SOME PROPERTIES OF CERTAIN FUNCTION SPACES AND SOME ASSOCIATED TOPICS IN THE THEORY OF FUNCTIONS OF A REAL VARIABLE

by

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ABSTRACT

This thesis is concerned with a general algebraic construction, applications of this construction, and related topics in the theory of functions of a real variable.

Chapter one is mainly concerned with an extension process related to that of the extension of a ring without zero divisors to its field of quotients, but which is applicable to systems of a more general character. The construction proves fruitful when applied to certain function spaces, and in chapter two, isomorphisms are established between extensions of a particular space $\mathfrak{T}$ (the slowly increasing functions), and certain classes of distributions. Chapter three is mainly devoted to various topological considerations, pertaining both to the general situation and to the particular case considered in chapter two. It is shown that the isomorphisms established in the second chapter provide homeomorphisms, when $\mathfrak{T}$ is endowed with the topology of pointwise convergence (amongst others) and when the distribution spaces in question have the weak dual topologies. Realisations of the algebraic theory in the realm of function spaces pose some questions in the theory of functions of a real variable, and in answer to these questions, a generalisation of a certain theorem of N. Wiener's is proven in chapter four. Besides being of some intrinsic interest, this enables further examples to

be constructed, some of which are related to the spaces of type $\mathfrak{S}$ which have been considered by Gel'fand and Shilov (10).

INTRODUCTION

In chapter one of this thesis, we show that a universal algebra $\mathfrak{A}$ which possesses an Abelian semigroup of injective endomorphisms, $\mathfrak{B}$, may be embedded in another universal algebra, $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, of the same type as $\mathfrak{A}$. There is a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ which is isomorphic with $\mathfrak{A}$, and in a number of cases this is a proper sub-algebra, and $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ then represents a genuine extension of $\mathfrak{A}$.

The construction of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ has an affinity with the familiar construction of the rationals from the integers, and an appropriate choice of $\mathfrak{A}$ and $\mathfrak{B}$ will yield the rationals, (though with a reduced algebraic structure). However the sort of embedding used here seems inappropriate for a simple algebraic system such as this. The construction tends to be fruitful when there are interesting self-mappings of $\mathfrak{A}$ which can be used to form the endomorphisms which constitute $\mathfrak{B}$. In the particular cases which are considered in this thesis, $\mathfrak{A}$ is usually a space of functions, and the endomorphisms which constitute $\mathfrak{B}$ will generally be provided by convolution with the members of another class of functions. However the extension process is, in principle, of a very general character, and in chapter one and parts of chapter three, the theory of the embedding is developed in a general form.[†]

The particular case most extensively studied is that in which $\mathfrak{A}$ is taken to be the class $\mathfrak{T}$ of slowly increasing functions, with certain operations, and $\mathfrak{B}$ '=' $\mathfrak{S} \cap \mathfrak{W}$, $\mathfrak{S}$ the space of rapidly decreasing functions and $\mathfrak{W}$ the Wiener class, i.e. the subclass of L^1 consisting of functions with nowhere vanishing Fourier transforms. It turns out that $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is isomorphic with $\mathfrak{Q}\mathfrak{Z}'$, the class of Fourier transforms of distributions of finite order.

In order to establish this isomorphism between $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ and $\mathfrak{Q}\mathfrak{Z}'$ it is necessary to establish a number of lemmas concerning distributions, and so a resumé of the apposite parts of distribution theory is included in chapter two.

In the development of the general theory of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ spaces, it is shown that there is a non-empty collection of sub-algebras of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, the Wiener sub-algebras, which may be topologised in a fairly natural fashion using any given topology on $\mathfrak{A}$. Theorems are proven which show that in certain situations some of the topological properties of a Wiener sub-algebra, $\mathfrak{G}'$, mimic those of $\mathfrak{A}$.

The character of the construction of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$

[†] There appears to be no reason why the theory could not be developed in a still more general form, by using the concept of a first order relational structure (see (5) Pg.189 and (16) Pg.7) in place of that of a universal algebra.

leads to a natural classification of an element of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ by means of its base class (see D.3.1.5.), and this in turn leads to consideration of certain sub-algebras of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, the spaces $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$ $(n = 0,1,2,\ldots)$. Some properties of the $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$ are established, in particular that they are Wiener sub-algebras and that $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}) \subseteq \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B})$; the question of their distinctness is also discussed.

Thus $\mathfrak{Q}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is a sub-algebra of $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$, and it is shown that $\mathfrak{Q}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is isomorphic with the space $\mathfrak{S}'$ of tempered distributions. It is also shown that $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is a Wiener sub-algebra of itself. For appropriate topologies on $\mathfrak{T}$, it is proved that the isomorphisms established between $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ and $\mathfrak{Q}\mathfrak{Z}'$, and between $\mathfrak{Q}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ and $\mathfrak{S}'$, are in fact homeomorphisms when the distribution spaces carry the weak dual topologies. It is shown by some examples that differing topologies on $\mathfrak{T}$ can lead to the same topology on $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$, and similarly for $\mathfrak{Q}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$.

The class $\mathfrak{W}$, of functions with nowhere vanishing Fourier transforms, intervenes naturally when seeking restrictions on a function g which will ensure that convolution with it will generate an injective endomorphism on some space of functions. Attention was first drawn to this class by Wiener, who showed,[†] as the crucial step in the proof of his Tauberian theorem, that if $f \in \mathfrak{W}$ then, given any finite interval $[a,b]$, there exists $g \in L^1$ such that $\hat{g}(t) = 1/\hat{f}(t)$ for all $t \in [a,b]$. It was a (very easily proved) result of this type which was used in proving that $\mathfrak{Q}(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$ is a Wiener sub-algebra of itself. The central result of chapter four is a generalisation of this theorem of Wiener's, in which assumptions stronger than Wiener's are made and stronger conclusions are drawn. The formulation and proof of this result depend on a very interesting theorem of Ingham's: a particular class, Λ, of functions emerges in connection with Ingham's theorem, and it is classes of functions dominated by $\exp[-\lambda(x)]$, for some $\lambda(x) \in \Lambda$, which replace the class L^1 of Wiener's theorem. Ingham's result is also used to establish the existence of a non-null function in the intersection of certain of the spaces of type $\mathfrak{S}$ considered by Gel'fand and Shilov.[‡]

There are heuristic reasons to expect that any $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ in which $\mathfrak{A}$ is a class of functions and $\mathfrak{B}$ a semigroup of endomorphisms generated by convolution, will possess at least a subclass corresponding to some family

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[†] See (24) Pg. 25. The result is not actually stated explicitly in the above form, but a corresponding result for Fourier series is given. (loc.cit. Pg. 14 .)

[‡] See (10) Pg.166 et seq.

of distributions: something corresponding to a 'δ-function', for example, arises automatically. On the other hand, in view of a result of Delsarte's (6) in which it is shown that for any compactly supported function, k, there exists a non-null function f such that $f * k \equiv 0$, it appears unlikely that the exact class $\mathfrak{D}'$ of Schwartz distributions can be represented as a $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ of this function space-convolution type. However, using the extension made of Wiener's theorem, we obtain a general result of the form 'if $f \in A$ and $g \in B$ and $f * g \equiv 0$, then $f \equiv 0$'. This latter result enables construction of a large class of convolution-based quotient spaces $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$. In particular, quotient spaces related to spaces of type $\mathfrak{S}$ are constructed and discussed briefly. The thesis ends with an appendix and some conjectures.

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ALPHABETS

1) German capitals

$\mathfrak{A}$ $\mathfrak{B}$ $\mathfrak{C}$ $\mathfrak{D}$ $\mathfrak{E}$ $\mathfrak{F}$ $\mathfrak{G}$ $\mathfrak{H}$ $\mathfrak{I}$ $\mathfrak{J}$ $\mathfrak{K}$ $\mathfrak{L}$ $\mathfrak{M}$

$\mathfrak{N}$ $\mathfrak{O}$ $\mathfrak{P}$ $\mathfrak{Q}$ $\mathfrak{R}$ $\mathfrak{S}$ $\mathfrak{T}$ $\mathfrak{U}$ $\mathfrak{V}$ $\mathfrak{W}$ $\mathfrak{X}$ $\mathfrak{Y}$ $\mathfrak{Z}$

2) The Greek alphabet

α β γ δ ϵ ζ η θ ι κ λ μ ν

A B Γ Δ E Z H Θ I K Λ M N

ξ o π ρ σ τ υ ϕ χ ψ ω

Ξ O Π P Σ T Y Φ X Ψ Ω

NOTATION

The following terms and symbols will be taken as understood throughout the text.

The terms 'injection', 'surjection', and 'bijection' are used for 'one-to-one mapping', 'onto mapping', and 'one-to-one onto mapping' respectively.

We use the symbols $\mathbb{R}$ and $\mathbb{C}$ to denote the real and complex number fields (with their usual topologies when this is relevant). The class of functions Lebesgue-integrable over $\mathbb{R}$ is denoted by L^1. Whenever we write $\int$ without stating the limits, we mean $\int_{-\infty}^{+\infty}$. We use $*$ to denote the convolution operation, of a function with another function or of a functional with a function.†
If $f \in L^1$, then $\hat{f}$ denotes the Fourier transform of f, defined by

$$\hat{f}(x) = \int f(t)e^{ixt}dt \quad \forall x \in \mathbb{R} \quad .$$

If $z = x + iy$, where x,y are both real, then we define $\mathfrak{Re}\, z$, $\mathfrak{Im}\, z$, and $\bar{z}$ to be x, y, and x-iy respectively. The symbol $\bar{\ }$ is used in other contexts, but not when any confusion is likely to arise.

The $\langle\ ,\ \rangle$ notation is used for linear functionals, so

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† The symbol $\star$ is occasionally used instead of $*$ as an aid to clarity. Its usage is defined in D.2.4.15. We also use $*$ in other contexts: see especially section 1.3.

that $\langle F,f\rangle$ denotes the number obtained by applying the functional F to the element f.

The binomial coefficient $n!/(n-r)!r!$ is denoted by $\binom{n}{r}$, and for each $x \in \mathbb{R}$, we use $[x]$ to denote the unique integer such that $[x] \leq x < [x] + 1$.

The symbol $\Rightarrow$ is used in some situations as a substitute for 'implies that'.

CONTENTS

INTRODUCTION TO CHAPTER 1.

In section 1.1. familiar material relating to semigroups and groups is brought together, in particular that any cancelling Abelian semigroup S may be isomorphically embedded in an Abelian group $\mathfrak{Q}(S)$, and it is shown that the group $\mathfrak{Q}(S)$ is defined up to an isomorphism by its mapping properties. Section 1.2. introduces the basic concepts of a universal algebra. In section 1.3. the material of the preceeding sections is used to define the notions 'pseudoring', 'quotient pair' and 'pseudofield'. There then follows the main result of chapter one (T.1.3.1.), namely that any quotient pair $(\mathfrak{A}, \mathfrak{B})$ may be isomorphically embedded in a pseudofield $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$. The pseudofield $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$ is shown to be determined up to an isomorphism by its mapping properties. The results of this section are believed to be new.

Finally in section 1.4. various applications are presented. These include the embedding of a ring without zero divisors in a field: this latter embedding embraces the Mikusinski Operational Calculus. Also a $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ space is constructed which, in effect, contains the Mikusinski system. The spaces $\mathfrak{T}$, $\mathfrak{S}$, $\mathfrak{M}$ are introduced, and under the assumption (to be proved in chapter two) that $\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}$ form a quotient pair, the pseudofield $(\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}), \mathfrak{Q}(\mathfrak{S} \cap \mathfrak{M}))$ is briefly discussed.

CHAPTER 1

1.1. Semigroups and groups †

D.1.1.1. A semigroup S is a set M together with an associative binary operation $\circ$ on M; that is an operation which assigns to any ordered pair of elements $(m_1, m_2) \in M \times M$ a further unique element of M denoted by $m_1 \circ m_2$, and is such that for any triple $(m_1, m_2, m_3) \in M \times M \times M$,

$$m_1 \circ (m_2 \circ m_3) = (m_1 \circ m_2) \circ m_3.$$

We shall denote the set M simply by S: no confusion is likely to arise from this convenient ambiguity. We shall also use different symbols e.g. $*$, $+$, to denote the binary operation of the semigroup when these are more appropriate than $\circ$.

D.1.1.2. A semigroup S with binary operation $\circ$ is said to be Abelian if for any pair $(s_1, s_2) \in S \times S$,

$$s_1 \circ s_2 = s_2 \circ s_1.$$

D.1.1.3. An Abelian semigroup S with binary operation $\circ$ is said to be cancelling if for any triple $(s, s_1, s_2) \in S \times S \times S$ the relation $s_1 \circ s = s_2 \circ s$ implies

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† For the material of this section see (8) and (14).

that $s_1 = s_2$.

D.1.1.4. A semigroup S with binary operation $\circ$ is said to be a group if there exists $e \in S$ such that

i) $s \circ e = s, \quad \forall s \in S$

ii) $\forall s \in S, \quad \exists s^{-1} \in S$ such that $s \circ s^{-1} = e$.

We call e the identity element of the group; it is unique.

D.1.1.5. Two semigroups S, T with binary operations $\circ$, $*$ respectively are said to be isomorphic if there exists a bijection ϕ , of S onto T such that, for any pair $(s_1, s_2) \in S \times S$, $\phi(s_1 \circ s_2) = \phi(s_1) * \phi(s_2)$. A bijection ϕ with this property is called an isomorphism. Clearly if ϕ is an isomorphism, so is ϕ^{-1}, and so the isomorphism relationship is symmetric: it is also transitive and reflexive. If S = T, ϕ is called an automorphism of S.

L.1.1.1. If S is a semigroup and T a group with binary operations $\circ$, $*$ respectively such that S and T are isomorphic, then S is a group.

Proof: If $\phi : S \to T$ is an isomorphism and $I \in T$ is the identity then there exists $e \in S$ such that $\phi(e) = I$. If $a \in S$, put $a^{-1} = \phi^{-1}([\phi(a)]^{-1})$, then

$\phi(a \circ e) = \phi(a) * I = \phi(a)$, and so $a \circ e = a$, and

$\phi(a \circ a^{-1}) = \phi(a) * [\phi(a)]^{-1} = I$, and so $a \circ a^{-1} = e$.

The result follows.

D.1.1.6. If S, T are two semigroups and ϕ is an injection of S into T, such that $\phi(S)$ is a sub-semigroup of T and ϕ is an isomorphism of S onto $\phi(S)$, then we say that ϕ is an isomorphic embedding of S in T.

T.1.1.1.[†] Every cancelling Abelian semigroup can be isomorphically embedded in an Abelian group.

Proof: If S is a cancelling Abelian semigroup, we consider the set of all ordered pairs $s_1//s_2$, $s_1, s_2 \in S$. We say that $s_1//s_2 = t_1//t_2$ if and only if $s_1 \circ t_2 = t_1 \circ s_2$. The binary relation =, so defined, is an equivalence relation since it is obviously reflexive and symmetric, and also transitive, for if $r_1//r_2 = s_1//s_2$, and $s_1//s_2 = t_1//t_2$, then

$$r_1 \circ s_2 = s_1 \circ r_2, \text{ and } s_1 \circ t_2 = t_1 \circ s_2,$$

and so $$(r_1 \circ t_2) \circ s_2 = s_1 \circ (r_2 \circ t_2) = (t_1 \circ r_2) \circ s_2,$$

hence $$r_1 \circ t_2 = t_1 \circ r_2.$$

The set of equivalence classes so formed we shall denote by $\mathfrak{Q}(S)$. If $\alpha = \{s_1//t_1\}$, $\beta = \{p_1//q_1\}$ are elements of $\mathfrak{Q}(S)$, we define $\alpha \circ \beta = \{(s_1 \circ p_1)//(t_1 \circ q_1)\} \in \mathfrak{Q}(S)$. If $\alpha, \beta \in \mathfrak{Q}(S)$, then $\alpha \circ \beta$ is well defined, for if $s_1//t_1 = s_2//t_2$, $p_1//q_1 = p_2//q_2$, then

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[†] See (14) Pgs.51-54.

$(s_1 \circ p_1)//(t_1 \circ q_1) = (s_2 \circ p_2)//(t_2 \circ q_2)$.

If $\alpha = \{s//t\} \in \mathfrak{Q}(S)$, we shall permit ourselves the useful ambiguity of writing $\alpha = s//t$, so that we represent an equivalence class by any one of its members: no confusion will arise from this practice. Clearly $\mathfrak{Q}(S)$ is an Abelian semigroup under the binary operation $\circ$, also if $s,t \in S$, then $s//s = t//t$ and if $\alpha = p//q \in \mathfrak{Q}(S)$, then $\alpha \circ (s//s) = \alpha$, and putting $\alpha^{-1} = q//p \in \mathfrak{Q}(S)$, we have that $\alpha \circ \alpha^{-1} = (p \circ q)//(q \circ p) = s//s$. It follows that $\mathfrak{Q}(S)$ is an Abelian group. We define $j:S \to \mathfrak{Q}(S)$ by $j(s) = s \circ s//s$. Then $j(s \circ t) = j(s) \circ j(t)$, and j is an injection, consequently j is an isomorphic embedding of S in $\mathfrak{Q}(S)$.

<u>D.1.1.7.</u> We shall call the map j, defined above, the natural embedding of S in $\mathfrak{Q}(S)$.

<u>D.1.1.8.</u> If S, T are semigroups with binary operations $\circ$, $\ast$ respectively and $\phi : S \to T$ is a map such that, for any pair $(s_1, s_2) \in S \times S$, $\phi(s_1 \circ s_2) = \phi(s_1) \ast \phi(s_2)$, then we say that ϕ is an homomorphism of S into T. If $S = T$, ϕ is called an endomorphism of S.

<u>L.1.1.2.</u> If $\phi : S \to T$ is an injective homomorphism of the cancelling Abelian semigroup S into the Abelian group T, then there exists precisely one homomorphism $\phi' : \mathfrak{Q}(S) \to T$ such that $\phi = \phi' j$, where j is the natural embedding of S in $\mathfrak{Q}(S)$. This ϕ' is an injection.

$$\begin{tikzcd} \mathfrak{Q}(S) \arrow[rr, dashed, "\phi'"] & & T \\ & S \arrow[ul, "j"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Proof: If S, T have binary operations $\circ$, $*$ respectively, and $\alpha = s_1//s_2 \in \mathfrak{Q}(S)$, define $\phi'(\alpha) = \phi(s_1) * [\phi(s_2)]^{-1}$. Then ϕ' is a well defined homomorphism of $\mathfrak{Q}(S)$ into T, for if $\alpha = s_1//s_2 = t_1//t_2$, then $\phi(s_1) * \phi(t_2) = \phi(t_1) * \phi(s_2)$, and if $\beta = r_1//r_2$, then

$$\begin{aligned} \phi'(\alpha \circ \beta) &= \phi(s_1 \circ r_1) * [\phi(s_2 \circ r_2)]^{-1} \\ &= \{\phi(s_1) * [\phi(s_2)]^{-1}\} * \{\phi(r_1) * [\phi(r_2)]^{-1}\} \\ &= \phi'(\alpha) * \phi'(\beta). \end{aligned}$$

Suppose $s \in S$ is any element, then $\phi' j(s) = \phi(s \circ s) * [\phi(s)]^{-1} = \phi(s)$ and so there exists an homomorphism ϕ' satisfying $\phi = \phi' j$. Suppose $\bar{\Phi}$ is any such homomorphism, e the identity in $\mathfrak{Q}(S)$, and I the identity in T, then $\bar{\Phi}(e) = I$. Thus if $\alpha \in \mathfrak{Q}(S)$, $\bar{\Phi}(\alpha^{-1}) = [\bar{\Phi}(\alpha)]^{-1}$, and so

$$\begin{aligned} \bar{\Phi}(s_1//s_2) &= \bar{\Phi}((s_1 \circ s_1)//s_1) * \bar{\Phi}(s_2//(s_2 \circ s_2)) \\ &= \bar{\Phi}(j(s_1)) * [\bar{\Phi}(j(s_2))]^{-1} \\ &= \phi(s_1) * [\phi(s_2)]^{-1} \\ &= \phi'(s_1//s_2), \end{aligned}$$

so that $\bar{\Phi} = \phi'$.

Finally we note that if $\phi'(s_1//s_2) = \phi'(r_1//r_2)$, then $\phi(s_1 \circ r_2) = \phi(r_1 \circ s_2)$, and since ϕ is an injection, $s_1 \circ r_2 = r_1 \circ s_2$; hence $s_1//s_2 = r_1//r_2$, and it follows that ϕ' is also an injection.

We have shown that any Abelian group T which includes an isomorphic image of the cancelling Abelian semigroup S, includes a consistent isomorphic image of $\mathfrak{Q}(S)$. The next lemma shows that we may use this situation to determine $\mathfrak{Q}(S)$ up to an isomorphism.

<u>L.1.1.3.</u>† Suppose S is a cancelling Abelian semigroup, Q_1, Q_2 Abelian groups and j_1, j_2 are injective homomorphisms of S into Q_1, Q_2 respectively. Suppose further that Q_1, Q_2 have the property that for every Abelian group T for which there is an injective homomorphism ϕ of S into T, there exist unique homomorphisms ϕ_1, ϕ_2 of Q_1, Q_2 respectively into T, such that $\phi = \phi_1 j_1 = \phi_2 j_2$. Then Q_1, Q_2 are isomorphic.

Proof: The assumptions may be summed up in the following diagrams:

$$\begin{tikzcd}
Q_1 \arrow[r, dashed, "\phi_1"] & T \\
& S \arrow[ul, "j_1"] \arrow[u, "\phi"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Q_2 \arrow[r, dashed, "\phi_2"] & T \\
& S \arrow[ul, "j_2"] \arrow[u, "\phi"']
\end{tikzcd}$$

(1) (2)

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†See also L.1.3.7.

In the first place, we take T to be Q_2 and Φ to be j_2 in diagram (1). Then there exists an homomorphism I_1 of Q_1 into Q_2 such that $j_2 = I_1 j_1$. Similarly from diagram (2), taking T to be Q_1 and Φ to be j_1, we obtain an homomorphism I_2 of Q_2 into Q_1 such that $j_1 = I_2 j_2$. It follows that

$$j_1 = (I_2 I_1) j_1$$

and

$$j_2 = (I_1 I_2) j_2.$$

Taking T to be Q_1 and Φ to be j_1 in diagram (1), Φ_1 will be the identity map, id_1, of Q_1 onto itself, and so from the uniqueness, it follows that

$$I_2 I_1 = id_1.$$

Similarly

$$I_1 I_2 = id_2,$$

where id_2 is the identity map of Q_2 onto itself. Now

id_1 surjective implies I_2 surjective,

id_2 surjective implies I_1 surjective,

id_1 injective implies I_1 injective,

id_2 injective implies I_2 injective.

Thus I_1, I_2 are each bijective, and mutually inverse, and it follows that Q_1, Q_2 are isomorphic.

1.2. Universal Algebras†

D.1.2.1. A Universal algebra U, is a set M together with a system $\mathfrak{D}$ of n-ary operations on M. Here $n \geq 0$, and may be different for different ϕ's in $\mathfrak{D}$. (An n-ary operation ϕ on M assigns to any ordered n-tple $(m_1, m_2, \ldots, m_n) \in M \times M \times \ldots \times M$ a further unique element of M denoted by $\phi(m_1, m_2, \ldots, m_n)$.) We shall denote the universal algebra U by $(M, \mathfrak{D})$, and when no confusion is likely we shall write U, for M. If $N \subseteq M$, and $V = (N, \mathfrak{D})$ is a universal algebra we shall say that V is a sub-algebra of U. If $(L, \mathfrak{D}_1)$, $(M, \mathfrak{D}_2)$ are universal algebras in which a one-to-one correspondence between $\mathfrak{D}_1$ and $\mathfrak{D}_2$ can be set up in such a way that every $\phi_1 \in \mathfrak{D}_1$ and the corresponding $\phi_2 \in \mathfrak{D}_2$ are n-ary with the same n, then we shall say that $(L, \mathfrak{D}_1)$, $(M, \mathfrak{D}_2)$ are of the same type. If $(L, \mathfrak{D}_1)$, $(M, \mathfrak{D}_2)$ are universal algebras of the same type we shall sometimes write $(L, \mathfrak{D}_2)$ for $(L, \mathfrak{D}_1)$ or $(M, \mathfrak{D}_1)$ for $(M, \mathfrak{D}_2)$, if it is convenient: no confusion is likely to arise from this.

D.1.2.2. If $(L, \mathfrak{D})$, $(M, \mathfrak{D})$ are universal algebras of the same type and $\chi : L \to M$ is a map such that for every

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† See (14) Pg.93 et seq.

n-ary operation $\phi \in \mathfrak{D}$, and every n-tple $(\ell_1, \ell_2, \ldots, \ell_n) \in L \times L \times \ldots \times L$,

$$\chi(\phi(\ell_1, \ell_2, \ldots, \ell_n)) = \phi(\chi(\ell_1), \chi(\ell_2), \ldots, \chi(\ell_n)),$$

then we shall say that χ is an homomorphism of $(L, \mathfrak{D})$ into $(M, \mathfrak{D})$. If in addition χ is a bijection, then we shall say that χ is an isomorphism of $(L, \mathfrak{D})$ onto $(M, \mathfrak{D})$, and that $(L, \mathfrak{D})$ is isomorphic to $(M, \mathfrak{D})$; in this case χ^{-1} is readily seen to be an isomorphism so that the isomorphism relationship is symmetric; it is also reflexive and transitive. An isomorphism of $(L, \mathfrak{D})$ onto itself is called an automorphism of $(L, \mathfrak{D})$. An homomorphism of $(L, \mathfrak{D})$ into itself is called an endomorphism of $(L, \mathfrak{D})$. An injective homomorphism is called an isomorphic embedding.

1.3. Pseudorings, pseudofields, quotient pairs; the embedding theorem

N.1.3.1. If $\mathfrak{U}$ is a universal algebra and $\mathfrak{B}$ a set of endomorphisms of $\mathfrak{U}$, then for $\phi \in \mathfrak{B}$, $f \in \mathfrak{U}$, we shall denote $\phi(f)$ by $f * \phi$. If $\phi_1, \phi_2 \in \mathfrak{B}$ we define the composition $\phi_1 \circ \phi_2$, of ϕ_1 with ϕ_2 to be the endomorphism of $\mathfrak{U}$ given by

$$f * (\phi_1 \circ \phi_2) = (f * \phi_1) * \phi_2, \quad \forall f \in \mathfrak{U}.$$

If $\mathfrak{B}$ is closed under composition, it forms a semigroup.

<u>L.1.3.1.</u> If $\mathfrak{A}$ is a universal algebra and $\mathfrak{B}$ an Abelian semigroup of injective endomorphisms of $\mathfrak{A}$, then $\mathfrak{B}$ is a cancelling Abelian semigroup.

Proof: If $\phi, \phi_1, \phi_2 \in \mathfrak{B}$ and $\phi_1 \circ \phi = \phi_2 \circ \phi$, then

$$f * (\phi_1 \circ \phi) = f * (\phi_2 \circ \phi), \quad \forall f \in \mathfrak{A}$$

and so $\quad (f * \phi_1) * \phi = (f * \phi_2) * \phi,$

hence $\quad f * \phi_1 = f * \phi_2, \quad \forall f \in \mathfrak{A}$

i.e. $\phi_1 = \phi_2$; the result follows.

<u>D.1.3.1.</u>[1] If $\mathfrak{A}$ $(\neq \emptyset)$ is a universal algebra and $\mathfrak{B}$ is an Abelian semigroup of endomorphisms of $\mathfrak{A}$, then we shall say that $\mathfrak{A}, \mathfrak{B}$ form a pseudoring; if in addition all the elements of $\mathfrak{B}$ are injections, then we shall say that $\mathfrak{A}, \mathfrak{B}$ form a quotient pair; and finally, if in addition to this $\mathfrak{B}$ forms an Abelian group, then we shall say that $\mathfrak{A}, \mathfrak{B}$ form a pseudofield.

<u>D.1.3.2.</u> If $(\mathfrak{A}, \mathfrak{B})$, $(\mathfrak{C}, \mathfrak{D})$ are two pseudorings with $\mathfrak{A}$ and $\mathfrak{C}$ of the same type, and μ, ν homomorphisms of $\mathfrak{A}$ into $\mathfrak{C}$ and $\mathfrak{B}$ into $\mathfrak{D}$ respectively, such that if $f \in \mathfrak{A}$ and $\phi \in \mathfrak{B}$ then $\mu(f * \phi) = \mu(f) * \nu(\phi)$, then we shall say that (μ, ν) is an homomorphism of $(\mathfrak{A}, \mathfrak{B})$ into $(\mathfrak{C}, \mathfrak{D})$. If μ and ν are bijective, we shall say

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[1] See N.1.3.2. and section 1.4. for examples.

that (μ,ν) is an isomorphism of $(\mathfrak{A},\mathfrak{B})$ onto $(\mathfrak{C},\mathfrak{D})$, and that $(\mathfrak{A},\mathfrak{B})$ is isomorphic to $(\mathfrak{C},\mathfrak{D})$; in this case (μ^{-1},ν^{-1}) is readily seen to be an isomorphism, so that the isomorphism relationship is symmetric, it is also reflexive and transitive. An isomorphism of $(\mathfrak{A},\mathfrak{B})$ onto itself is called an automorphism of $(\mathfrak{A},\mathfrak{B})$. An homomorphism of $(\mathfrak{A},\mathfrak{B})$ into itself is called an endomorphism of $(\mathfrak{A},\mathfrak{B})$. If μ and ν are both injective, both surjective, or both bijective, then correspondingly we say that (μ,ν) is injective, surjective, or bijective. An injective homomorphism (μ,ν) is called an isomorphic embedding.

<u>T.1.3.1.</u> <u>The embedding theorem</u> †

Any quotient pair can be isomorphically embedded in a pseudofield.

Proof: If $\mathfrak{A}$, $\mathfrak{B}$ form a quotient pair, we consider the set of all ordered pairs $f//\phi$, $f \in \mathfrak{A}$, $\phi \in \mathfrak{B}$. We say that $f//\phi = g//\psi$ if and only if $f \ast \psi = g \ast \phi$. The binary relation =, so defined is an equivalence relation, since it is obviously reflexive and symmetric, and also transitive, for if $f//\phi = g//\psi$, and $g//\psi = h//\chi$, then

$$f \ast \psi = g \ast \phi \text{ , and } g \ast \chi = h \ast \psi,$$

and so $\quad (f \ast \chi) \ast \psi = g \ast (\phi \circ \chi) = (h \ast \phi) \ast \psi$:

and hence $f \ast \chi = h \ast \phi$.

† See also (14) Pgs.54-56.

The set of equivalence classes so formed we shall denote by $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$.

If $\alpha = \{f//\phi\} \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, we shall permit ourselves the useful ambiguity of writing $\alpha = f//\phi$, so that we represent an equivalence class by any one of its members: no confusion will arise from this practice.

Now $\mathfrak{A} = (M, \mathfrak{D})$ is a universal algebra and if $\Xi \in \mathfrak{D}$ is an n-ary operation on M, and if $(f_1//\phi_1, f_2//\phi_2, \ldots, f_n//\phi_n) \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B}) \times \mathfrak{Q}(\mathfrak{A}, \mathfrak{B}) \times \ldots \times \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ is any n-tple, we define $\Xi(f_1//\phi_1, f_2//\phi_2, \ldots, f_n//\phi_n)$ to be $(\Xi(f_1 * (\phi_2 \circ \phi_3 \circ \ldots \circ \phi_n), f_2 * (\phi_1 \circ \phi_3 \circ \ldots \circ \phi_n), \ldots, f_n * (\phi_1 \circ \phi_2 \circ \ldots \circ \phi_{n-1})))//(\phi_1 \circ \phi_2 \circ \ldots \circ \phi_n)$. (If $n = 0$, the nullary operation Ξ on M is a fixed element $e \in M$, and we define the operation Ξ on $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ to be $e * \phi // \phi$, this element is easily shown to be independent of the choice of $\phi \in \mathfrak{B}$.) If $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B}) \times \mathfrak{Q}(\mathfrak{A}, \mathfrak{B}) \times \ldots \times \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, then $\Xi(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is well defined, for if $f_i//\phi_i = g_i//\psi_i$, $i = 1, 2, \ldots, n$, then

$$\Xi\,(f_1 * (\phi_2 \circ \phi_3 \circ \cdots \circ \phi_n), f_2 * (\phi_1 \circ \phi_3 \circ \cdots \circ \phi_n), \ldots,$$

$$f_n * (\phi_1 \circ \phi_2 \circ \cdots \circ \phi_{n-1})) * (\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n)$$

$$= \Xi\,(g_1 * (\psi_2 \circ \psi_3 \circ \cdots \circ \psi_n), g_2 * (\psi_1 \circ \psi_3 \circ \cdots \circ \psi_n), \ldots,$$

$$g_n * (\psi_1 \circ \psi_2 \circ \cdots \circ \psi_{n-1})) * (\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n),$$

since

$$(f_i * \psi_i) * (\phi_1 \circ \psi_1 \circ \phi_2 \circ \psi_2 \circ \cdots \circ \phi_{i-1} \circ \psi_{i-1} \circ \phi_{i+1} \circ \psi_{i+1} \circ \cdots \circ \phi_n \circ \psi_n)$$

$$= (g_i * \phi_i) * (\phi_1 \circ \psi_1 \circ \phi_2 \circ \psi_2 \circ \cdots \circ \phi_{i-1} \circ \psi_{i-1} \circ \phi_{i+1} \circ \psi_{i+1} \circ \cdots \circ \phi_n \circ \psi_n), \quad i = 1,2,\ldots,n.$$

It follows that $\mathfrak{Q}(\mathfrak{U}, \mathfrak{B})$ is a universal algebra with the same system of operations $\mathfrak{D}$, as $\mathfrak{U}$.

If $\beta = \phi_1 // \psi_1 \in \mathfrak{Q}(\mathfrak{B})$ and $\alpha = f // \phi \in \mathfrak{Q}(\mathfrak{U}, \mathfrak{B})$, then we define $\chi_\beta(\alpha)$ to be $(f * \phi_1) // (\phi \circ \psi_1) \in \mathfrak{Q}(\mathfrak{U}, \mathfrak{B})$. $\chi_\beta(\alpha)$ is well defined since if $\phi_1 // \psi_1 = \phi_2 // \psi_2$, and $f // \phi = g // \psi$, we have that $\phi_1 \circ \psi_2 = \phi_2 \circ \psi_1$ and $f * \psi = g * \phi$, and so

$$(f * \phi_1) * (\psi \circ \psi_2) = (g * \phi_2) * (\phi \circ \psi_1).$$

Now if $\beta_1 = \phi_1 // \psi_1$ and $\beta_2 = \phi_2 // \psi_2$ and $\chi_{\beta_1} = \chi_{\beta_2}$, then

$$f * (\phi_1 \circ \psi_2) = f * (\phi_2 \circ \psi_1) \quad \forall f \in \mathfrak{U},$$

and so $\beta_1 = \beta_2$. Furthermore if $\beta_1, \beta_2 \in \mathfrak{Q}(\mathfrak{B})$ then $\chi_{\beta_1} \chi_{\beta_2} = \chi_{(\beta_1 \circ \beta_2)}$, and in view of this and the preceeding observation we see that $\{\chi_\beta\}_{\beta \in \mathfrak{Q}(\mathfrak{B})}$ is isomorphic to $\mathfrak{Q}(\mathfrak{B})$. Thus $\mathfrak{Q}(\mathfrak{B})$ is (isomorphic to) an Abelian group of maps of $\mathfrak{Q}(\mathfrak{U}, \mathfrak{B})$ into itself. We will now show that for each $\beta \in \mathfrak{Q}(\mathfrak{B})$, the map $\chi_\beta : \mathfrak{Q}(\mathfrak{U}, \mathfrak{B}) \to \mathfrak{Q}(\mathfrak{U}, \mathfrak{B})$ is an endomorphism. We will permit ourselves the ambiguity of writing $\alpha * \beta$ for $\chi_\beta(\alpha)$, identifying the element β with the map of $\mathfrak{Q}(\mathfrak{U}, \mathfrak{B})$ which it provides. If $\beta = \phi//\psi \in \mathfrak{Q}(\mathfrak{B})$, $\Xi \in \mathfrak{D}$ is an n-ary operation of the universal algebra $\mathfrak{U} = (M, \mathfrak{D})$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathfrak{Q}(\mathfrak{U}, \mathfrak{B}) \times \mathfrak{Q}(\mathfrak{U}, \mathfrak{B}) \times \ldots \times \mathfrak{Q}(\mathfrak{U}, \mathfrak{B})$, $\alpha_i = f_i // \phi_i$, then

$$\begin{aligned}
&\Xi(\alpha_1, \alpha_2, \ldots, \alpha_n) * \beta \\
&\quad = (\Xi(f_1 * (\phi_2 \circ \phi_3 \circ \ldots \circ \phi_n), f_2 * (\phi_1 \circ \phi_3 \circ \ldots \circ \phi_n), \\
&\qquad \ldots, f_n * (\phi_1 \circ \phi_2 \circ \ldots \circ \phi_{n-1})) * \phi) // \\
&\qquad (\phi_1 \circ \phi_2 \circ \ldots \circ \phi_n) \circ \psi \\
&\quad = \Xi((f_1 * \phi) * (\phi_2 \circ \phi_3 \circ \ldots \circ \phi_n), (f_2 * \phi) * \\
&\qquad (\phi_1 \circ \phi_3 \circ \ldots \circ \phi_n), \ldots, (f_n * \phi) * \\
&\qquad (\phi_1 \circ \phi_2 \circ \ldots \circ \phi_{n-1})) // (\phi_1 \circ \phi_2 \circ \ldots \circ \phi_n \circ \psi) \\
&\quad = \Xi(\alpha_1 * \beta, \alpha_2 * \beta, \ldots, \alpha_n * \beta).
\end{aligned}$$

(If $n = 0$ the nullary operation Ξ on $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ is a fixed element $\xi \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, and the above reduces to observing that $\xi * \beta = \xi * \beta$.)

It follows that $\beta \in \mathfrak{Q}(\mathfrak{B})$ is an endomorphism of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$; it is also injective, for if $\alpha_1 = f_1//\phi_1$, $\alpha_2 = f_2//\phi_2$ are elements of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, and $\beta = \phi//\psi \in \mathfrak{Q}(\mathfrak{B})$, and if $\alpha_1 * \beta = \alpha_2 * \beta$, then $(f_1 * \phi) * (\phi_2 \circ \psi) = (f_2 * \phi) * (\phi_1 \circ \psi)$, and so $f_1 * \phi_2 = f_2 * \phi_1$, i.e. $\alpha_1 = \alpha_2$. It follows now that $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, $\mathfrak{Q}(\mathfrak{B})$ form a pseudofield.

We observe that if $f \in \mathfrak{A}$, $\phi, \psi \in \mathfrak{B}$, then $f * \phi//\phi = f * \psi//\psi$. Define $i : \mathfrak{A} \to \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ by

$$i(f) = f * \phi//\phi, \quad f \in \mathfrak{A}, \quad \phi \in \mathfrak{B},$$

then i is independent of the choice of $\phi \in \mathfrak{B}$, and moreover if $f_1, f_2 \in \mathfrak{A}$ and $i(f_1) = i(f_2)$ then $f_1 = f_2$, and so i is injective. If $\Xi \in \mathfrak{D}$ is an n-ary operation and $(f_1, f_2, \ldots, f_n) \in \mathfrak{A} \times \mathfrak{A} \times \ldots \times \mathfrak{A}$, then $i(\Xi(f_1, f_2, \ldots, f_n)) = \Xi(i(f_1), i(f_2), \ldots, i(f_n))$, and so i is an homomorphism of $\mathfrak{A}$ into $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$. We now recall the definition of j given in D.1.1.7., and that j is an injective homomorphism of $\mathfrak{B}$ into $\mathfrak{Q}(\mathfrak{B})$; we also note that if $f \in \mathfrak{A}$, $\phi \in \mathfrak{B}$ then

$$i(f \ast \phi) = ((f \ast \phi) \ast (\phi \circ \phi))//(\phi \circ \phi)$$
$$= ((f \ast \phi)//\phi) \ast ((\phi \circ \phi)//\phi)$$
$$= i(f) \ast j(\phi).$$

Hence (i,j) is an isomorphic embedding of $(\mathfrak{A}, \mathfrak{B})$ into $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$ and the result follows. We shall call i the natural embedding of $\mathfrak{A}$ into $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, and (i,j) the natural embedding of $(\mathfrak{A}, \mathfrak{B})$ into $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$.

<u>N.1.3.2.</u> There are cases of a universal algebra $\mathfrak{A}$ with an Abelian group, $\mathfrak{B}$, of endomorphisms of $\mathfrak{A}$ in which the endomorphisms are not injective. For example, take $\mathfrak{A} = (M, \mathfrak{D})$ where $M = \{f \mid \exists g$, continuous on $\mathbb{R}$ such that $f = g$ a.e. on $\mathbb{R}\}$, $\mathfrak{D} = \emptyset$. Then if e is the map which takes $f \in M$ to the continuous function to which it is equal a.e., then putting $\mathfrak{B} = \{e\}$, in view of the relation

$$(f \ast e) \ast e = f \ast e, \quad \forall f \in M,$$

we have $e \circ e = e$, and so $\mathfrak{B}$ is an Abelian group. It is obvious however, that e is not injective.

<u>L.1.3.2.</u> If $\mathfrak{A}$, $\mathfrak{B}$ form a pseudoring and $\mathfrak{B}$ is an Abelian group with identity e, then the elements of $\mathfrak{B}$ are injections if and only if $f \ast e = f$, $\forall f \in \mathfrak{A}$.

Proof: If all elements of $\mathfrak{B}$ were injections, then since $e \circ e = e$, we should have

$$(f * e) * e = f * e, \quad \forall f \in \mathfrak{U},$$

and so $\quad f * e = f, \quad \forall f \in \mathfrak{U}$.

Conversely if $\phi \in \mathfrak{B}$, $f_1, f_2 \in \mathfrak{U}$, $f_1 * \phi = f_2 * \phi$ and $f * e = f$, $\forall f \in \mathfrak{U}$, then we have that

$$(f_1 * \phi) * \phi^{-1} = (f_2 * \phi) * \phi^{-1},$$

and so $\quad f_1 * e = f_2 * e,$

hence $f_1 = f_2$, and the result follows.

<u>L.1.3.3.</u> If $\mathfrak{U}$, $\mathfrak{B}$ form a pseudofield and $\phi \in \mathfrak{B}$, then ϕ is surjective.

Proof: Take $f \in \mathfrak{U}$, put $g = f * \phi^{-1} \in \mathfrak{U}$, then

$$\begin{aligned} g * \phi &= f * e \\ &= f, \end{aligned}$$

and so ϕ is surjective.

<u>L.1.3.4.</u> If $\mathfrak{U}$, $\mathfrak{B}$ form a quotient pair and (i,j) is the natural embedding of $(\mathfrak{U}, \mathfrak{B})$ into $(\mathfrak{Q}(\mathfrak{U}, \mathfrak{B}),$ $\mathfrak{Q}(\mathfrak{B}))$ then

(1) i,j are injections.

(2) $\mathfrak{B}$ is a group if and only if j is surjective.

(3) $\mathfrak{B}$ is a group implies that i is surjective.

(4) It is possible to have i surjective and $\mathfrak{B}$ not a group.

Proof: (1) This we have already shown in T.1.3.1.

(2) If $\mathfrak{B}$ is a group, take $\beta = \phi // \psi \in \mathfrak{Q}(\mathfrak{B})$,

and put $\chi = \phi \circ \psi^{-1} \in \mathfrak{B}$; then $j(\chi) = ((\phi \circ \psi^{-1}) \circ \psi)//\psi = \beta$, and so j is surjective. Conversely if j is surjective, then it follows from L.1.1.1. that $\mathfrak{B}$ is a group.

(3) If $\mathfrak{B}$ is a group and $\alpha = f//\phi \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, put $g = f * \phi^{-1}$; then we have that

$$\begin{aligned} i(g) &= ((f * \phi^{-1}) * \phi)//\phi \\ &= (f * e)//\phi \\ &= f//\phi , \end{aligned}$$

and so $i(g) = \alpha$; it follows that i is surjective.

(4) Take $\mathfrak{A}$ to be $(M, \mathfrak{D})$, where $M = \{f | f \in L^1$ and $\hat{f}$ has compact support.$\}$, $\mathfrak{D} = \emptyset$, (with $f_1 = f_2$ in M taken to mean that $f_1 = f_2$ a.e.). Then if $g \in L^1$ is such that $|\hat{g}(t)| > 0$ everywhere on $\mathbb{R}$, define $\chi_g(f)$, for $f \in M$, to be $\int f(x-t)g(t)dt$. Take $\mathfrak{B} = \{\chi_g | g \in L^1$ and $|\hat{g}(t)| > 0, \forall t \in \mathbb{R}\}$. Then $\mathfrak{B}$ is an Abelian semigroup of injective endomorphisms of $\mathfrak{A}$, and so $\mathfrak{A}$, $\mathfrak{B}$ forms a quotient pair. Since $\mathfrak{B}$ has no unit element, it is not a group. However, if $\alpha = f//\chi_g \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, then by Wiener's theorem†, $\exists h \in \mathfrak{A}$ such that

$$\int h(x-t)g(t)dt = f(x)$$

---

† See (11) section 9J, and also chapter four of this thesis.

and so

$$\begin{aligned} i(h) &= (h \times \chi_g)//\chi_g \\ &= f//\chi_g \\ &= \alpha, \end{aligned}$$

and it follows that i is surjective.

<u>L.1.3.5.</u> If $\mathfrak{A}$, $\mathfrak{B}$ form a quotient pair, then $\mathfrak{A}$, $\mathfrak{B}$ form a pseudofield if and only if $(\mathfrak{A}, \mathfrak{B})$ is isomorphic to $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$.

Proof: If $\mathfrak{A}$, $\mathfrak{B}$ form a pseudofield, then $\mathfrak{B}$ is a group, and so from the previous lemma, i, j are bijective; consequently $(\mathfrak{A}, \mathfrak{B})$ is isomorphic to $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$. Conversely if $\mathfrak{A}$, $\mathfrak{B}$ is isomorphic to $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$, then $\mathfrak{B}$ and $\mathfrak{Q}(\mathfrak{B})$ are isomorphic, and it follows from L.1.1.1. that $\mathfrak{B}$ is a group, so that $\mathfrak{A}$, $\mathfrak{B}$ form a pseudofield.

<u>Corollary</u> If $\mathfrak{A}$, $\mathfrak{B}$ form a quotient pair, then $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$ is isomorphic to $(\mathfrak{Q}(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B})), \mathfrak{Q}(\mathfrak{Q}(\mathfrak{B})))$.

<u>L.1.3.6.</u> If (μ, ν) is an injective homomorphism of the quotient pair $(\mathfrak{A}, \mathfrak{B})$ into the pseudofield $(\mathfrak{C}, \mathfrak{D})$, then there exists precisely one homomorphism (μ', ν') of $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$ into $(\mathfrak{C}, \mathfrak{D})$ with the property that $\mu = \mu' i$, $\nu = \nu' j$, where (i, j) is the natural embedding of $(\mathfrak{A}, \mathfrak{B})$ into $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$. This (μ', ν') is an injection.

$$
\begin{tikzcd}
\mathfrak{Q}(\mathfrak{A},\mathfrak{B}) \arrow[rrrr, dashed, "\mu'"] & & & & \mathfrak{C} \\
& \mathfrak{Q}(\mathfrak{B}) \arrow[rr, dashed, "\nu'"'] & & \mathfrak{D} & \\
& & \mathfrak{B} \arrow[ul, "j"'] \arrow[ur, "\nu"] & & \\
& & \mathfrak{A} \arrow[uuull, "i"] \arrow[uuurr, "\mu"'] & &
\end{tikzcd}
$$

Proof: By L.1.1.2. there exists precisely one homomorphism ν' of $\mathfrak{Q}(\mathfrak{B})$ into $\mathfrak{D}$ such that $\nu = \nu' j$, and this ν' is injective. If $\alpha = f//\phi \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, we define $\mu'(\alpha)$ to be $\mu(f) \ast [\nu(\phi)]^{-1}$; μ' is well defined on $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ since if $f_1//\phi_1 = f_2//\phi_2$, we have that

$$f_1 \ast \phi_2 = f_2 \ast \phi_1,$$

and so $\quad \mu(f_1 \ast \phi_2) = \mu(f_2 \ast \phi_1),$

hence $\quad \mu(f_1) \ast \nu(\phi_2) = \mu(f_2) \ast \nu(\phi_1).$

Also because (μ,ν) is an homomorphism, it is easy to verify that μ' is one as well. If $f \in \mathfrak{A}$, then

$$i(f) = (f \ast \phi)//\phi \quad \text{for any } \phi \in \mathfrak{B},$$

and so
$$
\begin{aligned}
\mu' i(f) &= \mu(f \ast \phi) \ast [\nu(\phi)]^{-1} \\
&= \mu(f),
\end{aligned}
$$

hence $\quad \mu = \mu' i.$

If $\alpha = f_1 // \phi_1 \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ and $\beta = \phi // \psi \in \mathfrak{Q}(\mathfrak{B})$, then

$$\begin{aligned} \mu'(\alpha * \beta) &= \mu(f_1 * \phi) * [\nu(\phi_1 \circ \psi)]^{-1} \\ &= [\mu(f_1) * \nu(\phi)] * [\nu(\phi_1) \circ \nu(\psi)]^{-1} \\ &= \mu'(\alpha) * \nu'(\beta), \end{aligned}$$

and so there exists an homomorphism (μ', ν') satisfying $\mu = \mu' i$, $\nu = \nu' j$.

Suppose that $(\bar{\mu}, \bar{\nu})$ is any homomorphism satisfying $\mu = \bar{\mu} i$, $\nu = \bar{\nu} j$. Then since ν' is unique, $\bar{\nu} = \nu'$, and if $\alpha = f // \phi \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, we have that

$$\begin{aligned} \bar{\mu}(f // \phi) * \nu(\phi) &= \bar{\mu}((f // \phi) * j(\phi)) \\ &= \bar{\mu}(i(f)) \\ &= \mu(f), \end{aligned}$$

and it follows that $\bar{\mu}(\alpha) = \mu'(\alpha)$. This holds for every $\alpha \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, and so the uniqueness of (μ', ν') follows.

Finally, we have already that ν' is injective and if $\alpha_1 = f_1 // \phi_1$, $\alpha_2 = f_2 // \phi_2$ are elements of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, and $\mu'(\alpha_1) = \mu'(\alpha_2)$, then

$$\mu(f_1) * \nu(\phi_2) = \mu(f_2) * \nu(\phi_1),$$

so that

$$\mu(f_1 * \phi_2) = \mu(f_2 * \phi_1).$$

Hence

$$f_1 * \phi_2 = f_2 * \phi_1,$$

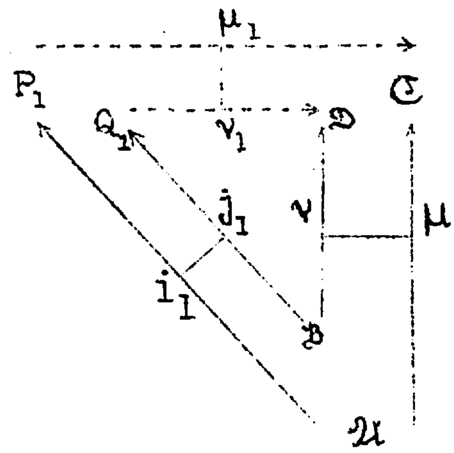
i.e. $\alpha_1 = \alpha_2$, and so μ' is injective.

We have shown that any pseudofield $(\mathfrak{C}, \mathfrak{D})$ which contains an isomorphic image of the quotient pair $(\mathfrak{A}, \mathfrak{B})$ contains a consistent isomorphic image of $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$. The next lemma shows that $(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$ is defined up to an isomorphism by this mapping property.

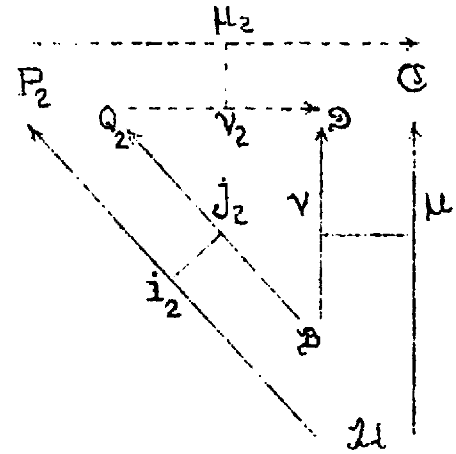
<u>L.1.3.7.</u>[†] Suppose $(\mathfrak{A}, \mathfrak{B})$ is a quotient pair, (P_1, Q_1), (P_2, Q_2) pseudofields and (i_1, j_1), (i_2, j_2) are injective homomorphisms of $(\mathfrak{A}, \mathfrak{B})$ into (P_1, Q_1), (P_2, Q_2) respectively. Suppose further that (P_1, Q_1), (P_2, Q_2) have the property that for every pseudofield $(\mathfrak{C}, \mathfrak{D})$ for which there is an injective homomorphism (μ, ν) of $(\mathfrak{A}, \mathfrak{B})$ into $(\mathfrak{C}, \mathfrak{D})$, there exist unique homomorphisms (μ_1, ν_1), (μ_2, ν_2) of (P_1, Q_1), (P_2, Q_2) respectively into $(\mathfrak{C}, \mathfrak{D})$ such that $\mu = \mu_1 i_1 = \mu_2 i_2$, $\nu = \nu_1 j_1 = \nu_2 j_2$. Then (P_1, Q_1), (P_2, Q_2) are isomorphic.

Proof: The assumptions may be summed up in the following diagrams.

[†] See also L.1.1.3.



(1)



(2)

In the first place, we take $(\mathfrak{C}, \mathfrak{D})$ to be (P_2, Q_2), and (μ, ν) to be (i_2, j_2) in diagram (1). Then there exists an homomorphism (I_1, J_1) of (P_1, Q_1) into (P_2, Q_2) such that $i_2 = I_1 i_1$ and $j_2 = J_1 j_1$. Similarly from diagram (2) taking $(\mathfrak{C}, \mathfrak{D})$ to be (P_1, Q_1) and (μ, ν) to be (i_1, j_1), we obtain an homomorphism (I_2, J_2) of (P_2, Q_2) into (P_1, Q_1) such that $i_1 = I_2 i_2$ and $j_1 = J_2 j_2$. It follows that

$$i_1 = (I_2 I_1) i_1, \qquad j_1 = (J_2 J_1) j_1,$$

and

$$i_2 = (I_1 I_2) i_2, \qquad j_2 = (J_1 J_2) j_2.$$

Taking $(\mathfrak{C}, \mathfrak{D})$ to be (P_1, Q_1) and (μ, ν) to be (i_1, j_1) in diagram (1), (μ_1, ν_1) will be the identity map, (id_{P_1}, id_{Q_1}), of (P_1, Q_1) onto itself, and so from uniqueness it follows that

$$I_2 I_1 = \mathrm{id}_{P_1}, \qquad J_2 J_1 = \mathrm{id}_{Q_1}.$$

Similarly

$$I_1 I_2 = \mathrm{id}_{P_2}, \qquad J_1 J_2 = \mathrm{id}_{Q_2},$$

where $(\mathrm{id}_{P_2}, \mathrm{id}_{Q_2})$ is the identity map of (P_2, Q_2) onto itself.

Now, id_{P_1} injective implies I_1 injective, and id_{P_2} surjective implies I_1 surjective, thus I_1 is bijective, and it follows by similar reasoning that I_2, J_1, J_2 are all bijections. It is now clear that (I_1, J_1), (I_2, J_2) are mutually inverse bijections and that (P_1, Q_1) and (P_2, Q_2) are isomorphic.

1.4. Examples

We now list some examples of the use of the preceeding theory; these examples vary widely in their content. Example E.1.4.5. is the one we have most extensively studied, and our results on this particular case will form chapter 2.

E.1.4.1. † Suppose that R is a commutative-associative ring with no zero divisors, regarded as a universal algebra with binary operation $(a,b) \to a + b$, unary operations of the form $a \to ac$, $c \in R$, and $a \to -a$, and

† See (1) Pg.43.

nullary operation 0. If $d \neq 0$ and if $a \in R$, we define $\phi_d(a) = ad$, and $\mathfrak{B} = \{\phi_d | d \in R/\{0\}^{\dagger}\}$. Then $\mathfrak{B}$ is an Abelian semigroup of endomorphisms of R, and in addition $\phi_d(a) = \phi_d(b)$, $d \neq 0$, implies that $a = b$, so that R, $\mathfrak{B}$ form a quotient pair. It is clear that $(\mathfrak{Q}(R, \mathfrak{B}), \mathfrak{Q}(\mathfrak{B}))$ corresponds to the field of quotients of R.

E.1.4.2. Suppose that $\mathbb{Q}^+$ is the set of all positive rational numbers and that $\mathbb{Z}$ is the set of all integers. If $r \in \mathbb{Z}/\{0\}$, and if $q \in \mathbb{Q}^+$, we define $\phi_r(q) = q^r$ and $\mathfrak{B} = \{\phi_r | r \in \mathbb{Z}/\{0\}\}$. Thus if $\mathbb{Q}^+$ is regarded as a universal algebra with binary operation $(p,q) \rightarrow pq$, unary operation $q \rightarrow q^{-1}$, and nullary operation 1, it is clear that $\mathfrak{B}$ is an Abelian semigroup of endomorphisms of $\mathbb{Q}^+$. Additionally if $q^r = p^r$, $r \neq 0$, then $p = q$, so that $\mathbb{Q}^+, \mathfrak{B}$ form a quotient pair. It is clear that $\mathfrak{Q}(\mathbb{Q}^+, \mathfrak{B})$ is isomorphic with the multiplicative group of all rational non-zero powers of the positive rationals.

E.1.4.3. Suppose that $\mathfrak{C}$ is the set of all continuous functions on the real line, regarded as a universal algebra with binary operation $(f,g) \rightarrow f + g$, unary operations of the form $f \rightarrow \lambda f$, λ any scalar, and $f \rightarrow \int_0^x f(t)g(x-t)dt$ for any locally integrable g, and nullary operation 0. For $n = 1,2,3,\ldots$, define $\phi_n(f) = \int_0^x dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} \ldots \int_0^{x_1} f(x_0)dx_0$, and $\phi_0(f) = f$. Put $\mathfrak{B} = \{\phi_n, n = 0,1,2,\ldots\}$. Then $\mathfrak{B}$ is an Abelian

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† i.e. r is any element except zero.

semigroup of endomorphisms of $\mathfrak{C}$, and in addition $\Phi_n(f) = 0$ implies that $f = 0$, so that $\mathfrak{C}$, $\mathfrak{B}$ form a quotient pair.

E.1.4.4. Take $\mathfrak{C}$ as in E.1.4.3. and consider the class

$$\mathfrak{C}^* = \{g \mid g \in \mathfrak{C} \text{ and } \exists\, u_1 > 0,\ u_2 < 0 \text{ such that } g(u_1) \neq 0 \text{ and } g(u_2) \neq 0\}.$$

For $f \in \mathfrak{C}$ and $g \in \mathfrak{C}^*$, define $\phi_g(f) = \int_0^x f(t)g(x-t)dt$. Then it follows from a result of Titchmarsh[†] that $\phi_g(f) \equiv 0$ implies that $f \equiv 0$, and that $\mathfrak{B} = \{\phi_g \mid g \in \mathfrak{C}^*\}$ is an Abelian semigroup of endomorphisms of $\mathfrak{C}$. Hence $\mathfrak{C}$, $\mathfrak{B}$ form a quotient pair. The resulting pseudofield is related to Mikusinski's Operational Calculus; for a discussion of this latter system, see 'Concluding Remarks'.

D.1.4.1.[‡] We denote by $\mathfrak{S}$ the space of functions k, of a real variable t, such that k(t) has derivatives of all orders and

$$\exists \lim_{|t| \to \infty} |t|^m |k^{(p)}(t)| = 0 \quad \text{for each } m,p = 0,1,2,\ldots .$$

D.1.4.2.[‖] We denote by $\mathfrak{T}$ the space of functions f, of a real variable t, such that f(t) has derivatives of all orders and for each $p = 0,1,2,\ldots$, there exists an $m > 0$ such that

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[†] See (21) and also (13).

[‡] The space $\mathfrak{S}$ was first introduced by L. Schwartz. See (20) Pg.89.

[‖] The space $\mathfrak{T}$ was also introduced by L. Schwartz. See (20) Pg.99, where $\mathfrak{T}$ is referred to under the name $\mathcal{O}_M$.

$$\exists \lim_{|t| \to \infty} |t|^{-m} |f^{(p)}(t)| = 0.$$

D.1.4.3.† We denote by $\mathfrak{W}$ the class consisting of those functions, integrable over the real line, which have nowhere-vanishing Fourier transforms.

E.1.4.5. Consider the space $\mathfrak{T}$ as a universal algebra with binary operation $(f,g) \to f + g$, unary operations of the form $f \to \lambda f$, λ any scalar; $f \to f^{(p)}$, p any non-negative integer; and $f \to \int f(t)g(x-t)dt$, g any element of $\mathfrak{S}$; and nullary operation 0. For $k \in \mathfrak{S} \cap \mathfrak{W}$, $f \in \mathfrak{T}$, define $\phi_k(f) = \int f(t)k(x-t)dt$; then $\mathfrak{B} = \{ \phi_k | k \in \mathfrak{S} \cap \mathfrak{W} \}$ is an Abelian semigroup of endomorphisms of $\mathfrak{T}$. We shall show in the following chapter that if $\phi_k(f) = 0$ then $f = 0$, so that $\mathfrak{T}$, $\mathfrak{B}$ form a quotient pair. We shall later establish an isomorphism between $\mathfrak{Q}(\mathfrak{T}, \mathfrak{B})$ and the class of Fourier transforms of distributions of finite order, and show how a topology may be introduced into $\mathfrak{Q}(\mathfrak{T}, \mathfrak{B})$ in a simple fashion, which will extend the algebraic isomorphism to an homeomorphism‡ .

A word on notation is perhaps in order at this point; the class $\mathfrak{S} \cap \mathfrak{W}$ is easily shown to be an Abelian semigroup under convolution, and is isomorphic to $\mathfrak{B}$ under the correspondence $k \to \phi_k$. Consequently we identify

† The space $\mathfrak{W}$ occurs in the context of Wiener's Tauberian theorem. See (24) Pg.25.

‡ $\Theta: X \to Y$ is said to be an homeomorphism if it is a bijection and both Θ and Θ^{-1} are continuous.

$\mathfrak{S} \cap \mathfrak{W}$ and $\mathfrak{B}$, writing $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ for $\mathfrak{Q}(\mathfrak{T}, \mathfrak{B})$ etc., and using the convolution notation we write $\phi_k \circ \phi_h$ as $h * k$, and $\phi_k(f)$ as $f * k$. No confusion is likely to arise from these conventions.

<u>Further examples</u>

1) Take $\mathfrak{C}$ as in E.1.4.3., and take $\mathfrak{B}$ to be the set consisting of all the fractional order integration operators ϕ_α $(\alpha \geq 0)$, given by

$$\phi_o(f) = f$$

$$\phi_\alpha(f) = \frac{1}{\Gamma(\alpha)} \int_o^x (x-t)^{\alpha-1} f(t)dt \quad (\alpha > 0),$$

for $f \in \mathfrak{C}$. Then it is not hard to verify that $\mathfrak{C}$, $\mathfrak{B}$ form a quotient pair. If $f \in \mathfrak{C}$ and $\phi_\alpha \in \mathfrak{B}$, then $f // \phi_\alpha$ corresponds in a loose sense to the α^{th} fractional derivative of f.

2) Consider the class B consisting of all bounded measurable functions. Then B and $\mathfrak{W}$ form a quotient pair under the convolution operation.

INTRODUCTION TO CHAPTER 2

In section 2.1., some results concerning the spaces $\mathfrak{T}$, $\mathfrak{S}$, and $\mathfrak{M}$ are proven. It is shown that $\mathfrak{T}$, with a certain set of operations, is a universal algebra, and that $\mathfrak{S} \cap \mathfrak{M}$ provides an Abelian semigroup of endomorphisms of $\mathfrak{T}$. In section 2.2. it is shown that $\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}$ form a quotient pair (T.2.2.1.); and in section 2.3. we review those properties of $(\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}), \mathfrak{Q}(\mathfrak{S} \cap \mathfrak{M}))$ deducible from the general theory of chapter one. In sections 2.5. and 2.6. some interesting connections are established between $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$ and certain classes of distributions: in this connection it is necessary to present a summary of those results of distribution theory which will be of use in sections 2.5. and 2.6., and this is done in section 2.4. This section (§ 2.4.) is concerned only with distributions in one dimension, some of the lengthier proofs involved are given in the appendix.

Section 2.5. is devoted to showing (T.2.5.1.) that there exists an isomorphism between $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$ and the class of Fourier transforms of distributions of finite order. Lemmas L.2.5.4. and L.2.5.5. contain the substance of this result, these lemmas being brought together to establish the isomorphism. In section 2.6. we identify a sub-algebra of $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$ which is isomorphic with the class of tempered distributions. Theorem T.2.6.1. is the key result[†] which enables us to establish this isomorphism, in T.2.6.2. The main results of these last two sections are entirely fresh.

† Theorem T.2.6.1. would appear to have some intrinsic interest, and I have been able to prove a variant (not given in the thesis) for continuous bounded functions and L^1-functions, rather than tempered and $\mathfrak{S} \cap \mathfrak{B}$ -functions.

CHAPTER 2

2.1. Review of example E.1.4.5.

L.2.1.1.[†] If $k \in \mathfrak{S}$, then $\hat{k} \in \mathfrak{S}$.

L.2.1.2.[‡] If $k \in \mathfrak{S}$, then $\hat{\hat{k}}(x) = 2\pi k(-x)$.

L.2.1.3.[‖] If $k_1, k_2 \in \mathfrak{S}$, then $(k_1 * k_2)^\wedge(t) = \hat{k}_1(t)\hat{k}_2(t)$.

L.2.1.4. If $f \in \mathfrak{T}$ and $k \in \mathfrak{S}$, then $(f * k)^{(p)}(x) = (f^{(p)} * k)(x) = (f * k^{(p)})(x)$ for $p = 0,1,2,\dots$.

Proof: The result follows easily from an application of the mean value theorem and Lebesgue's convergence theorem.

L.2.1.5. If $f \in \mathfrak{T}$ and $k \in \mathfrak{S}$, then $f * k \in \mathfrak{T}$.

Proof: The result follows from the previous lemma together with the fact that for each non-negative integer p, $f^{(p)}(x)$ is dominated by some polynomial.

L.2.1.6. If $f \in \mathfrak{T}$ and $k_1, k_2 \in \mathfrak{S}$ then $(f * k_1) * k_2 = f * (k_1 * k_2) = (f * k_2) * k_1$.

Proof: We observe that f is dominated by a polynomial, and we then apply Fubini's theorem. The result follows immediately.

We consider the space $\mathfrak{T}$ to be a universal algebra with the following set of operations;

(i) the binary operation $(f,g) \rightarrow f + g$,

(ii) the unary operations $f \rightarrow \lambda f$, where λ is any complex number,

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[†] See (20) Pg.105.

[‖] See(22) Pg.41.

[‡] See (22) Pg.42.

(iii) the unary operations $f \to f^{(p)}$, $p = 0,1,2,\ldots,$

(iv) the unary operations $f \to \int f(t)\ell(x-t)dt$, where ℓ is any element of $\mathfrak{S}$,

(v) the nullary operation 0.

The only one of these requiring any comment is (iv), and it follows from L.2.1.5. that (iv) does describe a class of unary operations on $\mathfrak{T}$. It follows from L.2.1.1., L.2.1.2., and L.2.1.3., that $\mathfrak{S}_\cap \mathfrak{M}$ is closed under convolution, and from L.2.1.4. and L.2.1.6. that $\mathfrak{B} = \mathfrak{S}_\cap \mathfrak{M}$ is an Abelian semigroup of endomorphisms of $\mathfrak{T}$.

We now proceed to the proof that if $f \in \mathfrak{T}$ and $k \in \mathfrak{S}_\cap \mathfrak{M}$ then $f * k$ is null only if f is null.

2.2. $\mathfrak{T}$, $\mathfrak{S}_\cap \mathfrak{M}$ form a quotient pair

L.2.2.1.[†] If k is integrable over the real line and $\hat{k}(t) = 0$ $\forall t$, then $k = 0$.

D.2.2.1.[‡] We denote by $\mathfrak{D}$ those functions k, of a real variable t, such that $k(t)$ has derivatives of all orders and has compact support.

It is not entirely obvious that $\mathfrak{D}$ contains any functions apart from the identically zero function. An example of a non-null function in $\mathfrak{D}$ is given by the

[†] This follows from a result given in (22) Pg.45.

[‡] The space $\mathfrak{D}$ was first considered by L. Schwartz. See (19) Pg.21.

function equal to $\exp[1/(t^2-1)]$ for $|t| < 1$ and zero for $|t| \geq 1$.

We note that $\mathfrak{D}$ is a subset of $\mathfrak{S}$.

<u>L.2.2.2.</u> If $\hat{g} \in \mathfrak{D}$ and $k \in \mathfrak{S} \cap \mathfrak{W}$, then there exists an $\ell \in \mathfrak{S}$ such that $\hat{\ell} \in \mathfrak{D}$ and $k * \ell = g$.

Proof: Since $\hat{g} \in \mathfrak{D}$, it follows that $\hat{g}/\hat{k} \in \mathfrak{D}$. Define ℓ to be the function whose Fourier transform is $\hat{g}/\hat{k}$. Then $\hat{\ell} \in \mathfrak{D}$, and so $\ell \in \mathfrak{S}$. Moreover $\hat{k}\hat{\ell} = \hat{g}$, which implies that $k * \ell = g$.

<u>T.2.2.1.</u> If $f \in \mathfrak{T}$, $k \in \mathfrak{S} \cap \mathfrak{W}$ and $f * k = 0$, then $f = 0$.

Proof: If g is any function such that $\hat{g} \in \mathfrak{D}$, then there exists an $\ell \in \mathfrak{S}$ such that $k * \ell = g$. Since $f * k = 0$, we have that

$$(f * k) * \ell = 0 \text{ and so } f * g = 0. \tag{1}$$

Now if $\int g(t)e^{ixt}dt \in \mathfrak{D}$, then $\int \{g(-t)e^{-itv}\}e^{ixt}dt \in \mathfrak{D}$ for every real v. Hence, since (1) holds for every g such that $\hat{g} \in \mathfrak{D}$, we have that

$$\int f(t)g(t-x)e^{i(t-x)v}dt = 0, \quad \text{for all real } x,\ v,$$

and it follows from L.2.2.1. that $f(t)g(t-x) = 0$ for every x and t; since g is subject only to the restriction that $\hat{g} \in \mathfrak{D}$, it follows that $f(t) = 0 \ \forall t$.

<u>Corollary</u> $\mathfrak{T}$, $\mathfrak{S} \cap \mathfrak{W}$ form a quotient pair.

2.3. Review of the properties of $(\mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W}), \mathfrak{Q}(\mathfrak{G} \cap \mathfrak{W}))$.

$\mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ is a universal algebra with the same set of operations as $\mathfrak{T}$, which are defined on $\mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ as follows;

(i) if $\alpha = f//g$, $\beta = h//k \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$, then $\alpha + \beta = (f * k + h * g)//(g * k)$,

(ii) if $\alpha = f//g \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ and p is any non-negative integer, then $\alpha^{(p)} = f^{(p)}//g$,

(iii) if $\alpha = f//g \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ and λ is any scalar, then $\lambda\alpha = (\lambda f)//g$,

(iv) if $\alpha = f//g \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ and ℓ is any element of $\mathfrak{G}$, then $\alpha * \ell = (f * \ell)//g$,

(v) $\exists\, 0 \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ given by $0//g$ for any $g \in \mathfrak{G} \cap \mathfrak{W}$.

It is easily seen that $\mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ is an Abelian group under addition, the inverse of α being $(-1)\alpha$, which we shall write as $-\alpha$. We shall call $\alpha^{(p)}$, the p-th derivative of α, and $\alpha * \ell$, where $\ell \in \mathfrak{G}$, the convolution of α and ℓ. It is clear that multiplication by a scalar, differentiation, and convolution by an element of $\mathfrak{G}$ are each distributive with respect to addition in $\mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$. We also have the following relations; $(\lambda\alpha)^{(p)} = \lambda\alpha^{(p)}$, $\alpha^{(p)} * \ell = (\alpha * \ell)^{(p)} = \alpha * \ell^{(p)}$, $(\alpha^{(p)})^{(q)} = \alpha^{(p+q)}$,

$(\lambda\alpha) * \ell = \lambda(\alpha * \ell)$, $(\alpha * \ell_1) * \ell_2 = \alpha * (\ell_1 * \ell_2) = (\alpha * \ell_2) * \ell_1$, $\lambda 0 = 0$, $0^{(p)} = 0$, $0 * \ell = 0$, $0\alpha = 0$, $\alpha * 0 = 0$; where $\alpha \in \mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$, p,q are non-negative integers, $\ell, \ell_1, \ell_2 \in \mathfrak{G}$ and λ is any complex number.

It follows easily from an argument given in Chapter 1 that $\mathfrak{Q}(\mathfrak{G} \cap \mathfrak{W})$ gives an Abelian group of automorphisms of $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$; if $\alpha = f//g \in \mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ and $\xi = \phi//\psi \in \mathfrak{Q}(\mathfrak{G} \cap \mathfrak{W})$, then $\alpha * \xi = (f * \phi)//(g * \psi)$. The identity automorphism I is $\phi//\phi$, where ϕ is any member of $\mathfrak{G} \cap \mathfrak{W}$. The natural embeddings of $\mathfrak{X}$ in $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ and $\mathfrak{G} \cap \mathfrak{W}$ in $\mathfrak{Q}(\mathfrak{G} \cap \mathfrak{W})$ will be denoted by i and j respectively as in Chapter 1.

The next section gives a summary of the familiar theory of distributions as developed by L. Schwartz and others. We include it here for the sake of completeness.

2.4. Digression on distributions

D.2.4.1.[†] If E is any set and if $\mathfrak{F}$ is a family of subsets of E such that

(i) $\emptyset \notin \mathfrak{F}$,

(ii) $F_1, F_2 \in \mathfrak{F} \Rightarrow F_1 \cap F_2 \in \mathfrak{F}$,

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[†] See also (2) Pg.57 and (23) Pg.6.

(iii) If $F' \subseteq F \subseteq E$ and $F' \in \mathfrak{F}$ then $F \in \mathfrak{F}$,

then we say that $\mathfrak{F}$ is a filter on E.

D.2.4.2.[†] If E is any set, $\mathfrak{F}$ a filter on E and if $\mathfrak{B} \subseteq \mathfrak{F}$ is such that, $F \in \mathfrak{F} \Rightarrow \exists F' \in \mathfrak{B}$ such that $F' \subseteq F$, then we shall say that $\mathfrak{B}$ is a basis of the filter $\mathfrak{F}$.

Any set $\mathfrak{B}$ of subsets of E is a basis of some filter, $\mathfrak{F}$, on E provided

(i) $\emptyset \notin \mathfrak{B}$

(ii) $B_1, B_2 \in \mathfrak{B} \Rightarrow \exists B \in \mathfrak{B}$ such that $B \subseteq B_1 \cap B_2$.

The filter $\mathfrak{F}$ corresponding to such a set $\mathfrak{B}$ consists of all sets having a subset belonging to $\mathfrak{B}$.

D.2.4.3.[‡] We say that a set E is a topological space if for each $x \in E$, a filter $\mathfrak{F}_x$ is given such that

(i) $F \in \mathfrak{F}_x \Rightarrow x \in F$,

(ii) $F \in \mathfrak{F}_x \Rightarrow \{\exists F' \in \mathfrak{F}_x$ such that, $y \in F' \Rightarrow$
$F \in \mathfrak{F}_y\}$.

In such a situation we call $\mathfrak{F}_x$ the filter, or family, of neighbourhoods at x, and if $\mathfrak{B}_x$ is a basis of $\mathfrak{F}_x$, then we say that $\mathfrak{B}_x$ is a basis of neighbourhoods of x.

D.2.4.4.[‖] If E is a linear space over $\mathbb{C}$ in which a topology τ is defined so that the maps

$$E \times E \ni (x, y) \rightarrow x + y \in E,$$

[†] See also (2) Pg.59 and (23) Pg.7.

[‡] See (23) Pg.8.

[‖] See(23) Pg.20 and (10) Pgs.1-11.

$$\mathbb{C} \times E \ni (\lambda, x) \to \lambda x \in E,$$

are continuous, then we say that E is a linear topological space over $\mathbb{C}$, and that τ is compatible with the linear structure of E.

L.2.4.1.† If E is a linear topological space over $\mathbb{C}$ and if $x \in E$, then N is a neighbourhood of x if and only if N-x is a neighbourhood of O.

In view of this lemma, if E is a linear topological space, to specify its topology it is clearly sufficient to give a basis of the family of neighbourhoods of zero for the topology of E.

D.2.4.5.† If E is a linear space and A is a subset of E such that for every $x \in E$ there exists $C_x > 0$ such that for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| \leq C_x$, $\lambda x \in A$, then we shall say that A is absorbing.

D.2.4.6.† If E is a linear space and A is a subset of E such that for every $\lambda \in \mathbb{C}$ satisfying $|\lambda| \leq 1$ and every $x \in A$, $\lambda x \in A$, then we shall say that A is balanced.

L.2.4.2.† A family of subsets $\mathfrak{N}$ of the linear space E (over $\mathbb{C}$) is the family of neighbourhoods of the origin in a topology compatible with the linear structure of E if and only if it has the following properties;

(i) if $N \in \mathfrak{N}$, then $0 \in N$;

(ii) $\forall N \in \mathfrak{N}$, $\exists M \in \mathfrak{N}$ such that $M + M \subseteq N$,

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†See (23) Pg.21.

(iii) $\forall N \in \mathfrak{N}$, $\forall \lambda \in \mathbb{C}$, $\lambda \neq 0$, $\lambda N \in \mathfrak{N}$,

(iv) every $N \in \mathfrak{N}$ is absorbing,

(v) every $N \in \mathfrak{N}$ includes a balanced $V \in \mathfrak{N}$.

D.2.4.7.† We have already defined the spaces $\mathfrak{D}$ and $\mathfrak{S}$ in definitions D.2.2.1. and D.1.4.1. These spaces may be regarded as linear spaces over $\mathbb{C}$, and we now introduce topologies on each by specifying a basis of neighbourhoods of the origin in each.

(i) The collection of all sets of the form

$$\{\phi \in \mathfrak{D} \,|\, |\phi^{(p)}(x)| < \epsilon_k \text{ whenever } p < m_k \text{ and } |x| > k,\ k = 0,1,2,\ldots\},$$

where $\{\epsilon_k\}$ is any sequence of positive numbers monotonically decreasing to zero, and $\{m_k\}$ is any sequence of positive numbers monotonically increasing to infinity, provides a basis of neighbourhoods of 0 in $\mathfrak{D}$.

(ii) The collection of all sets of the form

$$\{\phi \in \mathfrak{S} \,|\, |x^k \phi^{(p)}(x)| < \epsilon \text{ for } k \leq K \text{ and } p \leq P\},$$

where ϵ, K, P are positive, provides a basis of neighbourhoods of 0 in $\mathfrak{S}$.

L.2.4.3. The topologies introduced in the previous defintion are compatible with the linear structures of $\mathfrak{D}$ and $\mathfrak{S}$ respectively, so that $\mathfrak{D}$, $\mathfrak{S}$ are linear topological spaces.

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† These topologies were introduced by L. Schwartz. See (19) Pg.67 and (20) Pg.90.

Proof: This result follows from L.2.4.2. when we observe that the sets forming our bases are balanced in each case.

<u>L.2.4.4.</u>[†] If $k \in \mathfrak{D}$, then $\hat{k}(x)$ is the restriction to real values of the argument of an entire function of a complex variable.

Proof: Put $\tilde{k}(z) = \int k(t)e^{izt}dt$ for $z \in \mathbb{C}$, then for real x, $\tilde{k}(x) = \hat{k}(x)$, and it is clear that $\tilde{k}(z)$ is an entire function. We will denote the analytic continuation, $\tilde{k}$, of $\hat{k}$ simply by $\hat{k}$.

<u>D.2.4.8.</u>[‡] We denote by $\mathfrak{Z}$ the space of entire functions k, of a complex variable z, which are the analytic continuations of the Fourier transforms of functions in $\mathfrak{D}$.

$\mathfrak{Z}$ is a linear space over $\mathbb{C}$. We introduce a topology into $\mathfrak{Z}$ by taking neighbourhoods of the origin to be sets of the form

$$N = \{k(z) \mid \int k(x)e^{ixt}dx \in M\},$$

where M is some neighbourhood of the origin in $\mathfrak{D}$.

<u>L.2.4.5.</u> The topology on $\mathfrak{Z}$ given by the previous definition is compatible with its linear structure and so $\mathfrak{Z}$ is a linear topological space.

Proof: This result follows easily from L.2.4.2. and L.2.4.3.

<u>D.2.4.9.</u>[‖] We denote by $\mathfrak{D}'$, $\mathfrak{S}'$, $\mathfrak{Z}'$ the spaces of continuous linear functionals on $\mathfrak{D}$, $\mathfrak{S}$, $\mathfrak{Z}$ respectively.

[†] See (9) Pg.153. [‡] The space $\mathfrak{Z}$ was introduced by I.M. Gel'fand and G.E. Shilov. ——— See (9) Pg.155.
[‖] See (19) Pgs.24 and 69, (20) Pg.93, (9) Pg.159.

Elements of $\mathfrak{D}'$, $\mathfrak{S}'$ and $\mathfrak{Z}'$ will be called distributions.

L.2.4.6.[†] The following three maps are continuous.

(i) $\mathfrak{D} \ni f \to \hat{f} \in \mathfrak{Z}$,

(ii) $\mathfrak{Z} \ni f \to \hat{f} \in \mathfrak{D}$,

(iii) $\mathfrak{S} \ni f \to \hat{f} \in \mathfrak{S}$.

Proof: (i) If N is a neighbourhood of 0 in $\mathfrak{Z}$, then there exists M, a neighbourhood of 0 in $\mathfrak{D}$ such that

$$N = \{k(z) \mid \hat{k}(t) \in M\},$$

and so if

$$M^* = \{f(t) \mid 2\pi f(-t) \in M\},$$

and if $f \in M^*$, then $\hat{f} \in N$. Since M^* is a neighbourhood of 0, this proves (i).

(ii) If M is a neighbourhood of 0 in $\mathfrak{D}$ and

$$N = \{k(z) \mid \hat{k}(t) \in M\},$$

then if $k \in N$, $\hat{k} \in M$. The result follows.

(iii) If M is a neighbourhood of 0 in $\mathfrak{S}$, then M has a subset N of the form

$$N = \{k(x) \mid |x^q k^{(p)}(x)| < \epsilon \text{ for } p \leq P,\ q \leq k\},$$

for some $\epsilon > 0$ and some positive integers P and K. Put $A = \int \frac{1}{1+t^2}\,dt$, and choose ϵ' such that $0 < \epsilon' < \frac{\epsilon}{P!\,2^{K+1}A}$.

[†]See (10) Pg.125 et seq.

Put $N^{*} = \{k(x) \mid |x^p k^{(q)}(x)| < \epsilon' \text{ for } p \leq P+2,\ q \leq K\}$. Then if $f \in N^{*}$, $q \leq K$ and $p \leq P$, we have, integrating by parts, that

$$
\begin{aligned}
|x^q \hat{f}^{(p)}(x)| &\leq \int |D^q[t^p f(t)]| \, dt, \\
&\leq \sum_{r=0}^{q} \binom{q}{r} \int |D^{q-r} t^p| \, |D^r f(t)| \, dt, \\
&\leq \sum_{r=\max(0,q-p)}^{q} \binom{q}{r} p! \int |t|^{p-q+r} |f^{(r)}(t)| \, dt, \\
&\leq \sum_{r=\max(0,q-p)}^{q} \binom{q}{r} p! \int \frac{|t|^{p-q+r} + |t|^{p+2-q+r}}{1+|t|^2} |f^{(r)}(t)| \, dt, \\
&\leq \sum_{r=0}^{q} \binom{q}{r} p! 2A\epsilon', \\
&\leq P! 2^{K+1} A\epsilon', \\
&< \epsilon,
\end{aligned}
$$

and so $\hat{f} \in M$.

<u>D.2.4.10.</u>† If $F \in \mathfrak{D}'$ we define $\hat{F}$ to be the linear functional on $\mathfrak{Z}$ given by $\langle \hat{F}, k \rangle = \langle F, \hat{k} \rangle$, $k \in \mathfrak{Z}$. In view of the preceeding lemma $\hat{F} \in \mathfrak{Z}'$. Likewise if $F \in \mathfrak{Z}'$ we define $\hat{F}$ to be the linear functional on $\mathfrak{D}$ given by $\langle \hat{F}, k \rangle = \langle F, \hat{k} \rangle$, $k \in \mathfrak{D}$, and in this case $\hat{F} \in \mathfrak{D}'$; if $F \in \mathfrak{S}'$ we define $\hat{F}$ to be the linear functional on $\mathfrak{S}$ given by $\langle \hat{F}, k \rangle = \langle F, \hat{k} \rangle$, $k \in \mathfrak{S}$, and here $\hat{F} \in \mathfrak{S}'$.

In each of the above cases $\hat{F}$ will be referred to as the Fourier transform of F.

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†See (10) Pg.128.

<u>L.2.4.7.</u>[†] The following maps are all continuous:

(i) $\mathfrak{D} \ni f(x) \to k(x)f(x) \in \mathfrak{D}$, where k is any function with derivatives of all orders everywhere,

(ii) $\mathfrak{S} \ni f(x) \to k(x)f(x) \in \mathfrak{S}$, where $k \in \mathfrak{T}$,

(iii) $\mathfrak{Z} \ni f(z) \to k(z)f(z) \in \mathfrak{Z}$, where $k \in \mathfrak{Z}$,

(iv) $\mathfrak{D} \ni f(x) \to \alpha f(\lambda x+\mu) \in \mathfrak{D}$ where $\lambda,\mu \in \mathbb{R}$, $\lambda \neq 0$, $\alpha \in \mathbb{C}$, and similarly for $\mathfrak{Z}$ and $\mathfrak{S}$, though μ may be complex in the case of $\mathfrak{Z}$.

(v) $\mathfrak{D} \ni f(x) \to f^{(p)}(x) \in \mathfrak{D}$, where $p = 0,1,2,\ldots,$ and similarly for $\mathfrak{Z}$ and $\mathfrak{S}$.

<u>D.2.4.11.</u>[‡] (i) If k(x) has derivatives of all orders everywhere, $\alpha \in \mathbb{C}$, p a non-negative integer and $F \in \mathfrak{D}'$ we define kF, αF and $F^{(p)}$ by

$$\langle kF,f\rangle = \langle F,kf\rangle,\ f \in \mathfrak{D},$$
$$\langle \alpha F,f\rangle = \langle F,\alpha f\rangle,\ f \in \mathfrak{D},$$
$$\langle F^{(p)},f\rangle = (-1)^p \langle F,f^{(p)}\rangle,\ f \in \mathfrak{D},$$

respectively. It follows from the previous lemma that kF, αF, $F^{(p)}$ are all continuous linear functionals on $\mathfrak{D}$.

(ii) In like manner if $k \in \mathfrak{T}$, $\alpha \in \mathbb{C}$, p a non-negative integer, we define kF, αF, $F^{(p)}$, for $F \in \mathfrak{S}'$ and it is clear that kF, αF, $F^{(p)}$ are all continuous.

(iii) Again as before kF, αF, $F^{(p)}$ are defined in $\mathfrak{Z}'$ if $k \in \mathfrak{Z}$, $\alpha \in \mathbb{C}$, p is a non-negative integer and if $F \in \mathfrak{Z}'$.

[†]See (10) Pgs.101,102,108,109. Also (19) Pg.35.
[‡]See (10) Pgs.65-66.

In each of the above cases $F^{(p)}$ will be called the p-th derivative of F.

If $F_1, F_2 \in \mathfrak{D}'$ then we define F_1+F_2 by

$$\langle F_1+F_2,k\rangle = \langle F_1,k\rangle + \langle F_2,k\rangle, \quad \forall k \in \mathfrak{D} ,$$

and it is clear that $F_1+F_2 \in \mathfrak{D}'$. Thus $\mathfrak{D}'$ is a linear space. Likewise we may consider $\mathfrak{S}'$ and $\mathfrak{Z}'$ as linear spaces.

<u>N.2.4.1.</u> We shall denote the null functionals in $\mathfrak{D}'$, $\mathfrak{S}'$, $\mathfrak{Z}'$ respectively by 0; we then have that if $k \in \mathfrak{D}$, $\mathfrak{S}$, $\mathfrak{Z}$ respectively, then $\langle 0,k\rangle = 0$.

<u>D.2.4.12.</u>† A function f of a real variable is said to be tempered if it is continuous and dominated by some polynomial.

<u>D.2.4.13.</u> a) If f is a continuous function and if p is a non-negative integer, then we define $D^p f$ to be the linear functional on $\mathfrak{D}$ given by

$$\langle D^p f,k\rangle = (-1)^p \int f(t)k^{(p)}(t)dt, \quad \forall k \in \mathfrak{D} .$$

b) If f is a tempered function and if p is a non-negative integer, then we may extend the functional $D^p f$, defined above, to be a linear functional on $\mathfrak{S}$ given by

$$\langle D^p f,k\rangle = (-1)^p \int f(t)k^{(p)}(t)dt, \quad \forall k \in \mathfrak{S} .$$

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† Using the terminology of Bremmerman, see (3) Pg.83.

We note that in both the above cases $D^p f$ is continuous; so that in part a) $D^p f \in \mathfrak{D}'$, while for part b) $D^p f \in \mathfrak{S}'$.

We will, in either situation, refer to $D^p f$ as the p-th generalised derivative of f. A distribution of the form $D^p f$ is said to be of finite order.[†]

<u>L.2.4.8.</u>[‡] If $F \in \mathfrak{S}'$, then there exists a tempered function f and a non-negative integer p such that $F = D^p f$.

<u>D.2.4.14.</u>[||] If $F \in \mathfrak{S}'$ and $k \in \mathfrak{S}$, or if $F \in \mathfrak{D}'$ and $k \in \mathfrak{D}$, then we denote by $(F * k)(x)$ the function of a real variable x, given by $\langle F_t, k(x-t)\rangle$. [The suffix t on the functional F is used to distinguish the variables x and t: it indicates that F operates on a function of t.] This function will be called the convolution of F and k.

<u>L.2.4.9.</u>[¶] If $F \in \mathfrak{S}'$ and $k \in \mathfrak{S}$, or if $F \in \mathfrak{D}'$ and $k \in \mathfrak{D}$, then $(F * k)(x) \in \mathfrak{T}$.

<u>L.2.4.10.</u>[¶] If $F \in \mathfrak{S}'$ and $k \in \mathfrak{S}$, or if $F \in \mathfrak{D}'$ and $k \in \mathfrak{D}$; and if p is a non-negative integer then $(F * k)^{(p)}(x) = (F^{(p)} * k)(x) = (F * k^{(p)})(x)$.

<u>L.2.4.11.</u>[¶] If $F \in \mathfrak{S}'$ and $k_1, k_2 \in \mathfrak{S}$, or if $F \in \mathfrak{D}'$ and $k_1, k_2 \in \mathfrak{D}$, then $(F * k_1) * k_2 = F * (k_1 * k_2) = (F * k_2) * k_1$.

<u>L.2.4.12.</u> If $F \in \mathfrak{S}'$, $k \in \mathfrak{S}$ and $\phi \in \mathfrak{S}$, or if $F \in \mathfrak{D}'$, $k \in \mathfrak{D}$ and $\phi \in \mathfrak{D}$, then

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[†] See (23) chapter 24.

[‡] See (23) Pg.272, and (20) Pg.95.

[||] See (3) Pg.105.

[¶] See the appendix.

$$\int (F * k)(t)\, \hat{\phi}\,(t)dt = \langle (\hat{k}\,\hat{F}),\ \phi \rangle.$$

Proof: If $F \in \mathfrak{S}'$, $k \in \mathfrak{S}$ and $\phi \in \mathfrak{S}$, put $\psi(t) = \hat{\phi}(-t)$, and then we have that

$$\begin{aligned}
\langle (\hat{k}\hat{F}),\ \phi \rangle &= \langle \hat{F},\ \hat{k}\phi \rangle \\
&= \langle \hat{F}_t,\ \tfrac{1}{2\pi}(k * \psi)^{\wedge}\,(t) \rangle \\
&= \langle F_z,\ (k * \psi)(-z) \rangle \\
&= (F * (k * \psi))(0) \\
&= ((F * k) * \psi)(0) \quad \text{by L.2.4.11.,} \\
&= \int (F * k)(t)\, \hat{\phi}\,(t)dt.
\end{aligned}$$

The proof of the other case follows in similar fashion.

<u>L.2.4.13.</u>† If $f_1, f_2 \in \mathfrak{T}$ and for every $k \in \mathfrak{Z}$,

$$\int f_1(t)k(t)dt = \int f_2(t)k(t)dt,$$

then $f_1 = f_2$.

Proof: It is sufficient to show that, if $f \in \mathfrak{T}$, and if for every $k \in \mathfrak{Z}$ we have

$$\int f(t)k(t)dt = 0,$$

then $f = 0$.

If $k \in \mathfrak{Z}$, $u \in \mathbb{R}$, then $e^{iuz}k(z) \in \mathfrak{Z}$, and so

$$\int f(t)k(t)e^{iut}dt = 0,$$

†See (10) Pg.235-236.

for every $u \in \mathbb{R}$. Hence by L.2.2.1., for every $k \in \mathfrak{Z}$,

$$f(t)k(t) = 0 \quad \text{for all real } t,$$

and it follows that $f = 0$, which proves the lemma.

<u>L.2.4.14.</u> If $k \in \mathfrak{G}$ then the map

$$\mathfrak{Z} \ni \phi(z) \to \int k(t)\phi(z+t)dt \in \mathfrak{Z}$$

is continuous.

Proof: The maps

$$\mathfrak{Z} \ni \phi(x) \to \hat{\phi}(x) \in \mathfrak{D},$$

$$\mathfrak{D} \ni \ell(x) \to \hat{k}(-x)\ell(x) \in \mathfrak{D}, \text{ where } k \in \mathfrak{G},$$

$$\mathfrak{D} \ni h(x) \to \frac{1}{2\pi}\hat{h}(-z) \in \mathfrak{Z},$$

are all continuous by L.2.4.6. and L.2.4.7. Hence if $k \in \mathfrak{G}$, applying these maps consecutively and using L.2.1.3., we obtain that

$$\mathfrak{Z} \ni \phi(z) \to \int k(t)\phi(z+t)dt \in \mathfrak{Z}$$

is also continuous.

<u>D.2.4.15.</u> If $F \in \mathfrak{Z}'$, $k \in \mathfrak{G}$ then we define $F \star k$ † to be the linear functional on $\mathfrak{Z}$ given by

$$\langle F \star k, \phi \rangle = \langle F_z, \int k(t)\phi(z+t)dt \rangle, \quad \phi \in \mathfrak{Z}.$$

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† Notation not in general use. This particular type of convolution yielding a functional is familiar. For the sake of clarity, we here use the symbol $\star$, to make a distinction from the other convolution (see D.2.4.14.), which yields a function.

In view of the previous lemma, $F \star k \in \mathfrak{Z}'$.

L.2.4.15. If $F \in \mathfrak{Z}'$, $k \in \mathfrak{G}$ and $\phi \in \mathfrak{Z}$, then

$$(F \star k) \ast \phi = (F \ast \phi) \ast k.$$

Proof: Take $F \in \mathfrak{Z}'$, $k \in \mathfrak{G}$, $\phi \in \mathfrak{Z}$, then if $\ell \in \mathfrak{Z}$,

$$
\begin{aligned}
((F \star k) \ast \phi) \ast \ell &= (F \ast (k \ast \phi)) \ast \ell && \text{by D.2.4.15.,} \\
&= F \ast (k \ast \psi \ast \ell) && \text{by L.2.4.11.,} \\
&= (F \ast \phi) \ast (k \ast \ell) && \text{by L.2.4.11.,} \\
&= ((F \ast \phi) \ast k) \ast \ell && \text{by L.2.1.6.,}
\end{aligned}
$$

and since this relation holds for every $\ell \in \mathfrak{Z}$, the result follows by L.2.4.13., using lemmas L.2.4.9. and L.2.1.5.

N.2.4.2. With the following set of operations, $\mathfrak{Z}'$ is a universal algebra:

(i) the binary operation $(F,G) \to F + G$,

(ii) the unary operations $F \to \lambda F$, where λ is any complex number,

(iii) the unary operations $F \to F^{(p)}$, $p = 0,1,2,\ldots$,

(iv) the unary operations $F \to F \star \ell$, where ℓ is any element of $\mathfrak{G}$,

(v) the nullary operation 0.

D.2.4.16.† Topologies may be introduced to the linear spaces $\mathfrak{D}'$, $\mathfrak{G}'$, $\mathfrak{Z}'$, by taking as a basis for neighbour-

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† See (23) Pg.197 and (10) Pg.46.

hoods of the origin in $\mathfrak{D}'$ all sets of the form

$$\{F \in \mathfrak{D}' \mid |\langle F, \phi_i \rangle| < \epsilon \ , \ i = 1,2,\ldots,n.\}$$

where $\epsilon > 0$, n is any positive integer and $\phi_1, \phi_2, \ldots, \phi_n$, are any elements of $\mathfrak{D}$; and similarly for $\mathfrak{G}'$ and $\mathfrak{Z}'$.

These topologies are compatible with the linear structures of $\mathfrak{D}'$, $\mathfrak{G}'$, $\mathfrak{Z}'$ respectively, as may be verified by use of L.2.4.2. We shall call them the weak dual topologies.

L.2.4.16.† With the above topologies each of the spaces $\mathfrak{D}'$, $\mathfrak{G}'$, $\mathfrak{Z}'$ is sequentially complete.‡

2.5. The isomorphism theorem

L.2.5.1. If $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ and $\ell \in \mathfrak{Z}$, then $\alpha * \ell \in i(\mathfrak{T}) \subseteq \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$.

Proof: If $\alpha = f//g$ and $\ell \in \mathfrak{Z}$, then $\hat{\ell}/\hat{g} \in \mathfrak{D}$, and so $\exists\, k \in \mathfrak{Z}$ such that $g * k = \ell$ (this follows from L.2.2.2.). Hence $\alpha * \ell = \alpha * (g * k) = (f * k) * g//g$ and so $\alpha * \ell = i(f * k)$, and the result follows

L.2.5.2. If $g(t)$ is a continuous function, then $\exists\, \ell(t) \in \mathfrak{G}$ such that $\ell(t) > 0\ \forall t$ and $g(t)\ell^{(p)}(t)$ is tempered for each non-negative integer p.

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†See the appendix.

‡For a definition of this term see (23) Pg.38.

Proof: Put $A_n = 1 + \operatorname{Sup}_{|x|<n+1} |g(x)|$ for $n = 0,1,2,\ldots$, and

$A_n = A_{-n}$ for $n = -1,-2,-3,\ldots$.

If

$$k(x) = \begin{cases} \exp[-1/(1-x^2)] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and $B_p = \operatorname{Sup}_{\substack{|x|<1 \\ 0\leq r\leq |p|}} \{|k^{(r)}(x)|\}$, for each integer p, and if

$k_n(x) = k(x-n)/B_n$, for each integer n, then we have that

(i) $k_n(x) = 0$ if $|x-n| > 1$

(ii) $k_n(x) > 0$ if $|x-n| < 1$

(iii) $|k_n{}^{(p)}(x)| \leq 1$ if $0 \leq p \leq |n|$.

Put $\ell(x) = \Sigma_{n=-\infty}^{+\infty} k_n(x)/\{(1+|n|)^{|n|} A_n\}$. The sum converges for each real x, since at most two terms are non-zero; furthermore $\ell(x)$ has derivatives of all orders everywhere; and $\ell(x) > 0\ \forall x$. If m,p are non-negative integers and if $|x| > p+2$, it follows that $|[x]| > p+1$, and so

$$|k_{[x]}{}^{(p)}(x)| \leq 1$$

and $$|k^{(p)}_{([x]+1)}(x)| \leq 1.$$

Furthermore we have that $A_n \geq 1$ for each n, and so

$$\begin{aligned} |x^m \ell^{(p)}(x)| &\leq |x|^m \{1/(|[x]|+1)^{|[x]|} \\ &\quad + 1/(|[x]+1|+1)^{|[x]+1|}\} \\ &\rightarrow 0 \quad \text{as} \quad x \rightarrow \infty. \end{aligned}$$

Hence $\ell \in \mathfrak{G}$. It follows similarly since $(1+|n|)^{|n|} \geq 1$ for $n = 1,2,3,\ldots$, that for any non-negative integer p, if $|x| > p+2$,

$$|g(x)\ell^{(p)}(x)| \leq |g(x)|\{1/A_{[x]} + 1/A_{([x]+1)}\};$$

however if $x \geq 0$, $|g(x)| \leq A_{[x]}$ and if $x < 0$, $|g(x)| \leq A_{([x]+1)}$, and so in either case if $|x| > p+2$,

$$|g(x)\ell^{(p)}(x)| \leq 2,$$

and the result follows.

<u>L.2.5.3.</u> If $g(t)$ is a continuous function, $\ell(t)$ has derivatives of all orders everywhere and N is a non-negative integer, then there exists a continuous function h such that† $D^N h = \ell(D^N g)$ in $\mathfrak{D}'$. If in addition $g(t)\ell^{(p)}(t)$ is tempered for $p = 0,1,2,\ldots,N$, then h may be taken to be tempered.

Proof: If $\phi \in \mathfrak{D}$, we have that

$$\begin{aligned} \langle \ell(D^N g), \phi \rangle &= \langle D^N g, \ell\phi \rangle \\ &= (-1)^N \int g(t) D^N\{\ell(t)\phi(t)\}dt \\ &= (-1)^N \sum_{n=0}^{N} \binom{N}{n} \int g(t)\ell^{(N-n)}(t)\phi^{(n)}(t)dt. \end{aligned}$$

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†For the definition of D^N, see D.2.4.13. Note that D^N operates on a continuous function to give a distribution.

For $n = 0,1,2,\ldots,N$, put

$$I_o^n(x) = g(x)\ell^{(N-n)}(x)$$

$$I_{r+1}^n(x) = \int_o^x I_r^n(t)dt \quad r = 0,1,2,3,\ldots,$$

so that if $g(x)\ell^{(p)}(x)$ is tempered for $p = 0,1,2,\ldots,N$, then $I_r^n(x)$ is tempered for $n = 0,1,2,\ldots,N$, and $r = 0,1,2,\ldots$. We have that

$$\int g(t)\ell^{(N-n)}(t)\,\phi^{(n)}(t)dt = (-1)^{(N-n)}\int I_{(N-n)}^n(t)\,\phi^{(N)}(t)dt,$$

and it follows that

$$\langle \ell(D^N g), \phi\rangle = (-1)^N \int\{\sum_{n=o}^{N} \binom{N}{n}(-1)^{(N-n)} I_{(N-n)}^n(t)\}\,\phi^{(N)}(t)dt,$$

and so if

$$h(t) = \sum_{n=o}^{N} \binom{N}{n}(-1)^{(N-n)} I_{(N-n)}^n(t),$$

then

$$\ell(D^N g) = D^N h \text{ in } \mathfrak{D}',$$

and if $g(t)\ell^{(p)}(t)$ is tempered for $p = 0,1,2,\ldots,N$, then $h(t)$ is a finite sum of tempered functions, and so is itself tempered.

D.2.5.1. We shall denote by $\mathfrak{Q}\mathfrak{Z}'$ the subclass of $\mathfrak{Z}'$ consisting of those functionals which are the Fourier transforms of functionals of finite order in $\mathfrak{D}'$.†

L.2.5.4. If $F \in \mathfrak{Q}\mathfrak{Z}'$ then there exists an $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ such that

$$i^{-1}(\alpha \ast \phi) = F \ast \phi \quad \forall \phi \in \mathfrak{Z}$$

Proof: Since $F \in \mathfrak{Q}\mathfrak{Z}'$, there exists a continuous function g and a non-negative integer N such that $F = (D^N g)^\wedge$ in $\mathfrak{D}'$. It follows from L.2.5.2. that there exists $k \in \mathfrak{S}$ such that $k(t) > 0 \ \forall t$, and $g(t)k^{(p)}(t)$ is tempered for each non-negative integer p. By L.2.5.3. there exists a tempered function h, such that $D^N h = k(D^N g)$ (in $\mathfrak{D}'$).

Put

$$\ell(t) = \frac{1}{2\pi} \hat{k}(t),$$

$$L = (D^N h)^\wedge \quad (\text{in } \mathfrak{S}'),$$

$$\alpha = (L \ast \ell)//(\ell \ast \ell) \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}).$$

Then if $\phi \in \mathfrak{Z}$, using lemma L.2.2.2., there exists $\psi \in \mathfrak{Z}$ such that $\phi = \ell \ast \psi$, and so, using L.2.4.11., we obtain that

$$i^{-1}(\alpha \ast \phi) = L \ast \psi.$$

Now if $\chi \in \mathfrak{D}$, we have that

†See D.2.4.13.

$$\int (L * \psi)(t)\hat{\chi}(t)dt = \langle \hat{\psi}\hat{L}, \chi \rangle \quad \text{by L.2.4.12.,}$$

$$= \langle D^N h, (\hat{\psi}\chi)^{\hat{}} \rangle$$

$$= \langle D^N g, k(\hat{\psi}\chi)^{\hat{}} \rangle$$

$$= \langle \hat{F}, \hat{\ell}\hat{\psi}\chi \rangle$$

$$= \langle \hat{\phi}\hat{F}, \chi \rangle$$

$$= \int (F * \phi)(t)\hat{\chi}(t)dt \quad \text{by L.2.4.12.,}$$

and so it follows from L.2.4.13. that $L * \psi = F * \phi$. Hence $i^{-1}(\alpha * \phi) = F * \phi \quad \forall \phi \in \mathfrak{Z}$, which is the required result.

<u>L.2.5.5.</u> If $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$ then there exists an $F \in \mathfrak{Q}\mathfrak{Z}'$ such that

$$i^{-1}(\alpha * \phi) = F * \phi \qquad \forall \phi \in \mathfrak{Z} .$$

Proof: If $\alpha = f//g$, then the linear functional H defined on $\mathfrak{S}$ by

$$\langle H, \phi \rangle = \int f(t)\phi(t)dt \quad \forall \phi \in \mathfrak{S} ,$$

is continuous and so $\hat{H} \in \mathfrak{S}'$ is well defined and, by L.2.4.8., there exists a tempered function h, and a non-negative integer N, such that $\hat{H} = D^N h$ in $\mathfrak{S}'$.

By L.2.5.3., since $1/\hat{g}$ has derivatives of all orders everywhere, there exists a continuous function k such that

$D^N k = (1/\hat{g})(D^N h)$ in $\mathfrak{D}'$. It follows that $D^N h = \hat{g}(D^N k)$ in $\mathfrak{D}'$.

If $\ell(t) = \frac{1}{2\pi}(-1)^N k(-t)$, and $F = (D^N \ell)^\wedge$ in $\mathfrak{Z}'$, then $F \in \mathfrak{Q}\mathfrak{Z}'$ and it is easily verified that $\hat{F} = D^N k$. For every $\phi \in \mathfrak{Z}$ and $\chi \in \mathfrak{D}$ we have that

$$\begin{aligned}
\int (H * \phi)(t)\hat{\chi}(t)dt &= \langle \hat{\phi}\hat{H}, \chi \rangle \quad \text{by L.2.4.12.,} \\
&= \langle D^N k, \hat{g}\hat{\phi}\chi \rangle \\
&= \langle \hat{F}, \hat{g}\hat{\phi}\chi \rangle \\
&= \langle \hat{\phi}\hat{F}, \hat{g}\chi \rangle \\
&= \int ((F * \phi) * g)(t)\hat{\chi}(t)dt, \\
&\qquad \text{by L.2.4.12. and L.2.1.6.,}
\end{aligned}$$

and so it follows from L.2.4.13. that $H * \phi = (F * \phi) * g$. Hence

$$\begin{aligned}
\alpha * \phi &= (f * \phi)//g \\
&= (H * \phi)//g \\
&= i(F * \phi),
\end{aligned}$$

and the result follows.

<u>T.2.5.1. The isomorphism theorem.</u>

A) The spaces $\mathfrak{Q}\mathfrak{Z}'$ and $\mathfrak{Q}(\mathfrak{T}, \mathfrak{E} \cap \mathfrak{M})$, with the following system of operations, are universal algebras of the same type. The operations are

(i) the binary operation $(f,g) \to f + g$,

(ii) the unary operations $f \to \lambda f$, where λ is any complex number,

(iii) the unary operations $f \to f^{(p)}$, $p = 0,1,2,\ldots$,

(iv) † the unary operations $f \to f * \ell$, where ℓ is any element of $\mathfrak{G}$,

(v) the nullary operation 0.

B) The spaces $\mathfrak{D}\mathfrak{Z}'$ and $\mathfrak{D}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ are isomorphic.

Proof: We show first that there is a bijection of $\mathfrak{D}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ onto $\mathfrak{D}\mathfrak{Z}'$.

It follows from L.2.5.5. that to each $\alpha \in \mathfrak{D}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ there corresponds at least one $F \in \mathfrak{D}\mathfrak{Z}'$ such that, $\forall \phi \in \mathfrak{Z}$, $i^{-1}(\alpha * \phi) = F * \phi$. If for a given α there were two such F, F_1 and F_2, say, then $F_1 * \phi = F_2 * \phi$ for every $\phi \in \mathfrak{Z}$, and so $F_1 = F_2$. We will denote the unique F, corresponding to α in the above manner, by $\Theta(\alpha)$.

If $\Theta(\alpha_1) = \Theta(\alpha_2)$, then we have that

$$(\alpha_1 - \alpha_2) * \phi = 0 \quad \forall \phi \in \mathfrak{Z},$$

and by use of L.2.4.13. it is clear this implies that $\alpha_1 = \alpha_2$, and so Θ is injective. It follows from L.2.5.4.

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† We map the unary operation $*\ell$ in $\mathfrak{D}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ onto the unary operation $\star\, \ell$ in $\mathfrak{D}\mathfrak{Z}'$.

that Θ is surjective (on $\mathfrak{Q}\mathfrak{Z}'$).

We have already shown that, with the given system of operations, $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ and $\mathfrak{Z}'$ are universal algebras of the same type. We now show Θ to be an homomorphism of $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ into $\mathfrak{Z}'$.

(i) If $\phi \in \mathfrak{Z}$, we have that for $\alpha_1, \alpha_2 \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$,

$$\begin{aligned}
\Theta(a_1+\alpha_2) \ast \phi &= i^{-1}((\alpha_1+\alpha_2) \ast \phi) \\
&= i^{-1}(\alpha_1 \ast \phi) + i^{-1}(\alpha_2 \ast \phi) \\
&= \Theta(\alpha_1) \ast \phi + \Theta(\alpha_2) \ast \phi \\
&= [\Theta(\alpha_1) + \Theta(\alpha_2)] \ast \phi ,
\end{aligned}$$

and so $\Theta(\alpha_1+\alpha_2) = \Theta(\alpha_1) + \Theta(\alpha_2)$.

(ii) If $\phi \in \mathfrak{Z}$, we have that for $\lambda \in \mathbb{C}$ and $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$

$$\begin{aligned}
\Theta(\lambda\alpha) \ast \phi &= i^{-1}((\lambda\alpha) \ast \phi) \\
&= i^{-1}(\lambda(\alpha \ast \phi)) \\
&= \lambda i^{-1}(\alpha \ast \phi) \\
&= \lambda(\Theta(\alpha) \ast \phi) \\
&= (\lambda\Theta(\alpha)) \ast \phi ,
\end{aligned}$$

and so $\Theta(\lambda\alpha) = \lambda\Theta(\alpha)$.

(iii) If $\phi \in \mathfrak{Z}$, we have that for each non-negative integer p and $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$,

$$\begin{aligned} \theta(\alpha^{(p)}) * \phi &= i^{-1}(\alpha^{(p)} * \phi) \\ &= i^{-1}((\alpha * \phi)^{(p)}) \\ &= (i^{-1}(\alpha * \phi))^{(p)} \\ &= (\theta(\alpha) * \phi)^{(p)} \\ &= (\theta(\alpha))^{(p)} * \phi , \end{aligned}$$

and so $\theta(\alpha^{(p)}) = (\theta(\alpha))^{(p)}$.

(iv) If $\phi \in \mathfrak{Z}$, we have that for every $\ell \in \mathfrak{G}$ and $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$,

$$\begin{aligned} \theta(\alpha * \ell) * \phi &= i^{-1}((\alpha * \ell) * \phi) \\ &= (i^{-1}(\alpha * \phi)) * \ell \\ &= (\theta(\alpha) * \phi) * \ell \\ &= (\theta(\alpha) \star \ell) * \phi , \text{ by L.2.4.15.,} \end{aligned}$$

and so $\theta(\alpha * \ell) = \theta(\alpha) \star \ell$.

(v) It follows from (ii) with λ equal to zero that $\theta(0) = 0$.

Finally we observe that the image of a universal algebra $\mathfrak{U}$, under an injective homomorphism θ, in a universal algebra $\mathfrak{G}$ of the same type, is a subalgebra of $\mathfrak{G}$. It follows from this that $\mathfrak{M}\mathfrak{Z}'$ is a universal algebra of the same type as, and isomorphic to, $\mathfrak{M}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$.

2.6. An identification of $\mathfrak{G}'$ in $\mathfrak{M}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$

If $f(z) \in \mathfrak{Z}$ then $f(x) \in \mathfrak{G}$, and so we may write, somewhat loosely, $\mathfrak{Z} \subseteq \mathfrak{G}$. Moreover it is not hard to verify that every neighbourhood of 0 in $\mathfrak{G}$ has a subset which is a neighbourhood of 0 in $\mathfrak{Z}$. Consequently every continuous linear functional on $\mathfrak{G}$ is also a continuous linear functional on $\mathfrak{Z}$, and is in fact, by D.2.4.10. and L.2.4.8., an element of $\mathfrak{M}\mathfrak{Z}'$. Now if we make the observation that $\mathfrak{G}'$ is a subalgebra of $\mathfrak{M}\mathfrak{Z}'$ with the usual system of operations, then it is clear that by the mapping θ^{-1} defined in T.2.5.1., $\mathfrak{G}'$ may be isomorphically embedded in $\mathfrak{M}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$.

We give in this section a precise identification of its image. We show that this image consists of those elements α, of $\mathfrak{M}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ which are such that for every $g \in \mathfrak{G} \cap \mathfrak{W}$ there exists $f[g] \in \mathfrak{X}$ such that $\alpha = f[g]//g$.

<u>D.2.6.1.</u>† If $\alpha \in \mathfrak{M}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$, then we shall put

$$\mathfrak{D}_\alpha = \{g \in \mathfrak{G} \cap \mathfrak{W} \mid \exists f \in \mathfrak{X} \text{ such that } f//g = \alpha\}.$$

†See also D.3.1.5. et seq.

$\mathfrak{B}_\alpha$ will be referred to as the base class of α.

L.2.6.1. If $\phi \in \mathfrak{S}$, then there exists $k \in \mathfrak{S} \cap \mathfrak{W}$ such that $\hat{k}(t) > |\hat{\phi}(t)|$, $\forall t \in \mathbb{R}$.

Proof: Put

$$M_n = \sup_{|x| \geq n-1} \{|\hat{\phi}(x)|\}, \quad n = 1,2,3,\ldots,$$

$$M_0 = M_1,$$

$$M_n = M_{-n}, \quad n = -1,-2,-3,\ldots .$$

Put

$$\tilde{k}(x) = \begin{cases} \exp[1/(x^2-1)] & \text{for } |x| < 1, \\ 0 \quad \text{for } |x| \geq 1, & \end{cases}$$

$$\lambda = \inf_{|x| \leq 1/2} \{\tilde{k}(x)\}, \quad \text{so that } \lambda > 0,$$

$$B_p = \sup_{|x| \leq 1} \{|\tilde{k}^{(p)}(x)|\} \quad \text{for each non-negative integer } p,$$

$$k_n(x) = \tilde{k}(x-n)/\lambda \quad \text{for each integer } n.$$

If m is a non-negative integer, then since $\hat{\phi} \in \mathfrak{S}$, $|x^m \hat{\phi}(x)| \to 0$ as $x \to \infty$, and so $(n-1)^m M_n \to 0$ as $n \to \infty$. Put $A_n = M_n + 1/(1+|n|)^{|n|}$ for each integer n, so that $A_n \geq A_{n+1}$ for $n = 0,1,2,\ldots$, $A_n > M_n$ for every integer n, and for each non-negative integer m, $(n+1)^m A_n \to 0$ as $n \to \infty$.

Put

$$\ell(x) = \sum_{n=-\infty}^{+\infty} A_n k_n(x).$$

The sum converges for each real x, since at most two terms are non-zero. Furthermore $\ell(x) > 0 \ \forall x$, and $\ell(x)$ has derivatives of all orders everywhere. If m and p are non-negative integers, then we have that

$$|x^m \ell^{(p)}(x)| \leq |x|^m \{A_{[x]} |k_{[x]}^{(p)}(x)| + A_{[x]+1} |k_{[x]+1}^{(p)}(x)|\}$$

$$\leq (B_p/\lambda)|x|^m \{A_{[x]} + A_{[x]+1}\}$$

$$\leq (B_p/\lambda)(|[x]|+1)^m \{A_{[x]} + A_{[x]+1}\}$$

$$\rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.$$

It follows that $\ell \in \mathfrak{S}$. Furthermore for each real x, either $k_{[x]}(x) \geq 1$ or $k_{[x]+1}(x) \geq 1$, and so if $x \geq 0$,

$$\ell(x) - |\hat{\Phi}(x)| = A_{[x]} k_{[x]}(x) + A_{[x]+1} k_{[x]+1}(x) - |\hat{\Phi}(x)|$$

$$\geq A_{[x]+1} - M_{[x]+1}$$

$$> 0,$$

while if $x < 0$,

$$\ell(x) - |\hat{\phi}(x)| = A_{[x]}k_{[x]}(x) + A_{[x]+1}k_{[x]+1}(x) - |\hat{\phi}(x)|$$
$$\geq A_{[x]} - M_{[x]}$$
$$> 0.$$

Taking $k \in \mathfrak{S} \cap \mathfrak{M}$ to be such that $\hat{k} = \ell$, the result now follows.

<u>Corollary</u> If $\phi_i \in \mathfrak{S}$ for $i = 1,2,\ldots,n$, then there exists $k \in \mathfrak{S} \cap \mathfrak{M}$ such that $(k-\phi_i) \in \mathfrak{S} \cap \mathfrak{M}$ for $i = 1,2,\ldots,n$.

Proof: We have that for each $i = 1,2,\ldots,n$, there exists $k_i \in \mathfrak{S} \cap \mathfrak{M}$ such that $\hat{k}_i(t) > |\phi_i(t)|$ $\forall t \in \mathbb{R}$.

Put $k(x) = \Sigma_{i=1}^{n} k_i(x)$, so that $k(x) \in \mathfrak{S}$ and $\hat{k}(t) = \Sigma_{i=1}^{n} \hat{k}_i(t) > 0$, i.e. $k \in \mathfrak{S} \cap \mathfrak{M}$. We have that

$$|\hat{k}(t) - \hat{\phi}_i(t)| \geq \hat{k}(t) - |\hat{\phi}_i(t)|$$
$$\geq \hat{k}_i(t) - |\hat{\phi}_i(t)|$$
$$> 0,$$

and so $(k-\phi_i) \in \mathfrak{S} \cap \mathfrak{M}$ for each $i = 1,2,\ldots,n$.

<u>T.2.6.1.</u> Suppose that for each $k \in \mathfrak{S} \cap \mathfrak{M}$, there exists a tempered function of x, $f[k](x)$, with the property that for any two functions $k_1, k_2 \in \mathfrak{S} \cap \mathfrak{M}$,

$$f[k_1] * k_2 = f[k_1 * k_2].$$

Then there exists a functional $F \in \mathfrak{S}'$ such that

$$(F \ast k)(x) = f[k](x) \quad \forall k \in \mathfrak{S} \cap \mathfrak{W} \quad .$$

Proof: If $\phi \in \mathfrak{S}$, then there exists $k \in \mathfrak{S} \cap \mathfrak{W}$ such that $(k - \phi) \in \mathfrak{S} \cap \mathfrak{W}$. Put $\bar{f}[\phi] = f[k] - f[k - \phi]$, then $\bar{f}[\phi]$ is tempered. If $k_1, k_2 \in \mathfrak{S} \cap \mathfrak{W}$ are such that $(k_1 - \phi), (k_2 - \phi) \in \mathfrak{S} \cap \mathfrak{W}$ and if $\ell \in \mathfrak{S} \cap \mathfrak{W}$ we have that

$$\begin{aligned} \{f[k_1] - f[k_1 - \phi]\} \ast \ell &= f[\ell] \ast k_1 - f[\ell] \ast (k_1 - \phi) \\ &= f[\ell] \ast \phi \\ &= \{f[k_2] - f[k_2 - \phi]\} \ast \ell, \end{aligned}$$

by symmetry,

and so

$$f[k_1] - f[k_1 - \phi] = f[k_2] - f[k_2 - \phi].^{\dagger}$$

It follows that $\bar{f}[\phi]$ is a well defined tempered function for each $\phi \in \mathfrak{S}$.

If $k_1 \in \mathfrak{S} \cap \mathfrak{W}$, then there exists $k_2 \in \mathfrak{S} \cap \mathfrak{W}$ such that $(k_2 - k_1) \in \mathfrak{S} \cap \mathfrak{W}$, and if $\ell \in \mathfrak{S} \cap \mathfrak{W}$, we have that

$$\begin{aligned} \bar{f}[k_1] \ast \ell &= f[k_2] \ast \ell - f[k_2 - k_1] \ast \ell \\ &= f[k_1] \ast \ell, \end{aligned}$$

so that $\bar{f}[k_1] = f[k_1]$ for each $k_1 \in \mathfrak{S} \cap \mathfrak{W}$.

† This depends on a slight extension of Th. 2.2.1.

If $\phi_1, \phi_2 \in \mathfrak{S}$, then there exists $k \in \mathfrak{S} \cap \mathfrak{M}$ such that $(k-\phi_1), (k-(\phi_1 * \phi_2)) \in \mathfrak{S} \cap \mathfrak{M}$; and if $\ell \in \mathfrak{S} \cap \mathfrak{M}$ then we have that

$$\begin{aligned}(\bar{f}[\phi_1] * \phi_2) * \ell &= \{f[k] - f[k-\phi_1]\} * \phi_2 * \ell \\ &= f[\ell] * \phi_1 * \phi_2 \\ &= \{f[k] - f[k-(\phi_1 * \phi_2]\} * \ell \\ &= \bar{f}[\phi_1 * \phi_2] * \ell,\end{aligned}$$

and so $\bar{f}[\phi_1] * \phi_2 = \bar{f}[\phi_1 * \phi_2] \quad \forall \phi_1, \phi_2 \in \mathfrak{S}$.

Take $\delta(x) \in \mathfrak{D}$ such that $\int \delta(x)dx = 1$, and put $\delta_n(x) = n\delta(nx)$ for each positive integer n. Then if $\phi \in \mathfrak{S}$, $\psi(x) = \phi(-x)$ and $y \in \mathbb{R}$,

$$\begin{aligned}\int \bar{f}[\psi](y-x)\delta_n(x)dx &= \int \bar{f}[\psi](y-\frac{x}{n})\delta(x)dx \\ &\to \bar{f}[\psi](y) \quad \text{as} \quad n \to \infty,\end{aligned}$$

by Lebesgues convergence theorem. It follows that

$$(\bar{f}[\delta_n] * \psi)(y) \to \bar{f}[\psi](y) \quad \text{as} \quad n \to \infty,$$

and in particular that

$$\int \bar{f}[\delta_n](t)\phi(t)dt \to \bar{f}[\psi](0) \quad \text{as} \quad n \to \infty.$$

If $F_n \in \mathfrak{S}'$ is the functional given by

$$\langle F_n, \phi \rangle = \int \bar{f}[\delta_n](t)\phi(t)dt, \qquad \phi \in \mathfrak{S},$$

then from the above, $\lim_{n\to\infty} \langle F_n, \phi \rangle$ exists for each $\phi \in \mathfrak{G}$, and so, from L.2.4.16., it follows that there exists an $F \in \mathfrak{G}'$ such that

$$(F * \psi)(y) = \bar{f}[\psi](y),$$

for each $\psi \in \mathfrak{G}$ and every $y \in \mathbb{R}$.

Hence if $k \in \mathfrak{G} \cap \mathfrak{W}$, we have that

$$(F * k)(x) = \bar{f}[k](x)$$

$$= f[k](x),$$

and the theorem is proven.

<u>T.2.6.2.</u> If $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ and $\mathfrak{B}_\alpha = \mathfrak{G} \cap \mathfrak{W}$, then $\Theta(\alpha) \in \mathfrak{G}'$ †, and conversely if $F \in \mathfrak{G}'$ and $\alpha = \Theta^{-1}(F)$, then $\mathfrak{B}_\alpha = \mathfrak{G} \cap \mathfrak{W}$.

[Reminder: Θ is the isomorphism of $\mathfrak{Q}\mathfrak{Z}'$ and $\mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ given in T.2.5.1.]

Proof: If $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ is such that $\mathfrak{B}_\alpha = \mathfrak{G} \cap \mathfrak{W}$, then for each $k \in \mathfrak{G} \cap \mathfrak{W}$, there exists a function $f[k](x) \in \mathfrak{T}$, such that $f[k]//k = \alpha$. If $k_1, k_2 \in \mathfrak{G} \cap \mathfrak{W}$, then $f[k_1]//k_1 = f[k_1 * k_2]//(k_1 * k_2)$, and it follows that $f[k_1] * k_2 = f[k_1 * k_2]$. It follows from T.2.6.1. that there exists an $F \in \mathfrak{G}'$ such that $F * k = f[k]$ for each $k \in \mathfrak{G} \cap \mathfrak{W}$. Hence $\alpha = (F * k)//k$, and we have that for

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† In the sense that $\Theta(\alpha)$ may be extended to a continuous linear functional on $\mathfrak{G}$.

every $\phi \in \mathfrak{Z}$,

$$i^{-1}(\alpha * \phi) = F * \phi .$$

Hence the restriction of F to $\mathfrak{Z}'$ is equal to $\Theta(\alpha)$, and it follows that $\Theta(\alpha) \in \mathfrak{S}'$, in the sense previously explained.

Conversely if $F \in \mathfrak{S}'$ and $k \in \mathfrak{S} \cap \mathfrak{M}$, then

$$((F * k)//k) * \phi = i(F * \phi) \quad \forall \phi \in \mathfrak{Z} ,$$

and so if we put $\alpha = \Theta^{-1}(F)$, we have that

$$\alpha = (F * k)//k,$$

for any $k \in \mathfrak{S} \cap \mathfrak{M}$. It follows that $\mathfrak{B}_\alpha = \mathfrak{S} \cap \mathfrak{M}$.

INTRODUCTION TO CHAPTER 3

The first four sections of this chapter are largely concerned with topological aspects of an arbitrary quotient pair $\mathfrak{A}$, $\mathfrak{B}$. In section 3.1., some new concepts are introduced. These allow us in section 3.2. (T.3.2.1.) to use any topology given on $\mathfrak{A}$ to derive topologies on certain subalgebras of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$. Section 3.3. contains a number of theorems which relate properties of $\mathfrak{A}$ and $\mathfrak{B}$ to topological properties of certain subalgebras of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$; theorem T.3.3.5. being of particular importance for establishing, in sections 3.5. and 3.6., the continuity of certain operations. In section 3.4. we consider an interesting sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$; the space $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$. (The space $\mathfrak{Q}_o(\mathfrak{T}, \mathfrak{S}_\cap \mathfrak{M})$ was that sub-algebra of $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S}_\cap \mathfrak{M})$ which was shown in chapter two to be isomorphic with the class of tempered distributions.) We show, firstly, that if $\mathfrak{A}$ is a complete Hausdorff linear topological space, and if each $\phi \in \mathfrak{B}$ is a continuous endomorphism of $\mathfrak{A}$, then $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$ is also a complete Hausdorff linear topological space. After this, other aspects of $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$ are considered, in particular a family of sub-algebras of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ is constructed: the spaces $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$, n = 1,2,3,... . Various aspects are looked at, including the question of distinctness of the $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$. The section concludes with some examples to illustrate different situations which may arise.

The final two sections of this chapter are concerned with topological properties of $\mathfrak{Q}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ and $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$. The principal result of section 3.5 is T.3.5.3., where it is shown that, equipped with a suitable topology σ_0, $\mathfrak{Q}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is homeomorphic to $\mathfrak{S}'$, the class of tempered distributions, $\mathfrak{S}'$ carrying the weak dual topology. We also show in T.3.5.4. that more than one topology on $\mathfrak{T}$ can lead to the σ_0 topology on $\mathfrak{Q}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$. In section 3.6., the main results are T.3.6.2. and T.3.6.3. The former shows that there are several topologies on $\mathfrak{T}$ leading to the same topology, $(\sigma, \mathfrak{Z})$, on $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$, while the latter shows that with the $(\sigma, \mathfrak{Z})$-topology $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is homeomorphic with the class of Fourier transforms of distributions of finite order (when this class has the weak dual topology). All the results of this chapter are believed to be new.

CHAPTER 3

Throughout this chapter, $\mathfrak{A}$ will denote a Universal Algebra $(M, \mathfrak{D})$, and $\mathfrak{B}$ will be an Abelian semigroup of injective endomorphisms of $\mathfrak{A}$, i.e. $\mathfrak{A}, \mathfrak{B}$ will form a quotient pair.

3.1. Preliminaries

D.3.1.1. We will denote by $\mathfrak{U}(\mathfrak{A}, \mathfrak{B})$ the class of unary operations ξ, on $\mathfrak{A}$ such that for every $\phi \in \mathfrak{B}$ and for every $f \in \mathfrak{A}$,

$$\xi(f) \divideontimes \phi = \xi(f \divideontimes \phi).$$

We have in all cases that $\mathfrak{B} \subseteq \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$.

D.3.1.2.[†] If $\chi \in \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$, then for each $\alpha \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, put

$$\chi(\alpha) = \chi(f)//\phi,$$

where f, ϕ are such that $f//\phi = \alpha$. Note that if $f_1//\phi_1 = f_2//\phi_2$, then $\chi(f_1)//\phi_1 = \chi(f_2)//\phi_2$, so that χ is well defined on $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$.

[†] This definition is consistent with that given in T.1.3.1. for the case $\chi \in \mathfrak{D}$.

D.3.1.3. Suppose that $\mathfrak{G}$ is a sub-algebra† of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$. Suppose also that there exists a subset $\mathfrak{C}$ of $\mathfrak{U}(\mathfrak{A}, \mathfrak{B})$, with the property that for every $\alpha \in \mathfrak{G}$, and for every $\xi \in \mathfrak{C}$, we have $\xi(\alpha) \in i(\mathfrak{A})$. Then we shall say that $\mathfrak{G}$ and $\mathfrak{C}$ are Wiener-like.

N.3.1.1. It is reasonable to demand some explanation of the term 'Wiener-like'. Suppose that the situation referred to in D.3.1.3. arises, and that $\mathfrak{G} = \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$‡. Then if $\phi \in \mathfrak{B}$ and $\xi \in \mathfrak{C}$, we have that for every $f \in \mathfrak{A}$,

$$\xi(f//\phi) \in i(\mathfrak{A}).$$

For each $f \in \mathfrak{A}$ define

$$\chi(f) = i^{-1}\xi(f//\phi).$$

It is easily verified that the map χ, is an element of $\mathfrak{U}(\mathfrak{A}, \mathfrak{B})$, and that for every $f \in \mathfrak{A}$,

$$\xi(f) = \chi(f) * \phi .$$

We may write this relationship as

$$\xi = \phi\chi .$$

We have therefore shown that under the given conditions, if $\phi \in \mathfrak{B}$ and if $\xi \in \mathfrak{C}$, then there exists

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† For the purposes of this chapter, we will regard $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ as a sub-algebra of itself, but we will assume that if $\mathfrak{G} = (N, \mathfrak{D})$, then $N \neq \emptyset$. See section 1.2.

‡ That this can happen, see L.3.6.1.

$\chi \in \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$ such that $\xi = \phi\chi$. This resembles the following theorem of Wiener.

[†] Wiener's Theorem Denote by B the class of functions integrable over the real line whose Fourier Transforms vanish nowhere. Denote by C the class of functions integrable over the real line whose Fourier Transforms have compact support. Then if $\phi \in B$ and if $\xi \in C$, there exists $\chi \in C$ such that $\xi = \phi * \chi$.

D.3.1.4. Suppose that $\mathfrak{F}$ is a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ and that $\mathfrak{C} \subseteq \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$ and is such that $\mathfrak{F}$ and $\mathfrak{C}$ are Wiener-like.

1) If, give any two distinct points $\alpha_1, \alpha_2 \in \mathfrak{F}$, there exists $\xi \in \mathfrak{C}$ such that $\xi(\alpha_1) \neq \xi(\alpha_2)$, then we shall say that $\mathfrak{C}$ separates $\mathfrak{F}$.

2) If every $\xi \in \mathfrak{C}$ is an endomorphism of $\mathfrak{F}$, then we shall say that $\mathfrak{C}$ is endomorphic on $\mathfrak{F}$.

D.3.1.5.[‡] If $\alpha \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, then we shall put

$$\mathfrak{B}_\alpha = \{ \phi \in \mathfrak{B} \mid \exists f \in \mathfrak{A} \text{ such that } f//\phi = \alpha \}.$$

$\mathfrak{B}_\alpha$ will be referred to as the base class of α .

D.3.1.6. We define $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$ to be

$$\{\alpha \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B}_\alpha = \mathfrak{B} \}.$$

[†] See (11) section 9J, from which this result is easily deducible.

[‡] D.2.6.1. may now be seen to be a special case of this definition

<u>L.3.1.1.</u> $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$ is a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$.

Proof: Suppose that Ξ is an n-ary operation of the Universal Algebra $\mathfrak{A}$, and that $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$. Choose $\phi \in \mathfrak{B}$ and take $f_i \in \mathfrak{A}$ such that

$$\alpha_i = f_i // \phi, \quad i = 1, 2, \ldots, n.$$

Then if $\beta = \Xi(\alpha_1, \alpha_2, \ldots, \alpha_n)$, we have, denoteing $\phi \circ \phi \circ \cdots \circ \phi$ (r terms) by $\phi^{[r]}$, that

$$\beta = \Xi(f_1 * \phi^{[n-1]}, f_2 * \phi^{[n-1]}, \ldots, f_n * \phi^{[n-1]}) // \phi^{[n]}.$$

And so,

$$\beta = \Xi(f_1, f_2, \ldots, f_n) * \phi^{[n-1]} // \phi^{[n]},$$

$$= \Xi(f_1, f_2, \ldots, f_n) // \phi .$$

Hence $\phi \in \mathfrak{B}_\beta$. Consequently $\mathfrak{B}_\beta = \mathfrak{B}$: the result follows.

<u>L.3.1.2.</u> $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{B}$ are Wiener-like, $\mathfrak{B}$ separates $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{B}$ is endomorphic on $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$.

Proof: We have that $\mathfrak{B} \subseteq \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$. Furthermore if $\xi \in \mathfrak{B}$ and $\alpha \in \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$, then there exists $f \in \mathfrak{A}$ such that $\alpha = f//\xi$, i.e. $\xi(\alpha) = i(f)$. It follows that $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{B}$ are Wiener-like. If $\alpha_1, \alpha_2, \in \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$ and $\alpha_1 \neq \alpha_2$, then for any $\xi \in \mathfrak{B}$, we have that $\xi(\alpha_1) \neq \xi(\alpha_2)$. Hence $\mathfrak{B}$ separates $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$. Finally, since $\mathfrak{A}, \mathfrak{B}$ form a quotient pair, $\mathfrak{B}$ is endomorphic on $\mathfrak{A}$, and so, on $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ and $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$.

3.2. The topology in the general case[†]

T.3.2.1. Suppose that $\mathfrak{A}, \mathfrak{B}$ form a quotient pair and that $\mathfrak{A}$ is a topological space[‡] with topology τ. Suppose also that $\mathfrak{F}$ is a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, and that $\mathfrak{C} \subseteq \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$ is such that $\mathfrak{F}$ and $\mathfrak{C}$ are Weiner-like.

For each $\alpha \in \mathfrak{F}$ define the family $\mathfrak{G}_\alpha$ of subsets of $\mathfrak{F}$ to consist of all sets of the form

$$\{\beta | \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in N_j^\alpha,\ j = 1,2,\ldots,n.\},$$

where n is a positive integer; ϕ_j, $j = 1,2,\ldots,n$ is an element of $\mathfrak{C}$ and N_j^α is a τ-neighbourhood of $i^{-1}\phi_j(\alpha)$.

Then for each $\alpha \in \mathfrak{F}$, $\mathfrak{G}_\alpha$ is a basis of a filter $\mathfrak{F}_\alpha$, on $\mathfrak{F}$. The family of filters, $\{\mathfrak{F}_\alpha | \alpha \in \mathfrak{F}\}$, provides a topology on $\mathfrak{F}$. If every element of $\mathfrak{C}$ is injective on $\mathfrak{F}$, then $\mathfrak{G}_\alpha = \mathfrak{F}_\alpha$, $\forall \alpha \in \mathfrak{F}$.

Proof: 1) Proof that if $G \in \mathfrak{G}_\alpha$, then $\alpha \in G$.

Suppose that $G \in \mathfrak{G}_\alpha$, then G is of the form

$$G = \{\beta | \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in N_j^\alpha,\ j = 1,2,\ldots,n.\},$$

and so $\alpha \in G$.

[†] For a treatment of general topology, see (2).

[‡] See D.2.4.3.

2) Proof that if $G_1, G_2 \in \mathfrak{G}_\alpha$, then $G_1 \cap G_2 \in \mathfrak{G}_\alpha$. Choose $G_1, G_2 \in \mathfrak{G}_\alpha$, then

$$G_1 = \{\beta \mid \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in N_j^\alpha,\ j = 1,2,\ldots,n.\}$$

$$G_2 = \{\beta \mid \beta \in \mathfrak{F} \text{ and } i^{-1}\psi_j(\beta) \in M_j^\alpha,\ j = 1,2,\ldots,m.\}, \text{ say.}$$

Put

$$\chi_j = \begin{cases} \phi_j, & j = 1,2,\ldots,n \\ \psi_{j-n}, & j = n+1, n+2, \ldots, n+m, \end{cases}$$

$$L_j^\alpha = \begin{cases} N_j^\alpha, & j = 1,2,\ldots,n \\ M_{j-n}^\alpha, & j = n+1, n+2, \ldots, n+m, \end{cases}$$

and $\quad \ell = n + m.$

Then we have that

$$G_1 \cap G_2 = \{\beta \mid \beta \in \mathfrak{F} \text{ and } i^{-1}\chi_j(\beta) \in L_j^\alpha,$$
$$j = 1,2,\ldots,\ell.\} \in \mathfrak{G}_\alpha.$$

We have shown by 1) and 2) that $\mathfrak{G}_\alpha$ is a basis of a filter $\mathfrak{F}_\alpha$ on $\mathfrak{F}$. We next show that the family of filters, $\{\mathfrak{F}_\alpha \mid \alpha \in \mathfrak{F}\}$, provides a topology, which we shall call the $(\tau, \mathfrak{J})$ topology, on $\mathfrak{F}$.

3) If $F \in \mathfrak{F}_\alpha$, then $\alpha \in F$; this follows directly from 1).

4) Proof that if $F \in \mathfrak{F}_\alpha$ then there exists $F' \in \mathfrak{F}_\alpha$ such that if $\gamma \in F'$ then $F \in \mathfrak{F}_\gamma$.

Suppose that $F \in \mathfrak{F}_\alpha$, then F has a subset G, of the form

$$G = \{\beta | \beta \in \mathfrak{E} \text{ and } i^{-1}\phi_j(\beta) \in N_j^\alpha,\ j = 1,2,\ldots,n.\}.$$

Since $\mathfrak{U}$ is a topological space, for each $j = 1,2,\ldots,n$, there exists M_j^α, a τ-neighbourhood of $i^{-1}\phi_j(\alpha)$, such that if $f_j \in M_j^\alpha$ then N_j^α is a τ-neighbourhood of f_j.

Put

$$F' = \{\beta | \beta \in \mathfrak{E} \text{ and } i^{-1}\phi_j(\beta) \in M_j^\alpha,\ j = 1,2,\ldots,n.\},$$

then $F' \in \mathfrak{F}_\alpha$. Suppose that $\gamma \in F'$, then $i^{-1}\phi_j(\gamma) \in M_j^\alpha$, $j = 1,2,\ldots,n$. Hence N_j^α is a τ-neighbourhood of $i^{-1}\phi_j(\gamma)$, $j = 1,2,\ldots,n$. Consequently $G \in \mathfrak{G}_\gamma$, and so $F \in \mathfrak{F}_\gamma$. The result follows.

We have now shown that the family of filters $\{\mathfrak{F}_\alpha | \alpha \in \mathfrak{E}\}$ provides a topology on $\mathfrak{E}$. To complete the theorem we prove:

5) If every element of $\mathfrak{C}$ is injective on $\mathfrak{E}$, then $\mathfrak{G}_\alpha = \mathfrak{F}_\alpha$. To prove this, we have only to show that any superset of a set which is an element of $\mathfrak{G}_\alpha$, is also an element of $\mathfrak{G}_\alpha$. Suppose that $G \subseteq F \subseteq \mathfrak{E}$ and $G \in \mathfrak{G}_\alpha$. Then G is of the form

$$G = \{\beta | \beta \in \mathfrak{E} \text{ and } i^{-1}\phi_j(\beta) \in N_j^\alpha,\ j = 1,2,\ldots,n.\}.$$

Put

$$M_j^\alpha = N_j^\alpha \cup i^{-1}\phi_j(F), \quad j = 1,2,\ldots,n,$$

then M_j^{α} is a τ-neighbourhood of $i^{-1}\phi_j(\alpha)$. Hence, if

$$N = \{\beta \mid \beta \in \mathcal{G} \text{ and } i^{-1}\phi_j(\beta) \in M_j^{\alpha},\ j = 1,2,\ldots,n.\},$$

then $N \in \mathcal{G}_{\alpha}$.

If $\beta \in F$, then $i^{-1}\phi_j(\beta) \in i^{-1}\phi_j(F) \subseteq M_j^{\alpha}$, $j = 1,2,\ldots,n$, and so $\beta \in N$. Hence $F \subseteq N$.

Suppose that $\beta \in N$ but $\beta \notin F$. Then we have that $\beta \notin G$. Therefore there exists p such that

$$i^{-1}\phi_p(\beta) \notin N_p^{\alpha}.$$

Also since $\beta \notin F$ and all elements of $\mathcal{C}$ are injective on $\mathcal{G}$, we have that

$$i^{-1}\phi_p(\beta) \notin i^{-1}\phi_p(F).$$

Hence

$$i^{-1}\phi_p(\beta) \notin N_p^{\alpha} \cup i^{-1}\phi_p(F),$$

and so $\beta \notin N$. But this is a contradiction. Consequently if $\beta \in N$, then we must have that $\beta \in F$, that is we must have that $N \subseteq F$.

It follows that $N = F$, and so $F \in \mathcal{G}_{\alpha}$.

This completes the proof of the theorem.

3.3. Some general results

Throughout this section (§ 3.3) we shall assume that $\mathfrak{A}$ is a topological space, with topology τ, and that $\mathfrak{F}$ is a sub-algebra of $\mathfrak{D}(\mathfrak{A}, \mathfrak{B})$. We shall further assume that $\mathfrak{C} \subseteq \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$ is such that $\mathfrak{F}$ and $\mathfrak{C}$ are Weiner-like, and that $\mathfrak{F}$ has the $(\tau, \mathfrak{C})$ topology. Additional assumptions will be stated explicitly whenever they are made.

T.3.3.1. Suppose that $i(\mathfrak{A}) \subseteq \mathfrak{F}$. Then the map

$$i: \mathfrak{A} \rightarrow \mathfrak{F}$$

is continuous at $f \in \mathfrak{A}$, if and only if for every $\phi \in \mathfrak{C}$, the map

$$\phi : \mathfrak{A} \rightarrow \mathfrak{A}$$

is continuous at $f \in \mathfrak{A}$.

Proof: Suppose that i is continuous at $f \in \mathfrak{A}$. Choose $\phi \in \mathfrak{C}$. Take any τ-neighbourhood N, of $\phi(f)$, and put

$$G = \{\beta | \beta \in \mathfrak{F} \text{ and } i^{-1}\phi(\beta) \in N\}.$$

Then G is a $(\tau, \mathfrak{C})$-neighbourhood of $i(f)$, and so there exists a τ-neighbourhood F, of f, such that $i(F) \subseteq G$. Hence if $g \in F$ then $i(g) \in G$, and so $\phi(g) \in N$. Thus $\phi(F) \subseteq N$, and it follows that ϕ is continuous at f.

Suppose conversely, that every $\phi \in \mathfrak{C}$ is continuous at $f \in \mathfrak{U}$. Take any $(\tau, \mathfrak{C})$-neighbourhood of $i(f)$; this will have a subset F of the form

$$F = \{\beta \mid \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in N_j,\ j = 1,2,\ldots,n.\},$$

where, for each j, N_j is a τ-neighbourhood of $\phi_j(f)$. Since ϕ_j is continuous at f, there exists, for each j, a τ-neighbourhood M_j, of f, such that $\phi_j(M_j) \subseteq N_j$. Put

$$M = \bigcap_{j=1}^{n} M_j,$$

then M is a τ-neighbourhood of f. If $g \in M$, then $\phi_j(g) \in N_j$, $j = 1,2,\ldots,n$, and so $i(g) \in F$. It follows that $i(M) \subseteq F$.

This completes the proof of the theorem.

<u>T.3.3.2.</u> Suppose that $\mathfrak{C}$ is endomorphic on $\mathfrak{F}$. Suppose also that $\Xi \in \mathfrak{D}$ and

$$\Xi : \mathfrak{U} \times \mathfrak{U} \times \cdots \times \mathfrak{U} \to \mathfrak{U}$$

is a continuous m-ary operation on $\mathfrak{U}$. Then

$$\Xi : \mathfrak{F} \times \mathfrak{F} \times \cdots \times \mathfrak{F} \to \mathfrak{F}$$

is a continuous m-ary operation on $\mathfrak{F}$.

Proof: Choose $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{F}$, and put $\alpha = \Xi(\alpha_1, \alpha_2, \ldots, \alpha_m)$. If F is any $(\tau, \mathfrak{C})$-neighbourhood of α, then F will have a subset G of the form

$$G = \{\beta \mid \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in N_j^{\alpha},\ j = 1,2,\ldots,n.\}.$$

For each $j = 1,2,\ldots,n$, there exists a set of τ-neighbourhoods, $\{M_{j,1},M_{j,2},\ldots,M_{j,m}\}$ such that each $M_{j,k}$ is a τ-neighbourhood of $i^{-1}\phi_j(\alpha_k)$, $k = 1,2,\ldots,m$, and

$$\Xi(M_{j,1},M_{j,2},\ldots,M_{j,m}) \subseteq N_j^{\alpha}.$$

Put

$$G_k = \{\beta \mid \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in M_{j,k},\ j = 1,2,\ldots,n.\},$$

then G_k is a $(\tau,\mathfrak{C})$-neighbourhood of α_k. Moreover if $\beta_k \in G_k$, $k = 1,2,\ldots,m$, then $i^{-1}\phi_j(\beta_k) \in M_{j,k}$, $j = 1,2,\ldots,n$. Hence

$$\Xi(i^{-1}\phi_j(\beta_1),i^{-1}\phi_j(\beta_2),\ldots,i^{-1}\phi_j(\beta_m)) \in N_j^{\alpha},$$
$$j = 1,2,\ldots,n.$$

That is

$$i^{-1}\phi_j(\Xi(\beta_1,\beta_2,\ldots,\beta_m)) \in N_j^{\alpha},\ j = 1,2,\ldots,n;$$

and so $\Xi(\beta_1,\beta_2,\ldots,\beta_m) \in G \subseteq F$. Therefore $\Xi(G_1,G_2,\ldots,G_m) \subseteq F$, and the result follows.

<u>T.3.3.3.</u> Suppose that $\mathfrak{C}$ is endomorphic on $\mathfrak{F}$. Then if $\mathfrak{U}$, with topology τ, is a linear topological space†; $\mathfrak{F}$, with topology $(\tau,\mathfrak{C})$, is also a linear topological space.

Proof: It follows from the discussions in chapter 1 that $\mathfrak{F}$ is a linear space, and from the previous theorem that the map

$$\mathfrak{F} \times \mathfrak{F} \ni (\alpha,\beta) \to \alpha + \beta \in \mathfrak{F}$$

is continuous. It remains to prove that the map

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†See D.2.4.4.

$$\mathfrak{C} \times \mathfrak{F} \ni (\lambda,\alpha) \to \lambda\alpha \in \mathfrak{F}$$

is also continuous. Choose $\lambda \in \mathfrak{C}$ and $\alpha \in \mathfrak{F}$. Suppose that F is any $(\tau, \mathfrak{C})$-neighbourhood of $\lambda\alpha$. Then F has a subset G of the form

$$G = \{\beta | \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in N_j,\ j = 1,2,\ldots,n.\},$$

where N_j is a τ-neighbourhood of $i^{-1}\phi_j(\lambda\alpha) = \lambda i^{-1}\phi_j(\alpha)$. For each $j = 1,2,\ldots,n$, there exists $\epsilon_j > 0$ and M_j, a τ-neighbourhood of $i^{-1}\phi_j(\alpha)$, such that if $|\mu-\lambda| < \epsilon_j$ and if $f \in M_j$, then $\mu f \in N_j$. Put

$$\epsilon = \min\{\epsilon_1,\epsilon_2,\ldots,\epsilon_n\},$$

and

$$U = \{\beta | \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in M_j,\ j = 1,2,\ldots,n.\}.$$

Then $\epsilon > 0$ and U is a $(\tau, \mathfrak{C})$-neighbourhood of α. Moreover if $|\mu-\lambda| < \epsilon$ and if $\beta \in U$, then for each j,

$$\mu i^{-1}\phi_j(\beta) \in N_j.$$

Hence $\mu\beta \in F$.

The result follows.

<u>T.3.3.4.</u>† Suppose that $\mathfrak{C}$ separates $\mathfrak{F}$. Then:

1) if $\mathfrak{M}$ is a T_0 space, so is $\mathfrak{F}$,

2) if $\mathfrak{M}$ is a T_1 space, so is $\mathfrak{F}$,

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† For definitions of T_0, T_1 and T_2 spaces see (18) Pgs.52 and 53.

3) if $\mathfrak{U}$ is a T_2 space (i.e. a Hausdorff space), so is $\mathfrak{F}$.

Proof: † 1) Choose $\alpha, \beta \in \mathfrak{F}$ such that $\alpha \neq \beta$. We must show that at least one of $\alpha.\beta$ has a $(\tau, \mathfrak{F})$-neighbourhood, not containing the other point. Choose $\phi \in \mathfrak{F}$ such that $\phi(\alpha) \neq \phi(\beta)$. Put $f = i^{-1}\phi(\alpha)$, $g = i^{-1}\phi(\beta)$. Then $f \neq g$, and so there exists a τ-neighbourhood N, of g, say, such that $f \notin N$. Put

$$V = \{\gamma | \gamma \in \mathfrak{F} \text{ and } i^{-1}\phi(\gamma) \in N\}.$$

Then V is a $(\tau, \mathfrak{F})$-neighbourhood of β, and $\alpha \notin V$.

2) Choose $\alpha, \beta \in \mathfrak{F}$ such that $\alpha \neq \beta$. We must show that there exists a $(\tau, \mathfrak{F})$-neighbourhood of α, not containing β, and a $(\tau, \mathfrak{F})$-neighbourhood of β, not containing α. Take ϕ, f, g as before and choose τ-neighbourhoods M,N of f,g respectively, such that $f \notin N$ and $g \notin M$. Put

$$U = \{\gamma | \gamma \in \mathfrak{F} \text{ and } i^{-1}\phi(\gamma) \in M\}$$

$$V = \{\gamma | \gamma \in \mathfrak{F} \text{ and } i^{-1}\phi(\gamma) \in N\}.$$

Then U is a $(\tau, \mathfrak{F})$-neighbourhood of α, and V is a $(\tau, \mathfrak{F})$-neighbourhood of β. Moreover $\alpha \notin V$ and $\beta \notin U$.

3) Choose $\alpha, \beta \in \mathfrak{F}$ such that $\alpha \neq \beta$. We must show that α and β possess disjoint $(\tau, \mathfrak{F})$-

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† If $\mathfrak{F}$ has only one element, then there is nothing to prove: we therefore assume that $\mathfrak{F}$ has at least two distinct elements.

neighbourhoods. Take ϕ, f, g as before and choose τ-neighbourhoods M, N of f, g respectively, such that $M \cap N = \emptyset$

Put

$$U = \{\gamma | \gamma \in \mathfrak{F} \text{ and } i^{-1}\phi(\gamma) \in M\}$$

$$V = \{\gamma | \gamma \in \mathfrak{F} \text{ and } i^{-1}\phi(\gamma) \in N\}.$$

Then U is a $(\tau, \mathfrak{C})$-neighbourhood of α, and V is a $(\tau, \mathfrak{C})$-neighbourhood of β. Moreover if $\delta \in U$, then $i^{-1}\phi(\delta) \in M$, and so $i^{-1}\phi(\delta) \notin N$. Hence $\delta \notin V$. Similarly if $\delta \in V$, then $\delta \notin U$. It follows that $U \cap V = \emptyset$.

<u>T.3.3.5.</u> Suppose that $\mathcal{X} \in \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$ and that $\mathcal{X}$ is such that for every $\phi \in \mathfrak{C}$, $\phi\mathcal{X} \in \mathfrak{C}$. Then if $\mathcal{X}$ maps $\mathfrak{F}$ into $\mathfrak{F}$, $\mathcal{X}$ is continuous on $\mathfrak{F}$.

Proof: Choose $\alpha \in \mathfrak{F}$, and suppose that F is a $(\tau, \mathfrak{C})$-neighbourhood of $\mathcal{X}(\alpha)$. Then F has a subset G, of the form

$$G = \{\beta | \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j(\beta) \in N_j,\ j = 1,2,\ldots,n.\},$$

where N_j is a τ-neighbourhood of $i^{-1}\phi_j(\mathcal{X}(\alpha))$. Put

$$U = \{\beta | \beta \in \mathfrak{F} \text{ and } i^{-1}\phi_j\mathcal{X}(\beta) \in N_j,\ j = 1,2,\ldots,n.\}.$$

Then U is a $(\tau, \mathfrak{C})$-neighbourhood of α, and if $\beta \in U$, then $\mathcal{X}(\beta) \in G \subseteq F$. Hence $\mathcal{X}(U) \subseteq F$, and the result follows.

3.4. Consideration of $\mathfrak{Q}_0(\mathfrak{A}, \mathfrak{B})$.

T.3.4.1. The Completeness theorem

Suppose that $\mathfrak{A}$ is a complete† Hausdorff linear topological space, and that every $\phi \in \mathfrak{B}$ is a continuous endomorphism of $\mathfrak{A}$. Then with the $(\tau, \mathfrak{B})$-topology, $\mathfrak{Q}_0(\mathfrak{A}, \mathfrak{B})$ is also a complete Hausdorff linear topological space. ‡

Proof: It follows directly from T.3.3.3. and T.3.3.4. that $\mathfrak{Q}_0(\mathfrak{A}, \mathfrak{B})$ is a Hausdorff linear topological space.

Suppose that $\mathfrak{K}$ is a Cauchy filter † on $\mathfrak{Q}_0(\mathfrak{A}, \mathfrak{B})$; i.e. given any $(\tau, \mathfrak{B})$-neighbourhood, V, of $0 \in \mathfrak{Q}_0(\mathfrak{A}, \mathfrak{B})$, there exists $U \in \mathfrak{K}$ such that

$$U - U \subseteq V.$$

Choose $\phi \in \mathfrak{B}$. Consider $i^{-1}\phi(\mathfrak{K})$; we claim that this is a filter base on $\mathfrak{A}$.

1) Proof that $\emptyset \notin i^{-1}\phi(\mathfrak{K})$.

If $F \in i^{-1}\phi(\mathfrak{K})$, then there exists $C \in \mathfrak{K}$ such that $F = i^{-1}\phi(C)$. Since C must be non-empty, F is non-empty.

2) Proof that if $F_1, F_2 \in i^{-1}\phi(\mathfrak{K})$ then there exists

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† For explanation of terms see (23) chapter 5.

‡ To see that this is not an 'empty' theorem, consider $\mathfrak{A}$ to be the space of bounded measurable functions, provided with the norm: $\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$. Take $\mathfrak{B}$ to be the class of maps provided by $\mathfrak{D}$-functions under convolution.

$F \in i^{-1}\phi(\mathfrak{K})$ such that $F \subseteq F_1 \cap F_2$.

If $F_1, F_2 \in i^{-1}\phi(\mathfrak{K})$ then there exist $C_1, C_2 \in \mathfrak{K}$ such that $F_1 = i^{-1}\phi(C_1)$ and $F_2 = i^{-1}\phi(C_2)$. And so if $F = i^{-1}\phi(C_1 \cap C_2)$, then $F \in i^{-1}\phi(\mathfrak{K})$ and $F \subseteq F_1 \cap F_2$.

Hence $i^{-1}\phi(\mathfrak{K})$ is a filter base on $\mathfrak{A}$.

We claim further that if $\mathfrak{H}_\phi$ is the filter generated by this filter base, then $\mathfrak{H}_\phi$ is a Cauchy filter on $\mathfrak{A}$.

Choose any τ-neighbourhood M, of $0 \in \mathfrak{A}$. Put

$$V = \{\beta \mid \beta \in \mathfrak{Q}_0(\mathfrak{A}, \mathfrak{B}) \text{ and } i^{-1}\phi(\beta) \in M\}.$$

Then since ϕ is an endomorphism of $\mathfrak{A}$, $\phi(0) = 0$, and so V is a $(\tau, \mathfrak{B})$-neighbourhood of $0 \in \mathfrak{Q}_0(\mathfrak{A}, \mathfrak{B})$. Hence there exists $U \in \mathfrak{K}$ such that

$$U - U \subseteq V.$$

Put

$$N = i^{-1}\phi(U),$$

then $N \in \mathfrak{H}_\phi$ and $N - N \subseteq M$. It follows that $\mathfrak{H}_\phi$ is a Cauchy filter on $\mathfrak{A}$.

Since $\mathfrak{A}$ is complete, it follows that $\mathfrak{H}_\phi$ converges; and since $\mathfrak{A}$ is Hausdorff, there is a unique $f_\phi \in \mathfrak{A}$ to which $\mathfrak{H}_\phi$ converges. Every τ-neighbourhood of f_ϕ belongs to $\mathfrak{H}_\phi$. We will denote the filter of τ-neighbourhoods of a point $f \in \mathfrak{A}$, by $\mathfrak{F}_f$. Thus for each $\phi \in \mathfrak{B}$,

$$\mathfrak{F}_{f_\phi} \subseteq \mathfrak{H}_\phi.$$

Consider any pair ψ, χ of elements of $\mathfrak{B}$. Choose any τ-neighbourhood F, of $f_\chi * \psi$. Since ψ is continuous on $\mathfrak{U}$, there exists a τ-neighbourhood E of f_χ such that $\psi(E) \subseteq F$. Moreover, since $\mathfrak{F}_{f_\chi} \subset \mathfrak{H}_\chi$, E has a subset of the form $i^{-1}\chi(C)$, where $C \in \mathfrak{K}$. Hence F has a subset of the form $\psi i^{-1}\chi(C) = i^{-1}(\chi \circ \psi)(C)$, where $C \in \mathfrak{K}$. Hence

$$\mathfrak{F}_{(f_\chi * \psi)} \subseteq \mathfrak{H}_{(\chi \circ \psi)}.$$

But we already have that

$$\mathfrak{F}_{f_{(\chi \circ \psi)}} \subseteq \mathfrak{H}_{(\chi \circ \psi)}.$$

And so we have both that $\mathfrak{H}_{(\chi \circ \psi)}$ converges to $f_\chi * \psi$, and that $\mathfrak{H}_{(\chi \circ \psi)}$ converges to $f_{(\chi \circ \psi)}$. Consequently

$$f_\chi * \psi = f_{(\chi \circ \psi)}, \quad \forall \chi, \psi \in \mathfrak{B}.$$

Put $\alpha = f_\phi // \phi$, $\phi \in \mathfrak{B}$. Then α is a well defined element of $\mathfrak{D}_o(\mathfrak{U}, \mathfrak{B})$. Suppose that V is any $(\tau, \mathfrak{B})$-neighbourhood of α. Then V is of the form

$$V = \{\beta \mid \beta \in \mathfrak{D}_o(\mathfrak{U}, \mathfrak{B}) \text{ and } i^{-1}\phi_j(\beta) \in N_j, \; j = 1,2,\ldots,n.\},$$

where $\phi_j \in \mathfrak{B}$ and N_j is a τ-neighbourhood of $i^{-1}\phi_j(\alpha)$. Put $f_{\phi_j} = i^{-1}\phi_j(\alpha)$, then N_j is a τ-neighbourhood of f_{ϕ_j}. Hence N_j contains a set of the form $i^{-1}\phi_j(U_j)$, where $U_j \in \mathfrak{K}$, $j = 1,2,\ldots,n$. Put

$$U = \{\beta \mid \beta \in \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B}) \text{ and } i^{-1}\phi_j(\beta) \in i^{-1}\phi_j(U_j), \quad j = 1,2,\ldots,n.\}$$

$$= \bigcap_{j=1}^{n} U_j.$$

Clearly $U \subseteq V$ and $U \in \mathfrak{K}$. Hence $V \in \mathfrak{K}$. It follows that $\mathfrak{K}$ converges to $\alpha \in \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$.

L.3.4.1. Suppose that $\mathfrak{S}$ is a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ and that every element of $\mathfrak{B}$ maps $\mathfrak{S}$ into $\mathfrak{S}$, so that $\mathfrak{S}$, $\mathfrak{B}$ form a quotient pair. Then every element of $\mathfrak{B}$ maps $\mathfrak{Q}_o(\mathfrak{S}, \mathfrak{B})$ into itself, and there is a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ isomorphic with $\mathfrak{Q}_o(\mathfrak{S}, \mathfrak{B})$. There is one and only one such sub-algebra with the properties that if

$$I : \mathfrak{Q}_o(\mathfrak{S}, \mathfrak{B}) \to \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$$

is the isomorphic embedding, then

1) $\forall \alpha \in \mathfrak{Q}_o(\mathfrak{S}, \mathfrak{B})$, $\forall \phi \in \mathfrak{B}$,

$$I(\alpha) * \phi = I(\alpha * \phi),$$

2) $\forall \beta \in \mathfrak{S}$,

$$I(i(\beta)) = \beta,$$

where i denotes the natural embedding of $\mathfrak{S}$ in $\mathfrak{Q}_o(\mathfrak{S}, \mathfrak{B})$. Denoting this unique sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ by $\mathfrak{S}^*$, we have in addition that every element of $\mathfrak{B}$ maps $\mathfrak{S}^*$ into itself.

Proof: We have immediately that $\mathfrak{Q}_0(\mathfrak{G}, \mathfrak{B})$ is a sub-algebra of $\mathfrak{Q}(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{B})$, and that every element of $\mathfrak{B}$ maps $\mathfrak{Q}_0(\mathfrak{G}, \mathfrak{B})$ into itself. Moreover under the natural embedding I_0, say, $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ is isomorphic with $\mathfrak{Q}(\mathfrak{Q}(\mathfrak{A}, \mathfrak{B}), \mathfrak{B})$. Furthermore if $\alpha \in \mathfrak{Q}_0(\mathfrak{G}, \mathfrak{B})$, and if $\phi \in \mathfrak{B}$, then

$$I_0^{-1}(\alpha) * \phi = I_0^{-1}(\alpha * \phi),$$

while if $\beta \in \mathfrak{G}$, then

$$I_0^{-1}(i(\beta)) = \beta.$$

Hence there is at least one sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ which is isomorphic with $\mathfrak{Q}_0(\mathfrak{G}, \mathfrak{B})$ and for which the isomorphism has the desired properties. Suppose that

$$I : \mathfrak{Q}_0(\mathfrak{G}, \mathfrak{B}) \to \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$$

is any isomorphic embedding having the properties 1) and 2) given above. Choose $\alpha \in \mathfrak{Q}_0(\mathfrak{G}, \mathfrak{B})$. Then if $\alpha = \beta // \phi$, where $\phi \in \mathfrak{B}$ and $\beta \in \mathfrak{G}$, and if $\beta = g//\psi$, where $g \in \mathfrak{A}$ and $\psi \in \mathfrak{B}$, we have that

$$I(\alpha) = I(\beta // \phi).$$

And so,

$$\begin{aligned} I(\alpha) * \phi &= I(\beta * \phi // \phi) \\ &= I(i(\beta)) \\ &= \beta \\ &= g//\psi. \end{aligned}$$

Hence $\qquad I(\alpha) = g//(\phi \circ \psi),$

and clearly therefore, I is unique. It follows that $\mathfrak{G}^{*}$ is unique. Finally choose $\alpha^{*} \in \mathfrak{G}^{*}$, then for some $\alpha \in \mathfrak{Q}_{0}(\mathfrak{G}, \mathfrak{B})$,

$$\alpha^{*} = I(\alpha).$$

Thus if $\phi \in \mathfrak{B}$,

$$\begin{aligned} \alpha^{*} \ast \phi &= I(\alpha \ast \phi) \\ &\in \mathfrak{G}^{*}. \end{aligned}$$

Hence each element of $\mathfrak{B}$ maps $\mathfrak{G}^{*}$ into itself.

<u>L.3.4.2.</u> If $\mathfrak{G}$ is a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ such that every element of $\mathfrak{B}$ maps $\mathfrak{G}$ into itself, and if $\mathfrak{G}^{*} = \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, then $\mathfrak{G} = \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$.

Proof: Choose $\alpha \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ and suppose that $\alpha = f//\Phi$, where $f \in \mathfrak{A}$ and $\phi \in \mathfrak{B}$. Put $\beta = f//(\phi \circ \phi) \in \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$. Since $\mathfrak{G}^{*} = \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, there exists $\gamma \in \mathfrak{G}$ such that $\gamma = \beta \ast \phi$. That is, $\gamma = \alpha$. It follows that $\mathfrak{G} = \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$.

<u>D.3.4.1.</u> For $n = 0,1,2,\ldots$, we will denote by $\mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B})$ the sub-algebra, $\mathfrak{Q}_{n}(\mathfrak{A}, \mathfrak{B})^{*}$, of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$. For completeness we will define $\mathfrak{Q}_{-1}(\mathfrak{A}, \mathfrak{B})$ to be $i(\mathfrak{A})$; clearly $i(\mathfrak{A})$ is a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$.

<u>N.3.4.1.</u> For $n = -1,0,1,2,\ldots$, we always have that $\mathfrak{Q}_{n}(\mathfrak{A}, \mathfrak{B}) \subseteq \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B})$. If $i(\mathfrak{A}) \neq \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, then it follows from L.3.4.2. that $\mathfrak{Q}_{n}(\mathfrak{A}, \mathfrak{B}) \neq \mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, $n = -1, 0,1,2,\ldots$.

<u>L.3.4.3.</u> Suppose that $n \geq -1$ and $m \geq n+1$, and that

$\mathfrak{Q}_m(\mathfrak{A},\mathfrak{B}) = \mathfrak{Q}_n(\mathfrak{A},\mathfrak{B})$. Then for every $r \geq n$, $\mathfrak{Q}_r(\mathfrak{A},\mathfrak{B}) = \mathfrak{Q}_n(\mathfrak{A},\mathfrak{B})$.

Proof: We have that

$$\mathfrak{Q}_n(\mathfrak{A},\mathfrak{B}) \subseteq \mathfrak{Q}_{n+1}(\mathfrak{A},\mathfrak{B}) \subseteq \mathfrak{Q}_m(\mathfrak{A},\mathfrak{B}).$$

It follows that $\mathfrak{Q}_n(\mathfrak{A},\mathfrak{B}) = \mathfrak{Q}_{n+1}(\mathfrak{A},\mathfrak{B})$. Suppose that for some $r \geq n$, $\mathfrak{Q}_r(\mathfrak{A},\mathfrak{B}) = \mathfrak{Q}_{r+1}(\mathfrak{A},\mathfrak{B})$. Then

$$\mathfrak{Q}_o(\mathfrak{Q}_r(\mathfrak{A},\mathfrak{B}),\mathfrak{B}) = \mathfrak{Q}_o(\mathfrak{Q}_{r+1}(\mathfrak{A},\mathfrak{B}),\mathfrak{B})$$

and so

$$\mathfrak{Q}_{r+1}(\mathfrak{A},\mathfrak{B}) = \mathfrak{Q}_{r+2}(\mathfrak{A},\mathfrak{B}).$$

The result follows by induction.†

<u>N.3.4.2.</u> If $n \neq -1$, it is clear that $\mathfrak{Q}_n(\mathfrak{A},\mathfrak{B})$ consists precisely of those elements $\alpha \in \mathfrak{Q}(\mathfrak{A},\mathfrak{B})$ whose base class is $\mathfrak{B} \circ \mathfrak{B} \circ \ldots \circ \mathfrak{B}$ (n+1 terms). That is, those $\alpha \in \mathfrak{Q}(\mathfrak{A},\mathfrak{B})$ for which, given any n+1 elements of $\mathfrak{B}$, $\phi_1, \phi_2, \ldots, \phi_{n+1}$, there exists $f \in \mathfrak{A}$ such that $\alpha = f//(\phi_1 \circ \phi_2 \circ \ldots \circ \phi_{n+1})$.‡

For the remainder of this section (§3.4.), we will assume that $\mathfrak{A}$ is a topological space, with topology τ. We will also assume that $\mathfrak{Q}_o(\mathfrak{A},\mathfrak{B})$ has the $(\tau,\mathfrak{B})$ topology, that $\mathfrak{Q}_1(\mathfrak{A},\mathfrak{B})$ has the $((\tau,\mathfrak{B}),\mathfrak{B})$ topology, and so on. That is, $\mathfrak{Q}_{n+1}(\mathfrak{A},\mathfrak{B})$, which is isomorphic to $\mathfrak{Q}_o(\mathfrak{Q}_n(\mathfrak{A},\mathfrak{B}),\mathfrak{B})$, carries the $(\tau_n,\mathfrak{B})$ topology, where τ_n denotes the topology on $\mathfrak{Q}_n(\mathfrak{A},\mathfrak{B})$. $(n \geq -1)$.

<u>N.3.4.3.</u> For $n \geq 0$, we will denote by τ_n the topology on $\mathfrak{Q}_n(\mathfrak{A},\mathfrak{B})$ described above; we will write τ instead of τ_{-1} for the topology on $\mathfrak{A}$.

† See also E.3.4.1. ‡ Clearly $\mathfrak{Q}_n(\mathfrak{A},\mathfrak{B})$ and $\mathfrak{B} \circ \mathfrak{B} \circ \ldots \circ \mathfrak{B}$ (n+1 terms) are Wiener-like.

We will denote by I the natural embedding of $\mathfrak{D}_n(\mathfrak{A}, \mathfrak{B})$ into $\mathfrak{D}_{n+m}(\mathfrak{A}, \mathfrak{B})$ ($n \geq -1$ and $m \geq 0$).

T.3.4.2. The embedding

$$I : \mathfrak{D}_o(\mathfrak{A}, \mathfrak{B}) \to \mathfrak{D}_1(\mathfrak{A}, \mathfrak{B})$$

is continuous.

Proof: Choose $\phi \in \mathfrak{B}$. By T.3.3.5. ϕ is continuous on $\mathfrak{D}_o(\mathfrak{A}, \mathfrak{B})$. Hence by T.3.3.1., the map

$$I : \mathfrak{D}_o(\mathfrak{A}, \mathfrak{B}) \to \mathfrak{D}_1(\mathfrak{A}, \mathfrak{B})$$

is continuous.

Corollary † If $n \geq 0$, then the embedding

$$I : \mathfrak{D}_n(\mathfrak{A}, \mathfrak{B}) \to \mathfrak{D}_{n+1}(\mathfrak{A}, \mathfrak{B})$$

is continuous.

T.3.4.3. Suppose that $n \geq 0$ and that the map

$$I^{-1} : I(\mathfrak{D}_{n-1}(\mathfrak{A}, \mathfrak{B})) \to \mathfrak{D}_{n-1}(\mathfrak{A}, \mathfrak{B})$$

is continuous. Then the map

$$I^{-1} : I(\mathfrak{D}_n(\mathfrak{A}, \mathfrak{B})) \to \mathfrak{D}_n(\mathfrak{A}, \mathfrak{B})$$

is also continuous. (Where we take for $m \geq -1$, $I(\mathfrak{D}_m(\mathfrak{A}, \mathfrak{B}))$ to have the topology induced on it by $\mathfrak{D}_{m+1}(\mathfrak{A}, \mathfrak{B})$.)

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† It can happen that $i : \mathfrak{A} \to \mathfrak{D}_o(\mathfrak{A}, \mathfrak{B})$ is discontinuous. To see this recall T.3.3.1. and consider the σ-topology on $\mathfrak{T}$ given in D.3.5.1.

Proof: Choose $\alpha \in \mathfrak{D}_n(\mathfrak{A}, \mathfrak{B})$ and take any τ_n-neighbourhood V, of α. Then V is of the form [‡]

$$V = \{\beta \mid \beta \in \mathfrak{D}_n(\mathfrak{A}, \mathfrak{B}) \text{ and } \phi_j(\beta) \in M_j, \quad j = 1,2,\ldots,J.\},$$

where each M_j is a τ_{n-1}-neighbourhood of $\phi_j(\alpha)$.[||] Since the map

$$I^{-1} : I(\mathfrak{D}_{n-1}(\mathfrak{A}, \mathfrak{B})) \to \mathfrak{D}_{n-1}(\mathfrak{A}, \mathfrak{B})$$

is continuous, for each j, there exists N_j, a τ_n-neighbourhood of $\phi_j(\alpha)$ such that

$$N_j \cap I(\mathfrak{D}_{n-1}(\mathfrak{A}, \mathfrak{B})) \subseteq I(M_j).$$

Put

$$U = \{\beta \mid \beta \in \mathfrak{D}_{n+1}(\mathfrak{A}, \mathfrak{B}) \text{ and } \phi_j(\beta) \in N_j, \quad j = 1,2,\ldots,J.\},$$

then U is a τ_{n+1}-neighbourhood of α. Moreover if $\beta \in U \cap \mathfrak{D}_n(\mathfrak{A}, \mathfrak{B})$, then $\phi_j(\beta)$[¶] $\in M_j$, $j = 1,2,\ldots,J$, and so $\beta \in V$. Hence

$$I^{-1}(U \cap \mathfrak{D}_n(\mathfrak{A}, \mathfrak{B})) \subseteq V,$$

[‡] For consistency of notation, we will, when convenient, ignore the precise distinction between $\mathfrak{A}$ and $i(\mathfrak{A})$. This will allow us to always write V as we have done, rather than make an exception and have to write, when $n = 0$, $i^{-1}\phi_j(\beta)$ instead of $\phi_j(\beta)$.

[||] $i^{-1}\phi_j(\alpha)$ if $n = 0$.

[¶] $i^{-1}\phi_j(\beta)$ if $n = 0$.

and it follows that the map

$$I^{-1} : I(\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})) \to \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$$

is continuous.

<u>Corollary</u> Suppose that $n \geq 0$, and that the map

$$I : \mathfrak{Q}_{n-1}(\mathfrak{A}, \mathfrak{B}) \longleftrightarrow I(\mathfrak{Q}_{n-1}(\mathfrak{A}, \mathfrak{B}))$$

is an homeomorphism. Then the map

$$I : \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}) \longleftrightarrow I(\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}))$$

is also an homeomorphism.

<u>T.3.4.4. An alternative representation of the τ_n topology on $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$.</u>

For each $n \geq 0$ and each $\alpha \in \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$, define $\mathfrak{G}^n_\alpha$ to be the class consisting of all sets G of the form

$$G = \{\beta \mid \beta \in \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}) \text{ and } i^{-1}\phi^n_j(\beta) \in M_j,\ j = 1,2,\ldots,J.\},$$

where J is a positive integer, $\phi^n_1, \phi^n_2, \ldots, \phi^n_J$ are elements of $\mathfrak{B} \circ \mathfrak{B} \circ \cdots \circ \mathfrak{B}$ (n+1 terms), and M_j is a τ-neighbourhood of $i^{-1}\phi^n_j(\alpha)$.

For each $n \geq 0$ $\mathfrak{G}^n_\alpha$ is precisely the filter of τ_n-neighbourhoods of $\alpha \in \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$.

Proof: By induction on n. Certainly $\mathfrak{G}^o_\beta$ is, for each $\beta \in \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$, the filter of τ_o-neighbourhoods of β.

Suppose that for each $\beta \in \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$, $\mathfrak{G}^n_\beta$ is the filter of τ_n-neighbourhoods of β.

Choose $\alpha \in \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B})$. Suppose that N is any τ_{n+1}-neighbourhood of α, then N is the form

$$N = \{\beta | \beta \in \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B}) \text{ and } \phi_j(\beta) \in N_j, \quad j = 1,2,\dots,J.\},$$

where N_j is a τ_n-neighbourhood of $\phi_j(\alpha)$. By hypothesis, for each j, N_j is of the form

$$N_j = \{\gamma | \gamma \in \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}) \text{ and } i^{-1}\phi^n_{j,k}(\gamma) \in M_{j,k}, \quad k = 1,2,\dots,K_j\},$$

where $\phi^n_{j,k} \in \mathfrak{B} \circ \mathfrak{B} \circ \dots \circ \mathfrak{B}$ (n+1 terms) and $M_{j,k}$ is a τ-neighbourhood of $i^{-1}\phi^n_{j,k}\phi_j(\alpha)$. Put

$$\xi^{n+1}_{j,k} = \phi^n_{j,k}\ \phi_j.$$

Then

$$N = \{\beta | \beta \in \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B}) \text{ and } i^{-1}\xi^{n+1}_{j,k}(\beta) \in M_{j,k}, \quad k = 1,2,\dots,K_j,\ j = 1,2,\dots,J.\}.$$

Define $K_o = 0$, and put

$$\phi^{n+1}_k = \xi^{n+1}_{j,(k-K_1-K_2-\dots-K_{j-1})},$$

for $k = (K_1+K_2+\dots+K_{j-1}+1), (K_1+K_2+\dots+K_{j-1}+2), \dots, (K_1+K_2+\dots+K_{j-1}+K_j)$;

and $j = 1,2,\dots,J$.

Similarly put

$$M_k = M_{j,(k-K_1-K_2-\dots-K_{j-1})},$$

for $k = (K_1+K_2+\dots+K_{j-1}+1),(K_1+K_2+\dots+K_{j-1}+2),$
$\dots,(K_1+K_2+\dots+K_{j-1}+K_j);$

and $j = 1,2,\dots,J.$

Finally put $K = K_1+K_2+\dots+K_J$. Then we have that

$$N = \{\beta \mid \beta \in \mathfrak{Q}_{n+1}(\mathfrak{U}, \mathfrak{B}) \text{ and } i^{-1}\phi_k^{n+1}(\beta) \in M_k, \quad k = 1,2,\dots,K.\}.$$

Hence $N \in \mathfrak{G}_\alpha^{n+1}$.

Conversely suppose that $G \in \mathfrak{G}_\alpha^{n+1}$, then G is of the form

$$G = \{\beta \mid \beta \in \mathfrak{Q}_{n+1}(\mathfrak{U}, \mathfrak{B}) \text{ and } i^{-1}\phi_k^{n+1}(\beta) \in M_k, \quad k = 1,2,\dots,K.\}.$$

Choose $\phi_k^n \in \mathfrak{B} \circ \mathfrak{B} \circ \dots \circ \mathfrak{B}$ (n+1 terms), and $\phi_k \in \mathfrak{B}$ such that

$$\phi_k^{n+1} = \phi_k^n \, \phi_k, \quad k = 1,2,\dots,K.$$

Put

$$N_k = \{\gamma \mid \gamma \in \mathfrak{Q}_n(\mathfrak{U}, \mathfrak{B}), \text{ and } i^{-1}\phi_k^n(\gamma) \in M_k\}.$$

Then N_k is a τ_n-neighbourhood of $\phi_k(\alpha)$, and we have that

$$G = \{\beta \mid \beta \in \mathfrak{Q}_{n+1}(\mathfrak{U}, \mathfrak{B}) \text{ and } \phi_k(\beta) \in N_k, \quad k = 1,2,\dots,K.\}.$$

Hence G is a τ_{n+1}-neighbourhood of α. It follows that for each $\alpha \in \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B})$, $\mathfrak{G}_{\alpha}^{n+1}$ is the filter of τ_{n+1}-neighbourhoods of α. The Theorem follows by induction.

<u>E.3.4.1.</u> Take $\mathfrak{A}$ to be the set of all integers. Take $\mathfrak{B}_1$ to be the set of all positive integers and $\mathfrak{B}_2$ to be the set of all positive even integers. If $\phi \in \mathfrak{B}_1$ or if $\phi \in \mathfrak{B}_2$, and if $f \in \mathfrak{A}$, we take $\phi(f) = f * \phi = f \times \phi$. (Here '$\times$' denotes ordinary multiplication.) Then $\mathfrak{A}, \mathfrak{B}_1$ form a quotient pair and so do $\mathfrak{A}, \mathfrak{B}_2$. (We may take the system of operations on $\mathfrak{A}$ in each case to be $\varnothing$, though the precise choice of a system will be seen to be irrelevant.)

We have the following properties.

<u>1. The case of</u> $\mathfrak{A}, \mathfrak{B}_1$.

a) $\mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B}_1) = i(\mathfrak{A})$.

Proof: We have that $i(\mathfrak{A}) \subseteq \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B}_1)$. Choose $\alpha \in \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B}_1)$ then for each $\phi \in \mathfrak{B}_1$, there exists $f \in \mathfrak{A}$ such that $\alpha = f//\phi$. Hence there exists $f_1 \in \mathfrak{A}$ such that $\alpha = f_1//1$. But we have that $f_1 = f_1 * 1$, and so $\alpha = f_1 * 1//1 = i(f_1)$.†

b) As a corollary, using L.3.4.3. we have that for each $n \geq -1$, $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}_1) = \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B}_1)$.

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† Clearly this result generalises; whenever $\mathfrak{A}, \mathfrak{B}$ form a quotient pair and $\mathfrak{B}$ has an identity element, $i(\mathfrak{A}) = \mathfrak{Q}_o(\mathfrak{A}, \mathfrak{B})$.

2. The case of $\mathfrak{A}, \mathfrak{B}_2$.

a) If $n \geq -1$ then $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}_2) \neq \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B}_2)$.

Proof: We claim $1//2^{n+2} \in \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B}_2)$, but $1//2^{n+2} \notin \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}_2)$. Choose $\phi_1, \phi_2, \ldots, \phi_{n+2} \in \mathfrak{B}_2$. Then $\phi_1 \circ \phi_2 \circ \ldots \circ \phi_{n+2}$ has a factor 2^{n+2}; suppose the other factor is ψ, so that ψ is an integer. Clearly we have that

$$1//2^{n+2} = \psi//(\phi_1 \circ \phi_2 \circ \ldots \circ \phi_{n+2}).$$

Hence $1//2^{n+2} \in \mathfrak{Q}_{n+1}(\mathfrak{A}, \mathfrak{B}_2)$.

Suppose on the other hand that there exists $f \in \mathfrak{A}$ such that

$$1//2^{n+2} = f//2^{n+1}.$$

Then we should have that $2f = 1$. But this is a contradiction. Hence there is no $f \in \mathfrak{A}$ such that $1//2^{n+2} = f//2^{n+1}$, i.e. $1//2^{n+2} \notin \mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B}_2)$. The result follows.

3.5. Consideration of $\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$

We have already discussed some aspects of the class $\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$, see for example T.2.6.2. In this section (§ 3.5), we will consider some results concerning this class, which arise from the content of the previous sections of this chapter. We will take $\mathfrak{S}'$ to have the topology given in D.2.4.16.

L.3.5.1. $\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ and $\mathfrak{S}$ are Wiener-like.

Proof: Certainly $\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is a sub-algebra of $\mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$, and $\mathfrak{S} \subseteq \mathfrak{U}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$. Choose $\alpha \in \mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ and $\phi \in \mathfrak{S}$. Then by the corollary to L.2.6.1., there exists $\phi_1, \phi_2 \in \mathfrak{S} \cap \mathfrak{W}$ such that $\phi = \phi_1 + \phi_2$. Take $f_1, f_2 \in \mathfrak{T}$ such that

$$\alpha = f_1 // \phi_1 = f_2 // \phi_2 .$$

Then we have that

$$\begin{aligned} \alpha * \phi &= \alpha * \phi_1 + \alpha * \phi_2 \\ &= f_1 * \phi_1 // \phi_1 + f_2 * \phi_2 // \phi_2 \\ &= i(f_1 + f_2). \end{aligned}$$

The result follows.

T.3.5.1. Suppose that τ is any topology on $\mathfrak{T}$, compatible with its linear structure. Then the resulting $(\tau, \mathfrak{S} \cap \mathfrak{W})$ and $(\tau, \mathfrak{S})$ topologies on $\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ are identical.

Proof: It follows from T.3.3.3. that with either of the above two topologies, $\mathfrak{D}_0(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{B})$ is a linear topological space. Since $\mathfrak{G} \cap \mathfrak{B} \subseteq \mathfrak{G}$, it follows at once that any $(\tau, \mathfrak{G} \cap \mathfrak{B})$-neighbourhood of $0 \in \mathfrak{D}_0(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{B})$ is also a $(\tau, \mathfrak{G})$-neighbourhood of 0.

Choose any $(\tau, \mathfrak{G})$-neighbourhood of $0 \in \mathfrak{D}_0(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{B})$, F, say. Then F has a subset G of the form

$$G = \{\beta \mid \beta \in \mathfrak{D}_0(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{B}) \text{ and } i^{-1}(\beta * \phi_j) \in N_j, \quad j = 1,2,\ldots,n.\},$$

where n is a positive integer; ϕ_j, $j = 1,2,\ldots,n$, is an element of $\mathfrak{G}$, and N_j is a τ-neighbourhood of 0. For each $j = 1,2,\ldots,n$, choose $\phi_j^1, \phi_j^2 \in \mathfrak{G} \cap \mathfrak{B}$ such that

$$\phi_j = \phi_j^1 + \phi_j^2 .$$

Choose, for each j, a τ-neighbourhood of 0, M_j, such that

$$M_j + M_j \subseteq N_j.$$

Put

$$H = \{\beta \mid \beta \in \mathfrak{D}_0(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{B}) \text{ and } i^{-1}(\beta * \phi_j^k) \in M_j, \quad k = 1,2, \quad j = 1,2,\ldots,n.\}.$$

Then H is a $(\tau, \mathfrak{G} \cap \mathfrak{B})$-neighbourhood of 0, and if $\beta \in H$, then clearly $\beta \in G \subseteq F$. Hence $H \subseteq F$.

The result now follows.

T.3.5.2. Suppose that τ is any topology on $\mathfrak{T}$, compatible with its linear structure. Then for each $\phi \in \mathfrak{S}$, and each positive integer p, the maps

$$\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \ni \alpha \longrightarrow \alpha * \phi \in \mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$$

$$\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \ni \alpha \longrightarrow \alpha^{(p)} \in \mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$$

are continuous in the $(\tau, \mathfrak{S} \cap \mathfrak{W})$ topology.

Proof: By the previous theorem, the $(\tau, \mathfrak{S} \cap \mathfrak{W})$ and $(\tau, \mathfrak{S})$ topologies on $\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ are identical. Considering the $(\tau, \mathfrak{S})$ topology on $\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$, and applying T.3.3.5., the result follows.

D.3.5.1. We will denote by σ the topology of pointwise convergence, on $\mathfrak{T}$. That is, σ is the topology generated by taking as a basis of neighbourhoods of the origin, the class consisting of all sets of the form

$$\{f \mid f \in \mathfrak{T} \text{ and } |f(x_k)| < \epsilon,\ k = 1,2,\ldots,K.\},$$

where $\epsilon > 0$, K is a positive integer, and $x_1, x_2, \ldots, x_K$ are real numbers.

It is familiar that with this topology, $\mathfrak{T}$ is an Hausdorff linear topological space.

It follows immediately that with the $(\sigma, \mathfrak{S} \cap \mathfrak{W})$ topology, $\mathfrak{D}_o(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is an Hausdorff linear topological space, and that for each $\phi \in \mathfrak{S}$, and each positive integer p, the maps

$$\mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \ni \alpha \longrightarrow \alpha * \phi \in \mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$$

$$\mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \ni \alpha \longrightarrow \alpha^{(p)} \in \mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$$

are continuous.

Note that $\exists\, \phi \in \mathfrak{S} \cap \mathfrak{W}$ which does not provide a continuous map of $\mathfrak{T}$ into itself with respect to the σ-topology. Hence $i : \mathfrak{T} \to \mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is not continuous with respect to the σ and $(\sigma, \mathfrak{S} \cap \mathfrak{W})$ topologies.

T.3.5.3. The map Θ, which was discussed in sections 2.5. and 2.6., provides an homeomorphism of $\mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ (with the $(\sigma, \mathfrak{S} \cap \mathfrak{W})$ topology) and $\mathfrak{S}'$.

Proof: We have shown in T.2.6.2., that Θ is a linear bijection of $\mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ onto $\mathfrak{S}'$. From T.3.5.1., it follows that the $(\sigma, \mathfrak{S} \cap \mathfrak{W})$ topology is identical with the $(\sigma, \mathfrak{S})$ topology on $\mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$.

Suppose that V is any neighbourhood of 0 in the topology of $\mathfrak{S}'$. Then V has a subset of the form

$$\{F \mid F \in \mathfrak{S}' \text{ and } |\langle F, \phi_r \rangle| < \epsilon,\ r = 1,2,\dots,R.\},$$

where $\epsilon > 0$ and $\phi_1, \phi_2, \dots, \phi_R$ are elements of $\mathfrak{S}$. Put

$$M = \{f \mid f \in \mathfrak{T} \text{ and } |f(0)| < \epsilon\},$$

$$\psi_r(t) = \phi_r(-t),\ r = 1,2,\dots,R,$$

$$N = \{\alpha \mid \alpha \in \mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\alpha * \psi_r) \in M,$$
$$r = 1,2,\dots,R.\}.$$

Then N is a $(\sigma, \mathfrak{G})$-neighbourhood of $0 \in \mathfrak{Q}_o(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$. If $\alpha \in N$, then we have that

$$|(i^{-1}(\alpha \ast \psi_r))(0)| < \epsilon, \; r = 1,2,\ldots,R,$$

that is

$$|\langle\theta(\alpha), \; \phi_r\rangle| < \epsilon, \; r = 1,2,\ldots,R.$$

Hence if $\alpha \in N$, then $\theta(\alpha) \in V$, i.e. $\theta(N) \subseteq V$. It follows that with respect to the $(\sigma, \mathfrak{G})$ topology on $\mathfrak{Q}_o(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$, the map

$$\theta : \mathfrak{Q}_o(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W}) \to \mathfrak{G}'$$

is continuous.

We next show that θ^{-1} is continuous. Suppose that N is a $(\sigma, \mathfrak{G})$-neighbourhood of $0 \in \mathfrak{Q}_o(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$. Then N is of the form

$$N = \{\alpha | \alpha \in \mathfrak{Q}_o(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W}) \text{ and } i^{-1}(\alpha \ast \psi_r) \in M_r, \; r = 1,2,\ldots,R.\},$$

where for each r, $\psi_r \in \mathfrak{G}$ and M_r is a σ-neighbourhood of 0. Put

$$M = \bigcap_{r=1}^{R} M_r,$$

then M is a σ-neighbourhood of 0. Hence M has a subset M', of the form

$$M' = \{f | f \in \mathfrak{T} \text{ and } |f(\xi_j)| < \epsilon, \; j = 1,2,\ldots,J\},$$

where $\epsilon > 0$ and $\xi_1, \xi_2, \ldots, \xi_J$ are real numbers. And so if

$$N' = \{\alpha | \alpha \in \mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\alpha * \psi_r) \in M',$$
$$r = 1,2,\ldots,R.\},$$

then N' is a $(\sigma, \mathfrak{S})$-neighbourhood of $0 \in \mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ and clearly $N' \subseteq N$. Put

$$\phi_k(x) = \psi_{k-(j-1)R}(\xi_j - x) \;, \quad k = R(j-1)+1,\ R(j-1)+2, \ldots, R_j,$$
$$\text{and } j = 1,2,\ldots,J.$$

Then for each k, $\phi_k(x) \in \mathfrak{S}$. Put

$$V = \{F | F \in \mathfrak{S}' \text{ and } |\langle F, \phi_k \rangle| < \epsilon,\ k = 1,2,\ldots,JR.\}.$$

Then V is a neighbourhood of 0 in the topology of $\mathfrak{S}'$. If $F \in V$ and $\alpha \in \mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is such that $\Theta(\alpha) = F$, then

$$|\langle \Theta(\alpha), \phi_k \rangle| < \epsilon,\ k = 1,2,\ldots,JR,$$

and so

$$|(i^{-1}(\alpha * \psi_r))(\xi_j)| < \epsilon,\ r = 1,2,\ldots,R,\ j = 1,2,\ldots,J.$$

Hence $i^{-1}(\alpha * \psi_r) \in M'$, $r = 1,2,\ldots,R$, and so $\alpha \in N'$. It follows that $\Theta^{-1}(V) \subseteq N$, and so with respect to the $(\sigma, \mathfrak{S})$ topology on $\mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$, the map

$$\Theta^{-1} : \mathfrak{S}' \to \mathfrak{D}_0(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$$

is continuous.

The result now follows.

D.3.5.2. We will denote by σ^a the topology, on $\mathfrak{X}$, of pointwise convergence at $a \in \mathbb{R}$. That is, σ^a is the topology generated by taking as a basis of neighbourhoods of the origin, the class consisting of all sets of the form

$$\{f \mid f \in \mathfrak{X} \text{ and } |f(a)| < \epsilon\},$$

where $\epsilon > 0$.

It may be easily verified by use of L.2.4.2. that this gives a topology compatible with the linear structure of $\mathfrak{X}$. It is clear that for each $a \in \mathbb{R}$, the resulting σ^a topology does not make $\mathfrak{X}$ a T_0-space.

T.3.5.4. For each $a \in \mathbb{R}$, the $(\sigma^a, \mathfrak{S} \cap \mathfrak{W})$ and $(\sigma, \mathfrak{S} \cap \mathfrak{W})$ topologies on $\mathfrak{Q}_0(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$ are identical.

Proof: It follows from T.3.3.3. that with either of the above topologies, $\mathfrak{Q}_0(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$ is a linear topological space. It is clear that any $(\sigma^a, \mathfrak{S} \cap \mathfrak{W})$-neighbourhood of $0 \in \mathfrak{Q}_0(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$ is also a $(\sigma, \mathfrak{S} \cap \mathfrak{W})$- neighbourhood of 0.

Choose any $(\sigma, \mathfrak{S} \cap \mathfrak{W})$-neighbourhood of $0 \in \mathfrak{Q}_0(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$, F, say. Then F has a subset G of the form

$$G = \{\beta \mid \beta \in \mathfrak{Q}_0(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \psi_j) \in N_j,$$
$$j = 1,2,\ldots,J\},$$

where J is a postive integer, $\psi_1, \psi_2, \ldots, \psi_J$ are elements of $\mathfrak{S} \cap_J \mathfrak{W}$, and each N_j is a σ-neighbourhood of 0. Put $N = \bigcap_{j=1} N_j$, then N is a σ-neighbourhood of 0 and so has a subset M of the form

$$M = \{f \mid f \in \mathfrak{X} \text{ and } |f(x_k)| < \epsilon,\ k = 1,2,\ldots,K\}.$$

Moreover if

$$H = \{\beta \mid \beta \in \mathfrak{L}_o(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \psi_j) \in M,\ j = 1,2,\ldots,J\},$$

then $H \subseteq G$ and H is a $(\sigma, \mathfrak{S} \cap \mathfrak{W})$-neighbourhood of 0. Put

$$\phi_r(x) = \psi_{r-(k-1)J}(x+x_k-a),\ r = J(k-1)+1, J(k-1)+2, \ldots, Jk, \text{ and } k = 1,2,\ldots,K.$$

Then for each r, $\phi_r(x) \in \mathfrak{S} \cap \mathfrak{W}$. Put

$$M' = \{f \mid f \in \mathfrak{X} \text{ and } |f(a)| < \epsilon\},$$

and

$$H' = \{\beta \mid \beta \in \mathfrak{L}_o(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \phi_r) \in M',\ r = 1,2,\ldots,KJ.\}.$$

Then H' is a $(\sigma^a, \mathfrak{S} \cap \mathfrak{W})$-neighbourhood of $0 \in \mathfrak{L}_o(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$. Moreover if $\beta \in H'$, we have that

$$|(i^{-1}(\beta * \phi_r))(a)| < \epsilon,\ r = 1,2,\ldots,KJ,$$

and so

$$|(i^{-1}(\beta * \psi_j))(x_k)| < \epsilon,\ j = 1,2,\ldots,J,\ k = 1,2,\ldots,K.$$

Hence $i^{-1}(\beta * \psi_j) \in M$, $j = 1,2,\ldots,J$, and so $\beta \in H$. It follows that any $(\sigma, \mathfrak{S} \cap \mathfrak{W})$-neighbourhood of $0 \in \mathfrak{Q}_o(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$ is also a $(\sigma^a, \mathfrak{S} \cap \mathfrak{W})$-neighbourhood of 0. Hence result.

Remarks on § 3.5.

It was shown in section 3.4. that for some quotient pairs $\mathfrak{A}$, $\mathfrak{B}$, the spaces $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$ $(n = -1,0,1,2,\ldots)$ are all distinct, and in other cases the spaces $\mathfrak{Q}_n(\mathfrak{A}, \mathfrak{B})$ are all the same. It is an intruiging question as to whether or not all the spaces $\mathfrak{Q}_n(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$ are distinct. I have not been able to answer this question. A sufficient condition for the algebraic and topological identity of all the $\mathfrak{Q}_n(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$ $(n \geq 0)$, would be that any element of $\mathfrak{S} \cap \mathfrak{W}$ should be expressible as the convolution of two elements of $\mathfrak{S} \cap \mathfrak{W}$. For, if this condition were satisfied, then we could use L.3.4.3., T.3.4.4. and the corollary to T.3.4.3. to prove what we require. I do not believe the above condition to be a necessary condition, but I am unable to establish this either.

Theorem T.3.5.4. suggests the question of what range of topologies on $\mathfrak{X}$ will lead to a particular topology on $\mathfrak{Q}_o(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{W})$. I suspect that a wide selection of topologies on $\mathfrak{X}$ will lead to the $(\sigma, \mathfrak{S} \cap \mathfrak{W})$ topology on $\mathfrak{Q}_0(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$. This problem raises the still more general question of what range of topologies on $\mathfrak{A}$, for a given quotient pair $\mathfrak{A}, \mathfrak{B}$, will lead to a particular topology on a Wiener sub-algebra[†] of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$. In the next section it is shown that $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ is a Wiener sub-algebra of itself, and in T.3.6.2. that a number of different topologies on $\mathfrak{X}$ all lead to the same topology on $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$.

3.6. Consideration of $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$

Throughout this section (§ 3.6.) we shall take $\mathfrak{Z}'$ to have its usual topology, that is, the topology given in D.2.4.16. We defined $\mathfrak{Q}\mathfrak{Z}'$ in D.2.5.1., and we will take it to have the topology induced on it as a subspace of $\mathfrak{Z}'$.

L.3.6.1. $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ and $\mathfrak{Z}$ are Wiener-like, $\mathfrak{Z}$ separates $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ and $\mathfrak{Z}$ is endomorphic on $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$.

Proof: Certainly $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ is a subalgebra of itself, and $\mathfrak{Z} \subseteq \mathfrak{U}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$.[‡] Choose

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[†] We say that $\mathfrak{S}$, a sub-algebra of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$, is a Wiener sub-algebra, if there exists $\mathfrak{C} \subseteq \mathfrak{U}(\mathfrak{A}, \mathfrak{B})$ such that $\mathfrak{S}$ and $\mathfrak{C}$ are Wiener-like.

[‡] In the sense that each element of $\mathfrak{Z}$ provides by convolution, a map of $\mathfrak{X}$ into itself.

$\alpha \in \mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ and $\phi \in \mathfrak{Z}$. Then by L.2.5.1., we have that $\alpha * \phi \in i(\mathfrak{X})$. Hence $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$ and $\mathfrak{Z}$ are Wiener-like. It follows from L.2.4.13. that $\mathfrak{Z}$ separates $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$, and since $\mathfrak{Z} \subseteq \mathfrak{G}$, $\mathfrak{Z}$ is endomorphic on $\mathfrak{X}$ and so also on $\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$.

<u>T.3.6.1.</u> Suppose that τ is any topology on $\mathfrak{X}$. Then for each $\phi \in \mathfrak{G}$, and each positive integer p, the maps

$$\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W}) \ni \alpha \longrightarrow \alpha * \phi \in \mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$$

$$\mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W}) \ni \alpha \longrightarrow \alpha^{(p)} \in \mathfrak{Q}(\mathfrak{X}, \mathfrak{G} \cap \mathfrak{W})$$

are continuous in the $(\tau, \mathfrak{Z})$ topology.

Proof: The result follows immediately by T.3.3.5.

<u>D.3.6.1.</u> We will denote by s the topology on $\mathfrak{X}$ generated by taking as a basis of neighbourhoods of the origin, the class consisting of all sets of the form

$$\{f \mid f \in \mathfrak{X} \text{ and } \left|\int f(t)\, \phi_j(t)dt\right| < \epsilon,\ j = 1,2,\ldots,J.\},$$

where $\epsilon > 0$, J is a positive integer and $\phi_1, \phi_2, \ldots, \phi_J$ are elements of $\mathfrak{G}$.

It may be easily verified that with this topology, $\mathfrak{X}$ is a linear topological space.

<u>D.3.6.2.</u> We will denote by z the topology on $\mathfrak{X}$ generated by taking as a basis of neighbourhoods of the origin, the class consisting of all sets of the form

$$\{f \mid f \in \mathfrak{T} \text{ and } |\int f(t)\phi_j(t)dt| < \epsilon,\ j = 1,2,\ldots,J.\},$$

where $\epsilon > 0$, J is a positive integer and $\phi_1, \phi_2, \ldots, \phi_J$ are elements of $\mathfrak{Z}$.

It may be easily verified that with this topology, $\mathfrak{T}$ is a linear topological space.

N.3.6.1. Apart from s and z, it is clear that (for each $a \in \mathbb{R}$) no two of the σ, σ^a, s, z topologies, on $\mathfrak{T}$, are identical. To prove that s and z are distinct, take N to be a set of the form

$$N = \{f \mid f \in \mathfrak{T} \text{ and } |\int f(t)\phi(t)dt| < \epsilon\},$$

where $\phi \in \mathfrak{S} \cap \mathfrak{W}$. Then N is an s-neighbourhood of 0. Suppose that N has a subset N' of the form

$$N' = \{f \mid f \in \mathfrak{T} \text{ and } |\int f(t)\psi_j(t)dt| < \delta,\ j = 1,2,\ldots,J\},$$

where $\delta > 0$ and $\psi_1, \psi_2, \ldots, \psi_J$ are elements of $\mathfrak{Z}$. Choose $A > 0$ such that $[-A,A]$ includes the support of each $\hat{\psi}_j(t)$, $j = 1,2,\ldots,J$. Take $g(t) \in \mathfrak{S}$, not identically zero, such that $\hat{g}(t) = 0$ on $[-A,A]$. Then for each $x \in \mathbb{R}$, and each $j = 1,2,\ldots,J$, we have that

$$\int g(x-t)\psi_j(t)dt = 0.$$

And so, $\forall\lambda \in \mathbb{C}$, $\forall x \in \mathbb{R}$, we have that $\lambda g(x-t) \in N'$. Hence $\forall\lambda \in \mathbb{C}$, $\forall x \in \mathbb{R}$, $\lambda g(x-t) \in N$. Therefore

$$|\lambda \int g(x-t)\phi(t)dt| < \epsilon \quad \forall \lambda \in \mathbb{C}, \forall x \in \mathbb{R}.$$

It follows that for every $x \in \mathbb{R}$,

$$\int g(x-t)\phi(t)dt = 0.$$

Hence, by T.2.2.1., we have that $g(x) = 0 \quad \forall x \in \mathbb{R}$, which is a contradiction. It follows that N is not a z-neighbourhood of $0 \in \mathfrak{T}$.

T.3.6.2. The following topologies on $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ are compatible with its linear structure, and are identical

1) $(\sigma, \mathfrak{Z})$, 3) $(s, \mathfrak{Z})$,

2)† $(\sigma^a, \mathfrak{Z})$, 4) $(z, \mathfrak{Z})$.

Proof: It follows at once that with each of the above topologies, $\mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$ is a linear topological space. Denote by $\mathfrak{N}_\tau$ the class of τ-neighbourhoods of $0 \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$. We have at once that

$$U \in \mathfrak{N}_{(\sigma^a, \mathfrak{Z})} \Rightarrow U \in \mathfrak{N}_{(\sigma, \mathfrak{Z})},$$

and

$$U \in \mathfrak{N}_{(z, \mathfrak{Z})} \Rightarrow U \in \mathfrak{N}_{(s, \mathfrak{Z})}.$$

To complete the proof we will show

a) $U \in \mathfrak{N}_{(\sigma, \mathfrak{Z})} \Rightarrow U \in \mathfrak{N}_{(z, \mathfrak{Z})}$

b) $U \in \mathfrak{N}_{(s, \mathfrak{Z})} \Rightarrow U \in \mathfrak{N}_{(\sigma^a, \mathfrak{Z})}$.

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† for any $a \in \mathbb{R}$.

a) Choose $U \in \mathfrak{N}(\sigma, \mathfrak{Z})$, then U has a subset V of the form

$$V = \{\beta \mid \beta \in \mathfrak{L}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \phi_j) \in M_j,$$
$$j = 1,2,\ldots,J.\},$$

where each M_j is a σ-neighbourhood of $0 \in \mathfrak{T}$ and each ϕ_j is an element of $\mathfrak{Z}$. Put $M = \bigcap_{j=1}^{J} M_j$, then M is a σ-neighbourhood of $0 \in \mathfrak{T}$. Hence M has a subset N of the form

$$N = \{f \mid f \in \mathfrak{T} \text{ and } |f(x_k)| < \epsilon,\ k = 1,2,\ldots,K.\}.$$

Put

$$U_j = \{\beta \mid \beta \in \mathfrak{L}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \phi_j) \in N\}$$
$$= \{\beta \mid \beta \in \mathfrak{L}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } |(i^{-1}(\beta * \phi_j))(x_k)| < \epsilon,$$
$$k = 1,2,\ldots,K.\}.$$

Then $\bigcap_{j=1}^{J} U_j \subseteq U$. Put $\psi_k^j(z) = \phi_j(x_k+z)$, for each j and k. Take $\phi \in \mathfrak{Z}$ such that $\hat{\phi}(t) = 1$ on the support of $\hat{\phi}_j(-t)$, $j = 1,2,\ldots,J.$[†] We have that for each j and k,

$$(i^{-1}(\beta * \phi_j))(x_k) = \int (i^{-1}(\beta * \psi_k^j))(t)\,\phi(t)\,dt.$$

Put

$$L = \{f \mid f \in \mathfrak{T} \text{ and } \left|\int f(t)\phi(t)\,dt\right| < \epsilon.\},$$

then L is a z-neighbourhood of $0 \in \mathfrak{T}$.

Put

[†] That this is possible is proved in chapter four.

$$W_j = \{\beta | \beta \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \psi_k^j) \in L,$$
$$k = 1,2,\ldots,K.\},$$

$$W = \bigcap_{j=1}^{J} W_j,$$

then W_j is a $(z, \mathfrak{Z})$-neighbourhood of 0, and so is W. If $\beta \in W$, then $i^{-1}(\beta * \psi_k^j) \in L$, $k = 1,2,\ldots,K$, $j = 1,2,\ldots,J$, and so

$$|(i^{-1}(\beta * \phi_j))(x_k)| < \epsilon, \; k = 1,2,\ldots,K, \; j = 1,2,\ldots,J.$$

Hence $\beta \in U$, and it follows that $W \subseteq U$ and so U is a $(z, \mathfrak{Z})$-neighbourhood of $0 \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W})$.

b) Choose $U \in \mathfrak{N}(s, \mathfrak{Z})$, then U has a subset V of the form

$$V = \{\beta | \beta \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \phi_j) \in M_j,$$
$$j = 1,2,\ldots,J.\},$$

where each ϕ_j is an element of $\mathfrak{Z}$ and each M_j is an s-neighbourhood of $0 \in \mathfrak{T}$. Put $M = \bigcap_{j=1}^{J} M_j$, then M is an s-neighbourhood of $0 \in \mathfrak{T}$ and so it has a subset N of the form

$$N = \{f | f \in \mathfrak{T} \text{ and } |\int f(t)\psi_k(t)dt| < \epsilon, \; k = 1,2,\ldots,K.\},$$

where each $\psi_k \in \mathfrak{S}$. Put

$$U_j = \{\beta \mid \beta \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \phi_j) \in N\}$$

$$= \{\beta \mid \beta \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and }$$

$$\left|\int (i^{-1}(\beta * \phi_j))(t)\psi_k(t)dt\right| < \epsilon,\ k = 1,2,\ldots,K.\}.$$

Then $\bigcap_{j=1}^{J} U_j \subseteq U$, and each U_j is an $(s, \mathfrak{Z})$-neighbourhood of 0. Put

$$\chi_{j,k}(z) = \int \phi_j(z-u)\psi_k(a-u)du,$$

then $\chi_{j,k} \in \mathfrak{Z}$, $j = 1,2,\ldots,J$ and $k = 1,2,\ldots,K$. Furthermore we have that for each j,k,

$$\int (i^{-1}(\beta * \phi_j))(t)\psi_k(t)dt = (i^{-1}(\beta * \chi_{j,k}))(a).$$

Put

$$L = \{f \mid f \in \mathfrak{T} \text{ and } |f(a)| < \epsilon\},$$

then L is a σ^a-neighbourhood of $0 \in \mathfrak{T}$.

Put

$$W_j = \{\beta \mid \beta \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \chi_{j,k}) \in L,$$
$$k = 1,2,\ldots,K\}$$

$$W = \bigcap_{j=1}^{J} W_j,$$

then W is a $(\sigma^a, \mathfrak{Z})$-neighbourhood of 0. If $\beta \in W$, then

$$\left|\int (i^{-1}(\beta * \phi_j))(t)\psi_k(t)dt\right| < \epsilon,\ j = 1,2,\ldots,J,$$
$$k = 1,2,\ldots,K.$$

Hence $\beta \in U$, and it follows that $W \subseteq U$ and so U is a

$(\sigma^a, \mathfrak{Z})$-neighbourhood of $0 \in \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$.

This completes the proof of the theorem.

With the $(\sigma, \mathfrak{Z})$ topology, it follows immediately that $\mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$ is an Hausdorff linear topological space. (See L.3.6.1., T.3.3.3. and T.3.3.4.) In addition, from T.3.6.1., we have that for each $\phi \in \mathfrak{S}$ and each positive integer p, the maps

$$\mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}) \ni \alpha \longrightarrow \alpha * \phi \in \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$$

$$\mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}) \ni \alpha \longrightarrow a^{(p)} \in \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$$

are continuous.

<u>T.3.6.3.</u> The map

$$\Theta : \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}) \longleftrightarrow \mathfrak{D}\mathfrak{Z}'$$

provide an homeomorphism of $\mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$, with the $(\sigma, \mathfrak{Z})$ topology, and $\mathfrak{D}\mathfrak{Z}'$.

Proof: We have shown in T.2.5.1., that Θ is a linear bijection of $\mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$ onto $\mathfrak{D}\mathfrak{Z}'$. Suppose that V is any neighbourhood of 0 in the topology of $\mathfrak{D}\mathfrak{Z}'$. Then V has a subset of the form

$$\{F \mid F \in \mathfrak{D}\mathfrak{Z}' \text{ and } |\langle F, \phi_r\rangle| < \epsilon,\ r = 1,2,\ldots,R.\},$$

where $\epsilon > 0$ and $\phi_1, \phi_2, \ldots, \phi_R$ are elements of $\mathfrak{Z}$. Put

$$M = \{f \mid f \in \mathfrak{T} \text{ and } |f(0)| < \epsilon\},$$

$$\psi_r(z) = \phi_r(-z), \quad r = 1,2,\ldots,R,$$

$$N = \{\beta \mid \beta \in \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}) \text{ and } i^{-1}(\alpha * \psi_r) \in M, \quad r = 1,2,\ldots,R.\}.$$

Then N is a $(\sigma, \mathfrak{Z})$ neighbourhood of $0 \in \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$. If $\beta \in N$, we have that

$$|(i^{-1}(\beta * \psi_r))(0)| < \epsilon, \quad r = 1,2,\ldots,R.$$

that is,

$$|\langle\Theta(\beta), \phi_r\rangle| < \epsilon, \quad r = 1,2,\ldots,R.$$

Hence if $\beta \in N$, then $\Theta(\beta) \in V$, i.e. $\Theta(N) \subseteq V$. It follows that with respect to the $(\sigma, \mathfrak{Z})$ topology on $\mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$, the map

$$\Theta : \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}) \to \mathfrak{D}\mathfrak{Z}'$$

is continuous.

We next show that Θ^{-1} is continuous. Suppose that N is a $(\sigma, \mathfrak{Z})$-neighbourhood of $0 \in \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M})$. Then by the previous theorem, N has a subset N' of the form

$$N' = \{\beta \mid \beta \in \mathfrak{D}(\mathfrak{T}, \mathfrak{S} \cap \mathfrak{M}) \text{ and } i^{-1}(\beta * \psi_r) \in M_r, \quad r = 1,2,\ldots,R.\},$$

where for each r, $\psi_r \in \mathfrak{Z}$ and M_r is a z-neighbourhood of 0. Put $M = \bigcap_{r=1}^{R} M_r$, then M is a z-neighbourhood of 0. Hence

M has a subset M' of the form

$$M' = \{f \mid f \in \mathfrak{T} \text{ and } |\int f(t)\chi_j(t)dt| < \epsilon,\ j = 1,2,\ldots,J\},$$

where $\epsilon > 0$ and $\chi_1, \chi_2, \ldots, \chi_J$ are elements of $\mathfrak{Z}$. And so if

$$N_r = \{\beta \mid \beta \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W}) \text{ and } i^{-1}(\beta * \psi_r) \in M'\},$$

then for each r, N_r is a $(z, \mathfrak{Z})$-neighbourhood of 0 and therefore a $(\sigma, \mathfrak{Z})$ neighbourhood of $0 \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$. Put $N'' = \bigcap_{r=1}^{R} N_r$, then N'' is a $(\sigma, \mathfrak{Z})$ neighbourhood of $0 \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ and $N'' \subseteq N$. For $r = 1,2,\ldots,R$ and $j = 1,2,\ldots,J$, put

$$\phi_{r,j}(z) = \int \psi_r(u-z)\chi_j(u)du,$$

then $\phi_{r,j}(z) \in \mathfrak{Z}$. Put

$$V_r = \{F \mid F \in \mathfrak{Q}\mathfrak{Z}' \text{ and } |\langle F, \phi_{r,j}\rangle| < \epsilon,\ j = 1,2,\ldots,J.\}.$$

Then $V = \bigcap_{r=1}^{R} V_r$ is a neighbourhood of 0 in the topology of $\mathfrak{Q}\mathfrak{Z}'$. Moreover if $F \in V$ and $\alpha \in \mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ is such that $\Theta(\alpha) = F$, then for each r, j,

$$|\langle\Theta(\alpha), \phi_{r,j}\rangle| < \epsilon,$$

and so

$$|\int (i^{-1}(\alpha * \psi_r))(t)\chi_j(t)dt| < \epsilon.$$

Hence $i^{-1}(\alpha * \psi_r) \in M'$, $r = 1,2,\ldots,R$, and so $\alpha \in N''$. It follows that $\Theta^{-1}(V) \subseteq N$. and so with respect to the

$(\sigma, \mathfrak{Z})$ topology, the map

$$\theta^{-1} : \mathfrak{L}\mathfrak{Z}' \to \mathfrak{L}(\mathfrak{X}, \mathfrak{S} \cap \mathfrak{B})$$

is continuous.

The result now follows.

INTRODUCTION TO CHAPTER 4

This chapter is concerned with a generalisation (T.4.2.1.) of a theorem of Wiener's, and with connections between this result and the material of the preceeding chapters.

The proof of theorem T.4.2.1. depends on a theorem of Ingham's concerning functions with compactly supported Fourier transforms, and in section 4.1. a proof of Ingham's theorem is given. Also in this section we prove a variant (T.4.1.3.) of Ingham's theorem, and a theorem (T.4.1.2.) concerning spaces of type $\mathfrak{S}$. The proof given of T.4.1.1. is essentially Ingham's original proof. Theorems T.4.1.2. and T.4.1.3. are believed to be new.

The conditions involved in Ingham's theorem lead rather naturally to a function class Λ, defined in section 4.2. It turns out that, for every $\lambda(t) \in \Lambda$,

$$\lambda(a + b) \leq \lambda(a) + \lambda(b)$$

for every real a and b, and because of this inequality it follows that the class, $\mathfrak{L}_\lambda^o$, of functions $f(x)$ such that $f(x)\exp[\lambda(x)] \in L^1$, is closed under convolution. Furthermore Ingham's theorem guarantees the existence of functions $f(x) \in \mathfrak{L}_\lambda^o$ whose Fourier transforms are compactly supported. It is particularly on these two properties of $\mathfrak{L}_\lambda^o$ that the proof of theorem T.4.2.1. depends. We prove, in addition to theorem T.4.2.1., a number of results of a similar type: theorems T.4.2.2., T.4.2.4. and T.4.2.5. Theorem T.4.2.3. is a partial converse of T.4.2.1. Section 4.2. concludes with L.4.2.8. and T.4.2.6., which provide the basic link with the material of the previous chapters. All the main results of this section are believed to be new.

In section 4.3. connections with the first three chapters are developed by means of a number of examples of quotient pairs and Wiener-style theorems. One of these examples concerns the spaces $\mathfrak{S}_\beta$ $(\beta > 1)$ considered by Gel'fand and Shilov in (10). To conclude, the existence is demonstrated of an extensive family of quotient pairs based on the function spaces $\mathfrak{L}^o_\lambda$ introduced in section 4.2. All the results of section 4.3. are believed to be new.

CHAPTER 4

4.1. Existence theorems

T.4.1.1. Ingham's Theorem[†]

Suppose that λ is a well-defined, non-negative even function of a real variable t, and is such that $\lambda(t)/t$ decreases monotonically to zero as $t \to \infty$. Suppose also that $\ell > 0$. Then there exists an integrable function f, not identically zero, such that $\hat{f}$ is supported by $[-\ell, \ell]$ and for which there exists a positive constant A such that

$$|f(x)| \leq A \exp[-\lambda(x)] \quad \forall x \in \mathbb{R} ,$$

if and only if the integral

$$\int_1^\infty \frac{\lambda(x)}{x^2}\, dx$$

converges.

Proof: 1) Suppose that $\int_1^\infty \frac{\lambda(x)}{x^2} dx$ converges. Put $\mu(x) = \lambda(x) + |x|^{1/2}$, so that $\int_1^\infty \frac{\mu(x)}{x^2} dx$ converges, and μ is an even non-negative function such that $\mu(t)/t$ decreases monotonically to zero as $t \to \infty$.

We have that the sum $\sum_{n=1}^{\infty} \mu(n)/n^2$ converges, and so there exists a positive integer n_0, such that

$$\sum_{n=n_0+1}^{\infty} \frac{e\mu(n)}{n^2} < \frac{\ell}{2} ,$$

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† See (12).

and

$$\frac{e\mu(n_o)}{n_o} < \frac{\ell}{2} .$$

Define

$$\rho_n = \begin{cases} \dfrac{e\mu(n_o)}{n_o^{\,2}} & \text{for } n = 1,2,\ldots,n_o, \\[2ex] \dfrac{e\mu(n)}{n^2} & \text{for } n = n_o+1, n_o+2, n_o+3, \ldots . \end{cases}$$

Then we have that

$$\sum_{n=1}^{\infty} \rho_n < \ell .$$

For $N = 1,2,3,\ldots,$ put

$$f_N(y) = \begin{cases} \prod\limits_{n=1}^{N} \dfrac{\text{Sin}\rho_n y}{\rho_{n} y} & \text{if } y \neq 0 \\[2ex] 1 & \text{if } y = 0. \end{cases}$$

Since $(\text{Sin}x)/x = 1 + O(|x|)$ as $x \to 0$, and since $\sum_{n=1}^{\infty} \rho_n$ is convergent, we have that in any finite interval the sequence $\{f_N(y)\}$ converges uniformly to a function f, which is even, not identically zero, and continuous. Moreover if $N \geq 2$, then

$$|f_N(x)| \leq \begin{cases} 1 & \text{if } |x| < 1 \\ \dfrac{1}{\rho_1\rho_2 x^2} & \text{if } |x| \geq 1. \end{cases}$$

Hence $f(x)$ is integrable and

$$\hat{f}_N(t) \to \hat{f}(t) \quad \text{as} \quad N \to \infty,$$

uniformly in $t \in \mathbb{R}$. Now for each $N \geq 2$, $\hat{f}_N$ is supported by $[-\sum_{n=1}^{N} \rho_n, \sum_{n=1}^{N} \rho_n]$; hence we have that $\hat{f}$ is supported by $[-\sum_{n=1}^{\infty} \rho_n, \sum_{n=1}^{\infty} \rho_n]$, and so also by $[-\ell, \ell]$.

Since $\mu(x) \to \infty$ as $x \to \infty$ and $\mu(x)/x \to 0$ as $x \to \infty$, there exists X such that for every $x \geq X$,

$$[\mu(x)] > n_o ,$$

and

$$[\mu(x)] < x .$$

Take $x > X$, and put $m = [\mu(x)]$. Then we have that

$$\begin{aligned} |f(x)| &\leq \prod_{n=1}^{m} \frac{1}{\rho_n x} \\ &\leq \left(\frac{1}{\rho_m x}\right)^m \\ &= \left(\frac{m^2}{e\mu(m)x}\right)^m \\ &\leq \left(\frac{m\mu(x)}{e\mu(m)x}\right)^m \\ &\leq e^{-m} \\ &< e^{-\mu(x)+1}. \end{aligned}$$

Since $f(x)$ is even, the result follows.

2) Suppose that $\int_1^\infty \frac{\lambda(x)}{x^2}dx$ does not converge. Put $\mu(x) = \lambda(x) + 2|x|^{1/2}$, so that $\int_1^\infty \frac{\mu(x)}{x^2}dx$ does not converge. Suppose that there exists an integrable function $f(x)$, not identically zero, and a positive constant A such that $\hat{f}$ is supported by $[-\ell,\ell]$, and

$$|f(x)| < A \exp[-\mu(x)] \quad \forall x \in \mathbb{R} \quad .$$

Then we have that

$$|f(x)| < A \exp[-2|x|^{1/2}] \quad \forall x \in \mathbb{R} \quad .$$

Hence for each positive integer n, and each real t,

$$\exists\, \hat{f}^{(n)}(t) = \int (ix)^n f(x) e^{ixt} dx.$$

It follows that

$$\begin{aligned} |\hat{f}^{(n)}(t)| &\leq \int |x|^n |f(x)|\, dx \\ &\leq 2A \int_0^\infty x^n \exp[-\mu(x)]dx. \end{aligned}$$

For each positive integer n, put

$$A_n = \int_0^\infty x^n \exp[-\mu(x)]dx.$$

We have that

$$A_n \le \int_0^{n^4} x^n \exp[-x\frac{\mu(n^4)}{n^4}]dx + \int_{n^4}^{\infty} x^n \exp[-2(x)^{1/2}]dx$$

$$\le n^4 \int_0^{n^4} x^{n-1}\exp[-x\frac{\mu(n^4)}{n^4}]dx + \int_{n^2}^{\infty} 2y^{2n+1}\exp[-2y]dy$$

$$< n^4\left(\frac{n^4}{\mu(n^4)}\right)^n \int_0^{\infty} y^{n-1}\exp[-y]dy + 2e^{-n^2}\int_0^{\infty} y^{2n+1}\exp[-y]dy$$

$$= n^4\left(\frac{n^4}{\mu(n^4)}\right)^n (n-1)! + 2e^{-n^2}(2n+1)!$$

$$< 2n^4\left(\frac{n^4}{\mu(n^4)}\right)^n (n-1)! \ , \quad \text{for all suitably large } n,$$

$$\le 2n^3\left(\frac{n^5}{\mu(n^4)}\right)^n$$

$$< \left(\frac{2n^5}{\mu(n^4)}\right)^n , \quad \text{for all suitably large } n.$$

Now since $\int_1^{\infty} \frac{\mu(x)}{x^2}\, dx$ is divergent, so also is $\int_1^{\infty} \frac{\mu(x^4)}{x^5}dx$, and therefore the sum $\sum_{n=1}^{\infty} \frac{\mu(n^4)}{n^5}$ diverges. It follows that $\sum_{n=1}^{\infty} (A_n)^{-\frac{1}{n}}$ is divergent. Hence, using the theory of quasi-analytic functions[†], since $\hat{f}(x)$ is zero outside $[-\ell,\ell]$, it is also zero throughout $[-\ell,\ell]$, which implies

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[†] See (4) and (7).

that f is identically zero. But this is a contradiction. Hence there is no integrable function f, not identically zero, whose transform is compactly supported, and for which there exists a positive constant A such that

$$|f(x)| \leq A \exp[-\mu(x)] \quad \forall x \in \mathbb{R} .$$

Suppose however that there exists an integrable function g, and a positive constant B such that $\hat{g}$ is s supported by $[-\ell,\ell]$, and

$$|g(x)| \leq B \exp[-\lambda(x)] \quad \forall x \in \mathbb{R} .$$

From part 1) of this proof, we may choose an integrable function h, not identically zero, whose transform is supported by $[-\ell,\ell]$, and for which there exists a positive constant C such that

$$|h(x)| \leq C \exp[-2|x|^{1/2}].$$

Put

$$F(t) = \int \hat{g}(u)\hat{h}(t-u)du.$$

Then F(t) is supported by $[-2\ell,2\ell]$. Moreover

$$\hat{F}(-x) = 4\pi^2 g(x)h(x),$$

and so if we put

$$f(x) = \frac{1}{2\pi} \hat{F}(-x),$$

we have that $f(x)$ is integrable and that $\hat{f}(t)$ is compactly supported. For each real x,

$$|f(x)| \leq 2\pi\ B\ C\ \exp[-\lambda(x) - 2|x|^{1/2}]$$
$$= 2\pi\ B\ C\ \exp[-\mu(x)].$$

Consequently $f(x)$ is identically zero. Hence $(g(x)h(x))$ is also null, and since $h(x)$ has only a countable number of zeros, it follows that $g(x)$ is null. Hence the result.

<u>Note on spaces of type $\mathfrak{S}'$.</u>

I.M. Gel'fand and G.E. Shilov have in (10) considered at some length spaces of type $\mathfrak{S}$. Using their notation†, we have that‡

$$\bigcap_{\beta>1} \mathfrak{S}^{o}_{\beta} = \{f \mid \hat{f} \in \mathfrak{D}\text{ , and } \forall\alpha < 1,\ \forall p = 0,1,2,\ldots,$$

$$\exists C_{\alpha,p} \text{ such that for all real } x,$$

$$|f^{(p)}(x)| < C_{\alpha,p}\exp[-|x|^{\alpha}].\}.$$

They show in (10) that, for each $\beta > 1$, $\mathfrak{S}^{o}_{\beta}$ contains a non-null function. However, using Ingham's theorem, we can easily obtain the following stronger result:

<u>T.4.1.2.</u> The intersection

$$\bigcap_{\beta>1} \mathfrak{S}^{o}_{\beta}$$

†See the appendix for details, and also (10) Pg.166 et seq.

‡This is shown in the appendix.

contains a non-null function.

Proof: Put

$$\mathcal{K}(t) = \begin{cases} \dfrac{|t|}{(\log|t|)^2} & \text{if } |t| \geq e^2 \\ |t|/4 & \text{if } |t| < e^2. \end{cases}$$

Then, using Ingham's theorem, we have that there exists a non-null function, f, such that $\hat{f}$ is compactly supported, and such that

$$f(t) = O(\exp[-\mathcal{K}(t)]) \quad \text{as } t \to \pm\infty.$$

Choose any non-null $\mathcal{D}$-function, χ. Then $f * \chi$ is non-null and for each non-negative integer p,

$$\exists\, (f * \chi)^{(p)}(x) = (f * \chi^{(p)})(x), \text{ by L.2.1.4.,}$$
$$= O(\exp[-\mathcal{K}(x)]) \text{ as } x \to \pm\infty\ ^{\dagger}.$$

Hence $f * \chi \in \bigcap_{\beta>1} \mathfrak{S}^{0}_{\beta}$.

<u>T.4.1.3.</u> Suppose that λ is a well defined non-negative even function of a real variable t, and is such that $\lambda(t)/t$ decreases monotonically to zero as $t \to \infty$. Suppose also that $\ell > 0$. Then there exists an integrable function g, not identically zero such that $\hat{g}$ is supported by $[-\ell, \ell]$ and such that

$$g(x)\exp[\lambda(x)] \in L^1,$$

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† For justification see the proof of L.4.2.3.

if and only if the integral

$$\int_1^\infty \frac{\lambda(x)}{x^2}\,dx$$

converges.

Proof: 1) Suppose that the integral $\int_1^\infty \frac{\lambda(x)}{x^2}\,dx$ exists. Then by Ingham's theorem, there exists an integrable function g, not identically zero, and a positive constant A such that $\hat{g}$ is supported by $[-\ell,\ell]$ and

$$|g(x)| \leq A \exp[-\lambda(x) - |x|^{1/2}].$$

But then $(g(x)\exp[\lambda(x)] \in L^1$, and the result follows.

2) Assume conversely that the integral does not converge. Suppose in addition that there exists a non-null function h, such that

$$h(x)\exp[\lambda(x)] \in L^1,$$

and $\hat{h}$ is supported by $[-\ell,\ell]$. Put $\Phi(x) = h(x)\exp[\lambda(x)]$. Select any $\chi \in \mathcal{D}$, not identically zero. Then $\hat{\chi}$ has only a countable number of zeros. Take $B > 0$ such that $[-B,B]$ includes the support of χ.

Put

$$g(x) = (h * \chi)(x),$$

then $g \in L^1$, and $\hat{g} = \hat{h}\hat{\chi}$, so that $\hat{g}$ is supported by $[-\ell,\ell]$, and g is not identically zero.

Define

$$\mu(x) = \frac{x\lambda(x+B)}{2(x+B)} \quad \text{for } x \geq 0,$$

and put $\mu(x) = \mu(-x)$ for $x < 0$. Then μ is an even, non-negative function, and clearly $\mu(x)/x$ is monotonic decreasing to zero as $x \to \infty$. Moreover $\int_1^\infty \frac{\mu(x)}{x^2}\, dx$ does not exist, for if it did then $\int_1^\infty \frac{\lambda(x+B)}{x(x+B)}$ would also exist, and so therefore would $\int_1^\infty \frac{\lambda(x)}{x^2}\, dx$.

Suppose $|x| \geq 2B$ and $|t| \leq B$. Then we have that

$$|x-t| \leq |x| + B,$$

and so

$$\frac{\lambda(x-t)}{|x-t|} \geq \frac{\lambda(|x|+B)}{|x|+B}$$

$$= \frac{2\mu(x)}{|x|} .$$

Hence

$$\lambda(x-t) \geq 2\mu(x) \left|1 - \frac{t}{x}\right|,$$

and since

$$\left|1 - \frac{t}{x}\right| = 1 - \left|\frac{t}{x}\right|$$

$$\geq \frac{1}{2},$$

we have that for $|x| \geq 2B$ and $|t| \leq B$,

$$\lambda(x-t) \geq \mu(x).$$

It follows that if $|x| \geq 2B$, then

$$|g(x)| \leq \int_{-B}^{B} |\chi(t)|\,|\phi(x-t)| \exp[-\lambda(x-t)]dt$$

$$\leq \exp[-\mu(x)] \int_{-B}^{B} |\chi(t)|\,|\phi(x-t)|dt$$

$$\leq \exp[-\mu(x)]\{\sup_{t\in\mathbb{R}} |\chi(t)|\}\int |\phi(u)|du.$$

But, by Ingham's theorem, g can satisfy no such inequality. The result follows.

4.2. A generalisation of Wiener's theorem†

The main result of this section is T.4.2.1.

D.4.2.1. We will denote by Λ the class of functions λ, of a real variable x, such that

a) $\lambda(x) = \lambda(-x) \quad \forall x \in \mathbb{R}$,

b) $\lambda(x) \geq 0 \quad \forall x \in \mathbb{R}$,

c) for $x \geq 0$, $\lambda(x)$ is monotonic increasing,

d) for $x > 0$, $\lambda(x)/x$ is monotonic decreasing and tends to zero as x tends to infinity,

e) the integral $\int_1^\infty \frac{\lambda(x)}{x^2}\,dx$ exists.

N.4.2.1. Conditions c) and d) are sufficient to ensure that Λ contains only functions which are, excepting

† See (24) for the original version which relates to Fourier series. The generalisation given here deals with Fourier transforms, see (11) section 9J. For other types of generalisation see (17) chapters 6 and 7.

possibly $x = 0$, everywhere continuous.

L.4.2.1. If $\lambda \in \Lambda$, and $a,b \in \mathbb{R}$, then

$$\lambda(a+b) \leq \lambda(a) + \lambda(b).$$

Proof: Suppose firstly that $a,b > 0$, then we have that

$$\frac{a\lambda(a+b)}{a+b} \leq \lambda(a),$$

and

$$\frac{b\lambda(a+b)}{a+b} \leq \lambda(b).$$

Hence in this case $\lambda(a+b) \leq \lambda(a) + \lambda(b)$. Since λ is even, if $a,b < 0$, we have again that $\lambda(a+b) \leq \lambda(a) + \lambda(b)$.

Since $\lambda(x)$ is monotonic increasing for $x \geq 0$, we have that if $a > 0$ and $b < 0$, then

$$\begin{aligned} \lambda(a+b) &\leq \lambda(a-b) \\ &\leq \lambda(a) + \lambda(-b) \\ &= \lambda(a) + \lambda(b). \end{aligned}$$

Finally if $b = 0$, then for any a, we have that

$$\begin{aligned} \lambda(a+b) &= \lambda(a) \\ &\leq \lambda(a) + \lambda(0) \\ &= \lambda(a) + \lambda(b). \end{aligned}$$

The result follows.

Extensive use will be made of this lemma throughout the remainder of chapter 4.[†]

D.4.2.2. For each $\lambda \in \Lambda$, we define the function classes $\mathfrak{B}^{o}_{\lambda}$, $\mathfrak{B}_{\lambda}$, $\mathfrak{L}^{o}_{\lambda}$, $\mathfrak{L}_{\lambda}$ as follows

a) $\mathfrak{B}^{o}_{\lambda} = \{f \mid f(x)$ is measurable and $(f(x)\exp[\lambda(x)])$ is bounded on $\mathbb{R}$.$\}$

b) $\mathfrak{B}_{\lambda} = \{f \mid f$ has continuous derivatives of all orders, and for each non-negative integer p, $f^{(p)} \in \mathfrak{B}^{o}_{\lambda}$.$\}$

c) $\mathfrak{L}^{o}_{\lambda} = \{f \mid (f(x)\exp[\lambda(x)]) \in L^{1}.\}$

d) $\mathfrak{L}_{\lambda} = \{f \mid f$ has continuous derivatives of all orders, and for each non-negative integer p, $f^{(p)} \in \mathfrak{L}^{o}_{\lambda}$.$\}$

L.4.2.2. If $\lambda(t) \in \Lambda$ and $\alpha \in \mathbb{R}$, then $\lambda(\alpha t) \in \Lambda$.

Proof: This is easily verified.

L.4.2.3. If $\lambda \in \Lambda$, then there exists $f \in \mathfrak{B}_{\lambda} \cap L^{1}$, not identically zero, and such that $\hat{f} \in \mathfrak{D}$, $\hat{f}$ is supported by $[-1,1]$, and $\hat{f}(t) \geq 0 \quad \forall t \in \mathbb{R}$.

Proof: Put $\mu(t) = \lambda(t) + |t|^{1/2}$, so that $\mu \in \Lambda$ and $\mathfrak{B}^{o}_{\mu} \subseteq L^{1}$. By Ingham's theorem, there exists a function $g \in \mathfrak{B}^{o}_{\mu}$, not identically zero such that $\hat{g}$ is supported by $[-1,1]$. Select $\mathcal{X} \in \mathfrak{D}$, not identically zero, and put

$$h(x) = (g * \mathcal{X})(x).$$

Then $h(t)$ possesses continuous derivatives of all orders,

[†] It is interesting to note that if μ is a locally bounded function such that $\mu(x+y) \leq \mu(x) + \mu(y)$ for every $x,y > 0$, then $\exists \lim_{x \to \infty} \mu(x)/x = \operatorname{Inf}_{x>0} \mu(x)/x$.

and for each $p = 0,1,2,\ldots,$

$$h^{(p)}(x) = (g * \chi^{(p)})(x).$$

Moreover, for each positive integer p,

$$|h^{(p)}(x)| \exp[\mu(x)] \leq \exp[\mu(x)]\int |g(x-t)|\,|\chi^{(p)}(t)|dt$$
$$\leq \int |g(x-t)| \exp[\mu(x-t)]\,|\chi^{(p)}(t)| \exp[\mu(t)]dt$$
$$\leq \{\operatorname*{Sup}_{u\in\mathbb{R}} |g(u)\exp[\mu(u)]|\} \int |\chi^{(p)}(t) \exp[\mu(t)]| dt.$$

Hence $h(x) \in \mathfrak{B}_{\mu}$. Since $\hat{\chi}$ has only a countable number of zeros, it follows that h is not identically zero. Also, $\hat{h}$ is supported by $[-1,1]$.

Put

$$f(x) = \int h(x-t)\bar{h}(-t)dt.$$

Then for each positive integer p, there exists a constant K_p such that

$$|f^{(p)}(x)| \leq K_p\int \exp[-\mu(x-t)]\exp[-\mu(t)]dt$$
$$= K_p \int \exp[-\lambda(x-t) - |x-t|^{1/2} - \lambda(t) - |t|^{1/2}]dt$$
$$\leq K_p\exp[-\lambda(x)] \int \exp[-|x-t|^{1/2} - |t|^{1/2}]dt.$$

Hence $f \in \mathfrak{B}_{\lambda}$. Moreover we have that

$$\hat{f}(t) = |\hat{h}(t)|^2,$$

so that f is not identically zero, but $\hat{f}$ is supported by $[-1,1]$ and is nowhere negative. In addition since $\hat{g}$ and $\hat{\chi}$ have continuous derivatives of all orders, so also does $\hat{f}$; hence $\hat{f} \in \mathfrak{D}$.

<u>L.4.2.4.</u> Suppose that $\lambda \in \Lambda$, and that α_1, α_2, β_1, β_2 are such that $-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < +\infty$. Then there exists $f \in \mathfrak{B}_\lambda \cap L^1$, such that $\hat{f} \in \mathfrak{D}$ and in addition $\hat{f}(t) = 1$ on $[\beta_1, \alpha_2]$, $\hat{f}(t) \geq 0$, $\hat{f}(t)$ is supported by $[\alpha_1, \beta_2]$.

Proof: For $j = 1,2$, put $A_j = 2/(\beta_j - \alpha_j)$, $B_j = -(\alpha_j + \beta_j)/(\beta_j - \alpha_j)$, so that $A_j > 0$. Define $\lambda_{A_j}(x) = \lambda(A_j x) \in \Lambda$, $j = 1,2$. For $j = 1,2$, choose $f_j \in \mathfrak{B}_{\lambda_{A_j}} \cap L^1$ such that $\hat{f}_j \in \mathfrak{D}$, $\hat{f}_j(t) \geq 0$ $\forall t \in \mathbb{R}$, $\hat{f}_j$ is supported by $[-1,1]$, and $\int \hat{f}_j(t)dt = A_j$. It follows from L.4.2.3. that such functions exist.

Consider

$$w_j(x) = \frac{1}{A_j} \exp[(ixB_j)/A_j] f_j(x/A_j), \quad j = 1,2.$$

We have that $w_j(x) \in \mathfrak{B}_\lambda \cap L^1$, and moreover

$$\hat{w}_j(t) = \hat{f}_j(A_j t + B_j),$$

so that for $j = 1,2$, $\hat{w}_j \in \mathfrak{D}$, $\hat{w}_j(t) \geq 0$ $\forall t \in \mathbb{R}$, $\hat{w}_j$ is supported by $[\alpha_j, \beta_j]$ and $\int \hat{w}_j(t)dt = 1$.

Put

$$F(t) = \int_{-\infty}^{t} \{\hat{w}_1(u) - \hat{w}_2(u)\} du.$$

Then $F \in \mathfrak{D}$. Define $f(x) = 1/2\pi\hat{F}(-x)$, so that $\hat{f}(t) = F(t)$. Then $\hat{f} \in \mathfrak{D}$ and, as may easily be verified, $\hat{f}(t) = 1$ on $[\beta_1, \alpha_2]$, $\hat{f}(t) \geq 0$, and $\hat{f}(t)$ is supported by $[\alpha_1, \beta_2]$. Moreover if $x \neq 0$,

$$f(x) = \frac{1}{ix}\{w_1(x) - w_2(x)\},$$

and so $f \in \mathfrak{B}_\lambda \cap L^1$.

D.4.2.3. For each $\lambda \in \Lambda$, choose a function of x, ${}^\lambda\theta(x)$, with the following properties:

a) ${}^\lambda\theta \in \mathfrak{B}_\lambda \cap L^1$,

b) ${}^\lambda\hat{\theta} \in \mathfrak{D}$,

c) for $|t| \leq 1$, ${}^\lambda\hat{\theta}(t) = 1$, while for $|t| \geq 2$, ${}^\lambda\hat{\theta}(t) = 0$,

d) ${}^\lambda\hat{\theta}(t) \geq 0 \quad \forall t \in \mathbb{R}$.

By the previous lemma, such a function does exist. For each positive integer n, define ${}^\lambda\theta_n(x) = \frac{1}{n}\,{}^\lambda\theta(\frac{x}{n})$ Note that for $|t| \leq \frac{1}{n}$, ${}^\lambda\hat{\theta}_n(t) = 1$, while for $|t| \geq \frac{2}{n}$, ${}^\lambda\hat{\theta}_n(t) = 0$.

L.4.2.5. Suppose that $\lambda \in \Lambda$, $\mu(t) = \lambda(t) + |t|^{1/2}$, and that $g \in \mathfrak{L}^o_\lambda$. Then, given $\epsilon > 0$, there exists a positive integer M, such that if $m > M$, then for every real u,

$$\int \exp[\lambda(\tfrac{x}{m})]|{}^\mu\theta_m(x)\hat{g}(u) - \int {}^\mu\theta_m(x-t)g(t)e^{iut}dt|dx < \epsilon.$$

Proof: Put

$$I_m = \sup_{u\in\mathbb{R}} \int \exp[\lambda(\tfrac{x}{m})]\,|\,{}^{\mu}\theta_m(x)\hat{g}(u) - \int {}^{\mu}\theta_m(x-t)g(t)\, e^{iut}dt\,|\,dx.$$

Then for each positive integer m, we have that

$$\begin{aligned}
I_m &= \sup_{u\in\mathbb{R}} \int \exp[\lambda(\tfrac{x}{m})]\,|\int \{{}^{\mu}\theta_m(x) - {}^{\mu}\theta_m(x-t)\}g(t)e^{iut}dt\,|\,dx \\
&\le \int \exp[\lambda(\tfrac{x}{m})]\{\int |\,{}^{\mu}\theta_m(x) - {}^{\mu}\theta_m(x-t)|\,|\,g(t)|\,dt\}dx \\
&= \int |g(t)|\,\{\int |\,{}^{\mu}\theta_m(x) - {}^{\mu}\theta_m(x-t)|\exp[\lambda(\tfrac{x}{m})]dx\}dt \\
&= J_m, \quad \text{say.}
\end{aligned}$$

(The inversion of the above double integral being justified without difficulty by the use of Fubini's theorem.)

Put

$$\begin{aligned}
K_m(t) &= \int |\,{}^{\mu}\theta_m(x) - {}^{\mu}\theta_m(x-t)|\exp[\lambda(\tfrac{x}{m})]dx \\
&= \int |\,{}^{\mu}\theta\ (y) - {}^{\mu}\theta\ (y - \tfrac{t}{m})|\exp[\lambda(y)]dy.
\end{aligned}$$

We have that for some constant A,

$$|\,{}^{\mu}\theta(y)|\exp[\lambda(y)] \le A\ \exp[-|y|^{1/2}],$$

and

$$|\,{}^{\mu}\theta(y - \tfrac{t}{m})|\exp[\lambda(y)] \le A\ \exp[-|y - \tfrac{t}{m}|^{1/2}]\exp[\lambda(\tfrac{t}{m})],$$

the latter since $\lambda(y) \le \lambda(y - \frac{t}{m}) + \lambda(\frac{t}{m})$. Since $|t|^{1/2} \in \Lambda$, we have that

$$|y|^{1/2} \leq |y - \tfrac{t}{m}|^{1/2} + |\tfrac{t}{m}|^{1/2},$$

and so

$$|{}^{\mu}\theta(y - \tfrac{t}{m})| \exp[\lambda(y)] \leq A \exp[-|y|^{1/2}] \exp[\lambda(\tfrac{t}{m}) + |\tfrac{t}{m}|^{1/2}]$$

$$\leq A \exp[-|y|^{1/2}] \exp[\lambda(t) + |t|^{1/2}].$$

Moreover since ${}^{\mu}\theta$ is continuous, we have that, for each real t and y,

$$|{}^{\mu}\theta(y) - {}^{\mu}\theta(y - \tfrac{t}{m})| \exp[\lambda(y)] \to 0 \quad \text{as} \quad m \to \infty.$$

Hence using Lebesgue's convergence theorem, we can deduce that for each t,

$$K_m(t) \to 0 \quad \text{as} \quad m \to \infty.$$

Now, we have that

$$K_m(t) \leq \int |{}^{\mu}\theta(y)| \exp[\lambda(y)]dy + \int |{}^{\mu}\theta(y - \tfrac{t}{m})| \exp[\lambda(y)]dy$$

$$\leq A \int \exp[-|y|^{1/2}]dy + \int |{}^{\mu}\theta(y)| \exp[\lambda(y + \tfrac{t}{m})]dy$$

$$\leq A \int \exp[-|y|^{1/2}]dy + \exp[\lambda(\tfrac{t}{m})] \int |{}^{\mu}\theta(y)| \exp[\lambda(y)]dy$$

$$\leq A\{1 + \exp[\lambda(\tfrac{t}{m})]\} \int \exp[-|y|^{1/2}]dy.$$

Hence there exists a constant B, such that

$$K_m(t) \leq B \exp[\lambda(\tfrac{t}{m})].$$

Consequently,

$$|g(t)|K_m(t) \leq B|g(t)|\exp[\lambda(\tfrac{t}{m})].$$

Since $g \in \mathfrak{L}^o_\lambda$, putting

$$\phi(t) = |g(t)|\exp[\lambda(t)],$$

we have that $\phi \in L^1$, and moreover,

$$\begin{aligned}|g(t)|K_m(t) &\leq B\,\phi(t)\exp[\lambda(\tfrac{t}{m}) - \lambda(t)]\\ &\leq B\,\phi(t).\end{aligned}$$

Therefore, using once again Lebesgue's convergence theorem, we have that

$$\int|g(t)|K_m(t)dt \to 0 \quad \text{as} \quad m \to \infty.$$

That is,

$$J_m \to 0 \quad \text{as} \quad m \to \infty.$$

Since, for each m, $I_m \leq J_m$, the result follows.

<u>L.4.2.6.</u> Suppose that $\lambda \in \Lambda$, $\mu(t) = \lambda(t) + |t|^{1/2}$, $g \in \mathfrak{L}^o_\lambda$, and u is a real number for which $\hat{g}(u) \neq 0$. Then if I_m is, for $m = 1,2,3,\ldots$, defined as in the previous lemma, we have that for each positive integer r,

$$\int\exp[\lambda(\tfrac{x}{m})]\,|(\ast^{(r)}\{\mu_{\theta_m}(y) - \frac{\int\mu_{\theta_m}(y-t)g(t)e^{iut}dt}{\hat{g}(u)}\})(x)|dx$$

$$\leq \left|\frac{I_m}{\hat{g}(u)}\right|^r,$$

where $\circledast^{(r)}$ denotes (r-1)-th self convolution;[†] with respect to y in this case.

Proof: By the definition of I_m, the result is true for $r = 1$. Denote

$$(\circledast^{(r)}\{\mu_{\theta_m}(y) - \frac{\int \mu_{\theta_m}(y-t)g(t)e^{iut}dt}{\hat{g}(u)}\})(x)$$

by $f_{r,m,u}(x)$. Assuming that the inequality holds for $r = R$, we have that

$$\begin{aligned}\int \exp[\lambda(\tfrac{x}{m})]|f_{R+1,m,u}(x)|dx &= \int \exp[\lambda(\tfrac{x}{m})]|\int f_{R,m,u}(x-t)\\ &\qquad f_{1,m,u}(t)dt|dx\\ &\leq \{\int \exp[\lambda(\tfrac{y}{m})]|f_{R,m,u}(y)|dy\}\\ &\qquad \{\int \exp[\lambda(\tfrac{t}{m})]|f_{1,m,u}(t)|dt\}\\ &\leq \left|\frac{I_m}{\hat{g}(u)}\right|^R \left|\frac{I_m}{\hat{g}(u)}\right|\\ &= \left|\frac{I_m}{\hat{g}(u)}\right|^{R+1}.\end{aligned}$$

The result now follows by induction.

<u>T.4.2.1. A generalisation[‡] of Wiener's Theorem</u>

Suppose that $\lambda \in \Lambda$, and, for $n = 1,2,3,\ldots$,

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[†] For example, $(\circledast^{(2)}f(y))(x) = (f * f)(x)$.

[‡] Note that if we take λ to be identically zero, then the result reduces to Wiener's theorem.

$\lambda_n(t) = \lambda(\frac{t}{n})$. Suppose further that $g \in \mathcal{L}_\lambda^o$ and $|\hat{g}(t)| > 0 \quad \forall t \in \mathbb{R}$ (i.e. $g \in \mathcal{L}_\lambda^o \cap \mathfrak{M}$), and that a,b are such that $-\infty < a < b < +\infty$. Then there exists a positive integer m and a function $h \in \mathcal{L}_{\lambda_m}$ such that $\hat{h}$ is compactly supported and such that for each $t \in [a,b]$, $\hat{h}(t) = 1/\hat{g}(t)$.

Proof: Put

$$\gamma = \inf_{u\in[a-1,b+1]} |\hat{g}(u)| ,$$

so that $\gamma > 0$. For $n = 1,2,3,\ldots$, define I_n as in L.4.2.5. Choose a positive integer m, such that

$$\left|\frac{I_m}{\gamma}\right| < 1.$$

For each real x and each $u \in [a-1,b+1]$, put

$$G(u,x) = \frac{e^{-iux}}{\hat{g}(u)} \Big[{}^{\mu}\theta_m(x) + \sum_{r=1}^{\infty} \Big(\ast^{(r)}\Big\{{}^{\mu}\theta_m(y) - \frac{\int {}^{\mu}\theta_m(y-t)g(t)e^{iut}dt}{\hat{g}(u)}\Big\}\Big)(x)\Big]$$

where $\mu(t) = \lambda(t) + |t|^{1/2}$.

Note that the summation converges absolutely a.e., and furthermore, that for each $u \in [a-1,b+1]$,

$$\int |G(u,x)| \exp[\lambda(\tfrac{x}{m})]dx \le \frac{1}{\gamma}\Big[\int |{}^{\mu}\theta_m(x)| \exp[\lambda(\tfrac{x}{m})]dx + \sum_{r=1}^{\infty} \left|\frac{I_m}{\gamma}\right|^r \Big].$$

Hence there exists a constant A such that for each $u \in [a-1,b+1]$,

$$\int |G(u,x)| \exp[\lambda(\tfrac{x}{m})]dx < A. \qquad ---(1)$$

Furthermore, taking Fourier transforms with respect to x, we have that for each $u \in [a-1,b+1]$,

$$\hat{G}(u,u+t) = \frac{1}{\hat{g}(u)} [{}^{\mu}\hat{\theta}_m(t) + \sum_{r=1}^{\infty} ({}^{\mu}\hat{\theta}_m(t) - \frac{{}^{\mu}\hat{\theta}_m(t)\hat{g}(u+t)}{\hat{g}(u)})^r]$$

$$= \begin{cases} \dfrac{1}{\hat{g}(u+t)} & \text{if } |t| \leq \frac{1}{m} \\ 0 & \text{if } |t| \geq \frac{2}{m} . \end{cases}$$

(Note that for $\forall t \in \mathbb{R}$, and $\forall u \in [a-1,b+1]$, we have that

$$|{}^{\mu}\hat{\theta}_m(t) - \frac{{}^{\mu}\hat{\theta}_m(t)\hat{g}(u+t)}{\hat{g}(u)}| \leq |\frac{I_m}{\gamma}|$$

$$< 1.)$$

Now $\mu(t) = \lambda(t) + |t|^{1/2}$, and so for n = 1,2,3,..., $\mathfrak{B}_{\mu}^{o} \subseteq \mathfrak{L}_{\lambda_n}^{o}$. Consequently using the result of L.4.2.4., we may choose $\chi \in \mathfrak{L}_{\lambda_m}^{o}$ such that $\hat{\chi} \in \mathfrak{D}$, $\hat{\chi}$ is supported by $[-\frac{1}{m}, \frac{1}{m}]$ and $\int \hat{\chi}(t)dt = 1$. Put

$$f(x) = \int_{a-1}^{b+1} \{ \int \chi(y)G(u,y+x)e^{iyu}dy\}du.$$

Then we have that

$$\int |f(x)| \exp[\lambda(\tfrac{x}{m})]dx$$

$$\leq \int_{a-1}^{b+1} \{ \int | \chi(y) | \{ \int | G(u,y+x) | \exp[\lambda(\tfrac{x}{m})]dx\}dy\}du$$

$$\leq \int_{a-1}^{b+1} \{ \int | \chi(y) | \exp[\lambda(\tfrac{y}{m})] \{ \int | G(u,y+x) | \exp[\lambda(\tfrac{y+x}{m})]dx\}dy\}du$$

$$\leq (b-a+2)A \int | \chi(y) | \exp[\lambda(\tfrac{y}{m})]dy, \quad \text{by use of (1),}$$

the change of order of integration being justified by Fubini's theorem. Hence $f \in \mathcal{L}^{o}_{\lambda_m}$. Furthermore, and again justifying changes in the order of integration by Fubini's theorem, we have that

$$\hat{f}(t) = \int e^{ixt} \{ \int_{a-1}^{b+1} \{ \int \chi(y) G(u,y+x) e^{iyu} dy\}du\}dx$$

$$= \int_{a-1}^{b+1} \{ \int \chi(y) e^{iy(u-t)} \{ \int G(u,y+x) e^{i(y+x)t} dx\}dy\}du$$

$$= \int_{a-1}^{b+1} \hat{\chi}(u-t) \hat{G}(u,t)du$$

$$= \begin{cases} \dfrac{1}{\hat{g}(t)} & \text{if} \quad t \in [a-1+\frac{1}{m},\ b+1-\frac{1}{m}] \\ 0 & \text{if} \quad t \leq a-1-\frac{1}{m} \text{ or if } t \geq b+1+\frac{1}{m}. \end{cases}$$

Hence $\hat{f}$ is compactly supported, and $\hat{f}(t) = 1/\hat{g}(t)$ for each $t \in [a,b]$.

Finally choose $\phi \in \mathfrak{L}_{\lambda_m}$ such that $\hat{\phi} \in \mathfrak{D}$ and $\hat{\phi}(t) = 1 \ \forall t \in [a,b]$. Put

$$h(x) = (f * \phi)(x),$$

so that

$$h^{(p)}(x) = (f * \phi^{(p)})(x).^{\dagger}$$

Then we have that

$$\int |h^{(p)}(x)| \exp[\lambda(\tfrac{x}{m})]dx$$

$$\leq \int \exp[\lambda(\tfrac{x}{m})]\{\int |f(x-t)|\,|\phi^{(p)}(t)|dt\}dx$$

$$\leq \{\int |\phi^{(p)}(t)| \exp[\lambda(\tfrac{t}{m})]dt\}\{\int |f(y)| \exp[\lambda(\tfrac{y}{m})]dy\}.$$

Hence $h \in \mathfrak{L}_{\lambda_m}$, and moreover since $\hat{h}(t) = \hat{f}(t)\hat{\phi}(t)$, we have that $\hat{h}$ is compactly supported, and $\hat{h}(t) = 1/\hat{g}(t)$ $\forall t \in [a,b]$.

The following theorem gives a version of T.4.2.1. for functions $g(x)$ which are 'exponentially bounded' rather than 'exponentially integrable'.

<u>T.4.2.2.</u> Suppose that $\lambda, \mu \in \Lambda$ are such that

$$\exp[\mu(x) - \lambda(x)] \in L^1.$$

For $n = 1,2,3,\ldots$, put $\mu_n(t) = \mu(\frac{t}{n})$. If $g \in \mathfrak{B}^{\circ}_{\lambda} \cap \mathfrak{M}$,

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† This is easily verified by use of Lebesgue's convergence theorem, and the first mean value theorem.

and a,b are such that $-\infty < a < b < +\infty$, then there exists a positive integer m and a function $f \in \mathfrak{B}_{\mu_m} \cap L^1$ such that $\hat{f}$ has compact support and $\hat{f}(t) = 1/\hat{g}(t) \quad \forall t \in [a,b]$.

Proof: We have that

$$g(x)\exp[\mu(x)] = g(x)\exp[\lambda(x)]\exp[\mu(x)-\lambda(x)] \in L^1.$$

Hence $g \in \mathfrak{L}^o_\mu$, and so by the previous theorem, there exists a positive integer m, and a function $h \in \mathfrak{L}_{\mu_m}$ such that $\hat{h}$ has compact support and $\hat{h}(t) = 1/\hat{g}(t) \quad \forall t \in [a,b]$. Choose $\chi \in \mathfrak{B}_{\mu_m} \cap L^1$ such that $\hat{\chi}(t) = 1 \quad \forall t \in [a,b]$. Put

$$f(x) = (h * \chi)(x),$$

then $f \in L^1$, $\hat{f}$ is compactly supported, and $\hat{f}(t) /= 1/\hat{g}(t)$ $\forall t \in [a,b]$. Moreover

$$|f^{(p)}(x)|\exp[\mu(\tfrac{x}{m})]$$

$$\leq \int |\chi^{(p)}(x-t)|\exp[\mu(\tfrac{x-t}{m})]|h(t)|\exp[\mu(\tfrac{t}{m})]dt$$

$$\leq \{\sup_{u \in \mathbb{R}} |\chi^{(p)}(u)|\exp[\mu(\tfrac{u}{m})]\}\int |h(t)|\exp[\mu(\tfrac{t}{m})]dt,$$

and so $f \in \mathfrak{B}_{\mu_m}$.

The following theorem is in the nature of being a partial converse of T.4.2.1.

T.4.2.3. Suppose that λ is an even, non-negative function of a real variable t, with the properties that $\lambda(t)$ is monotonic increasing for $t \geq 0$, and that $\lambda(t)/t$ $(t > 0)$ decreases monotonically to zero as $t \to \infty$. Suppose also that g is a function such that

$$g(x)\exp[\lambda(x)] \in L^1,$$

and that $-\infty < a < b < +\infty$.

In this situation, if there exists a positive integer m and a function h with the properties

1) $h(x)\exp[\lambda(\frac{x}{m})] \in L^1$

2) $\hat{h}(t) = 1/\hat{g}(t) \quad \forall t \in [a,b],$

then the integral

$$\int_1^\infty \frac{\lambda(x)}{x^2}\,dx$$

converges.

Proof: We shall assume that the integral

$$\int_1^\infty \frac{\lambda(x)}{x^2}\,dx$$

does not converge, and derive a contradiction. Under this assumption, if $\mu(t) = \lambda(\frac{t}{m}) + 2|t|^{1/2}$, the integral

$$\int_1^\infty \frac{\mu(x)}{x^2}\,dx$$

does not converge. Put

$$f(x) = (h * g)(x).$$

It is not difficult to show that

$$f(x)\exp[\lambda(\tfrac{x}{m})] \in L^1,$$

and that $\hat{f}(t) = 1 \quad \forall t \in [a,b]$. Select $\delta > 0$ such that $b-\delta > a+\delta$, and take[†] χ such that $\hat{\chi} \in \mathfrak{D}$, $\hat{\chi}$ is supported by $[-\delta,\delta]$, $\int \hat{\chi}(t)dt = 1$, and

$$\chi(x) = O(\exp[-2|x|^{1/2}]) \quad \text{as} \quad x \to \pm\infty.$$

Put

$$k(x) = 2\pi f(x)\chi(x), \ \forall x \in \mathbb{R},$$

so that $\hat{k}(t) = 1 \quad \forall t \in [a-\delta,b+\delta]$, and $\hat{k}$ possesses derivatives of all orders. We have that

$$\hat{k}^{(n)}(t) = \int (ix)^n k(x)e^{ixt}dx \quad \forall t \in \mathbb{R}.$$

It follows that for each positive integer n, and each real t,

$$|\hat{k}^{(n)}(t)| \leq \int |x|^n |k(x)| dx$$
$$= A_n, \text{ say.}$$

Define $\phi(t) = |k(t)|\exp[\mu(t)]$, so that $\phi \in L^1$, and put $\psi(t) = \phi(t) + \phi(-t)$. Then, remembering that $\mu(x)/x$ is

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[†] Using L.4.2.4.

decreasing,

$$
\begin{aligned}
A_n &= \int |x|^n \phi(x)\exp[-\mu(x)]dx \\
&= \int_0^\infty x^n \psi(x)\exp[-\mu(x)]dx \\
&\leq \int_0^{n^4} x^n \psi(x)\exp[-x\,\frac{\mu(n^4)}{n^4}]dx + \int_{n^4}^\infty x^n \psi(x)\exp[-2(x)^{1/2}]dx \\
&\leq \int_0^\infty x^n \psi(x)\exp[-x\,\frac{\mu(n^4)}{n^4}]dx + e^{-n^2}\int_0^\infty x^n \psi(x)\exp[-x^{1/2}]dx .
\end{aligned}
$$

Now it is easily shown by differentiation that for each positive integer r, and each a > 0,

$$\exists \operatorname*{Sup}_{y>o} \{y^r \exp[-ay]\} = (\tfrac{r}{a})^r\ e^{-r}.$$

Hence

$$
\begin{aligned}
A_n &\leq \{(\frac{n^5}{\mu(n^4)})^n\ e^{-n} + e^{-n^2}(2n)^{2n}e^{-2n}\}\int_0^\infty \psi(x)dx \\
&\leq A(\frac{n^5}{e\mu(n^4)})^n \text{, for some positive constant A.}
\end{aligned}
$$

Since $\int_1^\infty \frac{\mu(x)}{x^2}\,dx$ is divergent, so also is $\int_1^\infty \frac{\mu(x^4)}{x^5}\,dx$, and therefore the sum $\sum_{n=1}^\infty \frac{\mu(n^4)}{n^5}$ diverges. It follows that $\sum_{n=1}^\infty (A_n)^{-\frac{1}{n}}$ is divergent. Hence, using the results of Carleman and Denjoy ((4) and (7)) concerning quasi-analytic functions, since

$$\hat{k}(t) = 1 \quad \forall t \in [a+\delta, b-\delta],$$

we have that

$$\hat{k}(t) = 1 \quad \forall t \in \mathbb{R} .$$

But since $k \in L^1$, this last result contradicts the Riemann-Lebesgue lemma †. It follows that the integral

$$\int_1^\infty \frac{\lambda(x)}{x^2}\, dx,$$

must converge.

The next lemma shows, amongst other things that the condition 'exp[$\nu(x)-\mu(x)$] is integrable', as imposed in T.4.2.2. is, for Λ -functions μ and ν, stronger than the statement 'exp[$\nu(x)-\mu(x)$] is bounded'.

<u>L.4.2.7.</u> Suppose $\mu,\nu \in \Lambda$, and

$$\exp[\nu(x)-\mu(x)] \in L^1.$$

Then $\forall A \in \mathbb{R}$, $\exists X$ such that if $x > X$, then

$$\nu(x) - \mu(x) < A.$$

Proof: Suppose that there exists a real number B with the property that for every X, there exists $x > X$ such that

$$\nu(x) - \mu(x) \geq B.$$

Take X_o so that whenever $x > X_o$, $\mu(x)/x < 1$. Choose $x_1 > X_o + 1$, such that

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† See (11) Pg.7.

$$\nu(x_1) - \mu(x_1) \geq B.$$

Choose, inductively, $x_{n+1} > x_n + 1$ such that

$$\nu(x_{n+1}) - \mu(x_{n+1}) \geq B. \quad (n \geq 1).$$

For $y \in [0,1]$, and each positive integer n, we have that

$$\mu(x_n + y)/(x_n + y) \leq \mu(x_n)/x_n,$$

and so

$$\begin{aligned}\mu(x_n + y) &\leq \mu(x_n) + y\mu(x_n)/x_n \\ &\leq \mu(x_n) + y \\ &\leq \nu(x_n) - B + y.\end{aligned}$$

Hence

$$\begin{aligned}\nu(x_n + y) - \mu(x_n + y) &\geq B - y \\ &\geq B - 1.\end{aligned}$$

Consequently

$$\int_{x_n}^{1+x_n} \exp[\nu(x)-\mu(x)]dx \geq \exp[B - 1],$$

and so

$$\begin{aligned}\int_{0}^{1+x_n} \exp[\nu(x)-\mu(x)]dx &\geq n \exp[B - 1] \\ &\to \infty \quad \text{as } n \to \infty.\end{aligned}$$

But this is a contradiction. The result follows.

The following two theorems are variants of T.4.2.1., relating to certain intersection classes of functions.

T.4.2.4. Suppose that M is a non-empty subset of Λ, with the properties that

1) $\forall \mu \in M$, $\exists \nu \in M$ such that for each $n = 1,2,3,\ldots$, $\exists A = A_n > 0$ with the property that

$$\exp[\mu(x)-\nu(\tfrac{x}{n})] < A \quad \forall x \in \mathbb{R} ,$$

2) $\exists \varkappa \in \Lambda$ such that $\forall \mu \in M$, $\exists B > 0$ with the property that

$$\exp[\mu(x)-\varkappa(x)] < B \quad \forall x \in \mathbb{R} .$$

Suppose also that $-\infty < a < b < +\infty$, that $g \in \bigcap_{\mu \in M} \mathfrak{L}^{o}_{\mu}$, and that $g \in \mathfrak{W}$. Then there exists $f \in \bigcap_{\mu \in M}(\mathfrak{L}_{\mu} \cap \mathfrak{B}_{\mu})$ such that $\hat{f}$ is compactly supported and $\hat{f}(t) = 1/\hat{g}(t) \quad \forall t \in [a,b]$.

Proof: Choose $\mu \in M$. Then there exists $\nu \in M$ such that for $n = 1,2,3,\ldots$, $\exp[\mu(x)-\nu(\frac{x}{n})]$ is bounded. It follows from T.4.2.1. that there exists a positive integer m and a function $h_{\mu} \in \mathfrak{L}_{\nu_m}$ $(\nu_m(t) = \nu(\frac{t}{m}))$ such that $\hat{h}_{\mu}$ is compactly supported and for each $t \in [a-1,b+1]$, $\hat{h}(t) = 1/\hat{g}(t)$. Since $\mathfrak{L}_{\nu_n} \subseteq \mathfrak{L}_{\mu}$ for $n = 1,2,3,\ldots$, we have that $h_{\mu} \in \mathfrak{L}_{\mu}$.

Choose $\chi \in \mathfrak{L}_{\varkappa} \cap \mathfrak{B}_{\varkappa}$ † such that $\hat{\chi}(t) \in \mathfrak{D}$ and

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† This may be done, for example, by taking $\chi \in \mathfrak{B}_{\lambda}$, where $\lambda(t) = \varkappa(t) + |t|^{1/2}$.

$$\hat{\chi}(t) = \begin{cases} 1 & \text{for } t \in [a,b] \\ 0 & \text{if } t \le a-1 \text{ or if } t \ge b+1. \end{cases}$$

Now if $\mu_1, \mu_2 \in M$, we have that

$$\begin{aligned} (h_{\mu_1} * \chi)^\wedge &= \hat{h}_{\mu_1} \hat{\chi} \\ &= \hat{\chi}/\hat{g} \\ &= (h_{\mu_2} * \chi)^\wedge . \end{aligned}$$

Hence $h_{\mu_1} * \chi = h_{\mu_2} * \chi$. We may therefore define a function f, by the relation

$$f(x) = (h_\mu * \chi)(x), \ \forall \mu \in M$$

Note that $\hat{f}$ is compactly supported and that for every $t \in [a,b]$, we have $\hat{f}(t) = 1/\hat{g}(t)$. Moreover for each $\mu \in M$, and each non-negative integer p, we have that

$$\begin{aligned} |f^{(p)}(x)| &\le \int |h_\mu(x-t)| \, |\chi^{(p)}(t)| dt \\ &\le B \exp[-\mu(x)] \int |h_\mu(x-t)| \exp[\mu(x-t)] \\ &\qquad |\chi^{(p)}(t)| \exp[\mathcal{K}(t)] dt. \end{aligned}$$

Hence $f \in \mathfrak{L}_\mu \cap \mathfrak{B}_\mu$, $\forall_\mu \in M$; That is $f \in \bigcap_{\mu \in M} (\mathfrak{L}_\mu \cap \mathfrak{B}_\mu)$.

<u>T.4.2.5.</u> Suppose that M is a non-empty subset of Λ with the following properties.

1) $\forall \mu \in M$, $\exists \nu \in M$ such that for each $n = 1,2,3,\ldots,$

$\exists A > 0$ with the property that

$$\exp[\mu(x) - \nu(\tfrac{x}{n})] < A \quad \forall x \in \mathbb{R} .$$

2) $\exists \mathcal{K} \in \Lambda$ such that $\forall \mu \in M$, $\exists B > 0$ with the property that

$$\exp[\mu(x) - \mathcal{K}(x)] < B \quad \forall x \in \mathbb{R} .$$

3) $\forall \mu \in M$, $\exists \nu \in M$ such that

$$\exp[\mu(x) - \nu(x)] \in L^1.$$

Suppose also that $-\infty < a < b < +\infty$, that $g \in \bigcap_{\mu \in M} \mathcal{B}^{o}_{\mu}$, and that $g \in \mathfrak{M}$. Then there exists $f \in \bigcap_{\mu \in M}(\mathcal{L}_{\mu} \cap \mathcal{B}_{\mu})$ such that $\hat{f}$ is compactly supported and $\hat{f}(t) = 1/\hat{g}(t) \quad \forall t \in [a,b]$.
Proof: This may be shown in a similar fashion to the preceeding proof. It may also be established by direct use of the previous result. The following proof represents the second alternative.

Choose $\mu \in M$. Then there exists $\nu \in M$ such that

$$\exp[\mu(x) - \nu(x)] \in L^1.$$

If $h \in \bigcap_{\lambda \in M} \mathcal{B}^{o}_{\lambda}$, then there exists $K > 0$ such that

$$|h(x)| \leq K \exp[-\nu(x)] \quad \forall x \in \mathbb{R} ,$$

and so

$$|h(x)| \exp[\mu(x)] \leq K \exp[\mu(x) - \nu(x)] \in L^1.$$

Hence $h \in \mathfrak{L}^{\circ}_{\mu}$, for each $\mu \in M$. It follows that

$$\bigcap_{\lambda \in M} \mathfrak{B}^{\circ}_{\lambda} \subseteq \bigcap_{\lambda \in M} \mathfrak{L}^{\circ}_{\lambda} .$$

Hence from the previous result, we have that there exists a function $f \in \bigcap_{\mu \in M} (\mathfrak{L}_{\mu} \cap \mathfrak{B}_{\mu})$, such that $\hat{f}$ is compactly supported and $\hat{f}(t) = 1/\hat{g}(t) \quad \forall t \in [a,b]$.

<u>L.4.2.8.</u> Suppose that $\lambda \in \Lambda$. Then we have the following.

1) If $g \in \mathfrak{L}^{\circ}_{\lambda}$, and if

$$f(x)\exp[-\lambda(x)] \in L^1,$$

then $\exists |f| * |g|$, and

$$(|f| * |g|)(x)\exp[-\lambda(x)] \in L^1.$$

2) If $g \in \mathfrak{B}^{\circ}_{\lambda}$, and if

$$f(x)\exp[-\lambda(x)] \in L^1,$$

then $\exists\, |f| * |g|$, and $\exists\, A > 0$, such that

$$(|f| * |g|)(x)\exp[-\lambda(x)] < A.$$

3) If $g \in \mathfrak{L}^{\circ}_{\lambda}$, and if $\exists\, B > 0$ such that

$$|f(x)|\exp[-\lambda(x)] < B,$$

then $\exists\, |f| * |g|$, and $\exists\, C > 0$ such that

$$(|f| * |g|)(x)\exp[-\lambda(x)] < C.$$

Proof: In all three cases we have that

$$\int |f(x-t)||g(t)|dt$$
$$= \int \exp[\lambda(x-t)-\lambda(t)]|f(x-t)|\exp[-\lambda(x-t)] |g(t)|\exp[\lambda(t)]dt$$
$$\leq \exp[\lambda(x)]\int |f(x-t)|\exp[-\lambda(x-t)]|g(t)|\exp[\lambda(t)]dt,$$

and the results now follow.

The following theorem will allow us to construct a whole family of quotient pairs† using classes of functions considered in this chapter.

<u>T.4.2.6.</u> Suppose that $\lambda \in \Lambda$ and that $g \in \mathfrak{L}^o_\lambda \cap \mathfrak{M}$. Suppose also that $f(x)$ is a function such that $f * g = 0$, and in addition either

$$\forall m,\ f(t)\exp[-\lambda(\tfrac{t}{m})] \in L^1,$$

or

$$\forall m,\ \exists A_m \text{ such that}$$
$$|f(t)|\exp[-\lambda(\tfrac{t}{m})] < A_m.$$

Then $f = 0$ (a.e.).

Proof: Choose $k \in \mathfrak{B}^o_\lambda \cap \mathfrak{L}^o_\lambda$, not identically zero, such that $\hat{k} \in \mathfrak{D}$. This may be done, for example, by taking $k \in \mathfrak{B}^o_\mu$ where $\mu(t) = \lambda(t)+|t|^{1/2}$. We have, using T.4.2.1., that there exists a positive integer m, and a function $\ell \in \mathfrak{L}_{\lambda_m}$ $(\lambda_m(t) = \lambda(\frac{t}{m}))$ such that for each t lying within

†See page 168 et seq.

the support of $\hat{k}$, $\hat{\ell}(t) = 1/\hat{g}(t)$. We have that $g \in \mathfrak{L}^{o}_{\lambda_m}$ and that $k \in \mathfrak{B}^{o}_{\lambda_m} \cap \mathfrak{L}^{o}_{\lambda_m}$, and so using the previous lemma, the multiple integral

$$\int |\ell(z-y)| \int |k(y-x)| \int |f(x-t)| \; |g(t)| \, dt dx dy$$

exists. Hence

$$\exists ((f * g) * k) * \ell = f * (g * k * \ell).$$

But we have that

$$\begin{aligned}(g * k * \ell)^{\wedge} &= \hat{g}\,\hat{k}\,\hat{\ell} \\ &= \hat{k},\end{aligned}$$

and so

$$((f * g) * k) * \ell = f * k.$$

Hence $f * k = 0$. It follows that for each k which obeys the conditions;

(i) $k \in \mathfrak{B}^{o}_{\lambda} \cap \mathfrak{L}^{o}_{\lambda}$,

(ii) $\hat{k} \in \mathfrak{D}$,

(iii) k is not identically zero,

we have that $(f * k)(x) = 0 \quad \forall x \in \mathbb{R}$.

Choose a function h obeying conditions (i) - (iii) above, and for each real y, put $h_y(t) = e^{iyt}h(t)$. Then for each real y, h_y obeys conditions (i) - (iii). Consequently

$$(f * h_y)(x) = 0 \quad \forall x,y \in \mathbb{R} .$$

That is,

$$\int f(t)h(x-t)e^{iyt}dt = 0 \quad \forall x,y \in \mathbb{R} .$$

But for each $x \in \mathbb{R}$, $f(t)h(x-t) \in L^1$, and so it follows using L.2.2.1., that for each real x,

$$f(t)h(x-t) = 0 \quad \text{a.e.}$$

Hence $f(t) = 0$ a.e.

4.3. Examples

In this section we give some applications of the preceeding theorems, and bring out some connections with the material of the first three chapters.

Consider T.4.3.3. with $\lambda(t) = A|t|^\alpha$, where $A > 0$ and $0 < \alpha < 1$. Suppose $0 < B < A$, and put $\mu(t) = B|t|^\alpha$, so that $\exp[\mu(x)-\lambda(x)] \in L^1$. Suppose that $-\infty < a < b < +\infty$ and that $g(x)$ is such that $g \in \mathfrak{B}_\lambda^o \cap \mathfrak{M}$. Then by T.4.2.2., there exists a constant C, $0 < C \leq B$, and a function f, such that

1) $\hat{f}(t) = 1/\hat{g}(t) \quad \forall t \in [a,b]$

2) f has continuous derivatives of all orders, and for each non-negative integer p, $\exists A_p > 0$ for which

$$|f^{(p)}(x)| < A_p \exp[-C|x|^\alpha], \quad \forall x \in \mathbb{R} .$$

In view of this result, if we follow Gel'fand and Shilov (10) and, for $\beta > 1$, put

$$\mathfrak{S}_\beta = \{g \mid \exists a > 0 \text{ such that } \forall q,\ \exists Cq \text{ with the property that } |g^{(q)}(x)| \leq Cq \exp[-a|x|^{1/\beta}]\},^{\dagger}$$

then we have the following parallel of Wiener's theorem:

<u>T.4.3.1.</u> If $g \in \mathfrak{S}_\beta \cap \mathfrak{B}$ $(\beta > 1)$, and if $-\infty < a < b < +\infty$, then there exists $f \in \mathfrak{S}_\beta$ such that $\hat{f}(t) = 1/\hat{g}(t)$ $\forall t \in [a,b]$.

We will assume throughout the remainder of this section that $\beta > 1$. Put

$$\mathfrak{T}_\beta = \{f \mid \forall a > 0,\ \forall q,\ \exists C_{q,a} \text{ such that } |f^{(q)}(x)| < C_{q,a} \exp[a|x|^{1/\beta}]\}.$$

The following results are not difficult to establish.

1) If $f \in \mathfrak{T}_\beta$ and p is a non-negative integer, then $f^{(p)} \in \mathfrak{T}_\beta$

2) If $k \in \mathfrak{S}_\beta$ and p is a non-negative integer, then $k^{(p)} \in \mathfrak{S}_\beta$

3) If $f \in \mathfrak{T}_\beta$ and if $k \in \mathfrak{S}_\beta$, then $\exists\, f * k \in \mathfrak{T}_\beta$, and for each non-negative integer p, $(f * k)^{(p)}(x) = (f^{(p)} * k)(x) = (f * k^{(p)})(x)$. (This is proved fairly easily using L.4.2.8., the mean value theorem and Lebesgue's convergence theorem.)

4) If $k_1, k_2 \in \mathfrak{S}_\beta$, then $k_1 * k_2 \in \mathfrak{S}_\beta$. (This may be proved by use of L.4.2.1.)

5) If $f \in \mathfrak{T}_\beta$ and if $k_1, k_2 \in \mathfrak{S}_\beta$, then $(f * k_1) * k_2 =$

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† See the appendix.

$f * (k_1 * k_2) = (f * k_2) * k_1$. (This has a straightforward proof, using L.4.2.8. and Fubini's theorem.)

Using the above results, it follows that $\mathfrak{T}_\beta$ may be taken to be a universal algebra with the following set of operations:

(i) the binary operation $(f,g) \to f + g$,

(ii) the unary operations $f \to \lambda f$, where λ is any complex number,

(iii) the unary operations $f \to f^{(p)}$, where p is any non-negative integer,

(iv) the unary operations $f \to \int f(t)\ell(x-t)dt$, where ℓ is any element of $\mathfrak{S}_\beta$,

(v) the nullary operation 0.

It follows from 4) above, and L.2.1.3., that $\mathfrak{S}_\beta \cap \mathfrak{W}$ is closed under convolution. Moreover it is clear that $\mathfrak{S}_\beta \cap \mathfrak{W}$ provides (by convolution) an Abelian semigroup of endomorphisms of $\mathfrak{T}_\beta$.

We can now establish the existence of a new class of quotient pairs:

<u>T.4.3.2.</u> If $\beta > 1$, then $\mathfrak{T}_\beta$, $\mathfrak{S}_\beta \cap \mathfrak{W}$ form a quotient pair. Furthermore $\mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{S}_\beta \cap \mathfrak{W})$ is a genuine extension of $\mathfrak{T}_\beta$; i.e. $i(\mathfrak{T}_\beta) \neq \mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{S}_\beta \cap \mathfrak{W})$.

Proof: After the results of the section above, to prove that $\mathfrak{T}_\beta$, $\mathfrak{S}_\beta \cap \mathfrak{W}$ form a quotient pair, it suffices to show that if $f \in \mathfrak{T}_\beta$ and if $k \in \mathfrak{S}_\beta \cap \mathfrak{W}$, then

$f * k$ is null if and only if f is null. Now $k \in \mathfrak{S}_\beta$ implies that there exists $a > 0$ and $C > 0$ such that

$$|k(x)| < C \exp[-a|x|^{1/\beta}].$$

Put $\lambda(t) = a|t|^{1/\beta}/2$. Then $k \in \mathfrak{L}^o_\lambda$. Moreover since $f \in \mathfrak{T}_\beta$, we have that for each n, $\exists C_n$ such that

$$|f(x)| < C_n \exp[\lambda(\tfrac{x}{n})].$$

Hence, by T.4.2.6., we have the required result.

To show that $i(\mathfrak{T}_\beta) \neq \mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{S}_\beta \cap \mathfrak{B})$, choose $\phi \in \mathfrak{S}_\beta \cap \mathfrak{B}$. Then $\phi \in \mathfrak{T}_\beta$, and so there exists an element $\phi // \phi \in \mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{S}_\beta \cap \mathfrak{B})$. Suppose that this element were of the form $i(f)$ for some $f \in \mathfrak{T}_\beta$, so that $f * \phi = \phi$. In this case $f * \psi = \psi \quad \forall \psi \in \mathfrak{S}_\beta \cap \mathfrak{B}$, and so we have that

$$\int f(t)\psi(t)dt = \psi(0) \quad \forall \psi \in \mathfrak{S}_\beta \cap \mathfrak{B} ,$$

and

$$\int f(t)\psi(t)e^{ixt}dt = \psi(0) \quad \forall \psi \in \mathfrak{S}_\beta \cap \mathfrak{B} \text{ and } \forall x \in \mathbb{R} .$$

Hence for each $\psi \in \mathfrak{S}_\beta \cap \mathfrak{B}$, $(f\psi)^\wedge(x) = \psi(0) \quad \forall x \in \mathbb{R}$. But $f\psi \in L^1$, and so this last result contradicts the Riemann-Lebesgue lemma[†] whenever $\psi(0) \neq 0$. The result follows.

<u>T.4.3.3.</u> If $\mathfrak{Z}_\beta = \mathfrak{S}_\beta \cap \mathfrak{Z}$ (i.e. $\mathfrak{Z}_\beta$ consists of

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[†]See (11) Pg.7.

those functions in $\mathfrak{G}_\beta$ whose transforms are compactly supported), then $\mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{G}_\beta \cap \mathfrak{W})$ and $\mathfrak{Z}_\beta$ are Wiener-like. Moreover $\mathfrak{Z}_\beta$ is endomorphic on $\mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{G}_\beta \cap \mathfrak{W})$, and $\mathfrak{Z}_\beta$ separates $\mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{G}_\beta \cap \mathfrak{W})$.

Proof: It follows from T.4.3.1. that $\mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{G}_\beta \cap \mathfrak{W})$ and $\mathfrak{Z}_\beta$ are Wiener-like, and it is easily established that $\mathfrak{Z}_\beta$ is endomorphic on $\mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{G}_\beta \cap \mathfrak{W})$. To see that $\mathfrak{Z}_\beta$ separates $\mathfrak{Q}(\mathfrak{T}_\beta, \mathfrak{G}_\beta \cap \mathfrak{W})$, suppose that $f \in \mathfrak{T}_\beta$ and

$$\int f(t)\chi(t)dt = 0 \quad \forall \chi \in \mathfrak{Z}_\beta.$$

Now choose $\chi \in \mathfrak{Z}_\beta$, such that χ is not identically zero. Then we also have that $\chi(t)e^{ixt} \in \mathfrak{Z}_\beta$ for each real x, and so

$$\int f(t)\chi(t)e^{ixt}dt = 0, \quad \forall x \in \mathbb{R} .$$

Since $f\chi \in L^1$, from this we obtain that $f(t)\chi(t) \equiv 0$ and so $f \equiv 0$.

We now use T.4.2.5. to obtain another result of the Wiener type. Take M to be set consisting of all the functions $\mu(x) = |x|^\alpha$ $(0 < \alpha < 1)$. Then $M \subset \Lambda$, and conditions 1), 2), 3) of T.4.2.5. are all satisfied by the class M. (We may take

$$\varkappa(t) = \begin{cases} \dfrac{|t|}{\log^2|t|} & \text{if} \quad |t| \geq e^2 \\ |t|/4 & \text{if} \quad |t| < e^2.) \end{cases}$$

Moreover it is easily established that

$$\bigcap_{\mu \in M} \mathfrak{B}_\mu = \bigcap_{\beta > 1} \mathfrak{S}_\beta ,$$

and so if we put

$$S = \bigcap_{\beta > 1} \mathfrak{S}_\beta,$$

we obtain the following theorem:

T.4.3.4. If $g \in S \cap \mathfrak{W}$ and if $-\infty < a < b < +\infty$, then there exists $f \in S$ such that $\hat{f}(t) = 1/\hat{g}(t)$ $\forall t \in [a,b]$.

Following on from the above, we can now construct another quotient pair related to the quotient pairs $\mathfrak{T}_\beta$, $\mathfrak{S}_\beta \cap \mathfrak{W}$. Put

$$T = \bigcup_{\beta > 1} \mathfrak{T}_\beta$$

$$= \{f \mid \exists \beta > 1 \text{ such that } \forall q,\ \exists C_{q,\beta} \text{ with the property that } |f^{(q)}(x)| < C_{q,\beta} \exp[|x|^{1/\beta}]\}.$$

From the results on page 169 it is easily established that T is a universal algebra with the following set of operations

(i) The binary operation $(f,g) \to f + g$.

(ii) The unary operations $f \to \lambda f$, where λ is any complex number.

(iii) The unary operations $f \to f^{(p)}$, where p is any non-negative integer.

(iv) The unary operations $f \rightarrow \int f(t)\ell(x-t)dt$, where ℓ is any element of S.

(v) The nullary operation 0.

Again, it is easily established that $S \cap \mathfrak{W}$ is closed under convolution, and that $S \cap \mathfrak{W}$ provides an Abelian semi-group of endomorphisms of T.

T.4.3.5. The function spaces T, $S \cap \mathfrak{W}$ form a quotient pair, and $i(T) \neq \mathfrak{Q}(T, S \cap \mathfrak{W})$.

Proof: The first part follows from the corresponding section of T.4.3.2., while the proof of the second part is along the same lines as the proof of the second part of that theorem.

T.4.3.6. If $Z = S \cap \mathfrak{Z}$, then $\mathfrak{Q}(T, S \cap \mathfrak{W})$ and Z are Wiener-like. Moreover Z is endomorphic on $\mathfrak{Q}(T, S \cap \mathfrak{W})$ and Z separates $\mathfrak{Q}(T, S \cap \mathfrak{W})$.

Proof: The proof of this theorem is along the same lines as the proof which was given for T.4.3.3.

In the preceeding cases, we have concentrated on 'large' Λ-functions; the next result involves 'small' Λ-functions.

T.4.3.7. If $(1+|t|^c)g(t) \in L^1$ $(c \geq 0)$, if $g \in \mathfrak{W}$ and if $-\infty < a < b < +\infty$, then there exists a function h such that

1) $(1+|t|^c)h(t) \in L^1$

2) $\hat{h}(t) = 1/\hat{g}(t) \quad \forall t \in [a,b]$.

Proof: In T.4.2.1. take

$$\lambda(t) = \begin{cases} c \log|t| & \text{if } |t| \geq e \\ c|t|/e & \text{if } |t| < e. \end{cases}$$

The result follows immediately.

This chapter has been concerned, among other things, with the construction of quotient pairs. To conclude it is worth remarking that with any $\lambda \in \Lambda$ we may associate a quotient pair $\mathfrak{A}, \mathfrak{B}$, the spaces $\mathfrak{A}, \mathfrak{B}$ being given by

$$\mathfrak{A} = \{f \mid \forall p = 0,1,2,\ldots, \quad \forall n = 1,2,3,\ldots, \quad \exists\, C_{p,n} \text{ such that } |f^{(p)}(x)| < C_{p,n}\exp[\lambda(\tfrac{x}{n})]\},$$

$$\mathfrak{B} = \mathfrak{L}_\lambda \cap \mathfrak{W} \quad ;$$

and the universal algebra $\mathfrak{A}$ having the usual system of operations.

APPENDIX

A.1. Notes concerning section 2.4.

In this section an outline is given of the proofs of lemmas L.2.4.9., L.2.4.10., L.2.4.11. and L.2.4.16. These proofs are quite straightforward, but rather lengthy, and so to avoid losing the theme of the text, they have been gathered together here. I can find no previous explicit statements of L.2.4.11., the $\mathfrak{Z}$ -case of L.2.4.9., and the $\mathfrak{S}$ -case of L.2.4.16.† However L.2.4.16. may be proved as an elementary deduction from certain general arguments such as those in (10) , and I suspect that lemmas L.2.4.9. and L.2.4.11. are also known. Alternative proofs of the $\mathfrak{S}$ -cases of L.2.4.9., L.2.4.10. and L.2.4.11. may be obtained by use of L.2.4.8., but here we give proofs which do not rely on this result.

A.1.1. To prove the $\mathfrak{S}$ -cases of L.2.4.9. and L.2.4.10., it is sufficient to show that if $F \in \mathfrak{S}'$ and $k \in \mathfrak{S}$, then $F \ast k$ is tempered and

$$\exists\,(F \ast k)'(x) = (F \ast k')(x).$$

Likewise for the $\mathfrak{Z}$ -cases.

(a) Suppose that $F \in \mathfrak{S}'$ and $k \in \mathfrak{S}$. We want to show that $F \ast k$ is tempered. We suppose this is not the case

† For the $\mathfrak{S}$ -case of L.2.4.9., see (25) Pg.141. For L.2.4.10., see (3) Pg.105. For L.2.4.16., see (9) Pgs.368-369 and (10) Pg.49.

and go on to derive a contradiction.

If $F * k$ is not tempered, then $\forall A > 0$, $\forall N > 0$, $\forall T > 0$, $\exists\, t$ such that $|t| > T$ and

$$|(F * k)(t)| \geq A|t|^N.$$

Choose a sequence $\{t_r\}$ such that $\{|t_r|\}$ is increasing and diverges, and such that $|t_r| > 1$ and

$$|(F * k)(t_r)| \geq |t_r|^r \quad r = 1,2,3,\ldots \,. \tag{1}$$

For $r = 1,2,3,\ldots,$ put

$$k_r(t) = \frac{k(t_r - t)}{(t_r)^r} \,.$$

Choose any $\mathfrak{S}$-neighbourhood, U, of 0. Then U has a subset V of the form

$$V = \{\phi \mid \phi \in \mathfrak{S} \text{ and } |t|^m |\phi^{(p)}(t)| < \epsilon \text{ for } m \leq M \text{ and } p \leq P\},$$

for some $\epsilon > 0$, and some positive integers M,P. Put

$$A = \max_{\substack{m \leq M \\ p \leq P}} \sup_{u \in \mathbb{R}} \{|u|^m |k^{(p)}(u)|\}.$$

Then if $m \leq M$, $p \leq P$, we have that

$$|t|^m |k_r^{(p)}(t)| = \frac{|t|^m |k^{(p)}(t_r-t)|}{|t_r|^r}$$

$$\leq \frac{2^m(|t_r-t|^m + |t_r|^m)|k^{(p)}(t_r-t)|}{|t_r|^r}$$

$$\leq \frac{2^M A}{|t_r|^r} + \frac{2^M A}{|t_r|^{r-M}} \quad .$$

Hence there exists R such that if $r > R$, then $k_r \in U$. Consequently, since F is continuous,

$$\langle F, k_r \rangle \to 0 \quad \text{as } r \to \infty.$$

But from (1) we have that for $r = 1,2,3,\ldots,$

$$|\langle F, k_r \rangle| \geq 1.$$

The result now follows.

(b) If $F \in \mathfrak{S}'$ and $k \in \mathfrak{S}$, then

$$\exists (F * k)'(x) = (F * k')(x) \quad \forall x \in \mathbb{R} \ .$$

Proof: Choose $x \in \mathbb{R}$, and suppose that $h \neq 0$. Then we have that

$$\frac{1}{h}\,[\langle F_t, k(x+h-t)\rangle - \langle F_t, k(x-t)\rangle] = \langle F_t, \frac{k(x+h-t)-k(x-t)}{h}\rangle.$$

Choose any $\mathfrak{S}$-neighbourhood, U of 0. Then U has a subset V of the form

$$V = \{\phi \mid \phi \in \mathfrak{S} \text{ and } |t|^m |\phi^{(p)}(t)| < \epsilon \text{ for } m \leq M \text{ and } p \leq P\},$$

for some $\epsilon > 0$, and some positive integers M,P. Put

$$A = \operatorname*{Max}_{\substack{m \leq M \\ p \leq P+2}} \ \operatorname*{Sup}_{t \in \mathbb{R}} \{(|x-t|+1)^m |k^{(p)}(t)|\},$$

$$\delta = \mathrm{Min}\{\epsilon/A, 1\}.$$

Then if $0 < |h| < \delta$ and $m \leq M$, $p \leq P$, we have, using the second mean value theorem on the real and imaginary parts of k, that for each real t,

$$|t|^m \left| \frac{k^{(p)}(x+h-t) - k^{(p)}(x-t)}{h} - k^{(p+1)}(x-t) \right|$$

$$= |t|^m \left| \mathfrak{Re}\, \frac{h}{2} k^{(p+2)}(x-t+\theta_1 h) + i\, \mathfrak{Im}\, \frac{h}{2} k^{(p+2)}(x-t+\theta_2 h) \right|,$$

for some $\theta_1, \theta_2 \in [0,1]$. Hence

$$|t|^m \left| \frac{k^{(p)}(x+h-t) - k^{(p)}(x-t)}{h} - k^{(p+1)}(x-t) \right|$$

$$\leq \frac{2|h|}{2} |t|^m \operatorname*{Sup}_{\theta \in [0,1]} \{|k^{(p+2)}(x-t+\theta h)|\}$$

$$\leq h \operatorname*{Sup}_{u \in \mathbb{R}} \{(|x-u|+1)^m |k^{(p+2)}(u)|\}$$

$$\leq A|h| < A\delta \leq \epsilon.$$

It follows that if $0 < |h| < \delta$, then

$$\left(\frac{k(x+h-t) - k(x-t)}{h} - k'(x-t)\right) \in U.$$

Since F is continuous, letting $h \to 0$, we have that, for each real x,

$$\exists (F * k)'(x) = (F * k')(x).$$

(c) Suppose that $F \in \mathcal{B}'$, $k \in \mathcal{B}$, but $F * k$ is not tempered. Choose a sequence $\{t_r\}$ obeying the conditions set out in (a), and put

$$k_r(z) = \frac{k(t_r - z)}{(t_r)^r} \quad \forall z \in \mathbb{C} .$$

Choose any $\mathcal{B}$-neighbourhood, U, of 0. Then U has a subset V of the form

$$V = \{\psi | \psi \in \mathcal{B} \text{ and } \hat{\psi}(t) \in N\},$$

for some $\mathcal{D}$-neighbourhood, N, of 0, of the form

$$N = \{\phi \mid \phi \in \mathcal{D} \text{ and } |\phi^{(p)}(t)| < \epsilon_j$$
$$\text{if } p < m_j \text{ and } |t| > j,\ j = 0,1,2,\ldots\},$$

where $\{\epsilon_j\}_0^\infty$ is a sequence of positive numbers decreasing monotonically to zero, and $\{m_j\}_0^\infty$ is another sequence of positive numbers diverging monotonically to $+\infty$.

Choose an integer K such that [-K,K] includes the support of $\hat{k}$. Put

$$A = \operatorname*{Max}_{p<m_K} \ \sup_{u\in\mathbb{R}} \{|\hat{k}^{(p)}(u)|\}.$$

Then if $|u| \leq K$, and if $p < m_K$, we have that

$$\hat{k}_r(u) = \frac{1}{(t_r)^r} \int k(t_r - t)e^{iut}dt$$

$$= \frac{e^{iut_r}}{(t_r)^r} \hat{k}(-u).$$

Hence

$$|k_r^{(p)}(u)| \leq \frac{1}{|t_r|^r} \sum_{j=o}^{p} \binom{p}{j} |\hat{k}^{(p-j)}(-u)|\,|t_r|^j$$

$$\leq A(1+|t_r|)^p/|t_r|^r$$

$$\leq A(1+|t_r|)^{m_k}/|t_r|^r.$$

On the other hand if $|u| > K$, then for any non-negative integer p, $\hat{k}_r^{(p)}(u) = 0$. Consequently there exists R such that if $r > R$, then $k_r \in U$. Since F is continuous we have that

$$\langle F, k_r\rangle \to 0 \quad \text{as} \quad r \to \infty.$$

But this is a contradiction. It follows that $(F * k)(x)$ is tempered.

(d) If $F \in \mathcal{B}'$ and $k \in \mathcal{B}$, then $\exists (F * k)'(x) = (F * k')(x)$.

Proof: Note that if $|a| \leq 1$ and if $|b| \leq c$, then for any non-negative integer m and any integer n such that $m \geq n$, we have that

$$\left| \sum_{j=m}^{\infty} \frac{a^{j-n} b^j}{j!} \right| \leq e^c. \tag{2}$$

Choose $x \in \mathbb{R}$ and suppose that $h \neq 0$ (h real). Then we have that

$$\frac{1}{h} [\langle F_t, k(x+h-t)\rangle - \langle F_t, k(x-t)\rangle] =$$

$$= \langle F_t, \frac{k(x+h-t)-k(x-t)}{h} \rangle .$$

Choose any $\mathfrak{Z}$-neighbourhood, U, of 0. Then U has a subset V of the form

$$V = \{\psi | \psi \in \mathfrak{Z} \text{ and } \hat{\psi}(t) \in N\},$$

where

$$N = \{ \phi \mid \phi \in \mathfrak{D} \text{ and } |\phi^{(p)}(t)| < \epsilon_j \text{ if } p < m_j \text{ and } |t| > j,\ j = 0,1,2,\ldots\},$$

for some $\{\epsilon_j\}_0^\infty$, a sequence of positive numbers converging monotonically to zero, and some $\{m_j\}_0^\infty$, a divergent monotonic increasing sequence of positive numbers. Put $k_x(t) = k(x-t)$, and choose an integer K such that $[-K,K]$ includes the support of $\hat{k}_x(w)$. Put

$$A = \underset{p<m_K}{\mathrm{Max}} \; \underset{w\in \mathbb{R}}{\mathrm{Sup}} \{|\hat{k}_x^{(p)}(w)|\},$$

$$\delta = \mathrm{Min}\{\epsilon_K/(Ae^K 2^{m_K}), 1\}.$$

Then if $0 < |h| < \delta$, and if $p < m_K$ and $|w| < K$, we have that

$$\left|\frac{d^p}{dw^p}\left[\int \left\{\frac{k(x+k-t) - k(x-t)}{h} - k'(x-t)\right\}e^{iwt}dt\right]\right|$$

$$= \left|\frac{d^p}{dw^p}\left[\hat{k}_x(-w)\left\{\frac{e^{iwh}-1}{h} - iw\right\}\right]\right|$$

$$= \left|\sum_{r=o}^{p}\binom{p}{r}\hat{k}_x^{(p-r)}(-w)(-1)^{p-r}\frac{d^r}{dw^r}\left[\frac{e^{iwh}-1}{h} - iw\right]\right|$$

$$\leq A\sum_{r=o}^{p}\binom{p}{r}\left|\frac{d^r}{dw^r}\left[\frac{e^{iwh}-1}{h} - iw\right]\right|$$

$$\leq A|h|e^K\sum_{r=o}^{p}\binom{p}{r}, \quad \text{by (2)},$$

$$< \epsilon_K.$$

On the other hand if $|w| > K$, then for any p

$$\left|\frac{d^p}{dw^p}\left[\int \left\{\frac{k(x+h-t) - k(x-t)}{h} - k'(x-t)\right\}e^{iwt}dt\right]\right| = 0.$$

Hence if $0 < |h| < \delta$, we have that

$$\left(\frac{k(x+h-t) - k(x-t)}{h} - k'(x-t)\right) \in U.$$

The result now follows as for case (b) above.

A.1.2. To prove the $\mathfrak{S}$-case of L.2.4.11., it is sufficient to show that if $F \in \mathfrak{S}'$, $k_1, k_2 \in \mathfrak{S}$, then $(F * k_1) * k_2 = F * (k_1 * k_2)$. Likewise for the $\mathfrak{Z}$-case. The proofs are quite direct.

(a) Suppose that $F \in \mathfrak{S}'$, $k_1, k_2 \in \mathfrak{S}$. Choose any $\mathfrak{S}$-neighbourhood, U, of 0. Then U has a subset V of the form

$$V = \{\phi \mid \phi \in \mathfrak{S} \text{ and } |t|^m |\phi^{(p)}(t)| < \epsilon \text{ for } m \leq M,\ p \leq P\},$$

for some $\epsilon > 0$ and some positive integers M,P.

Denote by Δ_r the subdivision of $[-r,r]$ into $2r^3$ equal parts, and let $x_{r,j}$ $j = 0,1,\ldots,2r^3$, denote the subdivision points. Put

$$g_r(w) = \sum_{j=0}^{2r^3} k_1(w-x_{r,j})k_2(x_{r,j})(x_{r,j+1}-x_{r,j}) - \int k_1(w-x)k_2(x)dx.$$

Put

$$A_j = \operatorname*{Max}_{\substack{m<M \\ p\leq P+1}} \ \operatorname*{Sup}_{u\in\mathbb{R}} \{(1+|u|)^m |k_j^{(p)}(u)|\},\ j = 1,2.$$

For each $p \leq P$, we have that

$$|g_r^{(p)}(w)|$$

$$\leq \sum_{j=o}^{2r^3} \int_{x_{r,j}}^{x_{r,j+1}} |k_1^{(p)}(w-x_{r,j})k_2(x_{r,j})$$

$$- k_1^{(p)}(w-x)k_2(x)|\,dx + \int_{|x|>r} |k_1^{(p)}(w-x)||k_2(x)|\,dx.$$

Applying the mean value theorem to the real and imaginary parts of the following expression, we have that for each $x > x_{r,j}$ $\exists\, \xi_1, \xi_2 \in [x_{r,j}, x]$ such that

$$|k_1^{(p)}(w-x_{r,j})k_2(x_{r,j}) - k_1^{(p)}(w-x)k_2(x)|$$

$$= |x-x_{r,j}||\,\mathfrak{Re}\,[\tfrac{d}{dt}\{k_1^{(p)}(w-t)k_2(t)\}]_{t=\xi_1}$$

$$+ i\,\mathfrak{Im}[\tfrac{d}{dt}\{k_1^{(p)}(w-t)k_2(t)\}]_{t=\xi_2}\,|$$

$$\leq \frac{2}{r^2} \sup_{\xi \in [x_{r,j}, x]} \{|k_1^{(p+1)}(w-\xi)k_2(\xi)|$$

$$+ |k_1^{(p)}(w-\xi)k_2'(\xi)|\}.$$

Noting that for each ξ, w we have the inequality

$$|w|^n \leq 2^n(|w-\xi|^n + |\xi|^n), \quad n = 1,2,3,\ldots,$$

it follows that if $x \in [x_{r,j}, x_{r,j+1}]$ then

$$|w|^m |k_1^{(p)}(w-x_{r,j})k_2(x_{r,j}) - k_1^{(p)}(w-x)k_2(x)|$$

$$\leq \frac{2^{m+1}}{r^2} [4A_1A_2].$$

Hence, for $p \leq P$ and $m \leq M$, we have that

$$|w|^m |g_r^{(p)}(w)|$$

$$\leq \frac{2^{m+4}}{r} A_1A_2 + \int_{|x|>r} |w|^m |k_1^{(p)}(w-x)|\, |k_2(x)|\, dx$$

$$\leq \frac{2^{M+4}}{r} A_1A_2 + 2^M A_1 \int_{|x|>r} (1+|x|)^M |k_2(x)|\, dx.$$

It follows that there exists $R > 0$ such that if $r > R$, then $g_r \in U$. Since F is continuous it follows that for each real w,

$$(F * g_r)(w) \to 0 \quad \text{as} \quad r \to \infty$$

Hence

$$\langle F_u, \sum_{j=0}^{2r^3} k_1(w-u-x_{r,j})k_2(x_{r,j})(x_{r,j+1}-x_{r,j})\rangle$$

$$\to (F * (k_1 * k_2))(w) \quad \text{as} \quad r \to \infty.$$

i.e.

$$\sum_{j=0}^{2r^3} (F * k_1)(w-x_{r,j})k_2(x_{r,j})(x_{r,j+1}-x_{r,j})$$

$$\to (F * (k_1 * k_2))(w) \quad \text{as} \quad r \to \infty.$$

But for each fixed w, $(F * k_1)(w-x)k_2(x) \in \mathfrak{S}$, and so the left hand side converges to $\int (F * k_1)(w-x)k_2(x)dx$ as $r \to \infty$. Hence $(F * k_1) * k_2 = F * (k_1 * k_2)$.

b) Suppose that $F \in \mathfrak{Z}'$, $k_1, k_2 \in \mathfrak{Z}$. Choose any $\mathfrak{Z}$-neighbourhood, U of 0. Then U has a subset V of the form

$$V = \{\psi | \psi \in \mathfrak{Z} \text{ and } \hat{\psi}(t) \in N\},$$

where

$$N = \{\phi \mid \phi \in \mathfrak{D} \text{ and } |\phi^{(p)}(t)| < \epsilon_j \text{ if } p < m_j$$
$$\text{and } |t| > j,\ j = 0,1,2,\ldots\},$$

for some $\{\epsilon_j\}_0^\infty$, a monotonic sequence of positive numbers converging to zero, and some $\{m_j\}_0^\infty$, a divergent monotonic sequence of positive numbers. Using the same definitions of Δ_r, $x_{r,j}$, g_r as before, we have that

$$\hat{g}_r(u) = \hat{k}_1(u)\{\sum_{j=0}^{2r^3} k_2(x_{r,j})e^{ix_{r,j}u}(x_{r,j+1}-x_{r,j}) - \hat{k}_2(u)\}$$
$$= \hat{k}_1(u)\ell_r(u), \text{ say.}$$

Take K, a positive integer, such that [-K,K] includes the support of $\hat{k}_1(u)$. Put

$$M_1 = \underset{p \le m_K}{\operatorname{Max}} \ \underset{u \in \mathbb{R}}{\operatorname{Sup}} \{|\hat{k}_1^{(p)}(u)|\},$$

$$M_2 = \underset{\substack{p \le m_K \\ q=1,2}}{\operatorname{Max}} \ \underset{x \in \mathbb{R}}{\operatorname{Sup}} \{|x|^p |k_2^{(q)}(x)|\}.$$

Then if $p \le m_K$ and if $|u| \le K$, we have that

$$|\hat{g}_r^{(p)}(u)| = |\sum_{m=o}^{p} \binom{p}{m} \hat{k}_1^{(p-m)}(u) \ell_r^{(m)}(u)|$$

$$\le M_1 \sum_{m=o}^{p} \binom{p}{m} |\ell_r^{(m)}(u)|. \qquad (3)$$

Note also that if $|u| > K$, then $\hat{g}_r^{(p)}(u) = 0$, $p = 0,1,2,\ldots$.

Now we have that if $m \le m_K$, then

$$\ell_r^{(m)}(u) = \sum_{j=o}^{2r^3} (i)^m \int_{x_{r,j}}^{x_{r,j+1}} \{(x_{r,j})^m k_2(x_{r,j}) e^{ix_{r,j}u}$$

$$- x^m k_2(x) e^{ixu}\} dx - (i)^m \int_{|x|>r} x^m k_2(x) e^{ixu} dx.$$

Applying the mean value theorem, we have that there exists $\xi_1, \xi_2 \in [x_{r,j}, x]$ such that

$$|(x_{r,j})^m k_2(x_{r,j}) e^{ix_{r,j}u} - x^m k_2(x) e^{ixu}|$$

$$= |x - x_{r,j}| |\mathcal{R}_e[\tfrac{d}{dt}\{t^m k_2(t) e^{iut}\}]_{t=\xi_1}$$

$$+ i\, \mathcal{I}_m[\tfrac{d}{dt}\{t^m k_2(t) e^{iut}\}]_{t=\xi_2}|$$

$$\le \frac{2}{r^2} \sup_{\xi \in [x_{r,j}, x_{r,j+1}]} \{m|\xi|^{m-1}|k_2(\xi)| + |\xi|^m |k_2'(\xi)|$$

$$+ |u||\xi|^m |k_2(\xi)|\},$$

for $x \in [x_{r,j}, x_{r,j+1}]$. Hence if $m \leq m_K$ and if $|u| \leq K$, then

$$|\ell_r^{(m)}(u)|$$

$$\leq \sum_{j=o}^{2r^3} \frac{4}{r^4}[mM_2 + M_2 + |u|M_2] + \int_{|x|>r} |x|^m |k_2(x)|dx$$

$$\leq \frac{8}{r}[m_K + K + 1]M_2 + \int_{|x|>r} (1+|x|)^{m_K}|k_2(x)|dx.$$

It follows from this and (3) that there exists R such that if $r > R$, then $g_r \in U$. The required result now follows as in the case a) given above.

A.1.3. In this sub-section a proof is given for the $\mathfrak{S}$ -case of lemma L.2.4.16. The $\mathfrak{D}$ and $\mathfrak{Z}$ -cases may either be proved similarly, or, once the $\mathfrak{D}$ -case is shown, the $\mathfrak{Z}$ -case may be proved by taking Fourier transforms. For a proof of the $\mathfrak{D}$ -case see (9) Pg.368.

Lemma Suppose that F is a discontinuous linear functional on $\mathfrak{S}$. Then for each $\mathfrak{S}$ -neighbourhood, U, of O, and for each $\lambda > 0$, there exists $k \in U$ such that $|\langle F,k\rangle| > \lambda$.

Proof: If U is an $\mathfrak{S}$ -neighbourhood of O, then U has a subset V of the form

$$V = \{\phi \mid \phi \in \mathfrak{S} \text{ and } |x|^m |\phi^{(p)}(x)| < \epsilon \text{ if } m \leq M \text{ and } p \leq P\},$$

for some $\epsilon > 0$, and some positive integers M,P. Since F is discontinuous, $\exists h > 0$ such that if W is any $\mathfrak{S}$-

neighbourhood of 0, then $\exists \ell \in W$ such that $|\langle F, \ell\rangle| > h$. Put

$$V^* = \{\phi \mid \phi \in \mathfrak{S} \text{ and } |x|^m |\phi^{(p)}(x)| < \epsilon h/\lambda \text{ if } m \leq M \text{ and } p \leq P\}.$$

Then $\exists \ell \in V^*$ such that $|\langle F, \ell\rangle| > h$. Put

$$k(x) = {}^{\lambda \ell(x)}/_{h}.$$

Then

$$|\langle F,k\rangle| = \frac{\lambda}{h} |\langle F, \ell\rangle| > \lambda.$$

Moreover if $m \leq M$ and $p \leq P$, then

$$|x|^m |k^{(p)}(x)| < (\epsilon h/\lambda)(\lambda/h) = \epsilon,$$

and so $k \in V$. This completes proof of the lemma.

Suppose that $\{F_r\}$ is a sequence of $\mathfrak{S}'$-functionals and that $\exists \lim_{r\to\infty} \langle F_r, \phi\rangle$, $\forall \phi \in \mathfrak{S}$. Define the linear functional F by the relation

$$\langle F, \phi\rangle = \lim_{r\to\infty} \langle F_r, \phi\rangle \quad \forall \phi \in \mathfrak{S} .$$

Then F is continuous.

Proof: Suppose that F were discontinuous. For n = 1,2, 3,..., put

$$U_n = \{\phi \mid \phi \in \mathfrak{S} \text{ and } |x|^m |\phi^{(p)}(x)| < \frac{1}{2^n} \text{ if } m,p \leq n\},$$

so that each U_n is an $\mathfrak{S}$-neighbourhood of 0. Suppose that for each n, $\ell_n \in U_n$. Put

$$\psi_n(x) = \sum_{r=n}^{\infty} \ell_r(x).$$

Then for each m,p we have that

$$|x|^m |\psi_n^{(p)}(x)| \leq \sum_{r=n}^{\max(m,p)-1} |x|^m |\ell_r^{(p)}(x)| + \sum_{r=\max(m,p)}^{\infty} |x|^m |\ell_r^{(p)}(x)| .$$

It follows that $\psi_n \in \mathfrak{S}$ n = 1,2,3,..., and that if $m,p \leq n$, then

$$|x|^m |\psi_n^{(p)}(x)| \leq \sum_{r=n}^{\infty} |x|^m |\ell_r^{(p)}(x)| \leq \frac{1}{2^{n-1}} .$$

Hence $\psi_n \in U_{n-1}$, n = 2,3,4,... . Clearly the sequence $\{\psi_n\}$ converges to zero in the topology of $\mathfrak{S}$. (4)

Take $V_1 = U_1$. Choose $k_1 \in V_1$ such that

$$|\langle F,k_1\rangle| > 2.$$

Choose m_1 such that

$$|\langle F_{m_1},k_1\rangle - \langle F,k_1\rangle| < 1.$$

Assume that for r = 1,2,...,n-1, neighbourhoods of 0, V_r, Functions k_r, and integers m_r, have been chosen. Choose a neighbourhood of 0, V_n, $\subseteq U_n$, such that if $\ell \in V_n$ then

$$|\langle F_{m_r}, \ell\rangle| < \frac{1}{2^n} \qquad r = 1,2,\ldots,n-1.$$

Choose $k_n \in V_n$ such that

$$|\langle F,k_n\rangle| > 2n + \sum_{r=1}^{n-1} |\langle F,k_r\rangle| .$$

Choose $m_n > m_{n-1}$ such that

$$|\langle F_{m_n},k_r\rangle - \langle F,k_r\rangle| < 1 \qquad r = 1,2,\ldots,n.$$

Put $k(x) = \sum_{r=1}^{\infty} k_r(x)$. Then $k \in \mathfrak{S}$, and by (4), the sum converges to k in the topology of $\mathfrak{S}$. Moreover, for each n,

$$|\langle F_{m_n},k\rangle| \geq |\langle F_{m_n},k_n\rangle| - \sum_{r=1}^{n-1} |\langle F_{m_n},k_r\rangle| - \sum_{r=n+1}^{\infty} |\langle F_{m_n},k_r\rangle|$$

$$> |\langle F,k_n\rangle| - 1 - \sum_{r=1}^{n-1} |\langle F,k_r\rangle| - (n-1) - \sum_{r=n+1}^{\infty} \frac{1}{2^r}$$

$$> n - \frac{1}{2^n} .$$

But, by assumption $\exists \lim_{n\to\infty} \langle F_n,k\rangle$. It follows that F must be continuous.

A.2. Spaces of type $\mathfrak{S}$.

We shall be concerned, throughout this section, only with functions possessing derivatives of all orders. The relevant definitions of Gel'fand and Shilov are, for $\beta > 1$, as follows:

1) $\mathfrak{S}_\beta^0 = \{\phi \mid \exists A,B,C, > 0$, such that for each $m,p = 0,1,\ldots,$

$$|x^m \phi^{(p)}(x)| \le CA^m m^{m\beta} B^p.\},$$

2) $\mathfrak{S}_\beta = \{\phi \mid \exists A > 0$ such that for each $p = 0,1,2,\ldots,$ $\exists C_p > 0$ such that for each $m = 0,1,2,\ldots,$

$$|x^m \phi^{(p)}(x)| \le C_p A^m m^{m\beta}.\}.$$

These definitions may be found in (10) Pg.167-8. Note that $m^{m\beta}$ is taken to be unity for $m = 0$.

Put

$$S = \{f \mid \forall \alpha < 1, \forall p = 0,1,2,\ldots, \exists C_{\alpha,p} \text{ such that}$$

$$|f^{(p)}(x)| < C_{\alpha,p} \exp[-|x|^\alpha]\},$$

and

$$S^0 = S \cap \mathfrak{Z}$$

Then S^0 contains precisely those functions in S with compactly supported transforms. We will now show that

a) $S = \bigcap_{\beta > 1} \mathfrak{S}_\beta$

b) $S^{o} = \bigcap_{\beta>1} \mathfrak{S}_{\beta}^{o}$.

Choose $\beta > 1$, and choose $\phi \in \mathfrak{S}_{\beta}^{o}$. Suppose that A,B,C are as in definition 1). Then by observing that for each real x,

$$|\phi^{(p)}(x)| < C\, B^{p}, \quad p = 0,1,2,\ldots,$$

it is clear that ϕ possesses a Taylor series which converges everywhere to ϕ. Moreover ϕ may be extended to an entire function of a complex variable, and we have the inequality

$$|x^{m}\phi^{(p)}(x+iy)| = |x|^{m}\, \left|\sum_{r=0}^{\infty} (iy)^{r}\, \frac{\phi^{(p+r)}(x)}{r!}\right|$$

$$\leq C\, B^{p}\, A^{m}\, m^{m\beta}\, e^{B|y|}\ .$$

Taking $m = m(x) = [\![(|x|/A\beta^{\beta})^{1/\beta}]\!]$, we have that

$$|x^{m}\phi^{(p)}(x+iy)| \leq C\, B^{p}\, \left|\frac{x}{\beta^{\beta}}\right|^{m} e^{B|y|},$$

and so

$$|\phi^{(p)}(x+iy)| \leq C\, B^{p}\, \beta^{-\beta m} e^{B|y|}$$

$$\leq C\, B^{p}\, \beta^{\beta}\exp[-|\tfrac{x}{A}|^{1/\beta}\log\beta]e^{B|y|}\ .$$

Hence, for some $a > 0$, and for $C' = C\,\beta^{\beta}$, we have that

$$|\phi^{(p)}(x+iy)| \leq C'B^{p}\exp[-a|x|^{1/\beta} + B|y|]. \qquad (1)$$

(Similarly, if $\phi \in \mathfrak{S}_{\beta}$ $(\beta > 1)$, then ϕ satisfies an inequality of the form

$$|\phi^{(p)}(x)| \leq C'_p \exp[-a|x|^{1/\beta}]. \qquad (2).)$$

If $\phi \in \mathfrak{S}^o_\beta$, then it follows from (1) by contour integration (see (3) Pg.97) that $\hat{\phi}$ is supported by $[-B,B]$, and so $\hat{\phi} \in \mathfrak{D}$. It follows that $\bigcap_{\beta>1} \mathfrak{S}^o_\beta \subseteq S^o$. In the other case, it follows directly from (2) that $\bigcap_{\beta>1} \mathfrak{S}_\beta \subseteq S$.

Conversely, suppose that $\phi \in S$. Then for each $\beta > 1$, and each $p = 0,1,2,\ldots$, there exists $C_{\beta,p}$ such that

$$|\phi^{(p)}(x)| \leq C_{\beta,p} \exp[-|x|^{1/\beta}].$$

Hence, for each $m \geq 0$,

$$\begin{aligned}|x^m \phi^{(p)}(x)| &\leq C_{\beta,p} \exp[m \log|x| - |x|^{1/\beta}] \\ &\leq C_{\beta,p} \exp[m\beta \log m\beta - m\beta],\end{aligned}$$

by considering $\sup_{u>o}\{m \log u - u^{1/\beta}\}$. It follows that

$$|x^m \phi^{(p)}(x)| \leq C_{\beta,p}(\beta^\beta e^{-\beta})^m m^{m\beta},$$

and so $\phi \in \mathfrak{S}_\beta$, $\forall \beta > 1$. Hence $S \subseteq \bigcap_{\beta>1} \mathfrak{S}_\beta$.

Suppose now that $\phi \in S^o$. Then, for each $\beta > 1$, $\exists C_\beta > 0$ such that

$$|\phi(x)| \leq C_\beta \exp[-|x|^{1/\beta}].$$

Hence, for $q = 0,1,2,\ldots$, we have that

$$|\hat{\phi}^{(q)}(t)| \leq C_\beta \int |x|^q \exp[-|x|^{1/\beta}]dx$$

$$\leq C_\beta \sup_{u>o}\{u^q \exp[-\tfrac{1}{2}(u)^{1/\beta}]\} \int \exp[-\tfrac{1}{2}|x|^{1/\beta}]dx$$

$$= C_\beta (2\beta q)^{q\beta} e^{-q\beta} \int \exp[-\tfrac{1}{2}|x|^{1/\beta}]dx$$

$$= C'_\beta\, q^{q\beta}(B_\beta)^q \;, \quad \text{say.} \qquad (3).$$

In relation (3), we may assume that for each $\beta > 1$, $B_\beta > 1$. Suppose that $A > 1$, and that $[-A,A]$ includes the support of $\hat{\phi}$.

The inequality

$$\left|\frac{d^q}{dt^q}\{\hat{\phi}(t)t^n\}\right| \leq C'_\beta\, q^{q\beta}(B_\beta)^q(2A)^n \qquad (4)$$

certainly holds for $n = 0$ and $q = 0,1,2,\ldots$. Suppose, inductively, that (4) holds for $n = 0,1,\ldots,N$, and $q = 0,1,2,\ldots$. Then we have that, for each non-zero q,

$$\left|\frac{d^q}{dt^q}\{\hat{\phi}(t)t^{N+1}\}\right| \leq \left|t\frac{d^q}{dt^q}\{\hat{\phi}(t)t^N\}\right| + q\left|\frac{d^{q-1}}{dt^{q-1}}\{\hat{\phi}(t)t^N\}\right|$$

$$\leq C'_\beta(2A)^N\{Aq^{q\beta}(B_\beta)^q + q(q-1)^{(q-1)\beta}(B_\beta)^{q-1}\}$$

$$\leq C'_\beta(2A)^N\{Aq^{q\beta}(B_\beta)^q + q^{q\beta-\beta+1}(B_\beta)^{q-1}\}$$

$$\leq C'_\beta(2A)^N\{Aq^{q\beta}(B_\beta)^q + q^{q\beta}(B_\beta)^q\}$$

$$\leq C'_\beta(2A)^{N+1}q^{q\beta}(B_\beta)^q \;,$$

since $\beta > 1$, $A > 1$, and $B_\beta > 1$. While for $q = 0$, clearly

$$\left|\frac{d^q}{dt^q}\{\hat{\phi}(t)t^{N+1}\}\right| \leq C'_\beta(2A)^{N+1}.$$

Hence, by induction, (4) holds for each $n,q = 0,1,2,\ldots$.

Therefore we have that

$$|x^q\,\phi^{(n)}(x)| = 2\pi\left|\int_{-A}^{A} e^{-ixt}\left[\frac{d^q}{dt^q}\{\hat{\phi}(t)t^n\}\right]dt\right|$$

$$\leq 4\pi A C'_\beta(2A)^n\, q^{q\beta}(B_\beta)^q ,$$

and so $\phi \in \mathfrak{S}^{\,o}_\beta$, $\forall\beta > 1$. It follows that $S^o \subseteq \bigcap_{\beta>1} \mathfrak{S}^{\,o}_\beta$.

For further details concerning spaces of type $\mathfrak{S}$, see (10) Pg.166-256.

CONJECTURES AND CONCLUDING REMARKS

As mentioned in the footnote on page 5, there seems to be no reason why the construction of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ should not be possible with $\mathfrak{A}$ a first order relational structure instead of a universal algebra. Less clear is the question of whether any embedding theory is possible with $\mathfrak{B}$ a non-Abelian semigroup; this would appear to be a rather difficult problem. It would, anyway, be interesting if some applications of the theory could be found in fields other than elementary algebra or functional analysis. However a number of conjectures are suggested by the applications in functional analysis already made in the thesis: in particular in the case of the $\mathfrak{Q}(\mathfrak{X}_\beta, \mathfrak{S}_\beta \cap \mathfrak{B})$ spaces of section 4.3., it appears not unlikely that $\mathfrak{Q}_0(\mathfrak{X}_\beta, \mathfrak{S}_\beta \cap \mathfrak{B})$ is isomorphic, and (with the topology of pointwise convergence on $\mathfrak{X}_\beta$) homeomorphic, with $\mathfrak{S}'_\beta$, the class of continuous linear functionals on $\mathfrak{S}_\beta$ (when $\mathfrak{S}'_\beta$ carries the weak dual topology).

It would appear that the sole precursor of the technique of applying an extension process to a space of functions, at all resembling that used in the thesis for the space $\mathfrak{X}$, is the Operational Calculus of J. Mikusinski.[†] In brief, Mikusinski's technique is based on the result of Titchmarsh that if f(x) and g(x) are supported by the

[†]See (15).

positive real line and are locally integrable, then if $f * g$ is null, at least one of f and g is null. It follows that the class of functions, continuous on $[0,\infty)$ and zero for $x < 0$, forms a ring under convolution, and that this ring has no zero divisors. Consequently this ring may be extended to a field of quotients. Mikusinski goes on to define differentiation and integration within this system and shows how the system may be used in solving various types of differential equations. A notion of convergence is introduced, applicable to sequences all of whose members can be expressed with a common denominator. It is clear that in certain ways Mikusinski's system overlaps with the class $\mathcal{D}'$ of Schwartz distributions, while on the other hand it is the case that neither system contains the other.

Mikusinski's system was constructed with some practical applications in view, and he himself drew [†] attention to some technical drawbacks of it, deriving mainly from the restriction to functions supported by a half line. He appears to have considered it impossible [‡] to construct a system such as his for functions not necessarily supported by a half line. This last view is essentially correct if the extension technique is restricted to the embedding of a ring in a field, but not correct for the more general

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[†] See (15) Pgs.124 et seq.

[‡] Loc. cit. Pg.126. Note that this is a translation of the second Polish edition, volume 30 of Monografie Matematyczne, 1957.

embedding process developed in this thesis, as the example of $\mathfrak{Q}(\mathfrak{T}, \mathfrak{G} \cap \mathfrak{W})$ shows.

It should not be difficult to extend some of the results for function spaces given in the body of this thesis to corresponding results for functions of n real variables and possibly to other sorts of functions, and there are various topological questions which might be considered. It is clear from the concluding pages of chapter four that there exists an extensive collection of $\mathfrak{Q}(\mathfrak{A}, \mathfrak{B})$ spaces based on function spaces and the convolution operation, and the possible identification of these, perhaps by some general method, as spaces of functionals would be one line for future investigation. It appears likely that there should be some summability applications of Theorem T.4.2.1., as was the case with Wiener's original theorem, and some modification of the class Λ might be possible. It is conjectured that theorem T.4.2.1. remains true with $\mathfrak{L}_{\lambda_m}$ replaced by $\mathfrak{L}_{\lambda}$.

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