to F(x,y)$$

is the zero map on V iff y = 0, i.e. $\forall x \in \underline{V}\ F(x,y) = 0 .\Rightarrow. y = 0$ and similarly is termed right nondegenerate if

$$F(y,\cdot) = 0 \Rightarrow y = 0.$$

F is called (bilaterally) nondegenerate if it is both right and left nondegenerate.

We remark that if V is finite dimensional, right and left nondegeneracy are equivalent, and further, that symmetry or skewness of F also implies this equivalence. We now prove a simple but important extension lemma for nondegenerate forms.

Lemma.

Let F be a bilinear form on a real vector space L, which is bilaterally nondegenerate, then any finite dimensional subspace N may be imbedded in a finite dimensional subspace N' on which F is nondegenerate.

Proof i) If L is finite dimensional the lemma is a triviality.

ii) Suppose then L to be of infinite dimension. We proceed by induction on the dimension of N.

a) $\mathbb{R}$ dim. N = 1 :- then $\underline{N} = \mathbb{R}n$ for some $n \in \underline{N}$

a1) Suppose $F(n,n) \neq 0$; then $F\restriction\underline{N}$ is already nondegenerate.

a2) Suppose $F(n,n) = 0$; by the following Sublemma there exists a $v \in \underline{L}$, such that both $F(v,n) \neq 0$, and $F(n,v) \neq 0$. Let $\underline{N}' = \mathbb{R}n + \mathbb{R}v$; now $F\restriction\underline{N}'$ is nondegenerate for $\lambda,\mu\in\mathbb{R}$ and $\forall x\in \underline{N}'$ $F(\lambda n+\mu v,x) = 0$

(whence) $\Longrightarrow$ $F(\lambda n+\mu v,n) = 0$

$= \lambda F(n,n) + \mu F(v,n)$

(Whence) $\Longrightarrow$ $\mu=0$

So $\forall x\in \underline{N}'$ $F(\lambda n,x)=0$ follows, which implies

$F(\lambda n,v)=0$, whence $\lambda=0$.

We insert here the required Sublemma.

<u>Sublemma</u>

Under the conditions of the lemma, it is possible for any $n\in\underline{L}$ to find a $v\in\underline{L}$ such that both

$F(n,v) \neq 0$ and $F(v,n) \neq 0$

<u>PROOF</u> We shall use 'reductio ad absurdum'; suppose then that there is a n for which it is not possible to find such a v. By the right nondegeneracy of F, there certainly exists a v such that $F(n,v)\neq 0$; we are assuming then that for every such v, $F(v,n)\neq 0$. But also by the left nondegeneracy of F, there exists a w such that $F(w,n)\neq 0$; we are again assuming that always for such w $F(n,w)=0$, otherwise we should have the required element both left and right non-F-orthogonal to n. But now consider v+w;

$F(n,v+w)$

$= F(n,v) + F(n,w)$

$\neq 0$

and $F(v+w,n)$

$=F(v,n) + F(w,n)$

$\neq 0.$

Thus (v+w) has the required property in any case, in contradiction to our hypothesis that there was none such.

▨ Sublemma

Continuing with the proof of the full Lemma, we go on to the induction step.

b) We make the hypothesis for induction, that any subspace $\underline{M}$ of $\underline{L}$ of dimension less than $\tilde{n}$ may be extended to a finite dimensional $\underline{M}'$, a subspace of $\underline{L}$ on which F is nondegenerate.

So let $\underline{N}$ be of dimension $\tilde{n}$; then $\underline{N}$ is a one dimensional extension of some $(\tilde{n}-1)$ dimensional subspace $\underline{M}$ of it, that is $\underline{N} = \mathbb{E}\mathrm{Vsp}\{\underline{M},n\}$

where $n \notin \underline{M}$ and dim. $\underline{M}=\tilde{n}-1$. By the induction hypothesis there is a finite dimensional $\underline{M}'$ containing $\underline{M}$ such that $F\restriction\underline{M}'$ is nondegenerate (moral?).

We continue by examining the two cases for location of n.

b1) $n\in\underline{M}'$:-If this be so $\underline{M}'$ is the sought extension of $\underline{N}$ for $\underline{N} = \mathrm{Vsp}\ \{\underline{M},n\}\subset\underline{M}'$

b2) $n \notin \underline{M}'$:-Then we may, without loss of generality, take $F(n,\underline{M}')=0$, by assuming the F projection of n onto $\underline{M}'$ to have been already subtracted. We subdivide the case b2) further:

b2') $F(n,n) \neq 0$:-Let $\underline{N}' = \underline{M}' \oplus \mathbb{E}n$. Note that this is certainly an F orthogonal direct sum. Then we have $F\restriction\underline{N}'$ certainly nondegenerate, by directness of the sum and nondegeneracy on the summands.

b2") $F(n,n) = 0$:-By the sublemma, there is a v in $\underline{L}$ such that both $F(n,v)\neq 0 \neq F(v,n)$. Again, without loss of generality, we may take $F(v,\underline{M}') = 0$ for if this were not so we could subtract the projection (left-F-projection) of v on $\underline{M}'$; this may not be all v for n is right-F-orthogonal to $\underline{M}'$.

F is then not degenerate on $\underline{N}' = \underline{M}' + \mathbb{R}n + \mathbb{R}v$, for if $\lambda, \mu \in \mathbb{R}$ and $m \in \underline{M}'$ and for every $x \in \underline{N}'$

$$F(m+\lambda n+\mu v, x) = 0$$

then in particular for every $m' \in \underline{M}'$

$$F(m+\lambda n+\mu v, m') = 0$$

$$= F(m,m') + \lambda F(n,m') + \mu F(v,m')$$

$$= F(m,m')$$

and since $F \restriction \underline{M}'$ is nondegenerate $m'=0$. Again we take the particular case

$$F(\lambda n+\mu v, n) = 0$$

$$= \lambda F(n,n) + \mu F(v,n)$$

$$= \mu F(v,n)$$

whence $\mu=0$; and proceeding similarly we find

$$F(\lambda n, v) = 0$$

$$= \lambda F(n,v)$$

whence $\lambda=0$.

Thus F is right nondegenerate on $\underline{N}'$ and since $\underline{N}'$ is finite dimensional by construction, F is (bilaterally) nondegenerate on $\underline{N}'$ an extension of $\underline{N}$.

Thus having achieved the induction step b) and proved an initial case a) we have shewn case ii) of the lemma and the proof is ended.

Lemma

We now proceed with the main theme of the proofs, and take K to be tame and such that for all z,z' in L.

$$\{K(z),z'\} + (-1)^{z}\{K(z'),z\} = 0$$

Then we may take the support of K without loss of generality (i.e. we may trivially extend if required by the above) to be a finite dimensional subspace of L on which F is not degenerate. Since this space is finite dimensional we may choose two bases of it (e_i) and (f_j) such that

$$F(f_i,e_j) = d_{ij}.$$

{(f_j) is the F contragredient basis to (e_i) and is given by

$$f_j = \textstyle\sum_i g_{ij} e_i$$

where g_{ij} is the transpose of the inverse of the matrix of F in the basis (e_i).}

Having chosen these bases, suppose $K(e_i) = k_i$. Then we may take

$$H_I = (-1)^{e_i}\, k_i f_i \quad ,$$

and then

$$\begin{aligned}\{n_j,e_i\} &= \{k_i f_i e_i\}\\ &= (-1)^{e_i}\,\{k_i,e_i\}f_i + (-1)^{2e} k_i\{f_i,e_j\}\\ &= k_i d_{ij}\end{aligned}$$

for from

$$\{K(z),z'\} + (-1)^{z}\{K(z'),z\} = 0$$

follows

$$\{K(e_i),e_j\} + (-1)^e\{K(e_j),e_i\} = 0$$
$$= \{k_i,e_j\} + (-1)^e\{k_j,e_i\}$$

and

$$\{k_j,e_i\} + (-1)^e \{k_j,e_i\} = 0$$

So we have

$$\{k_i,e_j\} = 0.$$

Thus

$$u = \sum u_i$$

has the property that on a basis for the support of K,

$$K(\cdot) - \{u,\cdot\} = 0.$$

Therefore

$$K(\cdot) = \{u,\cdot\}$$

by tameness.

Having set up a commutator map on finite dimensional subspaces we now continue to construct the required additive renormalization N connecting the states ω, and ω_0 where $\omega_0(e) = 1$, which has the characteristic properties

(i) $\{N(u),z\} = N\{u,z\}$

(ii) $\omega(N(u)) = \omega_0(u)$

for

$$(-1)^u = +1$$

i.e. for u of even degree.

On the scalars $\mathbb{R}e$ we let

$$N(\lambda e) = \frac{\lambda\omega_0(e)e}{\omega(e)} \quad ;$$

We recollect that $\omega(e) \neq 0$ by assumption. Then for these elements of degree zero (i) and (ii) are clearly satisfied. We proceed by iterative induction. Assuming that an N has been defined for all u of degree smaller than k, so that (i) and (ii) hold, we define a function K. For w, an element of degree k, let

$$K(x) = N\{w,z\} \quad .$$

This is well defined for

$$\deg\{w,z\} = \deg.\ w-1<k.$$

But then K is tame for it depends on the commutator with an element of bounded degree and further by (i) it satisfies

$$\{K(z),z'\} + (-1)^z\{K(z'),z\}$$
$$= \{N\{w,z\},z'\} + (-1)^z N\{\{w,z'\},z\}$$
$$= N(\{\{w,z\},z'\} + (-1)^z\{\{w,z'\},z\})$$
$$= N(0)$$
$$= 0$$

so that there is an element v such that

$$K(\cdot) = (v,\cdot) \quad .$$

We may normalize v by requiring

$$\omega(v) = \omega_0(w)$$

so that it is then uniquely given from w. We now define

$$N(w) = v \quad .$$

The required properties (i) and (ii) may easily be verified.[88]

§4 A SUCCESSION OF EXISTENCE THEOREMS

For the sake of necessary brevity, we shall adopt the course of laying out only the definitions and theorems of Segal's approach[89], commenting on their resemblances to others' practices. (This might be termed the 'Satz ohne Beweis' policy.)

We have encountered thus far in §II-1 operational processes (OProc) and canonical operational processes (COProc), and quantum processes (QProc) and their canonical counter parts (CQProc). We shall proceed to restrict all canonical processes considered to be skew systems, that is, we are considering only Weyl systems, and we shall continue however to abbreviate skew quantum process to SQProc.

The most obvious lack in the structure at present is any geometry of the underlying space. This is introduced in the form of a covariance group is a unitary representation, that provides an automorphism group of the process by conjugations. So we define a G-Covariant Skew (Quantum) Process (GSQProc) as a quadruple $(\Psi, \underline{\underline{K}}, v, \Gamma)$ where $(\Psi, \underline{\underline{K}}, v)$ is a skew quantum process over a topological vector space $\underline{L}$ which carries a non-degenerate skew bilinear form A, and V is a continuous linear representation of G on L leaving A invariant and Γ is a continuous unitary representation of G in K such that

(i) for every a in G $\Gamma(a)v = v$

(ii) for every a in G and for every x in L

$$\Gamma(a)\Psi(x)\Gamma(a)^{-1} = \Psi(V(a)x)$$

We see then that there are six constituents (above the common underlying algebra and topology) to this structure $\underline{L}$,A,G,$\underline{K}$,v and their relations (connecting morphisms)

$V : G \to \underline{\text{Aut}}\ \underline{L}$ (cont. hom.),

$\Psi : \underline{L} \to \underline{\text{SAOpr}}\ \underline{K}$ (cont.lin.)

and $\Gamma : G \to \underline{\text{UOpr}}\ \underline{K}$ (cont. hom.), satisfying specific relations. $\underline{L}$ will be the space of test functions for both the field and its conjugate. In fact the natural form that this will always occur in is with

$$\underline{L} = \underline{M} \oplus \underline{M}^*$$

where $\underline{M}$ is a locally convex topological vector space, $\underline{M}^*$ its dual, and U is a continuous representation of G on $\underline{M}$ and we form A and V by

$$A(x \oplus f, x' \oplus f') = f'x - fx'$$

$$V(a) = U(a) \oplus U(a)^{*-1} \quad ;$$

For a G-covariant skew quantum process the usual relations on $(\Psi,\underline{K},v,\Gamma)$ are satisfied. One will say $(\Psi,\underline{K},v,\Gamma)$ is built on $(\underline{M},G,U)$. Ψ splits into two parts, $\Psi \restriction M$ and $\Psi \restriction M^*$ which will be referred to as the Basic and the Conjugate process respectively. One may view $\Psi \restriction M$ as a field Φ, and the conjugate process as its conjugate Π; covariance is then built in.

Such a system as $(\Psi,\underline{K},v)$ is in part characterized by the associated Generating Functional on $\underline{K}$

$$x \to \langle \exp(i\Psi(x))v|v\rangle \qquad ;$$

indeed if the process is cyclic

$$(\underline{K} = \underline{\text{HilbSp}}\ \{(\Psi(L))''v\})$$

that is the algebra generated by the fields (and conjugates) when applied to the vacuum produces a dense set of $\underline{K}$, then a generating functional defines a unitary equivalence class of processes. The generating functional has a sequence of moments of which the second is the covariance form C of the process

$$C(x,y) = \langle \Psi(x)v|\Psi(y)\rangle$$

which is defined only on

$$\{(x,Y) \text{ in } \underline{K}^2 : v \text{ is in } \underline{\text{Dom}}\ \Psi(x) \text{ and } \underline{\text{Dom}}\ \Psi(y)\} .$$

A process is called Normal or Gaussian if there is a symmetric form Q on $\underline{L}$ such that

$$\langle \exp(i\Psi(x))v|v\rangle = \exp(-Q(x,x)/4) \quad ;$$

if a process is normal then

$$C(x,y) = Q(x,y)$$

so that Q is a positive semi-definite. In physical language a normal process is a free system and is determined by C the two point function; actually as noted above Ψ comprehends both the field and its conjugate. The uniqueness of free field systems is here expressed as "if there is a normal cyclic process over $(\underline{L},A)$ with covariance Q then it is unique." [90]

We have an important special case, the Isonormal process, which is a cyclic process over a prehilbert (unitary) space $\underline{L}$ such that

$$A(x,y) = \mathrm{Im}\langle x|y\rangle \text{ and } Q(x,y) = \mathrm{Re}\,\langle x|y\rangle \quad .$$

The property peculiar to this type of process is that there is a unique continuation of the representation Γ of G to a representation of all automorphisms of $\underline{L}$ on $\underline{K}$, which still stabilizes the vacuum and 'covaries' Ψ.

We continue now to the case of G as a measure preserving transformation group on a regular locally compact measure space $(\underline{S},m)$. If we have a positive self-adjoint operator C in $\mathbb{R}L_2(M)$ we may define

$$\underline{M} = \mathbb{R}\underline{\mathrm{HilbSp}}\,\{\underline{\mathrm{Dom}}\ C\}.$$

with inner product

$$\langle x|y\rangle_M = \langle Cx|Cy\rangle_2 \quad .$$

Then the regular representation U_0 of G on $\mathbb{R}L_2(\underline{S})$ is given by

$$U_0(g) : f(\cdot) \longrightarrow f(g^{-1}\cdot) : \mathbb{R}L_2(\underline{S}) \to \mathbb{R}L_2(\underline{S})$$

and if it intertwines C

$$\text{i.e. } U_0C = CU_0$$

there is an unique continuous representation U of G as orthogonal transformations on $\underline{M}$ such that $U = U_0$ where they are both defined. The state of affairs above just described is an abstraction of that where

$\underline{S}$ is a Euclidean space, G the group of Euclidean motions, and C is a G-invariant function such as energy in momentum space. There is a unique normal process $(\Psi,\underline{\underline{K}},v,\Gamma)$ over $\underline{M}$,G,(iC) with a given covariance operator C^2 by a result in the last paragraph. If one sets E to be the vacuum expectation functional $\langle\cdot v|v\rangle$, $\Phi = \Psi|\underline{M}$ and $\dot{\Phi} = \Psi|\underline{M}^*$ then

$$2E(\Phi(x)\Phi(y)) = \langle Cx|Cy\rangle$$

$$\& \; 2E(\dot{\Phi}(x)\dot{\Phi}(y)) = \langle C^{-1}x|C^{-1}y\rangle$$

$$\& \; 2E(\Phi(x)\dot{\Phi}(y)) = i\langle x|y\rangle$$

since the interchange $\Phi,\dot{\Phi} := \dot{\Phi},\Phi$ and $C:=C^{-1}$ induces

$$E(\Phi(x)\dot{\Phi}(y)) := -E(\Phi(x)\dot{\Phi}(y))$$

for since we have a Weyl system

$$\Phi(x)\dot{\Phi}(y) - \dot{\Phi}(y)\Phi(x) \subset i\langle x|y\rangle \quad .$$

This case may also be slightly reformulated as an isonormal process with the enhanced covariance that implies. First let

$$\Psi_1(x) = \Phi(C^{-1}x) \quad \text{and} \quad \Psi_2(x) = \dot{\Phi}(Cx) \quad ;$$

this transformation in the case that C is

$$(k^2 + m^2)^{-\frac{1}{4}} \quad \text{or} \quad (-\Delta + m^2)^{-\frac{1}{4}}$$

is the non-local one that transforms classical localization into quantum localization, or as far as position is concerned the inverse of the Foldy-Wouthuysen transformation taking Newton-Wigner localization to classical. We now make

$$\underline{H} = \underline{M} \oplus \underline{M}^*$$

into a complex space by the introduction of the map of order 4

$$j : x \oplus f \rightarrow -C^{-2}f \oplus C^2x \quad ,$$

so that the inner product is

$$\langle x\oplus f | x'\oplus f'\rangle = \langle Cx|Cx'\rangle + \langle C^{-1}x, C^{-1}x'\rangle + j(fx'-f'x) \; .$$

Thus the real and imaginary parts are of the form for an isonormal process, and the above Standard Normal Process over $\underline{M}$,G,C is the Isonormal Process over $\underline{H}$. Any free neutral scalar quantum field may be represented at a fixed time by a standard normal process built on a Euclidean space, the Euclidean group thereon with

$$C = (cI - \Delta)^{\frac{1}{4}} \quad \text{with } c > 0 \quad .$$

We continue by making a small modification of construction in the previous paragraph, to take into account equations of evolution with more general energy operators B. We start with a Hilbert space $\underline{H}$ and a positive self adjoint linear operator B on $\underline{H}$ with kernel {O}, and a domain $\underline{D}$ contained in which is invariant for B in the sense that

(i) $\underline{D} \subset \underline{\mathrm{Dom}}\; B^{\pm\frac{1}{2}}$

(ii) $B^2\underline{D} \subset \underline{D}$

(iii) $(\cos t B)\; \underline{D} \subset \underline{D}$

(iv) $B^{-1}(\sin t B)\; \underline{D} \subset \underline{D}$.

$\underline{D}$ is a domain on which the Duhamel form[9 2] of solution of the equation of motion below is defined :-

$$\partial_t^2 \phi + B^2\phi = 0 \; .$$

The normal process associated with the above equation relative to $\underline{D}$ is the isonormal process (Ψ, K, v, Γ) over $\underline{H}$ with respect to $C = B^{\frac{1}{2}}$ where $\underline{H}$ is the completion of $\underline{D} \times \underline{D}$ with inner product (for $u = (x,y)$)

$$S(u,u') = \langle B^{\frac{1}{2}} x | B^{\frac{1}{2}} x' \rangle + \langle B^{-\frac{1}{2}} y | B^{-\frac{1}{2}} y' \rangle \quad ,$$

and $\underline{H}$ is this space viewed as a complex Hilbert space with complex involution

$$J : (x,y) \to (-B^{-1}y, Bx)$$

and complex inner product

$$\langle u | u' \rangle = S(u,u') + iS(ju,u')$$

Then the time evolution is

$$U : \mathbb{R} \to \underline{\text{UOpr}}\ \underline{H} : t \to U(t)$$

where

$$U(t) : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos(t\,B)x + B^{-1}\sin(t\,B)y \\ -B\sin(t\,B)x + \cos(t\,B)y \end{pmatrix} \quad .$$

Ψ is then a Weyl system over $\underline{H}$ on $\underline{K}$; for the vacuum v in $\underline{K}$, normalized so that $||v|| = 1$ we have a cyclic vector since

$$W^*\text{-alg}\ \{\Psi(x) : x \text{ in } \underline{H}\}\ v \text{ is dense in } \underline{K} \ .$$

Being an isonormal process (Ψ, K, v, Γ) one has the extension of its Γ to a continuous homomorphism of the unitary operators of $\underline{H}$ to those of $\underline{K}$, with

$$\Gamma(U)\Psi(x)\Gamma(U)^{-1} = \Psi(Ux)$$

$$\& \ \Gamma(U)\ v = v \quad .$$

It is a remarkable fact that this whole structure is determined to within equivalence by the condition that for all self adjoint A on $\underline{H}$

$$d\Gamma(A) \geq 0$$

where $d\Gamma(A)$ is the infinitesimal generator of

$$\Gamma(\exp(i\ t\ A)) \quad ,$$

so $d\Gamma$ is the differential of the representation Γ. So this spectrum condition provides a sort of uniqueness, given a cyclic v and a one parameter group with a self adjoint generator which is non-negative. However for infinite dimensional $\underline{H}$ none of

(i) irreducibility of W*-alg$\{\Phi(x,t), \dot{\Phi}(y,t) : x \text{ in } \underline{D}\}$

(ii) Poincaré invariance

(iii) there is a stationary v, cyclic for

$$\{\exp(i\ \Psi(x,t))\ :\ x \text{ in } \underline{D}\}$$

suffice for a uniqueness result.

The connection of the above formalism with the Fock representation is provided by recovery of creators as

$$C(x) = 2^{-\frac{1}{2}}(\Psi(x) - i\ \Psi(Jx))$$

or rather the closure of it and the fact that if P_x is the projection on $\mathbb{E}_x$ then

$$d\Gamma(P_x) = C(x)C(x)^*$$

and

$$[C(x), C(y)] \in \langle x|y\rangle \quad .$$

This is reasonably reminiscent of the usual formula for the number operator given as a sum over a basis (x_i) for the test function space (or perhaps a completion of it in energy norm) $\underline{H}$:-

$$N = \sum_i a^*(x_i)a(x_i) \quad .$$

There is no claim here that such an operator exists for any basis at all or is basis independent if it does; in general it does not and we have only a sort of particular state count. Really $N = d\Gamma(I)$.

We proceed now to the current high point of Segal's programme. We take as basis for our skew quantum process, i.e. as our classical system for instance a field, a $\mathbb{C}$ Hilbert space $\underline{H}$ and a positive self adjoint operator $A \geq \varepsilon I$, with $\varepsilon > 0$, in $\underline{H}$. Then in the isonormal process $(\Psi,\underline{\underline{K}},v,\Gamma)$ over $\underline{H}$ we take

$$d\Gamma(A) = H \quad \text{and} \quad d\Gamma(I) = N \quad .$$

We have then a free system with classical driving term A and H the Hamiltonian and N the number operator. We have as before for any vector x,

$$d\Gamma(x) = C(x)C(x)^*$$

and

$$I + d\Gamma(P_x) = C^*(x)C(x) \quad .$$

It is designed to consider the fields and powers on their natural domains and so we define

$$\underline{D}_\infty(A) = \bigcap \{\underline{\text{Dom}}\ A^n : n \text{ in } N\} \subset \underline{H}$$

$$\underline{D}_\infty(H) = \bigcap \{\underline{\text{Dom}}\ H^n : n \text{ in } N\} \subset \underline{\underline{K}} \quad .$$

We also require the notion of differentiability for a map F between two topological vector spaces $\underline{V}$ and $\underline{W}$; F is said to be continuously differentiable, or in $C(\underline{V},\underline{W})$, if for every x in $\underline{V}$ there is a linear map F_x in $\underline{\text{Lin}}\ (\underline{V},\underline{W})$ such that for all y in $\underline{V}$ $\lim_{\varepsilon\to 0} \varepsilon^{-1}\{F(x+\varepsilon y) - F(x)\} = F_x y$

and

$$F,(y) : x \to F_x y$$

is continuous. Higher derivatives of C^n of F are of course given by iteration of the condition of being C^1 on the map

$$DF : x \to Fx : \underline{V} \to \underline{\mathrm{Lin}}\ (\underline{V},\underline{W})$$

and

$$D^nF : x \to D^nF(x) : \underline{V} \to \underline{\mathrm{Sym}}\ \underline{\mathrm{Lin}}\ (\underline{V}^n,\underline{W})\ .$$ [23]

The first result of interest is

$$(x,u) \to e^{i\psi(x)}u : \underline{D}_\infty(A)\times\underline{D}_\infty H \to \underline{D}_\infty(H)$$

is a C^∞ map; this is a fairly delicate result in this exact formulation. The final climax is the following theorem that applies to the situation of Glimm and Jaffe.

<u>THEOREM</u>[93]

Suppose G is a locally compact abelian group and B a self adjoint operator on $\mathbb{L}_2(G)$ which is translation invariant. Let b be the spectral function of B on G* (the Fourier transform of B). Suppose b^{-1} is in $L_p(G^*)$ for all p (including ∞) greater than some p'. Let $((\phi,\dot\phi),K,v,\Gamma)$ be the normal symmetric on $(L_2(G),G,B)$ and $H = d\Gamma(B)$.

then

for every n in $\mathbb{N}$ and for every f in $L_1(G)$, there is a continuous sesquilinear form $\phi^{(n)}(f)$ on $\underline{\underline{D}}_\infty H$, which is $\underline{D}_\infty$ H endowed with its inductive limit topology as $\mathbb{N}$-$\underline{\mathrm{ind.lim}}$. $\underline{\mathrm{Dom}}$ (H^n) where $\underline{\mathrm{Dom}}$ (H^n) has the restriction topology as a subset of $\underline{\underline{K}}$, such that

(i) $\phi^{(n)}(\cdot) : f \to \phi^{(n)}$ is linear and

$(f;x,y) \to \phi^{n}(f)(x,y) : L_1(G)\times\underline{\underline{D}}_\infty(H)\times\underline{\underline{D}}_\infty(H) \to \mathbb{C}$

is continuous.

& for all f in $L_1(G)$ and x & y in $\underline{\underline{D}}_\infty H$

(ii) $\phi^{(0)}(f)(x,y) = (\int f)\langle x|y\rangle$

and $\phi^{(n)}(f)(v,v) = 0$

(iii) for all g in $\underline{\underline{D}}_\infty(B)$

$\phi^{(n)}(f)(e^{i\phi(g)}x, e^{i\phi(g)}y) = \phi^{(n)}(f)(x,y)$

and $\phi^{(n)}(f)(e^{i\dot{\phi}(g)}x, e^{i\dot{\phi}(g)}y)$

$= \sum_{r=0}^{n} \binom{n}{r}\phi^{(n-r)}(fg^{r})(x,y)$.

Further the $\phi^{(n)}(f)$ are uniquely determined by the above conditions.

This theorem asserts the existence of well defined objects $\phi^{(n)}$ which are 'distributions' on $L^1(G)$ with values sesquilinear forms on the C^∞ vectors of the Hamiltonian that behave algebraically like powers of the free field. There is no restriction on dimension of G and it might have compact parts (e.g. be a torus) or even, be discrete. The condition is on the Fourier transform of the energy operator. Further there is a corollary asserting the existence of powers $\phi^{(n)}(a)$ at a point a of G such that

$\phi^{(n)} : a \to \phi^{(n)}(a) : G \to \underline{\text{Sq}}\ \underline{\text{Lin}}\ \underline{\text{Form}}\ \underline{\underline{D}}_\infty H$

is continuous. This sort of theorem will go over to the interacting case of a non-normal process though point powers will not exist and $\phi^{(n)}$ will not be defined on $L_1(G)$ but on $L_1(G\times\mathbb{R})$. It might be said in criticism that the existence of Wick powers for free fields has been known to physicists for some time, but the value of the theorem is that it is patently mathematically strong and may be widely applied and, further, that it should generalize to the interacting results. As previously noted the physical fields (certainly not in Fock space therefore) are supposed to satisfy say,

$$\partial_t^2 \phi + B^2\phi = \lambda\phi^{2p-1}$$

and so a meaning for powers in general is required, though this does not provide it.

There are further interesting corollaries to this theorem which indicate when the above form $\phi^{(n)}(f)$ is defined by an operator which is not of trivial domain. If F is a sesquilinear form on $\underline{\underline{D}}_\infty H$, there is associated to F an operator T_F from $\underline{\underline{D}}_\infty(H)$ to its anti-dual$*\underline{\underline{D}}_\infty(H)$, defined by

$$\langle T_F x \,|\, y\rangle = F(x,y) \;.$$

We note we have a Gel'fand trinity

$$\underline{\underline{D}}_\infty(H) \subset \underline{\underline{K}} \subset *\underline{\underline{D}}_\infty(H)$$

and define T'_F to be

$$T'_F = T_F \restriction \{x \text{ in } \underline{\underline{D}}_\infty(H) \;:\; T_F\, x \text{ is in } \underline{\underline{K}}\}$$

and call T'_F the operator associated to F in $\underline{\underline{K}}$. There are then the following results

a) if T'_F is densely defined and invariant with respect to all $e^{i\phi(g)}$ it is essentially normal

b) T'_F is densely defined if $e^{i\phi(g)}$ invariant and with v in its domain

so c) $\phi^n(f)$ exists (as an operator) if b^{-1} is in $L_p(G)$ for all $p > 1$ and for all y in $\underline{D}_\infty H$

$$|\phi^{(n)}(f)(x,y)| \leq \text{cst } ||y||$$

d) for the conditions of c) $\phi^{(n)}(f)$ for real f is essentially self adjoint on any dense $e^{i\phi(g)}$ invariant subspace

e) the conclusion of d) holds for $G = \mathbb{R}^1$ and

$$b(k) = (k^2 + m^2)^{\frac{1}{2}}$$

f) Dom $T_{\phi^{(n)}} = \{0\}$ for $G = \mathbb{R}^n$ and $n > 1$ and

$$b(k) = (|k|^2 + m^2)^{\frac{1}{2}} \quad .$$

§5 RELATION WITH PART I and CONCLUSION

What has been the relation of part II and particularly of the theorem at the end of II-4 to the work of Glimm and Jaffe? Briefly it is that the final theorem assures one of the existence of proper sesquilinear form operators such as $\phi^4(f)$ or $\phi^{2p}(f)$ on a given domain and corollary e) shows them essentially self adjoint on $\underline{\underline{D}}_\infty(H)$ for real (f). Corollary f) agrees that for theories out of two dimensional space-time there is no hope of operators representing field powers.

We shall end by showing that by this last theorem in their special case Glimm and Jaffe, after a slight natural extension of definition, agree with Segal over definition of ϕ^n.

Glimm and Jaffe are in the case $G = \mathbb{R}^1$,

$$b(k) = (k^2 + m^2)^{\frac{1}{2}},$$

and $$\underline{\underline{K}} = \underline{\underline{F}} = \widetilde{O}\underline{\underline{F}}_n \text{ where } \underline{\underline{F}}_n = SL_2(\mathbb{R}^2).$$

$\underline{H}$ should by rights be $L_2(\mathbb{R}^1)$ and then Γ which by the isonormality of the process extends to

$$U = \underline{UOpr}\ L_2(\mathbb{R}^1)$$

is the sum of tensor products of U representing itself,[94]

$$\Gamma = \oplus\{U^{\otimes n} : n \text{ in } \mathbb{N}\}$$

The vacuum $v = (1,0,0,\ldots\ldots)$ and $H = d\Gamma(B)$ is the infinitesimal generator of $\exp(i\ t\ \Gamma(-\Delta + m^2)^{\frac{1}{2}})$. But the action of this group is exactly that of the well beloved $\underline{H}_0$ for $\underline{F}$ is being interpreted as

$$\underline{\oplus}\ \{SL_n(G^{*n}) : n \text{ in } \mathbb{N}\} \quad .$$

$\underline{D}_\infty(H)$ is then the $\underline{D}_0$ of part I for this is the known domain of C^∞ vectors of H_0 and they expressly put the inductive limit topology on it. The only obvious difference is that the sesquilinear forms $\phi^n(f)$, and operators associated by corollary e), are defined by Segal for $L_1(\mathbb{R})$ and by Glimm and Jaffe only for $\underline{S}(\mathbb{R}^*)$, but they smear the fields ϕ with $\underline{S}(\mathbb{R}^2)$. However, they agree that their results hold in the case of space dimension 1 for smearing only with $\underline{S}(\mathbb{R})$ and use $\underline{S}(\mathbb{R}^2)$ on general grounds with a view to higher dimensions and the practice of Wightman. However, the fact that their Weyl system is continuous and $\underline{S}$ is dense in L^1 suffices to extend their fields. All of properties i) - iii) are then standard verifications and we may conclude that ϕ^n in two space-time dimensions has a meaning agreed upon by both parties.

As to the localization result, Segal has this too in stronger form. With $\mathbb{R}(\underline{O})$ the W*-algebra of an open set as before he can show (in Glimm Jaffe terminology) that if f is in $L_1 \cap L_2$ and supp $f = \underline{K}$ is measurable then for any neighbourhood $\underline{N}$ of O

$$\phi(f) \ @ \ \underline{R}\ (\underline{K} + \underline{N}) \quad .^{95}$$

Segal has an elaborate but general theory; Glimm and Jaffe have a few specific models derived by great ingenuity. It is interesting that there is so much similar in their work when interpretive translation has been carried out. I am of the opinion that both paths must be made to converge for there is powerful technique and mathematical perspicacity underlying Segal's work and directing physical intuition and great ingenuity behind that of Glimm and Jaffe.

EPILOGUE

Brevis esse laboro,

Obscurus fio.[96]

APPENDIX

A NAIVE POTTED HISTORY

of

SOME OF THE RIGOR

in

QUANTUM FIELD THEORY

The submitted title of this essay, "The Mathematical Foundations of Quantum Field Theory" is much too grand to be other than a subject classification and is in fact born of academic needs. The subject matter here treated is part of the recent attempts to secure, on a rigorous basis in mathematically well defined practice, the paradigms of the quantum field theory of physics, a study seemingly past its acme, quantum electrodynamics. It is the successes of quantum electrodynamics in producing, theoretical numbers in very good agreement with experimental values, on the basis of most ingenious ad hoc prescriptions of little or no definite mathematical validity, that have formed the background of lore and hope, that has sustained the use of the language of field theory in the physics of high energies and elementary particles. It should be added that the techniques of Q.F.T. (quantum field theory) mostly those referred to under the heading of Second Quantization (which it is not) have been imported, with great success, into Many-Body Theory and Statistical Mechanics.

cf. for instance L. Van Hove, N.M. Hugenholtz, L.P.Howland, Quantum Theory of Many Particle Systems, Benjamin Inc. New York, 1961 (a lecture note and reprint volume) pp.249; and much recent work especially Russian, e.g. that of Abrikosov, and Gorkov; and that of D. Ruelle, Rigorous Results in Statistical Mechanics, Benjamin, New York, 1969-; A. Wightman, Lectures on Statistical Mechanics I, mimeographed notes by G. Svetlichny, & A.Z. Capri, Princeton 1966- ;Lectures on Statistical Mechanics II, notes by J.E. Marden, Princeton 1967.

These successes, which do not seem to have successors, are remarkable for the looseness of definition of the methods used. It is commonplace to wonder at the handling with such skill of the arithmetic of infinite quantities by the early and productive workers; it is often ignored that there were many other questionable manoevers resorted to in attempting to 'renormalize the perturbation series', it is also even true that some modern approaches that eschew the pitfalls of perturbation theory also ignore the decencies of mathematics.

cf. objections to Regge polology in: Hung Cheng & Tai Tsin Wu, A symptotic Form of the S Matrix for large Angular Momentum in the left Half Plane, Phys. Rev. 144 1966, 1232-36.
Hung Cheng, Representation of the S Matrix by Regge parameters,Phys. Rev. 144, 1966, 1237-34.
Hung Cheng & Tai Tsin Wu, High Energy Collision Processes in Quantum Electrodynamics I, Harvard Physics Preprint, pp. 61.
High Energy Elastic Scattering in Q.E.D., Phys. Rev. Lett. 22, 1969 666-669.

For some time now there has been mathematical interest in securing the foundations of both quantum mechanics and quantum field theory, the latter with or without relativity. J. von Neumann's book

In English translation: Mathematical Foundations of Quantum Mechanics, Investigations in Physics, Study 2 Princeton U.P., 1955 pp.445.

is generally (unfortunately, probably erroneously) considered to have made clear the position of quantum mechanics, and later work concentrated on the quantum theory of fields, especially on the aspects associated with the physically interesting relativistic quantum field theory (R.Q.F.T.-i.e. incorporating Einsteinian special relativity). Some of the earliest considerations were those of K.O. Friedrichs,

> (collected in the book Mathematical Aspects of Quantum Theory of Fields, Interscience New York 1953; being papers published in Compr.PAM IV 1951 161-224;V 1952, 1-56, 349-41;VI 1953 1-72.

I.E. Segal and his student J.M. Cook, L. van Hove and among concerned physicists those of R. Haag, and A.S. Wightman. Segal, who has worked on these matters for many years, and remains in the forefront, started in 1947 with Postulates for general quantum mechanics (Ann. of Math. (2) 48, 1947, 930-948) and published on the subject in 1956 (2), 1957, 1958, 1959(2) and with increasing frequency later. Expositions of his general views are contained in his 1960 lectures to an American Mathematical Society Summer Seminar on Applied Mathematics (published as Mathematical Problems of Relativistic Physics, pp.112AMS, Lect in Ppp 1 Math II, Providence, 1963) and his lecture at the conference in honour of Marshall H. Star, Chicago, May 1968 (MIT math preprint pp.43). His student J.M. Cook (in The Mathematics of Second Quantization, Trans. Amer.Math.Soc. 74 1953, 222-245) put on a sound basis that standard tool, Fock space introduced in 1932 by V. Fock

(Konfigurationsraum und zweite Quanteilung, Z.Physik 75 622-647, 76 952) .

Van Hove started on the mathematical problems brought up by physics in

Representations irreductibles d'un groupe de lie infini

and continued with the seminal papers

Les difficulte de divergence pour un modele particulier de champ quantifie   Physien 18 145-159

and

Energy Corrections and Persistent Perturbation Effects in Continuous Spectra   I Physien 21 1955 901-23
II Physien 22 1956 343-54

R. Haag set out in his monograph

On Quantum Field Theories, pp.37
Mat.-Fys Medd. Danske Vid Selsk 24 1953 number 12

a careful approach to foundational questions and emphasized the phenomenon that now goes by the name of Haag's theorem, though it appears in Friedrich's book (p.139 ff). He has since become very active as a proponent of the approach to RQFT through algebras (C* or W*) of local observables; it is generally ignored that this approach had been previously suggested by Segal. Wightman, who early in his incursion into mathematical physics effectively discovered Mackey's Imprimitivity Theorem, set up an approach based on an axiom system he adopted, beginning in

Quantum Field Theory in terms of vacuum expectation values Phys Rev (2) 91 1953 1551-1660 .

This became a whole field of investigation, called Axiomatic Field Theory of which the first full development was published in

R.F.Streater & A.S.Wightman, PCT,Spin, Statistics and All That, pp.181, Benjamin, New York, 1964.

A good treatment of this is also to be found in

R.Jost, General Theory of Quantized Fields, pp 157 Lectures in Applied Math. vol. IV, AMS Providence 1965,

notes from lectures given at the same time as Segal's. Wightman's and Jost's students who have been very active in recent developments include H.Araki, D.Ruelle, K.Hepp, A.Jaffe, D.Lanford, J.Cannon, K.Osterwalder. Prominent 'outsiders' are R.F.Streater, H.Epstein,V.Glaeser, and A.Martin.

Recently, Axiomatic Field Theory has suffered a decline due to the persistent lack of any example verifying the axioms of Wightman (or even slightly weakened forms) which could be given a physical interpretation as describing any system with sensible interaction. The constructive approach--take a heuristic field theory and try to make a simile of it work--, which was long ago the view of Segal, was then given new life by Wightman and others of his school and has led to considerable hope of setting up a rigorous non-trivial field theory. Leaders in this recent onrush have been the mathematicians

James Glimm and A.Jaffe, a former student of Wightman. Recent independent work of Segal is fairly close in results, though neither in spirit nor in form and has led to continued controversy over who is adequately rigorous. There are lectures of Wightman (Cargese 1964) at the start of this resurgence that point the way plain. Recent references and reviews that should have shaped this presentation are the theses of A.Jaffe, O.E.Lanford, & J. Cannon, notes of K.Osterwalder on a summer 1968 course of A.Jaffe @ E.T.H.,Zurich, a course of A.Jaffe at Harvard 1968/69, a course of K.Hepp at L'Ecole Polytechnique 1968/69, a review of O.E.Lanford at Strasbourg 1968 and the lectures at the 1968 Varenna Summer School on local Quantum Theory by J.Glimm, A.Jaffe, K.Symanzik and M.Guenin.

The Bibliography as a whole should be considered as the reference for this appendix.

NOTES

References to the bibliography will be of the form Author (s) followed by the last two digits of the publication year and possibly a small roman letter distinguishing works of the same year (this is unambiguous for no listed item predates 1870); underlining of the author will signify a book and a final + that the work is listed in the addendum.

1. Horace, Ars Poetica 139; Nelson 65a
2. Appendix; rapporteur's talk of Hepp at Vienna 1967, Wightman 65,68
3. Jaffe 67,69a,68+, Glimm 67,67a,68,68a,68b,69,69a
   Glimm & Jaffe 68+,68a+,69a+,69b+,69c+,69,69a
   Jaffe & Powers 68, Jaffe,Lanford & Wightman 68
   Rosen 69, Simon 69
   Prosser 63 seems spurious.
4. Weinless 69
   Segal 67a,68,68a,68b,68c,69,69a
5. Wightman 65+
6. Glimm67,68; Hepp 69,69a+,69b+; <u>Friedrichs</u> 65
7. Guenin 67+,66,69
8. <u>Streater</u> & <u>Wightman</u> 64
9. Haag & Kastler 64; Haag & Swieca 65
10. Glimm & Jaffe 68a+, Jaffe 68+
11. Glimm & Jaffe 69c cf. Guenin 67+
12. Glimm & Jaffe 69b
13. Segal 67a

14. Segal all entries with 'physical' titles
15. Segal 69,69a,69b
16. Segal all entries esp. 65b,68,68c
17. Under Glimm, Jaffe,Hepp,Simon,Eckmann,Lanford, Høegh-Krohn
18. Private communications from both protagonists; Segal 69a final remarks
19. Rosen 69 had started this
20. I thank R.F. Streater and I.F. Wilde for attracting my attention to the possible use of the theorems of Sakai 57+& Dell-Antonion 67+, after the fashion of Guenin 67+, which they came across in the course of their work on perturbations by local densities, cf. Guenin 67+ & forthcoming papers of theirs.
21. Segal 68
22. It would seem from some of Segal's work, such as 56 and 56a, that the analysis in both cases should be unified, not just the algebra.
23. Palais 65, Maurin 67
24. In this regard see the approach of a number theorist in Weil 64.
25. Schwart, L. 57, Simon 69a
26. pedantic clarity is almost immediately dispensed with
27. a coining of Friedricks, I believe
28. Glimm & Jaffe 69a+
29. Friedricks 65 or Glimm 67
30. consider the production of a Klein bottle in two dimensions and other identification quotients.

31. Glimm 67,68
32. Guenin 66,67+
33. Segal 67a
34. unfortunately we do not eventually have time to
35. Glimm 68b
36. Nelson 65a,65b
37. Kac 54;Wax 64; Gross 66,67;Feldman 62
38. Glimm & Jaffe 69a
39. as in probability theory;Nelson 65a;Kahane 68+
40. Kahane 68+, Nelson 67
41. Glimm 68b
42. Bogoliubov and Shirkov 59
43. Segal 63; Rosen 69 & ref. to Bargmann therein
44. Yosida 65, Nelson 67
45. Doob
46. Simon 69a
47. Kac 59; Nelson 66
48. Bourbaki 58,60
49. Rosen 69
50. Glimm 69b, Rosen 69
51. Methods from functional analysis herein may be found in Nelson 65 or Yosida 65
52. note 47; Martin & Segal 64
53. in fact uniform:-Rosen 69 & Glimm 68b govern what follows
54. Glimm & Jaffe 69b+
55. Nelson 59, Maurin 67, Jaffe 68+
56. Araki 64a
57. Glimm & Jaffe 69a

58. Glimm & Jaffe 69a+ whose estimates will continually be used in the sequel

59. for the details are long cf. the preprints and notes.

60. Jaffe 65+,Maurin 67, Kreyn 67, Riesz-Nagy 65

61. Guenin 66,67+;Glimm & Jaffe 69b+, Jaffe 69,69a

62. Bourbaki-Alaoglu theorem e.g. Guenin 68+

63. Guenin 67+

64. Sakai 57, Guenin 68+

65. Dell'Antonio 67+

66. Dixmier 69,Guenin 68+,67+, Schwartz J.T. 67

67. Araki 64b after 64a,64,63

68. Griffin 55 from 54

69. Jaffe 69a

70. Guenin 67+

71. Jost- The General Theory of Quantized Fields A.M.S.1965

72. Riesz-Nagy 65, number 146 or ref. 74

73. Kreyn 67, chap V

74. Ruelle 69+

75. Glimm & Jaffe 69c; I am indebted to Dr. I.Halliday for calling my attention to this during the recent time when I have not been keeping up with journals.

76. Hepp 69, 69a+

77. Segal passim but especially 68a,68b,69a

78. Spanier 66+, de Rham 60

79. Segal 69, 69a

80. Nelson 67, Nelson 66; Kakutani 68, Wiener 64, Kahane 68+

81. Segal 53

82. Chevalley 55,56

83. Segal and Ruelle in Lurçat 67

84. Segal 63a

85. Segal 68

86. Segal 68c

87. Segal 63b,58,53,51a; Tarski 67

88. cf. N. Bowditch's remark on 'the C'est aisé à voir' of Laplace-R.E.Moritz, Memorabilia Mathematica,no.985

89. Segal 59,59a,61,62,64,67a,68,69,69a

90. Segal 62

91. Segal & Goodman 65

92. Segal 63,65,65a

93. Segal 69, Theorem 4.1

94. Mackey 63

95. @ signifies affiliation, that commutation with all unitaries in the commutant

96. Horace, Ars Poetica, 25

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