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# SUPERCONVERGENT SUM-RULES AND THE ELECTROMAGNETIC FORM-FACTORS OF ELEMENTARY PARTICLES.

by

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ABSTRACT

The covarient formalism of Scadron et. al. is extended to cover processes involving virtual photons, and is used to discuss the $O(3,1) \otimes SU(2)$ decomposition into kinematic singularity free form-factors of hadron-virtual photon three and four-point functions. An S-matrix theory of inelastic hadron-lepton electromagnetic scattering is employed to develop techniques whereby superconvergent sum-rules on such four-point functions may be derived.

Attention is focused on the virtual photoproduction off nucleons of non-strange pseudoscalar and vector mesons with isospin zero or unity. Charge-conjugation invarience of hadron-virtual photon interactions is assumed and eighty new sum-rules obtained. An alternative set of new sum-rules is derived on the assumption that such interactions are not in fact charge-conjugation invarient.

A finite width resonance approximation is used in an attempt to saturate the sum-rules for pion and $\eta$ production. This yields a large number of predictions concerning the structure of the form-factors parameterising the electromagnetic excitation of the nucleon into the $\Delta(1236)$, N(1525), N(1550), N(1680), and N(1688).

The sum-rules and predictions are valid for all non time-like values of the squared four-momentum of the virtual photon.

The predictions are in good agreement with the experimental data in cases where a comparison has proved possible.

## PREFACE

The research reported in this thesis was conducted under the supervision of Professor P.T. Matthews at the Imperial College of Science and Technology between October 1965 and October 1968.

The material contained herein is original except where stated, is not the result of collaboration with any other author, and has not been submitted to this or any other university for any other degree.

The author wishes to thank Professor Matthews and Dr. K.J. Barnes for suggesting electromagnetic form-factors as worthy of further investigation, and Dr. H.F. Jones for drawing his attention to on-shell superconvergence as a possible mode of attack. This suggestion led to the author's formulation of the present off-shell technique.

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# CONTENTS

# CHAPTER I

# INTRODUCTION

## 1.1 INTRODUCTORY REMARKS AND A SKETCH OF THE MATERIAL PRESENTED IN THIS THESIS.

We are concerned in this thesis with the application of superconvergent sum-rules to the study of the form-factors parameterising the dynamics of certain electromagnetic interactions involving hadrons.

More specifically we are interested in the form-factors into which one decomposes matrix elements of the electromagnetic current taken between an initial nucleon and a final nucleon[1] or isobar.[2] In section 1.2 we remind the reader of the definition and importance of electromatic form-factors, taking those of the nucleon as examples.

The derivation of superconvergent sum-rules for elements of the T-matrix taken between two initial and two final hadrons is now a wellknown technique for obtaining relations between hadronic coupling constants,[(3)-(7)] and is reviewed in section 1.3. By replacing one of the hadrons by a real photon, several authors have successfully extended the range of application of this technique.[8] They deduce relations between hadronic electromagnetic form-factors evaluated at zero squared photon four-momentum.

It is attractive to try to generalise the formalism further by taking the photon off the mass-shell. The relations obtained will then hold for some range of non-zero values of the argument of the form-factors involved. In section 1.4 we discuss this motivation further and pose the obvious question : Is such a generalisation possible, and if so, is it valid? On the basis of a few plausible assumptions this question is answered in the affirmative in Chapter 3.

Chapter 2 falls naturally into two distinct parts. Part I reviews the covarient "spinology" formalism advocated by Scadron,[(9),(10)] employing Dirac-Rarita-Schwinger wave-functions[(11)] and contracted propogators, and the covarient Reggeisation technique of Scadron and Jones.[(12)] Although this formalism was originally developed in Lorentz-space, it is a trivial matter to extend it to Lorentz$\otimes$SU(2) space. We give details of this extension. Our discussion of kinematical singularities is a little more detailed than that appearing in the various papers of the authors cited.

In Chapter 2 Part II we generalise the covarient formalism to enable it to be used for the analysis of three and four-point vertices involving virtual photons. We derive $O(3,1) \otimes SU(2)$ covarient form-factor decompositions for a wide range of virtual photonic three-point functions involving pairs of baryons or mesons, and again indicate clearly how one ensures that the form-factors are free from kinematical singularities. The contracted propogation formalism allows one to obtain generalised Rosenbluth formulae[(1), (2)] for unpolarised lowest order electron-hadron scattering cross-sections in a particularly simple and fully covarient manner. We give an example of such a calculation.

Our virtual photon formalism is designed to reduce to one valid for real photons on taking the appropriate limit. This latter is parallel in spirit to the real photon approach of Scadron and Jones.[(13)] Consequently the work of these authors in this direction is not reviewed.

As mentioned above, Chapter 3 is concerned with the validity of our generalised superconvergence programme. Section 3.1 discusses our assumptions concerning Regge behaviour in virtual photonic four-point functions. Section 3.2 investigates the extent to which one can deduce the analytic structure of such functions from general S-matrix theoretical postulates about non-perturbative two lepton-three hadron scattering processes. It uses a generalisation of Dresden and Chou's S-matrix theory of quantum electrodynamics.(14)

The formalism thus developed is used in Chapter 4 to derive superconvergent sum-rules for all possible interactions of the form: real or virtual photon + nucleon $\longrightarrow$ nucleon + meson, in which the meson is pseudoscalar or vector, has zero strangeness, isospin zero or one, and C-parity plus or minus one. Half of the combinations of these quantum numbers are hypothetical to date,(15) but the corresponding sum-rules are included for completeness since their derivation involves little or no extra work. We also indicate the modifications necessary to these sum-rules if virtual photon-hadron interactions are not in fact charge-conjugation invarient.

Finally, in Chapter 5, we attempt to saturate our sum-rules for the virtual photoproduction of the $\eta$-meson and the pion, using the resonance approximation discussed in section 1.3. We do not find it necessary to treat the resonances as stable particles, and are able to make a crude correction for their finite decay widths.

An important feature of our technique is that the number of sum-rules for a given four-point function is generally greater in the virtual photonic case than in the real photon limit. However, our formalism is so designed that provided the form-factors are analytic at zero argument, the predictions of these additional sum-rules remain valid and non-trivial in this limit. That is, by treating the real photon as the on-shell limit of a virtual particle we are able to derive real photonic predictions which cannot be obtained by methods[8] which treat the photon as real from the outset. Thus for example, our investigations double the number of available sum-rules for pion photoproduction.

With the increased number of sum-rules at our disposal we are able to attempt more ambitious saturations than hitherto possible. In the case of the $\eta$ sum-rules only one clearly established resonance is likely to contribute,[15] but we have a wide range of possibilities in the pion case.[15] We accordingly attempt several different approaches to the saturation of these latter sum-rules. The most complicated of these involves the nucleon Born-term and four pion-nucleon resonances.

We make some attempt to compare our predictions with phenomenological fits to the experimental data. The agreement is generally good, sometimes excellent, and in a few cases spectacular.

Kinematical definitions and relations, useful equivalence theorems, computations of coupling constants for strong decays of baryonic resonances, and some fits to photoproduction data are relegated to a series of nine appendices.

Finally, we ask the reader to bear in mind the following notation. We do not distinguish between equalities and identities; the symbol $\equiv$ is always to be read: "is defined to be". The symbol $\cong$ means: "is equivalent to, in virtue of the subsidiary conditions on the wave-functions with which it is contracted". The symbols $(m-n)$ or $(Am-n)$ following an equation denote the $n^{th}$ equation of section $m$ or appendix $m$ respectively. Sections are numbered decimally, the most significant digit being the chapter number.

## 1.2 THE MEANING AND IMPORTANCE OF THE ELECTROMAGNETIC FORM-FACTORS OF THE HADRONS.

In the study of the electromagnetic interactions of hadrons, a central role is played by matrix elements of hadron electromagnetic current operators. The formalism allowing one to parameterise the dynamic behaviour of such quantities in terms of sets of Lorentz scalar functions of scalar arguments is fully described in the second part of Chapter 2. But as an introduction we review here one of the simplest and best known examples, the electromagnetic form-factors of the nucleon.(1)

One is concerned with the matrix element $\langle K\Lambda | j_\alpha(0) | p\lambda\rangle$ of the proton (neutron) electromagnetic current operator, $j_\alpha(x)$, taken between an initial proton (neutron) state with momentum $p$, helicity $\lambda$, and a final proton (neutron) state with momentum K and helicity $\Lambda$. This will be contracted with an external electromagnetic field source, or, via a virtual photon propogator, with another electromagnetic current. The interaction is assumed to be translationally invarient and consequently one is only interested in the evaluation of the matrix element at the origin of the space-time coordinates.

If the nucleon behaved as a point spin one-half Dirac particle carrying bare charge $e_0$, one could write by analogy with the unrenormalised quantum electrodynamics of electrons:

$$\langle K\Lambda | j_\alpha(0) | p\lambda\rangle = e_0\, \bar{u}^\Lambda(K)\gamma_\alpha u^\lambda(p). \qquad (1{\cdot}2-1)$$

Unfortunately this simple realisation fails completely the test of comparison with experiment, even if one tries to take proper account of radiative corrections. It is certainly not true that the neutron is unable to take part in the electromagnetic interaction, indeed its magnetic moment is one of the most accurately established constants in elementary particle physics. The representation fails equally miserably when applied to protons. It cannot account for the anomalous magnetic moment of this particle, and leads to incorrect predictions for elastic electron-proton scattering and for proton Compton scattering.

The reason for this failure is not hard to see. The realisation 1.2-1 ignores the fact that the nucleon is a strongly interacting particle, and neglects the possibility of its possessing a finite spatial structure. The bare nucleon will be surrounded by a cloud of virtual pions, (and possibly other virtual particles). The virtual photon may interact with these as well as with the bare nucleon, thus modifying the electromagnetic interaction. Whether or not the bare nucleon is endowed with a spatial structure further modifying this interaction is an open question, but the virtual pion cloud will certainly cause the physical nucleon to behave as a structured particle.

There exist an infinity of Feynman graphs corresponding to strong interaction corrections to equation 1.2-1, and no method of summing these is known. So instead one adopts a different approach which at once takes account of all possible corrections to this equation.

In analogy with the previous equation one first factors out the helicity dependence of the matrix element, defining a "vertex function", $\mathcal{V}_\alpha$, by:

$$\langle K\Lambda | j_\alpha(0) | p\lambda \rangle = \bar{u}^\Lambda(K)\, \mathcal{V}_\alpha\, u^\lambda(p). \qquad (1{\cdot}2-2)$$

This a completely general Lorentz-group theoretic operation, the nucleon spinors corresponding to matrix elements of relativistic boosts.

The vertex function will be a 4 x 4 matrix in the space of four-component spinors. In addition it is required to satisfy certain constraints imposed by the assumed Lorentz, P, C, and T-invariences of the electromagnetic interaction and the fact that $j_\alpha$ is an hermitian operator. A particular consequence of these is the requirement that $\mathcal{V}_\alpha$ should have the same Lorentz transformation properties as $j_\alpha$, that is, it should be a Lorentz proper vector. Furthermore, the interaction is required to be gauge-invarient (current conserving) when the photon involved is real (virtual). The vertex function must therefore vanish on contraction with $q_\alpha$, the momentum of the photon.

The next step is to expand $\mathcal{V}_\alpha$ in terms of a set of linearly independent basis functions, (called "kinematic covarients"), satisfying these same constraints. They must remain linearly independent when sandwiched between the nucleon spinors, and those that do will be said to be "linearly inequivalent". The fact that the nucleon spinors satisfy the Dirac equation turns out to imply that no more than two kinematic covarients satisfying the required constraints can be linearly inequivalent. This result can be shown to be related to the spins and intrinsic parities of the particles involved. The expansion coefficients are called "electromagnetic form-factors". Being Lorentz scalars they can only depend on scalar variables. Since the nucleons are on-shell, only one linearly independent scalar variable can be constructed from the available momenta: it is convenient and conventional to choose to work with $q^2$, the squared four-momentum of the photon.

Our expansion of the vertex function will be the most general compatible with the various kinematical constraints and the electrodynamical one of current-conservation. The remaining dynamics is contained entirely in the functional dependence on $q^2$ of the two form-factors, and a study of the dynamics is reduced to a study of this dependence. It is clearly desirable that the form-factors should not be subject to any spurious kinematical dependence, that is, they should be "kinematic singularity free".

As the reader will no doubt be aware, the two conventional decompositions of the nucleon electromagnetic vertex function are:

$$v_\alpha = \left[F_1(q^2)\gamma_\alpha + \frac{i}{2m}F_2(q^2)\sigma_{\alpha\beta}q_\beta\right], \tag{1.2-3}$$

$$v_\alpha = \frac{2m}{P'^2}\left[G_e(q^2)P'_\alpha + G_m(q^2)\frac{1}{2m}\varepsilon_\alpha(P'q\gamma)\gamma_5\right]. \tag{1.2-4}$$

The momentum $P'$ is defined by:

$$P' \equiv p + K, \tag{1.2-5}$$

and m is the nucleon mass. At this point we better mention that our conventions regarding the metric tensor, scalar products, Dirac matrices, spin one-half wave-functions, and contracted Levi-Cevita tensors are to be found in Appendices 1, 2 and 3.

The form-factors are related as follows:

$$G_e = F_1 + q^2F_2/4m^2\ , \tag{1.2-6}$$

$$G_m = F_1 + F_2 . \tag{1.2-7}$$

and are assumed to carry superscripts p or $n$ according as we are dealing with the proton or the neutron. As a consequence of the hermiticity of the current operator, they can be shown to be purely real.

As mentioned above, we are now treating the nucleons as structured particles. The values of the form-factors at vanishing $q^2$ may be related to this structure in the following manner.

One compares the predictions of equations 1.2-1 and 3 for nucleon scattering by an external field. Working in a special frame, the Breit frame, in which:

$$\not{p} = -\underline{K} \tag{1.2-8}$$

and taking the static limit (vanishing q) at the conclusion of the two calculations, one is able to makethe identifications:

$$G_e(0) = e \ , \tag{1.2-9}$$

$$G_m(0) = 2m\mu . \tag{1.2-10}$$

where $e$ and $\mu$ are respectively the physically observed charge and magnetic moment of the appropriate nucleon. Defining the anomalous moment $\mu_a$ in an obvious way by:

$$\mu = \frac{e}{2m} + \mu_a \ , \tag{1.2-11, 12}$$

one then deduces:

$$F_1(0) = e \quad , \quad F_2(0) = 2m\mu_a . \tag{1.2-13, 14}$$

$F_1(q^2)$, $F_2(q^2)$, $G_e(q^2)$ and $G_m(q^2)$ are accordingly called the charge, moment, electric, and magnetic form-factors of the nucleon.

The various derivatives of $F_1$ ($F_2$) evaluated at zero $q^2$ may be similarly related to the Fourier transforms of the various moments of the spatial charge (magnetisation) distributions of the physical nucleon. In particular, the first derivatives are related to the mean-square radii of the corresponding distributions. It should be stressed, however, that the form factors are not to be interpreted as Fourier transforms of the spatial charge and moment distributions, rather, the former quantities at zero argument are related to the latter in one special frame.

The above discussion applies for kinematical reasons only to non time-like photons. In the time-like case, that is, virtual photoproduction of nucleon-antinucleon pairs, one defines the vertex function in an analogous fashion:

$$\langle K\Lambda, -\bar{p}\bar{\lambda} | j_\alpha(0) | 0 \rangle = \bar{u}^\Lambda(K) V_\alpha v^{\bar{\lambda}}(-\bar{p}). \tag{1.2-15}$$

One may again adopt equations 1.2-3 and 4 as suitable decompositions of this vertex function, and one then assumes that the form-factors for time-like $q^2$ may be obtained from those for non-time-like $q^2$ by analytic continuation. In other words, corresponding form-factors for the time-like and non-time-like interactions are assumed to be different sectors of the same analytic function.

It can be shown that the charge and moment form-factors are kinematic singularity free for all $q^2$. Hence equations 1.2-6 and 7 imply either that the electric and magnetic form-factors are non-independent at the pair-production threshold:

$$G_e(4m^2) = G_m(4m^2), \tag{1.2-16}$$

or that $F_1$ and $F_2$ have a dynamical pole at this point. This question has been discussed in detail by Bergia and Brown,(16) and also by Barger and Carhart.(17) The conclusion is that 1.2-16 should indeed be taken as operative; (it is then a purely kinematical constraint).

The disadvantage of working with form-factors subject to such a constraint is generally considered outweighed by the fact that the lowest order unpolarised cross-section for elastic electron-nucleon scattering involves only the squares of $G_e$ and $G_m$, not the cross-term $G_e G_m$. Although the charge and moment form-factors are free of kinematical singularities and constraints, the above cross-section involves the three combinations: $F_1^2$, $F_2^2$, and $F_1 F_2$.

In their assessment of the experimental data on nucleon form-factors, Chan et.al.[18] conclude that for non-positive definite $q^2$ this data is best fitted by ignoring the threshold constraint 1.2-16. For values of $-q^2$ up to about $5(GeV/c)^2$ the data is then very well fitted by the "scaling laws":

$$G_e^p(q^2) = \frac{G_m^p(q^2)}{1+\kappa^p} = \frac{G_m^n(q^2)}{\kappa^n} = e\left(1 - \frac{q^2}{0.71}\right)^{-2}. \qquad (1.2\text{-}17)$$

In these equations:

$$\kappa^p \equiv \frac{2m}{e}\mu_a^p = 1.79276\ , \qquad (1.2\text{-}18)$$

$$\kappa^n \equiv \frac{2m}{e}\mu_a^n = -1.91315\ , \qquad (1.2\text{-}19)$$

and the quantity 0.71 has units $(GeV/c)^2$. There is no objection to a pole in the form-factors at this latter value of $q^2$ since it lies outside the physical regions. However, the fact that the first equality of 1.2-17 violates 1.2-16 indicates that this scaling law fails when continued unmodified to time-like $q^2$. Data on the neutron electric form-factor is relatively sparse,[18] but is available for space-like $q^2$ down to about $-4(GeV/c)^2$. It is roughly consistent with the scaling law:

$$G_e^n(q^2) = \frac{q^2}{4m^2} G_m^n(q^2)\ , \qquad (1.2\text{-}20)$$

but the percentage experimental errors are very large. Note that 1.2-20 satisfies 1.2-16 when continued to time-like $q^2$.

To date the experimental data on electromagnetic pair-production and annihilation is insufficient to allow anything useful to be said about the behaviour of the form-factors in the time-like region.[18]

We have so far dealt only with the $O(3,1)$ decomposition of matrix elements of the proton and neutron current operators, treating these as unrelated problems. They are connected by invoking $SU(2)$ invarience and the assumption that the photon, (whether real or virtual), behaves like the superposition of an isoscalar and the third component of an isovector. Matrix elements of a single nucleon current operator may then be decomposed in $O(3,1) \otimes SU(2)$ space. Conventionally this decomposition is simply obtained from the previous ones by writing:

$$F^{t't}(q^2) = \chi^{\dagger t'}\left[F^s(q^2) + F^v(q^2)\tau_3\right]\chi^t, \tag{1.2-21}$$

where $t(t')$ is the isospin projection of the initial (final) nucleon, and $\chi^t$ ($\chi^{\dagger t'}$) is its two-component spinor wave-function in isospace, as discussed in section 2.12. F stands for any one of $F_{1,2}$, $G_{e,m}$, and $F^s$ ($F^v$) is the corresponding isoscalar (isovector) form-factor of the nucleon. $F^{t't}(q^2)$ vanishes unless $t'$ and $t$ are equal, in which case:

$$F^{1/2,1/2} = F^p, \tag{1.2-22}$$

$$F^{-1/2,-1/2} = F^n, \tag{1.2-23}$$

and it follows (from the explicit structure of the isospace wave-functions) that:

$$F^s = \frac{1}{2}\left(F^p + F^n\right), \tag{1.2-24}$$

$$F^v = \frac{1}{2}\left(F^p - F^n\right). \tag{1.2-25}$$

In the case of the nucleon form-factors the extension from $O(3,1)$ to $O(3,1) \otimes SU(2)$ is neither a simplification nor a complication from the point of view of pheonomenology, but it is an essential ingredient in any theoretical investigation of the dynamics.

To summarise, the nucleon form-factors are scalar functions which parameterise all corrections to the basic electromagnetic interaction of this particle whether they be radiative, strong, or due to a spatial structure of the bare nucleon.

At zero argument they are related to the effective structure of the physical nucleon.

More generally, any arbitary vertex may be expanded in terms of a set of linearly inequivalent kinematic basis covarients. The symmetries operative only constrain the expansion coefficients to be (coupling) constants for three-point vertices connecting three on-shell particles. In all other cases they are allowed to be scalar functions, (form-factors). For three-point vertices connecting one or more off-shell particles their arguments are the squared off-shell momenta. It is not true, as is sometimes stated, that form-factors are phenomenological variables put into fit the empirical data in a simple manner. In cases where they are kinematically allowed to be variable, they may only be taken as constant if one makes an extremely restrictive assumption about the dynamics of the interaction.

We had better point out that the form-factors corresponding to matrix elements of electromagnetic currents are only related to the static electric and magnetic multipole moments of the particles involved for matrix elements taken between identical initial and final single particles. This is simply due to the fact that in any other situation the static limit lies outside the physical region for scattering.

Thus in section 2.71 we shall see that a matrix element of the current taken between a pair of unequal mass spin one-half hadrons may be decomposed into a pair of form-factors which are closely analogous to the charge and moment form-factors of the nucleon. The difference is that the "charge" form-factor now disappears in the real-photon limit. In view of the previous paragraph this is perfectly consistent with the possibility that the hadrons carry non-zero static charges.

Finally we wish to mention another important difference between arbitrary three-point vertices and the special case reviewed in this section. We said earlier that the decomposition of the nucleon electromagnetic vertex-function had to be consistent with a constraint imposed by the assumed T-invarience of the interaction. In fact this is not strictly true.

For general three-point vertices, (matrix elements of some interaction Lagrangian taken between three particles), the combined constraints of hermiticity of the Lagrangian and T-invarience (or PT-invarience, if this is applicable whilst P and T are separately violated) imply that the kinematic covarients may be chosen in such a way that the coupling-constants or form-factors are purely real. The same is true in the electromagnetic case, where one is usually concerned with matrix elements of the current operator, provided that the initial and final on-shell particles are not identical. In the identical particle situation, (as for example the nucleon case reviewed here), the reality condition follows directly from the hermiticity of the current operator and the T or PT constraints become redundant.

It has recently been suggested(19) that the electromagnetic interactions of the hadrons may violate T, (and therefore PT since P is conserved), for non-vanishing $q^2$. This fact cannot be tested in the identical particle case, but will lead to complex form-factors if it obtains in the inelastic situation.

In this thesis we allow for both possibilities when deriving superconvergent sum-rules, but in order to obtain any useful predictions we find it necessary to assume T-invarience when attempting to saturate these.

## 1.3 ON-SHELL SUPERCONVERGENCE

Having obtained a set of form-factors for a vertex it remains to investigate their functional form. To date the methods at one's disposal fall broadly into three classes viz:

1) Dispersion relations on the form-factors,
2) Higher unitary symmetries,
3) Current algebra sum-rules.

Lack of space prevents us reviewing these here, instead we refer the reader to the literature.

On the basis of this thesis we propose adding a fourth candidate to the list, namely off-shell superconvergence. To obtain insight into how such a programme would give us the required information we first review on-shell superconvergence:[(3),(4)]

For simplicity we defer the generalisation to processes involving non-zero spins and isospins to the next chapter, and assume here that all the particles involved have both these quantum numbers zero. We do not wish to imply however that any superconvergent sum-rules would actually be found in such a case, indeed it is well known that they would not.[(3),(4)] All the particles involved are hadrons, and the reaction is of the type $1 + 2 \longrightarrow 3 + 4$, with momenta and masses $p_i$ and $m_i$ where $i = 1,2,3,4$.

We define Mandelstam variables:

$$s \equiv (p_1+p_2)^2, \quad t \equiv (p_1-p_3)^2 \quad , \quad u \equiv (p_1-p_4)^2. \qquad (1.3\text{-}1)$$

So the channels are defined to be:

$$s: \quad 1 + 2 \longrightarrow 3 + 4 \qquad (1.3\text{-}2)$$

$$t: \quad 1 + \bar{3} \longrightarrow \bar{2} + 4 \qquad (1.3\text{-}3)$$

$$u: \quad 1 + \bar{4} \longrightarrow 3 + \bar{2}, \qquad (1.3\text{-}4)$$

and with:

$$K \equiv \sum_{i=1}^{4} m_i^2 \ , \qquad (1.3\text{-}5)$$

we have:

$$s + t + u = \mathcal{K} . \qquad (1.3\text{-}6)$$

Since all the particles are spinless and isospinless the T-matrix elements are given by a single scalar "invarient amplitude", A:

$$T_{fi} \equiv \langle 43 \mid T \mid 21 \rangle \equiv A\,(s,t,u) = A\,(s,t). \qquad (1.3\text{-}7)$$

In the homogeneous stu-plane the s,t, and u branches of the physical region are given by the inequality:[20]

$$\mathcal{E}_{\mu}(p_1 p_2 p_3)\,\mathcal{E}_{\mu}(p_1 p_2 p_3) \leq 0 \qquad (1.3\text{-}8)$$

which is a homogeneous cubic in s,t, and u. The notation of this equation is explained in Appendix 3. If one of the particles has a mass greater than the sum of the masses of the other three, then in addition to the above three physical regions, equation 1.3-8 will lead to a fourth physical region bounded by a closed loop lying inside the reference triangle. This corresponds to decay of the heaviest particle into the lighter three. Its boundary is just the boundary of the Dalitz decay plot.

As for the analytic properties of A(s,t,u), one assumes that it has no singularities other than Born-term poles and those cuts specifically required by unitarity and crossing.[21]

Thus A(s,t,u) has a pole in s whenever this variable is equal to the squared mass of a stable particle or bound state having the same conserved quantum numbers as the initial and final s-channel states.

The amplitude has a superposition of cuts in the s-plane running along the positive real axis from $s_o$ to infinity, and given by the s-channel unitarity relation together with hermitian analyticity:

$$\partial isc_s T_{fi}(s,t) \equiv \lim_{\varepsilon\to 0^+} [T_{fi}(s+i\varepsilon,t) - T_{fi}(s-i\varepsilon,t)]$$
$$= \lim_{\varepsilon\to 0^+} [T_{fi}(s+i\varepsilon,t) - T^*_{if}(s+i\varepsilon,t)]$$
$$= -i \lim_{\varepsilon\to 0^+} \sum_N \delta^4(p_N - p_i) T_{fN}(s+i\varepsilon,t) T^*_{iN}(s+i\varepsilon,t)$$
$$= -i \lim_{\varepsilon\to 0^+} \sum_N \delta^4(p_N - p_i) T^*_{Nf}(s+i\varepsilon,t) T_{Ni}(s+i\varepsilon,t). \qquad (1.3\text{-}9)$$

In this equation N runs over all possible states containing more than one free stable particle and having the same conserved quantum numbers as the initial and final states; $p_N$ is the total momentum of the $N^{th}$ such state, and $p_i$ is the total initial momentum. The infinite set of multiparticle states may be divided into subsets containing the same particles. The $n^{th}$ such subset then gives rise to a cut running along the positive real axis from $s_0^{(n)}$ to infinity, where $s_0^{(n)}$ is the squared sum of the masses of the particles comprising that subset. Since all such cuts are superimposed, the effective branch-point (s-channel threshold) is given by:

$$s_0 = \min_n s_0^{(n)} \qquad (1.3\text{-}10)$$

Note that the line $s = s_0$ may lie entirely outside the s-channel physical region.

Unitarity has actually given us an expression for $\lim_{\varepsilon\to 0^+} [T_{fi}(s+i\varepsilon,t) - T^*_{if}(s+i\varepsilon,t)]$, and assuming that the process is CTP-invarient we have related this to $[\mathrm{disc}_s\, T_{fi}(s,t)]$. The hermitian analyticity theorem of Olive[22] states that for a CTP-invarient reaction:

$$\lim_{\varepsilon\to 0^+} T^*_{if}(s\pm i\varepsilon,t) = \lim_{\varepsilon\to 0^+} T_{fi}(s\mp i\varepsilon,t). \qquad (1.3\text{-}11)$$

The poles and cuts of A(s,t,u) in t and u are given by identical considerations in the respective channels.

Suppose then, that the amplitude has poles in s and u at some $s_j$ and $u_k$ respectively, and s and u channel thresholds at $s_o$ and $u_o$, (all $s_j$, $u_k$, $s_o$, and $u_o$ positive). Then treated as a function of s and t, the amplitude will have in the s-plane: poles at $s_j$ and $s_k$, a right-hand cut from $s_o$ to infinity and a left-hand cut from $s_o'$ to minus infinity, where:

$$s_k = \kappa - t - u_k , \qquad (1.3\text{-}12)$$

and:

$$s_o' = \kappa - t - u_o . \qquad (1.3\text{-}13)$$

Similarly in t and u the amplitude has not only those singularities coming from Born terms and unitarity in the channel under consideration, but also, (due to crossing), the singularities coming from that channel for which the total energy Mandelstam variable is being treated as the dependent variable.

Returning to the analytic structure of A(s,t), one notes that the left and right hand cuts in s do not overlap provided:

$$t > t'' \equiv \kappa - s_o - u_o . \qquad (1.3\text{-}14)$$

In most practical cases t" is a negative quantity.

Now suppose we know, for example from considerations of t-channel Regge behaviour,(3) Froissart bounds in s,(4) or the kinematical singularities of non-reduced helicity amplitudes,(4) that for:

$$t''' \geqslant t > t'' , \qquad (1.3\text{-}15)$$

the amplitude has the asymptotic behaviour:

$$\left| A(s,t) \right| \underset{|s|\to\infty}{\sim} |s|^{-n-\varepsilon} , \qquad (1.3\text{-}16)$$

where n is a positive definite integer, and $\varepsilon$ is a real number such that:

$$0 < \varepsilon < 1 . \qquad (1.3\text{-}17)$$

Then the amplitude is said to be superconvergent, and $s^{\beta}A(s,t)$ will satisfy a fixed-t unsubtracted dispersion relation for:

$$\beta = 1, 2, \ldots, n, \tag{1.3-18}$$

and t lying in the range indicated by 1.3-15. The lower bound to this range is needed to ensure that the left-and right-hand cuts do not overlap.

We may therefore write:

$$s^{\beta}A(s,t) = -i\int_{s_0}^{\infty}\frac{ds'\,s'^{\beta}\,\partial isc_s A(s',t)}{s'-s} - i\int_{-\infty}^{s'_0}\frac{ds'\,s'^{\beta}\,\partial isc_s A(s',t)}{s'-s}$$

$$-\sum_j \frac{s_j^{\beta}\,B(s_j,t)}{s_j-s} - \sum_k \frac{s_k^{\beta}\,B(s_k,t)}{s_k-s}, \tag{1.3-19}$$

where $B(s_j,t)$ and $B(s_k,t)$ denote the residues of $A(s,t)$ at the indicated poles. With the further proviso that t be chosen in such a way that the amplitude remains finite at vanishing s, we may set s equal to zero and obtain:

$$\int_{s_0}^{\infty} ds'\,s'^{m}\,\partial isc_s A(s',t) + \int_{-\infty}^{s'_0} ds'\,s'^{m}\,\partial isc_s A(s',t) - 2\pi i\sum_j s_j^{m}\,B(s_j,t)$$

$$-2\pi i\sum_k s_k^{m}\,B(s_k,t) = 0, \tag{1.3-20}$$

for: $m = 0, 1, 2, \ldots, (n-1)$. (1.3-20A)

For given m this equation is called an $m^{th}$.-moment superconvergent sum-rule. Zeroth moment sum-rules are often simply called ordinary sum-rules.

The Born-term residues are given by perturbation theory, and continuing to neglect spin and isospin one has:

$$B(s_j,t) = g_{fj}\,g_{ji}, \tag{1.3-21}$$

$$B(s_k,t) = -g_{fk}\,g_{ki}. \tag{1.3-22}$$

Here $g_{fj}$ and $g_{ji}$ are the respective coupling constants representing the interaction of the jth. s-channel stable single-particle intermediate state with the s-channel final and initial states. The $g_{fk}$ and $g_{ki}$ are the corresponding quantities for the vertices of the $k^{th}$ u-channel pole graph. The minus sign in equation 1.3-22 arises when one expresses the denominators of the u-channel pole graphs in terms of s and t.

Evaluation of the right- and left-hand discontinuity functions is of course much less straightforward, and a number of approximation procedures are possible. We shall only discuss the resonance approximation as used in this thesis.

Here one makes use of the empirical fact that $T_{fN}(s,t)$ and $T_{Ni}(s,t)$ are only simultaneously relatively large when the value of s is such that the particles comprising the state N may resonate, that is, when s is close to the squared mass of a resonance having the same conserved quantum numbers as this state. In this approximation the s-channel unitarity relation, (1.3-9), reads:

$$\partial isc_s A(s,t) \simeq -2i \sum_R \frac{\theta(s-s_0^{(R)}) M_R \Gamma_R(s)\, g_{fR}(s)\, g_{Ri}(s)}{(s-M_R^2)^2 + M_R^2 \Gamma_R^2(s)} , \qquad (1.3\text{-}23)$$

where:

$$\theta(s-s_0^{(R)}) \equiv \begin{cases} 1, & s \geqslant s_0^{(R)}, \\ 0, & s < s_0^{(R)}. \end{cases} \qquad (1.3\text{-}24)$$

Here R denotes an allowed resonating state with mass distribution centered on $M_R$ and total width $\Gamma_R(s)$, whilst $g_{fR}(s)$ and $g_{Ri}(s)$ are the scalar form-factors representing the interaction of this resonance with the final and initial s-channel states. The quantity $s_0^{(R)}$ is the branch point of A(s,t) in s due to those particles whose effect one is trying to approximate with the resonance R.

The above equation assumes that all form-factors are real and satisfy:

$$g_{fR}(s) = g_{Rf}(s) \quad , \quad g_{iR}(s) = g_{Ri}(s) \, . \tag{1.3-25}$$

This will be the case if all interactions involved are time-reversal invarient and describable in terms of (hermitian) interaction-Lagrangians. Similar considerations apply to the coupling-constants arising in the Born-term residues.

Form-factors rather than coupling-constants are required in equation 1.3-23 to take account of the mass distribution of each resonance. In the limit as the width of each resonance tends to zero, these form-factors become coupling-constants:

$$g_{fR}(s) \xrightarrow[\Gamma_R \to 0]{} g_{fR}(M_R^2) \, , \tag{1.3-26}$$

$$g_{Ri}(s) \xrightarrow[\Gamma_R \to 0]{} g_{Ri}(M_R^2) \, . \tag{1.3-27}$$

Equation 1.3-23 may also be derived from an isobaric model of the scattering amplitude. Again assuming hermitian analyticity and time-reversal invarience one has:

$$A(s \pm i\varepsilon, t) = A^*(s \mp i\varepsilon, t) \, , \tag{1.3-28}$$

so:

$$\partial isc_s A(s,t) = \lim_{\varepsilon \to 0^+} 2i \, \mathrm{Im} \, A(s+i\varepsilon, t) \, . \tag{1.3-29}$$

The isobaric model asserts that above the s-channel threshold:

$$A(s,t) \simeq \sum_R \frac{\theta(s - s_0^{(R)}) \, g_{fR}(s) \, g_{Ri}(s)}{s - M_R^2 + i M_R \Gamma_R(s)} \quad , \tag{1.3-30}$$

which in view of the previous equation again reproduces 1.3-23.

At relatively low energies, where the resonant peaks in the cross-section are known empirically to be large compared with the non-resonant background, equation 1.3-23 should be a reasonable approximation to the truth. As the energy is increased one knows that it becomes progressively more difficult to distinguish between resonances and background, whilst for very large values of s the discontinuity function should be computable from considerations of t-channel Regge behaviour. Indeed, this Regge behaviour is normally used to derive equation 1.3-16, and any approximation to the discontinuity function should certainly satisfy:

$$\max_{\pm} \left| A(s \pm i\varepsilon, t) \right| - \min_{\pm} \left| A(s \pm i\varepsilon, t) \right| \leq \left| \mathrm{disc}_s A(s,t) \right|$$
$$\leq \left| A(s+i\varepsilon,t) \right| + \left| A(s-i\varepsilon,t) \right| \qquad (1.3\text{-}31)$$

for all s,t.

In practice one normally makes a further approximation before using 1.3-23 to evaluate the first term on the left-hand side of 1.3-20. In order that this integral may be computed in closed form oneneglects the s-dependence of the $\Gamma_R(s)$, $g_{fR}(s)$ and $g_{Ri}(s)$, replacing these by $\Gamma_R$, $g_{fR}$, $g_{Ri}$ defined to be $\Gamma_R(M_R^2)$, $g_{fR}(M_R^2)$ and $g_{Ri}(M_R^2)$, respectively. Unfortunately the approximation is now certainly inconsistent with equations 1.3-16 and 31 except in cases where $n$ is equal to unity. This is reflected in the fact that the integral one is trying to evaluate diverges at its upper limit for non-vanishing m.

In order to properly improve the approximation so as to achieve consistency with equations 1.3-16 and 31, and the elimination of divergence difficulties, one ought to keep the s-dependence of the widths and coupling constants whilst adding background and possibly Regge t erms to the right-hand side of 1.3-23. If the duality hypothesis is to be believed, then Regge terms will notbe required. The resonances and background terms will conspire to reproduce exactly the required high energy Regge behaviour. This in itself will yield constraint equations on the unkmowns involved. Alternatively, one might use the resonance-plus-background approximation only for s less than some value corresponding to the upper bound of the "resonance region". For larger values of s the discontinuity function would then be computed from Regge behaviour. Again, the requirementthat the transition from resonance to Regge behaviour be smooth would yield constraint equations. The validity of this latter approximation procedure would not depend on the truth or otherwise of the duality hypothesis.

On the other hand, such sophisticated approximation procedures would certainly introduce large numbers of additional unknowns into the sum-rules greatly reducing their potential predictive power. Accordingly, it is customary to circumvent the divergence difficulties by somewhat cruder means, which do not involve the introduction of background or Regge terms.

One uses equation 1.3-23 to evaluate the required integral, but neglects all s' dependence except that occuring in the resonant denominators. Elsewhere s' is replaced by the relevant $M_R^2$. That is, one writes:

$$\int_{s_0}^{\infty} ds' s'^m \,\mathrm{disc}_s A(s',t) \simeq -2i\sum_R \int_{s_0^{(R)}}^{\infty} \frac{ds' s'^m M_R \Gamma_R(s') g_{fR}(s') g_{Ri}(s')}{(s'-M_R^2)^2 + M_R^2\Gamma_R^2(s')}$$

$$\simeq -2i\sum_R M_R^{2m+1}\Gamma_R \int_{s_0^{(R)}}^{\infty} \frac{ds' \; g_{fR}\, g_{Ri}}{(s'-M_R^2)^2 + M_R^2\Gamma_R^2}$$

$$= -2i\sum_R M_R^{2m}\left\{\frac{\pi}{2} + \tan^{-1}\left(\frac{M_R^2 - S_0^{(R)}}{M_R\Gamma_R}\right)\right\} g_{fR}\, g_{Ri} \;. \tag{1.3-32}$$

If one lets all $\Gamma_R$ tend to zero, 1.3-32 yields:

$$\int_{s_0}^{\infty} ds' s'^m \partial isc_s A(s',t) \xrightarrow[\Gamma_R \to 0]{} -2\pi i \sum_R M_R^{2m} g_{fR} g_{Ri} . \tag{1.3-33}$$

This corresponds to the much more drastic (and unnecessary) approximation in which the cut is replaced by a superposition of Born-like poles, or equivalently, is simulated by a super-superposition of δ-functions.

Thus the approximation of equation 1.3-32 is at worst an improvement on the pole approximation. In as far as it removes from the sum-rules divergences which would be incompatible with their known existence, it is perhaps an improvement on the resonance approximation as well. The point here is that one is now attempting to approximate the integrals which actually appear in the sum-rules, rather than the discontinuity functions themselves.

The integral over theleft-hand cut may be similarly approximated, and labelling the u-channel resonances by R' one has:

$$\int_{-\infty}^{-s_0'} ds' s'^m \partial isc_s A(s',t) = -\int_{u_0}^{\infty} du' (\kappa - t - u')^m \partial isc_u A(u',t)$$

$$\simeq 2i \sum_{R'} \int_{u_0^{(R')}}^{\infty} \frac{du' (\kappa - t - u')^m M_{R'} \Gamma_{R'}(u') g_{fR'}(u') g_{R'i}(u')}{(u' - M_{R'}^2)^2 + M_{R'}^2 \Gamma_{R'}^2(u')}$$

$$\simeq 2i \sum_{R'} (\kappa - t - M_{R'}^2)^m \left\{ \frac{\pi}{2} + \tan^{-1}\left( \frac{M_{R'}^2 - u_0^{(R')}}{M_{R'} \Gamma_{R'}} \right) \right\} g_{fR'} g_{R'i} , \tag{1.3-34}$$

where $\Gamma_{R'}$ , $g_{fR'}$ and $g_{R'i}$ again denote $\Gamma_{R'}(M_{R'}^2)$, $g_{fR'}(M_{R'}^2)$ and $g_{R'i}(M_{R'}^2)$ respectively. The $m^{th}$ moment sum-rule (equation 1.3-20) thus reads in this approximation:

$$\sum_{R} M_R^{2m}\left\{\frac{\pi}{2}+\tan^{-1}\left(\frac{M_R^2-s_0^{(R)}}{M_R\Gamma_R}\right)\right\}g_{fR}g_{Ri}+\pi\sum_{j}M_j^{2m}g_{fj}g_{ji}$$

$$-\sum_{R'}(\kappa-t-M_{R'}^2)^m\left\{\frac{\pi}{2}+\tan^{-1}\left(\frac{M_{R'}^2-u_0^{(R')}}{M_{R'}\Gamma_{R'}}\right)\right\}g_{fR'}g_{R'i}$$

$$-\pi\sum_{k}(\kappa-t-M_k^2)^m g_{fk}g_{ki}=0 \quad . \tag{1.3-35}$$

One is often concerned with sum-rules for processes known to exhibit s$\leftrightarrow$u crossing symmetry, that is, one has:

$$A(s,t,u)=\xi A(u,t,s)\ , \tag{1.3-36}$$

where:
$$\xi=\pm 1\ . \tag{1.3-37}$$

In such cases it proves convenient to treat the amplitude as a function of $\nu$ and t, with $\nu$ defined by:

$$\nu\equiv\tfrac{1}{4}(s-u)\ . \tag{1.3-38}$$

The amplitude then satisfies:

$$A(\nu,t)=\xi A(-\nu,t). \tag{1.3-39}$$

In the $\nu$-plane it has a right-hand cut due to s-channel unitarity running along the real axis from $\nu_0$ to infinity, and poles due to the s-channel Bron-terms at some $\nu_j$. These points are given by:

$$\nu_0=\tfrac{1}{4}(2s_0-\kappa+t)\ , \tag{1.3-40}$$

$$\nu_j=\tfrac{1}{4}(2s_j-\kappa+t)\ . \tag{1.3-41}$$

In view of equation 1.3-39, the left-hand cut due to u-channel unitarity runs from minus $\nu_0$ to minus infinity, and the u-channel Born-term poles occur at the points: minus $\nu_j$ . Moreover, in our previous notation one has:

$$\partial isc_\nu A(\nu,t)=-\xi\,\partial isc_\nu A(-\nu,t)\ , \tag{1.3-42}$$

$$B_j(\nu_j, t) = -\xi B_j(-\nu_j, t) . \tag{1.3-43}$$

Since $s_o$ and $u_o$ are now necessarily equal, equation 1.3-14 is again the condition to be satisfied if the two cuts are not to overlap.

The high $|\nu|$ behaviour of $A(\nu, t)$ may again be derived from (e.g.) considerations of t-channel Regge behaviour, and superconvergent sum-rules thereby deduced. In view of the previous discussion, an m.th moment such sum-rule will be trivially satisfied due to crossing symmetry for:

$$\xi = (-1)^m , \tag{1.3-44}$$

whilst for: $$\xi = -(-1)^m , \tag{1.3-45}$$

it will reduce to:

$$\int_{\nu_o}^{\infty} d\nu' \nu'^m \mathrm{disc}_\nu A(\nu', t) - 2\pi i \sum_j \nu_j^m B_j(\nu_j, t) = 0 . \tag{1.3-46}$$

This only involves the s-channel cut and poles, and in the approximation discussed above reads:

$$\sum_R (2M_R^2 - \kappa + t)^m \left\{ \frac{\pi}{2} + \tan^{-1}\left( \frac{M_R^2 - S_o^{(R)}}{M_R \Gamma_R} \right) \right\} g_{fR} g_{Ri}$$
$$+ \pi \sum_j (2M_j^2 - \kappa + t)^m g_{fj} g_{ji} = 0 . \tag{1.3-47}$$

The approximation procedure described above is often called attempted saturation. A sum-rule is said to be saturated if one has used sufficient resonances in its approximate evaluation that the predictions yielded are expected to be as accurate as is required. It is clearly impractical to attempt saturation with an infinite superposition of resonances, but unfortunately there exists no well defined prescription for determining how well a given finite superposition will saturate a particular sum-rule.

One simply has to make some sensible, but nevertheless largely arbitary choice of resonances. We return to this point again in a moment.

The complications introduced by the presence of non-vanishing spins and/or isospins are fully discussed in the next chapter. Each vertex may then involve several linearly independent couplings, so the $g_{fj}g_{ji}$, $g_{fk}g_{ki}$, $g_{fR}g_{Ri}$ and $g_{fR'}g_{R'i}$ appearing in equations 1.3-35 and 47 are each replaced by a quantity which is linear in products of pairs of "final" and "initial" coupling-constants or form-factors. In addition, these quantities are homogeneous polynomials in Mandelstam variables. The degree of each such polynomial depends on the spins and isospins both of the external particles and of the relevant intermediate state. The variables s and u appearing in these polynomials are again replaced as appropriate by the squared mass of an intermediate state.

In most practical cases the left- and right-hand cuts in s(or $\nu$) do not overlap at zero t. In such cases one usually looks for sum-rules which are valid for vanishing t since one can then separately equate to zero the coefficient of each power of t appearing after attempted saturation. One may thereby obtain several relations from each sum-rule. Assuming that one is deducing the high energy asymptotic behaviour of the amplitudes from a consideration of t-channel Regge behaviour, working at zero t has a further advantage. At this t-value there should be no manifestation of multi-Reggeon exchange with its attendant complication of non-linear effective trajectories.[23]

We see, then, that in this approximation superconvergent sum-rules lead to homogeneous linear equations relating products of pairs of "initial" and "final" Born-term coupling-constants to products of pairs of "coupling-constants" corresponding to the interaction with the initial and final particles of the resonances utilised in the attempted saturation. The predictions of a sum-rule are therefore sensitive to this choice of resonances.

In deciding which resonances should be employed in an attempt to saturate a sum-rule one must be guided by experimental information (when available) regarding which resonances are actually observed in the process under consideration. In the absence of anything better, one normally assumes on the basis of general empirical experience that lighter resonances will dominate the sum-rule compared with heavier ones. Finally, one has to bear in mind the number of final equations resulting from a given saturation attempt, and the number of unknowns that these will involve. Too few equations for the number of unknowns, and the final predictions may not be very useful; too many equations, and these are likely to prove inconsistent.

In cases where such an inconsistent set of final equations is obtained, it is often found that these reduce to a consistent set in some equal-mass limit. This is frequently the u(6,6) mass limit, and the consistent set of equations then sometimes reproduces u(6,6) symmetry predictions.[6] This has been suggested as indicative of some close connection between superconvergence and higher unitary symmetries. To date, however, such a connection remains completely obscure.[7]

## 1.4 INTRODUCTION TO OFF-SHELL SUPERCONVERGENCE AND ITS MOTIVATION.

In the previous section we reviewed the derivation and usefulness of superconvergent sum-rules for purely hadronic scattering processes involving two initial and two final particles.

The arguments may be extended without modification to processes in which only three of the particles are hadrons, the remaining particle being a (real) photon. Suppose for the sake of definiteness that particle 1 is the photon. Then the coupling-constants: $g_{ji}$, $g_{ki}$, $g_{Ri}(s=M_R^2)$ and $g_{R'i}(u=M_{R'}^2)$ of the previous section will now become electromagnetic form-factors evaluated at zero argument:

$$g_{ji} \longrightarrow f_{ji}(p_1^2=0) \qquad (1.4\text{-}1)$$

$$g_{ki} \longrightarrow f_{ki}(p_1^2=0) \qquad (1.4\text{-}2)$$

$$g_{Ri}(s=M_R^2) \longrightarrow f_{Ri}(s=M_R^2, p_1^2=0) \qquad (1.4\text{-}3)$$

$$g_{R'i}(u=M_{R'}^2) \longrightarrow f_{R'i}(u=M_{R'}^2, p_1^2=0) \qquad (1.4\text{-}4)$$

We have assumed that the three channels are again defined by equations 1.3-2,3,4, so that the photon is never "crossed".

When particle 1 is replaced by a photon, the number of conserved quantum numbers is reduced at the initial vertices, but remains unchanged at the final vertices, so the number of Born and resonance graphs to be considered is still usefully restricted.

The above programme is expected to yield useful relations between products of purely strong interaction coupling-constants and hadronic electromagnetic form-factors evaluated at zero argument. In particular, several authors,[8] including the present one, have considered with some success the derivation and approximate saturation of sum-rules for the wellknown pion photo-production process: photon + nucleon $\rightarrow$ nucleon + pion.

It would be exceedingly useful if one could generalise the above theory in such a way as to obtain similar relations involving hadronic electromagnetic form-factors evaluated at non-zero values of their arguments. To do this one would have to be able to derive and approximately saturate super-convergent sum-rules, not for T-matrix elements, but rather for perturbation theoretic four-point vertex functions representing the coupling of a virtual photon to (for example) one initial and two final on-shell hadrons. If the computation was performed in a way which assumed that the other end of the photon propogator was coupled to an initial and a final on-shell electron, for example, then the relations obtained would hold for all space-like arguments of the electromagnetic form-factors involved.

If particles 2 and 4 are nucleons, whilst particle 3 is a non-strange meson, then the form-factors involved are just those in which we are interested, corresponding to the interactions: virtual photon + nucleon $\rightarrow$ nucleon, and: virtual photon + nucleon $\rightarrow$ isobar. In addition, as we shall see later, the "amplitudes" involved are all either even or odd under "s $\leftrightarrow$ u crossing", which as discussed previously greatly simplifies the sum-rules. If the meson has zero spin, each final purely strong vertex involves only a single coupling-constant. In particular, if this particle is a pion the coupling-constants involved are symbolically: $g(N \rightarrow N\pi)$ which is known with fair accuracy, and some $g$(resonance $\rightarrow N\pi$), which are readily calculable in terms of the observed partial widths for decay of the resonances into $N\pi$.

In the case of other mesons, the strong-interaction coupling-constants involved are far less readily accessible experimentally, and it would be useful to look at these vertices as well to see if any predictions can be made.

Before proceeding blindly with such a programme however, one has to ask whether it is valid or even possible. As mentioned earlier, we return to this question in Chapter 3, where we conclude that with certain assumptions it appears to be both possible and valid.

In view of the nature of the photon involved, we call the technique "off-shell superconvergence".

## CHAPTER 2. PART I.

## REVIEW OF THE COVARIENT FORMALISM OF SCADRON et. al.

### 2.1 RARITA-SCHWINGER WAVE-FUNCTIONS

### 2.11 O(3,1) WAVE-FUNCTIONS.[9]

Our basic spin one-half four component spinor wave-functions and spin one four-vector wave functions are defined in Appendix 2. Rarita-Schwinger wave-functions[11] for particles with momentum p and helicity $\Lambda$ may be generated from these as follows, where J is an integer.

For an incoming particle or anti-particle with spin J the wave-function is:

$$\mathcal{E}^{\Lambda}_{(\mu)^J}(p) \equiv \sum_{(\lambda)^J} \langle (\lambda)^J | J\Lambda \rangle \left[ \mathcal{E}^{\lambda}_{\mu}(p) \right]^J, \tag{2.11-1}$$

and an outgoing particle or anti-particle of spin J has the wave-function: $\mathcal{E}^{\Lambda *}_{(\mu)^J}(p)$. (2.11-2)

The wave-function for an incoming particle of spin $J + \frac{1}{2}$ is:

$$u^{\Lambda}_{(\mu)^J}(p) \equiv \sum_{(\lambda)^J,\sigma} \langle (\lambda)^J, \sigma | J+\tfrac{1}{2}, \Lambda \rangle \left[ \mathcal{E}^{\lambda}_{\mu}(p) \right]^J u^{\sigma}(p), \tag{2.11-3}$$

whilst for an outgoing anti-particle of spin $J + \frac{1}{2}$ the wave-functions is:

$$v^{\Lambda}_{(\mu)^J}(p) \equiv \sum_{(\lambda)^J,\sigma} \langle (\lambda)^J, \sigma | J+\tfrac{1}{2}, \Lambda \rangle \left[ \mathcal{E}^{\lambda *}_{\mu}(p) \right]^J v^{\sigma}(p). \tag{2.11-4}$$

For an outgoing particle and an incoming antiparticle, both with spin $J + \frac{1}{2}$, the wave-functions are respectively:

$$\bar{u}^{\Lambda}_{(\mu)^J}(p) \equiv u^{\Lambda \dagger}_{(\mu)^J}(p)\, \gamma_0, \tag{2.11-5}$$

and

$$\bar{v}^{\Lambda}_{(\mu)^J}(p) \equiv v^{\Lambda \dagger}_{(\mu)^J}(p)\, \gamma_0. \tag{2.11-6}$$

In these equations we use the shorthand notations:

$$(\mu)^J \equiv \mu_1 \mu_2 \cdots \mu_J , \tag{2.11-7}$$

$$(\lambda)^J \equiv \lambda_1, \lambda_2, \ldots, \lambda_J , \tag{2.11-8}$$

and:

$$\left[\varepsilon_\mu^\lambda(p)\right]^J \equiv \varepsilon_{\mu_1}^{\lambda_1}(p)\varepsilon_{\mu_2}^{\lambda_2}(p)\ldots\varepsilon_{\mu_J}^{\lambda_J}(p). \tag{2.11-9}$$

The Clebsh-Gordan coefficients, ("parallel coupling coefficients" in this case), are given by:

$$\left\langle(\lambda)^J\middle|J,\Lambda\right\rangle = \left[2^{J-\bar{\Lambda}}(J+\Lambda)!(J-\Lambda)!/(2J)!\right]^{1/2}\delta_{\Lambda,\sum_{i=1}^{J}\lambda_i} , \tag{2.11-10}$$

and:

$$\left\langle(\lambda)^J,\sigma\middle|J+\tfrac{1}{2},\Lambda\right\rangle = \left[\frac{2^{J-\bar{\Lambda}}(J+\frac{1}{2}+\Lambda)!(J+\frac{1}{2}-\Lambda)!}{(2J+1)!}\right]^{1/2}\delta_{\Lambda,\sigma+\sum_{i=1}^{J}\lambda_i} , \tag{2.11-11}$$

where:

$$\bar{\Lambda} \equiv \sum_{i=1}^{J}|\lambda_i| . \tag{2.11-12}$$

With the realisations of equations 2.11-1 to 6 these wave-functions satisfy the Rarita-Schwinger subsidiary conditions, as required. We remind the reader that these are as follows. The wave-functions are traceless, symmetric tensors, and vanish on contraction with p:

$$\psi^\Lambda_{\mu_1\ldots\mu\ldots\mu\ldots\mu_J}(p) = 0 , \tag{2.11-13}$$

$$\psi^\Lambda_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_J}(p) = \psi^\Lambda_{\mu_1\ldots\mu_j\ldots\mu_i\ldots\mu_J}(p) \tag{2.11-14}$$

$$p_\mu \psi^\Lambda_{\mu_1\ldots\mu\ldots\mu_J}(p) = 0 . \tag{2.11-15}$$

In addition, the half-integer spin wave-functions vanish on contraction with $\gamma$:

$$\gamma_\mu \psi^\Lambda_{\mu_1\ldots\mu\ldots\mu_J}(p) = 0 = \bar{\psi}^\Lambda_{\mu_1\ldots\mu\ldots\mu_J}(p)\gamma_\mu , \tag{2.11-16}$$

and satisfy the Dirac equation:

$$(\not{p}-m)u^\Lambda_{(\mu)^J}(p) = (\not{p}+m)v^\Lambda_{(\mu)^J}(p) =$$

$$= \bar{u}^\Lambda_{(\mu)^J}(p)(\not{p}-m) = \bar{v}^\Lambda_{(\mu)^J}(p)(\not{p}+m) = 0 . \tag{2.11-17}$$

We should perhaps also mention that the wave-functions have been chosen in accordance with the phase conventions of Jacob and Wick.(24) In detail one has the following useful relations:

$$u^{-\Lambda}_{(\mu)J}(\mp\not p) = \pm(-1)^J \xi^{J+1/2}_{\Lambda} g^J(\mu)\,\gamma_0\, u^{\Lambda}_{(\mu)J}(\pm\not p) \qquad (2.11\text{-}18)$$

$$\varepsilon^{-\Lambda}_{(\mu)J}(\mp\not p) = (-1)^J \xi^{J}_{\Lambda} g^J(\mu)\,\varepsilon^{\Lambda}_{(\mu)J}(\pm\not p) \qquad (2.11\text{-}19)$$

$$\bar{u}^{\Lambda\,T}_{(\mu)J}(\mp\not p) = \pm\, g^J(\mu)\, T\, u^{\Lambda}_{(\mu)J}(\pm\not p) \qquad (2.11\text{-}20)$$

$$\varepsilon^{\Lambda *}_{(\mu)J}(\mp\not p) = g^J(\mu)\,\varepsilon^{\Lambda}_{(\mu)J}(\pm\not p) \qquad (2.11\text{-}21)$$

$$u^{\Lambda}_{(\mu)J}(p) = C\,\bar{v}^{\Lambda T}_{(\mu)J}(p) = -i\gamma_5(-1)^J \xi^{J+1/2}_{\Lambda} v^{-\Lambda}_{(\mu)J}(p) \qquad (2.11\text{-}22)$$

$$v^{\Lambda}_{(\mu)J}(p) = C\,\bar{u}^{\Lambda T}_{(\mu)J}(p) = i\gamma_5(-1)^J \xi^{J+1/2}_{\Lambda} u^{-\Lambda}_{(\mu)J}(p) \qquad (2.11\text{-}23)$$

$$\varepsilon^{\Lambda *}_{(\mu)J}(p) = (-1)^J \xi^{J}_{\Lambda}\,\varepsilon^{-\Lambda}_{(\mu)J}(p). \qquad (2.11\text{-}24)$$

In these equations we have used the following notations and definitions:

$$\psi(\pm\not p) \equiv \psi(p_0, \pm\not p), \qquad (2.11\text{-}25)$$

$$\xi^{S}_{\Lambda} \equiv (-1)^{S-\Lambda}, \qquad (2.11\text{-}26)$$

$$g^J(\mu) \equiv g(\mu_1)\,g(\mu_2)\ldots g(\mu_J), \qquad (2.11\text{-}27)$$

where:

$$g(\mu) \equiv \begin{cases} 1, & \mu = 0, \\ -1, & \mu = 1,2,3, \end{cases} \qquad (2.11\text{-}28)$$

the superscript $^T$ denotes the transposition operation in four-component spinor space, and C and T are four-by-four matrices acting in this space.

Specifically, with our choice of Dirac matrix realisation (Appendix 1), C is the matrix such that:

$$C\gamma_\mu C^{-1} = -\gamma^T_\mu = (-1)^\mu \gamma_\mu\,, \qquad (2.11\text{-}29)$$

and therefore:

$$C\gamma_5 C^{-1} = \gamma_5^T = \gamma_5 . \tag{2.11-30}$$

It has the properties:

$$C = C^* = -C^T = -C^\dagger = -C^{-1}, \tag{2.11-31}$$

and in cases where all three-momenta involved lie in the 13-plane, may be realised by:

$$C = \gamma_5 \otimes \sigma_2 = -i\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \tag{2.11-32}$$

T is the matrix defined by:

$$T \equiv i\gamma_0\gamma_5 C^{-1}, \tag{2.11-33}$$

and has the properties:

$$T = T^* = -T^T = -T^\dagger = -T^{-1}, \tag{2.11-34}$$

$$T\gamma_\mu T^{-1} = g(\mu)\,\gamma_\mu^T, \tag{2.11-35}$$

$$T\gamma_5 T^{-1} = -\gamma_5^T . \tag{2.11-36}$$

## 2.12 SU(2) WAVE-FUNCTIONS

These are closely, but not exactly, analogous to the Lorentz-space wave-functions of the previous section.

We take as our basic two-component spinor wave-functions for an incoming particle of isospin one-half, isospin projection:

$$T = \pm 1/2 , \tag{2.12-1}$$

the quantities $\chi^T$ realised by:

$$\chi^{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi^{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.12-2}$$

so that our isospin one-half normalisation is:

$$\chi^{T_1\dagger}\chi^{T_2} = \mathbb{1}_2\,\delta_{T_1,T_2} . \tag{2.12-3}$$

Our basic three-vector wave-functions for incoming particles or anti-particles of isospin one, isospin projection:

$$T = 0, \pm 1 , \tag{2.12-4}$$

are the quantities $\phi_i^T$ , i = 1,2,3, realised by:

$$\phi^0 = (0,0,1), \quad \phi^{\pm 1} = \frac{-1}{\sqrt{2}}(\pm 1, i, 0), \tag{2.12-5}$$

so the normalisation is in this case:

$$\phi_{i'}^{T_1 *}\phi_{i'}^{T_2} = \delta_{T_1,T_2} . \qquad (2.12\text{-}6)$$

In direct analogy with the previous section we then construct arbitrary isospin wave-functions as follows, where I is a positive integer.

An incoming particle or antiparticle with isospin I and isospin projection (third component of isospin) T, has the wave-function:

$$\phi_{(i)^I}^{T} = \sum_{(t)^I} \langle (t)^I | I, T \rangle \left[\phi_i^t\right]^I , \qquad (2.12\text{-}7)$$

whilst if the same particle or antiparticle is outgoing, the wave-function is: $\phi_{(i)^I}^{T *}$ .

For an incoming particle with isospin I + ½, and isospin projection T, the wave-function is:

$$\chi_{(i)^I}^{T} = \sum_{(t)^I,\tau} \langle (t)^I, \tau | I+\tfrac{1}{2}, T \rangle \left[\phi_i^t\right]^I \chi^{\tau} . \qquad (2.12\text{-}8)$$

In view of the local isomorphism between su(2) and O(3), the parallel coupling coefficients of equations 2.12-7 and 8 are again given by equations 2.11-10 and 11.

The wave functions for an outgoing particle, outgoing antiparticle, and incoming antiparticle, each with isospin I + ½ and isospin projection T, are $\chi_{(i)^I}^{T\dagger}$ , $\omega_{(i)^I}^{T}$ , and $\omega_{(i)^I}^{T\dagger}$ , respectively, where we again choose to define $\omega_{(i)^I}^{T}$ via the Jacob and Wick[24] phase convention:

$$\omega_{(i)^I}^{T} = \mathcal{C}\,\widetilde{\chi_{(i)^I}^{T\dagger}} = (-1)^I \xi_T^{I+\frac{1}{2}} \chi_{(i)^I}^{-T} . \qquad (2.12\text{-}9)$$

Here $\mathcal{C}$ is the two-by-two matrix acting in two-component spinor space and having the properties:

$$\mathcal{C}\tau_i\mathcal{C}^{-1} = -\tilde{\tau}_i = (-1)^i \tau_i , \qquad (2.12\text{-}10)$$

$$\mathcal{C} = \mathcal{C}^* = -\tilde{\mathcal{C}} = -\mathcal{C}^\dagger = -\mathcal{C}^{-1} . \qquad (2.12\text{-}11)$$

In these latter three sets of equations the tilde denotes the transposition operation in two-component spinor space, and the $\tau_i$, i = 1,2,3, are the Pauli matrices acting in this space. A realisation of $\mathcal{C}$ is:

$$\mathcal{C} = -i\tau_2 \, . \tag{2.12-12}$$

$\mathcal{C}$ is the isospace analogue of the matrix C in Lorentz-space. In isospace, however, we do not have an analogue of the matrix T of the previous section and the analogue of the remaining phase-convention relations are:

$$\tilde{\chi}^{T\dagger}_{(i)^I} = g^I(i)\chi^T_{(i)^I} , \tag{2.12-13}$$

$$\phi^{T*}_{(i)^I} = g^I(i)\phi^T_{(i)^I} , \tag{2.12-14}$$

$$\chi^T_{(i)^I} = \mathcal{C}\,\tilde{\omega}^{T\dagger}_{(i)^I} = -(-1)^I \xi_T^{I+1/2}\,\chi^{-T}_{(i)^I} , \tag{2.12-15}$$

$$\phi^{T*}_{(i)^I} = (-1)^I \xi^I_T\,\phi^{-T}_{(i)^I} , \tag{2.12-16}$$

where:

$$g^I(i) \equiv g(i_1)g(i_2)\ldots g(i_I) , \tag{2.12-17}$$

and

$$g(i) \equiv -(-1)^i \, . \tag{2.12-18}$$

These wave-functions satisfy the following Rarita-Schwinger subsidiary conditions: they are symmetric, traceless, tensors (or tensor-spinors):

$$\psi^T_{i_1\ldots i_j\ldots i_k\ldots i_I} = \psi^T_{i_1\ldots i_k\ldots i_j\ldots i_I} , \tag{2.12-19}$$

$$\psi^T_{i_1\ldots i\ldots i\ldots i_I} = 0 , \tag{2.12-20}$$

and for half-integer isospin vanish on contraction with the $\tau_i$:

$$\tau_i\,\psi^T_{i_1\ldots i\ldots i_I} = 0 = \psi^{T\dagger}_{i_1\ldots i\ldots i_I}\tau_i \, . \tag{2.12-21}$$

These three equations are the analogues of 2.11-13, 14, and 16, respectively. There exist no analogues in isospace of equations 2.11-15 and 17.

## 2.2 PROPOGATOR NUMERATORS

### 2.21 O(3,1) PROPOGATOR NUMERATORS.[9]

The Lorentz space propogator for a particle of spin s, mass m, and momentum K, is the quantity:

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{P}^{s}_{(\mu)^J (\nu)^J}(K)}{K^2 - m^2 + i\varepsilon} \tag{2.21-1}$$

where:

$$\mathcal{P}^{s}_{(\mu)^J (\nu)^J}(K) = \sum_{\Lambda} \psi^{\Lambda}_{(\mu)^J}(K)\, \overline{\psi}^{\Lambda}_{(\nu)^J}(K) . \tag{2.21-2}$$

Here $\psi^{\Lambda}_{(\mu)^J}(K)$ is the wave-function of the particle, J is as usual the largest integer less than or equal to s, and $\overline{\psi}(K)$ is to be understood to stand for $\phi^{*}(K)$ if s is integral.

Computation of $\mathcal{P}^{s}_{(\mu)^J (\nu)^J}(K)$ is naturally very tedious for general J, but it turns out to be relatively simple to calculate instead the "fully contracted propogator" defined by:

$$\mathcal{P}^{s}(p', p ; K) \equiv (p'_{\mu})^J \mathcal{P}^{s}_{(\mu)^J (\nu)^J}(K) (p_{\nu})^J , \tag{2.21-3}$$

where p' and p are respectively any momenta independent of K arising at the vertices with which the $\mu$ and $\nu$ labels of the propogator are contracted.

Scadron derives the expressions:

$$\mathcal{P}^{J}(p', p ; K) = C_J\, \mathcal{P}_J\left(-p'(K)\cdot p(K)\right) , \tag{2.21-4}$$

and:

$$\mathcal{P}^{J+1/2}(p', p ; K) = \frac{C_{J+1}}{J+1}\Big[ (\not{K} + m)\, \mathcal{P}'_{J+1}\left(-p'(K)\cdot p(K)\right)$$

$$- \not{p}'(K)(\not{K} - m)\not{p}(K)\, \mathcal{P}'_{J}\left(-p'(K)\cdot p(K)\right)\Big] . \tag{2.21-5}$$

In these two equations the symbols are defined as follows:

$$C_J \equiv 2^J (J!)^2 \big/ (2J)! \quad , \tag{2.21-6}$$

and for any pair of four-vectors a, b we define:

$$a_{\mu}(b) \equiv a_{\mu} - a\cdot b\, b_{\mu}/b^2 \tag{2.21-7}$$

so that:

$$a(b)\cdot c(b) = a\cdot c(b) = a(b)\cdot c = a\cdot c - a\cdot b\, c\cdot b / b^2 . \tag{2.21-8}$$

The solid harmonic $\mathcal{P}_J$, and its various derivatives with respect to its argument are then given by:

$$\mathcal{P}_J^{(n)}\left(-p'(K)\cdot p(K)\right) \equiv (-1)^{J-n}\left[p'^2(K)\,p^2(K)\right]^{\frac{1}{2}(J-n)} \times$$
$$\times\, P_J^{(n)}\left\{p'(K)\cdot p(K)\Big/\left[p'^2(K)\,p^2(K)\right]^{1/2}\right\}. \qquad (2.21\text{-}9)$$

We shall see in section 2.31 that in practice, having computed a suitably fully contracted propogator, only a few of the initial and final labels need to be freed in order that one may obtain the propogator needed for a particular graph calculation. The required labels may be freed by employing an O(3,1) generalisation of Zemach's O(3) differential technique, and we refer the reader to the above cited paper of Scadron[9] for details. This same paper lists all the partially contracted propogator numerators needed for this thesis.

Finally, we should perhaps mention that a considerable simplification occurs in the special case:

$$p' = p. \qquad (2.21\text{-}10)$$

The argument of the derivative of the Legendre polynomial appearing in equation 2.21-9 reduces to unity, and we therefore have:

$$\mathcal{P}_J^{(n)}(\xi) = \frac{(J+n)!\,\xi^{J-n}}{(J-n)!\,n!\,2^n}, \qquad (2.21\text{-}11)$$

where:

$$\xi \equiv -p^2(K). \qquad (2.21\text{-}12)$$

This greatly simplifies the structure of partially or fully contracted "forward propogators", that is, propogators whose initial and final non-free Lorentz indices are all contracted with the same momentum.

An extensive list of contracted forward propogators is also to be found in the paper of Scadron.[9]

## 2.22 SU(2) PROPOGATOR NUMERATORS

In SU(2) space there exists no analogue of the four-momentum contraction property in Lorentz-space, but since we shall only be concerned with propogators for particles with isospin not greater than three-halves, the porpogator numerators are easily computed directly from the defining equation:

$$\mathcal{P}^{I \text{ or } I+\frac{1}{2}}_{(i)^I (j)^I} \equiv \sum_{T} \psi^{T}_{(i)^I} \psi^{T\dagger}_{(j)^I} , \qquad (2.22\text{-}1)$$

which is just the isospace analogue of equation 2.21-2. We find:

$$\mathcal{P}^{0} = 1 , \quad \mathcal{P}^{1/2} = \mathbb{1}_2 , \quad \mathcal{P}^{1}_{ij} = \delta_{ij} , \qquad (2.22\text{-}2 \text{ to } 4)$$

and:

$$\mathcal{P}^{3/2}_{ij} = \frac{1}{6}\left\{4\delta_{ij}\mathbb{1}_2 - [\tau_i , \tau_j]\right\} . \qquad (2.22\text{-}5)$$

## 2.3 COUPLING-FUNCTIONS CONNECTING THREE MASSIVE PARTICLES.

### 2.31 O(3,1) COUPLING FUNCTIONS

For the sake of argument we assume the interaction is of the type: $1 + 2 \rightarrow 3$, and let particles 1,2,3 have momenta q, p, K; spins: $S_1, S_2, S_3$; helicities: $\lambda, \lambda', \Lambda$; normalities: $n_1, n_2, n_3$ respectively. As usual we define:

$$J_i \equiv \begin{cases} S_i, \text{particle i is a boson}, \\ S_i - \tfrac{1}{2}, \text{particle i is a fermion.} \end{cases} \qquad (2.31\text{-}1)$$

The normality, n, of a particle is then defined to be $(-1)^J \eta_P$ where $\eta_P$ is its intrinsic parity.

In order to derive a useful covarient momentum-space representation for the matrix elements of the interaction Lagrangian it is useful to invoke the Wigner-Eckart theorem and factor out the helicity dependence by writing:

$$\langle K\Lambda | \mathcal{L} | p\lambda', q\lambda \rangle = \overline{\Psi}^{\Lambda}_{(\nu)^{J_3}}(K)\, \mathcal{C}_{(\nu)^{J_3}(\mu)^{J_2}(\alpha)^{J_1}}(S_3; S_2, S_1)\, \psi^{\lambda'}_{(\mu)^{J_2}}(p)\, \psi^{\lambda}_{(\alpha)^{J_1}}(q). \qquad (2.31\text{-}2)$$

Here the $\psi$'s are the Rarita-Schwinger wave-functions, (matrix elements of Lorentz boosts), of the three particles. $\overline{\Psi}$ is to be understood as standing for $\phi^*$ if particle 3 is a boson. The quantity $\mathcal{C}$ is called a "coupling function". It is independent of the helicities of the three particles, and has simple transformation properties, being a Lorentz $(J_1 + J_2 + J_3)$th. rank tensor. For FFB vertices it is at the same time a 4 x 4 matrix in four-component spinor space.

The matrix elements may be further decomposed by expanding the coupling function with respect to a set of linearly inequivalent basis tensors, (tensor-matrices in the FFB case), called "kinematic covarients". The expansion coefficients are called "coupling-constants". Specifically, one writes:

$$\mathcal{C}_{(\nu)^{J_3}(\mu)^{J_2}(\alpha)^{J_1}}(S_3; S_2, S_1) = \sum_{j=1}^{N} g_j \mathcal{K}^{j}_{(\nu)^{J_3}(\mu)^{J_2}(\alpha)^{J_1}}(S_3; S_2, S_1), \qquad (2.31\text{-}3)$$

where the $g_j$ are the coupling constants, and the $\mathcal{K}^j$ are the kinematic covarients. N is just the number of linearly independent ways in which the three particles may couple. It is given by elementary considerations of quantum number conservation, and is, of course, representation independent.

The kinematic covarients have the same general structure and Lorentz transformation properties as the coupling-function. The maximum constructable number of linearly inequivalent covarients for a given vertex is reduced as required from $4(J_1 + J_2 + J_3)$ to N by "equivalence relations". That is, $\mathcal{K}^j$ which are linearly independent when standing alone may give rise to quantities which are no longer independent when they are contracted with, placed adjacent to, or sandwiched between the wave-functions of the three particles. This arises out of the Dirac-Rarita-Schwinger subsidiary conditions on these wave-functions.

The $g_j$ are Lorentz scalars. In view of momentum conservation and the fact that all particles are on-shell, all scalar products constructable from the momenta involved are constants. Hence there exist no scalar variables on which the $g_j$ can depend, and these too must be constants.

The number, N, of linearly independent couplings at a general 3-point vertex with all particles on-shell may be shown to be given as follows.

Let $S_I$ and $S_{III}$ be respectively the lowest and highest spin involved, and let $S_{II}$ be the remaining spin. Then one has either:

$$S_I + S_{II} \leq S_{III} \tag{2.31-4}$$

or:

$$S_I + S_{II} > S_{III}\ , \quad (S_I, S_{II} \leq S_{III}). \tag{2.31-5}$$

In the latter case define:

$$S \equiv S_I + S_{II} - S_{III}\ . \tag{2.31-6}$$

Finally define:

$$n \equiv n_1 n_2 n_3\ . \tag{2.31-7}$$

(This quantity is called the normality of the vertex, which is said to be normal or abnormal according as n equals plus or minus one.)

Conservation of angular momentum then implies that for both FFB and BBB vertices:

$$N = \begin{cases} (2s_I+1)(2s_{II}+1) & , s_I+s_{II} \leq s_{III}, \\ (2s_I+1)(2s_{II}+1)-s(s+1) & , s_I+s_{II} > s_{III}. \end{cases} \quad (2.31\text{-}8)$$

If the interaction is in addition space-reflection invarient, then conservation of parity further subdivides the vertices into normal and abnormal parity classes, and one has for FFB vertices:

$$N = \begin{cases} \frac{1}{2}(2s_I+1)(2s_{II}+1) & , s_I+s_{II} \leq s_{III}, \\ \frac{1}{2}[(2s_I+1)(2s_{II}+1)-s(s+1)] & , s_I+s_{II} > s_{III}. \end{cases} \quad (2.31\text{-}9)$$

whilst for BBB vertices:

$$N = \begin{cases} \frac{1}{2}[(2s_I+1)(2s_{II}+1)+n] & , s_I+s_{II} \leq s_{III}, \\ \frac{1}{2}[(2s_I+1)(2s_{II}+1)-s(s+1)+n] & , s_I+s_{II} > s_{III}. \end{cases} \quad (2.31\text{-}10)$$

Time-reversal invarience does not modify the number of couplings, but taken in conjunction with the postulated hermiticity of the interaction Lagrangian it does imply that in any particular representation of covarients may be chosen in a way which makes the coupling constants real.

Except in the special cases listed below, charge-conjugation invarience also leaves the number of couplings unchanged, merely relating the matrix elements of the interaction Legrangian for different processes. The special cases in which this invarience does modify the number of couplings are as follows.

Firstly, if particles 1,2,3 are self-conjugate bosons with C-parities: $C_1$, $C_2$, $C_3$, then one has a selection rule. The interaction is allowed, i.e. N is non-zero if and only if:

$$C_1C_2C_3 = 1 \quad , \quad (2.31\text{-}11)$$

which is just a special case of Furry's theorem. In cases where this equation is satisfied, the number of couplings is unchanged.

Next, if particles 1 and 2 are pair-conjugate bosons, whilst 3 is again a self-conjugate boson, then for:

$$s_1 (= s_2) = 0 \qquad (2.31\text{-}12)$$

one finds the selection rule:

$$n_{c_3} \equiv C_3 (-1)^{J_3} = 1 \qquad (2.31\text{-}13)$$

irrespective of whether or not the vertex is space-reflection invarient. But if parity is conserved one finds the additional selection rule:

$$n_3 = 1 \,. \qquad (2.31\text{-}14)$$

For:

$$s_1 (= s_2) > 0 \qquad (2.31\text{-}15)$$

either value of $n_{c_3}$ (called the C-normality of particle 3) and $n_3$ is allowed, but irrespective of parity considerations, charge-conjugation invarience further subdivides the couplings into two classes, one for each value of $C_3$.

Finally, if particle 1 is a fermion, 2 is the corresponding anti-fermion, and 3 is once more a self-conjugate boson one finds no selection rule on $C_3$ except in the special case where: the fermions have spin one-half, parity is conserved, and the neutral boson is (P) normal. In such a case it then has to be C-normal as well. For all other cases the couplings are again further divided into two classes corresponding to the two possible $C_3$ values.

The quantities at one's disposal for the construction of the kinematic covarients comprise the momenta: p, q, and K; the metric tensor: $g_{\pi\rho}$ ; the fourth rank Levi-Cevita tensor: $\varepsilon_{\pi\rho\sigma\tau}$, as defined and discussed in Appendix 3; and in the case of FFB vertices, the sixteen Dirac matrices of Appendix 1.

As mentioned previously, momentum conservation coupled with the existence of the Dirac-Rarita-Schwinger subsidiary conditions on the wave-functions of the particles severely restricts the range of possible linearly inequivalent co-varients constructable for a given vertex.

As expected from the above discussion, invarience under the discrete transformations of space-reflection, time-reversal, and charge-conjugation, further restricts this range, and we now briefly indicate how this comes about.

The effect on one particle states of the unitary operators $\mathcal{U}_P, \mathcal{U}_T$, and $\mathcal{U}_c$ is:

$$\mathcal{U}_P\,|p\lambda\rangle = \eta_P\,\xi^{s}_{\lambda}\,|-p,-\lambda\rangle \qquad (2.31\text{-}16)$$

$$\mathcal{U}_T\,|p\lambda\rangle = \eta_T(-1)^{2s}\langle -p,\lambda| \qquad (2.31\text{-}17)$$

$$\mathcal{U}_c\,|p\,\lambda\rangle = \eta_c\,|\bar{p},\bar{\lambda}\rangle\,, \qquad (2.31\text{-}18)$$

where $\eta_P$, $\eta_T$ and $\eta_c$ are the intrinsic P,T, and C-phases. P,T, and C-invariences of the Lagrangian:

$$\mathcal{U}_X^{-1}\,\mathcal{L}\,\mathcal{U}_X = \mathcal{L}\;, \qquad (2.31\text{-}19)$$

where X denotes P, T, or C as appropriate, then have the following implications.

P-invarience implies:

$$\langle K\Lambda|\mathcal{L}|p\lambda', q\lambda\rangle = \eta_P\,\xi^{s_3+s_2+s_1}_{\Lambda+\lambda'+\lambda}\langle -\underline{K},-\Lambda|\mathcal{L}|-p,-\lambda';-q,-\lambda\rangle\,, \qquad (2.31\text{-}20)$$

T-invarience implies:

$$\langle K\Lambda|\mathcal{L}|p\lambda', q\lambda\rangle = \eta_T\langle -p,\lambda';-q,\lambda|\mathcal{L}|-\underline{K},\Lambda\rangle\,, \qquad (2.31\text{-}21)$$

andC-invarience implies:

$$\langle K\Lambda|\mathcal{L}|p\lambda', q\lambda\rangle = \eta_c\langle \bar{K}\bar{\Lambda}|\mathcal{L}|\bar{p}\,\bar{\lambda}', \bar{q}\,\bar{\lambda}\rangle\,, \qquad (2.31\text{-}22)$$

where: $$\eta_X \equiv \eta^{*}_{X_3}\,\eta_{X_2}\,\eta_{X_1}\;. \qquad (2.31\text{-}23)$$

By means of the known phase relations satisfied by the wave-functions involved, (equations 2.11-18 to 24), one may readily convert the above equations into relations between coupling functions. Specifically, one obtains for FFB vertices: for a P-invarient Lagrangian:

$$\mathcal{C}_{\beta\alpha}(f,i) = \eta\, g(\beta) g(\alpha)\, \gamma_0\, \mathcal{C}_{\beta\alpha}(Pf, Pi)\, \gamma_0 \,, \tag{2.31-24}$$

for a T-invarient Legrangian:

$$\mathcal{C}_{\beta\alpha}(f,i) = \eta_T\, g(\beta) g(\alpha)\, T^{-1} \mathcal{C}^T_{\alpha\beta}(Ti, Tf)\, T \,, \tag{2.31-25}$$

and for a C-invarient Lagrangian:

$$\mathcal{C}_{\beta\alpha}(f,i) = \eta_c (-C)\, \mathcal{C}^T_{\beta\alpha}(Cf, Ci)\, C^{-1} . \tag{2.31-26}$$

In these three equations β denotes $(\nu)^{J_3}$, α denotes $(\mu)^{J_2}(\alpha)^{J_1}$, and we have defined:

$$g(\beta) \equiv g^{J_3}(\nu) \,, \tag{2.31-27}$$

$$g(\alpha) \equiv g^{J_2}(\mu)\, g^{J_1}(\alpha) . \tag{2.31-28}$$

The "arguments" of the coupling-functions indicate the final and initial states involved, and the letters P,T, or C in front of f or i denote the corresponding space-reflected, time-reversed, or charge-conjugate states. In particular, remembering that the coupling-functions are helicity independent one has:

$$\mathcal{C}_{\beta\alpha}(Tf, Ti) = \mathcal{C}_{\beta\alpha}(Pf, Pi) = \hat{\mathcal{C}}_{\beta\alpha}(f,i) \,, \tag{2.31-29}$$

where $\hat{\mathcal{C}}_{\beta\alpha}(f,i)$ is defined to be the quantity obtained from $\mathcal{C}_{\beta\alpha}(f,i)$ by reversing the signs of all 3-momenta appearing whilst leaving unchanged all other quantities, (including the coupling-constants).

The corresponding equations for BBB vertices are obtained from those above, (and in all that follows), by omitting the matrices $\gamma_0$, T, and C. In addition, the superscript indicating transposition becomes redundant, and the term (-C) in equation 2.31-26 is meant to indicate that in the corresponding BBB equation an additional minus sign is introduced.

Equation 2.31-24 thus leads to a constraint on the covarients for P-invarient vertices:

$$\mathcal{K}^{j}_{\beta\alpha}(f,i) = n\, g(\beta) g(\alpha) \gamma_0 \hat{\mathcal{K}}^{j}_{\beta\alpha}(f,i) \gamma_0 \,, \tag{2.31-30}$$

where the circumflex again indicates the reversal of all 3-momenta appearing. In agreement with our previous discussion, this equation divides the covarients into "normal" and "abnormal" classes.

The equation for T-invarient Lagrangians, (2.31-35), relates the "forward" ($i \to f$) and "reverse" ($f \to i$) interactions. But these are already related by hermiticity of the Lagrangian, which converted to an equation on the coupling function reads:

$$\mathcal{C}_{\beta\alpha}(f,i) = \overline{\mathcal{C}}_{\alpha\beta}(i,f) \equiv \gamma_0 \mathcal{C}^{\dagger}_{\alpha\beta}(i,f) \gamma_0 \,. \tag{2.31-31}$$

The coupling-constants for a time-reversal invarient interaction are therefore purely real if the covarients are chosen to satisfy:

$$\mathcal{K}^{j}_{\beta\alpha}(f,i) = \eta_T\, g(\beta) g(\alpha) T^{-1} \gamma_0 \hat{\mathcal{K}}^{*j}_{\beta\alpha}(f,i) \gamma_0 T. \tag{2.31-32}$$

Combining this equation with 2.31-30, one has a corresponding reality condition for interactions invarient under P T, (and therefore under C, assuming CPT-invarience), but not necessarily under P and T separately:

$$\mathcal{K}^{j}_{\beta\alpha}(f,i) = n\,\eta_T T^{-1} \mathcal{K}^{*j}_{\beta\alpha}(f,i) T. \tag{2.31-32A}$$

Note that in view of Luder's theorem, (2.41-18), $n\,\eta_T$ is equal to $n_C$, the overall C-normality of the vertex.

Except in the three special cases mentioned earlier, equation 2.31-26 merely relates the coupling-constants for "charge-conjugate" vertices. If all three particles are self-conjugate bosons, one has:

$$\mathcal{C}_{\beta\alpha}(Cf, Ci) = \mathcal{C}_{\beta\alpha}(f,i) \,, \tag{2.31-33}$$

whilst in both the other special cases the coupling functions satisfy:

$$\mathcal{C}_{(\nu)^{\sigma_3}(\mu)^{\sigma_2}(\alpha)^{\sigma_1}}(Cf, Ci) = \mathcal{C}_{(\nu)^{\sigma_3}(\alpha)^{\sigma_1}(\mu)^{\sigma_2}}(f,i)\Big|_{p \leftrightarrow q} \,. \tag{2.31-34}$$

Together, equations 2.31-26 and 33 or 34 as appropriate lead to constraints on the covarients which agree with our previous discussion.

In constructing a set of covarients for a given vertex it is necessary to invoke momentum conservation, the fact that all three particles are on-shell, the Dirac-Rarita-Schwinger subsidiary conditions on the wave-functions, and the implications on the covarients of any discrete symmetries of the Lagrangian. In addition one often has to make use of the three basic relations of Appendix 3, (equations A3-2,3, and 4), and the various relations derivable from these by contraction with momenta and/or Dirac matrices. Finally, the Dirac algebra itself must always be borne in mind.

Using the above principles, it is easy to set up a collection of basic rules which if followed will lead one a considerable way towards a linearly inequivalent set of covarients for any given vertex. One simply constructs all possible covarients according to these rules. In very simple cases this yields just the required number; but in more complicated (i.e. higher spin) cases, the number of covarients thus constructed is too large and one then has to search for equivalence relations amongst them reducing their number to the correct value. These rules now follow. We define:

$$\Lambda \equiv \tfrac{1}{2}\,(p - q), \qquad (2.31\text{-}35)$$

and let $\alpha, \mu$, and $\nu$ denote any one of the Lorentz indices of the wave-functions of particles 1,2, and 3 respectively. In addition, $\alpha'$, $\mu'$, and $\nu'$ each denote any second index of these same respective wave-functions.

General rules for any vertex.

(i) Any pair of covarients are equivalent if they differ only by the interchange of a pair of indices referring to the same wave-function.

(ii) If one of the spins is greater than the sum of the other two, then the covarients are given symbolically by:

$$\mathcal{K}^{j}(S_3,S_2,S_1)=\begin{cases}(\Lambda_\alpha)^{J_1-J_2-J_3}\,\mathcal{K}^{j}(S_3,S_2,S_2+S_3), & S_1>S_2+S_3,\\ (\Lambda_\mu)^{J_2-J_3-J_1}\,\mathcal{K}^{j}(S_3,S_3+S_1,S_1), & S_2>S_3+S_1,\\ (\Lambda_\nu)^{J_3-J_1-J_2}\,\mathcal{K}^{j}(S_1+S_2,S_2,S_1), & S_3>S_1+S_2.\end{cases} \qquad (2.31\text{-}36)$$

The covarients on the right hand side of this equation are those for a vertex which differs from the one under consideration only in that the highest spin is equal to the sum of the lower two spins of this original vertex.

(iii) The rules which follow deal separately with the covarients for parity-conserving normal and abnormal vertices. If parity is not conserved one is to use the covarients which would have been obtained had parity been conserved, together with those which would have arisen under the same circumstances had the vertex been of opposite normality. If the vertex is time-reversal invarient then each of these "opposite normality" covarients must be multiplied by an additional factor of i. For PT-invarient (and therefore T-violating) vertices, no such additional factors are required. All coupling-constants will then be real.

(iv) Covarients constructed according to the rules which follow lead to real coupling-constants for all T-invarient vertices except those involving an odd number of C-abnormal particles with observable C-parity, (for example the $A_1^0$ ). In these exceptional cases an additional factor of i must be included in each covarient if the coupling-constants are to be real.

This rule arises in the following way. One can prove that the covarients referred to satisfy equations 2.31-32 and 32A provided:

$$n\eta_T = 1 \, . \tag{2.31-37}$$

This equation is satisfied, or one can consistently choose $\eta_T$ to satisfy it, for all individual particles except the special ones mentioned. For these latter oneknows from Luder's theorem that:

$$n\eta_T = -1 \, . \tag{2.31-38}$$

and the covarients therefore require an additional i factor.

Equation 2.31-37 still holds for all covarients in the case of P-violating PT-invarient vertices, but for P-violating T-invarient interactions it only holds for those covarients which satisfy 2.31-30. For the opposite normality covarients of rule iii, it has to be replaced in this latter case by 2.31-38, again leading to an additional factor of i.

Special rules for parity-conserving BBB vertices.

(v) The covarients for normal vertices are to be constructed from the momenta: $\Lambda_\nu, \Lambda_\mu, \Lambda_\alpha$ ; and the metric tensors: $g_{\nu\mu}, g_{\mu\alpha}, g_{\alpha\nu}$ , (but not $g_{\nu\nu'}$ , $g_{\mu\mu'}$ , or $g_{\alpha\alpha'}$ ).

(vi) For abnormal vertices the covarients are to be constructed as in rule v, but in additi on each covarient is to include a single overall factor chosen from: $\varepsilon_{\nu\mu}(K\Lambda)$, $\varepsilon_{\mu\alpha}(K\Lambda)$, $\varepsilon_{\alpha\nu}(K\Lambda)$ , $\varepsilon_{\nu\mu\alpha}(K)$, and $\varepsilon_{\nu\mu\alpha}(\Lambda)$ ; (but not for example: $\varepsilon_{\nu\mu}(KK)$, $\varepsilon_{\alpha}(\Lambda KK)$, $\varepsilon_{\alpha\alpha'}(K\Lambda)$ , or $\varepsilon_{\alpha\alpha'\mu}(\Lambda)$ ). One is to bear in mind the following five equivalence relations:

$$2\Lambda_\mu \varepsilon_{\alpha\nu}(K\Lambda) + 2\Lambda_\alpha \varepsilon_{\nu\mu}(K\Lambda) \cong K\cdot\left[K\varepsilon_{\nu\mu\alpha}(\Lambda) - \Lambda\varepsilon_{\nu\mu\alpha}(K)\right] \tag{2.31-39}$$

$$2\Lambda_\nu \varepsilon_{\mu\alpha}(K\Lambda) \cong (2\Lambda - K)\cdot\left[K\varepsilon_{\nu\mu\alpha}(\Lambda) - \Lambda\varepsilon_{\nu\mu\alpha}(K)\right] \tag{2.31-40}$$

$$\varepsilon_{\nu\mu}(K\Lambda) g_{\nu'\alpha} - \varepsilon_{\nu\alpha}(K\Lambda) g_{\nu'\mu} \cong -\varepsilon_{\nu\mu\alpha}(K)\Lambda_{\nu'} \tag{2.31-41}$$

$$\varepsilon_{\mu\nu}(K\Lambda) g_{\mu'\alpha} - \varepsilon_{\mu\alpha}(K\Lambda) g_{\mu'\nu} \cong \left[\varepsilon_{\nu\mu\alpha}(K - 2\Lambda)\right]\Lambda_{\mu'} \tag{2.31-42}$$

$$\mathcal{E}_{\alpha\nu}(K\Lambda)g_{\alpha'\mu}-\mathcal{E}_{\alpha\mu}(K\Lambda)g_{\alpha'\nu}\cong\mathcal{E}_{\nu\mu\alpha}(2\Lambda-K)\Lambda_{\alpha'} \ . \qquad (2.31\text{-}43)$$

Special rules for parity conserving FFB vertices.

(vii) One is to construct the covarients for normal vertices following rule v, but in addition each is to include either an overall 4 X 4 unit matrix, or an overall factor $\gamma_\rho$ where $\rho$ is a single fixed index referring to the wave-function of the boson. No other Dirac matrices are to be used.

(viii) The covarients for abnormal vertices are to be constructed as though the vertex were normal, (i.e. rule vii is to be used). At the end of the calculation all covarients are to be either pre- or post-multiplied by $\gamma_5$ .

The above eight rules assume that the number of independent couplings is not modified by C-invarience. If this is not the case, one simply uses the same rules and then drops those covarients which violate the appropriate constraint equations.

We have chosen for sakeof argument to work in terms of components of the momentum $\Lambda$ , and contractions of the Levi-Cevita tensor with the momenta $\Lambda$ and K. The covarients may be written in terms of other momenta by means of the relations:

$$\Lambda_\nu\cong p_\nu\cong -q_\nu \ , \qquad (2.31\text{-}44)$$

$$2\Lambda_\mu\cong K_\mu\cong -q_\mu \ , \qquad (2.31\text{-}45)$$

$$2\Lambda_\alpha\cong K_\alpha\cong p_\alpha \ , \qquad (2.31\text{-}46)$$

$$K_\nu\cong p_\mu\cong q_\alpha\cong 0 \ , \qquad (2.31\text{-}47)$$

$$\mathcal{E}_{\sigma\tau}(K\Lambda)=\mathcal{E}_{\sigma\tau}(qp), \text{ etc.} \qquad (2.31\text{-}48)$$

An extensive list of coupling functions has been given by Scadron.[9]

## 2.32 O(3,1) ⊗ SU(2) COUPLING FUNCTIONS.

In this section T and t denote respectively the total isospin and third component of isospin of a particle. As usual we define an integer I by:

$$I \equiv \begin{cases} T, & T \text{ integral}, \\ T-\frac{1}{2}, & T \text{ half-integral}. \end{cases} \qquad (2.32\text{-}1)$$

According as T is integral or half-integral we call the particle an isoboson (b) or isofermion (f).

Given a set of SU(2) invarient three-point functions, each involving a different t configuration of the same SU(2) multiplets, one has that the coupling constants: $g^j(T_3t_3,T_2t_2,T_1t_1)$ corresponding to the different configurations, are related by the Wigner-Eckart theorem to a set of t-independent coupling constants: $g^j(T_3,T_2,T_1)$ . Specifically:

$$g^j(T_3t_3,T_2t_2,T_1t_1)=C(T_3t_3,T_2t_2,T_1t_1)g^j(T_3,T_2,T_1), \qquad (2.32\text{-}2)$$

where the C's are SU(2) Clebsh-Gordan coefficients, and are independent of j. This latter superscript has the same meaning as in the previous section, labelling the linearly independent couplings in Lorentz space.

For the purposes of this thesis, it will prove convenient to determine the C's up to an overall normalisation factor by means of an isospin-decomposition in SU(2)-space analogous to the Lorentz-space spin-decomposition of the previous section. We therefore define a t-independent isospace covarient, $\mathcal{K}_{(k)^{I_3}(j)^{I_2}(i)^{I_1}}(T_3,T_2,T_1)$, by:

$$C(T_3t_3,T_2t_2,T_1t_1)\equiv \phi^{t_3\dagger}_{(k)^{I_3}}\mathcal{K}_{(k)^{I_3}(j)^{I_2}(i)^{I_1}}(T_3,T_2,T_1)\phi^{t_2}_{(j)^{I_2}}\phi^{t_1}_{(i)^{I_1}}. \qquad (2.32\text{-}3)$$

The $\phi$'s are Rarita-Schwinger wave-functions in isospace, as discussed in section 2.12. The number of linearly inequivalent covarients resulting from the isospin decomposition of an

SU(2)-symmetric n-point function is just equal to the number of allowed values of total initial (equals total final) isospin. Thus an SU(2)-symmetric three-point function always involves a single isospace covarient.

We shall abbreviate equation 2.32-3 to:

$$C(T_3t_3,T_2t_2,T_1t_1)=\phi_b^{t_3\dagger}\mathcal{K}_{b,a_2a_1}(T_3,T_2,T_1)\phi_{a_2}^{t_2}\phi_{a_1}^{t_1}\,, \tag{2.32-4}$$

and the full spin $\otimes$ isospin decomposition in Lorentz $\otimes$ SU(2) space then reads:

$$\langle K\Lambda T_3t_3|\mathcal{L}|p\lambda' T_2t_2,q\lambda T_1t_1\rangle=\phi_b^{t_3\dagger}\bar{\psi}_\beta^\Lambda(K)\mathcal{C}_{\beta,\alpha_1\alpha_2}^{b,a_1a_2}(f,i)\times$$
$$\times\psi_{\alpha_2}^{\lambda'}(p)\phi_{a_2}^{t_2}\psi_{\alpha_1}^{\lambda}(q)\phi_{a_1}^{t_1}\,, \tag{2.32-5}$$

where:

$$\mathcal{C}_{\beta,\alpha_1\alpha_2}^{b,a_1a_2}(f,i)\equiv\mathcal{C}_{\beta,\alpha_1\alpha_2}(f,i)\mathcal{K}_{b,a_1a_2}(T_3,T_2,T_1)\,. \tag{2.32-6}$$

The implications of discrete symmetries of the Lagrangian on this coupling-function in Lorentz $\otimes$ SU(2) space are as follows. We again give them for ffb-FFB vertice. The corresponding equations for the other possible configurations are given by leaving out the appropriate matrices.

Space-reflection leaves $\mathcal{K}_{b,a_1a_2}$ unchanged, and so for a P-invarient Lagrangian 2.31-24 just generalises to:

$$\mathcal{C}_{\beta,\alpha}^{b,a}(f,i)=\eta g(\beta)g(\alpha)\gamma_0\hat{\mathcal{C}}_{\beta,\alpha}^{b,a}(f,i)\gamma_0\,, \tag{2.32-7}$$

splitting the spin part of the coupling function into normal and abnormal parity classes as previously.

Since time-reversal involves an interchange of initial and final states, the isospace covarient is affected by this operation, and 2.31-25 now becomes:

$$\mathcal{C}_{\beta,\alpha}^{b,a}(f,i)=\eta_T g(\beta)g(\alpha)g(b)g(a)T^{-1}\hat{\tilde{\mathcal{C}}}{}^{Ta,b}_{\alpha,\beta}(i,f)T\,. \tag{2.32-8}$$

In this equation the tilde denotes transposition of the isospace part, and we have defined:

$$g(b)\equiv g^{I_3}(k)\;, \tag{2.32-9}$$
$$g(a)\equiv g^{I_2}(j)g^{I_1}(i)\,. \tag{2.32-10}$$

Hermiticity of the Lagrangian now implies

$$\mathcal{C}^{ba}_{\beta\alpha}(f,i)=\gamma_0\tilde{\mathcal{C}}^{T*ab}_{\alpha\beta}(i,f)\gamma_0\equiv\bar{\mathcal{C}}^{ab}_{\alpha\beta}(i,f). \qquad (2.32\text{-}11)$$

Combining equations 2.32-8 and 11, we see that the condition for real coupling-constants is still provided by equation 2.31-32 on the Lorentz-space covarients, provided that the isospace covarient is chosen to satisfy:

$$\mathcal{K}_{ba}(f,i)=g(b)g(a)\mathcal{K}^{*}_{ba}(f,i). \qquad (2.32\text{-}12)$$

As far as charge-conjugation is concerned, the calculations are most closely analogous to the treatment in Lorentz-space alone if one considers the implications of invarience of the Lagrangian under the combined operation, ("G-parity operation"), of charge-conjugation followed by a rotation through $\pi$ about the 2-axis in isospace. This operation transforms a member of an SU(2) multiplet into that member of the corresponding anti-multiplet having the same third component of isospin. Denoting the intrinsic G-phases of the three particles by $\eta_{G_{1,2,3}}$ one finds that 2.31-26 generalises to:

$$\mathcal{C}^{ba}_{\beta\alpha}(f,i)=\eta^{*}_{G_3}\eta_{G_2}\eta_{G_1}(-C)(-\mathcal{C})\tilde{\mathcal{C}}^{T\,ba}_{\beta\alpha}(Gf,Gi)\mathcal{C}^{-1}C^{-1}. \qquad (2.32\text{-}13)$$

Of course, since the coupling-function is t-independent, $\mathcal{C}^{ba}_{\beta\alpha}(Gf,Gi)$ is just $\mathcal{C}^{ba}_{\beta\alpha}(Cf,Ci)$ .

Once again, this equation only constrains the covarients in special cases. These are obvious generalisations of those of the previous section. Further subdivision of the covarients into classes of opposite G-parity, and/or G-parity selection rules, arise if either all three multiplets are self-conjugate, or if multiplet 3 is self-conjugate whilst multiplets 1 and 2 are mutually pair-conjugate. A particular realisation of this latter case is needed later in this thesis, and we give the required results at the end of this section.

Bearing in mind the Rarita-Schwinger subsidiary conditions on the isospace wave-functions, and the quantities available, one has the following rules for the construction of isospace covarients.

(i) General rule

Any two covarients are equivalent if they differ only by the interchange of a pair of indices to be contracted with the same wave-function.

(ii) Rule for bbb vertices.

The covarient is to be constructed from SU(2) metric tensors of the types: $\delta_{ij}$ , $\delta_{jk}$ , and $\delta_{ki}$ , and SU(2) Levi-Cevita tensors of the type: $\varepsilon_{ijk}$ .

(iii) Rule for ffb vertices.

The covarient is to be constructed in the same manner as for bbb vertices, but in addition each covarient is to involve an overall 2 X 2 unit matrix, or a single overall Pauli matrix $\tau_\ell$ where $\ell$ is an isoboson label. In addition, one is to bear in mind the relations of Appendix 4.

Using these rules it is easy to deduce expressions for the isospin covarient of an arbitrary vertex. Irrespective of which particles are initial or final, denote their isospins by T, T', and T'', such that these satisfy:

$$T \leq T' \leq T'' . \qquad (2.32\text{-}14)$$

Then if T and T' are half-integral and:

$$T'' = T + T', \qquad (2.32\text{-}15)$$

one finds that the covarient may be realised in an obvious notation by:

$$\mathcal{K}_{(i)^I (i')^{I'} (i'')^{I''}}(T,T',T'') = (\delta_{ii''})^{I} (\delta_{i'i''})^{I'} \tau_{i''} . \qquad (2.32\text{-}16)$$

In all other cases the covarient is realised by:

$$\mathcal{K}_{(i)^I (i')^{I'} (i'')^{I''}}(T,T',T'') = (i\,\varepsilon_{ii'i''})^{(I+I'+I'')} (\delta_{ii''})^{(I''-I')} \times$$

$$\times (\delta_{i'i''})^{(I''-I)} \times \begin{cases} 1 & \text{for bbb vertices,} \\ 1\!\!1_2 & \text{for ffb vertices.} \end{cases} \qquad (2.32\text{-}17)$$

The notation of these equations is just a generalisation of that used previously. To make it absolutely clear we give an example:

$$\mathcal{K}_{i(i')^2(i'')^4}(3/2, 5/2, 4) = \delta_{i i''_1} \delta_{i'_1 i''_2} \delta_{i'_2 i''_3} \tau_{i''_4} . \qquad (2.32\text{-}18)$$

Covarients given by equations 2.32-16 and 17 automatically satisfy 2.32-12, and it was with this end in view that we included a factor $i^{(I+I'-I'')}$ in the right-hand side of 2.32-17.

We have not previously seen these two equations in print, but they are so obvious that we feel sure they must be well known to most authors.

To conclude this section we consider an example in which equation 2.32-13 does lead to selection rules and constraints on the covarients. The vertex is purely strong and conserves P, C, and T. Multiplet 1 has spin one-half and half-integer isospin, multiplet 2 is the corresponding anti-multiplet, and multiplet 3 has integer spin and isospin. From elementary considerations of conservation of observable quantum numbers, one deduces the selection rule:

$$G_3 = (-1)^{J_3 + I_3} \text{ if } \quad n_3 = +1. \qquad (2.32\text{-}19)$$

If multiplet 3 is abnormal, either value of $G_3$ is allowed.

The coupling function must satisfy:

$$\mathcal{C}_\beta^{b a_2 a_1}(Gf, Gi) = \mathcal{C}_\beta^{b a_1 a_2}(f, i)\Big|_{p \leftrightarrow q} , \qquad (2.32\text{-}20)$$

which on combination with 2.32-13 yields:

$$\mathcal{C}_\beta^{b a_2 a_1}(f,i) = G_3 \eta_{G_2} \eta_{G_1} (-C)(-\mathcal{T}) \tilde{\mathcal{C}}^{T\, b a_1 a_2}_\beta (f,i)\Big|_{p \leftrightarrow q} \mathcal{T}^{-1} C^{-1} . \qquad (2.32\text{-}21)$$

In all possible cases, isospin covarients given by equations 2.32-16 and 17 satisfy:

$$(-\mathcal{T}) \tilde{\mathcal{K}}_{b a_1 a_2}(b \bar{f} f) \mathcal{T}^{-1} = -(-1)^{I_3} \mathcal{K}_{b a_2 a_1}(b \bar{f} f) , \qquad (2.32\text{-}22)$$

and the spin part of the coupling function is therefore subject to the constraint:

$$\mathcal{C}_\beta(f,i) = (-1)^{I_3} \eta_{G_1} \eta_{G_2} G_3 C\, \mathcal{C}^T_\beta(f,i)\Big|_{p \leftrightarrow q} C^{-1} . \qquad (2.32\text{-}23)$$

Denoting normal and abnormal coupling functions by $\mathcal{C}^{+}_{\ldots}$ and $\mathcal{C}^{-}_{\ldots}$ respectively, we have fron Scadron's[9] paper that:

$$\mathcal{C}^{+}_{(\nu)J}(J,\bar{\tfrac{1}{2}},\tfrac{1}{2}) = (\Lambda_{\nu})^{J-1}(g_1\Lambda_{\nu_1} + g_2\gamma_{\nu_1}), \quad (n_3=1), \tag{2.32-24}$$

$$\mathcal{C}^{-}_{(\nu)J}(J,\bar{\tfrac{1}{2}},\tfrac{1}{2}) = (\Lambda_{\nu})^{J-1}(g_3\Lambda_{\nu_1} + g_4\gamma_{\nu_1})\gamma_5, \quad (n_3=-1). \tag{2.32-25}$$

Compatibility of equations 2.32-19, 23, and 24 implies that the intrinsic G-phases must in this case always satisfy:

$$\eta_{G_1}\eta_{G_2} = 1, \tag{2.32-26}$$

and we note that normal bosons with allowed G-parity couple to the fermion-antifermion system via both $g_1$ and $g_2$. Equations 2.32-23, 25, and 26 further imply that abnormal bosons with:

$$G_3 = (-1)^{J_3+I_3} \tag{2.32-27}$$

couple only via $g_3$, whilst those with:

$$G_3 = -(-1)^{J_3+I_3} \tag{2.32-28}$$

couple only via $g_4$. The spin covarients in the abnormal boson case are thus divided into two further classes depending on the G-parity of the boson multiplet.

## 2.4 T-MATRIX ELEMENTS CONNECTING FOUR MASSIVE PARTICLES

### 2.41 O(3,1) M-FUNCTIONS.[10]

As is wellknown, one may decompose such T-matrix elements into sets of scalar variables, (invarient amplitudes), embodying all the dynamics of the processes. The techniques involved are very similar to those employed in section 2.31 for the decomposition of on-shell three-point vertices into sets of coupling-constants. Consequently, we shall mainly concern ourselves in this section with emphasising the differences between these two techniques.

We shall restrict ourselves to a review of scattering processes involving two initial and two final hadrons with channels defined by:

$$s:\ 1(q)+2(p)\longrightarrow 3(q')+4(p')\ , \qquad (2.41\text{-}1)$$

$$t:\ 1(q)+\overline{3}(-q')\longrightarrow \overline{2}(-p)+4(p')\ , \qquad (2.41\text{-}2)$$

$$u:\ 1(q)+\overline{4}(-p')\longrightarrow 3(q')+\overline{2}(-p)\ . \qquad (2.41\text{-}3)$$

Particles 1 and 3 will be bosons, whilst 2 and 4 are fermions. The equations we give are readily extended to the four boson case by substituting the relevant wave-functions, and dropping all 4 X 4 spinor-space matrices. Four fermion scattering has been treated in considerable detail by Kellet,[26] and will not be reviewed here. The only additional complication in that case is the need to define an ordering convention for the spinors involved, different conventions being related by Fiertz transformations.

As in section 2.31 one first factors out the helicity dependence of the T-matrix elements, defining "M-functions": $M^{s,t,u}_{\mu'\nu'\mu\nu}$, having simple Lorentz transformation properties, by:

s-channel:

$$\langle q'\lambda_1', p'\lambda_2'|T|q\lambda_1, p\lambda_2\rangle \equiv \varepsilon^{*\lambda_1'}_{\mu'}(q')\,\overline{u}^{\lambda_2'}_{\nu'}(p')\,M^{s}_{\mu'\nu'\mu\nu}\,\varepsilon^{\lambda_1}_{\mu}(q)\,u^{\lambda_2}_{\nu}(q), \qquad (2.41\text{-}4)$$

t-channel:

$$\langle -\overline{p}\overline{\lambda}_2, p'\lambda_2'|T|q\lambda_1, -\overline{q}'\overline{\lambda}_1'\rangle \equiv \varepsilon^{\lambda_1'}_{\mu'}(-q')\,\overline{u}^{\lambda_2'}_{\nu'}(p')\,M^{t}_{\mu'\nu'\mu\nu}\,\varepsilon^{\lambda_1}_{\mu}(q)\,v^{\lambda_2}_{\nu}(-p), \qquad (2.41\text{-}5)$$

u-channel:

$$\langle q'\lambda_1', -\overline{p}\overline{\lambda}_2|T|q\lambda_1, -\overline{p}'\overline{\lambda}_2'\rangle \equiv \varepsilon^{*\lambda_1'}_{\mu'}(q')\,\overline{v}^{\lambda_2'}_{\nu'}(-p')\,M^{u}_{\mu'\nu'\mu\nu}\,\varepsilon^{\lambda_1}_{\mu}(q)\,v^{\lambda_2}_{\nu}(-p). \qquad (2.41\text{-}6)$$

In these equations particles: 1,2,3, 4 respectively have spins: $J_1$, $J_2+\frac{1}{2}$. $J_1'$, $J_2'+\frac{1}{2}$, and helicities: $\lambda_1, \lambda_2, \lambda_1', \lambda_2'$ . We have introduced the shorthand notation:

$$\mu\equiv(\mu)^{\sigma_1}\ ,\ \nu\equiv(\nu)^{\sigma_2}\ ,\ \mu'\equiv(\mu')^{\sigma_1'}\ ,\ \nu'\equiv(\nu')^{\sigma_2'}. \qquad (2.41\text{-}7,8,9,10)$$

These three M-functions transform as Lorentz tensor-multi-spinors, and are related by crossing. Although it has not been proved for arbitrary spin processes, one normally postulates the crossing rule:

$$M_{\mu_4\mu_3\mu_2\mu_1}\left(1(p_1)+2(p_2)\rightarrow 3(p_3)+4(p_4)\right)=$$
$$=\xi_s\, M_{\mu_4\mu_3\mu_2\mu_1}\left(1(p_1)+\overline{4}(-p_4)\longrightarrow 3(p_3)+\overline{2}(-p_2)\right), \qquad (2.41\text{-}11)$$

$$\text{where } \xi_s=\begin{cases}+1 \text{ for BB and FB crossing,}\\ -1 \text{ for FF crossing.}\end{cases} \qquad (2.41\text{-}12)$$

Thus by explicitly introducing an additional factor of minus one into the right-hand side of equation 2.41-6, (but leaving it unchanged in the four boson case), we may use the same M-function in all three channels. As usual one expands this in terms of a set of linearly inequivalent basis tensors (kinematic covarients):

$$M_{\mu'\nu'\mu\nu}=\sum_{j=1}^{N} A_j(s,t)\,\mathcal{K}^j_{\mu'\nu'\mu\nu}. \qquad (2.41\text{-}13)$$

The $\mathcal{K}^j$ are these kinematic covarients, and the $A_j$ are called "invarient amplitudes". In contrast to the vertex case, there are now two linearly independent scalar variables constructable from the momenta, and the $A_j$ are complex scalar functions of these. As we have indicated, one may conveniently choose to use any two of the three Mandelstam variables s, t, and u, as defined in section 1.3.

The crossing rule thus states that after the introduction of relative minus signs between M-functions which differ by the crossing of a pair of fermions, the invarient amplitudes $A_j$ in the three channels are, (for fixed $j$ ), different physical sectors of the same function of the scalar variables. As in the spinless case, one then postulates that this function is analytic apart from Born-term poles and unitarity cuts.

Apart from certain exceptional cases in which their number is further reduced, the number, N, of linearly inequivalent covarients for processes involving particles with spins, $S_{1,2,3,4}$ is easily deduced to be given as follows.

If parity is not conserved:

$$N=\prod_{i=1}^{4}(2S_i+1), \quad \text{for all processes.} \qquad (2.41\text{-}14)$$

If parity is conserved:

$$N=\frac{1}{2}\prod_{i=1}^{4}(2s_i+1)+\begin{cases}0, & \text{FB}\to\text{FB and FF}\to\text{FF},\\ \frac{1}{2}n, & \text{BB}\to\text{BB}.\end{cases} \qquad (2.41-15)$$

where n is the normality of the process, that is, the product of the normalities of the four particles involved.

For a given process one can construct an infinity of covarients which will satisfy all constraints imposed by the various symmetries of the T-matrix, but only certain sets of N of these will be linearly inequivalent. In performing the reduction to a linearly inequivalent set, it is possible to introduce into the final finite set of amplitudes poles which were not present in the original infinite set. One strictly makes the above analyticity postulate for this latter infinite set of amplitudes. Any additional poles then introduced are assumed to be spurious "kinematic singularities". If the amplitudes are to have only those singularities required by dynamics, one must be careful to perform the reduction to a linearly independent set in a way which leaves them "kinematic singularity free", (henceforth abbreviated to K.S.F.). We return to this point at the end of this section.

The statements that the T-matrix is P,T, or C-invarient may be readily converted into constraints on the M-function. For BF→BF scattering one just obtains equations 2.31 - 24, 25, or 26, as appropriate, with $\mathcal{T}$ replaced by M, $\alpha$ now standing for $\mu\nu$, and $\beta$ for $\mu'\nu'$. In the BB→BB case the same equations apply, but with the 4 X 4 matrices removed.

The parity-conservation equation again tells one that the M-function is to be expanded in terms of a set of proper-tensors if the process is normal, and a set of pseudo-tensors, each containing one overall $\gamma_5$ or Levi-Cevita tensor, if it is abnormal.

In general, the T and C-invarience equations just relate the M-functions for different processes. The amplitudes are not required to be real for T-invarient processes, since the T-matrix is not hermitian. However, for processes which are elastic in the s-channel: PT-invarience in the s-channel and C-invarience in the t-channel both impose the following constraint on the covarients:

$$\mathcal{K}^{j}_{\beta\alpha}(p'q',pq)=\gamma_0 T^{-1}\mathcal{K}^{j\,T}_{\alpha\beta}(pq,p'q')T\gamma_0 . \qquad (2.41\text{-}16)$$

This reduces the number of covarients to:

$$N=\tfrac{1}{2}(2s_1+1)(2s_2+1)\left[\tfrac{1}{2}(2s_1+1)(2s_2+1)+1\right]+\begin{cases}0, & BF\to BF,\ FF\to FF\\ \tfrac{1}{4}, & BB\to BB.\end{cases} \qquad (2.41\text{-}17)$$

In combination with the crossing rule, the above invarience principles reduce the number of covarients in certain further cases involving identical particles, and in other such cases relate the invarient amplitudes at different values of their arguments. We do not insist on the details here, as all cases are listed by Scadron and Jones.(10)

Even if the T-matrix violates P,C, and T individually, one may still relate different processes by CTP-invarience or by considering the corssing of all four particles. Both principles relate the same pair of processes, and consistency of the two results requires that the three overall discrete transformation phases satisfy:

$$\eta_C\,\eta_T\,\eta_P = 1. \qquad (2.41\text{-}18)$$

The result is then:

$$M_{\beta\alpha}(f,i)=(-1)^{J_1+J_2+J_3+J_4}\gamma_5 M_{\alpha\beta}(Ci,Cf)\gamma_5 . \qquad (2.41\text{-}19)$$

Since 2.41-18 must hold irrespective of which four particles are considered, we have a proof of Luder's theorem.

The covarient formalism also provides one with a simple proof of Olive's hermitian analyticity theorem(22) for general CTP-invarient processes which are also invarient under T (and therefore CP) and/or C (and therefore PT). One then has equation 1.3-11, together with:

$$\lim_{\varepsilon\to 0^+} A_j(s\pm i\varepsilon, t) = \lim_{\varepsilon\to 0^+} A_j^*(s\mp i\varepsilon, t), \tag{2.41-20}$$

so that:

$$\partial isc_s A_j(s,t) = \lim_{\varepsilon\to 0^+} 2i\,\mathrm{Im}\,A_j(s+i\varepsilon, t). \tag{2.41-21}$$

But the essential point which comes out of the proof is that these equations only hold provided the covarients are chosen to satisfy:

$$\mathcal{K}^j_{\beta\alpha}(f,i) = \eta_T\, g(\beta) g(\alpha) T^{-1}\gamma_0 \hat{\mathcal{K}}^{*j}_{\beta\alpha}(f,i)\gamma_0 T, \tag{2.41-22}$$

for T-invarient processes, and:

$$\mathcal{K}^j_{\beta\alpha}(f,i) = \eta_c T^{-1}\mathcal{K}^{*j}_{\beta\alpha}(f,i) T, \tag{2.41-23}$$

if the process is C (i.e. PT)-invarient, $\eta_c$ (equal to $\eta\eta_T$) being its overall C-normality. These equations are called discontinuity conditions. In view of the close similarity between equations 2.31-32 and 2.41-22, and between equations 2.31-32A and 2.41-23, one may adopt the same rules for the inclusion of overall factors of i in the covarients as one did in section 2.31. The covarients will then satisfy either one or both of the discontinuity conditions as appropriate, and the various implied choices of charge-conjugation phase will be the same as those made in order to obtain real coupling-constants for three-particle vertices.

Provided the appropriate discontinuity conditions are satisfied, the unitarity relation, 1.3-9, may be written in a form in which all terms involve the same external wave-functions and the same set of kinematic covarients. Factoring out the former, and equating coefficients of the latter, one obtains:

$$\partial isc_s A_j(s,t) = -i\sum_N A_j^N(s,t)\,\delta^4(p_N - p_i), \tag{2.41-24}$$

where the amplitudes $A_j^N$ are defined symbolically by:

$$M_{\beta\sigma}(fN)\mathcal{P}_{\sigma\tau}(N)\overline{M}_{\alpha\tau}(iN) \equiv \sum_{j=1}^{N} A_j^N \mathcal{K}^j_{\beta\alpha}. \tag{2.41-25}$$

In this equation: $M_{\beta\sigma}(fN)$ and $M_{\alpha\tau}(iN)$ denote the M-functions corresponding to the T-matrix elements: $T_{fN}(s,t)$ and $T_{iN}(s,t)$, $\mathcal{P}_{\sigma\tau}(N)$ denotes the set of on-shell propogator numerators for the particles comprising state N, and as usual:

$$\overline{M}_{\alpha\tau}(iN) \equiv \gamma_0 M^{\dagger}_{\alpha\tau}(iN)\gamma_0 \; . \qquad (2.41\text{-}26)$$

When one is using the above equations in the resonance approximation of section 1.3, the M-functions of 2.41-25 are replaced by coupling-functions which are assumed to satisfy 2.31-31. That is, one assumes that the couplings of the resonances to the initial and final states may be approximately represented in terms of hermitian interaction Lagrangians. Equation 1.3-32, for example, then reads for arbitrary spin processes:

$$\int_{s_0}^{\infty} ds' s'^m \,\mathrm{disc}_s A_j(s',t) \simeq -2i \sum_R M_R^{2m} \left\{ \frac{\pi}{2} + \tan^{-1}\left( \frac{M_R^2 - s_0^{(R)}}{M_R \Gamma_R} \right) \right\} A_j^R(t), \qquad (2.41\text{-}27)$$

where $A_j^R(t)$ is given by:

$$\mathcal{C}_{\beta\sigma}(fR)\mathcal{P}_{\sigma\tau}(R)\mathcal{C}_{\tau\alpha}(Ri)\Big|_{s=M_R^2} \equiv \sum_{j=1}^{N} A_j^R(t)\mathcal{K}^j_{\beta\alpha} \; , \qquad (2.41\text{-}28)$$

and $\mathcal{P}_{\sigma\tau}(R)$ now denotes: $\mathcal{P}^{S_R}_{(\sigma)^{J_R}(\tau)^{J_R}}(p+q)$, the on-shell propogator numerator for a spin $S^R$ particle with mass $M_R$ and momentum $(p+q)$.

The covarients for a given process may again be constructed by following the rules of section 2.31, provided that these rules are modified to take into account the fact that one is now dealing with four external particles. Let $\mu$, $\nu$, $\mu'$ and $\nu'$ denote any one of the Lorentz indices of the wave-functions of particles 1,2,3, and 4 respectively. Also, let $\pi$ denote any one of these four indices, $\rho$ and one of the remaining three, and $\tau$ either of the final pair. Then for four-boson and for two-boson/two-fermion processes, the required modifications are as follows.

Rules i, iii, iv, and viii remain unchanged, except that they now refer to M-functions, and viii is now the rule for abnormal two-boson/two-fermion processes. Rule ii is not applicable to M-functions.

Rule v applies to normal four-boson processes. There are now six possible types of metric tensor to choose from, namely the various $g_{\pi\rho}$. It is now possible to construct three linearly independent momentum combinations from p, q, p', and q'; denote these by a, b, and c. Any two of these will remain linearly inequivalent when contracted with the wave-functions, and we denote these by a and b. Thus as far as momenta are concerned, one now has the eight possible types: $a_\pi$ and $b_\pi$.

Rule vi now refers to abnormal four-boson processes. It is unchanged except that one now has up to thirty-five possible types of overall Levi-Cevita tensor. These are: $\varepsilon_{\mu'\nu'\mu\nu}$; four each of the types: $\varepsilon_{\pi\rho\sigma}(a)$, $\varepsilon_{\pi\rho\sigma}(b)$, $\varepsilon_{\pi\rho\sigma}(c)$, and $\varepsilon_{\pi}(abc)$; and six each of the types: $\varepsilon_{\pi\rho}(ab)$, $\varepsilon_{\pi\rho}(bc)$, and $\varepsilon_{\pi\rho}(ca)$. The equivalence relations between the possible covarients constructable in this fashion are much more involved and numerous than those for the corresponding three-point vertices, but are readily obtained in any specific case by the application of the basic equations of Appendix 3.

Rule vii is applicable to two-boson/two-fermion processes. It is unchanged except that the overall Dirac matrix factors to be included in the covarients are now to be chosen from the eight: $1\!\!1_4$, $\not{d}$, $\gamma_{\mu_1}$, $\gamma_{\mu_1'}$, $\gamma_{\mu_1}\not{d}$, $\gamma_{\mu_1'}\not{d}$, $\gamma_{\mu_1'}\gamma_{\mu_1}$ and $\gamma_{\mu_1'}\gamma_{\mu_1}\not{d}$. The momentum d is to be any fixed linear combination of the two boson momenta: q and q'.

Finally, a note about kinematic singularities. As mentioned above we adopt the viewpoint of Hearn[27] that if an M-function is expanded in terms of all possible covarients allowed by the symmetries of the T-matrix:

$$M = \sum_{j=1}^{m} A_j \mathcal{K}^j , \qquad (2.41\text{-}29)$$

where m is very large, (indeed presumably infinite), then the corresponding $A_j$ will all be K.S.F.. This was phrased in more rigorous terms earlier. We argued that it is for this (hypothetical) set of amplitudes that one should postulate "dynamical analyticity", and kinematical singularities are then defined as any additional singularities introduced by the reduction to a linearly inequivalent set of N covarients.

Suppose for the sake of argument that there exists an equivalence relation, (hereafter abbreviated to "E.R."), which reads:

$$\sum_{j=1}^{n} a_j \mathcal{K}^j \cong 0 , \qquad (2.41\text{-}30)$$

for some finite $n$ $(< m)$. Each $a_j$ may be either a scalar constant, (i.e. a pure number or a function of the masses), or a scalar variable, (i.e. a function of the Mandelstam variables). Suppose for the further sake of argument that 2.41-30 is used to eliminate $\mathcal{K}^1$ from 2.41-29. Then this latter equation becomes:

$$M \cong \sum_{j=2}^{n} A'_j \mathcal{K}^j + \sum_{j=n+1}^{m} A_j \mathcal{K}^j , \qquad (2.41\text{-}31)$$

where:

$$A'_j = A_j - (a_j/a_1) A_1 . \qquad (2.41\text{-}32)$$

The M-function now involves one less amplitude, but the $A'_j$ will only be K.S.F. if $a_1$ is a constant. Otherwise, each will have a kinematic pole at vanishing $a_1$; (except, of course, that it may happen for some particular j that $a_j$ also vanishes at that point).

Let us define a "type 1 E.R." to be one in which all the $a_j$ are constants, and a "type 2 E.R." to be one in which at least one of the $a_j$ is a variable. We further define a pair of type 2 E.R.'s to be "equivalent" or "inequivalent" according as one can or cannot be transformed into the other by means only of type 1 E.R.'s.

Thus in the reduction from an infinite linearly equivalent to a finite linearly inequivalent set of covarients, the final set of amplitudes will all be K.S.F. provided that type 2 E.R.'s are used only to eliminate covarients which appear in them with constant coefficients. The use of type 1 E.R.'s is not subject to restriction, since these can never introduce kinematic singularities.

It might seem that to obtain a set of K.S.F. amplitudes for a given process, one must eliminate an infinity of covarients by means of an infinity of E.R.'s., - a time consuming series of manipulations to say the least! Fortunately this is not the case. The crucial point is that the number of inequivalent type 2 E.R.'s constructable for any given process is finite, and in all practical cases rather small. Indeed, one needs quite a lot of spin before this number ceases to be zero.

The prescription for obtaining a K.S.F. set of amplitudes is therefore as follows. First construct a maximal set of inequivalent type 2 E.R.'s for the process. (In practice this comes with experience, and is not as difficult as it sounds). Let $r$ be the number of E.R.'s in this set, whilst as usual N is the required number of final covarients. If the set of E.R.'s contains more than (N + r) covarients, operate with type 1 E.R.'s until only (N + r) appear. Otherwise, obtain (N + r) covarients by constructing additional ones which are linearly inequivalent both to one another and to those appearing in the E.R.'s. Since all the inequivalent type 2 E.R.'s for the process relate only some or all of these (N + r) covarients, these latter must be related to all other possible covarients only through type 1 E.R.'s. Thus the corresponding (N + r) non-linearly-independent amplitudes must all be K.S.F.. Finally, select r of these covarients in such a way that each appears with a constant coefficient in a different E.R., and use each E.R. in turn to eliminate the respective covarient. Each of the final N amplitudes must then also be K.S.F., as required.

Note that the existence, for a given process, of a type 2 E.R. in which all coefficients were variables, would be sufficient to guarantee that no K.S.F. spin-decomposition was possible. This would in turn violate the usual assumption that the so-called "reduced helicity amplitudes"(28) for any process are K.S.F.. Happily, no examples of this pathological situation have yet been discovered.

## 2.42 O(3,1) ⊗ SU(2) M-FUNCTIONS.

As in the case of three particle vertices, it is convenient to build SU(2) invarience into the spin-decomposition of the previous section by a further isospin-decomposition, writing (in shorthand notation):

$$A_j(t_1' t_2', t_1 t_2) = \phi^{+t_1'}_{i_1'} \phi^{+t_2'}_{i_2'} \sum_{k=1}^{\mathcal{N}} A_j^k \mathcal{K}^k_{i_1' i_2' i_1 i_2} \phi^{t_1}_{i_1} \phi^{t_2}_{i_2} . \qquad (2.42\text{-}1)$$

Here the $\phi$'s are isospace wave-functions, the $\mathcal{K}^k$ are "kinematic" covarients in isospace, the $A_j$ are invarient amplitudes in Lorentz-space alone, (now t-dependent), and the $A_j^k$ are t-independent invarient amplitudes in Lorentz ⊗ SU(2) space. The number, $\mathcal{N}$, of linearly inequivalent isospace covarients is just equal to the total initial (equals total final) isospin.

The full spin ⊗ isospin decomposition in the s-channel, for example, then reads: $\langle q'\lambda_1' t_1', p'\lambda_2' t_2' | T | q\lambda_1 t_1, p\lambda_2 t_2 \rangle = \varepsilon^{*\lambda_1'}_{\mu'}(q') \phi^{+t_1'}_{i_1'} \times$

$$\times \bar{u}^{\lambda_2'}_{\nu'}(p') \phi^{+t_2'}_{i_2'} M^{i_1' i_2' i_1 i_2}_{\mu'\nu'\mu\nu} \varepsilon^{\lambda_1}_{\mu}(q) \phi^{t_1}_{i_1} u^{\lambda_2}_{\nu}(p) \phi^{t_2}_{i_2} , \qquad (2.42\text{-}2)$$

where:

$$M^{i_1' i_2' i_1 i_2}_{\mu'\nu'\mu\nu} = \sum_{j=1}^{N} \sum_{k=1}^{\mathcal{N}} A_j^k(s,t) \mathcal{K}^j_{\mu'\nu'\mu\nu} \mathcal{K}^k_{i_1' i_2' i_1 i_2} . \qquad (2.42\text{-}3)$$

The M-functions in the three channels are again related by the crossing rule, and equation 2.41-11 generalises in Lorentz $\otimes$ SU(2) space to: $M^{i_4 i_3 i_2 i_1}_{\mu_4\mu_3\mu_2\mu_1}\left(1(p_1)+2(p_2)\to 3(p_3)+4(p_4)\right)=$

$$=\xi_s\xi_T(-1)^{I_2+I_4}M^{i_4 i_3 i_2 i_1}_{\mu_4\mu_3\mu_2\mu_1}\left(1(p_1)+\bar{4}(-p_4)\to 3(p_3)+\bar{2}(-p_2)\right), \tag{2.42-4}$$

where $$\xi_T=\begin{cases}+1 & \text{for bb crossing,}\\ -1 & \text{for bf and ff crossing.}\end{cases} \tag{2.42-5}$$

This result follows on using 2.41-11 to generalise the standard crossing relation for spinless invarient amplitudes in SU(2) space alone:[29]

$$A\left(T_1t_1+T_2t_2\to T_3t_3+T_4t_4\right)=\xi_{24}A\left(T_1t_1+\bar{T}_4\bar{t}_4\to T_3t_3+\bar{T}_2\bar{t}_2\right), \tag{2.42-6}$$

where:
$$\xi_{24}=(-1)^{t_2-t_4}\times\begin{cases}(-1)^{-T_2-T_4}, & \text{for bb and fb crossing,}\\ (-1)^{T_2-T_4}, & \text{for ff crossing.}\end{cases} \tag{2.42-7}$$

One may choose to use the same SU(2) covarients in each channel, in which case isospin crossing matrices do not arise, or one may choose to use different covarients in each channel. This latter choice requires the use of crossing matrices[29] to pass from one channel to another, but enables one to decompose in terms of eigenamplitudes of total isospin (T) in each channel. In a given channel one then writes:

$$A_j(t_1't_2',t_1t_2)=\phi^{+t_1'}_{i_1'}\phi^{+t_2'}_{i_2'}\sum_T A_j^T\mathcal{K}^T_{i_1'i_2'i_1i_2}\phi^{t_1}_{i_1}\phi^{t_2}_{i_2}. \tag{2.42-8}$$

Each $\mathcal{K}^T$ is then the projection operator for total isospin T in that channel and must be normalised so that:

$$(\mathcal{K}^T)^2=\mathcal{K}^T\quad,\quad\sum_T\mathcal{K}^T=1. \tag{2.42-9, 10}$$

The structure of such projection operators may be determined to within a normalisation constant by considering the isospace part of the pole graph corresponding to the reaction: 1 + 2 $\to$ (particle with isospin T) $\longrightarrow$ 3 + 4. That is, one has: $\mathcal{K}^T_{i_1'i_2',i_1i_2}(T_1'T_2',T_1T_2)\propto\mathcal{K}_{i_1'i_2',i'}(T_1'T_2',T)\times$

$$\times\rho^T_{i',i}\mathcal{K}_{i,i_1i_2}(T,T_1T_2)\quad. \tag{2.42-11}$$

The $A_j^T$ for a given channel may be used as channel-independent amplitudes, but will not in general be eigen-amplitudes of total isospin in the other two channels, nor will they necessarily represent the simplest or most useful choice from the channel-independence point of view. Nevertheless, the determination of a set of unnormalised $\mathcal{K}^T_{i_1'i_2'i_1i_2}$ in at least one channel provides the best initial step in the construction of any set of $\mathcal{K}^k_{i_1'i_2'i_1i_2}$. Since the $\mathcal{K}^T$ will automatically be lineraly inequivalent, one avoids in this way any need to manipulate equivalence relations.

Writing $M^{b,a}_{\beta,\alpha}$ for $M^{i_1'i_2',i_1i_2}_{\mu'\nu',\mu\nu}$, the implications on the M-function of P, T, and G-invarience are given by substituting $M^{ba}_{\beta\alpha}$ for $\mathcal{C}^{ba}_{\beta\alpha}$ in equations 2.32-7, 8, and 13, respectively.

As usual, the P-invarience constraint on the Lorentz covarients is not affected by the extension of this equation to Lorentz $\otimes$ SU(2) space.

Similarly, equations 2.41-22 and 23 remain the respective discontinuity conditions for T and PT-invarient processes, provided that the SU(2) covarients satisfy equation 2.32-12. Equations 2.41-24 and 25 then generalise to:

$$\partial isc_s A_j^k(s,t) = -i\sum_N A_j^{kN}(s,t)\delta^4(p_N - p_i) \,, \qquad (2.42\text{-}12)$$

where the $A_j^{kN}$ are defined by:

$$M^{bc}_{\beta\sigma}(fN)\mathcal{P}_{\sigma\tau}(N)\mathcal{P}_{cd}(N)\bar{M}^{ad}_{\alpha\tau} \equiv \sum_{j=1}^{N}\sum_{k=1}^{\mathcal{N}} A_j^{kN}\mathcal{K}^j_{\beta\alpha}\mathcal{K}^k_{ba} \,. \qquad (2.42\text{-}13)$$

One readily proves that the right-hand side of equation 2.42-11 satisfies equation 2.32-12 for arbitrary $T_1$, $T_2$, $T_1'$, $T_2'$, and T. Hence, provided the isospin covarients are constructed by taking linear combinations with real coefficients of sets of $\mathcal{K}^T$, these former will also satisfy 2.32-12.

In certain special cases, particularly those involving identical multiplets, G-invarience and crossing together further constrain the covarients or relate the amplitudes at different values of their arguments.

One such special case will be needed later in this thesis. Multiplets 1 and 3 have integer spin and isospin, whilst 2 and 4 are identical multiplets having half-integer spin and isospin. The result is that if the covarients are chosen to satisfy:

$$\mathcal{K}^{j}_{\mu'\nu'\mu\nu}(q'p'qp)=\xi_j C\,\mathcal{K}^{j\,T}_{\mu'\nu\,\mu\nu'}(q',-p,q,-p')C^{-1}, \tag{2.42-14}$$

and:
$$\mathcal{K}^{k}_{i_1' i_2' i_1 i_2}=\xi_k \mathcal{C}\,\tilde{\mathcal{K}}^{k}_{i_1' i_2 i_1 i_2'}\,\mathcal{C}^{-1}, \tag{2.42-15}$$

with
$$\xi_{j,k}=\pm 1, \tag{2.42-16}$$

then the amplitudes will satisfy:

$$A^{k}_{j}(s,t,u)=G_1 G_3 \xi_j \xi_k A^{k}_{j}(u,t,s). \tag{2.42-17}$$

## 2.5 THE COVARIENT REGGEISATION TECHNIQUE.[12]

It is well known that the high energy asymptotic behaviours of the amplitudes for a strong interaction scattering process are determined in a given channel by the contributions each receives from "intermediate" Regge poles in the appropriate crossed channel. Until recently one knew of no easy means by which covarient partial-wave expansions might be obtained. It was therefore customary to Reggeise the crossed channel centre of mass frame helicity amplitudes.[28][30] These had then to be related to the direct channel amplitudes whose asymptotic behaviours were required.

In the formalism under review in Part I of this chapter a covarient partial-wave expansion presents no difficulties and one is therefore able to Reggeise invarient amplitudes directly. The essential point is simply that in a given channel the Jth partial-wave of a given amplitude is proportional to the contribution that amplitude receives from a spin J (or J + ½) on-shell one-particle intermediate state in that channel. From the point of view of Reggeisation the constant of proportionality is not explicitly required since it may be absorbed into the Regge couplings.

For the sake of argument, suppose one has a strong interaction two particle to two particle scattering process with kinematics and channels as defined in section 2.41, and one wishes to determine the high-s asymptotic behaviours of a set of K.S.F. invarient amplitudes for this process by covarient Reggeisation in the t-channel. We shall assume that this latter is a "boson channel". If this is not the case one simply replaces the spin J propogators in the argument following by the corresponding ones for spin J + $\frac{1}{2}$, and the $A_j^{Jn}$ become invarient eigenamplitudes for t-channel initial total angular momentum J + $\frac{1}{2}$.

Working for the moment in Lorentz space alone one has, then, an M-function $M_{\mu'\nu'\mu\nu}$ with K.S.F. spin decomposition:

$$M_{\mu'\nu'\mu\nu} = \sum_{j=1}^{N} A_j(s,t)\mathcal{K}^j_{\mu'\nu'\mu\nu} \, . \qquad (2.5\text{-}1)$$

Since the process is assumed to be P-invarient, one wishes to make a (covarient) "partial-wave" decomposition in the t-channel:

$$A_j(s,t) = \sum_{J=0}^{\infty} \sum_{n=\pm 1} (2J+1) A_j^{Jn}(s,t) \, , \qquad (2.5\text{-}2)$$

where the "invarient eigenamplitude" $A_j^{Jn}$ is that part of $A_j$ which corresponds to a t-channel initial state with total angular momentum J and normality:

$$n \equiv (\text{total parity})(-1)^J = \pm 1 . \qquad (2.5\text{-}3)$$

$A_j^{Jn}$ is then given by:

$$\sum_{j=1}^{N} A_j^{Jn}(s,t)\mathcal{K}^j_{\mu'\nu'\mu\nu} = C(J,n) M^{Jn}_{\mu'\nu'\mu\nu} \, , \qquad (2.5\text{-}4)$$

where $C(J,n)$ is a proportionality constant which will remain undetermined. $M^{Jn}_{\mu'\nu'\mu\nu}$ is the numerator of the pole graph for a t-channel on-shell single particle intermediate state with spin J and normality n, and is given by:

$$M^{Jn}_{\mu'\nu'\mu\nu} = \mathcal{C}^{n}_{\nu'\nu(\tau)^J}(s_4\bar{s}_2,J)\,\mathcal{P}^{J}_{(\tau)^J(\sigma)^J}(\Delta)\,\mathcal{C}^{n}_{(\sigma)^J\mu'\mu}(J,\bar{S}_3S_1)\ , \qquad (2.5\text{-}5)$$

with:

$$\Delta \equiv p'-p = q-q'\,. \qquad (2.5\text{-}6)$$

The coupling functions depend on n, since this affects their normality. In view of the construction rules of section 2.31, the structure of these functions may be exhibited in the form:

$$\mathcal{C}^{n}_{(\sigma)^J\mu'\mu}(J,\bar{S}_3S_1) = \sum_{r'=0}^{\mathrm{Min}(J,\bar{S}_3+S_1)}\ \sum_{j'=N'(r'-1)+1}^{N'(r')} f^{Jn}_{j'}\,\mathcal{K}^{j'n}_{(\sigma)^{r'}\mu'\mu}\left(-Q_\sigma\right)^{J-r'}, \qquad (2.5\text{-}7)$$

$$\mathcal{C}^{n}_{\nu'\nu(\tau)^J}(S_4\bar{S}_2,J) = \sum_{r''=0}^{\mathrm{Min}(J,S_4+\bar{S}_2)}\ \sum_{j''=N''(r''-1)+1}^{N''(r'')} g^{Jn}_{j''}\,\mathcal{K}^{j''n}_{\nu'\nu(\tau)^{r''}}\left(P_\tau\right)^{J-r''}. \qquad (2.5\text{-}8)$$

In these equations:

$$P \equiv \tfrac{1}{2}(p+p')\ , \qquad Q \equiv \tfrac{1}{2}(q+q')\ , \qquad (2.5\text{-}9,10)$$

$$N'(-1) \equiv 0 \equiv N''(-1)\ , \qquad (2.5\text{-}11)$$

and N'(r') and N"(r") are defined to be the respective number of independent couplings at the vertices: $S_1+\bar{S}_3 \rightarrow r'$ and $r'' \rightarrow \bar{S}_2+S_4$, when the spin r' and r" particles have normality n. The covarients $\mathcal{K}^{j'n}$ and $\mathcal{K}^{j''n}$ are J-independent, and contain no factors of the type $Q_\sigma$ and $P_\tau$ respectively.

Equation 2.5-5 may thus be written:

$$M^{Jn}_{\mu'\nu'\mu\nu} = \sum_{r'=0}^{\mathrm{Min}(J,\bar{S}_3+S_1)}\ \sum_{j'=N'(r'-1)+1}^{N'(r')}\ \sum_{r''=0}^{\mathrm{Min}(J,S_4+\bar{S}_2)}\ \sum_{j''=N''(r''-1)+1}^{N''(r'')} f^{Jn}_{j'}\,g^{Jn}_{j''}\ \times$$

$$\times\ \mathcal{K}^{j''n}_{\nu'\nu(\tau)^{r''}}\,\mathcal{P}^{J}_{(\tau)^{r''}(\sigma)^{r'}}(P,-Q;\Delta)\,\mathcal{K}^{j'n}_{(\sigma)^{r'}\mu'\mu}\ . \qquad (2.5\text{-}12)$$

Provided one specifies:

$$f^{Jn}_{j'} = 0 \quad \text{for all } j' > N'(J), \qquad (2.5\text{-}13)$$

and

$$g^{Jn}_{j''} = 0 \quad \text{for all } j'' > N''(J), \qquad (2.5\text{-}14)$$

the upper r' and r" summation limits, Min $(J,\bar{S}_3+S_1)$ and Min $(J,S_4+\bar{S}_2)$, in equation 2.5-12 may be replaced by $(\bar{S}_3+S_1)$ and $(S_4+\bar{S}_2)$ respectively. The specifications 2.5-13 and 14 just remove from this modified equation those terms which involve covarients and propogators contracted via more indices than the available intermediate spin allowes. After performing all indicated contractions, one may obtain terms which superficially appear to have poles at certain low integer values of J. However, on closer inspection one notices that such poles always have their origin in these same "nonsensical" contractions, and the terms in which they appear are eliminated by specifications 2.5-13 and 14.

The "tensorial structure" of 2.5-12 is now J-independent. This modified equation depends on J only through the coupling-constants and scalar functions of J involving solid harmonics (or derivatives thereof) with argument $P(\Delta)\cdot Q(\Delta)$. Comparison of equations 2.5-4 and 12 (modified) thus yields expressions for the $A_j^{\tau n}$ valid, (in combination with 2.5-13 and 14), for arbitrary non-negative definite integer J. Substitution of these expressions into equation 2.5-2 yields a covarient partial-wave expansion for each $A_j$.

The "angular factors" appearing in each such expansion are just Legendre polynomials and their derivatives. Using the orthogonality properties of such functions, these expansions may be inverted to give Froissart-Gribov expressions for the partial-wave amplitudes. After converting the summation in 2.5-2 to a contour integral one can therefore perform a Sommerfeld-Watson transform, picking up t-channel Regge pole contributions to $A_j(s,t)$ given by:

$$\sum_{j=1}^{N} A_j^{R}(s,t)\mathcal{K}^{j}_{\mu'\nu'\mu\nu} = \sum_{n=\pm 1}\sum_{\tau=\pm 1}\tfrac{1}{2}\left(1+\tau e^{i\pi\alpha_n^\tau(t)}\right)\left(2\alpha_n^\tau(t)+1\right)\times$$

$$\times\frac{\pi\alpha_n^{\tau\prime}(t)}{\sin\pi\alpha_n^\tau(t)}\sum_{r'=0}^{\bar{S}_3+S_1}\;\sum_{j'=N'(r'-1)+1}^{N'(r')}\;\sum_{r''=0}^{\bar{S}_2+S_4}\;\sum_{j''=N''(r''-1)+1}^{N''(r'')} f_{j'}(n,\tau;t)\times$$

$$\times g_{j''}(n,\tau;t)\mathcal{K}^{j''n}_{\nu'\nu(\tau)r''}\,\hat{\mathcal{P}}^{J\to\alpha_n^\tau(t)}_{(\tau)r''(\sigma)r'}(P,-Q;\Delta)\mathcal{K}^{j'n}_{(\sigma)r'\mu'\mu}. \qquad (2.5\text{-}15)$$

In this equation: $A_j^{R}(s,t)$ is the total t-channel Regge contribution (neglecting isospin), to the amplitude $A_j(s,t)$;

and $\alpha^{\tau}_{n}(t)$ is the Regge trajectory with signature $\tau$ and normality n. $\hat{\mathcal{P}}^{J\to\alpha^{\tau}_{n}(t)}_{(\tau)r''(\sigma)r'}$ is to be obtained from $\mathcal{P}^{J}_{(\tau)r''(\sigma)r'}$ by reversing the sign of the arguments of all Legendre polynomials (or derivatives) appearing, and after performing all contractions with the initial and final covarients one is to make the continuation: $J \longrightarrow \alpha^{\tau}_{n}(t)$. The "Regge coupling-constants" $f_{j'}(n,\tau;t)$ and $g_{j''}(n,\tau;t)$ are to be obtained from the corresponding $f^{Jn}_{j'}$ and $g^{Jn}_{j''}$ by making this same continuation, after first absorbing a factor $\sqrt{C(J,n)}$ into each. They have "nonsense zeros" at those values of t for which $\alpha^{\tau}_{n}(t)$ is equal to an (integer) J value satisfying 2.5-13 or 14 as appropriate. Notice from equation 2.5-15: firstly, the extremely simple way that parity is incorporated into the formalism; and secondly, that all Regge couplings involved are automatically factorised, that is, one only deals with products of pairs of "initial" and "final" Regge coupling-constants.

From 2.5-15 one obtains, then, an expression for each $A^{\pi}_{j}$ in terms of a linear combination of solid harmonic derivatives of the general form: $\hat{\mathcal{P}}^{(n)}_{\alpha(t)-m}\big(P(\Delta)\cdot Q(\Delta)\big)$. Each combination coefficient is the product of an "initial" Regge coupling-constant, a "final" Regge coupling-constant, and a polynomial in the masses and Mandelstam variables. The solid harmonic derivatives have detailed structure:

$$\hat{\mathcal{P}}^{(n)}_{\alpha(t)-m}\big(P(\Delta)\cdot Q(\Delta)\big) = \left[P^2(\Delta)Q^2(\Delta)\right]^{\frac{1}{2}(\alpha(t)-m-n)} \times$$

$$\times P^{(n)}_{\alpha(t)-m}\left[-P(\Delta)\cdot Q(\Delta)\big/\left(P^2(\Delta)Q^2(\Delta)\right)^{1/2}\right], \qquad (2.5\text{-}16)$$

where:

$$P(\Delta)\cdot Q(\Delta) = \frac{1}{4}\left[s-u+\frac{1}{t}(m_3^2-m_1^2)(m_4^2-m_2^2)\right], \qquad (2.5\text{-}17)$$

$$P^2(\Delta) = \frac{1}{4t}\left[t-(m_4+m_2)^2\right]\left[t-(m_4-m_2)^2\right], \qquad (2.5\text{-}18)$$

$$Q^2(\Delta) = \frac{1}{4t}\left[t-(m_3+m_1)^2\right]\left[t-(m_3-m_1)^2\right]. \qquad (2.5\text{-}19)$$

The high s leading asymptotic behaviour of the $A_j$ can thus be picked out for any fixed t. Notice that the correct "threshold factors", $[P^2(\Delta)Q^2(\Delta)]^{\frac{1}{2}(\alpha(t)-m-n)}$, appear quite automatically.

The $1/t$ terms in equations 2.5-17 to 19 arise out of the t-channel boost prescription:

$$\underline{a}\cdot\underline{b} \rightarrow -a(\Delta)\cdot b(\Delta) = -a\cdot b + \frac{a\cdot\Delta\, b\cdot\Delta}{\Delta^2} = -a\cdot b + \frac{a\cdot\Delta\, b\cdot\Delta}{t}\,, \qquad (2.5\text{-}20)$$

and lead to poles at zero t in the expressions for the $A_j^R$ if:

$$m_1 \neq m_3 \text{ and/or } m_2 \neq m_4. \qquad (2.5\text{-}21)$$

This is the so-called "unequal mass problem".(31) For processes with sufficiently high external spin the above mentioned polynomial coefficients may also have poles at vanishing t. These again have their origin in the boost prescription, and arise out of factors such as: $g_{\tau\sigma}(\Delta)$, $P_\sigma(\Delta)$, $Q_\sigma(\Delta)$, $P_\tau(\Delta)$, $Q_\tau(\Delta)$, etc. in the partially contracted propogators. Here one has the "high spin problem".(31) Note the common origin of both types of problem in this formalism.

Both types of unwanted pole in t can be simultaineously removed in any of three ways, viz:

i) The "fixed pole" solution.(12)(32) Instead of continuing directly to zero t by means of equation 2.5-15, one uses this equation only down to the t-channel threshold. The continuation to zero t is then performed by means of an unsubtracted fixed-s dispersion relation in which the contour remains at a safe distance from all singularities. The continued $A_j^R$ defined in this way remain finite below threshold, especially at vanishing t. However, this prescription introduces into the amplitudes additional fixed (i.e. t-independent) poles in the J-plane, and its validity therefore relies on these being consistent with Mandelstam analyticity.(33)

Whether this is in fact the case would still seem to be an open question.(12)

ii) The "evasive" solution.(34) At zero t the Reggeon simulates a massless particle, and the initial and final vertices ought therefore to be internally gauge-invarient at this point. That is, at zero t the initial and final Regge couplings should vanish on contraction with $\Delta_\sigma$ and $\Delta_\tau$ respectively. The Regge coupling-constants corresponding to vertex covarients which fail to behave in this way should therefore be proportional to t. These t-factors then cancel the unwanted poles.

iii) The "conspiratorial" solution.(34) One associates with each trajectory leading to unwanted poles an additional "conspirator trajectory" having the same conserved quantum numbers. The corresponding couplings of this latter trajectory are related to those of the former, and in addition have just those singularities at zero t which cause the total contribution to a given $A_j$ due to the two trajectories together to remain finite at that point.

Details of the fixed pole solution have been given by Scadron and Jones,(12) and of evasive and conspiratorial solutions by Gault.(35) The essential result of these detailed treatments is that whichever solution is adopted, the leading high-s asymptotic behaviour of each invarient amplitude remains the same after pole elimination as it was before this operation. Throughout the remainder of this thesis we shall therefore ignore all poles at integer J and at zero t arising during covarient Reggeisation.

So far we have neglected isospin and G-parity working in Lorentz-space alone and characterising the Regge trajectories by their normality and signature. If one is only concerned with the Reggeisation of boson channels of zero strangeness processes, one may make the argument fully general by working in Lorentz $\otimes$ SU(2) space and characterising the Regge trajectories in addition by their isospin (T), and G-parity (G). The total Regge contribution, $A_j^{kR}$(s,t), to the Lorentz $\otimes$ SU(2) invarient amplitude, $A_j^k$(s,t), is then given by modifying equation 2.5-15 as follows:

$$\sum_{j=1}^{N} A_j^R(s,t)\mathcal{K}^j_{\mu'\nu'\mu\nu} \rightarrow \sum_{j=1}^{N}\sum_{k=1}^{\mathcal{N}} A_j^{kR}(s,t)\mathcal{K}^j_{\mu'\nu'\mu\nu}\mathcal{K}^k_{i_1' i_2' i_1 i_2}\,, \qquad (2.5\text{-}22)$$

$$\sum_{n=\pm1}\sum_{\tau=\pm1} \rightarrow \sum_{n=\pm1}\sum_{\tau=\pm1}\sum_{G=\pm1}\sum_{T\ \text{allowed}} \qquad (2.5\text{-}23)$$

$$\alpha_n^\tau(t) \rightarrow \alpha_{nT}^{\tau G}(t)\,, \qquad (2.5\text{-}24)$$

$$f_{j'}(n,\tau;t) \rightarrow f_{j'}(n,\tau,T,G;t)\,, \qquad (2.5\text{-}25)$$

$$g_{j''}(n,\tau;t) \rightarrow g_{j''}(n,\tau,T,G;t)\,, \qquad (2.5\text{-}26)$$

and finally each term on the right-hand side is multiplied by the appropriate isospace pole graph factor:

$$\mathcal{K}_{i_2' i_2 i'}(T_4\overline{T}_2,T)\rho^T_{i'i}\mathcal{K}_{i i_1' i_1}(T,\overline{T}_3 T_1)\,. \qquad (2.5\text{-}27)$$

Some reduction in the range of the four-fold summation 2.5-23 will result if G-parity selection rules are operative at the initial and/or final vertices, and in addition the spin covarients will depend on T and G if the vertices are subject to G-parity constraints.

# CHAPTER 2, PART II.

## THE GENERALISATION TO REAL AND VIRTUAL PHOTONIC PROCESSES.

### 2.6 $O(3,1) \otimes SU(2)$ DECOMPOSITION OF VIRTUAL PHOTONIC THREE AND FOUR-POINT FUNCTIONS.

The formalism reviewed in Part I of this chapter was set up with purely hadronic processes in mind. With an eye towards important classes of reaction such as hadron photo-production and Compton scattering, and vertices involving hadron electromagnetic form-factors at zero argument, it is useful to generalise this formalism to include the possibility of one or more of the particles being real photons. The essential additional ingredient is gauge-invarience, and a suitable generalisation has been given by Scadron and Jones.[13]

If one wishes to study the electromagnetic form-factors at non-zero argument, and the electroproduction of hadrons, it is necessary to go a stage further and include the possibility that the photons are virtual. In this second part of Chapter 2 we give a generalisation to three-and four-point "vertex-functions" involving a virtual photon, the remaining particles being on-shell hadrons. These are current-conserving generalisations of our previous coupling- and M-functions. They correspond to matrix-elements of the electromagnetic current operator taken betwen on-shell states containing a total of two and three hadrons respectively. We remind the reader that as a consequence of space-time translational invarience, it is only necessary to work with matrix elements of the current evaluated at the origin of the space-time coordinates.

Our treatment is equally applicable to real photons. That is, it is designed to reduce to a valid real photon formalism in the limit as the squared four-momentum of the virtual photon tends to zero. In this limit it parallels the real photon approach of Scardon and Jones, which consequently will not be reviewed here.

In this present section we formulate sets of rules for the Lorentz $\otimes$ SU(2) space decomposition into kinematic singularity free form-factors of three- and four-point hadron/virtual photon vertices. In section 2.7 such decompositions are derived in Lorentz-space alone, (the extension to Lorentz$\otimes$ SU(2) space being relatively trivial), for all three-point vertices encountered later in this thesis. We relate those in which we are primarily interested to unpolarised cross-sections. Decompositions of a number of four-point vertices are deduced in Chapter 4.

We are concerned, then, with matrix elements: $\langle f|j_\alpha(0)|i\rangle$ in which $|i\rangle$ and $|f\rangle$ are respectively initial and final on-shell hadron states. For the sake of argument we shall assume that $|i\rangle$ contains a single hadron. The state $|f\rangle$ will then contain either one or two hadrons.

In practice such a matrix element will always be contracted via a virtual photon propogator with a second matrix element of the current operator; so if q is the virtual photon four-momentum:

$$q \equiv p_f - p_i = p_{i'} - p_{f'} \ , \tag{2.6-1}$$

we are dealing with a quantity which looks like:

$$\lim_{\epsilon\to 0^+} \langle f'|j_\beta(0)|i'\rangle \frac{g_{\beta\alpha}}{q^2+i\epsilon} \langle f|j_\alpha(0)|i\rangle . \tag{2.6-2}$$

It happens in this thesis that the states $|i'\rangle$ and $|f'\rangle$ will contain respectively one initial and one final on-shell electron, but they can, of course, be quite general states. Current conservation implies that:

$$q_\alpha\langle f'|j_\alpha(0)|i'\rangle = 0 = q_\alpha\langle f|j_\alpha(0)|i\rangle . \tag{2.6-3}$$

It will prove a useful shorthand notation to define a "virtual photon wave-function" by:

$$\mathcal{E}_\alpha(q) \equiv \lim_{\epsilon \to 0^+} \langle f' | j_\beta(0) | i' \rangle \frac{g_{\beta\alpha}}{q^2 + i\epsilon} , \tag{2.6-4}$$

and in virtue of 2.6-3 this satisfies:

$$q_\alpha \mathcal{E}_\alpha(q) = 0 . \tag{2.6-5}$$

In the absence of 2.6-5, $\mathcal{E}_\alpha(q)$ behaves like the wave-function for a superposition of a normal spin-one ($J^P = 1^-$) particle and a normal spin-zero ($J^P = 0^+$) particle. In polarisation language and with $\lambda$ denoting helicity (dependent on the helicities of the particles comprising states $|i'\rangle$ and $|f'\rangle$ ):

$(J^P, \lambda) = (1^-, \pm 1)$ correspond to transversely polarised virtual photons, (2.6-6)

$(J^P, \lambda) = (1^-, 0)$ corresponds to a longitudinally polarised virtual photon, (2.6-7)

$(J^P, \lambda) = (0^+, 0)$ corresponds to a virtual photon with scalar polarisation. (2.6-8)

Equation 2.6-5 then tells us that not all types of polarised virtual photon can have independent physical effects. Specifically, the observable effects of longitudinally and "scalarly" polarised virtual photons are linearly related.

In a manner exactly analogous to that adopted in sections 2.31 and 2.41, we factor the helicity dependence out of the matrix element, defining a "vertex function", $\mathcal{V}_{\nu\mu\alpha}(f,i)$ by (symbolically):

$$\langle f | j_\alpha(0) | i \rangle \equiv \overline{\Phi}_\nu^{\lambda_f}(f) \mathcal{V}_{\nu\mu\alpha}(f,i) \Phi_\mu^{\lambda_i}(i) . \tag{2.6-9}$$

These three- and four-point vertex functions are off-shell generalisations of our previous coupling and M-functions, and may be similarly expanded in terms of a set of linearly inequivalent Lorentz basis tensors or tensor-spinors (kinematic covarients):

$$\mathcal{V}_{\nu\mu\alpha}(f,i) = \sum_{j=1}^{N} f_j(q^2, \ldots) \mathcal{K}^j_{\nu\mu\alpha}(f,i) . \tag{2.6-10}$$

But in view of 2.6-3 the vertex function and covarients now are required to satisfy:

$$\mathcal{V}_{\nu\mu\alpha}(f,i)q_\alpha = 0 = \mathcal{K}^j_{\nu\mu\alpha}q_\alpha \quad . \tag{2.6-11}$$

The expansion coefficients, $f_j$ are now electromagnetic form-factors. They are scalar functions, (in general complex), of the scalar variables constructable from the momenta involved at the vertex. Since the photon is off-shell we now have a single such variable, $q^2$, for three-point vertices. In the case of four-point functions three linearly independent variables are now available, and we may conveniently choose to use $q^2$ and any two of the Mandelstam variables defined as though the photon were a real initial particle. If $m_{2,3,4}$ are the masses of the hadrons, these variables satisfy:

$$s + t + u - q^2 = m_2^2 + m_3^2 + m_4^2 \, . \tag{2.6-12}$$

Since the observable effects of the scalar and longitudinal polarisations of the virtual photon are linearly related, the number, N, of linearly independent form-factors is given by:

$$N = N(1^-) \tag{2.6-13}$$

where $N(1^-)$ is defined to be the number of linearly inequivalent covarients for an on-shell interaction: $i + \Gamma \rightarrow f$ subject to the same conservation laws.

From the SU(2) point of view the virtual photon behaves like a superposition of an isoscalar and the third component of an isovector, these two components behaving in such a way that they individually conserve total isospin in hadronic electromagnetic interactions. Thus although we have so far worked only in Lorentz space, we may again usefully exploit SU(2) invarience by extending the argument to Lorentz $\otimes$ SU(2) space. We thus write symbolically:

$$f_j(q^2,..;T_f t_f, T_i t_i) = \phi^{+t_f}_{i'}(f) \sum_{k=1}^{\mathcal{N}} f^k_j(q^2,..;T_f,T_i)\mathcal{K}^k_{i'i}(T_f,T_i)\phi^{t_i}_i(i) . \tag{2.6-14}$$

We have (in general) two isospin covarients for three-point vertices. One corresponds to the couplings to the isoscalar part of the photon, and the other to the couplings to the isovector part. In the four-point case, $\mathcal{N}$ is equal to the number of isospin covarients for the reaction: $i+0\to f$, (isoscalar form-factors), plus the number for the reaction: $i+1\to f$, (isovector form-factors). Here i,f,o and 1 refer of course to the isospins involved.

Our original virtual photon wave-function can of course be decomposed in this same fashion. The fact that the spin wave-functions corresponding to the states $|i'\rangle$ and $|f'\rangle$ satisfy the Jacob and Wick phase conventions, then ensures that the virtual photon wave-function satisfies these same phase conventions. For example, if $|i'\rangle$ and $|f'\rangle$ are both on-shell single-electron states with momenta $q_1, q_2$, and helicities $\lambda_1, \lambda_2$ respectively, we may define:

$$\mathcal{E}^{0}_{\alpha}(q) \equiv \mathcal{E}^{1/2,-1/2}_{\alpha}(q) = \mathcal{E}^{-1/2,1/2}_{\alpha}(q), \tag{2.6-15}$$

$$\mathcal{E}^{\pm1}_{\alpha}(q) \equiv \mathcal{E}^{\pm 1/2,\pm 1/2}_{\alpha}(q), \tag{2.6-16}$$

where:

$$\mathcal{E}^{\lambda_1,\lambda_2}_{\alpha}(q) \equiv \frac{e}{q^2}\,\bar{u}^{\lambda_2}(q_2)\,\gamma_\alpha\, u^{\lambda_1}(q_1), \tag{2.6-17}$$

and:

$$q = q_1 - q_2 . \tag{2.6-18}$$

Working in the Breit frame, with the z-axis parallel to $\underline{q}$, so that:

$$q = (0,0,0,|\underline{q}|), \tag{2.6-19}$$

we then easily deduce that:

$$\mathcal{E}^{0}_{\alpha}(q) = \mathcal{E}^{0}_{\alpha}(q_0, -\underline{q}) = \frac{2em_e}{q^2}(1,0,0,0), \tag{2.6-20}$$

$$\mathcal{E}^{\pm1}_{\alpha}(q) = \mathcal{E}^{\mp1}_{\alpha}(q_0, -\underline{q}) = \frac{e|\underline{q}|}{q^2}(0,\pm1,i,0). \tag{2.6-21}$$

Thus in the Breit frame these wave-functions do indeed satisfy the phase conventions 2.11-19, 21, and 24 as required.

Since the phase conventions are frame-independent for wave-functions having the correct Lorentz transformation properties, they will automatically be satisfied in any general frame. Note that in the special frame above, the non-transverse polarisation of the virtual photon is purely scalar. This is a necessary consequence of equations 2.6-5 and 19, which together imply:

$$\mathcal{E}_3^\lambda = 0. \tag{2.6-22}$$

The analytic structure of hadronic electromagnetic form-factors as functions of $q^2$ is not fully understood to date. To see what happens in the important real photon limit, (vanishing $q^2$), we now consider for a moment coupling and M-functions corresponding to the same final states $|f\rangle$ as previously, but with initial states which in addition to the particles comprising the states $|i\rangle$ now contain a real photon with momentum q. These functions are then defined in Lorentz space by:

$$\langle f|\begin{Bmatrix}\mathcal{L}\\ T\end{Bmatrix}|\gamma(q),i\rangle \equiv \bar{\phi}_\nu^{\lambda_f}(f)\begin{Bmatrix}\mathcal{C}_{\nu\mu\alpha}(f,i)\\ M_{\nu\mu\alpha}(f,i)\end{Bmatrix}\phi_\mu^{\lambda_i}(i)\mathcal{E}_\alpha^\lambda(q), \tag{2.6-23}$$

and have the spin decompositions:

$$\begin{Bmatrix}\mathcal{C}_{\nu\mu\alpha}(f,i)\\ M_{\nu\mu\alpha}(f,i)\end{Bmatrix} = \sum_{j=1}^{N}\begin{Bmatrix}g_j\\ A_j(s,t)\end{Bmatrix}\mathcal{K}'^{j}_{\nu\mu\alpha}(f,i) . \tag{2.6-24}$$

The $g_j$ are now photon-hadron coupling-constants, and the $A_j(s,t)$ are invarient amplitudes for Hadron photo-production processes. $\mathcal{E}_\alpha^\lambda(q)$ is now a real photon wave-function, and is required to satisfy the Rarita-Schwinger subsidiary condition:

$$q_\alpha \mathcal{E}_\alpha^\lambda(q) = 0 . \tag{2.6-25}$$

We require the theory to be invarient under the gauge-transformation:

$$\mathcal{E}_\alpha^\lambda(q) \to \mathcal{E}_\alpha'^\lambda(q) \equiv \mathcal{E}_\alpha^\lambda(q) + \xi(q^2)q_\alpha , \tag{2.6-26}$$

where $\xi$ is any scalar function of $q^2$ such that:

$$\lim_{q^2\to 0^\pm} \xi(q^2)q^2 = 0 . \tag{2.6-27}$$

This requirement is necessary because since the real photon is an on-shell massless particle, $\mathcal{E}'^{\lambda}_{\alpha}(q)$ is a perfectly valid real photon wave-function provided that the same is true of $\mathcal{E}^{\lambda}_{\alpha}(q)$. That is, $\mathcal{E}'^{\lambda}_{\alpha}(q)$ has the same Lorentz transformation properties as $\mathcal{E}^{\lambda}_{\alpha}(q)$, and also satisfies 2.6-17. As a consequence of this gauge-invarience requirement, the coupling functions, M-functions, and kinematic covarients are required to satisfy:

$$q_\alpha \mathcal{C}_{\nu\mu\alpha}(f,i) = q_\alpha \mathcal{K}^{j}_{\nu\mu\alpha}(f,i) = q_\alpha M_{\nu\mu\alpha}(f,i) = 0 \,. \qquad (2.6\text{-}28)$$

As a further consequence of the masslessness of the real photon, equation 2.6-17 reduces to a transversality condition. It says that the real photon can only be transversally polarised, or more precisely, that the observable effects of the longitudinal and scalar polarisations must exactly cancel one another. It is thus clear that in the real photon case the number, N, of linearly inequivalent spin covarients is given in our previous notation by:

$$N = N(1^-) - N(0^+) \,. \qquad (2.6\text{-}29)$$

Note that for space-reflection invarient interactions $N(0^+)$ is by no means always equal to $\frac{1}{3}N(1^-)$; indeed, $N(1^-)$ is often not even a multiple of three.

Since real and virtual photons have identical isospin structure, we may again extend to Lorentz $\otimes$ SU(2) space by making the isospin decompositions:

$$\begin{Bmatrix} g_j(T_f t_f, T_i t_i) \\ A_j(s,t;T_f t_f, T_i t_i) \end{Bmatrix} = \phi^{+t_f}_{i'}(f) \sum_{k=1}^{\mathcal{N}} \begin{Bmatrix} g^k_j(T_f,T_i) \\ A^k_j(s,t;T_f,T_i) \end{Bmatrix} \mathcal{K}^k_{i'i}(T_f,T_i)\, \phi^{t_i}_i(i) . \qquad (2.6\text{-}30)$$

For given states $|i\rangle$ and $|f\rangle$, the isospin covarients may be chosen to be the same as those employed in the corresponding virtual photonic case, (equation 2.6-14).

Returning to the virtual photon case, we thus see that in the real photon limit just $(N(1^-) - N(0^+))$ of our original

$N(1^-)$ spin covarients will reamin linearly inequivalent. In order to preserve linear independence of the couplings we must therefore arrange that just $N(0^+)$ of our $N(1^-)$ covarients are proportional to $q^2$, and we must do this in a way that does not endow the corresponding form-factors with poles at vanishing $q^2$.

Now in corresponding real and virtual photon cases, (same states $|i\rangle$ and $|f\rangle$): the spin covarients have the same Lorentz transformation properties, vanish on contraction with $q_\alpha$, and in view of our previous discussion concerning the phase-conventions satisfied by the virtual photon wave-functions, are subject to the same constraints due to P, C, and T-invarience. In addition, they are contracted with the same hadron wave-functions, and the real and virtual photon wave-functions both vanish on contraction with $q_\alpha$. Thus those virtual photonic spin covarients which remain finite at zero $q^2$ will constitute a valid set of covarients for the corresponding real photonic coupling- or M-function. One therefore assumes that the form-factors and coupling-constants or invarient-amplitudes corresponding to these covarients satisfy:

$$\lim_{q^2 \to 0} f_j(q^2) = g_j \,, \qquad (2.6\text{-}31)$$

or:

$$\lim_{q^2 \to 0} f_j(q^2, s, t) = A_j(s,t) \,, \qquad (2.6\text{-}32)$$

as appropriate.

Having discussed the basic underlying theory, it remains to set up rules forthe construction of spin and isospin covarients for a given vertex. We quickly deal first with the relatively simple problem of isospin covarient construction.

As discussed above, the isospin covarients corresponding to isoscalar form-factors will be any set suitable for the isospin decomposition of the coupling/M-function corresponding to the reaction: $T_i + 0 \to T_f$. That is:

$$\mathcal{K}^{k(s)}_{i'i}(T_f; T_i) = \mathcal{K}^{k}_{i'i}(T_f; T_i, 0) \,, \quad k = 1, 2, \ldots, \mathcal{N}(T_i + 0 \to T_f) \,. \qquad (2.6\text{-}33)$$

The isovector covarients will be given by constructing a suitable set of covarients for the isospin decomposition of the reaction: $T_i+1 \to T_f$, and projecting out the couplings to the third component of the isospin one wave-function. Thus:

$$\mathcal{K}^{k(v)}_{i',i}(T_f;T_i)=\mathcal{K}^{k}_{i',ii''}(T_f;T_i,1)\delta_{i''3}\ ,\quad k=1,2,\dots,\mathcal{N}(T_i+1\to T_f)\ . \tag{2.6-34}$$

As a simple (and wellknown) example, we consider matrix elements of the current taken between initial and final single hadrons with isospin one-half. From equations 2.32-17 and 16 respectively, we have:

$$\mathcal{K}(\tfrac{1}{2}+0\to\tfrac{1}{2})=\mathbb{1}_2\ , \tag{2.6-35A}$$

and:

$$\mathcal{K}_{i''}(\tfrac{1}{2}+1\to\tfrac{1}{2})=\tau_{i''}\ . \tag{2.6-35B}$$

Hence if the spin decomposition of the vertex leads to form-factors $f_j(q^2)$, we have:

$$f_j(q^2;t_f,t_i)=\chi^{+t_f}\left[f^s_j(q^2)+f^v_j(q^2)\tau_3\right]\chi^{t_i}\ . \tag{2.6-36}$$

As mentioned previously, it is unnecessary to modify the isospin decomposition when passing to the real photon limit.

We now turn to the more complicated problem of photonic spin decomposition, assuming parity conservation but neglecting for the moment complications due to C and T invarience. Let $j$ run over the range: $1,2,\dots,\infty$ , and let $\mathcal{K}^j_{\nu\mu\alpha}$ be the infinity of valid (but not linearly inequivalent) covarients for the parity conserving purely on-shell hadronic reaction: $i+1^-(q)\to f$ . Let $\mathcal{K}^j_{\nu\mu}$ be the infinity of covarients for the similar reaction: $i+0^+(q)\to f$ . As usual, $\mu$ and $\nu$ are the sets of Lorentz indices for the wave-functions of the particles comprising states $|i\rangle$ and $|f\rangle$ respectively, and $\alpha$ is the index of the $1^-$ wave-function. Of the $\mathcal{K}^j_{\nu\mu\alpha}$, just $N(1^-)$ will be linearly inequivalent, and in virtue of the subsidiary condition on the $1^-$ wave-function, none can have the structure:

$$\mathcal{K}^j_{\nu\mu\alpha}=\mathcal{K}^j_{\nu\mu}q_\alpha\ . \tag{2.6-37}$$

Just $N(0^+)$ of the $\mathcal{K}^j_{\nu\mu}$ will be linearly inequivalent, and thus we shall have $N(0^+)$ linearly inequivalent covarients with structure:

$$\mathcal{K}^j_{\nu\mu\alpha} = \mathcal{K}^j_{\nu\mu} b_\alpha \tag{2.6-38}$$

where $b$ is any momentum other than $q$ constructable from those available at the vertex.

The $\mathcal{K}^j_{\nu\mu\alpha}$ will satisfy all constraints required on covarients valid for spin decomposition of $\langle f | j_\alpha(0) | i \rangle$, except that not all of them will vanish on contraction with $q_\alpha$ and neither will the correct number vanish at zero $q^2$. Let us therefore partially follow Scadron and Jones,(13) and define a "gauge projection operator" $\mathcal{G}_{\alpha'\alpha}(b)$ by:

$$\mathcal{G}_{\alpha'\alpha}(b) \equiv g_{\alpha'\alpha} - (q_{\alpha'} b_\alpha / b\cdot q) \quad , \tag{2.6-39}$$

where $b$ is now any momentum constructable from those available at the vertex. In contrast to Scadron and Jones, we do not exclude the possibility:

$$b = q . \tag{2.6-40}$$

For any Lorentz tensor or tensor-spinor, $T_\alpha$, carring a four-vector index $\alpha$, we define:

$$T'_\alpha(b) \equiv T_{\alpha'} \mathcal{G}_{\alpha'\alpha}(b) = T_\alpha - (T\cdot q / b\cdot q) b_\alpha \, , \tag{2.6-41}$$

so:

$$T'_\alpha(b) q_\alpha = 0 \, , \tag{2.6-42}$$

$$b'_\alpha(b) = 0 \, , \tag{2.6-43}$$

$$T'_\alpha(b) = T_\alpha \quad \text{if} \quad T_\alpha q_\alpha = 0 \, , \tag{2.6-44}$$

and:

$$q'_\alpha(b) = q_\alpha - (q^2 / b\cdot q) b_\alpha . \tag{2.6-45}$$

Thus the infinity of covarients $\mathcal{K}'^j_{\nu\mu\alpha}(b)$ satisfy all the constraints satisfied by the $\mathcal{K}^j_{\nu\mu\alpha}$, and in addition vanish on contraction with $q_\alpha$ for all $j$. From 2.6-44 we note that those $\mathcal{K}^j_{\nu\mu\alpha}$ which already vanish on contraction with $q_\alpha$ are left unchanged by the gauge projection operation. Provided $b\cdot q$ is non-vanishing, the number of linearly inequivalent

$\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ will be:

$$N(1^-) \text{ for } b=q, \tag{2.6-46}$$

and:

$$[N(1^-)-N(0^+)] \text{ for } b\neq q, \tag{2.6-47}$$

in consequence of equation 2.6-43 and the discussion preceeding equations 2.6-37 and 38. For any $b$ other than q, a further infinity of suitable current conserving covarients is furnished by the set of $\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b)$ . $N(0^+)$ of these will be linearly inequivalent for non-vanishing b.q, and from 2.6-45 we have:

$$\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b) \cong -(q^2/b.q)\mathcal{K}^{j}_{\nu\mu}b_{\alpha} \xrightarrow[q^2\to 0]{} 0 \ . \tag{2.6-48}$$

If we can construct a momentum b such that b.q is a function only of the hadron masses, the problem is therefore solved. The $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ and $\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b)$ will be non-singular and will satisfy the same respective equivalence relations as the corresponding $\mathcal{K}^{j}_{\nu\mu\alpha}$ and $\mathcal{K}^{j}_{\nu\mu}$. This latter statement follows on contracting the equivalence relations on the $\mathcal{K}^{j}_{\nu\mu\alpha'}$ with $g_{\alpha'\alpha}(b)$ and multiplying those on the $\mathcal{K}^{j}_{\nu\mu}$ by $q'_{\alpha}(b)$. One postulates that the two-fold infinity of $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ and $\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b)$ correspond to K.S.F. form-factors, this now being necessary even for three-point vertices where the form-factors now depend on $q^2$. If one then takes a set of $N(1^-)$ $\mathcal{K}^{j}_{\nu\mu\alpha}$ and $N(0^+)$ $\mathcal{K}^{j}_{\nu\mu}$ corresponding to K.S.F. spin decompositions of the appropriate hadronic reactions, the corresponding $\mathcal{K}^{j'}_{\nu\mu\alpha}(b)$ and $\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b)$ will furnish a K.S.F. spin decomposition of the matrix element $\langle f|j_{\alpha}(0)|i\rangle$ . Since b will necessarily be unequal to q, we shall have $[N(1^-)-N(0^+)]$ $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ (which remain finite at zero $q^2$), and $N(0^+)$ $\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b)$ (which are equivalent to zero at vanishing $q^2$). Thus the necessary number of covarients will automatically vanish when one passes to the real photon limit.

No suitable momentum b can be constructed for four-point vertices. In the case of three-point vertices, however, there exists just one momentum whose scalar product with q is independent of $q^2$, namely:

$$P' \equiv p_f + p_i \ . \qquad (2.6\text{-}49)$$

This satisfies:

$$P' \cdot q = m_f^2 - m_i^2 \ , \qquad (2.6\text{-}50)$$

so provided the initial and final masses are unequal, the problem is solved for three-point vertices by the choice:

$$b = P'. \qquad (2.6\text{-}51)$$

Even in this special case, the above choice will not lead to such a simple solution to the problem if one subsequently wishes to Reggeise or take off shell the initial and/or final hadron.

In all situations where b.q is a function of the scalar variables for all b which are linear combinations of the available momenta, the $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ and $\mathcal{K}^{j}_{\nu\mu}\, q'_{\alpha}(b)$ are singular at vanishing b.q. It is therefore necessary to construct non-singular linear combinations of these covarients. Since we are starting with an infinity of singular covarients, there exist an infinity of different ways in which the singularities may be removed. It is thus possible to construct out of the $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ and $\mathcal{K}^{j}_{\nu\mu} q'_{\alpha}(b)$ an infinite-fold infinity of non-singular covarients suitable for the spin decomposition of our matrix element. To this set of covarients will correspond an infinite-fold infinity of form-factors, and it is on these that we ought, if possible to make our requisite postulate concerning freedom from kinematic singularities. The particular elimination procedure we choose will lead to a one-fold infinity of non-singular covarients and form-factors, and is thus equivalent to the elimination of all but these from the above infinite-fold infinity. We hope to be able to make a choice for which the equivalent reduction in numbers of form-factors does not endow those remaining with any additional singularities. Our infinity of singular covarients are subject to equivalence relations, and these will impose corresponding relations amongst the non-singular covarients. A simple criterion for achieving our aim is that irrespective of the values of the scalar variables, the non-singular covarients we obtain should not be subject to any equivalence relations in addition to those specifically required by the relations amongst the $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ and $\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b)$. That is, for all values of the scalar variables just $N(1^-)$ of our non-singular covarients should be linearly inequivalent, and just $N(0^+)$ of these should be proportional to $q^2$.

If such a singularity elimination procedure proves possible, we can postulate that the infinite-fold infinity of form-factors was free of kinematic singularities. Should a suitable elimination prove non-existent, we shall have to choose one which introduces the least number of additional singularities, and then postulate freedom of kinematic singularities for the corresponding one-fold infinity of form-factors. We shall then have to assume that the additional singularities introduced are electrodynamical in origin, being a necessary consequence of gauge-invarience and/or current-conservation.

Finally, it will be necessary to reduce our infinity of non-singular covarients to a linearly inequivalent set in a way which does not endow the final form-factors with any additional singularities. Whether or not it will be possible in practice to bypass this step by starting with a set of $N(1^-)$

$\mathcal{K}^{j}_{\nu\mu\alpha}$ and N($0^+$) $\mathcal{K}^{j}_{\nu\mu}$ corresponding to K.S.F. spin decom-positions of the purely hadronic reactions: $i+1^- \to f$ and: $i+0^+ \to f$ , will depend on the extent to which the equivalence relations between the infinity of $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ and $\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b)$ are modified by the singularity removal operation.

In order to investigate these problems further, we first notice that since the infinity of $\mathcal{K}^{j}_{\nu\mu\alpha}$ and $\mathcal{K}^{j}_{\nu\mu}$ are necessarily finite, the $\mathcal{K}^{j}_{\nu\mu\alpha'}q_{\alpha'}$ and $\mathcal{K}^{j}_{\nu\mu}$ have no poles for any finite values of the scalar variables, although they may possess zeros. We also note that each of the $\mathcal{K}^{j}_{\nu\mu\alpha'}q_{\alpha'}$ must be a linear combination of the $\mathcal{K}^{j}_{\nu\mu}$ . Just N($0^+$) of these are linearly inequivalent, hence the $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ and $\mathcal{K}^{j}_{\nu\mu}q'_{\alpha}(b)$ involve just N($0^+$) linearly inequivalent terms with simple poles at vanishing b.q, and these latter terms possess no further singularities other than zeros. It will therefore always be possible to pick out from amongst the primed covarients N($0^+$) covarients, which we now redenote simply by: $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$, j = 1,2,...,N($0^+$), having the structure:

$$\mathcal{K}'^{j}_{\nu\mu\alpha}(b) = \mathcal{K}^{j}_{\nu\mu\alpha} - a_j S^{j}_{\nu\mu\alpha}(b), \qquad j = 1,2,\ldots,N(0^+), \tag{2.6-52}$$

where

$$S^{j}_{\nu\mu\alpha}(b) = T^{j}_{\nu\mu}\, b_{\alpha}/b\cdot q \ . \tag{2.6-53}$$

In these equations the $a_j$ are scalar functions of the scalar variables, possibly possessing zeros, but having no poles, whilst the $T^{j}_{\nu\mu}$ are a set of linearly inequivalent tensors (or tensor-spinors) which are free of both poles and zeros in the scalar variables. We call the $S^{j}_{\nu\mu\alpha}(b)$ "singular-tails". (Scadron and Jones have a similar definition, but do not explicitly exhibit any overall scalar factor.) The remaining primed covarients must then have the structure:

$$\left.\begin{aligned} \mathcal{K}'^{i}_{\nu\mu\alpha}(b) = \mathcal{K}^{i}_{\nu\mu\alpha} - \sum_{j=1}^{N(0^+)} a_{ij} S^{j}_{\nu\mu\alpha}(b)\ , \\ [i - N(0^+)] = 1,2,\ldots,\infty\ , \end{aligned}\right\} \tag{2.6-54}$$

where again, the $a_{ij}$ are pole-free scalar variables, and some of the $\mathcal{K}'^{i}_{\nu\mu\alpha}(b)$ will be of the form: $\mathcal{K}^{i}_{\nu\mu} q'_{\alpha}(b)$.

We see immediately that it will be of no use simply to eliminate the singular tails by multiplying all primed covarients by b.q, since:

$$b\cdot q\, \mathcal{K}'^{j}_{\nu\mu\alpha}(b) \xrightarrow[b\cdot q \to 0]{} -a_j T^{j}_{\nu\mu} b_{\alpha} \quad , \qquad (2.6\text{-}55)$$

and:
$$b\cdot q\, \mathcal{K}'^{i}_{\nu\mu\alpha}(b) \xrightarrow[b\cdot q \to 0]{} -\sum_{j=1}^{N(0^+)} a_{ij} T^{j}_{\nu\mu} b_{\alpha} , \qquad (2.6\text{-}56)$$

so that only $N(0^+)$ of the resulting covarients will remain linearly inequivalent at vanishing b.q.

Instead we must choose $N(0^+)$ primed covarients, each involving a different linearly inequivalent singular tail, and by taking linear combinations use these to remove the singular tails from the remaining primed covarients. Each of the former covarients may then be safely multiplied by b.q to remove its own tail. The reason for our above change in notation now becomes clear; we can always choose for this purpose the $N(0^+)$ $\mathcal{K}'^{j}_{\nu\mu\alpha}(b)$ , since any other choice just reduces to an alternative choice of linearly inequivalent singular tails. We therefore define tail-free covarients by:

$$\tilde{\mathcal{K}}^{j}_{\nu\mu\alpha} \equiv b\cdot q\, \mathcal{K}'^{j}_{\nu\mu\alpha}(b) = b\cdot q\, \mathcal{K}^{j}_{\nu\mu\alpha} - a_j T^{j}_{\nu\mu} b_{\alpha} \, , \qquad (2.6\text{-}57)$$

$$\left.\begin{aligned} \tilde{\mathcal{K}}^{i}_{\nu\mu\alpha} &\equiv \left\{\prod_{j\in\{j\}_i} a_j\right\} \left[\mathcal{K}'^{i}_{\nu\mu\alpha}(b) - \sum_{j\in\{j\}_i} \frac{a_{ij}}{a_j} \mathcal{K}'^{j}_{\nu\mu\alpha}(b)\right] \\ &= \left\{\prod_{j\in\{j\}_i} a_j\right\} \left[\mathcal{K}^{i}_{\nu\mu\alpha} - \sum_{j\in\{j\}_i} \frac{a_{ij}}{a_j} \mathcal{K}^{j}_{\nu\mu\alpha}\right] , \end{aligned}\right\} \qquad (2.6\text{-}58)$$

where $\{j\}_i$ is the set of $j$ values for which $a_{ij}$ is non-zero at least for one set of values of the scalar variables.

The $\mathcal{K}^{j}_{\nu\mu\alpha}$ and $\mathcal{K}^{i}_{\nu\mu\alpha}$ appearing in the "tildered" covarients cannot involve $b_\alpha$ as a factor, since we have already seen that $b'_\alpha(b)$ is zero. So if all $a_j$ are non-vanishing, just $N(1^-)$ of the tildered covarients will be linearly inequivalent for non-vanishing $q^2$, irrespective of whether b.q or any of the $a_{ij}$ $(j\epsilon\{j\}_i)$ happen to be zero. Furthermore, in view of the structure of the primed covarients, just $[N(1^-) - N(0^+)]$ of these tildered covarients will remain linearly inequivalent at zero $q^2$. However, if one of the $a_j$ vanishes, we have:

$$\lim_{a_j\to 0} \tilde{\mathcal{K}}^j_{\nu\mu\alpha} = b\cdot q\, \mathcal{K}^j_{\nu\mu\alpha} , \tag{2.6-59}$$

$$\lim_{a_j\to 0} \tilde{\mathcal{K}}^i_{\nu\mu\alpha} = -\left\{\prod_{j\epsilon\{j\}_i} a_j\right\} \frac{a_{ij}}{a_j} \mathcal{K}^j_{\nu\mu\alpha} , \quad j\epsilon\{j\}_i , \tag{2.6-60}$$

all other $\tilde{\mathcal{K}}$ remaining unaffected. Thus at zero $a_j$ we have an additional proportionality between all those tildered covarients whose definition involves the elimination of the singular tail $S^j_{\nu\mu\alpha}$ from a primed covarient. Note that the problem is purely one of an additional unwanted proportionality; although it is not obvious at a first glance, the right-hand sides of equations 2.6-59 and 60 do in fact vanish on contraction with $q_\alpha$ at zero $a_j$. To see this one only has to notice that:

$$\tilde{\mathcal{K}}^j_{\nu\mu\alpha} q_\alpha = 0 \tag{2.6-61}$$

implies:
$$\mathcal{K}^j_{\nu\mu\alpha} q_\alpha = a_j T_{\nu\mu} \quad \text{or} \quad b\cdot q = 0 , \tag{2.6-62}$$

so that:
$$b\cdot q\, \mathcal{K}^j_{\nu\mu\alpha} q_\alpha \xrightarrow[a_j\to 0]{} 0 ; \tag{2.6-63}$$

and:
$$\tilde{\mathcal{K}}^i_{\nu\mu\alpha} q_\alpha = 0 \tag{2.6-64}$$

implies:
$$\left.\begin{aligned} &\left\{\prod_{j\epsilon\{j\}_i} a_j\right\} \frac{a_{ij}}{a_j} \mathcal{K}^j_{\nu\mu\alpha} q_\alpha = \\ &= \left\{\prod_{j\epsilon\{j\}_i} a_j\right\} \Big[\mathcal{K}^i_{\nu\mu\alpha} q_\alpha - \sum_{\substack{k\epsilon\{j\}_i \\ k\neq j}} \frac{a_{ik}}{a_k} \mathcal{K}^k_{\nu\mu\alpha} q_\alpha\Big] \xrightarrow[\substack{a_j\to 0\\ j\epsilon\{j\}_i}]{} 0 \end{aligned}\right\} \tag{2.6-65}$$

Our criterion for choosing a suitable singularity eleimination procedure is therefore that all the $a_j$ for the choice we make should be functions only of the masses. In certain cases, (which would appear to be restricted to four-point vertices), such a choice proves impossible. That is, there exists no b for which we can pick out N($0^+$) $\mathcal{K}'(b)$ having linearly inequivalent residues at vanishing b.q which are free of kinematic zeros. In such cases we clearly have to choose an elimination for which the minimum possible number of $a_j$ are functions of the scalar variables.

We now turn to a discussion of the structure of equivalence relations (E.R.'S) on the tildered covarients. The E.R.'s on the unprimed covarients have the general structure:

$$\sum_k c_k \mathcal{K}^k_{\nu\mu\alpha} \cong 0 , \qquad (2.6\text{-}66)$$

and, as mentioned in section 2.41, we call these type 1 or type 2 according as none or at least one of the $c_k$ are functions of the scalar variables. Operating on this E.R. with $\mathcal{G}_{\alpha'\alpha}(b)$ yields:

$$\sum_k c_k \mathcal{K}'^k_{\nu\mu\alpha}(b) \cong 0 , \qquad (2.6\text{-}67)$$

and subtraction of 2.6-66 from 2.6-67 then gives us an E.R. on the singular parts of the primed covarients:

$$\sum_k c_k \left[\mathcal{K}'^k_{\nu\mu\alpha}(b) - \mathcal{K}^k_{\nu\mu\alpha}\right] \cong 0 . \qquad (2.6\text{-}68)$$

In our previous discussion of singularity elimination procedures we chose N($0^+$) primed covarients with linearly inequivalent singular tails, and then expressed the singular parts of all other primed covarients in terms of these tails. For a proper discussion of E.R.'s, the way in which these latter expressions are arrived at is rather crucial, and leads to a further subdivision in the classification of E.R.'s. We can always choose the E.R.'s as typified by 2.6-66 so that, irrespective of whether they are of type 1 or 2, they fall into one of two further classes. For a given E.R., it may be that all the $\mathcal{K}'$ appearing in the corresponding equation 2.6-67 have singularities which are already determined in terms of the N($0^+$) $S^j_{\nu\mu\alpha}(b)$ by other equations. Such E.R.'s are in fact comparatively rare, and we call them type 1B or 2B as appropriate. A much more common situation is that all but one the $\mathcal{K}'$ in 2.6-67 have singularities already determined in terms of the $S^j_{\nu\mu\alpha}(b)$, whilst the singularity of the remaining $\mathcal{K}'$ is similarly determined by no equation other than the corresponding 2.6-68. In this case we say that the equivalence relation is of type 1A or 2A.

We first consider type 1A and 2A E.R.'s. These have the general structure:

$$c\mathcal{K}_{\nu\mu\alpha}+\sum_{i\in\{i\}}c_i\mathcal{K}^i_{\nu\mu\alpha}+\sum_{j\in\{j\}}c_j\mathcal{K}^j_{\nu\mu\alpha}\cong 0\ , \qquad (2.6\text{-}69)$$

where $\{i\}$ denotes a set of i-values, and the $\mathcal{K}^i_{\nu\mu\alpha}$ are such that the singularities of the $\mathcal{K}'^i_{\nu\mu\alpha}(b)$ are already known from equations of the form 2.6-54. The $\mathcal{K}^j_{\nu\mu\alpha}$ are such that the corresponding $\mathcal{K}'^j_{\nu\mu\alpha}(b)$ are known to be given by equations 2.5-52, and $\{j\}$ is the set of j-values for which $c_j$ is non-zero. The singularities of $\mathcal{K}'_{\nu\mu\alpha}(b)$ are supposed to be given by no equation other than 2.6-68 which in conjunction with 2.6-57 and 58 yields:

$$c(\mathcal{K}'-\mathcal{K})\cong\sum_{i\in\{i\}}\sum_{j\in\{j\}_i}c_i a_{ij}S^j+\sum_{j\in\{j\}}c_j a_j S^j\ , \qquad (2.6\text{-}70)$$

where we have suppressed the Lorentz indices and the argument b of the gauge projection operator. The important point is that this is the only equation which tells us how to eliminate the singularity from $\mathcal{K}'$ and hence define $\tilde{\mathcal{K}}$. Following our previous elimination procedure, we therefore define:

$$c\tilde{\mathcal{K}} \equiv \left\{\prod_{j\in[j]} a_j\right\}\left[c\mathcal{K}' + \sum_{j\in\{j\}} c_j \mathcal{K}'^j + \sum_{i\in\{i\}} \sum_{j\in\{j\}_i} \frac{c_i a_{ij}}{a_j} \mathcal{K}'^j\right], \quad (2.6\text{-}71)$$

where $[j]$ is the set of all j-values included in at least one of $\{j\}_i$ for $i\in\{i\}$. In view of 2.6-67, this reduces to:

$$c\tilde{\mathcal{K}} \cong -\left\{\prod_{j\in[j]} a_j\right\}\sum_{i\in\{i\}} c_i\left[\mathcal{K}'^i - \sum_{j\in\{j\}_i} \frac{a_{ij}}{a_j}\mathcal{K}'^j\right], \quad (2.6\text{-}72)$$

so from 2.6-58, we have finally:

$$c\tilde{\mathcal{K}} + \sum_{i\in\{i\}} c_i \left\{\prod_{j\in[j]_i} a_j\right\}\tilde{\mathcal{K}}^i \cong 0, \quad (2.6\text{-}73)$$

where for given i, $[j]_i$ is the set of all j-values contained in $[j]$ but not in $\{j\}_i$.

We see that 2.6-73 involves only $\tilde{\mathcal{K}}$ and the $\tilde{\mathcal{K}}^i$. Provided $a_j$ is a function only of the masses for all $j\in[j]$, the structure of this E.R. as far as $\tilde{\mathcal{K}}$ and the $\tilde{\mathcal{K}}^i$ are concerned is essentially the same as that of 2.6-69. That is, if $\mathcal{K}$ or a given $\mathcal{K}^i$ can be eliminated by means of this latter equation, then 2.6-73 may be used to eliminate $\tilde{\mathcal{K}}$ or the corresponding $\tilde{\mathcal{K}}^i$. However, if $a_j$ is a function of the scalar variables for at least one element of $[j]_i$, then $\tilde{\mathcal{K}}^i$ can no longer be eliminated without the introduction of kinematic singularities, even if $c_i$ is a constant. But the crucial point is that it will always be possible to eliminate $\tilde{\mathcal{K}}$ for constant c, irrespective of whether any of the $a_j$ are variables.

To summarise, then, the important point about a type A E.R. on unprimed covarients is that it leads to an E.R. on tildered covarients which itself defines one of these. The new E.R. may always be used to eliminate this latter covarient without the introduction of kinematic singularities, provided that the original E.R. can be used to similarly eliminate the corresponding unprimed covarient.

Let us now turn to type B E.R.'s which in our previous notation have the general structure:

$$\sum_{i\in\{i\}} c_i \mathcal{K}^i + \sum_{j\in\{j\}} c_j \mathcal{K}^j \cong 0 \;. \tag{2.6-74}$$

Operating with $\mathcal{G}(b)$ yields an e quation, which we will denote by 2.6-74', in which the $\mathcal{K}^i$ and $\mathcal{K}^j$ of 2.6-74 are replaced by $\mathcal{K}'^i(b)$ and $\mathcal{K}'^j(b)$. The structure of these latter covarients is already determined by other equations to be of the form 2.6-54 and 52, so the corresponding $\tilde{\mathcal{K}}^i$ and $\tilde{\mathcal{K}}^j$ are defined independently of 2.6-74' by equations 2.6-58 and 57. Inverting these latter equations and substituting $\tilde{\mathcal{K}}$ for $\mathcal{K}'$ in 2.6-74' therefore yields:

$$\sum_{i\in\{i\}} \frac{c_i \tilde{\mathcal{K}}^i}{\left\{\prod_{j\in\{j\}_i} a_j\right\}} + \sum_{i\in\{i\}} \sum_{j\in\{j\}_i} \frac{c_i a_{ij} \tilde{\mathcal{K}}^j}{a_j\, b\cdot q} + \sum_{j\in\{j\}} \frac{c_j \tilde{\mathcal{K}}^j}{b\cdot q} \cong 0. \tag{2.6-75}$$

This unfortunately involves the $a_{ij}$, which will frequently be variable even though all the $a_j$ may be constants. All is not lost however, since 2.6-68 now reads:

$$\sum_{i\in\{i\}} \sum_{j\in\{j\}_i} c_i a_{ij} S^j + \sum_{j\in\{j\}} c_j a_j S^j \cong 0 \,. \tag{2.6-76}$$

In order to separately equate to zero the coefficients of each $S^j$, we need to recast this relation in the form of an exact equality rather than an equivalence. The $S^j_{\nu\mu\alpha}(b)$ cannot vanish on contraction with the hadron wave-functions, but $q^2 S^j_{\nu\mu\alpha}(q)$ does vanish on contraction with the wave-function of the photon. Hence we may replace the equivalence by an equality if we replace the right-hand side by: $\delta_{bq} \sum_{j\in\langle j\rangle} q^2 d_j S^j$ in which $d_j$ is a function of the scalar variables and/or masses, and $\langle j\rangle$ is defined to be the set of all j-values contained in $[j]$ and/or $\{j\}$. Equation 2.6-75 then reduces to:

$$\sum_{i\in\{i\}} c_i \left\{ \prod_{j\in[j]_i} a_j \right\} \tilde{\mathcal{X}}^i \cong 0 \qquad , b \neq q , \tag{2.6-77}$$

$$\sum_{i\in\{i\}} c_i \left\{ \prod_{j\in\langle j\rangle_i} a_j \right\} \tilde{\mathcal{X}}^i + \sum_{j\in\langle j\rangle} \frac{d_j}{a_j} \left\{ \prod_{j\in\langle j\rangle} a_j \right\} \tilde{\mathcal{X}}^j \cong 0 , b = q , \tag{2.6-78}$$

where $\langle j\rangle_i$ is the set of all j-values contained in $\langle j\rangle$ but not in $\{j\}_i$.

Thus as in the case of type A E.R.'s, the structure of type B E.R.'s is considerably modified when these latter are converted into E.R.'s on tildered covarients. For b different from q, the $\tilde{\mathcal{X}}^i$ in 2.6-77 can only be eliminated for constant $c_i$ if $a_j$ is a constant for all $j\in[j]_i$. If b is equal to q, the same is true of the $\tilde{\mathcal{X}}^i$ in 2.6-78, except that we now require $a_j$ to be a constant for all $j\in\langle j\rangle_i$. Irrespective of whether $c_j$ vanishes, the $\tilde{\mathcal{X}}^j$ in 2.6-78 can also be eliminated provided $d_j$ and $\frac{1}{a_j}\prod_{j\in\langle j\rangle} a_j$ are constants. We stress again that the differing properties of type A and B equivalence relations results from the fact that each of the former define one of the tildered covarients appearing in them, whereas in the case of the latter all such covarients appearing are defined independently.

We are now in a position to state the rules for the reduction of the tildered covarients to a linearly inequivalent set corresponding to kinematic singularity free form-factors. We define a pair of type B E.R.'s to be inequivalent if one cannot be transformed into the other by means of type A E.R.'s only. One starts with a set of $N(1^-)$ $\mathcal{K}_{\nu\mu\alpha}$ and $N(0^+)$ $\mathcal{K}_{\nu\mu}$ corresponding to K.S.F. decompositions of the purely hadronic reactions: $i + 1^- \rightarrow f$ and $i + 0^+ \rightarrow f$. If one can find a momentum b different from q and choose a singularity elimination procedure for which all $a_j$ are constants, then the $N(1^-)$ tildered covarients obtained will correspond to a K.S.F. set of form-factors. Just $[N(1^-) - N(0^+)]$ of these covarients will remain linearly inequivalent at vanishing $q^2$, so by taking linear combinations of the covarients in such a way that no additional singularities are introduced, one can arrive at a final set of covarients, just $N(0^+)$ of which vanish at zero $q^2$. In deducing K.S.F. spin decompositions for the purely hadronic reactions, one must remember that the squared masses of the $1^-$ and $0^+$ particles are now variables (equal to $q^2$ ). This means that even in the case of three-point vertices, these decompositions will now involve the use of type 2 equivalence theorems.

If the above procedure proves impossible, one follows the reduction rules of section 2.41 but the inequivalent type A and B E.R.'s (whether of type 1 or 2) now assume the respective roles of the type 1 and 2 E.R.'s of that section. That is, if there exist $r(1^-)$ and $r(0^+)$ inequivalent type B E.R.'s for the respective reactions $i+1^-\rightarrow f$ and $i+0^+\rightarrow f$, then one starts with $[N(1^-) + r(1^-)]$ covarients for the former reaction together with $(1-\delta_{bq})[N(0^+)+r(0^+)]$ covarients for the latter. The $\delta_{bq}$ symbol arises because equation 2.6-57 implies that all covarients for this latter reaction vanish on contraction with $g_{\alpha'\alpha}(q)$. The choice $b=q$ has another advantage. With this choice, just the $N(0^+)$ $\tilde{\mathcal{K}}^j_{\nu\mu\alpha}$ will be equivalent to zero at vanishing $q^2$, and it will therefore be unnecessary to take further linear combinations of the tildered covarients after performing the reduction to a linearly inequivalent set. Of course, if this choice is made, care must be taken that the reduction does not eliminate any of the $\tilde{\mathcal{K}}^j_{\nu\mu\alpha}$.

To conclude this section, we devote a few words to the implications for the vertex functions of P, T, and C-invariences, hermiticity, and crossing of the interaction Lagrangian.

Since the real and virtual photon wave-functions satisfy the Jacob and Wick phase conventions, these various implications are again given by the appropriate equations of sections 2.31, 32, 41, and 42, with the coupling and M-functions replaced by the corresponding three and four-point vertices. For a given four-point vertex one may define s, t, and u channels in the same way as for M-functions. One has a direct channel as discussed above, and two further channels obtained from this by crossing the initial hadron with each of the final hadrons. In Chapter 3 we show that the vertex-function continues to satisfy the crossing rules 2.41-11 and 2.42-4. Thus the various implications for the M-function of the crossing rules again apply equally to four-point vertex functions.

In particular, the P-invarience constraint on our tildered covarients reads:

$$\tilde{\mathcal{K}}_{\nu\mu\alpha}(f,i) = \eta g(\nu)g(\mu)g(\alpha)\gamma_0 \hat{\tilde{\mathcal{K}}}_{\nu\mu\alpha}(f,i)\gamma_0 \ , \qquad (2.6\text{-}79)$$

whilst the condition for real form-factors in the case of a T-invarient three-point interaction is:

$$\tilde{\mathcal{K}}_{\nu\mu\alpha}(f,i) = \eta_T g(\nu)g(\mu)g(\alpha)T^{-1}\gamma_0 \hat{\tilde{\mathcal{K}}}^{*}_{\nu\mu\alpha}(f,i)\gamma_0 T \ . \qquad (2.6\text{-}80)$$

As usual, the circumflex accent denotes the sign reversal of all 3-momenta appearing. Since this operation leaves invarient the scalar products of pairs of 4-momenta, these same equations must be satisfied by the primed covarients. Now:

$$g(\alpha)\hat{\mathcal{K}}_{\nu\mu\alpha'}(f,i+1^-)\hat{\mathcal{G}}_{\alpha'\alpha}(b) =$$

$$= g(\alpha)\hat{\mathcal{K}}_{\nu\mu\alpha}(f,i+1^-) - g(\alpha')\hat{\mathcal{K}}_{\nu\mu\alpha'}q_{\alpha'}b_\alpha/b\cdot q \ , \qquad (2.6\text{-}81)$$

$$g(\alpha)\hat{\mathcal{K}}_{\nu\mu}(f,i+0^+)\hat{q}'_\alpha(b) = \hat{\mathcal{K}}_{\nu\mu}(f,i+0^+)q_\alpha(b) \ , \qquad (2.6\text{-}82)$$

and the photon has both normality and time-reversal phase equal to plus one. Hence 2.6-79 implies:

$$\begin{Bmatrix} \mathcal{K}_{\nu\mu\alpha}(f,i+1^-) \\ \mathcal{K}_{\nu\mu}(f,i+0^+) \end{Bmatrix} = n_f n_i g(\nu) g(\mu) \gamma_0 \begin{Bmatrix} g(\alpha)\hat{\mathcal{K}}_{\nu\mu\alpha}(f,i+1^-) \\ \hat{\mathcal{K}}_{\nu\mu}(f,i+0^+) \end{Bmatrix} \gamma_0 \quad , \qquad (2.6\text{-}83)$$

whilst 2.6-80 requires:

$$\begin{Bmatrix} \mathcal{K}_{\nu\mu\alpha}(f,i+1^-) \\ \mathcal{K}_{\nu\mu}(f,i+0^+) \end{Bmatrix} = \eta_{T_f} \eta_{T_i} g(\nu) g(\mu) T^{-1} \gamma_0 \begin{Bmatrix} g(\alpha)\hat{\mathcal{K}}^*_{\nu\mu\alpha}(f,i+1^-) \\ \hat{\mathcal{K}}^*_{\nu\mu}(f,i+0^+) \end{Bmatrix} \gamma_0 T . \qquad (2.6\text{-}84)$$

The $1^-$ and $0^+$ particles are both normal, so equation 2.6-83 is identical to 2.31-24. Equation 2.6-84 is identical to 2.31-32 provided we specify that the $1^-$ and $0^+$ particles are to be treated as though they both have time-reversal phase equal to plus one. This requires that they both be treated as C-normal particles, hence the $0^+$ particle is to be considered as having opposite C-parity to that of the photon. The unprimed covarients are then to be constructed following the rules of sections 2.31 and 2.41 as appropriate.

The tildered covarients resulting from these unprimed covarients will then automatically satisfy the P-invarience constraint, 2.6-79. They will, in addition, carry just those overall i-factors needed to ensure purely real form-factors and satisfaction of the discontinuity condition in the respective cases of T-invarient three-point and four-point interactions. (We show in Chapter 3 that the discontinuity condition on four-point vertex functions is the same as that on M-functions, and we pointed out previously that this latter is formally the same as the reality condition for coupling functions.)

It is well known that the form-factors for matrix elements of the current taken between identical initial and final single particles may be chosen to be purely real even for T-violating interactions. This arises out of the hermiticity of the current operator, and the reality condition reads:

$$\tilde{\mathcal{K}}_{\nu\mu\alpha}(p_f, p_i) = \bar{\tilde{\mathcal{K}}}_{\mu\nu\alpha}(p_i, p_f) \quad , \qquad (2.6\text{-}85)$$

where the bar has its usual significance.

We shall need the generalisation to four-point vertex functions of equations 2.42-14 to 17. This is trivial; the isoscalar and isovector parts of the photon both have C-parity equal to minus one, and this corresponds to G-parity minus one (plus one) for the isoscalar (isovector) parts. Hence, assuming particle 1 is the photon, one simply replaces $G_1$ in equation 2.42-17 by minus one (plus one) for isoscalar (isovector) form-factors.

Finally, we wish to stress that although we have called quantities of the form $\mathcal{K}\cdot q\, q_\alpha / q^2$ singular tails, they are not really singular at all. In fact since $q_\alpha$ vanishes on contraction with the photonic wave-function for all $q^2$, such terms are themselves equivalent to zero even at vanishing $q^2$. The purpose of eliminating such terms as though they are singular is simply to ensure that the correct number of covarients are proportional to $q^2$.

## 2.7 THE SPIN DECOMPOSITION OF SOME PHOTONIC THREE-POINT VERTICES.

In the following sections we derive Lorentz-space (spin) decompositions for the real and virtual photonic three-point vertices: $(\gamma, \frac{1}{2}, \frac{1}{2})^{\pm}$, $(\gamma, \frac{1}{2}, J+\frac{1}{2})^{\pm}$, $(\gamma, 0, J)^{\pm}$, and $(\gamma, 1, J)^{\pm}$. The symbol $(\gamma, S_1, S_2)^{n}$ denotes a vertex with overall normality coupling a real or virtual photon to an initial hadron with spin $s_1$ and a final hadron with spin $S_2$. The various kinematic quantities involved will be denoted by the symbols which are used when these vertices are encountered later in this thesis. The $\gamma$FF vertices appear in Chapter 5, and the $\gamma$BB vertices in Chapter 4. A differing set of kinematic symbols will be used for these two types of vertex since they appear in one-particle intermediate state graphs corresponding respectively to the s and t channels of the same four-point function.

For $\gamma$FF vertices we define the momentum (mass) of the initial and final hadrons to be $p(m)$ and $K(M)$ respectively. The momentum of the photon is then:

$$q \equiv K - p \quad , \tag{2.7-1}$$

and as usual we further define:

$$P' \equiv K + p \quad . \tag{2.7-2}$$

Further useful kinematic relations are then listed in Appendix 6.

The kinematic notation for $\gamma$BB vertices is defined in terms of that above by the substitutions:

$$p \to -k \;,\; K \to \Delta \;,\; m \to \mu \;,\; M \to M \;,\; P' \to P'' . \tag{2.7-3}$$

So:

$$P'' = \Delta - k \quad , \tag{2.7-4}$$

and in view of the kinematic relations of Appendix 5, which are still applicable here, the momentum of the photon, $(\Delta + k)$, is still equal to q.

The decompositions we derive are only strictly valid when both of the hadrons are on the mass-shell. If one or both of these particles are taken off-shell, it is necessary to include additional "off-shell" couplings to take account of the relaxation of the appropriate Dirac-Rarita-Schwinger subsidiary conditions. Neglect of these terms in the off-shell hadron case is equivalent to making the dynamical assumption that the "off-shell" form-factors vanish. On the other hand, when such vertices appear in the Born-terms for four-point functions, only the "on-shell" couplings will actually contribute to the pole-like behaviour, the "off-shell" couplings in the numerator each including as a factor the denominator of the Born-term.

To a certain extent, then, it is useful to perform the spin decompositions in a way which whilst treating the hadrons as on-shell, does not rely for freedom of kinematical singularities on their masses being constant. This also renders the decompositions suitable for use in covarient Reggeisation calculations, where the Reggeon simulates a superposition of on-shell particles with variable mass.

## 2.71 $(\gamma, \frac{1}{2}, \frac{1}{2})^{\pm}$ VERTICES

The $\gamma$-nucleon-nucleon vertex is well known and has been studied in great detail.[1] As mentioned in section 1.2, it decomposes in Lorentz-space into a pair of linearly independent couplings which may be chosen, for example, to be the "charge" and "moment" or "electric" and "magnetic" couplings as given by equations 1.2-3 and 4. These couplings remain independent in the real photon limit.

What is less generally known is the fact that this structure is a direct consequence of the fact that one is dealing with identical initial and final hadrons. The structure of the general $(\gamma, \frac{1}{2}, \frac{1}{2})^{+}$ vertex involving non-identical hadrons is necessarily quite different. Indeed, although one still has two linearly inequivalent couplings in the virtual photonic case, the covarients become proportional to one another in the real photon limit. This behaviour is in agreement with the "counting rules" of section 2.6 for numbers of Lorentz-space couplings, and the $\gamma$-nucleon-nucleon vertex must be considered as exceptional case.

We therefore consider first the unequal mass $(\gamma, \frac{1}{2}, \frac{1}{2})^{+}$ vertex, and then show why and how the spin decomposition has to be modified in the equal mass case. The $(\gamma, \frac{1}{2}, \frac{1}{2})^-$ vertex, (where the masses are necessarily unequal) is treated at the same time.

From equation 2.31-9 we have:

$$N^{\pm}(1^-, \tfrac{1}{2}, \tfrac{1}{2}) = 2 \; , \tag{2.71-1}$$

$$N^{\pm}(0^+, \tfrac{1}{2}, \tfrac{1}{2}) = 1 \; , \tag{2.71-2}$$

so we expect in view of equations 2.6-13 and 29 that:

$$N^{\pm}(\gamma^V, \tfrac{1}{2}, \tfrac{1}{2}) = 2 \; , \tag{2.71-3}$$

$$N^{\pm}(\gamma^R, \tfrac{1}{2}, \tfrac{1}{2}) = 1 \; . \tag{2.71-4}$$

Here $N^+(N^-)$ is the number of couplings at the normal (abnormal) vertex indicated by the parentheses, and $\gamma^V$ $(\gamma^R)$ denotes a virtual (real) photon.

From Scadron's paper(9) we take the spin decompositions for on-shell hadronic vertices:

$$\mathcal{C}^{\pm}_{\alpha}(1^-, \tfrac{1}{2}, \tfrac{1}{2}) = (g_1 \gamma_\alpha + g_2 P'_\alpha) I^{\pm} , \tag{2.71-5}$$

$$\mathcal{C}^{\pm}(0^+, \tfrac{1}{2}, \tfrac{1}{2}) = g_3 I^{\pm} , \tag{2.71-6}$$

where as usual:

$$I^{\pm} \equiv \begin{cases} \mathbb{1}_4 \; , \\ \gamma_5 \; . \end{cases} \tag{2.71-7}$$

When the $1^-$ and $0^+$ particles are given variable squared mass, $q^2$, the $g_{1,2,3}$ become form-factors depending on this quantity but are K.S.F. This is because the only E.R.'s which can become type 2 in the variable mass case are those which relate covarients involving the contraction of Levi-Cevita tensors with momenta, $\gamma$-matrices, and possibly one another, to the covarients: $(\gamma_\alpha, P'_\alpha, \mathbb{1}_4) I^{\pm}$. From dimensional considerations, the coefficient of the former covarients in such E.R.'s is always unity, and hence their elimination in favour of the latter covarients leads to K.S.F. form-factors. Similar arguments indicate that no type B E.R.'s are involved. Hence, we can go ahead and operate on the $(\gamma_{\alpha'}, P'_{\alpha'}, q_{\alpha'}) I^{\pm}$ with a gauge projection operator: $\mathcal{G}_{\alpha'\alpha}(b)$. In this section alone we shall make the most general choice:

$$b = aP' + cq , \tag{2.71-8}$$

where a and c are functions of the hadron masses. This is to illustrate that the final result is really independent of the choice of b. We therefore have:

$$\gamma'_\alpha(b)I^{\pm} = \left[\gamma_\alpha - \frac{\not{q}(aP'_\alpha + cq_\alpha)}{aP'\cdot q + cq^2}\right]I^{\pm} , \tag{2.71-9}$$

$$(P'_\alpha)'(b)I^{\pm} = \left[P'_\alpha - \frac{P'\cdot q(aP'_\alpha + cq_\alpha)}{aP'\cdot q + cq^2}\right]I^{\pm} , \tag{2.71-10}$$

$$q'_\alpha(b)I^{\pm} = \left[q_\alpha - \frac{q^2(aP'_\alpha + cq_\alpha)}{aP'\cdot q + cq^2}\right]I^{\pm} . \tag{2.71-11}$$

Remembering that in virtue of the Dirac equation:

$$\not{q}I^{\pm} \cong (M \mp m)I^{\pm} , \tag{2.71-12}$$

and:

$$\not{q}\gamma_\alpha I^{\pm} \cong [(M \pm m)\gamma_\alpha - P'_\alpha + q_\alpha]I^{\pm} , \tag{2.71-13}$$

two possible K.S.F. tail elimination procedures are possible. We may choose to define either:

$$\tilde{\mathcal{K}}^1_\alpha I^{\pm} \equiv [(M \pm m)\gamma'_\alpha(b) - (P'_\alpha)'(b)]I^{\pm} \cong (\not{q}\gamma_\alpha - q_\alpha)I^{\pm} = i\sigma_{\alpha\beta}q_\beta I^{\pm} , \tag{2.71-14}$$

$$\tilde{\mathcal{K}}^2_\alpha I^{\pm} \equiv [q^2\gamma'_\alpha(b) - (M \mp m)q'_\alpha(b)]I^{\pm} \cong (q^2\gamma_\alpha - \not{q}q_\alpha)I^{\pm} , \tag{2.71-15}$$

and: $$\tilde{\mathcal{K}}^3_\alpha I^{\pm} \equiv b\gamma'_\alpha(b)I^{\pm} = [a(P'\cdot q\gamma_\alpha - P'_\alpha\not{q}) - c\tilde{\mathcal{K}}^2_\alpha]I^{\pm} , \tag{2.71-16}$$

or:

$\tilde{\mathcal{K}}^1_\alpha I^{\pm}$ as defined by 2.71-14,

$$\tilde{\mathcal{K}}^4_\alpha I^{\pm} \equiv [q^2(P'_\alpha)'(b) - P'\cdot q\, q'_\alpha(b)]I^{\pm} = (q^2P'_\alpha - P'\cdot q\, q_\alpha)I^{\pm} , \tag{2.71-17}$$

and: $$\tilde{\mathcal{K}}^5_\alpha I^{\pm} \equiv b(P'_\alpha)'(b)I^{\pm} = c\tilde{\mathcal{K}}^4_\alpha I^{\pm} . \tag{2-71-18}$$

Now it is easy to derive from 2.71-12 and 13 the E.R.:

$$(P'\cdot q\gamma_\alpha - P'_\alpha\not{q})I^{\pm} \cong (M \mp m)\tilde{\mathcal{K}}^1_\alpha I^{\pm} , \tag{2.71-19}$$

so both elimination procedures lead to a pair of linearly inequivalent covarients, one of which is equivalent to zero at vanishing $q^2$, as required. Furthermore, we can similarly prove that:

$$\tilde{\mathcal{K}}^4_\alpha I^{\pm} \cong \left[(M \pm m)\tilde{\mathcal{K}}^2_\alpha - q^2 \tilde{\mathcal{K}}^1_\alpha\right] I^{\pm}, \tag{2.71-20}$$

from which $\tilde{\mathcal{K}}^4_\alpha$ may be eliminated in favour of $\tilde{\mathcal{K}}^2_\alpha$. Thus both procedures lead to the same K.S.F. spin decomposition:

$$V^{\pm}_\alpha(\gamma, \tfrac{1}{2}, \tfrac{1}{2}) \equiv \left[F_1(q^2)(q^2\gamma_\alpha - \not{q}\, q_\alpha) + F_2(q^2)\, i\sigma_{\alpha\beta} q_\beta\right] I^{\pm}. \tag{2.71-21}$$

Note that had we chosen:

$$b = q \,, \tag{2.71-22}$$

then we should have had just two primed covarients: $\gamma'_\alpha(q)$ and $(P'_\alpha)'(q)$. Only one elimination procedure would then have been possible, again leading to the covarients $\tilde{\mathcal{K}}^1_\alpha I^{\pm}$ and $\tilde{\mathcal{K}}^2_\alpha I^{\pm}$. Had we chosen instead:

$$b = P' \,, \tag{2.71-22A}$$

then we should have again had two primed covarients:

$$\gamma'_\alpha(P') I^{\pm} = (\gamma_\alpha - \not{q}\, P'_\alpha / P'.q) I^{\pm} \,, \tag{2.71-23}$$

and:

$$q'_\alpha(P') I^{\pm} = (q_\alpha - q^2 P'_\alpha / P'.q) I^{\pm} \,. \tag{2.71-24}$$

These are already non-singular, and the second is equivalent to zero at vanishing $q^2$. They furnish a suitable set of covarients for the decomposition of the vertex, and in view of 2.71-19 and 20 are equivalent to the pair appearing in 2.71-21. Finally we consider the covarient $\mathcal{E}_\alpha(P'q\gamma)\gamma_5 I^{\pm}$ which also vanishes on contraction with $q_\alpha$. We have:

$$\mathcal{E}_\alpha(P'q\gamma)\gamma_5 I^{\pm} = 2\mathcal{E}_\alpha(K\gamma p)\gamma_5 I^{\pm} \,, \tag{2.71-25}$$

and on expanding the right-hand side of this by means of equation A3-29, we find:

$$\mathcal{E}_\alpha(P'q\gamma)\gamma_5 I^{\pm} \cong \left[\tilde{\mathcal{K}}^2_\alpha - (M \pm m)\tilde{\mathcal{K}}^1_\alpha\right] I^{\pm}. \qquad (2.71\text{-}26)$$

Thus without introducing kinematical singularities into the form-factors, we may choose, if we so desire, to eliminate either $\mathcal{K}^1_\alpha$ or $\mathcal{K}^2_\alpha$ in favour of $\mathcal{E}_\alpha(P'q\gamma)\gamma_5$.

We have so far assumed that the hadrons are non-identical. In the identical hadron case $F_2(q^2)$ remains K.S.F., but $F_1(q^2)$ as defined by 2.71-21 has a kinematic pole at zero $q^2$. This arises because we have tried to eliminate non-existent terms. Taking the equal mass limit of equations 2.71-9, 10, and 11 we have:

$$\lim_{M \to m} \gamma'_\alpha(b) = \gamma_\alpha \quad , \qquad (2.71\text{-}27)$$

$$\lim_{M \to m} (P'_\alpha)'(b) = P'_\alpha \quad , \qquad (2.71\text{-}28)$$

$$\lim_{M \to m} q'_\alpha(b) = -aP'_\alpha/c. \qquad (2.71\text{-}29)$$

corresponding to the fact that in the identical hadron case, (where the vertex is necessarily normal), $\gamma_\alpha$ and $P'_\alpha$ both vanish on contraction with $q_\alpha$. Thus in this special case we still have two linearly inequivalent covarients, but these now remain inequivalent in the real photon limit.

Our equivalence relations read in the identical hadron case:

$$i\sigma_{\alpha\beta}q_\beta \cong 2m\gamma_\alpha - P'_\alpha \quad , \qquad (2.71\text{-}30)$$

$$\mathcal{E}_\alpha(P'q\gamma)\gamma_5 \cong 2mP'_\alpha - P'^2\gamma_\alpha. \qquad (2.71\text{-}31)$$

Without introducing kinematic singularities into the form-factors we may choose to use any two of the four covarients appearing in these equations except the pairs $\left(\mathcal{E}_\alpha(P'q\gamma)\gamma_5,\ i\sigma_{\alpha\beta}q_\beta\right)$ and $\left(\mathcal{E}_\alpha(P'q\gamma)\gamma_5, P'_\alpha\right)$. These latter correspond to pairs of form-factors with kinematic poles at vanishing $q^2$

and vanishing $P'^2\ (=4m^2-q^2)$ respectively.

It often proves convenient, particularly in connection with unpolarised cross-sections for lowest order elastic electron-nucleon scattering, to decompose the γ-nucleon-nucleon vertex in terms of the covarients $\mathcal{E}_\alpha(P'q\gamma)\gamma_5$ and $P'_\alpha$. One may avoid poles in the form-factors by explicitly factoring out the singular term $1/P'^2$. This then forces the form-factors to satisfy the "threshold constraint" that they be proportional to one another at vanishing $P'^2$, (the nucleon-antinucleon pair-production threshold).

In this way one arrives at equations 1.2-3 and 4.

## 2.72 $(\gamma,\ \frac{1}{2},\ J+\frac{1}{2})^{\pm}$ VERTICES.

Assuming that J is non-zero, we have in our previous notation:

$$N^{\pm}(1^-,\tfrac{1}{2},J+\tfrac{1}{2})=3 \quad , \quad N^{\pm}(0^+,\tfrac{1}{2},J+\tfrac{1}{2})=1 \ , \qquad (2.72\text{-}1)$$

so our general rules yield:

$$N^{\pm}(\gamma^V,\tfrac{1}{2},J+\tfrac{1}{2})=3 \quad , \quad N^{\pm}(\gamma^R,\tfrac{1}{2},J+\tfrac{1}{2})=2 \ . \qquad (2.72\text{-}2)$$

We know experimentally that the hadron masses will be necessarily unequal, so we expect to find no exceptions to equations 2.72-2.

In this case we shall choose to operate with $\mathcal{G}_{\alpha'\alpha}(q)$ on a suitable set of covarients for the coupling function $\mathcal{C}^{\pm}_{(\mu)^J,\alpha'}(1^-,\tfrac{1}{2},J+\tfrac{1}{2})$. Our final set of covarients will then remain free of kinematic singularities even when the hadrons are allowed to have variable squared mass.

We may conveniently choose:[9]

$$\mathcal{C}^{\pm}_{(\mu)^J,\alpha}(1^-,\tfrac{1}{2},J+\tfrac{1}{2})=(q_\mu)^{J-1}\Big[\sum_{j=1}^{3} g_j\,\mathcal{K}^j_{\mu_1\alpha}\Big]I^{\pm} \ , \qquad (2.72\text{-}3)$$

where:

$$\mathcal{K}^1_{\mu_1\alpha}=q_{\mu_1}\gamma_\alpha \quad , \quad \mathcal{K}^2_{\mu_1\alpha}=q_{\mu_1}p_\alpha \quad , \quad \mathcal{K}^3_{\mu_1\alpha}=g_{\mu_1\alpha} \ . \ (2.72\text{-}4)$$

Then:
$$\mathcal{K}'^{1}_{\mu_1\alpha}(q) = q_{\mu_1}(\gamma_\alpha - \not{q}\, q_\alpha/q^2) \,, \tag{2.72-5}$$
$$\mathcal{K}'^{2}_{\mu_1\alpha}(q) = q_{\mu_1}(p_\alpha - p\cdot q\, q_\alpha/q^2) , \tag{2.72-6}$$
$$\mathcal{K}'^{3}_{\mu_1\alpha}(q) = g_{\mu_1\alpha} - q_{\mu_1} q_\alpha/q^2 \quad . \tag{2.72-7}$$

(Again, the $g_j$ are K.S.F. when we set the squared mass of the $1^-$ hadron equal to $q^2$, and no type B E.R.'s are involved.)

Bearing in mind once again equation 2.71-12, we have just a single linearly inequivalent singular tail in agreement with the second of equations 2.72-1, and we see that a suitable tail elimination is achieved by defining:

$$\tilde{\mathcal{K}}^{1}_{\mu_1\alpha} I^{\pm} \equiv \left[\mathcal{K}'^{1}_{\mu_1\alpha}(q) - (M \mp m)\mathcal{K}'^{3}_{\mu_1\alpha}(q)\right] I^{\pm} \cong (q_{\mu_1}\gamma_\alpha - \not{q}\, g_{\mu_1\alpha}) I^{\pm} , \tag{2.72-8}$$
$$\tilde{\mathcal{K}}^{2}_{\mu_1\alpha} \equiv \mathcal{K}'^{2}_{\mu_1\alpha}(q) - p\cdot q\, \mathcal{K}'^{3}_{\mu_1\alpha}(q) = q_{\mu_1} p_\alpha - p\cdot q\, g_{\mu_1\alpha} \quad , \tag{2.72-9}$$
$$\tilde{\mathcal{K}}^{3}_{\mu_1\alpha} \equiv q^2 \mathcal{K}'^{3}_{\mu_1\alpha}(q) = q^2 g_{\mu_1\alpha} - q_{\mu_1} q_\alpha . \tag{2.72-10}$$

Thus the decomposition:
$$V^{\pm}_{(\mu)_J^-\alpha}(\gamma, \tfrac{1}{2}, J+\tfrac{1}{2}) = (q_\mu)^{J-1}\Big[G_1(q^2)\tilde{\mathcal{K}}^{1}_{\mu_1\alpha} \pm G_2(q^2)\tilde{\mathcal{K}}^{2}_{\mu_1\alpha} \pm G_3(q^2)\tilde{\mathcal{K}}^{3}_{\mu_1\alpha}\Big] I^{\pm} \tag{2.72-11}$$
satisfies the counting rules 2.72-2, and involves only K.S.F. form-factors. Without introducing kinematic singularities one could equally well replace p by P' throughout.

The plus/minus signs in front of $G_2$ and $G_3$ have been introduced in order to simplify certain Feynman graphs involving this vertex and appearing in Chapter 5. They allow such graphs to be written in a form which is invarient under a change of normality of the spin-($J + \frac{1}{2}$) particle, to the extent that no plus/minus signs are involved.

A decomposition alternative to that of 2.72-11 is derived in section 2.8.

A further alternative is provided by the decomposition:

$$\mathcal{V}^{\pm}_{(\mu)^J\alpha}(\gamma,\tfrac{1}{2},J+\tfrac{1}{2})=(q_\mu)^{J-1}\Big[G_7(q^2)q_{\mu_1}\gamma_{\alpha'} + G_8(q^2)g_{\mu_1\alpha'}+G_9(q^2)q_{\mu_1}q_{\alpha'}\Big]g_{\alpha'\alpha}(P')I^{\pm}. \qquad (2.72\text{-}12)$$

This is only of use if the hadron masses are kept fixed and unequal, but it does have the advantage of remaining valid at zero J where just $g'_{\mu_1\alpha}(P')$ disappears. The covarient multiplying $G_9$ is equivalent to zero for vanishing $q^2$.

Spin decompositions equivalent to ours have been obtained for special cases by Gourdin and Salin,(36) and Mathews(37). Bjorken and Walecka(2) treat the general case, but are mainly concerned with the non-covarient approach based on helicity "amplitudes". They relate these to the covarient form-factors of the previous three authors and these latter are related to ours in Appendix 9.

## 2.73 THE NORMAL $(\gamma, 0, J)$ VERTEX.

Assuming that J is non-zero we have:

$$N^+(1^-,0,J)=2 \quad , \quad N^+(0^+,0,J)=1 \, , \qquad (2.73\text{-}1)$$

so:

$$N^+(\gamma^V,0,J)=2 \quad , \quad N^+(\gamma^R,0,J)=1 \, . \qquad (2.73\text{-}2)$$

We may conveniently choose:(9)

$$\mathcal{E}^+_{(\mu)^J,\alpha}(1^-,0,J)\equiv(-q_\mu)^{J-1}\sum_{j=1}^{2} g_j \mathcal{K}^j_{\mu_1\alpha} \, , \qquad (2.73\text{-}3)$$

where:

$$\mathcal{K}^1_{\mu_1\alpha}=q_{\mu_1}\Delta_\alpha \quad , \quad \mathcal{K}^2_{\mu_1\alpha}=g_{\mu_1\alpha}. \qquad (2.73\text{-}4)$$

Once again, no problem due to kinematic singularities of the $g_j$ arise, and no type B E.R.'s are involved. To save repeating ourselves, we will state here and now that these same observations apply in the following two sections.

We therefore have:

$$\mathcal{K}'^{1}_{\mu_1\alpha}(q) = q_{\mu_1}\left(\Delta_\alpha - \Delta\cdot q\, q_\alpha/q^2\right), \tag{2.73-5}$$

and:

$$\mathcal{K}'^{2}_{\mu_1\alpha}(q) = g_{\mu_1\alpha} - q_{\mu_1}q_\alpha/q^2 \quad , \tag{2.73-6}$$

and on eliminating the singular tail in the only suitable manner we obtain the K.S.F. decomposition:

$$\begin{aligned} V^{+}_{(\mu)^J\alpha}(\gamma,0,J) \equiv (-q_\mu)^{J-1}\big[&f_2(q^2)(q_{\mu_1}\Delta_\alpha - \Delta\cdot q\, g_{\mu_1\alpha}) \\ &+ f_3(q^2)(q^2 g_{\mu_1\alpha} - q_{\mu_1}q_\alpha)\big]. \end{aligned} \tag{2.73-7}$$

We have denoted the form-factors by the symbols which will be used when this vertex appears in Chapter 4. Again, $\Delta$ could equally well be replaced throughout by P''.

For zero J we only have the single primed covarient: $\mathcal{K}'^{1}_{\alpha}$ so eliminating its tail in the only way possible, that is, by multiplication by $q^2$, we obtain:

$$V^{+}_{\alpha}(\gamma,0,0) = f_4(q^2)(q^2\Delta_\alpha - \Delta\cdot q\, q_\alpha). \tag{2.73-8}$$

Alternatively, we can operate with $g_{\alpha'\alpha}(P'')$ on the covarients: $(-q_\mu)^{J-1}(q_{\mu_1}P''_\alpha,\ g_{\mu_1\alpha},\ q_{\mu_1}q_{\alpha'})$, the final covarient arising from $\mathcal{C}^{+}_{(\mu)^J}(0^+,0,J)q_{\alpha'}$. Provided the hadron masses are kept fixed and different, this yields the K.S.F. decomposition:

$$\begin{aligned} V^{+}_{(\mu)^J\alpha}(\gamma,0,J) = (-q_\mu)^{J-1}\big[&f_5(q^2)q_{\mu_1}(q_\alpha - q^2 P''_\alpha/P''\cdot q) \\ &+ f_6(q^2)(g_{\mu_1\alpha} - q_{\mu_1}P''_\alpha/P''\cdot q)\big], \end{aligned} \tag{2.73-9}$$

which automatically reduces in the zero-J case to:

$$V^{+}_{\alpha}(\gamma,0,0) = f_5(q^2)(q_\alpha - q^2 P''_\alpha/P''\cdot q). \tag{2.73-10}$$

Equations 2.73-8 and 10 agree with the counting rules:

$$N^{+}(\gamma^{V},0,0) = 1 \quad , \quad N^{+}(\gamma^{R},0,0) = 0 \ , \tag{2.73-11}$$

obtained from:

$$N^{+}(1^-,0,0) = 1 = N^{+}(0^+,0,0) \tag{2.73-12}$$

We thus have, apparently, the rather surprising situation that the real photon cannot couple to a pair of spin-zero hadrons having different masses but the same normality. This is not indicative of a failure of the theory; rather, it should be interpreted as a statement that such a coupling cannot be described in gauge-invarient fashion with a K.S.F. form-factor. (Remember that it is only this requirement which leads to the second of equations 2.73-10, and only by trying to impose it have we forced the coupling to vanish.) The coupling can be restored in a gauge-invarient manner provided we are willing to postulate that the form-factor has a pole at zero $q^2$.

In the equal mass case we apparently have further problems due to the vanishing of $P''.q$. But for this very reason it will now prove sufficient to use the covarient $P''_\alpha$ (c.f. the $\gamma$-nucleon-nucleon vertex). Of course, if one of the hadrons is subsequently taken off-shell we lose both gauge-invarience and current-conservation. (This applies equally to the $\gamma$-nucleon vertex). However, it is wellknown that such problems can generally be overcome, at least for Born-terms involving a real photon, by requiring only that the sum of the Born-terms in all three channels be gauge-invarient.

### 2.74 THE ABNORMAL ($\gamma$, 0, J) VERTEX.

In this case we have:

$$N^-(1^-,0,J)=\begin{cases}1\ , & J\geqslant 1\ ,\\ 0\ , & J=0\ ,\end{cases} \qquad (2.74\text{-}1)$$

$$N^-(0^+,0,J)=0\ , \quad \text{all } J. \qquad (2.74\text{-}2)$$

So:

$$N^-(\gamma^V,0,J)=N^-(\gamma^R,0,J)=\begin{cases}1, & J\geqslant 1\ ,\\ 0, & J=0\ .\end{cases} \qquad (2.74\text{-}3)$$

The decomposition:[9]

$$\mathcal{C}^{-}_{(\mu)^J\alpha}(1^-,0,J)=(-q_\mu)^{J-1}g\,\varepsilon_{\mu_1\alpha}(\Delta q) \qquad (2.74\text{-}4)$$

already vanishes on contraction with $q_\alpha$ and so a suitable K.S.F. decomposition of the corresponding photonic vertex is just:

$$V^{-}_{(\mu)^J\alpha}(\gamma,0,J)=(-q_\mu)^{J-1}f_1(q^2)\varepsilon_{\mu_1\alpha}(\Delta q). \qquad (2.74\text{-}5)$$

For zero J, no coupling is possible since no $\mu_1$ index is available, so equation 2.74-3 is satisfied for all J.

## 2.75 THE NORMAL ($\gamma$, 1, J) VERTEX.

Here we have:

$$N^+(1^-,1,J)=\begin{cases}5, & J\geqslant 2,\\ 4, & J=1,\end{cases} \qquad (2.75\text{-}1)$$

and:

$$N^+(0^+,1,J)= 2,\ J\geqslant 1, \qquad (2.75\text{-}2)$$

from which it follows that:

$$N^+(\gamma^V,1,J)=\begin{cases}5, & J\geqslant 2,\\ 4, & J=1,\end{cases} \qquad (2.75\text{-}3)$$

and:

$$N^+(\gamma^R,1,J)=\begin{cases}3, & J\geqslant 2,\\ 2, & J=1.\end{cases} \qquad (2.75\text{-}4)$$

The counting-rules and spin decomposition for zero J have already been given in section 2.73.

We are going to let $\alpha,\mu$ and $(\sigma)^J$ be the Lorentz indices of the respective wave-functions of the photon, spin-1 hadron, and spin-J hadron, so we may choose:[9]

$$\mathcal{C}^{+}_{(\sigma)^J\mu\alpha}(1^-,1,J)=(-q_\sigma)^{J-2}\sum_{j=1}^{5}g_j\,\mathcal{K}^j_{\sigma_1\sigma_2\mu\alpha}\ , \qquad (2.75\text{-}5)$$

and:

$$\mathcal{C}^{+}_{(\sigma)^J\mu}(0^+,1,J)=(-q_\sigma)^{J-2}\sum_{j=6}^{7}g_j\,\mathcal{K}^j_{\sigma_1\sigma_2\mu}\ , \qquad (2.75\text{-}6)$$

where:

$$\mathcal{K}^1=g_{\sigma_1\mu}g_{\sigma_2\alpha}\ ,\qquad \mathcal{K}^2=g_{\sigma_1\mu}q_{\sigma_2}\Delta_\alpha\ , \qquad (2.75\text{-}7,8)$$

$$\mathcal{K}^3=g_{\sigma_1\alpha}q_{\sigma_2}q_\mu\ ,\qquad \mathcal{K}^4=g_{\mu\alpha}q_{\sigma_1}q_{\sigma_2}\ , \qquad (2.75\text{-}9,10)$$

$$\mathcal{K}^5 = q_{\sigma_1} q_{\sigma_2} q_\mu \Delta_\alpha , \quad \mathcal{K}^6 = g_{\sigma_1\mu} q_{\sigma_2} , \tag{2.75-11,12}$$

$$\mathcal{K}^7 = q_{\sigma_1} q_{\sigma_2} q_\mu . \tag{2.75-13}$$

Thus:

$$\mathcal{K}'^1(q) = \mathcal{K}^1 - \mathcal{K}^6 q_\alpha / q^2 , \tag{2.75-14}$$

$$\mathcal{K}'^2(q) = \mathcal{K}^2 - \Delta\cdot q\, \mathcal{K}^6 q_\alpha / q^2 , \tag{2.75-15}$$

$$\mathcal{K}'^3(q) = \mathcal{K}^3 - \mathcal{K}^7 q_\alpha / q^2 , \tag{2.75-16}$$

$$\mathcal{K}'^4(q) = \mathcal{K}^4 - \mathcal{K}^7 q_\alpha / q^2 , \tag{2.75-17}$$

and:

$$\mathcal{K}'^5(q) = \mathcal{K}^5 - \Delta\cdot q\, \mathcal{K}^7 q_\alpha / q^2 . \tag{2.75-18}$$

For J equal to unity, $\mathcal{K}^1$ and $\mathcal{K}'^1(q)$ no longer appear. Eliminating the two singular tails, we deduce the K.S.F. spin decompositions:

$$V^+_{(\sigma)^J\mu\alpha}(\gamma 1 J) = (-q_\sigma)^{J-2} \sum_{j=1}^{5} f_j(q^2) \tilde{\mathcal{K}}^j_{\sigma_1\sigma_2\mu\alpha} , J \geqslant 2 , \tag{2.75-19}$$

$$V^+_{\sigma_1\mu\alpha}(\gamma 1 1) = \sum_{j=1,2,4;6} f_j(q^2) \tilde{\mathcal{K}}^j_{\sigma_1\mu\alpha} , \tag{2.75-20}$$

where:

$$\tilde{\mathcal{K}}^1_{\sigma_1\sigma_2\mu\alpha} = \mathcal{K}'^3(q) - \mathcal{K}'^4(q) = q_{\sigma_2}(g_{\sigma_1\alpha} q_\mu - g_{\mu\alpha} q_{\sigma_1}) , \tag{2.75-21}$$

$$\tilde{\mathcal{K}}^2_{\sigma_1\sigma_2\mu\alpha} = \mathcal{K}'^5(q) - \Delta\cdot q\, \mathcal{K}'^4(q) = q_{\sigma_1} q_{\sigma_2}(q_\mu \Delta_\alpha - \Delta\cdot q\, g_{\mu\alpha}) , \tag{2.75-22}$$

$$\tilde{\mathcal{K}}^3_{\sigma_1\sigma_2\mu\alpha} = \mathcal{K}'^2(q) - \Delta\cdot q\, \mathcal{K}'^1(q) = g_{\sigma_1\mu}(q_{\sigma_2} \Delta_\alpha - \Delta\cdot q\, g_{\sigma_2\alpha}) , \tag{2.75-23}$$

$$\tilde{\mathcal{K}}^4_{\sigma_1\sigma_2\mu\alpha} = q^2 \mathcal{K}'^4(q) = q_{\sigma_1} q_{\sigma_2}(q^2 g_{\mu\alpha} - q_\mu q_\alpha) , \tag{2.75-24}$$

$$\tilde{\mathcal{K}}^5_{\sigma_1\sigma_2\mu\alpha} = q^2 \mathcal{K}'^1(q) = g_{\sigma_1\mu}(q^2 g_{\sigma_2\alpha} - q_{\sigma_2} q_\alpha) , \tag{2.75-25}$$

$$\tilde{\mathcal{K}}^{1,2,4}_{\sigma_1\mu\alpha} q_{\sigma_2} = \tilde{\mathcal{K}}^{1,2,4}_{\sigma_1\sigma_2\mu\alpha} , \tag{2.75-26}$$

and:

$$\tilde{\mathcal{K}}^6_{\sigma_1\mu\alpha} q_{\sigma_2} = q^2 \mathcal{K}'^2(q) , \tag{2.75-27}$$

so that:

$$\tilde{\mathcal{K}}^6_{\sigma_1\mu\alpha} = g_{\sigma_1\mu}(q^2 \Delta_\alpha - \Delta\cdot q\, q_\alpha) . \tag{2.75-28}$$

No special problems arise for identical hadrons, equations 2.75-20, 21, 22, 24, 26, and 28 still providing a perfectly valid decomposition in this case.

As usual, $\Delta_\alpha$ may be replaced throughout by $P''_\alpha$, and having made this substitution in 2.75-8 and 11, one may choose for fixed unequal hadron masses to decompose instead in terms of the five covarients: $\mathcal{K}'^{1}(P'')$, $\mathcal{K}'^{3}(P'')$, $\mathcal{K}'^{4}(P'')$, $\mathcal{K}^6 q'_\alpha(P'')$, and $\mathcal{K}^7 q'_\alpha(P'')$. These final two covarients then vanish at zero $q^2$, $\mathcal{K}'^{1}(P'')$ does not appear to J equal to unity or zero, and $\mathcal{K}'^{3}(P'')$ and $\mathcal{K}^6 q'_\alpha(P'')$ disappear as well for zero J.

## 2.76 THE ABNORMAL $(\gamma,1,J)$ VERTEX.

In this case we have:

$$N^-(1^-,1,J)=\begin{cases}4, & J\geqslant 2\,,\\ 3, & J=1\,,\end{cases} \tag{2.76-1}$$

and:

$$N^-(0^+,1,J)=1\,,\ J\geqslant 1\,, \tag{2.76-2}$$

so:

$$N^-(\gamma^V,1,J)=\begin{cases}4, & J\geqslant 2\,,\\ 3, & J=1\,,\end{cases} \tag{2.76-3}$$

$$N^-(\gamma^R,1,J)=\begin{cases}3, & J\geqslant 2\,,\\ 2, & J=1\,.\end{cases} \tag{2.76-4}$$

The zero J case has already been treated in section 2.74.

We have some type B equivalence theorems here. With Lorentz indices defined as in section 2.75, and writing:[9]

$$\mathcal{C}^-_{(\sigma)^J\mu\alpha}(1^-,1,J)=(-q_\sigma)^{J-2}\,\mathcal{C}^-_{\sigma_1\sigma_2\mu\alpha}(1^-,1,2)\,, \tag{2.76-5}$$

we have seven "obvious" covarients in terms of which $\mathcal{C}^-_{\sigma_1\sigma_2\mu\alpha}(1^-,1,2)$ can be decomposed, namely:

$$\tilde{\mathcal{K}}^1\equiv\varepsilon_{\mu\alpha}(q\Delta)\,q_{\sigma_1}q_{\sigma_2}\,,\qquad \tilde{\mathcal{K}}^2\equiv\varepsilon_{\mu\alpha\sigma_1}(q)\,q_{\sigma_2}\,, \tag{2.76-6,7}$$

$$\mathcal{K}^3\equiv\varepsilon_{\mu\alpha\sigma_1}(\Delta)\,q_{\sigma_2}\,,\qquad \tilde{\mathcal{K}}^4\equiv\varepsilon_{\alpha\sigma_1}(q\Delta)\,g_{\sigma_2\mu}\,, \tag{2.76-8,9}$$

$$\tilde{\mathcal{K}}^5 \equiv \varepsilon_{\alpha\sigma_1}(q\,\Delta)q_{\sigma_2}q_\mu \quad , \qquad \mathcal{K}^6 \equiv \varepsilon_{\mu\sigma_1}(q\,\Delta)g_{\sigma_2\alpha} \quad , \quad (2.76\text{-}10,11)$$

and
$$\mathcal{K}^7 \equiv \varepsilon_{\mu\sigma_1}(q\,\Delta)q_{\sigma_2}\Delta_\alpha \,. \qquad (2.76\text{-}12)$$

The tildes denote those covarients which vanish on contraction with $q_\alpha$. Only certain sets of four of these seven covarients are linearly inequivalent, three type B E.R.'s being operative. Equation A3-11 yields:

$$\mathcal{K}^3 + \mathcal{K}^6 \cong \tilde{\mathcal{K}}^4 , \qquad (2.76\text{-}7)$$

whilst from A3-12 we deduce:

$$q^2\mathcal{K}^3 \cong \tilde{\mathcal{K}}^1 + \Delta\cdot q\,\tilde{\mathcal{K}}^2 + \tilde{\mathcal{K}}^5 , \qquad (2.76\text{-}8)$$

and:

$$\mathcal{K}^7 + \Delta\cdot q\,\mathcal{K}^3 \cong \Delta^2\tilde{\mathcal{K}}^2 + \tilde{\mathcal{K}}^5 . \qquad (2.76\text{-}9)$$

The gauge projection operation yields:

$$\mathcal{K}'^3(q) = \mathcal{K}^3 + \mathcal{K}^8 q_\alpha/q^2 \quad , \qquad (2.76\text{-}10)$$

$$\mathcal{K}'^6(q) = \mathcal{K}^6 - \mathcal{K}^8 q_\alpha/q^2 \quad , \qquad (2.76\text{-}11)$$

$$\mathcal{K}'^7(q) = \mathcal{K}^7 - \Delta\cdot q\,\mathcal{K}^8 q_\alpha/q^2 , \qquad (2.76\text{-}12)$$

where
$$\mathcal{K}^8 = \varepsilon_{\mu\sigma_1}(q\,\Delta)q_{\sigma_2} \,. \qquad (2.76\text{-}13)$$

The single singular tail may be eliminated in two possible ways. The most useful elimination is provided by:

$$\tilde{\mathcal{K}}^3 \equiv q^2\mathcal{K}'^3(q) = \left[\varepsilon_{\mu\sigma_1}(q\,\Delta)q_\alpha - \varepsilon_{\mu\sigma_1\alpha}(\Delta)q^2\right]q_{\sigma_2} \quad , \qquad (2.76\text{-}14)$$

$$\tilde{\mathcal{K}}^6 \equiv \mathcal{K}'^6(q) + \mathcal{K}'^3(q) = \varepsilon_{\mu\sigma_1}(q\,\Delta)g_{\sigma_2\alpha} - \varepsilon_{\mu\sigma_1\alpha}(\Delta)q_{\sigma_2} \quad , \qquad (2.76\text{-}15)$$

$$\tilde{\mathcal{K}}^7 \equiv \mathcal{K}'^7(q) + \Delta\cdot q\,\mathcal{K}'^3(q) = \left[\varepsilon_{\mu\sigma_1}(q\,\Delta)\Delta_\alpha - \Delta\cdot q\,\varepsilon_{\mu\sigma_1\alpha}(\Delta)\right]q_{\sigma_2} \qquad (2.76\text{-}16)$$

and the three equivalence relations then read:

$$\tilde{\mathcal{K}}^4 \cong \tilde{\mathcal{K}}^6 \ , \tag{2.76-17}$$

$$\tilde{\mathcal{K}}^3 \cong \tilde{\mathcal{K}}^1 + \Delta\cdot q\, \tilde{\mathcal{K}}^2 + \tilde{\mathcal{K}}^5 \ , \tag{2.76-18}$$

and:

$$\tilde{\mathcal{K}}^7 \cong \Delta^2 \tilde{\mathcal{K}}^2 + \tilde{\mathcal{K}}^5 \ . \tag{2.76-19}$$

We may eliminate $\tilde{\mathcal{K}}^5$, $\tilde{\mathcal{K}}^6$, and $\tilde{\mathcal{K}}^7$ without the introduction of kinematic singularities, yielding the spin decomposition:

$$\bar{v}_{(\sigma)^{J}\mu\alpha}(\gamma, 1, J) = (-q_{\sigma})^{J-1} \sum_{j=1}^{4} F_j(q^2) \tilde{\mathcal{K}}^j_{\sigma_1\sigma_2\mu\alpha} \ . \tag{2.76-20}$$

This is a particularly useful expression in that it holds for all J (including zero), and remains valid both for variable and for equal masses. In agreement with our counting rules: $\tilde{\mathcal{K}}^3$ vanishes for zero $q^2$ and disappears for zero J, $\tilde{\mathcal{K}}^2$ also disappears for zero J, and $\tilde{\mathcal{K}}^4$ disappears for J less than two. For zero J, $\tilde{\mathcal{K}}^1$ is the same covarient as was derived in section 2.74.

## 2.8 COVARIENT DERIVATION, IN TERMS OF FORM-FACTORS, OF AN UNPOLARISED CROSS-SECTION INVOLVING THE $(\gamma, \frac{1}{2}, J + \frac{1}{2})^{\pm}$ VERTEX.

We consider here the unpolarised cross-section for the process: electron + spin-$\frac{1}{2}$ hadron $\longrightarrow$ electron + spin-$(J + \frac{1}{2})$ hadron, treated to lowest (i.e. second) quantum electrodynamical order, and in the approximation that the hadrons are treated as stable particles.

This problem has already been considered by a number of authors,(2)(38)(39) but their methods tend to be somewhat awkward, involving various non-covarient operations and a certain amount of trial and error in order to arrive at an initial vertex decomposition which will lead to a final expression free of cross-terms between different form-factors.

We shall repeat the calculation making use of the contracted forward propogators of section 2.2 together with equations A3-22 to 26. These allow us to compute the cross-section very simply and in a fully covarient manner. Moreover, we are able to deduce the required initial vertex decomposition.

The momenta (masses) of the initial electron, final electron, initial hadron, and final hadron are defined to be: $q_1(m_e)$, $q_2(m_e)$, $p(m)$, and $K(M)$ respectively. The momentum of the virtual photon is defined by:

$$q \equiv K - p = q_1 - q_2 \,. \tag{2.8-1}$$

It is assumed that the reader knows how to calculate to second order the unpolarised cross-section(38) given $\sum_f \overline{\sum_i} |T^{(2)}|^2$, the squared modulus of the second-order T-matrix element averaged over initial helicities and summed over final helicities. Thus we shall only compute this latter quantity.

If $J_\alpha(x)$ and $j_\alpha(x)$ denote respectively the hadronic and electronic electromagnetic current operators, we have in view of the hermiticity of these quantities:

$$\sum_f \overline{\overline{\sum_i}} |T^{(2)}|^2 = \frac{1}{q^4} t_{\alpha\beta} T_{\alpha\beta} \,, \tag{2.8-2}$$

where:

$$t_{\alpha\beta} = \sum_f \overline{\overline{\sum_i}} \langle q_2 | j_\alpha(0) | q_1 \rangle \langle q_1 | j_\beta(0) | q_2 \rangle \,, \tag{2.8-3}$$

and:

$$T_{\alpha\beta} = \sum_f \overline{\overline{\sum_i}} \langle K | J_\alpha(0) | p \rangle \langle p | J_\beta(0) | K \rangle \,. \tag{2.8-4}$$

We have: $$\langle q_2 | j_\alpha(0) | q_1 \rangle = e\,\bar{u}(q_2)\gamma_\alpha u(q_1) \,, \tag{2.8-5}$$

and we define, (factoring out the electronic charge);

$$\langle K | J_\alpha(0) | p \rangle \equiv e\,\overline{U}_{(\mu)\mp}(K) V^{\pm}_{(\mu)\mp\alpha} U(p) \,. \tag{2.8-6}$$

The lower (upper) case u's are the electron (hadron) wave-functions, and the plus/minus sign on $V^{\pm}_{(\mu)\mp\alpha}$ indicates the overall normality of that vertex.

Hence:

$$t_{\alpha\beta}=\tfrac{1}{2}e^2 tr\left[\gamma_\alpha(\not{q}_1+m_e)\gamma_\beta(\not{q}_2+m_e)\right] , \tag{2.8-7}$$

and:

$$T_{\alpha\beta}=\tfrac{1}{2}e^2 tr\left[v^{\pm}_{(\mu)\mp\alpha}(\not{p}+m)\overline{v}^{\pm}_{(\nu)\mp\beta}\,\rho^{\mp+1/2}_{(\nu)\mp,(\mu)\mp}(K)\right] , \tag{2.8-8}$$

where as usual:

$$\overline{v}^{\pm}_{(\nu)\mp\beta}\equiv\gamma_0 v^{\pm\dagger}_{(\nu)\mp\beta}\gamma_0 . \tag{2.8-9}$$

So:

$$t_{\alpha\beta}=e^2\left[2(q_{1\alpha}q_{2\beta}+q_{2\alpha}q_{1\beta})+q^2 g_{\alpha\beta}\right] , \tag{2.8-10}$$

and since this tensor turns out to be symmetric, only the symmetric part of $T_{\alpha\beta}$ need be calculated. We note as a check that:

$$q_\alpha t_{\alpha\beta}=0=t_{\alpha\beta}q_\beta , \tag{2.8-11}$$

as required by current conservation.

Instead of calculating $T_{\alpha\beta}$ directly fnom 2.8-8, it is convenient as an intermediate step to use a computational trick due to von Gehlen.[39] One performs a decomposition, into kinematic covarients and form-factors, of that part of $T_{\alpha\beta}$ which is symmetric and, (in view of 2.8-11), contains no overall factors of $q_\alpha$ or $q_\beta$. In this case two form-factors will be involved, and these can depend only on $q^2$. The simplest pair of kinematic covarients are $g_{\alpha\beta}$ and $p_\alpha p_\beta$, but we wish to satisfy the current-conservation equation:

$$q_\alpha T_{\alpha\beta}=0=T_{\alpha\beta}q_\beta , \tag{2.8-12}$$

so in the spirit of our previous discussions we write:

$$T^{\text{SYM.}}_{\alpha\beta}/e^2\equiv\mathcal{G}_{\alpha\alpha'}(q)\left[T_1(q^2)g_{\alpha'\beta'}+T_2(q^2)p_{\alpha'}p_{\beta'}\right]\mathcal{G}_{\beta'\beta}(q)$$

$$=T_1(q^2)\left(g_{\alpha\beta}-\frac{q_\alpha q_\beta}{q^2}\right)+T_2(q^2)\left(p_\alpha-\frac{p\cdot q\,q_\alpha}{q^2}\right)\left(p_\beta-\frac{p\cdot q\,q_\beta}{q^2}\right). \tag{2.8-13}$$

This is only an intermediate step in the calculation, and we shall not need to make any postulates about the analytic structure of the $T_{1,2}(q^2)$; it will not therefore be necessary to eliminate the singular tails from 2.8-13. The gauge-projection operators in the first equality of 2.8-13 only become important when we relate the form-factors of the hadronic vertex to the $T_{1,2}(q^2)$. As far as relating these latter to the cross-section is concerned, we have from 2.8-11 that:

$$\mathcal{G}_{\alpha'\alpha}(q)t_{\alpha\beta}\mathcal{G}_{\beta\beta'}(q)=t_{\alpha'\beta'} \quad , \tag{2.8-14}$$

so we deduce immediately that:

$$\sum_f \overline{\overline{\sum_i}}\,|T^{(2)}|^2 = 4(e^4/q^4)\Big[(q_1\cdot q_2+q^2)T_1(q^2) + (p\cdot q_1\, p\cdot q_2+\tfrac{1}{4}m^2q^2)T_2(q^2)\Big] \;. \tag{2.8-15}$$

Note this well known[2] result that the dynamics of the lowest order unpolarised cross-section is parameterised entirely by two scalar functions of $q^2$. These will be linear combinations of the three (in general complex) form-factors parameterising the dynamics of the hadronic vertex. Thus only a limited amount of information about these latter form-factors is available from a study of this particular class of cross-sections.

The power of the von Gehlen trick now becomes clear: we only have to calculate any two independent components of the tensor $T^{SYM}_{\alpha\beta}$. We shall choose to keep the calculation covarient by computing $p_\alpha T^{SYM}_{\alpha\beta}p_\beta$ and $g_{\alpha\beta}T^{SYM}_{\alpha\beta}$. Relating these quantities to $T_1$ and $T_2$ by means of the second equality of 2.8-13 we have:

$$T_1=\frac{1}{2\zeta}\left(p_\alpha T^{SYM}_{\alpha\beta}p_\beta+\zeta g_{\alpha\beta}T^{SYM}_{\alpha\beta}\right), \tag{2.8-16}$$

and:

$$T_2=\frac{1}{2\zeta^2}\left(3p_\alpha T^{SYM}_{\alpha\beta}p_\beta+\zeta g_{\alpha\beta}T^{SYM}_{\alpha\beta}\right), \tag{2.8-17}$$

where: $\zeta \equiv -p^2(q)$. (2.8-18)

From 2.8-8 it follows that:

$$p_\alpha T^{SYM}_{\alpha\beta} p_\beta = \tfrac{1}{2} tr\left[p_\alpha v^{\pm}_{(\mu)^J\alpha}(\not{p}+m) p_\beta \bar{v}^{\pm}_{(\nu)^J\beta} \rho^{J+1/2}_{(\nu)^J,(\mu)^J}(K)\right], \quad (2.8\text{-}19)$$

and:

$$g_{\alpha\beta} T^{SYM}_{\alpha\beta} = \tfrac{1}{2} tr\left[v^{\pm}_{(\mu)^J\alpha}(\not{p}+m) \bar{v}^{\pm}_{(\nu)^J\alpha} \rho^{J+1/2}_{(\nu)^J,(\mu)^J}(K)\right] . \quad (2.8\text{-}20)$$

Using these equations to compute $p_\alpha T^{SYM}_{\alpha\beta} p_\beta$ and $g_{\alpha\beta} T^{SYM}_{\alpha\beta}$ in terms of the $G_{1,2,3}(q^2)$ of section 2.72, then yields the unpolarised cross-section in terms of these form-factors. One soon sees, however, that the expression is going to invove cross terms of the form $G_i^* G_j$ between the various pairs of different form-factors. In order to obtain an expression involving only the squared moduli of form-factors, we must look for a more suitable decomposition of the hadronic vertex.

Consider the covarients: $(q_\mu)^{J-1}\varepsilon_{\mu_1\sigma}(pq)\varepsilon_{\sigma\alpha}(pq)I^{\pm}$, $(q_\mu)^J\varepsilon_\alpha(pq\gamma)\gamma_5 I^{\pm}$ and $(q_\mu)^{J-1}\varepsilon_{\mu_1\alpha}(pq)\gamma_5 I^{\pm}$ . These are linearly inequivalent, vanish on contraction with $q_\alpha$, and have the correct parity behaviour. In addition they all vanish on contraction with $p_\alpha$, and will not therefore contribute to $p_\alpha T^{SYM}_{\alpha\beta} p_\beta$ . They will give rise to cross-terms in $g_{\alpha\beta} T^{SYM}_{\alpha\beta}$ between their own respective form-factors, but not between these and a form-factor corresponding to any additional covarient involving a factor of the form: $(ap_\alpha + cq_\alpha)$. This latter covarient should be linearly equivalent to a combination of the former three; should vanish on contraction with $q_\alpha$; should have the required parity behaviour; and to satisfy the counting-rules, should be equivalent to zero at vanishing $q^2$. A suitable covarient is furnished in: $(q_\mu)^J(p\cdot q\, q_\alpha - q^2 p_\alpha)I^{\pm}$. We can thus usefully choose to use this covarient and any two of the previous three. Since we shall ultimately wish to relate the covarients we pick to those appearing in equation 2.72-11 , we shall choose to leave out: $(q_\mu)^{J-1}\mathcal{E}_{\mu_1\alpha}(pq)\gamma_5 I^\pm$. Of the above four covarients, this one has the most complicated expansion, (equation A3-27), in terms of those of 2.72-11. In order to eliminate the remaining cross-terms, we shall multiply $(q_\mu)^J\mathcal{E}_\alpha(pq\gamma)\gamma_5 I^\pm$ and $(q_\mu)^{J-1}\mathcal{E}_{\mu_1\sigma}(pq)\mathcal{E}_{\sigma\alpha}(pq)I^\pm$ by an as yet undetermined pair of linear combinations of the same pair of form-factors. Thus we write:

$$V^\pm_{(\mu)^J\alpha}\equiv(q_\mu)^{J-1}\Big[G_4(q^2)\tilde{\mathcal{K}}^4_{\mu_1\alpha}+\big(a\,G_5(q^2)+b\,G_6(q^2)\big)\tilde{\mathcal{K}}^5_{\mu_1\alpha}$$
$$+\big(c\,G_5(q^2)+d\,G_6(q^2)\big)\tilde{\mathcal{K}}^6_{\mu_1\alpha}\Big]I^\pm, \qquad (2.8\text{-}21)$$

where:

$$\tilde{\mathcal{K}}^4_{\mu_1\alpha}=q_{\mu_1}(p\cdot q\,q_\alpha-q^2p_\alpha), \qquad (2.8\text{-}22)$$

$$\tilde{\mathcal{K}}^5_{\mu_1\alpha}=\mathcal{E}_{\mu_1\sigma}(pq)\mathcal{E}_{\sigma\alpha}(pq), \qquad (2.8\text{-}23)$$

$$\tilde{\mathcal{K}}^6_{\mu_1\alpha}=q_{\mu_1}\mathcal{E}_\alpha(pq\gamma)\gamma_5 \quad , \qquad (2.8\text{-}24)$$

and a,b,c,d are scalar constants, (which may be complex), to be determined so that cross-terms between $G_5$ and $G_6$ vanish.

We then have:

$$\bar{V}^\pm_{(\nu)^J\beta}=(q_\nu)^{J-1}I^\pm\Big[G_4^*\tilde{\mathcal{K}}^4_{\nu_1\beta}+(a^*G_5^*+b^*G_6^*)\tilde{\mathcal{K}}^5_{\nu_1\beta}$$
$$+(c^*G_5^*+d^*G_6^*)q_{\nu_1}\gamma_5\mathcal{E}_\beta(pq\gamma)\Big], \qquad (2.8\text{-}25)$$

so:

$$p_\alpha T^{SYM}_{\alpha\beta}p_\beta=\tfrac{1}{2}\zeta^{\prime 2}|G_4|^2\,tr\big[(\not p\pm m)\mathcal{P}^{J+1/2}(q,q;K)\big], \qquad (2.8\text{-}26)$$

and:

$$g_{\alpha\beta}T^{SYM}_{\alpha\beta}=\tfrac{1}{2}tr\Big[-q^2\zeta'|G_4|^2(\not p\pm m)\mathcal{P}^{J+1/2}(q,q;K)$$
$$+|aG_5+bG_6|^2\mathcal{E}_{\mu_1\sigma}(pq)\mathcal{E}_{\sigma\alpha}(pq)\mathcal{E}_{\nu_1\tau}(pq)\mathcal{E}_{\tau\alpha}(pq)(\not p\pm m)\mathcal{P}^{J+1/2}_{\nu_1;\mu_1}(q,q;K)$$
$$+(aG_5+bG_6)(c^*G_5+dG_6^*)\mathcal{E}_{\mu_1\sigma}(pq)\mathcal{E}_{\sigma\alpha}(pq)(\not p\pm m)\gamma_5\mathcal{E}_\alpha(pq\gamma)\times$$
$$\times\mathcal{P}^{J+1/2}_{;\mu_1}(q,q;K)+(cG_5+dG_6)(a^*G_5^*+b^*G_6^*)\mathcal{E}_\alpha(pq\gamma)\gamma_5(\not p\pm m)\times$$
$$\times\mathcal{E}_{\nu_1\tau}(pq)\mathcal{E}_{\tau\alpha}(pq)\mathcal{P}^{J+1/2}_{\nu_1;}(q,q;K)+|cG_5+dG_6|^2\mathcal{E}_\alpha(pq\gamma)\times$$
$$\times(\not p\mp m)\mathcal{E}_\alpha(pq\gamma)\mathcal{P}^{J+1/2}(q,q;K)\Big] \quad ,$$
$$(2.8\text{-}27)$$

where:
$$\zeta' \equiv q^2\zeta = -q^2 p^2(q)$$
$$= \tfrac{1}{4}\left[q^2-(M+m)^2\right]\left[q^2-(M-m)^2\right] . \qquad (2.8\text{-}28)$$

Now all freed indices in the partially contracted forward propogators are contracted with quantities of the forms: $\mathcal{E}_{\mu_1\cdot}(pq)$ and $\mathcal{E}_{\nu_1\cdot}(pq)$ . Thus we can immediately drop from these propogators all terms involving at least one of the factors: $p_{\mu_1}, q_{\mu_1}, K_{\mu_1}, p_{\nu_1}, q_{\nu_1}$ and $K_{\nu_1}$. For our purposes we then have from Scadron's paper:[9]

$$\mathcal{P}^{J+1/2}(q,q;K) \cong C_{J+1}\xi^{J}(\not{K}+M) , \qquad (2.8\text{-}29)$$

$$\mathcal{P}^{J+1/2}_{;\mu_1}(q,q;K) \cong -\tfrac{1}{2}C_{J+1}\xi^{J-1}\not{q}(K)(\not{K}-M)\gamma_{\mu_1} , \qquad (2.8\text{-}30)$$

$$\mathcal{P}^{J+1/2}_{\nu_1;}(q,q;K) \cong -\tfrac{1}{2}C_{J+1}\xi^{J-1}\gamma_{\nu_1}(\not{K}-M)\not{q}(K) , \qquad (2.8\text{-}31)$$

and:
$$\mathcal{P}^{J+1/2}_{\nu_1;\mu_1}(q,q;K) \cong -\frac{1}{2J}C_{J+1}\xi^{J-1}\Big[(J+2)g_{\nu_1\mu_1}(\not{K}+M) + \gamma_{\nu_1}(\not{K}-M)\gamma_{\mu_1}\Big] , \qquad (2.8\text{-}32)$$

where
$$\xi \equiv -q^2(K) = \zeta'/M^2 . \qquad (2.8\text{-}33)$$

Next, we deduce from equations A3-22, 24, 25, and 26 that:

$$\mathcal{E}_\alpha(pq\gamma)\mathcal{E}_\alpha(pq\gamma) = 2\zeta' , \qquad (2.8\text{-}34)$$

$$\mathcal{E}_{\alpha\sigma}(pq)\mathcal{E}_\alpha(pq\gamma)\gamma_5\mathcal{E}_\sigma(pq\gamma) = 2\zeta'(\not{p}\not{q}-p\cdot q) , \qquad (2.8\text{-}35)$$

$$\mathcal{E}_{\mu\sigma}(pq)\mathcal{E}_{\sigma\alpha}(pq)\mathcal{E}_{\alpha\tau}(pq)\mathcal{E}_{\tau\mu}(pq) = 2\zeta'^2 , \qquad (2.8\text{-}36)$$

and:
$$\mathcal{E}_{\tau\alpha}(pq)\mathcal{E}_{\alpha\sigma}(pq)\mathcal{E}_\tau(pq\gamma)\mathcal{E}_\sigma(pq\gamma) = -2\zeta'^2 . \qquad (2.8\text{-}37)$$

Finally we notice that $\mathcal{E}_\cdot(pq\gamma)$ anti-commutes with $\not{p}$ and $\not{q}$ and therefore with $\not{K}$ .

Computation of the traces in 2.8-26 and 27 is now trivial, and we find:

$$p_\alpha T^{SYM}_{\alpha\beta} p_\beta = \frac{C_{J+1}}{N^{2J}}\zeta'^{J+2}\left[(N+m)^2-q^2\right]|G_4|^2 , \qquad (2.8\text{-}38)$$

whilst the condition that $g_{\alpha\beta} T^{SYM}_{\alpha\beta}$ be free of cross-terms between $G_5$ and $G_6$ turns out to be:

$$Ma = -2c, \quad b = o, \quad d \text{ arbitrary.} \tag{2.8-39}$$

In 2.8-38 we have replaced M by a new variable, N, defined by:

$$N = nM, \tag{2.8-40}$$

where n is the normality of the hadronic vertex. This removes a plus/minus sign from this equation, and also from 2.8-43, 44, 45, 49, 50, and 51. The simplest expression for $g_{\alpha\beta} T^{SYM}_{\alpha\beta}$ results from the choice:

$$a = -2\sqrt{\frac{J}{J+2}}\,, \quad b = 0\,, \quad c = M\sqrt{\frac{J}{J+2}}\,, \quad d = M\,. \tag{2.8-41}$$

We choose to introduce an overall plus/minus sign into the definition of the vertex, again in order to remove similar signs from subsequent equations. We then have:

$$V^{\pm}_{(\mu)^J\alpha} \equiv \pm (q_\mu)^{J-1} \Big[ G_4(q^2) \tilde{\mathcal{K}}^4_{\mu,\alpha}$$

$$+ \sqrt{\frac{J}{J+2}}\, G_5(q^2)\left(M \tilde{\mathcal{K}}^6_{\mu,\alpha} - 2\tilde{\mathcal{K}}^5_{\mu,\alpha}\right) + M G_6(q^2) \tilde{\mathcal{K}}^6_{\mu,\alpha} \Big] I^{\pm}, \tag{2.8-42}$$

and:

$$g_{\alpha\beta} T^{SYM}_{\alpha\beta} = \frac{-C_{J+1}}{N^{2J}} \zeta^{/J+1} \left[(N+m)^2 - q^2\right] \Big[ q^2 |G_4|^2$$

$$+ 2N^2\left(|G_5|^2 + |G_6|^2\right)\Big]\,. \tag{2.8-43}$$

Equations 2.8-16, 17, 38, and 43 together yield:

$$T_1 = \frac{-C_{J+1}}{N^{2(J-1)}} \zeta^{/J+1} \left[(N+m)^2 - q^2\right] \left(|G_5|^2 + |G_6|^2\right), \tag{2.8-44}$$

and:

$$T_2 = \frac{C_{J+1}}{N^{2J}} \zeta^{/J} q^2 \left[(N+m)^2 - q^2\right] \left[ q^2 |G_4|^2 - N^2\left(|G_5|^2 + |G_6|^2\right)\right], \tag{2.8-45}$$

which in conjunction with 2.8-15 express $\sum_f \overline{\sum_i} |T^{(2)}|^2$ in terms of the linearly independent combinations of form-factors:

$q^2 |G_4|^2$ and $\left(|G_5|^2 + |G_6|^2\right)$.

It remains to determine $G_{4,\,5,\,6}$ in terms of the $G_{1,\,2,\,3}$

of section 2.72. We immediately have:

$$\tilde{\mathcal{K}}^4 = -q^2\tilde{\mathcal{K}}^2 - \tfrac{1}{2}(M^2 - m^2 - q^2)\tilde{\mathcal{K}}^3 . \tag{2.8-46}$$

From A3-18:

$$\tilde{\mathcal{K}}^5 = \tfrac{1}{2}(M^2 - m^2 + q^2)\tilde{\mathcal{K}}^2 + \tfrac{1}{2}(M^2 + m^2 - q^2)\tilde{\mathcal{K}}^3 . \tag{2.8-47}$$

Equation A3-29 and the Dirac equation together yield:

$$\tilde{\mathcal{K}}^6 I^{\pm} \cong \left\{ \tfrac{1}{2}\left[q^2 - (M \pm m)^2\right]\tilde{\mathcal{K}}^1 + (M \pm m)\tilde{\mathcal{K}}^2 \mp m\tilde{\mathcal{K}}^3 \right\} I^{\pm} . \tag{2.8-48}$$

Thus:

$$2\zeta' G_4 = 2NG_1 + (N^2 + m^2 - q^2)G_2 - (N^2 - m^2 + q^2)G_3 \quad , \tag{2.8-49}$$

$$4\zeta'\sqrt{\frac{J}{J+2}}\,G_5 = -2(N-m)G_1 - (N^2 - m^2 - q^2)G_2 + 2q^2 G_3 \quad , \tag{2.8-50}$$

$$4N\zeta' G_6 = 2\left[q^2 + m(N-m)\right]G_1 + N(N^2 - m^2 - q^2)G_2 - 2Nq^2 G_3 \quad , \tag{2.8-51}$$

and irrespective of the normality of the hadronic vertex, the $G_{4,5,6}$ each have kinematic poles at:

$$q^2 = (M+m)^2 \tag{2.8-52}$$

and at:

$$q^2 = (M-m)^2 . \tag{2.8-53}$$

As usual one may substitute threshold constraints for these poles by explicitly exhibiting a double-pole factor $1/\zeta'$ on the right-hand side of the definition 2.8-42.

# CHAPTER 3

# OFF-SHELL SUPERCONVERGENT SUM-RULES AND THEIR SATURATION.

## 3.1 INTRODUCTION AND BASIC ASSUMPTIONS; DERIVATION OF SUM-RULES.

The superconvergence programme reviewed in section 1.3 was only applicable to scattering processes involving two initial and two final on-shell hadrons and/or real photons. We saw in section 1.4 that it would be useful to try to extend such a programme to four-point functions, (as discussed in section 2.6), involving three on-shell hadrons and a virtual photon. In this chapter we discuss the additional assumptions needed to make such an extension possible.

Although our arguments will be obviously much more general, we restrict for the sake of definiteness to the four-point function corresponding to the "interaction": virtual photon + baryon$\rightarrow$baryon + meson. Such a four-point function will arise when we treat to lowest quantum electrodynamical order the process: lepton + baryon$\rightarrow$identical lepton + baryon + meson.

Following section 2.6 we define the s, t, and u channels of the four-point function, and then perform a decomposition into invarient amplitudes or, more strictly, three-variable form-factors. Let us assume for the moment that these continue to satisfy the crossing rules 2.41-11 and 2.42-4. We then postulate that the high sub-energy, (i.e. s), asymptotic behaviour of these amplitudes is determined by exchanged, (i.e. t-channel intermediate), Regge trajectories. We further postulate that the only trajectories which appear are those which would have been involved had the photon been real.

Let us be a little more specific. The process: lepton + "anti"-meson$\rightarrow$lepton + baryon + antibaryon is treated to second quantum eletrodynamical order, and we perform a partial-wave expansion in terms of the relative total angular momentum between the baryon and the anti-baryon. We keep fixed the difference between this quantity and the total angular momentum of the three final particles. A Sommerfeld-Watson transformation is performed, and the contour is expanded in the usual way. Our assumption is then that the only additional singularities picked up are those which would have been encountered had the photon been on-shell. In the language of the covarient formalism this corresponds to the calculation of t-channel graphs of the form: virtual photon + "anti"-meson $\rightarrow$ strong interaction Reggeon $\rightarrow$ baryon + anti-baryon. The process as a whole may well involve additional Regge trajectories of purely electrodynamical origin, but it is assumed that these will only manifest themselves in a partial-wave decomposition in terms of that angular momentum which we have held constant.

To justify this we argue that the invarient amplitudes for the four-point function are each the product of an overall "scale factor" and a function of the three "Mandelstam" variables. The former represents the coupling of the electromagnetic lepton current to the bare hadrons, whilst the latter describes the strong and radiative corrections to this coupling. The strong interaction corrections presumably dominate over the radiative ones so it is not unreasonable to assume that the high sub-energy asymptotic behaviour of these functions, and therefore of the invarient amplitudes themselves, is determined by some characteristic behaviour of the strong interaction, namely the existence of the strong interaction Regge poles. It is hard to see how any purely electrodynamical Regge trajectories can be involved without these continuing to manifest themselves when we extrapolate to the real photon limit.

Having postulated a method whereby we can determine the high sub-energy asymptotic behaviour of the amplitudes, we have to see whether the statement that an amplitude superconverges can be converted into a useful sum-rule. This requires that we know the analytic structure of the amplitude as a function of the sub-energy, and have a prescription for computing, at least approximately, the discontinuity across its cuts. If our previous postulate is to be meaningful we must also be sure that the amplitudes indeed satisfy the same s,t,u crossing relations as would be operative were the photon on-shell.

There are two ways of proceeding. Following Chew et. al.[40] we may say that since s,t, and $q^2$ are independent variables, the s,t,u crossing rules and analytic structure should remain unchanged when we take the zero $q^2$ limit. They are therefore the same whether the photon is real or virtual, apart from slight kinematic modifications in the latter case due to the sum of s,t, and u being linearly dependent on the variable $q^2$. In particular, the cuts are to be calculated by extrapolation to non-vanishing $q^2$ of those unitarity relations which hold when the photon is on-shell. Essentially, this is just a statement that the strong-interaction does not distinguish between real and virtual photons. Our off-shell superconver--gence programme is then to be carried out in direct analogy with the corresponding on-shell programme for a real photon.

Alternatively, we may try to apply general S-matrix theory arguments to the overall five-particle scattering process treated non-perturbatively. We first postulate that there exists for this process a unitary S-operator and a corresponding T-operator defined in the usual way. Matrix elements of this latter operator may certainly be decomposed into invarient amplitudes and kinematic covarients since this is a purely kinematic and group-theoretic operation. These invarient amplitudes are then postulated

1) to satisfy the obvious generalisations of the crossing rules for four-particle T-matrix elements and

2) to be analytic functions of the renormalised electronic charge at the point where that quantity vanishes. The assumptions of the previous paragraph may then be deduced as a consequence of these three postulates and a comparison with the field-theoretic perturbation expansion. This is demonstrated in the following section.

## 3.2 THE ANALYTIC STRUCTURE OF VIRTUAL-PHOTONIC FOUR-POINT FUNCTIONS; SATURATION OF SUM-RULES.

In this section we are motivated by some ideas of Dresden and Chou[14] concerning an S-matrix theory of quantum electrodynamics, but we shall apply them to reactions between arbitrary numbers of initial and final particles which involve both electromagnetic and strong interactions. We shall be concerned with electromagnetic interactions between leptons and hadrons which are modified by the strong interactions of these latter particles. We postulate that even though such reactions are primarily electromagnetic, they may be described by the relevant matrix elements, $S_{fi}$, of a unitary operator S, in the same way that one describes purely strong interactions. A T-matrix may then be defined in the usual way by:

$$S_{fi} \equiv \delta_{fi} - i\delta^4(p_f - p_i)T_{fi} \quad , \qquad (3.2\text{-}1)$$

where $p_f$ and $p_i$ are respectively the total final and total initial momenta, and we further postulate that $T_{fi}$ may be expanded in a power series in e, the magnitude of the renormalised electronic charge:

$$T_{fi} = \sum_{n=0}^{\infty} e^n T_{fi}^{(n)} , \tag{3.2-2}$$

and is therefore analytic in e in a region surrounding the point where e vanishes. As usual we factor out the helicity dependence of $T_{fi}$ by defining an M-function (Lorentz tensor-spinor) by:

$$T_{fi} \equiv \overline{\psi}(f) : M : \psi(i) , \tag{3.2-3}$$

and decompose M into invarient amplitudes, $A_i$, depending on suitably defined generalised Mandelstam variables, according to:

$$M \equiv \sum_i A_i \mathcal{K}^i . \tag{3.2-4}$$

In these equations $\overline{\psi}(f)$ and $\psi(i)$ stand symbolically for the Rarita-Schwinger wave-functions of the initial and final particles, colons denote contraction over (suppressed) Lorentz indices, and the $\mathcal{K}^i$ are kinematic basis covarients having the same Lorentz transformation properties as M.

Since $T_{fi}$ and $T_{fi}^{(n)}$ involve the same external particles we then have:

$$T_{fi}^{(n)} = \overline{\psi}(f) : M^{(n)} : \psi(i) , \tag{3.2-5}$$

and:

$$M^{(n)} = \sum_i A_i^{(n)} \mathcal{K}^i , \tag{3.2-6}$$

where:

$$M = \sum_{n=0}^{\infty} e^n M^{(n)} , \tag{3.2-7}$$

and:

$$A_i = \sum_{n=0}^{\infty} e^n A_i^{(n)} . \tag{3.2-8}$$

Equations 3.2-2, 7, and 8 differ from the corresponding ones in Dresden and Chou's theory in that due to the presence of strong as well as electromagnetic interactions, we no longer have that $T_{fi}^{(0)}$, $M^{(0)}$, and $A_i^{(0)}$ necessarily vanish. This turns out to be crucial to our argument.

In view of the postulated unitarity of S we have:

$$2\,\mathrm{Im}\,T_{fi} = -\sum_N \delta^4(p_i - p_N)\,T_{fN}T_{iN}^* = -\sum_N \delta^4(p_i - p_N)\,T_{Nf}^* T_{Ni}\,, \tag{3.2-9}$$

where $p_N$ denotes the total momentum of the "intermediate state" labelled by N. After expanding both sides of this equation by means of 3.2-2, our postulate of analyticity in e at zero e enables us to equate coefficients of $e^n$ obtaining:

$$2\,\mathrm{Im}\,T_{fi}^{(n)} = -\sum_N \sum_{m=0}^{n} \delta^4(p_i - p_N)\,T_{fN}^{(m)} T_{iN}^{(n-m)*} = -\sum_N \sum_{m=0}^{n} \delta^4(p_i - p_N)\,T_{Nf}^{(m)*} T_{Ni}^{(n-m)}\,. \tag{3.2-10}$$

By further postulating that $T_{fi}$ is hermitian analytic:

$$T_{fi}^{\pm *}(V) = T_{if}^{\mp}(V)\,, \tag{3.2-11}$$

where:

$$T_{fi}^{\pm}(V) \equiv \lim_{\epsilon \to 0^+} T_{fi}(V \pm i\epsilon, W)\,, \tag{3.2-12}$$

V denotes the total energy Mandelstam variable, and W denotes the remaining linearly independent Mandelstam variables, we similarly obtain:

$$T_{fi}^{(n)\pm *}(V) = T_{if}^{(n)\mp}(V)\,. \tag{3.2-13}$$

Defining:

$$\mathrm{disc}_V\, T_{fi}(V) \equiv T_{fi}^{+}(V) - T_{fi}^{-}(V)\,, \tag{3.2-14}$$

equations 3.2-9 and 10 respectively may then be written:

$$\mathrm{disc}_V\, T_{fi}(V) = -i\sum_N \delta^4(p_i - p_N)\,T_{fN}^{\pm}(V)\,T_{Ni}^{\mp}(V)\,, \tag{3.2-15}$$

and:

$$\mathrm{disc}_V\, T_{fi}^{(n)}(V) = -i\sum_N \sum_{m=0}^{n} \delta^4(p_i - p_N)\,T_{fN}^{(m)\pm}(V)\,T_{Ni}^{(n-m)\mp}(V)\,. \tag{3.2-16}$$

Next, if $p_1,\ldots,p_k,\ldots,p_r$ denote the momenta of the initial particles, and $q_1,\ldots,q_\ell,\ldots,q_s$ are the momenta of the final particles, we postulate the crossing relation:

$$M(q_1,\ldots,q_\ell,\ldots,q_s;p_1,\ldots,p_k,\ldots,p_r)=$$
$$=\xi M(q_1,\ldots,-\bar{p}_k,\ldots,q_s;p_1,\ldots,-\bar{q}_\ell,\ldots,p_r)\ , \qquad (3.2\text{-}17)$$

where: $\xi = \begin{cases} +1 \text{ for BB and FB crossing,} \\ -1 \text{ for FF crossing.} \end{cases}$ (3.2-18)

We then obtain from 3.2-7:

$$M^{(n)}(q_1,\ldots,q_\ell,\ldots,q_s;p_1,\ldots,p_k,\ldots,p_r)=$$
$$=\xi M^{(n)}(q_1,\ldots,-\bar{p}_k,\ldots,q_s;p_1,\ldots,-\bar{q}_\ell,\ldots,p_r)\ . \qquad (3.2\text{-}19)$$

Thus if our initial postulates are correct, we now have unitarity, hermitian analyticity, and crossing relations on our $T^{(n)}_{fi}$, and these are exact to all orders in e.

Finally we connect with field theoretic perturbation theory of the electromagnetic interaction, by assuming that $e^n T^{(n)}_{fi}$ may be partially computed, using our arbitrary spin Feynman rules, by taking the sum of all topologically different graphs involving $\frac{1}{2}(n-j)$ virtual-photon propogators, where j is the number of external (real) photons. These graphs are to be such that they connect the external particle lines by means only of: virtual-photon propogators, virtual-lepton propogators, lepton-photon-lepton vertices, many hadron vertices, and many hadron-many photon vertices. Each vertex is to carry one power of e for each real or virtual photon coupled to it. This is a fairly natural assumption since we have made a series expansion of $T_{fi}$ in powers of e, but are continuing to treat its strong interaction structure non-perturbatively.

We further assume that $e^n T^{(n)}_{fi}$ has a (Born-term) pole at $V = m^2$ if any of the hadronic or hadronic-photonic vertices involved in its computation can be subdivided into a pair of vertices connected only by an on-shell, stable, single-particle intermediate state with mass m and squared four-momentum V. Here, V is again any one of the Mandelstam variables involved.

We now specialise to the process in which we are interested, namely: electron $(q_1, m_e)$ + nucleon $(p, m) \longrightarrow$ electron $(q_2, m_e)$ + nucleon $(p', m)$ + meson $(k, \mu)$, where the first and second quantities in the parentheses following each particle are respectively the momentum and mass of that particle. Let the set $\{p_1, p_2, p_3, p_4, p_5\}$ be any permutation of the set $\{q_1, p, -q_2, -p', -k\}$. Then since the external particles are on shell we can construct ten variable scalar products, namely:

$$p_i \cdot p_j = p_j \cdot p_i \;;\; i \neq j \;;\quad i, j = 1, 2, 3, 4, 5. \tag{3.2-20}$$

Thus we have ten Mandelstam variables:

$$S_{ji} = S_{ij} \equiv (p_i + p_j)^2 \;,\; i \neq j \;, \tag{3.2-21}$$

and therefore expect that the crossing relation (3.2-17) should relate the M-function for our basic process to nine other processes. This indeed turns out to be the case, since we may define the following ten channels and corresponding total energy Mandelstam variables.

| | Channel | Mandelstam variable | | |
|---|---|---|---|---|
| 1) | $eN \longrightarrow eNM$ | $s$ | $\equiv$ | $(p + q_1)^2$ |
| 2) | $e\bar{e} \longrightarrow \bar{N}NM$ | $q^2$ | $\equiv$ | $(q_1 - q_2)^2$ |
| 3) | $e\bar{N} \longrightarrow e\bar{N}M$ | $U$ | $\equiv$ | $(q_1 - p')^2$ |
| 4) | $e\bar{M} \longrightarrow eN\bar{N}$ | $T$ | $\equiv$ | $(q_1 - k)^2$ |
| 5) | $\bar{e}N \longrightarrow \bar{e}NM$ | $u'$ | $\equiv$ | $(p - q_2)^2$ |
| 6) | $\bar{N}N \longrightarrow e\bar{e}M$ | $t$ | $\equiv$ | $(p - p')^2$ |

| | | | | |
|---|---|---|---|---|
| 7) | $\bar{M}N \longrightarrow eN\bar{e}$ | $u$ | $\equiv$ | $(p - k)^2$ |
| 8) | $\bar{e}\bar{N} \longrightarrow \bar{e}\bar{N}M$ | $t'$ | $\equiv$ | $(p' + q_2)^2$ |
| 9) | $\bar{N}\bar{M} \longrightarrow e\bar{e}\bar{N}$ | $s$ | $\equiv$ | $(p' + k)^2$ |
| 10) | $\bar{e}\bar{M} \longrightarrow \bar{e}N\bar{N}$ | $s'$ | $\equiv$ | $(q_2 + k)^2$. |

(3.2-22)

Channel (1) is our basic reaction, (2) to (7) are obtained by crossing a single pair of particles, and (8) to (10) by crossing two pairs of particles.

Only correctly chosen sets of five Mandelstam variables are linearly independent, since momentum conservation states:

$$p_1 + p_2 + p_3 + p_4 = - p_5 \tag{3.2-23}$$

This allows us to express each of the $S_{15}$, $S_{25}$, $S_{35}$, and $S_{45}$ in terms of the $S_{12}$, $S_{13}$, $S_{14}$, $S_{23}$, $S_{24}$, and $S_{34}$. But these latter six variables are related according to:

$$S_{12}+S_{13}+S_{14}+S_{23}+S_{24}+S_{34}=2(m_1^2+m_2^2+m_3^2+m_4^2)+m_5^2 , \tag{3.2-24}$$

so only any five of them are linearly independent. Equation 3.2-24 is obtained by taking the scalar product of each side of 3.2-23 with itself. On subtracting $p_4$ from both sides of 3.2-23 and again squaring the result we obtain a second useful constraint equation:

$$S_{12}+S_{23}+S_{31}=S_{45}+(m_1^2+m_2^2+m_3^2) . \tag{3.2-25}$$

In particular, from 3.2-24 we may derive:

$$\mathcal{S}+T+\mathcal{U}+s+t+u=4m^2+2\mu^2+3m_e^2 , \tag{3.2-26}$$

$$\mathcal{S}+q^2+u'+\mathcal{U}+t+t'=4m^2+\mu^2+4m_e^2 , \tag{3.2-27}$$

and:

$$\mathcal{S}+q^2+u'+T+u+s'=3m^2+2\mu^2+4m_e^2 , \tag{3.2-28}$$

whilst 3.2-25 yields amongst other things:

$$s+t+u=q^2+2m^2+\mu^2 \,, \tag{3.2-29}$$

and:

$$S+q^2+u'=s+m^2+2m_e^2 \,. \tag{3.2-30}$$

Thus for instance we may choose as our five linearly independent Mandelstam variables any one of the sets: $\{S,t,u,T,U\}$, $\{S,t,q^2,U,u'\}$ , and $\{S,u,q^2,T,u'\}$ . The discontinuity of a given amplitude in s when any two of t,u, $q^2$ are held constant is then related to the corresponding discontinuity in S by:

$$\mathrm{disc}_s\, A(s,t,u,T,U)=-\mathrm{disc}_S\, A(S,t,u,T,U) \,, \tag{3.2-31}$$

$$\mathrm{disc}_s\, A(s,t,q^2,U,u')=\mathrm{disc}_S\, A(S,t,q^2,U,u') \,, \tag{3.2-32}$$

$$\mathrm{disc}_s\, A(s,u,q^2,T,u')=\mathrm{disc}_S\, A(S,u,q^2,T,u') \,. \tag{3.2-33}$$

The discontinuities in t and u are similarly determined once one knows the corresponding ones in T and U .

The S, T, and U -channels of our overall process are shown graphically in figures 3.2-1a, b, and c. In figures 3.2-2a, b, and c we show the only possible graphs contributing to the corresponding $e^2T^{(2)}$ for each of these three channels.

Suppressing the Lorentz indices of the hadronic wave-functions, we have:

$$e^2M^{(2)}(q_2,p',k;q_1,p)=\frac{V_\alpha(q_2,q;q_1)}{q^2+i\varepsilon}M_\alpha(p',k;p,q) \,, \tag{3.2-34}$$

$$e^2M^{(2)}(q_2,p',-\bar{p};q_1,-\bar{k})=\frac{V_\alpha(q_2,q;q_1)}{q^2+i\varepsilon}M_\alpha(p',-\bar{p};-\bar{k},q) \,, \tag{3.2-35}$$

$$e^2M^{(2)}(q_2,-\bar{p},k;q_1,-\bar{p}')=\frac{V_\alpha(q_2,q;q_1)}{q^2+i\varepsilon}M_\alpha(-\bar{p},k;-\bar{p}',q) \,, \tag{3.2-36}$$

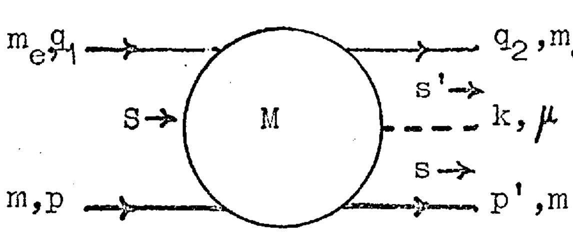


Figure 3.2-1a
S-channel of T

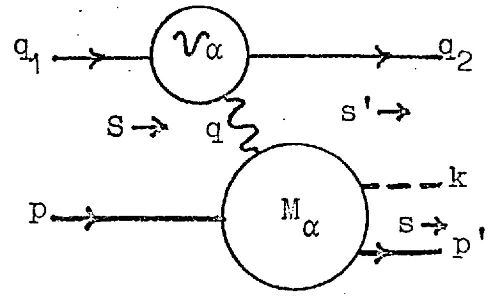


Figure 3.2-2a
S-channel of $e^2T^{(2)}$

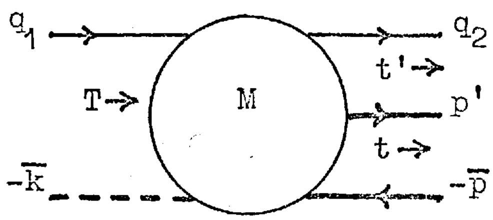


Figure 3.2-1b
T-channel of T

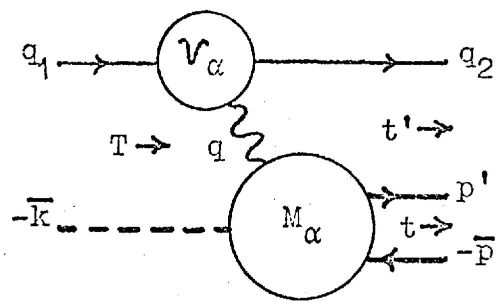


Figure 3.2-2b
T-channel of $e^2T^{(2)}$

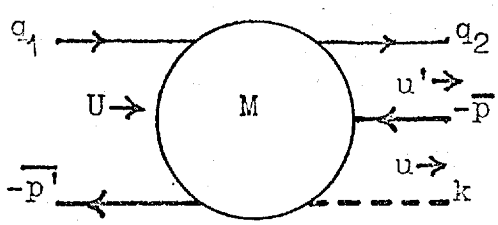


Figure 3.2-1c
U-channel of T

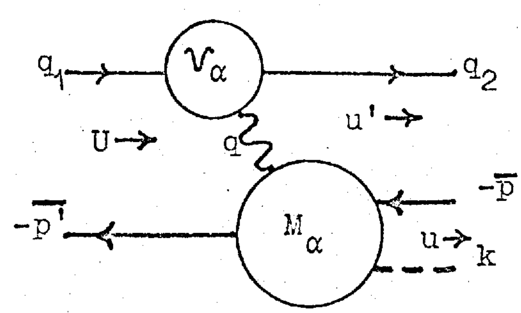


Figure 3.2-2c
U-channel of $e^2T^{(2)}$

where $v_\alpha$ and the $M_\alpha$ are the respective vertex functions for the leptonic and hadronic vertices. These equations are subject to the spinor ordering convention that in each channel $v_\alpha$ is to be sandwiched between the lepton spinors and $M_\alpha$ between the baryon spinors.

The vertex functions $v_\alpha$ and $M_\alpha$ can only depend respectively on $q^2$ and on any three of s,t,u, and $q^2$, so we note that the $M^{(2)}$ depend on only three linearly independent Mandelstam variables. The dependence of the overall M-function on a further pair of variables only arises from terms of higher order in e.

Equations 3.2-18, 19, 34, 35, and 36 together imply that:

$$M_\alpha(p',k;p,q) = M_\alpha(p',-\bar{p};-\bar{k},q) = -M_\alpha(-\bar{p},k;-\bar{p}',q) \ . \qquad (3.2\text{-}37)$$

Thus the $M_\alpha$ satisfy the same crossing rule as would obtain were the virtual photon propogator replaced by a real photon wave-function. We may therefore perform a channel independent spin decomposition:

$$M_\alpha = \sum_j A_j(s,t,q^2)\tilde{\mathcal{K}}^j_\alpha \ , \qquad (3.2\text{-}38)$$

as detailed in section 2.6, and conclude that the S,T,U channels of $e^2M^{(2)}$ are given by taking the s,t,u physical sectors of the $A_j$.

Let us now specialise to the case in which we are interested where both the baryons are nucleons.

The Born-term poles of the three $e^2M^{(2)}$ are shown in figures 3.2-3a,b, and c. These endow the $A_j$ with poles at:

$$s = m^2 \ , \quad t = \mu^2 \ , \quad u = m^2 \ , \qquad (3.2\text{-}39)$$

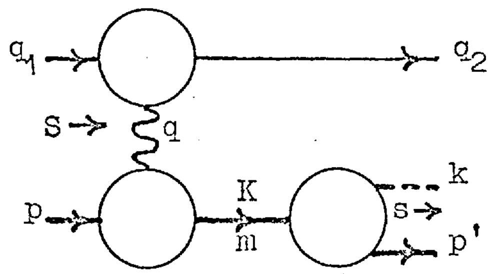


Figure 3.2-3a
Born-term pole at
$s=m^2$ in $e^2T^{(2)}$

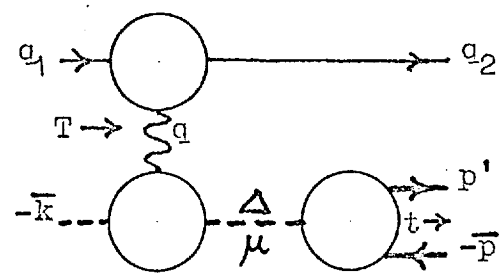


Figure 3.2-3b
Born-term pole at
$t=\mu^2$ in $e^2T^{(2)}$

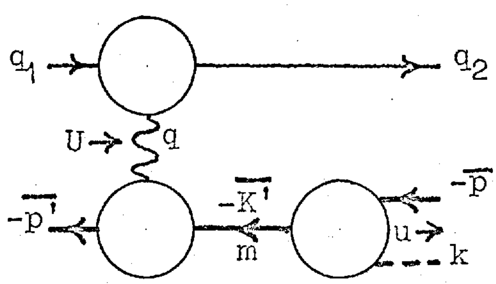


Figure 3.2-3c
Born-term pole at
$u=m^2$ in $e^2T^{(2)}$

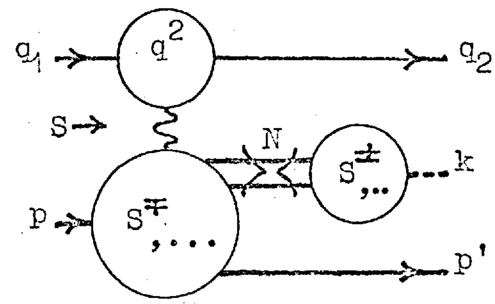


Figure 3.2-4
A disallowed contribution
to $e^2T^{(o)\pm}_{fN}(s)T^{(2)\mp}_{Ni}(s)$.

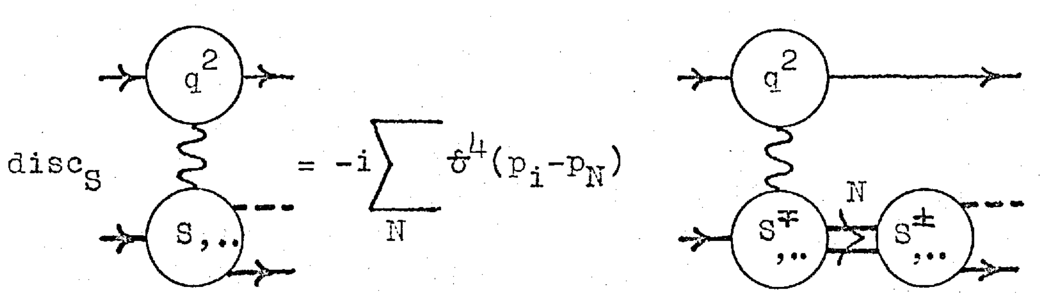


Figure 3.2-5: Graphical realisation of equation 3.2-40.

as would have been the case had the photon been real.

Turning finally to the unitarity relations, we have on applying 3.2-16 in the S-channel:

$$\mathrm{disc}_{S}\, T^{(2)}_{fi}(S) = -i\sum_{N}\delta^4(p_i - p_N)\Big[T^{(0)\pm}_{fN}(S)\,T^{(2)\mp}_{Ni}(S)$$

$$+T^{(1)\pm}_{fN}(S)\,T^{(1)\mp}_{Ni}(S) + T^{(2)\pm}_{fN}(S)\,T^{(0)\mp}_{Ni}(S)\Big]. \qquad (3.2\text{-}40)$$

In partially computing the right-hand side of this equation we have to observe the following constraints. Firstly, $T^{(2)}_{fi}$ vanishes by definition unless $q_1$ and $q_2$ are unequal. Secondly, the "intermediate states" cannot involve any virtual particles. Thirdly, since we are assuming the absence of weak interactions, hadrons and leptons can only be coupled via virtual photon propogators. Finally, we must everywhere satisfy the momentum-mass inequality:

$$\Big(\sum_j p_j\Big)^2 \begin{cases} = \Big(\sum_j m_j\Big)^2, & \text{all } \underline{p}_j \text{ equal,} \\ > \Big(\sum_j m_j\Big)^2, & \text{otherwise.} \end{cases} \qquad (3.2\text{-}41)$$

These constraints serve to limit the right-hand side of 3.2-40 to graphs with the structure shown in figure 3.2-5, all of which arise from the term $T^{(0)\pm}_{fN}(S)\,T^{(2)\mp}_{Ni}(S)$ . In this figure the symbol: $\Rightarrow$ denotes a multiparticle "intermediate state" with baryon number one and lepton number zero.

Denoting by: $\Rightarrow\!\Leftarrow$ , a similar intermediate state with baryon number zero, we show in figure 3.2-4 another graph apparently having the general structure: $T^{(0)\pm}_{fN}(S)\,T^{(2)\mp}_{Ni}(S)$. If our final meson is a pion 3.2-41 implies that the state $\Rightarrow\!\Leftarrow$ is also a pion in which case $T^{(0)\pm}_{fN}(S)$ involves no interaction and vanishes. In cases where the final meson is a resonance, $\Rightarrow\!\Leftarrow$ could certainly be any multiparticle state into which it is observed to decay and 3.2-41 would then be satisfied. However, if the resonance decays electromagnetically into this state the right-hand vertex must implicitly involve a virtual photon propogator. The graph then contirbutes to $T^{(2)\pm}_{fN}(s)T^{(2)\mp}_{Ni}(s)$, not to $T^{(0)\pm}_{fN}(s)T^{(2)\mp}_{Ni}(s)$. If the resonance is observed to possess any strong decay modes it is doubtful if we are justified in treating it as an external particle. We can only do so if we neglect the existence of its strong decay products. The state $\Rightarrow$ can then only be the final resonance treated as a stable particle and the graph is eliminated as in the pion case.

After factoring out the external wave-functions, the leptonic vertex function, and the virtual photon propogator, the equation represented graphically by figure 3.2-5 reads:

$$\mathrm{disc}_s M_\alpha(p+q \to p'+k; s,t,q^2) = -i \lim_{\epsilon\to 0^+} \sum_N \delta^4(p_i - p_N)\times$$

$$\times M_\sigma(p_N \to p'+k; s\pm i\epsilon, t, q^2)\mathcal{P}_{\sigma\tau}(N) M_{\tau\alpha}(p+q \to p_N; s\mp i\epsilon, t, q^2)\,, \tag{3.2-42}$$

where $\mathcal{P}_{\sigma\tau}(N)$ denotes the set of propogator numerators for the particles comprising the Nth. state $\Rightarrow$. Adopting the spin decomposition:

$$M_\alpha(p+q \to p'+k; s,t,q^2) \equiv \sum_j A_j(s,t,q^2)\tilde{\mathcal{K}}^j_\alpha\,, \tag{3.2-43}$$

we may similarly write:

$$\lim_{\epsilon\to 0^+} M_\sigma(p_N \to p'+k; s\pm i\epsilon, t, q^2)\mathcal{P}_{\sigma\tau}(N) M_{\tau\alpha}(p+q \to p_N;$$

$$s\mp i\epsilon, t, q^2) \equiv \sum_j A^N_j(s^\pm, t, q^2)\tilde{\mathcal{K}}^j_\alpha\,. \tag{3.2-44}$$

In view of 3.2-32, equations 3.2-42, 43, and 44 together yield:

$$\mathrm{disc}_s A_j(s,t,q^2) = -i\sum_N \delta^4(p_i - p_N) A^N_j(s^\pm, t, q^2)\,. \tag{3.2-45}$$

This is precisely the unitarity relation which would have been obtained had the photon been real, except that it now applies in addition to those $A_j$ whose corresponding $\tilde{\mathcal{K}}^j_\alpha$ vanish at zero $q^2$. Similar considerations apply to the discontinuities of the $A_j$ in t and u determined by application of 3.2-16 to $T_{fi}^{(2)}$ in the T and U channels.

The discontinuity condition, guaranteeing the hermitian (rather than anti-hermitian) analyticity of $T_{fi}^{(2)}$ for T-invarient processes, continues to be given by 2.41-23 which reads in this context:

$$v_\alpha \tilde{\mathcal{K}}^j_{\mu\alpha} = \eta_T g(\mu) g(\alpha) T^{-1} \gamma_0 \hat{v}_\alpha \hat{\tilde{\mathcal{K}}}^{*j}_{\mu\alpha} \gamma_0 T, \qquad (3.2\text{-}46)$$

where $\mu$ denotes the Lorentz indices of the meson wave-function. For the obvious choices:

$$v_\alpha = \gamma_\alpha \quad , \qquad (3.2\text{-}47)$$

and:

$$v_\alpha = \left[\gamma_\alpha F_1(q^2) + \frac{i}{2m_e} \sigma_{\alpha\beta} q_\beta F_2(q^2)\right] , \qquad (3.2\text{-}48)$$

this reduces to:

$$\tilde{\mathcal{K}}^j_{\mu\alpha} = \eta_T g(\mu) g(\alpha) T^{-1} \gamma_0 \hat{\tilde{\mathcal{K}}}^{*j}_{\mu\alpha} \gamma_0 T , \qquad (3.2\text{-}49)$$

where $\eta_T$ is now the product of the time-reversal phases of the virtual photon and the three hadrons. Again this is the same equation as that obtaining in the corresponding real photon case.

To summarise, the results of this section coupled with the Reggeisation assumption of the previous one indicate that we may derive and saturate superconvergent sum-rules for our virtual photonic four-point functions by utilising exactly the same techniques as would be employed were the photon real.

# CHAPTER 4

## DERIVATION OF SUPERCONVERGENT SUM RULES FOR THE PHOTO-AND ELECTROPRODUCTION OF NON-STRANGE MESONS OFF NUCLEONS.

In this chapter we utilise the formalism and results of Chapters 2 and 3 to derive superconvergent sum-rules for the four-point function corresponding to the process: real or virtual photon + nucleon $\rightarrow$ non-strange meson + nucleon. We consider the eight cases in which the meson has all possible combinations of the quantum numbers: $(J^P;\ I;\ C_n) = (0^-$ or $1^-;\ 0$ or $1;\ +1$ or $-1)$. (4-1)

The kinematical definitions and relations we shall use throughout this and the final chapter are listed in Appendix 5.

Throughout the remainder of this thesis we shall often be dealing simultaneously with both three-point and four-point vertex functions. For the sake of clarity we shall henceforth use the term "M-function" when referring to these latter, and will speak of decomposing them into "invarient amplitudes". The use of the terms "vertex-function" and "form-factor" will be restricted to three-point functions.

### 4.1 SPIN DECOMPOSITIONS.

### 4.11 SPIN DECOMPOSITION FOR THE PRODUCTION OF PSEUDOSCALAR MESONS.

The vertex is abnormal overall, and we write the M-function as $M_\alpha$ where $\alpha$ is the Lorentz index of the real or virtual photon wave-function.

We have:

$$N^-(1^- + \tfrac{1}{2} \rightarrow \tfrac{1}{2} + 0) = 6 \quad , \quad N^-(0^+ + \tfrac{1}{2} \rightarrow \tfrac{1}{2} + 0) = 2 \quad , \qquad (4.11\text{-}1)$$

so:

$$N^-(\gamma^V + \tfrac{1}{2} \rightarrow \tfrac{1}{2} + 0) = 6 \quad , \quad N^-(\gamma^R + \tfrac{1}{2} \rightarrow \tfrac{1}{2} + 0) = 4 \ . \qquad (4.11\text{-}2)$$

We therefore require a K.S.F. spin decomposition:

$$M_\alpha = \sum_{i=1}^{6} A_i(\nu, t, q^2)\tilde{\mathcal{K}}^i_\alpha \;, \qquad (4.11\text{-}3)$$

in which the $\tilde{\mathcal{K}}^i_\alpha$ vanish on contraction with $q_\alpha$, and just $\tilde{\mathcal{K}}^5_\alpha$ and $\tilde{\mathcal{K}}^6_\alpha$ are proportional to $q^2$. The vertex is $s \leftrightarrow u$ crossing symmetric and the $\tilde{\mathcal{K}}^i_\alpha$ are therefore required to satisfy 2.42-14. This will ensure that each $A_i$ is an even or odd function of $\nu$.

Since we are going to work with the gauge projection operator $\mathcal{G}_{\alpha'\alpha}(q)$, we require as a starting point a suitable set of covarients corresponding to the coupling function: $\mathcal{C}^-_\alpha(1^- + \frac{1}{2} \to \frac{1}{2} + 0)$. No type B equivalence theorems are involved, and a suitable choice of initial covarients is furnished[10] in the set: $([\gamma_\alpha, \not{q}], P_\alpha, \Delta_\alpha, \gamma_\alpha, P_\alpha\not{q}, \Delta_\alpha\not{q})\gamma_5$.

Several other possible choices are available, and the reasons for prefering this particular one are as follows. To exploit $s \leftrightarrow u$ crossing symmetry we require that the initial covarients be even or odd under the substitutions:

$$-p \leftrightarrow p', \quad q \leftrightarrow q, \quad k \leftrightarrow k. \qquad (4.11\text{-}4)$$

This dictates that we choose as our linearly inequivalent "indexed" momenta: $P_\alpha$ and either $\Delta_\alpha$ or $Q_\alpha$, (these latter two being respectively equivalent to $-k_\alpha$ and $+\frac{1}{2}k_\alpha$). We choose $P_\alpha$ and $\Delta_\alpha$ since $\Delta$ will be the momentum of the t-channel Reggeons. We choose $\not{q}$ to be our single "slashed" momentum in order to exploit the useful equivalence relation:

$$\{\gamma_\alpha, \not{q}\} \cong 0 \,. \qquad (4.11\text{-}5)$$

The covarient: $[\gamma_\alpha, \not{q}]\gamma_5$ already vanishes on contraction with $q_\alpha$, and from the remaining five covarients we obtain:

$$P'_\alpha(q) = P_\alpha - \nu q_\alpha/q^2 \;, \quad \Delta'_\alpha(q) = \Delta_\alpha - \Delta\cdot q\, q_\alpha/q^2 \;,$$

$$\gamma'_\alpha(q) = \gamma_\alpha - \not{q} q_\alpha/q^2 \;, \quad P'_\alpha(q)\not{q} = P_\alpha\not{q} - \nu\not{q} q_\alpha/q^2 \;,$$

and:

$$\Delta'_\alpha(q)\not{q} = \Delta_\alpha\not{q} - \Delta\cdot q\,\not{q} q_\alpha/q^2 \,. \qquad (4.11\text{-}6 \text{ to } 10)$$

Two singular-tails: $q_\alpha/q^2$ and $\not{q}\, q_\alpha/q^2$ are involved in agreement with the second of equations 4.11-1. Elimination of the second of these need not introduce any singularities, but elimination of the first will necessarily introduce electrodynamical poles into two of the amplitudes at the vanishing of either $\nu$ or $\Delta\cdot q$. We shall choose the following elimination:

$$\tilde{\mathcal{K}}^1_\alpha \equiv [\gamma_\alpha, \not{q}]\gamma_5 \,,$$
$$\tilde{\mathcal{K}}^2_\alpha \equiv [\Delta\cdot q\, P'_\alpha(q) - \nu\Delta'_\alpha(q)]\gamma_5 = (\Delta\cdot q\, P_\alpha - \nu\Delta_\alpha)\gamma_5 \,,$$
$$\tilde{\mathcal{K}}^3_\alpha \equiv [\Delta'_\alpha(q)\not{q} - \Delta\cdot q\,\gamma'_\alpha(q)]\gamma_5 = (\Delta_\alpha\not{q} - \Delta\cdot q\,\gamma_\alpha)\gamma_5 \,,$$
$$\tilde{\mathcal{K}}^4_\alpha \equiv [P'_\alpha(q)\not{q} - \nu\gamma'_\alpha(q)]\gamma_5 = (P_\alpha\not{q} - \nu\gamma_\alpha)\gamma_5 \,,$$
$$\tilde{\mathcal{K}}^5_\alpha \equiv q^2\Delta'_\alpha(q)\gamma_5 = (q^2\Delta_\alpha - \Delta\cdot q\, q_\alpha)\gamma_5 \,,$$
$$\tilde{\mathcal{K}}^6_\alpha \equiv q^2\gamma'_\alpha(q)\gamma_5 = (q^2\gamma_\alpha - \not{q}\, q_\alpha)\gamma_5 \,. \qquad (4.11\text{-}11 \text{ to } 16)$$

The covarients $\tilde{\mathcal{K}}^5_\alpha$ and $\tilde{\mathcal{K}}^6_\alpha$ are equivalent to zero at vanishing $q^2$ as required, and the amplitudes $A_2$ and $A_5$ have electrodynamical poles at vanishing $\Delta\cdot q$.

Equivalent sets of covarients for this vertex have already been given by a variety of authors(40) using slightly different techniques.

The spin decomposition for scalar meson production is given simply by dropping the $\gamma_5$'s, but this we shall not require.

## 4.12 SPIN DECOMPOSITION FOR THE PRODUCTION OF VECTOR MESONS.

This vertex is normal overall and we denote the M-function by $M_{\mu\alpha}$ where $\alpha$ again refers to the photon and $\mu$ is the Lorentz index of the vector meson wave-function.

In this case we have:

$$N^+(1^- + \tfrac{1}{2} \to \tfrac{1}{2} + 1) = 18 \quad , \quad N^+(0^+ + \tfrac{1}{2} \to \tfrac{1}{2} + 1) = 6 \quad , \qquad (4.12\text{-}1)$$

so:

$$N^+(\gamma^V+\tfrac{1}{2}\to\tfrac{1}{2}+1)=18\quad,\quad N^+(\gamma^R+\tfrac{1}{2}\to\tfrac{1}{2}+1)=12\quad,\qquad(4.12\text{-}2)$$

and we write:

$$M_{\mu\alpha}\equiv\sum_{i=1}^{18}A_i(\nu,t,q^2)\tilde{\mathcal{K}}^i_{\mu\alpha}\ .\qquad(4.12\text{-}3)$$

The $\tilde{\mathcal{K}}^i_{\mu\alpha}$ are again to vanish on contraction with $q_\alpha$. and just $\tilde{\mathcal{K}}^{13}_{\mu\alpha}$ to $\tilde{\mathcal{K}}^{18}_{\mu\alpha}$ are to be proportional to $q^2$. In order to exploit $s\leftrightarrow u$ crossing symmetry we again require that the $\tilde{\mathcal{K}}^i_{\mu\alpha}$ satisfy equation 2.42-14.

We are going to determine the $\tilde{\mathcal{K}}^i_{\mu\alpha}$ by operation with $\mathcal{G}_{\alpha'\alpha}(q)$ on a suitable set of covarients for the coupling function: $\mathcal{C}^+_{\mu\alpha}(1^- + \tfrac{1}{2}\to\tfrac{1}{2} + 1)$. Two inequivalent type B E.R.'s are involved. To see this we notice that the infinity of possible $\tilde{\mathcal{K}}_{\mu\alpha}$ fall into three classes. Firstly we have the infinity of "factorised" covarients: $\mathcal{K}_\mu\tilde{\mathcal{K}}_\alpha$, where $\mathcal{K}_\mu$ and $\tilde{\mathcal{K}}_\alpha$ are any of the infinity of covarients suitable for the spin decomposition of the functions $\mathcal{C}^+_\mu(0 + \tfrac{1}{2}\to\tfrac{1}{2} + 1)$ and $\mathcal{V}^+_\alpha(\gamma + \tfrac{1}{2}\to\tfrac{1}{2} + 0)$ respectively. Using only type A E.R.'s these can all be expressed in terms of eighteen "obvious" covarients suitable for decomposition of $\mathcal{V}^+_{\mu\alpha}(\gamma + \tfrac{1}{2}\to\tfrac{1}{2} + 1)$, for example the $(P,q,\gamma)_\mu\tilde{\mathcal{K}}^{1,\dots,6}_\alpha$ where the $\tilde{\mathcal{K}}^{1,\dots,6}_\alpha\gamma_5$ are the final covarients of the previous section. Secondly we have the infinity of $(q_\mu\mathcal{K}_\alpha - \mathcal{K}\cdot q\, g_{\mu\alpha})$, where $\mathcal{K}_\alpha$ is any covarient suitable for the decomposition of $\mathcal{C}^+_\alpha(1^- + \tfrac{1}{2}\to\tfrac{1}{2} + 0)$. These are related by type A E.R.'s so that only any six are linearly independent. For example one has the six covarients obtained by choosing for the $\mathcal{K}_\alpha$ the set: $(P,\Delta,\gamma)_\alpha\,(\mathbb{1}_4,\not{q})$. As in the previous section there are then just two linearly independent $\mathcal{K}\cdot q$, one involving an overall unit matrix and the other an overall $\not{q}$. Thus by taking linear combinations of these six covarients we may eliminate the metric tensor from all but two of them. The four which are free of metric tensors may then be expressed in terms of the $(P,q,\gamma)_\mu \tilde{\mathcal{K}}_\alpha^{1,\dots,6}$ using only type A E.R.'s. We then have twenty covarients, the previous eighteen and for example: $(q_\mu \Delta_\alpha - \Delta\cdot q\, g_{\mu\alpha})$ and $(q_\mu \gamma_\alpha - \not{q}\, g_{\mu\alpha})$. Thirdly we have an infinity of covarients involving terms of the type: $\varepsilon_{\mu\alpha}(\cdot\ \cdot)\gamma_5$. These are related to the previous twenty covarients by means of equivalence relations derived from the equations of Appendix 3. All such E.R.'s are of type A, and the "Levi-Cevita" covarients may always be eliminated in favour of the former twenty without the introduction of kinematic singularities. Finally, the two covarients involving $g_{\mu\alpha}\mathbb{1}_4$ and $g_{\mu\alpha}\not{q}$ can only be related to the $(P,q,\gamma)_\mu \tilde{\mathcal{K}}_\alpha^{1,\dots,6}$ through a pair of inequivalent type B E.R.'s derived from equations A7-6 and 7.

We are required, then, to take as our starting point twenty covarients suitable for the decomposition of $\mathcal{C}^+_{\mu\alpha}(1^- + \frac{1}{2} \to \frac{1}{2} + 1)$ and related only through the above pair of equations. An obvious choice is the set: $[(P,\Delta,\gamma)_\mu (P,\Delta,\gamma)_\alpha,\ g_{\mu\alpha}](\mathbb{1}_4, \not{q})$, but we shall choose:

$$
\begin{aligned}
\mathcal{K}^1 &\equiv PP & \mathcal{K}^{11} &\equiv PP\not{q}\\
\mathcal{K}^2 &\equiv qP & \mathcal{K}^{12} &\equiv qP\not{q}\\
\mathcal{K}^3 &\equiv P\Delta & \mathcal{K}^{13} &\equiv P\Delta\not{q}\\
\mathcal{K}^4 &\equiv q\Delta & \mathcal{K}^{14} &\equiv q\Delta\not{q}\\
\mathcal{K}^5 &\equiv P\gamma & \mathcal{K}^{15} &\equiv P[\gamma,\not{q}]\\
\mathcal{K}^6 &\equiv \gamma P & \mathcal{K}^{16} &\equiv [\gamma,\not{q}]P\\
\mathcal{K}^7 &\equiv q\gamma & \mathcal{K}^{17} &\equiv q[\gamma,\not{q}]\\
\mathcal{K}^8 &\equiv \gamma\Delta & \mathcal{K}^{18} &\equiv [\gamma,\not{q}]\Delta\\
\mathcal{K}^9 &\equiv g & \mathcal{K}^{19} &\equiv g\not{q}\\
\mathcal{K}^{10} &\equiv [\gamma,\gamma] & \mathcal{K}^{20} &\equiv [\gamma\not{q}\gamma],
\end{aligned}
$$

(4.12-4 to 23)

where we have adopted the shorthand notation of appendix 7 with in addition:

$$\mathcal{K} \equiv \mathcal{K}_{\mu\alpha} \quad , \qquad g \equiv g_{\mu\alpha} \quad . \qquad (4.12\text{-}24,25)$$

We have chosen to use $[\gamma,\gamma]$ and $[\gamma \not{q} \gamma]$ rather than $\gamma\gamma$ and $\gamma\gamma\not{q}$ since only the former two covarients satisfy the $s \leftrightarrow u$ crossing relation 2.42-14. The choices $q_\mu$ and $[\gamma,\not{q}]_\mu$ rather than $\Delta_\mu$ and $\gamma_\mu \not{q}$ are then dictated by the requirement that for simplicity the gauge projection operation should lead to just six linearly inequivalent singular tails. As in the previous section we choose $[\gamma,\not{q}]_\alpha$ rather than $\gamma_\alpha \not{q}$ since the former already vanishes on contraction with $q_\alpha$.

The gauge projection operation yields:

$$
\begin{array}{ll}
\mathcal{K}'^{1}(q) = PP - \nu S^2 & \mathcal{K}'^{11}(q) = PP\not{q} - \nu S^5 \\
\mathcal{K}'^{2}(q) = qP - \nu S^3 & \mathcal{K}'^{12}(q) = qP\not{q} - \nu S^6 \\
\mathcal{K}'^{3}(q) = P\Delta - \Delta\cdot q\, S^2 & \mathcal{K}'^{13}(q) = P\Delta\not{q} - \Delta\cdot q\, S^5 \\
\mathcal{K}'^{4}(q) = q\Delta - \Delta\cdot q\, S^3 & \mathcal{K}'^{14}(q) = q\Delta\not{q} - \Delta\cdot q\, S^6 \\
\mathcal{K}'^{5}(q) = P\gamma - S^5 & \mathcal{K}'^{15}(q) = P[\gamma,\not{q}] \\
\mathcal{K}'^{6}(q) = \gamma P - \nu S^4 & \mathcal{K}'^{16}(q) = [\gamma,\not{q}]P - \nu S^1 \\
\mathcal{K}'^{7}(q) = q\gamma - S^6 & \mathcal{K}'^{17}(q) = q[\gamma,\not{q}] \\
\mathcal{K}'^{8}(q) = \gamma\Delta - \Delta\cdot q\, S^4 & \mathcal{K}'^{18}(q) = [\gamma,\not{q}]\Delta - \Delta\cdot q\, S^1 \\
\mathcal{K}'^{9}(q) = g - S^3 & \mathcal{K}'^{19}(q) = g\not{q} - S^6 \\
\mathcal{K}'^{10}(q) = [\gamma,\gamma] - S^1 & \mathcal{K}'^{20}(q) = [\gamma\not{q}\gamma] \quad ,
\end{array}
\qquad (4.12\text{-}26 \text{ to } 45)
$$

where the singular tails are given by:

$$
\begin{array}{ll}
S^1 = [\gamma,\not{q}]q/q^2 & S^4 = \gamma q/q^2 \\
S^2 = Pq/q^2 & S^5 = Pq\not{q}/q^2 \\
S^3 = qq/q^2 & S^6 = qq\not{q}/q^2 \quad .
\end{array}
$$

(4.12-46 to 51)

Eliminating these in the usual manner so as to introduce the least number of electrodynamical poles, we obtain:

$$\tilde{\mathcal{K}}^1 \equiv \Delta\cdot q\,\mathcal{K}'^1 - \nu\mathcal{K}'^3 = \Delta\cdot q\,PP - \nu P\Delta$$
$$\tilde{\mathcal{K}}^2 \equiv \mathcal{K}'^2 - \nu\mathcal{K}'^9 = qP - \nu g$$
$$\tilde{\mathcal{K}}^3 \equiv \mathcal{K}'^4 - \Delta\cdot q\,\mathcal{K}'^9 = q\Delta - \Delta\cdot q\, g$$
$$\tilde{\mathcal{K}}^4 \equiv \mathcal{K}'^{16} - \nu\mathcal{K}'^{10} = [\gamma,\phi]P - \nu[\gamma,\gamma]$$
$$\tilde{\mathcal{K}}^5 \equiv \mathcal{K}'^{11} - \nu\mathcal{K}'^5 = PP\phi - \nu P\gamma$$
$$\tilde{\mathcal{K}}^6 \equiv \mathcal{K}'^{13} - \Delta\cdot q\,\mathcal{K}'^5 = P\Delta\phi - \Delta\cdot q\,P\gamma$$
$$\tilde{\mathcal{K}}^7 \equiv \mathcal{K}'^7 - \mathcal{K}'^{19} = q\gamma - g\phi$$
$$\tilde{\mathcal{K}}^8 \equiv \Delta\cdot q\,\mathcal{K}'^6 - \nu\mathcal{K}'^8 = \Delta\cdot q\,\gamma P - \nu\gamma\Delta$$
$$\tilde{\mathcal{K}}^9 \equiv \mathcal{K}'^{18} - \Delta\cdot q\,\mathcal{K}'^{10} = [\gamma,\phi]\Delta - \Delta\cdot q\,[\gamma,\gamma]$$
$$\tilde{\mathcal{K}}^{10} \equiv \mathcal{K}'^{15} = P[\gamma,\phi]$$
$$\tilde{\mathcal{K}}^{11} \equiv \mathcal{K}'^{17} = q[\gamma,\phi]$$
$$\tilde{\mathcal{K}}^{12} \equiv \mathcal{K}'^{20} = [\gamma\phi\gamma]$$
$$\tilde{\mathcal{K}}^{13} \equiv q^2\mathcal{K}'^3 = q^2 P\Delta - \Delta\cdot q\,Pq$$
$$\tilde{\mathcal{K}}^{14} \equiv q^2\mathcal{K}'^9 = q^2 g - qq$$
$$\tilde{\mathcal{K}}^{15} \equiv q^2\mathcal{K}'^8 = q^2\gamma\Delta - \Delta\cdot q\,\gamma q$$
$$\tilde{\mathcal{K}}^{16} \equiv q^2\mathcal{K}'^5 = q^2 P\gamma - Pq\phi$$
$$\tilde{\mathcal{K}}^{17} \equiv q^2\mathcal{K}'^{19} = q^2 g\phi - qq\phi$$
$$\tilde{\mathcal{K}}^{18} \equiv q^2\mathcal{K}'^{10} = q^2[\gamma,\gamma] - [\gamma,\phi]q$$
$$\tilde{\mathcal{K}}^{19} \equiv \mathcal{K}'^{12} - \nu\mathcal{K}'^{19} = qP\phi - \nu g\phi$$
$$\tilde{\mathcal{K}}^{20} \equiv \mathcal{K}'^{14} - \Delta\cdot q\,\mathcal{K}'^{19} = q\Delta\phi - \Delta\cdot q\, g\phi \,.$$

(4.12-52 to 71)

In terms of the covarients of equations 4.12-4 to 23 equation A7-6 reads: $\Delta\cdot q\,\mathcal{K}^5-\Delta\cdot q\,\mathcal{K}^6-\nu\mathcal{K}^7+\nu\mathcal{K}^8$

$$+m\nu\mathcal{K}^{10}+\mathcal{K}^{12}-\mathcal{K}^{13}+m\mathcal{K}^{15}-m\mathcal{K}^{16}+\left(m^2-\frac{t}{4}\right)\mathcal{K}^{20}\cong 0, \quad (4.12\text{-}72)$$

whilst A7-7 states:

$$(q^2-\Delta\cdot q)\mathcal{K}^2-q^2\mathcal{K}^3+\nu\mathcal{K}^4+m(\Delta\cdot q-q^2)\mathcal{K}^7+mq^2\mathcal{K}^8$$

$$+\frac{1}{4}\left[4\nu^2-(\Delta\cdot q)^2+q^2t\right]\mathcal{K}^{10}-m\mathcal{K}^{14}+\nu\mathcal{K}^{15}-\nu\mathcal{K}^{16}$$

$$+\frac{1}{4}(t-\Delta\cdot q)\mathcal{K}^{17}+\frac{1}{4}\Delta\cdot q\,\mathcal{K}^{18}+m\nu\mathcal{K}^{20}\cong 0\ . \quad (4.12\text{-}73)$$

Operating on these two equations with $\mathcal{G}_{\alpha'\alpha}(q)$ yields a pair of equations which we will denote by 4.12-72' and 4.12-73' in which each $\mathcal{K}^i$ is replaced by $\mathcal{K}'^i(q)$ . As a check we verify that equations [(4.12-72')-(4.12-72)] and [(4.12-73')-(4.12-73)] are indeed satisfied in the sense that they each reduce to the trivial result: zero equivalent to zero. Inverting equations 4.12-52 to 71 and substituting the results into 4.12-72' and 4.12-73', we obtain from 4.12-72':

$$m\tilde{\mathcal{K}}^4+\tilde{\mathcal{K}}^6+\nu\tilde{\mathcal{K}}^7+\tilde{\mathcal{K}}^8-m\tilde{\mathcal{K}}^{10}+\left(\frac{t}{4}-m^2\right)\tilde{\mathcal{K}}^{12}-\tilde{\mathcal{K}}^{19}\cong 0\ , \quad (4.12\text{-}74)$$

whilst 4.12-73' yields: $(\Delta\cdot q-q^2)\tilde{\mathcal{K}}^2-\nu\tilde{\mathcal{K}}^3+\nu\tilde{\mathcal{K}}^4$

$$+m(q^2-\Delta\cdot q)\tilde{\mathcal{K}}^7-\frac{1}{4}\Delta\cdot q\,\tilde{\mathcal{K}}^9-\nu\tilde{\mathcal{K}}^{10}+\frac{1}{4}(\Delta\cdot q-t)\tilde{\mathcal{K}}^{11}-m\nu\tilde{\mathcal{K}}^{12}$$

$$+\tilde{\mathcal{K}}^{13}-\nu\tilde{\mathcal{K}}^{14}-m\tilde{\mathcal{K}}^{15}+m\tilde{\mathcal{K}}^{17}-\frac{t}{4}\tilde{\mathcal{K}}^{18}+m\tilde{\mathcal{K}}^{20}\cong 0\ . \quad (4.12\text{-}75)$$

Thus without introducing kinematic singularities into the amplitudes we can eliminate any one of $\tilde{\mathcal{K}}^{4,6,8,10,19}$, using 4.12-74, and any one of $\tilde{\mathcal{K}}^{13,15,17,20}$ by means of 4.12-75. We do not wish to eliminate any of the six covarients proportional to $q^2$, nor any of those which by virtue of the tail elimination procedure correspond to amplitudes necessarily endowed with electrodynamical poles. Such a latter step would introduce these poles into more final amplitudes than the minimum number required to have them. The covarients $\tilde{\mathcal{K}}^{13,14,15,16,17,18}$ vanish at zero $q^2$, and $\tilde{\mathcal{K}}^{1,8,13,15}$ correspond to amplitudes having electrodynamical poles at vanishing $\Delta \cdot q$.

We therefore choose to eliminate $\tilde{\mathcal{K}}^{19}$ and $\tilde{\mathcal{K}}^{20}$. Our final spin decomposition is then given by equation 4.12-3 with the eighteen $\tilde{\mathcal{K}}^{i}_{\mu\alpha}$ defined by equations 4.12-52 to 69. The amplitudes $A_{1,8,13,15}$ are subject to the above mentioned poles.

Scadron and Jones(13) have also obtained twelve covarients for the process: real photon + nucleon $\rightarrow$ nucleon + vector meson. They use a similar technique but apply their gauge projection operator to twelve covarients for the elastic (!) reaction: vector meson + nucleon $\rightarrow$ nucleon + vector meson. This method seems to us to be rather hard to justify, and we prefer our own approach.

## 4.13 S↔U CROSSING SYMMETRY OF THE SPIN DECOMPOSITIONS.

For the covarients of the previous two sections, equation 2.42-14 reduces to:

$$\tilde{\mathcal{K}}^{i}_{\mu\alpha}(P,\Delta,q,\nu,t,q^2,\{\gamma\}\gamma_5)$$

$$= \xi_i \tilde{\mathcal{K}}^{i}_{\mu\alpha}(-P,\Delta,q,-\nu,t,q^2,\gamma_5\{-\gamma\}^{rev}) \quad (4.13\text{-}1)$$

where $\{\gamma\}$ denotes any product of " $\gamma_\nu$'s ", and $\{-\gamma\}^{rev}$ denotes the same product with each $\gamma_\nu$ multiplied by minus unity, and the order of $\gamma_\nu$'s reversed. For the covarients of section 4.11 we then have:

$$\xi_i = \begin{cases} +1, & i = 3,5,6, \\ -1, & i = 1,2,4, \end{cases} \quad (4.13\text{-}2)$$

whilst for the covarients of section 4.12:

$$\xi_i = \begin{cases} +1 \,, & i = 1,3,4,6,8,10,12,14,16, \\ -1 \,, & i = 2,5,7,9,11,13,15,17,18. \end{cases} \qquad (4.13\text{-}3)$$

## 4.2 ISOSPIN DECOMPOSITIONS AND ALLIED TOPICS

Following the methods outlined in sections 2.42 and 2.6 we first use the isospin M-functions corresponding to t-channel pole diagrams to construct to within normalisation constants the covarients (projection operators) corresponding to eigenvalues of t-channel total isospin. We then invoke equation 2.42-15 to pick out those linear combinations of these projection operators which when adopted as channel independent isospin covarients will lead to $O(3,1) \otimes SU(2)$ amplitudes which are even or odd under s⇔u crossing.

We could equally well start by constructing the s or u channel isospin projection operators, but we work in the t-channel because our covarient Reggeisation calculations will require us to know which combinations of invarient amplitudes correspond to eigenvalues of t-channel total isospin and third component of total isospin.

### 4.21 PRODUCTION OF ISOSCALAR MESONS.

We have in the t-channel:

$$\mathcal{K}^{0,S}(\tfrac{1}{2}\bar{\tfrac{1}{2}},\bar{0}\gamma) \propto \mathcal{K}(\tfrac{1}{2}\tfrac{1}{2},0)\,\mathcal{P}^{0}\,\mathcal{K}^{S}(0,0\gamma)\,, \qquad (4.21\text{-}1)$$

$$\mathcal{K}^{1,V}(\tfrac{1}{2}\bar{\tfrac{1}{2}},\bar{0}\gamma) \propto \mathcal{K}_k(\tfrac{1}{2}\tfrac{1}{2},1)\,\mathcal{P}^{1}_{k\ell}\,\mathcal{K}^{V}_{\ell}(1,0\gamma)\,. \qquad (4.21\text{-}2)$$

From section 2.22:

$$\mathcal{P}^{0} = 1 \;, \qquad \mathcal{P}^{1}_{k\ell} = \delta_{k\ell} \;, \qquad (4.21\text{-}3,4)$$

and from sections 2.32 and 2.6:

$$\mathcal{K}(\tfrac{1}{2}\tfrac{1}{2},0) = \mathbb{1}_2 \;, \qquad \mathcal{K}_k(\tfrac{1}{2}\tfrac{1}{2},1) = \tau_k \;, \qquad (4.21\text{-}5,6)$$

$$\mathcal{K}^{s}(0,0\gamma) = \mathcal{K}(0,00) = 1 \;, \qquad (4.21\text{-}7)$$

$$\mathcal{K}^{v}_{\ell}(1,0\gamma) = \mathcal{K}_{\ell m}(1,01)\,\delta_{m3} = \delta_{\ell 3} \;. \qquad (4.21\text{-}8)$$

So:
$$\mathcal{K}^{0,s}(\tfrac{1}{2},\bar{\tfrac{1}{2}},\bar{0}\,\gamma) \propto \mathbb{1}_2 \;, \qquad (4.21\text{-}9)$$

and:
$$\mathcal{K}^{1,v}(\tfrac{1}{2}\,\bar{\tfrac{1}{2}},\bar{0}\,\gamma) \propto \tau_3 \;. \qquad (4.21\text{-}10)$$

Note that in equations 4.21-1 to 10, superscripts 0 and 1 denote the values of t-channel total isospin, whilst s and v denote respectively isoscalar and isovector transitions. The covarients $\mathbb{1}_2$ and $\tau_3$ both satisfy 2.42-15, so it is unnecessary to take linear combinations and we adopt as our channel independent decompositions in Lorentz $\otimes$ SU(2) space:

$$M_{\alpha} \equiv \sum_{i=1}^{6} (A_i^s \mathbb{1}_2 + A_i^v \tau_3)\tilde{\mathcal{K}}^i_{\alpha} \;, \qquad (4.21\text{-}11)$$

or:
$$M_{\mu\kappa} \equiv \sum_{i=1}^{18} (A_i^s \mathbb{1}_2 + A_i^v \tau_3)\tilde{\mathcal{K}}^i_{\mu\kappa} \;, \qquad (4.21\text{-}12)$$

as appropriate. The isospace "$\xi$-factors" of 2.42-15 are then given by:

$$\xi_s = +1 \;, \qquad \xi_v = -1 \;. \qquad (4.21\text{-}13,14)$$

It is important to realise that isospin crossing matrices do not "mix" isoscalar and isovector amplitudes. That is, amplitudes which are isoscalar (isovector) in a given channel are also isoscalar (isovector) in all other channels. Hence our notation for the amplitudes. For the same reason, 4.21-13, 14 may be usefully written:

$$\eta_{G_{\gamma s}}\xi_s = \eta_{G_{\gamma v}}\xi_v = -1 \;. \qquad (4.21\text{-}15,16)$$

Finally, we note from 4.21-9 and 10 that when we Reggeise, the $A_i^s$ will only get contributions from trajectories with: I = 0, whilst the $A_i^v$ will only get contributions from trajectories with: I = 1, $I_3$ = 0. (Rule 4.21-17)

4.22 PRODUCTION OF ISOVECTOR MESONS.

With the notation of the previous section, we now have in the t-channel:

$$\mathcal{K}_j^{0,V}(\tfrac{1}{2}\tfrac{\bar{1}}{2},\bar{1}\gamma) \propto \mathcal{K}(\tfrac{1}{2}\tfrac{1}{2},0)\rho^0\mathcal{K}_j^V(0,1\gamma) \quad , \qquad (4.22\text{-}1)$$

$$\mathcal{K}_j^{1,S}(\tfrac{1}{2}\tfrac{\bar{1}}{2},\bar{1}\gamma) \propto \mathcal{K}_k(\tfrac{1}{2}\tfrac{1}{2},1)\rho^1_{k\ell}\,\mathcal{K}^S_{\ell j}(1,1\gamma), \qquad (4.22\text{-}2)$$

and:

$$\mathcal{K}_j^{1,V}(\tfrac{1}{2}\tfrac{\bar{1}}{2},\bar{1}\gamma) \propto \mathcal{K}_k(\tfrac{1}{2}\tfrac{1}{2},1)\rho^1_{k\ell}\,\mathcal{K}^V_{\ell j}(1,1\gamma). \qquad (4.22\text{-}3)$$

In addition to equations 4.21-3 to 8 we now need:

$$\mathcal{K}^S_{\ell j}(1,1\gamma) = \mathcal{K}_{\ell j}(1,10) = \delta_{\ell j} \quad , \qquad (4.22\text{-}4)$$

$$\mathcal{K}^V_{\ell j}(1,1\gamma) = \mathcal{K}_{\ell j m}(1,11)\delta_{m3} = i\varepsilon_{\ell j 3} \; . \qquad (4.22\text{-}5)$$

So:

$$\mathcal{K}_j^{0,V}(\tfrac{1}{2}\tfrac{\bar{1}}{2},\bar{1}\gamma) \propto \delta_{j3} \quad , \qquad (4.22\text{-}6)$$

$$\mathcal{K}_j^{1,S}(\tfrac{1}{2}\tfrac{\bar{1}}{2},\bar{1}\gamma) \propto \tau_j \quad , \qquad (4.22\text{-}7)$$

and:

$$\mathcal{K}_j^{1,V}(\tfrac{1}{2}\tfrac{\bar{1}}{2},\bar{1}\gamma) \propto \tfrac{1}{2}[\tau_j,\tau_3] \; . \qquad (4.22\text{-}8)$$

Again it is unnecessary to take linear combinations of these three covarients since each one already satisfies 2.42-15, so we write in Lorentz $\otimes$ SU(2) space:

$$M^j_\alpha \equiv \sum_{i=1}^{6}\left(A^0_i\tau_j + A^+_i\delta_{j3} + A^-_i\tfrac{1}{2}[\tau_j,\tau_3]\right)\tilde{\mathcal{K}}^i_\alpha \quad , \qquad (4.22\text{-}9)$$

for production of pseudoscalar mesons, and:

$$M^j_{\mu\alpha} \equiv \sum_{i=1}^{18}\left(A^0_i\tau_j + A^+_i\delta_{j3} + A^-_i\tfrac{1}{2}[\tau_j,\tau_3]\right)\tilde{\mathcal{K}}^i_{\mu\alpha}$$

for vector meson production. (4.22-10)

The $\xi$ factors of equation 2.42-15 are then given by:

$$\eta_{G_{\gamma S}}\xi_0 = 1 = \eta_{G_{\gamma V}}\xi_+ \quad , \quad \eta_{G_{\gamma V}}\xi_- = -1 \quad . \qquad (4.22\text{-}11, 12, 13).$$

From equations 4.22-6,7,8 we see that when we Reggeise, trajectories with I = 0,(1) will only contribute to $A^+$, ($A^0$ and $A^-$). Furthermore, on inserting the appropriate isospace wave functions we find:

$$A_i(M^\circ\gamma\to p\bar{p}) = -(A_i^\circ + A_i^+) \,, \tag{4.22-14}$$

$$A_i(M^\circ\gamma\to n\bar{n}) = -(A_i^\circ - A_i^+) \,, \tag{4.22-15}$$

$$A_i(M^-\gamma\to n\bar{p}) = -\sqrt{2}(A_i^\circ + A_i^-) \,, \tag{4.22-16}$$

$$A_i(M^+\gamma\to p\bar{n}) = -\sqrt{2}(A_i^\circ - A_i^-) \,, \tag{4.22-17}$$

where M denotes the meson. Equations 4.22-14 and 15 each correspond to linear combinations of states with $I^t = 0$ and states with $I^t = 1$, $I_3^t = 0$; whilst equations 4.22-16 and 17 have $I^t = 1$, and $I_3^t = -1$ and $+1$ respectively. In more detail, 4.22-14 and 15 read:

$$A_i(M^\circ\gamma\to p\bar{p}) = -\begin{cases} A_i^+ \,, I^t=0 \,, \\ A_i^\circ \,, I^t=1 \,, \end{cases} \tag{4.22-18}$$

$$A_i(M^\circ\gamma\to n\bar{n}) = -\begin{cases} -A_i^+ \,, I^t=0 \,, \\ A_i^\circ \,, I^t=1 \,. \end{cases} \tag{4.22-19}$$

So the above rule is more precisely stated: $A^+$ gets contributions only from trajectories with $I = 0$; $A^-$ gets contributions only from trajectories with $I = 1$ and $I_3 \neq 0$; and $A^\circ$ gets contributions only from trajectories with $I = 1$, but any value of $I_3$ is allowed. However, if conservation of C-parity at the photonic vertex disallowes a particular isovector trajectory from coupling when $I_3^t = 0$, then this trajectory does not contribute to $A^\circ$ even for $I_3^t \neq 0$, contributing in this case only to $A^-$. (Rule 4.22-20)

The detailed mechanism by which the latter part of this rule comes about is easily seen if one ignores spin and denotes by $A_R^\circ(1)$ and $A_R^-(1)$ the contributions to $A^\circ$ and $A^-$ respectively of a given isovector trajectory. One then has in isospace:

$$A_R^\circ(1)\tau_j + A_R^-(1)\tfrac{1}{2}[\tau_j,\tau_3] = \mathcal{C}_k(\tfrac{1}{2}\tfrac{1}{2},1)\rho^1_{kl}\,\mathcal{C}_{lj}(1,1\,\gamma) \,, \tag{4.22-21}$$

where: $$\mathcal{C}_k(\tfrac{1}{2}\tfrac{1}{2},1) = g_R(\tfrac{1}{2}\tfrac{1}{2},1)\,\mathcal{K}_k(\tfrac{1}{2}\tfrac{1}{2},1) \,, \tag{4.22-22}$$

$$\mathcal{C}_{\ell j}(1,1\gamma) = g^S_R(1,1\gamma)\mathcal{K}^S_{\ell j}(1,1\gamma) + g^V_R(1,1\gamma)\mathcal{K}^V_{\ell j}(1,1\gamma)\,, \qquad (4.22\text{-}23)$$

and $g_R, g^S_R, g^V_R$ denote (factorised) Regge couplings. Now $\chi^\dagger(N)\underline{\tau}\cdot\underline{\phi}(M)\omega(\bar{N})$ is non-vanishing for all SU(2)-allowed configurations of isospin-projection, but $\chi^\dagger(N)[\underline{\tau}\cdot\underline{\phi}(M),\tau_3]\omega(\bar{N})$ is non-vanishing only for allowed configurations with $I^t_3 \neq 0$, so:

$$A^0_R(1) = g_R(\tfrac{1}{2}\tfrac{1}{2},1)\, g^S_R(1,1\gamma)\,, \quad \text{any } I^t_3 \qquad (4.22\text{-}24)$$

$$A^-_R(1) = g_R(\tfrac{1}{2}\tfrac{1}{2},1)\, g^V_R(1,1\gamma)\,, \quad I^t_3 \neq 0. \qquad (4.22\text{-}25)$$

Hence the statement that C-parity conservation at the photonic vertex forbids the trajectory from coupling when $I^t_3 = 0$ is equivalent to saying:

$$g^S_R(1,1\gamma) = 0\,, \quad \text{for } I^t_3 = 0. \qquad (4.22\text{-}26)$$

But $g^S_R(1,1\gamma)$ is independent of isospin-projection as is equation 4.22-24, so 4.22-26 forces $A^0_R(1)$ to vanish even for $I^t_3 \neq 0$. However, conservation of C-parity does not require: $g^V_R(1,1\gamma) = 0$, and so for $I^t_3 \neq 0$ the trajectory is still able to contribute to $A^-$.

## 4.3 M-FUNCTIONS FOR COVARIENT REGGEISATION AND FURTHER REGGEON SELECTION RULES.

We write the matrix element of the e.m. current-operator corresponding to the t-channel process: (real or virtual photon, momentum: q) + (meson, $J^P = (J^P)_M$, momentum: $-$ k) $\longrightarrow$ (on-shell particle, spin: J, normality: $\pm 1$, momentum: $\Delta$) $\longrightarrow$ (nucleon, momentum: p') + (anti-nucleon, momentum: $-$ p), as:

$$\langle p', -\bar{p}\,|\,J^{J\pm}_\alpha(0)\,|\,-\bar{k}\rangle \equiv \begin{cases} \bar{u}(p')M^{J\pm}_\alpha v(-p) & ,\ (J^P)_M = 0^-, \\ \bar{u}(p')M^{J\pm}_{\mu\alpha} v(-p)\varepsilon_\mu(-k) & ,\ (J^P)_M = 1^-, \end{cases} \qquad (4.3\text{-}1,2)$$

where:

$$\left.\begin{matrix} M^{J\pm}_\alpha \\ M^{J\pm}_{\mu\alpha} \end{matrix}\right\} = \mathcal{C}^{\pm}_{(\nu)^J}(\tfrac{1}{2}\tfrac{1}{2}J)\,\mathcal{P}^J_{(\nu)^J(\sigma)^J}(\Delta) \begin{cases} V^{\mp}_{(\sigma)^J\alpha}(J0\gamma) & , \\ V^{\pm}_{(\sigma)^J\mu\alpha}(J1\gamma)\,. \end{cases} \qquad (4.3\text{-}3,4)$$

Note that we are decomposing the matrix-element in spin-space only, and that we shall hereafter refer to the intermediate particle as a Reggeon, even though we have yet to continue to complex J.

The coupling function $\mathcal{C}^{\pm}_{(\nu)^J}(\frac{1}{2}\ \frac{1}{2}\ J)$ was given in section 2.32, (equations 2.32-24 and 25), but in terms of a different set of momenta coupling at the vertex. In terms of the momenta involved here, these two equations read:

$$\mathcal{C}^{+}_{(\nu)^J}(\tfrac{1}{2}\tfrac{1}{2}J)=(P_\nu)^{J-1}(g_1P_{\nu_1}+g_2\gamma_{\nu_1}) \quad , \qquad (4.3\text{-}5)$$

$$\mathcal{C}^{-}_{(\nu)^J}(\tfrac{1}{2}\tfrac{1}{2}J)=(P_\nu)^{J-1}(g_3P_{\nu_1}+g_4\gamma_{\nu_1})\gamma_5 \, . \qquad (4.3\text{-}6)$$

In section 2.32 we saw that normal trajectories only couple to the nucleon-antinucleon system if they have:

$$G = (-1)^{J+I} \qquad (4.3\text{-}7)$$

Either sign of G is allowed for abnormal trajectories, but they only couple via $g_3$ or $g_4$ according as they have:

$$G = (-1)^{J+I} \quad \text{or} \quad G = -(-1)^{J+I} \qquad (4.3\text{-}8)$$

For non-vanishing $I_3^t$ no C or G parity selection rules are operative at the photonic vertex, but for zero $I_3^t$ we have the selection rule:

$$C_n\,_{\text{Reggeon}} = C_\gamma\, C_n\,_{\text{Meson}} = -\,C_n\,_{\text{Meson}}. \qquad (4.3\text{-}9)$$

No "splitting" of the couplings takes place at this vertex for any $I_3^t$.

The combined implications of equations 4.3-7, 8, 9 and rules 4.21-17, 20 of the previous section are summarised in table 4.3-II.

In table 4.3-I we list the quantum numbers and zero-t values of the well established trajectories. All trajectories having sets of quantum numbers other than those appearing in table 4.3-I are indicated in table 4.3-II as "not known". They are assumed to have negative definite $\alpha(o)$ since if this were not so they would by now be well established.

Having deduced table 4.3-II we are in a position to complete the covarient Reggeisation calculations by working in Lorentz-space alone.

Table 4.3-III combines the results of tables 4.3-I and II. It lists, for trajectories coupling via $g_{1,2}$, $g_3$, and $g_4$, the least rapid asymptotic fall-off that trajectory can contribute to a given amplitude if the latter is to satisfy an ordinary (i.e. zeroth moment) sum-rule at vanishing t. At the present time there exists some doubt as to whether C-parity is ever conserved in virtual photonic interactions involving hadrons.[19] For completeness we therefore list these minimum fall-off requirements for both of the cases: C-parity conserved and C-parity violated at the photonic vertex.

In table 4.3-IV we combine the results of equations: 2.42-14 to 17, 4.13-2,3: 4.21-15,16; 4.22-11 to 13; and 1.3-44 to 46, and list those Lorentz $\otimes$ SU(2)-space amplitudes which will give rise to non-trivial sum-rules if sufficiently superconvergent. This table is only applicable if the four-point functions are charge-conjugation invarient. If this is not the case, all superconvergent amplitudes result in non-trivial sum-rules.

TABLE 4.3-I THE KNOWN REGGE-TRAJECTORIES.

| $n_R$ | TRAJ. | $\tau_R$ | $P_R$ | $I_R$ | $C_R$ | $G_R$ | $\alpha_R(0)$ |
|---|---|---|---|---|---|---|---|
| + | P,P' | + | + | 0 | + | + | 1.00 |
| | $\rho$ | − | − | 1 | − | + | 0.57 |
| | $\omega,\phi$ | − | − | 0 | − | − | 0.52 |
| | $A_2$ | + | + | 1 | + | − | 0.40 |
| − | $A_1$ | − | + | 1 | + | − | $0<\alpha_R(0)<1$ |
| | $\pi$ | + | − | 1 | + | − | $\alpha_R(0)<0$ |

TABLE 4.3-II: REGGE-TRAJECTORIES CONTRIBUTING TO EACH ISOSPIN AMPLITUDE VIA THE VARIOUS REGGEON-NUCLEON-ANTINUCLEON COUPLINGS.

| $n_R$ ↓ | REGGEON-N-$\bar{N}$ COUPLING-CONSTANT ↓ | $\tau_R$ ↑ | $P_R$ ↑ | KNOWN TRAJECTORIES | | | | | $G_R^\dagger$ FOR $I_R=0$ ↓ | $G_R^\dagger$ FOR $I_R=1$ ↓ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $I_R$ → | | 0 | | 1 | | | | |
| | | AMPLITUDE | | $A^S$ & $A^+$ | | $A^V$ & $A^0$ | | $A^-$ | | |
| | | $C_M$ → | | + | − | + | − | ± | | |
| | | $C_R^*$ → | | − | + | − | + | ± | | |
| | | $G_R^*$ → | | − | + | + | − | ± | | |
| + | $g_{1,2}$ | + | + | | P,P' | | $A_2$ | $A_2$ | + | − |
| | | − | − | $\omega,\phi$ | | $\rho$ | | $\rho$ | − | + |
| − | $g_3$ | + | − | | N | | $\pi$ | $\pi$ | + | − |
| | | − | + | N | | N | | N | − | + |
| | $g_4$ | + | − | N | | N | | N | − | + |
| | | − | + | | N | | $A_1$ | $A_1$ | + | − |

*Allowed by C-parity selection rule at $\gamma M\underline{R}$ vertex.
†Allowed by G-parity selection rule at $RN\bar{N}$ vertex.
▨ indicates disallowed,
N indicates "none known".

TABLE 4.3-III: MINIMUM REQUIREMENTS FOR THE REGGE CONTRIBUTIONS DUE TO THE FOUR TYPES OF COUPLING TO GIVE RISE TO A SUPERCONVERGENT AMPLITUDE AT $t = o$.

| ISOSPIN INDEX OF AMPLITUDE | | | REGGEON-N-$\bar{N}$ COUPLING CONSTANT | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | C CONSERVED AT PHOTONIC VERTEX | | | C VIOLATED AT PHOTONIC VERTEX | | |
| $I_M=0$ | $I_M=1$ | $(C_n)_M$ | $g_{1,2}$ | $g_3$ | $g_4$ | $g_{1,2}$ | $g_3$ | $g_4$ |
| S | + | + | 2 | 1 | 1 | 3 | 1 | 1 |
| | | − | 3 | 1 | 1 | | | |
| V | o | + | 2 | 1 | 1 | 2 | 1 | 2 |
| | | − | 2 | 1 | 2 | | | |
| NONE | − | ± | 2 | 1 | 2 | 2 | 1 | 2 |

Entry N indicates that contribution must fall off as least as fast as $\nu^{\alpha(0)-N}$ when $|\nu| \to \infty$.

TABLE 4.3-IV: AMPLITUDES GIVING RISE TO NON-TRIVIAL SUM-RULES IF SUFFICIENTLY SUPERCONVERGENT.

| TYPE OF SUM-RULE | | EVEN MOMENT | | ODD MOMENT | |
|---|---|---|---|---|---|
| SET OF O(3,1) AMPLITUDES | | $\{A\}_+$ | $\{A\}_-$ | $\{A\}_+$ | $\{A\}_-$ |
| 0 | + | S,V | NONE | NONE | S,V |
| | − | NONE | S,V | S,V | NONE |
| 1 | + | o,+ | − | − | o,+ |
| | − | − | o,+ | o,+ | − |
| $I_M$ | $(C_n)_M$ | ISOSPIN INDEX | | | |

$\{A\}_\pm$ indicates the set of $A_i$ for which the $\xi_i$ factor of equations 4.13-2 and 3 is equal to $\pm 1$.

## 4.4 LORENTZ SPACE COVARIENT REGGEISATION CALCULATION FOR PRODUCTION OF PSEUDOSCALAR MESONS.

The relevant M-function, $M_\alpha^{J\pm}$ is defined by equation 4.3-3, with the coupling-functions $\mathcal{C}^{\pm}_{(\nu)\mathcal{J}}(\frac{1}{2}\ \frac{1}{2}\ J)$ given by equations 4.3-5 and 6, and involving two coupling constants each: $g_{1,2}$ for $\mathcal{C}^+$ and $g_{3,4}$ for $\mathcal{C}^-$. The vertex function $\mathcal{V}^-_{(\sigma)\mathcal{J}\alpha}(J\ 0\ \gamma)$ is given by equation 2.74-5 and involves a single form-factor $f_1(q^2)$. $\mathcal{V}^+_{(\sigma)\mathcal{J}\alpha}(J\ 0\ \gamma\ )$ is given by equation 2.73-7 and involves two form-factors $f_{2,3}(q^2)$, the covarient corresponding to $f_3(q^2)$ being proportional to $q^2$. Thus in agreement with our counting rules for this process: $M_\alpha^{J+}$ and $M_\alpha^{J-}$ together involve six factorised couplings: $g_1f_1$, $g_2f_1$, $g_3f_2$, $g_4f_2$, $g_3f_3$, and $g_4f_3$, of which just the final two correspond to covarients which are proportional to $q^2$. Each of these coupling constants and form-factors should strictly carry an index J, so that on Reggeisation, $(J\to\alpha(t))$, it will gain a dependence on t. It is convenient to define:

$$M_\alpha^{\mathcal{J}+}\equiv\left(g_1M^{\mathcal{J}+}_{11\alpha}+g_2M^{\mathcal{J}+}_{21\alpha}\right)f_1(q^2)\ , \tag{4.4-1}$$

$$M_\alpha^{\mathcal{J}-}\equiv\left(g_3M^{\mathcal{J}-}_{32\alpha}+g_4M^{\mathcal{J}-}_{42\alpha}\right)f_2(q^2) + \left(g_3M^{\mathcal{J}-}_{33\alpha}+g_4M^{\mathcal{J}-}_{43\alpha}\right)f_3(q^2)\ . \tag{4.4-2}$$

We then have:

$$M^{\mathcal{J}+}_{11\alpha}=\mathcal{P}^{\mathcal{J}}_{;\sigma}\,\varepsilon_{\sigma\alpha}(\Delta q)\ , \tag{4.4-3}$$

$$M^{\mathcal{J}+}_{21\alpha}=\gamma_\nu\,\mathcal{P}^{\mathcal{J}}_{\nu;\sigma}\,\varepsilon_{\sigma\alpha}(\Delta q)\ , \tag{4.4-4}$$

$$M^{\mathcal{J}-}_{32\alpha}=-\left(\mathcal{P}^{\mathcal{J}}\Delta_\alpha+\Delta\cdot q\,\mathcal{P}^{\mathcal{J}}_{;\alpha}\right)\gamma_5\ , \tag{4.4-5}$$

$$M^{\mathcal{J}-}_{33\alpha}=\left(q^2\mathcal{P}^{\mathcal{J}}_{;\alpha}+\mathcal{P}^{\mathcal{J}}q_\alpha\right)\gamma_5\cong q^2\,\mathcal{P}^{\mathcal{J}}_{;\alpha}\gamma_5\ , \tag{4.4-6}$$

$$M^{J-}_{42\alpha} = -(\gamma_\nu \mathcal{P}^{J}_{\nu;}\Delta_\alpha + \Delta\cdot q\,\gamma_\nu \mathcal{P}^{J}_{\nu;\alpha})\gamma_5 , \qquad (4.4\text{-}7)$$

$$M^{J-}_{43\alpha} = (q^2 \gamma_\nu \mathcal{P}^{J}_{\nu;\alpha} + \gamma_\nu \mathcal{P}^{J}_{\nu;} q_\alpha)\gamma_5 \cong q^2 \gamma_\nu \mathcal{P}^{J}_{\nu;\alpha}\gamma_5 , \qquad (4.4\text{-}8)$$

where the argument of each partially contracted propogator numerator is: $(P,-q;\Delta)$.

We take these propogators from Scadron's paper,(9) but in view of the following facts, their structure is in each case equivalent to a considerably simplified form.

In view of equations A5-54, 55, and 47:

$$P(\Delta) \cong P \quad , \qquad P(\Delta)\cdot q(\Delta) \cong \nu \quad ,$$

$$\text{and} \quad P^2(\Delta) \cong P^2 = m^2 - \tfrac{1}{4}t \; , \qquad (4.4\text{-}9,10,11)$$

in all propogators.

$\mathcal{P}^{J}_{;\sigma}$ and $\mathcal{P}^{J}_{\nu;\sigma}$ are both contracted with $\varepsilon_{\sigma\alpha}(\Delta\, q)$ , so:

$$\Delta_\sigma \cong 0 , \qquad q_\sigma(\Delta) \cong 0 \quad ,$$

$$q_\sigma \cong 0 \; , \text{ and } \qquad g_{\nu\sigma}(\Delta) \cong g_{\nu\sigma} . \qquad (4.4\text{-}12,13,14,15)$$

As usual we have: $\qquad q_\alpha \cong 0$ . $\qquad$ (4.4-16)

Finally, in view of the Dirac equation on the nucleon and anti-nucleon spinors and the fact that the $\nu$ index is always contracted with $\gamma_\nu$ in the case of $\mathcal{P}^{J}_{\nu;\sigma}$ and $\gamma_\nu\gamma_5$ in the case of $\mathcal{P}^{J}_{\nu;}$ and $\mathcal{P}^{J}_{\nu;\alpha}$ , we have:

$$P_\nu(\Delta) \cong \begin{cases} m \text{ in } \mathcal{P}^{J}_{\nu;\sigma} \; , & (4.4\text{-}17) \\ 0 \text{ in } \mathcal{P}^{J}_{\nu;} \text{ and } \mathcal{P}^{J}_{\nu;\alpha} , & (4.4\text{-}18) \end{cases}$$

$$q_\nu(\Delta) \cong \begin{cases} \not{q} \text{ in } \mathcal{P}^{J}_{\nu;\sigma} \; , & (4.4\text{-}19) \\ \not{q} - \dfrac{2m\Delta\cdot q}{t} \text{ in } \mathcal{P}^{J}_{\nu;} \text{ and } \mathcal{P}^{J}_{\nu;\alpha} , & (4.4\text{-}20) \end{cases}$$

and: $$g_{\nu\alpha}(\Delta) \cong \gamma_\alpha - \frac{2m\Delta\cdot q}{t} \text{ in } \mathcal{P}^{J}_{\nu;\alpha} . \qquad (4.4\text{-}21)$$

Thus the propogator numerators are given by:

$$\mathcal{P}^{J} = c_J \mathcal{P}_J \,, \qquad \mathcal{P}^{J}_{;\sigma} \cong \frac{-c_J}{J} P_\sigma \mathcal{P}'_J \,, \tag{4.4-22, 23}$$

$$\gamma_\nu \mathcal{P}^{J}_{\nu;\sigma} \cong \frac{-c_J}{J^2}\left[\gamma_\sigma \mathcal{P}'_J + P_\sigma \not{q}\, \mathcal{P}''_J - m q^2(\Delta) P_\sigma \mathcal{P}''_{J-1}\right] \,, \tag{4.4-24}$$

$$\mathcal{P}^{J}_{;\alpha} \cong \frac{-c_J}{J}\left(P_\alpha \mathcal{P}'_J + \tfrac{1}{t} P^2 \Delta\cdot q\, \Delta_\alpha \mathcal{P}'_{J-1}\right) \,, \tag{4.4-25}$$

$$\gamma_\nu \mathcal{P}^{J}_{\nu;}\gamma_5 \cong \frac{c_J}{J}\left(\not{q} - \tfrac{2}{t} m\, \Delta\cdot q\right) \mathcal{P}'_J \gamma_5 \,, \tag{4.4-26}$$

$$\gamma_\nu \mathcal{P}^{J}_{\nu;\alpha}\gamma_5 \cong \frac{c_J}{J^2}\Big[\left(\tfrac{2}{t} m \Delta_\alpha - \gamma_\alpha\right)\mathcal{P}'_J + P_\alpha\left(\tfrac{2}{t} m\, \Delta\cdot q - \not{q}\right)\mathcal{P}''_J + \tfrac{1}{t} P^2 \Delta\cdot q \left(\tfrac{2}{t} m\, \Delta\cdot q - \not{q}\right)\Delta_\alpha \mathcal{P}''_{J-1}\Big]\gamma_5 \,. \tag{4.4-27}$$

The argument of all solid harmonic derivatives in the above equations is $P(\Delta).q(\Delta)$. When we make the continuation $J \to \alpha(t)$ a term $\mathcal{P}^{(n)}_J$ will therefore have leading asymptotic behaviour $\nu^{\alpha(t)-n}$. As we mentioned in section 2.5, this leading asymptotic behaviour is not affected as far as the dominant contributions to the amplitudes are concerned by any mechanism invoked to remove the poles at zero t.

After substitution of the above expressions into equations 4.4-3 to 8, it remains to relate the nine covarients: $(\gamma, P, \Delta)_\alpha (\mathbb{1}_4, \not{q})\gamma_5$, $(\mathbb{1}_4, \not{q})\mathcal{E}_\alpha(P\Delta q)$ and $\mathcal{E}_\alpha(\gamma\Delta q)$ to our six $\tilde{\mathcal{K}}^i_\alpha$ and two singular tails. This is achieved for the first six covarients by inversion of equations 4.11-11 to 16. We expand the final three covarients in terms of the initial six by means of equations A3-29 and 30, and the Dirac equation. On converting these expansions into expansions in terms of the $\tilde{\mathcal{K}}^i_\alpha$ and singular tails we obtain:

$$\mathcal{E}_\alpha(\gamma\Delta q) \cong m\, \tilde{\mathcal{K}}^1_\alpha - 2\tilde{\mathcal{K}}^4_\alpha \,, \tag{4.4-28}$$

$$\mathcal{E}_\alpha(P\Delta q) \cong -P^2 \tilde{\mathcal{K}}^1_\alpha - \tilde{\mathcal{K}}^2_\alpha + 2m\, \tilde{\mathcal{K}}^4_\alpha \,, \tag{4.4-29}$$

$$\not{q}\, \mathcal{E}_\alpha(P\Delta q) \cong -m\nu\, \tilde{\mathcal{K}}^1_\alpha - \tfrac{1}{2}\Delta\cdot q\, \tilde{\mathcal{K}}^3_\alpha + 2\nu \tilde{\mathcal{K}}^4_\alpha + m\, \tilde{\mathcal{K}}^5_\alpha - \tfrac{t}{2}\tilde{\mathcal{K}}^6_\alpha \,. \tag{4.4-30}$$

We note as a check that these three equations involve no singular-tails or terms in $\tilde{\mathcal{K}}^{5,6}_\alpha/q^2$, in agreement with the fact that $\mathcal{E}_\alpha(\gamma\Delta q)$ and $\mathcal{E}^*_\alpha(P\Delta q)$ vanish on contraction with $q_\alpha$.

This same check is available when we similarly expand the $M^{J\pm}_{jk\alpha}$, and as a further check we have that the only allowed singularities will be $1/t$ poles due to Reggeisation and $1/\Delta\cdot q$ electrodynamic poles. The former may occur in the coefficients of all covarients, the latter only in the coefficients of $\tilde{\mathcal{K}}^2_\alpha$ and $\tilde{\mathcal{K}}^5_\alpha$. To effect the necessary cancellation of all unwanted singularities it is necessary to invoke the recurrence relations on the solid harmonic derivatives.

The two linearly independent recurrence relations on the Legendre polynomials read:

$$z P^{(n+1)}_J(z) - (J-n) P^{(n)}_J(z) = P^{(n+1)}_{J-1}(z), \tag{4.4-31}$$

and:
$$P^{(n+1)}_{J+1}(z) - P^{(n+1)}_{J-1}(z) = (2J+1) P^{(n)}_J(z), \tag{4.4-32}$$

so:
$$P(\Delta)\cdot q(\Delta)\mathcal{P}^{(n+1)}_J[P(\Delta)\cdot q(\Delta)] - (J-n)\mathcal{P}^{(n)}_J[P(\Delta)\cdot q(\Delta)] =$$
$$= P^2(\Delta)q^2(\Delta)\mathcal{P}^{(n+1)}_{J-1}[P(\Delta)\cdot q(\Delta)], \tag{4.4-33}$$

and:
$$\mathcal{P}^{(n+1)}_{J+1}[P(\Delta)\cdot q(\Delta)] - P^2(\Delta)q^2(\Delta)\mathcal{P}^{(n+1)}_{J-1}[P(\Delta)\cdot q(\Delta)] =$$
$$= (2J+1)\mathcal{P}^{(n)}_J[P(\Delta)\cdot q(\Delta)]. \tag{4.4-34}$$

After some use of these recurrence relations we finally obtain:

$$M^{J+}_{11\alpha} = -\frac{C_J}{J}\mathcal{P}'_J\left[P^2\tilde{\mathcal{K}}^1_\alpha + \tilde{\mathcal{K}}^2_\alpha - 2m\tilde{\mathcal{K}}^4_\alpha\right], \tag{4.4-35}$$

$$M^{J+}_{21\alpha} = \frac{C_J}{J^2}\Big[m\left(\mathcal{P}'_J - \nu\mathcal{P}''_J + P^2q^2(\Delta)\mathcal{P}''_{J-1}\right)\tilde{\mathcal{K}}^1_\alpha + mq^2(\Delta)\mathcal{P}''_{J-1}\tilde{\mathcal{K}}^2_\alpha$$
$$-2\left(\mathcal{P}'_J + \nu\mathcal{P}''_J + m^2q^2(\Delta)\mathcal{P}''_{J-1}\right)\tilde{\mathcal{K}}^4_\alpha - \mathcal{P}''_J\left(\frac{\Delta\cdot q}{2}\tilde{\mathcal{K}}^3_\alpha - m\tilde{\mathcal{K}}^5_\alpha + \frac{t}{2}\tilde{\mathcal{K}}^6_\alpha\right)\Big], \tag{4.4-36}$$

$$M^{J-}_{32\alpha} = \frac{C_J}{J}\left[\mathcal{P}'_J\tilde{\mathcal{K}}^2_\alpha + P^2\mathcal{P}'_{J-1}\tilde{\mathcal{K}}^5_\alpha\right], \tag{4.4-37}$$

$$M^{J-}_{33\alpha} = \frac{-C_J}{J\Delta\cdot q}\left[q^2\mathcal{P}'_J\tilde{\mathcal{K}}^2_\alpha + \left(J\mathcal{P}_J + P^2q^2\mathcal{P}'_{J-1}\right)\tilde{\mathcal{K}}^5_\alpha\right], \tag{4.4-38}$$

$$M^{J-}_{42\alpha} = \frac{-C_J}{J^2}\Big[\Delta\cdot q\,\mathcal{P}''_J\Big(\frac{2m}{t}\tilde{\mathcal{X}}^2_\alpha - \tilde{\mathcal{X}}^4_\alpha\Big) + \big(\mathcal{P}'_J + \nu\mathcal{P}''_J - P^2q^2\mathcal{P}''_{J-1}\big)\tilde{\mathcal{X}}^3_\alpha$$
$$+ P^2\Delta\cdot q\,\mathcal{P}''_{J-1}\Big(\frac{2m}{t}\tilde{\mathcal{X}}^5_\alpha - \tilde{\mathcal{X}}^6_\alpha\Big)\Big], \qquad (4.4\text{-}39)$$

$$M^{J-}_{43\alpha} = \frac{C_J}{J^2}\Big[q^2\mathcal{P}''_J\Big(\frac{2m}{t}\tilde{\mathcal{X}}^2_\alpha - \tilde{\mathcal{X}}^4_\alpha\Big) - \frac{1}{t}P^2q^2\Delta\cdot q\,\mathcal{P}''_{J-1}\,\tilde{\mathcal{X}}^3_\alpha$$
$$+ \big(J\mathcal{P}'_J + P^2q^2\mathcal{P}''_{J-1}\big)\Big(\frac{2m}{t}\tilde{\mathcal{X}}^5_\alpha - \tilde{\mathcal{X}}^6_\alpha\Big)\Big]. \qquad (4.4\text{-}40)$$

In table 4.4-I we pick out the dominant asymptotic contribution each invarient amplitude receives from trajectories coupling via each of the six $g_j f_k$. An entry N in the $A_i$th. row and $g_j f_k$th. column indicates that after the continuation $J \to \alpha(t)$ the coefficient of $g_j(t)f_k(q^2,t)\tilde{\mathcal{X}}^i_\alpha$ has leading high $|\nu|$ asymptotic behaviour: $\nu^{\alpha_{jk}(t)-N}$, where $\alpha_{jk}(t)$ is the leading trajectory which is allowed by the selection rules to contribute via that coupling. A dot indicates that the amplitude receives no contribution via the particular coupling.

If several amplitudes receive the same leading asymptotic behaviour via a given coupling, it is often possible by taking linear combinations of these to construct a new amplitude with improved behaviour. We find in view of table 4.3-III that only one such combination is superconvergent and we list this in table 4.4-I as well. It is denoted by $A_7$ and defined by:

$$A_7 \equiv 2(2A_1 + mA_4) + tA_2\ . \qquad (4.4\text{-}41)$$

It has an electrodynamical pole at vanishing $\Delta\cdot q$.

Picking out the dominant contribution each amplitude receives via the three pairs of couplings $(g_{1,2})f_1$, $g_3(f_{2,3})$, and $g_4(f_{2,3})$, we deduce from tables 4.3-III and IV that we have no first or higher moment sum-rules. But provided C-parity is conserved at the photonic vertex for vanishing $I^t_3$, we have the following non-trivial ordinary (i.e. zeroth moment) sum-rules. They are valid for all $q^2$ and all non-positive definite t, and $(J^P, I, C_n)_M$ denotes the quantum numbers of the final meson.

$(J^P, I, C_n)_M = (0^-, 0, +)$ $\quad$ $[\eta(549), \eta'(958), E(1420)]$

Ordinary sum-rules on the four amplitudes:

$A_3^{S,V}$, $A_6^{S,V}$. (List 4.4-42)

---

$(J^P, I, C_n)_M = (0^-, 0, -)$ $\quad$ (No known examples)

Ordinary sum-rules on the two amplitudes:

$A_7^{S,V}$ (List 4.4-43)

---

$(J^P, I, C_n)_M = (0^-, 1, +)$ $\quad$ $(\pi)$

Ordinary sum-rules on the five amplitudes:

$A_3^{0,+}$, $A_6^{0,+}$, $A_7^-$ . (List 4.4-44)

---

$(J^P, I, C_n)_M = (0^-, 1, -)$ $\quad$ (No known examples)

Ordinary sum rules on the two amplitudes:

$A_7^{0,+}$. (List 4.4.-45)

---

If interactions between virtual photons and hadrons are not in fact charge-conjugate invarient then only $A_7$ is super-convergent for non-zero $q^2$. But $s \leftrightarrow u$ crossing symmetry will no longer force the amplitudes to be even or odd functions of $\nu$ , so we then have the following sum-rules valid for non-vanishing $q^2$ and non-positive definite t.

$(J^P,I)_M = (0^-,0)$ $(\eta, \eta', E)$

Ordinary sum-rules on the two amplitudes:

$A_7^{S,V}$. (List 4.4-46)

---

$(J^P,I)_M = (0^-,1)$ $(\pi)$

Ordinary sum-rules on the three amplitudes:

$A_7^{o,+,-}$. (List 4.4-47)

---

The sum-rules for pion photoproduction have already been obtained via rather different methods by a variety of authors.[8] Pande has obtained the sum-rule on $A_3^o$; Choudhury and Nussinov, and Altarelli and Colocci, those on $A_3^o$ and $A_3^+$; Halpern, the one on $A_7^-$; and Musto and Nicodemi, all three.

Subject to charge-conjugation invarience at the virtual photonic vertex we have shown that these sum-rules remain valid for non-vanishing $q^2$. We stress that this does not follow merely from the assumption that electro-production Reggeises in the same manner as photo-production. The amplitudes could easily have been given poorer asymptotic behaviour in the electroproduction case due to additional contributions proportional to $q^2$. In particular, such contributions might have come via the couplings $(g_3, g_4)f_3$, since all terms of the form $(g_3,g_4)f_3\,\tilde{\mathcal{K}}_\alpha^{1,\dots,4}$ must appear with coefficients proportional to $q^2$.

We have also deduced two further sum-rules on the amplitudes $A_6^o$ and $A_6^+$. These appear only in electroproduction. Since $\mathcal{K}_\alpha^6$ is proportional to $q^2$ it is not necessary, (as is sometimes erroneously supposed), for these two amplitudes to vanish at zero $q^2$. So on saturating these two sum-rules and then continuing to zero $q^2$, we shall obtain additional relations between form-factors evaluated at zero argument. These relations cannot be obtained by the purely on-shell methods of the above cited authors.[8]

TABLE 4.4-I

CONTRIBUTIONS TO THE AMPLITUDES FOR PRODUCTION OF PSEUDO-SCALAR MESONS DUE TO NORMAL TRAJECTORIES COUPLING VIA $(g_{1,2})f_1$, AND ABNORMAL TRAJECTORIES COUPLING VIA $(g_{3,4})(f_{2,3})$.

| AMPLITUDE | CROSSING FACTOR | COUPLING INDEX jk, i.e., CONTRIBUTION DUE TO: $g_j(t)f_k(q^2,t)$. | | | | | |
|---|---|---|---|---|---|---|---|
| | | 11 | 21 | 32 | 33 | 42 | 43 |
| $A_1$ | − | 1 | 1 | . | . | . | . |
| $A_2$ | − | 1 | 3 | 1 | 1 | 2 | 2 |
| $A_3$ | + | . | 2 | . | . | 1 | 3 |
| $A_4$ | − | 1 | 1 | . | . | 2 | 2 |
| $A_5$ | + | . | 2 | 2 | 0 | 3 | 1 |
| $A_6$ | + | . | 2 | . | . | 3 | 1 |
| $A_7$ | − | . | . | 1 | 1 | 2 | 2 |

## 4.5 LORENTZ-SPACE COVARIENT REGGEISATION CALCULATION FOR PRODUCTION OF VECTOR MESONS.

This calculation is carried out in direct analogy with that of the previous section. The M-function $M^{J\pm}_{\mu\alpha}$ is defined by equation 4.3-4, and we use the decomposition of equations 2.75-19, 21, 22, 23, 24, and 25 for the vertex function $V^{+}_{(\sigma)J\mu\alpha}(J1\gamma)$.

For the vertex function $V^{-}_{(\sigma)J\mu\kappa}(J1\gamma)$ we shall use the decomposition:

$$\begin{aligned} V^{-}_{(\sigma)J\mu\kappa}(J1\gamma) = \Big[ & f_6(q^2)\mathcal{E}_{\mu\sigma_1}(q\Delta)(\Delta_\alpha q_{\sigma_2} - \Delta\cdot q\, g_{\sigma_2\alpha}) \\ & + f_7(q^2)\mathcal{E}_{\alpha\sigma_1}(q\Delta)q_\mu q_{\sigma_2} + f_8(q^2)\mathcal{E}_{\alpha\sigma_1}(q\Delta)g_{\sigma_2\mu} \\ & + f_9(q^2)\mathcal{E}_{\mu\sigma_1}(q\Delta)(q^2 g_{\alpha\sigma_2} - q_{\sigma_2}q_\alpha)\Big](-q_\sigma)^{J-2} . \end{aligned} \qquad (4.5\text{-}1)$$

This differs from the decomposition 2.76-20 derived in section 2.76, but has the advantage that:

$$q_{\sigma_1} \cong 0 \;,\; \Delta_{\sigma_1} \cong 0 \;,\; q_{\sigma_1}(\Delta) \cong 0 \;, \text{ and } g_{\cdot\sigma_1}(\Delta) \cong g_{\cdot\sigma_1} \quad (4.5\text{-}2 \text{ to } 5)$$

in all propogator numerators contracted with the vertex.

The form-factors of 4.5-1 are related to those of 2.76-20 by:

$$f_6 = \frac{1}{t}(F_2 - \Delta\cdot q F_1) \,, \qquad (4.5\text{-}6)$$

$$f_7 = \left(\frac{\Delta\cdot q}{t} - 1\right)F_1 - \frac{1}{t}F_2 \,, \qquad (4.5\text{-}7)$$

$$f_8 = \left[q^2 - \frac{(\Delta\cdot q)^2}{t}\right]F_1 + \frac{\Delta\cdot q}{t}F_2 + q^2 F_3 + F_4 \,, \qquad (4.5\text{-}8)$$

$$f_9 = -(F_1 + F_3) \,. \qquad (4.5\text{-}9)$$

Thus the $f_{6,7,8}$ are subject to kinematic poles at zero t. But since the $f_{6,7,8,9}$ are free of kinematic singularities in $\nu$, our results for the dominant asymptotic contribution to each amplitude from a given abnormal trajectory will be unaffected by our working with $f_{6,\ldots,9}$ rather than $F_{1,\ldots,4}$.

Again, with a suffix J implied on each of the $g_{1 \text{ to } 4}$ and $f_{1 \text{ to } 9}$, we define:

$$M^{J+}_{\mu\alpha} \equiv \sum_{j=1}^{2} \sum_{k=1}^{5} M^{J+}_{jk\mu\alpha}\, g_j f_k \quad , \tag{4.5-10}$$

$$M^{J-}_{\mu\alpha} \equiv \sum_{j=3}^{4} \sum_{k=6}^{9} M^{J-}_{jk\mu\alpha}\, g_j f_k \quad . \tag{4.5-11}$$

Then:

$$M^{J+}_{11\mu\alpha} = \mathcal{P}^{J} g_{\mu\alpha} + \mathcal{P}^{J}_{;\alpha} q_{\mu} \quad , \tag{4.5-12}$$

$$M^{J+}_{12\mu\alpha} = \mathcal{P}^{J}(q_{\mu}\Delta_{\alpha} - \Delta\cdot q\, g_{\mu\alpha}) \quad , \tag{4.5-13}$$

$$M^{J+}_{13\mu\alpha} = -(\mathcal{P}^{J}_{;\mu}\Delta_{\alpha} + \Delta\cdot q\, \mathcal{P}^{J}_{;\mu\alpha}) \quad , \tag{4.5-14}$$

$$M^{J+}_{14\mu\alpha} = \mathcal{P}^{J}(q^2 g_{\mu\alpha} - q_{\mu} q_{\alpha}) \quad , \tag{4.5-15}$$

$$M^{J+}_{15\mu\alpha} = q^2 \mathcal{P}^{J}_{;\mu\alpha} + \mathcal{P}^{J}_{;\mu} q_{\alpha} \quad , \tag{4.5-16}$$

$$M^{J+}_{21\mu\alpha} = \gamma_{\nu}(\mathcal{P}^{J}_{\nu;} g_{\mu\alpha} + \mathcal{P}^{J}_{\nu;\alpha} q_{\mu}) \quad , \tag{4.5-17}$$

$$M^{J+}_{22\mu\alpha} = \gamma_{\nu}\mathcal{P}^{J}_{\nu;}(q_{\mu}\Delta_{\alpha} - \Delta\cdot q\, g_{\mu\alpha}) \quad , \tag{4.5-18}$$

$$M^{J+}_{23\mu\alpha} = -\gamma_{\nu}(\mathcal{P}^{J}_{\nu;\mu}\Delta_{\alpha} + \Delta\cdot q\, \mathcal{P}^{J}_{\nu;\mu\alpha}) \quad , \tag{4.5-19}$$

$$M^{J+}_{24\mu\alpha} = \gamma_{\nu}\mathcal{P}^{J}_{\nu;}(q^2 g_{\mu\alpha} - q_{\mu} q_{\alpha}) \quad , \tag{4.5-20}$$

$$M^{J+}_{25\mu\alpha} = \gamma_{\nu}(q^2 \mathcal{P}^{J}_{\nu;\mu\alpha} + \mathcal{P}^{J}_{\nu;\mu} q_{\alpha}) \quad , \tag{4.5-21}$$

$$M^{J-}_{36\mu\alpha} = -(\mathcal{P}^{J}_{;\sigma_1}\Delta_{\alpha} + \Delta\cdot q\, \mathcal{P}^{J}_{;\sigma_1\alpha})\varepsilon_{\mu\sigma_1}(q\Delta)\gamma_5 \quad , \tag{4.5-22}$$

$$M^{J-}_{37\mu\alpha} = -q_{\mu}\mathcal{P}^{J}_{;\sigma_1}\varepsilon_{\alpha\sigma_1}(q\Delta)\gamma_5 \quad , \tag{4.5-23}$$

$$M^{J-}_{38\mu\alpha} = \mathcal{P}^{J}_{;\sigma_1\mu}\,\varepsilon_{\alpha\sigma_1}(q\,\Delta)\gamma_5 \;, \tag{4.5-24}$$

$$M^{J-}_{39\mu\alpha} = (q^2 \mathcal{P}^{J}_{;\sigma_1\alpha} + \mathcal{P}^{J}_{;\sigma_1} q_\alpha)\,\varepsilon_{\mu\sigma_1}(q\,\Delta)\gamma_5 \;, \tag{4.5-25}$$

$$M^{J-}_{46\mu\alpha} = -\gamma_\nu(\mathcal{P}^{J}_{\nu;\sigma_1}\Delta_\alpha + \Delta\cdot q\,\mathcal{P}^{J}_{\nu;\sigma_1\alpha})\,\varepsilon_{\mu\sigma_1}(q\,\Delta)\gamma_5 \;, \tag{4.5-26}$$

$$M^{J-}_{47\mu\alpha} = -q_\mu\gamma_\nu\mathcal{P}^{J}_{\nu;\sigma_1}\,\varepsilon_{\alpha\sigma_1}(q\,\Delta)\gamma_5 \;, \tag{4.5-27}$$

$$M^{J-}_{48\mu\alpha} = \gamma_\nu\mathcal{P}^{J}_{\nu;\sigma_1\mu}\,\varepsilon_{\alpha\sigma_1}(q\,\Delta)\gamma_5 \;, \tag{4.5-28}$$

$$M^{J-}_{49\mu\alpha} = \gamma_\nu(q^2\mathcal{P}^{J}_{\nu;\sigma_1\alpha} + \mathcal{P}^{J}_{\nu;\sigma_1} q_\alpha)\,\varepsilon_{\mu\sigma_1}(q\,\Delta)\gamma_5 \;. \tag{4.5-29}$$

The propogators are again equivalent in view of simplifying relations to considerably less complicated forms than those listed by Scadron[9] for the general case.

We have already noted the simplifying relations of equations 4.5-2 to 5, and in addition equations 4.4-9, 10, 11, and 16 are again operative. As the analogues of equations 4.4-17 to 21 we now have:

$$P_\nu(\Delta) \cong m \;, \text{ and } \; q_\nu(\Delta) \cong \not{q}$$
$$\text{in } \mathcal{P}^{J}_{\nu;} \;, \mathcal{P}^{J}_{\nu;\alpha} \;, \text{ and } \mathcal{P}^{J}_{\nu;\mu} \;, \tag{4.5-30,31}$$

$$P_\nu(\Delta) \cong 0 \;, \text{ and } \; q_\nu(\Delta) \cong \not{q} - \frac{2m\Delta\cdot q}{t}$$
$$\text{in } \mathcal{P}^{J}_{\nu;\sigma_1} \;, \mathcal{P}^{J}_{\nu;\sigma_1\alpha} \;, \text{ and } \mathcal{P}^{J}_{\nu;\sigma_1\mu} \;, \tag{4.5-32,33}$$

and:

$$g_{\nu\rho}(\Delta) \cong \begin{cases} \gamma_\rho & , \text{ in } \mathcal{P}^{J}_{\nu;\rho} \;, \quad (4.5\text{-}34) \\ \gamma_\rho - \dfrac{2m}{t}\Delta_\rho , & \text{in } \mathcal{P}^{J}_{\nu;\sigma_1\rho} \;, \quad (4.5\text{-}35) \end{cases}$$

for $\rho \equiv \alpha$ or $\mu$ .

Thus the propogators are given by:

$$\mathcal{P}^{J} = c_J \mathcal{P}_J \quad (4.5\text{-}36) \;, \qquad \gamma_\nu\mathcal{P}^{J}_{\nu;} \cong \frac{c_J}{J}(\not{q}\,\mathcal{P}'_J - m q^2(\Delta)\mathcal{P}'_{J-1}) \;, \quad (4.5\text{-}37)$$

$$\mathcal{P}^{J}_{;\rho} \cong \frac{-C_J}{J}\left(P_\rho \mathcal{P}'_J - P^2 q_\rho(\Delta)\mathcal{P}'_{J-1}\right) , \qquad (4.5\text{-}38)$$

$$\gamma_\nu \mathcal{P}^{J}_{\nu;\rho} \cong \frac{-C_J}{J^2}\Big[\gamma_\rho \mathcal{P}'_J - (2J+1) m q_\rho(\Delta)\mathcal{P}'_{J-1} + \left(m q_\rho(\Delta) + P_\rho \not{q}\right)\mathcal{P}''_J$$
$$- \left(m q^2(\Delta) P_\rho + P^2 q_\rho(\Delta)\not{q}\right)\mathcal{P}''_{J-1}\Big] , \qquad (4.5\text{-}39)$$

$$\mathcal{P}^{J}_{;\mu\alpha} \cong \frac{C_J}{J(J-1)}\Big[P_\mu P_\alpha \mathcal{P}''_J - P^2 g_{\mu\alpha}(\Delta)\mathcal{P}'_{J-1} - P^2\left(P_\mu q_\alpha(\Delta)\right.$$
$$\left. + q_\mu(\Delta) P_\alpha\right)\mathcal{P}''_{J-1} + P^4 q_\mu(\Delta) q_\alpha(\Delta)\mathcal{P}''_{J-2}\Big] , \qquad (4.5\text{-}40)$$

$$\gamma_\nu \mathcal{P}^{J}_{\nu;\mu\alpha} \cong \frac{-C_J}{J^2(J-1)}\Big\{(2J+1) m g_{\mu\alpha}(\Delta)\mathcal{P}'_{J-1} - \left[\gamma_\mu P_\alpha + P_\mu \gamma_\alpha + m g_{\mu\alpha}(\Delta)\right]\mathcal{P}''_J$$
$$+ \Big[(2J+1) m\left(P_\mu q_\alpha(\Delta) + q_\mu(\Delta) P_\alpha\right) + P^2\left(\gamma_\mu q_\alpha(\Delta) + q_\mu(\Delta)\gamma_\alpha + g_{\mu\alpha}(\Delta)\not{q}\right)\Big]\mathcal{P}''_{J-1}$$
$$- (2J+1) m P^2 q_\mu(\Delta) q_\alpha(\Delta)\mathcal{P}''_{J-2} - \left[m\left(P_\mu q_\alpha(\Delta) + q_\mu(\Delta) P_\alpha\right) + P_\mu P_\alpha \not{q}\right]\mathcal{P}'''_J$$
$$+ \left[m q^2(\Delta) P_\mu P_\alpha + P^2\left(P_\mu q_\alpha(\Delta)\not{q} + q_\mu(\Delta) P_\alpha \not{q} + m q_\mu(\Delta) q_\alpha(\Delta)\right)\right]\mathcal{P}'''_{J-1}$$
$$- P^4 q_\mu(\Delta) q_\alpha(\Delta)\not{q}\,\mathcal{P}'''_{J-2}\Big\} , \qquad (4.5\text{-}41)$$

$$\mathcal{P}^{J}_{;\sigma_1} \cong \frac{-C_J}{J} P_{\sigma_1}\mathcal{P}'_J , \qquad (4.5\text{-}42)$$

$$\gamma_\nu \mathcal{P}^{J}_{\nu;\sigma_1}\gamma_5 \cong \frac{-C_J}{J^2}\left(\gamma_{\sigma_1}\mathcal{P}'_J + \hat{\not{q}} P_{\sigma_1}\mathcal{P}''_J\right)\gamma_5 , \qquad (4.5\text{-}43)$$

$$\mathcal{P}^{J}_{;\sigma_1\rho} \cong \frac{C_J}{J(J-1)}\left(P_{\sigma_1} P_\rho \mathcal{P}''_J - P^2 g_{\sigma_1\rho}\mathcal{P}'_{J-1} - P^2 P_{\sigma_1} q_\rho(\Delta)\mathcal{P}''_{J-1}\right) , \qquad (4.5\text{-}44)$$

$$\gamma_\nu \mathcal{P}^{J}_{\nu;\sigma_1\rho}\gamma_5 \cong \frac{-C_J}{J^2(J-1)}\Big[-\left(\gamma_{\sigma_1} P_\rho + \hat{\gamma}_\rho P_{\sigma_1}\right)\mathcal{P}''_J + P^2\left(\gamma_{\sigma_1} q_\rho(\Delta) + g_{\sigma_1\rho}\hat{\not{q}}\right)\mathcal{P}''_{J-1}$$
$$- P_{\sigma_1} P_\rho \hat{\not{q}}\,\mathcal{P}'''_J + P^2 P_{\sigma_1} q_\rho(\Delta)\hat{\not{q}}\,\mathcal{P}'''_{J-1}\Big]\gamma_5 . \qquad (4.5\text{-}45)$$

In these relations: $\rho$ again stands for $\mu$ or $\alpha$, and we have defined:

$$\hat{\not{q}} \equiv \not{q} - 2m\,\Delta\cdot q/t \quad , \quad \hat{\gamma}_\rho \equiv \gamma_\rho - 2m\Delta_\rho/t . \qquad (4.5\text{-}46,47)$$

The reader is also reminded that:

$$q_\mu(\Delta) \cong q_\mu(1 - \Delta\cdot q/t) \quad , \quad q_\alpha(\Delta) \cong -\Delta\cdot q\,\Delta_\alpha/t , \qquad (4.5\text{-}48,49)$$

$$g_{\mu\alpha}(\Delta) \cong g_{\mu\alpha} - q_\mu \Delta_\alpha/t , \quad q^2(\Delta) = q^2 - (\Delta\cdot q)^2/t . \qquad (4.5\text{-}50,51)$$

The expressions 4.5-36 to 45 for the propogators are substituted into equations 4.5-12 to 19, and it is then necessary to express the thirty-six covarients: $[(P,q,\gamma)_\mu(P,\Delta,\gamma)_\alpha, g_{\mu\alpha}](1\!\!1_4, \not{q})$ and: $\{[(P,q)_\mu\mathcal{E}_\alpha(Pq\Delta), \mathcal{E}_\mu(Pq\Delta)(P,\Delta)_\alpha, \mathcal{E}_{\mu\alpha}(q\Delta)](1\!\!1_4, \not{q}), \gamma_\mu\mathcal{E}_\alpha(Pq\Delta), \mathcal{E}_\mu(Pq\Delta)\gamma_\alpha, (P,q)_\mu\mathcal{E}_\alpha(\gamma q\Delta), \mathcal{E}_\mu(\gamma q\Delta)(P,\Delta)_\alpha\}\gamma_5$ in terms of the eighteen $\mathcal{K}^i_{\mu\alpha}$ and six singular tails. Inversion of equations 4.12-52 to 71 yields the required relations for eighteen of the first twenty covarients, and since the remaining two of them are related to the others through equations 4.12-72 and 73 they too can be similarly expanded. Expansion of the final sixteen covarients is achieved by relating them to the initial twenty through equations A3-27, 29, and 30. As a check on the calculation one uses the fact that covarients involving $\mathcal{E}_\alpha(Pq\Delta)$, $\mathcal{E}_\alpha(\gamma q\Delta)$, and $\mathcal{E}_{\mu\alpha}(q\Delta)$ vanish on contraction with $q_\alpha$. Their expansions cannot therefore involve any singular tails.

Finally, one obtains expansions for each of the eighteen $M^{\pi+}_{(1,2)(1,\dots,5)\mu\alpha}$ and $M^{\pi-}_{(3,4)(6,\dots,9)\mu\alpha}$ in terms of the eighteen $\mathcal{K}^i_{\mu\alpha}$. Again, the fact that all unwanted singularities must cancel serves as a check on the calculations. To effect such a cancellation of singularities it is necessary to make extensive use of the recurrence relations 4.4-33 and 34.

We do not propose to give here the vector meson analogues of equations 4.4-28 to 30 and 35 to 40. Not only do we have fifty-four such equations, but many of these are extremely lengthy and complicated. Instead we merely give the analogue of table 4.4-I.

In tables 4.5-I and II respectively we list the leading asymptotic contributions to the amplitudes from normal and abnormal trajectories. The notation is the same as that employed in table 4.4-I. We again list those linear combinations of amplitudes which have better asymptotic behaviour than the individual amplitudes involved, and which will, in view of table 4.3-III, give rise to superconvergent sum-rules. These linear combinations are defined as follows:

$$A_{19} \equiv A_4 + A_{10} \ , \tag{4.5-52}$$

$$A_{20} \equiv mA_4 - \tfrac{t}{4} A_6 + A_{12} \ , \tag{4.5-53}$$

$$A_{21} \equiv A_6 - A_8 \ , \tag{4.5-54}$$

$$A_{22} \equiv \Delta\cdot q\,(A_6 - A_8) - q^2 A_{16} \ , \tag{4.5-55}$$

$$A_{23} \equiv tA_9 - \Delta\cdot q\, A_{18} \ , \tag{4.5-56}$$

$$A_{24} \equiv A_9 + A_{11} - A_{18} \ . \tag{4.5-57}$$

Remembering that $A_{1,8,13,}$ and $_{15,}$ have electrodynamical poles at vanishing $\Delta\cdot q$, we see that $A_{21}$ is subject to a similar pole.

Picking out the dominant contribution to each amplitude due to each of the three sets of couplings: $(g_{1,2})(f_{1,\ldots,5})$ $g_3(f_{6,\ldots,9})$, and $g_4(f_{6,\ldots,9})$, we find in view of tables 4.3-III and IV the following non-trivial sum-rules. They are valid for non-positive definite t, all $q^2$, and are subject to C-parity being conserved at the photonic vertex for vanishing $I_3^t$.

$(J^P, I, C_n)_M = (1^-, 0, +)$ (No known examples)

Ordinary sum rules on the twelve amplitudes:

$A_1^{S,V}$, $A_6^{S,V}$, $A_{16}^{S,V}$, $A_{19}^{S,V}$, $A_{20}^{S,V}$, $A_{21}^{S,V}$,

and first moment sum rules on the two amplitudes:

$A_5^{S,V}$ (List 4.5-58)

$(J^P, I, C_n)_M = (1^-, 0, -)$ $(\omega, \phi)$

Ordinary sum-rules on the six amplitudes:

$A_5^{S,V}$, $A_{23}^{S,V}$, $A_{24}^{S,V}$

and first moment sum-rules on the two amplitudes:

$A_{19}^{S,V}$. (List 4.5-59)

---

$(J^P, I, C_n)_M = (1^-, 1, +)$ (No known examples)

Ordinary sum-rules on the fifteen amplitudes:

$A_1^{0,+}$, $A_5^-$, $A_6^{0,+}$, $A_{16}^{0,+}$, $A_{19}^{0,+}$, $A_{20}^{0,+}$, $A_{21}^{0,+}$, $A_{23}^-$, $A_{24}^-$,

and first moment sum-rules on the three amplitudes:

$A_5^{0,+}$, $A_{19}^-$. (List 4.5-60)

---

$(J^P, I, C_n)_M = (1^-, 1, -)$ $(\rho)$

Ordinary sum rules on the nine amplitudes:

$A_1^-$, $A_5^{0,+}$, $A_{19}^-$, $A_{22}^-$, $A_{23}^{0,+}$, $A_{24}^{0,+}$,

and first moment sum-rules on the two amplitudes:

$A_{19}^{0,+}$. (List 4.5-61)

---

If interactions between virtual photons and hadrons are not in fact charge-conjugate invarient, the number of superconvergent amplitudes is again somewhat reduced. But the amplitudes are no longer forced to be even or odd under $s \longleftrightarrow u$ crossing, so the sum-rules for electroproduction are then as follows. They are again valid for all non-positive definite t.

$(J^P, I)_M = (1^-, 0)$ $(\omega, \phi)$

Ordinary sum-rules on the ten amplitudes:

$A_1^V$, $A_5^{S,V}$, $A_{19}^{S,V}$, $A_{22}^{S,V}$, $A_{23}^{S,V}$, $A_{24}^{S,V}$,

and first moment sum-rules on the two amplitudes:

$A_{19}^{S,V}$. (List 4.5-62)

---

$(J^P, I)_M = (1^-, 1)$ $(\rho)$

Ordinary sum-rules on the sixteen amplitudes:

$A_1^{o,-}$, $A_5^{o,+,-}$, $A_{19}^{o,+,-}$, $A_{22}^{o,-}$, $A_{23}^{o,+,-}$, $A_{24}^{o,+,-}$

and first moment sum-rules on the three amplitudes:

$A_{19}^{o,+,-}$. (List 4.5-63)

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TABLE 4.5-I

CONTRIBUTIONS TO THE AMPLITUDES FOR PRODUCTION OF VECTOR MESONS DUE TO NORMAL TRAJECTORIES COUPLING VIA: $(g_{1,2})(f_{1,2,3,4,5})$.

| AMPLITUDE | CROSSING FACTOR | COUPLING INDEX jk, i.e., CONTRIBUTION DUE TO: $g_j(t)f_k(q^2,t)$. | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 11 | 12 | 13 | 14 | 15 | 21 | 22 | 23 | 24 | 25 |
| $A_1$ | + | . | . | 2 | . | 2 | . | . | 4 | . | 4 |
| $A_2$ | − | 1 | . | 3 | . | 3 | 3 | 1 | 3 | . | 3 |
| $A_3$ | + | 2 | 0 | 2 | . | 2 | 2 | 0 | 2 | . | 2 |
| $A_4$ | + | . | . | . | . | . | 2 | 0 | 2 | . | 2 |
| $A_5$ | − | . | . | . | . | . | . | . | 3 | . | 3 |
| $A_6$ | + | . | . | . | . | . | 2 | . | 2 | . | 4 |
| $A_7$ | − | . | . | . | . | . | 1 | 1 | 3 | . | 3 |
| $A_8$ | + | . | . | . | . | . | 2 | . | 2 | . | 2 |
| $A_9$ | − | . | . | . | . | . | 3 | 1 | 3 | . | 3 |
| $A_{10}$ | + | . | . | . | . | . | 2 | 0 | 2 | . | 2 |
| $A_{11}$ | − | . | . | . | . | . | 3 | 1 | 3 | . | 3 |
| $A_{12}$ | + | . | . | . | . | . | 2 | 0 | 2 | . | 2 |
| $A_{13}$ | − | . | . | 3 | . | 1 | 3 | 1 | 3 | . | 3 |
| $A_{14}$ | + | 2 | . | 3 | 0 | 2 | 2 | 0 | 2 | 2 | 2 |
| $A_{15}$ | − | . | . | . | . | . | 3 | 1 | 3 | . | 1 |
| $A_{16}$ | + | . | . | . | . | . | . | . | 4 | . | 2 |
| $A_{17}$ | − | . | . | . | . | . | 3 | 1 | 3 | 1 | 3 |
| $A_{18}$ | − | . | . | . | . | . | 3 | 1 | 3 | . | 3 |
| $A_{19}$ | + | . | . | . | . | . | . | . | . | . | . |
| $A_{20}$ | + | . | . | . | . | . | . | . | 2 | . | 2 |
| $A_{21}$ | + | . | . | . | . | . | . | . | 2 | . | 2 |
| $A_{22}$ | + | . | . | . | . | . | . | . | 2 | . | 2 |
| $A_{23}$ | − | . | . | . | . | . | 3 | . | 3 | . | 3 |
| $A_{24}$ | − | . | . | . | . | . | 3 | . | 3 | . | 3 |

TABLE 4.5-II

CONTRIBUTIONS TO THE AMPLITUDES FOR PRODUCTION OF VECTOR MESONS DUE TO ABNORMAL TRAJECTORIES

COUPLING VIA: $(g_{3,4})(f_{6,7,8,9})$.

| AMPLITUDE | CROSSING FACTOR | COUPLING INDEX jk, i.e., CONTRIBUTION DUE TO: $g_j(t)f_k(q^2,t)$. | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 36 | 37 | 38 | 39 | 46 | 47 | 48 | 49 |
| $A_1$ | + | 2 | . | 2 | 2 | 3 | . | 3 | 3 |
| $A_2$ | − | 1 | 1 | 3 | 1 | 2 | 2 | 2 | 2 |
| $A_3$ | + | . | . | . | . | 1 | 1 | 1 | 1 |
| $A_4$ | + | 0 | 0 | 2 | 2 | 1 | 1 | 1 | 1 |
| $A_5$ | − | . | . | . | . | 2 | . | 2 | 2 |
| $A_6$ | + | 2 | . | 2 | 2 | 1 | 1 | 3 | 3 |
| $A_7$ | − | 1 | 1 | 2 | 1 | 0 | 2 | 2 | 2 |
| $A_8$ | + | . | . | . | . | 1 | 1 | 3 | 1 |
| $A_9$ | − | 1 | 1 | 3 | 3 | 2 | 2 | 2 | 2 |
| $A_{10}$ | + | 0 | 0 | 2 | 2 | 1 | 1 | 1 | 1 |
| $A_{11}$ | − | 1 | 1 | 3 | 3 | 2 | 2 | 2 | 2 |
| $A_{12}$ | + | 0 | 0 | 2 | 2 | 1 | 1 | 1 | 1 |
| $A_{13}$ | − | 1 | 1 | 3 | 1 | 2 | 2 | 2 | 2 |
| $A_{14}$ | + | 0 | 0 | 2 | 0 | 1 | 1 | 1 | 1 |
| $A_{15}$ | − | 1 | 1 | 3 | 1 | 2 | 2 | 2 | 0 |
| $A_{16}$ | + | . | . | . | . | 3 | . | 3 | 1 |
| $A_{17}$ | − | 1 | 1 | 3 | 1 | 2 | 2 | 2 | 2 |
| $A_{18}$ | − | 1 | 1 | 3 | 1 | 2 | 2 | 2 | 2 |
| $A_{19}$ | + | 2 | . | 2 | 2 | . | . | . | . |
| $A_{20}$ | + | 2 | . | 2 | 2 | 1 | . | 3 | 3 |
| $A_{21}$ | + | 2 | . | 2 | 2 | 3 | . | 3 | 1 |
| $A_{22}$ | + | 2 | . | 2 | 2 | 3 | . | 3 | 3 |
| $A_{23}$ | − | 1 | . | . | . | . | . | 3 | . |
| $A_{24}$ | − | 1 | 1 | 3 | 1 | . | . | 3 | 3 |

# CHAPTER 5

## APPROXIMATE SATURATION OF THE SUM-RULES FOR REAL AND VIRTUAL PHOTONIC PRODUCTION OF PSEUDOSCALAR MESONS.

### 5.1 INTRODUCTORY REMARKS

### 5.11 THE BASIC SATURATION FORMULA.

Let $\Omega$ stand for either of the superscripts s and v if the meson is isoscalar, and for any one of the superscripts o,+, and -, if the meson is isovector. Then in the approximation of equation 1.3-47 as modified by the spin considerations of equations 2.41-27 and 28, a non-trivial mth. moment sum-rule on the amplitude $A_i^{\Omega}(\nu,t,q^2)$ reads:

$$\sum_R \left(2M_R^2-2m^2-\mu^2-q^2+t\right)^m\left[\frac{\pi}{2}+\tan^{-1}\left(\frac{M_R^2-S_0^{(R)}}{M_R\Gamma_R}\right)\right]A_i^{\Omega R}(t,q^2)$$
$$+\pi\left(t-\mu^2-q^2\right)^m A_i^{\Omega B}(t,q^2)=0, \qquad (5.11-1)$$

where R denotes an s-channel resonance, and B indicates the s-channel (nucleon) Born-term residue. With N and M denoting respectively a nucleon and the meson, the $A_i^{\Omega R}$ and $A_i^{\Omega B}$ are defined for isoscalar mesons by:

$$\sum_i\left(A_i^{sR}\mathbb{1}_2+A_i^{vR}\tau_3\right)\tilde{\mathcal{X}}_\alpha^i\equiv M_\alpha(\gamma+N\to R\to N+M)\Big|_{s=M_R^2}, \qquad (5.11-2)$$

$$\sum_i\left(A_i^{sB}\mathbb{1}_2+A_i^{vB}\tau_3\right)\tilde{\mathcal{X}}_\alpha^i\equiv M_\alpha(\gamma+N\to N\to N+M)\Big|_{s=m^2}, \qquad (5.11-3)$$

and for isovector mesons by:

$$\sum_i\left(A_i^{oR}\tau_j+A_i^{+R}\delta_{j3}+\tfrac{1}{2}A_i^{-R}[\tau_j,\tau_3]\right)\tilde{\mathcal{X}}_\alpha^i\equiv M_\alpha^j(\gamma+N\to R\to N+M)\Big|_{s=M_R^2}, \qquad (5.11-4)$$

$$\sum_i\left(A_i^{oB}\tau_j+A_i^{+B}\delta_{j3}+\tfrac{1}{2}A_i^{-B}[\tau_j,\tau_3]\right)\tilde{\mathcal{X}}_\alpha^i\equiv M_\alpha^j(\gamma+N\to N\to N+M)\Big|_{s=m^2} \qquad (5.11-5)$$

The $M_\alpha$ and $M_\alpha^j$ are the numerators of the pole graphs for the processes indicated in parentheses, and the intermediate states are to be treated as single stable particles.

## 5.15 IMPLICATIONS OF THE ISOSPIN STRUCTURE OF THE S-CHANNEL POLE GRAPH NUMERATORS.

The s-channel pole graph numerator for a single stable intermediate particle of spin $(J + \frac{1}{2})$ and normality $\pm 1$, has the general Lorentz-space structure:

$$M_\alpha^{J\pm}\Big|_{O(3,1)} \equiv \sum_i A_i^{J\pm}(t,q^2)\tilde{\mathcal{K}}_\alpha^i =$$

$$= \mathcal{C}^{\mp}(0\tfrac{1}{2},J+\tfrac{1}{2}):\mathcal{P}^{J+1/2}(K):\mathcal{V}_\alpha^{\pm}(J+\tfrac{1}{2},\tfrac{1}{2}\gamma)\ , \qquad (5.12\text{-}1)$$

where the right-hand side of the second equality is to be evaluated at s equal to the squared mass of the intermediate particle.

The full Lorentz $\otimes$ SU(2) structure of this pole graph numerator is then given by:

$$\left.\begin{matrix} M_\alpha^{J\pm} \\ M_\alpha^{jJ\pm} \end{matrix}\right\} \equiv \sum_i \left\{\begin{matrix} A_i^{S,J\pm}\mathbb{1}_2 + A_i^{V,J\pm}\tau_3 \\ A_i^{0,J\pm}\tau_j + A_i^{+,J\pm}\delta_{j3} + \frac{1}{2}A_i^{-,J\pm}[\tau_j,\tau_3] \end{matrix}\right\}\tilde{\mathcal{K}}_\alpha^i =$$

$$= \left[\mathcal{C}^{\mp}\left\{\begin{matrix}\mathcal{K}(0,\tfrac{1}{2};I+\tfrac{1}{2}) \\ \mathcal{K}_j(1,\tfrac{1}{2};I+\tfrac{1}{2})\end{matrix}\right\}\right]:\left[\mathcal{P}^{J+1/2}(K)\mathcal{P}^{I+1/2}\right]:\left[\mathcal{V}_\alpha^{S,\pm}\mathcal{K}^S(I+1/2;1/2,\gamma)\right.$$

$$\left. + \mathcal{V}_\alpha^{V,\pm}\mathcal{K}^V(I+1/2;1/2,\gamma)\right],\ \text{for } I_M = \begin{cases}0\\1\end{cases}. \qquad (5.12\text{-}2)$$

In this equation $(I + \frac{1}{2})$ is the isospin of the intermediate particle, and $\mathcal{V}_\alpha^{S,\pm}$ $(\mathcal{V}_\alpha^{V,\pm})$ is to be obtained from $\mathcal{V}_\alpha^{\pm}$ by substituting the corresponding isoscalar (isovector) O(3,1)$\otimes$SU(2) form-factor for each O(3,1) form-factor appearing.

If the final meson is isoscalar we can only have intermediate particles with isospin one half, and from sections 2.22, 2.32, and 2.6:

$$\mathcal{K}(0\tfrac{1}{2},\tfrac{1}{2}) = \mathcal{K}^S(\tfrac{1}{2},\tfrac{1}{2}\gamma) = \mathcal{P}^{1/2} = \mathbb{1}_2\ , \qquad (5.12\text{-}3,4,5)$$

$$\mathcal{K}^V(\tfrac{1}{2},\tfrac{1}{2}\gamma) = \tau_3\ . \qquad (5.12\text{-}6)$$

Thus:

$$M_\alpha^{J\pm} = \mathcal{C}^{\mp}:\mathcal{P}^{J+1/2}(K):\left(\mathcal{V}_\alpha^{S,\pm}\mathbb{1}_2 + \mathcal{V}_\alpha^{V,\pm}\tau_3\right). \qquad (5.12\text{-}7)$$

Comparing with equations 5.12-1 and 2 we see that if F denotes any form-factor resulting from the spin decomposition of $\mathcal{V}_\alpha^{\pm}$, then the $A_i^{\Omega,J\pm}$ may be obtained from $A_i^{J\pm}$ by the substitutions:

$$A_i^{S,J\pm} = A_i^{J\pm}(F)\Big|_{F\to F^S} , \qquad (5.12\text{-}8)$$

$$A_i^{V,J\pm} = A_i^{J\pm}(F)\Big|_{F\to F^V} , \qquad (5.12\text{-}9)$$

for all F.

Intermediate particles with isospins one-half or three-halves are allowed if the final meson is isovector. So in addition to equations 5.12-4,5,6 we need:

$$\mathcal{K}_j(1\tfrac{1}{2},\tfrac{1}{2}) = \tau_j \quad , \quad \mathcal{K}_{jk}(1\tfrac{1}{2},\tfrac{3}{2}) = \delta_{jk}\mathbb{1}_2 \quad , \qquad (5.12\text{-}10,11)$$

$$\mathcal{K}_\ell^V(\tfrac{3}{2},\tfrac{1}{2}\gamma) = \delta_{\ell 3}\mathbb{1}_2 \, , \quad \mathcal{P}_{k\ell}^{3/2} = \tfrac{1}{3}\left(2\delta_{k\ell} - \tfrac{1}{2}[\tau_k,\tau_\ell]\right), \qquad (5.12\text{-}12,13)$$

yielding:

$$M_\alpha^{j\,J\pm}(I=\tfrac{1}{2}) = \mathcal{C}^{\mp} : \mathcal{P}^{J+1/2}(K) : \left\{ \mathcal{V}_\alpha^{S,\pm}\tau_j + \mathcal{V}_\alpha^{V,\pm}\left(\delta_{j3} + \tfrac{1}{2}[\tau_j,\tau_3]\right)\right\}, \qquad (5.12\text{-}14)$$

and:

$$M_\alpha^{j\,J\pm}(I=\tfrac{3}{2}) = \mathcal{C}^{\mp} : \mathcal{P}^{J+1/2}(K) : \mathcal{V}_\alpha^{V,\pm}\tfrac{1}{3}\left(2\delta_{j3} - \tfrac{1}{2}[\tau_j,\tau_3]\right) . \qquad (5.12\text{-}15)$$

The isospace covarient $\mathcal{K}_\ell^S(\tfrac{3}{2},\tfrac{1}{2}\gamma)$ corresponds to a disallowed coupling and hence $M_\alpha^{jJ\pm}(I=\tfrac{3}{2})$ receives no contribution from $\mathcal{V}_\alpha^{S,\pm}$. Comparison with 5.12-1 and 2 yields the following isovector meson analogues of equations 5.12-8 and 9:

$$A_i^{0,J\pm}(I=\tfrac{1}{2}) = A_i^{J\pm}(F)\Big|_{F\to F^S} , \qquad (5.12\text{-}16)$$

$$A_i^{0,J\pm}(I=\tfrac{3}{2}) = 0 , \qquad (5.12\text{-}17)$$

$$A_i^{+,J\pm}(I=\tfrac{1}{2}) = A_i^{J\pm}(F)\Big|_{F\to F^V} , \qquad (5.12\text{-}18)$$

$$A_i^{+,J\pm}(I=\tfrac{3}{2}) = \tfrac{2}{3}A_i^{J\pm}(F)\Big|_{F\to F^V} , \qquad (5.12\text{-}19)$$

$$A_i^{-,J\pm}(I=\tfrac{1}{2}) = A_i^{J\pm}(F)\Big|_{F\to F^V} , \qquad (5.12\text{-}20)$$

$$A_i^{-,J\pm}(I=\tfrac{3}{2}) = -\tfrac{1}{3}A_i^{J\pm}(F)\Big|_{F\to F^V} . \qquad (5.12\text{-}21)$$

for all F.

Having obtained equations 5.12-8,9, and 16 to 21 we may drop the isospin dependence from our saturation graphs and work in spin-space alone.

## 5.13 A NOTE ON THE O(3,1) STRUCTURE OF THE S-CHANNEL POLE GRAPH NUMERATORS.

In the following section we compute in spin-space alone the s-channel pole graph numerators needed for saturation of our sum-rules. These come out naturally in terms of the structure:

$$M_\alpha = \sum_{i=1}^{8} B_i(\text{masses, coupling constants, form-factors, } s[\longrightarrow \text{squared intermediate mass}], t, q^2)\mathcal{L}^i_\alpha \equiv \Big[B_1[\gamma_\alpha, \not{q}] + B_2 p'_\alpha + B_3 p_\alpha + B_4\gamma_\alpha + B_5 p'_\alpha \not{q} + B_6 p_\alpha \not{q} + B_7 q_\alpha + B_8 q_\alpha \not{q}\Big]\gamma_5 \quad , \quad (5.13\text{-}1)$$

where the $B_i$ do not involve any poles in s,t, or $q^2$.

If $M_\alpha$ vanishes on contraction with $q_\alpha$, the $B_i$ must satisfy:

$$p'\cdot q B_2 + p\cdot q B_3 = -q^2 B_7 \quad , \quad (5.13\text{-}2)$$

$$B_4 + p'\cdot q B_5 + p\cdot q B_6 = -q^2 B_8 \quad , \quad (5.13\text{-}3)$$

and from equations 4.11-11 to 16 we then have:

$$M_\alpha = B_1 \tilde{\mathcal{K}}^1_\alpha + (B_2+B_3)\tilde{\mathcal{K}}^2_\alpha/\Delta\cdot q + \tfrac{1}{2}(B_5-B_6)\tilde{\mathcal{K}}^3_\alpha + (B_5+B_6)\tilde{\mathcal{K}}^4_\alpha + (p'\cdot q B_2 + p\cdot q B_3)\tilde{\mathcal{K}}^5_\alpha/(q^2\Delta\cdot q) + (B_4 + p'\cdot q B_5 + p\cdot q B_6)\tilde{\mathcal{K}}^6_\alpha/q^2 \,. \quad (5.13\text{-}4)$$

This satisfies our counting rules for all $q^2$, since in view of 5.13-2 and 3 the coefficients of $\tilde{\mathcal{K}}^5_\alpha$ and $\tilde{\mathcal{K}}^6_\alpha$ are free of poles in $1/q^2$.

In practice we shall drop terms in $q_\alpha$ whenever they appear in the calculation, but since our initial expression will always contain a photonic vertex function $V_\alpha$ which vanishes on contraction with $q_\alpha$, $(p'.qB_2 + p.qB_3)$ and

$(B_4 + p'.qB_5 + p.qB_6)$ will still be proportional to $q^2$.

Even if $V.q$ does not vanish, we can still write $M_\alpha$ in the form 5.13-4 by using the fact that $q_\alpha/q^2$ and $q_\alpha \not{q}/q^2$ are both equivalent to zero for all $q^2$. That is, the equality in this equation is then replaced by an equivalence. However, 5.13-2 and/or 5.13-3 will no longer be satisfied and our final expression for $M_\alpha$ will violate the counting rules in the real photon limit.

Note that at zero $q^2$ 5.13-2 reads:

$$\left.\frac{(B_2+B_3)}{\Delta\cdot q}\right|_{q^2=0} = \left.\frac{2(B_2+B_3)}{t-\mu^2}\right|_{q^2=0} = \left.\frac{-2B_2}{s-m^2}\right|_{q^2=0} = \left.\frac{2B_3}{(s-m^2)+(t-\mu^2)}\right|_{q^2=0}. \quad (5.13\text{-}4)$$

Thus if $M.q$ vanishes, the coefficient of $\tilde{\mathcal{K}}^2_\alpha$ remains finite in the real photon limit for all t provided s is not equal to $m^2$. This is the situation for our resonance pole graph numerators where s is equal to $M_R^2$. The fact that $M.q$ is zero then forces $B_3$ to be proportional to $(t+M_R^2-m^2-\mu^2)$.

In the case of the s-channel Born-term residue $B_2$ fortunately vanishes and the coefficient of $\tilde{\mathcal{K}}^2_\alpha$ just has the pole at zero $(t-\mu^2)$ in the real photon limit.

For non-vanishing $q^2$ the coefficient of $\tilde{\mathcal{K}}^2_\alpha$ has a pole even in the case of the resonance graphs, and the coefficient of $\tilde{\mathcal{K}}^5_\alpha$ has a similar pole for all $q^2$.

Except in the case of the photoproduction Born terms these poles have no obvious dynamical origin, and would seem to be kinematical. They are essential, however, if the correct number of covarients are to be proportional to $q^2$.

The dynamical interpretation of the $(t-\mu^2)^{-1}$ pole in the amplitude of the photoproduction Born-terms is well known. The photon-nucleon-nucleon vertex relies for its gauge invarience on the Dirac equation, and the s and u channel Born-terms are therefore only gauge-invarient at their respective poles. The same applies to the pion Born term since the photon-pion-pion vertex is proportional to $\Delta_\alpha$. The $(t-\mu^2)^{-1}$ pole in the $\tilde{\mathcal{K}}^2_\alpha$ amplitudes ensures that the sum of the three Born terms remains gauge invarient for arbitrary s,t,u. Failure of gauge invarience is accompanied, if one is treating photoproduction as a limit of electroproduction, by a violation of the counting rules. Away from their respective poles the three Born-graphs gain terms in $\tilde{\mathcal{K}}^5_\alpha/q^2$. The $(t-\mu^2)^{-1}$ poles in the coefficients of these terms ensures that they cancel from the sum of the three Born-graphs.

Our graphs for intermediate resonances with spin greater than one half turn out to have rather complicated structures. We shall therefore compute them in terms of the $\mathcal{L}^{1,\dots,6}_\alpha$ and will pick out only those combinations of the corresponding $\mathcal{B}^{1,\dots,6}_\alpha$ needed for sum rule saturation.

## 5.2 COMPUTATION OF THE O(3,1) S-CHANNEL POLE GRAPH NUMERATORS NEEDED FOR SATURATION.

### 5.21 THE BORN-TERM.

This is well known. One has:

$$M^B_\alpha = \mathcal{V}^-(0\tfrac{1}{2},\tfrac{1}{2})\,\mathcal{P}^{1/2}(K)\,\mathcal{V}^+_\alpha(\tfrac{1}{2},\tfrac{1}{2}\gamma) \,, \tag{5.21-1}$$

where:

$$\mathcal{V}^-(0\tfrac{1}{2},\tfrac{1}{2}) = g\gamma_5 \,, \tag{5.21-2}$$

$$\mathcal{P}^{\frac{1}{2}}(K) = \not{K} + m \,, \tag{5.21-3}$$

$$\mathcal{V}^+_\alpha(\tfrac{1}{2},\tfrac{1}{2}\gamma) = e\left[F_1(q^2)\gamma_\alpha + \frac{i}{2m}F_2(q^2)\sigma_{\alpha\beta}q_\beta\right] . \tag{5.21-4}$$

After using the Dirac equation to express $\not{K}$ in terms of m and $\not{q}$ one easily obtains:

$$M^B_\alpha\Big|_{s=m^2} = \frac{eg}{2m}\left[-m(F_1+F_2)\mathcal{L}^1_\alpha + 4mF_1\mathcal{L}^3_\alpha + 2F_2\mathcal{L}^6_\alpha\right] \,, \tag{5.21-5}$$

and it then follows from 5.13-4 that:

$$M_\alpha^B\Big|_{s=m^2} = \frac{eg}{2m}\Big[-m(F_1+F_2)\tilde{\mathcal{K}}_\alpha^1 + \frac{4m}{\Delta\cdot q}F_1\tilde{\mathcal{K}}_\alpha^2 - F_2\tilde{\mathcal{K}}_\alpha^3 + 2F_2\tilde{\mathcal{K}}_\alpha^4 - \frac{2m}{\Delta\cdot q}F_1\tilde{\mathcal{K}}_\alpha^5 - F_2\tilde{\mathcal{K}}_\alpha^6\Big] . \qquad (5.21\text{-}6)$$

## 5.22 INTERMEDIATE RESONANCES WITH SPIN $\frac{1}{2}$.

In this case we have:

$$M_\alpha^{\frac{1}{2}\pm} = \mathcal{V}^\mp(0\tfrac{1}{2},\tfrac{1}{2})\mathcal{P}^{\frac{1}{2}}(K)\mathcal{V}_\alpha^\pm(\tfrac{1}{2},\tfrac{1}{2}\gamma) , \qquad (5.22\text{-}1)$$

where: $$\mathcal{V}^\mp(0\tfrac{1}{2},\tfrac{1}{2}) = gI^\mp , \qquad (5.22\text{-}2)$$

$$\mathcal{P}^{\frac{1}{2}}(K) = K\!\!\!/ + M , \qquad (5.22\text{-}3)$$

$$\mathcal{V}_\alpha^\pm(\tfrac{1}{2},\tfrac{1}{2}\gamma) = e\Big[F_1(q^2)(q^2\gamma_\alpha - q\!\!\!/ q_\alpha) \pm F_2(q^2)i\sigma_{\alpha\beta}q_\beta\Big]I^\pm . \qquad (5.22\text{-}4)$$

In these equations the plus/minus superscript on the M-function indicates the normality, n, of the intermediate resonance, and M is the mass of this resonance. The decomposition of the photonic vertex is taken from 2.71-21, except that for later convenience we have introduced a plus/minus sign into the definition of $F_2$, and have explicitly exhibited a factor e. As in section 2.8 we define:

$$N \equiv nM , \qquad (5.22\text{-}5)$$

and 5.22-1 may then be written:

$$M_\alpha^{\frac{1}{2}N} = eg(K\!\!\!/ - N)\Big[F_1(q^2\gamma_\alpha - q\!\!\!/ q_\alpha) + F_2 i\sigma_{\alpha\beta}q_\beta\Big]\gamma_5 . \qquad (5.22\text{-}6)$$

After a little Dirac algebra we obtain:

$$M_\alpha^{\frac{1}{2}N}\Big|_{s=N^2} \cong eg\Big\{-\tfrac{1}{2}\big[q^2F_1 + (N+m)F_2\big]\mathcal{L}_\alpha^1 + 2q^2F_1\mathcal{L}_\alpha^3 - (N-m)\big[q^2F_1 + (N+m)F_2\big]\mathcal{L}_\alpha^4 + 2F_2\mathcal{L}_\alpha^6\Big\} , \qquad (5.22\text{-}7)$$

from which we have finally:

$$M_\alpha^{\frac{1}{2}N}\Big|_{s=N^2} \cong eg\Big\{-\tfrac{1}{2}\big[q^2F_1 + (N+m)F_2\big]\tilde{\mathcal{K}}_\alpha^1 + \frac{2q^2}{\Delta\cdot q}F_1\tilde{\mathcal{K}}_\alpha^2 - F_2\tilde{\mathcal{K}}_\alpha^3 + 2F_2\tilde{\mathcal{K}}_\alpha^4 + \frac{1}{\Delta\cdot q}(N^2-m^2-q^2)F_1\tilde{\mathcal{K}}_\alpha^5 - \big[(N-m)F_1 + F_2\big]\tilde{\mathcal{K}}_\alpha^6\Big\} . \qquad (5.22\text{-}8)$$

We note as a check that this reduces to equation 5.21-6 under the substitutions:

$$F_1 \to F_1/q^2 \;, \quad F_2 \to F_2/2m \;, \quad N \to m \;. \qquad (5.22\text{-}9,10,11)$$

## 5.23 INTERMEDIATE RESONANCES WITH SPIN $(J+\frac{1}{2}) \geq 3/2$.

We define n, M, and N as in the previous section and now have:

$$M_\alpha^{J\pm} = \mathcal{C}^{\mp}_{(\nu)J}(0\tfrac{1}{2}, J+\tfrac{1}{2})\,\mathcal{P}^{J+1/2}_{(\nu)J;(\sigma)J}(K)\,\mathcal{V}^{\pm}_{(\sigma)J\alpha}(J+\tfrac{1}{2}, \tfrac{1}{2}\gamma) \;, \qquad (5.23\text{-}1)$$

where:
$$\mathcal{C}^{\mp}_{(\nu)J}(0\tfrac{1}{2}, J+\tfrac{1}{2}) = g\,(p'_\nu)^J I^{\mp} \;, \qquad (5.23\text{-}2)$$

and the decomposition of $\mathcal{V}^{\pm}_{(\sigma)J\alpha}(J+\frac{1}{2};\frac{1}{2},\gamma)$ is given by equations 2.72-8 to 11. We shall modify 2.72-11 by again explicitly exhibiting a factor of e on the right-hand side. As we are only interested in evaluating $M_\alpha^{J\pm}$ at the point s equals $M^2$ we may, in view of the structure of the propogator, make the substitution:

$$\not{q}\, I^{\pm} \to (M \mp m) I^{\pm} \qquad (5.23\text{-}3)$$

in $\mathcal{V}^{\pm}_{(\sigma)J\alpha}$.

Dropping terms in $q_\alpha$, equation 5.23-1 then reads:

$$M_\alpha^{J\pm}\Big|_{s=M^2} \cong e g I^{\mp}\Big\{\mathcal{P}^{J+1/2}(p',q;K)\big(G_1\gamma_\alpha \pm G_2 p_\alpha\big) - \mathcal{P}^{J+1/2}_{;\alpha}(p',q;K)\big[(M\mp m)G_1 \pm p\cdot q\, G_2 \mp q^2 G_3\big]\Big\} I^{\pm} \;. \qquad (5.23\text{-}4)$$

From Scadron's paper[9] we have at $s=M^2$:

$$\mathcal{P}^{J+1/2}(p',q;K) = \frac{C_{J+1}}{M^2(J+1)}\Big[M^2(\not{K}+M)\mathcal{P}'_{J+1} - (M\not{p}'+p'\cdot K)(\not{K}-M)(M\not{q}+q\cdot K)\mathcal{P}'_J\Big] \;, \qquad (5.23\text{-}5)$$

and:

$$\mathcal{P}^{J+1/2}_{;\alpha}(p',q;K) = \frac{C_{J+1}}{M^2 J(J+1)}\Big\{(M\not{p}'+p'\cdot K)(\not{K}-M)(M\not{q}+q\cdot K)\big[p'_\alpha(K)\mathcal{P}''_J + p'^2(K)q_\alpha(K)\mathcal{P}''_{J-1}\big] - M^2(\not{K}+M)\big[p'_\alpha(K)\mathcal{P}''_{J+1} + p'^2(K)q_\alpha(K)\mathcal{P}''_J\big] - (M\not{p}'+p'\cdot K)(\not{K}-M)(M\gamma_\alpha+K_\alpha)\mathcal{P}'_J\Big\} \;, \qquad (5.23\text{-}6)$$

where each of the solid harmonic derivatives has argument: $-p'(K)\cdot q(K)$.

After some tedious algebra, we therefore obtain in view of the Rarita-Schwinger subsidiary conditions and the Dirac equation:

$$
\begin{aligned}
M_{\alpha}^{JN}\Big|_{S=N^2} \cong \frac{egC_{J+1}}{N^4 J(J+1)}\Big[ & N^4 J\mathcal{P}'_{J+1}\Big\{\big[(N+m)\mathcal{L}_{\alpha}^{3}-\mathcal{L}_{\alpha}^{6}\big]G_2-\big[\tfrac{1}{2}\mathcal{L}_{\alpha}^{1}-2\mathcal{L}_{\alpha}^{3}+(N-m)\mathcal{L}_{\alpha}^{4}\big]G_1\Big\} \\
& +N^2 J\mathcal{P}'_{J}(p'\cdot K-Nm)\Big\{\big[\tfrac{1}{2}(q\cdot K-N^2+Nm)\mathcal{L}_{\alpha}^{1}-2q\cdot K\mathcal{L}_{\alpha}^{3}+(Np\cdot q \\
& -mq\cdot K)\mathcal{L}_{\alpha}^{4}-2N\mathcal{L}_{\alpha}^{6}\big]G_1-\big[(Np\cdot q+mq\cdot K)\mathcal{L}_{\alpha}^{3}+(N^2+Nm-q\cdot K)\mathcal{L}_{\alpha}^{6}\big]G_2\Big\} \\
& -\Big\{N^2\Big\{\big[p'\cdot K\mathcal{P}''_{J+1}+p'^2(K)q\cdot K\mathcal{P}''_{J}\big]\big[(N+m)\mathcal{L}_{\alpha}^{3}-\mathcal{L}_{\alpha}^{6}\big]-N^2\mathcal{P}''_{J+1}\big[(N \\
& +m)\mathcal{L}_{\alpha}^{2}-\mathcal{L}_{\alpha}^{5}\big]\Big\}+(p'\cdot K-Nm)\Big\{N^2\mathcal{P}'_{J}\big[\tfrac{1}{2}N\mathcal{L}_{\alpha}^{1}-(N+m)\mathcal{L}_{\alpha}^{3}-N(N+m)\mathcal{L}_{\alpha}^{4} \\
& +\mathcal{L}_{\alpha}^{6}\big]+N^2\mathcal{P}''_{J}\big[(Np\cdot q+mq\cdot K)\mathcal{L}_{\alpha}^{2}+(N^2+Nm-q\cdot K)\mathcal{L}_{\alpha}^{5}\big] \\
& -\big[p'\cdot K\mathcal{P}''_{J}+p'^2(K)q\cdot K\mathcal{P}''_{J-1}\big]\big[(Np\cdot q+mq\cdot K)\mathcal{L}_{\alpha}^{3}+(N^2+Nm \\
& -q\cdot K)\mathcal{L}_{\alpha}^{6}\big]\Big\}\Big\}\big[(N-m)G_1+p\cdot q\,G_2-q^2G_3\big]\Big]. \qquad (5.23\text{-}7)
\end{aligned}
$$

The solid harmonic derivatives are given by:

$$\mathcal{P}_{J}^{(n)}[-p'(K)\cdot q(K)]=(-1)^{J-n}\left[p'^2(K)q^2(K)\right]^{\frac{J-n}{2}}P_{J}^{(n)}\left\{\frac{p'(K)\cdot q(K)}{[p'^2(K)q^2(K)]^{1/2}}\right\}, \qquad (5.23\text{-}8)$$

so to evaluate $M_{\alpha}^{JN}\Big|_{S=N^2}$ we need the following relations obtained from Appendix 5:

$$p'\cdot K\Big|_{S=N^2}=\tfrac{1}{2}R\,, \qquad p\cdot q\Big|_{S=N^2}=\tfrac{1}{2}(N^2-m^2-q^2)\,, \qquad (5.23\text{-}9,10)$$

$$q\cdot K\Big|_{S=N^2}=\tfrac{1}{2}(N^2-m^2+q^2)\,, \qquad p'^2(K)\Big|_{S=N^2}=\frac{1}{4N^2}(4N^2m^2-R^2)\,, \qquad (5.23\text{-}11,12)$$

$$q^2(K)\Big|_{S=N^2}=\frac{-1}{4N^2}\left[(N+m)^2-q^2\right]\left[(N-m)^2-q^2\right]\,, \qquad (5.23\text{-}13)$$

$$q(K)\cdot p'(K)\Big|_{S=N^2}=\frac{1}{4N^2}\left[R(N^2+m^2-q^2)-4N^2m^2+2N^2t\right]\,, \qquad (5.23\text{-}14)$$

where we have defined:

$$R\equiv N^2+m^2-\mu^2\,. \qquad (5.23\text{-}15)$$

Since $M_{\alpha}^{JN}\big|_{S=N^2}$ depends on $\mu$ only through its dependence on R, our expressions for specific $M_{\alpha}^{JN}\big|_{S=N^2}$ are considerably simplified if we work with R rather than $\mu$. When saturating sum-rules, however, we must not lose sight of the fact that R also depends on $N^2$ and $m^2$.

The computation of $M_{\alpha}^{JN}\big|_{S=N^2}$ for a given value of J is achieved by expanding out the solid harmonic derivatives and then invoking equations 5.23-9 to 14. Before doing this it is useful to deduce in a general way the basic structure of $M_{\alpha}^{JN}\big|_{S=N^2}$ as a polynomial in t and $q^2$. This is straightforward, and from the standard equation for the expansion of the nth. derivative of a Legendre polynomial we find:

$$M_{\alpha}^{JN}\Big|_{S=N^2} \sim eg\Big\{(a+bq^2+ct)^J(G_1\mathcal{L}_{\alpha}^{1,3,4}+G_2\mathcal{L}_{\alpha}^{3,6}) \\ +(d+fq^2+ht)^{J-1}\big[G_1\mathcal{L}_{\alpha}^{2,5,6}+(j+kq^2)G_2\mathcal{L}_{\alpha}^{1,2,4,5} \\ +q^2G_3\mathcal{L}_{\alpha}^{1,2,3,4,5,6}\big]\Big\} \quad , \qquad (5.23\text{-}16)$$

where a,b,c,d,f,h,j,k denote functions only of the masses. The notation of this equation is manifestly loose and inexact, but its meaning should be clear to the reader. It is meant to indicate the values of the integers r and s for all terms of the type $(q^2)^r(t)^s$ appearing in the coefficient of each of the eighteen $egG_{1,..,3}\mathcal{L}_{\alpha}^{1,...,6}$. A knowledge of the powers of t appearing in each such coefficient is particularly useful, since we are eventually going to separately equate to zero the coefficient of each power of t appearing in the sum-rules. We are now able in advance to deduce, to within polynomials in the masses, the structure of these equations for any attempted saturation of a particular sum-rule.

## 5.24 CONTRIBUTIONS TO THE SUM-RULES ON $A_{3,6}$ FROM RESONANCES WITH SPIN THREE-HALVES.

From equation 5.23-16 we may usefully define in the case of intermediate resonances with spin three-halves:

$$M_\alpha^{1N}\Big|_{s=N^2} \equiv \frac{eg}{12N^2}\sum_{i=1}^{6}\sum_{k=1}^{3}\sum_{r=0}^{1} a_{ik}^r t^r G_k \tilde{\mathcal{K}}_\alpha^i \quad , \tag{5.24-1}$$

where the $a_{ik}^r$ are functions only of $q^2$ and the masses. From equation 2.21-6 we have:

$$\frac{c_{J+1}}{J(J+1)}\Big|_{J=1} = \frac{1}{3} \quad , \tag{5.24-2}$$

and we have chosen for convenience to further factor out the quantity: $eg/4N^2$.

In this case we require the expansions:

$$\mathcal{P}_1' = 1 \quad , \qquad \mathcal{P}_2' = -3p'(K)\cdot q(K) \quad , \tag{5.24-3,4}$$

$$\mathcal{P}_0'' = 0 = \mathcal{P}_1'' \quad , \qquad \mathcal{P}_2'' = 3 \quad . \tag{5.24-5,6,7}$$

Substitution of these into equation 5.23-7, followed by the use of equations 5.23-9 to 14 and 5.13-4 enables us to determine all thirty-six $a_{ik}^r$.

Some of these are quite complicated and we certainly don't propose to bore the reader by listing them all here. Instead we merely give those which will be needed later, namely the $a_{3k}^r$ and $a_{6k}^r$. These are as follows:

$$a_{31}^1 = 0 = a_{33}^1 \quad , \qquad a_{32}^1 = -3N^2 \quad , \tag{5.24-8,9,10}$$

$$a_{31}^0 = 2\left[mR - N(3N^2 - m^2)\right] \quad , \tag{5.24-11}$$

$$a_{32}^0 = \left[2R + N(3N+2m)\right]q^2 - N\left[(2N-m)R + N(3N^2+2Nm-7m^2)\right] \quad , \tag{5.24-12}$$

$$a_{33}^0 = 2\left[R + N(3N+m)\right]q^2 \quad , \tag{5.24-13}$$

$$a^1_{61} = 0 = a^1_{62} \ , \qquad a^1_{63} = 6N^2 = -2a^1_{32} \ , \qquad (5.24\text{-}14 \text{ to } 17)$$

$$a^0_{61} = -4(N-m)(R+Nm) \ , \qquad (5.24\text{-}18)$$

$$a^0_{62} = 2\left[(R+Nm)q^2 - (N^2-m^2)(2R+Nm)\right] \ , \qquad (5.24\text{-}19)$$

$$a^0_{63} = 2\left[(R+Nm)q^2 + (N^2-Nm+m^2)R + Nm(N^2-4Nm+m^2)\right] \ . \qquad (5.24\text{-}20)$$

## 5.25 CONTRIBUTIONS TO THE SUM-RULES ON $A_{3,6}$ FROM RESONANCES WITH SPIN FIVE-HALVES.

Here we may usefully define:

$$M^{2N}_{\alpha}\Big|_{S=N^2} \equiv \frac{eg}{80N^4}\sum_{i=1}^{6}\sum_{k=1}^{3}\sum_{r=0}^{2} b^r_{ik}\, t^r G_k \tilde{\mathcal{K}}^i_\alpha \ , \qquad (5.25\text{-}1)$$

where the $b^r_{ik}$ are again functions only of $q^2$,R,N, and m. We have used the fact that:

$$\frac{C_{J+1}}{J(J+1)}\Big|_{J=2} = \frac{1}{15} \ , \qquad (5.25\text{-}2)$$

and have also chosen to explicitly exhibit a factor: $3eg/16N^4$.

In addition to the expansions of $\mathcal{P}'_2$, $\mathcal{P}''_1$, and $\mathcal{P}''_2$ given in the previous section we now need:

$$\mathcal{P}'_3 = \frac{3}{2}\left[5\left(p'(K)\cdot q(K)\right)^2 - p'^2(K)q^2(K)\right] \ , \qquad (5.25\text{-}3)$$

$$\mathcal{P}''_3 = -15\, p'(K)\cdot q(K) \ . \qquad (5.25\text{-}4)$$

The fifty-four $b^r_{ik}$ may then be obtained in the same manner as were the $a^r_{ik}$ of the previous section.

Again we list only those $b^r_{3k}$ and $b^r_{6k}$ which we shall need later. They are as follows:

$$b^2_{31} = 0 = b^2_{33} \ , \qquad b^2_{32} = 10N^4 \ , \qquad (5.25\text{-}5,6,7)$$

$$b^1_{31} = 4N^2\left[N(5N^2-m^2) - 2mR\right] \ , \qquad (5.25\text{-}8)$$

$$b^{0}_{31}=2\Big\{2\big[mR^{2}-N(2N^{2}+Nm-m^{2})R-N^{2}m(N-m)^{2}\big]q^{2}$$
$$-m(3N^{2}+m^{2})R^{2}+2N\big[(N+m)(2N^{3}-2N^{2}m+5Nm^{2}-m^{3})R$$
$$+Nm(N^{4}-10N^{3}m+2Nm^{3}-m^{4})\big]\Big\}\,, \qquad (5.25\text{-}9)$$

$$b^{2}_{61}=0=b^{2}_{62}\,, \qquad b^{2}_{63}=-20N^{4}=-2b^{2}_{32}\,, \qquad (5.25\text{-}10 \text{ to } 13)$$

$$b^{1}_{61}=4N^{2}\big[4(N-m)R+Nm(3N-m)\big]\,, \qquad (5.25\text{-}14)$$

$$b^{0}_{61}=2(N-m)\Big\{(3N^{2}-2Nm+m^{2})R^{2}+4Nm\big[(N^{2}-3Nm+m^{2})R$$
$$-2Nm^{2}(N-m)\big]-R(3R+4Nm)q^{2}\Big\}\,. \qquad (5.25\text{-}15)$$

## 5.3 PRELIMINARY CONSIDERATIONS REGARDING POSSIBLE SATURATIONS.

Before plunging into an attempted saturation of a particular sum-rule with the Born-term plus a given superposition of resonances, one would like to know whether such a saturation is likely to prove fruitful. We now investigate the extent to which such pre-cognition is furnished in the general results of sections 5.1 and 2.

Firstly we show that sum-rules on $A_7$ as defined by equation 4.4-41 are not saturable with the Born-term plus a superposition of resonances of finite spin.

Equation 5.21-6 indicates that in the real photon limit $t\ \mathrm{disc}_\nu A_2$ receives a contribution:

$$\frac{4eg\,F_1(0)t}{t-\mu^{2}}=-4eg\,F_1(0)\sum_{r=1}^{\infty}\left(\frac{t}{\mu^{2}}\right)^{r},\quad |t|<\mu^{2}\,, \qquad (5.3\text{-}1)$$

from the Born-term numerator. On the other hand, we saw in section 5.13 that the contribution to $t\ \mathrm{disc}_\nu A_2$ from any resonance graph was non-infinite for all $t$ at vanishing $q^2$. We have also seen that $t\ \mathrm{disc}_\nu(2A_1+mA_{11})$ does not receive $1/\Delta\cdot q$ poles from the Born-term or the resonance graphs. Thus if

$(J + \frac{1}{2})$ is the spin of the highest spin resonance(s) used in the saturation, we see from equations 5.13-4 and 5.23-16 that the highest power of t appearing in the contribution to $disc_\nu A_7$ of the superposition of resonances will be $t^{J+1}$. After differentiating the sum-rule (J+2) times with respect to t, we shall therefore obtain on setting t and $q^2$ equal to zero:

$$eg F_1(0) = 0 \, , \tag{5.3-2}$$

where $F_1(o)$ will carry the superscript s or v according as our sum-rule is on $A_7^{S,0}$ or $A_7^{V,+,-}$. But our nucleon form-factors are normalised to:

$$F_1^{S,V}(0) = \frac{1}{2} \quad , \tag{5.3-3}$$

so for finite J the sum-rule is not saturable at vanishing $q^2$ unless g vanishes. This would certainly not appear to be the case for the pion or for the pionic resonances.

For non-vanishing $q^2$ the above argument does not immediately apply since the resonance graphs then contribute kinematical $1/\Delta\cdot q$ poles to t $disc_\nu A_2$. However, the form-factors are supposed to be analytic in $q^2$ at zero $q^2$, so any attempt to saturate the sum-rule for non-vanishing $q^2$ will yield predictions which will tend smoothly to nonsense as $q^2$ tends to zero.

We therefore scrap our sum-rules on $A_7$ and turn our attention to those on $A_3$ and $A_6$. The above considerations do not apply to these latter since no $1/\Delta\cdot q$ poles are involved.

On defining:

$$M_\alpha^{JN}\Big|_{S=N^2} \equiv ge \sum_{i=1}^{6} A_i^{JN}(t,q^2)\tilde{\mathcal{X}}_\alpha^i \, , \text{ for } J \geq 1 \, , \tag{5.3-4}$$

we note from equations 5.13-4 and 5.23-16 that

$$A_3^{JN} = \sum_{r=0}^{J-1} t^r \left(A_{31}^{JN,r} G_1 + q^2 A_{33}^{JN,r} G_3\right) + \sum_{r=0}^{J} t^r A_{32}^{JN,r} G_2 \quad , \tag{5.3-5}$$

$$A_6^{JN}=\sum_{r=0}^{J-1} t^r A_{61}^{JN,r} G_1+\sum_{r=0}^{J} t^r\left(A_{62}^{JN,r} G_2+A_{63}^{JN,r} G_3\right), \qquad (5.3\text{-}6)$$

where the $A^{JN,r}_{3,6;1,2,3}$ are polynomials in $q^2$ and the masses but are independent of t. In particular, only $G_2$ contributes to the coefficient of $t^J$ in $A_3^{JN}$ whilst the same coefficient in $A_6^{JN}$ receives contributions only from $G_2$ and $G_3$. Also, $G_3$ gives no contribution to $A_3^{JN}$ at vanishing $q^2$ but continues to contribute to $A_6^{JN}$ at that point.

Except in as far as resonances with isospin three-halves contribute only to sum-rules on $A^{+,-}_{3,6}$, let us assume that we utilise the same set of particles (i.e. resonances plus the nucleon) in attempting to saturate all sum-rules on $A_{3,6}$ corresponding to the production of a given pseudoscalar meson. Let s' and s" denote the highest and next highest spins of all those isospin one-half particles utilised, and let s"' and s"" be the corresponding respective quantities for the isospin three-halves resonances used in the case of sum-rules on $A^{+,-}_{3,6}$.

Suppose that for a set of sum-rules on $A^{S,V}_{3,6}$:

$$s' \geqslant 3/2 \qquad (5.3\text{-}7)$$

and only one of the isospin one-half particles utilised has this spin. Then on separtely equating to zero each power of t appearing in the sum-rules we shall obtain:

$$G^{S,V}_{2,3}(s',q^2)=0, \qquad (\text{coeffs. of } t^{s'-\frac{1}{2}}); \quad (5.3\text{-}8)$$

$$G^{S,V}_{1}(s',q^2)=0, \text{if } s'-2 \geqslant s'', \qquad (\text{coeffs. of } t^{s'-3/2}). \quad (5.3\text{-}9)$$

If a similar situation obtains in the case of a set of sum-rules on $A^{0,+,-}_{3,6}$ these will yield:

$$G^{S}_{2,3}(s',q^2)=0, \qquad (\text{coeffs. of } t^{s'-1/2} \text{ in } A^{0}_{3,6});$$

$$G^{S}_{1}(s',q^2)=0, \text{ if } s'-2 \geqslant s'', \qquad (\text{coeffs. of } t^{s'-3/2} \text{ in } A^{0}_{3,6});$$

(5.3-10,11)

$$G^{V}_{2,3}(s',q^2)=0, \text{ if } s'-1 \geq s''' , \qquad (\text{coeffs. of } t^{s'-1/2} \text{ in } A^{+,-}_{3,6});$$

$$G^{V}_{1}(s',q^2)=0, \text{ if } s'-2 \geq Max(s'',s'''), (\text{coeffs. of } t^{s'-3/2} \text{ in } A^{+,-}_{3,6}).$$

(5.3-12,13)

Similarly, if for a set of sum-rules on $A^{+,-}_{3,6}$ :

$$s''' \geq 3/2 \qquad (5.3\text{-}14)$$

and only one of the isospin three-halves resonances has this spin we shall obtain:

$$G^{V}_{2,3}(s''',q^2)=0, \text{ if } s'''-1 \geq s' , \qquad (\text{coeffs. of } t^{s'''-1/2} \text{ in } A^{+,-}_{3,6});$$

$$G^{V}_{1}(s''',q^2)=0, \text{ if } s'''-2 \geq Max(s',s''), (\text{coeffs. of } t^{s'''-3/2} \text{ in } A^{+,-}_{3,6}).$$

(5.3-15,16)

The spin in the argument of each form-factor indicates the resonance involved.

Equations 5.3-8 to 13 and 15,16 are identities in $q^2$ and are useful in two ways. Firstly they tell us whether a given superposition of resonances is likely to saturate the sum-rules. Clearly the sum-rules are not well-saturated if they are forced to predict the vanishing of all form-factors corresponding to a given resonance. Secondly, if we know in advance that a given saturation is going to predict that all $G_{2,3}$ form-factors corresponding to a given resonance vanish, we only need to compute the coefficient of $G_1$ in the $A^{J,N}_{3,6}$ corresponding to that resonance.

Since resonances with isopsin three-halves contribute only to sum-rules on $A^{+,-}_{3,6}$ we may finally enquire whether it is reasonable to try and saturate sum-rules on $A^{0,+,-}_{3,6}$ with the Born-term plus a superposition of isospin three-halves resonances only.

From 5.21-6 and 5.12-16 we see that the contributions of the Born-term numerator to $\mathrm{disc}_\nu A_3^0$ and $\mathrm{disc}_\nu A_6^0$ are given by:

$$A_3^{0,B} = A_6^{0,B} = -eg F_2^S(q^2)/2m \; . \tag{5.3-17}$$

So with this saturation the sum-rules on $A_3^0$ and $A_6^0$ both imply the identity:

$$F_2^S(q^2) = 0 \quad , \tag{5.3-18}$$

that is:

$$F_2^p(q^2) = -F_2^n(q^2) \; . \tag{5.3-19}$$

These equations appear to be satisfied experimentally to within about 5% at all values of $q^2$ for which they have been tested. At vanishing $q^2$ 5.3-19 relates the anomalous magnetic moments of the proton and neutron according to:

$$\kappa^p = -\kappa^n . \tag{5.3-20}$$

The presence of the isospin three-halves resonances in the saturation prevent the sum-rules on $A_{3,6}^{+,-}$ from implying the contradictory result:

$$F_2^V(q^2) = 0 \quad , \tag{5.3-21}$$

so we are not forced to the erroneous conclusion:

$$F_2^p(q^2) = 0 = F_2^n(q^2) \; . \tag{5.3-22}$$

In the following sections, whilst bearing in mind the results of this one, we shall try to saturate the sum-rules on $A_{3,6}$ for the production of given mesons with the Born-term together with those resonances which are clearly seen experimentally in the process under consideration.

The squared coupling constants for the decays into the final state of the various resonances utilised are related to the observed partial widths in Appendix 8. Such computations leave undetermined the sign of these coupling constants, and we have taken them to be positive in all cases.

Each coupling constant appears in the various sum-rules multiplied by a form-factor. If improved experimental evidence determines a given coupling constant to be negative, the corresponding form-factor in our predictions must be multiplied by an additional factor of minus unity.

Similar remarks apply to the pion-nucleon coupling constant which we have taken to be positive, but the relative signs of this and the $\eta$ -nucleon coupling constant are determined by SU(6) symmetry.

## 5.4 APPROXIMATE SATURATION OF THE SUM-RULES FOR $\eta$-PRODUCTION.

For production of pseudoscalar mesons with zero isospin we have ordinary sum-rules on $A_{3,6}^{S,V}$ for those with positive C-parity, (the $\eta$, $\eta'$, and E); and on $A_7^{S,V}$ for those with negative C-parity. This latter case is hypothetical to date, but in any case we have already seen that sum-rules on $A_7$ cannot be saturated using the resonance approximation. In the former case, only the photo-production of the $\eta$ has been studied in any detail, and we shall accordingly restrict ourselves to a saturation of the sum-rules for production of this particle.

The only resonance which has been clearly seen in $\eta$-photoproduction is the N(1550) with:[15]

$$(I, J^P) = (\tfrac{1}{2}, \tfrac{1}{2}^-) \quad , \tag{5.4-1}$$

a situation in qualitative agreement with the fact that this is the only known resonance with an appreciable width for decay into $N\eta$. Indeed with:[15]

$$\Gamma(1550 \to N\eta) \simeq 0{\cdot}70\, \Gamma_{total}(1550) \quad , \tag{5.4-2}$$

and:

$$\Gamma_{total}(1550) \simeq 130 \text{ MeV} \quad , \tag{5.4-3}$$

this partial width is quite large. We shall try to saturate the sum-rules for photo- and electroproduction of the with just the nucleon and the N(1550). With M and $\Gamma$ denoting the mass and total width of this resonance, and $\mu$ denoting the pion mass, we then have in view of equations 5.11-1, 5.12-8 and 9, 5.21-6, and 5.22-8 that the predictions of the sum-rules are as follows. From the sum-rules on $A_3^{S,V}$:

$$Y(1550,130)g(1550\to N\eta)F_2^{S,V}(\gamma N\to 1550)=\frac{-g(N\to N\eta)}{2m}F_2^{S,V}(\gamma N\to N)\,, \qquad (5.4\text{-}4,5)$$

and from the sum-rules on $A_6^{S,V}$:

$$Y(1550,130)g(1550\to N\eta)\left[F_2^{S,V}(\gamma N\to 1550)-(M+m)F_1^{S,V}(\gamma N\to 1550)\right]$$
$$=\frac{-1}{2m}g(N\to N\eta)F_2^{S,V}(\gamma N\to N)\,. \qquad (5.4\text{-}6,7)$$

In these four equations we have introduced the shorthand notation:

$$Y(M,\Gamma)\equiv\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left[\frac{M^2-(m+\mu)^2}{M\Gamma}\right]\,. \qquad (5.4\text{-}8)$$

The pion mass appears because although the final meson is an $\eta$ the s-channel cut still starts at $(m+\mu)^2$.

The solution of equations 5.4-4 to 7 is a trivial matter and they yield:

$$F_1(\gamma p\to 1550^+)=0=F_1(\gamma n\to 1550^0)\,, \qquad (5.4\text{-}9,10)$$

$$\frac{F_2(\gamma n\to 1550^0)}{F_2(\gamma p\to 1550^+)}=\frac{F_2(\gamma n\to n)}{F_2(\gamma p\to p)}\,, \qquad (5.4\text{-}11)$$

$$F_2(\gamma p\to 1550^+)=\frac{-g(N\to N\eta)F_2(\gamma p\to p)}{2mY(1550,130)g(1550\to N\eta)}\,. \qquad (5.4\text{-}12)$$

We stress that these equations hold for all non-positive definite $q^2$.

The first two equations predict the vanishing for all non-positive $q^2$ of the pair of $\gamma N\to 1550$ form-factors which appear only in the virtual photonic case. These two equations cannot be obtained from pure photoproduction sum-rules.

Equation 5.4-11 predicts that the "moment" form-factors for the $\gamma N \to 1550$ vertex are in the same ratio as the corresponding nucleon ones. In as far as the empirical scaling relations:

$$G_e^p(q^2) \simeq \frac{G_m^p(q^2)}{1+\kappa^p} \simeq \frac{G_m^n(q^2)}{\kappa^n} \quad , \quad G_e^n(q^2) \simeq 0 \quad , \tag{5.4-13,14,15}$$

are valid the right-hand side of 5.4-11 may be replaced for all $q^2$ by:

$$\kappa^n / \kappa^p \simeq -1{\cdot}067 \, . \tag{5.4-16}$$

Taking the respective mean masses of the nucleon and the pion to be 939 MeV and 138 MeV we obtain:

$$Y(1550, 130) \simeq 0{\cdot}949 \quad , \tag{5.4-17}$$

and from Appendix 8, Table A8-I:

$$g\,(1550 \to N\eta) = 2{\cdot}11 \quad . \tag{5.4-18}$$

Thus 5.4-12 predicts:

$$F_2(\gamma p \to 1550^+) = -(0{\cdot}266)\left[g(N \to N\eta) F_2(\gamma p \to p)\right] GeV^{-1} \, . \tag{5.4-19}$$

If we decide to relate $g(N \to N\eta)$ to $g(N \to N\pi)$ by unitary symmetry we must go at least to SU(6), the F/D coupling ratio and $\eta - \eta'$ mixing angle being involved at the SU(3) level. With the SU(6) predictions of negligible mixing and an F/D coupling ratio of +2/3, we have:

$$g(N \to N\eta) = +\frac{\sqrt{3}}{5}\, g(N \to N\pi) \quad , \tag{5.4-20}$$

so with $g(N \to N\pi)$ given by equation 5.51-6, 5.4-19 reads finally:

$$F_2(\gamma p \to 1550^+) = -(1{\cdot}26)\, F_2(\gamma p \to p) \; GeV^{-1} \, . \tag{5.4-21}$$

In case the reader is mystified by the dimensions of $F_2(\gamma N \to 1550)$, we remind him that although we are using the conventional dimensionless nucleon form-factors, our $F_2(\gamma N \to 1550)$ as defined by 2.71-21 has dimensions of mass$^{-1}$. These dimensions are then carried by the coefficient of $F_2(\gamma N \rightarrow N)$ in equations 5.4-12, 19, and 21.

## 5.5 SOME APPROXIMATE SATURATIONS OF THE SUM-RULES FOR PION PRODUCTION.

### 5.51 INTRODUCTORY REMARKS AND DEFINITIONS

In the case of the production of pseudoscalar mesons with isospin unity we have ordinary sum-rules on: $A_3^{o,+}$, $A_6^{o,+}$, and $A_7^-$ for the pion, and on: $A_7^{o,+}$ for the hypothetical case where the meson has negative C-parity. Having already seen that sum-rules on $A_7$ cannot be saturated in the resonance approximation we now restrict ourselves to the $A_{3,6}^{o,+}$ sum-rules for pion production.

The situation here is a little less certain than that of the previous section. A large number of baryonic resonances with appreciable partial widths for decay into $N\pi$ are now known[15] and might be expected to contribute to this production process. Whilst the $\Delta(1236)$ with

$$(I, J^P, \Gamma)_{\Delta(1236)} = (3/2,\ 3/2^+,\ 120) \qquad (5.51\text{-}1)$$

is very clearly seen in photoproduction, the higher resonances would not appear as yet to be fully disentangled. The experimental evidence favours the view that this process is dominated at low energies by, (apart from the $\Delta(1236)$), the N(1525) with

$$(I, J^P, \Gamma)_{N(1525)} = (1/2,\ 3/2^-,\ 115), \qquad (5.51\text{-}2)$$

and the N(1688) with

$$(I, J^P, \Gamma)_{N(1688)} = (1/2, 5/2^+,\ 130). \qquad (5.51\text{-}3)$$

On the other hand, the possibility of an appreciable contribution from the N(1680) with

$$(I, J^P, \Gamma)_{N(1680)} = (1/2, 5/2^-, 170) \qquad (5.51\text{-}4)$$

is not completely ruled out. Low energy pion photoproduction does not appear to receive any appreciable contributions from resonances with spin-one-half. In particular, the presence of the Roper resonance, N(1470), with

$$(I, J^P, \Gamma)_{N(1470)} = (1/2,\ 1/2^+,\ 210) \qquad (5.51\text{-}5)$$

has not been detected experimentally.

In the following section we accordingly attempt to saturate the sum-rules with the nucleon together with the $\Delta(1236)$, N(1525), N(1680), and N(1688). The predictions obtained are not very illuminating at present and in the subsequent sections we attempt to gain more useful (and approximate) predictions by progressively leaving out the higher mass resonances.

But first, let us define some further symbols to simplify the notation of the following sections.

We keep the symbols $m$ and $\mu$ for the nucleon and pion masses, and define $M_1$, $M_2$, $M_3$, and $M_4$ to be the respective masses of the $\Delta(1236)$, N(1525), N(1680), and N(1688). The standard notation is used for the nucleon form-factors and g(N) will stand for the pion-nucleon coupling constant. We shall adopt the value:

$$g^2(N)/4\pi = 14\cdot 8 \qquad (5.51\text{-}6)$$

corresponding in standard notation to

$$f^2/4\pi = 0\cdot 080\ . \qquad (5.51\text{-}6A)$$

The symbol g(1236) will denote the coupling constant for the interaction: $\Delta(1236) \rightarrow N\pi$, and $G^V_{1,2,3}(1236)$ will be the $O(3,1) \otimes SU(2)$ form-factors for: $\gamma N \rightarrow \Delta(1236)$. Similar notation will be used for the remaining coupling constants and form-factors. The relevant coupling constants are computed in Appendix 8 using equations 5.51-1 to 4, (taken from the January 1968 Rosenfeld tables), as input data.

The finite width correction factors, $Y(M,\Gamma)$, are defined as in the previous section, the s-channel cut again starting at the point $(m+\mu)^2$. With the above input data we find:

$$Y(1236,120)=0\cdot878\ , \qquad Y(1525,115)=0\cdot953\ ,$$

$$Y(1680,170)=0\cdot946\ , \qquad Y(1688,130)=0\cdot959\ , \quad (5.51\text{-}7 \text{ to } 10)$$

so as in the previous section, the predictions of the sum-rules will differ but slightly from those which would have been obtained had we neglected the widths of the resonances.

It will prove convenient to define:

$$\mathcal{G}_{1,2,3}(q^2)\equiv\frac{m\,g(1236)}{9M_1^2\,g(N)}Y(1236,120)\,G^{V}_{1,2,3}(1236)\ , \qquad (5.51\text{-}11)$$

$$H^{S,V}_{1,2,3}(q^2)\equiv\frac{m\,g(1525)}{6M_2^2\,g(N)}Y(1525,115)\,G^{S,V}_{1,2,3}(1525)\ , \qquad (5.51\text{-}12)$$

$$K^{S,V}_{1,2,3}(q^2)\equiv\frac{m\,g(1680)}{40M_3^4\,g(N)}Y(1680,170)\,G^{S,V}_{1,2,3}(1680)\ , \qquad (5.51\text{-}13)$$

$$L^{S,V}_{1,2,3}(q^2)\equiv\frac{m\,g(1688)}{40M_4^4\,g(N)}Y(1688,130)\,G^{S,V}_{1,2,3}(1688)\ , \qquad (5.51\text{-}14)$$

in which the various coupling constants, masses, and numerical factors are suggested by the structure of equations 5.11-1, 5.12-16 to 19, 5.21-6, 5.24-1, 5.25-1, and 5.4-8.

Finally, in view of equations 5.24-1 and 5.25-1 we shall define quantities $(W,X,Y,Z)^r_{ik}$ which are functions only of $q^2$ and the relevant masses, by:

$$M^j_\alpha(1236)\equiv e\sum_{i=1}^{6}\sum_{k=1}^{3}\sum_{r=0}^{1}W^r_{ik}\,t^r\,\tilde{\mathcal{X}}^i_\alpha\,\mathcal{G}_k\,\delta_{j3}\ , \qquad (5.51\text{-}15)$$

$$M^j_\alpha(1525)\equiv e\sum_{i=1}^{6}\sum_{k=1}^{3}\sum_{r=0}^{1}X^r_{ik}\,t^r\,\tilde{\mathcal{X}}^i_\alpha\left(H^S_k\tau_j+H^V_k\delta_{j3}\right), \qquad (5.51\text{-}16)$$

$$M^j_\alpha(1680) \equiv e \sum_{i=1}^{6} \sum_{k=1}^{3} \sum_{r=0}^{2} Y^r_{ik} t^r \tilde{\mathcal{K}}^i_\alpha (K^s_k \tau_j + K^v_k \delta_{j3}) \quad , \qquad (5.51\text{-}17)$$

$$M^j_\alpha(1688) \equiv e \sum_{i=1}^{6} \sum_{k=1}^{3} \sum_{r=0}^{2} Z^r_{ik} t^r \tilde{\mathcal{K}}^i_\alpha (L^s_k \tau_j + L^v_k \delta_{j3}) \quad , \qquad (5.51\text{-}18)$$

where the quantities on the left-hand sides are the $O(3,1) \otimes SU(2)$ space M-functions corresponding to the pole-graph numerators for the intermediate states indicated. The $(W,X,Y,Z)^r_{ik}$ that we shall need may then be computed from equations 5.24-8 to 20 or 5.25-5 to 15 as appropriate by inserting the relevant mass x normality product for N.

Specifically, with:

$$R_{1,2,3,4} \equiv M^2_{1,2,3,4} + m^2 - \mu^2 \quad , \qquad (5.51\text{-}19)$$

we have:

$$W^r_{ik} = a^r_{ik}\Big|_{\substack{N=-M_1 \\ R=R_1}} \quad , \qquad X^r_{ik} = a^r_{ik}\Big|_{\substack{N=M_2 \\ R=R_2}} \quad , \qquad (5.51\text{-}20,21)$$

$$Y^r_{ik} = b^r_{ik}\Big|_{\substack{N=-M_3 \\ R=R_3}} \quad , \qquad Z^r_{ik} = b^r_{ik}\Big|_{\substack{N=M_4 \\ R=R_4}} \quad . \qquad (5.51\text{-}22,23)$$

## 5.52 ATTEMPTED SATURATION USING THE N(939), Δ(1236), N(1525), N(1680), and N(1688).

In terms of the quantities defined in the previous section, we obtain the following equations on separately equating to zero the coefficient of each power of t appearing in each of the four sum-rules after attempted saturation. Repeated k indices are meant to imply summation over k = 1,2,3. In deriving these relations we have made use of equations: 5.24-8,9, and 14 to 17, and: 5.25-5,6, and 10 to 13.

Sum-rule on $A_3^0$

Coefficient of $t^2$:

$$Y_{32}^2 K_2^s + Z_{32}^2 L_2^s = 0 , \tag{5.52-1}$$

coefficient of $t$:

$$X_{32}^1 H_2^s + Y_{3k}^1 K_k^s + Z_{3k}^1 L_k^s = 0 , \tag{5.52-2}$$

coefficient of $t^0$:

$$X_{3k}^0 H_k^s + Y_{3k}^0 K_k^s + Z_{3k}^0 L_k^s = F_2^s . \tag{5.52-3}$$

Sum-rule on $A_6^0$

Coefficient of $t^2$:

$$Y_{32}^2 K_3^s + Z_{32}^2 L_3^s = 0 , \tag{5.52-4}$$

coefficient of $t$:

$$-2 X_{32}^1 H_3^s + Y_{6k}^1 K_k^s + Z_{6k}^1 L_k^s = 0 , \tag{5.52-5}$$

coefficient of $t^0$:

$$X_{6k}^0 H_k^s + Y_{6k}^0 K_k^s + Z_{6k}^0 L_k^s = F_2^s . \tag{5.52-5}$$

Sum-rule on $A_3^+$

Coefficient of $t^2$:

$$Y_{32}^2 K_2^v + Z_{32}^2 L_2^v = 0 , \tag{5.52-7}$$

coefficient of $t$:

$$W_{32}^1 \mathcal{G}_2 + X_{32}^1 H_2^v + Y_{3k}^1 K_k^v + Z_{3k}^1 L_k^v = 0 , \tag{5.52-8}$$

coefficient of $t^0$:

$$W_{3k}^0 \mathcal{G}_k + X_{3k}^0 H_k^v + Y_{3k}^0 K_k^v + Z_{3k}^0 L_k^v = F_2^v . \tag{5.52-9}$$

Sum-rule on $A_6^+$

Coefficient of $t^2$:

$$Y_{32}^2 K_3^V + Z_{32}^2 L_3^V = 0 \quad , \tag{5.52-10}$$

coefficient of t:

$$-2W_{32}^1 \mathcal{G}_3 - 2X_{32}^1 H_3^V + Y_{6k}^1 K_k^V + Z_{6k}^1 L_k^V = 0 \, , \tag{5.52-11}$$

coefficient of $t^0$:

$$W_{6k}^0 \mathcal{G}_k + X_{6k}^0 H_k^V + Y_{6k}^0 K_k^V + Z_{6k}^0 L_k^V = F_2^V \, . \tag{5.52-12}$$

From equations 5.52-1,4,7, and 10 we have immediately the predictions:

$$\frac{K_2^S}{L_2^S} = \frac{K_3^S}{L_3^S} = \frac{K_2^V}{L_2^V} = \frac{K_3^V}{L_3^V} = -\frac{Z_{32}^2}{Y_{32}^2} = -\frac{M_4^4}{M_3^4} \, . \tag{5.52-13}$$

This set of relations holds independently of the other resonances used in the saturation provided none of these has spin exceeding three-halves. In view of equations 5.51-13 and 14, 5.52-13 reduces to:

$$\frac{G_2^S(1680)}{G_2^S(1688)} = \frac{G_3^S(1680)}{G_3^S(1688)} = \frac{G_2^V(1680)}{G_2^V(1688)} = \frac{G_3^V(1680)}{G_3^V(1688)} = -\frac{g(1688)Y(1688,130)}{g(1680)Y(1680,170)} \simeq -39.4 \, . \tag{5.52-14}$$

One may equally well replace the superscripts s and v in these equations by superscripts o and + referring to the charge of the resonance.

In conjunction with 5.52-13 we may use equations 5.52-2, 3,5, and 6 to express $G_1^S(1680)$ and $G_{1,2,3}^S(1688)$ in terms of $G_{1,2,3}^S(1525)$ and $F_2^S$. Similarly, from 5.52-8,9,12, and 13 we may obtain $G_1^V(1680)$ and $G_{1,2,3}^V(1688)$ in terms of $G_{1,2,3}^V(1525)$, $G_{1,2,3}^V(1236)$, and $F_2^V$. The present scarcity of experimental data on the $G_{1,2,3}^{s,v}(1525)$ does not render such information very useful at the present time, and we shall not pursue this particular attempt at saturation any further.

## 5.53 INCLUSION OF THE N(939), Δ(1236), N(1525) and N(1688).

If the contributions from the N(1680) are left out of the equations of the previous section, 5.52-1, 4, 7, and 10 become:

$$L_2^S = L_3^S = L_2^V = L_3^V = 0 \quad , \tag{5.53-1}$$

that is, we have the prediction that $G_2$ and $G_3$ vanish identically for both charge states of the N(1688).

Equations 5.52-2 and 5.52-5 may therefore be written:

$$H_2^S = -Z_{31}^1 L_1^S / X_{32}^1 \quad , \tag{5.53-2}$$

$$H_3^S = Z_{61}^1 L_1^S / X_{32}^1 \quad , \tag{5.53-3}$$

and on defining:

$$V_1 \equiv X_{32}^1 Z_{31}^0 - X_{32}^0 Z_{31}^1 + \tfrac{1}{2} X_{33}^0 Z_{61}^1 \quad , \tag{5.53-4}$$

$$V_2 \equiv X_{32}^1 Z_{61}^0 - X_{62}^0 Z_{31}^1 + \tfrac{1}{2} X_{63}^0 Z_{61}^1 \quad , \tag{5.53-5}$$

equations 5.52-3 and 6 become:

$$X_{32}^1 X_{31}^0 H_1^S + V_1 L_1^S = X_{32}^1 F_2^S \quad , \tag{5.53-6}$$

$$X_{32}^1 X_{61}^0 H_1^S + V_2 L_1^S = X_{32}^1 F_2^S \quad . \tag{5.53-7}$$

These equations may be solved for $L_1^S$ and $H_{1,2,3}^S$ in terms of $F_2^S$ yielding:

$$L_1^S = \left(X_{61}^0 V_1 - X_{31}^0 V_2\right)^{-1} X_{32}^1 \left(X_{61}^0 - X_{31}^0\right) F_2^S \quad , \tag{5.53-8}$$

$$H_1^S = \left(X_{61}^0 V_1 - X_{31}^0 V_2\right)^{-1} \left(V_1 - V_2\right) F_2^S \quad , \tag{5.53-9}$$

$$H_2^S = -\left(X_{61}^0 V_1 - X_{31}^0 V_2\right)^{-1} Z_{31}^1 \left(X_{61}^0 - X_{31}^0\right) F_2^S \quad , \tag{5.53-10}$$

$$H_3^S = \tfrac{1}{2}\left(X_{61}^0 V_1 - X_{31}^0 V_2\right)^{-1} Z_{61}^1 \left(X_{61}^0 - X_{31}^0\right) F_2^S \quad . \tag{5.53-11}$$

Now $X_{32}^1$, $Z_{31}^1$, $Z_{61}^1$, $X_{61}^0$, and $X_{31}^0$ depend only on the masses, but $(V_1 - V_2)$ and $(X_{61}^0 V_1 - X_{31}^0 V_2)$ are linear functions of $q^2$. It turns out that:

$$(V_1 - V_2) = -(3\cdot03)(10^2)(4\cdot48 - q^2) \quad , \tag{5.53-12}$$

$$(X^0_{61} V_1 - X^0_{31} V_2) = (3\cdot24)(10^3)(6\cdot48 - q^2) \quad , \tag{5.53-13}$$

where in these, and in all subsequent equations, the evaluated functions of the masses are expressed in units of GeV/c raised to the appropriate power. In connection with these equations we remind the reader that our choice of metric corresponds to positive $q^2$ for time-like photons. There is certainly nothing very startling about the prediction that $G_1^S(1688)$ includes a factor $(4.48-q^2)$, and since the $N(1688)$ - $\bar{N}(939)$ pair production threshold is situated at the time-like point:

$$q^2 = (M_4 + m)^2 \simeq 6\cdot91 \text{ GeV}^2 \quad , \tag{5.53-14}$$

there is no objection to a pole in this form-factor at

$$q^2 \simeq 6\cdot48 \text{ GeV}^2 \,. \tag{5.53-15}$$

However, since this point lies above the $N(1525)$-$\bar{N}(939)$ production threshold:

$$q^2 = (M_2 + m)^2 \simeq 6\cdot08 \text{ GeV}^2 \quad , \tag{5.53-16}$$

we require that the $G^S_{1,2,3}$ (1525) should be finite at the point given by 5.53-15. Hence we have the additional prediction:

$$F_2^S(q^2 \simeq 6\cdot48) = 0 \,. \tag{5.53-17}$$

There is really no experimental evidence to confirm or contradict this prediction. Although it does not satisfy the combined empirical scaling laws 1.2-17 and 20, we have already seen that the former of these violates the threshold constraint 1.2-16 if continued unmodified to time-like $q^2$. It becomes a plausible prediction if one bears in mind that the equation:

$$F_2^p(0) = -F_2^n(0) \tag{5.53-18}$$

if satisfied to within about 5%.

Let us now turn to the remaining "isovector" equations. On leaving out the N(1680) contributions, equations 5.52-8 and 11 now read in view of 5.53-1:

$$H_2^V = -\frac{1}{X_{32}^1}\left(Z_{31}^1 L_1^V + W_{32}^1 \mathcal{G}_2\right) , \tag{5.53-19}$$

$$H_3^V = \frac{1}{2X_{32}^1}\left(Z_{61}^1 L_1^V - 2W_{32}^1 \mathcal{G}_3\right) . \tag{5.53-20}$$

On defining:

$$V_3 \equiv \left(W_{32}^1 X_{32}^0 - W_{32}^0 X_{32}^1\right) , \tag{5.53-21}$$

$$V_4 \equiv \left(W_{32}^1 X_{62}^0 - W_{62}^0 X_{32}^1\right) , \tag{5.53-22}$$

$$V_5 \equiv \left(W_{32}^1 X_{33}^0 - W_{33}^0 X_{32}^1\right) , \tag{5.53-23}$$

$$V_6 \equiv \left(W_{32}^1 X_{63}^0 - W_{63}^0 X_{32}^1\right) , \tag{5.53-24}$$

equations 5.52-9 and 12 therefore reduce to:

$$X_{31}^0 X_{32}^1 H_1^V + V_1 L_1^V = X_{32}^1\left(F_2^V - W_{31}^0 \mathcal{G}_1\right) + V_3 \mathcal{G}_2 + V_5 \mathcal{G}_3 , \tag{5.53-25}$$

$$X_{61}^0 X_{32}^1 H_1^V + V_2 L_1^V = X_{32}^1\left(F_2^V - W_{61}^0 \mathcal{G}_1\right) + V_4 \mathcal{G}_2 + V_6 \mathcal{G}_3 . \tag{5.53-26}$$

We may use 5.53-19, 20, 25 and 26 to obtain $L_1^V$ and $H_{1,2,3}^V$ in terms of $F_2^V$ and $\mathcal{G}_{1,2,3}$. The solutions are:

$$L_1^V = \left(X_{61}^0 V_1 - X_{31}^0 V_2\right)^{-1}\Big[X_{32}^1\left(X_{61}^0 - X_{31}^0\right)F_2^V + X_{32}^1\left(W_{61}^0 X_{31}^0 - W_{31}^0 X_{61}^0\right)\mathcal{G}_1$$
$$+ \left(X_{61}^0 V_3 - X_{31}^0 V_4\right)\mathcal{G}_2 + \left(X_{61}^0 V_5 - X_{31}^0 V_6\right)\mathcal{G}_3\Big] , \tag{5.53-27}$$

$$H_1^V = \left[X_{32}^1\left(X_{61}^0 V_1 - X_{31}^0 V_2\right)\right]^{-1}\Big[X_{32}^1\left(V_1 - V_2\right)F_2^V - X_{32}^1\left(W_{61}^0 V_1 - W_{31}^0 V_2\right)\mathcal{G}_1$$
$$+ \left(V_1 V_4 - V_2 V_3\right)\mathcal{G}_2 + \left(V_1 V_6 - V_2 V_5\right)\mathcal{G}_3\Big] , \tag{5.53-28}$$

$$H_2^V = -Z_{31}^1\left[X_{32}^1\left(X_{61}^0 V_1 - X_{31}^0 V_2\right)\right]^{-1}\Big\{X_{32}^1\left(X_{61}^0 - X_{31}^0\right)F_2^V + X_{32}^1\left(W_{61}^0 X_{31}^0 - W_{31}^0 X_{61}^0\right)\mathcal{G}_1 + \Big[\left(X_{61}^0 V_3\right.$$
$$\left. - X_{31}^0 V_4\right) + \left(W_{32}^1/Z_{31}^1\right)\left(X_{61}^0 V_1 - X_{31}^0 V_2\right)\Big]\mathcal{G}_2 + \left(X_{61}^0 V_5 - X_{31}^0 V_6\right)\mathcal{G}_3\Big\} , \tag{5.53-29}$$

$$H_3^V = \frac{1}{2}Z_{61}^1\left[X_{32}^1\left(X_{61}^0 V_1 - X_{31}^0 V_2\right)\right]^{-1}\Big\{X_{32}^1\left(X_{61}^0 - X_{31}^0\right)F_2^V + X_{32}^1\left(W_{61}^0 X_{31}^0 - W_{31}^0 X_{61}^0\right)\mathcal{G}_1$$
$$+ \left(X_{61}^0 V_3 - X_{31}^0 V_4\right)\mathcal{G}_2 + \Big[\left(X_{61}^0 V_5 - X_{31}^0 V_6\right) - \left(\frac{2W_{32}^1}{Z_{61}^1}\right)\left(X_{61}^0 V_1 - X_{31}^0 V_2\right)\Big]\mathcal{G}_3\Big\} . \tag{5.53-30}$$

There are no consistency problems here; we merely require that the numerator of the right-hand side of each equation should vanish when $q^2$ is equal to 6.48 GeV$^2$. In particular, due to the contributions of the $\Delta(1236)$ to the isovector sum-rules we are not lead to predict the vanishing of $F_2^V(6.48)$.

On evaluating the various mass polynomials appearing in equations 5.53-8 to 11 and 27 to 30, we obtain finally in view of 5.51 -11 to 14:

$$G_1^S(1525) = -(1.84)(6.48-q^2)^{-1}(4.48-q^2)F_2^S \ \text{GeV}^{-1}, \tag{5.53-31}$$

$$G_2^S(1525) = -(1.79)(6.48-q^2)^{-1}F_2^S \ \text{GeV}^{-2}, \tag{5.53-32}$$

$$G_3^S(1525) = (1.043)(6.48-q^2)^{-1}F_2^S \ \text{GeV}^{-2}, \tag{5.53-33}$$

$$G_1^S(1688) = -(0.0850)(6.48-q^2)^{-1}F_2^S \ \text{GeV}^{-2}, \tag{5.53-34}$$

$$G_2^S(1688) = 0 \ \text{GeV}^{-3}, \qquad G_3^S(1688) = 0 \ \text{GeV}^{-3}, \tag{5.53-35,36}$$

$$\begin{aligned} G_1^V(1525) = \frac{-(1.84)}{(6.48-q^2)}\Big[&(4.48-q^2)F_2^V-(0.724)(7.76-q^2)G_1^V(1236)+(2.45)(0.290\\ &+q^2)(7.65-q^2)G_2^V(1236)+(2.45)(0.264+q^2)(4.57-q^2)G_3^V(1236)\Big] \ \text{GeV}^{-1}, \end{aligned} \tag{5.53-37}$$

$$\begin{aligned} G_2^V(1525) = \frac{-(1.79)}{(6.48-q^2)}\Big[&F_2^V+(0.552)G_1^V(1236)+(0.246)(4.53-q^2)G_2^V(1236)\\ &+(0.245)(1.212+q^2)G_3^V(1236)\Big] \ \text{GeV}^{-2}, \end{aligned} \tag{5.53-38}$$

$$\begin{aligned} G_3^V(1525) = \frac{(1.043)}{(6.48-q^2)}\Big[&F_2^V+(0.552)G_1^V(1236)-(0.245)(8.44-q^2)G_2^V(1236)\\ &-(1.088)(4.74-q^2)G_3^V(1236)\Big] \ \text{GeV}^{-2}, \end{aligned} \tag{5.53-39}$$

$$\begin{aligned} G_1^V(1688) = \frac{-(0.0850)}{(6.48-q^2)}\Big[&F_2^V+(0.552)G_1^V(1236)-(0.245)(8.44-q^2)G_2^V(1236)\\ &+(0.245)(1.212+q^2)G_3^V(1236)\Big] \ \text{GeV}^{-2}, \end{aligned} \tag{5.53-40}$$

$$G_2^V(1688) = 0 \ \text{GeV}^{-3}, \qquad G_3^V(1688) = 0 \ \text{GeV}^{-3}. \tag{5.53-41,42}$$

The isoscalar solutions speak for themselves. $\left|G_{1,2,3}^{S}(1525)\right|$ and $\left|G_{1}^{S}(1688)\right|$ whilst non-vanishing are predicted to be relatively small compared with the corresponding isovector form-factors, since each is equal to the product of $F_2^S$ (which is very small compared with $F_2^V$) and a term whose modulus is less than unity for all non-time-like $q^2$. In particular, $\left|G_1^S(1688)\right|$ is predicted to be relatively small even in comparison with $\left|G_{1,2,3}^{S}(1525)\right|$.

Not very much can be said about the isovector solutions, since of the $G_{1,2,3}^{V}$ (1236) only $G_1^V(1236,\ q^2 = o)$ is known with any accuracy empirically. However the factor (0.0850) in equation 5.53-40 does suggest that $\left|G_1^V(1688)\right|$ is relatively small in comparison with $\left|G_{1,2,3}^{V}(1525)\right|$.

## 5.54 INCLUSION OF THE N(939), Δ(1236), AND N(1525).

On omitting from the previous section the contributions from the N(1688), the equations and results are modified as follows.

Dealing first with the isoscalar equations, 5.53-2 and 3 now predict:

$$H_2^s = 0 = H_3^s \quad , \tag{5.54-1,2}$$

whilst 5.53-6 and 7 read:

$$H_1^s = F_2^s / X_{31}^o \quad , \tag{5.54-3}$$

$$H_1^s = F_2^s / X_{61}^o \quad . \tag{5.54-4}$$

Since:

$$X_{31}^o \simeq -(12\cdot 67)\,GeV^3 , \tag{5.54-5}$$

and:

$$X_{61}^o \simeq -(10\cdot 96)\,GeV^3 , \tag{5.54-6}$$

these latter two equations are inconsistent, but their predictions only differ by about 14%. It is interesting to note that $X^0_{31}$ and $X^0_{61}$ both vanish in the limit:

$$M_2 \to m \quad , \quad \mu \to 0 \ . \tag{5.54-7}$$

So in this "equal mass" limit equations 5.54-3 and 4 are consistent. Both predict the identical vanishing of $F_2^S$, which as discussed in section 5.3 is a prediction which holds experimentally to within about 5%. Nothing can then be said about $H_1^S$ which does not contribute to the sum-rules, but equations 5.54-1 and 2 remain valid in this limit.

With the N(1688) contributions absent from the previous isovector equations, these can be solved for $H^V_{1,2,3}$ and $\mathcal{G}_3$ in terms of $F_2^V$ and $\mathcal{G}_{1,2}$. We obtain:

$$\mathcal{G}_3 = -\left(X^0_{61}V_5 - X^0_{31}V_6\right)^{-1}\Big[X^1_{32}\left(X^0_{61}-X^0_{31}\right)F_2^V + X^1_{32}\left(W^0_{61}X^0_{31} - W^0_{31}X^0_{61}\right)\mathcal{G}_1 + \left(X^0_{61}V_3 - X^0_{31}V_4\right)\mathcal{G}_2\Big] \ , \tag{5.54-8}$$

$$H_1^V = \left(X^0_{61}V_5 - X^0_{31}V_6\right)^{-1}\Big[\left(V_5-V_6\right)F_2^V - \left(W^0_{61}V_5 - W^0_{31}V_6\right)\mathcal{G}_1 + \left(X^1_{32}\right)^{-1}\left(V_4V_5 - V_3V_6\right)\mathcal{G}_2\Big] \ , \tag{5.54-9}$$

$$H_2^V = -\left(W^1_{32}/X^1_{32}\right)\mathcal{G}_2 \ , \tag{5.54-10}$$

$$H_3^V = -\left(W^1_{32}/X^1_{32}\right)\mathcal{G}_3 \ . \tag{5.54-11}$$

Again, no particular consistency problems arise; we simply require that the numerators of the right-hand sides of 5.54-8 and 9 vanish when $(X^0_{61}V_5 - X^0_{31}V_6)$ vanishes. This turns out to occur at the spacelike point:

$$q^2 \simeq -(1{\cdot}212)\ \mathrm{GeV}^2 \ . \tag{5.54-12}$$

In view of the detailed structure of the mass polynomials appearing, these two constraints turn out to be the same. On approximate evaluation they both yield:

$$\left[F_2^V+(0{\cdot}550)G_1^V(1236)-(2{\cdot}365)G_2^V(1236)\right]\Big|_{q^2=-1{\cdot}212}=0. \qquad (5.54\text{-}13)$$

This prediction becomes plausible if we suppose, as is not unreasonable, that the $G^V_{1,2}$ (1236) are proportional to $F_2^V$ for all space-like $q^2$. We then require:

$$\left[F_2^V+(0{\cdot}550)G_1^V(1236)-(2{\cdot}365)G_2^V(1236)\right]\Big|_{q^2=0}=0. \qquad (5.54\text{-}14)$$

In view of the results of Appendix 9, we find that this equation agrees with the empirical data on pion-photoproduction in the 33-resonance region if one assumes an $E_1^+/M_1^+$ ratio of about $-2{\cdot}3\%$ . As mentioned in the said appendix, the data on $E_1^+/M_1^+$ is subject to very large percentage experimental errors, and the value of $-2{\cdot}3\%$ certainly lies inside this error range.

On evaluating equations 5.54-1 to 4 and 8 to 11 we obtain the predictions:

$$G_1^S(1525)=-\begin{cases}(1{\cdot}80)F_2^S\ \mathrm{GeV}^{-1}, & \text{(sum-rule on } A_3^0\text{)},\\ (1{\cdot}56)F_2^S\ \mathrm{GeV}^{-1}, & \text{(sum-rule on } A_6^0\text{)},\end{cases} \qquad (5.54\text{-}15)$$

$$G_2^S(1525)=0\ \mathrm{GeV}^{-2}, \qquad G_3^S(1525)=0\ \mathrm{GeV}^{-2}, \qquad (5.54\text{-}16,17)$$

$$G_3^V(1236)=\frac{-4{\cdot}08}{(1{\cdot}212+q^2)}\Big[F_2^V+(0{\cdot}552)G_1^V(1236)$$
$$-(0{\cdot}245)(8{\cdot}44-q^2)G_2^V(1236)\Big]\ \mathrm{GeV}^{-2}, \qquad (5.54\text{-}18)$$

$$G_1^V(1525)=\frac{-1{\cdot}88}{(1{\cdot}212+q^2)}\Big[F_2^V-(1{\cdot}218)(0{\cdot}762+q^2)G_1^V(1236) \qquad (5.54\text{-}19)$$
$$-(1{\cdot}56)(0{\cdot}306-q^2)G_2^V(1236)\Big]\ \mathrm{GeV}^{-1},$$

$$G_2^V(1525)=-(0{\cdot}878)G_2^V(1236)\ \mathrm{GeV}^{-2}, \qquad (5.54\text{-}20)$$

$$G_3^V(1525)=\frac{3{\cdot}58}{(1{\cdot}212+q^2)}\Big[F_2^V+(0{\cdot}552)G_1^V(1236)$$
$$-(0{\cdot}245)(8{\cdot}44-q^2)G_2^V(1236)\Big]\ \mathrm{GeV}^{-2}. \qquad (5.54\text{-}21)$$

The isoscalar solutions again speak for themselves. They are in qualitative agreement with the corresponding results of the previous section, and this is not surprising since we have neglected the form-factor $G_1^S(1688)$ which was predicted as being relatively small compared with $G_{1,2,3}^S(1525)$. However, the results 5.53-31 to 36 are to be preferred to 5.54-15 to 17 due to their increased predictive power and the fact that they (presumably) correspond to a better saturation of the isoscalar sum-rules. On the other hand, 5.53-34 at least will have to be scrapped should it be discovered that $F_2^S(6.48)$ is non-vanishing.

The isovector solutions again suffer from the lack of reliable data on the $G_{1,2,3}^V(1236)$, and the reader is referred to Appendix 9 for a discussion of the empirical data on $G_{1,2}^V(1236,\ q^2 = 0)$. Although a reasonably accurate estimate of $G_1^V(1236,\ q^2 = 0)$ is available, the predictions 5.54-18 to 21 depend rather violently on the (relatively unreliable) value of the parameter $\rho$ of that appendix.

We tabulate below the values predicted by 5.54-18 to 21 for $G_3^V(1236,\ q^2 = 0)$ and $G_{1,2,3}^V(1525,\ q^2 = 0)$ using input data based on three different values of $\rho$ . The value

$$\rho = 0 \tag{5.54-22}$$

corresponds to pure magnetic dipole excitation at the $\gamma N(939) \rightarrow \Delta(1236)$ vertex; the value

$$\rho = -0{\cdot}064 \tag{5.54-23}$$

corresponds to the vanishing of $G_2^V(1236,\ q^2 = 0)$, and therefore of $G_2^V(1525,\ q^2 = 0)$ in this case; and the value:

$$\rho = -0{\cdot}018\,, \tag{5.54-24}$$

(which is well within the error range of the empirical data), is that for which the sum-rules predict the vanishing of $G_3^V$ $(1236,\ q^2 = o)$ and $G_3^V(1525,\ q^2 = o)$. It is obtained by setting $G_3^V(1236, q^2=0)$ equal to zero in 5.54-18 and then solving the resulting equation simultaneously with A9-3,16 and 21 of Appendix 9. The corresponding empirical solutions for $G_{1,2}^V$ $(1236,\ q^2 = o)$ are:

$$G_1^V(1236, q^2=0) = 2{\cdot}63\ \mathrm{GeV}^{-1}\ , \tag{5.54-25}$$

$$G_2^V(1236, q^2=0) = 1{\cdot}94\ \mathrm{GeV}^{-2}\ . \tag{5.54-26}$$

We have not bothered to compute the value of $\rho$ for which $G_1^V(1525,\ q^2 = o)$ will be predicted as vanishing since such a value will lie outside the error range of the empirical data.

We remind the reader that in practical applications $G_3^V(1236)$ and $G_3^V(1525)$ will be damped by kinematical factors proportional to $q^2$, thus the relatively high values of $|G_3^V(1236,\ q^2 = o)|$ and $|G_3^V(1525,\ q^2 = o)|$ corresponding to a $\rho$ value of $-0.064$ are not superficially unreasonable.

TABLE 5.54-I

| | $(E_1^+/M_1^+)(\gamma N \to \Delta)$ | | |
|---|---|---|---|
| FORM-FACTOR ↘ | 0 | -0.018 | -0.064 |
| $G_3^V(1236,q^2=0)$ | 3.88 | 0 | -10.87 |
| $G_1^V(1525,q^2=0)$ | 2.58 | 2.35 | 0.705 |
| $G_2^V(1525,q^2=0)$ | -1.90 | -1.70 | 0 |
| $G_3^V(1525,q^2=0)$ | -3.40 | 0 | 9.54 |

Predicted values of $G_3^V(1236,\ q^2 = o)$ and $G_{1,2,3}^V(1525,\ q^2 = o)$ corresponding to input data based on various assumed values for $(E_1^+/M_1^+)(\gamma N \to \Delta)$.

## 5.55 INCLUSION OF THE N(939) AND $\Delta(1236)$ ONLY.

In this case we have only two isoscalar equations. They are consistent, and both imply:

$$F_2^S = 0 \; . \qquad (5.55\text{-}1)$$

As discussed in section 5.3, this is a remarkably sound prediction in view of the crudity of the approximation. It holds more generally for any attempted saturation in which all the resonances utilised have isospin three-halves.

Equations 5.54-10 and 11 reduce to:

$$G_2^V(1236) = 0 \; , \qquad (5.55\text{-}2)$$

$$G_3^V(1236) = 0 \; . \qquad (5.55\text{-}3)$$

As we demonstrate in Appendix 9, a very wide range of values for the ratio $M_1 G_2^V(1236,\ q^2 = 0)/G_1^V(1236,\ q^2 = 0)$ are in qualitative agreement with the experimental data on the $E_1^+/M_1^+$ ratio for pion photoproduction in the $\Delta(1236)$ resonance region. For finite $G_1^V(1236,\ q^2 = 0)$ the vanishing ratio predicted by equation 5.55-2 corresponds to a value:

$$E_1^+/M_1^+ \simeq -0{\cdot}064 \; . \qquad (5.55\text{-}4)$$

In view of the widespread uncertainty concerning the correct empirical value for this quantity, it is in good agreement with Gourdin and Salin's value of -0.045.

If we accept 5.55-2 and 3, the remaining pair of isovector equations read:

$$W_{31}^0 g_1 = F_2^V \; , \text{ (sum-rule on } A_3^+), \qquad (5.55\text{-}5)$$

$$W_{61}^0 g_1 = F_2^V \; , \text{ (sum-rule on } A_6^+). \qquad (5.55\text{-}6)$$

Since:

$$W_{31}^0 \simeq (13{\cdot}62)\,\mathrm{GeV}^3 \; , \qquad (5.55\text{-}7)$$

and:

$$W_{61}^0 \simeq (10{\cdot}70)\,\mathrm{GeV}^3 \; , \qquad (5.55\text{-}8)$$

these two equations are inconsistent, although their respective predictions will only differ by about 23%. In the "equal mass" limit:

$$M_1 \to m \;, \quad \mu \to 0 \;, \tag{5.55-9}$$

$W^0_{31}$ and $W^0_{61}$ become equal and non-vanishing, rendering the equations consistent and non-trivial. (C.f. the equal mass limit of $X^0_{31}$ and $X^0_{61}$. These differing behaviours arise out of the opposite normalities of the resonances concerned.)

On evaluation of equations 5.55-5 and 6 we obtain:

$$G_1^V(1236) \simeq \begin{cases} (1{\cdot}078)F_2^V \ \mathrm{GeV}^{-1}, \ (\text{sum-rule on } A_3^+), & (5.55\text{-}10) \\ (1{\cdot}373)F_2^V \ \mathrm{GeV}^{-1}, \ (\text{sum-rule on } A_6^+). & (5.55\text{-}11) \end{cases}$$

At vanishing $q^2$ this becomes:

$$G_1^V(1236, q^2=0) \simeq \begin{cases} 2{\cdot}00 \ \mathrm{GeV}^{-1}, \ (A_3^+), & (5.55\text{-}12) \\ 2{\cdot}54 \ \mathrm{GeV}^{-1}, \ (A_6^+). & (5.55\text{-}13) \end{cases}$$

A number of authors[8] have obtained results equivalent to equation 5.55-12 by means of the sum-rules for pure photoproduction. On comparison with our four fits to the photoproduction data given in Appendix 9, we see that whichever fit is adopted this prediction is between about 20% and 30% too low. In view of the drastic nature of the approximation this is nevertheless a reasonable result.

However, equation 5.55-13 is in spectacular agreement with the three lower fits and in good agreement even with the highest one. It differs from the fits corresponding to $E_1^+/M_1^+$ ratios of +6.4%, zero, -4.5%, and -6.4% by about 11%, 5%, 0.4%, and 2% respectively. Since the prediction is based on the vanishing of $G_2^V(1236)$, the final fit possibly provides the most justified comparison.

This prediction cannot of course be obtained by the methods of the authors cited above. Its accuracy may simply arise out of the happy coincidence that all the errors introduced by the approximation procedure exactly compensate one another as far as this equation is concerned. On the other hand, it could indicate an almost exact cancellation, in the vanishing $q^2$ continuation, of all contributions to the coefficient of t in the $A_6^{+}$ sum-rule other than those due to $G_1^V(1236)$ and $F_2^V$. This could well include the contributions from $G_{2,3}^V(1236)$, thus eliminating the reliance of the result on equations 5.55-2 and 3. We are unable to offer any explanation for the mechanism responsible for such a cancellation.

Finally, we wish to indicate a possible alternative approach to the sum-rule on $A_3^{+}$. At vanishing $q^2$ this can receive no contribution from $G_3^V(1236)$, and as a check on the calculations we note that $W_{33}^{o}$ is indeed proportional to $q^2$. Thus in this limit the vanishing of the coefficient of $t^o$ in the $A_3^{+}$ sum-rule implies:

$$W_{31}^{o}\mathcal{G}_1(0) + W_{32}^{o}\Big|_{q^2=0}\mathcal{G}_2(0) = F_2^V(0) . \qquad (5.55\text{-}14)$$

We may argue that a great deal of faith cannot be placed in equation 5.55-2 since as it involves only a single form-factor it is unlikely to correspond to a well-saturated coefficient of t in the sum-rule. If we then scrap this equation as unreliable we may keep the $\mathcal{G}_2(o)$ term in 5.55-14. By substituting into this equation an empirical value for the $\mathcal{G}_2(o)/\mathcal{G}_1(o)$ ratio we may try to improve the $A_3^{+}$ prediction for $G_1(o)$, or vice versa.

On substituting for $\mathcal{G}_2(o)$ the value:

$$\mathcal{G}_2(0) = \mathcal{G}_1(0)/M_1 \quad , \tag{5.55-15}$$

corresponding to a pure magnetic dipole transition, we find:

$$G_1^V(1236, q^2=0) = 2.64\ GeV^{-1} \quad , \tag{5.55-16}$$

which is within $1\frac{1}{2}\%$ of the empirical fit obtained by assuming such a vanishing $E_1^+/M_1^+$ ratio. If on the other hand we substiture the value:

$$\mathcal{G}_2(0) = (0{\cdot}308)\,\mathcal{G}_1(0)/M_1 \ , \tag{5.55-17}$$

which corresponds to Gourdin and Salin's value of $E_1^+/M_1^+$, we find:

$$G_1^V(1236, q^2=0) = 2{\cdot}16\ GeV^{-1} \quad . \tag{5.55-18}$$

This is in much poorer agreement with the corresponding empirical fit form which it differs by about 14%.

## SUMMARY OF RESULTS AND CONCLUSIONS

Using the (original) $O(3,1)\otimes SU(2)$ invariant off-shell techniques developed in Chapter 2 Part II and in Chapter 3, and assuming charge-conjugation invarience of hadron-virtual photon interactions, we have obtained the following super-convergent sum-rules. They are valid for non-positive definite t and for all non time-like $q^2$; $\gamma$ denotes a real or virtual photon.

i) Four sum-rules, (list 4.4-42), for each of the processes:

$$\gamma N \rightarrow N\eta \quad , \tag{1}$$
$$\gamma N \rightarrow N\eta' \quad , \tag{2}$$
$$\gamma N \rightarrow N E \quad . \tag{3}$$

ii) Five sum-rules, (list 4.4-44), for the process:

$$\gamma N \rightarrow N\pi \quad . \tag{4}$$

iii) Two sum-rules, (list 4.4-43), for the production of hypothetical mesons with:

$$(J^P, I, C_n) = (0^-, 0, -) \quad . \tag{5}$$

iv) Two sum-rules, (list 4.4.45), for the production of hypothetical mesons with:

$$(J^P, I, C_n) = (0^-, 1, -) \quad . \tag{6}$$

v) Eight sum-rules, (list 4.5-59), for each of the processes:

$$\gamma N \rightarrow N\omega \quad , \tag{7}$$
$$\gamma N \rightarrow N\phi \quad . \tag{8}$$

vi) Eleven sum-rules, (list 4.5-61), for the process:

$$\gamma N \rightarrow N\rho \quad . \tag{9}$$

vii) Fourteen sum-rules, (list 4.5-58), for the production of hypothetical mesons with:

$$(J^P, I, C_n) = (1^-, 0, +) \quad . \tag{10}$$

viii) Eighteen sum-rules, (list 4.5-60), for the production of hypothetical mesons with:

$$(J^P, I, C_n) = (1^-, 1, +) . \tag{11}$$

Of the above eighty sum-rules, fourteen refer to amplitudes having electrodynamical poles as a necessary consequence of gauge-invarience and/or current-conservation. We conclude that these cannot be saturated in the resonance approximation.

All eighty sum-rules remain non-trivial in the vanishing $q^2$ limit, but only fifty-three of these can be obtained if one treats the photon as an on-shell particle from the outset. Thus we conclude that even if one is only interested in obtaining sum-rules for a real photoproduction process, the correct way to proceed is to treat the photon as a virtual particle and only take the vanishing $q^2$ limit at the conclusion of the calculation.

All the above sum-rules are original, but in the real photon limit three of the sum-rules for pion production have been obtained independently of our own investigations by a variety of authors. They all employ a rather different non-covarient approach.

On assuming instead that hadron-virtual photon interactions are not charge-conjugation invarient we have obtained the following sum-rules for space-like virtual photoproduction processes. All are original.

ix) Two sum-rules for each of the processes 1,2,3 and 5, and three sum-rules on each of the processes 4 and 6, of which none can be saturated in the resonance approximation, (lists 4.4-46 and 47).

x) Twelve sum-rules, of which eleven can be saturated in the resonance approximation, for each of the processes 7,8, and 10, (list 4.5-62).

xi) Nineteen sum-rules, of which seventeen can be saturated in the resonance approximation, for each of the processes 9 and 11, (list 4.5-63).

The amount and complexity of the algebra involved in a proper saturation of the sum-rules for the production of vector mesons is so great that we have postponed these calculations until list programming techniques have been developed to enable this algebra to be carried out by computer.

We have instead restricted our saturation attempts to the sum-rules for the processes 1 and 4. Here we have a reasonable idea of which resonances should dominate the sum-rules, and since none of these has spin exceeding five-halves the algebra is just about manageable when carried out by hand. It has been necessary to assume charge-conjugation invarience of the hadron-virtual photon interactions since otherwise we obtain sum-rules for the two virtual photoproduction processes which cannot be saturated in the resonance approximation. Whilst alternative approximation procedures are available, only this particular approach will yield predictions about the form-factors for electromagnetic nucleon $\rightarrow$ isobar excitation.

Saturation of the sum-rules for $\eta$-production is a relatively trivial matter since only the N(1550) is expected to contribute strongly, (in addition to the nucleon Born-term of course). We have predicted the values for all non time-like $q^2$ of all four form-factors parameterising the $\gamma N(939) \longrightarrow N(1550)$ excitation, (equations 5.4-9,10,11, and 21). We have yet to compare these predictions with the experimental data on $\eta$ photo- and electro-production in the N(1550) resonance region.

On saturating the sum-rules for pion production with the

N(1688), N(1680), N(1525), $\Delta$(1236), and nucleon Born-term, we have obtained relations between the nucleon moment form-factors and those parameterising the excitation of the nucleon into these various isobars, (equations 5.52-2,3,5,6,8,9,12, and 13). These may be solved for the $G^S_{1,2,3}(1688)$ and $G^S_{1,2,3}(1680)$ in terms of the $G^S_{1,2,3}(1525)$ and $F^S_2$, and for the $G^V_{1,2,3}(1688)$ and $G^V_{1,2,3}(1680)$ in terms of the $G^V_{1,2,3}(1525)$, $G^V_{1,2,3}(1236)$ and $F^V_2$. Thus all twelve form-factors for N(1688) and N(1680) production may be obtained in terms of the eleven remaining form-factors. In particular, the $G^{S,V}_{2,3}(1688)$ and $G^{S,V}_{2,3}(1680)$ are related to one another through the four equations 5.52-14. The rather large dimensionless constant, -39.4, appearing on the right-hand side of these equations becomes plausible once one bears in mind the fact that with our choice of coupling constants for the isobar-nucleon-pion vertices, we have:

$$\frac{g\,[N(1680)\to N\pi]}{g\,[N(1688)\to N\pi]} \simeq \frac{1}{38\cdot 9}\ . \qquad (12)$$

The empirical data on the above inelastic form-factors was very sparse at the time when the research reported in this thesis was initiated. The author was in fact only well acquainted with the data on $G^V_{1,2}(1236,\ q^2 = 0)$. A detailed comparison of the above predictions and the present experimental information will be carried out in the near future.

On omitting the N(1680) contributions from the sum-rules we can now longer discuss the $G^{S,V}_{1,2,3}(1680)$, but are now able to predict the $G^{S,V}_{1,2,3}(1525)$. That is, we can still predict twelve form-factors but the input data required is reduced from eleven to five form-factors.

In this saturation attempt we have predicted the vanishing of the four $G^{S,V}_{2,3}(1688)$, and have obtained $G^S_1(1688)$ and $G^S_{1,2,3}(1525)$

in terms only of $F_2^S$, (equations 5.53-31 to 34). If the expressions for the $G_{1,2,3}^S(1525)$ are required to remain finite when continued unmodified into the physical time-like region, we require in addition that $F_2^S$ should vanish when $q^2$ has the time-like value 6.48 $GeV^2$. We have concluded that this constraint is plausible. The isovector equations have been solved for $G_1^V(1688)$ and $G_{1,2,3}^V(1525)$ in terms of $G_{1,2,3}^V(1236)$ and $F_2^V$, (equations 5.53-37 to 40). Finiteness of the $G_{1,2,3}^V(1525)$ in the time-like physical region again requires the $G_{1,2,3}^V(1236)$ and $F_2^V$ to satisfy a constraint equation for $q^2$ equal to 6.48 $GeV^2$. Lack of data in the time-like region has prevented our discussing the plausibility of this latter constraint.

Qualitatively, this saturation attempt has led to the predictions that the isoscalar form-factors are small in comparison with the isovector ones, and that the $\gamma N \rightarrow N(1688)$ form-factors are small in comparison with the corresponding ones for the $\gamma N \rightarrow N(1525)$ transition. These are not unreason--able results. Again, a detailed comparison with the latest experimental data will be attempted in a subsequent article.

On leaving out the N(1688) contributions as well, the predictive power of the isovector equations is reduced but $G_3^V(1236)$ can now be predicted rather than being needed as input. In this way we have obtained expressions for $G_{1,2,3}^V(1525)$ and $G_3^V(1236)$ in terms of $G_{1,2}^V(1236)$ and $F_2^V$, (equations 5.54-18 to 21). Finiteness of these solutions at $q^2$ equal to the space-like value -1.212 $GeV^2$ requires the latter three form-factors to satisfy a constraint equation at this point. We have demonstrated the plausibility of this constraint. Using experimental data with which we were well acquainted, we have evaluated these solutions at vanishing $q^2$, (Table

5.54-I), and found them to be particularly sensitive to the value adopted for the ratio $G_2^V(1236, q^2 = 0)/G_1^V(1236, q^2 = 0)$. Unfortunately this ratio is not well determined by the present data and these particular predictions may possibly prove more useful as a means of predicting it in terms of empirical information on, say, $G_1^V(1525)$.

There is not really much point in omitting the K(1688) contributions from the isoscalar sum-rules since all form-factors appearing are already expressible in terms of $F_2^S$ only. On doing this for the sake of completeness, however, we have predicted the vanishing of the $G_{2,3}^S(1525)$ and have obtained two inconsistent equations, (5.54-15), relating $G_1^S(1525)$ to $F_2^S$. One may insist that these together predict the vanishing of both $G_1^S(1525)$ and $F_2^S$, this latter prediction at least being in agreement to within about 5% with all available experimental data. Alternatively one may note that the predictions of the two equations treated separately only differ by about 14%, and are quite close at vanishing $q^2$ to corresponding result of the previous saturation attempt. In the equal mass limit the two equations become consistent; $G_1^S(1525)$ no longer contributes to the sum-rules and both then predict the vanishing of $F_2^S$.

Finally we have investigated the possibility of attempting to saturate these sum-rules with the Born-term and the $\Delta(1236)$ alone. In this case we have predicted the vanishing of $F_2^S$, $G_2^V(1236)$, and $G_3^V(1236)$. The first of these predictions is in good agreement with the data, as discussed previously. The second corresponds at vanishing $q^2$ to an $E_1^+/M_1^+$ ratio of about -6.4%, in qualitative agreement with the experimental result that this ratio is of the order of a few percent and probably negative. We have also obtained two inconsistent equations, (5.5-10 and 11), relating $G_1^V(1236)$ to $F_2^V$. We cannot use them to predict the vanishing of these two form-factors since we know that $F_2^V(0)$ is non-vanishing and also require $G_1^V(1236)$ to be non-zero if $G_{2,3}^V(1236)$ both vanish. In the equal mass limit the two equations become consistent and remain non-trivial. With physical masses their respective predictions differ by about twenty-three percent. Evaluating these equations in the vanishing $q^2$ limit we have obtained two inconsistent predictions for $G_1^V(1236, q^2 = 0)$.

One of these can be obtained by superconvergence of real photoproduction. It differs from the empirical data by between about twenty and thirty percent, depending on the value adopted for the $E_1^+/M_1^+$ ratio. On the other hand, the particular sum-rule from which this equation is obtained receives no contribution from $G_3^V(1236)$ at vanishing $q^2$ so one can try to fit it to the empirical data by adopting a non-zero value for $G_2^V(1236, q^2 = 0)$. (The equation predicting the vanishing of this latter form-factor is expected to be rather poorly saturated.) In this way we have found that the sum-rule satisfies the experimental data to within about $1\frac{1}{2}\%$, (equation 5.5-16), if one adopts the value $E_1^+/M_1^+$ equal to zero, as predicted for example by $SU(6)_W$ symmetry. On adopting Gourdin and Salin's value of -4.5%, however, the equation (5.5-18) differs from the data by about 14%.

The second equation can only be obtained by means of our off-shell approach. It agrees excellently with experiment as it stands, differing by 2% from the fit corresponding to the vanishing of $G_2^V(1236, q^2 = 0)$, and by 0.4% from the fit corresponding to Gourdin and Salin's value for $E_1^+/M_1^+$.

Thus the predictions of this final saturation attempt are in surprisingly good agreement with experiment, especially when the crudity of the approximation is borne in mind. This gives us confidence that the predictions of the more realistic saturation attempts will prove to be substantially correct when more detailed comparisons with experiment are available.

We conclude that the derivation and saturation of off-shell superconvergent sum-rules for hadron-virtual photon scattering processes provides a useful and powerful means of investigating the hadron electromagnetic form-factors.

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# APPENDICES

## APPENDIX 1 DEFINITION OF OUR METRIC AND DIRAC MATRICES.

We use the Lorentz-space metric defined by:

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 . \qquad \text{(A1-1)}$$

Rather than distinguish between covarient and contra-varient four-vectors, we simply write all such vectors in the form:

$$a = a_\mu = (a_0, a_1, a_2, a_3) = (a_0, \underline{a}) = (a_0, a_i) , \qquad \text{(A1-2)}$$

with the summation convention for repeated Lorentz (Greek) indices defined by:

$$a \cdot b \equiv a_\mu b_\mu \equiv a_0 b_0 - \underline{a} \cdot \underline{b} = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 , \qquad \text{(A1-3)}$$

so:

$$g_{\mu\nu} a_\nu = a_\mu . \qquad \text{(A1-4)}$$

Our Dirac matrices are then required to satisfy:

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} , \qquad \text{(A1-5)}$$

and we define:

$$\gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 , \qquad \text{(A1-6)}$$

$$\sigma_{\mu\nu} \equiv \frac{i}{2} [\gamma_\mu, \gamma_\nu] , \qquad \text{(A1-7)}$$

$$\sigma_{\mu 5} \equiv \frac{i}{2} [\gamma_\mu, \gamma_5] . \qquad \text{(A1-8)}$$

It follows from A1-5 and 6 that:

$$\{\gamma_\mu, \gamma_5\} = 0 . \qquad \text{(A1-9)}$$

In cases where our work is simplified by using an explicit realisation of these matrices, we shall always choose:

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} , \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} , \qquad \text{(A1-10)}$$

with the usual Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad \text{(A1-11)}$$

We then have: $$\gamma_5 = \begin{pmatrix} 0 & -i\mathbb{1}_2 \\ -i\mathbb{1}_2 & 0 \end{pmatrix}, \tag{A1-12}$$
and:
$$\gamma_0^2 = -\gamma_1^2 = -\gamma_2^2 = -\gamma_3^2 = -\gamma_5^2 = 1 \ . \tag{A1-13}$$
The eight matrices: $\mathbb{1}_4$, $\gamma_0$, $\sigma_{ij}$, and $\sigma_{i5}$ are hermitian in this realisation, whilst the eight: $\gamma_i$, $\gamma_5$, $\sigma_{0i}$, and $\sigma_{05}$ are anti-hermitian. In this same realisation, if $\Gamma$ denotes any one of these sixteen matrices, then all sixteen satisfy:
$$\overline{\Gamma} \equiv \gamma_0 \Gamma^\dagger \gamma_0 = \Gamma, \tag{A1-14}$$
that is, the sixteen $\gamma_0\Gamma$ are all hermitian.

## APPENDIX 2. EXPLICIT REALISATION OF THE BASIC SPIN ONE-HALF AND SPIN ONE WAVE FUNCTIONS.

Our basic spin one-half four-component spinor wave-function is realised by:
$$u^{\frac{1}{2}}(p) = \frac{1}{\sqrt{p_0+m}} \begin{bmatrix} (p_0+m)\, w^{\frac{1}{2}}(\underline{p}/|\underline{p}|) \\ \underline{\sigma}\cdot\underline{p}\ \ w^{\frac{1}{2}}(\underline{p}/|\underline{p}|) \end{bmatrix}, \tag{A2-1}$$
where the two-component spinor $w^{1/2}(\underline{p}/|\underline{p}|)$ is given for $\underline{p}$ parallel to the 3-axis by:
$$w^{\frac{1}{2}}(0,0,1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A2-2}$$
and for general orientation of $\underline{p}$ by:
$$w^{\frac{1}{2}}(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) = \frac{1}{\sqrt{2(1+\cos\theta)}} \begin{pmatrix} 1+\cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix}. \tag{A2-3}$$
The wave-functions $u^{-1/2}(p)$, $\overline{u}^{\pm 1/2}(p)$, $v^{\pm 1/2}(p)$, and $\overline{v}^{\pm 1/2}(p)$ are then generated from $u^{1/2}(p)$ by means of the appropriate equations of section 2.11. The normalisation of these wave-functions is:
$$\overline{u}^{\lambda'}(p)\, u^{\lambda}(p) = 2m\,\delta_{\lambda'\lambda} \ . \tag{A2-4}$$

We realise the four-vector wave-functions for a massive spin-one particle by:
$$\varepsilon^0(p) = \frac{1}{m}\left(|\underline{p}|,\ p_0\,\underline{p}/|\underline{p}|\right), \tag{A2-5}$$

$$\varepsilon^{\pm 1}(p_0, 0, 0, |p|) = \frac{-1}{\sqrt{2}}(0, \pm 1, i, 0)\,, \tag{A2-6}$$

$$\varepsilon^{\pm 1}(p_0, |p|\cos\phi\sin\theta, |p|\sin\phi\sin\theta, |p|\cos\theta) = \frac{-1}{\sqrt{2}}(0,$$
$$\pm\cos\phi\cos\theta - i\sin\phi\,, \pm\sin\phi\cos\theta + i\cos\phi\,, \mp\sin\theta)\,, \tag{A2-7}$$

so the normalisation in this case is:

$$\varepsilon^{*\lambda'}_{\mu}(p)\,\varepsilon^{\lambda}_{\mu}(p) = -\delta_{\lambda'\lambda}\;. \tag{A2-8}$$

## APPENDIX 3 USEFUL RELATIONS INVOLVING THE FOURTH-RANK LEVI-CEVITA TENSOR

We define:

$$\varepsilon_{\mu\nu\lambda\rho} \equiv \begin{cases} 1, & (\mu,\nu,\lambda,\rho) = \text{even permutation of } (0,1,2,3), \\ -1 & (\mu,\nu,\lambda,\rho) = \text{odd permutation of } (0,1,2,3), \\ 0, & \text{any two indices equal.} \end{cases} \tag{A3-1}$$

This numerical tensor then satisfies the useful basic relations:

$$\varepsilon_{\mu\nu\lambda\rho}\, g_{\alpha\beta} = \varepsilon_{\alpha\nu\lambda\rho}\, g_{\mu\beta} + \varepsilon_{\mu\alpha\lambda\rho}\, g_{\nu\beta} + \varepsilon_{\mu\nu\alpha\rho}\, g_{\lambda\beta} + \varepsilon_{\mu\nu\lambda\alpha}\, g_{\rho\beta}\,, \tag{A3-2}$$

and:

$$\varepsilon_{\mu\nu\lambda\rho}\,\varepsilon_{\mu'\nu'\lambda'\rho'} = -\begin{vmatrix} g_{\mu\mu'} & g_{\mu\nu'} & g_{\mu\lambda'} & g_{\mu\rho'} \\ g_{\nu\mu'} & g_{\nu\nu'} & g_{\nu\lambda'} & g_{\nu\rho'} \\ g_{\lambda\mu'} & g_{\lambda\nu'} & g_{\lambda\lambda'} & g_{\lambda\rho'} \\ g_{\rho\mu'} & g_{\rho\nu'} & g_{\rho\lambda'} & g_{\rho\rho'} \end{vmatrix}\,. \tag{A3-3}$$

A third basic relation follows from equations A1-5, A1-6, and A3-1, which together imply:

$$\gamma_5 = \frac{-1}{4!}\,\varepsilon_{\mu\nu\lambda\rho}\,\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\;, \tag{A3-4}$$

from which after some tedious algebra one obtains:

$$\begin{aligned} -\varepsilon_{\mu\nu\lambda\rho}\,\gamma_5 = {} & g_{\mu\nu}\gamma_\lambda\gamma_\rho + g_{\lambda\rho}\gamma_\mu\gamma_\nu + g_{\mu\rho}\gamma_\nu\gamma_\lambda + g_{\nu\lambda}\gamma_\mu\gamma_\rho \\ & - g_{\mu\lambda}\gamma_\nu\gamma_\rho - g_{\nu\rho}\gamma_\mu\gamma_\lambda - g_{\mu\nu}\,g_{\lambda\rho} \\ & - g_{\nu\lambda}\,g_{\mu\rho} + g_{\mu\lambda}\,g_{\rho\nu} - \gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\;. \end{aligned} \tag{A3-5}$$

Equations A1-1,5,6,7,8, and A3-1,2,3,5 are together sufficient for the derivation of all possible relations involving the Dirac matrices, the metric tensor, and the Levi-Cevita tensor.

We shall generally use the shorthand notation made clear by the following examples:

$$\left.\begin{aligned} &\mathcal{E}(abcd) \equiv \mathcal{E}_{\mu\nu\lambda\rho}\, a_\mu b_\nu c_\lambda d_\rho \,, \\ &\mathcal{E}_{\mu\nu}(ab) \equiv \mathcal{E}_{\mu\nu\lambda\rho}\, a_\lambda b_\rho \,, \\ &\mathcal{E}(a)_\nu(bc) \equiv \mathcal{E}_{\mu\nu\lambda\rho}\, a_\mu b_\lambda c_\rho \,. \end{aligned}\right\} \qquad \text{(A3-6)}$$

and in particular for non-commutative four-vectors:

$$\mathcal{E}_{\mu\nu}(\gamma\gamma) \equiv \mathcal{E}_{\mu\nu\lambda\rho}\gamma_\lambda\gamma_\rho \begin{cases} \neq \mathcal{E}_{\mu\nu\lambda\rho}\gamma_\rho\gamma_\lambda \,, \\ \neq 0 \,. \end{cases} \qquad \text{(A3-7)}$$

Note, however, that if b is just a four-momentum, it follows from the antisymmetry property:

$$\mathcal{E}_{\mu\nu\lambda\rho} = -\mathcal{E}_{\mu\nu\rho\lambda} \qquad \text{(A3-8)}$$

that: $\mathcal{E}_{\mu\nu}(bb) = 0 = \mathcal{E}_\mu(b)_\rho(b) = \ldots$ etc. . (A3-9)

Similarly if $S_{\lambda\rho}$ is a symmetric tensor,

$$\mathcal{E}_{\mu\nu\lambda\rho} S_{\lambda\rho} = 0 = \mathcal{E}_{\mu\nu\lambda\rho} S_{\mu\rho} = \text{etc.} \,. \qquad \text{(A3-10)}$$

We now list some useful relations; a,b,c,d,... will always stand for four-momenta and not for Dirac matrices unless this is explicitly stated.

From A3-2 we have: $\mathcal{E}_{\mu\nu}(ab) g_{\lambda\rho} + \mathcal{E}_{\nu\lambda}(ab) g_{\mu\rho}$

$$+ \mathcal{E}_{\lambda\mu}(ab) g_{\nu\rho} = a_\rho \mathcal{E}_{\mu\nu\lambda}(b) - b_\rho \mathcal{E}_{\mu\nu\lambda}(a) \,, \qquad \text{(A3-11)}$$

so in particular: $\mathcal{E}_{\mu\nu}(ab) c_\lambda + \mathcal{E}_{\nu\lambda}(ab) c_\mu$

$$+ \mathcal{E}_{\lambda\mu}(ab) c_\nu = (a\cdot c)\mathcal{E}_{\mu\nu\lambda}(b) - (b\cdot c)\mathcal{E}_{\mu\nu\lambda}(a) . \qquad \text{(A3-12)}$$

Equations A3-11 and 12 continue to hold if any of the a,b,c are Dirac $\gamma$-matrices, provided one writes the equations with these factors appearing in the same order throughout, that is, provided one writes:

$$\left.\begin{aligned} &\text{r.h.s. (A2-11)} = a_\rho \mathcal{E}_{\mu\nu\lambda}(b) - \mathcal{E}_{\mu\nu\lambda}(a) b_\rho \,, \\ &\text{and:} \\ &\text{r.h.s. (A2-12)} = a_\sigma \mathcal{E}_{\mu\nu\lambda}(b) c_\sigma - \mathcal{E}_{\mu\nu\lambda}(a)(b\cdot c) \,. \end{aligned}\right\} \qquad \text{(A3-13)}$$

In particular, if a,b,c are all $\gamma$-matrices, we have, using:

$$\{\varepsilon_{\mu\nu\lambda}(\gamma), \gamma_\rho\} = 2\varepsilon_{\mu\nu\lambda\rho} \tag{A3-14}$$

and, (in our realisation): $\gamma \cdot \gamma = 4$, (A3-15)

that:

$$\varepsilon_{\mu\nu}(\gamma\gamma)g_{\lambda\rho} + \varepsilon_{\nu\lambda}(\gamma\gamma)g_{\mu\rho} + \varepsilon_{\lambda\mu}(\gamma\gamma)g_{\nu\rho} = 2\left[\gamma_\rho \varepsilon_{\mu\nu\lambda}(\gamma) - \varepsilon_{\mu\nu\lambda\rho}\right], \tag{A3-16}$$

and:

$$\varepsilon_{\mu\nu}(\gamma\gamma)\gamma_\lambda + \varepsilon_{\nu\lambda}(\gamma\gamma)\gamma_\mu + \varepsilon_{\lambda\mu}(\gamma\gamma)\gamma_\nu = -6\varepsilon_{\mu\nu\lambda}(\gamma). \tag{A3-17}$$

Further relations are easily derived by saturating free indices in A3-11 and 12 with additional four-momenta or $\gamma$-matrices.

From equation A3-3 one may derive the following relations:

$$\varepsilon_{\mu\nu\lambda\rho}\varepsilon_{\mu\nu'\lambda'\rho'} = -1! \begin{vmatrix} g_{\nu\nu'} & g_{\nu\lambda'} & g_{\nu\rho'} \\ g_{\lambda\nu'} & g_{\lambda\lambda'} & g_{\lambda\rho'} \\ g_{\rho\nu'} & g_{\rho\lambda'} & g_{\rho\rho'} \end{vmatrix}, \tag{A3-18}$$

$$\varepsilon_{\mu\nu\lambda\rho}\varepsilon_{\mu\nu\lambda'\rho'} = -2! \begin{vmatrix} g_{\lambda\lambda'} & g_{\lambda\rho'} \\ g_{\rho\lambda'} & g_{\rho\rho'} \end{vmatrix}, \tag{A3-19}$$

$$\varepsilon_{\mu\nu\lambda\rho}\varepsilon_{\mu\nu\lambda\rho'} = -3!\, g_{\rho\rho'}, \tag{A3-20}$$

and:

$$\varepsilon_{\mu\nu\lambda\rho}\varepsilon_{\mu\nu\lambda\rho} = -4!. \tag{A3-21}$$

Contracting free indices in A3-3 and A3-18 to 21 with four-momenta and/or $\gamma$-matrices then yields additional useful relations. If two or more $\gamma$-matrices are involved they are best distinguished by superscripts, since it is important that when the determinants are expanded the $\gamma$-matrices occur in each term in the order in which they appeared in the original "double-epsilon" product. At least, one must not re-order the matrices in a particular term without taking proper account of the anti-commutation relations.

We have derived from the above equations the following interesting relations which greatly simplify the calculation of lowest order unpolarised cross-sections for processes, such as: electron + nucleon $\rightarrow$ electron + isobar:

$$\mathcal{E}_{\mu}(ab\gamma)\mathcal{E}_{\mu}(ab\gamma)=-2\left[a^2b^2-(a\cdot b)^2\right], \tag{A3-22}$$

$$\mathcal{E}(\gamma ab\gamma)\mathcal{E}(\gamma ab\gamma)=4\left[a^2b^2-(a\cdot b)^2\right], \tag{A3-23}$$

$$\mathcal{E}_{\mu\nu}(ab)\mathcal{E}_{\mu}(ab\gamma)\gamma_5\mathcal{E}_{\nu}(ab\gamma)=2\left[a^2b^2-(a\cdot b)^2\right](a\cdot b-\not{a}\not{b}), \tag{A3-24}$$

$$\mathcal{E}_{\mu\nu}(ab)\mathcal{E}_{\nu\lambda}(ab)\mathcal{E}_{\lambda\rho}(ab)\mathcal{E}_{\rho\mu}(ab)=2\left[a^2b^2-(a\cdot b)^2\right]^2, \tag{A3-25}$$

$$\mathcal{E}_{\mu}(ab\gamma)\mathcal{E}_{\nu}(ab\gamma)\mathcal{E}_{\mu\lambda}(ab)\mathcal{E}_{\lambda\nu}(ab)=-2\left[a^2b^2-(a\cdot b)^2\right]^2. \tag{A3-26}$$

A large number of relations can be derived from equation A3-5 by contraction with $\gamma$-matrices and/or four-momenta and possibly invoking equations A1-7 and 8. The only ones needed for this thesis are as follows.

Contracting A3-5 with $a_\lambda$ and $b_\rho$, we obtain after some anti-commutation:

$$\begin{aligned}\mathcal{E}_{\mu\nu}(ab)\gamma_5 &= \not{a}\gamma_\mu\gamma_\nu\not{b}-a_\mu\gamma_\nu\not{b}\\ &+\gamma_\mu a_\nu\not{b}+\not{a}b_\mu\gamma_\nu-\not{a}\gamma_\mu b_\nu-g_{\mu\nu}\not{a}\not{b}\\ &+g_{\mu\nu}a\cdot b-a\cdot b\gamma_\mu\gamma_\nu+a_\mu b_\nu-b_\mu a_\nu .\end{aligned} \tag{A3-27}$$

The reason for anti-commuting the right-hand-side into the form above, is that it is then particularly simple to invoke the Dirac equation if $\mathcal{E}_{\mu\nu}(ab)\gamma_5$ is sandwiched between half-integer spin Dirac spinors: $\overline{\psi}(a)$ and $\psi(b)$.

Contraction of A3-5 with $\gamma_\rho$ yields:

$$\mathcal{E}_{\mu\nu\lambda}(\gamma)\gamma_5=g_{\mu\nu}\gamma_\lambda+g_{\nu\lambda}\gamma_\mu-g_{\mu\lambda}\gamma_\nu-\gamma_\mu\gamma_\nu\gamma_\lambda, \tag{A3-28}$$

which on contraction with $a_\mu$ and $b_\lambda$ gives:

$$\varepsilon_\nu(a\gamma b)\gamma_5 = b_\nu \not{a} + a_\nu \not{b} - \not{a}\gamma_\nu \not{b} - a\cdot b\,\gamma_\nu \ . \tag{A3-29}$$

Finally, contracting A3-27 with $c_\nu$ we obtain:

$$\begin{aligned}\varepsilon_\mu(acb)\gamma_5 = -\not{a}\gamma_\mu\not{c}\not{b} - a\cdot c\,\gamma_\mu\not{b} + b\cdot c\not{a}\gamma_\mu + a\cdot b\,\gamma_\mu\not{c}\\ + a_\mu\not{c}\not{b} - b_\mu\not{a}\not{c} + c_\mu\not{a}\not{b}\\ - b\cdot c\,a_\mu + a\cdot c\,b_\mu - a\cdot b\,c_\mu \ .\end{aligned} \tag{A3-30}$$

## APPENDIX 4. USEFUL RELATIONS INVOLVING THE THIRD RANK LEVI-CEVITA TENSOR.

We similarly define:

$$\varepsilon_{ijk} \equiv \begin{cases} 1, & (i,j,k) = \text{even permutation of } (1,2,3), \\ -1, & (i,j,k) = \text{odd permutation of } (1,2,3), \\ 0, & \text{any two indices equal.} \end{cases} \tag{A4-1}$$

We then have the useful relations:

$$\varepsilon_{ijk}\delta_{\ell m} = \varepsilon_{\ell jk}\delta_{im} + \varepsilon_{i\ell k}\delta_{jm} + \varepsilon_{ij\ell}\delta_{km} \ , \tag{A4-2}$$

$$\varepsilon_{ijk}\varepsilon_{i'j'k'} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} & \delta_{ik'} \\ \delta_{ji'} & \delta_{jj'} & \delta_{jk'} \\ \delta_{ki'} & \delta_{kj'} & \delta_{kk'} \end{vmatrix} \ , \tag{A4-3}$$

$$\varepsilon_{ijk}\varepsilon_{i'j'k} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} \\ \delta_{ji'} & \delta_{jj'} \end{vmatrix} \ , \tag{A4-4}$$

$$\varepsilon_{ijk}\varepsilon_{i'jk} = 2\delta_{ii'} \ , \tag{A4-5}$$

$$i\varepsilon_{jk}(\tau) = \tfrac{1}{2}[\tau_j, \tau_k] \ , \tag{A4-6}$$

$$\varepsilon_{ij}(\tau)\tau_k = \varepsilon_{kj}(\tau)\tau_i + \varepsilon_{ik}(\tau)\tau_j + 3\varepsilon_{ijk}1\!\!1_2 \ , \tag{A4-7}$$

$$\varepsilon_j(\tau\tau) = 2i\tau_j \ . \tag{A4-8}$$

## APPENDIX 5 KINEMATICAL DEFINITIONS AND RELATIONS FOR CHAPTERS 4 and 5.

With N denoting a nucleon, M a meson, and $\gamma$ a real or virtual photon, we define the s,t, and u channels of the processes considered in Chapters 4 and 5 by:

s channel: $N(p) + \gamma(q) \rightarrow N(p') + M(k)$ (A5-1)

t channel: $\bar{M}(-k) + \gamma(q) \rightarrow N(p') + \bar{N}(-p)$ (A5-2)

u channel: $\bar{N}(-p') + \gamma(q) \rightarrow \bar{N}(-p) + M(k)$. (A5-3)

The parentheses in these channel definitions define the momenta of the particle involved, and we define: m, m', and $\mu$ to be the masses of the initial nucleon, final nucleon, and meson respectively. We have distinguished between m and m' in order to keep the relations of this appendix as general as possible, but throughout the remainder of this thesis we always take m and m' to be equal.

We define:

$$s \equiv K^2, \text{ where } K \equiv p+q = p'+k, \quad \text{(A5-4,5)}$$

$$t \equiv \Delta^2, \text{ where } \Delta \equiv q-k = p'-p, \quad \text{(A5-6,7)}$$

$$u \equiv K'^2, \text{ where } K' \equiv p'-q = p-k, \quad \text{(A5-8,9)}$$

$$\Lambda \equiv \tfrac{1}{2}(p-q), \quad Q \equiv \tfrac{1}{2}(k+q), \quad \Lambda'' \equiv \tfrac{1}{2}(p'+q), \quad \text{(A5-10,11,12)}$$

$$\Lambda' \equiv \tfrac{1}{2}(p'-k), \quad P \equiv \tfrac{1}{2}(p'+p), \quad \Lambda''' \equiv \tfrac{1}{2}(p+k), \quad \text{(A5-13,14,15)}$$

$$\nu \equiv \tfrac{1}{4}(s-u), \text{ and } \kappa \equiv m'^2+m^2+\mu^2+q^2. \quad \text{(A5-16,17)}$$

The sets: $(K, \Lambda, \Lambda')$, $(\Delta, -Q, P)$, and $(-K', -\Lambda'', -\Lambda''')$ are sets of "natural" momenta for the s,t, and u channels respectively. Under s$\leftrightarrow$t crossing at fixed u:

$$p' \leftrightarrow p', \quad q \leftrightarrow q, \quad p \leftrightarrow -k, \quad K \leftrightarrow \Delta, \quad \Lambda \leftrightarrow -Q, \quad \Lambda' \leftrightarrow P, \quad \text{(A5-18)}$$

whilst under s$\leftrightarrow$u crossing at fixed t:

$$q\leftrightarrow q,\quad k\leftrightarrow k,\quad p\leftrightarrow -p',\quad K\leftrightarrow -K',\quad \Lambda\leftrightarrow -\Lambda'',\quad \Lambda'\leftrightarrow -\Lambda''',\quad \nu\leftrightarrow -\nu\,. \tag{A5-19}$$

Equations 2.42-14 to 17 are applicable to the processes under consideration here, and the "natural" pair of Mandelstam variables is therefore $\nu$ and t.

With the above definitions we have the following relations between the Mandelstam variables:

$$s+t+u=\mathcal{K}\ ,\qquad s=\tfrac{1}{2}(\mathcal{K}-t+4\nu)\ , \tag{A5-20,21}$$

and:

$$u=\tfrac{1}{2}(\mathcal{K}-t-4\nu)\ . \tag{A5-22}$$

Note that $\mathcal{K}$ is a function of $q^2$ and not a constant in the virtual photon case, and $q^2$ is then itself a Mandelstam variable as discussed in Chapter 3. So we strictly have three independent Mandelstam variables in the off-shell case, the natural set being: $\nu$, t, and $q^2$.

Equations A5-5,7,9, and 10 to 15 yield on inversion:

$$2p=K+2\Lambda=2P-\Delta=2\Lambda'''+K'\ , \tag{A5-23,24,25}$$

$$2q=K-2\Lambda=2Q+\Delta=2\Lambda''-K'\ , \tag{A5-26,27,28}$$

$$2p'=K+2\Lambda'=2P+\Delta=2\Lambda''+K'\ , \tag{A5-29,30,31}$$

$$2k=K-2\Lambda'=2Q-\Delta=2\Lambda'''-K'\ . \tag{A5-32,33,34}$$

The scalar products between the six pairs of different momenta choosable from p, q, p', and k, are given by:

$$2p\cdot q=s-m^2-q^2=m'^2+\mu^2-t-u\ , \tag{A5-35,36}$$

$$2p'\cdot q=m'^2+q^2-u=t+s-m^2-\mu^2\ , \tag{A5-37,38}$$

$$2p\cdot k=m^2+\mu^2-u=t+s-m'^2-q^2\ , \tag{A5-39,40}$$

$$2p'\cdot k=s-m'^2-\mu^2=m^2+q^2-t-u\ , \tag{A5-41,42}$$

$$2p'\cdot p = m'^2 + m^2 - t = s + u - \mu^2 - q^2 \quad , \qquad \text{(A5-43,44)}$$

$$2k\cdot q = \mu^2 + q^2 - t = s + u - m'^2 - m^2 \quad . \qquad \text{(A5-45,46)}$$

Similarly, in terms of for example the "natural" t-channel momenta we have:

$$4P^2 = 2(m'^2 + m^2) - t \quad , \quad P\cdot Q = \nu \quad , \qquad \text{(A5-47,48)}$$

$$2P\cdot\Delta = (m'^2 - m^2) \quad , \quad 2Q\cdot\Delta = (q^2 - \mu^2) \quad , \qquad \text{(A5-49,50)}$$

and: $4Q^2 = 2(\mu^2 + q^2) - t$ . (A5-51)

Having taken the particle with momentum q off-shell, we shall in fact find it more convenient to work in the t-channel with momenta: $\Delta$ , $-q$, and P, and we then need the relations:

$$P\cdot q = \nu + \tfrac{1}{4}(m'^2 - m^2) \quad , \qquad \text{(A5-52)}$$

and: $2\Delta\cdot q = t + q^2 - \mu^2$ . (A5-53)

Note that for $m' = m$:

$$P\cdot\Delta = 0, \quad P\cdot q = \nu, \qquad \text{(A5-54,55)}$$

$$p\cdot q\big|_{s=x} = -p'\cdot q\big|_{u=x} \quad , \qquad \text{(A5-56)}$$

$$p\cdot q\big|_{u=x} = -p'\cdot q\big|_{s=x} \quad , \qquad \text{(A5-57)}$$

$$p\cdot k\big|_{s=x} = -p'\cdot k\big|_{u=x} \quad , \qquad \text{(A5-58)}$$

$$p\cdot k\big|_{u=x} = -p'\cdot k\big|_{s=x} \quad . \qquad \text{(A5-59)}$$

## APPENDIX 6: KINEMATICS OF THE THREE-PARTICLE VERTEX.

We assume that the vertex couples an initial on-shell particle with mass m, momentum p, to a final on-shell particle with mass M, momentum K. The third particle is assumed to be initial with momentum q, but its squared four momentum will always simply be written as $q^2$ to allow for the possibility of its being a virtual photon.

Then:

$$q = K-p, \tag{A6-1}$$

and we define:

$$P' \equiv K + p\ . \tag{A6-2}$$

The following useful relations then hold:

$$K = \tfrac{1}{2}\,(P' + q) \tag{A6-3}$$

$$p = \tfrac{1}{2}\,(P' - q) \tag{A6-4}$$

$$2p.q = M^2-m^2-q^2 \tag{A6-5}$$

$$2p.K = M^2 + m^2 - q^2 \tag{A6-6}$$

$$2K.q = M^2 - m^2 + q^2 \tag{A6-7}$$

$$P'.q = M^2 - m^2 \tag{A6-8}$$

$$P'^2 = 2(M^2 + m^2)-q^2. \tag{A6-9}$$

Note in particular that since any momentum, Q, involved at the vertex may be written in the form:

$$Q = aP' + bq, \tag{A6-10}$$

where a and b are constant coefficients, then if q refers to the only off-shell particle, $Q.q$ is a constant only when

$$Q = P', \tag{A6-11}$$

the general relation being:

$$Q.q = a(M^2-m^2) + bq^2. \tag{A6-12}$$

Note also the useful relations:

$$-4q^2p^2(q) = -4M^2q^2(K) = q^4-2(M^2+m^2)q^2+(M^2-m^2)^2$$
$$= \left[q^2-(M+m)^2\right]\left[q^2-(M-m)^2\right]. \tag{A6-13}$$

## APPENDIX 7. SOME TWO INDEX TYPE 2 EQUIVALENCE RELATIONS FOR FB→FB PROCESSES.

Consider covarients of the general form:

$\mathcal{E}_\mu(a\gamma b)\,\mathcal{E}_\alpha(c\gamma d)$, where a, b, c, and d are four-momenta. Such covarients arise for example in the study of normal reactions with the spin configuration: $1 + \frac{1}{2} \to 1 + \frac{1}{2}$. We may expand $\mathcal{E}_\mu(a\gamma b)\,\mathcal{E}_\alpha(c\gamma d)$ in two ways: firstly by means of equation A3-3, and secondly by using equation A3-29 and the fact that:

$$\mathcal{E}_\mu(a\gamma b)\mathcal{E}_\alpha(c\gamma d) = \left[\mathcal{E}_\mu(a\gamma b)\gamma_5\right]\left[\mathcal{E}_\alpha(c\gamma d)\gamma_5\right] \quad . \qquad \text{(A7-1)}$$

Equating the right hand sides of the two expansions leads to the trivial result (o = o) only in the special case:

$$a = c, \; b = d, \qquad \text{(A7-2)}$$

and in other cases one obtains type 2 equivalence theorems between normal FB→FB two-index covarients.

In particular, one may derive in this way type 2 E.R's. relating the non-gauge-invarient covarients of section 4.12. Choosing p', p, and q to be the three independent four-momenta one may apply the above treatment to the covarients:

$$\mathcal{K}^{(i)}_{\mu\alpha} \equiv \mathcal{E}_\mu(p'\gamma q)\mathcal{E}_\alpha(p'\gamma p) , \qquad \text{(A7-3)}$$

$$\mathcal{K}^{(ii)}_{\mu\alpha} \equiv \mathcal{E}_\mu(p'\gamma p)\mathcal{E}_\alpha(q\gamma p) , \qquad \text{(A7-4)}$$

and:

$$\mathcal{K}^{(iii)}_{\mu\alpha} \equiv \mathcal{E}_\mu(p'\gamma q)\mathcal{E}_\alpha(q\gamma p) . \qquad \text{(A7-5)}$$

where as usual we have chosen and ordered the momenta in each case in that way which reduces to a minimum the number of anti-commutation operations required prior to invokation of the Dirac equation on the nucleon spinors. We find that in view of the subsidiary conditions, the above operation performed on $\mathcal{K}^{(i)}_{\mu\alpha}$ and $\mathcal{K}^{(ii)}_{\mu\alpha}$ leads to the same E.R. in both cases, whilst $\mathcal{K}^{(iii)}_{\mu\alpha}$ leads to a second independent E.R.. No further independent E.R.'s are generated by considering the three covarients which may be obtained from the previous three by interchange of the indices. In the special case m' = m, the E.R. coming from $\mathcal{K}^{(i)}_{\mu\alpha}$ or $\mathcal{K}^{(ii)}_{\mu\alpha}$ reads:

$$m\nu[\gamma,\gamma]+P^2[\gamma\not{q}\gamma]+\nu[\gamma,\Delta]-\Delta\cdot q[\gamma,P]$$
$$-[P,\Delta]\not{q}+m[P,[\gamma,\not{q}]]\cong 0\ , \qquad \text{(A7-6)}$$

whilst $\mathcal{K}^{(iii)}_{\mu\alpha}$ yields:

$$[2\nu^2-\tfrac{1}{2}(\Delta\cdot q)^2][\gamma,\gamma]+2m\nu[\gamma\not{q}\gamma]-\Delta\cdot q\{P,\Delta\}+2\nu\Delta\Delta$$
$$+(\Delta\cdot q-2q^2)[P,\Delta]+m\,\Delta\cdot q\{\gamma,\Delta\}+m(2q^2-\Delta\cdot q)[\gamma,\Delta]$$
$$-2m\Delta\Delta\not{q}+2\nu[P,[\gamma,\not{q}]]+\tfrac{1}{4}t\{\Delta,[\gamma,\not{q}]\}$$
$$+\tfrac{1}{4}(t-2\Delta\cdot q)[\Delta,[\gamma,\not{q}]]\cong 0\ . \qquad \text{(A7-7)}$$

Here we have adopted the shorthand notation that:

$$AB\equiv A_\mu B_\alpha\ , \qquad [A,B]\equiv A_\mu B_\alpha - B_\alpha A_\mu\ , \qquad \text{(A7-8,9)}$$
$$\{A,B\}\equiv A_\mu B_\alpha + B_\alpha A_\mu\ , \qquad \text{(A7-10)}$$

and: $$[\gamma\not{b}\gamma]\equiv\gamma_\mu\not{b}\gamma_\alpha-\gamma_\alpha\not{b}\gamma_\mu\ , \qquad \text{(A7-11)}$$

where A,B are momenta or $\gamma$-matrices, and b is a momentum.

One might try to deduce further type 2 E.R.'s by considering for example the forms: $\mathcal{E}_\mu(a\ b\ c)\ \mathcal{E}_\alpha(a\ b\ c)$, $\mathcal{E}_\mu(a\gamma\gamma)\,\mathcal{E}_\alpha(b\gamma c)$, $\mathcal{E}_\mu(a\gamma b)\,\mathcal{E}_\alpha(\gamma\gamma c)$, $\mathcal{E}_\mu(a\gamma\gamma)\ \mathcal{E}_\alpha(\gamma\gamma b)$, $\mathcal{E}_{\mu\sigma}(AB)\times$ $\times\,\mathcal{E}_{\alpha\sigma}(CD)$, and $\mathcal{E}_{\mu\sigma\tau}(A)\ \mathcal{E}_{\alpha\sigma\tau}(B)$.

However, we have succeeded in showing that irrespective of whether or not $m' = m$, none of these forms leads to a type 2 E.R. inequivalent to those already obtained for the process under consideration.

## APPENDIX 8 RELATION BETWEEN THE O(3,1)⊗SU(2) COUPLING CONSTANTS AND PARTIAL WIDTHS FOR DECAYS OF PION-NUCLEON RESONANCES INTO THE NUCLEON PLUS A PSEUDOSCALAR MESON.

In this appendix we compute the coupling constant for the decay: $(J+\frac{1}{2})^{\pm}\longrightarrow\frac{1}{2}^{\pm}+0^{\pm}$ in terms of the partial width.

After deducing the general relation we use it to calculate the coupling constants encountered in sections 5.4 and 5.5. We adopt the notation that the (momentum, mass, helicity, isospin, isospin projection) of the resonance, final baryon, and meson are defined to be: $(K, M, \Lambda, I_R, t_R)$, $(p', m, \lambda', I_B, t_B)$, and $(k, \mu, 0, I_M, t_M)$ respectively.

The $O(3,1) \otimes SU(2)$ invarient coupling constant, g, is defined in our usual notation by:

$$\langle p', \lambda', t_B; k, t_M | T | K, \Lambda, t_R \rangle^{\pm} = g \bar{u}^{\lambda'}(p')(p'_\mu)^J I^{\pm} u^{\Lambda}_{(\mu)^J}(K) \phi^{* t_M}_{(i'')^{I_M}} \times$$
$$\times \chi^{\dagger t_B}_{(i')^{I_B}} \mathcal{K}_{(i'')^{I_M}(i')^{I_B}(i)^{I_R}}(I_R \to I_B + I_M) \chi^{t_R}_{(i)^{I_R}} \quad , \text{(A8-1)}$$

where as usual:

$$I^{\pm} \equiv \begin{cases} \mathbb{1}_4 \, , \\ \gamma_5 \, , \end{cases} \tag{A8-2}$$

and the plus (minus) signs are to be adopted for decays which are normal (abnormal) overall. The partial width for decay in a particular configuration of isospin projection is then:

$$\Gamma^{\pm}(t_R \to t_B + t_M) = \frac{\rho_f}{2M} \sum_{\lambda} \bar{\sum_{\Lambda}} \left| \langle p' \lambda' t_B, k t_M | T | K \Lambda t_R \rangle^{\pm} \right|^2 \Bigg|_{K=(M,\underline{0})} \, , \tag{A8-3}$$

where the final-state phase-space factor is given by:

$$\rho_f \Big|_{K=(M,\underline{0})} = \frac{1}{4\pi M} |\underline{p}'| \Big|_{K=(M,\underline{0})} \, . \tag{A8-4}$$

The partial width, $\Gamma^{\pm}$, conventionally tabulated by experimentalists is then defined by:

$$\Gamma^{\pm} = \sum_{\substack{\text{all allowed} \\ t_B, \, t_M}} \Gamma^{\pm}(t_R \to t_B + t_M) \quad , \tag{A8-5}$$

and has the same value for all members of the initial multiplet. Hence:

$$\Gamma^{\pm} = X(I_R \to I_B + I_M) \sum_{\lambda} \bar{\sum_{\Lambda}} |T^{\lambda\Lambda}_{\pm}|^2 \frac{|\underline{p}'|}{8\pi M^2} \Bigg|_{K=(M,\underline{0})} \, , \tag{A8-6}$$

where: $$T^{\lambda\Lambda}_{\pm} \equiv g \bar{u}^{\lambda}(p')(p'_\mu)^J I^{\pm} u^{\Lambda}_{(\mu)^J}(K) \, , \tag{A8-7}$$

and:

$$X(I_R \to I_B + I_M) = \sum_{\substack{\text{all allowed} \\ t_B, t_M \text{ for} \\ \text{any fixed } t_R}} \left| \chi^{\dagger t_B}_{(i')^{I_B}} \mathcal{K}_{(i'')^{I_M}(i')^{I_B}(i)^{I_R}}(I_R \to I_B + I_M) \chi^{t_R}_{(i)^{I_R}} \phi^{* t_M}_{(i'')^{I_M}} \right|^2, \tag{A8-8}$$

this latter quantity being the same for all $t_R$. The factor X arises because we have not bothered to normalise our isospin covarients; that is, they yield un-normalised Clebsh-Gordan coefficients when contracted with the external SU(2) wave-functions. They may be normalised by multiplication by $X^{-\frac{1}{2}}$.

Thus:

$$\frac{\Gamma^{\pm}}{g^2 X} = \frac{1}{16\pi M^2(J+1)} tr\left[(\not{p}' \pm m)\mathcal{P}^{J+1/2}(p', p'; K)\right] |\not{p}'| \Big|_{K=(M,\underline{0})}, \tag{A8-9}$$

where the forward contracted propogator is given by:[9]

$$\mathcal{P}^{J+1/2}(p', p'; K) = -C_{J+1}\left[p'^2(K)\right](\not{K} + M) \,. \tag{A8-10}$$

In view of the kinematical relations:

$$\not{p}'^2 \Big|_{K=(M,\underline{0})} = -p'^2(K) = \frac{1}{4M^2}\left[(M+m)^2 - \mu^2\right]\left[(M-m)^2 - \mu^2\right] \,, \tag{A8-11}$$

and:

$$2(p' \cdot K \pm Mm) = \left[(M \pm m)^2 - \mu^2\right] \,, \tag{A8-12}$$

we easily obtain finally:

$$\frac{\Gamma^{\pm}}{g^2 X} = \frac{C_{J+1}\left[(M+m)^2 - \mu^2\right]^{J+1/2}\left[(M-m)^2 - \mu^2\right]^{J+1/2}\left[(M \pm m)^2 - \mu^2\right]}{4^{J+2}(J+1)\pi M^{2J+3}} \,. \tag{A8-13}$$

Note that our coupling constants have the dimensions of $\text{mass}^{-J}$.

Using equation A8-13 we compute the coupling constants for the decays tabulated below. The X-factors are obtained from equation A8-8, and the input data is taken from the January 1968 Rosenfeld tables.[15] For the masses of the nucleon, the pion, and the $\eta$ we take the respective values: 939 MeV, 138MeV, and 549 MeV. Note that the partial decay

TABLE A8-I

COMPUTATION OF THE COUPLING CONSTANTS FOR BARYON DECAY APPEARING IN SECTIONS 5.4 and 5.5.

| RESONANCE (MASS, $\Gamma$) (MeV) | I | $J^P$ | DECAY PRODUCTS | DECAY NORMALITY | X-FACTOR | BRANCHING RATIO | PARTIAL $\Gamma$ (MeV) | DECAY COUPLING CONSTANT g |
|---|---|---|---|---|---|---|---|---|
| $\Delta(1236,120)$ | 3/2 | $3/2^+$ | $N\pi$ | + | 1 | 1.00 | 120 | 15.5 $GeV^{-1}$ |
| N(1525,115) | 1/2 | $3/2^-$ | $N\pi$ | - | 3 | 0.55 | 63.3 | 10.8 $GeV^{-1}$ |
| N(1550,130) | 1/2 | $1/2^-$ | $N\eta$ | + | 1 | 0.70 | 91.0 | 2.11 $GeV^0$ |
| N(1680,170) | 1/2 | $5/2^-$ | $N\pi$ | + | 3 | 0.40 | 68.0 | 5.48 $GeV^{-2}$ |
| N(1688,130) | 1/2 | $5/2^+$ | $N\pi$ | - | 3 | 0.65 | 84.5 | 213 $GeV^{-2}$ |

width only determines the relevant coupling constant to within an overall sign factor. We have assumed that all the coupling constants listed are positive, and the reader is referred to section 5.3 for a discussion of the implications of this assumption.

## APPENDIX 9. PHENOMENOLOGICAL FITS TO PION PHOTOPRODUCTION IN THE 33-RESONANCE REGION.

The empirical values of the coupling-constants $G^V_{1,2}(1236, q^2 = 0)$, hereafter abbreviated to $G^V_{1,2}(o)$, may be obtained from a phenomenological fit to the data on the resonant scattering process:

$$\gamma^{(R)} + p \to \Delta^+(1236) \to p + \pi^0 . \qquad (A9-1)$$

The centre-of-mass frame angular distribution for this process is proportional to:

$$A + C\cos^2\theta + \alpha \sin^2\theta \cos 2\phi \quad , \tag{A9-2}$$

where $\theta$ is the scattering angle and $\phi$ is the angle subtended by the production plane and the plane of polarisation of the incident (real) photon. The constants A, C, and $\alpha$ are polynomial in the masses and homogeneous quadratic in $G^V_{1,2}(o)$. Assuming the reality of these coupling constants, the ratio

$$f \equiv M_1 G_2^V(o) / G_1^V(o) \tag{A9-3}$$

may be determined from either of the ratios C/A or $\alpha$/C. In practice the data is usually analysed in terms of the ratio:

$$\rho(f) \equiv E_1^+ / M_1^+ \quad , \tag{A9-4}$$

where $E_1^+(M_1^+)$ is the non-covarient multipole amplitude corresponding to a purely electric quadrupole (magnetic dipole) induced transition.

Once $\rho$ and therefore f is known, the value of $G_1^V(0)$ may be obtained from the empirical value of the product $\Gamma_{tot}\, \sigma_{res}$ where $\Gamma_{tot}$ is the total width of the $\Delta(1236)$ and $\sigma_{res}$ is the total unpolarised corss-section for the process A9-1.

All empirical fits and theoretical calculations indicate that $|\rho|$ is very small, probably not more than a few percent. Thus, for example, $SU(6)_W$ symmetry predicts $\rho = 0$, whilst U(6,6) implies a $|\rho|$ value of a few per-cent. The dispersion theoretic treatment of CGLN[40] indicates that $|\rho|$ is probably not more than 2%, and the empirical data of McDonald et. al.[43] is consistent with a value:

$$\rho = 0{\cdot}00 \pm 0{\cdot}06 \ . \tag{A9-5}$$

The combined investigations of Drickey and Mozley[42], Berkelman and Waggoner[44], and Vasilikov et. al.[45] yield values of $\alpha/C$ at three energies near to resonance. Gourdin and Salin(36) fit these to a best value:

$$\rho = -0.045 \quad , \tag{A9-6}$$

but do not give an error estimate.

The error range in the empirical values of both C/A and $\alpha/C$ is about $\pm 10\%$, and this unfortunately leads to a much larger fractional error in $\rho$. We shall show in a moment that f is itself a rather violently varying function of $\rho$ for $|\rho|$ of the order of a few per-cent. Consequently, analysis of the present data does not lead to a very precise estimate for f.

The phenomenological value of $G_1^V(o)$ has been computed by Gourdin and Salin,(36) and by Mathews.(37) Their methods are essentially identical, but they obtain widely differing results. Dalitz and Sutherland(41) have pointed out that this is due to an error in the fit of Gourdin and Salin arising mainly out of neglect of an SU(2) Clebsh-Gordan coefficient. This error affects their estimates of $E_1{}^+$ and $M_1{}^+$ by identical overall factors, so their value for $\rho$ would appear to be substantially correct.

Mathews' estimate is free from computational errors, but his value for $\Gamma_{tot}$ needs to be updated from 110 to 120 MeV. In addition, he takes for $\sigma_{res}$ the value:

$$\sigma_{res}(\gamma p \to \Delta^+ \to p\pi^0) = \sigma_{tot}(\gamma p \to p\pi^0)\Big|_{S=M_1^2} \simeq 269\,\mu b \quad . \tag{A9-7}$$

A more up to date value is now available,(41) namely:

$$\sigma_{res} = \sigma_{tot}\Big|_{S=M_1^2} - \sigma_{background}\Big|_{S=M_1^2} = [(267\pm 5)-(7\pm 3)]\,\mu b$$
$$= (260\pm 6)\,\mu b \quad . \tag{A9-8}$$

It is therefore necessary to update Mathews' estimate of $G_1^V(o)$ by a factor of 1.025.

(We show in a moment that $G_1^V(o)$ is proportional to $\Gamma_{tot}^{1/2}\,\sigma_{res}^{1/2}$.)

This estimate is based on the assumption that $\rho$ vanishes, and it will be useful to see how the updated fit is affected by assuming instead: a) Gourdin and Salin's value for $\rho$, b) the value

$$\rho \approx -0.064 \,, \tag{A9-9}$$

corresponding to the vanishing of $G_2^V(o)$, and c) the value

$$\rho \approx +0.064 \,, \tag{A9-10}$$

which should give some idea of the upper bound on the values of $G_{1,2}^V(o)$.

For the benefit of the reader we first mention that our $G_{1,2,3}^V(o)$ are related to the $C_{3,4,5}(o)$ of Gourdin and Salin (denoted by $C_{3,4,5}^G(o)$) and the $C_{3,4,5}(o)$ of Mathews (denoted by $C_{3,4,5}^M(o)$) as follows.

$$\sqrt{2/3}\, G_1^V(0) = C_3^M(0) = C_3^G(0)/\mu \,, \tag{A9-11}$$

$$\sqrt{2/3}\, G_2^V(0) = -C_4^M(0) = -\left[C_4^G(0) + C_5^G(0)\right]/\mu^2 \,, \tag{A9-12}$$

$$\sqrt{2/3}\, G_3^V(0) = C_4^M(0) + C_5^M(0) = C_4^G(0)/\mu^2 \,. \tag{A9-13}$$

Note that Mathews does not define his $C_{3,4,5}$ in the same way as do Gourdin and Salin. Also, these authors work in terms of $O(3,1)$ decompositions of the matrix element $\langle \Delta^+ | j_\alpha(0) | p \rangle$, whereas we define our $G_{1,2,3}^V$ by means of an $O(3,1) \otimes SU(2)$ decomposition of $\langle \Delta | j_\alpha(0) | N \rangle$. This is responsible for the $\sqrt{2/3}$ factors appearing in equations A9-11, 12 and 13. They arise because:

$$\chi_\ell^{\dagger\, \frac{1}{2}}(\Delta)\, \delta_{\ell 3}\, \chi^{\frac{1}{2}}(N) = \sqrt{2/3} \,. \tag{A9-14}$$

From expressions given in Gourdin and Salin's paper, coupled with equations A9-11 and 12 we have:

$$\rho \equiv \frac{E_1^+}{M_1^+} = \frac{-(M_1 - m)\left(G_1^V(0) - M_1 G_2^V(0)\right)}{\left[(3M_1 + m)G_1^V(0) - M_1(M_1 - m)G_2^V(0)\right]} \,. \tag{A9-15}$$

Hence:

$$f = \frac{1}{(1+\rho)}\left[1 + \frac{(3M_1+m)}{(M_1-m)}\rho\right] \simeq \frac{[1+(15.65)\rho]}{(1+\rho)} , \qquad \text{(A9-16)}$$

and in particular:

$\rho = 0$, (pure $M_1^+$ transition), implies $f = 1$, (A9-17)

$\rho = -0.045$, (Gourdin and Salin's estimate), implies $f \simeq 0.308$, (A9-18)

$$f = 0, \left(G_2^V(0) = 0\right), \text{implies } \rho = \frac{-(M_1-m)}{(3M_1+m)} \simeq -0.064 , \qquad \text{(A9-19)}$$

$$\rho = \frac{(M_1-m)}{(3M_1+m)} \left(\simeq +0.064\right), \text{ implies } f \simeq 1.88 . \qquad \text{(A9-20)}$$

So f varies over the range zero to (1.88) when $\rho$ varies over the range (-0.064) to (+ 0.064), and consequently f is not well determined by the present empirical data.

In view of equations A9-11 and 12 we have from Mathews' paper:

$$G_1^V(0) = \frac{3M_1^{3/2}\,\Gamma_{tot}^{1/2}\,\sigma_{res}^{1/2}}{e(M_1-m)^{1/2}(M_1+m)^{3/2}} \cdot \frac{(1+\rho)}{(1+3\rho^2)^{1/2}} , \qquad \text{(A9-21)}$$

so:

$$G_1^V(0) = G_1^V(0)\Big|_{\rho=0}\left[1+\rho-\tfrac{3}{2}\rho^2+O(\rho^3)\right] , \qquad \text{(A9-22)}$$

and as expected, the empirical error in $\rho$ only affects the determination of $G_1^V(o)$ by a few per-cent.

Multiplying Mathews' estimate of $C_3^M(o)$ by $\sqrt{3/2}\,(1.025)$ to obtain $G_1^V(o)\Big|_{\rho=o}$, and then invoking equation A9-22, we obtain the fits tabulated below. If $\rho$ is assumed known with perfect accuracy the error range for $G_{1,2}^V(o)$ is about $\pm 3\%$.

TABLE A9-I

VALUES OF $G^V_{1,2}(0)$ OBTAINED BY FITTING THE EXPERIMENTAL DATA ON $\Gamma_{tot}\,\sigma_{res}$ FOR VARIOUS VALUES OF $E_1^+/M_1^+$.

| $\rho \equiv \frac{E_1^+}{M_1^+}$ | $f \equiv \frac{M_1 G_2^V(0)}{G_2^V(0)}$ | $G_1^V(0)$ ($GeV^{-1}$) | $G_2^V(0)$ ($GeV^{-2}$) |
|---|---|---|---|
| +0.064 | 1.88 | 2.84 | 5.34 |
| 0 | 1 | 2.68 | 2.17 |
| -0.045 | 0.308 | 2.55 | 0.635 |
| -0.064 | 0 | 2.49 | 0 |