POLYTROPES AND OTHER SPHERES IN GENERAL RELATIVITY

A Thesis Presented by

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ABSTRACT

The general theory of relativity is used to analyse, to the first post-Newtonian approximation, the stability of various spherically symmetrical bodies against radial perturbations.

First, static composite bodies are investigated that consist of a core, composed of ideal gas and radiation, in which the ratio β of the gas-pressure to the total pressure is constant, and of an envelope of adiabatic gas. Numerical analysis indicates that the stability of such a body depends strongly on the position of the interface separating core from envelope, the body being stable for a greater range of values of σ (the ratio of the central pressure to the central rest-energy density) the closer the interface is to the centre.

The ratio of the critical radius R_c (at which instability sets in) to the Schwarzschild radius R_s, for various small values of $\beta (0 \leq \beta \leq 0.1)$ in the core, is also investigated. It is found that this ratio too depends strongly on the position of the interface, being almost independent of β for bodies in which the interface is near the centre; but the

farther the interface is from the centre the more the ratio R_c/R_s depends on β. Also, for <u>all</u> positions of the interface, the ratio R_c/R_s increases as β decreases.

In the case of radially oscillating adiabatic gas-spheres, a method different from those used by previous investigators is used to obtain a criterion for instability in the form

$$\gamma - 4/3 \leq K \quad ,$$

equality occurring for marginal stability, where γ is the ratio of the specific heats and K is a constant depending on the density distribution. Conflicting results due to previous investigators are assessed in the light of the present investigation, for the validity of which an independent check is obtained.

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CHAPTER I

A S T R O P H Y S I C A L   I N T R O D U C T I O N

The present work deals with the effects of general relativity on the structure and stability of various stellar models. Consequently, although it would be out of place to give a comprehensive account of the development of theoretical and observational astrophysics, it will be useful to give an outline of some of the advances made and also to state one or two principal results, since they will have a bearing on the following chapters.

(I) CLASSICAL THEORY OF NON-COMPOSITE STELLAR MODELS

In the pioneer researches on stellar structure a star was assumed to be a spherically symmetrical object in an equilibrium state in which the internal pressure is just sufficient to balance the gravitational forces. Such an equilibrium configuration was characterized by three parameters, namely, the mass, the radius, and the luminosity. Order of magnitude estimates were derived for the more important physical variables, i.e. the central pressure, the average temperature, etc. (1) The ultimate object was to derive the march of the many physical variables throughout the star, and to determine the processes taking place therein which would account for the observational facts, such as the Hertzsprung-Russell diagram and the mass-luminosity law.

In 1870, Homer Lane introduced into the theory of stellar structure the concept of quasi-static adiabatic changes in which the heat energy of the model remains unchanged. For these processes the equation connecting the pressure p and the density ρ is of the form

$$p = K\rho^{\gamma} , \qquad (1.1)$$

where K is a constant and γ is the ratio of the specific heats c_p/c_v . Early this century this concept was generalized by Emden and led to one in which the change in heat energy dQ is proportional to the change in absolute temperature dT , i.e. $dQ = cdT$. This is now known as a polytropic change, and the equation of state connecting the pressure p and the density ρ is given by

$$p = K\rho^{\gamma'} , \quad \gamma' = \frac{c_p - c}{c_v - c} , \qquad (1.2)$$

where c is the above constant of proportionality, K is another constant, and c_p and c_v are the specific heats of the material at constant pressure and constant volume, respectively. Equation (1.2) is usually written in the form

$$p = K\rho^{1+\frac{1}{n}}, \qquad (1.3)$$

where $n = {}^1/\gamma'-1$, is known as the polytropic index.

With the equation of state in the form (1.3) the equations governing gravitational (hydrostatic) equilibrium, in terms of the usual polytropic variables ξ and θ defined by

$$r = \alpha\xi, \quad \rho = \rho_c\theta^n, \quad \text{where} \quad \alpha^2 = \frac{n+1}{4\pi G} K\rho_c^{\frac{1}{n}-1}, \qquad (1.4)$$

and where ρ_c is the central density and r the distance from the centre of the configuration, reduce to the Lane-Emden equation[1]

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \xi^2\theta^n = 0, \qquad (1.5)$$

which is to be solved subject to the boundary conditions

$$\theta = 1, \quad \frac{d\theta}{d\xi} = 0, \quad \text{at} \quad \xi = 0. \qquad (1.6)$$

This equation can be solved analytically (in terms of known functions) only for the particular cases $n=0, 1$, and 5.

(II) COMPOSITE STELLAR MODELS

Great progress was made after nuclear physics was introduced into the theory of stellar structure in the late nineteen-thirties, when stellar energy was attributed to specific nuclear reactions occurring in the central regions (at least) of the star. Without going into details, we may state that even typical stars on the main sequence in the Hertzsprung-Russell diagram contain certain inhomogenities in the sense that the inner part, where the thermo-nuclear transmutation of hydrogen occurs, is represented by one set of equations, whereas the outer part(extending to the surface) is characterized by another set. The inner part is called the core, and the outer part the envelope. Stars in the upper-main sequence derive energy from the carbon cycle and consist of a convective core and a radiative envelope. The lighter stars, which are in the lower-main sequence and derive energy from the proton-proton chain, consist, on the other hand, of a radiative core and a convective envelope.

In a similar way, red giants are considered to be composite configurations consisting of two distinct parts - a degenerate helium core of high density and an extensive convective envelope of hydrogen of low density. Between these two parts there occurs a radiative transition zone which occupies only a minute fraction of the mass and contains a hydrogen burning shell.

White-dwarf stars are believed to consist mainly of degenerate matter of densities of the order of 10^6 gm/cm^3 and in general, as we shall see, can be regarded as composite. In the interior of these stars we may encounter either relativistic degeneracy, non-relativistic degeneracy, or a combination of both. In the first case the electron-pressure is connected to the density by a relation of the form of a polytrope of index 3, whereas in the second case this relation is of the form of a polytrope of index 1.5. Although a white-dwarf could conceivably consist of relativistic degenerate matter throughout, in general we expect the degeneracy in the outer parts, at least, to be non-relativistic. The boundary inside the star dividing the degenerate regions into relativistic and non-relativistic parts can be defined as occurring where the electron-pressure for both regions is identical and it has been found that the corresponding interfacial density is $\rho \sim 1.916 \times 10^6 \mu_E$.[2] Thus white dwarf-stars may be expected in general to consist of a core and an envelope, the core being a polytrope of index 3 and the envelope a polytrope of index 1.5.

Since an equation of state has been obtained which for small densities tends to the form applicable to non-relativistic degeneracy, and for higher densities goes over into the equation of state for relativistic degeneracy[1], it is feasible and more realistic to regard a white-dwarf star as a composite configuration rather than to assume that a single polytropic equation holds throughout. In this thesis composite models, whether applicable to white dwarfs or not, will be analysed in terms of general relativity.

(III) STABILITY AND ADIABATIC RADIAL PULSATIONS

Turning to the question of stability, it is well known that considerations involving the total energy of a star lead to the result that, if γ (the ratio of the specific heats) is less than $4/3$, then the star is unstable. Neutral equilibrium occurs for

$\gamma = 4/3$, the total energy of the star in this case being equal to that when all the mass is dispersed to infinity.[3]

For adiabatic radial pulsations (of stellar models) in which displacements from equilibrium positions are assumed to be proportional to the distances from the centre (according to Milne[4] this assumption of homologous displacements cannot be far from the truth), the non-relativistic equation of motion, to the first order in the motions, is given by[5]

$$\frac{d}{dr}(\gamma p \operatorname{div} \xi) + [\sigma^2 + \frac{4GM_r}{r^3}]\rho\xi = 0 , \qquad (1.7)$$

where ξ is given by $V = \frac{dr}{dt} = \frac{\partial}{\partial t}[\xi(r)e^{i\sigma t}]$.

If $\gamma = 4/3$, the frequency σ of these radial oscillations is zero, and hence, any homologous expansion or contraction brings the model into a new equilibrium configuration. For $\gamma < 4/3$, the frequency of homologous radial oscillations is found to be imaginary and hence the model must be unstable, expanding or contracting at an exponentially accelerated rate after any radial disturbance. The same results hold for a uniform configuration ($n=0$) as well as for a polytrope ($n \neq 0$). For $\gamma > 4/3$, the star is stable against small radial disturbances.

(IV) APPLICATION OF GENERAL RELATIVITY TO STATIC MODELS

Following this brief summary of that part of classical theory of stellar structure relevant to the work in this thesis, I shall now indicate how the above results are modified when general relativity replaces Newtonian theory. The effects of general relativity become significant for models in which the ratio of the pressure to the energy-density at the centre cannot be neglected. As we shall see, this ratio (denoted by σ) plays an important role when considering the conditions for stability or instability of a given model. Incidentally, it should be noted that, since in relativity-theory mass and energy are equivalent, the density function that appears in the gravitational field equations must include the density of the internal energy as well as the mass-density.

In 1963, in studying the stability of a succession of static configurations, Iben[6] drew attention to the importance of the binding energy (rest-mass energy minus total energy) in determining the behaviour of a given model. Since then problems of the stability of quasi-static configurations have been analysed from this point of view. Indeed, in 1964 Tooper[7], when considering static general-relativistic polytropic fluid spheres, found that, although a negative binding energy is a necessary condition for instability, it is not a sufficient condition. Assuming the usual polytropic equation of state, he derived the following general-relativistic generalization of the Lane-Emden equation for a polytope of index n (for derivation see Appendix V),

$$\xi^2 \frac{d\theta}{d\xi} \frac{1-2(n+1)\sigma v/\xi}{1+\sigma\theta} + v + \sigma\xi^3\theta^{n+1} = 0, \text{ where } \frac{dv}{d\xi} = \xi^2\theta^n . \quad (1.8)$$

On solving this equation, subject to the usual boundary conditions, by numerical methods (for $n \neq 0$ there seems to be no analytical solution in terms of known functions), Tooper found that it is possible, for given n, for there to be more than one configuration of the same mass and radius, but with widely different internal structures, however, those models with a high value of σ are unstable. He also showed that, for a given value of the rest-mass, it is possible for there to be two distinct values of the total mass, the model with the higher value of σ being unstable.

In 1965, so as to ensure that the speed of sound is always less than that of light, Tooper[8] considered adiabatic spherically symmetrical fluid spheres obeying a more truly relativistic pressure-density relation of the form

$$p = K\rho_g^{1+\frac{1}{n}} \quad , \quad \rho c^2 = \rho_g c^2 + np \ , \qquad (1.9)$$

where ρ_g is the density of the rest-mass of the matter (gas). He derived a new general-relativistic generalization of the Lane-Emden equation that differed slightly from equation (1.8), namely

$$\frac{d\theta}{d\xi} \ \frac{1-2(n+1)\sigma v/\xi}{1+(n+1)\sigma\theta} + v + \sigma\xi^3\theta^{n+1} = 0, \text{ where } \frac{dv}{d\xi} = \xi^2\theta^n(1+n\sigma\theta). \qquad (1.10)$$

The values of σ, (now defined as $p_c/\rho_{g_c}c^2$), at which instability against radial perturbations sets in were found by using a variational principle due to Chandrasekhar[9] that will be described below. For $n<3$, Tooper showed that, regarded a functions of σ, both the mass and the binding energy reach their first maxima for these values of σ. As σ increases, the binding energy eventually becomes negative. It was also found that, unlike the case of a classical model, there are unstable relativistic configurations with positive binding energy, instability occurring for smaller values of σ as n is increased. For n=3, the configurations are unstable for all values of σ, the binding energy being always negative.

A graphical method was given by Tooper for determining σ and hence the internal structure of a configuration of specified mass and radius, but it was found that for a particular value of σ no more than one stable configuration exists (and in some cases no stable configuration exists at all).

Tooper made some applications of this work to degenerate stars, in particular to limiting cases of white-dwarfs in which the electron gas is extremely relativistic (corresponding to n=3 and $\gamma={}^{4}/3$). These objects are unstable in general relativity (being marginally stable in Newtonian theory). On the other hand. white-dwarf configurations in which the pressure-density relation is non-relativistic or moderately relativistic over most of the star were shown by Chandrasekhar and Tooper[10] to be stable provided that the radius of the star is many times the Schwarzschild radius, the actual factor depending on the density distribution.

In Chapter 3 of the present thesis, one of the topics that will be dealt with concerns the stability of composite static spherically-symmetrical stellar models consisting of a core and an envelope, the core being a mixture of ideal gas and radiation, the equation of state being similar in form to the adiabatic equation (1.9) with $n=3$ but including the internal energy of the radiation as well as that of the gas. The envelope is assumed to be composed of material for which the equation of state is given by

$$p = K\rho_g^{1+\frac{1}{n_1}} \quad , \quad \rho c^2 = \rho_g c^2 + A_1 p \quad , \qquad (1.11)$$

where A_1 is a constant depending on the actual constituents of the envelope. Expressions for the physical parameters (the mass, the radius, etc.) will be given, but the principal result obtained is that the binding energy (and hence the stability) of the configurations depends not only on the parameter σ but also strongly on the position of the interface dividing the core and the envelope.

Although the numerical work (for various values of A_1 and n_1 for which $\beta \sim 0$ in the core and $\beta \sim 1$ in the envelope) is not particularly precise (especially for large σ), it serves to exhibit the main features of these composite models and appears to agree with our intuition. It is shown for these models that, given the position of the interface, instability sets in for <u>smaller</u> values of σ than those predicted by Tooper for the instability of complete (i.e. all envelope) models. It is also shown that the binding energy goes through a maximum (i.e. instability occurs) for <u>smaller</u> values of σ when the interface is <u>closer</u> to the surface.

Since early in 1963, much interest has been shown in the properties of large spherical masses (of the order of 10^8 solar masses), following the suggestion of Hoyle and Fowler that star-like condensations of this order of mass may be possible sources of the large energies (up to about 10^{62} ergs) required to account for the strong discrete radio sources (assuming that they are at cosmological distances). For masses of this order, general relativistic effects tend to be significant. Hoyle and Fowler assumed that a polytrope of index 3 might provide a suitable model for their investigations.[11]

In 1964, following the work of Feynman[12] and Iben[6] (who were the first to point out that general relativistic instabilities set in at a very early stage in the condensation of massive objects), Fowler[13,14], using a post-Newtonian approximation to the first order in $\frac{GM}{Rc^2}$ and also taking β to be zero in the post-Newtonian terms, obtained for the binding energy the formula

$$E_b = 6\pi \int_0^R \beta p r^2 dr - \frac{8\pi G}{c^2} \int_0^R p r M_r dr - \frac{6\pi G^2}{c^2} \int_0^R \rho M_r^2 dr, \tag{1.12}$$

where R is the radius of the configuration and the other symbols have their usual meanings.

For a polytrope of index n, expression (1.12) may be reduced to

$$\frac{E_b}{Mc^2} = \frac{3\bar{\beta}}{4(5-n)} \left(\frac{R_s}{R}\right) - \zeta_n \left(\frac{R_s}{R}\right)^2, \tag{1.13}$$

where R_s is the Schwarzschild radius, ζ_n is given by

$$\zeta_n = \frac{3}{8(n+1)} \frac{\xi_s^2}{v(\xi_s)^3} \left[\int_0^{\xi_s} \theta^{2n+1} \xi^4 d\xi + \frac{10}{n+2} \int_0^{\xi_s} \theta^{n+2} \xi^2 d\xi \right] \tag{1.14}$$

and $\bar{\beta}$ is the average value of β throughout the configuration. It was shown by Hoyle and Fowler[11] for massive polytropes that β is given by

$$\beta = \frac{1}{\mu}\left[\frac{3}{4\pi}(n+1)^3\left(\frac{k}{H}\right)^4 \frac{1}{aG^3}\right]^{1/4}\left(\frac{v(\xi_s)}{M}\right)^{1/2} \theta^{(n-3)/4}, \qquad (1.15)^*$$

and $\bar{\beta}$ is obtained by averaging β with respect to the distribution of mass. Regarding the binding energy in expression (1.13) as a function of the radius R Fowler[14] showed that it goes through a maximum at a critical radius R_c given by

$$\frac{R_c}{R_s} = \frac{8(5-n)}{3}\frac{\xi_n}{\bar{\beta}}, \qquad (1.16)$$

thus showing that R_c is inversely proportional to $\bar{\beta}$, which is small for massive stars — being a constant of the order 10^{-3} for a polytrope of index 3 and mass about $10^8 M_\odot$. Fowler discussed the onset of instability for values of R below this critcal value.

In 1965, Tooper[16] studied models of massive stars composed of a mixture of ideal gas and radiation in which β, the ratio of the gas pressure to the total pressure, was assumed to be a constant throughout. For those models in which β is not small compared with unity the equations of equilibrium were integrated numerically by Tooper to give a two-parameter family

*This expression will be discussed in Chapter 3. (See also Tooper[15]).

of solutions depending on the values of the constants σ and β. It was shown that instability sets in when the binding energy, as a function of σ, has a maximum for a fixed value of β. For small β, the maximum in the binding energy occurs for small σ and a post-Newtonian approximation to the first order in $\frac{GM}{Rc^2}$ is adequate to describe these models. In the case $\beta << 1$, corresponding to the most massive objects, the equations of equilibrium have the same form as those for an adiabatic fluid sphere of index $n=3$, and are thus described by a one-parameter family of solutions (depending on σ only), but these models are unstable since their binding energy is always negative.

The method of approach used by Fowler[13,14], which is described above, will be adopted in Chapter 4 of the present thesis to derive an expression for the critical radius R_c for composite models in which the core is taken to be a mixture of ideal gas and radiation (with constant β), and the envelope an adiabatic fluid for which the equation of state is given by (1.11). An expression for the binding energy of these models is derived which, in the appropriate limit, becomes the particular formula obtained by Fowler. From the former we obtain an expression for the critical radius that is much more complicated than Fowler's relation (1.16). It is found that this critical radius depends not only on the value of β but also very strongly on the position of the interface dividing the core and the envelope.

The numerical work (for $n_1=1$) serves to show that, for a given position of the interface, the ratio of the critical radius R_c to the Schwarzschild radius R_s is strongly dependent on the value of β in the core, and also, for a given value of β, this ratio increases more and more rapidly the farther the interface is from the centre.

(V) APPLICATION OF GENERAL RELATIVITY TO NON-STATIC MODELS

The effects of general relativity on contracting spheres was discussed by Bondi[17] in 1964. As a preliminary, he suggested that the condition for the neutral equilibrium of a spherically symmetrical configuration, which in classical theory is simply $\gamma = {}^4/_3$, is likely to be much more complicated in general relativity, probably depending on the detailed structure of the model and on distance from the centre.

In the particular case of a uniform sphere contracting adiabatically, he showed that this is indeed the case.

The method used by Bondi (in this particular case) was to consider a one-parameter family of uniform static spheres having the same mass M and gradually to deform a model through this sequence of configurations in such a way that the only time-dependent field equation is $\frac{8\pi G}{c^4} T_4^1 = \frac{-e^{-\lambda}}{r} \frac{\partial \lambda}{\partial t}$, the time-dependence of the other field equations being neglected (so that they are therefore identical with those for a static sphere)*. Bondi found that the value of γ correspon-ding to neutral equilibrium for such quasi-static spheres is in fact greater than $4/3$, the actual value depending on the surface potential $2GM/Rc^2$. He also found that the value of γ varied according to the position inside the configuration, being greatest at the centre.

In 1964 Chandrasekhar(9) investigated, by means of the time-dependent field equations of general relativity, static spheres subject to radial per-turbations. In order to obtain an equation for the characteristic frequencies of the oscillations which

*The time-dependent field equations may be found in Chapter 2 (equations (2.6)-(2.10)).

ensue, he neglected second and higher order terms in the motions. The perturbation increments of all physical quantities were assumed to be harmonic in time and were expressed in terms of the unperturbed variables and a Lagrangian displacement ζ (defined by $V = \frac{\partial \zeta}{\partial t}$, where V is the velocity of the perturbed system). These were connected by one of the relations arising from the identity $(T_i^{\ j})_{;j} = 0$.

In the particular case of a uniform sphere for which the surface potential is small, Chandrasekhar integrated the equations with the aid of a 'trial function' $\zeta = \xi e^{\nu/2}$, where ξ has its usual significance (polytropic dimensionless radius). The condition for dynamical instability that he obtained (for such a sphere) was

$$\gamma - \frac{4}{3} < \left(\frac{19}{14}\right) \frac{4p_c}{3\rho_c c^2} = \left(\frac{19}{14}\right) \left(\frac{2GM}{3Rc^2}\right) , \tag{1.17}$$

thus confirming Bondi's result that the Newtonian lower limit of $^4/_3$ for the ratio of the specific heats γ required to ensure dynamical stability is increased by the effects of general relativity.

Re-writing (1.17) in the form

$$R < \frac{19}{14(3\gamma-4)} \left(\frac{2GM}{c^2}\right) , \tag{1.18}$$

Chandrasekhar concluded that if γ should exceed the value $^4/_3$ by only a small amount dynamical instability will occur should the radius contract to the value R_c given by

$$R_c = \frac{19}{14(3\gamma-4)} \left(\frac{2GM}{c^2}\right) . \qquad (1.19)$$

Using the first post-Newtonian approximation, Chandrasekhar obtained similar results for polytropic spheres, the numerical factor multiplying $^{2GM}/_{3Rc^2}$ and $\frac{1}{(3\gamma-4)} \frac{2GM}{c^2}$, respectively, of the above formulae increasing for values of the polytropic index increasing from n=0 (value for a uniform sphere).

In 1965, Kaplan and Lupanov[18], investigated the effects of general relativity on the stability of radially oscillating <u>polytropic spheres</u>. They used a simple method (originally devised by Kaplan) for the analysis of the field equations in the case of not too dense configurations. The method involving rewriting the relativistic equations of hydrostatic equilibrium, namely[6,19]

$$\frac{dp}{dr} = -\frac{GM_r}{r^2}\rho \frac{\left(1+\frac{p}{\rho c^2}\right)\left(1+\frac{4\pi p r^3}{c^2 M_r}\right)}{\left(1-\frac{2GM_r}{rc^2}\right)} , \quad \frac{dM_r}{dr} = 4\pi r^2 \rho \qquad (1.20)$$

in the form

$$\frac{dp}{dr} = -\frac{GM_r}{r^2}\rho\left[1 + \frac{p_c}{\rho_c c^2} \quad g_n(r)\right], \quad \frac{dM_r}{dr} = 4\pi r^2\rho \ , \quad (1.21)$$

and then replacing the function $g_n(r)$ by a particular numerical constant, namely 4. (This approximation is in fact exact throughout a uniform sphere and at the centre of any polytrope. In general, however, $g_n(r) \neq 4$ for a polytrope when $r \neq 0$, e.g. if n=3, we find that $2.5 < g_3(r) < 5.5$). Using the approximation $g_n(r) \equiv 4$, the gravitational constant G was replaced by $G' = G[1+4p_c/\rho_c c^2]$, from which Kaplan and Lupanov obtained the following formulae for the mass and the radius of the polytropic model:-

$$M = 4\pi\left[\frac{n+1}{4\pi}\right]^{3/2}\left(\frac{K}{G}\right)^{3/2}\frac{\rho_c^{(3-n)/2n}}{\left(1+\frac{4K\rho_c^{1/n}}{c^2}\right)^{3/2}}\left(\xi^2\frac{d\theta}{d\xi}\right)_s \ , \quad (1.22)$$

and

$$R = \left[\frac{n+1}{4\pi}\right]^{1/2}\left(\frac{K}{G}\right)^{1/2}\frac{\rho_c^{(1-n)/2n}}{\left(1+\frac{4K\rho_c^{1/n}}{c^2}\right)^{1/2}}\xi_s, \quad (1.23)$$

where θ and ξ are the usual classical polytropic variables. On putting c formally infinite, equations (1.19) and (1.20) reduce to the usual classical formulae, namely

$$M = 4\pi\left[\frac{n+1}{4\pi}\right]^{3/2}\left(\frac{K}{G}\right)^{3/2}\rho_c^{(3-n)/2n}\left(\xi^2\frac{d\theta}{d\xi}\right)_s ,$$

and $$R = \left[\frac{n+1}{4\pi}\right]^{1/2}\left(\frac{K}{G}\right)^{1/2}\rho_c^{(1-n)/2n}\,\xi_s .$$

From equation (1.22) Kaplan and Lupanov found that the mass of a model of given central density initially increases with increasing central density and then decreases, and that the values of the central pressure and central density ($\tilde{p}_c$ and $\tilde{\rho}_c$ respectively) for which the mass has its maximum value are related by the equation

$$\frac{3-n}{4n} = \frac{K\rho_c^{\frac{1}{n}}}{c^2} = \frac{\tilde{p}_c}{\tilde{\rho}_c c^2} ,$$

$$\text{i.e.}\quad \gamma - {}^4/_3 = \frac{4\tilde{p}_c}{3\tilde{\rho}_c c^2} . \qquad (1.24)$$

Using a perturbation method, the perturbations being harmonic in time, Kaplan and Lupanov obtained an equation of motion from which, when integrated with the aid of a further approximation, yielded a condition for dynamical instability in the form

$$\frac{3-n}{4n} < \frac{K\rho_c^{1/n}}{c^2} = \frac{p_c}{\rho_c c^2} ,$$

i.e. $$\gamma - 4/3 < \frac{4}{3} \frac{p_c}{\rho_c c^2} \qquad (1.25)$$

which for a uniform sphere gives

$$\gamma - 4/3 < \frac{2GM}{3Rc^2} . \qquad (1.26)$$

On comparing condition (1.26) with equation (1.24), it is easily seen that the descending branch of $M(\rho_c)$ is unstable, marginal stability occurring when the mass goes through its maximum value.

Comparing the above results with those obtained by Chandrasekhar, it is seen that the factor $(\frac{19}{14})$ appearing in the inequality (1.17) derived by Chandrasekhar for the condition of dynamical instability of a uniform sphere is not present in the corresponding inequality (1.26) obtained by Kaplan and Lupanov for the same type of sphere. Also, for polytropes ($n \neq 0$), instead of Chandrasekhar's result that the factor replacing $(\frac{19}{14})$ in (1.17) and (1.18) increases for increasing values of the polytropic index n, Kaplan and Lupanov found that inequality (1.26) holds for all values of n. (Their result depends of course, on their method of approximation). One object of this thesis is to investigate these discrepancies (see Chapter 5).

In Chapter 5 of the present thesis, the problem of the stability of slowly oscillating spherically-symmetrical adiabatic fluid spheres will be considered on the basis of the time-dependent field equations of general relativity. Unlike the work of Kaplan and Lupanov(18) and Chandrasekhar(9), it has not been found necessary to introduce perturbations and the technique used here involves fewer assumptions. Using the two relations obtained from the vanishing of the covariant derivative of the energy-momentum tensor i.e. $(T_i^{\ j})_{;j} = 0$, the equations of motion of polytropes in radial motion will be derived in the post-Newtonian approximation, and to the first order in the motions, in the form

$$\frac{1}{\rho}\frac{\partial}{\partial r}(\gamma p \operatorname{div}\zeta) + \left\{(1-\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^2} + \frac{4\pi G\gamma p r}{c^2}\right\}\zeta'(r)$$
$$+\left\{\sigma^2 e^{\nu_0-\lambda_0} + \frac{4GM_r}{r^3} + (9+\gamma)\frac{G^2M_r^2}{c^2r^4} + \frac{4\pi Gp(2+\gamma)}{c^2} + \right.$$
$$\left. + 2(1+\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^3}\right\}\zeta(r) = 0, \qquad (1.27)$$

where a prime denotes differentiation with respect to r, $\zeta(r)$ is defined by $V = \frac{\partial}{\partial t}(\zeta(r)e^{i\sigma t})$, and σ is the frequency of the oscillations. A similar equation is also derived for a uniform sphere. In the classical limit each of the equations of motion reduces to the corresponding equation in Newtonian theory derived by Rosseland[5]. On integrating (1.27) and the corresponding equation for a uniform sphere, conditions for dynamical instability will be obtained:-

$$\text{(i)} \quad \gamma - 4/3 < 4/3 \frac{p_c}{\rho_c c^2}, \qquad \text{(Uniform sphere)} \tag{1.28}$$

$$\text{(ii)} \quad \gamma - 4/3 < 2.25 \frac{p_c}{\rho_c c^2}. \qquad \text{(Polytrope of index 3)} \tag{1.29}$$

The corresponding results obtained by Chandrasekhar[9] involve an additional factor of $19/14$ in the right hand side of (1.28) and about 2.63 in place of 2.25 in formula (1.29), whereas Kaplan and Lupanov[18] give the same factor $4/3$ in both formulae.

For the case of a uniform sphere, it will be shown that the mass, as a function of the central density, has a maximum at the value of $p_c/\rho_c c^2$ at which instability sets in, confirming the result obtained by Kaplan and Lupanov[18]. In the case of a polytrope with index n slightly less than 3, expression (1.28) will be checked by means of the first post-Newtonian approximation to the relativistic values of the polytropic variables ξ, θ, and $v(\xi)$, defined in Appendix V, and it will also be shown that dynamical instability sets in if the radius contracts to the value R_c given by

$$R_c = \frac{0.96}{\gamma - 4/3} R_s \,. \qquad (1.30)$$

REFERENCES

(1) Chandrasekhar, S., An Introduction to the Study of Stellar Structure, (Chicago, University of Chicago Press), 1938.

(2) Schwarzschild, M., Structure and Evolution of the Stars, (Princeton, Princeton University Press), 1958.

(3) Milne, E.A., Thermodynamics of the Stars, (Extracted from Handbuch der Astrophysik III), 1930.

(4) Milne, E.A., Monthly Notices of the Royal Astronomical Society, 109, No.5, 35, 1949.

(5) Rosseland, S., The Pulsation Theory of Variable Stars, (Oxford, Clarendon Press), 1949.

(6) Iben, I., Astrophysical Journal, 138, 1090, 1963.

(7) Tooper, R.F., Astrophysical Journal, 140, No.2, 434, 1964.

(8) Tooper, R.F., Astrophysical Journal, 142, 1541, 1965.

(9) Chandrasekhar, S., Astrophysical Journal, 140, 417, 1964.

(10) Chandrasekhar, S., and Tooper, R.F., Astrophysical Journal, 139, 1396, 1964.

(11) Hoyle, F., and Fowler, W.A., Astrophysical Journal Supplement, 9, 201, 1964.

(12) Feynman, R.P., Private communication (1963), mentioned in Reviews of Modern Physics, 36, 545, 1964.

(13) Fowler, W.A., Reviews of Modern Physics, 36, 545, 1964.

(14) Fowler, W.A., Presented in part at the JOINT DISCUSSION ON RADIO GALAXIES during the XIIth General Assembly of the INTERNATIONAL ASTROPHYSICAL UNION, Hamburg, Germany, August 1964.

(15) Tooper, R.F., *Astrophysical Journal*, 155, 145, 1969.

(16) Tooper, R.F., *Astrophysical Journal*, 143, 465, 1965.

(17) Bondi, H., *Proceedings of the Royal Society*, 281, 39, 1964.

(18) Kaplan, S.A., and Lupanov, G.A., *Soviet Astronomy*, 9, No.2, 233, 1965.

(19) Oppenheimer, J.R., and Volkoff, G.M., *Physical Review*, 55, 374, 1939.

CHAPTER 2

MATHEMATICAL INTRODUCTION

I. EINSTEIN'S FIELD EQUATIONS

The derivation of Einstein's law of gravitation may be found in almost any treatise on general relativity(1,2) and so it is only necessary to give a very simple outline of the basic results.

In the absence of matter and energy, Einstein chose for his law of gravitation

$$R_{\alpha}^{\beta} = 0 , \tag{2.1}$$

where R_{α}^{β} is the Riemann-Christoffel tensor. This law is, of course, independent of any particular coordinate system and thus we may take for the coordinate system (x_1, x_2, x_3, x_4) and the line element in the form

$$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} , \tag{2.2}$$

where ds is the separation of two events whose coordinate separation is (dx_1, dx_2, dx_3, dx_4), and $g_{\alpha\beta}$ is the metric tensor.

In the presence of matter and energy, the law of gravitation takes a different form, and involves the use of the energy-momentum tensor T_α^β, this tensor expresses the energy content and the state of motion of the medium at the point considered. In formulating the law of gravitation in the presence of matter and energy, appeal was made to Newtonian mechanics in the presence of a weak static gravitational field since in the first approximation the field equations must reduce to Poisson's equation. Considerations of this type led Einstein to state his law

$$R_\alpha^\beta - \frac{1}{2}g_\alpha^\beta R = -KT_\alpha^\beta \qquad (2.3)$$

the constant K being given by

$$K = \frac{8\pi G}{c^4}, \qquad (2.4)$$

where G is the constant of gravitation and c is the velocity of light in free space. Obviously, when the energy-momentum tensor vanishes the law (2.3) reduces to that for empty space-time (2.1).

Although no general solution of Einstein's field equations is known, we can nevertheless make certain logical assumptions concerning the form of the solutions which correspond to the physical problem considered. In particular, since spherically symmetrical configurations are considered throughout the present work, coordinates can be chosen in such a way that the line-element for the system will exhibit spherical symmetry.

Without loss of generality it can be shown[1] that for space-like coordinates (r, θ, ϕ) and time-like coordinate ct, the line-element for a spherically symmetrical configuration reduces to the form

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta + \sin^2\theta \, d\phi^2) + c^2 e^{\nu} dt^2, \quad (2.5)$$

where $\lambda = \lambda(r,t)$, $\nu = \nu(r,t)$ are functions of r and t only. Using these coordinates, the components of the energy-momentum tensor which do not vanish are found to be[1],

$$-\frac{8\pi G}{c^4} T_1{}^1 = R_1{}^1 - \frac{1}{2} g_1{}^1 R = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \quad (2.6)$$

$$-\frac{8\pi G}{c^4} T_2{}^2 = -\frac{8\pi G}{c^4} T_3{}^3 = R_2{}^2 - \frac{1}{2} g_2{}^2 R = R_3{}^3 - \frac{1}{2} g_3{}^3 R$$

$$= e^{-\lambda}\left(\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 + \frac{1}{2r}(\nu' - \lambda')\right) - e^{-\nu}\left(\frac{1}{2}\ddot{\lambda} + \frac{1}{4}\dot{\lambda}^2 - \frac{1}{4}\dot{\lambda}\dot{\nu}\right),$$
$$(2.7)$$

$$-\frac{8\pi G}{c^4} T_4{}^4 = R_4{}^4 - \frac{1}{2} g_4{}^4 R = -e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2}, \quad (2.8)$$

$$-\frac{8\pi G}{c^4} T_4{}^1 = R_4{}^1 - \tfrac{1}{2} g_4{}^1 R = e^{-\lambda} \frac{\dot{\lambda}}{r} , \qquad (2.9)$$

$$-\frac{8\pi G}{c^4} T_1{}^4 = R_1{}^4 - \tfrac{1}{2} g_1{}^4 R = -e^{-\nu} \frac{\dot{\lambda}}{r} , \qquad (2.10)$$

where a prime denotes differentiation with respect to the radial coordinate r, and a dot differentiation with respect to t. These field equations are not all independent, since the covariant derivative of $R_\alpha^\beta - \frac{1}{2} g_\alpha^\beta R$ vanishes identically, and so

$$(T_\alpha^\beta)_{;\beta} = 0 . \qquad (2.11)$$

In the case of a perfect fluid, which is defined as a mechanical medium incapable of exerting transverse stresses, the components of the energy-momentum tensor with respect to the actual coordinate system that is being used may be put in the form

$$T_\alpha^\beta = (p + \rho c^2) \frac{dx^\beta}{ds} \cdot \frac{dx_\alpha}{ds} - g_\alpha^\beta p , \qquad (2.12)$$

where p is the proper macroscopic pressure of the fluid (arising from all causes), ρ is the proper macroscopic density, being the sum of the rest-mass density and the mass-density equivalent of the internal energy, and $\frac{dx^\beta}{ds}$ are the components of the macroscopic velocity of the fluid.

II. SCHWARZSCHILD EXTERIOR SOLUTION

In accordance with the spherically symmetrical nature of the field surrounding any spherical distribution of matter and energy, the solution required will be a solution of the equations (2.6)-(2.10). Furthermore, since we require that the energy-momentum tensor vanishes in the free space surrounding the matter, $T_{\alpha}^{\beta} = 0$ outside the distribution of matter and energy, and this assumption forms the basis of Birkhoff's theorem[1,3], which states that spherical symmetry alone is a sufficient condition for a static solution of the field equations in the empty space-time surrounding a sphere of material. This solution was first given by Schwarzschild[4] in 1916 and is known as the Schwarzschild exterior solution. It may be written in the form

$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta d\phi^2)+c^2\left(1-\frac{2GM}{rc^2}\right)dt^2 , \qquad (2.13)$$

where M is the total mass of the system. From the form of this metric, it is evident that the sphere $R_s = \frac{2GM}{c^2}$ constitutes a place where the field is singular; for the rate of a clock on this sphere is obviously zero. The radius R_s is usually called the Schwarzschild limit, appropriate to the mass M.

This singularity in the metric has been studied intensively by many authors,[5,6,7] and there has been much speculation as to whether it has any physical significance. This question need not concern us here since in 1959 Buchdahl[6] was able to show quite generally that, for configurations in which the density does not increase outwards, the coordinate radius R of the sphere of matter and energy is necessarily restricted by the inequality

$$R \geq {}^{9}/8R_s \; , \tag{2.14}$$

equality holding only for constant density with $e^{\nu} = 0$ at the centre of the configuration.

III. THE STATIC (TIME-INDEPENDENT) FIELD EQUATIONS

Unlike the external solution the internal solutions for static and non-static systems differ, being dependent on the pressure and density and on how they vary with time t. We shall consider static systems first.

Using a co-moving coordinate system (at rest with respect to the fluid), the components of the energy-momentum tensor (2.12) can be written,

$$T_1^{\ 1} = T_2^{\ 2} = T_3^{\ 3} = -p \ , \quad T_4^{\ 4} = \rho c^2 \ , \tag{2.15}$$

and the metric reduces to the form

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu}dt^2 \ , \tag{2.16}$$

where $\lambda = \lambda(r)$, $\nu = \nu(r)$ are functions of r only. The time-independent field equations reduce to

$$\frac{8\pi G p}{c^4} = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \ , \tag{2.17}$$

$$\frac{8\pi G p}{c^4} = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{1}{4}\lambda'\nu' + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{r}\right) \ , \tag{2.18}$$

$$\frac{8\pi G \rho}{c^2} = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \ . \tag{2.19}$$

Also, from equation (2.11) for the covariant derivative of the energy-momentum tensor, the only component which does not vanish identically is the (r,r) component and this reduces to

$$\frac{dp}{dr} = -(p + \rho c^2)\frac{\nu'}{2} \ . \tag{2.20}$$

As already stated, the above field equations are not all independent, and indeed equation (2.20) may be obtained by setting (2.17) and (2.18) equal to each other and using (2.19)[1]. Thus in what follows, we shall drop equation (2.18) and use only equations (2.17), (2.19) and (2.20).

Any solution must satisfy certain conditions in order to have physical significance:-

(i) The pressure and density are finite everywhere.

(ii) Outside a finite region of radius R space-time is empty.

(iii) At the outer boundary of the system ($r=R$) the pressure must vanish.

(iv) At the outer boundary the solution must be continuous with the usual Schwarzschild exterior metric (2.13).

Equation (2.19) may be integrated immediately by writing it in the form

$$\frac{8\pi G\rho}{c^2} r^2 = 1-(re^{-\lambda})' ,$$

whence

$$\left[re^{-\lambda}\right]_0^r = r - \frac{8\pi G}{c^2}\int_0^r \rho r^2 dr ,$$

and consequently

$$e^{-\lambda} = 1 - \frac{2G}{rc^2}\int_0^r 4\pi r^2 \rho dr . \qquad (2.21)$$

Defining the mass inside the radius r (arising from all causes) as measured by an external observer to be M_r, so that

$$M_r = \int_0^r 4\pi\rho r^2 dr , \qquad (2.22)$$

equation (2.21) may be written as

$$e^{-\lambda} = 1 - \frac{2GM_r}{rc^2} . \qquad (2.23)$$

As already stated, equations (2.17), (2.19) and (2.20) are three independent equations in four unknowns λ, ν, p and ρ, and thus in order to solve them completely it is necessary to introduce a further condition. This usually takes the form of an equation of state $p = p(\rho)$ connecting the pressure p with the density ρ. However, other approaches have been used [5,8,9] which either supplement or replace the equation of state but will not be considered in this work.

IV. THE EQUATION OF STATE

(i) The simplest condition that can be imposed on the distribution of the density of the configuration is that ρ is constant throughout (i.e. that the system is a uniform sphere). This condition enables us to integrate the field equations analytically, and we obtain what is known as the Schwarzschild interior solution. It will suffice here merely to state the results obtained. If the pressure vanishes at a coordinate radius R we have

$$\begin{cases} \rho = \text{constant}, & 0 \leq r, \leq R , \\ \rho = 0, & r > R. \end{cases}$$

Also,

$$M_r = \frac{4}{3}\pi\rho r^3 ,$$

$$e^{-\lambda} = 1 - \frac{2GM_r}{rc^2} ; \quad e^{\nu} = \left\{\frac{1}{2}[3(1-\frac{2GM}{Rc^2})^{\frac{1}{2}} - (1-\frac{2GM_r}{rc^2})^{\frac{1}{2}}]\right\}^2 ,$$

and

$$p = \rho \frac{(1-\frac{2GM_r}{rc^2})^{\frac{1}{2}} - (1-\frac{2GM}{Rc^2})^{\frac{1}{2}}}{3(1-\frac{2GM}{Rc^2})^{\frac{1}{2}} - (1-\frac{2GM_r}{rc^2})^{\frac{1}{2}}} .$$

In order that the pressure be positive everywhere we must have

$$R \geq \frac{9}{8} \cdot \frac{2GM}{c^2} ,$$

equality holding for a configuration in which $e^{\nu} = 0$ at the centre, and is in complete agreement with Buchdahl's result (2.14).

(ii) Another important equation of state is the <u>polytropic</u> equation according to which the pressure and density are connected by a power law of the form

$$p = K\rho^{1+\frac{1}{n}} , \qquad (2.24)$$

where K and n are positive constants. The constant n is known as the polytropic index and is usually assumed to have some definite value in a given problem. The constant K, on the other hand, has usually been calculated from the thermal characteristics of a given fluid sphere, but it can also be evaluated given the mass and radius of the sphere and the ratio of the central pressure to the central density[10]. In the equation of state, p is the total pressure arising from the pressure of the gas and radiation, and ρ is the total density arising from all causes, including the internal energy. A particular case of the polytropic equation of state is the classical adiabatic relation

$$p = K\rho^{\gamma} , \qquad (2.25)$$

where γ is the ratio of the principal specific heats c_p/c_v .

Although, as is well known, the polytropic equation of state (2.24) has been of fundamental importance in the study of stellar structure, for configurations in which the central density is extremely high the velocity of sound at the centre can exceed the velocity of light (in free space) for all values of n[10].

(iii) A relativistic equation of state proposed by Tooper[11] for a perfect gas undergoing an adiabatic process is given by

$$p = K\rho_g^{1+\frac{1}{n}} , \quad \rho c^2 = \rho_g c^2 + np , \qquad (2.26)$$

where ρ_g is the density of the rest-mass of the gas, and $n = \frac{1}{\gamma-1}$. The velocity of sound for this equation of state, unlike that for the polytropic equation of state, is always less than that of light provided that the index $n \geq 1$.

(iv) Another equation of state which will be used frequently in this thesis is that for a mixture of perfect gas isotropic radiation at a temperature T. The total pressure may be expressed as

$$p = p_g + p_r ,$$

where

$$p_g = (\frac{k}{\mu H})\rho_g T \quad \text{and} \quad p_r = \frac{1}{3}aT^4 ,$$

are the pressures of the gas and radiation respectively. Here ρ_g is the gas density, k is Boltzmann's constant, μ is the molecular weight, and H is the mass of a proton. If ρ is the total density, i.e. the sum of the densities of the rest-mass of the gas, the energy content of the microscopic kinetic energy of the gas, and the energy of the radiation, it immediately follows that

$$\rho c^2 = \rho_g c^2 + \frac{p_g}{\gamma-1} + 3p_r ,$$

where γ is the ratio of the specific heats of the gas. Now, if we define β as the ratio of the gas pressure to the total pressure we have

$$p_g = \beta p \quad \text{and} \quad p_r = (1-\beta)p ,$$

and consequently

$$\beta p = (\frac{k}{\mu H})\rho_g T , \ (1-\beta)p = \frac{1}{3}aT^4. \quad (2.27)$$

If β is constant, elimination of T between the above equations gives

$$p = K(\beta)\rho_g{}^{4/3} , \quad \text{where} \quad K(\beta) = [(\frac{k}{\mu H})^4 \frac{3}{a} \cdot \frac{1-\beta}{\beta^4}]^{\frac{1}{3}} ,$$

and so the equation of state in parametric form becomes[12]

$$p = K(\beta)\rho_g{}^{4/3} , \ \rho c^2 = \rho_g c^2 + \frac{\beta}{\gamma-1} K(\beta)\rho_g{}^{4/3} + 3(1-\beta)K(\beta)\rho_g{}^{4/3}, \quad (2.28)$$

giving the total energy-density ρc^2 in terms of the pressure p .

For variable β , a similar treatment was given by Milne[13]. From equation (2.27) we get

$$\frac{\beta^{4-s}}{1-\beta} p^{3-s} = \frac{\left(\frac{k}{\mu H}\right)^{4-s}}{\frac{1}{3}a} \rho_g^{4-s} T^{-s} ,$$ where s is a constant, and Milne assumed that β varies with temperature through the star according to the law

$$\frac{1-\beta}{\beta^{4-s}} = \frac{1-\beta_c}{\beta_c^{4-s}} \left(\frac{T}{T_c}\right)^s , \qquad (2.29)$$

where the subscript c denotes central values. Hence, on Milne's assumption,

$$p = K\rho_g^{\frac{4-s}{3-s}} , \quad \text{where} \quad K^{3-s} = \frac{\left(\frac{k}{\mu H}\right)^{4-s}}{\frac{1}{3}a} \frac{1-\beta_c}{\beta_c^{4-s}} \frac{1}{T_c^s} ,$$

and thus in parametric form the equation of state becomes

$$p = K\rho_g^{1+\frac{1}{n}} , \quad \rho c^2 = \rho_g c^2 + \frac{\beta}{\gamma-1} K\rho_g^{1+\frac{1}{n}} + 3(1-\beta)K\rho_g^{1+\frac{1}{n}} , \qquad (2.30)$$

where $n = 3-s$. When $s=0$ equation (2.30) obviously reduces to (2.28). It is seen that if β is a small constant throughout the configuration, equation (2.28) reduces to the form of equation (2.26) with $n=3$, whereas if β is approximately unity equation (2.30) reduces to (2.26).

V. THE GENERAL RELATIVISTIC FORM OF THE LANE-EMDEN EQUATION

In order to avoid unnecessary repetition, the equation of state will be taken in the general form

$$p = K\rho_g^{1+\frac{1}{n}}, \quad \rho c^2 = \rho_g c^2 + Ap, \quad (2.31)$$

where the appropriate values of the constants A and n will be chosen to correspond to the particular equation of state under consideration.

At this point it is convenient to follow Tooper[11,12] and introduce a new variable $\theta = \theta(r)$ related to the gas density ρ_g at a given point in the configuration and the central gas-density ρ_{g_c} by the formula

$$\rho_g = \rho_{g_c}\theta^n, \qquad (2.32)$$

the value of n being the same as the appearing in equation (2.31). In terms of this new variable, the pressure is given by

$$p = K\rho_{g_c}^{1+\frac{1}{n}}\theta^{n+1}. \qquad (2.33)$$

From equation (2.32) it is seen that θ takes the value unity at the centre of the configuration, i.e. $\theta(0) = 1$. Also from equation (2.33), if the pressure vanishes at the surface $r = R$, then $\theta(R) = 0$. From (2.33) the central pressure is given by $p_c = K\rho_{g_c}^{1+\frac{1}{n}}$, and so we may write (2.33) in the equivalent form

$$p = p_c \theta^{n+1} . \tag{2.34}$$

With these expressions for the pressure and density in terms of θ, equation (2.20) may be written

$$p_c(n+1)\theta^n \frac{d\theta}{dr} = -\frac{1}{2}[p_c\theta^{n+1}+\rho_{g_c}\theta^n c^2+Ap_c\theta^{n+1}]\frac{d\nu}{dr} . \tag{2.35}$$

Introducing a parameter σ defined by

$$\sigma = \frac{p_c}{\rho_{g_c} c^2} = \frac{K\rho_{g_c}^{1/n}}{c^2} , \tag{2.36}$$

equation (2.35) becomes

$$2\sigma(n+1)\frac{d\theta}{dr} + [1+(1+A)\sigma\theta]\frac{d\nu}{dr} = 0 . \tag{2.37}$$

On integrating this equation and letting ν take the value $\nu(0)$ at the centre, we get

$$e^{\nu} = e^{\nu(0)} \left[\frac{1+(1+A)\sigma}{1+(1+A)\sigma\theta}\right]^{\frac{2(n+1)}{1+A}} . \tag{2.38}$$

Since the internal solution of the field equations must be continuous with the external solution, the value of ν at the surface of the configuration $\nu = \nu(R)$ must be identical with that obtained from equation (2.13), and so

$$e^{\nu(R)} = 1 - \frac{2GM}{Rc^2} = e^{\nu(0)}[1+(1+A)\sigma]^{\frac{2(n+1)}{1+A}} .$$

Hence, equation (2.38) for e^{ν} in terms of the variable θ becomes

$$e^{\nu} = [1+(1+A)\sigma\theta]^{\frac{-2(n+1)}{1+A}} (1 - \frac{2GM}{Rc^2}) . \qquad (2.39)$$

We now have expressions for the density, the pressure, and e^{ν} in terms of the variable θ. The expression for e^{λ} in terms of θ can be written down immediately using equations (2.22) and (2.23), and hence

$$e^{-\lambda} = 1 - \frac{2GM_r}{rc^2} , \quad \text{where} \quad M_r = 4\pi\rho_{g_c} \int_0^r \theta^n[1+A\theta\sigma]r^2 dr .$$

In order to determine the above quantities as functions of the radius r, we need an equation connecting θ with r. To obtain this we make use of the remaining field equation (2.17) and substitute for $\frac{d\nu}{dr}$ and $e^{-\lambda}$ from equations (2.37) and (2.23).

We get

$$\frac{\sigma(n+1)}{1+(1+A)\sigma\theta} \quad r\frac{d\theta}{dr}\left(1-\frac{2GM_r}{rc^2}\right) + \frac{GM_r}{rc^2} + \frac{4\pi G p_c r^2 \theta^{n+1}}{c^4} = 0 , \tag{2.40}$$

where, from equation (2.22), we have

$$\frac{dM_r}{dr} = 4\pi\rho r^2 = 4\pi\rho_{g_c} r^2\theta^n(1+A\theta\sigma) . \tag{2.41}$$

We now introduce the dimensionless variables ξ and $v(\xi)$ defined by

$$r = a\,\xi , \tag{2.42}$$

and

$$M_r = 4\pi\rho_{g_c} a^3 v(\xi), \tag{2.43}$$

where

$$a = \left[\frac{(n+1)\sigma c^2}{4\pi G\rho_{g_c}}\right]^{\frac{1}{2}} , \tag{2.44}$$

and therefore has the dimensions of length. In terms of these variables, equations (2.40) and (2.41) become

$$\frac{1-2(n+1)\sigma v(\xi)/\xi}{1+(1+A)\sigma\theta} \quad \xi^2\frac{d\theta}{d\xi} + v(\xi) + \sigma\xi^3\theta^{n+1} = 0 , \tag{2.45}$$

and $$\frac{dv(\xi)}{d\xi} = \xi^2\theta^n(1+A\sigma\theta) . \tag{2.46}$$

These are the desired equations connecting θ (and its derivatives) with the (dimensionless) radius variable ξ, and together will be referred to as the general-relativistic generalization of the Lane-Emden equation of index n, since in the classical limit, given by $\sigma \to 0$ and $\rho_g \to \rho$, these equations reduce to

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \xi^2\theta^n = 0 , \qquad (2.47)$$

which is just the Lane-Emden equation for a polytrope of index n.

We define a 'complete' configuration as one that is non-composite in the sense that a single equation of state holds throughout. For such a configuration, equations (2.45) and (2.46) are to be solved subject to the boundary conditions

$$\theta(0) = 1 , \quad v(0) = 0 . \qquad (2.48)$$

Since $v(\xi) = C(\xi^3)$, it follows from (2.45) that

$$\frac{d\theta}{d\xi} \to 0 \quad \text{as}\ \xi \to 0 . \qquad (2.49)$$

The surface of the sphere is taken as the smallest positive value ξ_s of ξ for which

$$\theta(\xi_s) = 0. \qquad (2.50)$$

Consequently, the radius R of the configuration is given by

$$R = a\,\xi_s , \qquad (2.51)$$

and the total mass M by

$$M = 4\pi\rho_{g_c} a^3 \nu(\xi_s) \ . \qquad (2.52)$$

Also,(11,12) the distributions of the density and pressure are given, respectively, by:-

$$\rho_c = \rho_{g_c}(1+A\sigma) \ , \qquad (2.53)$$

$$\rho = \rho_c \theta^n \frac{(1+A\sigma\theta)}{(1+A\sigma)} \ , \qquad (2.54)$$

$$\rho_g = \frac{\rho_c \theta^n}{1+A\sigma} \ , \qquad (2.55)$$

and

$$p = \rho_{g_c} \sigma c^2 \theta^{n+1} \ . \qquad (2.56)$$

Finally, the speed of sound in the model is

$$V_s = \sqrt{\left(\frac{dp}{d\rho}\right)} = c\sqrt{\left\{\frac{n+1}{n} \cdot \frac{\sigma\theta}{1 + \frac{A}{n}(n+1)\sigma\theta}\right\}} . \qquad (2.57)$$

REFERENCES

(1) Tolmon, R.C., _Relativity Thermodynamics and Cosmology_, (Oxford: Clarendon Press), 1939.

(2) Eddington, A.S., _The Internal Constitution of the Stars_, (Cambridge: University Press), 1926.

(3) Birkhoff, G.D., _Relativity and Modern Physics_, (Harvard University Press), 1923.

(4) Schwarzschild, K., _Berl. Ber._, 1916.

(5) Einstein, A., _Annals of Mathematics_, _40_, No.4, 922, 1939.

(6) Buchdahl, H.A., _Physical Review_, _116_, No.4, 1027, 1959.

(7) McCrea, W.H., _Nature_, _201_, 589, 1964.

(8) Tolman, R.C., _Physical Review_, _33_, 364, 1939.

(9) Thompson, I.H., and Whitrow, G.J., _Monthly Notices of the Royal Astronomical Society_, _136_, 207, 1967.

(10) Tooper, R.F., Astrophysical Journal, _140_, No.2, 434, 1964.

(11) Tooper, R.F., Astrophysical Journal, _142_, 1541, 1965.

(12) Tooper, R.F., Astrophysical Journal, _143_, 465, 1965.

(13) Milne, E.A., _Thermodynamics of the Stars_, (Extracted from Handbuch der Astrophysik. III,), 1930.

CHAPTER 3

STABILITY OF COMPOSITE MODELS

(I) INTRODUCTION

In this chapter, we shall examine spherical models with a core and an envelope, the core being a mixture of ideal gas and radiation for which β, the ratio of the gas pressure to the total pressure, is taken to be a small constant. The envelope is a shell of adiabatic gas.

(i) Core The equation of state for the core is given by (2.28). As mentioned in Chapter 1 (see pages 11 and 13), an equation of state of this form (with $\beta \sim 0$) is expected to hold throughout massive stars and white-dwarfs in which the electron-gas is extremely relativistic.

(ii) Envelope The equation of state in the envelope is taken to be of the general form (2.31), with $n < 3$ In general, the parameter A depends on the constitution of the material concerned, and in particular if we put $A = n$ in the equation of state in the envelope, we obtain the equation (2.26) derived by Tooper for an adiabatic fluid[1].

The distance from the centre at which the equation of state (2.28) must be replaced by (2.31) will be called the interfacial radius (Df.) Clearly, the envelope as such would not exist if we were to assume that β is the same constant throughout the configuration, for in this case the index n in equation (2.31) would be equal to 3, and A would be given by $A = (\beta/\gamma-1)+3(1-\beta)$. However, except possibly in the case of extremely massive objects ($\geq 10^8 M_\odot$), β is unlikely to be constant throughout the model. In fact, as mentioned in Chapter 1 (page 15), Fowler and Hoyle[2] have shown that, for polytropes <u>in which β is small</u>, it will depend on the polytropic variable θ according to the relation

$$\beta \sim \frac{1}{\mu}\left[\frac{3}{4\pi}(n+1)^3\left(\frac{k}{H}\right)^4 \frac{1}{aG^3}\right]^{\frac{1}{4}} \left(\frac{v(\xi_s)}{M}\right)^{\frac{1}{2}} \theta^{(n-3)/4} , \tag{3.1}$$

where the symbols have their customary meanings. It follows that <u>if β is small</u>, then only for a polytrope of index 3 is β a constant throughout the model, being in fact given by

$$\beta \sim \frac{4.3}{\mu}\left(\frac{M_\odot}{M}\right)^{\frac{1}{2}} . \tag{3.2}$$

The assumption that β is small means that the object concerned must be massive e.g. if $M = 10^6 M_\odot$, then β is of the order of 10^{-3}.

An alternative way to obtain equation (3.1) is to use equation (2.29) and the corresponding expression for the total mass, as derived by Milne[3], namely

$$M = \left[\frac{(n+1)^3}{4\pi G^3}\left(\frac{k}{\mu H}\right)^4 \frac{3}{a} \frac{1-\beta_c}{\beta_c^4}\right]^{\frac{1}{2}} v(\xi_s) \ . \qquad (3.3)$$

It follows that, taking $n \leq 3$,

$$\frac{\beta^4}{1-\beta} = \frac{3(n+1)^3}{4\pi}\left(\frac{k}{\mu H}\right)^4 \frac{1}{aG^3}\left(\frac{V(\xi_s)}{M}\right)^2 \theta^{n-3} \ . \quad (3.4)$$

In the particular case of a massive sphere for which we may expect β to be small this equation reduces to the form (3.1). For $n < 3$, it follows that near the surface (where $\theta \to 0$) the right hand side of (3.4) is very large, which of course means that β is close to unity, and hence the radiation pressure becomes small compared with the gas pressure. The equation of state may then be taken as that for an adiabatic sphere (2.26), since $\beta=1$ at the surface. Indeed, in a recent paper by Tooper[4] in which he considered massive configurations composed of a mixture of ideal gas and radiation with the assumption that the temperature gradient is equal to adiabatic temperature gradient, Tooper showed that β is approximately constant except in a thin layer near the surface, and also for these models $\frac{d\beta}{dr} = 0$ at the centre as well as at the surface, provided $\gamma > \frac{3}{2}$. (This result may be obtained from equation (3.4)).

(II) CHARACTERISTIC EQUATIONS FOR CORE AND ENVELOPE

(i) Core

In the core, which is assumed to be characterized by the equation of state for a mixture of ideal gas and radiation in which the ratio β of the gas pressure to the total pressure is a small constant, we have

$$p = K(\beta)\rho_g^{4/3}, \quad \rho c^2 = \rho_g c^2 + \frac{\beta}{\gamma-1}K(\beta)\rho_g^{4/3} + 3(1-\beta)K(\beta)\rho_g^{4/3}, \tag{3.5}$$

where the constant $K(\beta)$ is given by

$$K(\beta)^3 = \left(\frac{k}{\mu H}\right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4}. \tag{3.6}$$

This equation of state is a particular case of equation (2.31) in which $n=3$, and in which A is a constant given by

$$A = \frac{\beta}{\gamma-1} + 3(1-\beta) = 3 + \beta\left(\frac{4-3\gamma}{\gamma-1}\right) . \qquad (3.7)^{*}$$

Although the parameters, variables, and the equations of equilibrium have been stated in Chapter 2 for general A and n, it is convenient to re-state them here for the above equation of state. Thus, if σ is defined by

$$\sigma = \frac{p_c}{\rho_{g_c} c^2} = \frac{K(\beta)}{c^2} \rho_{g_c}^{\frac{1}{3}} , \qquad (3.8)$$

then, in terms of the variables ξ, θ and $v(\xi)$, the density of rest-mass of the gas and the total pressure are given by

$$\rho_g = \rho_{g_c} \theta^3 , \qquad (3.9)$$

and

$$p = p_c \theta^4 = K(\beta) \rho_{g_c}^{4/3} \theta^4 . \qquad (3.10)$$

The radius r and the mass inside radius r can be expressed as

$$r = a \xi , \qquad (3.11)$$

and

$$M_r = 4\pi \rho_{g_c} a^3 v(\xi) , \qquad (3.12)$$

where

$$a^2 = \frac{\sigma c^2}{\pi G \rho_{g_c}} \qquad (3.13)$$

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*The symbol A used corresponds to $(f-1)$ used by Tooper[5].

Also, the equations of hydrostatic equilibrium become

$$\frac{1-8\sigma v(\xi)/\xi}{1+(1+A)\sigma\theta}\,\xi^2\,\frac{d\theta}{d\xi} + v(\xi) + \sigma\xi^3\theta^4 = 0\ , \tag{3.14}$$

and $$\frac{dv}{d\xi} = \xi^2\theta^3(1+A\sigma\theta)\ , \tag{3.15}$$

and are to be solved subject to the usual boundary conditions

$$\theta(0) = 1\ ,\ v(0) = 0\ . \qquad \left(\frac{d\theta}{d\xi} \to 0 \text{ as } \xi \to 0\right)\ . \tag{3.16}$$

The solutions are relativistic generalizations of the usual Lane-Emden solutions but, unlike the case of complete models, the surface, the total mass, the radius, etc., can only be defined when the equation of state (3.5) holds throughout. We can, however, define the interfacial values (denoted by subscript i) of these quantities.

At the interface, (where the envelope joins onto the core) the radius r_i is given by

$$r_i = \alpha\,\xi_i\ , \tag{3.17}$$

and the pressure and density of the rest-mass of the gas by

$$\rho_{g_i} = \rho_{g_c}\,\theta_i^{\,3}\,, \tag{3.18}$$

and

$$p_i = p_c\theta_i^{\,4} = K(\beta)\rho_{g_c}^{\;4/3}\theta_i^{\,4}\,. \tag{3.19}$$

Hence, the total energy-density at the interface is given by

$$\rho_i c^2 = \rho_{g_i}c^2 + Ap_i = \rho_{g_c}\theta^3{}_i c^2 + \frac{\beta}{\gamma-1}K(\beta)\rho_{g_c}^{\;4/3}\theta^4{}_i + 3(1-\beta)K(\beta)\rho_{g_c}^{\;4/3}\theta_i^4\,, \tag{3.20}$$

and the mass inside this interface is

$$M_i = 4\pi\rho_{g_c}\alpha^3 v(\xi_i)\,. \tag{3.21}$$

(ii) Envelope

For the envelope, we shall take the equation of state to be (2.31) with a general $n = n_1 \leq 5$, and with A replaced by A_1. To avoid confusion with the corresponding quantities in the core, the variables ξ, θ and $v(\xi)$ in the envelope will be replaced by $\eta, \phi, v_1(\eta)$ respectively, and the envelope values of the parameters σ, n, α will be indicated by the subscript 1.

By analogy with the analysis for the core, it is convenient to introduce a new variable ϕ defined by

$$\rho_g = \rho_{g_c} \phi^{n_1} , \qquad (3.22)$$

where the value ρ_{g_c} is identical with that in equation (3.9). We also define

$$\sigma_1 = \frac{K_1 \rho_{g_c}{}^{1/n_1}}{c^2} , \qquad (3.23)$$

and write

$$p = K_1 \rho_{g_c}{}^{1+1/n_1} \phi^{n_1+1} , \qquad (3.24)$$

and

$$r = \alpha_1 \eta . \qquad (3.25)$$

For the mass inside radius r, we have

$$M_r = 4\pi\rho_{g_c} \alpha_1{}^3 v_1(\eta) , \qquad (3.26)$$

where

$$\alpha_1{}^2 = \frac{(n_1+1)\sigma_1 c^2}{4\pi G \rho_{g_c}} . \qquad (3.27)$$

The equations of hydrostatic equilibrium for the envelope become

$$\frac{1-2(n_1+1)\sigma_1 v_1(\eta)/\eta}{1+(1+A_1)\sigma_1 \phi} \eta^2 \frac{d\phi}{d\eta} + v_1(\eta) + \sigma_1 \eta^3 \phi^{n_1+1} = 0 , \qquad (3.28)$$

and

$$\frac{dv_1}{d\eta} = \eta^2 \phi^{n_1}(1+A_1\sigma_1\phi) \; . \qquad (3.29)$$

Although in general, the required solutions of the differential equations (3.28) and (3.29) will not, in this case, be the usual generalizations of the Lane-Emden solutions, since they do not extend to the centre, and hence need not be subject to the usual boundary conditions there - except, of course, in the limiting case when there is no distinction between envelope and core, we can, nevertheless, readily define the total mass, the radius, etc., of the model. The outer surface is taken to be that radius $r=R$ where the pressure vanishes. In other words, the surface corresponds to the smallest positive value η_s for which

$$\phi(\eta_s) = 0 \; , \qquad (3.30)$$

and its radius is given by

$$R = \alpha_1 \eta_s \; . \qquad (3.31)$$

Similarly the total mass will be given by

$$M = 4\pi\rho_{g_c}\alpha_1^3 v_1(\eta_s) \; . \qquad (3.32)$$

At the interface, the value r_i of the radius will be

$$r_i = a_1 \eta_i , \tag{3.33}$$

and the interfacial values of the pressure, rest-mass density of the gas, and the total energy density will be, respectively,

$$p_i = K_1 \rho_{g_c}^{1+\frac{1}{n_1}} \phi_i^{n_1+1} , \tag{3.34}$$

$$\rho_{g_{i_1}} = \rho_{g_c} \phi_i^{n_1} , \tag{3.35}$$

and

$$\rho_i c^2 = \rho_{g_{i_1}} c^2 + A_1 p_i = \rho_{g_c} \phi_i^{n_1} c^2 + A_1 K_1 \rho_{g_c}^{1+\frac{1}{n_1}} \phi_i^{n_1+1} \tag{3.36}$$

We may also express the mass inside the interfacial radius r_i by

$$M_i = 4\pi \rho_{g_c} a_1^3 v_1(\eta_i) . \tag{3.37}$$

(III) INTERFACIAL BOUNDARY CONDITIONS

Since the pressure and the density are to be continuous everywhere, and in particular at the interface, the values of these quantities, given by equations (3.19) and (3.20) must be identical, respectively, with those given by (3.34) and (3.36). Thus, for the continuity of the pressure

$$p_i = K(\beta)\rho_{g_i}^{4/3} = K_1 \rho_{g_{i_1}}^{1+\frac{1}{n_1}} . \tag{3.38}$$

Also, from the definitions of σ and σ_1 given by equations (3.8) and (3.23), we have

$$\frac{\sigma_1}{\sigma} = \frac{K_1 \rho_{g_c}^{1/n_1}}{K(\beta)\rho_{g_c}^{1/3}} = \frac{K_1}{K(\beta)}\left(\rho_{g_c}\right)^{1/n_1 - 1/3} . \tag{3.39}$$

Hence, from equation (3.38) we obtain

$$\frac{\sigma_1}{\sigma} = \frac{\rho_{g_i}^{4/3}}{\rho_{g_{i_1}}^{1+1/n_1}} \rho_{g_c}^{1/n_1 - 1/3} , \tag{3.40}$$

which becomes, on using the definitions of θ and ϕ,

$$\frac{\sigma_1}{\sigma} = \frac{\theta_i^{4}}{\phi_i^{n_1+1}} . \tag{3.41}$$

From equations (3.20) and (3.36), for the continuity of the density, it follows that

$$\rho_i c^2 = \rho_{g_i} c^2 + Ap_i = \rho_{g_{i_1}} c^2 + A_1 p_i \ . \tag{3.42}$$

Hence,

$$\rho_{g_c} \theta_i^{\,3} c^2 + Ap_i = \rho_{g_c} \phi_i^{\,n_1} c^2 + A_1 p_i \ ,$$

and so

$$\theta_i^{\,3} = \phi_i^{\,n_1} + [A_1 - A] \frac{p_i}{\rho_{g_c} c^2} \ ,$$

and consequently, from equation (3.19),

$$\theta_i^{\,3} = \phi_i^{\,n_1} + [A_1 - A] \sigma \theta_i^{\,4} \ . \tag{3.43}$$

From equation (3.43) it follows that in the classical limit ($\sigma \to 0$), equation (3.41) becomes

$$\frac{\sigma_1}{\sigma} = \frac{1}{\theta_i} \ \frac{3-n_1}{n_1} \ . \tag{3.44}$$

Since, at the interface, the respective values of r and M given by equations (3.17) and (3.33), and (3.21) and (3.37), must be identical, it follows that

$$r_i = a \xi_i = a_1 \eta_i \ , \tag{3.45}$$

and

$$M_i = 4\pi \rho_{g_c} a^3 v(\xi_i) = 4\pi \rho_{g_c} a_1^{\,3} v_1(\eta_i) \ ,$$

and hence

$$\alpha^3 v(\xi_i) = \alpha_1{}^3 v_1(\eta_i) \ . \qquad (3.46)$$

From the definitions of α and α_1, equations (3.45) and (3.46) become, respectively,

$$\eta_i = \frac{\alpha}{\alpha_1} \xi_i = \left[\frac{4\ \sigma}{(n_1+1)\sigma_1}\right]^{1/2} \xi_i \ , \qquad (3.47)$$

and

$$v_1(\eta_i) = \frac{\alpha}{\alpha_1}^3 v(\xi_i) = \left[\frac{4\ \sigma}{(n_1+1)\sigma_1}\right]^{3/2} v(\xi_i) . \qquad (3.48)$$

Thus, for given values of ξ_i and σ, the interfacial values $\eta_i, \phi_i, v_1(\eta_i)$ and also σ_1 can be determined. These values provide the necessary (interfacial) boundary conditions to be satisfied in solving equations (3.28) and (3.29).

The method of solution involved can be summarized as follows. The equations of hydrostatic equilibrium for the core (3.14) and (3.15) are solved for particular values of σ. Then, given a value ξ_i of ξ (and thus given θ_i and $v(\xi_i)$), the boundary conditions to be imposed upon the equation of equilibrium for the envelope (3.28) and (3.29), are

(i) $\phi_i^{n_1} = \theta_i^3 + (A-A_1)\sigma\theta_i^4$,

(ii) $\sigma_1 = \frac{\theta_i^4}{\phi_i^{n_1+1}}\sigma$,

(iii) $\eta_i = \left[\frac{4\sigma}{(n_1+1)\sigma_1}\right]^{1/2} \xi_i$,

(iv) $v_i(\eta_i) = \left[\frac{4\sigma}{(n_1+1)\sigma_1}\right]^{3/2} v(\xi_i)$.

Numerical results are given in Table I at the end of this chapter (pages 102-105).

(IV) PHYSICAL PARAMETERS

(i) Mass and Radius

Assuming that the equations (3.28) and (3.29) have been solved subject to the above mentioned boundary conditions, then, as already stated in equations (3.31) and (3.32), the total mass and radius are

$$M = 4\pi\rho_{g_c}\alpha_1^3 v_1(\eta_s), \qquad (3.49)$$

$$R = \alpha_1\eta_s . \qquad (3.50)$$

From these two quantities a mass-radius relation may be derived, thus

$$M = 4\pi\rho_{g_c}\frac{R}{\eta_s}\alpha_1^2 v_1(\eta_s) ,$$

and using equation (3.27) for a_1 we obtain

$$R_{s/R} = \frac{2GM}{Rc^2} = \frac{2(n_1+1)\sigma_1 v_1(\eta_s)}{\eta_s} , \qquad (3.51)$$

for the ratio of the Schwarzschild radius to the coordinate radius.

(ii) Components of the Metric Tensor

For the envelope, equation (2.20) may be reduced to (cf. equation (2.37)),

$$2\sigma_1(n_1+1)\frac{d\phi}{dr} + [1+(1+A_1)\sigma_1\phi]\frac{d\nu}{dr} = 0 ,$$

and integrating this equation, with respect to r, between the limits r=r and r=R we obtain,

$$e^{\nu} = e^{\nu(R)}[1+(1+A_1)\sigma_1\phi]^{\frac{-2(n_1+1)}{1+A_1}} ,$$

which becomes, since there is continuity with the Schwarzschild exterior solution,

$$e^{\nu} = [1+(1+A_1)\sigma_1\phi]^{\frac{-2(n_1+1)}{1+A_1}}\left(1 - \frac{2GM}{Rc^2}\right) . \qquad (3.52)$$

Similarly for the core, equation (2.20) gives,

$$8\sigma\frac{d\theta}{dr} + [1+(1+A)\sigma\,\theta]\frac{d\nu}{dr} = 0\ ,$$

and on integrating with respect to r, between the limits $r=0$ and $r=r$,

$$e^{\nu} = e^{\nu(o)}\left[\frac{1+(1+A)\sigma}{1+(1+A)\sigma\theta}\right]^{8/1+A}, \tag{3.53}$$

where $\nu(o)$ is the value of ν at the centre. But, unlike the case of a complete model, for which the constant of integration was determined by inserting surface values into (3.53), we appeal instead to the continuity of ν across the interface. Thus, using (3.52) and (3.53), we have

$$e^{\nu(r_i)} = [1+(1+A_1)\,\sigma_1\phi_i]^{\frac{-2(n_1+1)}{1+A_1}}\ 1-\frac{2GM}{Rc^2} = e^{\nu(o)}\left[\frac{1+(1+A)\sigma}{1+(1+A)\sigma\,\theta_i}\right]^{8/1+A}, \tag{3.54}$$

and so

$$e^{\nu(o)} = [1+(1+A_1)\,\sigma_1\phi_i]^{\frac{-2(n_1+1)}{1+A_1}}\left(1-\frac{2GM}{Rc^2}\right)\left[\frac{1+(1+A)\,\sigma\,\theta_i}{1+(1+A)\,\sigma}\right]^{8/1+A}. \tag{3.55}$$

Consequently, in the core we have,

$$e^{\nu} = [1+(1+A_1)\,\sigma_1\phi_i]^{\frac{-2(n_1+1)}{1+A_1}}\left[\frac{1+(1+A)\ \theta_i}{1+(1+A)\ \ \theta}\right]^{8/1+A}\left(1-\frac{2GM}{Rc^2}\right),\ (\theta\geq\theta_i) \tag{3.56}$$

and in the envelope,

$$e^{\nu} = [1+(1+A_1)\,\sigma_1\phi]^{\frac{-2(n_1+1)}{1+A_1}}\left(1-\frac{2GM}{Rc^2}\right). \qquad (3.57)$$

From these equations it is seen that $e^{\nu} < 1$ for all values of ξ_i, and $e^{\nu(o)}$ is a minimum value, i.e. $e^{\nu(o)} \overset{i}{<} e^{\nu} < 1$ for all ξ_i.

From equation (2.23) for $e^{-\lambda}$ we have,

$$e^{-\lambda} = 1 - \frac{2GM_r}{rc^2},$$

which becomes, for the core,

$$e^{-\lambda} = 1 - 8\sigma\frac{v(\xi)}{\xi}, \qquad (3.58)$$

and for the envelope,

$$e^{-\lambda} = 1-2(n_1+1)\,\sigma_1\frac{v_1(\eta)}{\eta}. \qquad (3.59)$$

The transformation equations (from the envelope variables to the core variables) (3.47) and (3.48) ensure the continuity of $e^{-\lambda}$ across the interface. Also it is clear that $e^{\lambda} \geq 1$ for every value of ξ_i, equality occurring at the centre of the configuration. Inserting surface values into equations (3.57) and (3.59) we find that

$$e^{\nu(R)} = e^{-\lambda(R)} = \left(1-\frac{2GM}{Rc^2}\right) = \frac{1-2(n_1+1)\,\sigma_1 v_1(\eta_s)}{\eta_s}, \qquad (3.60)$$

(V) GRAVITATIONAL ENERGY

(i) Total Energy

In accordance with the equivalence of matter and energy in the theory of relativity, the total energy E of a body, including the internal energy, is Mc^2, where M is the total mass of the body. (It may be determined, in principle, by measuring the force exerted on a unit mass at a large distance from the system and then using Newton's inverse square law of gravitational attraction). Thus,

$$E = Mc^2 = \int_0^R 4\pi\rho c^2 r^2 dr , \qquad (3.61)$$

where R is the coordinate radius of the body and ρc^2 is the total energy-density.

(ii) Proper Energy

The proper energy E_0 of a body is defined as the integral of the total energy-density taken over elements of proper volume $e^{\lambda/2} r^2 \sin\theta dr d\theta d\phi$. For a distribution of matter and energy of coordinate radius R, this energy is given by

$$E_0 = \int_0^R 4\pi\rho c^2 e^{\lambda/2} r^2 dr . \qquad (3.62)$$

Physically, we may interpret this quantity as the total energy exclusive of gravitational potential energy. For, on expanding the right hand side of equation (3.62) to the first order in $\frac{GM_r}{rc^2}$, it follows that[6]

$$E_o \simeq 4\pi \int_o^R \rho c^2 [1 + \frac{2GM_r}{rc^2}] r^2 dr = E + \int_o^R \frac{GM_r dM_r}{r} , \tag{3.63}$$

and the second term on the right hand side is just the work that would have to be done on the system to disperse the total matter and energy to infinity against gravitational forces. In fact we define the gravitational potential energy Ω by

$$E = E_o + \Omega \tag{3.64}$$

In terms of the core and envelope variables $(\xi, \theta, v(\xi))$ and $(\eta, \phi, v, (\eta))$ we obtain for the total energy E of the system, using equations (3.5), (3.11), (3.25), (3.32), (3.58), (3.59),

$$E = 4\pi\rho_{g_c} a_1^3 v_1(\eta_s) c^2 , \tag{3.65}$$

and for the proper energy E_o of the system,

$$E_o = 4\pi\rho_{g_c} c^2 a^3 \int_o^{\xi_i} \frac{\theta^3 \xi^2 [1+A\sigma\theta] d\xi}{[1-8\sigma\frac{v(\xi)}{\xi}]^{1/2}} + 4\pi\rho_{g_c} c^2 a_1^3 \int_{\eta_i}^{\eta_s} \frac{\phi^{n_1}\eta^2[1+A_1\sigma_1\phi]d\eta}{[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}]^{1/2}}, \tag{3.66}$$

where $A = \frac{\beta}{\gamma-1} + 3(1-\beta)$.

(iii) Binding Energy

The energy of all the constituent particles of the gas dispersed to infinity with zero internal energy is given by

$$E_{o_g} = M_{o_g} c^2 = \int_o^R 4\pi\rho_g c^2 e^{\lambda/2} r^2 dr , \tag{3.67}$$

where M_{o_g} , the rest-mass of the gas can, at least in principle, be calculated by counting the constituent particles and multiplying by the appropriate rest-mass.

In terms of the core and envelope variables, equation (3.67) becomes

$$E_{o_g} = 4\pi\rho_{g_c}\alpha^3c^2\int_0^{\xi_i}\frac{\theta^3\xi^2 d\xi}{[1-\frac{8\sigma v(\xi)}{\xi}]^{1/2}} + $$

$$+ 4\pi\rho_{g_c}\alpha_1{}^3c^2\int_{\eta_i}^{\eta_s}\frac{\phi^n\eta^2 d\eta}{[1-\frac{2(n_1+1)v_1(\eta)\sigma}{\eta}]^{1/2}} . \tag{3.68}$$

We define the binding energy E_b as the difference between the energy of the unbound particles dispersed to infinity with zero internal energy and the total energy of the bound system. Hence,

$$E_b = E_{o_g} - E ,$$

or using (3.61) and (3.67)

$$E_b = (M_{o_g}-M)c^2 = \int_0^R 4\pi\rho_g c^2 e^{\lambda/2} r^2 dr - \int_0^R 4\pi\rho c^2 r^2 dr . \tag{3.69}$$

In terms of the dimensionless variables $\xi, \theta, v(\xi)$, $\eta, \phi, v_1(\eta)$, equation (3.69) for the binding energy, on using (3.68), becomes

$$E_b = 4\pi\rho_{g_c}\alpha^3c^2\int_0^{\xi_i} \frac{\theta^3\xi^2 d\xi}{[1-8\sigma v(\xi)/\xi]^{1/2}} +$$

$$+ 4\pi\rho_{g_c}\alpha_1{}^3c^2\int_{\eta_i}^{\eta_s} \frac{\phi^{n_1}\eta^2 d\eta}{[1-2(n_1+1)v_1(\eta)\sigma/\eta]^{1/2}} - Mc^2 \quad . \tag{3.70}$$

It is not apparent from inspection of equation (3.69) whether the binding energy is a positive or a negative quantity. For, although the gas density ρ_g is smaller than the total density, the factor e^{λ} is, in general, greater than unity. Consequently, the sign of the binding energy can only be ascertained by detailed calculation.

It was pointed out in Chapter 1 that the binding energy plays a fundamental role in determining the stability (or instability) of a given configuration, but before we consider this question we shall analyse in detail the functional dependence of E_b on the central density and the position of the interface.

We begin by noting that, in the particular case when the interface is at the outer surface, so that there is no envelope, equation (3.70) becomes

$$(E_b)_{\xi_s} = 4\pi\rho_{g_c}\alpha^3c^2\int_0^{\xi_s} \frac{\theta^3\xi^2 d\xi}{[1-8\sigma \frac{v(\xi)}{\xi}]^{1/2}} - mc^2, \tag{3.71}$$

where m is the total mass of this model. In the expression, $(E_b)_{\xi_s}$ is just the binding energy of the complete model the equation of state throughout being given by (3.5). The total mass m is thus given by

$$m = 4\pi\rho_{g_c}\alpha^3 v(\xi_s) \ . \qquad (3.72)$$

For the difference in the binding energies of the composite model and the complete model we find, using equations (3.70), (3.71) and (3.72),

$$E_b-(E_b)_{\xi_s} = 4\pi\rho_{g_c}\alpha^3c^2\int_0^{\xi_s}\frac{\theta^3\xi^2 d\xi}{\left[1-\frac{8\sigma v(\xi)}{\xi}\right]^{1/2}} +$$

$$+ 4\pi\rho_{g_c}c^2\left[\alpha_1^3\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2 d\eta}{\left[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}\right]^{1/2}}-\alpha^3\int_{\xi_i}^{\xi_s}\frac{\theta^3\xi^2 d\xi}{\left[1-\frac{8\sigma v(\xi)}{\xi}\right]^{1/2}}\right]$$

$$-Mc^2 - \left[4\pi\rho_{g_c}\alpha^3c^2\int_0^{\xi_s}\frac{\theta^3\xi^2 d\xi}{\left[1-\frac{8\sigma v(\xi)}{\xi}\right]^{1/2}}-4\pi\rho_{g_c}\alpha^3c^2 v(\xi_s)\right] ,$$

and hence, using (3.49),

$$E_b-(E_b)_{\xi_s} = 4\pi\rho_{g_c}\alpha^3c^2v(\xi_s) - 4\pi\rho_{g_c}\alpha_1^3v_1(\eta_s)$$

$$-4\pi\rho_{g_c}c^2\alpha^3\int_{\xi_i}^{\xi_s}\frac{\theta^3\xi^2 d\xi}{[1-\frac{8\sigma v(\xi)}{\xi}]^{1/2}} + 4\pi\rho_{g_c}c^2\alpha_1^3\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2 d\eta}{[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}]^{1/2}} \tag{3.73}$$

In this equation we see that, corresponding to each term that refers to the composite configuration, there is a term (in the core variables) that applies to the complete configuration (with a change in sign). Thus the result of any transformation of a composite configuration term can immediately be written down in terms of a similar transformation of the corresponding complete configuration term with the appropriate change of variables (and sign). And so, using equation (3.29), we have

$$4\pi\rho_{g_c}c^2\alpha_1^3\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2 d\eta}{[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}]^{1/2}}$$

$$= 4\pi\rho_{g_c}c^2\alpha_1^3\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\frac{d\eta}{[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}]^{1/2}(1+A_1\sigma_1\phi)}$$

$$= 4\pi\rho_{g_c} c^2 \alpha_1^3 \int_{\eta_i}^{\eta_s} \frac{dv_1}{d\eta}\, d\eta$$

$$+ 4\pi\rho_{g_c} c^2 \alpha_1^3 \int_{\eta_i}^{\eta_s} \frac{dv_1}{d\eta} \left\{ \frac{1}{[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}]^{1/2}(1+A_1\sigma_1\phi)} - 1 \right\} d\eta$$

which becomes, after simple manipulation,

$$4\pi\rho_{g_c} c^2 \alpha_1^3 \int_{\eta_i}^{\eta_s} \frac{\phi^{n_1}\eta^2 d\eta}{[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}]^{1/2}} = 4\pi\rho_{g_c} c^2 \alpha_1^3 [v_1(\eta_s) - v_1(\eta_i)]$$

$$+ 4\pi\rho_{g_c} c^2 \alpha_1^3 \int_{\eta_i}^{\eta_s} \frac{dv_1}{d\eta} \left\{ \frac{1}{[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}]^{1/2}} - 1 \right\} d\eta$$

$$- 4\pi\rho_{g_c} c^2 \alpha_1^3 \int_{\eta_i}^{\eta_s} \frac{A_1\sigma_1\phi\, {}^{dv_1}/_{d\eta}\, d\eta}{[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}]^{1/2}(1+A_1\sigma_1\phi)} \tag{3.74}$$

Using this equation and the corresponding equation for the complete model terms together with the interfacial boundary condition (3.45), equation (3.73) for the difference in the binding energies becomes

$$E_b-(E_b)_{\xi_s} = 4\pi\rho_{g_c} c^2 \left\{ a_1^3 \int_{\eta_i}^{\eta_s} \frac{dv_1}{d\eta} \left[\frac{1}{\left(1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}\right)^{1/2}} -1 \right] d\eta \right.$$

$$- a^3 \int_{\xi_i}^{\xi_s} \frac{dv}{d\xi} \left[\frac{1}{\left(1-\frac{8\sigma v(\xi)}{\xi}\right)^{1/2}} -1 \right] d\xi$$

$$\left. -a_1^3 \int_{\eta_i}^{\eta_s} \frac{A_1\sigma_1\phi \, {dv_1}/{d\eta} \, d\eta}{\left[1-\frac{2(n_1+1)\sigma_1 v_1(\eta)}{\eta}\right]^{1/2}(1+A_1\sigma_1\phi)} + a^3 \int_{\xi_i}^{\xi_s} \frac{A\sigma\theta \, {dv}/{d\xi} \, d\xi}{\left[1-\frac{8\sigma v(\xi)}{\xi}\right]^{1/2}(1+A\sigma\theta)} \right\}.$$

(3.75)

On expanding, we find that, in the classical limit,

$$E_b^{(1)}-(E_b)^{(1)}_{\xi_s} = 4\pi\rho_{g_c} c^2 \left\{ a_1^3(n_1+1)\sigma_1 \int^{\eta_s} \frac{v_1}{\eta}\frac{dv_1}{d\eta} d\eta \right.$$

$$\left. - a^3 4\sigma \int_{\xi_i}^{\xi_s} \frac{v}{\xi}\frac{dv}{d\xi} d\xi - a_1^3 A_1 \sigma_1 \int_{\eta_i}^{\eta_s} {}^{\eta_i} \phi \frac{dv_1}{d\eta} + a^3 A\sigma \int_{\xi_i}^{\xi_s} \theta \frac{dv}{d\xi} d\xi \right\}$$

(3.76)

the superscript 1 denoting classical values. This formula gives (in the classical limit), the excess in the binding energy of a composite model over that of the complete (no envelope) model with the same central density, the internal energy being included in the mass density.

To evaluate this expression we consider the quantity $I_{n_1}(\eta_s)$ defined by

$$I_{n_1}(\eta_s) = (n_1+1)\int_{\eta_i}^{\eta_s} \frac{v_1}{\eta}\frac{dv_1}{d\eta}\,d\eta - A_1\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\,d\eta \,. \quad (3.77)$$

Using equation (3.29) in the classical limit,

i.e. $\frac{dv_1}{d\eta} = \phi^{n_1}\eta^2$, we have

$$A_1\int_{\eta_i}^{\eta_s}\phi\,\frac{dv_1}{d\eta}d\eta = A_1\int_{\eta_i}^{\eta_s}\phi^{n_1+1}\,\eta^2 d\eta \,,$$

and on integrating by parts we find that

$$A_1\int_{\eta_i}^{\eta_s}\phi\,\frac{dv_1}{d\eta}d\eta = \frac{A_1}{3}\left[\eta^3\phi^{n_1+1}\right]_{\eta_i}^{\eta_s} - A_1\frac{(n_1+1)}{3}\int_{\eta_i}^{\eta_s}\phi^{n_1}\eta^3\,\frac{d\phi}{d\eta}\,d\eta \,.$$

Using the generalization of the Lane-Emden equation (3.28) in the classical limit we obtain

$$A_1\int_{\eta_i}^{\eta_s} \phi \frac{dv_1}{d\eta} d\eta = -\frac{A_1}{3}[\eta_i{}^3\phi_i{}^{n_1+1}] + A_1\frac{(n_1+1)}{3}\int_{\eta_i}^{\eta_s} \frac{v_1}{\eta}\frac{dv_1}{d\eta} d\eta , \tag{3.78}$$

where we have used the condition that $\phi(\eta_s) = 0$ at the surface. Hence equation (3.77) becomes, on using (3.78),

$$I_{n_1} = (n_1+1)\left[1-\frac{A_1}{3}\right]\int_{\eta_i}^{\eta_s} \frac{v_1}{\eta}\frac{dv_1}{d\eta} d\eta + \frac{A_1}{3}\eta_i{}^3\phi_i{}^{n_1+1} . \tag{3.79}$$

Using this formula, together with a similar formula in terms of the core variables, in equation (3.76) we obtain

$$E_b{}^{(1)} - (E_b)^{(1)}_{\xi s} = 4\pi\rho_{g_c}c^2\left\{\alpha_1{}^3\sigma_1\frac{(n_1+1)}{3}(3-A_1)\times\right.$$

$$\times\int_{\eta_i}^{\eta_s} \frac{v_1}{\eta}\frac{dv_1}{d\eta} d\eta - \alpha^3\sigma\frac{4}{3}(3-A)\int_{\xi_i}^{\xi_s} \frac{v}{\xi}\frac{dv}{d\xi}\, d + \alpha_1{}^3\sigma_1\frac{A_1}{3}\eta_i{}^3\phi_i{}^{n_1+1}$$

$$\left. - \alpha^3\sigma\frac{A}{3}\xi_i{}^3\theta_i{}^4\right\} . \tag{3.80}$$

From the interfacial boundary conditions (3.41) and (3.45) it follows that

$$E_b^{(1)}-(E_b)^{(1)}_{\xi_s} = 4\pi\rho_{g_c}c^2\left\{a_1^3\sigma_1\frac{(n_1+1)}{3}(3-A_1)\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\frac{dv_1}{d\eta}d\eta - a^3\sigma\frac{4}{3}(3-A)\int_{\xi_i}^{\xi_s}\frac{v}{\xi}\frac{dv}{d\xi}d\xi + a_1^3\sigma_1\eta_i^3\phi_i^{n_1+1}\left(\frac{A_1}{3}-\frac{A}{3}\right)\right\}. \tag{3.81}$$

If the interface is at the centre of the configuration, equation (3.81) gives

$$(E_b)^{(1)}-(E_b)^{(1)}_{\xi_s} = 4\pi\rho_{g_c}c^2\left\{\sigma_1 a_1^3(n_1+1)\frac{(3-A_1)}{3}\int_0^{\eta_s}\frac{v_1}{\eta}\frac{dv_1}{d\eta}d\eta - a^3\sigma\frac{4}{3}(3-A)\int_0^{\xi_s}\frac{v}{\xi}\frac{dv}{d\xi}d\xi\right\}, \tag{3.82}$$

and this is just the difference in binding energies of two complete models, one being a configuration for which the equation of state is given by (2.31) and the other being a configuration whose equation of state is given by (3.5). If $\beta \simeq 0$, then from (3.7) it follows that $A \simeq 3$ and hence the equation of state of the core is approximately identical in the form with that of an adiabatic fluid of index 3. Consequently,

$$(E_b)^{(1)}_{\xi_s} = 0 \, , \tag{3.83}$$

in accordance with the usual classical result[1]. Thus for an adiabatic fluid sphere of index n_1, we have $A_1 = \frac{1}{\gamma-1} = n_1$, and (3.83) becomes

$$E_b^{(1)} = 4\pi\rho_{g_c} c^2 \sigma_1 a_1^3 (n_1+1) \frac{(3-n_1)}{3} \int_0^{\eta_s} \frac{v_1}{\eta} \frac{dv_1}{d\eta} d\eta \, . \tag{3.84}$$

Hence, in terms of the mass inside coordinate radius r, we have

$$E_b^{(1)} = \frac{3-n_1}{3} \int_0^R \frac{GM_r dM_r}{r} \, ,$$

and this becomes on using (3.63) and (3.64),

$$E_b^{(1)} = \frac{n_1-3}{3} \Omega \, , \tag{3.85}$$

which is just the usual expression for the binding energy (in the classical limit) in terms of the gravitational potential energy.[1,6]

If $\beta \simeq 0$ in the core (so that (3.83) holds), it follows from equation (3.81) that

$$E_b^{(1)} = 4\pi\rho_{g_c}\sigma_1 a_1^3 c^2 \left\{ \frac{(n_1+1)}{3}(3-A_1)\int_{\eta_i}^{\eta_s} \frac{v_1}{\eta}\frac{dv_1}{d\eta}d\eta + \eta_i^3 \phi_i^{n_1+1}\left(\frac{A_1}{3}-1\right) \right\} . \tag{3.86}$$

In particular, if the envelope corresponds to that of an adiabatic fluid of index n_1 so that $A_1 = \frac{1}{\gamma-1} = n_1$, equation (3.86) gives the following expression for the (classical) binding energy of a complete model

$$E_b^{(1)} = 4\pi\rho_{g_c}\sigma_1 a_1^3 c^2 \left\{ \frac{n_1+1}{3}(3-n_1)\int_{\eta_i}^{\eta_s} \frac{v_1}{\eta}\frac{dv_1}{d\eta}d\eta + \eta_i^3 \phi_i^{n_1+1}\left(\frac{n_1}{3}-1\right) \right\} . \tag{3.87}$$

In terms of the dimensionless envelope variables $(\eta, \phi, v_1(\eta))$ equation (I.7) of Appendix I becomes

$$\frac{5-n_1}{3} G\,(4\pi\rho_{g_c}a_1^3)^2 \frac{1}{a_1}\int_{\eta_i}^{\eta_s} \frac{v_1(\eta)}{\eta}\frac{dv_1(\eta)}{d\eta}d\eta$$
$$= G(4\pi\rho_{g_c}a_1^3)^2\left(\frac{v_1(\eta_s)^2}{a_1\eta_s} - \frac{v_1(\eta_i)^2}{a_1\eta_i}\right)$$

$$+(n_1+1)4\pi\rho g_c a_1{}^3\sigma_1 c^2 v_1(\eta_i)\phi_i-(n_1+1)\frac{4\pi\rho}{3}g_c a_1{}^3\sigma_1 c^2\phi_i{}^{n_1+1}\eta_i{}^3 ,$$

which may be written,

$$\frac{5-n_1}{3}\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\frac{dv_1}{d\eta} = [\frac{v_1(\eta_s)^2}{\eta_s} - \frac{v_1(\eta_i)^2}{\eta_i}] + v_1(\eta_i)\phi_i - \frac{\eta_i{}^3}{3}\phi_i{}^{n_1+1} . \quad (3.88)$$

Consequently, equation (3.87) for the binding energy (in the classical limit) of a composite model in which the equation of state in the core is such that equation (3.83) holds, the equation of state for the envelope being that of an adiabatic fluid of index n_1 , becomes

$$E_b{}^{(1)} = 4\pi\rho g_c\sigma_1 a_1{}^3 c^2 J\eta_i , \quad (3.89)$$

where

$$J\eta_i = \frac{(n_1+1)(3-n_1)}{5-n_1}\left\{\left(\frac{v_1(\eta_s)^2}{\eta_s} - \frac{v_1(\eta_i)^2}{\eta_i}\right) + v_1(\eta_i)\phi_i\right\} + \frac{2(n_1-3)}{5-n_1}\phi_i{}^{n_1+1}\eta_i{}^3 . \quad (3.90)$$

We may consider $J\eta_i$ as a 'measure' of the classical binding energy (of the composite model) as a function of the position of the interface, given the central rest-density and the central pressure.* The graph of $J\eta_i$ (for various indices n_1) as a function of the position of the interface is shown in Fig.1. For a given index n_1 of the envelope, we see that the binding energy decreases as the position of the interface lies farther from the centre. In Newtonian theory the condition for marginal stability of an adiabatic fluid is $\gamma = 4/3$ (which corresponds to a polytrope of index $n=3$) and the condition for instability is $\gamma < 4/3$ (or $n_1 > 3$).(3) The condition for instability is equivalent to $E_b \leq 0$. In other words in Newtonian theory, a negative binding energy is a necessary and sufficient condition for the instability of an adiabatic fluid sphere, and the higher the binding energy the more stable the model.† This follows from the fact that the binding energy is the amount of energy required to disperse the constituent particles of the system to infinity against gravity. Thus a system with zero binding energy corresponds to marginal stability, and a tightly bound system has

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*In passing, we see from equations (3.68) and (3.70) that $J\eta_i$ appears in the post-Newtonian term for the difference in the internal proper energies of the models, i.e.

$$E_{o_g} - (E_{o_g})_{\xi_s} = 4\pi\rho_{g_c} c^2 [\alpha_1{}^3 v_1(\eta_s) - \alpha^3 v(\xi_s)] + 4\pi\rho_{g_c} c^2 \alpha_1{}^3 \sigma_1 J\eta_i \, .$$

†As mentioned in Chapter 1, this is not so in general relativity.

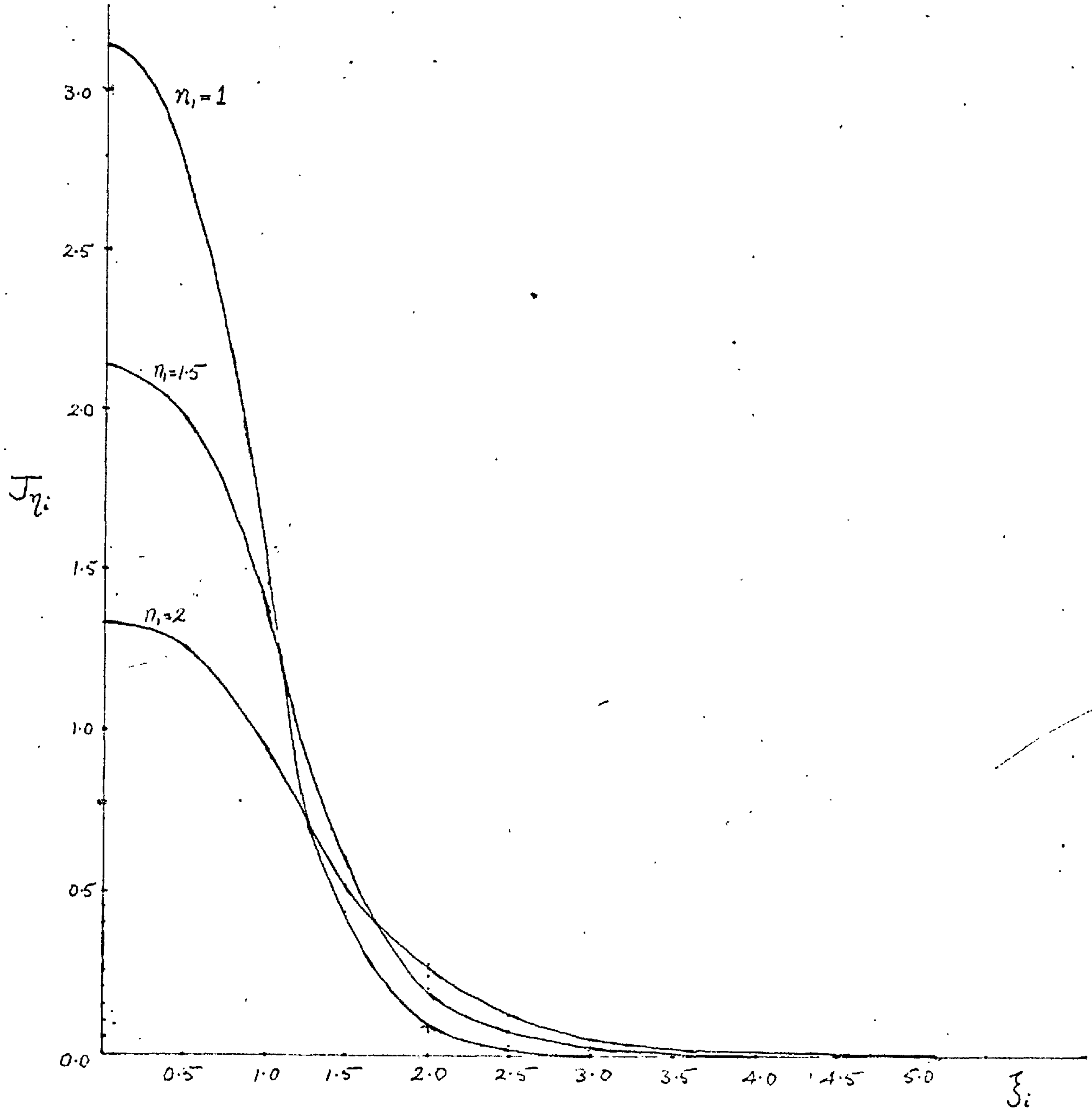

FIG. I.— The 'measure' of the classical binding energy J_{η_i} defined in equation (3.98) versus the position of the interface (ξ_i), for various values of n_1. The maximum value of each curve is seen to correspond with $\xi_i = 0$, i.e. when the interface is at the centre and hence the configuration has no core.

a high binding energy. From Fig.1 we see that, for a given index n_1 in the envelope, a model for which the interface is nearer the centre is more stable than a similar model (same central pressure and density) with the interface farther from the centre.

When the post-Newtonian terms are taken into consideration, instabilities can occur even when the binding energy is positive.(1,7) The effect of an envelope on the magnitude of the binding energy will now be considered from the standpoint of general relativity. Equation (3.75) becomes, in the post-Newtonian approximation,

$$(E_b)-(E_b)_{\xi_s}=4\pi\rho_{g_c}c^2\left\{\alpha_1^3(n_1+1)\sigma_1\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\frac{dv_1}{d\eta}\right.$$

$$+\alpha_1^3\,\tfrac{3}{2}(n_1+1)^2\sigma_1^2\int_{\eta_i}^{\eta_s}\frac{v_1^2}{\eta^2}\frac{dv_1}{d\eta}d\eta+\alpha^3A\sigma\int_{\xi_i}^{\xi_s}\frac{dv}{d}\theta[4\sigma\frac{v}{\xi}-A\sigma\,\theta]d\xi$$

$$-\alpha^3 4\sigma\int_{\xi_i}^{\xi_s}\frac{v}{\xi}\frac{dv}{d\xi}d\xi-\alpha^3 24\,\sigma^2\int_{\xi_i}^{\xi_s}\frac{v^2}{\xi^2}\frac{dv}{d\xi}d\xi+\alpha^3A\sigma\int_{\xi_i}\frac{dv}{d\xi}\theta d\xi$$

$$-\alpha_1{}^3 A_1 \sigma_1 \int_{\eta_i}^{\eta_o} \frac{dv_1}{d\eta}\phi\, d\eta - \alpha_1{}^3 A_1 \sigma_1 \int_{\eta_i}^{\eta_s} \frac{dv_1}{d\eta}\phi\left[(n_1+1)\frac{\sigma_1 v_1}{\eta} - A_1\sigma_1\phi\right]d\eta \Bigg\}, \tag{3.91}$$

from which we obtain*

$$(E_b)-(E_b)_{\xi_s} = 4\pi\rho_{g_c} c^2 \alpha_1{}^3 \sigma_1 \Bigg\{ \frac{n_1}{3}\eta_i{}^3 \phi_i{}^{n_1+1} + (n_1+1)\left(\frac{n_1}{3}-1\right)$$

$$\times \int_{\eta_i}^{\eta_s} \frac{\phi^{n_1}\eta^2}{\eta}\left(\eta^2 \frac{d\phi}{d\eta}\right) d\eta - \frac{\alpha^3}{\alpha_1{}^3}\frac{\sigma}{\sigma_1}\xi_i{}^3\theta_i{}^4 \Bigg\} + 4\pi\rho_{g_c}{}^2 \alpha_1{}^3 \sigma_1{}^2 \Bigg\{ \frac{n_1+1}{2}\phi_i{}^{n_1+1}$$

$$\times \eta_i{}^4 \left(\frac{d\phi}{d\eta}\right)_i - (n_1+1)\eta_i{}^3 \phi_i{}^{n_1+2} - \frac{3(n_1+1)}{2}\int_{\eta_i}^{\eta_s} \phi^{2n_1+1}\eta^4 d\eta - 3(n_1+1)$$

$$\times \int_{\eta_i}^{\eta_s} \phi^{n_1+2}\eta^2 d\eta \Bigg\} - 4\pi\rho_{g_c} c^2 \alpha^3 \sigma^2 \Bigg\{ 2\theta_i{}^4 \xi_i{}^4 \left(\frac{d\theta}{d\xi}\right)_i - 4\xi_i{}^3\theta_i{}^5$$

$$-6\int_{\xi_i}^{\xi_s} \theta^7 \xi^4 d\xi - 12\int_{\xi_i}^{\xi_s} \theta^5 \xi^2 d\xi \Bigg\}. \tag{3.92}$$

This may be written in the form

$$(E_b-(E_b)_{\xi_s} = 4\pi\rho_{g_c}\sigma_1 \alpha_1{}^3 c^2 J\eta_i{}^{(1)}$$

$$-4\pi\rho_{g_c}\alpha_1{}^3\sigma_1{}^2 c^2 \Bigg\{ \frac{n_1+1}{2}\phi_i{}^{n_1+1}\eta_i{}^2 v_1(\eta_i) + (n_1+1)\eta_i{}^3\phi_i{}^{n_1+2}$$

*The derivation of formula (3.92) will be found in Appendix II.

$$+\frac{3}{2}(n_1+1)\int_{\eta_i}^{\eta_s} \phi_i^{2n_1+1}\eta_i^4 d\eta + 3(n_1+1)\int_{\eta_i}^{\eta_s} \phi^{n_1+2}\eta^2 d\eta \Bigg\}$$

$$+4\pi\rho_{g_c} a^3\sigma^2 c^2 \left\{2\theta_i^4 \xi_i^2 v(\xi_i) + 4\xi_i^3\theta_i^5 + 6\int_{\xi_i}^{\xi_s} \theta^7\xi^4 d\xi + 12\int_{\xi_i}^{\xi_s} \theta^5\xi^2 d\xi\right\},$$

(3.93)

where

$$J\eta_i^{(1)} = \left\{\frac{n_1}{3}\eta_i^3\phi_i^{n_1+1} + (n_1+1)\left(\frac{n_1}{3}-1\right)\int_{\eta_i}^{\eta_s} \frac{\phi^{n_1}\eta^2}{\eta}\left(\eta^2\frac{d\phi}{d\eta}\right)d\eta - \frac{a^3}{a_1^3}\frac{\sigma}{\sigma_1}\xi_i^3\theta_i^4\right\}$$

(3.94)

It is easily verified that in the classical limit equation (3.93) reduces to equation (3.89).

Using the interfacial boundary conditions (3.43), (3.44), (3.45) and (3.46), equations (3.93) and (3.94) yield

$$E_b-(E_b)_{\xi_s} = 4\pi\rho_{g_c} a_1^3\sigma_1 c^2 J\eta_i^{(1)} - 4\pi\rho_{g_c} a_1^3\sigma_1^2 c^2 (n_1-3)\eta_i^3\phi_i^{n_1+2}$$

$$-4\pi\rho_{g_c} a_1^3\sigma_1^2 c^2 \left\{\frac{3}{2}(n_1+1)\int_{\eta_i}^{\eta_s} \phi^{2n_1+1}\eta^4 d\eta + 3(n_1+1)\int_{\eta_i}^{\eta_s} \phi^{n_1+2}\eta^2 d\eta\right\}$$

$$+4\pi\rho_{g_c} a^3\sigma^2 c^2 \left\{6\int_{\xi_i}^{\xi_s} \theta^7\xi^4 d\xi + 12\int_{\xi_i}^{\xi_s} \theta^5\xi^2 d\xi\right\}, \qquad (3.95)$$

where

$$J\eta_i^{(1)} = (n_1+1)\left(\frac{n_1}{3}-1\right)\int_{\eta_i}^{\eta_s} \frac{\phi^{n_1}\eta^2}{\eta}\left(\eta^2\frac{d\phi}{d\eta}\right)d\eta + \left(\frac{n_1}{3}-1\right)\eta_i^3\phi_i^{n_1+1}. \tag{3.96}$$

This is the desired expression for the difference in the binding energies of (1) a composite model for which the equation of state in the core corresponds to that for a fluid sphere whose equation of state is (3.5) with $A=3$, and for which the equation of state in the envelope is that for an adiabatic fluid of index n_1; and (2) a complete model for which the equation of state throughout is the same as that of the core in (1).

Before giving any numerical results, we shall check the above against that obtained by Fowler[7] for a complete model consisting of a mixture of ideal gas and radiation in which the ratio of the gas pressure to the total pressure is extremely small. Clearly, from equations (3.95) and (3.96), when either (i) the interface extends to the surface, or (ii) $n_1=3$, it follows that $E_b-(E_b)_{\xi_s}=0$, as expected. When the interface extends to the centre, i.e. $\xi_i \to 0$, $\eta_i \to 0$, we have, for the difference between the binding energies of two complete models, one being a sphere with equation of state of the form $p=K_1\rho_g^{1+\frac{1}{n_1}}$ and the other an adiabatic sphere of index 3,

$$E_b-(E_b)_{\xi_s} = 4\pi\rho_{g_c}\alpha_1^3\sigma_1 c^2 J\eta_s^{(1)}$$

$$-4\pi\rho_{g_c}\alpha_1^3\sigma_1^2c^2\left\{\tfrac{3}{2}(n_1+1)\int_0^{\eta_s}\phi^{2n_1+1}\eta^4 d\eta + 3(n_1+1)\int_0^{\eta_s}\phi^{n_1+2}\eta^2 d\eta\right\}$$

$$+4\pi\rho_{g_c}\alpha^3\sigma^2c^2\left\{6\int_0^{\xi_s}\theta^7\xi^4 d\xi + 12\int_0^{\xi_s}\theta^5\xi^2 d\xi\right\} . \qquad (3.97)$$

A massive sphere in which $\beta\sim 0$ corresponds to one with equation of state (3.5), A (defined in equation (3.7)) being equal to 3 and hence the classical binding energy is zero. Since the third term in equation (3.97) does not depend on n_1 , we should expect that the first and second terms correspond to E_b and the third to $(E_b)_{\xi_s}$, viz.

$$-(E_b)_{\xi_s}=4\pi\rho_{g_c}\alpha^3\sigma^2c^2\left\{6\int_0^{\xi_s}\theta^7\xi^4 d\xi+12\int_0^{\xi_s}\theta^5\xi^2 d\xi\right\}. \qquad (3.98)$$

Now, Fowler's expression for the total energy E of a fluid sphere with $\beta \sim 0$

$$E = \frac{8\pi G}{c^2}\int_0^R prM_r dr + \frac{6\pi G^2}{c^2}\int_0^R \rho M_r^2 dr \,, \qquad (3.99)$$

which becomes on introducing the dimensionless variables defined in equations (3.8), (3.9), (3.10), (3.11), and (3.12),

$$E = \frac{8\pi G}{c^2} 4\pi\rho_{g_c}\alpha^3\sigma c^2\rho_{g_c}\alpha^2 \int_0^{\xi_s} \theta^4 \xi v(\xi) d\xi + \frac{6\pi G^2}{c^2}\rho_{g_c}(4\pi\rho_{g_c}\alpha^3)^2\alpha \int_0^{\xi_s} \theta^3 v^2 d\xi$$

$$= 4\pi\rho_{g_c}\alpha^3\sigma^2 c^2 \left[\frac{8\pi G}{\sigma c^2}\rho_{g_c}\frac{\sigma c^2}{\pi G\rho_{g_c}} \int_0^{\xi_s} \theta^4 \xi v(\xi) d\xi + \frac{6\pi G^2}{c^2}\rho_{g_c}\frac{4\pi\rho_{g_c}\alpha^4}{\sigma^2 c^2} \int_0^{\xi_s} \theta^3 v^2 d\xi \right]$$

Hence, to the first post-Newtonian approximation,

$$E = 4\pi\rho_{g_c}\alpha^3\sigma^2 c^2 \left\{ -8 \int_0^{\xi_s} \xi^3\theta^4 \frac{d\theta}{d\xi} d\xi + 24 \int_0^{\xi_s} \theta^3 \xi^4 \left(\frac{d\theta}{d\xi}\right)^2 d\xi \right\}. \qquad (3.100)$$

Now

$$\int_0^{\xi_s} \theta^4 \xi^3 \frac{d\theta}{d\xi} d\xi = -\frac{1}{4}\int_0^{\xi_s} \xi^4 \left[\theta^4 \frac{d^2\theta}{d\xi^2} + 4\theta^3 \left(\frac{d\theta}{d\xi}\right)^2\right] d\xi \,,$$

and, to the order of approximation considered,

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2\theta^3 ,$$

and thus

$$-\xi^2\frac{d^2\theta}{d\xi^2} = \xi^2\theta^3 + 2\xi\frac{d\theta}{d\xi} .$$

Consequently,

$$\int_0^{\xi_s} \theta^4\xi^2\frac{d\theta}{d\xi}d\xi = -\int_0^{\xi_s} \theta^3\xi^4\left(\frac{d\theta}{d\xi}\right)^2 d\xi + \frac{1}{4}\int_0^{\xi_s} \theta^7\xi^4 d\xi + \frac{1}{2}\int_0^{\xi_s} \xi^3\theta^4\frac{d\theta}{d\xi}d\xi ,$$

and hence

$$\int_0^{\xi_s} \theta^3\xi^4\left(\frac{d\theta}{d\xi}\right)^2 d\xi = \frac{1}{4}\int_0^{\xi_s} \theta^7\xi^4 d\xi - \frac{1}{2}\int_0^{\xi_s} \xi^3\theta^4\frac{d\theta}{d\xi}\, d\xi . \qquad (3.101)$$

Using this result, equation (3.100) can be rewritten in the form

$$E = 4\pi\rho_{g_c} a^3\sigma^2 c^2 \left\{ 6\int_0^{\xi_s} \theta^7\xi^4 d \;\; -20\int_0^{\xi_s} \xi^3\theta^4\frac{d\theta}{d\xi}\, d\xi \right\} ,$$

which becomes, on integrating the last integral by parts,

$$E = 4\pi\rho_{g_c} a^3\sigma^2 c^2 \left\{ 6\int_0^{\xi_s} \theta^7\xi^4 d\xi + 12\int_0^{\xi_s} \xi^2\theta^5 d\xi \right\}, \qquad (3.102)$$

which is identical with equation (3.98), the required result. Thus, as expected, equation (3.102) represents the negative binding energy $-(E_b)_{\xi_s}$ of the complete configuration considered.

Using this result, we can readily obtain the binding energy of the composite model under consideration. For, from equations (3.95) and (3.102), the binding energy of this model is given by

$$E_b = 4\pi\rho_{g_c}\alpha_1{}^3\sigma_1 c^2 J\eta_i{}^{(1)} - 4\pi\rho_{g_c}\alpha_1{}^3\sigma_1{}^2 c^2(n_1-3)\eta_i{}^3\phi_i{}^{n_1+2}$$

$$-4\pi\rho_{g_c}\alpha_1{}^3\sigma_1{}^2 c^2\left\{\tfrac{3}{2}(n_1+1)\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4 d\eta + 3(n_1+1)\int_{\eta_i}^{\eta_s}\phi^{n_1+2}\eta^2 d\eta\right\}$$

$$-4\pi\rho_{g_c}\alpha^3\sigma^2 c^2\left\{6\int_0^{\xi_i}\theta^7\xi^4 d\xi + 12\int_0^{\xi_i}\xi^2\theta^5 d\right\}, \qquad (3.103)$$

which may be conveniently written in the form

$$\frac{E_b}{4\pi\rho_{g_c}\alpha_1{}^3\sigma_1 c^2} = J\eta_i{}^{(1)} - \sigma_1\prod_{n_1}. \qquad (3.104)$$

Using expression (3.32) for the total mass M and equation (3.41), we finally obtain the following useful formula for the binding energy of the composite model considered,

$$\frac{E_b}{Mc^2} = \frac{\sigma}{v_1(\eta_s)} \left(\frac{\theta_i^{\,4}}{\phi_i^{\,n_1+1}}\right) \left[J\eta_i^{(1)} - \sigma\left(\frac{1}{\theta_i}\right)^{\frac{3-n_1}{n_1}} \Big] \Big|_{n_1}\right] . \tag{3.105}$$

(VI) NUMERICAL RESULTS

In solving the equations of equilibrium for various positions of the interface and various values of n_1 , we shall here assume that β , the ratio of the gas pressure to the total pressure, is extremely small in the core*. From consideration of the graphs given by Tooper[1] for the binding energy of complete models, the maximum binding energy of a composite configuration as a function of the parameter σ may be expected to occur for small values of σ , and so we solve numerically (with the aid of the trapezoidal rule) the equations of hydrostatic equilibrium (3.28) and (3.29), assuming that $\sigma \ll 1$, for various positions of the interface ξ_i , subject to the boundary conditions (3.43), (3.47) and (3.48). The surface

*Various non-zero values of β will be considered in Chapter 4.

value η_s for which $\phi(\eta_s) = 0$ is determined subject to the approximation $\phi(\eta_s) < 10^{-3}$. Although this approximate method is rather rough, the results obtained give a general picture of the models under consideration, and seem to be intuitively reasonable. A selection of the values obtained in this way for η, ϕ and $v_1(\eta)$ for various ξ_i is given in Table I.

Fig.2 shows graphs of the post-Newtonian term $\int\Big|_{n_1}$, in the formula (3.104) for the dimensionless binding energy as a function of ξ_i. We see that, for a given n_1, $\int\Big|_{n_1}$ decreases as ξ_i increases, which means that this term has a greater effect on the binding energy of the composite model the nearer the interface is to the centre.

On page 81 above, $J\eta_i$ was defined as the classical 'measure' of the binding energy, since this quantity indicates the change in the classical binding energy given the central rest-density and central pressure, for various values of n_1 and various positions of the interface. In the same way, from equation (3.104), we may define $J\eta_i^{(1)} - \sigma_1 \int\Big|_{n_1}$ as the <u>relativistic 'measure' of the binding energy</u>. Since $\int\Big|_{n_1}$ decreases rapidly as ξ_i increases for $\xi_i \leq 2$,

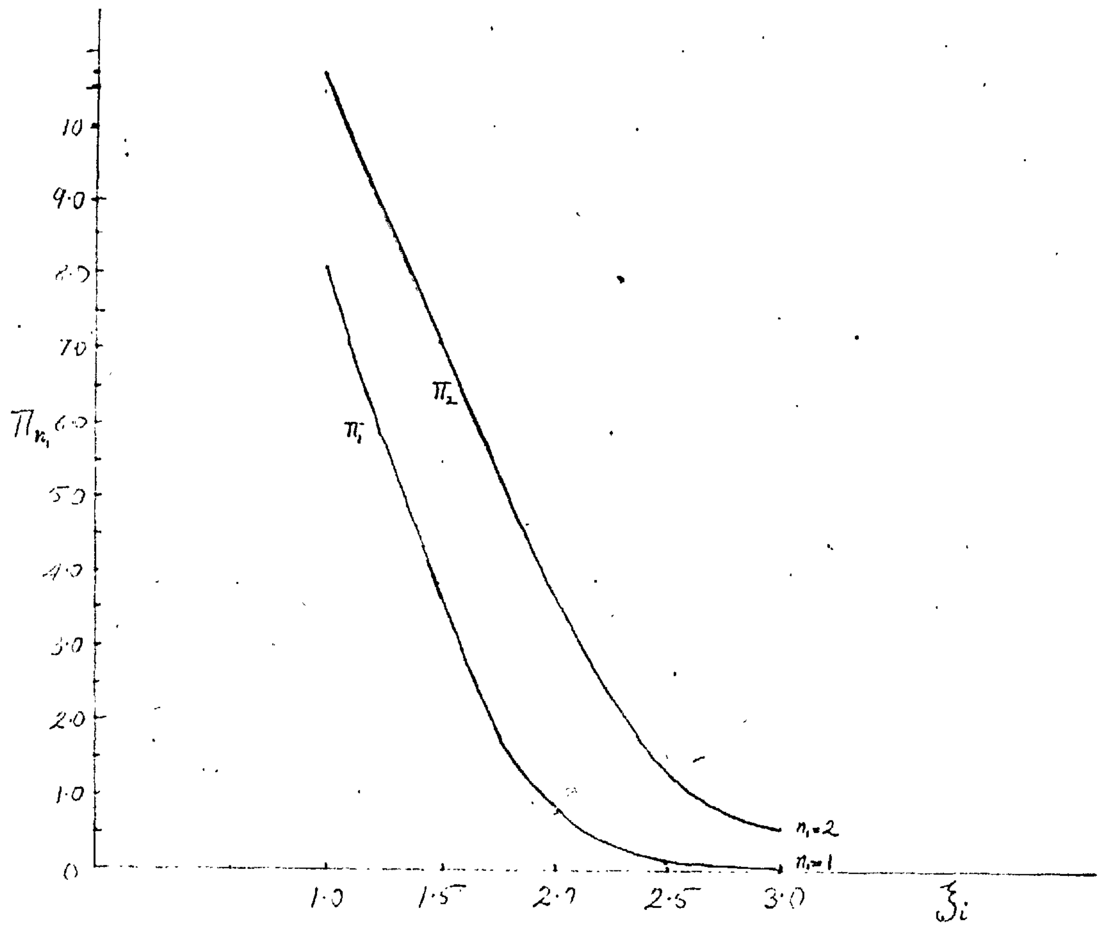

Fig. 2 — The first post-Newtonian term in the dimensionless form of the binding energy defined in equation (3.105). As in Fig. 1 (for the classical term), it is seen that Π_1 is a decreasing function of ξ_i.

and tapers off for larger values of ζ_i , and since $\Pi_{n_1} > \Pi_{n_2}$ for $n_1 > n_2$ for a given value of ζ_i , it appears that, for small values of ζ_i , $J\eta_i^{(1)} - \sigma_1 \Pi_{n_1}$ decreases with increasing n_1 as in the classical case, whereas for larger values of ζ_i it is not possible to make this inference since the value of $J\eta_i^{(1)} - \sigma_1 \Pi_{n_1}$ is more sensitive to the value of σ_1 . But we can say that for large σ_1 , and $\zeta_i \gtrsim 2.5$ (which implies $J\eta_i$ small), this 'measure' of the binding energy is negative.

In Fig.3 and Fig.4 the binding energy per unit mass is displayed as a function of the parameter σ for various positions of the interface (ζ_i), n_1 being 1 in Fig.3 and n_1 being 2 in Fig.4. We see that for a given n_1 in the envelope, the binding energy decreases with increasing ζ_i for the range of values considered. In other words, for a given value of n_1 , the nearer the interface is to the centre the larger the binding energy. The above conclusions concerning the so-called 'measure' of the binding energy can be extended to the actual binding energy, and indeed for $\zeta_i \geq 2.5$ (at least for $n_1 = 1$) the value of the binding energy is sensitive to the value of σ , even for

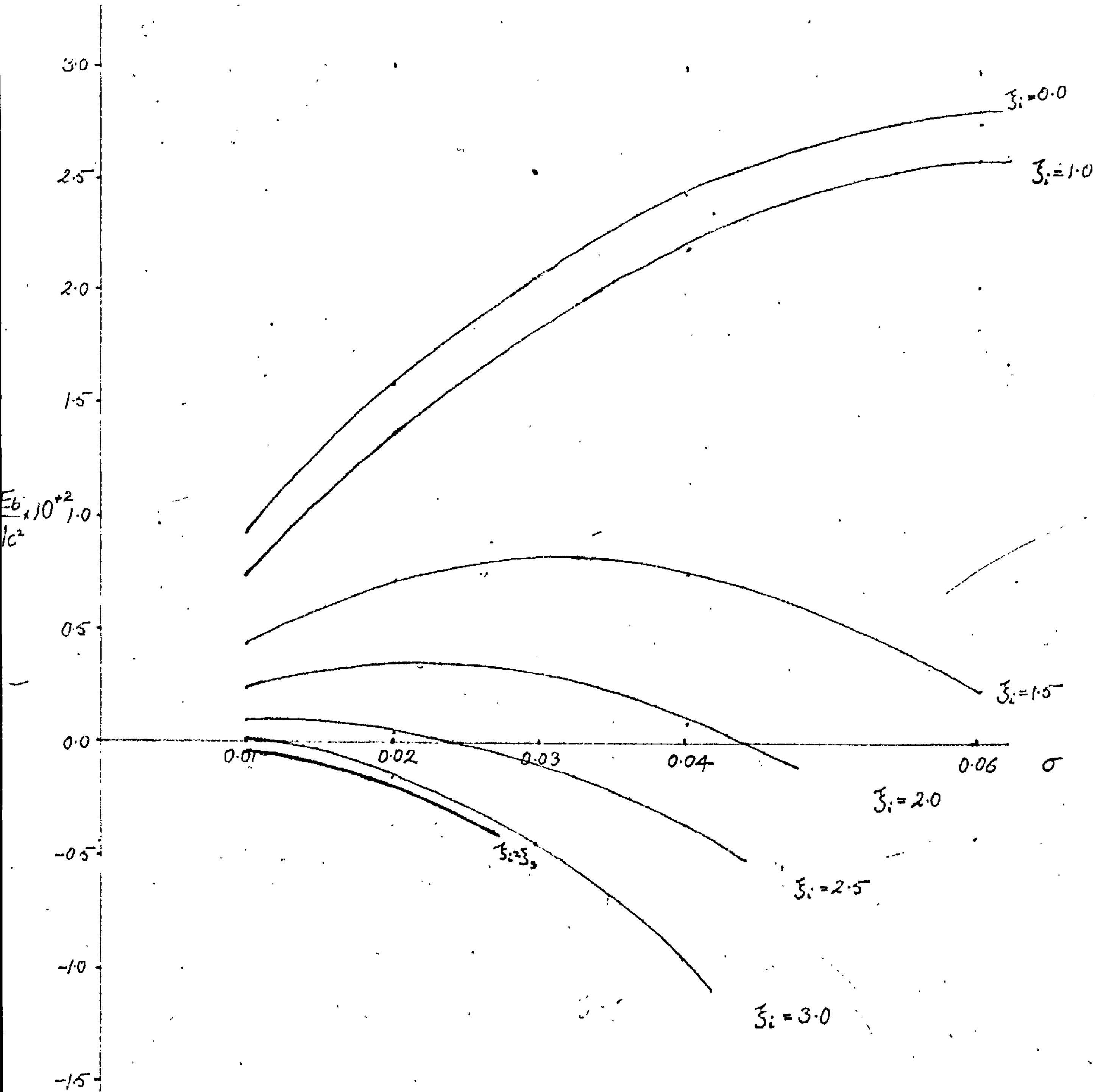

Fig. 3 — The dimensionless binding energy given by equation (3.114) versus the parameter σ, for $n_1 = 1$ in the envelope, for various positions of the interface ξ_i.

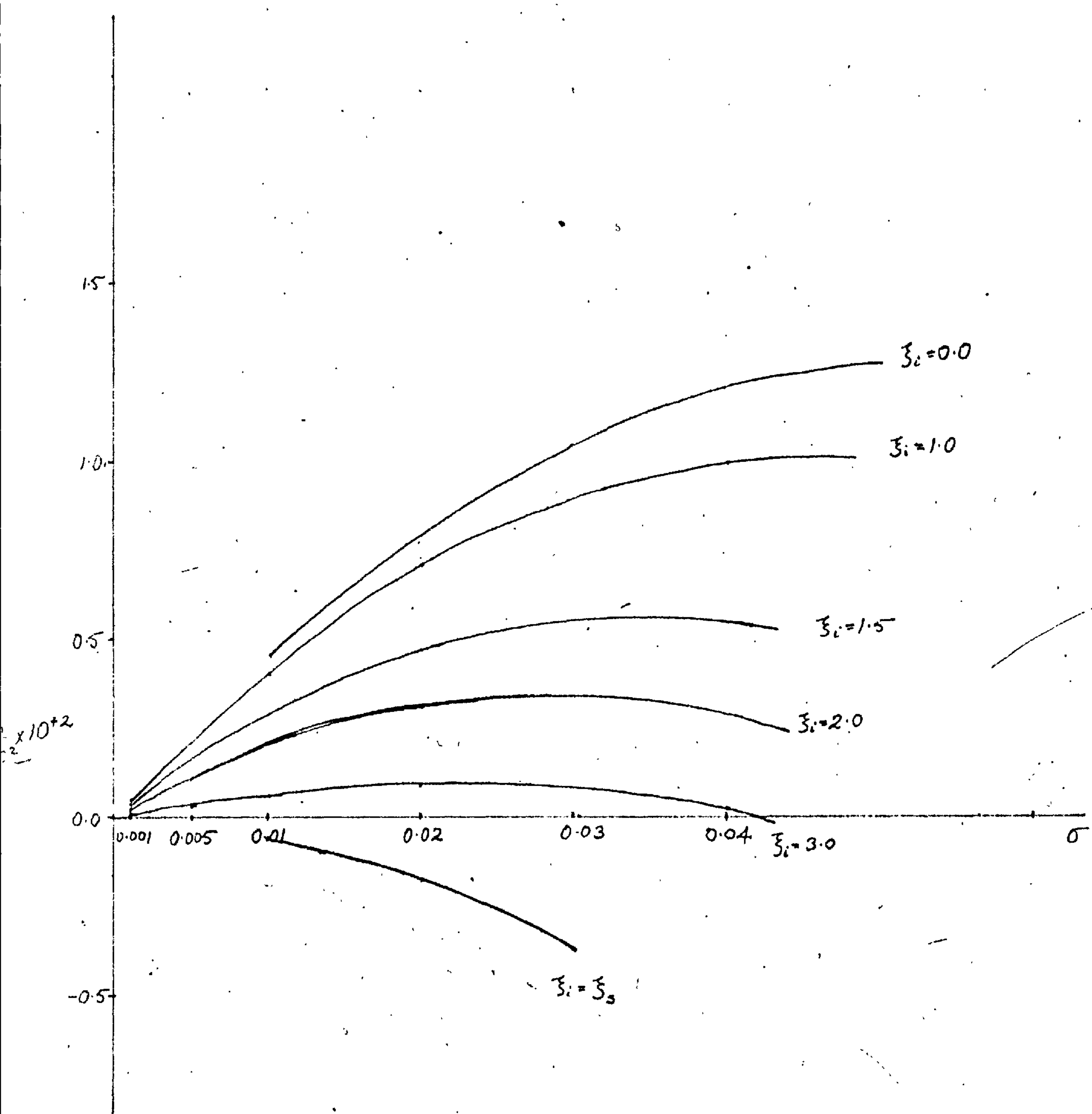

Fig. 4 – The dimensionless binding energy given by equation (3.114) versus the parameter σ, for $n_1 = 2$ in the envelope, for various positions of the interface.

small values of this parameter.

For complete models, using Chandrasekhar's variational principle[8], Tooper[4,5,6] has shown that instability sets in at the first peak of the binding energy as a function of σ. If the same were true for composite models, it would mean that in Fig. 3 instability would occur at the value of σ for which the binding energy is a maximum for a given model, and the model is unstable for larger values of σ, even though the binding energy is positive.

The total energy of a fluid sphere exclusive of the rest-mass energy when infinitely dispersed from its equilibrium state is equal in magnitude but opposite in sign to the binding energy, and this allows us to give a simple explanation of the onset of instability at the maximum of the binding energy regarded as a function of σ. Suppose σ_m is the value of σ at which the binding energy is a maximum or at which the internal energy required for hydrostatic equilibrium is a minimum. Then, if we consider the adiabatic expansion of a model for which $\sigma > \sigma_m$, the binding energy would be increased; in other words, the equilibrium energy required after expansion would be less than that required before the expansion, and so further expansion would ensue. On the other hand, for a model for which $\sigma < \sigma_m$, the opposite is true: after expansion more internal energy would be required to maintain equilibrium, but since this is not forthcoming (it being assumed that there is no energy generation in the core) the expansion stops. Consider next adiabatic contraction. In a configuration for which $\sigma > \sigma_m$, the binding energy would be reduced and hence the energy required for hydrostatic equilibrium would be increased; since this energy is not made available in the adiabatic contraction, further collapse must ensue. Again, for a configuration for which $\sigma < \sigma_m$, the opposite would be the case. Following contraction, less internal energy would be required to maintain equilibrium, and since this excess energy cannot be emitted contraction stops. Thus we see that σ_m , the value of σ corresponding to maximum binding energy, may be regarded as the critical value of σ at which instability sets in.

From Fig.3 and Fig.4, we can also see how the position of the interface affects stability. For a given n_1 in the envelope, as ξ_i increases, (i.e. as the model consists of more and more core) the maximum in the binding energy as a function of σ moves to the left of the diagram, i.e. occurs for smaller values of σ. Moreover, for large values of ξ_i (i.e. ξ_i close to ξ_s), the binding energy is always negative. For $n_1=3$ (or equivalently $\xi_i = \xi_s$), the classical binding energy is zero, as can be seen from equations (3.89) and (3.90) or (3.96) and (3.104), and the post-Newtonian terms are negative. Thus, in this case, the binding energy is always negative, and these objects are unstable over the full range of values of σ. But even in the case of small ξ_i , the models can become unstable, for sufficiently large values of σ, even when the binding energy is positive.

The application of an envelope to a core (for which $n_1=3$) has the effect of increasing the binding energy and produces a peak in the graph representing it as a function of σ. The smaller the interfacial radius, the higher is this peak and the larger the value of σ_m at which it occurs. For a given ξ_i , we find that, the smaller the value of n_1 , the larger the value of σ_m at which the peak in the binding energy occurs.

From the above considerations we can draw the following general conclusions. Given a core consisting of matter and radiation in which β, the ratio of the gas pressure to the total pressure, is an extremely small constant (such a core may be regarded as a classical polytrope of index $n=3$) and an envelope fitted onto this core subject to the usual interfacial continuity conditions, the envelope being an adiabatic spherical shell of index $n_1 < 3$, we conclude that the envelope has a significant influence on the stability of the whole system in the sense that, the smaller the interfacial radius, the greater the range of central density compatible with stability.

TABLE I

SUMMARY OF SOLUTIONS OF EQUATIONS (3.28) AND (3.29)
FOR VARIOUS ξ_i .

$n_1=1$

ξ_i	θ_i	$v(\xi_i)$	η_i	ϕ_i	$v_1(\eta_i)$	η	ϕ	$v_1(\eta)$
0.0	1.0	0.0	0.0	1.0	0.0	1.0	0.84	0.30
0.0						2.0	0.45	1.74
0.0						3.0	0.05	3.11
0.0						3.14	0.0	3.14
0.5	0.96	0.04	0.68	0.88	0.09	0.9	0.84	0.2
0.5						1.5	0.64	0.82
0.5						2.0	0.44	1.6
0.5						3.1	0.02	3.0
1.0	0.85	0.25	1.21	0.62	0.44	1.3	0.59	0.54
1.0						1.5	0.52	0.76
1.0						2.0	0.35	1.4
1.0						3.0	0.03	2.5
1.5	0.72	0.63	1.53	0.37	0.66	1.6	0.34	0.75
1.5						2.0	0.23	1.1
1.5						2.5	0.10	1.5
1.5						2.9	0.01	1.7
2.0	0.58	1.05	1.65	0.20	0.58	1.74	0.17	0.6
2.0						2.0	0.12	0.8
2.0						2.4	0.04	0.9
2.0						2.7	0.003	1.0

TABLE I continued

$n_1=1$

ξ_i	θ_i	$v(\xi_i)$	η_i	ϕ_i	$v_1(\eta_i)$	η	ϕ	$v_1(\eta)$
2.5	0.46	1.40	1.63	0.10	0.39	1.7	0.08	0.4
2.5						2.0	0.04	0.48
2.5						2.2	0.02	0.51
2.5						2.4	0.004	0.53
3.0	0.36	1.66	1.52	0.05	0.22	1.6	0.04	0.22
3.0						1.7	0.03	0.23
3.0						1.8	0.02	0.24
3.0						2.1	0.002	0.25

$n_1=1.5$

ξ_i	θ_i	$v(\xi_i)$	η_i	ϕ_i	$v_1(\eta_i)$	η	ϕ	$v_1(\eta)$
0.0	1.0	0.0	0.0	1.0	0.0	1.0	0.84	0.29
0.0						2.0	0.49	1.49
0.0						3.0	0.16	2.56
0.0						3.65	0.0	2.71
0.5	0.96	0.04	0.6	0.92	0.07	1.0	0.80	0.28
0.5						2.0	0.48	1.40
0.5						3.0	0.15	2.50
0.5						3.6	0.01	2.67
1.0	0.85	0.25	1.06	0.73	0.3	1.2	0.69	0.40
1.0						2.0	0.45	1.20
1.0						3.0	0.17	2.2
1.0						3.7	0.02	2.4

TABLE I continued

n_1=1.5

ξ_i	θ_i	$v(\xi_i)$	η_i	ϕ_i	$v_1(\eta_i)$	η	ϕ	$v_1(\eta)$
1.5	0.72	0.63	1.31	0.52	0.42	1.5	0.46	0.55
1.5						2.0	0.34	0.96
1.5						3.0	0.13	1.64
1.5						3.9	0.001	1.8
2.0	0.58	1.05	1.39	0.34	0.34	1.6	0.3	0.42
2.0						2.0	0.22	0.65
2.0						3.0	0.11	1.0
2.0						4.1	0.006	1.2
2.5	0.46	1.40	1.32	0.21	0.2	1.5	0.2	0.24
2.5						2.0	0.15	0.33
2.5						3.0	0.08	0.53
2.5						4.4	0.005	0.68
3.0	0.36	1.66	1.2	0.13	0.10	2.0	0.08	0.17
3.0						3.0	0.05	0.27
3.0						4.0	0.02	0.35
3.0						5.2	0.001	0.38

n_1=2

ξ_i	θ_i	$v(\xi_i)$	η_i	ϕ_i	$v_1(\eta_i)$	η	ϕ	$v_1(\eta)$
0.0	1.0	0.0	0.0	1.0	0.0	1.0	0.85	0.27
0.0						2.0	0.52	1.30
0.0						3.0	0.24	2.16
0.0						4.35	0.0	2.41

TABLE I continued

$n_1=2$

ζ_i	θ_i	$v(\zeta_i)$	η_i	ϕ_i	$v_1(\eta_i)$	η	ϕ	$v_1(\eta)$
0.5	0.96	0.04	0.5	0.94	0.04	1.0	0.8	0.3
0.5						2.0	0.5	1.2
0.5						3.0	0.25	2.1
0.5						4.3	0.003	2.4
1.0	0.85	0.25	0.99	0.8	0.24	2.0	0.5	1.2
1.0						3.0	0.23	2.0
1.0						4.0	0.06	2.2
1.0						4.48	0.006	2.27
1.5	0.72	0.63	1.25	0.61	0.36	2.0	0.42	0.9
1.5						3.0	0.23	1.5
1.5						4.0	0.09	1.86
1.5						4.9	0.001	1.9
2.0	0.58	1.05	1.34	0.44	0.32	2.0	0.3	0.6
2.0						3.0	0.2	1.0
2.0						4.0	0.1	1.3
2.0						5.7	0.002	1.44
2.5	0.46	1.40	1.30	0.31	0.21	3.0	0.17	0.60
2.5						4.0	0.1	0.80
2.5						5.0	0.06	0.98
2.5						7.1	0.001	1.05
3.0	0.36	1.66	1.2	0.21	0.12	3.0	0.12	0.3
3.0						5.0	0.06	0.6
3.0						7.0	0.02	0.78
3.0						9.04	0.003	0.81

REFERENCES

(1) Tooper, R.F., *Astrophysical Journal*, 142, 1541, 1965.

(2) Hoyle, F., and Fowler, W.A., *Astrophysical Journal Supplement*, 9, 201, 1964.

(3) Milne, E.A., *Thermodynamics of the Stars*, (Extracted from Handbuch der Astrophysik III), 1930.

(4) Tooper, R.F., *Astrophysical Journal*, 155, 145, 1969.

(5) Tooper, R.F., *Astrophysical Journal*, 143, 465, 1965.

(6) Tooper, R.F., *Astrophysical Journal*, 140, No.2, 434, 1964.

(7) Fowler, W.A., *Reviews of Modern Physics*, 36, 545, 1964.

(8) Chandrasekhar, S., *Astrophysical Journal*, 140, 417, 1964.

CHAPTER 4

THE CRITICAL RADIUS FOR COMPOSITE MODELS

(I) INTRODUCTION

In this chapter we shall determine, for various values of β (ratio of the gas pressure to total pressure), in the core and for different positions of the interface, the critical radius R_c at which instability sets in. We shall again base our analysis on the binding energy of the model, but it will now be considered as a function of R, the total radius of the configuration, instead of σ, the ratio of the central pressure to the central energy-density.

As before, we assume that the core is a mixture of ideal gas and radiation, and so the parameters, dimensionless variables and the equations of hydrostatic equilibrium are given by (3.7), (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14) and (3.15). Also, the envelope is again taken to be an adiabatic shell of index n_1 and so is characterized by the equations (2.31), (3.22), (3.23), (3.25), (3.26), (3.28) and (3.29) with $A_1 = n_1$. Thus the equation of state of the core is given in parametric form by the equations

$$p = K(\beta)\rho_g^{4/3}, \quad \rho c^2 = \rho_g c^2 + \frac{\beta p}{\gamma-1} + 3(1-\beta)p, \tag{4.1}$$

where

$$K(\beta) = \left[\left(\frac{k}{\mu H}\right)^4 \frac{1-\beta}{\beta^4} \frac{3}{a}\right]^{1/3}, \tag{4.2}$$

and where β is now assumed to be a constant greater than zero. In the envelope the equation of state is given by equation (2.26) or (2.31) with $A_1=n_1$, and so in parametric form we have,

$$p = K(\beta)\rho_g^{1+1/n_1}, \tag{4.3}$$

where the energy-density is given by

$$\rho c^2 = \rho_g c^2 + n_1 p . \tag{4.4}$$

(II) BINDING ENERGY

As before, the interfacial boundary conditions are given by equations (3.43), (3.47) and (3.48), and the total binding energy is defined by equation (3.69), namely

$$E_b = E_{o_g} - Mc^2, \tag{4.5}$$

where Mc^2 is the total energy of the system, and

E_{o_g} is given by

$$E_{o_g} = \int_0^R 4\pi\rho_g c^2 e^{\lambda/2} r^2 dr \,. \qquad (4.6)$$

Thus, in terms of the envelope and the core we have,

$$E_{o_g} = 4\pi\int_0^{r_i} [\rho c^2 - \frac{\beta p}{\gamma-1} + 3(1-\beta)p] e^{\lambda/2} r^2 dr + 4\pi\int_{r_i}^R [\rho c^2 - n_1 p] e^{\lambda/2} r^2 dr. \qquad (4.7)$$

Substituting for $e^{\lambda/2}$ from equation (21), i.e. $e^{-\lambda} = 1 - 2GM_r/rc^2$, we have,

$$E_{o_g} = 4\pi\int_0^{r_i} \rho c^2 (1 - \frac{2GM_r}{rc^2})^{-1/2} r^2 dr - 4\pi\int_0^{r_i} [\frac{\beta p}{\gamma-1} - 3(1-\beta)p](1 - \frac{2GM_r}{rc^2})^{-1/2} r^2 dr$$

$$+ 4\pi\int_{r_i}^R \rho c^2 (1 - \frac{2GM_r}{rc^2})^{-1/2} r^2 dr - 4\pi\int_{r_i}^R n_1 p (1 - \frac{2GM_r}{rc^2})^{-1/2} r^2 dr \,. \qquad (4.8)$$

Consequently, the binding energy for the configuration becomes, from equations (4.5) and (4.8) in the post-Newtonian approximation,

$$E_b = 4\pi\int_0^{r_i} \rho c^2 (1 + \frac{GM_r}{rc^2} + \frac{3}{2}\frac{G^2 M_r^2}{r^2 c^4}) r^2 dr - 4\pi\int_0^{r_i} [\frac{\beta p}{\gamma-1} + 3(1-\beta)p](1 + \frac{GM_r}{rc^2}) r^2 dr$$

$$+4\pi\int_{r_i}^{R} \rho c^2\left(1+\frac{GM_r}{rc^2}+\frac{3}{2}\frac{G^2M_r^{\,2}}{r^2c^4}\right)r^2dr-4\pi\int_{r_i}^{R} n_1 p\left(1+\frac{GM_r}{rc^2}\right)r^2dr$$

$$-4\pi\int_{0}^{R} \rho c^2 r^2 dr \; ,$$

and so

$$E_b=\int_0^R \rho\frac{GM_r}{r}dV-\int_0^{r_i}\left[\frac{\beta}{\gamma-1}+3(1-\beta)\right]p\left(1+\frac{GM_r}{rc^2}\right)dV-\int_{r_i}^{R} n_1 p\left(1+\frac{GM_r}{rc^2}\right)dV$$

$$+6\pi\int_0^R \rho\frac{G^2M_r^{\,2}}{c^2}dr \; . \qquad (4.9)$$

(III) CLASSICAL TERM

Since we will be considering models for which β, although no longer negligible, is not greater than about 0.1 in the core, we will make the approximation $\beta=0$ in the post-Newtonian terms (involving the factor $\frac{1}{c^2}$), which will therefore be identical with the post-Newtonian terms for the binding energy given by equation (3.103). With this consideration in mind, and to facilitate numerical integration, the formula (4.9) for E_b will be considered first of all in the classical limit. Denoting the classical binding energy by $E_b^{(1)}$, it follows that

$$E_b(1) = \int_o^R \rho \frac{GM_r}{r} dV - \int_o^{r_i} [\frac{\beta}{\gamma-1} + 3(1-\beta)] p dV - \int_{r_i}^R n_1 p dV .$$

(4.10)

Since β is assumed to be a constant and also from equation (2.20) we find that

$$E_b(1) = \int_o^R \rho \frac{GM_r}{r} dV - [\frac{\beta}{\gamma-1} + 3(1-\beta)][pV]_o^{r_i} - [\frac{\beta}{\gamma-1} + 3(1-\beta)]$$

$$\times \int_o^{r_i} V\rho \frac{GM_r}{r^2} dr - [n_1 pV]_{r_i}^R - n_1 \int_{r_i}^R V\rho \frac{GM_r}{r^2} dr ,$$

and since $V = (4/3)\pi r^3 = 1/3 r \frac{dV}{dr}$,

$$E_b(1) = \int_o^R \rho \frac{GM_r}{r} dV - [\frac{\beta}{\gamma-1} + 3(1-\beta)] p_i V_i - [\frac{\beta}{\gamma-1} + 3(1-\beta)]$$

$$\times \int_o^{r_i} \rho \frac{GM_r}{r} \frac{dV}{3} + n_1 p_i V_i - \frac{n_1}{3} \int_{r_i}^R \rho \frac{GM_r}{r} dV . \quad (4.11)$$

Thus

$$E_b(1) = \beta\left[\frac{3\gamma-4}{3(\gamma-1)}\right] \int_o^{r_i} \rho \frac{GM_r}{r} dV + (1-\frac{n_1}{3}) \int_{r_i}^R \rho \frac{GM_r}{r} dV$$

$$-[\frac{\beta}{\gamma-1} + 3(1-\beta) - n_1] p_i V_i , \quad (4.12)$$

The second integral in (4.12) has already been determined in Appendix I and is given by (I.7), namely

$$-\left(\frac{5-n_1}{3}\right)\Omega_i = G\left(\frac{M^2}{R} - \frac{M_i^{\,2}}{r_i}\right) + (n_1+1)\frac{p_i}{\rho_i}M_i - (n_1+1)p_iV_i ,$$
(4.13)

where Ω_i is defined by, (cf. equation (I.1) of Appendix I)

$$-\Omega_i = \int_{r_i}^{R} \frac{GM_r dM_r}{r} = \int_{r_i}^{R} \frac{GM_r}{r}\rho dV , \qquad (4.14)$$

and dV is an element of volume. The first integral in equation (4.12) is evaluated in Appendix III and is given by

$$-\frac{1}{3}\Omega_c = \frac{1}{3}\int_0^{r_i}\frac{G\rho M_r}{r}dV = \frac{1}{2}\frac{GM_i^{\,2}}{r_i} - \frac{2p_i}{\rho_i}M_i + 2p_iV_i .$$
(4.15)

Defining ω and ψ, respectively, by

$\omega^3 = \dfrac{\rho_i}{\bar{\rho}(r_i)}$, and $\psi = \dfrac{\xi^{1/3}}{4v(\xi_i)^{2/3}}$, where $\bar{\rho}(r_i)$, the average density of the core, is defined by

$$M_i = {}^4/_3\pi\frac{\rho_i}{\bar{\rho}(r_i)} ,$$

it follows from (4.1) that, in the classical limit,

$$\frac{p_i}{\rho_i} = \sigma c^2 \left(\frac{\rho_i}{\rho_c}\right)^{1/3} = \frac{\sigma c^2}{\rho_c^{1/3}} \, \omega\bar{\rho}(r_i)^{1/3} , \qquad (4.16)$$

where $\sigma = {}^{p_c}/\rho_c c^2$. From equations (3.12) and (3.13) it follows that

$$M_i = 4\pi\rho_c a^3 v(\xi_i) \quad \text{and} \quad a^2 = \frac{\sigma c^2}{\pi G \rho_c} ,$$

and hence equation (4.16) becomes,

$$\frac{p_i}{\rho_i} = \frac{\sigma c^2}{\rho_c^{1/3}} \, \omega\left[\frac{4\pi\rho_c a^3 v(\xi_i)}{4/3\pi r_i^3}\right]^{1/3} ,$$

and so

$$\frac{p_i}{\rho_i} = \frac{GM_i}{r_i}\omega\psi . \qquad (4.17)$$

Also, from equation (4.17) we obtain

$$p_i = \frac{GM_i}{r_i}\omega\psi\rho_i = \frac{GM_i}{r_i}\omega\psi . \omega^3\bar{\rho}(r_i) = \frac{GM_i}{r_i}\omega^4\psi \, \frac{M_i \, 3}{4\pi r_i^3} ,$$

and hence

$$p_i = \frac{3}{4\pi} \frac{GM_i^2}{r_i^4}\psi\omega^4 , \qquad (4.18)$$

and also

$$p_i V_i = \frac{GM_i^2}{r_i}\omega^4\psi . \qquad (4.19)$$

On substituting (4.13) and (4.15) in equation (4.12), we obtain

$$E_b^{(1)} = \beta\,\frac{3\gamma-4}{3(\gamma-1)}\,3\left[\frac{1}{2}G\frac{M_i^2}{r_i} - \frac{2p_iM_i}{\rho_i}\right]$$

$$+\left(1-\frac{n_1}{3}\right)\frac{6}{5-n}\left[\frac{1}{2}\left(\frac{GM^2}{R} - \frac{GM_i^2}{r_i}\right) + \frac{(n_1+1)}{2}\,\frac{p_i}{\rho_i}M_i\right]$$

$$+\beta\,\frac{3\gamma-4}{3(\gamma-1)}\,6p_iV_i - \left(1-\frac{n_1}{3}\right)\frac{6}{5-n_1}\,\frac{(n_1+1)}{2}p_iV_i - \left[\frac{\beta}{\gamma-1}+3(1-\beta)-n_1\right]p_iV_i,$$

and so

$$E_b^{(1)} = \frac{3p_i}{\rho_i}\,\frac{M_i(n_1+1)}{5-n_1}\left\{\left(1-\frac{n_1}{3}\right)-\beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{2}{n_1+1}\,5-n_1\right\}$$

$$-\left\{\left(1-\frac{n_1}{3}\right)-\beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{5-n_1}{2}\right\}\frac{3}{5-n_1}\,\frac{GM_i^2}{r_i} + \left(1-\frac{n_1}{3}\right)\frac{6}{5-n_1}\,\frac{1}{2}\,\frac{GM^2}{R}$$

$$+\beta\,\frac{3\gamma-4}{\gamma-1}\,2p_iV_i - \left(1-\frac{n_1}{3}\right)\frac{3}{5-n_1}(n_1+1)p_iV_i - \left[3-n_1+\beta\frac{4-3\gamma}{\gamma-1}\right]p_iV_i\ .$$

Hence

$$E_b^{(1)} = \left\{\left(1-\frac{n_1}{3}\right)-\beta\frac{3\gamma-4}{3(\gamma-1)}\,\frac{2}{n_1+1}5-n_1\right\}\frac{3(n_1+1)}{5-n_1}\,\frac{M_ip_i}{\rho_i}$$

$$-\left\{\left(1-\frac{n_1}{3}\right)-\beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{5-n_1}{2}\right\}\frac{3}{5-n_1}\,\frac{GM_i^2}{r_i} + \beta\,\frac{3\gamma-4}{\gamma-1}\cdot 2p_iV_i$$

$$+\left(1-\frac{n_1}{3}\right)\frac{3}{5-n_1}\,\frac{GM^2}{R} - \left\{\frac{6(3-n_1)}{5-n_1} - \beta\,\frac{3\gamma-4}{\gamma-1}\right\}p_iV_i\ . \qquad (4.20)$$

This equation may be re-written in the form

$$E_b^{(1)} = \left\{(1-\frac{n_1}{3}) - \beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{5-n_1}{2}\right\}\left\{\frac{3(n_1+1)}{5-n_1}\,\frac{M_i p_i}{\rho_i} - \frac{3}{5-n_1}\,\frac{GM_i^2}{r_i}\right\} + (1-\frac{n_1}{3})\frac{3}{5-n_1}\,\frac{GM^2}{R}$$

$$-[\frac{6(3-n_1)}{5-n_1} - \beta\,\frac{3\gamma-4}{\gamma-1}]p_i V_i + \beta\,\frac{3\gamma-4}{\gamma-1}[2p_i V_i - \frac{(3-n_1)}{2}\,\frac{M_i p_i}{\rho_i}] \; . \tag{4.21}$$

Using equations (4.22) and (4.24) in equation (4.26), we obtain

$$E_b^{(1)} = \left\{(1-\frac{n_1}{3}) - \beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{5-n_1}{2}\right\}\left\{(n_1+1)\omega\psi - 1\right\}\frac{3}{5-n_1}\,\frac{GM_i^2}{r_i}$$
$$+(1-\frac{n_1}{3})\frac{3}{5-n_1}\,\frac{GM^2}{R} - [\frac{6(3-n_1)}{5-n_1} - \beta\,\frac{3\gamma-4}{\gamma-1}]\omega^4\psi\frac{GM_i^2}{r_i}$$
$$+\beta\,\frac{3\gamma-4}{\gamma-1}[2\omega^4\psi\frac{GM_i^2}{r_i} - \frac{3-n_1}{2}\omega\psi\frac{GM_i^2}{r_i}] \; . \tag{4.22}$$

On defining a new quantity q as the ratio of the interfacial radius to the total radius R, i.e.

$$q = \frac{r_i}{R} \; , \tag{4.23}$$

equation (4.22) becomes,

$$E_b^{(1)} = \Bigg\{[(1-\frac{n_1}{3}) - \beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{5-n_1}{2}][(n_1+1)\omega\psi - 1] + (1-\frac{n_1}{3})(\frac{M}{M_i})^2 q$$
$$-[2(3-n_1) - \beta\,\frac{(3\gamma-4)}{3(\gamma-1)}(5-n_1)]\omega^4\psi + \beta\,\frac{3\gamma-4}{3(\gamma-1)}\,2(5-n_1)\omega^4\psi$$
$$+\frac{3\gamma-4}{3(\gamma-1)}\,\frac{(3-n_1)(5-n_1)}{2}\,\omega\psi\Bigg\}\frac{3}{5-n_1}\,\frac{GM_i^2}{r_i} \; . \tag{4.24}$$

Denoting the Schwarzschild radius R_s by

$$R_s = \frac{2GM}{c^2},$$

we obtain

$$\frac{GM_i^2}{r_i} = \frac{1}{2}\ \frac{R_s\ M_i^2 c^2}{Mr_i},$$

and so equation (4.24) for the classical binding energy of the composite model (per core mass M_i) gives

$$\frac{E_b^{(1)}}{M_i c^2} = \frac{3}{2(5-n_1)}\left\{[\beta\,\frac{3\gamma-4}{3(\gamma-1)}\ \frac{5-n_1}{2} - (1-\frac{n_1}{3})][1-(n_1+1)\omega\psi]\right.$$
$$+(1-\frac{n_1}{3})(\frac{M}{M_i})^2 q-[2(3-n_1)-\beta\,\frac{3\gamma-4}{(\gamma-1)}\ (5-n_1)]\omega^4\psi$$
$$\left.+\beta\,\frac{3\gamma-4}{3(\gamma-1)}\ \frac{(3-n_1)(5-n_1)}{2}\omega\psi\right\}(\frac{M_i}{M})\frac{R_s}{qR}\ . \qquad (4.25)$$

Formula (4.25) allows us to write down, almost immediately, the classical binding energies of two important complete models, the former, studied by Fowler(1), a mixture of gas and radiation with $\beta<<1$ forming a polytrope of index 3, and the latter, studied by Tooper(2), an adiabatic gas sphere of index n_1 :-

(1) Fowler's model can be obtained from the

above work by letting the interface extend to the surface, so that $M_i \to M$. Consequently, (4.25) gives

$$\frac{E_b^{(1)}}{Mc^2} = \frac{3}{2(5-n_1)} \quad \beta \frac{3\gamma-4}{3(\gamma-1)} \frac{5-n_1}{2} \frac{R_s}{R} ,$$

and hence

$$\frac{E_b^{(1)}}{Mc^2} = \beta \frac{3\gamma-4}{4(\gamma-1)} \frac{R_s}{R} , \qquad (4.26)$$

which is identical with Fowler's expression for the binding energy of a massive star if the ratio of the specific heats γ is equal to $5/3$.[1]

(2) Tooper's model can be obtained from the above work by letting the interface shrink to the centre, so that $M_i \to 0$. Hence,

$$E_b^{(1)} = \lim_{M_i \to 0} \left\{ \frac{3}{2(5-n_1)} \left[[\beta \frac{3\gamma-4}{3(\gamma-1)} \frac{5-n_1}{2} - (1-\frac{n_1}{3})][1-(n_1+1)\psi\omega] \right. \right.$$

$$-[2(3-n_1)-\beta \frac{3\gamma-4}{\gamma-1}(5-n_1)]\omega^4\psi + \beta \frac{3\gamma-4}{3(\gamma-1)} \frac{(5-n_1)(3-n_1)}{2}\omega\psi$$

$$\left. \left. +(1-\frac{n_1}{3})(\frac{M}{M_i})^2 q \right] \frac{M_i^2}{M} \frac{R_s c^2}{qR} \right\} ,$$

and hence

$$E_b^{(1)} = \lim_{M_i \to 0} \frac{3}{2(5-n_1)} \quad (1-\frac{n_1}{3})\frac{MR_s}{R} ,$$

and so

$$E_b^{(1)} = \frac{3-n_1}{5-n_1} \frac{GM^2}{R} , \qquad (4.27)$$

which is identical with that obtained by Tooper for an adiabatic sphere of index n_1.[2]

(IV) CRITICAL RADIUS

The above results can be regarded as useful checks on the validity of the more general formula (4.25) for the binding energy of our composite model. From the relativistic point of view, this classical expression can be regarded as the first term in a power series in the dimensionless parameter $R_s/qR = 2GM/qRc^2$, the post-Newtonian terms being given by the corresponding terms in equation (3.103), since β although not zero is being taken sufficiently small to be replaced by zero in the post-Newtonian terms. Thus the ratio of the binding energy E_b of the composite model to the mass of its core is given, in the first post-Newtonian approximation, by a formula of the type

$$\frac{E_b}{M_i c^2} = \frac{E_b^{(1)}}{M_i c^2} - \int n_1 q^2 (\frac{M}{M_i})(\frac{R_s}{qR})^2 , \qquad (4.28)$$

where the actual form of ζ_{n_1} will be determined later. Before evaluating ζ_{n_1}, we deduce from (4.28) that the binding energy has a maximum at a critical radius R_c given by

$$0=\frac{-3}{2(5-n_1)}\left\{[\beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{5-n_1}{2}-(1-\frac{n_1}{3})][1-(n_1+1)\omega\psi]+(1-\frac{n_1}{3})(\frac{M}{M_i})^2 q\right.$$
$$\left.-[2(3-n_1)-\beta\,\frac{3\gamma-4}{(\gamma-1)}(5-n_1)]\omega^4\psi+\beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{(3-n_1)(5-n_1)}{2}\omega\psi\right\}\frac{M_i}{Mq}\,\frac{R_s}{R_c^2}$$
$$+2\zeta_{n_1}q^2(\frac{M}{M_i})\frac{R_s^2}{q^2R_c^3},$$

i.e. by

$$\frac{R_c}{R_s}=4(5-n_1)\zeta_{n_1}\Big/\frac{3}{q}\left\{[\beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{5-n_1}{2}-(1-\frac{n_1}{3})][1-(n_1+1)\psi\omega]\right.$$
$$+(1-\frac{n_1}{3})(\frac{M}{M_i})^2 q-[2(3-n_1)-\beta\,\frac{3\gamma-4}{\gamma-1}(5-n_1)]\psi\omega^4$$
$$\left.+\beta\,\frac{3\gamma-4}{3(\gamma-1)}\,\frac{(3-n_1)(5-n_1)}{2}\omega\psi\right\}(\frac{M_i}{M})^2 . \tag{4.29}$$

The quantity in curly brackets in the denominator of (4.29), regarded as a function of β and position of the interface will be denoted by $G_{n_1}(\beta,\zeta_i)$. It is tabulated in Table II for $n_1=1$. It is seen that $G_1(\beta,\zeta_i)$ no longer proportional to β as in the corresponding expression in the case of the model studied by Fowler[3], depends strongly on the position of the interface and (for a given β) decreases with increasing values of ξ_i . Also it is seen that $G_1(\xi_i, \beta)$ increases steadily with increasing β .

(V) POST-NEWTONIAN TERM

We shall now evaluate the quantity ζ_{n_1} defined in equation (4.28) by considering the post-Newtonian terms of equation (3.103) for the binding energy of the composite model discussed in Chapter 3. The post-Newtonian terms, denoted by $E_b^{(2)}$, in equation (3.103) are given by

$$E_b^{(2)} = -4\pi\rho_{g_c}\alpha_1^3\sigma_1^2c^2(n_1-3)\eta_i^3\phi_i^{n_1+2}$$
$$-4\pi\rho_{g_c}\alpha_1^3\sigma_1^2c^2\left[\frac{3}{2}(n_1+1)\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4d\eta+3(n_1+1)\int_{\eta_i}^{\eta_s}\phi^{n_1+2}\eta^2d\eta\right]$$
$$-4\pi\rho_{g_c}\alpha^3\ {}^2c^2\left[6\int_0^{\xi_i}\theta^7\xi^4d\xi+12\int_0^{\xi_i}\theta^5\xi^2d\xi\right], \quad (4.30)$$

where $\sigma_1\eta, \phi, \xi, \theta$, etc., have been defined in Section II of Chapter 3. Using these definitions we see that

$$\frac{R_s}{R} = \frac{2GM}{Rc^2} = \frac{2G.4\pi\rho_{g_c}\alpha_1^3v_1(\eta_s)}{c^2\alpha_1\eta_s} ,$$

and hence, on using equation (3.27) for a_1 ,

$$\frac{R_s}{R} = 2G.\frac{4\pi\rho_{g_c}(n_1+1)\sigma_1 c^2}{4\pi G\rho_{g_c} c^2}\frac{v_1(\eta_s)}{\eta_s} = 2(n_1+1)\sigma_1\frac{v_1(\eta_s)}{\eta_s}$$

Consequently,

$$\sigma_1{}^2 = (\frac{R_s}{R})^2 \frac{\eta_s}{v_1(\eta_s)^2 4(n_1+1)^2} \tag{4.31}$$

Also

$$M_i = 4\pi\rho_{g_c} a_1{}^3 v_1(\eta_i) = 4\pi\rho_{g_c} a^3 v(\xi_i) \cdot \tag{4.32}$$

Hence, from equations (4.31) and (4.32) it follows that

$$4\pi\rho_{g_c} a_1{}^3\sigma_1{}^2 c^2 = \frac{M_i}{v_1(\eta_i)} \cdot \frac{\eta_s{}^2 c^2}{v_1(\eta_s)^2 4(n_1+1)^2}(\frac{R_s}{R})^2 \cdot \tag{4.33}$$

Thus equation (4.30) becomes, on using equations (4.31) and (4.33),

$$\frac{E_b{}^{(2)}}{M_i c^2} = -\frac{\eta_s{}^2 \quad (n_1-3)}{v_1(\eta_i)v_1(\eta_s)^2 4(n_1+1)^2}\eta_i{}^3\phi_i{}^{n_1+2}(\frac{R_s}{R})^2$$

$$-\frac{\eta_s{}^2}{v_1(\eta_i)v_1(\eta_s)^2} \cdot \frac{3}{8(n_1+1)}\left[\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4 d\eta + 2\int_{\eta_i}^{\eta_s}\phi^{n_1+2}\eta^2 d\eta\right](\frac{R_s}{R})^2$$

$$-\frac{\eta_s^2}{v_1(\eta_i)v_1(\eta_s)^2}\cdot\frac{3}{2(n_1+1)^2}\left(\frac{a}{a_1}\right)^3\left(\frac{\sigma}{\sigma_1}\right)^2\left[\int_0^{\xi_i}\theta^7\xi^4 d\xi + 2\int_0^{\xi_i}\theta^5\xi^2 d\xi\right]\left(\frac{R_s}{R}\right)^2 . \quad (4.34)$$

But, from equation (4.28), we obtain

$$\zeta_{n_1}\frac{M}{M_i}\left(\frac{R_s}{R}\right)^2 = -\frac{E_b^{(2)}}{M_i c^2},$$

and so

$$\zeta_{n_1} = -\frac{E_b^{(2)}}{Mc^2}\left(\frac{R}{R_s}\right)^2 = -\frac{E_b^{(2)}}{M_i c^2}\left(\frac{R}{R_s}\right)^2\frac{v_1(\eta_i)}{v_1(\eta_s)}. \quad (4.35)$$

Hence, from equation (4.34), it follows that ζ_{n_1} can be expressed as

$$\zeta_{n_1} = \frac{\eta_s^2}{4(n_1+1)^2 v_1(\eta_s)^2}(n_1-3)\eta_i^3\phi_i^{n_1+2}$$

$$+\frac{3\,\eta_s^2}{8(n_1+1)v_1(\eta_s)^3}\left\{\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4 d\eta + 2\int_{\eta_i}^{\eta_s}\phi^{n_1+2}\eta^2 d\eta\right\}$$

$$+\frac{3}{2(n_1+1)^2}\cdot\frac{\eta_s^2}{v_1(\eta_s)^3}\left(\frac{a}{a_1}\right)^3\left(\frac{\sigma}{\sigma_1}\right)^2\left[\int_0^{\xi_i}\theta^7\xi^4 d\xi + 2\int_0^{\xi_i}\theta^5\xi^2 d\xi\right]. \quad (4.36)$$

When the interface extends to the centre, the model becomes a complete model characterized by the equation of state (4.3) and (4.4), and ζ_{n_1} reduces to

$$\zeta_{n_1} = \frac{3\eta_s^2}{8(n_1+1)v_1(\eta_s)^3}\left[\int_{\eta_i}^{\eta_s} \phi^{2n_1+1}\eta^4 d\eta + 2\int_{\eta_i}^{\eta_s} \phi^{n_1+2}\eta^2 d\eta\right]. \tag{4.37}$$

This expression is almost identical with that obtained by Fowler[3]. In the notation used here equation (6) of Fowler's paper becomes

$$\zeta_{n_1} = \frac{3}{8(n_1+1)}\frac{\eta_s^2}{v_1(\eta_s)^3}\left[\int_0^{\eta_s} \phi^{2n_1+1}\eta^4 d\eta + \frac{10}{n_1+2}\int_0^{\eta_s} \phi^{n_1+2}\eta^2 d\eta\right], \tag{4.38}$$

which, except when $n_1=3$, differs from our expression (4.37) by the factor $10/(n_1+2)$ in place of 2 in front of the second integral.

The reason for this slight difference can be readily explained. In Fowler's model the energy density is given by

$$\rho c^2 = \rho_0 c^2 + 3(1-\beta/2)p ,$$

whereas in the present work it is given by

$$\rho c^2 = \rho_g c^2 + A_1 p ,$$

where our ρ_g corresponds to his ρ_0, and $A_1=n_1$

for an adiabatic fluid sphere. Fowler's model was composed of a mixture of gas and radiation (with β taken to be zero in the post-Newtonian terms), whereas the corresponding model here is an adiabatic fluid sphere. When $A_1=3$, there is complete agreement in the post-Newtonian terms.

When the interface extends to the surface, equation (4.36) reduces to

$$\zeta_3 = \frac{3}{2(n_1+1)^2}\,\frac{\eta_s^{\,2}}{v_1(\eta_s)^3}\left(\frac{a}{a_1}\right)^3\left(\frac{\sigma}{\sigma_1}\right)^2\left[\int_0^{\xi_s}\theta^7\xi^4 d\xi + 2\int_0^{\xi_s}\theta^5\xi^2 d\xi\right],$$

which becomes, on using equations (3.47) and (3.48),

$$\zeta_3 = \frac{3}{32}\,\frac{\xi_s^{\,2}}{v(\xi_s)^3}\left[\int_0^{\xi_s}\theta^7\xi^4 d\xi + 2\int_0^{\xi_s}\theta^5\xi^2 d\xi\right], \qquad (4.39)$$

which (apart from the difference in notation) is identical with the particular case of Fowler's formula (4.38) for a complete polytrope of index 3 as expected. Using this result together with equation (4.29) when the interface is at the centre, and taking $\gamma={}^5/_3$, we obtain Fowler's result(3), namely

$$\frac{R_c}{R_s} = \frac{16\zeta_3}{3\beta}. \qquad (4.40)$$

From the definition of $\prod_{n_1}$ in equation (3.104), together with (4.31), we have

$$\frac{E_b}{M_i c^2} = \frac{E_b^{(1)}}{M_i c^2} - \frac{\prod_{n_1} \sigma_1^2}{v_1(\eta_i)} ,$$

and hence, from equation (4.32)

$$\frac{E_b}{M_i c^2} = \frac{E_b^{(1)}}{M_i c^2} - \frac{\prod_{n_1} \eta_s^2}{4(n_1+1)^2 v_1(\eta_s)^2 v_1(\eta_i)} \left(\frac{R_s}{R}\right)^2 . \tag{4.41}$$

Comparing this expression with (4.28) we find that

$$\zeta_{n_1} \frac{M}{M_i} \left(\frac{R_s}{R}\right)^2 = \frac{\prod_{n_1} \eta_s^2}{4(n_1+1)^2 v_1(\eta_s)^2 v_1(\eta_i)} \left(\frac{R_s}{R}\right)^2 ,$$

and hence

$$\zeta_{n_1} = \frac{\prod_{n_1} \eta_s^2}{4(n_1+1)^2 v_1(\eta_s)^3} . \tag{4.42}$$

On using the results of Table I together with the values of $\prod_{n_1}$ obtained in Chapter 3, equation (4.42) permits easy calculation of ζ_{n_1} for various positions of the interface.

(VI) NUMERICAL RESULTS

Using the values obtained from (4.42) in equation (4.29), we can obtain the ratio of the

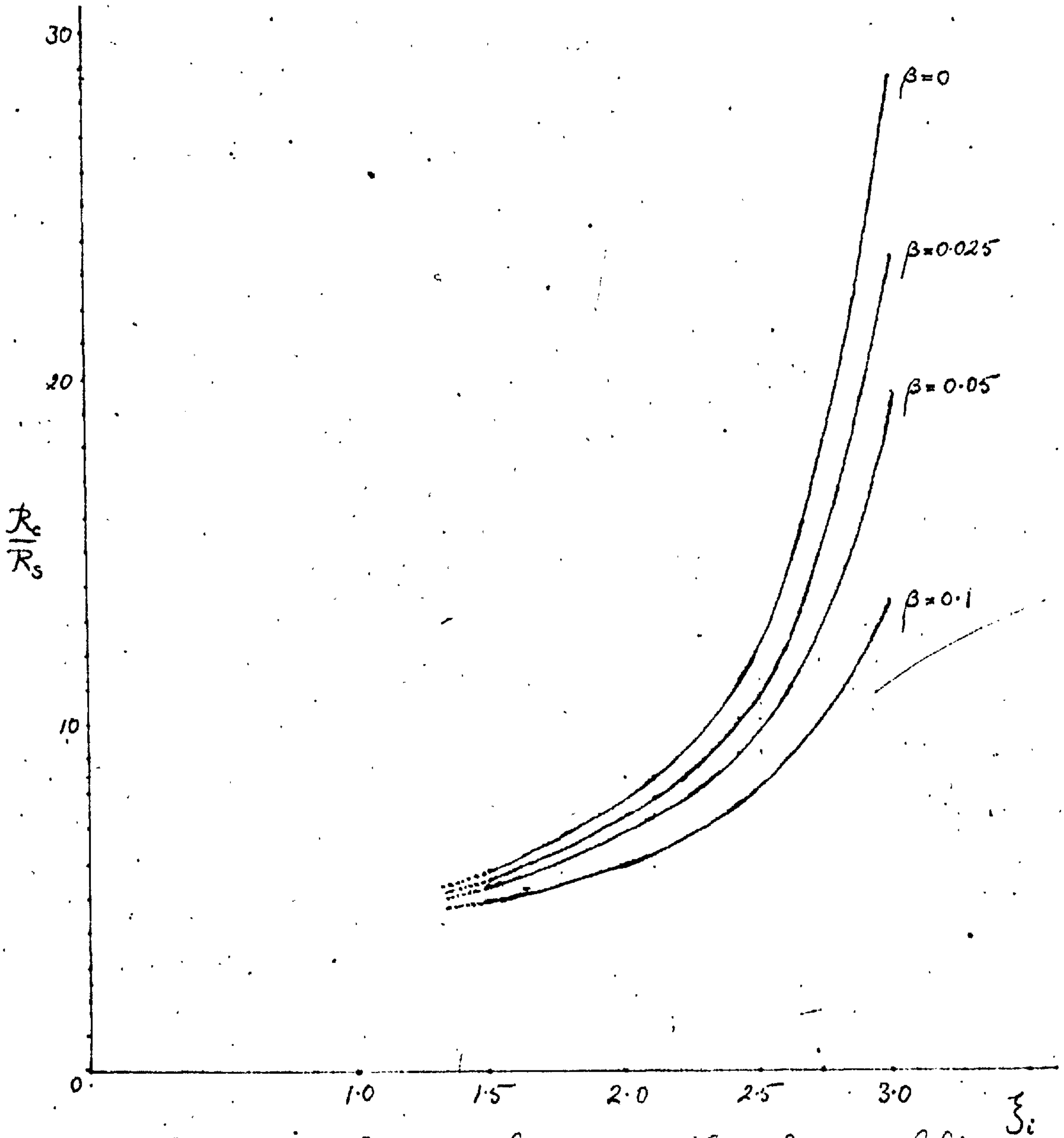

Fig. 5 — The ratio of the critical radius to the Schwarzschild radius versus the position of the interface ξ_i, for $n_i = 1$, for various values of the parameter β.

critical radius to the Schwarzschild radius. Graphs for this ratio for $n_1=1$ as a function of β and ξ_i (dimensionless interfacial radius) are shown in Fig.5. It is seen that this ratio depends strongly on the position of the interface and on the value of β in the core. For a particular value of β, the ratio increases steadily for small value of ξ_i and then more rapidly as ξ_i increases. For small values of ξ_i (i.e. when the interface is close to the centre and hence the structure of the core only slightly affects the model as a whole) it is seen that the critical radius is almost independent of β in the core, as expected. For larger values of ξ_i it is seen that the critical radius depends more strongly on the value of β, and it appears that for such values of ξ_i and for some values of β ~~for~~ there ~~to be~~ are no stable configurations at all.

TABLE II

SUMMARY OF VALUES THE DENOMINATOR OF EQUATION (4.29) FOR VARIOUS ζ_i AND β.

$G_1(\zeta_i, \beta)$				
ζ_i	β=0.0	β=0.025	β=0.05	β=0.1
0.9	9.13	9.28	9.43	9.73
1.2	2.80	2.89	2.97	3.15
1.5	1.24	1.30	1.36	1.47
1.8	0.67	0.71	0.75	0.84
2.1	0.41	0.45	0.48	0.55
2.7	0.17	0.20	0.23	0.29
3.0	0.12	0.14	0.17	0.22

REFERENCES

(1) Fowler, W.A., Reviews of Modern Physics, 36, 545, 1964.

(2) Tooper, R.F., Astrophysical Journal, 142, 1541, 1965.

(3) Fowler, W.A., Presented in part at the JOINT DISCUSSION ON RADIO GALAXIES during the XIIth. General Assembly of the INTERNATIONAL ASTROPHYSICAL UNION, Hamburg, Germany, August 1964.

CHAPTER 5

THE STABILITY OF PULSATING ADIABATIC SPHERES

I. BASIC EQUATIONS

In this chapter conditions for the stability of slowly oscillating adiabatic fluid spheres in General Relativity will be investigated.

Since we shall consider spherically symmetrical systems with oscillations taking place in the radial direction, we can take for the metric

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\Theta^2 + \sin^2\Theta d\phi^2) + c^2 e^{\nu} dt^2 \; , \qquad (5.1)$$

where $\lambda = \lambda(r,t)$, $\nu = \nu(r,t)$ are functions of r and t only. The field equations associated with this metric are given by (2.6) to (2.10), and for convenience will again be stated here. Thus we have

$$\frac{-8\pi G}{c^4} T_1^{\ 1} = e^{-\lambda}\left(\frac{1}{r}\frac{\partial\nu}{\partial r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \; , \qquad (5.2)$$

$$\frac{-8\pi G}{c^4} T_2^{\ 2} = \frac{-8\pi G}{c^4} T_3^{\ 3} = e^{-\lambda}\left(\frac{1}{2}\frac{\partial^2\nu}{\partial r^2} - \frac{1}{4}\frac{\partial\nu}{\partial r}\frac{\partial\lambda}{\partial r} + \frac{1}{4}\left(\frac{\partial\nu}{\partial r}\right)^2 + \frac{1}{2r}\left(\frac{\partial\nu}{\partial r} - \frac{\partial\lambda}{\partial r}\right)\right)$$
$$- e^{-\nu}\left(\frac{1}{2}\frac{\partial^2\lambda}{\partial t^2} + \frac{1}{4}\left(\frac{\partial\lambda}{\partial t}\right)^2 - \frac{1}{4}\frac{\partial\lambda}{\partial t}\frac{\partial\nu}{\partial t}\right), \qquad (5.3)$$

$$-\frac{8\pi G}{c^4} T_4{}^4 = -e^{-\lambda}\left(\frac{1}{r}\frac{\partial\lambda}{\partial r} - \frac{1}{r^2}\right) - \frac{1}{r^2} , \tag{5.4}$$

$$-\frac{8\pi G}{c^4} T_4{}^1 = +e^{-\lambda}\,\frac{1}{r}\frac{\partial\lambda}{\partial t} , \tag{5.5}$$

$$-\frac{8\pi G}{c^4} T_1{}^4 = \frac{-e^{-\nu}}{c^2}\,\frac{1}{r}\frac{\partial\lambda}{\partial t} , \tag{5.6}$$

where $T_\alpha{}^\beta$ denotes the energy-momentum tensor and is taken in the form

$$T_\alpha{}^\beta = (p+\rho c^2)u^\beta u_\alpha - g_\alpha{}^\beta p , \tag{5.7}$$

where p is the pressure, ρ the density (arising from all causes), and

$$u^\beta = \frac{dx^\beta}{ds} \tag{5.8}$$

is the contravariant four-velocity.

As stated in Chapter 2, the field equations (5.2) to (5.6) are not all independent, but are connected by the identity

$$(T_\alpha{}^\beta)_{;\beta} = 0 . \tag{5.9}$$

With the metric in the form (5.1), equation (5.9) for the covariant derivative of the energy-momentum tensor reduces to two relations[1]

$$\frac{\partial T_4{}^4}{\partial t} + \frac{\partial T_4{}^1}{\partial r} + \frac{1}{2}(T_4{}^4 - T_1{}^1)\frac{\partial\lambda}{\partial t} + T_4{}^1\left[\frac{1}{2}\frac{\partial}{\partial r}(\lambda+\nu) + \frac{2}{r}\right] = 0, \tag{5.10}$$

and

$$\frac{\partial T_1{}^4}{\partial t} + \frac{\partial T_1{}^1}{\partial r} + \frac{1}{2} T_1{}^4 \frac{\partial}{\partial t}(\lambda+\nu) + \frac{1}{2}(T_1{}^1 - T_4{}^4)\frac{\partial \nu}{\partial r} + \frac{2}{r}(T_1{}^1 + p) = 0 \; . \qquad (5.11)^*$$

Neglecting all quantities of the second and higher orders in the motions, we obtain from the metric (5.1)

$$u^1 = e^{-\nu_0/2} \frac{V}{c} , \qquad u_1 = -e^{\lambda_0 - \nu_0/2} \frac{V}{c} ,$$
$$u^4 = \frac{e^{-\nu_0/2}}{c} , \qquad u_4 = c e^{\nu_0/2} , \qquad (5.12)$$

where $V = dr/dt$, and the subscripts zero denote quantities that would describe the system if it were in equilibrium. Then, to the same order as equations (5.12), the components of the energy-momentum tensor (5.7) become

$$T_1{}^1 = T_2{}^2 = T_3{}^3 = -p , \quad T_4{}^4 = \rho c^2 , \qquad (5.13)$$

and

$$T_1{}^4 = (p+\rho c^2) u_1 u^4 = -e^{\lambda_0 - \nu_0} V(\rho + \frac{p}{c^2}), \; T_4{}^1 = (p+\rho c^2)V \; . \qquad (5.14)$$

With the components of the energy-momentum tensor given by equations (5.13) and (5.14), the field

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* The sign of p differs from that of Chandrasekhar because of his expression for $T_\alpha{}^\beta$.

equations (5.2)-(5.6) become

$$+\frac{8\pi Gp}{c^4} = e^{-\lambda}\left(\frac{1}{r}\frac{\partial\nu}{\partial r} + \frac{1}{r^2}\right) - \frac{1}{r^2}\ , \tag{5.15}$$

$$\frac{8\pi Gp}{c^4} = e^{-\lambda}\left(\frac{1}{2}\frac{\partial^2\nu}{\partial r^2} - \frac{1}{4}\frac{\partial\lambda}{\partial r}\frac{\partial\nu}{\partial r} + \frac{1}{4}\left(\frac{\partial\nu}{\partial r}\right)^2 + \frac{1}{2r}\left(\frac{\partial\nu}{\partial r} - \frac{\partial\lambda}{\partial r}\right)\right)$$

$$-e^{-\nu}\left(\frac{1}{2}\frac{\partial^2\lambda}{\partial t^2} + \frac{1}{4}\left(\frac{\partial\lambda}{\partial t}\right)^2 - \frac{1}{4}\frac{\partial\lambda}{\partial t}\frac{\partial\nu}{\partial t}\right)\ , \tag{5.16}$$

$$\frac{8\pi G\rho}{c^2} = e^{-\lambda}\left(\frac{1}{r}\frac{\partial\lambda}{\partial r} - \frac{1}{r^2}\right) + \frac{1}{r^2}\ , \tag{5.17}$$

$$\frac{8\pi G}{c^2}\left(\rho + \frac{p}{c^2}\right)V = -e^{-\lambda}\,\frac{1}{r}\frac{\partial\lambda}{\partial t}\ , \tag{5.18}$$

$$\frac{8\pi G}{c^2}\left(\rho + \frac{p}{c^2}\right)V = -e^{-\nu}\,\frac{1}{r}\frac{\partial\lambda}{\partial t}\ . \tag{5.19}$$

Also equations (5.10) and (5.11) may be written, on using equations (5.13) and (5.14),

$$\frac{\partial}{\partial t}[\rho c^2] + \frac{\partial}{\partial r}[(\rho c^2+p)V] + \frac{1}{2}(\rho c^2+p)\frac{\partial\lambda}{\partial t} + (p+\rho c^2)V\left[\frac{1}{2}\frac{\partial}{\partial r}(\lambda+\nu)+\frac{2}{r}\right]=0, \tag{5.20}$$

and

$$+\frac{\partial}{\partial t}\left[e^{\lambda_0-\nu_0}V\left(\rho+\frac{p}{c^2}\right)\right] + \frac{\partial p}{\partial r} + \frac{1}{2}e^{\lambda_0-\nu_0}V\left(\rho+\frac{p}{c^2}\right)\frac{\partial}{\partial t}(\lambda+\nu)+\frac{1}{2}(p+\rho c^2)\frac{\partial\nu}{\partial r}=0. \tag{5.21}$$

Equation (5.17) may be integrated immediately, and on defining

$$M_r = 4\pi\int_0^r \rho r^2 dr\ , \tag{5.22}$$

we obtain

$$e^{-\lambda} = 1 - \frac{2GM_r}{rc^2} . \qquad (5.23)$$

To the order of approximation required for ensuring that equations (5.20) and (5.21) are correct in the first post-Newtonian approximation, equation (5.15) gives, using equation (5.23),

$$\frac{\partial \nu}{\partial r} = \frac{8\pi Gp}{c^4} re^{\lambda} + \frac{2GM_r}{r^2c^2} + \frac{4G^2M_r^{\,2}}{r^3c^4} . \qquad (5.24)$$

Equation (5.18) may be written as

$$\frac{\partial \lambda}{\partial t} = -re^{\lambda} \frac{8\pi G}{c^2}(\rho + \frac{p}{c^2})V . \qquad (5.25)$$

On rewriting equation (5.21) in the form

$$e^{\lambda_o - \nu_o} V(p + \frac{p}{c^2})\frac{\partial \nu}{\partial t} + 2\frac{\partial p}{\partial t} + (p + \rho c^2)\frac{\partial \nu}{\partial r} + 2Ve^{\lambda_o - \nu_o}\frac{\partial}{\partial t}(\rho + \frac{p}{c^2})$$

$$+ 2e^{\lambda_o - \nu_o}(\rho + \frac{p}{c^2})\frac{\partial V}{\partial t} + e^{\lambda_o - \nu_o}V(\rho + \frac{p}{c^2})\frac{\partial \lambda}{\partial t} = 0 , \qquad (5.26)$$

and substituting for $\frac{\partial \nu}{\partial r}$ from equations (5.24) and (5.25), we obtain

$$e^{\lambda_o - \nu_o}V(\rho + \frac{p}{c^2})\frac{\partial \nu}{\partial t} + 2\frac{\partial p}{\partial r} + (p + \rho c^2)[\frac{2GM_r}{r^2c^2} + \frac{4G^2M_r^{\,2}}{r^3c^2} + \frac{8\pi Gpe^{\lambda}r}{c^4}]$$

$$-e^{\lambda_o - \nu_o}V(\rho + \frac{p}{c^2})re^{\lambda}\frac{8\pi G}{c^2}V(\rho + \frac{p}{c^2}) + 2e^{\lambda_o - \nu_o}V\frac{\partial}{\partial t}(\rho + \frac{p}{c^2})$$

$$+2e^{\lambda_o - \nu_o}(\rho + \frac{p}{c^2})\frac{\partial V}{\partial t} = 0 .$$

Hence, in the first post-Newtonian approximation,

$$e^{\lambda_0-\nu_0}V(\rho+\frac{p}{c^2})\frac{\partial\nu}{\partial t} + 2\frac{\partial p}{\partial r} + \frac{2GM_r}{r^2}\rho + \frac{2GM_r}{r^2c^2}p + \frac{4G^2M_r^2}{r^3c^2}\rho + \frac{8\pi Gp}{c^2}\rho r$$

$$+2e^{\lambda_0-\nu_0}V\frac{\partial}{\partial t}(\rho+\frac{p}{c^2}) + 2e^{\lambda_0-\nu_0}(\rho+\frac{p}{c^2})\frac{\partial V}{\partial t} = 0 \ . \quad (5.27)$$

On dividing throughout by $(\rho+\frac{p}{c^2})e^{\lambda_0-\nu_0}$ and differentiating with respect to t, we obtain

$$\frac{dV}{dt}\frac{\partial\nu}{\partial t} - V\frac{d}{dt}(\frac{d\nu}{dt} - V\frac{\partial\nu}{\partial r}) = 2e^{\nu_0-\lambda_0}\frac{d}{dt}\left[\frac{1}{(\rho+\frac{p}{c^2})}\frac{\partial p}{\partial r} + \frac{GM_r}{r^2} + \frac{2G^2M_r^2}{r^3c^2} + \frac{4\pi Gpr}{c^2}\right]$$

$$+ 2\frac{d}{dt}(\frac{1}{\rho+\frac{p}{c^2}})\frac{\partial}{\partial t}(V(\rho+\frac{p}{c^2})) + \frac{2}{(\rho+\frac{p}{c^2})}\frac{\partial}{\partial t}\left[(\rho+\frac{p}{c^2})\frac{\partial V}{\partial t} + V\frac{\partial}{\partial t}(\rho+\frac{p}{c^2})\right],$$

which may be written in the form

$$-V\frac{d}{dt}(\frac{d\nu}{dt} - V\frac{\partial\nu}{\partial r}) - \frac{dV}{dt}\frac{\partial\nu}{\partial t} = 2e^{\nu_0-\lambda_0}\left[\frac{1}{\rho+\frac{p}{c^2}}\frac{d}{dt}(\frac{\partial p}{\partial r}) + \frac{\partial p}{\partial r}\frac{d}{dt}(\frac{1}{\rho+\frac{p}{c^2}}) - \frac{2GM_r}{r^3}V - \frac{6G^2M_r^2}{r^4c^2}V + \frac{4\pi G}{c^2}(\frac{rdp}{dt} + pV)\right]$$

$$+2\frac{d}{dt}(\frac{1}{\rho+\frac{p}{c^2}})\frac{\partial}{\partial t}[V(\rho+\frac{p}{c^2})] + \frac{2}{(\rho+\frac{p}{c^2})}\frac{d}{dt}\left[(\rho+\frac{p}{c^2})(\frac{dV}{dt} - V\frac{\partial V}{\partial r}) + V\frac{\partial}{\partial t}(\rho+\frac{p}{c^2})\right]$$

$$+ 2e^{\nu_0-\lambda_0}\left[\frac{G}{r^2}\frac{dM_r}{dt} + \frac{4G^2M_r}{r^3c^2}\frac{dM_r}{dt}\right].$$

Hence, in the first post-Newtonian approximation,

$$-V\frac{d^2\nu}{dt^2} - \frac{dV}{dt}\frac{\partial\nu}{\partial t} = 2e^{\nu_o-\lambda_o}\left[\frac{1}{(\rho+\frac{p}{c^2})}(\frac{\partial p}{\partial r}) + \frac{\partial p}{\partial r}\frac{d}{dt}(\frac{1}{\rho+\frac{p}{c^2}}) - \frac{2GM_rV}{r^3} - \frac{6G^2M_r{}^2V}{r^4c^2} + \frac{4\pi G}{c^2}(pV+r\frac{dp}{dt})\right]$$

$$+2V\frac{d}{dt}(\frac{1}{\rho+\frac{p}{c^2}})\frac{\partial}{\partial t}(\rho+\frac{p}{c^2}) + 2\frac{d}{dt}(\frac{1}{\rho+\frac{p}{c^2}})[\frac{dV}{dt} - V\frac{\partial V}{\partial r}]$$

$$+ \frac{2}{(\rho+\frac{p}{c^2})}\frac{d}{dt}\left[(\rho+\frac{p}{c^2})(\frac{dV}{dt} - V\frac{\partial V}{\partial r}) + V\frac{\partial}{\partial t}(\rho+\frac{p}{c^2})\right] + 2e^{\nu_o-\lambda_o}\left[1 + \frac{4GM_r}{rc^2}\right]\frac{G}{r^2}\frac{dM_r}{dt} . \qquad (5.28)$$

II. EQUATIONS OF CONTINUITY AND EQUATION OF STATE

At this point it is necessary to derive the equations of continuity and to introduce the equation of state which will be used extensively in obtaining the equation of motion required. Thus, from equations (5.15) and (5.17) we obtain

$$\frac{\partial}{\partial r}(\lambda+\nu) = \frac{8\pi G}{c^4}e^{\lambda}r(p+\rho c^2) . \qquad (5.29)$$

Using equation (5.29) in equation (5.20), we deduce that

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial r}[(\rho+\frac{p}{c^2})V] - \frac{1}{2}(\rho+\frac{p}{c^2})^2 re^{\lambda}\frac{8\pi G}{c^2}V + V(\rho+\frac{p}{c^2})\left[\frac{2}{r}+\frac{4\pi Ge^{\lambda}r}{c^2}(\rho+\frac{p}{c^2})\right] = 0,$$

and hence

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r}[(\rho+\frac{p}{c^2})V] + \frac{2}{r}(\rho+\frac{p}{c^2})V = 0 \; . \qquad (5.30)$$

Since we are considering motions in a purely radial direction, equation (5.30) may be written as

$$\frac{\partial\rho}{\partial t} + \mathrm{div}[(\rho+\frac{p}{c^2})V] = 0 \; . \qquad (5.31)$$

Also

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + V\frac{\partial\rho}{\partial r} = -\frac{\partial}{\partial r}[(\rho+\frac{p}{c^2})V] - \frac{2}{r}V(\rho+\frac{p}{c^2}) + V\frac{\partial\rho}{\partial r} \; ,$$

and consequently

$$\frac{d\rho}{dt} = -(\rho+\frac{p}{c^2})\frac{\partial V}{\partial r} - \frac{V}{c^2}\frac{\partial p}{\partial r} - V\frac{\partial\rho}{\partial r} - \frac{2}{r}V(\rho+\frac{p}{c^2}) + V\frac{\partial\rho}{\partial r} \; .$$

Hence,

$$\frac{d\rho}{dt} = -(\rho+\frac{p}{c^2})\mathrm{div}\, V - \frac{V}{c^2}\frac{\partial p}{\partial r} \; . \qquad (5.32)$$

The above equations reduce to the usual continuity relations(2) in the classical limit.

Assuming an equation of state of the form

$$p = K\rho^{\gamma} \; , \qquad (5.33)$$

where γ is the ratio of the specific heats and K is constant, we obtain

$$\frac{dp}{dt} = \frac{\gamma p}{\rho}\frac{d\rho}{dt} \; .$$

Thus, using equation (5.32)

$$\frac{dp}{dt} = -\frac{\gamma p}{\rho}\left[(\rho+\frac{p}{c^2})\operatorname{div} V + \frac{V}{c^2}\frac{\partial p}{\partial r}\right]. \tag{5.34}$$

Similarly,

$$\frac{\partial p}{\partial t} = -\frac{\gamma p}{\rho}\operatorname{div}(\rho+\frac{p}{c^2})V . \tag{5.35}$$

Now

$$\frac{d}{dt}\left(\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{dp}{dt}\right) - \frac{\partial V}{\partial r}\frac{\partial p}{\partial r}$$

$$= -\frac{\partial}{\partial r}\left[\frac{\gamma p}{\rho}\left((\rho+\frac{p}{c^2})\operatorname{div} V + \frac{V}{c^2}\frac{\partial p}{\partial r}\right)\right] - \frac{\partial V}{\partial r}\frac{\partial p}{\partial r}, \tag{5.36}$$

also from equations [(5.31) and (5.35)] and [(5.32) and (5.34)],

$$\frac{\partial}{\partial t}(\rho+\frac{p}{c^2}) = -(1+\frac{\gamma p}{\rho c^2})\operatorname{div}[(\rho+\frac{p}{c^2})V] , \tag{5.37}$$

and

$$\frac{d}{dt}(\rho+\frac{p}{c^2}) = -(\rho+\frac{p}{c^2})(1+\frac{\gamma p}{\rho c^2})\operatorname{div} V - \frac{V}{c^2}(1+\frac{\gamma p}{\rho c^2})\frac{\partial p}{\partial r} . \tag{5.38}$$

III. THE EQUATION OF MOTION

Using the relations (5.37) and (5.38), we easily obtain

$$V\frac{d}{dt}(\rho+\frac{p}{c^2})\frac{\partial}{\partial t}(\rho+\frac{p}{c^2}) = O(V^2) , \tag{5.39}$$

$$\frac{d}{dt}\left(\frac{1}{\rho+p/c^2}\left(\rho+\frac{p}{c^2}\right)\left[\frac{dV}{dt} - V\frac{\partial V}{\partial r}\right]\right) = O(V^2) \; , \qquad (5.40)$$

and

$$\frac{1}{\left(\rho+\frac{p}{c^2}\right)}\frac{d}{dt}\left[\left(\rho+\frac{p}{c^2}\right)\left(\frac{dV}{dt} - V\frac{\partial V}{\partial r}\right) + V\frac{\partial}{\partial t}\left(\rho+\frac{p}{c^2}\right)\right] = \frac{d^2V}{dt^2} + O(V^2), \qquad (5.41)$$

which, on substitution into equation (5.28) give

$$-V\frac{d^2\nu}{dt^2} - \frac{dV}{dt}\frac{\partial\nu}{\partial t} - 2\frac{d^2V}{dt^2} = 2e^{\nu_0-\lambda_0}\left[\frac{1}{\left(\rho+\frac{p}{c^2}\right)}\frac{d}{dt}\left(\frac{\partial p}{\partial r}\right) + \frac{\partial p}{\partial r}\frac{d}{dt}\left(\frac{1}{\rho+\frac{p}{c^2}}\right) - \frac{2GM_rV}{r^3} - \frac{6G^2M_r^{\,2}V}{r^4c^2} + \frac{4\pi G}{c^2}\left(pV+\frac{rdp}{dt}\right)\right] + 2e^{\nu_0-\lambda_0}\left[1+\frac{4GM_r}{rc^2}\right]\frac{G}{r^2}\frac{dM_r}{dt} \; . \qquad (5.42)$$

On using equations (5.36) and (5.38), equation (5.42) becomes

$$-\frac{d^2V}{dt^2} - \frac{V}{2}\frac{d^2\nu}{dt^2} - \frac{1}{2}\frac{dV}{dt}\frac{d\nu}{dt} = e^{\nu_0-\lambda_0}\left\{\left[\frac{\partial}{\partial r}\left[\frac{\gamma p}{\rho}\left(\left(\rho+\frac{p}{c^2}\right)\mathrm{div}V+\frac{V}{c^2}\frac{\partial p}{\partial r}\right)\right] + \frac{\partial V}{\partial r}\frac{\partial p}{\partial r}\right] \times \frac{-1}{\rho+p/c^2}\right.$$

$$+ \frac{\partial p}{\partial r}\frac{1}{\left(\rho+\frac{p}{c^2}\right)^2}\left[\left(\rho+\frac{p}{c^2}\right)\left(1+\frac{\gamma p}{\rho c^2}\right)\mathrm{div}V + \frac{V}{c^2}\left(1+\frac{\gamma p}{\rho c^2}\right)\frac{\partial p}{\partial r}\right] - \frac{2GM_rV}{r^3} - \frac{6G^2M_r^{\,2}V}{r^4c^2}$$

$$\left. + \frac{4\pi G}{c^2}\left(pV+\frac{rdp}{dt}\right) + \left(1+\frac{4GM_r}{rc^2}\right)\frac{G}{r^2}\frac{dM_r}{dt}\right\} ,$$

which may be written as

$$\frac{d^2V}{dt^2} + \frac{V}{2}\frac{d^2\nu}{dt^2} + \frac{1}{2}\frac{dV}{dt}\frac{\partial\nu}{\partial t} - \frac{e^{\nu_o-\lambda_o}}{(\rho+\frac{p}{c^2})}\frac{\partial}{\partial r}[\frac{\gamma p}{\rho}(\rho+\frac{p}{c^2})\mathrm{div}V]$$

$$-\frac{e^{\nu_o-\lambda_o}}{(\rho+\frac{p}{c^2})}\left[\frac{\partial}{\partial r}(\frac{\gamma p}{\rho c^2}V\frac{\partial p}{\partial r}) + \frac{\partial V}{\partial r}\frac{\partial p}{\partial r}\right] + [(\rho+\frac{p}{c^2})\mathrm{div}V+\frac{V}{c^2}\frac{\partial p}{\partial r}]\frac{(1+\frac{\gamma p}{\rho c^2})}{(\rho+\frac{p}{c^2})^2}\frac{\partial p}{\partial r}e^{\nu_o-\lambda_o}$$

$$-\left[\frac{2GM_r}{r^3}V + \frac{6G^2M_r^2}{r^4c^2}V - \frac{4\pi Gp}{c^2}(V+\frac{r}{p}\frac{dp}{dt})-(1+\frac{4GM_r}{rc^2})\frac{G}{r^2}\frac{dM_r}{dt}\right]e^{\nu_o-\lambda_o} = 0. \tag{5.43}$$

On substituting for $\frac{dp}{dt}$ and $\frac{\partial p}{\partial r}$ from equations (5.34) and (5.27), equation (5.43) becomes in the first post-Newtonian approximation, (cf. equation (IV.5))

$$e^{\nu_o-\lambda_o}(\frac{d^2V}{dt^2} + \frac{V}{2}\frac{d^2\nu}{dt^2} + \frac{1}{2}\frac{dV}{dt}\frac{\partial\nu}{\partial t}) = \frac{1}{\rho}[\frac{\partial}{\partial r}(\gamma p\,\mathrm{div}V)]$$

$$+\left[(1-\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^2} + \frac{4\pi Gpr\gamma}{c^2}\right]\frac{\partial V}{\partial r} + \left[\frac{4GM_r}{r^3} + (9+\gamma)\frac{G^2M_r^2}{c^2r^4}\right.$$

$$\left.+\frac{4\pi Gp}{c^2}(1+\gamma) + 2(1+\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^3}\right]V - \left[1+\frac{4GM_r}{rc^2}\right]\frac{G}{r^2}\frac{dM_r}{dt}\,. \tag{5.44}$$
[*]

To obtain the rate of change of the mass inside radius r, we return to equation (5.22), i.e.

$$M_r = 4\pi\int_0^r \rho r^2 dr\,.$$

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* The derivation of this equation may be found in Appendix IV.

Hence,

$$\frac{dM_r}{dt} = \frac{\partial M_r}{\partial t} + V\frac{\partial M_r}{\partial r} ,$$

and consequently,

$$\frac{dM_r}{dt} = 4\pi\int_0^r \frac{\partial \rho}{\partial t} r^2 dr + V\, 4\pi\rho r^2 . \tag{5.45}$$

On using equation (5.31) in (5.45),

$$\frac{dM_r}{dt} = -4\pi\int_0^r \left\{\frac{\partial}{\partial r}[(\rho+\frac{p}{c^2})V] + \frac{2}{r}(\frac{p}{c^2}+\rho)V\right\} r^2 dr + 4\pi\rho r^2 V ,$$

and on integrating by parts it follows that

$$\frac{dM_r}{dt} = -4\pi[(\rho+\frac{p}{c^2})Vr^2]_0^r + 4\pi\int_0^r (\rho+\frac{p}{c^2})V\, 2r dr - 4\pi\int_0^r \frac{2}{r}(\rho+\frac{p}{c^2})r^2 dr$$

$$+ 4\pi\rho r^2 V ,$$

and hence,

$$\frac{dM_r}{dt} = -4\pi p\frac{r^2}{c^2}V . \tag{5.46}$$

This result is identical with that obtained by Bondi(3) for the case when there is no loss of radiation across any shell of radius r during the pulsation. It may be noted that in the Newtonian limit, the mass inside radius r is constant in time, in accordance with classical theory.[2] In the post-Newtonian approximation, as well as in the Newtonian limit, the total mass of the model is a constant, i.e. there is no mass lost to the surrounding space, which is consistent with our assumption of an adiabatic sphere.

Using equation (5.46), the equation of motion (5.44) becomes

$$e^{\nu_0-\lambda_0}\left(\frac{d^2V}{dt^2} + \frac{V}{2}\frac{d^2\nu}{dt^2} + \frac{1}{2}\frac{dV}{dt}\frac{\partial\nu}{\partial t}\right) = \frac{1}{\rho}\left[\frac{\partial}{\partial r}(\gamma p \,\mathrm{div} V)\right]$$

$$+\left[(1-\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^2} + \frac{4\pi Gpr\gamma}{c^2}\right]\frac{\partial V}{\partial r}$$

$$+\left[\frac{4GM_r}{r^3} + (9+\gamma)\frac{G^2M_r^{\,2}}{c^2r^4} + \frac{4\pi Gp}{c^2}(2+\gamma) + 2(1+\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^3}\right]V = 0. \tag{5.47}$$

This is the equation of motion for the small oscillations of an adiabatic fluid sphere, correct to order $\frac{1}{c^2}$ and to first order in the motions. This equation may be compared with that obtained by Kaplan and Lupanov[4] and also that obtained by Chandrasekhar,[1] but unlike their methods of derivation it has not been necessary in the present analysis to introduce perturbations with time dependence $e^{i\sigma t}$ (the same for all physical quantities).

It is easily shown that $V\frac{d^2\nu}{dt^2}$ and $\frac{dV}{dt}\frac{\partial\nu}{\partial t}$ are of order V^2, and so may be neglected. Hence, if we put

$$V = \frac{\partial\psi}{\partial t}\text{, where } \psi = \zeta(r)e^{i\sigma t}\text{,}$$

so that $V = i\sigma\,\zeta(r)e^{i\sigma t}$; $\frac{dV}{dt} = (i\sigma)^2\zeta(r)e^{i\sigma t}$;

$$\frac{d^2V}{dt^2} = (i\sigma)^3\zeta(r)e^{i\sigma t}\text{,} \tag{5.48}$$

then equation (5.47) gives

$$\frac{1}{\rho}\frac{\partial}{\partial r}(\gamma p\,\mathrm{div}\,\zeta) + \left[(1-\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^2} + \frac{4\pi G\gamma pr}{c^2}\right]\zeta'(r)$$
$$+\left[\sigma^2 e^{\nu_0-\lambda_0} + \frac{4GM_r}{r^3} + (9+\gamma)\frac{G^2M_r^{\,2}}{c^2r^4} + \frac{4\pi Gp}{c^2}(2+\gamma) + 2(1+\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^3}\right]\zeta(r) = 0, \tag{5.49}$$

where the prime symbol denotes differentiation with respect to r.

For a uniform sphere, i.e. $\rho = \rho(t)$, we obtain, in a similar manner,

$$\frac{1}{\rho}\frac{\partial}{\partial r}(\gamma p\,\mathrm{div}\,\zeta) + \left[(\ -\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^2} + \frac{4\pi G\gamma pr}{c^2}\right]\zeta'(r)$$
$$+\left[\sigma^2 e^{\nu_0-\lambda_0} + \frac{4GM_r}{r^3} + (9+\gamma)\frac{G^2M_r^{\,2}}{c^2r^4} + \frac{4\pi Gp(2+\gamma)}{c^2} + \frac{2\gamma p}{\rho c^2}\frac{GM}{r^3}\right]\zeta(r) = 0\text{ .} \tag{5.50}$$

In the classical limit equations (5.49) and (5.50) each reduce to the well-known equation of motion, given by Rosseland,[2] for the oscillations of an adiabatic sphere,

$$\frac{1}{\rho}\frac{d}{dr}(\gamma p \operatorname{div} \xi) + (\sigma^2 + \frac{4GM_r}{r^3})\xi = 0 . \qquad (5.51)$$

(IV) THE HOMOGENEOUS SPHERE

We shall first obtain the condition for dynamical instability in the case of a homogeneous sphere of constant density ρ and constant ratio of specific heats γ. Following Chandrasekhar,[1] we shall write

$$y^2 = 1 - \frac{r^2}{a^2} , \text{ where } a^2 = \frac{3c^2}{8\pi G\rho} . \qquad (5.52)$$

Thus, from the Schwarzschild interior solution discussed in Section IV of Chapter II, we obtain

$$p = \rho c^2 \frac{y - y_s}{3y_s - y} , \quad e^{\lambda} = \frac{1}{y^2} \text{ and } e^{\nu} = \frac{1}{4}(3y_s - y)^2 , \qquad (5.53)$$

where $y_s^2 = 1 - \frac{R^2}{a^2}$, and R is the radius of the model. Hence, on integration with respect to r, equation (5.50) becomes

$$\frac{\gamma}{\rho}\int_0^{\zeta_s}\frac{\partial}{\partial r}(p\,\mathrm{div}\,\zeta)dr + \left[-\frac{\gamma G M_r}{r^3 \rho c^2} + \frac{4\pi G\gamma}{c^2}\right]\int_0^{\eta_s} c^2 \alpha\eta\rho\left(\frac{y-y_s}{3y_s-y}\right)\frac{d}{d\eta}\left(\frac{1}{\eta}(3y_s-y)\right)d\eta$$

$$+\sigma^2\int_0^{\eta_s}\frac{1}{y^2}\frac{4\alpha}{(3y_s-y)^2}\frac{1}{2}\eta(3y_s-y)d\eta + \frac{4GM_r}{r^3}\int_0^{\eta_s}\frac{\alpha}{2}\eta(3y_s-y)d\eta$$

$$+\left[-\frac{\gamma}{\rho c^2}\frac{GM_r}{r^3} + \frac{4\pi G(2\gamma+2)}{c^2}\right]\int_0^{\eta_s}\rho c^2\frac{y-y_s}{3y_s-y}\frac{\alpha\eta}{2}(3y_s-y)d\eta$$

$$+\frac{4\pi}{3}\rho(9+\gamma)\frac{G^2 M_r}{c^2 r^3}\int_0^{\eta_s}\frac{\alpha^3\eta^3}{2}(3y_s-y)d\eta = 0 \qquad (5.54)$$

where $\eta = \frac{r}{\alpha}$, $\eta_s = \frac{R}{\alpha}$ and $\zeta(r) = \frac{\eta}{2}(3y_s-y)$.

To the first post-Newtonian order of approximation, on putting

$$y = \cos\theta \quad \text{and} \quad \eta = \sin\theta ,$$

where $\theta_s = \sin^{-1}\frac{R}{\alpha}$, we find that

$$\int_0^{\eta_s}\eta(y-y_s)d\eta = \frac{1}{8}\theta_s^4 , \qquad (5.55)$$

$$\int_0^{\eta_s}\eta^3(3y_s-y)d\eta = \frac{1}{2}\theta_s^4 , \qquad (5.56)$$

$$\int_0^{\eta_s} \eta \, \frac{y-y_s}{3y_s-y} \, \frac{d}{d\eta}[\tfrac{1}{2}\eta(3y_s-y)]d\eta = \frac{1}{16}\theta_s^{\,4} \;, \qquad (5.57)$$

$$\int_0^{\eta_s} \frac{\eta}{y^2(3y_s-y)} \, d\eta = \frac{\theta_s^{\,2}}{4} + \frac{19}{2.48}\,\theta_s^{\,4} \;, \qquad (5.58)$$

$$\int_0^{\eta_s} \tfrac{1}{2}\eta(3y_s-y)d\eta = \tfrac{1}{2}\theta_s^{\,2} - \frac{23}{48}\,\theta_s^{\,4} \;, \qquad (5.59)$$

and

$$\int_0^{\xi_s} \frac{\gamma}{\rho}\frac{\partial}{\partial r}(p \operatorname{div} \xi)dr = -\frac{3}{4}\frac{\gamma c^2}{\alpha}[\theta_s^{\,2} - \frac{\theta_s^{\,4}}{12}] \;. \qquad (5.60)$$

Using these results in equation (5.54), we obtain

$$-\frac{3}{4}\frac{\gamma c^2}{\alpha}[\theta_s^{\,2} - \frac{\theta_s^{\,4}}{12}] + \left[-\frac{\gamma GM}{R^3\rho c^2} + \frac{4\pi\gamma G}{c^2}\right]\rho\alpha^2\frac{1}{16}\theta_s^{\,4} + \alpha\sigma^2[\frac{\theta_s^{\,2}}{2} + \frac{19}{48}\theta_s^{\,4}]$$

$$\alpha\theta_s^{\,2}\,\frac{2GM}{R^3} - \frac{2.23}{24}\,\frac{GM}{R^3}\theta_1^{\,4}\alpha + \left[-\frac{\gamma GM}{\rho c^2 R^3} + \frac{(2+2\gamma)}{c^2}\,\frac{3GM}{R^3}\right]\frac{c^2\alpha}{16}\theta_s^{\,4}\rho$$

$$+ \frac{4}{3}\,(9+\gamma)\,\frac{G^2M}{R^3c^2}\,\theta_s^{\,4}\,\frac{\alpha^3}{4}\pi\rho = 0 \;,$$

which simplifies tc

$$2\sigma^2[1+\frac{19}{24}\theta_s^{\,2}] + \frac{2GM}{R^3}\left[(4-3\gamma)+\theta_s^{\,2}[\frac{11}{8}\gamma-\frac{5}{6}]\right] = 0 \;,$$

and hence

$$\sigma^2 + \frac{GM}{R^3}\left[(4-3\gamma) + (\frac{15}{4}\gamma - 4)\theta_s^{\,2}\right] = 0 \;. \qquad (5.61)$$

The condition for dynamical <u>instability</u> is $\sigma^2 < 0$, and hence

$$(4-3\gamma) + (\tfrac{15}{4}\gamma-4)\theta_s^2 > 0 ,$$

i.e.

$$\gamma-\tfrac{4}{3} < (\tfrac{5}{4}\gamma-\tfrac{4}{3})\theta_s^2 ,$$

consequently

$$\gamma-\tfrac{4}{3} < \tfrac{1}{3}\theta_s^2 . \qquad (5.62)$$

From the definition of θ_s we have, for small θ_s,

$$\theta_s^2 = \frac{R^2}{\alpha^2} = \frac{R^2}{3c^2}8\pi G\rho ,$$

and from equations (5.52) and (5.53),

$$p_c = \rho_c c^2 \frac{1-(1-\frac{R^2}{3c^2}8\pi\rho G)^{1/2}}{3(1-\frac{R^2}{3c^2}8\pi\rho G)^{1/2}-1} ,$$

which gives, to the order of approximation considered,

$$\frac{p_c}{\rho_c c^2} = \frac{\frac{1}{2}(\frac{R^2}{3c^2}8\pi\rho G)}{2} = \tfrac{1}{4}\theta_s^2 .$$

Hence, the condition for instability (5.62) becomes

$$\gamma - \tfrac{4}{3} < \tfrac{4}{3}\frac{p_c}{\rho_c c^2} , \qquad (5.63)$$

marginal stability occurring when $\sigma = 0$, i.e. when

$$\gamma - \frac{4}{3} = \frac{1}{3}\theta_s{}^2 = \frac{4}{3}\frac{p_c}{\rho_c c^2} . \qquad (5.64)$$

It will be noted that in the classical limit expressions (5.63) and (5.64) reduce to the usual classical conditions imposed on the ratio of the specific heats for instability and marginal stability, respectively.

In the post-Newtonian approximation, the sphere is unstable for values of γ smaller than $(\frac{4}{3} + \frac{4}{3}\frac{p_c}{\rho_c c^2})$, and will expand or contract at an exponentially accelerated rate. From equation (5.61), when γ is close to $\frac{4}{3}$, σ^2 is given approximately by

$$\sigma^2 = \frac{4}{3}\pi\rho G\left[(3\gamma-4)-\frac{4p_c}{\rho_c c^2}\right] , \qquad (5.62)$$

Consequently, in the case of a uniform sphere, the Newtonian lower limit $\frac{4}{3}$, for the ratio of the specific heats γ ensuring dynamical stability, is increased when general relativistic effects are taken into account, and will be significant even for configurations in which the ratio of the central pressure to density is small, provided that γ is close to $\frac{4}{3}$.

Before proceeding to investigate the stability of more general models (non-uniform density), it will first be shown that, in the case of a uniform sphere, instability occurs at a maximum of the mass regarded as a function of $p_c/\rho_c c^2$.

Consider a uniform sphere in hydrostatic equilibrium. Applying equations (2.36), (2.44) and (2.52) applied to such a sphere (n=0), we obtain

$$M = 4\pi\rho_c \alpha^3 v(\xi_s) ,$$

where $\alpha^2 = \dfrac{p_c}{4\pi G\rho_c^2} = \dfrac{K\rho_c^{\gamma-1}}{4\pi G\rho_c}$.

Hence,

$$M = 4\pi\rho_c \left(\frac{K\rho_c^{\gamma-2}}{4\pi G}\right)^{3/2} v(\xi_s) ,$$

and therefore

$$M = \frac{1}{(4\pi)^{1/2}} \left(\frac{K}{G}\right)^{3/2} \rho_c^{\frac{3\gamma-4}{2}} v(\xi_s) , \tag{5.65}$$

where $v(\xi)$, ξ and θ satisfy the generalization of the Lane-Emden equation given by (2.45) and (2.46), with n=0, namely

$$\frac{1-\dfrac{2K\rho_c^{\gamma-1}}{c^2} v(\xi)/\xi}{1+\dfrac{K\rho_c^{\gamma-1}}{c^2}\theta} \xi^2 \frac{d\theta}{d\xi} + v(\xi) + \frac{K\rho_c^{\gamma-1}}{c^2} \xi^3\theta = 0 , \tag{5.66}$$

where $\dfrac{dv}{d\xi} = \xi^2$. (5.67)

On integrating (5.67), we obtain

$$v(\xi_s) = \frac{\xi_s^3}{3} \quad . \tag{5.68}$$

The solution of the generalized Lane-Emden equation has been given by Tooper[5] and in the present notation becomes

$$\frac{K\rho_c^{\gamma-1}}{c^2}\,\theta = \frac{\left(1+\frac{3K\rho_c^{\gamma-1}}{c^2}\right)\left(1-\frac{2}{3}\frac{K\rho_c^{\gamma-1}}{c^2}\xi^2\right)^{1/2}-\left(1-\frac{K\rho_c^{\gamma-1}}{c^2}\right)}{3\left(1+\frac{K\rho_c^{\gamma-1}}{c^2}\right)-\left(1+\frac{3K\rho_c^{\gamma-1}}{c^2}\right)\left(1-\frac{2}{3}\frac{K\rho_c^{\gamma-1}}{c^2}\xi^2\right)^{1/2}} .$$

Hence, at the surface $(\theta=0)$, it follows that

$$\xi_s^2 = \frac{6\left(1+\frac{2K\rho_c^{\gamma-1}}{c^2}\right)}{\left(1+\frac{3K\rho_c^{\gamma-1}}{c^2}\right)^2} \quad . \tag{5.69}$$

Thus, equation (5.68) may be written as

$$v(\xi_s) = \frac{6^{3/2}}{3}\left[\frac{1+\frac{2K\rho_c^{\gamma-1}}{c^2}}{1+\frac{3K\rho_c^{\gamma-1}}{c^2}}\right]^{3/2} ,$$

and hence, to the order of approximation considered,

$$v(\xi_s) = \frac{6^{3/2}}{3\left(1+\frac{4K\rho_c^{\gamma-1}}{c^2}\right)^{3/2}} \quad . \tag{5.70}$$

Consequently, formula (5.65) for the total mass becomes

$$M = \frac{1}{(4\pi)^{1/2}} \left(\frac{K}{G}\right)^{3/2} \frac{\rho_c^{\frac{3\gamma-4}{2}}}{\left(1+\frac{4K\rho_c^{\gamma-1}}{c^2}\right)^{3/2}} \, \frac{6}{3}^{\,2/3} , \qquad (5.71)$$

which is identical with that obtained by Kaplan and Lupanov[(4)] in the case of a uniform sphere.

By equating $\frac{dM}{d\rho_c}$ to zero, we can obtain the value of ρ_c for which the mass has its maximum value. Thus,

$$\frac{dM}{d\rho_c} = 0 = \frac{3\gamma-4}{2}\rho_c^{\frac{3\gamma-4}{2}}\left(1+\frac{4K\rho_c^{\gamma-1}}{c^2}\right)^{3/2} - \rho_c^{\frac{3\gamma-4}{2}}\frac{3}{2}\left(1+\frac{4K\rho_c^{\gamma-1}}{c^2}\right)^{1/2}\frac{4K\rho_c^{\gamma-2}(\gamma-1)}{c^2}$$

and so $\quad (3\gamma-4)\left(1+\frac{4K\rho_c^{\gamma-1}}{c^2}\right) - 3.4(\gamma-1)\frac{K}{c^2}\rho_c^{\gamma-1} = 0 ,$

and hence,

$$3\gamma-4 + \frac{4K\rho_c^{\gamma-1}}{c^2}[(3\gamma-4)-3(\gamma-1)] = 0 .$$

Consequently,

$$3\gamma-4 = \frac{4K\rho_c^{\gamma-1}}{c^2} ,$$

which, from the equation of state, may be written in the form

$$3\gamma-4 = \frac{4p_c}{\rho_c c^2} . \qquad (5.72)$$

But from equation (5.64) we see that equation (5.72) is just the condition for the onset of dynamical instability. Hence, it follows that marginal instability occurs when the mass, regarded as a function of ρ_c, is a maximum. Also, for values of γ and of the ratio of central-pressure to density such that inequality (5.63) holds, we conclude that the descending branch $\left(\frac{dM}{d\rho_c} < 0\right)$ of M is unstable, a result which is in complete agreement with that obtained by Kaplan and Lupanov.[4]

On re-writing the condition (5.62) for dynamical instability in the form

$$\gamma - \frac{4}{3} < \frac{1}{3} \cdot \frac{8\pi\rho G R^2}{3c^2} ,$$

and using the formula for the total mass,

$$M = \frac{4}{3}\pi\rho R^3 ,$$

we obtain

$$\gamma - \frac{4}{3} < \frac{1}{3} \cdot \frac{2GM}{Rc^2} .$$

Consequently, $$R < \frac{1}{3} \cdot \frac{1}{(\gamma - \frac{4}{3})} \cdot \frac{2GM}{c^2} ,$$

and hence

$$\frac{R}{R_s} < \frac{1}{3\gamma - 4} , \qquad (5.73)$$

where R_s is the Schwarzschild radius. Thus instability will occur if the ratio of the actual radius of the configuration to the Schwarzschild radius falls below $1/(3\gamma-4)$.

V. NON-UNIFORM SPHERES

Turning now to the problem of stability of non-uniform adiabatic spheres, attention will be concentrated on these with a density distribution similar to that in a polytrope of index 3, with the object of elucidating the discrepancy between the results obtained by Chandrasekhar[1] on the one hand and Kaplan and Lupanov[4] on the other, for this particular type of sphere.

It is known that in the classical limit the solution of equation (5.49) corresponding to marginal stability is proportional to ξ, where ξ is defined by equation (V.9) of Appendix V[1].

Consequently, on putting $\zeta = \xi$ in the equation of motion (5.49), we should obtain the condition for marginal stability in the post-Newtonian approximation. Hence, writing (5.49) in the form

$$\gamma\frac{\partial}{\partial r}(p\,\mathrm{div}\,\xi) + \left[(1-\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^2} + \frac{4\pi G\gamma p r}{c^2}\right]\rho\,\xi'(r)$$

$$+\left[\frac{4GM_r}{r^3} + (9+\gamma)\frac{G^2M_r^2}{c^2r^4} + \frac{4\pi Gp}{c^2}(2+\gamma) + 2(1+\gamma)\frac{p}{\rho c^2}\cdot\frac{GM_r}{r^3}\right]\rho\,\xi(r) = 0\,, \tag{5.74}$$

where,

$$n = a\xi , \quad a^2 = \frac{p_c}{\pi G \rho_c^{\,2}} \quad . \qquad (5.75)$$

Integration with respect to ξ yields

$$\left[\frac{\gamma}{a} p \,\mathrm{div}\, \xi\right]_0^{\xi_s} + (1-\gamma)\frac{G}{c^2}\int_0^{\xi_s} \frac{pM_r}{r^2}\xi' d\xi + \frac{4\pi G\gamma}{c^2}\int_0^{\xi_s} p\rho r \,\xi' d\xi$$

$$+4G\int_0^{\xi_s} \frac{M_r}{r^3}\rho\,\xi\, d\xi \;+(9+\gamma)\frac{G^2}{c^2}\int_0^{\xi_s} \frac{M_r^{\,2}}{r^4}\rho\,\xi\, d\xi + \frac{4\pi G}{c^3}(2+\gamma)\int_c^{\xi_s} p\rho\,\xi\, d\xi$$

$$+ 2(1+\gamma)\frac{G}{c^2}\int_0^{\xi_s}\frac{M_r}{r^3}p\,\xi\, d\xi = 0 \; . \qquad (5.76)$$

By straightforward integration we obtain in the first post-Newtonian approximation,

$$\left[\frac{\gamma}{a}p\,\mathrm{div}\,\xi\right]_0^{\xi_s} = -3\pi G\gamma\rho_c^{\,2} \; , \qquad (5.77)$$

$$(1-\gamma)\frac{G}{c^2}\int_0^{\xi_s}\frac{pM_r}{r^2}\xi' d\xi = \frac{4}{5}\pi G(1-\gamma)\frac{p_c\rho_c}{c^2} \; , \qquad (5.78)$$

$$\frac{4\pi G\gamma}{c^2}\int_0^{\xi_s} p\rho r\,\xi' d\xi = 4\pi\gamma\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\theta^7\xi\, d\xi \; , \qquad (5.79)$$

$$2(1+\gamma)\frac{G}{c^2}\int_0^{\xi_s}\frac{M_r}{r^3}p \quad d \quad = \frac{8}{5}(1+\gamma)\;\pi G\frac{p_c\rho_c}{c^2} \; , \qquad (5.80)$$

$$(9+\gamma)\frac{G^2}{c^2}\int_0^{\xi_s}\frac{M_r^{\,2}}{r^4}\rho\,\xi\,d\xi = (9+\gamma)\;16\pi G\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\xi\theta^3\left(\frac{d\theta}{d\xi}\right)^2 d\xi\,, \tag{5.81}$$

$$4\pi G\frac{(2+\gamma)}{c^2}\int_0^{\xi_s} p\,\xi\rho\,d\xi = 4\pi G(2+\gamma)\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\theta^7\xi\,d\xi\,, \tag{5.82}$$

and

$$4G\int_0^{\xi_s}\frac{M_r}{r^3}\rho\,\xi\,d\xi = 4\pi G\rho_c^{\,2}-\frac{16}{5}\pi G\frac{p_c\rho_c}{c^2} - 4.32\pi G\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\theta^3\left(\frac{d\theta}{d\xi}\right)^2 d\xi$$
$$-16\pi G\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\xi\theta^7 d\xi\,, \tag{5.83}$$

where subscript c denotes central values, and θ is defined (Appendix V) by

$$\rho = \rho_c\theta^3\,. \tag{5.84}$$

Hence, using equations (5.77)-(5.83), equation (5.76) becomes

$$-3\pi G\gamma\rho_c^{\,2}+\frac{4}{5}\pi G(1-\gamma)\frac{p_c\rho_c}{c^2}+4\pi G\gamma\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\theta^7\xi\,d\xi+4\pi G\rho_c^{\,2}-\frac{16}{5}\pi G\frac{p_c\rho_c}{c^2}$$
$$-4.32\pi G\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\xi\theta^3\left(\frac{d\theta}{d\xi}\right)^2 d\xi-16\pi G\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\xi\theta^7 d\xi$$
$$+(9+\gamma)\;16\pi G\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\xi\theta^3\left(\frac{d\theta}{d\xi}\right)^2 d\xi$$
$$+4\pi G(2+\gamma)\frac{p_c\rho_c}{c^2}\int_0^{\xi_s}\theta^7\xi\,d\xi+\frac{8}{5}(1+\gamma)\pi G\frac{p_c\rho_c}{c^2} = 0\,. \tag{5.85}$$

From equation (5.85) it is immediately seen, on neglecting the post-Newtonian terms, that the condition for marginal stability is

$$\gamma = 4/3 \ ,$$

which is the usual classical result. We shall now take $\gamma = 4/3$ in the post-Newtonian terms of equation (5.85), because the error involved in the value of γ, being itself of order $1/c^2$, will lead to errors of order $1/c^4$ in these terms and so can be neglected to the order of approximation to which we are working. Hence,

$$(4-3\gamma) + \frac{p_c}{\rho_c c^2}\left[-\frac{4}{15} + \frac{16}{3}\int_0^{\xi_s} \theta^7 \xi d\xi - \frac{16}{5} - 4.32\int_0^{\xi_s} \xi\theta^3\left(\frac{d\theta}{d\xi}\right)^2 d\xi \right.$$

$$\left. -16\int_0^{\xi_s} \xi\theta^7 d\xi + 16\,\frac{31}{3}\int_0^{\xi_s} \xi\theta^3\left(\frac{d\theta}{d\xi}\right)^2 d\xi + \frac{40}{3}\int_0^{\xi_s} \theta^7 \xi d\xi + \frac{56}{15}\right] = 0 \ ,$$

and so

$$(4-3\gamma) + \left[\frac{4}{15} + \frac{8}{3}\int_c^{\xi_s} \theta^7 \xi d\xi + \frac{7.16}{3}\int_0^{\xi_s} \xi\theta^3\left(\frac{d\theta}{d\xi}\right)^2 d\xi\right]\frac{p_c}{\rho_c c^2} = 0 \ . \qquad (5.86)$$

On taking approximate numerical values of the integrals in (5.86) we obtain

$$4-3\gamma + \left[0.27 + 1.27 + 5.22\right]\frac{p_c}{\rho_c c^2} = 0 ,$$

i.e. $$4-3\gamma + 6.76\frac{p_c}{\rho_c c^2} = 0 ,$$

and hence the condition for marginal stability may be written as

$$\gamma - {}^4/_3 = 2.25\frac{p_c}{\rho_c c^2} . \qquad (5.87)$$

Thus as in the case of a uniform sphere, we deduce that for the type of non-uniform sphere here considered the Newtonian lower limit of ${}^4/_3$ for the value of γ compatible with dynamical stability is increased by the effects of general relativity. This result was also found by Chandrasekhar but he obtained a numerical factor 2.63 on the right hand side of equation (5.87), whereas Kaplan and Lupanov obtained 1.33 (i.e. ${}^4/_3$).

The numerical discrepancy between (5.87) and the corresponding result obtained by Kaplan and Lupanov can be traced to their method of approximation (see Chapter I, pp. 20). It has not been possible to pin-point the reason for difference between equation (5.87) and the corresponding result in Chandrasekhar's work. Therefore, to decide the point an independent check on the validity of equation (5.87) will now be given.

It has already been shown (see pp. 151 and 152 of Chapter 5) for a uniform sphere that the graph of the mass M as a function of the ratio of the central pressure to the density consists of two branches: ascending $(dM/d\rho_c > 0)$ and descending $(dM/d\rho_c < 0)$, instability occurring at the maximum value of M. Kaplan and Lupanov[4] showed that this result is also approximately true for the type of non-uniform sphere considered above. Also, for spheres in which the equation of state is of the form (2.26) Tooper[6] has shown that for general n, instability occurs when the mass regarded as a function of $p_c/\rho_c c^2$ attains its maximum value.

With these general considerations in mind, we now proceed to justify our result (5.87) above.

To the first post-Newtonian approximation, the relativistic generalization of the Lane-Emden equation is (see Appendix V)

$$\xi^2 \frac{d\theta}{d\xi} \frac{1-8\sigma v/\xi}{1+\sigma\theta} + v + \sigma\xi^3\theta^4 = 0 \, . \qquad (5.88)$$

where

$$\sigma = p_c/\rho_c c^2 = K\rho_c^{\gamma-1}/c^2 \quad , \qquad (5.89)$$

from which we obtain

$$v(\xi) = \frac{-\xi^2\frac{d\theta}{d\xi} - \sigma\xi^3\theta^4}{1+\sigma\theta-8\sigma(\xi\frac{d\theta}{d\xi})} . \tag{5.90}$$

Hence, the surface value of $v(\xi)$ is given by

$$v(\xi_s) = \frac{-(\xi^2\frac{d\theta}{d})_s}{1-8\sigma(\xi\frac{d\theta}{d\xi})_s} , \tag{5.91}$$

and the mass M may be written as

$$M = 4\pi\rho_c a^3 v(\xi_s) , \tag{5.92}$$

where

$$a^2 = \frac{c^2}{\pi G\rho_c} .$$

Consequently, from equation (5.91), we obtain

$$M = \frac{-4\pi\rho_c(\sigma c^2)^{3/2}}{\pi^{3/2}G^{3/2}\rho_c^{3/2}} \frac{(\xi^2\frac{d\theta}{d\xi})_s}{1-8\sigma(\xi\frac{d\theta}{d\xi})_s} ,$$

and, from (5.93),

$$M = \frac{-4K^{3/2}}{\pi^{1/2}G^{3/2}} \rho_c^{\frac{3\gamma-4}{2}} \frac{(\xi^2\frac{d\theta}{d\xi})_s}{1-8\sigma(\xi\frac{d\theta}{d\xi})_s} , \tag{5.93}$$

giving the mass as a function of ρ_c, since $(\xi^2\frac{d\theta}{d\xi})_s$ and $(\xi\frac{d\theta}{d\xi})_s$ are functions of ρ_c.

On equating $\frac{dM}{d\rho_c}$ to zero, we obtain the value of ρ_c for which the mass is a maximum. Thus, from equation (5.93)

$$\frac{dM}{d\rho_c} = 0 = [(\frac{3\gamma-4}{2})\rho_c^{\frac{3\gamma-4}{2}-1}(\xi^2\frac{d\theta}{d\xi})_s + \rho_c^{\frac{3\gamma-4}{2}}\frac{d}{d\rho_c}(\xi^2\frac{d\theta}{d\xi})_s]$$

$$[1-\frac{8K\rho_c^{\gamma-1}}{c^2}(\xi\frac{d\theta}{d\xi})_s]$$

$$+[8(\gamma-1)\frac{K\rho_c^{\gamma-2}}{c^2}(\xi\frac{d\theta}{d\xi})_s + 8\frac{K\rho_c^{\gamma-1}}{c^2}\frac{d}{d\rho_c}(\xi\frac{d\theta}{d\xi})_s](\xi^2\frac{d\theta}{d\xi})_s\rho_c^{\frac{3\gamma-4}{2}} . \tag{5.94}$$

Classically, $\frac{d}{d\rho_c}(\xi\frac{d\theta}{d\xi})_s = 0$, and hence to the first order in $1/c^2$ equation (5.94) becomes

$$\frac{3\gamma-4}{2}(\xi^2\frac{d\theta}{d\xi})_s + \rho_c\frac{d}{d\rho_c}(\xi^2\frac{d\theta}{d\xi})_s - \frac{3\gamma-4}{2}\frac{8K\rho_c^{\gamma-1}}{c^2}(\xi\frac{d\theta}{d\xi})_s(\xi^2\frac{d\theta}{d\xi})_s$$

$$+8(\gamma-1)\frac{K\rho_c^{\gamma-1}}{c^2}(\xi\frac{d\theta}{d\xi})_s(\xi^2\frac{d\theta}{d\xi})_s = 0 .$$

Consequently,

$$\frac{3\gamma-4}{2} + \rho_c\frac{d}{d\rho_c}[\log(\xi^2\frac{d\theta}{d\xi})_s] - (\frac{3\gamma-4}{2})\frac{8K\rho_c^{\gamma-1}}{c^2}(\xi\frac{d\theta}{d\xi})_s$$

$$+8(\gamma-1)\frac{K\rho_c^{\gamma-1}}{c^2}(\xi\frac{d\theta}{d\xi})_s = 0 . \tag{5.95}$$

Assuming now in accordance with the work of previous investigators, that marginal instability does indeed occur at the maximum of the mass (as a function of ρ_c), and that the condition of marginal stability is of the form

$$\gamma - 4/3 = \frac{Cp_c}{\rho_c c^2} = C\sigma = \frac{CK\rho_c^{\gamma-1}}{c^2} , \qquad (5.96)$$

where C is a numerical constant, it will be shown, by inserting (5.96) in (5.95) that, for the particular type of sphere in question, C is approximately 2.25, in agreement with equation (5.87).

Substituting (5.96) in equation (5.95), we obtain

$$\frac{3C\sigma}{2} + \rho_c \frac{d}{d\rho_c}[\log(\xi^2\frac{d\theta}{d\xi})_s] + {}^8/_3\sigma\frac{(\xi^2\frac{d\theta}{d\xi})_s}{\xi_s} = 0 \quad . \qquad (5.97)$$

Using the tables for the classical Lane-Emden functions,

$$\frac{-(\xi^2\frac{d\theta}{d\xi})_s}{\xi_s} = 0.2926 \ .$$

Hence, to the first order in $1/c^2$, (5.97) yields

$$\rho_c\frac{d}{d\rho_c}[\log(\xi^2\frac{d\theta}{d\xi})_s] + [\tfrac{3}{2} C - 0.7803] = 0 ,$$

and therefore

$$\frac{d}{d\rho_c}[\log(\xi^2\frac{d\theta}{d\xi})_s] + [\tfrac{3}{2} C - 0.7803]\frac{K\rho_c^{\gamma-2}}{c^2} = 0 .$$

On integrating this expression we obtain

$$\log\left(\xi^2\frac{d\theta}{d\xi}\right)_s = A-\left[\tfrac{3}{2}\ C-0.7803\right]\frac{\sigma}{\gamma-1}\ , \tag{5.98}$$

where A is a constant of integration. To the first post-Newtonian approximation, it therefore follows that

$$\left(\xi^2\frac{d\theta}{d\xi}\right)_s = A\ \exp\left[-\left[\tfrac{9}{2}\ C-2.3410\right]\sigma\right]. \tag{5.99}$$

To determine A , we note that, when $\sigma=0$, $\left(\xi^2\frac{d\theta}{d\xi}\right)_s$ takes the classical value, namely

$$\left(\xi^2\frac{d\theta}{d\xi}\right)_s = -2.0182\ ,$$

and so equation (5.99) can be replaced by

$$\left(-\xi^2\frac{d\theta}{d\xi}\right)_s = 2.0182\ \exp\left[-\left[\tfrac{9}{2}\ C-2.3410\right]\sigma\right]. \tag{5.100}$$

Using Appendix VI, it follows that

$$\left(-\xi^2\frac{d\theta}{d\xi}\right)_s = 2.0182-15.44\,\sigma\ , \tag{5.101}$$

to the first post-Newtonian approximation.

Hence, equation (5.100) gives

$$2.0182-15.44\sigma = 2.0182\left[1-\left[\tfrac{9}{2}\ C-2.3410\right]\sigma\right]$$

and consequently,

$$\frac{9}{2} C-2.341 = 7.6512 ,$$

giving

$$C = 2.22 , \tag{5.102}$$

in reasonably good agreement with the value 2.25 of the constant C in equation (5.87).

Finally we note that, on using equations (V.15), (V.16) and (V.5) with $n=3$, we obtain to order $1/c^2$,

$$\frac{GM}{Rc^2} = \frac{4\pi G\rho_c}{c^2} \alpha^2 \frac{v(\xi_s)}{\xi_s} , \tag{5.103}$$

where

$$\alpha^2 = \frac{p_c}{\pi G\rho c^2} ,$$

and hence

$$\frac{GM}{Rc^2} = \frac{4p_c}{\rho_c c^2} \frac{v(\xi_s)}{\xi_s} . \tag{5.104}$$

Consequently,

$$\frac{GM}{Rc^2} = \frac{4p_c}{\rho_c c^2} \frac{2.0182}{6.3968} ,$$

and therefore

$$\frac{2GM}{Rc^2} = 2.34\frac{p_c}{\rho_c c^2} . \tag{5.105}$$

Substituting (5.105) in (5.87), we obtain

$$\gamma^{-4/3} = \frac{2.25}{2.34}\left(\frac{2GM}{Rc^2}\right) ,$$

which gives

$$R = \frac{0.96}{\gamma^{-4/3}} R_s , \qquad (5.106)$$

and conclude that instability occurs if the mass contracts to a radius given by (5.106).

REFERENCES

(1) Chandrasekhar, S., Astrophysical Journal, 140, 417, 1964.

(2) Rosseland, S., The Pulsation Theory of Variable Stars, (Oxford, Clarendon Press), 1949.

(3) Bondi, H., Proceedings of the Royal Society, 281, 39, 1964.

(4) Kaplan, S.A., and Lupanov, G.A., Soviet Astronomy, 9, No.2, 233, 1965.

(5) Tooper, R.F., Astrophysical Journal, 140, No.2, 434, 1964.

(6) Tooper, R.F., Astrophysical Journal, 142, 1541, 1965.

Appendix I

Derivation of the Gravitational Potential Energy in the Envelope of a Composite Model.

We calculate the gravitational potential energy (in the classical limit) Ω_i of the outer part of the model (the envelope) between the interface $r=r_i$ and the surface $r=R$. Thus,

$$-\Omega_i = G\int_{r_i}^{R} \frac{M_r dM_r}{r} = \frac{1}{2}G\left(\frac{M^2}{R} - \frac{M_i^2}{r_i}\right) + \frac{1}{2}G\int_{r_i}^{R} \frac{M_r^2}{r^2}dr \ . \qquad (I.1)$$

Defining S_r by

$$\frac{dS_r}{dr} = \frac{GM_r}{r^2} \ ,$$

we obtain

$$-\Omega_i = \frac{1}{2}G\left(\frac{M^2}{R} - \frac{M_i^2}{r_i}\right) + \frac{1}{2}\int_{r_i}^{R} \frac{dS_r M_r d_r}{dr} \ ,$$

and hence,

$$-\Omega_i = \frac{1}{2}G\left(\frac{M^2}{R} - \frac{M_i^2}{r_i}\right) - \frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}S_i M_i - \frac{1}{2}\int_{r_i}^{R} S_r dM_r \ ,$$

where we have used the formula $S_R = -\frac{GM}{R}$, and where subscript i, as before, denotes interfacial values. Consequently,

$$-\Omega_i = -\frac{1}{2}G\frac{M_i^2}{r_i} - \frac{1}{2}S_iM_i - \frac{1}{2}\int_{r_i}^{R} S\,dM_r . \qquad (I.2)$$

In the classical limit,

$$-\frac{dS_r}{dr} = -\frac{GM_r}{r^2} = \frac{1}{\rho}\frac{dp}{dr} = (n_1+1)\frac{d}{dr}\left(\frac{p}{\rho}\right) ,$$

and so, on integrating, we have

$$-S_r+S_R = (n_1+1)\frac{p}{\rho} .$$

Hence,

$$-S_r = (n_1+1)\frac{p}{\rho} + \frac{GM}{R} , \qquad (I.3)$$

and thus,

$$-S_i = (n_1+1)\frac{p_i}{\rho_i} + \frac{GM}{R} . \qquad (I.4)$$

Consequently, on using equations (I.2), (I.3) and (I.4), we obtain

$$-\Omega_i = -\frac{1}{2}\frac{GM_i^2}{r_i} + \frac{1}{2}(n_1+1)\frac{p_i}{\rho_i}M_i + \frac{1}{2}G\frac{M}{R}M_i + \frac{1}{2}(n_1+1)\int_{r_i}^{R}\frac{p}{\rho}dM_r + \frac{1}{2}\frac{GM}{R}\int_{r_i}^{R} dM_r ,$$

and so

$$-\Omega_i = -\frac{1}{2}\frac{GM_i^{2}}{r_i} + \frac{1}{2}\frac{GM^2}{R} + \frac{1}{2}(n_1+1)\frac{p_i}{\rho_i}M_i + \frac{1}{2}(n_1+1)\int_{r_i}^{R} p dV \ , \tag{I.5}$$

where $dV = 4\pi r^2 dr$.

Also,

$$-\Omega_i = G\int_{r_i}^{R}\frac{M_r dM_r}{r} = -4\pi\int_{r_i}^{R}\frac{dp}{dr} r^3 dr \ ,$$

$$= -4\pi[pr^3]_{r_i}^{R} + 4\pi . 3\int_{r_i}^{R} pr^2 dr \ .$$

Hence

$$-\Omega_i = 3p_i v_i + 3\int_{r_i}^{R} pdV \ . \tag{I.6}$$

Using (I.6) in (I.5) we have

$$-\left(\frac{5-n_1}{3}\right)\Omega_i = G\left(\frac{M^2}{R} - \frac{M_i^{2}}{r_i}\right) + (n_1+1)\frac{p_i}{\rho_i}M_i - (n_1+1)p_i V_i \ . \tag{I.7}$$

This is the desired formula for the gravitational potential energy of the envelope. It may be noted that, in the particular case when the interface is at the centre of the model,

$$\Omega = \frac{3}{n_1 - 5} \frac{GM^2}{R} \quad , \tag{I.8}$$

which is the usual expression for the gravitational potential energy of an adiabatic fluid sphere (or a polytrope) of index n_1. (Chapter 3, refs.[6 and 7]).

Appendix II

Derivation of Formula (3.92) for $E_b-(E_b)_{\xi s}$

In the first post-Newtonian approximation, equation (3.91) for the difference in the binding energies $E_b-(E_b)_{\xi s}$ is

$$E_b-(E_b)_{\xi s} = 4\pi\rho_{g_c}c^2 \quad a_1^3(n_1+1)\sigma_1\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\frac{dv_1}{d\eta}\,d\eta$$

$$+ a_1^3\,\tfrac{3}{2}(n_1+1)^2\sigma_1^2\int_{\eta_i}^{\eta_s}\frac{v_1^2}{\eta^2}\frac{dv_1}{d\eta}d\eta$$

$$-a^3 4\sigma\int_{\xi i}^{\xi s}\frac{v}{\xi}\frac{dv}{d\xi}d\xi - a^3 24\sigma^2\int_{\xi i}^{\xi s}\frac{v^2}{\xi^2}\frac{dv}{d\xi}d\xi$$

$$-a_1^3A_1\sigma_1\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\phi[(n_1+1)\sigma_1\frac{v_1}{\eta}-A_1\sigma_1\phi]\,d\eta-a_1^3A_1\sigma_1\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\phi d\eta$$

$$+a^3A\sigma\int_{\xi i}^{\xi s}\frac{dv}{d\xi}\theta[4\sigma\frac{v}{\xi}- A\sigma\theta]d\xi+a^3A\sigma\int_{\xi i}^{\xi s}\frac{dv}{d\xi}\theta d\xi \quad .$$

Hence,

$$E_b-(E_b)_{\xi s} = 4\pi\rho_{g_c}c^2\left\{a_1^3(n_1+1)\sigma_1\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\frac{dv_1}{d\eta}d\eta-a_1^3A_1\sigma_1\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\phi d\eta\right.$$

$$\left.-a^3 4\sigma\int_{\xi i}^{\xi s}\frac{v}{\xi}\frac{dv}{d\xi}d\xi+a^3A\sigma\int_{\xi i}^{\xi s}\frac{dv}{d\xi}\theta d\xi\right\}$$

$$+4\pi\rho_{g_c}c^2\left\{\frac{3}{2}(n_1+1)^2\sigma_1{}^2a_1^3\int_{\eta_i}^{\eta_s}\frac{v_1{}^2}{\eta^2}\frac{dv_1}{d\eta}d\eta\right.$$

$$-a_1{}^3A_1\sigma_1{}^2\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\phi[(n_1+1)\sigma_1\frac{v_1}{\eta}-A_1\sigma_1\phi]d\eta$$

$$\left.-24\sigma^2a^3\int_{\xi_i}^{\xi_s}\frac{v^2}{\xi^2}\frac{dv}{d\xi}d\xi+a^3A\sigma^2\int_{\xi_i}^{\xi_s}\frac{dv}{d\xi}\theta[4\sigma\frac{v}{\xi}-A\sigma\theta]d\xi\right\}. \qquad (II.1)$$

We note that, apart from a change of sign, the above expression is symmetrical in the core and envelope variables and parameters, provided that in the core n_1 is replaced by 3. Thus, in the following, we need only consider only the envelope variables, with the knowledge that the corresponding expressions for the core can immediately be written down.

In the expression

$$I_\eta=(n_1+1)\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\frac{dv_1}{d\eta}d\eta-A_1\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\phi d\eta\ ,$$

we use equation (3.29), i.e. $\frac{dv_1}{d\eta}=\eta^2\phi^{n_1}(1+A_1\sigma_1\phi)$ to obtain

$$A_1\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\phi d\eta=A_1\int_{\eta_i}^{\eta_s}\phi^{n_1+1}\eta^2d\eta+\sigma_1A_1{}^2\int_{\eta_i}^{\eta_s}\phi^{n_1+2}\eta^2d\eta$$

$$=\frac{A_1}{3}[\phi^{n_1+1}\eta^3]_{\eta_i}^{\eta_s}-\frac{A_1(n_1+1)}{3}\int_{\eta_i}^{\eta_s}\eta^3\phi^{n_1}\frac{d\phi}{d\eta}d\eta+{}_1A_1{}^2\int_{\eta_i}^{\eta_s}\phi^{n_1+2}\eta^2d\eta,$$

and hence

$$A_1\int_{\eta_i}^{\eta_s}\frac{dv_1}{d\eta}\phi d\eta = -\frac{A_1}{3}[\phi_i^{n_1+1}\eta_i^3]-\frac{A_1(n_1+1)}{3}\int_{\eta_i}^{\eta_s}\frac{\eta^2\phi^{n_1}}{\eta}(\eta^2\frac{d\phi}{d\eta})d\eta$$

$$+\sigma_1 A_1^2\int_{\eta_i}^{\eta_s}\phi^{n_1+2}\eta^2 d\eta . \qquad (II.2)$$

From the equation of hydrostatic equilibrium (3.28) we obtain, to the first order in σ_1,

$$v_1(\eta) = -\eta^2\frac{d\phi}{d\eta} - 2\sigma_1(n_1+1)\eta^3(\frac{d\phi}{d\eta})^2 - \sigma_1(A_1+1)\phi v_1(\eta) - \sigma_1\eta^3\phi^{n_1+1},$$

from which it follows that

$$(n_1+1)\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\frac{dv_1}{d\eta}d\eta = (n_1+1)\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\phi^{n_1}\eta^2(1+A_1\sigma_1\phi)d\eta$$

$$= (n_1+1)\int_{\eta_i}^{\eta_s}\frac{v_1}{\eta}\phi^{n_1}\eta^2 d\eta + A_1(n_1+1)\sigma_1\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1+1}}{\eta}\eta^2 v_1 d\eta$$

$$= -(n_1+1)\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2}{\eta}(\eta^2\frac{d\phi}{d\eta})d\eta - 2(n_1+1)^2\sigma_1$$

$$\times\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2}{\eta}\eta^3(\frac{d\phi}{d\eta})^2 d\eta$$

$$-(A_1+1)(n_1+1)\sigma_1\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1+1}}{\eta}\eta^2 v_1(\eta)d\eta - \sigma_1(n_1+1)\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4 d\eta$$

$$+A_1(n_1+1)\sigma_1\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1+1}}{\eta}\eta^2 v_1(\eta)d\eta \ . \quad (II.3)$$

Hence,

$$I_\eta = \frac{A_1}{3}\eta_i^3\phi_i^{n_1+1}+(n_1+1)(\frac{A_1}{3}-1)\int_{\eta_i}^{\eta_s}\frac{\eta^2\phi^{n_1}}{\eta}(\eta^2\frac{d\phi}{d\eta})d\eta-\sigma_1 A_1^2\int_{\eta_i}^{\eta_s}\phi^{n_1+2}\eta^2 d\eta$$

$$-2(n_1+1)^2\sigma_1\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2}{\eta}\eta^3(\frac{d\phi}{d\eta})^2 d\eta-\sigma_1(n_1+1)(A_1+1)\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1+1}}{\eta}\eta^2 v_1(\eta)d\eta$$

$$-\sigma_1(n_1+1)\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4 d\eta+A_1(n_1+1)\sigma_1\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1+1}\eta^2}{\eta}v_1 d\eta \ . \quad (II.4)$$

Using this result in equation (II.1), and confining attention to terms depending only on η, we obtain

$$[E_b-(E_b)_{\zeta s}]_\eta = 4\pi\rho g_c a_1^3\sigma_1\left[\frac{A_1}{3}\eta_i^3\phi_i^{n_1+1}+(n_1+1)(\frac{A_1}{3}-1)\times\int_{\eta_i}^{\eta_s}\eta^2\frac{\phi^{n_1}}{\eta}(\eta^2\frac{d\phi}{d\eta})d\eta\right]$$

$$+4\pi\rho g_c c^2 a_1^3\sigma_1^2\left[\frac{3}{2}(n_1+1)^2\int_{\eta_i}^{\eta_s}\phi^{n_1}\eta^4(\frac{d\phi}{d\eta})^2 d\eta+A_1\int_{\eta_i}^{\eta_s}\eta^2\phi^{n_1+1}[+(n_1+1)\eta\frac{d\phi}{d\eta}+A_1\phi]d\eta\right.$$

$$\left.-A_1^2\int_{\eta_o}^{\eta_s}\phi^{n_1+2}\eta^2 d\eta-2(n_1+1)^2\int_{\eta_i}^{\eta_s}\phi^{n_1}\eta^4(\frac{d\phi}{d\eta})^2 d\eta+(n_1+1)(A_1+1)\times\int_{\eta_i}^{\eta_s}\phi^{n_1+1}\eta^3\frac{d\phi}{d\eta}d\eta\right]$$

$$-(n_1+1)\int_{\eta_i}^{\eta_s} \phi^{2n_1+1}\eta^4 d\eta - A_1(n_1+1)\int_{\eta_i}^{\eta_s} \phi^{n_1+1}\eta^3\frac{d\phi}{d\eta}d\eta \Big] ,$$

to the first post-Newtonian approximation, which reduces to

$$(E_b-(E_b)_{\xi a}]_\eta = 4\pi\rho_{g_c} c^2 \alpha_1{}^3\sigma_1 \Big[\frac{A_1}{3}\eta_i{}^3\phi_i{}^{n_1+1}$$

$$+(n_1+1)\left(\frac{A_1}{3}-1\right)\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2}{\eta}\left(\eta^2\frac{d\phi}{d\eta}\right)d\eta \Big]$$

$$+4\pi\rho_{g_c}\alpha_1{}^3\sigma_1{}^2c^2\Big[(n_1+1)(A_1+1)\int_{\eta_i}^{\eta_s}\phi^{n_1+1}\eta^3\frac{d\phi}{d\eta}d\eta$$

$$-\frac{1}{2}(n_1+1)^2\int_{\eta_i}^{\eta_s}\phi^{n_1}\eta^4\left(\frac{d\phi}{d\eta}\right)^2 d\eta-(n_1+1)\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4 d\eta\Big] . \quad \text{(II.5)}$$

Considering now the post-Newtonian terms in equation (II.5), we find that to the first order in $1/c^2$, equation (3.28) becomes

$$\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right) = -\eta^2\phi^{n_1} ,$$

and hence,

$$\eta^2\frac{d^2\phi}{d\eta^2} = -\eta^2\phi^{n_1}-2\eta\left(\frac{d\phi}{d\eta}\right) .$$

Consequently,

$$\int_{\eta_i}^{\eta_s} \phi^{n_1+1} \eta^3 \frac{d\phi}{d\eta} d\eta = [\phi^{n_1+1} \frac{\eta^4}{4} \frac{d\phi}{d\eta}]_{\eta_i}^{\eta_s} - \frac{1}{4}\int_{\eta_i}^{\eta_s} \eta^4 [(n_1+1)\phi^{n_1} (\frac{d\phi}{d\eta})^2 + \phi^{n_1+1} \frac{d^2\phi}{d\eta^2}] d\eta$$

becomes

$$\int_{\eta_i}^{\eta_s} \phi^{n_1+1} \eta^3 \frac{d\phi}{d\eta} d\eta = -\frac{1}{4}\phi_i^{n_1+1} \eta_i^4 (\frac{d\phi}{d\eta})_i - \frac{n_1+1}{4}\int_{\eta_i}^{\eta_s} \phi^{n_1} \eta^4 (\frac{d\phi}{d\eta})^2 d\eta + \frac{1}{4}\int_{\eta_i}^{\eta_s} \phi^{2n_1+1} \eta^4 d\eta + \frac{1}{2}\int_{\eta_i}^{\eta_s} \eta^3 \phi^{n_1+1} \frac{d\phi}{d\eta} d\eta ,$$

and hence

$$\frac{n_1+1}{2}\int_{\eta_i}^{\eta_s} \phi^{n_1} \eta^4 (\frac{d\phi}{d\eta})^2 d\eta = -\frac{1}{2}\phi_i^{n_1+1} \eta_i^4 (\frac{d\phi}{d\eta})_i + \frac{1}{2}\int_{\eta_i}^{\eta_s} \phi^{2n_1+1} \eta^4 d\eta - \int_{\eta_i}^{\eta_s} \phi^{n_1+1} \eta^3 \frac{d\phi}{d\eta} d\eta . \qquad \text{(II.6)}$$

Therefore, using (II.6) the post-Newtonian terms in expression (II.5) become

$$B = (n_1+1)(A_1+1)\int_{\eta_i}^{\eta_s} \phi^{n_1+1} \eta^3 \frac{d\phi}{d\eta} d\eta - \frac{1}{2}(n_1+1)^2 \int_{\eta_i}^{\eta_s} \phi^{n_1} \eta^4 (\frac{d\phi}{d\eta})^2 d\eta - (n_1+1)\int_{\eta_i}^{\eta_s} \phi^{2n_1+1} \eta^4 d\eta$$

$$=\frac{n_1+1}{2}\phi_i^{\,n_1+1}\eta_i^{\,4}\left(\frac{d\phi}{d\eta}\right)_i-\frac{3(n_1+1)}{2}\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4 d\eta$$

$$+(n_1+1)(A_1+2)\int_{\eta_i}^{\eta_s}\phi^{n_1+1}\eta^3\frac{d\phi}{d\eta}\ ,$$

and since

$$(n_1+1)(n_1+2)\int_{\eta_i}^{\eta_s}\phi^{n_1+1}\eta^3\left(\frac{d\phi}{d\eta}\right)d\eta=-(n_1+1)\eta_i^{\,3}\phi_i^{\,n_1+2}-3(n_1+1)\int_{\eta_i}^{\eta_s}\eta^2\phi^{n_1+2}d\eta$$

it follows that

$$B=\frac{(n_1+1)}{2}\phi_i^{\,n_1+1}\eta_i^{\,4}\left(\frac{d\phi}{d\eta}\right)_i-\frac{(A_1+2)(n_1+1)}{(n_1+2)}\eta_i^{\,3}\phi_i^{\,n_1+2}$$
$$-\frac{3(n_1+1)}{2}\int_{\eta_i}^{\eta_s}\eta^4\phi^{2n_1+1}d\eta-\frac{3(A_1+2)(n_1+1)}{(n_1+2)}\int_{\eta_i}^{\eta_s}\eta^2\phi^{n_1+2}d\eta\ . \qquad \text{(II.7)}$$

Using equation (II.7) in (II.5) we obtain

$$[E_b-(E_b)_{\text{Ⅎ}s}]_\eta=4\pi\rho_{g_c}c^2\alpha_1^{\,3}\sigma_1\Bigg[\frac{A_1}{3}\eta_i^{\,3}\phi_i^{\,n_1+1}+(n_1+1)\left(\frac{A_1}{3}-1\right)$$

$$\times\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2}{\eta}\left(\eta^2\frac{d\phi}{d\eta}\right)d\eta\Bigg]$$

$$+4\pi\rho_{g_c}\alpha_1^{\,3}\sigma_1^{\,2}c^2\Bigg[\frac{n_1+1}{2}\phi_i^{\,n_1+1}\eta_i^{\,4}\left(\frac{d\phi}{d\eta}\right)_i-\frac{(n_1+1)(A_1+2)}{n_1+2}\eta_i^{\,3}\phi_i^{\,n_1+2}$$
$$-\frac{3(n_1+i)}{2}\int_{\eta_i}^{\eta_s}\eta^4\phi^{2n_1+1}d\eta-\frac{3(A_1+2)(n_1+1)}{n_1+2}\int_{\eta_i}^{\eta_s}\eta^2\phi^{n_1+2}d\eta\Bigg] \qquad \text{(II.8)}$$

Since we are taking the equation of state to be that of an adiabatic fluid we take $A_1=n_1$ and $A=3$, and hence equation (II.1) reduces to

$$E_b-(E_b)_{\xi_s}=4\pi\rho_{g_c}c^2a_1^3\sigma_1\left[\frac{n_1}{3}\eta_i^3\phi_i^{n_1+1}+(n_1+1)\left(\frac{n_1}{3}-1\right)\right.$$

$$\left.\times\int_{\eta_i}^{\eta_s}\frac{\phi^{n_1}\eta^2}{\eta}\left(\eta^2\frac{d\phi}{d\eta}\right)d\eta-\frac{a^3}{a_1^3}\frac{\sigma}{\sigma_1}\xi_i^3\theta_i^4\right]$$

$$+4\pi\rho_{g_c}a_1^3\sigma_1^2c^2\left[\frac{n_1+1}{2}\phi_i^{n_1+1}\eta_i^4\left(\frac{d\phi}{d\eta}\right)_i-(n_1+1)\eta_i^3\phi_i^{n_1+2}\right.$$

$$\left.-\frac{3(n_1+1)}{2}\int_{\eta_i}^{\eta_s}\phi^{2n_1+1}\eta^4d\eta-3(n_1+1)\int_{\eta_i}^{\eta_s}\eta^2\phi^{n_1+2}d\eta\right]$$

$$-4\pi\rho_{g_c}c^2a^3\sigma^2\left[2\theta_i^4\xi_i^4\left(\frac{d\theta}{d\xi}\right)_i-4\xi_i^3\theta_i^5-6\int_{\xi_i}^{\xi_s}\theta^7\xi^4d\xi-12\int_{\xi_i}^{\xi_s}\theta^5\xi^2d\xi\right].$$

(II.9)

Appendix III

Derivation of Formula (4.15) for the Gravitational Potential Energy in the Core.

We calculate, in the classical limit, the gravitational potential energy Ω_c of the core.

Since

$$-\Omega_c = G\int_0^{r_i} \frac{\rho M_r}{r}\, dV = G\int_0^{r_i} \frac{M_r dM_r}{r} \tag{III.1}$$

it follows that

$$-\Omega_c = \frac{1}{2}G\,\frac{M_i^2}{r_i} + \frac{1}{2}G\int_0^{r_i} \frac{M_r^2}{r^2}dr\ ,$$

and hence on defining S_r by

$$\frac{dS_r}{dr} = \frac{GM_r}{r^2}\ ,$$

we obtain

$$-\Omega_c = \frac{1}{2}G\frac{M_i^2}{r_i} + \frac{1}{2}\int_0^{r_i}\frac{dS_r}{dr}\, M_r d_r\ .$$

Consequently,

$$-\Omega_c = \frac{1}{2}\,\frac{GM_i^2}{r_i} + \frac{1}{2}S_iM_i - \frac{1}{2}\int_0^{r_i} S_r dM_r\ . \tag{III.2}$$

In the classical limit,

$$-\frac{dS_r}{dr} = -\frac{GM_r}{r^2} = \frac{1}{\rho}\frac{dp}{dr} = 4\frac{d}{dr}\left(\frac{p}{\rho}\right) ,$$

and so, on integrating between r and r_i, we obtain

$$\left[-S_r\right]_r^{r_i} = 4\left[\frac{p}{\rho}\right]_r^{r_i} ,$$

and hence,

$$S_r = S_i + \frac{4p_i}{\rho_i} - \frac{4p}{\rho} . \qquad \text{(III.3)}$$

On substituting for S_r in (III.2) we obtain

$$-\Omega_c = \frac{1}{2}G\frac{M_i^2}{r_i} + \frac{1}{2}S_iM_i - \frac{1}{2}S_iM_i - \frac{2p_i}{\rho_i}M_i + 2\int_0^{r_i} \frac{p}{\rho}dM_r ,$$

and hence,

$$-\Omega_c = \frac{1}{2}G\frac{M_i^2}{r_i} - \frac{2p_i}{\rho_i}M_i + 2\int_0^{r_i} pdV . \qquad \text{(III.4)}$$

From (III.1) we see that

$$-\Omega_c = -4\pi\int_0^{r_i} \frac{dp}{dr}r^3dr ,$$

in the classical limit, and thus

$$-\Omega_c = -3p_iV_i + 3\int_0^{r_i} pdV . \qquad \text{(III.5)}$$

From equations (III.4) and (III.5) we deduce that

$$-\Omega_c = \frac{1}{2}\frac{GM_i^2}{r_i} - \frac{2p_i}{\rho_i}M_i - \frac{2}{3}[3p_iV_i - \Omega_c] ,$$

and hence that

$$-\frac{1}{3}\Omega_c = \frac{1}{3}\int_0^{r_i}\frac{G\rho M_r dV}{r} = \frac{1}{2}\frac{GM_i^2}{r_i} - \frac{2p_i}{\rho_i}M_i + 2p_iV_i . \qquad \text{(III.6)}$$

Appendix IV

Derivation of Formula (5.44) for the Equation of Motion

The equation of motion (5.43) is

$$\frac{d^2V}{dt^2} + \frac{V}{2}\frac{d^2\nu}{dt^2} + \frac{1}{2}\frac{dV}{dt}\frac{\partial\nu}{\partial t} - \frac{e^{\nu_o-\lambda_o}}{\rho+p/c^2}\cdot\frac{\partial}{\partial r}[\frac{\gamma p}{\rho}(\rho+p/c^2)\text{div } V]$$

$$-\frac{e^{\nu_o-\lambda_o}}{\rho+p/c^2}\left\{\frac{\partial}{\partial r}(\frac{\gamma p}{\rho c^2}V\frac{\partial p}{\partial r})+\frac{\partial V}{\partial r}\frac{\partial p}{\partial r}\right\}+[(\rho+p/c^2)\text{div}V+\frac{V}{c^2}\frac{\partial p}{\partial r}]\frac{(1+\frac{\gamma p}{\rho c^2})}{(\rho+p/c^2)^2}\frac{\partial p}{\partial r}e^{\nu_o-\lambda_o}$$

$$-\left[\frac{2GM_r}{r^3}V-\frac{6G^2M_r^2V}{r^4c^2}-\frac{4\pi Gp}{c^2}(V+\frac{r}{p}\frac{dp}{dt})-(1+\frac{4GM_r}{rc^2})\frac{G}{r^2}\frac{dM_r}{dt}\right]e^{\nu_o-\lambda_o} = 0 \,. \tag{IV.1}$$

On substituting for $\frac{dp}{dt}$ and $\frac{\partial p}{\partial r}$ from equations (5.34) and (5.27), we obtain

$$\frac{d^2V}{dt^2} + \frac{V}{2}\frac{d^2\nu}{dt^2} + \frac{1}{2}\frac{dV}{dt}\frac{\partial\nu}{dt} - \frac{e^{\nu_o-\lambda_o}}{\rho+p/c^2}\cdot\frac{\partial}{\partial r}[\frac{\gamma p}{\rho}(\rho+p/c^2)\text{div}V]$$

$$-[(\rho+p/c^2)(\frac{\partial V}{\partial r}+\frac{2V}{r})]\frac{(1+\frac{\gamma p}{\rho c^2})}{(\rho+p/c^2)}[\frac{V}{2}\frac{\partial\nu}{\partial t}+(\frac{GM_r}{r^2}+\frac{2G^2M_r^2}{r^3c^2}+\frac{4\pi Gpr}{c^2})e^{\nu_o-\lambda_o}]$$

$$+\frac{V}{c^2}(1+\frac{\gamma p}{\rho c^2})e^{\nu_o-\lambda_o}\frac{G^2M_r^2}{r^4} - e^{\nu_o-\lambda_o}\left[\frac{2GM_r}{r^3}V+\frac{6G^2M_r^2V}{r^4c^2}-\frac{4\pi Gp}{c^2}V\right]$$

$$-e^{\nu_o-\lambda_o}\frac{4\pi Gpr}{c^2}[\frac{\gamma p}{\rho}[(\rho+p/c^2)\text{div}V+\frac{V}{c^2}\frac{\partial p}{\partial r}]]+(1+\frac{4GM_r}{rc^2})\frac{G}{r^2}\frac{dM_r}{dt}e^{\nu_o-\lambda_o}$$

$$- \frac{e^{\nu_o - \lambda_o}}{(\rho + p/c^2)} \left[\frac{\gamma}{\rho c^2} V \left(\frac{\partial p}{\partial r}\right)^2 - \frac{\gamma p V}{\rho^2 c^2} \frac{\partial p}{\partial r} \frac{\partial \rho}{\partial r} + \frac{\gamma p}{\rho c^2} \frac{\partial V}{\partial r} \frac{\partial p}{\partial r} + \frac{\gamma p}{\rho c^2} V \frac{\partial^2 p}{\partial r^2} + \frac{\partial V}{\partial r} \frac{\partial p}{\partial r} \right] = 0 ,$$

and on using equation (5.27) again we obtain

$$\frac{d^2 V}{dt^2} + \frac{1}{2} V \frac{d^2 \nu}{dt^2} + \frac{1}{2} \frac{dV}{dt} \frac{\partial \nu}{\partial t} - \frac{e^{\nu_o - \lambda_o}}{(\rho + p/c^2)} \cdot \frac{\partial}{\partial r} \left[\frac{\gamma p}{\rho} (\rho + p/c^2) \operatorname{div} V \right]$$

$$- \left(\frac{\partial V}{\partial r} + \frac{2V}{r} - \frac{V}{c^2} \frac{GM_r}{r^2}\right) e^{\nu_o - \lambda_o} \left[\frac{GM_r}{r^2} + \frac{2G^2 M_r^2}{r^3 c^2} + \frac{4\pi G p r}{c^2} \right] + \frac{\partial V}{\partial r} e^{\nu_o - \lambda_o} \left[\frac{GM_r}{r^2} + \frac{2G^2 M_r^2}{r^3 c^2} + \frac{4\pi G p r}{c^2} \right]$$

$$- \frac{\gamma p}{\rho c^2} \left(\frac{\partial V}{\partial r} + \frac{2V}{r}\right) e^{\nu_o - \lambda_o} \frac{GM_r}{r^2} - e^{\nu_o - \lambda_o} \left[\frac{2GM_r V}{r^3} + \frac{6G^2 M_r^2 V}{c^2 r^4} - \frac{4\pi G p}{c^2} \times \left[V - r\gamma \left(\frac{\partial V}{\partial r} + \frac{2V}{r}\right) \right] \right]$$

$$- \frac{e^{\nu_o - \lambda_o}}{(\rho + p/c^2)} \left[\frac{\gamma V}{\rho c^2} \frac{G^2 M_r^2 \rho^2}{r^4} - \frac{V \rho}{c^2} \frac{G^2 M_r^2}{r^4} - \frac{\gamma p}{\rho c^2} \frac{\partial V}{\partial r} \rho \frac{GM_r}{r^2} + \frac{\gamma p V}{\rho c^2} \frac{\partial^2 p}{\partial r^2} \right] + \left[1 + \frac{4GM_r}{rc^2} \right] \frac{G}{r^2} e^{\nu_o - \lambda_o} \frac{dM_r}{dt} = 0 ,$$

where, since there is spherical symmetry, we have taken

$$\operatorname{div} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right) .$$

After some algebraic rearrangement of terms we obtain

$$\frac{d^2V}{dt^2} + \frac{1}{2}V\frac{d^2\nu}{dt^2} + \frac{1}{2}\frac{dV}{dt}\frac{\partial\nu}{\partial t} - \frac{e^{\nu_o-\lambda_o}}{(\rho+p/c^2)}\frac{\partial}{\partial r}[\frac{\gamma p}{\rho}(\rho+p/c^2)\mathrm{div}V]$$

$$- \frac{4\pi Gpr\gamma}{c^2}e^{\nu_o-\lambda_o}\frac{\partial V}{\partial r}$$

$$-Ve^{\nu_o-\lambda_o}\left[\frac{4GM_r}{r^3}+(8+\gamma)\frac{G^2M^2}{c^2r^4} + \frac{4\pi Gp}{c^2}(1+2\gamma)+\frac{\gamma p}{\rho c^2}\frac{2GM_r}{r^3} + \frac{\gamma p}{\rho^2c^2}\times\right.$$

$$\left.[-\frac{GM_r'\rho}{r^2} + \frac{2GM_r\rho}{r^3} - \frac{GM_r}{r^2}\frac{\partial\rho}{\partial r}]\right]$$

$$+\left[1+\frac{4GM_r}{rc^2}\right]\frac{G}{r^2}e^{\nu_o-\lambda_o}\frac{dM_r}{dt} = 0 \ ,$$

and hence

$$e^{\lambda_o-\nu_o}(\frac{d^2V}{dt^2} + \frac{1}{2}V\frac{d^2\nu}{dt^2} + \frac{1}{2}\frac{dV}{dt}\frac{\partial\nu}{\partial t}) = \frac{1}{(\rho+p/c^2)}[\frac{\partial}{\partial r}(\gamma p\mathrm{div}V)]$$

$$+\left\{(1-2\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^2} + \frac{4\pi Gpr\gamma}{c^2}\right\}\frac{\partial V}{\partial r}$$

$$+\left[\frac{4GM_r}{r^3} + (9+\gamma)\frac{G^2M_r^2}{c^2r^4} + \frac{4\pi Gp}{c^2}(1+2\gamma) + \frac{2p}{\rho c^2}\frac{GM}{r^3} - \frac{\gamma p}{\rho c^2}\frac{GM'}{r^2}\right]V$$

$$+\frac{\gamma p^2}{\rho^2c^2}\frac{\partial}{\partial r}(\mathrm{div}V) - \left[1+\frac{4GM_r}{rc^2}\right]\frac{G}{r^2}\frac{dM_r}{dt} = 0 \ . \tag{IV.2}$$

Since

$$\frac{1}{c^2}\frac{\partial}{\partial r}(p\mathrm{div}V) = \frac{p}{c^2}\frac{\partial}{\partial r}(\mathrm{div}V) + \frac{\mathrm{div}V}{c^2}\frac{\partial p}{\partial r} \ ,$$

and so on substituting for $\partial p/\partial r$ from equation (5.27),

$$\frac{p}{c^2}\frac{\partial}{\partial r}(\mathrm{div}V) = \frac{1}{c^2}\frac{\partial}{\partial r}(p\,\mathrm{div}V)$$

$$+\left(\frac{\partial V}{\partial r} + \frac{2V}{r}\right)\left[\frac{GM_r}{r^2}\rho + \frac{GM_r}{r^2c^2}p + \frac{2G^2M_r^{\,2}}{r^3c^2}\rho + \frac{4\pi Gp\rho r}{c^2}\right]1/c^2.$$

(IV.3)

Consequently, to the first order, in $1/c^2$,

$$\frac{\gamma p^2}{\rho^2c^2}\frac{\partial}{\partial r}(\mathrm{div}V) = \frac{\gamma p}{\rho^2c^2}\frac{\partial}{\partial r}(p\,\mathrm{div}V) + \left(\frac{\partial V}{\partial r} + \frac{2V}{r}\right)\left[\frac{\gamma p}{\rho c^2}\frac{GM_r}{r^2}\right].$$

(IV.4)

Hence, on using (IV.4) in equation (IV.2) we obtain

$$e^{\lambda_0-\nu_0}\left(\frac{d^2V}{dt^2} + \frac{1}{2}V\frac{d^2\nu}{dt^2} + \frac{1}{2}\frac{dV}{dt}\frac{\partial\nu}{\partial t}\right) = \frac{1}{\rho}\left[\frac{\partial}{\partial r}(\gamma p\,\mathrm{div}V)\right]$$

$$+\left[(1-\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^2} + \frac{4\pi Gpr\gamma}{c^2}\right]\frac{\partial V}{\partial r}$$

$$+\left[\frac{4GM_r}{r^3} + (9+\gamma)\frac{G^2M_r^{\,2}}{c^2r^4} + \frac{4\pi Gp}{c^2}(1+\gamma) + 2(1+\gamma)\frac{p}{\rho c^2}\frac{GM_r}{r^3}\right]V$$

$$-\left[1 + \frac{4GM_r}{rc^2}\right]\frac{G}{r^2}\frac{dM_r}{dt}\ldots$$

(IV.5)

Appendix V

Summary of Useful Formulae for Relativistic Polytropes (after Tooper)

For fluid spheres with equation of state of the form

$$p = K\rho^{1+\frac{1}{n}} , \qquad (V.1)$$

where K and n are constants and p, ρ denote pressure and density, respectively, if we introduce the variable θ defined by

$$\rho = \rho_c \theta^n , \qquad (V.2)$$

we obtain,

$$p = p_c \theta^{n+1} = K\rho_c^{1+\frac{1}{n}} \theta^{n+1} . \qquad (V.3)$$

In terms of θ, equation (2.20) becomes

$$2(\sigma)(n+1)\frac{d\theta}{dr} = -(1+\sigma\theta)\frac{d\nu}{dr} , \qquad (V.4)$$

where

$$\sigma = \frac{K\rho_c^{\frac{1}{n}}}{c^2} = \frac{p_c}{\rho_c c^2} . \qquad (V.5)$$

From equations (2.22) and (2.23) we have

$$e^{-\lambda} = 1 - \frac{2GM_r}{rc^2} , \tag{V.6}$$

and

$$M_r = 4\pi\rho_c \int_0^r \theta^n r^2 dr , \tag{V.7}$$

and hence from equation (V.4) we obtain, on using equation (2.17),

$$\frac{\sigma(n+1)}{1+\sigma\theta} r\frac{d\theta}{dr}\left(1-\frac{2GM_r}{rc^2}\right) + \frac{GM_r}{rc^2} + \frac{4\pi G\rho_c r^2\theta^{n+1}}{c^4} = 0 . \tag{V.8}$$

On introducing dimensionless variables ξ and $v(\xi)$ defined by

$$r = \alpha\xi , \tag{V.9}$$

and

$$M_r = 4\pi\rho_c \alpha^3 v(\xi) , \tag{V.10}$$

where

$$\alpha^2 = \left[\frac{(n+1)\sigma c^2}{4\pi G\rho c}\right] , \tag{V.11}$$

equations (V.8) and (V.7) become

$$\xi^2\frac{d\theta}{d\xi} \frac{1-2(n+1)\sigma v(\xi)/\xi}{1+\sigma\theta} + v(\xi) + \sigma\xi^3\theta^{n+1} = 0 , \tag{V.12}$$

and

$$\frac{dv}{d\xi} = \xi^2 \theta^n , \tag{V.13}$$

respectively. These equations, which constitute the general-relativistic generalization of the classical Lane-Emden equation, are to be solved subject to the boundary conditions,

$$\theta(0) = 1, \ v(0) = 0 \ . \qquad \text{(V.14)}$$

The boundary of the sphere is given by the smallest positive value ξ_s of ξ for which

$$\theta(\xi_s) = 0 \ .$$

And from equations (V.9) and (V.10), it follows that the mass and radius are given by

$$M = 4\pi\rho_c a^3 v(\xi_s) \ , \qquad \text{(V.15)}$$

and

$$R = a\,\xi_s \ . \qquad \text{(V.16)}$$

Appendix VI

Derivation of Formula (5.101)

We shall derive equation (5.101) using equation (5.92) in the first post-Newtonian approximation.

Writing $\theta = \theta^{(1)} + \theta^{(2)}$, $v = v^{(1)} + v^{(2)}$,

$$(VI.1)$$

where superscripts (1) and (2) denote the classical and the post-Newtonian terms respectively, $\theta^{(1)}$ and $v^{(1)}$ satisfy the usual Lane-Emden equations,

$$\xi^2 \frac{d\theta^{(1)}}{d\xi} + v^{(1)} = 0 , \quad \frac{dv^{(1)}}{d\xi} = \xi^2 \theta^{(1)3} , \qquad (VI.2)$$

and the equation satisfied by $\theta^{(2)}$ and $v^{(2)}$ is

$$\xi^2 \frac{d\theta^{(2)}}{d\xi} + \frac{8v^{(1)2}}{\xi} + v^{(2)} + \xi^3 \theta^{(1)4} = 0 . \qquad (VI.3)$$

If ξ_s is the first zero of θ , and $\xi_s^{(1)}$ is the first zero of $\theta^{(1)}$, then

$$\theta(\xi_s) = 0 , \qquad \theta^{(1)}(\xi_s^{(1)}) = 0 , \qquad (VI.4)$$

where

$$\xi = \xi^{(1)} + \sigma \xi^{(2)} . \qquad (VI.5)$$

Hence,

$$\theta^{(1)}(\zeta_s) = \theta^{(1)}(\zeta_s^{(1)}+\sigma\zeta_s^{(2)}) \sim \theta^{(1)}(\zeta_s^{(1)} + \sigma\zeta_s^{(2)}\left(\frac{d\theta^{(1)}}{d\zeta}\right)_{\zeta_s^{(1)}} ,$$

and so

$$\theta^{(1)}(\zeta_s) \sim \frac{\sigma\zeta_s^{(2)}\ \zeta_s^{(1)^2}\left(\frac{d\theta}{d\zeta}\right)_{\zeta_s^{(1)}}}{\zeta_s^{(1)^2}} .$$

Consequently, using (VI.2),

$$\theta^{(1)}(\zeta_s) \sim \frac{-\sigma\zeta_s^{(2)}v^{(1)}(\zeta_s^{(1)})}{\zeta_s^{(1)^2}} . \qquad \text{(VI.6)}$$

Similarly,

$$\theta^{(2)}(\zeta_s) \simeq \theta^2(\zeta_s^{(1)})+\sigma\zeta_s^{(2)}\left(\frac{d\theta^{(2)}}{d\zeta}\right)_{\zeta_s^{(1)}} . \qquad \text{(VI.7)}$$

Thus, from equations (VI.4), (VI.6), and (VI.7), it follows that

$$0 = \theta(\zeta_s) = \theta^{(1)}(\zeta_s) + \theta^{(2)}(\zeta_s) \sim \frac{-\sigma\zeta_s^{(2)}v^{(1)}(\zeta_s^{(1)})}{\zeta_s^{(1)^2}} + \theta^{(2)}(\zeta_s^{(1)}) ,$$

and hence,

$$\xi_s^{(2)} \sim \frac{\theta^{(2)}(\xi_s^{(1)})\,\xi_s^{(1)^2}}{v^{(1)}(\xi_s^{(1)})} \,. \tag{VI.8}$$

From equation (VI.3), we see that, in the classical limit,

$$v^{(2)}(\xi_s) \sim -(\xi^2 \frac{d\theta^{(2)}}{d\xi})_{\xi_s^{(1)}} \; \frac{-8v^{(1)}(\xi_s^{(1)})^2}{\xi_s^{(1)}} \,. \tag{VI.9}$$

Since $v^{(1)}(\xi_s) \sim v^{(1)}(\xi_s^{(1)})$, equations (VI.1), (VI.2) and (VI.8) give

$$v(\xi_s) \sim -\xi_s^2 (\frac{d\theta^{(1)}}{d\xi})_{\xi_s} + \sigma v^{(2)}(\xi_s)$$

$$\sim -\xi_s^{(1)^2} (\frac{d\theta^{(1)}}{d\xi})_{\xi_s^{(1)}} [1 + \frac{\sigma \xi_s^{(1)} \theta^{(2)}(\xi_s^{(1)})}{v^{(1)}(\xi_s^{(1)})}]^2 + \sigma v^{(2)}(\xi_s). \tag{VI.10}$$

Hence, using (VI.9), we obtain

$$v(\xi_s) = -(\xi_s^2 \frac{d\theta^{(1)}}{d\xi})_{\xi_s^{(1)}} [1 + \frac{\sigma 2 \xi \theta^{(2)}}{v^{(1)}(\xi)}]_{\xi_s^{(1)}} + \sigma[-\xi^2 \frac{d\theta^{(2)}}{d\xi} - \frac{8(v^{(1)}(\xi)^2}{\xi_s}]_{\xi_s^{(1)}} \,. \tag{VI.11}$$

From equation (5.91),

$$\left(-\xi^{2}\frac{d\theta}{d\xi}\right)_{\xi_s} = v(\xi_s)\left[1-8\sigma\left(\xi\frac{d\theta}{d\xi}\right)_{\xi_s^{(1)}}\right]. \quad \text{(VI.12)}$$

Consequently, from equations (VI.11) and (VI.12),

$$\left(-\xi^{2}\frac{d\theta}{d\xi}\right)_{\xi_s} = \left(-\xi^{2}\frac{d\theta^{(1)}}{d\xi}\right)_{\xi_s^{(1)}} - \sigma\left[-2\xi\theta^{(2)} + \xi^{2}\frac{d\theta^{(2)}}{d\xi}\right]_{\xi_s^{(1)}} . \quad \text{(VI.13)}$$

On using the table of the post-Newtonian functions for a polytrope of index 3 given by Chandrasekhar[1], equation (VI.13) becomes

$$\left(-\xi^{2}\frac{d\theta}{d\xi}\right)_{s} = 2.0182 - 15.44\sigma . \quad \text{(VI.14)}$$