STABILITY OF FLOWS

OF

NON-NEWTONIAN FLUIDS

by

Randal Roger Cousins, M.A.

ABSTRACT

The phenomenon of turbulent drag reduction by the addition of certain polymer additives is described. The role of molecular elongation is discussed and we interpret its effect from a continuum viewpoint in terms of normal stresses differences. The development of turbulent flow is described, and the relevance of stability theory to an understanding of turbulence is argued. A review of work on the stability of parallel flows is given, and we discuss methods for solving the Orr-Sommerfeld equation, which underlies most of the work. The particular theoretical models used for the fluids are derived from a general theory of continuum mechanics.

For plane Poiseuille flow of a second-order fluid a linear theory is developed. Unlike the Newtonian case Squire's theorem is not valid, and a three-dimensional analysis is required. The non-Newtonian terms are in general destabilising. Under certain conditions the first growing disturbance will propagate at an angle to the basic flow, giving a longitudinal vortex structure close to the channel boundaries not present at the onset of instability in a Newtonian fluid. The analysis is extended to finite amplitude disturbances by introducing a time-dependent amplitude. The interaction of three fundamental wave forms with each other, their harmonics and the main flow is considered, and the existence of states of finite amplitude equilibrium is established. Detailed calculations are confined to the simpler two-dimensional case. Disturbances which would decay under linear theory may in fact grow provided the initial amplitude is sufficiently large. A threshold amplitude for instability is found as a function of Reynolds number. A further viscoelastic property, that of stress relaxation is introduced. For infinitesimal disturbances the relaxation terms lead to a modified form of the Orr-Sommerfeld equation, and the net result is to destabilise the flow.

ACKNOWLEDGEMENTS

The author wishes to thank

The Director of the National Physical Laboratory for permitting this work to be presented as a thesis,

Dr. F.J. Lockett, his supervisor at NPL, for originally stimulating his interest in the subject of this thesis and for much helpful advice,

Professor J.T. Stuart, his director of studies at Imperial College, for great encouragement and many helpful conversations,

Mr. D.H. Ferriss of NPL for undertaking much of the computation described in sections 4.9 and 4.10.

PUBLICATIONS

Parts of this thesis have been published in the International Journal of Engineering Science:

most of chapter 3	Int. Jnl. Eng. Sci., 8, 595-607 (1970)
sections 4.8 - 4.10	Int. Jnl. Eng. Sci., 10, 511-524 (1972)
most of chapter 5	Int. Jnl. Eng. Sci., 10, 301-308 (1972)

TABLE OF CONTENTS

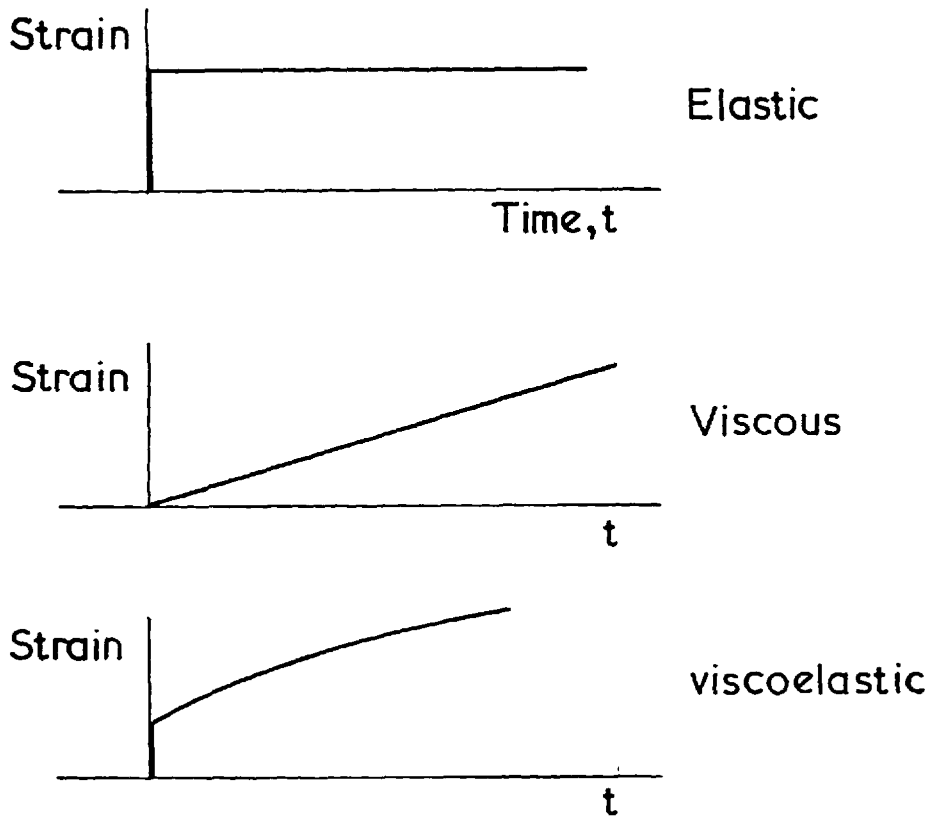
CHAPTER 1: INTRODUCTION

1.1 The Toms' effect

In many practical situations involving fluid flow, turbulence or eddying occurs near solid surfaces, and resistance to motion is largely associated with this turbulence. Thus much of the power needed to drive a ship through the water is dissipated in the turbulence which originates in the boundary layer close to the hull. Similarly, the pressure drop along a pipe is much greater than it would be if laminar conditions could be maintained.

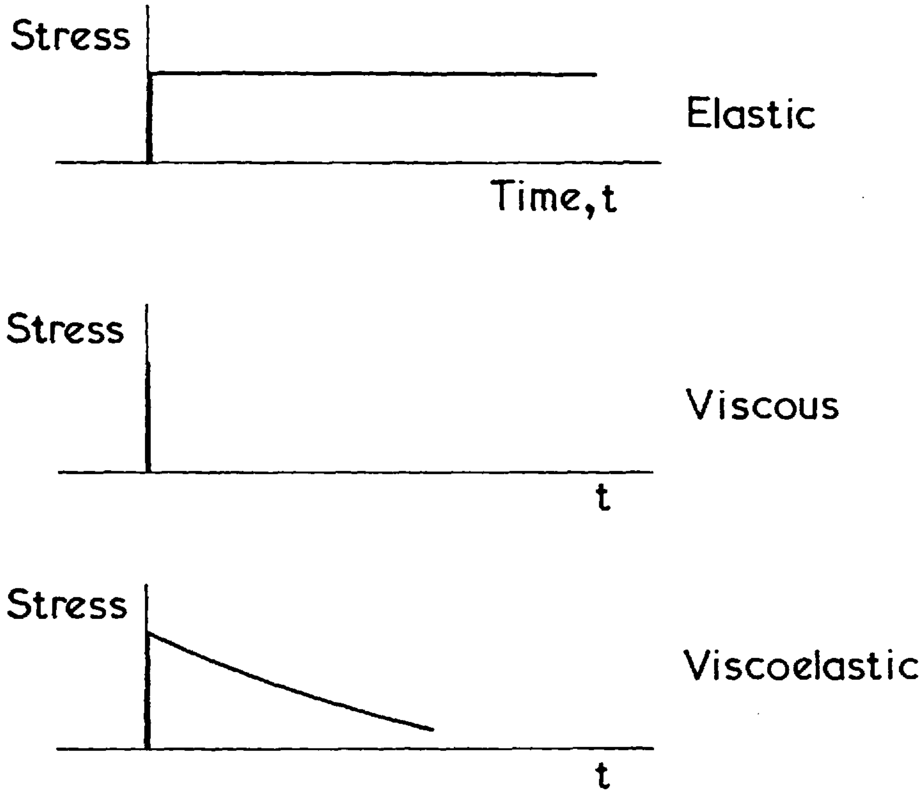
Large reductions in turbulent frictional resistance, for example, of the pressure drop in pipe flows, can sometimes be achieved by dissolving small quantities of certain substances in the liquid. The first clear scientific description of the phenomenon which now bears his name was given by Toms (1949), who investigated the flow of various dilute solutions through pipes of different diameters. Toms dissolved polymethyl methacrylate in monochlorobenzene, but since then many other polymers and solvents have been used to achieve drag reduction. See, for example, Hoyt and Fabula (1964), who examined a wide range of water soluble polymers and showed that the most effective could produce as much as 40% reduction in turbulent friction in concentrations as low as ten parts per million by weight. A frequent combination that has been used is a solution of polyethylene oxide in water as it is inexpensive and effective (Virk et al, 1967). The additive substances usually have very high molecular weights, of the order of 10^6, and are effective in concentrations of the order of ten to one hundred parts per million by weight. In these very low concentrations the solution has a viscosity, μ, which is independent of shear rate and indistinguishable from that of the solvent alone. Here, then, are fluids which are essentially the same as the solvent (usually water) in their values of density and viscosity, which are normally regarded as the relevant fluid parameters, yet they behave in a radically different way.

In order to avoid confusion we define (Lumley, 1969) 'drag reduction' as the reduction in skin friction in turbulent flow below that of the solvent alone. This excludes substances such as polyvinyl alcohol and ammonium alginate which delay the transition from laminar to turbulent flow (Shaver, 1957), but after transition the drag is higher than that of the solvent alone (Hershey and Zakin, 1967). These polymer solutions are often described as non-Newtonian, and in this work a Newtonian fluid is defined as one for which the extra-stress tensor, that is stress with the pressure part subtracted out, is proportional with a scalar coefficient to the rate of strain tensor. This is merely the three-dimensional extension of Newton's law of viscosity in simple shear. These tensors are defined in more detail in chapter 2. Any departure from this simple relationship is referred to as non-Newtonian. We shall also, use the word 'viscoelastic'. Viscoelastic materials exhibit some of the properties of an elastic solid and some of the properties of a viscous fluid. For instance, if a tensile load is applied to an elastic solid the material will extend only a certain amount to accommodate the load, whereas a viscoelastic material will continue to stretch (until it breaks), a flow phenomenon called creep. Again, to produce a constant strain in a viscous fluid we should need to apply an instanteous stress, in an elastic solid we should require a constant stress, while in a viscoelastic material the stress would gradually decay. This last property is called stress relaxation. For a one-dimensional strain creep and stress relaxation are illustrated in figures 1.1 and 1.2.



Behaviour of materials under a constant stress applied for $t \geq 0$

FIG. 1.1 CREEP



Behaviour of materials under a constant strain applied for $t \geq 0$

FIG. 1.2 STRESS RELAXATION

Polymers effective in drag reduction are linear and flexible (this will be discussed below). Such polymers, if isolated in a solution at rest, would take on a random walk configuration if the bonds were ideally flexible and interference of the links with each other could be ignored. In practice polymers do assume a tangled ball configuration which approaches this ideal (Lumley, 1969). An effective diameter can be assigned to this ball (roughly proportional to the square root of the number of monomer units), which may be determined by light scattering measurements. The effective diameter is, of course, much less than the extended length of the molecular chain. Merrill et al (1966) defined a critical concentration in terms of this effective diameter as the concentration at which the polymer molecules, if they were spheres of the appropriate diameter, would be in dense spherical packing. Concentrations less than this are termed dilute. We shall be discussion the very dilute solutions for which drag reduction is most marked. For example, Merrill et al (1966) found that a 38% drag reduction was produced by a 50 parts per million aqueous solution of polyethylene oxide of molecular weight 6×10^5. The critical concentration in this case is 2450 parts per million, so the solution is very dilute.

Hoyt and Fabula (1964) examined many aqueous solutions of natural and synthetic polymers and found that the most effective polymers were those with a linear structure having few, or no, side chains and those of simple form. Molecules with many branched side chains were relatively ineffective. Merrill et al (1966) compared polyethylene oxide with the less flexible polyisobutylene and concluded that effectiveness increased with flexibility. From further measurements they found that the number of monomer units in the main chain was the important factor rather than molecular weight, and also that increasing the chain length improved the effectiveness of the polymer. Hershey and Zakin (1967) used different solvents and found that the drag reduction was better for 'good' solvents in which polymer-solvent interactions were favoured over polymer-polymer interactions. In good solvents the polymer is relatively extended, and we shall discuss the significance of molecular extension later.

It is convenient to introduce a Reynolds number, $\rho UL/\mu$, where U and L are typical values of the speed and length of the flow situation. When a dilute polymer solution is made to pass through a sufficiently large pipe the curve of friction coefficient, a non-dimensional parameter proportional to the pressure drop down the pipe, against Reynolds number (see figure 1.3) follows the curve for water until a certain threshold Reynolds number is reached, after which drag reduction occurs (Gadd, 1966). In a larger pipe the threshold Reynolds number is higher, but the shear stress at the wall is then approximately the same as for the threshold condition in the smaller pipe. If two different drag reducing fluids are compared in the same pipe as in figure 1.4 (Gadd, 1966) it will be seen that it is impossible to give an unequivocal rating of the effectiveness of different additives. The drag reduction for a given additive in a given pipe becomes greater up to some limit (figure 1.5). It should be stressed that along the line of greatest effectiveness drag reduction does not occur in the region of laminar flow.

A possible explanation of the threshold shear stress is provided by theories (Ericksen, 1962 and Tulin, 1966) which predict an elongation and orientation of the molecules for shear rates greater than some critical value. These theories imply that molecular elongation is the essential requirement for reduction of turbulent drag. The phenomenon of degradation, to which some polymer

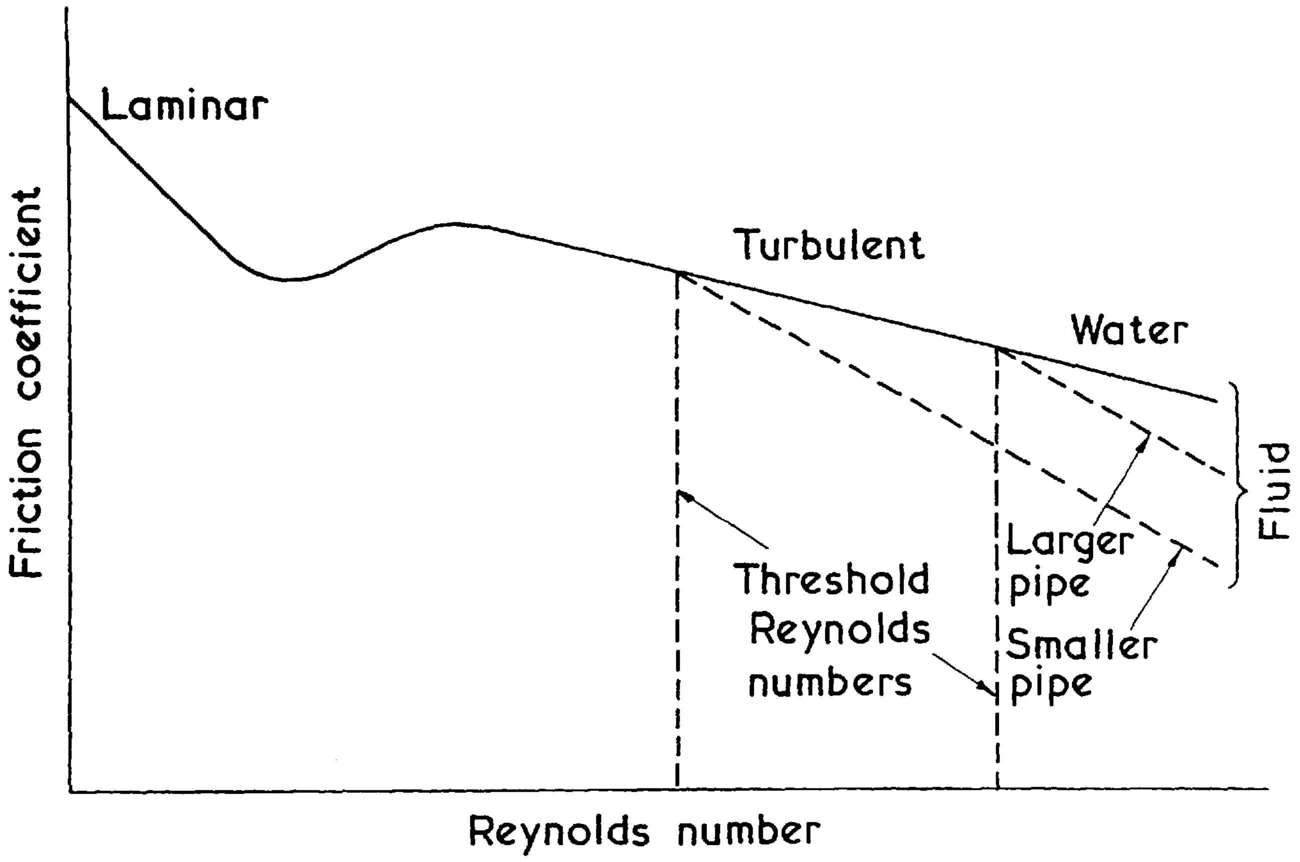

Laminar
Turbulent
Water
Fluid
Threshold
Reynolds
numbers
Larger
pipe
Smaller
pipe
Friction coefficient
Reynolds number

FIG. 1.3

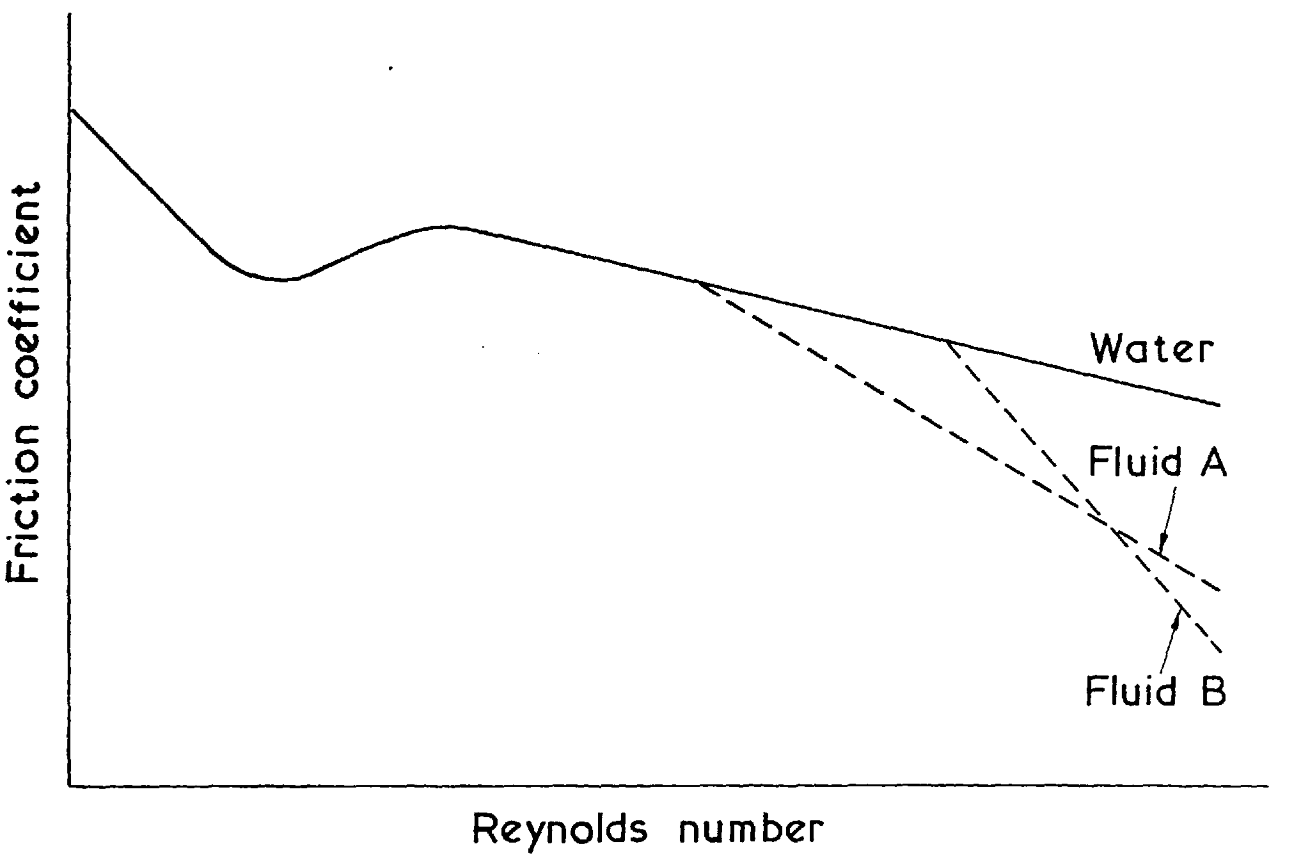

Water
Fluid A
Fluid B
Friction coefficient
Reynolds number

FIG. 1.4

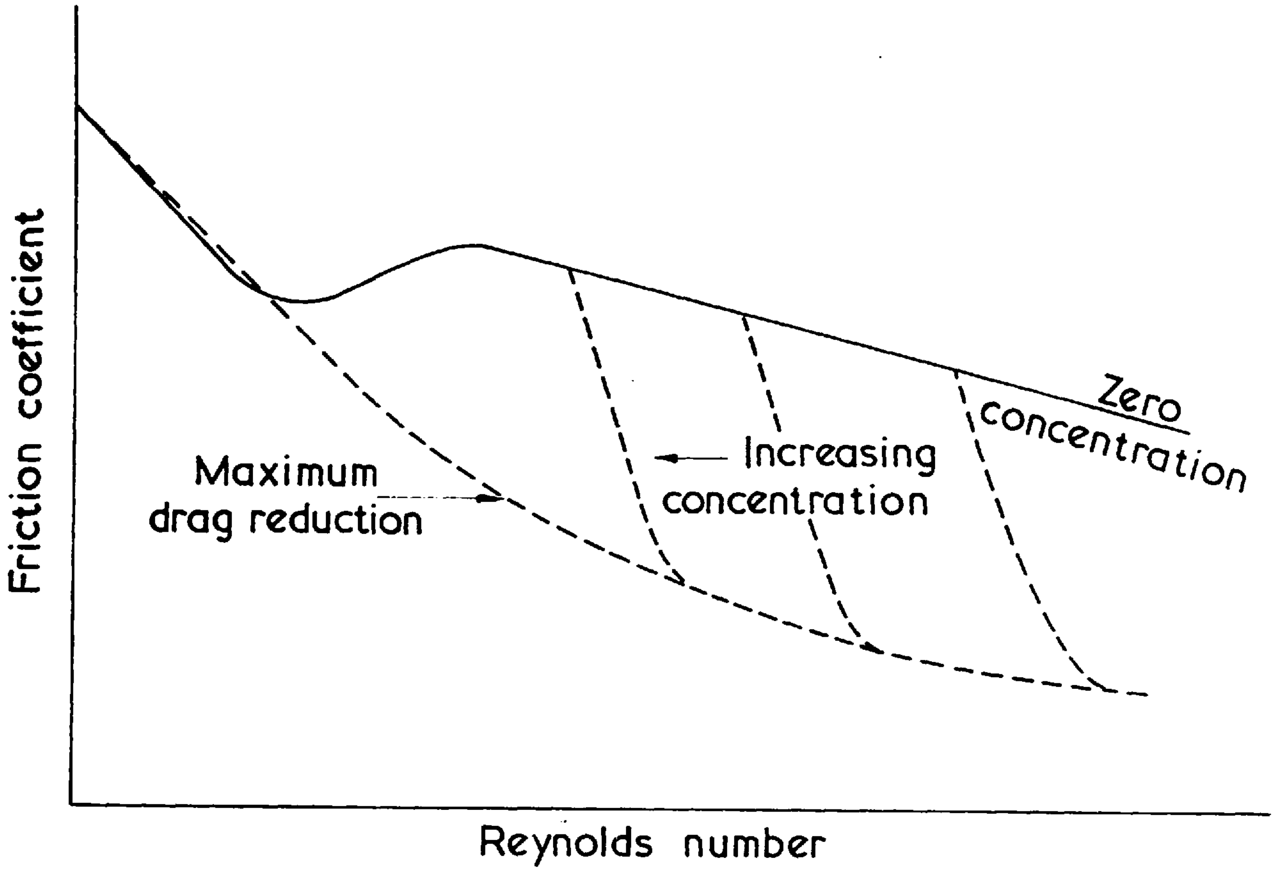

Friction coefficient
Maximum
drag reduction
Increasing
concentration
Zero
concentration
Reynolds number

FIG.1.5

solutions are susceptible when subjected to continued shearing action, also supports this theory. For example, polyethylene oxide loses its effectiveness (Gadd, 1965), and it is observed that the molecular chains are broken up. On the other hand guar gum shows very little mechanical degradation, probably because any broken molecules rapidly reform, and the solution maintains its effectiveness.

If molecular elongation, or something equivalent to it, is the factor common to all fluids which reduces drag, Gadd (1966) has suggested that a primary mechanism may be a thickening on the laminar sublayer, a thin region close to wall where the flow is essentially smooth or laminar. The presence of the surface suppresses turbulent eddies near to it. Any elongated molecular filaments would tend to become aligned in the flow direction due to the action of the high shear. In chapter 3 we shall show that, in contrast to a Newtonian fluid, longitudinal vorticity may be established close to the wall in a non-Newtonian solution. The molecules will be 'unwrapped' in a helical manner and have a resulting elongation in the direction of the main flow. The theory only relates to conditions at the breakdown of laminar flow, but it does suggest a mechanism which <u>may</u> result in a different turbulent structure in the important region close to the wall. Tulin (1966) argues that elongated molecules would lead to an increase in dissipation of turbulent energy. A new balance of energy production and dissipation would need to be established, leading to a reduction in turbulence close to the wall and hence a thickening of the laminar sublayer. The change in energy balance need not be large to make a large difference to the surface friction. Whatever its cause, thickening of the sublayer does occur (Goren, 1966 and Elata et al, 1966), and the aligned molecules tend to suppress transverse motions. Instead of a molecular description we could employ the continuum concept of normal stress differences to provide forces which would oppose transverse motions. Normal stress differences will be properly defined at the end of chapter 2, but in simple terms the phrase means that a modified form of the pressure does not act equally in all directions in situations when it would do so for a Newtonian liquid. Normal stress differences have been observed in drag reducing solutions by Metzner and Park (1964) who found that drag reduction increases with the ratio of a normal stress difference to shear stress. We shall use a continuum approach throughout this thesis and work mainly with a constitutive equation that models fluids exhibiting normal stress effects.

Lumley (1969) considers the effects of agglomerations of polymer molecules. Adapting some results on the coalescence of raindrops (Saffman and Turner, 1956), he concludes that although agglomerations have been observed they are unlikely to be dynamically important since the time scale for their formation is much larger than the relevant time scale of the strain rate. Gadd (1966) supports this view. He passed a solution through filter paper to remove the agglomerations and found that the effectivenss was not impaired. Lumley also discusses the possible effect of the polymer molecules on the structure of turbulence. Gadd (1965) shows that some additives may have a dramatic effect on the small scale turbulence of jets squirted into ambient liquid, but other additives do not affect the jets at all. Lumley reviews attempts to relate appropriate length and time scales of the molecules to scales arising from turbulent eddies, but the situation is very confusing. Turbulent scaling creates more problems than it solves, and Lumley concludes that some important aspect is not yet identified or understood. At present, molecular elongation and its affect on the laminar sublayer are the only physical facts which point to improving our understanding of the Toms' effect.

Since these observations can be interpreted in terms of a continuum theory as discussed earlier, we regard this as motivation for examining the effect of various viscoelastic properties and their possible relevance to drag reduction. In chapters 3 and 4 we investigate the influence of normal stress effects on the stability of channel flow, and in chapter 5 examine channel flow of fluids exhibiting stress relaxation. The particular constitutive equations used are derived in chapter 2 from a general rational approach to continuum behaviour, and in the next section we discuss the relevance of stability theory to turbulence.

1.2 Transition from laminar to turbulent flow in a boundary layer

Although the drag reducing properties of polymer additives apply to turbulent flow there is no satisfactory mathematical theory for turbulent flow of a Newtonian fluid, let alone a non-Newtonian fluid. In attempting to develop a theory towards explaining the Toms' effect we restrict our attention to the early stages of transition from laminar to turbulent flow and show how viscoelasticity affects these early stages. We hope that it may shed some light on how the additives affect the later stages of transition and fully developed turbulence, but the calculations are too daunting to proceed far. Stuart (1965a, b) identifies the main stages of transition from laminar to turbulent flow in the boundary layer of a Newtonian fluid flowing past a flat plate.

I Tollmien-Schlichting waves

When the flow is wholly laminar any small disturbances to the flow that may be present decay. As the Reynolds number reaches a certain critical value two-dimensional waves, called Tollmien-Schlichtling waves, cease to decay. In this region the disturbances may be regarded as small, and, neglecting products of disturbance quantities, a linearised analysis is possible. These waves have been observed experimentally (Schubauer and Skramstad, 1943), and there is excellent agreement between their observations and the linear theory (for example, Lin 1945 and 1955). We should note that the waves observed by Schubauer and Skramstad grow spatially, while the theoretical waves grow in time. Comparisons are made by relating the two using group velocity as a transformation velocity (Watson, 1962 and Gaster 1962, 1965a, b).

II Three-dimensional wave amplification

Minor irregularities in the flow can give rise to a rate of wave growth which varies with spanwise position (Klebanoff and Tidstrom, 1959), leading an initially two-dimensional wave into a three-dimensional form. In many cases the flow is nearly periodic in the spanwise direction. Controlled experiments have therefore been performed in which the three-dimensional wave~~s~~ is introduced by a vibrating ribbon (Klebanoff et al, 1962 and Kovasznay et al, 1962).

III Peak-valley development

As the three-dimensional waves progress downstream the boundary layer develops a much more pronounced three-dimensional structure with an associated streamwise vortex system. At certain spanwise locations called peaks the velocity fluctuations develop extremely strongly, and the streamwise vortex component of flow is away from the wall; at the neighbouring valleys where the flow is towards the wall the fluctuations develop more slowly (Klebanoff et al, 1962). These longitudinal peaks and valleys are visible in the pictures of Meyer and Kline (1961). In this region non-linear effects become important, that is, the disturbances can no longer be regarded as infinitesimal and products of disturbance quantities must be taken into account. The strong development of the three-dimensional structure appears to be a property of the wave motion itself, unprovoked by any irregularity. Benney and Lin (1960) and Benney (1961 and 1964) give a partial explanation for the range where the disturbance amplitude is finite though small enough for a perturbation theory to be applied. Their study is based on the interaction of a two-dimensional wave with a three-dimensional wave of the same streamwise wave number. Among other components to emerge at second order in amplitude is a slowly varying streamwise vortex component of flow. In order to match one form observed in the experiments of Klebanoff et al (1962) it is assumed that the frequencies of the waves are the same. Stuart (1962) questions this assumption and shows that if the frequencies are different an additional slowly varying interaction term is introduced.

IV Shear layer development

Downstream from spanwise stations corresponding to peaks the instantaneous velocity profile develops a region of large shear with an associated inflexion in the profile. (Klebanoff et al, 1962, Kovasznay et al, 1962). Stuart (1965) develops a linear theory to describe convection of vorticity and stretching of vortex lines. A boundary layer velocity profile (primary flow) is allowed to interact with a streamwise vortex flow field periodic in the spanwise direction. This secondary (vortex) flow may be obtained from some nonlinear analysis of the type described in III. At spanwise stations corresponding to peaks the vortex lines of the primary flow are convected outwards and are stretched in the outer regions of the boundary layer where the streamlines are divergent. The net result is to develop a strong shear layer (vorticity concentration) in the boundary layer profile at such spanwise stations. The interaction time required to produce vorticity concentrations comparable with those observed is of the same order as the times required experimentally. The velocity profiles obtained are in reasonable qualitative agreement whose those observed in experiments.

V Breakdown

When the shear developed in IV is sufficiently strong a velocity fluctuation develops from the shear layer at a much higher frequency than that of the basic wave. Greenspan and Benney (1963) have calculated the growth of amplitude of oscillations in a shear layer which models that observed experimentally by Kovasznay et al (1962). Their theory explains the growth of such high frequency bursts, but they do not consider three-dimensional or non-linear effects.

VI Turbulent-spot development

The bursts described in V travel faster than the primary wave, and as they travel downstream the eddies spread spanwise and towards the wall. The agglomeration of these turbulent spots leads to fully developed turbulent flow further downstream. Meyer and Kline (1961) show many photographs of this stage of transition.

The account above describes the development of intense shear layers from which turbulent spots develop. Runstadler, Kline and Reynolds (1963) express the view that the structure of the layer close to the wall, and its interaction with the outer flow play a dominant role in maintaining a fully developed turbulent flow. They argue that viscosity acts to amplify fluid disturbances and that the production of turbulent energy is due to instability in the wall layer.

It follows, then, that the methods of studying the growth of fluid disturbances, i.e. hydrodynamic stability theory, are relevant to an understanding of turbulence. Runstadler et al observe a structure of longitudinal streaks in the viscous dominated wall layer, and the break up of this structure is very similar to the flow structure observed in the peak-valley and breakdown stages of transition outlined above. Kline (1967) reiterates the argument that it is an instability mechanism that plays a vital part in maintaining turbulence. In the discussion following Kline's paper T.J. Black drew attention to his experimental evidence (1966) which supports the hypothesis that the basic turbulence mechanism is one of periodic instability. His results indicate that turbulent momentum transfer measured near the edge of the wall layer exhibited definite and fairly regular peaks (in time). The flow is dominated viscous stresses and the stability-governed energy exchange. Dr Black interprets turbulent flow as a developing laminar flow that is repeatedly modified by this instability. ~~Kline~~ Kim et al (1971) examine the instability process in more detail and describe the bursting phenomenon in three stages: (i) lifting of low speed longitudinal streaks from the innermost layer; this forms unstable (inflexional) instantaneous velocity profiles; (ii) growth of an oscillatory motion in the region of flow downstream of the inflexional zone; (iii) breakup of the oscillatory motion into more random motions accompanied by a return to the wall of the longitudinal low speed streak. The third stage shows velocity profiles that have returned approximately to the mean profile. This bursting cycle is intermittent, but has a well-defined mean frequency.

1.3 Stability of parallel flows

In the previous section the relevance of stability theory to an understanding of turbulent flow is argued. The bulk of experimental evidence relates to boundary layers, but from a theoretical viewpoint several difficulties arise. In the linearised theory it is assumed that the undisturbed flow is parallel, and the velocity profile at a given station is a function of one coordinate only, namely that in the direction of shear. The component of velocity normal to the boundary is ignored and the growth of the boundary layer is neglected. According to boundary layer theory these processes are small when the Reynolds number is large, and it seems reasonable to neglect them. However, when nonlinear effects are considered it is no longer valid to ignore them (Stuart, 1971), since terms neglected in the linearised parallel flow approximation may be comparable with terms arising from products of velocity fluctuations, and the growth or development of an oscillation in space or time will be strongly affected by the simultaneous spread of the boundary layer. Although this interaction is present in most boundary layer experiments it is not understood theoretically.

The difficulties outlined above disappear if we consider the stability of plane Poiseuille flow, a prototype parallel flow which describes laminar motion in a channel. The basic flow is given by

$$\underset{\sim}{v} = (U(x_2), 0, 0) \tag{1.3.1}$$

$$U(x_2) = 1-x_2^{\,2}, \tag{1.3.2}$$

where $\underset{\sim}{v}$ is the velocity whose components are expressed with respect to rectangular Cartesian coordinates, x_1 being measured down the channel $(-1 \leq x_2 \leq 1)$, x_2 across the channel and x_3 in the spanwise direction. This flow has another advantage for our analysis: it satisfies the equation of motion for both Newtonian fluids and the particular viscoelastic fluids considered in this thesis. The basic boundary layer flow, however, is different for the various fluid models. Much of the subsequent analysis is given in terms of the general parallel flow velocity $U(x_2)$ as in (1.3.1). The equations derived will be applicable to the stability of jets, wakes and other shear layers, including Couette flow (U proportional to x_2) between parallel planes. Detailed calculations, however, are confined to the stability of Poiseuille flow, when U takes the form (1.3.2).

Reynolds (1884) is generally credited with the first description of turbulent flow. He regarded turbulence as arising from an instability of laminar flow and by dimensional analysis he uncovered (1895) the important number that bears his name. He pointed out that when this number exceeds a certain critical value the disorder begins. The key equation to the linear theory of stability of parallel flows was derived independently by Orr (1907) and Sommerfeld (1908). The critical Reynolds number can be obtained from this equation and appropriate boundary conditions. Much analytical and numerical work has been done with this equation, and since it forms the basis for most of the calculations in this thesis a more detailed exposition is given in the next section. A linearised theory is developed in chapter 3 and 5 for non-Newtonian fluids, and it is shown that in some circumstances the first disturbance to grow in time will be three-dimensional. This is in contrast to the case (stage I) for a Newtonian fluid described in the previous section. In general the critical Reynolds number is lower for a non-Newtonian fluid (chapters 3 and 5).

Linear theory for a Newtonian fluid leads to a critical Reynolds number of about 5000-6000, while it is observed that plane Poiseuille flow breaks into a turbulent flow at a much lower Reynolds number (1000-2500, Davies and White, 1928). Linear analysis describes conditions prevailing at the initial breakdown of laminar flow, providing the disturbances are infinitesimal. Meksyn and Stuart (1951) suggested that nonlinear effects may permit the existence of a threshold amplitude above which oscillations can grow in time even though the Reynolds number may be less than the (linear) critical value. They obtained a minimum critical Reynolds number of 2900, and oscillations of amplitude greater than 8 per cent of the maximum speed of the undisturbed flow would grow in time at this value of the Reynolds number. More accurate Calculations by Grohne (1969)~~, who includes~~ of the ~~some~~ nonlinear effects considered ~~neglected~~ by Meksyn and Stuart, give a critical Reynolds number of 2500. In chapter 4 a nonlinear analysis is developed for a non-Newtonian fluid. Three-dimensional disturbances are considered, leading to a formidable sequence of equations, which are only solved in the much simpler two-dimensional case.

1.4 The Orr-Sommerfeld equation

It is convenient to give a brief description of the Orr-Sommerfeld equation at this stage. Modified forms of it for viscoelastic fluids are derived in chapters 3, 4 and 5. Throughout this section we shall work with variables expressed in non-dimensional form with respect to characteristic length and time scales and density. Let us introduce a two-dimensional disturbance to the original flow (1.3.1) so that the velocity takes the form

$$\left.\begin{aligned} v_1 &= U(x_2) + u_1(x_2)\exp\{i\alpha(x_1-ct)\}, \\ v_2 &= u_2(x_2)\exp\{i\alpha(x_1-ct)\}, \\ v_3 &= 0. \end{aligned}\right\} \qquad (1.4.1)$$

The disturbance represents a wave travelling downstream with speed c. The equation for two-dimensional motion in an incompressible Newtonian fluid is

$$\frac{\partial v_i}{\partial t} + v_1 \frac{\partial v_i}{\partial x_1} + v_2 \frac{\partial v_i}{\partial x_2} = -\frac{\partial p}{\partial x_i} + \frac{1}{R}\left\{\frac{\partial^2 v_i}{\partial x_1^2} + \frac{\partial^2 v_i}{\partial x_2^2}\right\} \qquad (1.4.2)$$

for $i=1,2$, where p is the (non-dimensional) pressure and R the Reynolds number. The condition of incompressibility is

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0. \qquad (1.4.3)$$

Equations (1.4.2) and (1.4.3) are obtained in more detail in the next chapter. If we substitute (1.4.1) into (1.4.2) and (1.4.3) and neglect products of the small quantities u_1, u_2 terms linear in the disturbance quantities yield

$$\{D^2-\alpha^2-i\alpha R(U-c)\}u_1 - RU'u_2 + i\alpha R\hat{p} = 0, \qquad (1.4.4)$$

$$\{D^2-\alpha^2-i\alpha R(U-c)\}u_2 + R\hat{p}' = 0, \qquad (1.4.5)$$

$$i\alpha u_1 + u_2' = 0, \qquad (1.4.6)$$

where D or the prime denote differentiation with respect to x_2 and the pressure, p, is expressed in the form

$$p = \bar{p} + \hat{p}(x_2)\exp\{i\alpha(x_1-ct)\}. \qquad (1.4.7)$$

Eliminating u_1 and $\hat{p}$ from (1.4.4)-(1.4.6) we obtain the Orr-Sommerfeld equation

$$\{(D^2-\alpha^2)^2 - i\alpha R(U-c)(D^2-\alpha^2) + i\alpha RU''\}u_2 = 0. \qquad (1.4.8)$$

This is a fourth-order differential equation to which we apply four boundary conditions. For instance, for plane Poiseuille flow (1.3.2) in a channel bounded by $x_2=\pm 1$ the conditions of no-slip at the walls $(u_1=0)$ and no flow across the boundaries $(u_2=0)$ yield via

(1.4.6) the boundary conditions

$$u_2 = u_2' = 0 \quad \text{at} \quad x_2 = \pm 1. \qquad (1.4.9)$$

Since (1.4.8) is linear in u_2 is has four independent solutions which may be combined linearly to produce a general solution. Boundary conditions (1.4.9) applied to this solution lead to four relations linking the unknown coefficients in the linear combination. These coefficients may be eliminated to yield a condition of the type

$$F(\alpha, R, c) = 0. \qquad (1.4.10)$$

For each pair of real values α and R there is a characteristic value c, which is, in general, complex. If the imaginary part c_i of c is positive the disturbance grows exponentially in time according to linear theory and the flow is unstable. If c_i is negative the disturbance decays and the flow is stable. If $c_i=0$ (1.4.10) leads to a relation between α and R, which may be drawn as a curve in the α-R plane. This curve is called the neutral stability curve. Alternatively, we may regard (1.4.8) and (1.4.9) as defining an eigen-problem to determine, given α and R, the eigenvalue c and the corresponding eigenfunction u_2.

Some of the difficulties encountered in solving (1.4.8) and (1.4.9) can be seen by first examining the inviscid form of the equation obtained by Rayleigh (1880). If we let $R \to \infty$ (1.4.8) reduces to

$$u_2'' - \alpha^2 u_2 - \frac{U''}{U-c} u_2 = 0. \qquad (1.4.11)$$

As the order of the equation is reduced we can no longer impose the no-slip condition, and the boundary conditions are now

$$u_2 = 0 \quad \text{at} \quad x_2 = \pm 1 \qquad (1.4.12)$$

for flow in a channel. For a general parallel,flow Rayleigh shows that an inflection point in the undisturbed velocity profile (given by $U''=0$) is necessary for instability of an inviscid flow. As plane Poiseuille flow (1.3.2) does not satisfy this condition we conclude that the presence of viscosity is essential for instability to occur. In a later paper (1913) Rayleigh shows that $U-c_r$, where c_r is the real part of the wave speed, must vanish somewhere within the flow field. The values of x_2 for which $U-c_r$ vanishes define positions where the solution of (1.4.11) is singular. In thin regions, called critical layers, centred on these positions we must take viscosity into account to remove the irregularity in the inviscid solution. As the inviscid solution does not satisfy the no-slip condition, we also require viscous regions close to the wall across which the inviscid solution can be matched to the (viscous) boundary condition. We may therefore divide the channel into various regions, using (1.4.8) in the critical layers and the boundary layers, but retaining the Rayleigh equation (1.4.11) as a good approximation elsewhere. In practice the critical layers lie close to the walls and it is convenient to work with viscous regions that include a critical layer and a wall layer.

In chapter 3 an asymptotic method due to Heisenberg (1924) is used to obtain the inviscid solution, which includes some integrals whose integrands are singular at the critical layer. Lin (1944, 1945 and 1967) shows that this difficulty can be overcome by regarding x_2 as a complex variable and integrating round the singularity along a suitable contour in the complex x_2 plane. Tollmien (1929) introduces a stretched coordinate to handle the viscous region, and shows how to modify the inviscid solution in the critical layer by a viscous correction so that the resulting function is regular. This point is discussed in some detail by Hughes and Reid (1965) and Reid (1965). In terms of the new coordinate, approximations to solutions of (1.4.8) are obtained. Tollmien (1947) improved these asymptotic expansions, but the method does not lead to higher approximations. Detailed analytical solutions to the Orr-Sommerfeld equation are obtained by Eagles (1969). He rationalises the Lin-Heisenberg approach by a careful consideration of inner, intermediate and outer approximations valid in particular regions of the complex x_2 plane near the critical value of x_2. By matching the asymptotic expansions he obtains solutions that are regular in the critical layer. Only the simpler approximate solutions obtained using the Lin-Heisenberg method are used in chapter 3.

A computer is used only in the final stages to evaluate the functions obtained. Although the analysis of chapter 4 is non-linear the first stage in the solution is to obtain the eigensolution of the linear problem. The analytical method of the previous chapter does not yield sufficiently accurate results for subsequent use in the sequence of equations which describe the effects of a finite-amplitude disturbance, so a numerical method is used from the outset.

Lee and Reynolds (1967) used a variational method to obtain a matrix eignevalue problem from the Orr-Sommerfeld equation and the boundary conditions. They employed a set of approximating functions to represent the eigenfunction, but this set was not complete. The method produced fast and accurate eigensolutions provided that the approximating functions were chosen well, but the adjoint function (defined in section 3.5), also required in subsequent calculations was not given accurately.

Another method is to utilise the linear properties of the equation. Lock (1955) has shown numerically that even solutions lead to greater instability than odd solutions, so for the stability analysis we need only consider even functions and restrict the eigenvalue problem to half the channel $0 \leq x_2 \leq 1$. The even eigensolution that is required is a linear combination of the two independent even solutions of the equation. Accordingly, one starts at the centre of the channel and integrates towards the wall, or vice versa, to obtain the required functions, which are then combined linearly to satisfy the boundary conditions. The major difficulty of this method is that one of the solutions grows very rapidly as the integration proceeds. In a typical case a growth of 10^{18} may be expected. It is clear that numerical generation of a second solution that is linearly independent of the rapidly growing one will be difficult, since any round-off error in the computer will in effect throw in a small multiple of the growing solution, which will swiftly dominate the calculation. The eigenfunction does not exhibit this rapid growth, which means that only a small part of the growing solution is required.

Nachtsheim (1964) chose to work in double precision arithmetic, but even so, he had to integrate both from the wall and the centre of the channel and match the solutions in between. He required a much shorter step-length than a finite-difference method to achieve the same accuracy.

Kaplan's scheme (1964) involved suppression of the growing solution during the calculation of the second function. The well-behaved solution is known, from asymptotic analysis, to satisfy the Rayleigh equation, that is, the inviscid form of the Orr-Sommerfeld equation, over much of the channel. At each stage of the integration Kaplan used the inviscid equation as a filter, and subtracted enough of the growing solution to ensure that the Rayleigh equation is satisfied at each stage. This procedure upsets points previously treated, but not significantly. The Rayleigh equation cannot be expected to be suitable where viscous effects are important, but Lee and Reynolds (1967) found no advantage in using other filters. There is a discontinuity in the inviscid solution which is smoothed by the finite intervals used as integration steps.

As the filtering technique is complicated and has uncertainties in the regions dominated by viscosity, it was decided to use a finite-difference method first used successfully by Thomas (1953). As only even order derivatives appear in the Orr-Sommerfeld equation we replace derivatives by central differences. By transforming to a new variable via a central difference function the error in the difference representation of the differential equation can be made very small. The Orr-Sommerfeld equation is now reduced to a matrix eigenvalue problem, and standard algebraic methods may be employed. Fast and accurate computer routines were available to tackle the calculation. The method is described fully in section 4.9. Pekeris and Shkoller (1967) also use Thomas's method and obtain a neutral stability curve in very good agreement with the one obtained here.

Since completing the calculations another method has come to the author's attention. Davey and Nguyen (1971) divide the half channel into intervals which may be of varying widths. At each grid point they form a vector, $\underset{\sim}{y}$, consisting of u_2 and its first three derivatives. Using the Orr-Sommerfeld equation they integrate to obtain $\underset{\sim}{y}$ at the next grid point and determine the matrix which defines the relation between the two values of $\underset{\sim}{y}$. Repeating the process for each interval leads to a sequence of matrices $A_{(n)}$ which may be multiplied together to form a matrix B such that

$$\underset{\sim}{y}(x_2=1) = B\,\underset{\sim}{y}(x_2=0). \tag{1.4.13}$$

B may be formed directly using Kaplan's method, but the approach outlined above avoids the problems encountered by Kaplan, who found it difficult to determine B accurately. An iterative technique may be used to determine the eigenvalue c and the eigenfunction u_2 from (1.4.13). Although the method is basically simple each matrix $A_{(n)}$ needs recalculating each time c is changed in the iteration procedure. Davey and Nguyen admitted that the resulting program could be faulted on grounds of efficiently, and it certainly requires more computer operations than the finite-difference method used in chapter 4.

CHAPTER 2: CONSTITUTIVE EQUATIONS

2.1 Introduction and notation

In describing a fluid it is convenient to regard it as consisting of a number of fluid elements or particles X. In some reference configuration the particles will be at position whose cartesian coordinates are

$$X_i = X_i(X), \qquad i=1,2,3. \tag{2.1.1}$$

As the material flows the position of particle X at time t is given by

$$\underline{x} = \underline{\chi}(X,t), \tag{2.1.2}$$

where χ is a smooth mapping of the particle space into three-dimensional Euclidean space. Velocity and acceleration at points in the fluid are given by time derivatives of the motion, namely

$$\underset{\sim}{v} = \dot{\underset{\sim}{x}} \equiv \frac{d}{dt}\underline{\chi}(X,t), \qquad \underset{\sim}{a} = \ddot{\underset{\sim}{x}} \equiv \frac{d^2}{dt^2}\underset{\sim}{\chi}(X,t). \tag{2.1.3}$$

It is usually more convenient when dealing with fluids to use a spatial description rather than a material one and to express any varible $\psi(X,t)$ as a function of $\underset{\sim}{x}$ and t. If we let

$$\frac{\partial\psi}{\partial t} = \frac{d}{dt}\psi(\underset{\sim}{x},t) \tag{2.1.4}$$

material and spatial derivatives are related by

$$\frac{d\psi(X,t)}{dt} = \frac{\partial\psi}{\partial t} + (\text{grad }\psi).\dot{\underset{\sim}{x}}, \tag{2.1.5}$$

where grad ψ denotes the spatial gradient of ψ.

If dV is a volume element of the fluid in the reference configuration and dv is a volume element at time t, then

$$dv = J\, dV, \tag{2.1.6}$$

where

$$J = \frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}, \qquad 0<J<\infty. \tag{2.1.7}$$

It can easily be shown that

$$\frac{dJ}{dt} = J \operatorname{div} \dot{\underset{\sim}{x}}, \tag{2.1.8}$$

from which we deduce, for any ψ, that

$$\frac{d}{dt}\int_{v(t)} \psi(\underset{\sim}{x},t)dv = \frac{d}{dt}\int_{V} \psi\, JdV = \int_{v(t)} \left(\frac{d\psi}{dt} + \psi\nabla.\dot{\underset{\sim}{x}}\right) dv. \tag{2.1.9}$$

In particular, choose ψ to be the density, ρ, of the fluid, and since the mass of a material volume is unaltered

$$0 = \frac{d}{dt}\int_{v(t)} \rho dv = \int_{v(t)} \left(\frac{d\rho}{dt} + \rho\nabla.\dot{\underset{\sim}{x}}\right) dv. \tag{2.1.10}$$

This equation holds for any volume v, and hence we have the continuity equation

$$\frac{d\rho}{dt} + \rho\nabla.\dot{\underset{\sim}{x}} = 0. \tag{2.1.11}$$

A motion is called isochoric if it preserves volume, that is dv/dt=0, which implies

$$\nabla.\dot{\underset{\sim}{x}} = 0. \tag{2.1.12}$$

We shall be dealing throughout with incompressible fluids, for which ρ is constant, and in this case the continuity equation reduces to (2.1.12), showing that all possible motions of an incompressible fluid are isochoric. We note that isochoric describes a class of motions while incompressible refers to a type of material.

If $\underline{n}$ is the outward normal to a surface S in the fluid, then there exists a stress vector $\underset{\sim}{t}(\underset{\sim}{x}, \underline{n}, t)$ such that the force exerted by the surrounding fluid on that enclosed by S is $\int_S \underset{\sim}{t}\, dS$. If $\underline{f}$ is the external body force per unit mass acting on the fluid, conservation of momentum yields

$$\frac{d}{dt}\int_v \rho\, \dot{\underset{\sim}{x}}\, dv = \int_v \rho\, \underset{\sim}{f}\, dv + \int_S \underset{\sim}{t}\, dS, \tag{2.1.13}$$

where S is the surface enclosing any volume v in the fluid. By letting $v \to 0$ we can see that the stress vectors are locally in equilibrium, and this enables us to prove that the stress vector may represented by

$$t_i = T_{ji} n_j, \tag{2.1.14}$$

where T_{ji} is a second order tensor called the stress. For most fluids there is no intrinsic angular momentum, that is, all the torques on any volume of fluid arise from macroscopic forces. In this case conservation of moment of momentum leads to the result that the stress tensor is symmetric. We shall be considering incompressible fluids on which no body forces act, and so equation (2.1.13) reduces to

$$\rho\, d\dot{x}_i/dt = T_{ij,j} \tag{2.1.15}$$

using the divergence theorem. Partial differentiation with respect to x_j is denoted by $,j$ and the usual summation convention is used.

Description of the fluid, neglecting thermal effects, is completed by a constitutive equation, which relates the stress to the motion of the fluid. Before discussing constitutive relations it is convenient to introduce some further notation.

If the motion of the fluid is such that the particle which reaches position $\underset{\sim}{x}$ at time t passes through position $\underline{\xi}$ at time τ, that is

$$\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{x}, t, \tau), \qquad -\infty < \tau \leq t, \tag{2.1.16}$$

we define the deformation function $F(t)$ and the relative deformation function $F_t(\tau)$ by

$$F(t) \equiv \nabla\underset{\sim}{x}(\underline{X}, t), \qquad F_t(\tau) \equiv \nabla\underline{\xi}(\underset{\sim}{x}, t, \tau). \tag{2.1.17}$$

We note that $F_t(t)$ is the unit tensor. Now

$$J(t) = |\det F(t)|, \qquad J_t(\tau) \equiv \det F_t(\tau) = \frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x_1,x_2,x_3)}, \tag{2.1.18}$$

and since $J_t(t)=1$ and $F_t(\tau)$ is invertible we see that $J_t(\tau)$ remains positive. By the polar decomposition theorem (see Ericksen, 1960) F has two unique decompositions,

$$F = RU = VR, \tag{2.1.19}$$

in which R is orthogonal and U and V are symmetric and positive definite. We now introduce the left and right Cauchy-Green tensors

$$B \equiv V^2 = FF^T, \quad C \equiv U^2 = F^TF, \tag{2.1.20}$$

where T denotes a transposed matrix. Similar tensors may be derived from $F_t(\tau)$. For any function ψ let

$$\dot{\psi} \equiv \frac{\partial}{\partial\tau}\psi(\tau)\Big|_{\tau=t}, \qquad \overset{(n)}{\psi} \equiv \frac{\partial^n}{\partial\tau^n}\psi(\tau)\Big|_{\tau=t}, \tag{2.1.21}$$

so that the nth acceleration gradient $L_n(t)$ is given by

$$L_n(t) = \text{grad}\ \overset{(n)}{\underset{\sim}{x}} = \overset{(n)}{F_t}(t), \tag{2.1.22}$$

and the nth spin and stretching tensors, W_n and D_n, are defined by

$$W_n(t) \equiv \overset{(n)}{R_t}(t), \qquad D_n(t) = \overset{(n)}{U_t}(t). \tag{2.1.23}$$

We shall also require the Rivlin-Ericksen tensors

$$A_n(t) \equiv \overset{(n)}{C_t}(t). \tag{2.1.24}$$

A_1 and A_2 are defined in a more obvious way on page 43.

By differentiating the expression

$$C_t(\tau) = F_t(\tau)^T F_t(\tau) \tag{2.1.25}$$

we obtain

$$\overset{(n)}{C_t}(\tau) = \sum_{i=0}^{n} \binom{n}{i} \overset{(i)}{F_t}(\tau)^T \overset{(n-i)}{F_t}(\tau), \tag{2.1.26}$$

which reduces for $\tau=t$ to

$$A_n = L_n + L_n^T + \sum_{i=1}^{n-1} \binom{n}{i} L_i^T L_{n-i}. \tag{2.1.27}$$

In this way the kinematic matrices A_n are given as polynomials in components of the acceleration gradients L_n. Rivlin and Ericksen (1955) have shown, under suitable conditions, that the inverse is true, namely that the acceleration gradients, L_n, can be expressed as polynomials in the components of $A_1, A_2, \ldots, A_n$. We shall need to express our constitutive equation in a form that is independent of the particular coordinates used, and since A_n is frame-indifferent, unlike L_n, these results enable a stress which is dependent on acceleration gradients to be expressed in a frame-indifferent form.

Finally in this section we introduce the history $\psi^{(t)}$ up to time t of the function ψ by

$$\psi^{(t)}(s) \equiv \psi(t-s), \qquad s\geq 0. \tag{2.1.28}$$

2.2 The general constitutive equation

In determining the stress we lay down three postulates (Truesdell and Noll, 1965, p.56):

(1) Principle of determinism: The stress in a body is determined by the history of motion of that body.

(2) Principle of local action: In determining the stress at a given particle X, the motion outside an arbitary neighbourhood of X may be disregarded.

(3) Principle of material frame-indifference: The form of the constitutive equation is to be written in a way that is independent of the choice of coordinates or frame of reference.

If $\underset{\sim}{x} = \underset{\sim}{\chi}(X, t)$ is the position of particle X at time t we define the localisation $\underset{\sim}{\chi}_X$ at X of the motion $\underset{\sim}{\chi}$ by

$$\underset{\sim}{\chi}_X(Z, \tau) \equiv \underset{\sim}{\chi}(Z, \tau) - \underset{\sim}{\chi}(X, \tau), \tag{2.2.1}$$

for $Z \in$ an arbitrary neighbourhood of X. The localisation gives the motion of a neighbourhood of X relative to X. Under the principles above it can be shown that most general form of the constitutive equation is

$$T(t) = \mathfrak{F}(\underset{\sim}{\chi}_X^{(t)}), \tag{2.2.2}$$

where $\mathcal{F}$ is the response functional at X and $\chi_X^{(t)}$ is the history up to time t of the localisation at X of the motion χ. From the second principle it can be shown that the value of the functional depends on $\chi_X^{(t)}(Z, s)$ only for Z in an arbitrarily small neighbourhood of the origin. But for small Z the localised motion can be approximated by its derivative, namely

$$\chi_X^{(t)}(Z, s) \approx \{\nabla\chi_X^{(t)}(0, s)\}Z. \tag{2.2.3}$$

Since χ_X differs from χ only by a translation (2.2.3) may be written

$$\chi_X^{(t)}(\underline{Z}, s) \approx F^{(t)}(s)\, Z, \tag{2.2.4}$$

and the approximation may be made as precise as is required by restricting Z to a small enough neighbourhood. We have now shown that for many materials the functional depends only on the history $F^{(t)}$ of the deformation gradient.

With this motivation we shall confine our attention to simple materials, which are defined by an exact constitutive relation of the form

$$T(t) = \mathop{G}_{s=0}^{\infty} \{F^{(t)}(s)\}. \tag{2.2.5}$$

Since this equation is invariant under rotation of the coordinate frame it may be written in the form

$$\bar{T}(t) \equiv F^T(t)\, T(t)\, F(t) = \mathop{\mathcal{F}}_{s=0}^{\infty} \{C^{(t)}(s)\} \tag{2.2.6}$$

using (2.1.19) and (2.1.20).

2.3 Incompressibility

Incompressibility is a form of internal constraint on the body, and the principle of determinism needs some modification to account for it. Simple constraints are defined by a scalar valued function of a tensor variable, and particle X in a body is subject to the constraint γ if the possible motions are restricted to those for which

$$\gamma\{F(\tau)\} = 0, \qquad -\infty<\tau<\infty. \tag{2.3.1}$$

(2.3.1) is a constitutive equation subject to the principle of frame-indifference and consequently it may be expressed in the form

$$\lambda\{C(\tau)\} = 0, \qquad -\infty<\tau<\infty. \tag{2.3.2}$$

If we denote the tensor $\partial\lambda/\partial C_{ij}$ by $\lambda_c(C)$, differentiating (2.3.2) with respect to τ gives

$$\mathrm{tr}\ \{\lambda_c(C)\ \dot{C}\} = 0. \tag{2.3.3}$$

It is possible from the definitions (2.1.20) and (2.1.23) to express $\dot{C}(t)$ in terms of the stretching tensor D ($\equiv D_1$), namely

$$\dot{C}(t) = 2F(t)^T D(t)\ F(t), \tag{2.3.4}$$

and so (2.3.3) reduces, putting $\tau=t$, to

$$\mathrm{tr}\ \{F\ \lambda_c(C)F^T D\} = 0. \tag{2.3.5}$$

We are now in a position to modify the first principle postulated in section (2.2):

Principle of determinism for simple materials subject to internal constraints:

The stress T at time t is determined by the history $F^{(t)}$ of the deformation gradient only to within a stress N that does no work in any motion satisfying the constraints. (Truesdell and Noll, 1965, p.70).

The rate at which stresses do work is given by $\operatorname{tr}(TD)$ per unit volume, so the principle above requires that N satisfies

$$\operatorname{tr}(ND) = 0 \tag{2.3.6}$$

for all symmetric tensors D which satisfy (2.3.5). Since $\operatorname{tr}(AB)$ defines an inner product in the space of symmetric tensors, N must be a scalar multiple of $F\,\lambda_c(C)F^T$. If there are several constraint functions $\lambda^{(i)}$ this result becomes

$$N = \sum_{i=1}^{n} q_i\, F\, \lambda_c^{(i)}(C)\, F^T, \tag{2.3.7}$$

where q_i are scalar coefficients. The history of the deformation gradient then determines the extra stress

$$T_E = T + \sum_{i=1}^{n} q_i\, F\, \lambda_c^{(i)}(C)\, F^T \tag{2.3.8}$$

by a constitutive relation of the form

$$T_E(t) = \mathop{\mathcal{G}}_{s=0}^{\infty} \{F^{(t)}(s)\}. \tag{2.3.9}$$

Finally, we consider the special constraint of incompressibility. For isochoric motion we have from (2.1.8), (2.1.12) and (2.1.20) that

$$\det C(\tau) = 1. \tag{2.3.10}$$

A material is incompressible if it is susceptible only of isochoric motions, which is equivalent to saying, from (2.1.11) and (2.1.12), that the density at a material point remains constant. A corresponding constraint function is

$$\lambda(C) = \det C - 1, \tag{2.3.11}$$

and the extra stress reduces to

$$T_E = T + pI. \tag{2.3.12}$$

The response functional is determined only up to an arbitrary scalar multiple of the unit tensor, though this indeterminacy may be removed by a normalisation such as

$$\mathrm{tr}\, T_E = 0. \tag{2.3.13}$$

2.4 Simple fluids

Although specific fluid models, such as the one leading to the Navier-Stokes equation, are well-defined and of wide application, the physical concept of a fluid is somewhat vague, but it includes the idea that fluid should not alter its material response after an arbitrary deformation that leaves the density unaltered. A simple fluid is defined as a simple material satisfying certain isotropy relations which express this property in mathematical terms. It can be shown (Noll, 1958) the general constitutive equation (2.2.5) for a simple fluid can be written in the form

$$T = -p(\rho)I + \mathop{\mathcal{D}}_{s=0}^{\infty} \{G(s);\, \rho\}, \tag{2.4.1}$$

where $p(\rho)$ is a scalar function of the density,

$$G(s) \equiv C_t(t-s) - I, \tag{2.4.2}$$

and the response functional $\mathcal{D}$ has value 0 when $G(s) \equiv 0$. The response functional does not depend on any reference configuration, which expresses the physical property that simple fluids have no preferred configurations, that is, they have no permanent memory for any particular state. A simple fluid remembers that past only through the tensor $G(s)$, which measures the deformation of the configuration at time $t-s$ with respect to the configuration at the ever-changing present time t.

For incompressible fluids the stress T in (2.4.1) must be replaced by the extra stress $T_E = T+pI$. But the density ρ at a particle cannot depend on time and hence may be omitted from the constitutive equations. (2.4.1) is then replaced by

$$T = -pI + \mathop{D}_{s=0}^{\infty} \{G(s)\}. \tag{2.4.3}$$

As discussed in the previous section, D is indeterminate, but may be fully defined by a normalisation (2.3.13) such as

$$\mathrm{tr} \mathop{D}_{s=0}^{\infty} \{G(s)\} = 0. \tag{2.4.4}$$

If the domain of D is a suitable function space it is possible (Green and Rivlin, 1957) to approximate the functional by integral polynomials:

$$\mathop{D}_{s=0}^{\infty} \{G(s)\} = \sum_{n=1}^{m} \int_0^{\infty} \cdots \int_0^{\infty} g\{s_1,\ldots,s_n;\ G(s_1),\ldots,G(s_n)\}ds_1\ldots ds_n, \tag{2.4.5}$$

where the tensor functions g are multilinear and isotropic in tensors $G(s_1),\ldots,G(s_n)$. The g's may be taken to be invariant under permutations of $1,\ldots,m$, and consequently each integrand may be replaced by a sum of terms of the form

$$\psi(s_1,\ldots,s_n)\mathrm{tr}\{G(s_1)\ldots G(s_{k_1})\}\mathrm{tr}\{G(s_{k_1}+1)\ldots G(s_{k_2})\}\ldots$$
$$\{G(s_{k_{r-1}}+1)\ldots G(s_{k_r})\}G(s_{k_r}+1)\ldots G(s_n), \tag{2.4.6}$$

where $1 \leq k_1 < k_2 < \ldots < k_r \leq n$, $k_1 \leq 6$, $k_j - k_{j-1} \leq 6$, $n - k_r \leq 5$. If $n = k_r$ the product of the tensors in (2.4.6) is replaced by the unit tensor, and for incompressible fluids all such terms can be incorporated in the pressure term pI. The first few terms in the integral representation of an incompressible fluid are

$$T = -pI + \int_0^\infty \zeta(s)\, G(s)ds + \int_0^\infty\int_0^\infty \left[\alpha(s_1,s_2)\mathrm{tr}\{G(s_1)\}G(s_2)+\beta(s_1,s_2)G(s_1)G(s_2)\right]ds_1 ds_2 + \ldots \tag{2.4.7}$$

In chapter 5 we shall examine fluids that may be represented in this way. The theories used in chapters 3 and 4 can also be obtained from (2.4.7) by using the Dirac δ-function and its derivatives in the kernels in the integrands.

2.5 Fading Memory

For simple materials the present stress depends on the history of the deformation gradient. Unfortunately the entire history can never be known, and it is usually assumed at the start of an experiment that the previous history has a negligible influence on the result. This assumption is expressed in the principle of fading memory, namely that deformations that occurred in the distant past should have less influence in determining the present stress than those that occurred in the recent past. The general mathematical definition is complicated but the principle can be easily illustrated in a special case of (2.4.7). If the kernels are decaying exponentials we see at once that the principle is satisfied. We shall examine kernels containing such terms in chapter 5.

The magnitude of a tensor A is defined as

$$|A| \equiv \surd(\mathrm{tr}\, AA^T), \tag{2.5.1}$$

and following Coleman and Noll (1961b) we now introduce the concept of an obliviator, $h(s)$, which decays to zero as rapidly as required as $s\to\infty$. An obliviator is a weighting function which greatly diminishes the influence of the distant past. A decaying exponential is an example of an obliviator. The recollection of a history $G(s)$ is now defined by

$$\|G(s)\|_h \equiv \left\{ \int_0^\infty \{h(s)|G(s)|\}^2 \, ds \right\} . \qquad (2.5.2)$$

The rest history, in which the fluid is always at rest, is given by $G=0$, and clearly this has zero recollection. If the deformation has deviated little from rest in the past it will have only a small recollection. Since the obliviator gives little weight to the distant past any deformation for which $G(s)$ is small in the recent past will also have small recollection although the motion may have had a violent ancient history.

The history of the deformation gradient is described by $G(s)$, and the static continuation of this history is defined by

$$G_n(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq n, \\ G(s-n) & \text{if } s > n, \end{cases} \qquad (2.5.3)$$

Then

$$\|G_n(s)\|_h^2 = \int_0^\infty |G(s+n)|^2 \{h(s)\}^2 \, ds \to 0 \quad \text{as} \quad n \to \infty \qquad (2.5.4)$$

provided G is bounded in magnitude and h decays fast enough (faster than $1/\sqrt{s}$). From (2.5.4) we have the following theorem of stress relaxation: If $G(s)$ is any bounded deformation history then the stress T_n corresponding to the static continuation of $G(s)$ ultimately becomes the equilibrium value of the stress, that is, the stress corresponding to the rest history $G=0$. For an incompressible fluid

$$T_n \to -pI \quad \text{as} \quad n \to \infty. \qquad (2.5.5)$$

The formulation of fading memory discussed above gives in effect only the equilibrium stress. In order to improve the approximation we restrict the functionals D to those that are n times continuously differentiable. By using a Taylor expansion we can approximate the constitutive equation (2.4.3) in the form (Coleman and Noll, 1960 and 1961a)

$$T = -pI + \sum_{j=1}^{n} \underset{s=0}{\overset{\infty}{P_j}} \{G(s)\}, \qquad (2.5.6)$$

where P_j is a bounded homogeneous polynomial functional of $G(s)$ of degree j. The principle of fading memory may be invoked by requiring that the remainder term R_n approach zero faster than the nth power of the recollection of the deformation history, that is,

$$R_n = 0(\|G\|^n). \qquad (2.5.7)$$

Equation (2.5.6) is an approximation to the general constitutive equation of a simple fluid when the recollection of the history is small. In physical terms, the recollection is small when the deformation was small in the recent past, even though it may have been large long before. This condition will be satisfied if the motion is slow in a sense to be discussed in the next section.

2.6 Slow motions and Rivlin-Ericksen fluids

For a given deformation history $G(s)$ we define its retardation by

$$G^{\alpha}(s) \equiv G(\alpha s), \qquad 0<\alpha<1. \qquad (2.6.1)$$

Physically, deformations corresponding to G^{α} are the same as those corresponding to G, but take place at a slower rate. Assuming G

is sufficiently differentiable it follows (Coleman and Noll, 1960) that

$$G^{\alpha}(s) = \sum_{j=0}^{n} \frac{s^j}{j!} \overset{(j)}{G}{}^{\alpha} + o(\alpha^n), \tag{2.6.2}$$

where the order symbol means that the recollection of $\alpha^{-n}\, o(\alpha^n)$ tends to zero as $\alpha \to 0$, and

$$\overset{(j)}{G}{}^{\alpha} = \frac{d^j}{ds^j} G^{\alpha}(s) \Big|_{s=0} = \alpha^j \frac{d^j}{ds^j} G(s) \Big|_{s=0}. \tag{2.6.3}$$

From (2.1.24)

$$G^{\alpha}(s) = \sum_{j=1}^{n} \frac{s^j}{j!} (-1)^j A_j^{\alpha} + o(\alpha^n), \tag{2.6.4}$$

where A_j^{α} is related to the jth Rivlin-Ericksen tensor by

$$A_j^{\alpha} = \alpha^j A_j. \tag{2.6.5}$$

We note that $\overset{(0)}{G}{}^{\alpha}=0$ from (2.4.2), and consequently $G^{\alpha} = O(\alpha)$. If we substitute G^{α} for G in (2.4.3) the error if of order $o(\alpha^n)$, and so to this order (2.4.3) may be replaced by

$$T_{\alpha} = -pI + \sum_{j=1}^{n} \underset{s=0}{\overset{\infty}{P_j}} \{G^{\alpha}(s)\}, \tag{2.6.6}$$

where T_{α} is the stress corresponding to the retarded motion. Since P_j is a bounded homogeneous polynomial it can be proved (Coleman and Noll, 1960) using (2.6.4) that

$$\underset{s=0}{\overset{\infty}{P_j}} \{G^{\alpha}(s)\} = \Sigma\, L_{k_1 \ldots k_j}(A_{k_1}^{\alpha}, \ldots, A_{k_j}^{\alpha}) + o(\alpha^n), \tag{2.6.7}$$

where L is a multilinear tensor function of j tensor variables. The summation is over all sets of j suffices $(k_1,\ldots,k_j)$ satisfying

$$1<k_1\leqslant\ldots\leqslant k_j\leqslant n, \qquad k_1+\ldots+k_j\; n. \tag{2.6.8}$$

The results (2.6.6) and (2.6.7) show that the general constitutive equation (2.4.3) of a simple fluid may be approximated by

$$T = -pI + \Sigma\, L_{k_1\ldots k_j}\,(A_{k_1},\ldots,A_{k_j}), \tag{2.6.9}$$

where the summation is over all sets of suffices $(k_1,\ldots,k_j)$, $j=1,2,\ldots,n$ satisfying (2.6.8). In this way the entire history is represented by derivatives of the deformation evaluated at the present time. We note, however, that the theory depends on the motion being slow. Fluids for which (2.6.9) is valid are called Rivlin-Ericksen fluids.

It has been shown (Rivlin-Ericksen, 1955, Spencer and Rivlin, 1960) that the property of isotropy limits the possible forms of (2.6.9). The stress is given in the form

$$T = -pI + \sum_{m=1}^{N} \alpha_m\,(\Pi_m+\Pi_m^{\,T}), \tag{2.6.10}$$

where Π_m and its transpose $\Pi_m^{\,T}$ are certain matrix products formed from the kinematic matrices $A_1,\ldots,A_n$, and α_m are polynomials in the traces of certain other matrix products formed from the kinematic matrices. In the particular case when the stress depends only on the first kinematic matrix A_1 equation (2.6.10) takes the exact form

$$T = -pI + \eta\, A_1, \tag{2.6.11}$$

and consequently for slow flows of any incompressible isotropic simple fluid

$$T_\alpha = -pI+\eta A_1{}^\alpha + o(\alpha), \tag{2.6.12}$$

(2.6.11) is the constitutive equation for a classical Navier-Stokes fluid and the constant η is the viscosity. Although this theory emerges at the first stage of approximation for slow motion it has given excellent results in general flows of many, though not all, physical fluids. Better approximations for slow flows are given by further terms in the Rivlin-Ericksen formulation (2.6.10). To order $o(\alpha^2)$ in the retardation factor α (2.6.10) becomes

$$T = -pI + \eta A_1 + \lambda A_2 + \mu A_1{}^2, \tag{2.6.13}$$

and to order $o(\alpha^3)$ it gives

$$T = -pI+\eta A_1+\lambda A_2+\mu A_1{}^2+\beta_1 A_3+\beta_2(A_2A_1+A_1A_2)+\beta_3(trA_2)A_1, \tag{2.6.14}$$

Expressions for A_1 and A_2 are given on page 43.

where the coefficients are constants. In the next two chapters we shall examine second-order fluids which are described by (2.6.13) in the hope that this theory will yield better results than the classical (Newtonian) theory for a particular flow of dilute polymer solutions. We note, however, first, that the retardation theorem refers to the constitutive equation itself and not to the equations of motion derived from it. Second, a small term introduced into a differential equation may have a large effect on its result, and, third, although (2.6.13) may be a better approximation than (2.6.11) for slow flows of dilute polymer solutions, it may not be better for general flows. Noting these possible objections to the theory we shall proceed in the next chapter to examine flow through a channel of a dilute polymer solution.

The physical significance of (2.6.13) may be seen in terms of normal stress differences. Let us consider a rectilinear shearing flow given by

$$\underset{\sim}{v} = (Kx_2, 0, 0), \tag{2.6.15}$$

where K is a constant. The normal stress differences are then

$$\left.\begin{aligned} T_{11}-T_{33} &= \mu K^2, \\ T_{22}-T_{33} &= (2\lambda+\mu)K^2, \end{aligned}\right\} \tag{2.6.16}$$

which clearly vanish for a Newtonian fluid. In section 1.1 we discussed the role of non-zero normal stress differences in drag reduction, and we can see from (2.6.16) that the second-order fluid (2.6.13) provides a comparatively simple model that includes normal stress effects.

CHAPTER 3: STABILITY OF PLANE POISEUILLE FLOW OF A SECOND-ORDER FLUID: LINEAR ANALYSIS

3.1 Introduction

In the previous chapter we derived, under certain conditions and assumptions, a particular constitutive relationship for viscoelastic fluids, namely the second-order Rivlin-Ericksen formulation (2.6.13). These idealised fluids do not exhibit stress relaxation or elastic recovery, but since they do have properties such as normal stress effects not found in a Newtonian fluid they do provide a convenient starting point for investigating stability of non-Newtonian fluids. In this chapter we shall examine disturbances of plane Poiseuille flows which are regarded as small, and in section 3.3 derive linear equations based on this assumption. It will be shown that the stability of the flow is governed by a modified version of the Orr-Sommerfeld equation used in the analysis for Newtonian fluids.

Squire (1933) has shown that for Newtonian fluids the analysis of three-dimensional disturbances is equivalent to that of two-dimensional in-plane disturbances, a result which is discussed in more detail in section 6. Apart from Gupta and Rai (1968), where three-dimensional disturbances to the flow down an inclined plane are considered, previous stability analyses for viscoelastic fluids, for example Gupta (1967), Chan Man Fong and Walters (1965), Listrov (1965), Gorodtsov and Leonov (1967), Gupta and Rai (1967), Mook (1967), Jones (1967), Jones and Walters (1968) and Schwarz and Chun (1968), have been confined to two-dimensional disturbances. These results give only a partial solution to the stability problem, for Lockett (1969a) has shown that Squire's theorem is not in general valid for viscoelastic fluids. In section 3.6 we shall therefore examine fully three-dimensional disturbances of the basic flow considered. The fluid is regarded as only slightly viscoelastic so that we may develop a perturbation analysis about the solution for a Newtonian fluid, the derivation of which is outlined in section 3.4.

As a consequence of Squire's theorem, at the onset of instability in a Newtonian fluid the component of vorticity parallel to the basic flow vanishes at the channel boundary. For a viscoelastic fluid, however, Squire's theorem no longer holds, and, under certain conditions, longitudinal vorticity remains non-zero at the walls. Thus viscoelasticity introduces a new structure into the flow at the onset of instability, and as the region close to the wall plays an important role in the subsequent development of turbulent flow this property may be significant.

The constitutive relation for a second-order fluid introduces three parameters, one of which is the (Newtonian) viscosity. Of the other two viscoelastic parameters one affects disturbances parallel to the original flow, causing the flow to be more unstable than the corresponding flow of a Newtonian fluid. Both viscoelastic parameters affect out-of-plane disturbances and may cause either stabilisation or destabilisation. We shall show in the final section of this chapter that under certain conditions there exists a direction not parallel to the plane of the original flow along which the first unstable disturbance will propagate.

3.2 Equations of motion for a second-order fluid

The constitutive relation for a second-order fluid takes the form

$$\underset{\sim}{S}^* = \eta^* \underset{\sim}{A}_1^* + \lambda^* \underset{\sim}{A}_2^* + \mu^* \underset{\sim}{A}_1^{*2}, \qquad (3.2.1)$$

where the stars denote dimensional quantities: $\underset{\sim}{s}^*$ is the stress additional to that produced by a hydrostatic pressure p^*, ρ^* denotes density, $\underset{\sim}{A}_1{}^*$ and $\underset{\sim}{A}_2{}^*$ are kinematic matrices defined below, η^*, λ^* and μ^* are fluid constants, and $v_i{}^*$ are components of velocity in a rectangular Cartesian coordinate system $x_i{}^*(i=1,2,3)$. We now define a set of dimensionless variables, denoted by removing the stars from the corresponding symbol, defined with respect to ρ^* and a characteristic length L and time T. The constitutive equation (2.1) now becomes

$$\underset{\sim}{s} = \frac{1}{R}\underset{\sim}{A}_1 + \lambda\underset{\sim}{A}_2 + \mu\underset{\sim}{A}_1{}^2, \qquad (3.2.2)$$

where R is a Reynolds number for the flow given by

$$R = L^2\rho^*/T\eta^*, \qquad (3.2.3)$$

and the viscoelastic constants λ and μ are given by

$$\lambda = \lambda^*/\rho^*L^2, \qquad \mu = \mu^*/\rho^*L^2. \qquad (3.2.4)$$

The kinematic matrices $\underline{A}_1$ and $\underline{A}_2$ are defined (Rivlin 1955) by

$$\underset{\sim}{A}_1 = \underset{\sim}{V}+\underset{\sim}{V}^T, \qquad (3.2.5)$$

$$\underline{A}_2 = \frac{\partial\underset{\sim}{A}_1}{\partial t} + v_k\frac{\partial\underset{\sim}{A}_1}{\partial x_k} + \underset{\sim}{A}_1\underset{\sim}{V} + (\underset{\sim}{A}_1\underset{\sim}{V})^T, \qquad (3.2.6)$$

where

$$\underset{\sim}{V} = \|\partial v_i/\partial x_j\|, \qquad (3.2.7)$$

and T denotes the transposed matrix. The equations of motion (2.1.15) are now

$$-\frac{\partial p}{\partial x_i} + \frac{\partial s_{ij}}{\partial x_j} = \frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}, \qquad (3.2.8)$$

and the continuity condition for an incompressible fluid (2.1.12) is

$$\partial v_k/\partial x_k \;=\; 0. \qquad (3.2.9)$$

3.3 Stability equations for disturbances to Poiseuille flow

We now consider velocity fields of the form

$$\left.\begin{aligned} v_1 &= U(x_2) + u_1(x_2)\exp\{i(\alpha x_1+\beta x_3-\alpha ct)\}, \\ v_2 &= u_2(x_2)\exp\{i(\alpha x_1+\beta x_3-\alpha ct)\}, \\ v_3 &= u_3(x_2)\exp\{i(\alpha x_1+\beta x_3-\alpha ct)\}. \end{aligned}\right\} \qquad (3.3.1)$$

These expressions represent a steady parallel flow U and a disturbance of the form required in stability analyses. A number of important situations, such as those found in boundary layer and lubrication theory, lead to velocity fields which are closely approximated by the form given here. In this chapter we shall confine our attention to Poiseuille flow between two parallel planes $x_2=\pm 1$ represented by

$$U(x_2) \;=\; 1-x_2^{\,2}, \qquad (3.3.2)$$

though the equations developed will be applicable to a wide range of functions U. It is worth noting at this stage that the undisturbed velocity $(1-x_2^{\,2}, 0, 0)$ is an exact solution of the equations of section 2 with boundary conditions $v_k=0$ at $x_2=\pm 1$. In the present linearised theory, we shall assume that squares and products of disturbance components u_k and their derivatives may be neglected. The flow and disturbance is expressed in the form (3.3.1) since the basic flow depends only on x_2, and more general disturbances may be written, in this linear theory, in terms of harmonic components in the other variables.

Since the basic flow U is regarded as known we need only consider equations governing disturbance quantities. If we write these quantities in the form

$$\hat{f}(x_2)\exp\{i(\alpha x_1+\beta x_3-\alpha ct)\}, \tag{3.3.3}$$

we obtain from (3.2.8)

$$\begin{aligned} i\alpha\hat{s}_{11}+\hat{s}'_{12}+i\beta\hat{s}_{13} &= i\alpha(U-c)u_1+U'u_2+i\alpha\hat{p}, \\ i\alpha\hat{s}_{12}+\hat{s}'_{22}+i\beta\hat{s}_{23} &= i\alpha(U-c)u_2+\hat{p}', \\ i\alpha\hat{s}_{13}+\hat{s}'_{23}+i\beta\hat{s}_{33} &= i\alpha(U-c)u_3+i\beta\hat{p}, \end{aligned} \tag{3.3.4}$$

where the prime denotes differentiation with respect to x_2. Substituting (3.3.1) in (3.2.5) and (3.2.6) we obtain

$$\hat{\underset{\sim}{A}}_1 = \begin{bmatrix} 2i\alpha u_1 & u_1'+i\alpha u_2 & i\beta u_1+i\alpha u_3 \\ u_1'+i\alpha u_2 & 2u_2' & i\beta u_2+u_3' \\ i\beta u_1+i\alpha u_3 & i\beta u_2+u_3' & 2i\beta u_3 \end{bmatrix}, \tag{3.3.5}$$

$$\hat{\underset{\sim}{A}}_1{}^2 = U' \begin{bmatrix} 2u_1'+2i\alpha u_2 & 2i\alpha u_1+2u_2' & i\beta u_2+u_3' \\ 2i\alpha u_1+2u_2' & 2u_1'+2i\alpha u_2 & i\beta u_1+i\alpha u_3 \\ i\beta u_2+u_3' & i\beta u_1+i\alpha u_3 & 0 \end{bmatrix}, \tag{3.3.6}$$

$$\hat{\underset{\sim}{A}}_2 = i\alpha(U-c)\hat{\underset{\sim}{A}}_1 + \begin{bmatrix} 0 & U''u_2 & 0 \\ U''u_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2i\alpha u_2 & 3i\alpha u_1+u_2' & i\beta u_2 \\ 3i\alpha u_1+u_2' & 2(2u_1'+i\alpha u_2) & 2i\beta u_1+i\alpha u_3 \\ i\beta u_2 & 2i\beta u_1+i\alpha u_3 & 0 \end{bmatrix} \tag{3.3.7}$$

The first order part of the extra stress, that is, stress other than that due to a pressure, is then given by

$$\left.\begin{aligned}
\hat{s}_{11} &= \{\tfrac{1}{R} +\lambda i\alpha(U-c)\}2i\alpha u_1+2(\lambda+\mu)U'(u_1'+i\alpha u_2)-2\lambda U'u_1' \\
\hat{s}_{12} &= \{\tfrac{1}{R} +\lambda i\alpha(U-c)\}(u_1'+i\alpha u_2)+\lambda U''u_2+(\lambda+2\mu)U'(i\alpha u_1+u_2')+2\lambda U'i\alpha u_1 \\
\hat{s}_{13} &= \{\tfrac{1}{R} +\lambda i\alpha(U-c)\}(i\beta u_1+i\alpha u_3)+\mu U'(i\beta u_2+u_3')+\lambda U'i\beta u_2 \\
\hat{s}_{22} &= \{\tfrac{1}{R} +\lambda i\alpha(U-c)\}2u_2'+2(\lambda+\mu)U'(u_1'+i\alpha u_2)+2\lambda U'u_1' \\
\hat{s}_{23} &= \{\tfrac{1}{R} +\lambda i\alpha(U-c)\}(i\beta u_2+u_3')+(\lambda+\mu)U'(i\beta u_1+i\alpha u_3)+\lambda U'i\beta u_1 \\
\hat{s}_{33} &= \{\tfrac{1}{R} +\lambda i\alpha(U-c)\}2i\beta u_3,
\end{aligned}\right\}\quad(3.3.8)$$

which yields, when substituted in the equations of motion (3.3.4), the following equations governing the perturbations:

$$\{D^2-\alpha^2-\beta^2-i\alpha R(U-c)\}u_1-RU'u_2-i\alpha R\hat{p} = -i\alpha R\lambda(U-c)(D^2-\alpha^2-\beta^2)u_1-R\lambda A_1-R\mu A_2, \quad (3.3.9)$$

$$\{D^2-\alpha^2-\beta^2-i\alpha R(U-c)\}u_2-R\hat{p}' = -i\alpha R\lambda(U-c)(D^2-\alpha^2-\beta^2)u_2-R\lambda B_1-R\mu B_2, \quad (3.3.10)$$

$$\{D^2-\alpha^2-\beta^2-i\alpha R(U-c)\}u_3-i\beta R\hat{p} = -i\alpha R\lambda(U-c)(D^2-\alpha^2-\beta^2)u_3-R\lambda C_1-R\mu C_2, \quad (3.3.11)$$

where D or a prime denotes differentiation with respect to x_2, and

$$\left.\begin{aligned}
A_1 &= -3\alpha^2u_2U'+3i\alpha u_1'U'+2i\alpha u_1U''-\beta^2u_2U'-i\beta u_3'U'-i\beta u_3U''+u_2'U''+u_2U''' \\
B_1 &= -2(\alpha^2+\beta^2)u_1U'+3i\alpha u_2U''+4i\alpha u_2'U'+4u_1''U'+4u_1'U'' \\
C_1 &= 2i\alpha u_3'U'-2\alpha\beta u_2U'+2i\beta u_1'U'+2i\beta u_1U''+i\alpha u_3U'' \\
A_2 &= 2i\alpha U'(u_1'+i\alpha u_2)+2U''(i\alpha u_1+u_2')+2U'(i\alpha u_1'+u_2'')+i\beta U'(u_3'+i\beta u_2) \\
B_2 &= 2i\alpha U'(i\alpha u_1+u_2')+2U''(u_1'+i\alpha u_2)+2U'(u_1''+i\alpha u_2')-\beta U'(\alpha u_3+\beta u_1) \\
C_2 &= i\alpha U'(u_3'+i\beta u_2)+U''(i\alpha u_3+i\beta u_1)+U'(i\beta u_1'+i\alpha u_3')
\end{aligned}\right\}\quad(3.3.12)$$

In addition (3.2.9) requires

$$i\alpha u_1+u_2'+i\beta u_3 = 0. \quad (3.3.13)$$

We now introduce the following transformations due to Squire (1933)

$$\left.\begin{aligned} \widetilde{\alpha u}_1 &= \alpha u_1+\beta u_3, \quad \tilde{u}_2 = u_2, \quad \widetilde{pR} = \hat{p}R, \\ \widetilde{\alpha R} &= \alpha R, \quad \tilde{c} = c, \quad \tilde{\alpha}^2 = \alpha^2+\beta^2, \end{aligned}\right\} \tag{3.3.14}$$

and obtain from (3.3.9) to (3.3.11) and (3.3.13)

$$\begin{aligned} \{D^2-\tilde{\alpha}^2-i\widetilde{\alpha R}(U-\tilde{c})\}\tilde{u}_1-\tilde{R}U'\tilde{u}_2-i\widetilde{\alpha R p} = &-i\widetilde{\alpha R}\lambda(U-\tilde{c})(D^2-\tilde{\alpha}^2)\tilde{u}_1 \\ &-\tilde{R}\lambda\left[A_1+\frac{\beta}{\alpha}C_1\right] \\ &-\tilde{R}\mu\left[A_2+\frac{\beta}{\alpha}C_2\right], \end{aligned} \tag{3.3.15}$$

$$\{D^2-\tilde{\alpha}^2-i\widetilde{\alpha R}(U-\tilde{c})\}\tilde{u}_2-\widetilde{Rp}' = -i\widetilde{\alpha R}\lambda(U-\tilde{c})(D^2-\tilde{\alpha}^2)\tilde{u}_2 - \frac{\widetilde{\alpha R}}{\alpha}\lambda B_1 - \frac{\widetilde{\alpha R}}{\alpha}\mu B_2, \tag{3.3.16}$$

$$i\widetilde{\alpha u}_1 + \tilde{u}_2' = 0 \tag{3.3.17}$$

Following Lockett (1969a) we eliminate pressure from (3.3.15) (3.3.17) and obtain a modified Orr-Sommerfeld equation

$$\{(D^2-\tilde{\alpha}^2)^2-i\widetilde{\alpha R}(U-\tilde{c})(D^2-\tilde{\alpha}^2)+i\widetilde{\alpha R}U''\}\tilde{u}_2 = \lambda N_1+\theta(2\lambda+\mu)N_2, \tag{3.3.18}$$

where

$$\left.\begin{aligned} N_1 &= i\widetilde{\alpha R}\{U''''\tilde{u}_2-(U-\tilde{c})(D^2-\tilde{\alpha}^2)^2\tilde{u}_2\}, \\ N_2 &= i\widetilde{\alpha R}\{U'(D^2-\tilde{\alpha}^2)-U'''\}(\tilde{u}_2'+i\tilde{\alpha}^2\tilde{g}), \end{aligned}\right\} \tag{3.3.19}$$

and

$$\theta = \beta^2/\alpha^2, \quad \tilde{g} = u_3/\beta. \tag{3.3.20}$$

We note that the right hand side of (3.3.18) involves u_3 in addition to u_2.

For a Newtonian fluid, with $\lambda=u=0$, equations (3.3.15)-(3.3.17) have the same structure as (3.3.9)-(3.3.11) and (3.3.13) with $u_3=\beta=0$. Thus the analysis of three-dimensional disturbances is equivalent to the analysis of two-dimensional in-plane disturbances, subject to the transformations (3.3.14). Further, (3.3.14) shows that the equivalent two-dimensional problem is associated with a lower Reynolds number. Hence the critical Reynolds number is given directly by the two-dimensional analysis. This result is known as Squire's theorem (1933). We note that the result is valid since, with $\lambda=\mu=0$, α and β occur in (3.3.15)-(3.3.18) only via the combination $\tilde{\alpha}$.

Now let us consider the case where λ and μ are not zero. In (3.3.18) dependence on β occurs in θ as well as via the combination $\tilde{\alpha}$. Lockett (1969a) has noted that if $2\lambda+\mu=0$ dependence on β and u_3 vanishes, and (3.3.18) is identical in form to the equation governing two-dimensional disturbances, namely

$$\{(D^2-\tilde{\alpha}^2)^2-i\tilde{\alpha}\tilde{R}(U-\tilde{c})(D^2-\tilde{\alpha}^2)+i\tilde{\alpha}\tilde{R}U''\}\tilde{u}_2 = \lambda i\tilde{\alpha}\tilde{R}\{U''''\tilde{u}_2-(U-\tilde{c})(D^2-\tilde{\alpha}^2)^2\tilde{u}_2\}. \tag{3.3.21}$$

For this class of fluids Squire's theorem remains valid. In general, however, these conditions will not apply, and we shall need to investigate three-dimensional disturbances for which the theorem does not hold. It has been shown (Lockett, 1969a) that a modified Squire's theorem is applicable to the first-order perturbation analysis for slightly viscoelastic fluids, namely, that under the additional transformations

$$\tilde{\lambda} = \lambda, \qquad 2\tilde{\lambda}+\tilde{\mu} = \theta(2\lambda+\mu) \tag{3.3.22}$$

all equations governing three-dimensional disturbances take the same form, and may therefore be handled in a single calculation.

For in-plane disturbances $\beta=0$ and the term in μ vanishes from (3.3.18), showing that the viscoelastic parameter μ does not affect stability considerations with respect to these disturbances.

It does, however, contribute to out-of-plane disturbances ($\beta\neq 0$), and λ affects both types of disturbance.

We now restrict our attention to those second-order fluids which are only slightly viscoelastic. The constants λ and μ are presumed to be small and will be used in section 3.6 as parameters in a perturbation analysis about the solution for a Newtonian fluid. This procedure is possible since the basic flow (3.3.2) is the same for both Newtonian and second-order fluids.

3.4 Solution of the Orr-Sommerfeld equation

In this section we shall recapitulate some results of stability theory for Newtonian fluids. If we set $\lambda=\mu=0$ equation (3.3.18) becomes

$$\{(D^2-\tilde{\alpha}^2)^2-i\tilde{\alpha}\tilde{R}(U-\tilde{c})(D^2-\tilde{\alpha}^2)+i\tilde{\alpha}\tilde{R}U''\}\tilde{u}_2 = 0, \qquad (3.4.1)$$

which is the familiar Orr-Sommerfeld equation. In general this equation, which is linear in $\tilde{u}_2$, has four independent solutions f_1,f_2,f_3 and f_4, and so the general solution may be written in the form

$$\tilde{u}_2 = \sum_{k=1}^{4} a_k f_k, \qquad (3.4.2)$$

where a_k are constants. For flow in a channel bounded by walls at $x_2=\pm 1$ the boundary conditions are

$$u_1=u_2=u_3=0 \quad \text{at} \quad x_2=\pm 1, \qquad (3.4.3)$$

which from (3.3.17) may be written in the form

$$\tilde{u}_2=\tilde{u}_2'=0 \quad \text{at} \quad x_2=\pm 1. \qquad (3.4.4)$$

Imposing these conditions on (3.4.2) we obtain four homogeneous equations in the coefficients a_k. Since the f_k are non-trivial solutions of (3.4.1) the coefficient determinant must vanish, that is

$$F(\tilde{\alpha},\tilde{R},\tilde{c}) \equiv \begin{vmatrix} f_1(-1) & f_2(-1) & f_3(-1) & f_4(-1) \\ f_1'(-1) & f_2'(-1) & f_3'(-1) & f_4'(-1) \\ f_1(1) & f_2(1) & f_3(1) & f_4(1) \\ f_1'(1) & f_2'(1) & f_3'(1) & f_4'(1) \end{vmatrix} = 0, \qquad (3.4.5)$$

which provides a complex eigenvalue relation between $\tilde{c}=c_r+ic_i$, $\tilde{\alpha}$ and $\tilde{R}$. We regard $\tilde{\alpha}$ and $\tilde{R}$ as given real quantities, and (3.4.5) can be resolved into two real simultaneous equations to determine c_r and c_i. If c_i is positive the disturbance grows exponentially, while if c_i negative the disturbance decays to zero, so the motion is stable. When $c_i=0$ we may eliminate c_r from the eigen-relations (3.4.5) to obtain a relationship between $\tilde{\alpha}$ and $\tilde{R}$. For a given value of $\tilde{R}$ the corresponding values of $\tilde{\alpha}$ represent neutral oscillations, and the curve of $\tilde{\alpha}$ against $\tilde{R}$ is called the neutral stability curve. The minimum value of $\tilde{R}$ on this curve is called the critical Reynolds number $\tilde{R}_c$, since for flows where $\tilde{R}<\tilde{R}_c$ all infinitesimal disturbances decay, and for flows where $\tilde{R}>\tilde{R}_c$ there are some unstable disturbances.

As discussed in section 1.2 we could have chosen to work with spatially growing waves and take the frequency $\widetilde{\alpha c}$ to be real instead of $\tilde{\alpha}$. From an analytical viewpoint it is more convenient for $\widetilde{\alpha c}$ to be complex than $\tilde{\alpha}$ since time derivatives appear in the equation of motion (3.2.8) in a much simpler manner than spatial derivatives. Further if we apply Squire's theorem to reduce the analysis of a three-dimensional disturbance to an equivalent two-dimensional case via the transformations (3.3.14) the corresponding two-dimensional flow will have a complex Reynolds number. Further consideration of spatially growing waves is given by Gaster (1965a,b) and Gaster and Davey (1968). Here, however, we shall only consider waves which grow in time.

The basic flow U is symmetric in x_2, so any solution of (3.4.1) may be split into an odd part, $f_o(x_2)$, and an even part, $f_e(x_2)$, and both f_o and f_e and their derivatives vanish separately at $x_2=\pm 1$. Thus both f_o and f_e are eigensolutions of (3.4.1) and it is sufficient to consider only odd and even functions in the eigenvalue problem. Lock (1955) has shown numerically that even solutions lead to greater instabilities than odd solutions, so in determining $\tilde{R}_c$ we need only consider even functions and restrict the eigenvalue problem to the intervals $0 \leq x_2 \leq 1$, with boundary conditions

$$\tilde{u}_2' = \tilde{u}_2''' = 0 \quad \text{at} \quad x_2=0 \qquad (3.4.6)$$

replacing the conditions at $x_2=-1$. We note that in much of the literature the interval $-1 \leq x_2 \leq 0$ is used instead of the upper half channel, though there is no difference in principle whichever is used.

As laminar flow begins to break down the Reynolds number is fairly large, at least of the order of 1000 if we take the experimental value, so for much of the channel we may neglect the viscous terms and replace (3.4.1) by the Rayleigh equation

$$(U-\tilde{c})(D^2-\tilde{\alpha}^2)\tilde{u}_2 - U''\tilde{u}_2 = 0, \qquad (3.4.7)$$

which may be written in the form

$$\tilde{u}_2'' - \frac{U''}{U-\tilde{c}}\tilde{u}_2 = \tilde{\alpha}^2\tilde{u}_2. \qquad (3.4.8)$$

The order of the equation has been reduced from four to two, and we can no longer impose all the boundary conditions at $x_2=0,1$. Applying iteratively the method of variation of parameters to (3.4.8) Heisenburg

(1924) obtained solutions as convergent series in $\tilde{\alpha}^2$, namely

$$f_e = (U-\tilde{c}) \sum_{0}^{\infty} \tilde{\alpha}^{2n} h_{2n}(x_2),$$
$$f_o = (U-\tilde{c}) \sum_{0}^{\infty} \tilde{\alpha}^{2n} k_{2n-1}(x_2), \tag{3.4.9}$$

where

$$\left.\begin{aligned} h_o &= 1; \quad h_{2n+2} = \int_0^{x_2} \frac{dx_2}{(U-\tilde{c})^2} \int_0^{x_2} (U-\tilde{c})^2 h_{2n}(x_2)dx_2, && n\geq 0; \\ k_1 &= \int_0^{x_2} \frac{dx_2}{(U-\tilde{c})^2}; \quad k_{2n+1} = \int_0^{x_2} \frac{dx_2}{(U-\tilde{c})^2} \int_0^{x_2} (U-\tilde{c})^2 k_{2n-1}(x_2)dx_2, && n\geq 1. \end{aligned}\right\} \tag{3.4.10}$$

For convenience we write y in place of x_2, and let y_o be the value of y at which $U-\tilde{c}$ vanishes. At $y=y_o$ the integrands in (3.4.10) are singular and to define the integrals we must fix a contour in the complex y-plane round the singular points. Heisenberg (1924) and Lin (1944 and 1967) discuss this difficulty and show that it is necessary to integrate along a contour with passes below $y=y_o$. Using this procedure Stuart (1954) obtains for $f_e(y)$ the approximate solution

$$\left.\begin{aligned} f_e &= (y_o^2-y^2)(1 + \tfrac{1}{10}\tilde{\alpha}^2 y^2) + \tfrac{4}{15} y_o^2\tilde{\alpha}^2 \left\{ y^2 + \tfrac{1}{2}(y_o^2-y^2)\log\left[\frac{y_o^2}{y_o^2-y^2}\right]\right\}, && y<y_o \\ f_e &= (y_o^2-y^2)(1 + \tfrac{1}{10}\tilde{\alpha}^2 y^2) + \tfrac{4}{15} y_o^2\tilde{\alpha}^2 \left\{ y^2 + \tfrac{1}{2}(y_o^2-y^2)\left[\log\frac{y_o^2}{y^2-y_o^2} + i\pi\right]\right\}, && y>y_o \end{aligned}\right\} \tag{3.4.11}$$

These two expressions have discontinuous derivatives at $y=y_o$, and to obtain a regular solution throughout the channel we must solve the full viscous equation (3.4.1) since the inviscid equation (3.4.8) breaks down at $y=y_o$.

The region close to $y=y_o$ is called the critical layer, for here viscosity plays an important part in determining the flow. To account fully for the action of viscosity we introduce a stretched coordinate

$$\eta = (y-y_o)/\varepsilon, \tag{3.4.12}$$

where ε is a small quantity defined by

$$\varepsilon = (U_o'\tilde{\alpha}\tilde{R})^{-1/3} \tag{3.4.13}$$

and U_o' represents the value of U' at $y=y_o$. We note that $U_o'<0$ in $0\leq x_2\leq 1$ so that ε is negative. We now expand $\tilde{u}_2$ in powers of ε:-

$$\tilde{u}_2 = \chi_o(\eta) + \varepsilon\,\chi_1(\eta) + \varepsilon^2\,\chi_2(\eta) + \dots . \tag{3.4.14}$$

The velocity U may be expressed in a similar way:-

$$U = \tilde{c} + U_o'\varepsilon\eta + \tfrac{1}{2}U_o''\varepsilon^2\eta^2 + \dots . \tag{3.4.15}$$

Substituting these expansions in (3.4.1), and equating coefficients of powers of ε we obtain

$$i\,\frac{d^4\chi_o}{d\eta^4} + \eta\,\frac{d^2\chi_o}{d\eta^2} = 0, \tag{3.4.16}$$

$$i\,\frac{d^4\chi_1}{d\eta^4} + \eta\,\frac{d^2\chi_1}{d\eta^2} = \frac{U_o''}{U_o'}\,\chi_o - \frac{1}{2}\,\frac{U_o''}{U_o'}\,\eta^2\,\frac{d^2\chi_o}{d\eta^2}, \tag{3.4.17}$$

and so on. (3.4.16) is the Airy equation if we regard $d^2\chi_o/d\eta^2$ as the independent variable. The four independent solutions to (3.4.16) are

$$\left.\begin{aligned}
\chi_o^{(1)} &= 1, \\
\chi_o^{(2)} &= \eta, \\
\chi_o^{(3)} &= \int_\infty^\eta d\eta \int_\infty^\eta \left\{\tfrac{2}{3}(i\eta)^{3/2}\right\}^{1/3} H_{1/3}^{(1)} \left[\tfrac{2}{3}(i\eta)^{3/2}\right] d\eta, \\
\chi_o^{(4)} &= \int_{-\infty}^\eta d\eta \int_{-\infty}^\eta \left\{\tfrac{2}{3}(i\eta)^{3/2}\right\}^{1/3} H_{1/3}^{(2)} \left[\tfrac{2}{3}(i\eta)^{3/2}\right] d\eta,
\end{aligned}\right\} \quad (3.4.18)$$

where $H_{1/3}^{(1)}$ and $H_{1/3}^{(2)}$ are Hankel functions of order 1/3. The first two solutions correspond to the first terms in the inviscid expansions (3.4.9). To obtain a function which matches (3.4.11) outside the critical layer and is regular across it we seek the next term in the expansion of the even solution $\chi^{(1)}$. First we set

$$\chi_{(1)}^{(1)} = \frac{1}{y_o} \bar{S}(\eta) \quad (3.4.19)$$

and substitute in (3.4.17) with $\chi_o = \chi_o^{(1)}$ to obtain

$$i\bar{S}'''' + \eta\bar{S}'' = 1. \quad (3.4.20)$$

Tollmien (1929) has shown that $\bar{S}(\eta)$ behaves like $\eta\log\eta$ for large positive η and as $\eta\log|\eta|-i\pi\eta$ for large negative η. If we express (3.4.11) in the form

$$\left.\begin{aligned}
f_e &= A_o+\varepsilon(A_1\eta+B_1\eta\log\eta)+O(\varepsilon^2), \quad \eta>0, \\
f_e &= A_o+\varepsilon\{A_1\eta+B_1(\eta\log|\eta|-i\pi\eta)\}+O(\varepsilon^2), \quad \eta<0,
\end{aligned}\right\} \quad (3.4.21)$$

we see that we can replace $\eta\log\eta$ or $\eta\log|\eta|-i\pi\eta$ as appropriate by $\bar{S}(\eta)$ to achieve a regular matching of (3.4.11) across the critical layer. We therefore take as one of the two independent even solutions of (3.4.1) the expressions (3.4.11) suitably modified by $\bar{S}(\eta)$ in the critical region, and let $f_e(y)$ now represent this solution.

The other even solution is obtained from a linear combination of $\chi^{(3)}$ and $\chi^{(4)}$. Use of asymptotic expansions of the Hankel functions shows that $\chi^{(4)}$ is large for large positive η, which corresponds to the centre of the channel. We therefore require only a very small multiple of $\chi^{(4)}$, and since it is small at the wall, it will have only a small effect on the eigenvalues, so in practice it is ignored. In the computation we shall take only the first term in the expansion of $\chi^{(3)}$, though Lock (1955) shows how to obtain the next term.

We can now represent the even solution of (3.4.1) by

$$\tilde{u}_2 = A f_e + B \chi_o^{(3)}, \tag{3.4.22}$$

where A and B are constants. Boundary conditions (3.4.4) yield

$$\left.\begin{aligned} A f_e + B\chi_o^{(3)} &= 0, \\ A \frac{df_e}{dy} + B \frac{d\chi_o^{(3)}}{dy} &= 0, \end{aligned}\right\} \text{ at } y=1 \tag{3.4.23}$$

Eliminating A and B and using (3.4.12) we have

$$\frac{\chi_o^{(3)}(\eta_1)}{\eta_1 \dfrac{d}{d\eta} \chi_o^{(3)}(\eta_1)} = \frac{f_e(1)}{(1-y_o) \dfrac{d}{dy} f_e(1)}, \tag{3.4.24}$$

where $\chi_o^{(3)}$ is expressed as a function of η, f_e as a function of y, and η_1 is the value of η at the wall. This equation reduces to two real simultaneous equations, which, for a given value of y_o, may be solved to obtain η_1 and $\tilde{\alpha}^2$, and, from (3.4.12) and (3.4.13), $\tilde{R}$. In the computation values of $\chi_o^{(3)}$, $\bar{S}$ and their derivatives were obtained from tables calculated by Holstein (1950). The eigenvalues $\tilde{\alpha}$, $\tilde{R}$ and y_o obtained in this way and corresponding eigensolutions of (3.4.1) are used in the perturbation analysis of section 3.6.

3.5 The adjoint function

In the analysis of the next section we shall require a function adjoint to $\tilde{u}_2$, which was obtained in section 3.4, and it is convenient to introduce it now. We consider the differential form

$$L(\phi) \equiv (D^2-\tilde{\alpha}^2)^2\phi - i\tilde{\alpha}\tilde{R}(U-\tilde{c})(D^2-\tilde{\alpha}^2)\phi - i\tilde{\alpha}\tilde{R}U''\phi, \qquad (3.5.1)$$

and the adjoint form, which we denote by $L^*(\phi^*)$, follows from the definition given by Ince (1944, p210), and is

$$L^*(\phi^*) \equiv (D^2-\tilde{\alpha}^2)\phi^* - i\tilde{\alpha}\tilde{R}(U-\tilde{c})(D^2-\tilde{\alpha}^2)\phi^* + 2i\tilde{\alpha}\tilde{R}U'D\phi^*. \qquad (3.5.2)$$

It now follows that

$$\int_0^1 \{\phi^*L(\phi)-\phi L^*(\phi^*)\}dy = \left[P(\phi,\phi^*)\right]_0^1, \qquad (3.5.3)$$

where

$$P(\phi,\phi^*) = \phi\left[\{2\tilde{\alpha}^2+i\tilde{\alpha}\tilde{R}(U-\tilde{c})\}D\phi^*+i\tilde{\alpha}\tilde{R}U'\phi^*-D^3\phi^*\right]$$
$$-\phi'\left[\{2\tilde{\alpha}^2+i\tilde{\alpha}\tilde{R}(U-\tilde{c})\}\phi^*-D^2\phi^*\right] - \phi''D\phi^* - \phi'''D\phi^*. \qquad (3.5.4)$$

In particular, if ϕ is a solution of $L(\phi)=0$ and ϕ^* is a solution of $L^*(\phi^*)=0$, then

$$\left[P(\phi,\phi^*)\right]_0^1 = 0. \qquad (3.5.5)$$

In our problem we first identify ϕ with the even solution $\tilde{u}_2$ of (3.4.1) which satisfies $\phi = D\phi = 0$ at $y=1$, and if we similarly restrict ϕ^* to be an even solution of the adjoint equation $L^*(\phi^*)=0$, we then have from (3.5.4) and (3.5.5)

$$\phi''D\phi^* = \phi'''\phi^* \text{ at } y=1. \qquad (3.5.6)$$

It was shown by Stuart (1960) that, if we denote the independent even solutions of (3.4.1) by ϕ_1 and ϕ_2, then the two independent even solutions of $L^*(\phi^*)=0$ are $(D^2-\tilde{\alpha}^2)\phi_1$ and $(D^2-\tilde{\alpha}^2)\phi_2$. Hence we can write ϕ^* as a linear combination of those functions, and we can formally satisfy one boundary condition on ϕ^* by suitable choice of the combination. It is most convenient to satisfy the condition $\phi^*=0$ at $y=1$, and we deduce at once from (3.5.6) that $D\phi^*=0$ at $y=1$. We cannot impose two arbitrary homogeneous boundary conditions on ϕ^* because the parameter $\tilde{c}$ has already been fixed in solving the eigenvalue problem (3.4.1) and (3.4.4), though we should comment that the adjoint problem has the same eigenvalues as the original problem provided we impose the boundary conditions $\phi^*=D\phi^*=0$ at $y=1$.

Following Michael (1964) we now identify ϕ with the even solution of the inhomogeneous Orr-Sommerfeld equation

$$L(\phi) \;=\; k(y); \quad \phi=D\phi=0 \quad \text{at} \quad y=1, \tag{3.5.7}$$

where $k(y)$ is an even function of y. With ϕ^* defined as in the previous paragraph, we still have the relation (3.5.5) holding, and hence

$$\int_0^1 \phi^* \, k(y)dy \;=\; 0. \tag{3.5.8}$$

We shall require this relation in the subsequent analysis.

3.6 Perturbations from the Newtonian case

As indicated earlier in the chapter we restrict the analysis for three-dimensional disturbances $(\beta\neq 0)$ to slightly viscoelastic fluids for which λ and μ are small. We express eigenfunctions, $\tilde{u}_2$, $\tilde{g}$, and eigenvalues $\tilde{\alpha}$, $\tilde{R}$ and $\tilde{c}$ in the form

$$f \;=\; f_o + \lambda f_1 + \theta(2\lambda+\mu)\, f_2 + \dots \, , \tag{3.6.1}$$

and neglect squares and products of λ and μ. The suffix zero refers to the solution for a Newtonian fluid. We now substitute these expressions in (3.3.18) to obtain from terms independent of λ and μ

$$\{(D^2-\tilde{\alpha}_o^2)^2 - i\tilde{\alpha}_o\tilde{R}_o(U-\tilde{c}_o)(D^2-\tilde{\alpha}_o^2) + i\tilde{\alpha}_o\tilde{R}_oU''\}\tilde{u}_{20} = 0, \qquad (3.6.2)$$

which is identical to (3.4.1). Terms proportional to λ and to $\theta(2\lambda+\mu)$ give the first order perturbation equations

$$\{(D^2-\tilde{\alpha}_o^2)^2 - i\tilde{\alpha}_o\tilde{R}_o(U-\tilde{c}_o)(D^2-\tilde{\alpha}_o^2) + i\tilde{\alpha}_o\tilde{R}_oU''\}\tilde{u}_{2n} = k_n(x_2) \qquad (3.6.3)$$

where $n=1,2$ refer to the components in (3.6.1) and

$$k_n(x_2) = N_{no} + [\{4\tilde{\alpha}_o\tilde{\alpha}_n+i(\tilde{\alpha}_n\tilde{R}_o+\tilde{\alpha}_o\tilde{R}_n)(U-\tilde{c}_o)-i\tilde{\alpha}_o\tilde{R}_o\tilde{c}_n\}(D^2-\tilde{\alpha}_o^2)$$
$$-2i\tilde{\alpha}_o^2\tilde{\alpha}_n\tilde{R}_o(U-\tilde{c}_o)-i(\tilde{\alpha}_o\tilde{R}_n+\tilde{\alpha}_n\tilde{R}_o)U'']\tilde{u}_{20}, \qquad (3.6.4)$$

N_{no} referring to the functions in (3.3.19) for a Newtonian fluid. To obtain the eigenvalues $\tilde{\alpha}_n$, $\tilde{R}_n$ and $\tilde{c}_n$ we use a method suggested by Stuart (1960) and Watson (1960, 1962) using the adjoint function ϕ^* obtained in the previous section. For the inhomogeneous Orr-Sommerfeld equation we have from (3.5.8)

$$\int_o^1 \phi^* k_n \, dx_2 = 0; \quad n=1,2. \qquad (3.6.5)$$

This is a complex equation which yields, for each value of n, two real equations linking $\tilde{\alpha}_n$, $\tilde{R}_n$ and $\tilde{c}_n$ which are all real on the neutral stability curve. To provide sufficient relations to determine the eigenvalues we must fix a direction for the perturbation of the Newtonian neutral curve by viscoelasticity in the form of a linear relation between $\tilde{\alpha}_n$, $\tilde{R}_n$ and $\tilde{c}_n$. We choose in this chapter to take perturbations at constant $\tilde{\alpha}$ (with both α and β constant), as this direction is approximately normal to the neutral stability curve in the region of greatest interest, namely near the critical Reynolds number.

The only unknown function involved in (3.6.5) is $\tilde{g}_o$, which appears in the term governing out-of-plane disturbances. Setting $\lambda=\mu=0$ in (3.3.9) and (3.3.11) we obtain

$$\{D^2-\tilde{\alpha}_o^2-i\tilde{\alpha}_o\tilde{R}_o(U-\tilde{c}_o)\}(\alpha_o u_{30}-\beta_o u_{10}) = -\tilde{\alpha}_o\tilde{R}_o U'\tilde{u}_{20}, \qquad (3.6.6)$$

which may be written

$$\{D^2-\tilde{\alpha}_o^2-i\tilde{\alpha}_o\tilde{R}_o(U-\tilde{c}_o)\}(\tilde{u}_{20}'+i\tilde{\alpha}_o^2\tilde{g}_o) = -i\tilde{\alpha}_o\tilde{R}_o U'\tilde{u}_{20} \qquad (3.6.7)$$

provided $\beta\neq0$. The boundary condition at the wall is $\tilde{g}_o=0$ at $x_2=1$. From (3.3.18) we deduce that, since $\tilde{u}_2$ and U are even functions of x_2, $\tilde{g}_o$ is necessarily odd, giving a second boundary condition $\tilde{g}_o=0$ at $x_2=0$. (3.6.7) is solved numerically for $f=\tilde{u}_{20}'+i\tilde{\alpha}_o^2\tilde{g}_o$ by representing

$$h^2D^2f(y) = f(y+h) - 2f(y) + f(y-h) + O(h^4), \qquad (3.6.8)$$

where $0\leq y\leq1$ is divided into N equal intervals of step-length h. This formula may be applied at $y=h, 2h, \ldots, (N-1)h$ and the resulting set of simultaneous equation may be solved by matrix methods, which will be discussed in greater detail in the next chapter. The boundary conditions give

$$f(0) = \tfrac{1}{2}f(1), \quad f(Nh) = 0, \qquad (3.6.9)$$

and finally $\tilde{g}_o$ is calculated from f.

3.7 Discussion of the results

The eigensolutions for a Newtonian fluid were obtained by the method outlined in section 4. The eigenvalues are given in table 3.1, and a typical example of $\tilde{u}_2$ is illustrated in figure 3.1. $\tilde{R}_1$ and $\tilde{R}_2$ are independent of β, but dependence of the Reynolds number of β may be obtained by using the transformations

$$\left.\begin{aligned} \tilde{R} &= \tilde{R}_o + \lambda\tilde{R}_1 + \theta(2\lambda+\mu)\tilde{R}_2, \\ \tilde{\tilde{\alpha R}} &= \alpha R. \end{aligned}\right\} \qquad (3.7.1)$$

The points labelled A-J in the table correspond to values obtained by Stuart (1955), but as they give little detail of the neutral stability curve in the critical region further points, labelled 1-5, were obtained using the method of section 3.4. With the eigenvalues and $\tilde{u}_{20}$ thus calculated the out-of-plane velocity component $\tilde{g}_o$ is obtained from (3.6.7). Step-lengths of $h=0.04$, 0.02, 0.01 and 0.005 were tried and there was little difference between values of $\tilde{g}_o$ for the last three values of h, so it was decided to use $h=0.01$ throughout the calculation. The behaviour of $\tilde{g}_o$ for several points on the neutral stability curve is shown in figures 3.2-3.5. The curves for points 1-5 are not shown, as they lie close to those for points A and J.

A result of particular interest obtained from the graphs is that $\tilde{g}_o' \neq 0$ at $x_2=1$. For a Newtonian fluid the first unstable disturbance to appear is in-plane with $u_3=\beta=0$, but Lockett (1969b) has shown that under certain conditions, which will be determined below, the first unstable disturbance in a viscoelastic fluid may be out-of-plane. The significance of this will be seen if we examine vorticity of the flow, whose components are given by

$$\left.\begin{aligned} \omega_1 &= (u_3'-i\beta u_2)\exp\{i(\alpha x_1+\beta x_3-\alpha ct)\}, \\ \omega_2 &= i(\beta u_1-\alpha u_3)\exp\{i(\alpha x_1+\beta x_3-\alpha ct)\}, \\ \omega_3 &= -U'+(i\alpha u_2-u_1')\exp\{i(\alpha x_1+\beta x_3-\alpha ct)\}. \end{aligned}\right\} \qquad (3.7.2)$$

For the first growing disturbance in a Newtonian fluid ω_1 and ω_2 vanish, whereas if the first perturbation in a viscoelastic fluid is

	E	D	C	B	A	1	2	3	4	5	J	I	H	G	F
$\tilde{R}_o$	44600	19400	10350	6320	5460	5300	5250	5250	5320	5700	7050	10150	21200	49600	140000
$\tilde{\alpha}_o$	0.58	0.68	0.80	0.94	1.02	1.04	1.06	1.08	1.10	1.14	1.18	1.17	1.10	1.00	0.86
y_o	0.93	0.91	0.89	0.87	0.86	0.858	0.857	0.856	0.855	0.855	0.86	0.87	0.89	0.91	0.93
$\tilde{R}_1 \times 10^{-7}$	-	3.099	1.543	1.114	1.305	1.369	1.593	2.009	2.610	8.310	- 9.025	- 7.173	-16.50	-	-
$\tilde{c}_1$	-	-55.6	-59.1	-73.9	-99.0	-107	-124	-152	-193	-546	455	243	179	-	-
$\tilde{R}_2 \times 10^{-6}$	-	-15.75	- 5.59	- 2.78	- 2.98	-2.51	-2.19	- 2.45	- 2.96	- 7.79	11.55	6.59	9.45	-	-
$\tilde{c}_2$	-	6.03	4.46	5.64	11.1	2.47	9.88	11.9	15.5	45.0	-70.9	-38.4	-28.9	-	-

Table 3.1

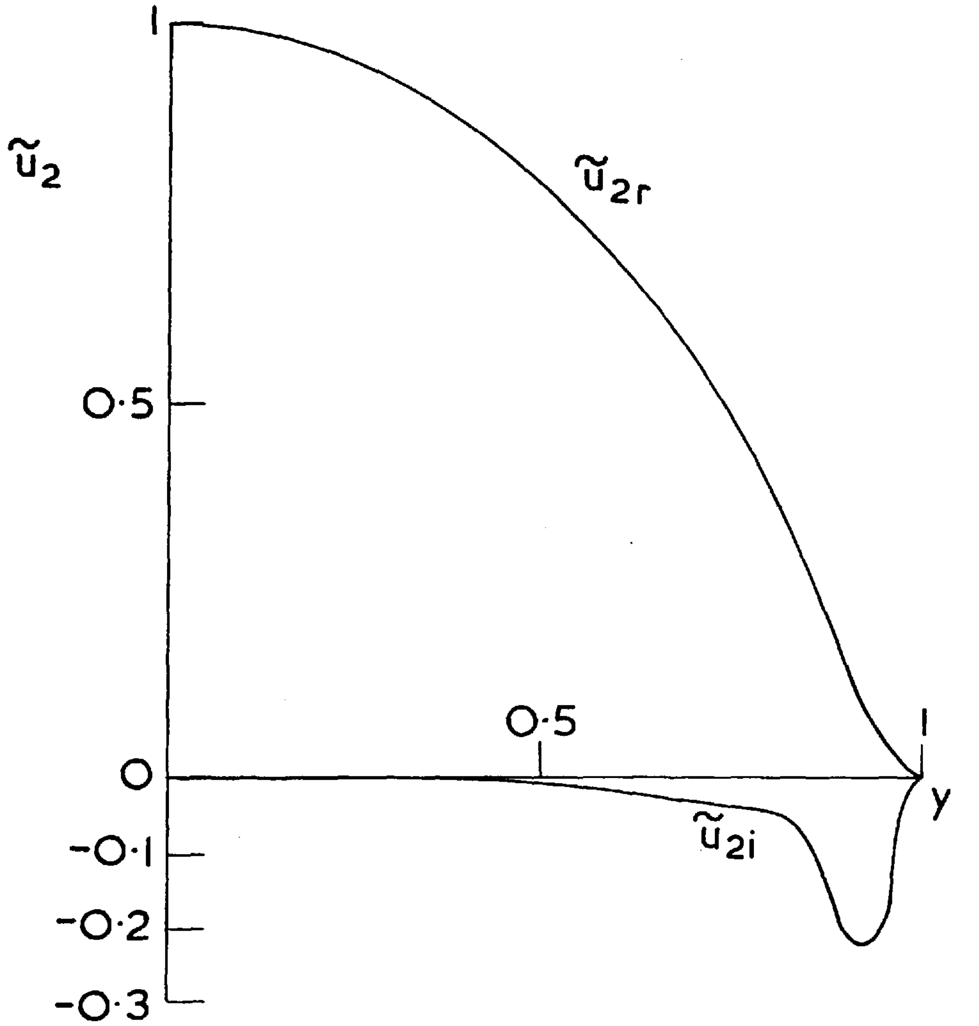


FIG. 3.1 A TYPICAL FORM OF THE FUNCTION $\tilde{u}_2$ CORRESPONDING TO POINT I

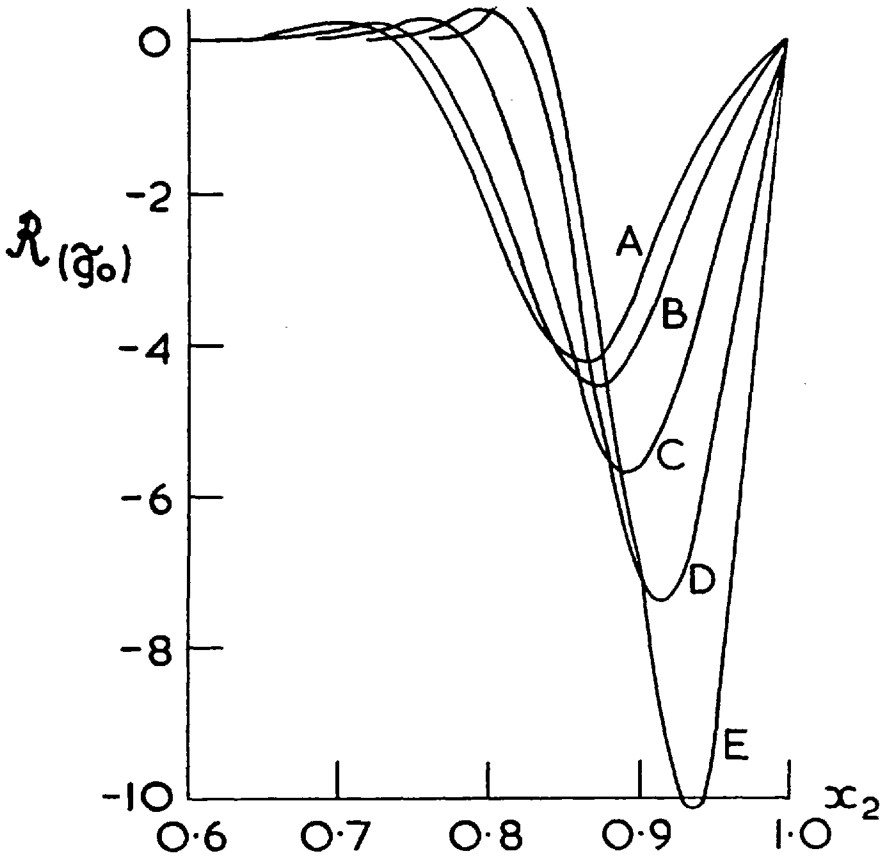

0
-2
-4
-6
-8
-10
ℛ(g̃o)
A
B
C
D
E
0·6
0·7
0·8
0·9
1·0
x2

FIG. 3.2

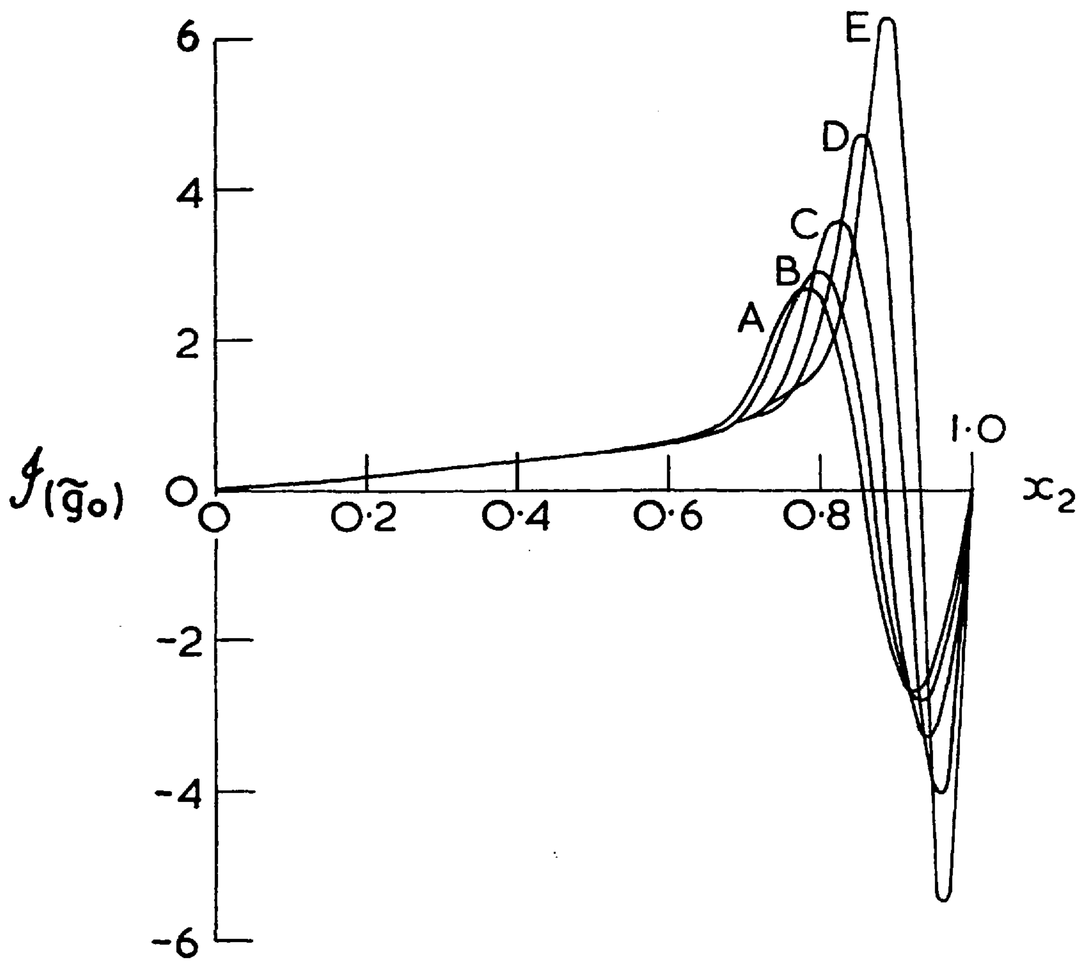
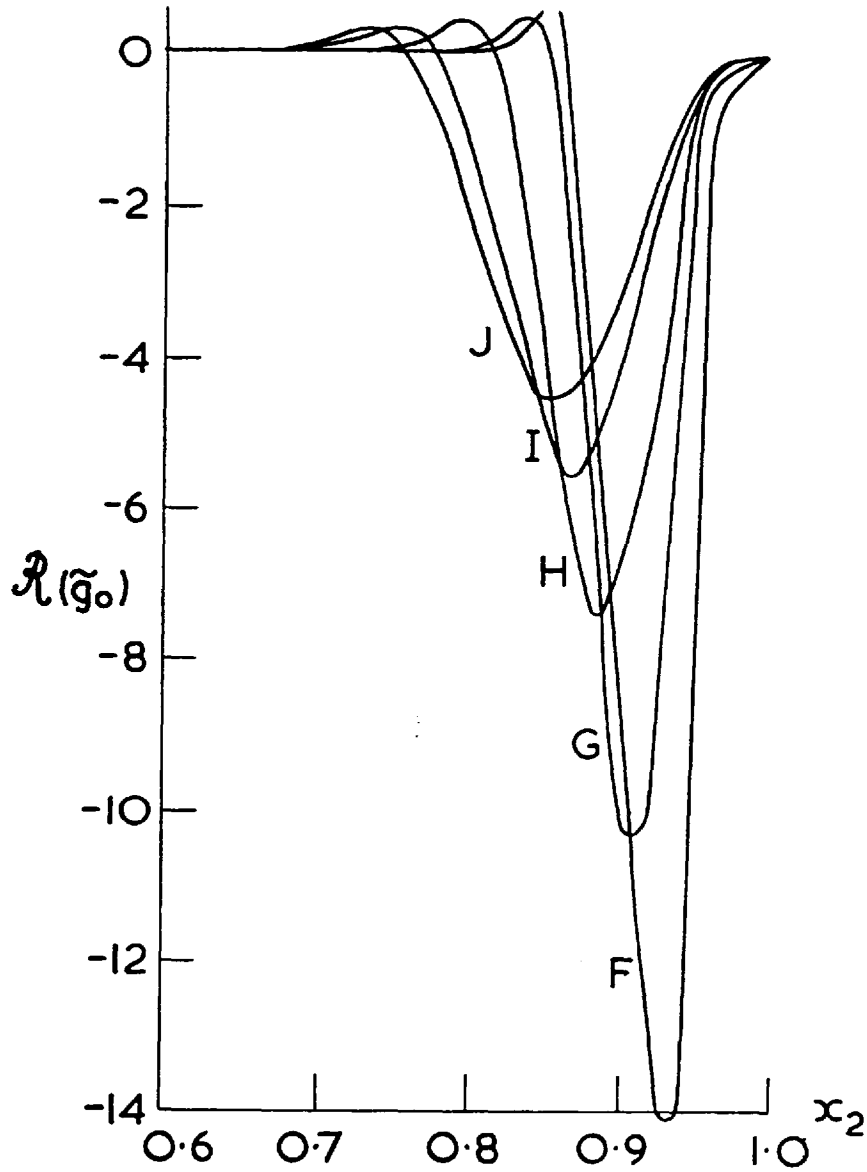
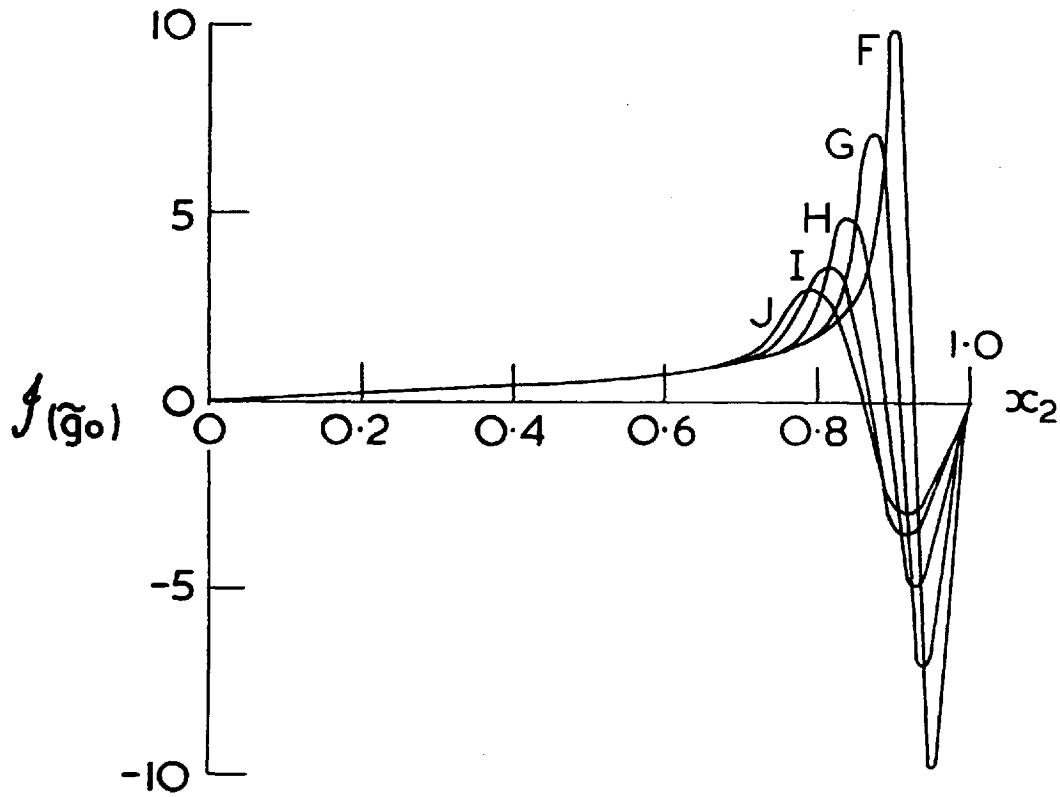

6
4
2
0
-2
-4
-6
ℐ(g̃o)
A
B
C
D
E
0
0·2
0·4
0·6
0·8
1·0
x2

FIG. 3.3



0
-2
-4
-6
-8
-10
-12
-14
J
I
H
G
F
R(g̃o)
0·6
0·7
0·8
0·9
1·0
x2

FIG. 3.4



10
5
0
-5
-10
F
G
H
I
J
I(g̃o)
0
0·2
0·4
0·6
0·8
1·0
x2

FIG. 3.5

out-of-plane ω_1 and ω_2 are non-zero, and, since $\tilde{g}_0'$ is non-zero at the walls, ω_1 is also non-zero there. Slight viscoelasticity may therefore introduce a longitudinal component of vorticity which persists in the region close to the wall; conditions under which this occurs are discussed in the next section. This vorticity component may play an important role in the subsequent development to turbulence, and the region near the wall is believed to play a major part in this transition. Lockett (1969b) has suggested this mechanism as a possible starting point for a theory to explain the drag-reduction properties of certain long-chain polymer solutions.

The adjoint function ϕ^* (corresponding to $\tilde{u}_{20}$ at each point on the neutral curve for which eigensolutions were calculated) was obtained from the two independent even solutions of (3.4.1) as indicated in section 3.5. Rather than calculate the required second derivatives numerically, which is an inaccurate process, they were obtained from direct differentiation of (3.4.11), $\zeta(\eta)$ and $\chi^{(3)}(\eta)$, using the tables of Holstein (1950) where necessary. A typical form of ϕ^* is shown in figure 3.6.

The perturbation terms $\tilde{R}_n$ and $\tilde{c}_n$ may now be calculated from (3.6.5), the integrals being evaluated by Simpson's rule. In table 3.1 the figures for points E,F and G were omitted as they were too large to be realistic; this results directly from seeking perturbations at constant $\tilde{\alpha}$. For large Reynolds numbers this direction is roughly parallel to the neutral stability curve and is therefore unsuitable. In the perturbed neutral curves, figures 3.7 and 3.8, there is some uncertainty in the region of maximum $\tilde{\alpha}$, and this again reflects the poor choice of direction in this area. Considering disturbances at constant $\tilde{\alpha}$ is, however, well suited to the critical region with which we are chiefly concerned. (Michael (1964) describes how to choose more

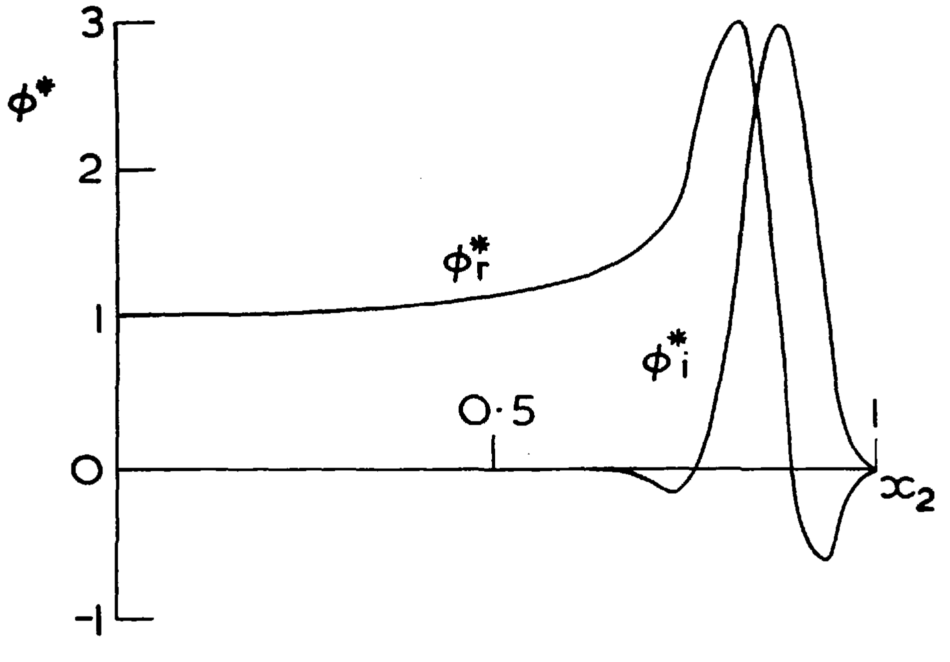


FIG.3.6 A TYPICAL FORM OF THE ADJOINT FUNCTION ϕ^* CORRESPONDING TO POINT I

appropriate directions for perturbations at other parts of the neutral stability curve). For certain values of λ and μ some of the larger coefficients $\tilde{R}_n$ and $\tilde{c}_n$ in table 1 may have to be discarded though the perturbation procedure is still valid at points close to the critical part of the curve. All the coefficients may be used for sufficiently small values of the viscoelastic parameters.

Figure 3.7 shows neutral stability curve for three different values of λ, when $\theta(2\lambda+\mu)=0$. The solid line refers to a Newtonian fluid. It is known from thermodynamic considerations that λ is negative (see Coleman and Markovitz, 1964), and the limited experimental evidence that exists tends to confirm this result. The graphs show that for negative values of λ the presence of viscoelasticity in the fluid is destabilising in the sense that it reduces the critical Reynolds number, in agreement with previous results (Chan Man Fong and Walters, 1965, Mook, 1967, Jones and Walters, 1968, Jones, 1967, and Schwarz and Chun, 1968). Although there is some discrepancy in the literature about the exact form of the perturbed curves, there is no doubt about the main result, that negative λ causes destabilisation. It will also be seen that the value of $\tilde{\alpha}$ at the critical point increases with $|\lambda|$, though it is difficult to determine this trend accurately. This uncertainty is reflected in the papers already cited, where $\tilde{\alpha}$ is found to increase with $|\lambda|$ by Chan Man Fong and Walters (method (ii)), Mook, and Schwarz and Chun, but to decrease by Chan Man Fong and Walters (method (i)), Jones, and Jones and Walters.

The effects of perturbations corresponding to the term in $\theta(2\lambda+\mu)$ are shown in figure 3.8, where the Newtonian case is again shown by a solid line. These curves are for constant values of θ. A plot of neutral stability curves at constant β is obtained via

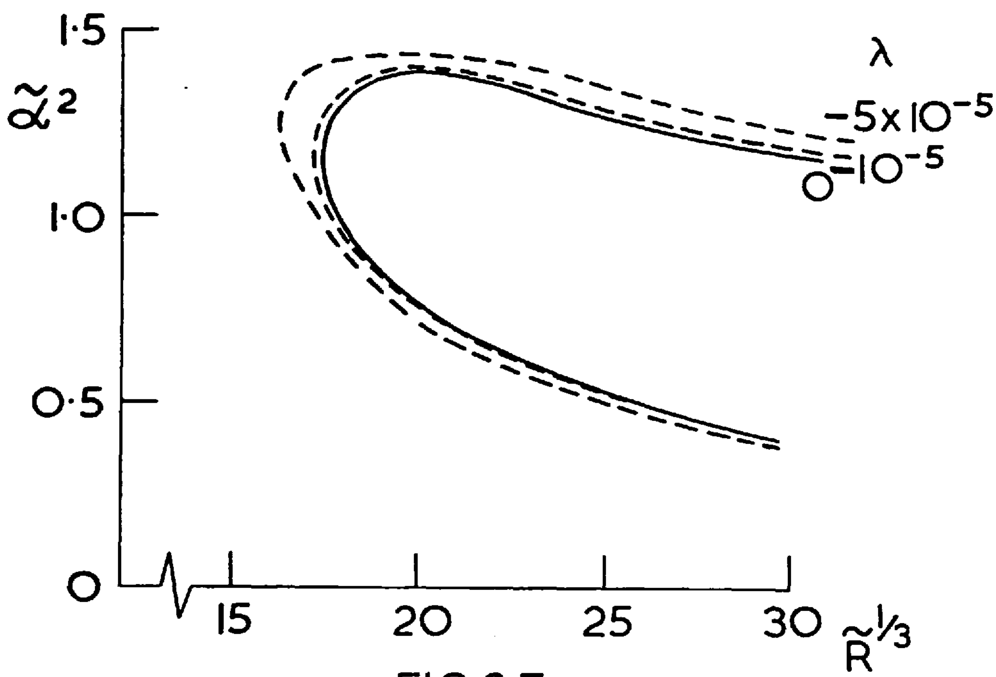

1·5
1·0
0·5
0
λ
−5x10−5
−10−5
0
15
20
25
30
R̃^1/3
α̃²

FIG.3.7

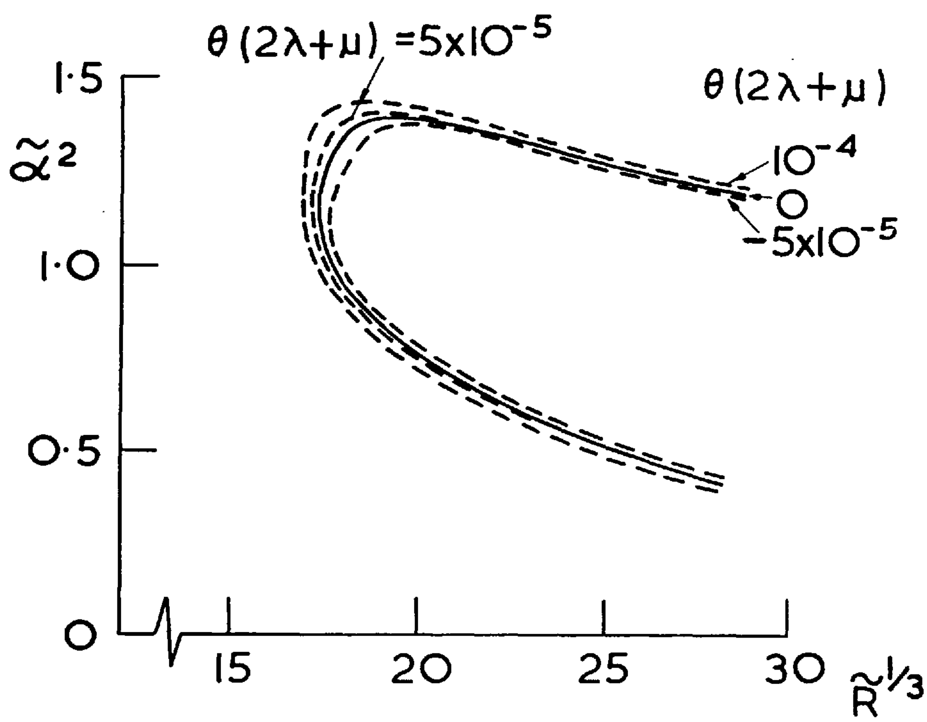

θ(2λ+μ) =5x10−5
θ(2λ+μ)
10−4
0
−5x10−5
1·5
1·0
0·5
0
15
20
25
30
R̃^1/3
α̃²

FIG.3.8

(3.7.1) and shown in figure 3.9. The parameter $2\lambda+\mu$ may be either positive or negative, the flow being destabilised for positive values or stabilised for negative values. Again, increasing values of the parameter cause the value of $\tilde{\alpha}$ at the critical point to increase. In a real fluid both viscoelastic parameters need to be considered, and in the linear perturbation theory of this chapter their separate effects can be added together. It is possible, with suitable choice of λ and μ, for the critical Reynolds number to be the same as that of a Newtonian fluid, though the neutral curves would only coincide at a few points.

Although the approximations made in section 3.4 in specifying $\tilde{u}_2$ are sufficiently good to determine whether the terms in λ and μ stabilise or destabilise the fluid, they do not enable us to determine the position of the neutral curve with any great accuracy. For instance, for $\lambda=\mu=0$, the present results give a critical Reynolds number of about 5240, in contrast to the usually quoted figure of 5780 obtained by Thomas (1953). In the next chapter, where it is far more important to specify the eigenfunctions accurately, a direct numerical approach to (3.4.1) is used, and better results are obtained, namely a value of 5774.4 for R_c and a corresponding wave number of $\alpha = 1.02024$, which compare well with the results of Porteous and Denn (1971) who used a Runge-Kutta method developed by Kaplan (1964) to obtain $R_c = 5775$, $\alpha = 1.0206$.

3.8 The possibility of a critical direction

The investigation so far has been in terms of eigenvalues defined by transformations (3.3.14), but to determine how the value of β/a affects the critical Reynolds number we must work with α and R directly. The neutral stability curve becomes a surface

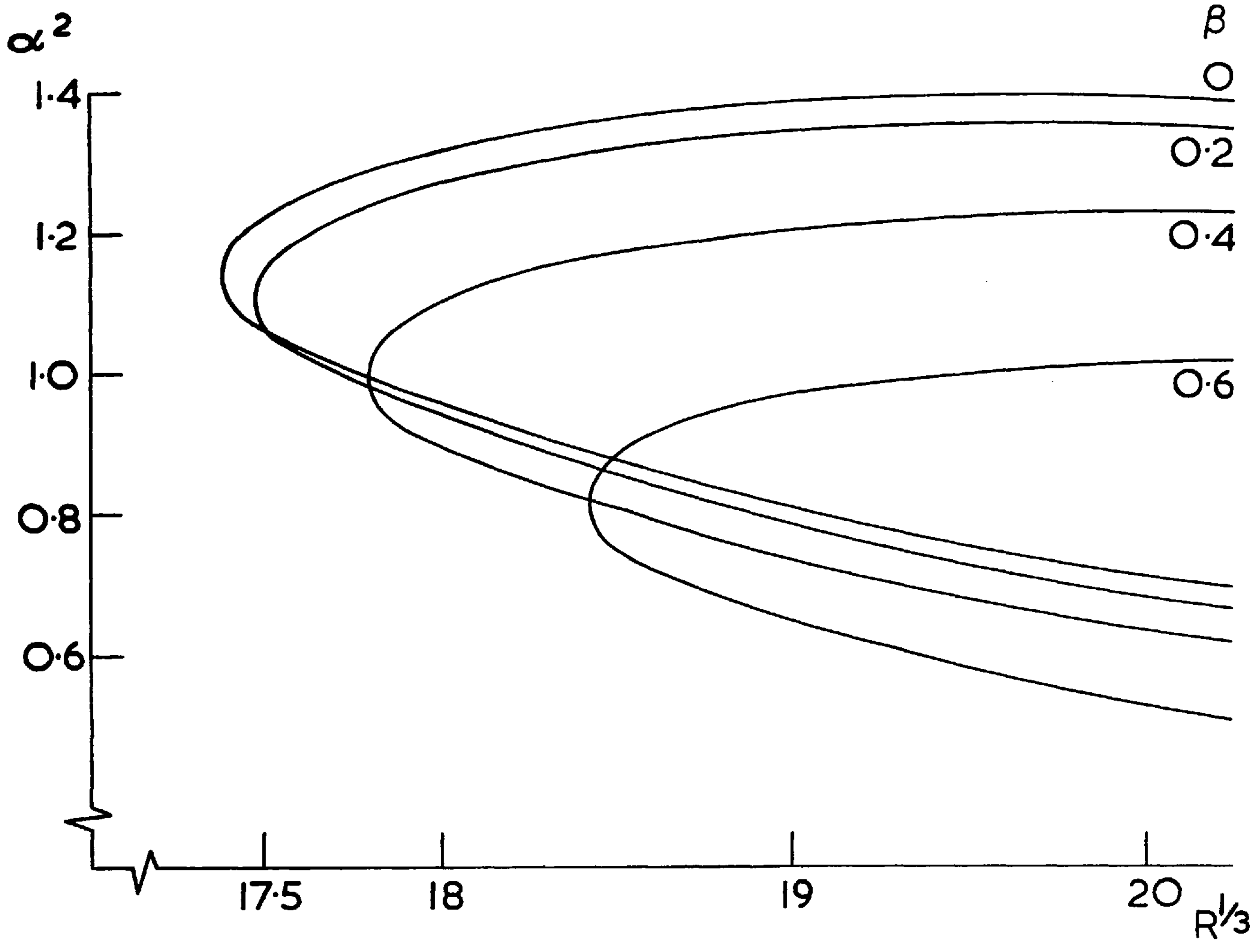


FIG. 3.9 NEUTRAL STABILITY CURVES FOR VARIOUS VALUES OF β. $\lambda=0$, $2\lambda+\mu=5\times10^{-5}$

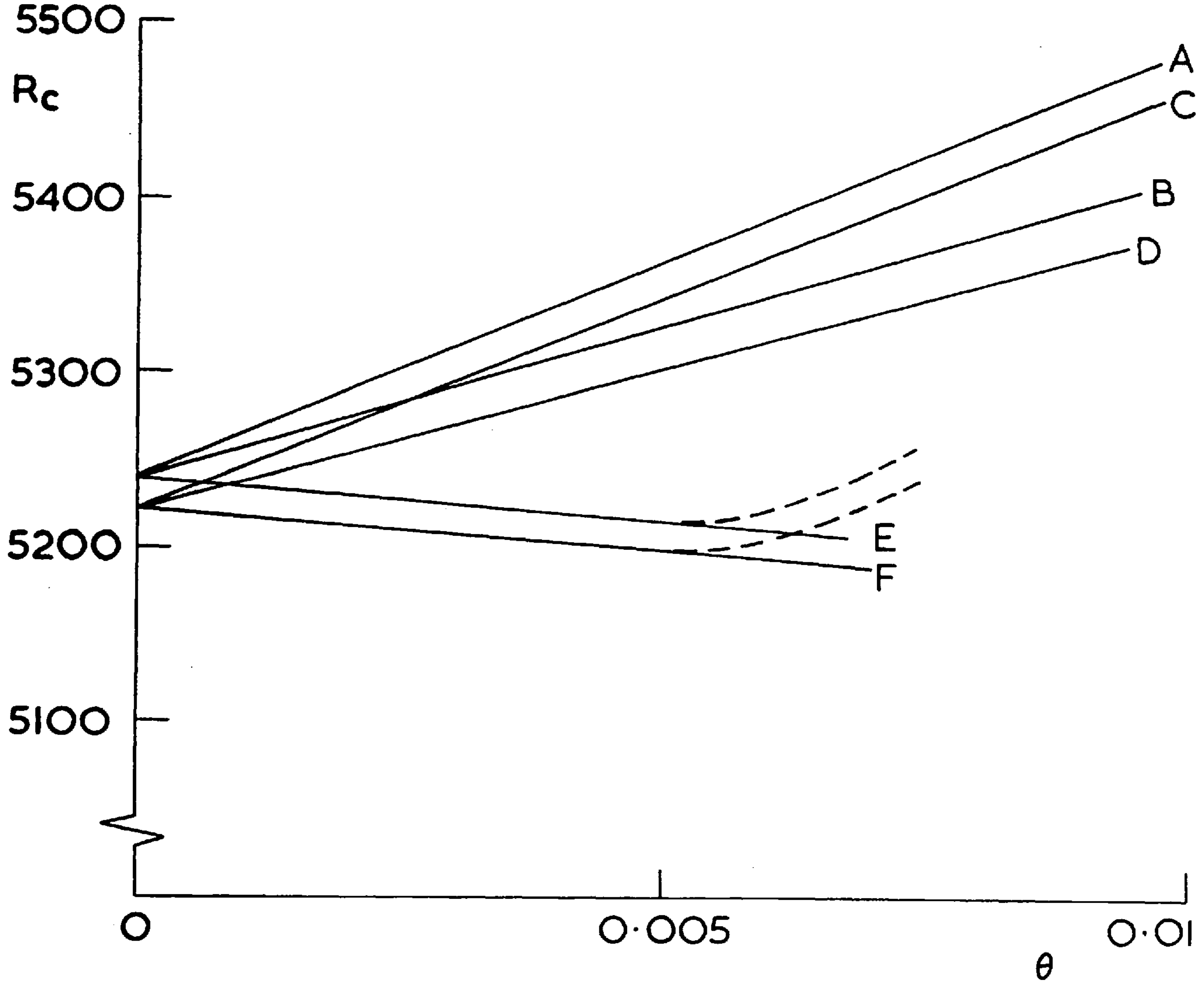
if we use as axes α, β and R. We may cut this surface with a plane parallel to the αR-plane and obtain a neutral curve and consequently a critical Reynolds number corresponding to a particular value of β. The values are obtained via (3.7.1). Variation of R_c with $\theta(=\beta^2/\alpha^2)$ is shown in figure 3.10. The curve for a Newtonian fluid is labelled A, and the effect of terms corresponding to $\lambda=-10^{-6}$ is shown by B. We observe that R_c increases with β, showing that the least stable disturbance occurs when $\beta=0$. We note also that the term in λ makes little difference to the slope of the curve, while perturbations corresponding to the $2\lambda+\mu$ term affect the gradient as indicated by C and D. For sufficiently large values of $2\lambda+\mu$ (>0.00106 approximately) the value of R_c decreases with θ as shown by E and F. The value 0.00106 is insensitive to variations in λ. Since we require the product $\theta(2\lambda+\mu)$ rather than the combination of parameters, $2\lambda+\mu$, to be small for the method of perturbing the Newtonian solution to be valid, we may choose μ as large as necessary, but remaining consistent with the second-order model of a fluid, though its application will be restricted to small values of θ. The physical significance of $2\lambda+\mu$ may be seen by considering a rectilinear shearing flow given by

$$v_1=Kx_2, \quad v_2=v_3=0, \tag{3.8.1}$$

where K is a constant. The second normal stress difference required to support this flow is, by (3.2.2)-(3.2.7),

$$S_{22}-S_{33} = (2\lambda+\mu)K^2. \tag{3.8.2}$$

Thus the type of behaviour exhibited in C-F is governed by the magnitude and sign of the second normal stress difference for the fluid in rectilinear shearing motion.



	λ	$2\lambda + \mu$
A	0	0
B	-10^{-6}	0
C	0	2×10^{-4}
D	-10^{-6}	2×10^{-4}
E	0	0·0012
F	-10^{-6}	0·0012

FIG.3.10

When $\alpha=0$ all disturbances of a Newtonian fluid are stable, and consequently the critical Reynolds number is infinite, as we shall now show. We write the velocity field in the form

$$\left.\begin{aligned} v_1 &= U + u_1(x_2)\exp\{i\beta(x_3-ct)\}, \\ v_2 &= u_2(x_2)\exp\{i\beta(x_3-ct)\}, \\ v_3 &= u_3(x_2)\exp\{i\beta(x_3-ct)\}. \end{aligned}\right\} \qquad (3.8.3)$$

Using (3.2.2) and (3.2.5) the resultant stress tensor for a Newtonian fluid has first order components

$$\hat{S} = \frac{1}{R}\begin{bmatrix} 0 & u_1' & i\beta u_1 \\ u_1' & 2u_2' & i\beta u_2+u_3' \\ i\beta u_1 & i\beta u_2+u_3' & 2i\beta u_3 \end{bmatrix}, \qquad (3.8.4)$$

and from the equations of motion (3.2.8) we obtain

$$\left.\begin{aligned} \hat{S}_{12}' + i\beta\,\hat{S}_{13} &= -i\beta c u_1 + U' u_2, \\ \hat{S}_{22}' + i\beta\,\hat{S}_{23} &= -i\beta c u_2 + \hat{p}', \\ \hat{S}_{23}' + i\beta\,\hat{S}_{33} &= -i\beta c u_3 + i\beta\hat{p}, \end{aligned}\right\} \qquad (3.8.5)$$

where the pressure perturbation is $\hat{p}\exp\{i\beta(x_3-ct)\}$. The continuity condition (3.2.9) gives

$$u_2'+i\,u_3 = 0, \qquad (3.8.6)$$

and this enables us to reduce (3.8.5) to

$$\left.\begin{aligned} (D^2-\beta^2)u_1 &= -i\beta R c u_1 + RU'u_2, \\ (D^2-\beta^2)u_2 &= -i\beta R c u_2 + R\hat{p}', \\ (D^2-\beta^2)u_3 &= -i\beta R c u_3 + i\beta R\hat{p}, \end{aligned}\right\} \qquad (3.8.7)$$

where D again denotes differentiation with respect to x_2. Eliminating p and u_3 the Orr-Sommerfeld equation becomes in the case $\alpha=0$

$$(D^2-\beta^2+i\beta cR)(D^2-\beta^2)u_2 = 0. \tag{3.8.8}$$

The two independent even solutions are

$$\left.\begin{aligned} f_1 &= \text{ch } \beta y, \\ f_2 &= \text{ch } ay \cos by + i\text{sh } ay \sin by, \end{aligned}\right\} \tag{3.8.9}$$

where a and b are real quantities satisfying

$$ab = -\tfrac{1}{2}\,\beta\, c_r R, \quad a^2-b^2 = \beta^2+\beta c_i R. \tag{3.8.10}$$

Boundary conditions $u_2=u_2'=0$ at $x_2=1$ lead to the eigenvalue relation

$$f_1 f_2' = f_2 f_1' \quad \text{at} \quad x_2=1, \tag{3.8.11}$$

which from (3.8.9) becomes,

$$\beta \text{th } \beta = \frac{a \text{ sh } a \cos b - b \text{ ch } a \sin b + i\, a \text{ ch } a \sin b + i\, b \text{ sh } a \cos b}{\text{ch } a \cos b + i \text{sh } a \sin b}. \tag{3.8.12}$$

The imaginary part of this equation is

$$0 = (a+b)\sin b \cos b \tag{3.8.13}$$

from which we deduce that

$$\beta^2 + \beta c_i R = 0, \tag{3.8.14}$$

showing that βc_i is negative for all values of R, and hence all disturbances of the form (3.8.3) are stable.

We have assumed that slight viscoelasticity makes only a small change in R_c, and we expect curves E and F to remain close to A and B for every value of θ. In particular $R_c \to \infty$ as $\theta \to \infty (\alpha \to 0)$ for both Newtonian and slightly viscoelastic fluids, so E and F will diverge from the straight lines shown in figure 3.10 in a manner indicated by the broken curves, and there will be a minimum value of R_c at some value of $\beta \neq 0$ for viscoelastic fluids. This is in marked contrast to Newtonian fluids where the minimum value of R_c is given by $\beta=0$. Linear perturbation theory leads to terms proportional to β^2, so R_c is linearly dependent on θ and hence there is no tendency for curves E and F to bend upwards. In the next chapter we shall examine non-linear effects, though the analysis does not lead to a critical direction along which the first (linearly) unstable wave would propagate. The first stage in performing a nonlinear calculation is to solve the linear Orr-Sommerfeld equation. As more accurate results are required than in this chapter for subsequent computation a finite difference method was used to obtain eigensolutions of the Orr-Sommerfeld equation. Details of the method and results which are applicable to linear theory will be found in sections 4.9 and 4.10.

CHAPTER 4: STABILITY OF PLANE POISEUILLE FLOW OF A SECOND-ORDER FLUID: NON-LINEAR ANALYSIS

4.1 Non-linear interactions

In the previous chapter conditions were obtained under which disturbances first become unstable. Linear theory predicts that such disturbances grow exponentially until they are sufficiently large for the transport of momentum by the finite fluctuations to be considerable, and the associated mean stress, the Reynolds stress, has an appreciable effect on the mean flow. This distortion of the mean flow modifies the rate of transfer of energy to the disturbance. As this energy transfer is the cause of the growth of the disturbance the rate of growth is itself modified. It is possible that there may exist an equilibrium state in which the rate of transfer of energy from the modified mean flow to the disturbance is exactly balanced by the rate of viscous dissipation of the energy of the disturbance. 'Equilibrium' in this sense means that the oscillations have a steady finite amplitude.

The effect of the non-linear terms in the momentum equations is shown in three ways, (1) the generation of harmonics of the basic disturbance, (2) modification of the mean flow by the disturbance, and (3) modification of the fundamental. To see how these effects arise we shall consider for simplicity a disturbance whose x-dependence is $\exp(i\alpha x)$, where α is a wave number. In the non-linear terms the product of the fundamental with itself immediately introduces the harmonic proportional to $\exp(2i\alpha x)$, which in turn interacts with the fundamental and with itself to generate higher harmonics, and so on. In physical terms we are dealing with real quantities, so corresponding to $\exp(i\alpha x)$ we must introduce a term proportional to its complex conjugate $\exp(-i\alpha x)$. The product of these expressions gives a term independent of x which represents the modification of the mean flow by the fundamental. The mean flow is modified by the harmonics and their conjugates in a similar way. The product of terms in $\exp(2i\alpha x)$ and $\exp(-i\alpha x)$ gives a term in $\exp(i\alpha x)$, and so represents a modification of the fundamental. The argument above suggests that we ought to Fourier analyse the velocity components in x.

4.2 Fourier analysis of the flow

As in the previous chapter we shall examine the stability of Poiseuille flow in a channel bounded by planes $y=\pm 1$, where the undisturbed flow (3.3.2) is

$$(u, v, w) = (1-y^2, 0, 0), \qquad (4.2.1)$$

all quantities being expressed in non-dimensional form. This flow is caused by a uniform and constant pressure gradient in the x-direction. When a disturbance is present the arguments of the previous section suggests that we may express the velocity as a Fourier series in x, and similarly in z, namely

$$u=u_o + \sum_{m=1}^{\infty} (u_{mo}e^{im\alpha x} + \tilde{u}_{mo}e^{-im\alpha x}) + \sum_{n=1}^{\infty} (u_{on}e^{in\gamma z} + \tilde{u}_{on}e^{-in\gamma z})$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (u_{mn}e^{im\alpha x+in\gamma z} + \tilde{u}_{mn}e^{-im\alpha x-in\gamma z} + u_{m-n}e^{im\alpha x-in\gamma z} + \tilde{u}_{m-n}e^{-im\alpha x+in\gamma z}), \qquad (4.2.2)$$

where u_o and u_{mn} are functions of y and t and the tilde denotes a complex conjugate. Similar expressions hold for v and w. In addition to the longitudinal pressure gradient there will exist pressure variations within the fluid arising from the Reynolds stresses of the non-linear interactions, so we must also express the pressure as a Fourier series:

$$p=xp_o(t)+p_1+\sum_{m=1}^{\infty}(p_{mo}e^{im\alpha x}+\tilde{p}_{mo}e^{-im\alpha x})+\sum_{n=1}^{\infty}(p_{on}e^{in\gamma z}+\tilde{p}_{on}e^{-in\gamma z})$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(p_{mn}e^{im\alpha x+in\gamma z}+\tilde{p}_{mn}e^{-im\alpha x-in\gamma z}+p_{m-n}e^{im\alpha x-in\gamma z}+\tilde{p}_{m-n}e^{-im\alpha x+in\gamma z}), \tag{4.2.3}$$

where p_1 and p_{mn} are functions of y and t. We note that the pressure gradient p_o will vary with time if the mass flux is held fixed. Pressure variations arising from Reynolds stresses are represented by p_1 and p_{mn}.

In suffix notation the momentum and continuity equations for a second-order fluid are

$$\frac{\partial u_i}{\partial t}+u_k u_{i,k}=-p_{,i}+\frac{1}{R}u_{i,jj}+\lambda\left(\frac{\partial u_{i,jj}}{\partial t}+u_k u_{i,kjj}\right)+(\lambda+\mu)(u_{i,k}u_{k,jj}+2u_{j,k}u_{i,jk})$$

$$+(2\lambda+\mu)(u_{k,i}u_{k,jj}+u_{j,k}u_{j,ik}+u_{j,k}u_{k,ij}), \tag{4.2.4}$$

$$u_{i,i}=0, \tag{4.2.5}$$

where $,j$ denotes differentiation with respect to x_j, and the viscoelastic parameters are λ and μ. We substitute (4.2.2) and (4.2.3) into these equations and separate similar harmonic components. We note immediately from (4.2.5) that

$$v_o'=0, \tag{4.2.6}$$

where the prime denotes differentiation with respect to y. There is no flow normal to the solid boundaries $y=\pm1$, so we deduce from (4.2.6) that v_o vanishes everywhere. We shall also use the operator D to denote differentiation with respect to y, and define

$$E_{mn} \equiv D^2 - m^2\alpha^2 - n^2\gamma^2. \qquad (4.2.7)$$

U_{mn}, V_{mn}, W_{mn} will be used to represent terms arising from non-linear interactions. We then obtain the following system of equations

$$\frac{\partial u_o}{\partial t} + p_o - \frac{1}{R} u_o'' - \lambda \frac{\partial u_o''}{\partial t} = U_{oo}, \qquad (4.2.8)$$

$$p_1' = 2(2\lambda+\mu)(u_o'u_o''+w_o'w_o'')+V_{oo}, \qquad (4.2.9)$$

$$\frac{\partial w_o}{\partial t} - \frac{1}{R} w_o'' - \lambda \frac{\partial w_o''}{\partial t} = W_{oo}, \qquad (4.2.10)$$

$$\left\{\frac{\partial}{\partial t} + k_{mn} - \frac{1}{R} E_{mn} - \lambda\left(\frac{\partial}{\partial t} + k\right)E_{mn} - (\lambda+\mu)(k_{mn}''+2k_{mn}'D)\right\} \begin{bmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \end{bmatrix}$$

$$+ \begin{bmatrix} im\alpha \\ D \\ in\gamma \end{bmatrix} \{p_{mn}-(2\lambda+\mu)(k_{mn}'v_{mn}+u_o''u_{mn}+u_o'u_{mn}'+w_o''w_{mn}+w_o'w_{mn}')\}$$

$$+ \begin{bmatrix} u_o'-\lambda u_o'''-(\lambda+\mu)(u_o'E_{mn}+2u_o''D) \\ 0 \\ w_o'-\lambda w_o'''-(\lambda+\mu)(w_o'E_{mn}+2w_o''D) \end{bmatrix} v_{mn} - (2\lambda+\mu) \begin{bmatrix} 0 \\ (u_o'E_{mn}-u_o''')u_{mn}+(w_o'E_{mn}-w_o''')w_{mo} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{bmatrix}, \qquad (4.2.11)$$

$$im\alpha u_{mn} + v_{mn}' + in\gamma w_{mn} = 0, \qquad (4.2.12)$$

where (4.2.11) is written in matrix form and

$$k_{mn} = im\alpha u_o + in\gamma w_o . \qquad (4.2.13)$$

If we write

$$h_{mn} = in\gamma u_o - im\alpha w_o , \qquad (4.2.14)$$

$$s_{mn} = in\gamma u_{mn} - im\alpha w_{mn} , \qquad (4.2.15)$$

and eliminate pressure terms from (4.2.11) using (4.2.12) we obtain in matrix form

$$\begin{bmatrix} \left(\frac{\partial}{\partial t} + k_{mn} - \frac{1}{R}E_{mn}\right)E_{mn} - k''_{mn} - \lambda\left\{\left(\frac{\partial}{\partial t} + k\right)E^2_{mn} - k^{(4)}_{mn}\right\} & -(2\lambda+\mu)(h'_{mn}E_{mn} - h'''_{mn}) \\ h'_{mn} - \lambda h''''_{mn} - (\lambda+\mu)(h'_{mn}E_{mn} + 2h''_{mn}D) & \frac{\partial}{\partial t} + k_{mn} - \frac{1}{R}E_{mn} - \lambda\left(\frac{\partial}{\partial t} + k_{mn}\right)E_{mn} - (\lambda+\mu)(k''_{mn} + 2k'_{mn}D) \end{bmatrix} \begin{bmatrix} v_{mn} \\ s_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} -im\alpha U'_{mn} - in\gamma W'_{mn} - (m^2\alpha^2 + n^2\gamma^2)V_{mn} \\ in\gamma U_{mn} - im\alpha W_{mn} \end{bmatrix} . \qquad (4.2.16)$$

The linear theory developed in the previous chapter is recovered by putting $m=n=1$ in (4.2.16) and neglecting non-linear terms. We see from (4.2.10) that w_o arises from non-linear interactions and so we may neglect it in the linear theory. Equation (4.2.16) then reduces to (3.3.18).

The mean flow equation (4.2.8) represents the distorted mean motion in the x-direction averaged with respect to x and z. Although there is no basic flow in the z-direction a flow parallel to the z-axis will be generated through Reynolds stresses in the fluid, and this flow is given by (4.2.10). The disturbance may be regarded as being composed of a number of fundamental waves, and their harmonics generated by non-linear interactions. In the next section we shall introduce a particular choice of fundamental waves. The fundamentals and harmonics are described by (4.2.16) for particular values of m and n.

4.3 Fundamental disturbances

We choose as fundamental disturbances three waves:-

$$\underset{\sim}{u}_{10}e^{i\alpha x}\ ,\ \underset{\sim}{u}_{11}e^{i\alpha x+i\gamma z}\ ,\ \underset{\sim}{u}_{1-1}e^{i\alpha x-i\gamma z}\ . \qquad (4.3.1)$$

These particular waves were chosen to cover several different possibilities. By setting $\underset{\sim}{u}_{11} = \underset{\sim}{u}_{1-1} = 0$ we have the interaction of a two-dimensional wave with itself. With $\underset{\sim}{u}_{10}=0$ and either $\underset{\sim}{u}_{11}=0$ or $\underset{\sim}{u}_{1-1}=0$ we may examine the self-interaction of a three-dimensional disturbance. Linear theory predicts that the first unstable disturbance may be either two- or three-dimensional and these special cases of (4.3.1) enable us to trace the subsequent development of this disturbance. In some circumstances there may be more than one periodic disturbance present, and (4.3.1) allows us to consider the interaction of two- and three-dimensional waves ($\underset{\sim}{u}_{1-1}=0$) or of two oblique waves ($u_{10}=0$). The case examined by Stuart (1962) is obtained when $\underset{\sim}{u}_{1-1}=\underset{\sim}{u}_{11}$, so that the three-dimensional disturbance is a standing wave in the z-direction.

It is possible to take $\underset{\sim}{u}_{01}\exp(i\gamma z)$ as a fundamental disturbance, but as linear theory does not predict a growing disturbance with $\alpha=0$ we have preferred to allow this wave to arise as a secondary effect due to non-linear interactions of the waves (4.3.1).

4.4 Finite amplitude expansions of the harmonics

The method of tackling (4.2.16) was developed by Stuart (1960) and Watson (1960), and in this section we extend the techniques used by Stuart (1962), converting the partial differential equations (4.2.16) into a sequence of ordinary differential equations which may be solved successively. Separation of variables is completed by introducing time-dependent amplitude functions, $A(t)$, $B(t)$ and $C(t)$, corresponding to the fundamental disturbances (4.3.1). Although these amplitudes are finite we assume that they are nevertheless small and seek expansions of the harmonics in powers of A, B and C. We now take as the fundamental waves

$$\left.\begin{aligned} &(\psi_{100}, \phi_{100}, \chi_{100})A(t)\exp(i\alpha x), \\ &(\psi_{110}, \phi_{110}, \chi_{110})B(t)\exp(i\alpha x+i\gamma z), \\ &(\psi_{1-10}, \phi_{1-10}, \chi_{1-10})C(t)\exp(i\alpha x-i\gamma z), \end{aligned}\right\} \qquad (4.4.1)$$

and their complex conjugates, where ψ, ϕ, χ are functions of y only. Functions $\underset{\sim}{u}_{10}$, $\underset{\sim}{u}_{11}$, $\underset{\sim}{u}_{1-1}$ in (4.3.1) contain the waves in (4.4.1) together with products and higher powers of the amplitude functions. Terms independent of x and z arise from products of harmonics with their conjugates. Terms proportional to $\exp(i\alpha x)$ come from products such as $AB\tilde{B}$, $\tilde{A}BC$, and the terms in $\exp(i\gamma z)$ involve $\tilde{A}B$, $A\tilde{C}$, etc. Proceeding in this way we obtain the following expansion scheme:-

$$u_o = f_o+|A|^2f_1+|B|^2f_2+|C|^2f_3+\ldots$$

$$w_o = |A|^2g_1+|B|^2g_2+|C|^2g_3+\ldots$$

$$u_{10} = A(\psi_{100}+|A|^2\psi_{101}+|B|^2\psi_{102}+|C|^2\psi_{103}+\ldots)+\tilde{A}BC\psi_{104}+\ldots$$

$$u_{11} = B(\psi_{110}+|A|^2\psi_{111}+|B|^2\psi_{112}+|C|^2\psi_{113}+\ldots)+A^2\tilde{C}\psi_{114}+\ldots$$

$$u_{1-1} = C(\psi_{1-10}+|A|^2\psi_{1-11}+|B|^2\psi_{1-12}+|C|^2\psi_{1-13}+\dots)+ A^2\tilde{B}\psi_{1-14}+\dots$$

$$u_{01} = \tilde{A}B\psi_{011}+A\tilde{C}\psi_{012}+\dots$$

$$u_{20} = A^2\psi_{201}+BC\psi_{202}+\dots$$

$$u_{02} = B\tilde{C}\psi_{020}+\dots$$

$$u_{21} = AB\psi_{210}+\dots$$

$$u_{2-1} = AC\psi_{2-10}+\dots$$

$$u_{22} = B^2\psi_{220}+\dots$$

$$u_{2-2} = C^2\psi_{2-20}+\dots \;, \qquad (4.4.2)$$

where ψ_{pqr}, f_j, g_j are functions of y only. Similar expressions hold for v_{mn}, w_{mn} and s_{mn} with ϕ, χ and σ replacing ψ respectively. We shall show later that it is consistent to work to third order in amplitude, and the expansions above have been written to this order in anticipation.

Under linear theory, the amplitude $A(t)$ satisfies

$$\frac{dA}{dt} = -i\alpha cA, \qquad (4.4.3)$$

which yields the exponential behaviour. The function A is associated with spatial dependence proportional to $\exp(i\alpha x)$, and in the non-linear extension it is consistent to add to the right hand side of (4.4.3) those products of the amplitudes that are also associated with $\exp(i\alpha x)$. We therefore write

$$\frac{dA}{dt} = A(a_o+a_1|A|^2+a_2|B|^2+a_3|C|^2+\dots)+a_4\tilde{A}BC+\dots \;. \qquad (4.4.4)$$

In a similar way we obtain

$$\frac{dB}{dt} = B(b_o+b_1|A|^2+b_2|B|^2+b_3|C|^2+\ldots)+b_4A^2\tilde{C}+\ldots\ , \qquad (4.4.5)$$

$$\frac{dC}{dt} = C(c_o+c_1|A|^2+c_2|B|^2+c_3|C|^3+\ldots)+c_4A^2\tilde{B}+\ldots\ . \qquad (4.4.6)$$

The coefficients a_o, b_o and c_o are determined from linear theory, and it will be shown later how the other coefficients may be calculated.

To establish that it is possible to work to third order in amplitude let us consider the simplified system where B=C=0. The order of magnitude argument which is presented below does carry through for (4.4.4)-(4.4.6) though it is rather complicated. For the special case, (4.4.4) reduces to

$$\frac{dA}{dt} = A(a_o+a_1|A|^2+a_5|A|^4+\ldots). \qquad (4.4.7)$$

If we multiply through by $\tilde{A}$, take the complex conjugate and add the two equations we obtain

$$\frac{d|A|^2}{dt} = 2|A|^2(\alpha c_i+a_{1r}|A|^2+a_{5r}|A|^4)+\ldots\ , \qquad (4.4.8)$$

where c_i is the imaginary part of the wave speed obtained by linear theory, and a_{kr} is the real part of a_k. We shall be examining disturbances close to the neutral curve, and in this region $|c_i|$ is small, showing that $|A|^2$ is slowly varying. There exists the possibility of an equilibrium amplitude, when $|A|$ is constant, which is given by

$$0 = \alpha c_i + a_{1r}|A|_e^2 + a_{5r}|A|_e^4 + \dots , \qquad (4.4.9)$$

where e denotes the equilibrium amplitude. To a first approximation

$$|A|_e^2 = -\frac{\alpha c_i}{a_{1r}} . \qquad (4.4.10)$$

We shall later show that as $c_i \to 0$, a_{1r} in general remains finite and non-zero, showing that $|A|$ is of order $|c_i|^{\frac{1}{2}}$. Similarly a_{5r} remains of order unity as $c_i \to 0$. In the right hand side of (4.4.8) the first two terms are both of order $|c_i|^2$, while the third and subsequent terms are at least of order $|c_i|^3$. It is therefore consistent, for small amplitude disturbances, to retain only the first two terms in (4.4.8). The terms retained in (4.4.4) to (4.4.6) as written down are of the same order of magnitude, and to this order equations (4.4.2) are consistent with amplitude equations as they stand.

It remains to express the non-linear terms in (4.2.16) in a way similar to (4.4.2) and we write

$$\begin{aligned}
U_{00} &= |A|^2U_{001}+|B|^2U_{002}+|C|^2U_{003}+\dots \\
U_{10} &= A(|A|^2U_{101}+|B|^2U_{102}+|C|^2U_{103}+\dots)+\tilde{A}BCU_{104}+\dots \\
U_{11} &= B(|A|^2U_{111}+|B|^2U_{112}+|C|^2U_{113}+\dots)+A^2\tilde{C}U_{114}+\dots \\
U_{1-1} &= C(|A|^2U_{1-11}+|B|^2U_{1-12}+|C|^2U_{1-13}+\dots)+A^2\tilde{B}U_{1-14}+\dots \\
U_{01} &= \tilde{A}BU_{011}+A\tilde{C}U_{012}+\dots \\
U_{20} &= A^2U_{201}+BCU_{202}+\dots \\
U_{02} &= B\tilde{C}U_{020}+\dots \\
U_{21} &= ABU_{210}+\dots
\end{aligned}$$

$$
\begin{aligned}
U_{2-1} &= ACU_{2-10}+\dots \\
U_{22} &= B^2U_{220}+\dots \\
U_{2-2} &= C^2U_{2-20}+\dots ,
\end{aligned} \tag{4.4.11}
$$

where U_{pqr} depend only on y. Similar expansions hold for V_{mn} and W_{mn}.

4.5 Equations governing the harmonics

The set of partial differential equations (4.2.8)-(4.2.10) and (4.2.16) may now be reduced to ordinary differential equations by substituting the expansions (4.4.2) and (4.4.11) into them, and separating the coefficients of products of A,B and C. We specify that in the mean motion equation (4.2.8), the pressure gradient is constant. Otherwise we might choose the mean motion to have a constant mass flux, in which case we should need to expand

$$
p_0(t) = p_{00} + p_{01}|A|^2 + p_{02}|B|^2 + p_{03}|C|^2 + \dots \tag{4.5.1}
$$

and retain the constants p_{on} in the equations below. These constants may be determined by integrating the mean flow across the channel. However, the condition of constant pressure gradient yields $p_{on}=0$ for $n\geq 1$.

To simplify the notation we define the following operators:-

$$
\left.
\begin{aligned}
L(a,\alpha,\gamma) &= \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \\
&\text{where} \\
L_{11} &= \{a+i\alpha f_o - \tfrac{1}{R}(D^2-\alpha^2-\gamma^2)\}(D^2-\alpha^2-\gamma^2) - i\alpha f_o'' - \lambda\{(a+i\alpha f_o)(D^2-\alpha^2-\gamma^2)^2 - i\alpha f_o^{(4)}\}, \\
L_{12} &= -(2\lambda+\mu)i\gamma\{f_o'(D^2-\alpha^2-\gamma^2) - f_o'''\}, \\
L_{21} &= i\gamma f_o' - \lambda i\gamma f_o''' - (\lambda+\mu)i\gamma\{f_o'(D^2-\alpha^2-\gamma^2) + 2f_o''D\}, \\
L_{22} &= a+i\alpha f_o - \tfrac{1}{R}(D^2-\alpha^2-\gamma^2) - \lambda(a+i\alpha f_o)D^2-\alpha^2-\gamma^2) - (\lambda+\mu)i\alpha(f_o''+2f_o'D),
\end{aligned}
\right\}
\tag{4.5.2}
$$

$$\left.\begin{aligned}
&\hat{L}(k,\theta_k,\alpha,\gamma) = \begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & L_{22} \end{bmatrix}, \\
&\text{where} \\
&\hat{L}_{11} = \{\theta_k + i\alpha f_k + i\gamma g_k - \frac{1}{R}(D^2-\alpha^2-\overset{2}{\gamma})\}(D^2-\alpha^2-\gamma^2) - i\alpha f_k'' - i\gamma g_k'' \\
&\qquad -\lambda\{(\theta_k+i\alpha f_k+i\gamma g_k)(D^2-\alpha^2-\gamma^2)^2 - i\alpha f_k^{(4)} - i\gamma g_k^{(4)}\}, \\
&\hat{L}_{12} = -(2\lambda+\mu)\{(i\gamma f_k'-i\alpha g_k')(D^2-\alpha^2-\gamma^2) - i\gamma f_k''' + i\alpha g_k''', \\
&\hat{L}_{21} = i\gamma f_k' - i\alpha g_k' - \lambda i\gamma f_k''' + \lambda i\alpha g_k''' - (\lambda+\mu)\{(i\gamma f_k' - i\alpha g_i')(D^2-\alpha^2-\gamma^2) + 2(i\gamma f_k'' - i\alpha g_k'')D\}, \\
&\hat{L}_{22} = \theta_k + i\alpha f_k + i\gamma g_k - \frac{1}{R}(D^2-\alpha^2-\gamma^2) - \lambda(\theta_k+i\alpha f_k+i\gamma g_k)(D^2-\alpha^2-\gamma^2) \\
&\qquad -(\lambda+\mu)\{i\alpha f_k'' + i\gamma g_k'' + (i\alpha f_k' + i\gamma g_k')D\},
\end{aligned}\right\} \tag{4.5.3}$$

and the function

$$M_{mnk} = \begin{bmatrix} -im\alpha U'_{mak} - in\gamma W'_{mnk} - (m^2\alpha^2+n^2\gamma^2)V_{mnk} \\ in\gamma U_{mnk} - im\alpha W_{mnk} \end{bmatrix}. \tag{4.5.4}$$

From the mean motion equations (2.8) and (2.10), we obtain

$$-\frac{1}{R} f_o'' + p_o = 0, \tag{4.5.5}$$

$$2a_{or}f_1 - \left(\frac{1}{R} + \lambda a_{or}\right) f_1'' = U_{001}, \tag{4.5.6}$$

$$2b_{or}f_2 - \left(\frac{1}{R} + \lambda b_{or}\right) f_2'' = U_{002}, \tag{4.5.7}$$

$$2c_{or}f_3 - \left(\frac{1}{R} + \lambda c_{or}\right) f_3'' = U_{003}, \tag{4.5.8}$$

$$2a_{or}g_1 - \left(\frac{1}{R} + \lambda a_{or}\right) g_1'' = W_{001}, \tag{4.5.9}$$

$$2b_{or}g_2 - \left(\frac{1}{R} + \lambda b_{or}\right) g_2'' = W_{002}, \tag{4.5.10}$$

$$2c_{or}g_3 - \left(\frac{1}{R} + \lambda c_{or}\right) g_3'' = W_{003}, \tag{4.5.11}$$

where a_{or}, b_{or} and c_{or} denote the real parts of a_o, b_o and c_o. (4.5.5) may be integrated immediately to give the undisturbed flow

$$f_o = 1-y^2, \tag{4.5.12}$$

so that the pressure gradient necessary to drive the flow is

$$p_o = -2/R. \tag{4.5.13}$$

We note from (4.5.5) that f_o''' and $f_o^{(4)}$ are both zero. For convenience let (P, Q) denote a column vector in the matrix equations which follow. From (4.2.16) coefficients of appropriate products of A, B and C yield the following system of equations:-

$$L(a_o,\alpha,0)(\phi_{100},\sigma_{100}) = 0, \tag{4.5.14}$$

$$L(b_o,\alpha,\gamma)(\phi_{110},\sigma_{110}) = 0, \tag{4.5.15}$$

$$L(c_o,\alpha,-\gamma)(\phi_{1-10},\sigma_{1-10}) = 0, \tag{4.5.16}$$

$$L(\tilde{a}_o+b_o,0,\gamma)(\phi_{011},\sigma_{011}) = M_{011}, \tag{4.5.17}$$

$$L(a_o+\tilde{c}_o,0,\gamma)(\phi_{012},\sigma_{012}) = M_{012}, \tag{4.5.18}$$

$$L(2a_o,2\alpha,0)(\phi_{201},\sigma_{201}) = M_{201}, \tag{4.5.19}$$

$$L(b_o+c_o,2\alpha,0)(\phi_{202},\sigma_{202}) = M_{202}, \tag{4.5.20}$$

$$L(b_o+\tilde{c}_o,0,2\gamma)(\phi_{020},\sigma_{020}) = M_{020}, \tag{4.5.21}$$

$$L(a_o+b_o,2\alpha,\gamma)(\phi_{210},\sigma_{210}) = M_{210}, \tag{4.5.22}$$

$$L(a_o+c_o,2\alpha,-\gamma)(\phi_{2-10},\sigma_{2-10}) = M_{2-10}, \tag{4.5.23}$$

$$L(2b_o,2\alpha,2\gamma)(\phi_{220},\sigma_{220}) = M_{220}, \tag{4.5.24}$$

$$L(2c_o,2\alpha,-2\gamma)(\phi_{2-20},\sigma_{2-20}) = M_{2-20}, \tag{4.5.25}$$

$$L(a_o+2a_{or},\alpha,0)(\phi_{101},\sigma_{101}) = \hat{L}(1,a_1,\alpha,0)(\phi_{100},\sigma_{100})+M_{101}, \tag{4.5.26}$$

$$L(a_o+2b_{or},\alpha,0)(\phi_{102},\sigma_{102}) = \hat{L}(2,a_2,\alpha,0)(\phi_{100},\sigma_{100})+M_{102}, \tag{4.5.27}$$

$$L(a_o+2c_{or},\alpha,0)(\phi_{103},\sigma_{103}) = \hat{L}(3,a_3,\alpha,0)(\phi_{100},\sigma_{100})+M_{103}, \tag{4.5.28}$$

$$L(\tilde{a}_o+b_o+c_o,\alpha,0)(\phi_{104},\sigma_{104}) = \hat{L}(4,a_4,\alpha,0)(\phi_{100},\sigma_{100})+M_{104}, \tag{4.5.29}$$

$$L(b_o+2a_{or},\alpha,\gamma)(\phi_{111},\sigma_{111}) = \hat{L}(1,b,\alpha,\gamma)(\phi_{110},\sigma_{110})+M_{111}, \tag{4.5.30}$$

$$L(b_o+2b_{or},\alpha,\gamma)(\phi_{112},\sigma_{112}) = \hat{L}(2,b_2,\alpha,\gamma)(\phi_{110},\sigma_{110})+M_{112}, \tag{4.5.31}$$

$$L(b_o+2c_{or},\alpha,\gamma)(\phi_{113},\sigma_{113}) = \hat{L}(3, b_3,\alpha,\gamma)(\phi_{110},\sigma_{110})+M_{113}, \tag{4.5.32}$$

$$L(2a_o+\tilde{c}_o,\alpha,\gamma)(\phi_{114},\sigma_{114}) = \hat{L}(4,b_4,\alpha,\gamma)(\phi_{110},\sigma_{110})+M_{114}, \tag{4.5.33}$$

$$L(c_o+2a_{or},\alpha,-\gamma)(\phi_{1-11},\sigma_{1-11}) = \hat{L}(1,c_1,\alpha,-\gamma)(\phi_{1-10},\sigma_{1-10})+M_{1-11}, \tag{4.5.34}$$

$$L(c_o+2b_{or},\alpha,-\gamma)(\phi_{1-12},\sigma_{1-12}) = \hat{L}(2,c_2,\alpha,-\gamma)(\phi_{1-10},\sigma_{1-10})+M_{1-12}, \tag{4.5.35}$$

$$L(c_o+2c_{or},\alpha,-\gamma)(\phi_{1-13},\sigma_{1-13}) = \hat{L}(3,c_3,\alpha,-\gamma)(\phi_{1-10},\sigma_{1-10})+M_{1-13}, \tag{4.5.36}$$

$$L(2a_o+\tilde{b}_o,\alpha,-\gamma)(\phi_{1-14},\sigma_{1-14}) = \hat{L}(4,c_4,\alpha,-\gamma)(\phi_{1-10},\sigma_{1-10})+M_{1-14}. \tag{4.5.37}$$

In (4.5.29), (4.5.33) and (4.5.37) we have introduced the functions f_4 and g_4 which are identically zero. The continuity equation (4.2.12) becomes

$$im\alpha\psi_{mnk} + \phi'_{mnk} + in\gamma\chi_{mnk} = 0 \tag{4.5.38}$$

for relevant values of m, n and k.

The boundary conditions to be applied to these equations are that the velocity components vanish on the walls $y=\pm 1$. (We have already used these conditions in obtaining (4.5.12)). By inspection of the operator L and equation (4.5.38) we see that it is possible, since each component of L is either wholly even or wholly odd, to examine separately the symmetric and antisymmetric parts of the disturbance. In chapter 3 we saw that waves whose x-component was antisymmetric about $y=0$ were less stable than symmetric waves, and in this chapter, too, we choose the fundamentals u_{10}, u_{11}, u_{1-1} to be odd functions of y and consider flow in the half-channel $0\leq y\leq 1$. Then v_{1n} will be an even function of y and w_{1n} odd for $n=0,1$ and -1. After examining the equations further we see that the boundary conditions become

$$\left.\begin{array}{lll} \text{at } y=1 & f_1=f_2=f_3=g_1=g_2=g_3=\psi_{mn}=\psi'_{mn}=\phi_{mn}=\chi_{mn}=\sigma_{mn}=0 & \\ \text{at } y=0 & \text{even function } f_k, g_k, \phi_{10k}, \phi_{11k}, \phi_{1-1k} \text{ for } k=0,\dots,4, & \\ & \psi_{011}, \psi_{012}, \psi_{020}, \psi_{201}, \psi_{202}, \psi_{210}, \psi_{2-10}, \psi_{220}, \psi_{2-20}, & \\ & \chi_{011}, \chi_{012}, \chi_{020}, \chi_{201}, \chi_{202}, \chi_{210}, \chi_{2-10}, \chi_{220}, \chi_{2-20}, & \\ & \text{odd functions } \psi_{10k}, \psi_{11k}, \psi_{1-1k}, \chi_{10k}, \chi_{11k}, \chi_{1-1,k} \text{ for } k=0,\dots,4, & \\ & \phi_{011}, \phi_{012}, \phi_{020}, \phi_{201}, \phi_{202}, \phi_{210}, \phi_{2-10}, \phi_{220}, \phi_{2-20}. & \end{array}\right\} \quad (4.5.39)$$

4.6 Method of solution

For given values of α and R equations (4.5.14)-(4.5.16) represent eigenvalue problems to determine the wave speeds of the fundamentals and the fundamentals themselves. In particular we note that (4.5.14) where γ is zero, decouples to give separate equations for ϕ_{100} and χ_{100}. The equation for ϕ_{100} is the eigen-problem which determines a_o, and is the equation obtained by linear analysis in the previous chapter. With a_o thus determined the equation for χ_{100} is satisfied only by

$$\chi_{100} = 0. \tag{4.6.1}$$

Having solved these eigen-problems, a_o, b_o and c_o and the fundamentals given by $(m,n,k) = (1,0,0),(1,\pm1,0)$ are known, so that the right hand sides of (4.5.6)-(4.5.11) and (4.5.17)-(4.5.25) are fully specified, and we may now solve these equations to obtain f_k, g_k, χ_{mnk}. At this stage the functions on the right hand side of (4.5.26)-(4.5.37) are known apart from the constants a_k, b_k and c_k. To obtain them we first introduce an operator adjoint to L:-

$$\left.\begin{aligned}
L^*(a,\alpha,\gamma) &= \begin{bmatrix} L^*_{11} & L^*_{12} \\ L^*_{21} & L^*_{22} \end{bmatrix}, \\
&\text{where} \\
L^*_{11} &= \{a+i\alpha f_o - \tfrac{1}{R}(D^2-\alpha^2-\gamma^2)\}(D^2-\alpha^2-\gamma^2)-\lambda(a+i\alpha f_o)(D^2-\alpha^2-\gamma^2)^2-4\lambda i\alpha f_o' D^3 \\
&\quad +2i\alpha f_o' D-6\lambda i\alpha f_o'' D^2+4\lambda i\alpha(\alpha^3+\gamma^2)f_o' D+2\lambda i\alpha(\alpha^2+\gamma^2)f_o'', \\
L^*_{12} &= i\gamma f_o'-(\lambda+\mu)i\gamma f_o'(D^2-\alpha^2-\gamma^2), \\
L^*_{21} &= -(2\lambda+\mu)i\gamma\{f_o'(D^2-\alpha^2-\gamma^2)+2f_o'' D\}, \\
L^*_{22} &= a+i\alpha f_o - \tfrac{1}{R}(D^2-\alpha^2-\gamma^2)+\mu i\alpha(2f_o' D+f_o'')-\lambda(a+i\alpha f_o)(D^2-\alpha^2-\gamma^2),
\end{aligned}\right\} \tag{4.6.2}$$

and adjoint functions ϕ^*, σ^* which satisfy

$$L^*(\phi^*,\sigma^*)=0; \quad \phi^*=\sigma^*=0 \text{ at } y=1; \quad \phi^* \text{ even, } \sigma^* \text{ odd at } y=0, \tag{4.6.3}$$

where corresponding suffices and functional dependence of the operators have been suppressed for simplicity. The value of $D\phi^*$ at $y=1$ is left unspecified for otherwise (4.6.3) would become an eigen-problem, and since the eigenvalues have already been determined in solving for the fundamentals the only solution to (4.6.3) would be $\phi^* = \sigma^* = 0$. However, we shall now prove from the definition of ϕ^* that $D\phi^* = 0$ at $y=1$. If we let ϕ and σ represent any solution of $L(\phi, \sigma) = 0$ subject to boundary conditions,

$$\phi=D\phi=\sigma=0 \quad \text{at} \quad y=1; \quad \phi \text{ even}, \quad \sigma \text{ odd} \quad \text{at} \quad y=0 \tag{4.6.4}$$

then

$$\begin{aligned}0 &= \int_0^1 \{ [\phi^*,\sigma^*]L(\phi,\sigma)-[\phi,\sigma]L^*(\phi^*,\sigma^*)\}dy \\ &= \Big[\Big\{\frac{1}{R} + \lambda(a+i\alpha f_o)\Big\} \{\sigma D\sigma^*-\sigma^*D\sigma+\phi D^3\phi^*-\phi D^3\phi^*+D\phi^*.D^2\phi-D\phi.D^2\phi^*+2(\alpha^2+\gamma^2)(\phi^*D\phi-\phi^*D\phi)\} \\ &\quad -(\lambda+2\mu)i\alpha f'_o\sigma\sigma^*+(2\lambda+\mu)i\gamma\{f'_o(\sigma D\phi^*-\phi^*D\sigma)+f''_o\sigma\phi^*\}+(\lambda+\mu)i\gamma\{f'_o(\phi D\sigma^*-\sigma^*D\phi)-f''_o\phi\sigma^*\} \\ &\quad +(a+i\alpha f_o)(\phi^*D\phi-\phi D\phi^*)-i\alpha f'_o\{(1-2\lambda(\alpha^2+\gamma^2))\phi\phi^*+\lambda(\phi^*D^2\phi-2D\phi.D\phi^*+3\phi D^2\phi^*)\} \\ &\quad +\lambda i\alpha f''_o(3\phi D\phi^*-\phi^*D\phi)\Big]_0^1\end{aligned} \tag{4.6.5}$$

where ϕ and σ are defined following equation (4.4.2). When we substitute the boundary conditions this reduces to

$$0 = \Big\{\frac{1}{R} + \lambda(a+i\alpha f_o)\Big\} D\phi^*.D^2\phi \quad \text{at} \quad y=1 \tag{4.6. }$$

In general $D^2\phi\neq0$, and we deduce the result

$$D\phi^*=0 \quad \text{at} \quad y=1. \tag{4.6.7}$$

Now functions ϕ and σ appearing in the left hand sides of (4.5.26)-(4.5.37) satisfy boundary conditions (4.6.4), from which, together with (4.6.7), it follows that for such ϕ and σ

$$\int_0^1 \{ [\phi^*,\sigma^*]L(\phi,\sigma)-[\phi,\sigma]L^*(\phi^*,\sigma^*)\}dy = 0. \tag{4.6.8}$$

Substituting from (4.5.26)-(4.5.37) this equation becomes

$$\int_0^1 [\phi^*,\sigma^*]\{\hat{L}(\phi,\sigma)+M\}dy = 0, \qquad (4.6.9)$$

where the suffices have again been suppressed. (4.6.9) may be written in the form

$$\theta_k \int_0^1 [\phi^*,\sigma^*]Fdy + \int_0^1 [\phi^*,\sigma^*]Gdy = 0, \qquad (4.6.10)$$

where θ_k is an unknown constant and F and G are functions that are known at this stage of the calculation. The adjoint functions appear linearly in (4.6.10), and as they are undefined to within an arbitrary scalar multiplier it is convenient to specify

$$\phi^*=1 \quad \text{at} \quad y=0. \qquad (4.6.11)$$

We shall show that in solving for the adjoint functions numerically we can take the known value of $\phi^*(0)$ to the right hand side of (4.6.3) expressed in finite difference form, and solve the resulting algebraic equations by normal matrix methods. In this way we may calculate the coefficients a_k, b_k and c_k without obtaining the unknown functions in (4.5.26)-(4.5.37).

There is, however, a difficulty that arises in determining the adjoint functions ϕ^*, σ^*. The argument above hinges on our ability to specify a non-zero solution of $L(\phi,\sigma)=0$, for otherwise ϕ^*,σ^* as defined above would necessarily vanish. For example, a_o+2a_{or}, α, 0, R do not form a system of eigenvalues of the operator $L(a_o+2a_{or},\alpha,0)$ in equation (4.5.26), and consequently there is no (non-zero) eigenfunction of this operator. Watson (1960) shows that we may overcome this difficulty by working to order $|a_{or}|$, which is small close to the neutral stability curve since it is proportional to the imaginary part of the wave speed determined by linear theory. Now $a_o,\alpha,0,R$ do form an eigenvalue system of operator L, and we therefore specify that ϕ^*,σ^* satisfy $L^*(a_o,\alpha,0)(\phi^*,\sigma^*)=0$, from which we can obtain ϕ^*,σ^* and hence a, to order $|a_{or}|$. Similar difficulties arise from equations (4.5.27), (4.5.28), (4.5.30)-(4.5.32), (4.5.34)-(4.5.36), but by ignoring terms of order $|a_{or}|$, $|b_{or}|$ and $|c_{or}|$ we can define the appropriate adjoint functions, and hence obtain the coefficients a_k, b_k, c_k for k=1,2,3. In this way we obtain solutions which remain regular for small values of $|a_{or}|$, $|b_{or}|$ and $|c_{or}|$. The work required to calculate the coefficients is considerably reduced as nine equations require only three different adjoint functions, namely $(\phi^*,\sigma^*)_{101}$, $(\phi^*,\sigma^*)_{111}$ and $(\phi^*,\sigma^*)_{1-11}$. We observe immediately from (4.6.2) that the last two are simply related, that is $(\phi^*,\sigma^*)_{1-11} = (\phi^*,-\sigma^*)_{111}$, since $b_o=c_o$ as we shall prove later.

Equations (4.5.29), (4.5.33) and (4.5.37) present a different problem as the coefficients in the left hand sides differ from those in (4.5.14)-(4.5.16) by large amounts. The adjoint functions defined by (4.6.3) are therefore necessarily zero, even to order $|a_{or}|$, and consequently a_4,b_4 and c_4 may take any value. As Stuart (1962) argues we may choose, in particular, that

$$a_4=b_4=c_4 = 0, \tag{4.6.12}$$

for even then the solutions of (4.5.29), (4.5.33) and (4.5.27) remain regular.

We note that (4.5.15) and (4.5.16) are essentially the same equations and we deduce that

$$b_o=c_o, \quad \phi_{110}=\phi_{1-10}, \quad \psi_{110}=\psi_{1-10}, \quad \chi_{110}=-\chi_{1-10} \tag{4.6.13}$$

The non-linear terms appearing in (4.5.6)-(4.5.11) and (4.5.17)-(4.5.26) were obtained using a computer program and the results thus obtained are given in the appendix. Using (4.6.1) and (4.6.13) we can deduce various relations between the non-linear terms, and consequently obtain several symmetries between the functions ψ_{mnk}, ϕ_{mnk}, χ_{mnk}, which are expressed in the table 4.1 below.

/ Table 4.1

relations between non-linear terms	applied to equations	gives relations between the harmonics
$W_{001} = 0$	5.9	$g_1 = 0$
$U_{003} = U_{002}$	5.7 , 5.8	$f_3 = f_2$
$W_{003} = -W_{002}$	5.10, 5.11	$g_3 = -g_2$
$U_{012} = \tilde{U}_{011}$, $V_{012} = \tilde{V}_{011}$, $W_{012} = -\tilde{W}_{011}$	5.17, 5.18	$\psi_{012} = \tilde{\psi}_{011}$, $\phi_{012} = \tilde{\phi}_{011}$, $\chi_{012} = -\tilde{\chi}_{011}$
$W_{201} = 0$	5.19	$\chi_{201} = 0$
$W_{202} = 0$	5.20	$\chi_{202} = 0$
$\tilde{U}_{020} = U_{020}$, $\tilde{V}_{020} = V_{020}$, $\tilde{W}_{020} = -W_{020}$	5.21	$\tilde{\psi}_{020} = \psi_{020}$, $\tilde{\phi}_{020} = \phi_{020}$, $\tilde{\chi}_{020} = -\chi_{020}$
$U_{2-10} = U_{210}$, $V_{2-10} = V_{210}$, $W_{2-10} = -W_{210}$	5.22, 5.23	$\psi_{2-10} = \psi_{210}$, $\phi_{2-10} = \phi_{210}$, $\chi_{2-10} = -\chi_{210}$
$U_{2-20} = U_{220}$, $V_{2-20} = V_{220}$, $W_{2-20} = -W_{220}$	5.24, 5.25	$\psi_{2-20} = \psi_{220}$, $\phi_{2-20} = \phi_{220}$, $\chi_{2-20} = -\chi_{220}$
$W_{101} = 0$	5.26	$\chi_{101} = 0$

Table 4.1

The non-linear functions in (4.5.27)-(4.5.37) may also be obtained using the computer program, but it is possible to deduce certain symmetries by inspection without evaluating the products themselves. For example M_{112} is obtained from the non-linear terms in (4.2.4) from products of functions with suffices $\sim 1, 1, 0$ and $2, 2, 0$, where the tilde denotes that the conjugate functions are used. Similarly M_{1-13} is obtained from products of functions with suffices $\sim 1, -1, 0$ and $2, -2, 0$. Now since

$$(\tilde{\psi},\tilde{\phi},\tilde{\chi})_{1-10} = (\tilde{\psi},\tilde{\phi},-\tilde{\chi})_{110}, \quad (\psi,\phi,\chi)_{2-20} = (\psi,\phi,-\chi)_{220}, \tag{4.6.14}$$

we can see from the non-linear terms in (4.2.4) that

$$(U,V,W)_{1-13} = (U,V,-W)_{112}. \tag{4.6.15}$$

We may further show that other non-linear functions $(U,V,W)_{mnk}$ are related as in (4.6.15) providing that both pairs of corresponding functions involved in the products forming $(U,V,W)_{mnk}$ are related as in (4.6.14). Using the results of table 4.1 we obtain in this way the results given in tables 4.2 and 4.3.

/ Tables 4.2 and 4.3

mnk	suffices of functions involved in the products forming $(U,V,W)_{mnk}$
1, 0, 2	~1, 1, 0; 2, 1, 0 and 1, 1, 0; ~0, 1, 1
1, 0, 3	~1, -1, 0; 2, -1, 0 and 1, -1, 0; 0, 1, 2
1, 1, 1	1, 0, 0; 0, 1, 1 and ~1, 0, 0; 2, 1, 0
1, -1, 1	1, 0, 0; ~0, 1, 2 and ~1, 0, 0; 2, -1, 0
1, 1, 2	~1, 1, 0; 2, 2, 0
1, -1, 3	~1, -1, 0; 2, -2, 0
1, -1, 2	1, 1, 0; ~0, 2, 0 and ~1, 1, 0; 2, 0, 2
1, 1, 3	1, -1, 0; 0, 2, 0 and ~1, -1, 0; 2, 0, 2
1, 1, 4	~1, -1, 0; 2, 0, 1 and 1, 0, 0; 0, 1, 2
1, -1, 4	~1, 1, 0; 2, 0, 1 and 1, 0, 0; ~0, 1, 1

Table 4.2

relations between non-linear terms	applied to equations	gives relations between coefficients	and relations between harmonics
$(U,V,W)_{103}=(U,V,-W)_{102}$	5.27, 5.28	$a_3=a_2$	$(\psi,\phi,\chi)_{103}=(\psi,\phi,-\chi)_{102}$
$(U,V,W)_{1-11}=(U,V,-W)_{111}$	5.30, 5.34	$c_1=b_1$	$(\psi,\phi,\chi)_{1-11}=(\psi,\phi,-\chi)_{111}$
$(U,V,W)_{1-13}=(U,V,-W)_{112}$	5.31, 5.36	$c_3=b_2$	$(\psi,\phi,\chi)_{1-13}=(\psi,\phi,-\chi)_{112}$
$(U,V,W)_{1-12}=(U,V,-W)_{113}$	5.32, 5.35	$c_2=b_3$	$(\psi,\phi,\chi)_{1-12}=(\psi,\phi,-\chi)_{113}$
$(U,V,W)_{1-14}=(U,V,-W)_{114}$	5.33, 5.37	$c_4=b_4$	$(\psi,\phi,\chi)_{1-14}=(\psi,\phi,-\chi)_{114}$

Table 4.3

4.7 Amplitudes of equilibrium states

The results of the previous section lead to considerable simplification of the amplitude equations (4.4.4)-(4.4.6), which reduce to

$$\frac{dA}{dt} = A(a_o+a_1|A|^2+a_2|B|^2+a_2|C|^2), \qquad (4.7.1)$$

$$\frac{dB}{dt} = B(b_o+b_1|A|^2+b_2|B|^2+b_3|C|^2), \qquad (4.7.2)$$

$$\frac{dC}{dt} = C(b_o+b_1|A|^2+b_3|B|^2+b_2|C|^2) \qquad (4.7.3)$$

to third order in the amplitudes. If we multiply these equations by $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ respectively, take complex conjugates and add the corresponding pairs of equations, we obtain

$$\frac{d|A|^2}{dt} = 2|A|^2(a_{or}+a_{1r}|A|^2+a_{2r}|B|^2+a_{2r}|C|^2), \qquad (4.7.4)$$

$$\frac{d|B|^2}{dt} = 2|B|^2(b_{or}+b_{1r}|A|^2+b_{2r}|B|^2+b_{3r}|C|^2), \qquad (4.7.5)$$

$$\frac{d|C|^2}{dt} = 2|C|^2(b_{or}+b_{1r}|A|^2+b_{3r}|B|^2+b_{2r}|C|^2). \qquad (4.7.6)$$

If equilibrium states are possible we obtain at once from (4.7.5) and (4.7.6) that

$$|B|=0 \quad \text{or} \quad |C|=0 \quad \text{or} \quad (b_{2r}-b_{3r})(|B|^2-|C|^2) = 0. \qquad (4.7.7)$$

If $b_{2r}=b_{3r}$ then (4.7.5) and (4.7.6) yield

$$\frac{d}{dt}\log|B|^2 = \frac{d}{dt}\log|C|^2, \quad \text{i.e.} \quad C = k|B| \qquad (4.7.8)$$

for all time, where k is a constant. We are now in a position to examine the possible equilibrium states.

1 A=B=C=0. This represents basic Poiseuille flow, which is stable to small disturbances if a_{or} and b_{or} are both negative.

2 B=C=0. The disturbance reduces to a single two-dimensional wave, which is examined in detail in the remaining sections of the chapter. The amplitude is given by

$$|A|^2 = \frac{Ka_{or}\exp(2a_{or}t)}{1-Ka_{1r}\exp(2a_{or}t)}, \qquad (4.7.9)$$

where K is a constant. For $a_{1r} < 0$ an equilibrium state exists when $a_{or} > 0$, and then

$$|A|^2 \to -a_{or}/a_{1r} \quad \text{as} \quad t\to\infty, \qquad (4.7.10)$$

while for $a_{1r} > 0$ equilibrium is possible when $a_{or} < 0$, and then

$$|A|^2 \to -a_{or}/a_{1r} \quad \text{as} \quad t\to-\infty. \qquad (4.7.11)$$

In the latter case instability arises if the initial amplitude lies above a threshold value given by $|A|^2=-a_{or}/a_{1r}$, for then

$$|A(t)|^2 \to \infty \quad \text{as} \quad t \to \frac{1}{2a_{or}} \log \frac{1}{Ka_{1r}} \qquad (4.7.12)$$

3 A=C=0. The disturbance reduces to a single three-dimensional wave, whose amplitude is given by

$$|B|^2 = \frac{Kb_{or}\exp(2b_{or}t)}{1-Kb_{2r}\exp(2b_{or}t)} . \qquad (4.7.13)$$

The equilibrium states are similar to those discussed above, when they exist

$$|B|^2 \to -b_{or}/b_{2r}. \qquad (4.7.14)$$

4 A=B=0. This is included for completeness. The analysis is identical to the last one.

5 $A=0$, $b_{2r}=b_{3r}$. This represents the interaction of two three-dimensional waves, and since (4.7.8) holds for all time the analysis proceeds as in case 2, with

$$|B|^2 = \frac{Kb_{or}\exp(2b_{or}t)}{1-Kb_{2r}(1+k^2)\exp(2b_{or}t)} , \quad |C|^2 = k^2|B|^2, \qquad (4.7.15)$$

where they exist the equilibrium amplitudes are given by

$$|B|^2 \to -b_{or}/b_{2r}(1+k^2), \quad |C|^2 \to -k^2b_{or}/b_{2r}(1+k^2). \qquad (4.7.16)$$

6 $A=0$, $|B|=|C|$. The analysis is again similar to that above, and

$$|B|^2 = |C|^2 = \frac{Kb_{or}\exp(2b_{or}t)}{1-K(b_{2r}+b_{3r})\exp(2b_{or}t)} , \qquad (4.7.17)$$

with equilibrium conditions given by

$$|B|^2 = |C|^2 \to -b_{or}/(b_{2r}+b_{3r}). \qquad (4.7.18)$$

A special case occurs when $B=\pm C$, for then the fundamental disturbance is proportional to $\exp(i\alpha x)\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\gamma z$, and so represents a standing wave in the z-direction.

7 $\quad 0 \neq |B| \neq |C| \neq 0, \; b_{2r} \neq b_{3r}$. No equilibrium states are possible since (4.7.7) cannot be satisfied.

In all the situations considered so far the analysis has reduced to that of a two-dimensional disturbance, and we shall now examine the more interesting cases which are fundamentally three-dimensional. It is not possible to obtain analytically the time-dependence of the amplitudes, and only the equilibrium amplitudes are given. We note that there are four possible equilibrium cases consistent with (4.7.7).

8 $\quad C=0$. From (4.7.4) and (4.7.5) we obtain

$$|A|_e^2 = \frac{b_{or}a_{2r}-b_{2r}a_{or}}{b_{2r}a_{1r}-a_{2r}b_{1r}}, \quad |B|_e^2 = \frac{b_{1r}a_{or}-a_{1r}b_{or}}{b_{2r}a_{1r}-a_{2r}b_{1r}}. \tag{4.7.19}$$

9 $\quad B=0$. This is identical to case 8 with $|C|_e^2$ replacing $|B|_e^2$.

10 $\quad |B|=|C|$. From (4.7.4)-(4.7.6) we have

$$|A|_e^2 = \frac{a_{2r}b_{or}-a_{or}b_{2r}}{a_{1r}(b_{2r}+b_{3r})-2a_{2r}b_{1r}}, \quad |B|_e^2=|C|_e^2 = \frac{a_{or}b_{1r}-a_{1r}b_{or}}{a_{1r}(b_{2r}+b_{3r})-2a_{2r}b_{1r}}. \tag{4.7.20}$$

As in case 6 the special case $B=\pm C$ represents standing waves in the z-direction.

11 $\quad b_{2r}=b_{3r}$. Equation (4.7.8) holds in this case, and equilibrium conditions are given by

$$|A|_e^2 = \frac{a_{2r}b_{or}-a_{or}b_{2r}}{a_{1r}b_{2r}-a_{2r}b_{1r}}, \quad |B|_e^2 = \frac{a_{or}b_{1r}-a_{1r}b_{or}}{(1+k^2)(a_{1r}b_{2r}-a_{2r}b_{1r})}, \quad |C|_e^2 = k^2|B|_e^2. \tag{4.7.21}$$

The situations above represent the possibility of an equilibrium state arising out a combination of two- and three-dimensional oscillations. An important feature of the analysis in this chapter lies in the determination of which of the possible equilibria 1-11 above is most likely to occur. If there are two competing disturbances we could determine which one was likely to dominate. The stability analysis is based on equations (4.7.4)-(4.7.6), and is therefore restrictive in not including all hydrodynamic disturbances.

Finally, we establish the self-consistency of the method developed in this chapter, and relate the arguments of section 4.4 to a three-dimensional context. If c_1 and c_2 are wave speeds of the two- and three-dimensional disturbances, then

$$a_{or} = \alpha c_{1i}, \quad b_{or} = \sqrt{\alpha^2+\gamma^2}\, c_{2i}. \qquad (4.7.2\ )$$

Our analysis is concerned with the region close to the neutral curve where c_{1i} and c_{2i} are small, and we assume that a_{1r} and a_{2r} have non-zero values as $c_{1i}\to 0$ and that b_{1r}, b_{2r} and b_{3r} remain non-zero as $c_{2i}\to 0$. This assumption may break down for isolated values of α and R but should be true in general. In each of the cases listed above we note that $|A|^2$, $|B|^2$ and $|C|^2$ are all of order $(c_{1i}^2+c_{2i}^2)^{\frac{1}{2}}$. In (4.7.4) and (4.7.5) the terms retained are of order $(c_{1i}^2+c_{2i}^2)$, while the neglected terms are of order $(c_{1i}^2+c_{2i}^2)^{3/2}$ and higher. In this way we justify truncating the expansion of amplitude equations and velocity components at third order in amplitude rather than including higher-order terms. We note that to truncate at first order is invalid as some of the neglected terms are of similar magnitude to those retained. However, at a (R) location where a solution changes character still higher order terms may be needed.

4.8 Two-dimensional flow

The linear analysis of the previous chapter suggested that three-dimensional disturbances may be fundamentally important in the development of turbulence in a viscoelastic fluid, but the computation of the functions necessary to calculate a_k, b_k and c_k is prohibitively lengthy, and calculations have been restricted to the considerably simpler two-dimensional problem. It is convenient at this stage to set $w=0$ and to work with a stream function, θ, defined by

$$u = \partial\theta/\partial y, \quad v = -\partial\theta/\partial x, \tag{4.8.1}$$

so that the continuity equation (4.2.5) is satisfied identically. The momentum equation (4.2.4) reduces to

$$\left(\frac{\partial}{\partial t} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\theta}{\partial x}\frac{\partial}{\partial y}\right)(1-\lambda\nabla^2)\nabla^2\theta = \frac{1}{R}\nabla^4\theta, \tag{4.8.2}$$

and we note immediately that two-dimensional motions are independent of μ, which is consistent with a more general result obtained by Rivlin (1955). The only disturbance to be considered is the one proportional to $A(t)\exp(i\alpha x)$. With notation similar to that already used in this chapter, the relevant equations are (4.5.14), (4.5.6), (4.5.19) and (4.5.26) which, on writing for simplicity

$$\theta_{100}=\theta_1, \quad \theta_{201}=\theta_2, \quad \theta_{101}=\theta_{11}, \tag{4.8.3}$$

become

$$L(a_o, \alpha, 0)(\theta_1, 0) = 0, \tag{4.8.4}$$

$$2a_{or}f_1 - \left(\frac{1}{R} + \lambda a_{or}\right)f_1'' = i\alpha(1+2\lambda\alpha^2)(\theta_1\tilde{\theta}_1'' - \theta_1''\tilde{\theta}_1) + i\alpha\lambda(\theta_1^{(4)}\tilde{\theta}_1 - \theta_1\tilde{\theta}_1^{(4)}), \tag{4.8.5}$$

$$2L(2a_o,\ 2\alpha,\ 0)(\theta_2,0) = \{\theta_1(1-\lambda E_{10})E_{10}\theta_1'-\theta_1'(1-\lambda E_{10})E_{11}\theta_1,\ 0\}, \tag{4.8.6}$$

$$\begin{aligned}L(a_o+2a_{or},\alpha,0)(\theta_{11},0) = \{&(ia_1/\alpha-f_1)(1-\lambda E_{10})E_{10}\theta_1+f_1''\theta_1-\lambda f_1^{(4)}\theta_1\\ &+\theta_2'(1-\lambda E_{10})E_{10}\tilde{\theta}_1+2\theta_2(1-\lambda E_{10})E_{10}\tilde{\theta}_1'\\ &-2\tilde{\theta}_1'(1-\lambda E_{20})E_{20}\theta_2-\tilde{\theta}_1(1-\lambda E_{20})E_{20}\theta_2',\ 0\},\end{aligned} \tag{4.8.7}$$

with boundary conditions

$$\theta_1=\theta_1'=\theta_{11}=\theta_{11}'=f_1=0 \quad \text{at} \quad y=1, \tag{4.8.8}$$

$$\theta_1,\theta_{11},f_1 \quad \text{even at} \quad y=0, \quad \theta_2 \quad \text{odd at} \quad y=0.$$

In contrast to chapter 3, we make no perturbations in λ. As in section 4.6 we introduce an adjoint function, θ^*, which satisfies

$$\left.\begin{aligned}&\{(a_o+i\alpha f_o)(1-\lambda E_{10})E_{10}-4i\alpha\lambda f_o'D^3+2(1+2\lambda\alpha^2)f_o'D-6\lambda f_o''D^2+2\alpha^2\lambda f_o''\}\theta^*=0,\\ &\qquad\qquad \theta^*=0 \quad \text{at} \quad y=1, \quad \theta^* \quad \text{even at} \quad y=0.\end{aligned}\right\} \tag{4.8.9}$$

The result obtained in section 4.6 that $D\theta^*=0$ at $y=1$ is used instead of the boundary condition at $y=1$ specified in (4.8.9) for the numerical determination of θ^*, which is described in the next section, and we also normalise θ^* so that $\theta^*=1$ at $y=0$. Equation (4.6.10) simplifies in the two-dimensional case to give

$$a_1 = i\alpha \int_0^1 \theta^* q\,dy \Big/ \int_0^1 \theta^*(1-E_{10})E_{10}\theta_1\,dy, \tag{4.8.10}$$

where q is the non-zero element of the right hand side of (4.8.7) with a_1 set equal to zero. We again note that this expression is valid only in the limit $a_{or}\to 0$ (Watson 1960).

4.9 Numerical method of solution

Instead of expressing equations (4.8.4)-(4.8.9) in finite difference form directly, Thomas (1953) introduced an auxiliary function, and he was then able to reduce the truncation errors contained in the finite difference approximations to the differential equations. We here follow Thomas's idea, though the auxiliary function g is defined slightly differently from his to give an even better form of the finite difference representation of the differential equation. We define g by

$$\theta_1 = \left\{ 1 + \frac{1}{6}\,\delta^2 - \frac{1}{720}\,\delta^4 \right\} g, \qquad (4.9.1)$$

where δ denotes a central difference. If we divide the interval $0 \leq y \leq 1$ into N equal intervals of step length h and let $^{(n)}$ denote the appopriate grid point so that

$$\theta_1^{(n)} = \theta_1(nh), \qquad (4.9.2)$$

we then obtain the following representations for derivatives of θ_1:

$$\left.\begin{aligned}
h^4 D^4 \theta_1^{(n)} &= \delta^4\, g^{(n)} + 0(\delta^{10}),\\
h^3 D^3 \theta_1^{(n)} &= \mu \left\{ \delta^3 - \frac{1}{12}\,\delta^5 \right\} g^{(n)} + 0(\delta^7),\\
h^2 D^2 \theta_1^{(n)} &= \left\{ \delta^2 + \frac{1}{12}\,\delta^4 \right\} g^{(n)} + 0(\delta^6),\\
h D \theta_1^{(n)} &= \mu\delta g^{(n)} + 0(\delta^5),
\end{aligned}\right\} \qquad (4.9.3)$$

where μ now denotes the usual average operator defined by

$$\mu g^{(n)} = \tfrac{1}{2}\{g^{(n+\frac{1}{2})} + g^{(n-\frac{1}{2})}\}. \qquad (4.9.4)$$

Boundary conditions, θ_1 even (hence $\theta_1' = \theta_1''' = 0$ at $y=0$) and $\theta_1 = \theta_1' = 0$ at $y=1$, are incorporated by setting

$$\theta_1^{(-1)}=\theta_1^{(1)}, \quad \theta_1^{(-2)}=\theta_1^{(2)}, \quad \theta_1^{(N)}=0, \quad \theta_1^{(N+1)}=\theta_1^{(N-1)}, \tag{4.9.5}$$

and (4.9.3) may now be applied for $n=0, 1, 2, \ldots, N-1$. Before expressing (4.8.4)-(4.8.9) in finite difference form it is convenient to divide the equations by $i\alpha$ and replace a_o by the wave speed c, which is given by

$$a_o = -i\alpha c. \tag{4.9.6}$$

We may now represent (4.8.4) by

$$\left\{\frac{i}{\alpha R} - \lambda(f_o-c)\right\} \delta^4 g^{(n)} + h^2 \left\{(f_o-c)(1+2\lambda\alpha^2) - \frac{i\alpha}{R}\right\} \left(\delta^2 + \frac{1}{12}\delta^4 \right)$$

$$g^{(n)} + h^4 \left\{ \frac{i\alpha^3}{R} - f_o'' - \alpha^2(f_o-c)(1+\lambda\alpha^2) \right\} g^{(n)}$$

$$= O(h^2\delta^6 g^{(n)}), \quad n=0,1,\ldots,N-1, \tag{4.9.7}$$

which we may write in the form

$$(B-cC)\underset{\sim}{g} = 0, \tag{4.9.8}$$

where B and C are matrices and $\underset{\sim}{g}$ is a column vector with components $g^{(n)}$. An eigensolution $(c, \underset{\sim}{g})$ is found iteratively using a method developed by Osborne (1967). To derive the iteration, we consider the problem

$$(B-cC)\underset{\sim}{g} = \beta(c)\underset{\sim}{x}, \tag{4.9.9}$$

$$\underset{\sim}{s}^T \underset{\sim}{g} = K, \tag{4.9.10}$$

where $\underset{\sim}{x}$ and $\underset{\sim}{s}$ are fixed vectors, K a fixed constant and β a multiplier which ensures that the vector $\underset{\sim}{g}$ obtained from (4.9.9) satisfies the scaling condition (4.9.10). As c varies, so must β, and when c passes through an eigenvalue of (4.9.8) then β in general vanishes. Use of Newton's method to determine the zeros of $\beta(c)$ provides a convenient way of obtaining the eigenvalues of (4.9.8).

To apply Newton's method it is necessary to know $d\beta/dc$, which can be found by differentiating (4.9.9) and (4.9.10):

$$\left.\begin{aligned} (B-cC)\frac{d\underset{\sim}{g}}{dc} - C\,\underset{\sim}{g} &= \frac{d\beta}{dc}\underset{\sim}{x}, \\ \underset{\sim}{s}^T \frac{d\underset{\sim}{g}}{dc} &= 0, \end{aligned}\right\} \qquad (4.9.11)$$

and obtaining

$$\underset{\sim}{s}^T \underset{\sim}{g}\,\frac{1}{\beta}\frac{d\beta}{dc} = -\underset{\sim}{s}^T(B-cC)^{-1}C\,\underset{\sim}{g}. \qquad (4.9.12)$$

The correction to the current estimate of c is then

$$-\frac{\beta}{d\beta/dc} = \frac{\underset{\sim}{s}^T\underset{\sim}{g}}{-\underset{\sim}{s}^T(B-cC)^{-1}C\,\underset{\sim}{g}}. \qquad (4.9.13)$$

Osborne has shown that this method gives a third order iteration provided that $\underset{\sim}{s}^T\underset{\sim}{g}$ is non-zero, which is readily achieved by an appropriate choice of $\underset{\sim}{s}$ and there we choose $\underset{\sim}{s} = \underset{\sim}{e}_p$, a unit vector whose only non-zero element is in position p, where p is the index of the component of maximum modulus in the vector $(B-cC)^{-1}C\,\underset{\sim}{g}$. In implementing the method it is unnecessary to calculate an inverse matrix, and the form of each iteration is as follows:

1 given the kth estimate $(c_k, \underset{\sim}{g}_k)$ of $(c, \underset{\sim}{g})$ obtain vector $\underset{\sim}{h}_{k+1}$ from

$$(B-c_kC)\underset{\sim}{h}_{k+1} = C\,\underset{\sim}{g}_k/g_k^{(p_k)}, \tag{4.9.13}$$

2 obtain $\underset{\sim}{g}_{k+1}$ from

$$(B-c_kC)\underset{\sim}{g}_{k+1} = C\underset{\sim}{h}_{k+1}, \tag{4.9.14}$$

3 form

$$c_{k+1} = c_k + h_{k+1}^{(p_{k+1})}\Big/ g_{k+1}^{(p_{k+1})}, \tag{4.9.15}$$

where p_k is the index of the component of maximum modulus in $\underset{\sim}{g}_k$. The eigenvectór $\underset{\sim}{g}$ is normalised so that $g^{(0)}=1$. B and C are both of band diagonal form, and advantage is taken of this structure in the matrix procedures used in solving (4.9.13) and (4.9.14). At each stage the matrix $B-c_kC$ is decomposed into LU form, where L and U are lower and upper triangular matrices respectively. An equation of the form

$$L U\underline{x} = \underline{y} \tag{4.9.16}$$

is solved by direct elimination in two stages,

$$L\underline{v}=\underline{y}, \quad U\underline{x}=\underline{v}. \tag{4.9.17}$$

A more direct method using inverse matrices could be used, but the band structure would be lost, with resulting storage problems and a considerable increase in computer time required. With an initial estimate of $c=0.24+0.008i$ the process converged in three or four iterations to an accuracy of five decimal places for the eigenvalue and four for the eigenvector $\underset{\sim}{\theta}$, in agreement with Thomas (1953) for the same values of α and R. Previous work suggested that this value of c was of the right order of magnitude, and the work of Lee and Reynolds (1966) shows that the higher eigenvalues are well separated from the one of particular interest where $|c_i|$ is small. It was found that the initial estimate of the eigenvalue did not affect the convergence process very strongly; with the same initial eigenvector doubling or halving the starting value of c only added one more iteration to give the same result. Choice of initial eigenvector may be important in the numerical procedure, but as we are only interested in one particular eigensolution of the system this dependence was not investigated, and throughout a parabolic profile was used with

$$g^{(n)} = 1-n^2h^2 + 0i \tag{4.9.18}$$

as starting values for the eigenvector. Step-lengths of $h=0.04$, 0.02, 0.01 and 0.005 were tried, and it was found that little further accuracy was obtained using $h=0.005$, so it was decided to use $h=0.01$ throughout. θ_1 and its derivatives are obtained from (4.9.1) and (4.9.3), and a check it was found that (4.8.4) was satisfied to eleven decimal places, the working accuracy of the computer, with θ_1 normalised so that $\theta_1(0)=1$.

Direct finite difference schemes are used to represent the left hand sides of (4.8.5), (4.8.6) and (4.8.9), namely

$$\left.\begin{aligned} hD &= \mu\left(\delta - \frac{1}{6}\delta^3\right) + 0(\delta^5), \\ h^2D^2 &= \left(\delta^2 - \frac{1}{12}\delta^4\right) + 0(\delta^6), \\ h^3D^3 &= \mu\delta^3 + 0(\delta^5), \\ h^4D^4 &= \delta^4 + 0(\delta^6). \end{aligned}\right\} \tag{4.9.19}$$

We note from (4.8.5) and the boundary conditions (4.8.8) that f_1 is a real function of y, which simplifies the numerical solution of the equation. As the right hand side of (4.8.9) is zero the condition

$\theta^*(1)=0$ is replaced by $d\theta^*(1)/dy=0$, this interchange being permitted by the argument leading to (4.6.6) and (4.6.7). The adjoint function was normalised by setting $\theta^*(0)=1$, and this value at the grid point $n=0$ may be written on the other side of the equation to give a non-zero right hand side. LU decompositions are used to solve the algebraic equations derived from (4.8.5), (4.8.6) and (4.8.9). It was found that $\psi^*(1)=0$ to the accuracy of the finite difference representation.

Finally, Simpson's rule, in which $\int_0^1 f(y)dy$ is calculated by

$$\int_0^1 f(y)dy \doteqdot \frac{1}{3}h \quad \{f^{(0)} + f^{(N)} + 4\sum_{n=1}^{\frac{1}{2}N} f^{(2n-1)} + 2\sum_{n=1}^{\frac{1}{2}(N-2)} f^{(2n)}\}, \tag{4.9.20}$$

is used to evaluate the integrals in (4.8.10). We note that both θ_1 and θ^* may be multiplied by any scalar without altering the value of a_1.

4.10 Discussion of results

Typical graphs of the various functions calculated are shown in figures 4.1-4.4, corresponding to $\alpha=1$, $R=10000$ and $\lambda=0$. There is excellent agreement with Thomas (1953) for the fundamental θ_1 and with Lee and Reynolds (1966) for the adjoint function θ^*. For other values of α and R functions θ_1, θ_2 and θ^* had the same general form, while f_1 sometimes had additional oscillations. Different values of λ made only small differences to the functions, and in particular the position of the extrema of θ_{1i}, θ_{2r}, θ_{2i} and the inflexion point of θ_{1r} are unaltered by changes in λ. As we should expect

$$\int_0^1 f_o\, f_1\, dy < 0, \tag{4.10.1}$$

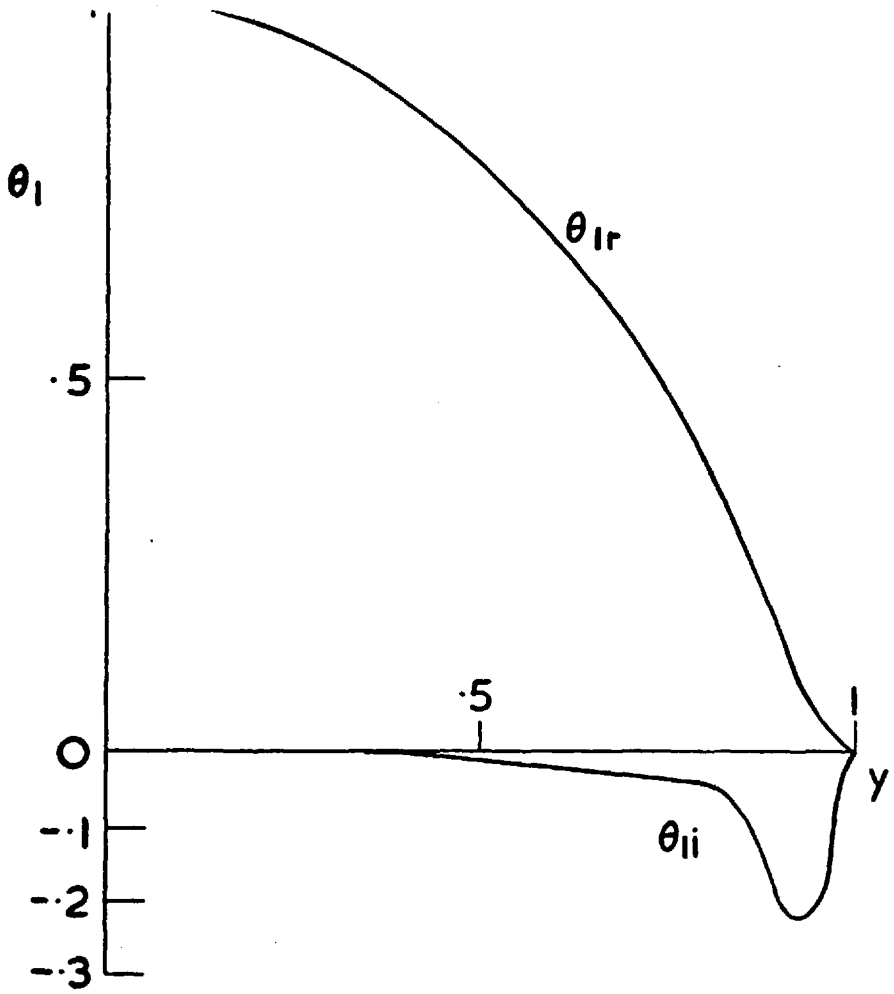


FIG. 4.1 THE FUNDAMENTAL DISTURBANCE θ_1 FOR $\alpha=1$, $R=10000$, $\lambda=0$

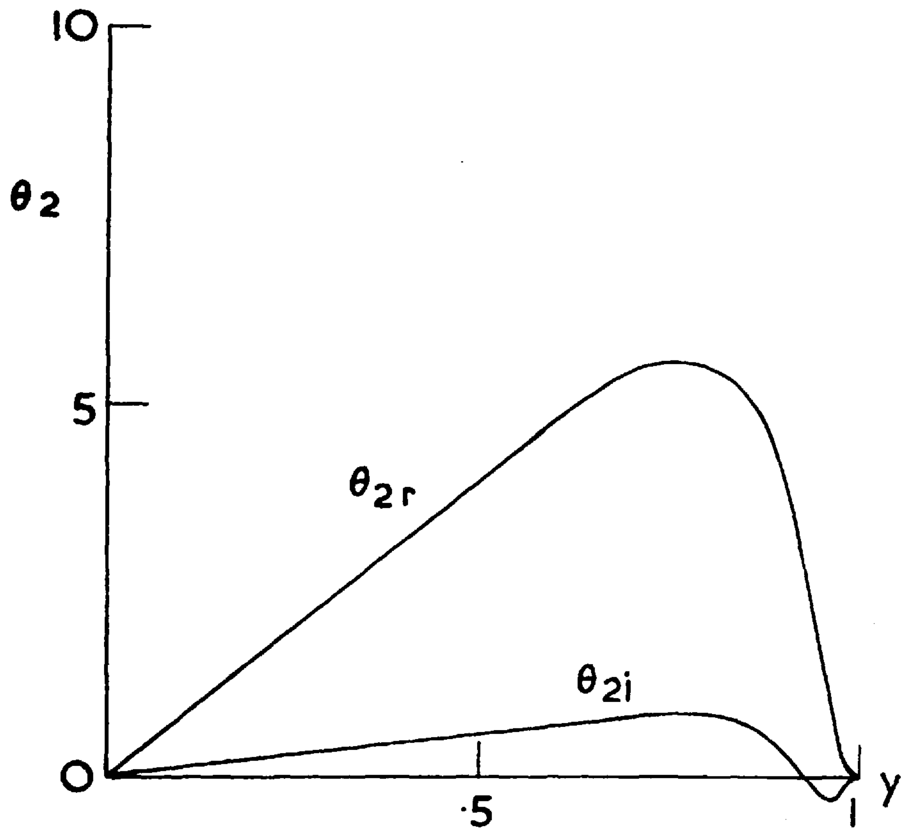


FIG. 4.2 THE FIRST HARMONIC θ_2 FOR $\alpha=1$, $R=10000$, $\lambda=0$

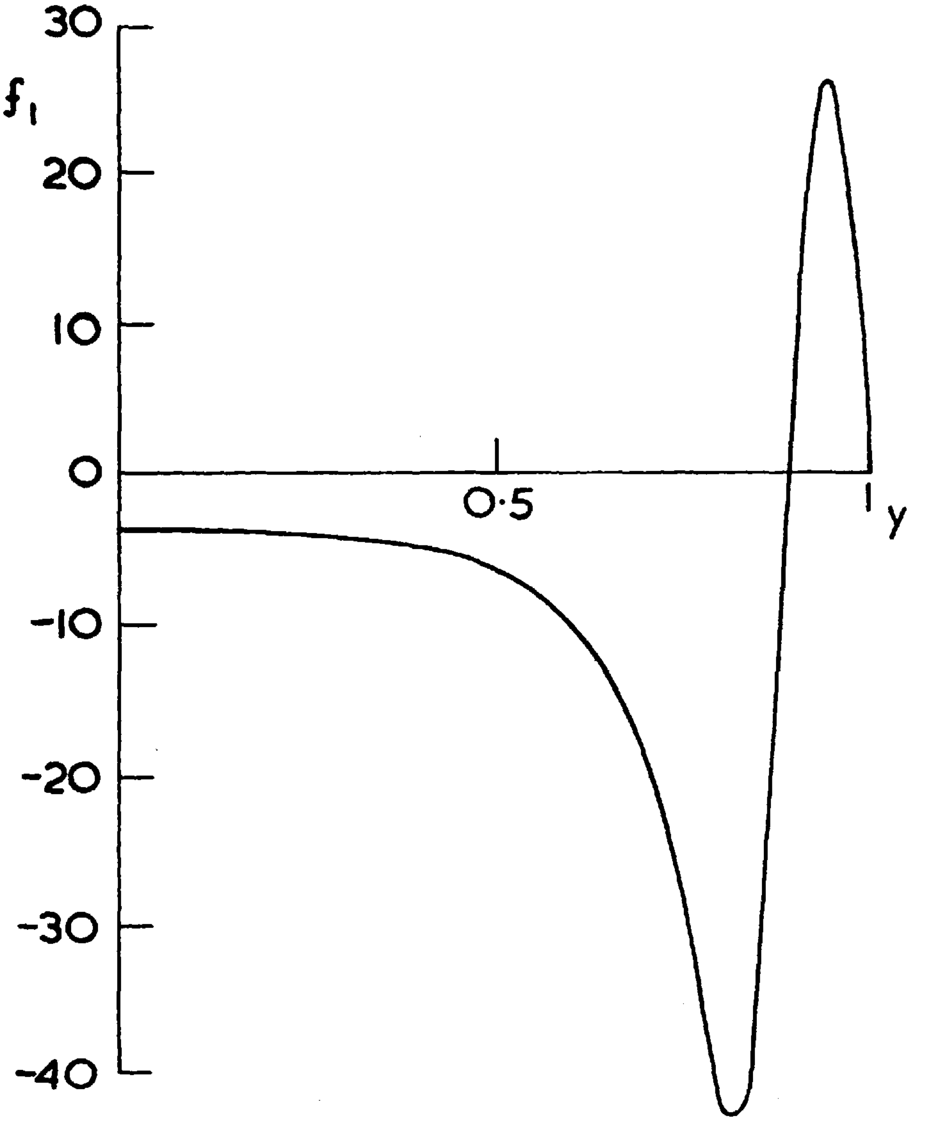


FIG. 4.3 f_1 FOR $\alpha=1$, R=10000, $\lambda=0$
THIS FUNCTION REPRESENTS MODIFICATION OF THE BASIC FLOW

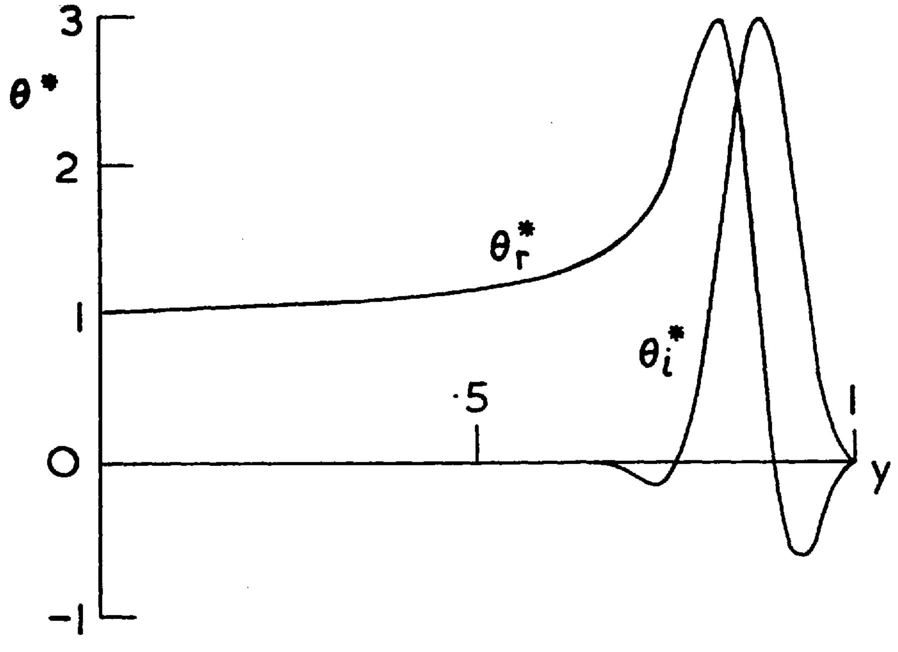


FIG. 4.4 THE ADJOINT FUNCTION θ^* FOR $\alpha = 1$, $R = 10000$, $\lambda = 0$

so that

$$\int_0^1 f_o^2\,dy \;>\; \int_0^1 (f_o+f_1|A|)^2\,dy, \qquad (4.10.2)$$

provided that we neglect $|A|^4$, indicating that energy is transferred from the basic flow to the disturbance.

Distortion of the mean motion can be seen in terms of the influence of the Reynolds stress. For two-dimensional flow of a Newtonian fluid, Stuart (1958) shows that if the disturbance velocities are u and v and the mean motion is $\bar{u}$ then, by integrating the momentum equation, we obtain an energy balance relation

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}(u^2+v^2)\,dxdy = \iint (\overline{-uv})\,\frac{\partial\bar{u}}{\partial y}\,dxdy - \frac{1}{R}\iint \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2 dxdy, \qquad (4.10.3)$$

where the integrals are evaluated over a volume bounded by the channel walls and one wavelength. The overbar denotes an average with respect to x. The first integral gives the rate of increase of the disturbance energy within the volume considered. The second term represents the rate of energy transfer from the mean flow to the disturbance. The final integral is the rate of viscous dissipation of energy of the disturbance. In the notation of section 4.8, the Reynolds stress is

$$\overline{-uv} = \overline{\frac{\partial\theta}{\partial y}\frac{\partial\theta}{\partial x}}, \qquad (4.10.4)$$

and its spatial dependence is given by

$$F = \frac{1}{\alpha}\,\mathrm{Re}\,\{i\alpha(\theta_1'\tilde{\theta}_1-\tilde{\theta}_1'\theta_1)\} \qquad (4.10.5)$$

to second order in amplitude, where a prime denotes differentiation with respect to y and a tilde denotes the complex conjugate. The Reynolds stress function F is therefore given by

$$F = 2(\theta'_{1r}\theta_{1i}-\theta_{1r}\theta'_{1i}). \qquad (4.10.6)$$

For a viscoelastic fluid, too, the function F gives the rate of energy transfer from the mean flow to the disturbance. A typical example of the Reynolds stress function for both Newtonian and second-order fluids is given in figure 4.5. The function for $\lambda=0$ is in good agreement with that obtained by Stuart (1958). It will be seen that the function is increased by viscoelasticity, suggesting that more energy is passed to the disturbance than in a Newtonian fluid, though the overall energy transfer will also depend on the amplitude of the disturbance. The effect of viscoelasticity on the amplitude is discussed below.

The second-order model of the fluid (2.6.13) is only likely to be valid for small values of λ and μ. In the linear analysis of the previous chapter, it has been possible to work with $0 \geq \lambda \geq -10^{-4}$. In the non-linear analysis developed in this chapter, we have worked with similar values of λ. For convenience, results were obtained for a Newtonian fluid ($\lambda=0$) and a viscoelastic fluid for which $\lambda=-10^{-5}$. Sample results for other values of λ in the range 0 to -10^{-4} were obtained, and the displacements of the curve $c_i=0$ and $a_{1r}=0$ from those for a Newtonian fluid are similar to those shown in figure 4.10. Values of the wave speed c for $\lambda=0$ and -10^{-5} are given in tables 4.4-4.7. The one result that can be compared with Porteous and Denn's (1971) agrees closely with theirs, namely, for $R=5000$, $\alpha=1$, $\lambda=-10^{-5}$ the wave speed obtained here is $c=0.268259-0.001581i$, whereas they calculated $c=0.2682465-0.0015824i$.

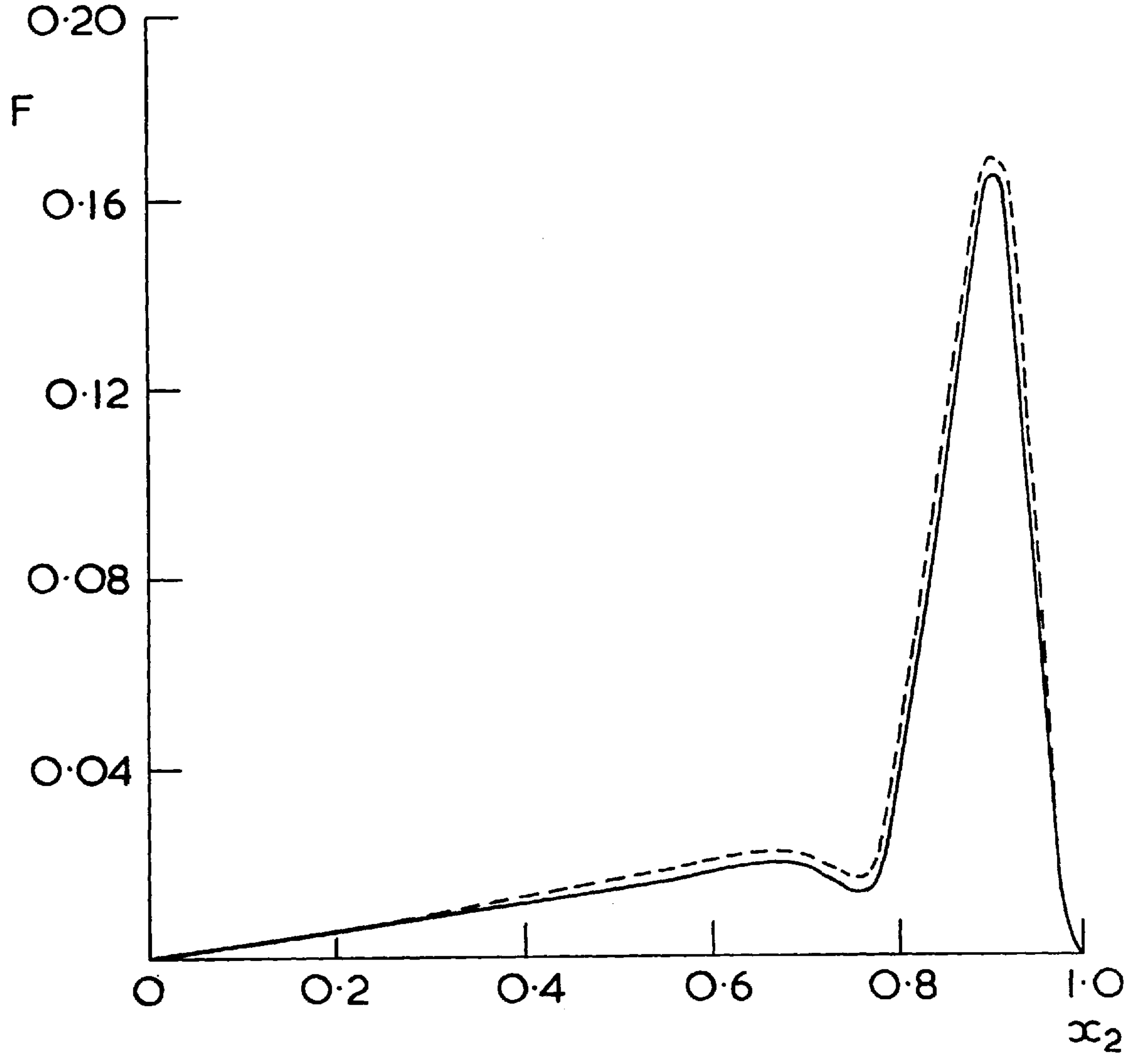


FIG. 4.5 REYNOLDS STRESS FUNCTION FOR $\lambda=0$ (FULL CURVE) AND $\lambda=-10^{-5}$ (BROKEN CURVE) AT $\alpha=1$, $R=10^4$

The behaviour of a_1 for $\lambda=0$ and -10^5 for various values of the Reynolds number is given in tables 4.8 and 4.9 and illustrated in figures 4.6–4.9. The curves of a_{1r} against α for $\lambda=0$ are in reasonable qualitative agreement with Pekeris and Shkoller (1967), who use a different parameter. The curve in figure 4.10 on which $a_{1r}=0$ is close to the one they obtained, though values of α differ by up to 7% for points not close to the neutral curve. This may be due to their neglect of the time derivative in the mean motion equations (4.2.8), but, as Reynolds and Potter (1967) point out, values of a_{1r} can only be expected to be accurate close to the neutral stability curve, and higher order terms should be included in the amplitude equation to determine a_{1r} more accurately elsewhere.

In figure 4.10 perturbations represented by points within the neutral stability curve (regions II and III) are unstable according to linear theory since $c_i>0$ and the time dependence $\exp(i\alpha ct)$ indicates a growing disturbance. In region III, however, the non-linear theory of this chapter permits equilibrium states of finite amplitude given by (4.4.10). The equilibrium amplitude is given by

$$|A| = (\alpha c_i/-a_{1r})^{\frac{1}{2}}, \qquad (4.10.7)$$

and its value for various Reynolds numbers is shown in figure 4.11, from which we see that viscoelasticity increases the equilibrium amplitude. ∠ It may be seen from figure 4.10 that region II for a viscoelastic fluid contains points from all four regions for a Newtonian fluid. Flows corresponding to those points from I and IV are destabilised by viscoelasticity since disturbances which decay under linear theory (I and IV) now grow (II) whether linear or non-linear theory is used. Points from region III for a Newtonian fluid which lie in region II for a viscoelastic fluid also correspond to

∠ In figure 4.12 the amplitudes are related to measurable physical quantities.

α/R	4000	5000	6000	7000	8000	10000	12000	14000
1.2	0.301052	0.289202	0.279674	0.271716	0.264881	0.253545	0.244321	0.236481
1.1	0.290744	0.279671	0.270821	0.263484	0.257217	0.246960	0.238743	0.231887
1.0	0.278541	0.268128	0.259820	0.252932	0.247071	0.237532	0.229940	0.223677
0.9	0.264659	0.254908	0.247086	0.240601	0.235087	0.226088	0.218952	0.213082
0.8	0.248980	0.240011	0.232783	0.226744	0.221582	0.213144	0.206417	0.200871
0.7	0.230981	0.223023	0.216554	0.211106	0.206434	0.198713	0.192506	0.187356
0.6	0.210357	0.203241	0.197581	0.192868	0.188825	0.182118	0.176691	0.172142
0.5	0.189204	0.181989	0.176455	0.172021	0.168320	0.162403	0.157742	0.153907
0.4	0.172062	0.163934	0.157754	0.152843	0.148792	0.142452	0.137628	0.133790

Table 4.4

Values of c_r when $\lambda=0$

α/R	4000	5000	6000	7000	8000	10000	12000	14000
1.2	-0.006420	-0.005619	-0.005501	-0.005760	-0.006238	-0.007533	-0.009017	-0.010534
1.1	-0.004013	-0.002021	-0.000935	-0.000381	-0.000160	-0.000308	-0.000892	-0.001699
1.0	-0.004948	-0.001753	+0.000320	+0.001711	+0.002659	+0.003733	+0.004155	+0.004206
0.9	-0.009575	-0.005204	-0.002157	+0.000063	+0.001732	+0.004014	+0.005429	+0.006327
0.8	-0.018085	-0.012648	-0.008682	-0.005666	-0.003298	+0.000164	+0.002548	+0.004266
0.7	-0.030126	-0.023979	-0.019305	-0.015621	-0.012636	-0.008088	-0.004784	-0.002277
0.6	-0.043910	-0.037803	-0.032988	-0.029055	-0.025762	-0.020531	-0.016541	-0.013385
0.5	-0.056950	-0.050946	-0.046524	-0.042900	-0.039835	-0.034852	-0.030902	-0.027652
0.4	-0.067620	-0.061942	-0.057610	-0.054140	-0.051263	-0.046695	-0.043159	-0.040287

Table 4.5

Values of c_i when $\lambda=0$

α/R	4000	5000	6000	7000	8000	10000	12000	14000
1.2	0.301178	0.289332	0.279807	0.271849	0.265016	0.253682	0.244452	0.236626
1.1	0.290863	0.279802	0.270957	0.263615	0.257355	0.247089	0.238862	0.231999
1.0	0.278661	0.268259	0.259949	0.253065	0.247212	0.237662	0.230073	0.223803
0.9	0.264766	0.255026	0.247217	0.240738	0.235225	0.226232	0.219098	0.213221
0.8	0.249056	0.240111	0.232889	0.226862	0.221713	0.213279	0.206564	0.201023
0.7	0.231012	0.223067	0.216616	0.211195	0.206532	0.198830	0.192642	0.187500
0.6	0.210314	0.203217	0.197574	0.192879	0.188850	0.182176	0.176770	0.172236
0.5	0.189098	0.181884	0.176365	0.171929	0.168237	0.162328	0.157693	0.153874
0.4	0.171942	0.163800	0.157616	0.152695	0.148647	0.142300	0.137475	0.133630

Table 4.6

Values of c_r when $\lambda=-10^{-5}$

α/R	4000	5000	6000	7000	8000	10000	12000	14000
1.2	-0.006133	-0.005254	-0.005063	-0.005249	-0.005656	-0.006811	-0.008156	-0.009535
1.1	-0.003810	-0.001753	-0.000614	-0.000003	+0.000272	+0.000228	-0.000254	-0.000960
1.0	-0.004824	-0.001581	+0.000538	+0.001973	+0.002964	+0.004117	+0.004616	+0.004740
0.9	-0.009528	-0.005119	-0.002036	+0.000219	+0.001921	+0.004265	+0.005740	+0.006692
0.8	-0.018114	-0.012649	-0.008657	-0.005613	-0.003220	+0.000290	+0.002719	+0.004480
0.7	-0.030223	-0.024062	-0.019375	-0.015674	-0.012672	-0.008090	-0.004753	-0.002213
0.6	-0.044042	-0.037942	-0.033131	-0.029197	-0.025900	-0.020657	-0.016650	-0.013476
0.5	-0.056714	-0.051080	-0.046673	-0.043061	-0.040007	-0.035040	-0.031102	-0.027858
0.4	-0.067717	-0.062049	-0.057728	-0.054267	-0.051400	-0.046850	-0.043323	-0.040474

Table 4.7

Values of c_i when $\lambda=-10^{-5}$

α/R	4000	5000	6000	7000	8000	10000	12000	14000
1.2	27.6-223i	34.1-184i	44.5-162i	59.6-85.7i	80.4-49.3i	115-37.6i	197-95.4i	304-132i
1.1	22.0-160i	21.9-128i	24.3-111i	27.7-57.2i	30.9-32.1i	45.2-23.8i	68.8-67.9i	110-97.2i
1.0	27.3-107i	15.3-79.3i	8.62-67.7i	2.36-30.4i	-1.67-15.8i	2.74-11.1i	11.9-44.0i	27.2-65.4i
0.9	35.8-59.6i	20.2-38.0i	7.31-28.5i	-5.08-7.43i	-16.2-0.45i	-20.4+0.65i	-21.7-21.6i	-21.4-37.8i
0.8	44.3-16.8i	27.8-3.80i	15.9+3.42i	3.54+13.7i	-5.31+13.2i	-18.0+11.2i	-30.3-3.71i	-40.1-11.9i
0.7	52.7+20.3i	35.3+25.3i	23.0+29.6i	11.9+26.2i	9.25+23.9i	4.15+20.4i	-5.72+14.4i	-13.3+8.44i
0.6	59.8+47.8i	42.0+39.8i	29.7+32.9i	18.1+27.1i	20.2+30.8i	21.3+27.8i	22.4+28.3i	17.9+24.7i
0.5	65.7+30.4i	46.6+27.5i	33.9+24.2i	24.0+24.0i	28.1+28.3i	33.6+31.3i	39.0+35.4i	38.5+36.9i
0.4	70.4+1.94i	50.1+6.38i	35.1+11.4i	28.3+18.4i	31.8+23.7i	40.0+29.4i	48.1+38.1i	52.0+44.2i

Table 4.8

Values of a_1 when $\lambda=0$

α/R	4000	5000	6000	7000	8000	10000	12000	14000
1.2	29.4-241i	35.2-203i	47.4-180i	62.8-90.4i	85.1-53.6i	121-40.3i	211-102i	347-148i
1.1	23.1-177i	22.8-141i	26.1-126i	29.0-60.3i	33.4-34.4i	47.3-25.7i	73.5-72.5i	126-104i
1.0	28.1-114i	15.5-88.5i	8.70-75.3i	3.62-32.5i	2.01-17.6i	4.94-12.1i	14.1-47.1i	34.2-68.3i
0.9	37.2-61.3i	21.0-39.1i	7.34-29.1i	-3.91-7.76i	17.2-1.63i	21.5-0.05i	-22.1-21.5i	-20.7-39.0i
0.8	46.0-16.8i	29.4-3.12i	16.5+3.96i	4.78+13.1i	-3.64+12.9i	-15.7+11.0i	-32.5-2.07i	-40.5-12.3i
0.7	55.1+20.9i	37.1+26.5i	23.9+31.0i	13.2+27.5i	10.6+24.6i	7.24+21.6i	-2.57+14.8i	-9.62+8.04i
0.6	63.7+48.4i	44.3+41.3i	30.9+34.7i	19.6+29.8i	22.7+32.1i	23.7+29.4i	24.5+28.9i	20.7+25.1i
0.5	69.4+30.9i	49.5+28.6i	35.3+26.1i	27.1+26.3i	30.6+29.5i	35.3+32.0i	42.3+36.1i	42.4+37.3i
0.4	73.8+2.41i	54.3+7.04i	36.8+12.6i	32.4+20.9i	34.2+25.2i	41.8+29.9i	51.9+38.9i	56.7+45.2i

Table 4.9

Values of a_1 when $\lambda=-10^{-5}$

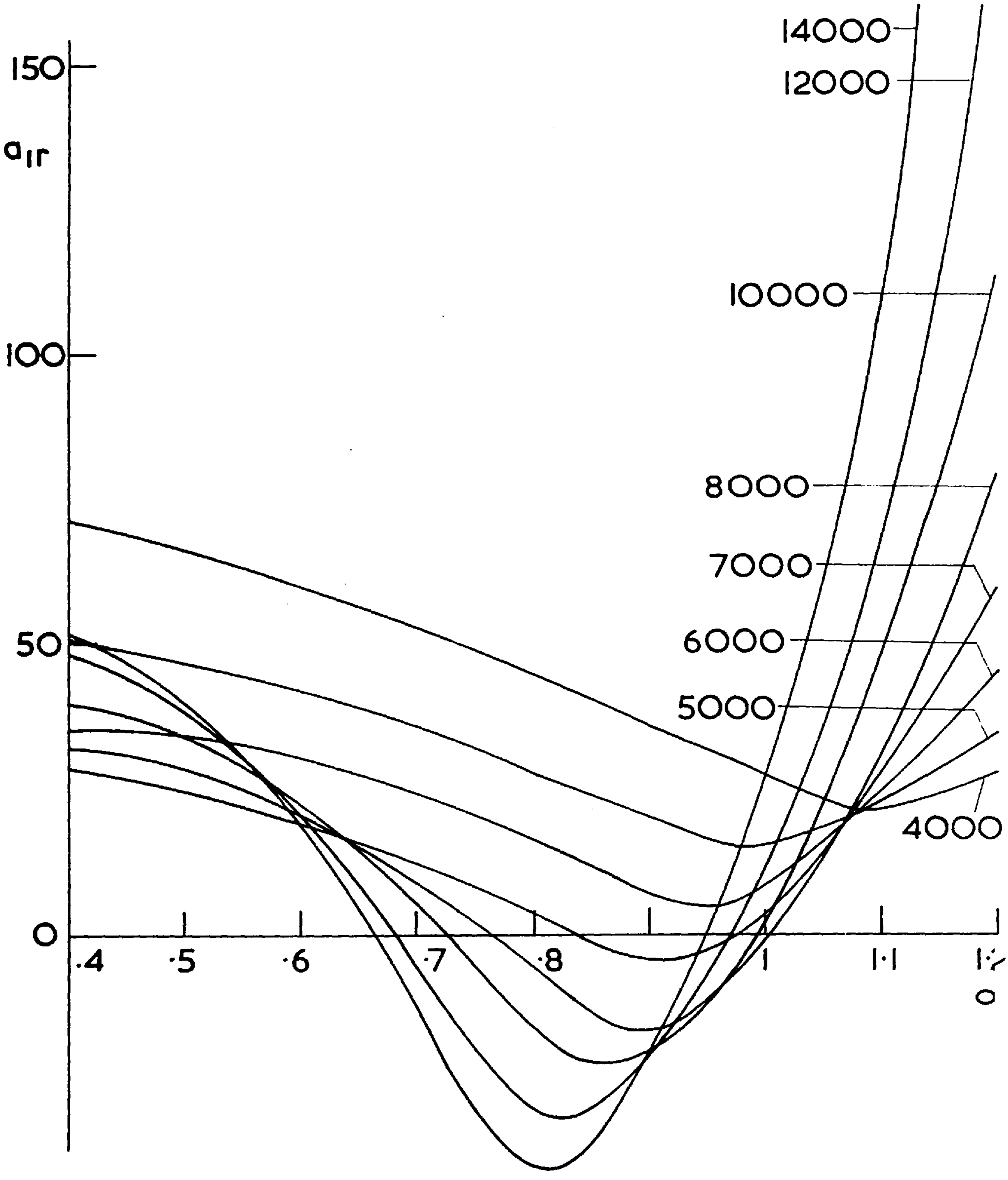


FIG. 4.6 a_{1r} FOR $\lambda = 0$ FOR VARIOUS VALUES OF THE REYNOLDS NUMBER

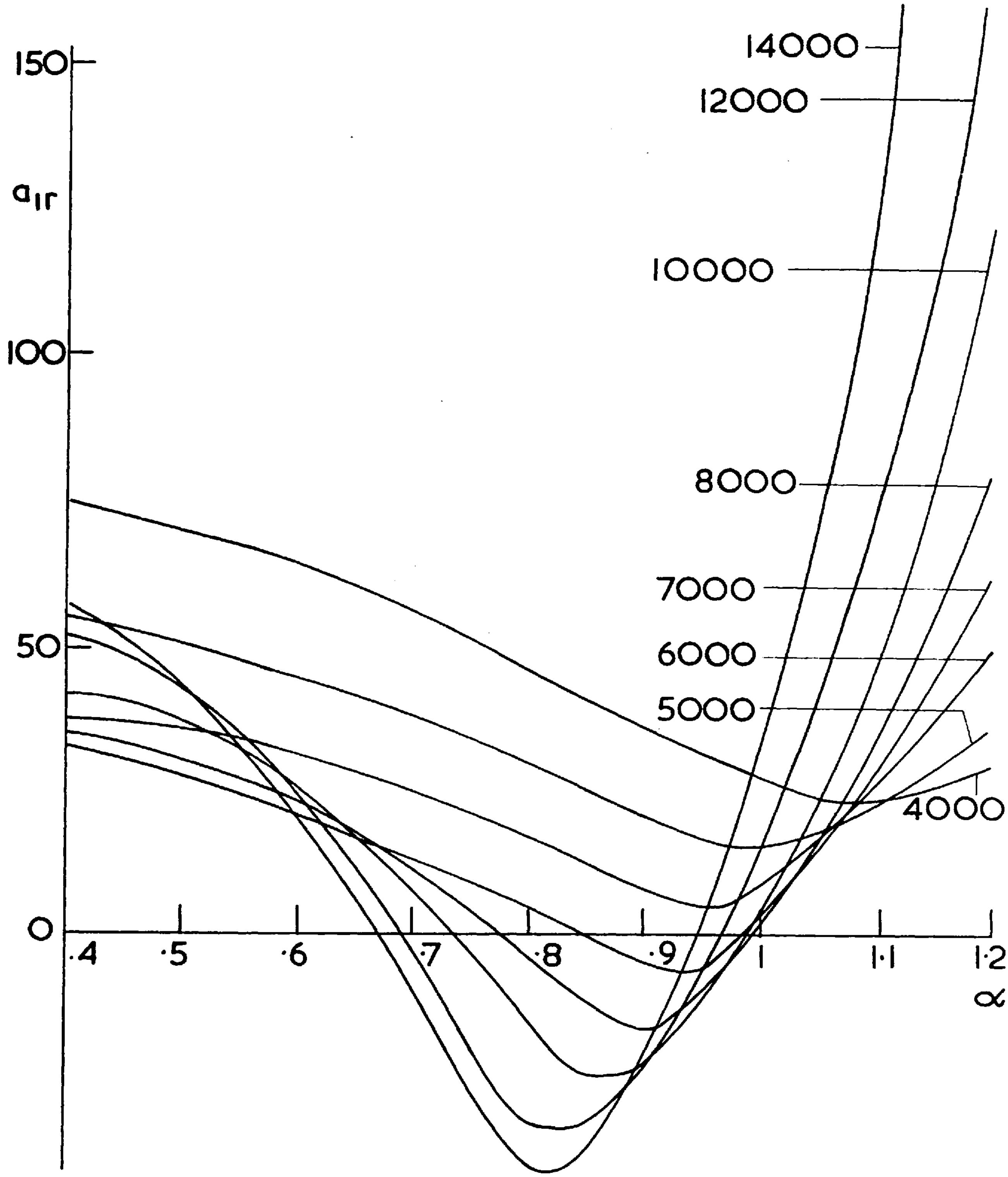


FIG. 4.7 a_{1r} FOR $\lambda = -10^{-5}$ FOR VARIOUS VALUES OF THE REYNOLDS NUMBER

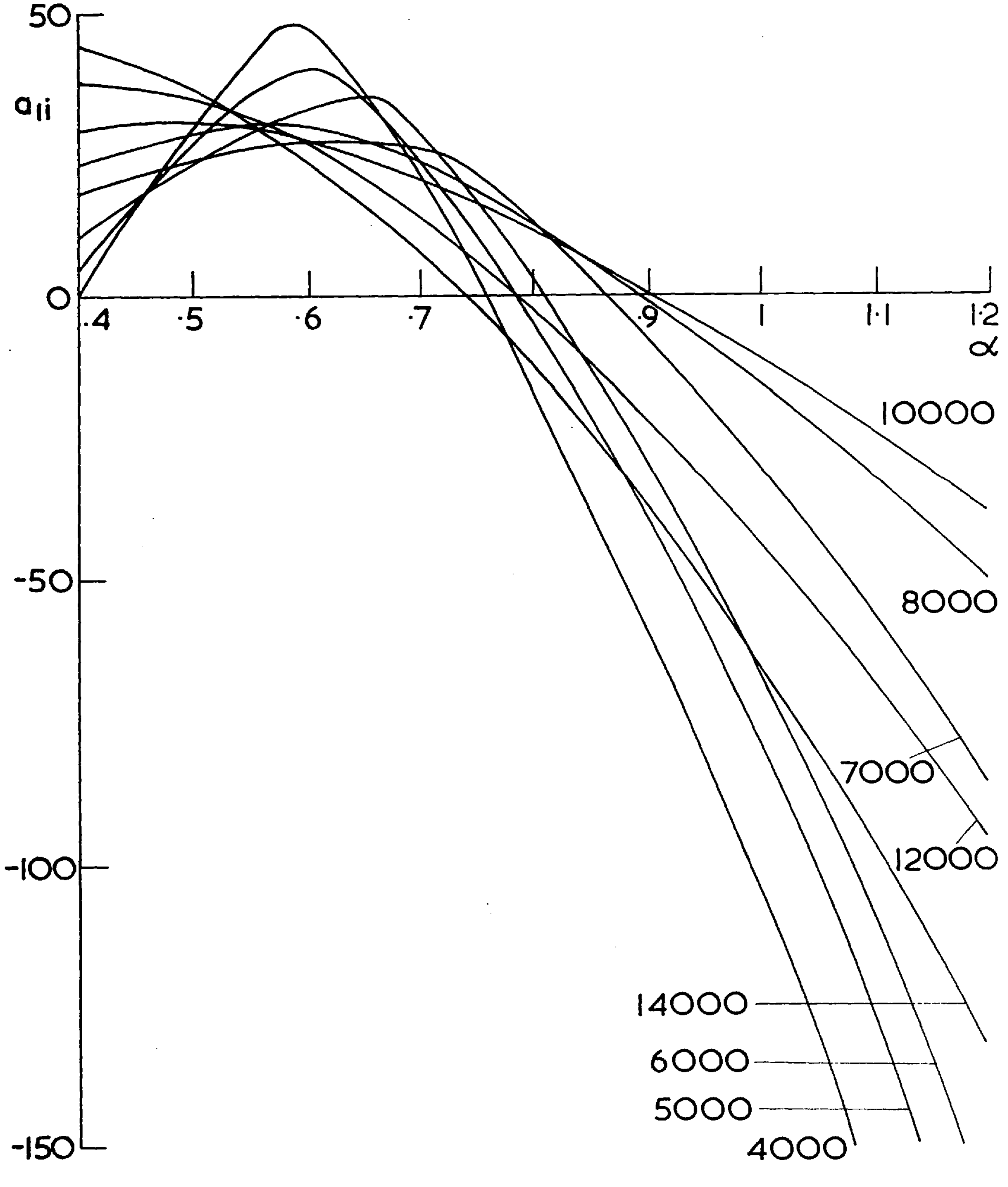


FIG. 4.8 a_{1i} FOR $\lambda = 0$ FOR VARIOUS VALUES OF THE REYNOLDS NUMBER

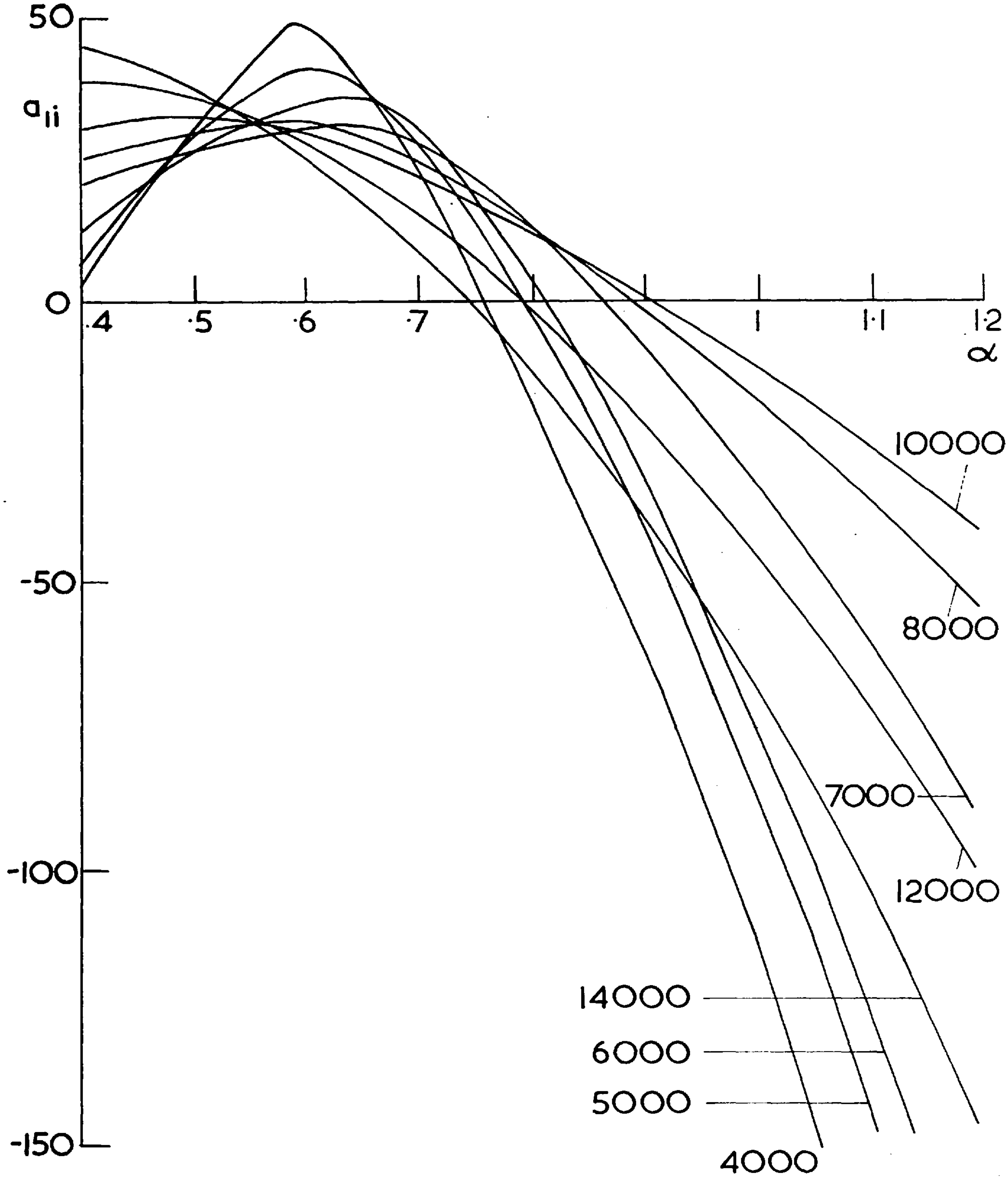
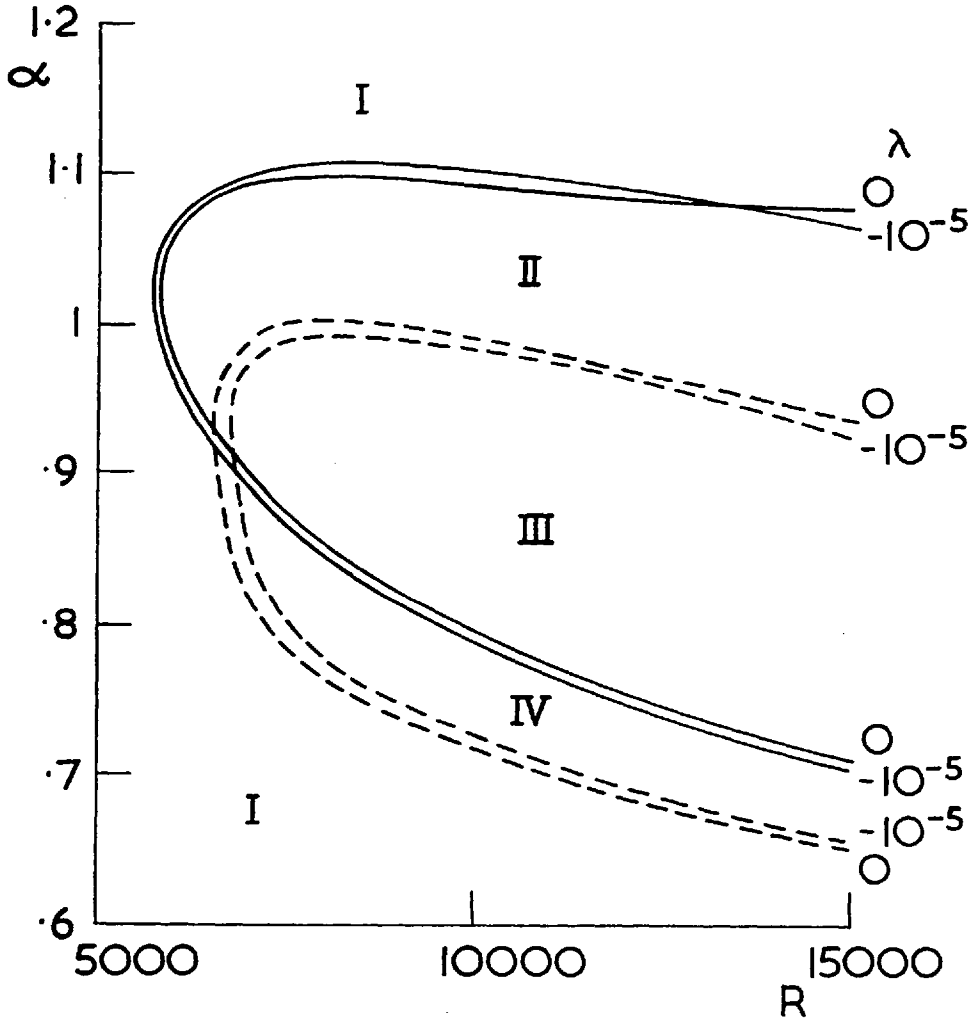


FIG. 4.9 a_{11} FOR $\lambda = -10^{-5}$ FOR VARIOUS VALUES OF THE REYNOLDS NUMBER



—— neutral stability curve ($C_i = 0$) for $\lambda = 0, -10^{-5}$

---- curve on which $a_{1r} = 0$ for $\lambda = 0, -10^{-5}$

In region I infinitesimal disturbances are stable, but finite amplitude instabilities exist. II is a region of instability. In IV all disturbances are stable. In region III disturbances are unstable under linear theory, but equilibrium states of finite amplitude exist, and viscoelasticity increases the equilibrium amplitude.

FIG. 4·10

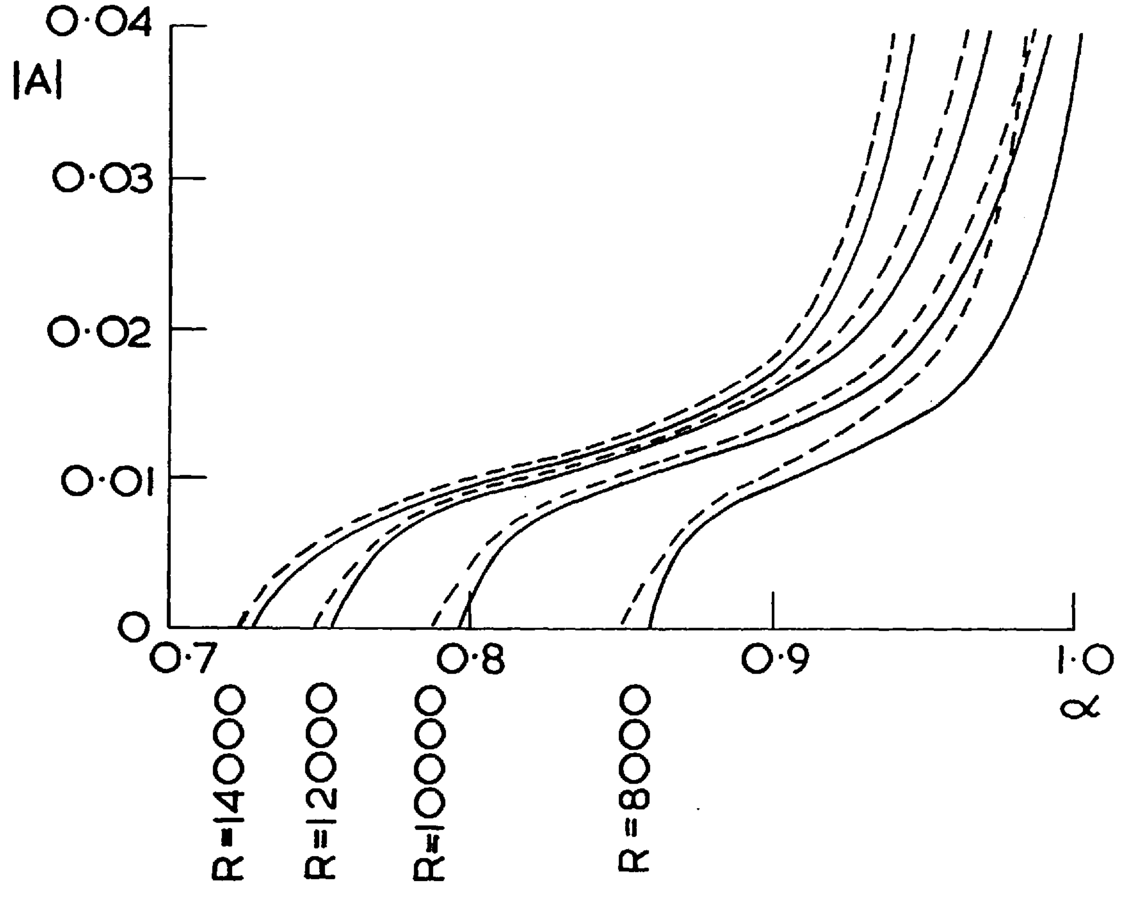


FIG. 4.11 VALUES OF THE SUPERCRITICAL EQUILIBRIUM AMPLITUDE IN REGION III OF FIG. 9 FOR $\lambda = 0$ (FULL CURVES) AND $\lambda = -10^{-5}$ (BROKEN CURVES)

disturbances destabilised by viscoelasticity, for disturbances in III would reach an equilibrium amplitude, but in II no such equilibrium is reached.

Points outside the neutral curve (I and IV) represent stable disturbances under linear theory, for c_i is negative, but non-linear effects result in instability in region I if the amplitude is greater than some finite value. We can estimate a reduction in the critical Reynolds number due to finite-amplitude disturbances. On the centre-line $(y=0)$ the fluctuation intensity is, to order $|A|$

$$\left(\overline{u_2^2}\right)^{\frac{1}{2}} \Big/ \bar{u}_1 = \sqrt{2}\,\alpha\,|A|\,|\psi_1|, \qquad (4.10.8)$$

where the overbar denotes an average with respect to x. We note that the perturbation in the x-direction, $u_1-\bar{u}_1$, is of order $|A|^2$ on $y=0$. Since $|\psi_1(0)|$ and $|\bar{u}_1(0)|$ are both unity the fluctuation intensity reduces to $\sqrt{2}\,\alpha\,|A|$ to order A. For a given value of the Reynolds number the minimum value of (4.10.8) was obtained from tables 4.1-4.6. Disturbances with fluctuation intensities below this minimum decay, while those with greater intensities are unstable, so this minimum value, plotted in figure 4.12, serves to define a relation between critical Reynolds number and the centre-line turbulence intensity present in a particular flow. The curve for $\lambda=0$ is similar in shape to that obtained by Reynolds and Potter (1967). The presence of viscoelasticity is destabilising in region I as the critical Reynolds number is reduced for a given intensity, as may be seen from figure 4.12. Porteous and Denn (1971) have solved an equation essentially the same as (4.8.2) and their results are broadly similar, as may be seen in figure 4.12.

The reasons given by Stuart (1960) and Watson (1960) for truncating the series expansion (4.4.7) after the second term are

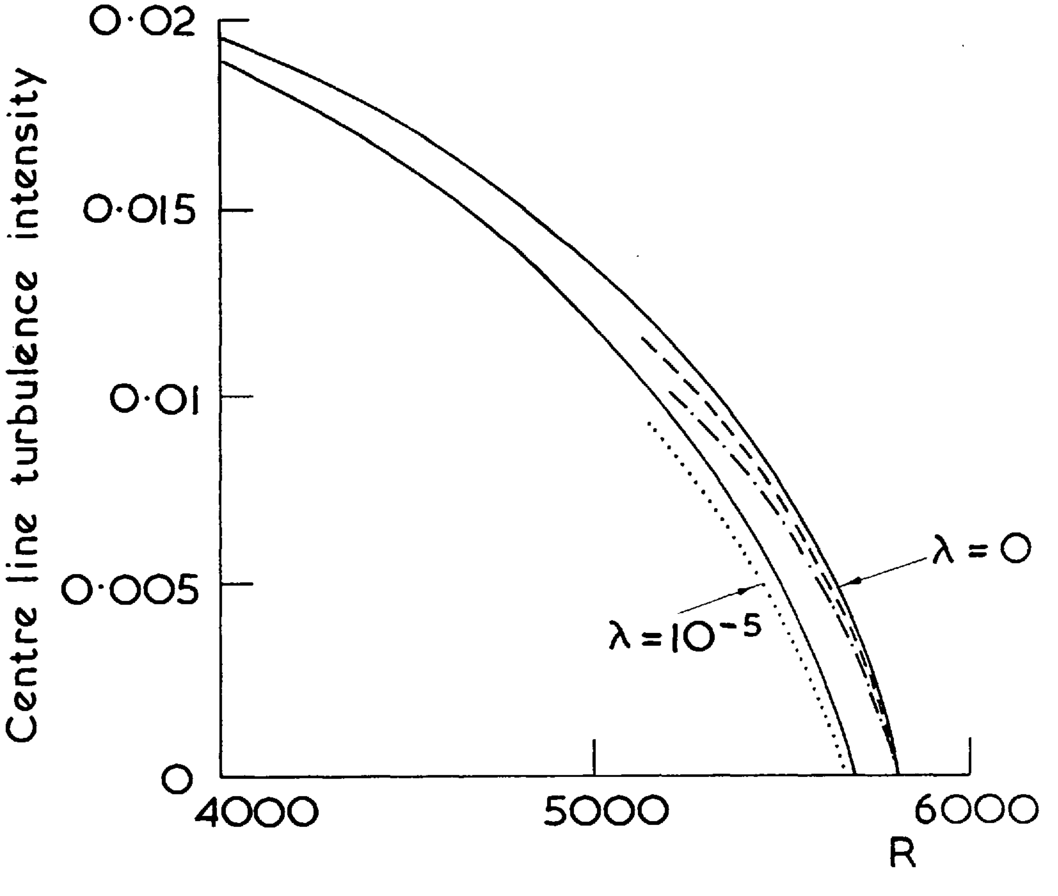


FIG. 4.12 VARIATION OF CRITICAL REYNOLDS NUMBER WITH CENTRE-LINE TURBULENCE INTENSITY FOR $\lambda = 0$ (— THIS PAPER, ---- REYNOLDS AND POTTER, -·-·- PORTEOUS AND DENN) AND $\lambda = -10^{-5}$ (— THIS PAPER, ········ PORTEOUS AND DENN)

based on certain magnitude estimates, and in particular require $c_i(\alpha R)^{1/3}$ to be small. We should therefore expect the method to be valid close to the neutral curve. To determine the region of validity we should need to calculate the next coefficient as in the amplitude equation (4.4.7), but this has not been undertaken as it would entail a prohibitive amount of computation.

Davey and Nguyen (1971) argue that for subcritical flows ($c_i<0$) it is necessary to impose the more restrictive condition $|c_i| \alpha R<<1$ and to suppose that the disturbance has already attained its equilibrium amplitude. If we set

$$n^2 = 2a_{or}R = -2\alpha c_i R \qquad (4.10.9)$$

then for $\cos n = 0$ the solution, f_1, of (4.8.5) for a Newtonian fluid may contain an arbitrary multiple of $\cos ny$ in the subcritical case, thus invalidating the subsequent calculations for these specific values of n. A similar argument holds for λ non-zero, though the definition of n is slightly different. It is not necessary to suppose that $|c_i| \alpha R<<1$ for the condition $|c_i| \alpha R<\pi^2/8$ will ensure that $\cos n \neq 0$ and the time-dependent approach of Stuart and Watson remains valid for subcritical flows, though the region of validity is restricted. For example, the values quoted in figure 4.12 for the lower range of Reynolds number must be viewed with some caution. However, provided we remain close enough to the neutral curve for subcritical flows it is clear that viscoelasticity has a destabilising influence, for at a given Reynolds number a lower turbulence intensity is required in the viscoelastic case for finite amplitude instability of the flow.

CHAPTER 5: EFFECT OF STRESS RELAXATION PARAMETERS ON THE STABILITY OF PLANE POISEUILLE FLOW

5.1 Limitations of Rivlin-Ericksen theory

Although the theory used in the previous two chapters does introduce non-Newtonian properties, especially normal stress effects, into the fluid, important features are neglected. The Rivlin-Ericksen description expresses stress in a fluid in terms of time derivatives of the rate of strain field evaluated at the time instant under consideration. The history of the motion is approximated, in a way essentially similar to a Taylor series expansion, by sufficient derivatives at one particular time, and there is no explicit use of the history in the final specification. Consequently properties such as stress relaxation are omitted. In this chapter some of the effects of stress relaxation are examined.

In the next section a Maxwell fluid is used. A constitutive relation introduced by Oldroyd is also included as it leads to similar stability equations. For both types of fluid the history is introduced by requiring the stress to satisfy a time dependent differential equation. Walters (1970) shows how these fluids are related to those of an integral type, and the remainder of this chapter is devoted to the integral representation of a fluid derived in section 2.4. The stress is expressed as integrals over past time, and specific dependence on history of the motion is determined by kernels in the integrands. By particular choices of the kernels both Newtonian and Rivlin-Ericksen theories can be extracted from this more general representation. In section 5.4 non-trivial stress relaxation behaviour is retained by means of decaying exponentials in the kernels, and linearised equations governing small disturbances to a parallel flow are derived. By comparing the resulting Orr-Sommerfeld equation with that obtained for a Newtonian fluid we deduce that the effect of stress relaxation is to decrease the Reynolds number at which instability sets in.

5.2 Maxwell and Oldroyd fluids

The Maxwell constitutive relation assumes that the material can be characterised by a single relaxation time θ and a constant kinematic viscosity ν. Dependence of stress on the history of the motion is achieved by making the stress T, extra to a hydrostatic pressure, satisfy a differential equation of the form

$$T_{ij} + E\frac{\vartheta\, T_{ij}}{\vartheta t} = \frac{1}{R}A_{1ij}, \tag{5.2.1}$$

where the elasticity number E is defined by

$$E = \nu\theta/L^2, \tag{5.2.2}$$

and L is a representative length scale, which we shall take as half the channel width as before. The other variables in (5.2.1) are expressed in non-dimensional form as in section (3.2). $\vartheta/\vartheta t$ is the convected derivative defined in (2.1.21). We note that as for a second order fluid there exists a solution of the form

$$v_1 = U \equiv 1-x_2^2, \quad v_2=v_3=0, \tag{5.2.3}$$

namely, undisturbed plane Poiseuille flow. As we are considering theories applicable to dilute polymer solutions we assume that elasticity effects are small and take $E \ll 1$, in which case (5.2.1) reduces to first order in E to

$$T = \frac{1}{R}\left(1 - E\frac{\vartheta}{\vartheta t}\right)A_1, \tag{5.2.4}$$

which is identical to the second-order fluid relation (3.2.2) with $\mu=0$. Platten and Schechter (1970) used a Maxwell model (5.2.1) to examine stability of the flow U=constant. Since the undisturbed flow may be reduced to rest by a suitable choice of axes we should expect the flow to be stable at all Reynolds numbers. Their analysis shows that for any Reynolds number there is an infinite set of frequencies for which the disturbances amplify, and therefore the flow is always unstable. Their argument assumes that elasticity effects are small and that the term $E\, \partial T_{ij}/\partial t$ is small compared with T_{ij}. The latter assumption breaks down for the high frequency disturbances which, according to their theory, amplify. A better constitutive equation is required to investigate such disturbances. High frequency disturbances would be damped by viscous dissipation and are not relevant to our discussion. The argument above, however, does illustrate the dangers of misusing an approximate theory, a point emphasised by Craik (1968).

Another constitutive relation that has attracted much attention in the literature of non-Newtonian fluids is that due to Oldroyd (1950), where the extra stress is given in non-dimensional form by

$$\left(1 + RE_1 \frac{\partial}{\partial t} \right) T = \left(\frac{1}{R} + E_2 \frac{\partial}{\partial t} \right) A_1 . \qquad (5.2.5)$$

This equation contains features of both Maxwell and second-order fluids, provided the term containing elasticity effects is small (5.2.5) reduces to

$$T = \frac{1}{R} A_1 + (E_2 - E_1) A_2 , \qquad (5.2.6)$$

and stability conditions are given by the theory for second-order fluids. A term in A_1^2 can easily be incorporated.

The interesting cases are those for which the viscoelastic terms in (5.2.1) or (5.2.5) are not small, although the parameters, E or E_1 and E_2, may themselves be small. These constitutive equations have been derived in a semi-empirical manner, though Walters (1970) has shown how they may be related to the more general integral forms derived in section 2.4. We shall use an integral representation for the succeeding analysis.

5.3 An integral representation of the stress

In an attempt to describe the history of the motion we shall use the integral representation of the stress tensor derived in section 2.4. Assuming that the response functional is sufficiently differentiable the stress in an incompressible isotropic fluid can be approximated in the form

$$T_{ij}=-p\delta_{ij} + \int_{-\infty}^{t} \phi(t-\tau)\left[C_t(\tau)\right]_{ij}d\tau + \int_{-\infty}^{t}\int_{-\infty}^{t}\psi(t-\tau_1,t-\tau_2)\left[C_t(\tau_1)C_t(\tau_2)\right]_{ij}d\tau_1 d\tau_2+\dots, \tag{5.3.1}$$

where δ_{ij} is the Knonecker delta, p a hydrostatic pressure and C_t the right Cauchy-Green tensor relative to time t. This equation is similar to (2.4.7), but it is more convenient here to work with C_t rather than the tensor G defined by (2.4.2). We shall redefine the (indeterminate) pressure at a later stage so that it includes terms from the integrals which arise from multiplies of the unit matrix contained within C_t. In their most general form the kernels of (5.2.1) are functions of the invariants of $\overset{(n)}{C_t}$ and $\overset{(n)}{C_t}\,\overset{(m)}{C_t}$ for m,n=1,2,3,..., but we shall only consider the dependence specifically indicated in (5.3.1).

It is convenient to use a reference configuration X_i introduced in section 2.1 and to allow the fluid particle X, whose reference coordinates are X_i, move to the point ξ_i after time τ and to point x_i after time t. For linear analysis of perturbations of a parallel flow we seek a velocity field of the form

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \{U(x_2),0,0\}+\{u_1(x_2),u_2(x_2),u_3(x_2)\}\exp(i\alpha x_1+i\beta x_3-i\omega t), \tag{5.3.2}$$

similar to equation (3.3.1), and we shall neglect products of the small quantities u_k. We may obtain this velocity field by taking

$$\left.\begin{aligned} X_1 &= x_1-U(x_2)t+\{a(x_2)+c(x_2)t\}\exp(z), \\ X_2 &= x_2+b(x_2)\exp(z), \\ X_3 &= x_3+d(x_2)\exp(z), \end{aligned}\right\} \tag{5.3.3}$$

where

$$\left.\begin{aligned} z &= i\alpha x_1+i\beta x_3-i\omega t, \\ a(s) &= -\left[u_1(s)\{i\alpha U(s)-i\omega\}+U'(s)u_2(s)\right]/\{i\alpha U(s)-i\omega\}^2, \\ b(s) &= -u_2(s)/\{i\alpha U(s)-i\omega\}, \\ c(s) &= U'(s)u_2(s)/\{i\alpha U(s)-i\omega\}, \\ d(s) &= -u_3(s)/\{i\alpha U(s)-i\omega\}, \end{aligned}\right\} \tag{5.3.4}$$

the prime denoting a derivative. Expressions similar to (5.3.2)-(5.3.4) hold with ξ_i, τ and $\zeta(\xi_i,\tau)$ replacing x_i, t and $z(x_i,t)$ respectively. It follows from (5.3.4) that

$$U'b+c = 0. \tag{5.3.5}$$

The continuity condition for an incompressible fluid (2.1.12) becomes

$$i\alpha u_1+u_2'+i\beta u_3 \;=\; 0, \tag{5.3.6}$$

or, from (5.3.4),

$$i\alpha a+b'+i\beta d \;=\; 0. \tag{5.3.7}$$

The right Cauchy-Green tensor $C_t(\tau)$ is defined by

$$\left[C_t(\tau)\right]_{ij} \;\equiv\; \frac{\partial\xi_k}{\partial x_i}\frac{\partial\xi_k}{\partial x_j} \;=\; \frac{\partial X_m}{\partial x_i}\frac{\partial\xi_k}{\partial X_m}\frac{\partial\xi_k}{\partial X_n}\frac{\partial X_n}{\partial x_j}\,, \tag{5.3.8}$$

and may be evaluated from (5.3.3) and the equivalent form $X_i=X_i(\underline{\xi},\tau)$. Before evaluating $C_t(\tau)$ it is convenient to obtain ξ_i as a function of $\underset{\sim}{x}$, t and τ. From the equation for particle paths, namely,

$$d\xi_i/d\tau \;=\; v_i(\underset{\sim}{\xi},\tau), \quad \xi_i(\tau{=}t) \;=\; x_i, \tag{5.3.9}$$

we obtain

$$\left.\begin{aligned} \xi_1 &= x_1-(t-\tau)U(x_2)+O(u_i), \\ \xi_2 &= x_2+b(x_2)\{\exp(z)-\exp(\zeta)\}+O(u_i^2), \\ \xi_3 &= x_3+O(u_i). \end{aligned}\right\} \tag{5.3.10}$$

Using Taylor expansions it then follows that

$$\left.\begin{aligned} U'(\xi_2) &= U'(x_2)+b(x_2)U''(x_2)\{\exp(z)-\exp(\zeta)\}+O(u_i^2), \\ u_i(\xi_2) &= u_i(x_2)+O(u_i^2), \\ f(\xi_2) &= f(x_2)+O(f^2) \quad \text{for} \quad f=a,b,c,d. \end{aligned}\right\} \tag{5.3.11}$$

Also

$$\exp(\zeta) \;=\; \exp\left[z-\{i\alpha U(x_2)-i\omega\}(t-\tau)\right] + O(u_i). \tag{5.3.12}$$

We now obtain

$$F^{-1}(t) \equiv \left\| \frac{\partial X_n}{\partial x_j} \right\| = \left\| \begin{matrix} 1+(i\alpha a+i\alpha ct)e^z & -U't+(a'+c't)e^z & (i\beta a+i\beta ct)e^z \\ i\alpha be^z & 1+b'e^z & i\beta be^z \\ i\alpha de^z & d'e^z & 1+i\beta de^z \end{matrix} \right\| , \tag{5.3.13}$$

where the functions U', a, b, c, d are evaluated at x_2. Now

$$\det F^{-1} = 1+(i\alpha a+i\alpha ct+b'+i\beta d+i\alpha U'bt)e^z+O(u_i^2), \tag{5.3.14}$$

which, by (5.3.4) and (5.3.7), becomes

$$\det F^{-1} = 1 + O(u_i^2), \tag{5.3.15}$$

We may therefore invert F^{-1} to obtain, to first order,

$$F(t) \equiv \left\| \frac{\partial x_k}{\partial X_m} \right\| = \left\| \begin{matrix} 1-i\alpha ae^z & U't-(a'+c't-i\beta U'dt)e^z & -i\beta ae^z \\ -i\alpha be^z & 1+(i\alpha ct-b')e^z & -i\beta be^z \\ -i\alpha de^z & -(i\alpha U'dt+d')e^z & 1-i\beta de^z \end{matrix} \right\| . \tag{5.3.16}$$

An expression similar in form holds for $\|\partial\xi_k/\partial X_m\|$, but with the functions evaluated at ξ_2. Using (5.3.11) and (5.3.12) we can rewrite $\|\partial\xi_k/\partial X_m\|$ in the form

$$F(\tau) \equiv \left\| \frac{\partial \xi_k}{\partial X_m} \right\| = \left\| \begin{matrix} 1-i\alpha ae^\zeta & U'\tau+(U''b\tau+i\beta U'd\tau-a'-c'\tau)e^\zeta-U''b\tau e^z & -i\beta ae^\zeta \\ -i\alpha be^\zeta & 1+(i\alpha c\tau-b')e^\zeta & -i\beta be^\zeta \\ -i\alpha de^\zeta & -(i\alpha U'd\tau+d')e^\zeta & 1-i\beta de^\zeta \end{matrix} \right\| \tag{5.3.17}$$

to first order, where U', U'', a, b, c, d are now evaluated at x_2. We are now in a position to express $C_t(\tau)$ in terms of $\exp(\zeta)$, $\exp(z)$, $t-\tau$ and functions evaluated at x_2. If we write

$$s = t-\tau, \tag{5.3.18}$$

equation (5.3.8) becomes, after some manipulation, to first order,

$$\left.\begin{aligned}
[C_t(\tau)]_{11} &= 1+2(i\alpha a-i\alpha U'bs)\exp(z)-2i\alpha a\exp(\zeta),\\
[C_t(\tau)]_{12} &= U's+(a'+i\alpha b+i\beta dU's-U''bs+i\alpha U'^2bs^2)\exp(z)\\
&\qquad +(-a'-i\alpha b+2i\alpha U'as)\exp(\zeta),\\
[C_t(\tau)]_{13} &= (i\beta a+i\alpha d-i\beta U'bs)\exp(z)-(i\beta a+i\alpha d)\exp(\zeta),\\
[C_t(\tau)]_{22} &= 1+U'^2s^2+2(b'-U'a's+U'^2b's^2+U'U''bs^2)\exp(z)\\
&\qquad +2(-b'+U'a's+i\alpha U'bs-i\alpha U'^2as^2)\exp(\zeta),\\
[C_t(\tau)]_{23} &= (i\beta b+d'-i\beta U'as+i\beta U'^2bs^2)\exp(z)\\
&\qquad +(-i\beta b-d'+i\beta U'as+i\alpha U'ds)\exp(\zeta),\\
[C_t(\tau)]_{33} &= 1+2i\beta d\exp(z)-2i\beta d\exp(\zeta),
\end{aligned}\right\} \tag{5.3.19}$$

where U', a, b, c, d and their derivatives are evaluated at x_2. In the next section determination of the stress will be completed by specifying particular kernels in the integrals of (5.3.1).

5.4 Effects of stress relaxation

Theories for Newtonian and Rivlin-Ericksen fluids can be incorporated in the constitutive equation (5.3.1) by use of the Dirac δ-function and its derivatives. For instance, a Newtonian fluid is described by

$$\phi(s) = \eta\delta'(s), \tag{5.4.1}$$

where δ' denotes the derivative of the δ-function, suitably defined,
and other kernels are zero, where η is the viscosity of the fluid.

Terms proportional to $\delta(s)$ may be included in the pressure. A second-order fluid is described by

$$\left.\begin{aligned} \phi(s) &= \eta\delta'(s) + \lambda\delta''(s), \\ \psi(s_1,s_2) &= \mu\delta(s_1)\,\delta(s_2), \end{aligned}\right\} \tag{5.4.2}$$

and other kernels zero, where λ and μ are the viscoelastic parameters used in previous chapters. To obtain the theories discussed in the last section it is convenient to integrate (5.3.1) by parts so that the integrands contain $\dot{C}_t(\tau)$ rather than $C_t(\tau)$. Walters (1970) then shows, by using decaying exponentials in the new kernels, that equations for the Maxwell (5.2.1) and Oldroyd (5.2.5) fluids may be recovered. We shall introduce stress relaxation behaviour by including decaying exponentials of the form $\nu\exp(-ks)$ in the kernels of (5.3.1), with ν and k positive. $1/k$ is a relaxation time. If the relaxation processes are characterised by a distribution of relaxation times their effect can be determined by replacing the single exponential by a sum $\sum_{n=1}^{N} \nu_n\exp(-k_n s)$. The form of the subsequent algebra is unaltered by additional exponentials, so for simplicity a kernel with a single decaying term is used, though it is possible at any stage of the calculation to observe the effect of more than one exponential. We shall examine in detail the fluid described by

$$\phi(s) = \eta\delta'(s) + \nu\exp(-ks), \tag{5.4.3}$$

with other kernels zero. The algebra is considerably simplified by omitting the second-order fluid terms, though their effect will be discussed later.

As in previous chapters it is convenient to work with non-dimensional quantities defined with respect to density ρ, channel width $2L$ and undisturbed centre-line velocity U^* (the characteristic time used in chapter 3 is given by $2L/U^*$). From hereon the variables refer to non-dimensional quantities. The Reynolds number of the flow is given by

$$R = U^*L\rho/\eta, \tag{5.4.4}$$

and we define a stress relaxation parameter

$$S = \nu/U^*L\rho. \tag{5.4.5}$$

If we separate the zero order and first order parts of the stress and pressure, namely

$$\left.\begin{aligned} p &= \bar{p}(\underline{x}) + S/k + \hat{p}(x_2)\ \exp(z), \\ T &= \bar{T}(x_2) + \hat{T}(x_2)\ \exp(z), \end{aligned}\right\} \tag{5.4.6}$$

we find, on substituting in the momentum equation

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial T_{ij}}{\partial x_j}, \tag{5.4.7}$$

that the undisturbed flow satisfies

$$\frac{1}{R} U'' + \frac{\partial \bar{p}}{\partial x_1} = 0 = \frac{\partial \bar{p}}{\partial x_2} = \frac{\partial \bar{p}}{\partial x_3}. \tag{5.4.8}$$

Flow between stationary parallel planes situated at $x_2=\pm 1$ is then given by

$$U(x_2) = 1-x_2^2. \tag{5.4.9}$$

It is worth remarking that the stress relaxation term does not alter the flow or the pressure gradient required to drive the flow. The second-order fluid terms, too, have no effect on the flow of the pressure gradient (see section 3.3). We note also that the part of the integrand in (5.2.1) which is proportional to the unit matrix leads to the term S/k in (5.4.6), which is incorporated in the redefined zero order pressure.

The first order terms in the momentum equation (5.4.7) give, using (5.2.19),

$$\begin{aligned}(i\alpha U-i\omega)u_1+U'u_2+i\alpha\hat{p} = {} & -(i\alpha U-i\omega)\left\{\frac{1}{R}-\frac{S}{k(k+i\alpha U-i\omega)}\right\}(a''-\alpha^2a-\beta^2a)\\ & -\frac{1}{R}(2i\alpha U'a'+i\alpha U''a)\\ & +\frac{1}{R}\{U'(b''-\alpha^2b-\beta^2b)+2U''b'\}\\ & -\frac{S}{k^2}\{U'(b''-2\alpha^2b-\beta^2b)-2U''b'+i\alpha U'a'-i\alpha U''a)\}\\ & +\frac{S}{(k+i\alpha U-i\omega)^2}(3i\alpha U'a'+2i\alpha U''a-\alpha^2U'b)\\ & +\frac{S}{k^3}(2i\alpha U'^2b'+4i\alpha U'U''b)\\ & +\frac{S}{(k+i\alpha U-i\omega)^3}4\alpha^2U'^2a,\end{aligned} \qquad (5.4.10)$$

$$\begin{aligned}(i\alpha U-i\omega)u_2+\hat{p}' = {} & -(i\alpha U-i\omega)\left\{\frac{1}{R}-\frac{S}{((k+i\alpha U-i\omega)}\right\}(b''-\alpha^2b-\beta^2b)\\ & -\frac{1}{R}(2i\alpha U'b'+i\alpha U''b)\end{aligned}$$

$$+ \frac{S}{k^2} \{U'(\beta^2 a-\alpha\beta d-2a'')+U''(-i\alpha b-2a')\}$$

$$+ \frac{S}{(k+i\alpha U-i\omega)^2} \{U'(2a''-\alpha^2 a-\beta^2 a+3i\alpha b')+U''(2i\alpha b+2a')\}$$

$$+ \frac{S}{k^3} \{U'^2(4b''-2\alpha^2 b-2\beta^2 b)+12U'U''b'+4U''^2 b\}$$

$$+ \frac{S}{(k+i\alpha U-i\omega)^3} \{U'^2(4\alpha^2 b-8i\alpha a')-8i\alpha U'U''a\}$$

$$- \frac{S}{(k+i\alpha U-i\omega)^2} 12\alpha^2 U'^2 a, \qquad (5.4.11)$$

$$(i\alpha U-i\omega)u_3+i\beta\hat{p} = -(i\alpha U-i\omega)\left\{\frac{1}{R} - \frac{S}{k(k+i\alpha U-i\omega)}\right\}(d''-\alpha^2 d-\beta^2 d)$$

$$- \frac{1}{R}(2i\alpha U'd'+i\alpha U''d)$$

$$+ \frac{S}{k^2}(\alpha\beta U'b-i\beta U'a'-i\beta U''a)$$

$$+ \frac{S}{(k+i\alpha U-i\omega)^2}(2i\alpha U'd'+i\alpha U''d-\alpha\beta U'b+i\beta U'a'+i\beta U''a)$$

$$+ \frac{S}{k^3}(2i\beta U'^2 b'+4i\beta U'U''b)$$

$$+ \frac{S}{(k+i\alpha U-i\omega)^3}(2\alpha\beta U'^2 a+2\alpha^2 U'^2 d). \qquad (5.4.12)$$

Since we are concerned with dilute polymer solutions for which the viscoelastic effects are small we expect the stress to decay rapidly. Accordingly we shall take

$$k \gg 1 \tag{5.4.13}$$

and neglect terms of order k^{-3}. Equations (5.4.10)-(5.4.12) then reduce to

$$(i\alpha U-i\omega)u_1+U'u_2+i\alpha\hat{p} = -\left(\frac{1}{R}-\frac{S}{k^2}\right)\{(i\alpha U-i\omega)(a''-\alpha^2-\beta^2 a)+2i\alpha U'a'+i\alpha U''a\}$$

$$+\left(\frac{1}{R}-\frac{S}{k^2}\right)\{U'(b''-\alpha^2 b-\beta^2 b)+2U''b'\}, \tag{5.4.14}$$

$$(i\alpha U-i\omega)u_2+\hat{p}' = -\left(\frac{1}{R}-\frac{S}{k^2}\right)\{(i\alpha U-i\omega)(b''-\alpha^2 b-\beta^2 b)+2i\alpha U'b'+i\alpha U''b\}, \tag{5.4.15}$$

$$(i\alpha U-i\omega)u_3+i\beta\hat{p} = -\left(\frac{1}{R}-\frac{S}{k^2}\right)\{(i\alpha U-i\omega)(d''-\alpha^2 d-\beta^2 d)+2i\alpha U'd'+i\alpha U''d\}. \tag{5.4.16}$$

If we introduce the following transformations, based on those being used by Squire (1933),

$$\left.\begin{array}{l} \tilde{\alpha}^2 = \alpha^2+\beta^2, \quad \alpha\tilde{R}^{-1} = \tilde{\alpha}(R^{-1}-S/k^2), \quad \tilde{\alpha}\tilde{u}_1 = \alpha u_1+\beta u_3 \\ \tilde{\alpha}^2\tilde{a} = \alpha(\alpha a+\beta d), \quad \alpha\tilde{p} = \tilde{\alpha}\hat{p}, \quad \tilde{u}_2 = u_2, \tilde{\alpha}\tilde{b}=\alpha b, \quad \alpha\tilde{\omega} = \tilde{\alpha}\omega, \end{array}\right\} \tag{5.4.17}$$

a linear combination of (5.4.14) and (5.4.16) gives

$$(i\tilde{\alpha}U-i\tilde{\omega})\tilde{u}_1+U'\tilde{u}_2+i\tilde{\alpha}\tilde{p} = -\tilde{R}^{-1}\{(i\tilde{\alpha}U-i\tilde{\omega})(\tilde{a}''-\tilde{\alpha}^2\tilde{a})+2i\tilde{\alpha}U'\tilde{a}'+i\tilde{\alpha}U''\tilde{a}\},$$

$$+\tilde{R}^{-1}\{U'(\tilde{b}''-\tilde{\alpha}^2\tilde{b})+2U''\tilde{b}'\}, \tag{5.4.18}$$

while (5.4.15) becomes

$$(i\tilde{\alpha}U-i\tilde{\omega})\tilde{u}_2+\tilde{p}' = -\tilde{R}^{-1}\{(i\tilde{\alpha}U-i\tilde{\omega})(\tilde{b}''-\tilde{\alpha}^2\tilde{b})+2i\tilde{\alpha}U'\tilde{b}'+i\tilde{\alpha}U''\tilde{b}\} \tag{5.4.19}$$

The last two equations are identical in form to (5.4.14) and (5.4.15), showing that, when (5.4.13) holds, Squire's theorem is applicable, and three-dimensional disturbances are equivalent to a two-dimensional flow under the transformations (5.4.17). Since $\tilde{\alpha} \geq \alpha$ critical conditions when the flow first becomes unstable are given by the analysis for two-dimensional disturbances. These results do not hold if we retain higher order terms in k^{-1}, and Lockett (1969a) has shown that they are not valid for second-order fluids. However, we are concerned in this section with the particular effect of a small measure of elasticity for which equations (5.4.14)-(5.4.16) do apply, and in this case Squire's theorem remains valid. Eliminating pressure from these equations and using continuity equations (5.3.6) and (5.3.7) we obtain

$$(i\alpha U-i\omega)(D^2-\alpha^2-\beta^2)u_2-i\alpha U''u_2 = -\left(\frac{1}{R}-\frac{S}{k^2}\right)\{(i\alpha U-i\omega)(D^2-\alpha^2-\beta^2)^2b+4i\alpha U'b''' +6i\alpha U''b''-4i\alpha(\alpha^2+\beta^2)U'b'-2i\alpha(\alpha^2+\beta^2)U''b\}, \tag{5.4.20}$$

which, from (5.3.4), reduces to

$$(i\alpha U-i\omega)(D^2-\alpha^2-\beta^2)u_2-i\alpha U''u_2 - \left(\frac{1}{R}-\frac{S}{k^2}\right)(D^2-\alpha^2-\beta^2)^2u_2 = 0. \tag{5.4.21}$$

This equation is the Orr-Sommerfeld equation for a Newtonian fluid with R^{-1} replaced by $R^{-1}-S/k^2$. We define a number

$$R^* = 1/(R^{-1}-S/k^2). \tag{5.4.22}$$

For given values of R^* and the wavenumbers α and β equation (5.4.21) together with boundary conditions $u_2=u_2'=0$ on the walls define an eigenvalue problem to determine the frequency ω. If the imaginary part of ω is positive the disturbance grows exponentially, indicating instability, while if it is negative the disturbance decays and the flow is stable. Neutral conditions are given by real values of ω. The least value of R^* for which ω is real determines conditions under which instability is about to set in. The results of Squire discussed above show that critical conditions are obtained when $\beta=0$. By comparison with (3.4.1) critical conditions occur when $R^* = R_c (\simeq 5774)$, the critical Reynolds number for a Newtonian fluid. The critical Reynolds number, R^+, for a fluid exhibiting stress relaxation is then given by

$$\frac{1}{R_c} = \frac{1}{R^+} - \frac{S}{k^2} . \qquad (5.4.23)$$

Since S is positive we deduce at once that $R^+ < R_c$. Viscoelastic effects that can be expressed in the particular form of stress relaxation discussed in this section are therefore destabilising.

Although Squire's theorem no longer holds when second-order Rivlin-Ericksen terms are included, a two-dimensional analyses will still suffice to determine critical conditions when $2\lambda+\mu$, which is a measure of the second normal stress difference, is less than a certain positive number (see section 3.8). It is convenient to retain terms of k^{-3}, and under these circumstances (5.4.21) is replaced by

$$(i\alpha U-i\omega)(D^2-\alpha^2)u_2-i\alpha U''u_2 - \left\{\frac{1}{R} - \frac{S}{k^2} + \left(\lambda + \frac{S}{k^3}\right)(i\alpha U-i\omega)\right\} (D^2-\alpha^2)^2u_2 = 0. \qquad (5.4.24)$$

As previously discussed the relaxation term proportional to S/k^2 is a destabilising influence as it increases the effective Reynolds number, R^* (5.4.22). The second-order fluid parameter, λ, is negative, and its influence is destabilising. Although this destabilising effect is reduced, and may even be reversed, by the term proportional to S/k^3, the greater influence of the term in S/k^2

dominates. Hence, the second-order fluid also is destabilised by the addition of stress relaxation terms. Squire's theorem is also invalid when the restriction $k \gg 1$ is removed, but in this case, too, we expect stress relaxation to be destabilising as some of the terms in the equations corresponding to (5.4.14)-(5.4.16) can be grouped to give the number, R^*, defined in (5.4.22).

There seems little prospect of extending the integral constitutive equation for disturbances of finite amplitude. Non-linear effects would make the determination of the particle paths (5.3.10) and matrices required to evaluate the Cauchy-Green tensor exceedingly difficult.

CHAPTER 6: SOME CONCLUSIONS

In the introductory chapter the phenomenon of turbulent drag reduction by the addition of minute quantities of certain polymer to the solvent was described. In some way, elongation of the polymer molecules in the direction of the mean flow is responsible for a thickening of the laminar sublayer of fluid close to the wall. The effect of molecular elongation can be interpreted from a continuum viewpoint in terms of normal stress differences. There is no satisfactory theory of turbulence for Newtonian fluids, let alone the non-Newtonian solutions that exhibit drag reduction. In order to make a start on an otherwise intractable problem a stability theory for non-Newtonian fluids is developed in chapters 3-5. The theory deals with conditions obtaining when laminar flow begins to break down. The initial and subsequent stages of transition from laminar to turbulent flow are described in order to establish the relevance of stability analyses to an understanding of turbulence.

A general theory of continuum mechanics is described in chapter 2, and the constitutive relations describing the particular fluids used in chapters 3-5 are placed in the context of this general theory. Under various assumptions models exhibiting (1) normal stress effects and (2) stress relaxation are derived.

For a second order fluid, which exhibits normal stress effects, a linear stability analysis is developed in chapter 3. Such an analysis is applicable to the first stage of transition. For a Newtonian fluid Squire's theorem is valid, and the analysis may consequently be restricted to two-dimensional disturbances. The first growing disturbance will lie in the plane defined by the streamwise and cross-channel directions. For a non-Newtonian fluid Squire's theorem is not valid, and a fully three-dimensional analysis is used to determine conditions under which the first growing disturbance will travel outside the plane defined above. Further (non-linear) analysis would be required to determine the direction of the first growing disturbance, but the theory developed in chapter 4 was too complicated for calculations to be made in this case. When the first growing disturbance is out-of-plane a longitudinal vortex structure is established in the region close to the wall. The importance of this structure in the subsequent transition to and maintenance of turbulent flow is discussed in section 1.2. This structure is not present at the initial breakdown of laminar flow in a Newtonian fluid, though it does appear at a later stage of transition. In order to assess the significance of this difference in flow fields of Newtonian and non-Newtonian fluids at the onset of instability we should need to follow the subsequent development of turbulence, but the prospect is daunting! In addition to predicting this change in structure of the flow field linear theory establishes that viscoelastic effects as expressed in the second-order fluid model destabilise the flow, that is, the critical Reynolds number, R_c, is less for dilute polymer solutions than for the solvent alone.

Linear theory deals only with the initial onset of instability, and in order to examine further stages of transition a non-linear theory is developed in chapter 4. The possibility of equilibrium states in which disturbances oscillate with a steady finite amplitude is established. A three-dimensional analysis governing the interaction of three fundamental modes of oscillation is described, and a formidable sequence of equations is obtained. The solution of these equations will have to await the development of more powerful computer, and calculations were only made in the special case of a single two-dimensional wave interacting with itself, its harmonics and the mean flow. For this case the existence of finite amplitude equilibrium states in the supercritical region (unstable under linear theory) is established. The equilibrium amplitude given wave number and Reynolds number is greater in a viscoelastic fluid than in a Newtonian fluid, indicating that there is a greater transfer of energy from the mean flow to the disturbance in the viscoelastic case. In the subcritical zone, where infinitesimal disturbances decay, the results show that finite amplitude waves can lead to instability of the flow provided the initial amplitude is sufficiently large. The critical Reynolds number is now dependent of the maximum amplitude of disturbances present in the flow, and this maximum is a measure of the intensity of turbulence. For a given level of turbulence the critical Reynolds number is less for a non-Newtonian fluid than for a Newtonian liquid, confirming the result of linear theory that viscoelasticity is in general a destabilising influence.

The two-dimensional analysis considered above does not, clearly, lead to the direction of propagation of the first growing disturbance suggested by linear theory. With considerable addition complications, one could examine the non-linear development of a three-dimensional wave ($A=C=0$, $B\neq0$ in equation 4.4.1), or the interaction of two three-dimensional waves ($A=0$, $B\neq0$, $C\neq0$). The latter may be more appropriate for future study, since it is the square of the spanwise wave number rather than the wavenumber itself that is involved in determining the critical direction of the initial instability. With B and C non-zero we have two waves equally inclined to the x_1x_2-plane that would be relevant for this analysis. Under suitable conditions their interaction would produce standing waves in the spanwise direction and generate streamwise vorticity and the longitudinal streak pattern that are important in the later stages of transition and the maintenance of fully developed turbulent flow. The additional difficulties in performing the calculation can be most easily seen by comparing the eigen-problems (4.5.14) and (4.5.15). For the two-dimensional case (4.5.14) the matrix operator decomposes to give a single fourth order differential equation for the eigenfunction. (4.5.15), however, gives two simultaneous differential equations in ϕ and σ, which leads to a sixth order equation in ϕ alone. The choice for future study is to tackle the involved and lengthy calculations that (4.5.15) and subsequent equations require or else to develop some simpler theory that is more tractable, yet includes the essential features of non-linear interactions.

Viscoelastic fluids have properties other than normal stress differences, and in chapter 5 the effect on stability of another property, stress relaxation is examined. The contribution of the history of the motion to the stress is incorporated in an integral representation of the stress. The formulation is unfortunately too complicated to consider anything other than a linear theory governing infinitesimal disturbances. The second-order fluid terms are neglected to isolate the stress relaxation effects. Provided the relaxation time is sufficiently small Squire's theorem remains valid and leads to a modified form of the Orr-Sommerfeld equation, from which we deduce that stress relaxation is a destabilising influence. It is difficult to see how to develop this approach further. The results of chapter 5 reinforce those obtained earlier, that viscoelastic properties are in general destabilising.

The work reported here has been confined to channel flow, and one area of future study would be to examine the stability of boundary layers. Since viscoelasticity modifies the undisturbed velocity profile comparisons with the Newtonian case are much harder to make, and it would be more difficult to assess the influence of non-Newtonian properties on the flow structure.

Although we have not explained the Toms' effect we have shown that the viscoelastic properties of normal stress differences and stress relaxation have a destabilising influence. The results point to a new flow structure at the onset of instability, and we have discussed its significance in the subsequent development and maintenance of turbulence flow. The results obtained make some contribution to the long process of understanding turbulent drag reduction.

REFERENCES

D.J. Benney, 1961, J. Fluid Mech. 10, 209.

D.J. Benney, 1964, Phys. Fluids 7, 319.

D.J. Benney and C.C. Lin, 1960, Phys. Fluids 3, 656.

T.P. Black, 1966, Proc. Heat Transfer and Fluid Mechanics Institute. Stanford U.P.

C.F. Chan Man Fong and K. Walters, 1965, J. Méc. 4, 439.

B.D. Coleman and H. Markovitz, 1964, J. Appl. Phys. 35, 1.

B.D. Coleman and W. Noll, 1960, Arch. Rat. Mech. Anal. 1, 1.

B.D. Coleman and W. Noll, 1961a, Ann. N.Y. Acad. Sci. 89, 672.

B.D. Coleman and W. Noll, 1961b, Rev. Mod. Phys. 33, 239.

A.A. Craik, 1968, J. Fluid Mech. 33, 33.

A. Davey and H.P.F. Nguyen, 1971, J. Fluid Mech. 45, 701.

S.J. Davies and C.M. White, 1928, Proc. Roy. Soc. A119, 92.

P.M. Eagles, 1969, Quart. J. Mech. Appl. Math. 22, 129.

C. Elata, J. Lehrer and A. Kahanovitz, 1966, Israel J. Tech. 4, 87.

J.L. Ericksen, 1960, Handbuch der Physik III/1, Springer, appendix to Classical Field Theories.

J.L. Ericksen, 1962, Trans. Soc. Rheol. 6, 275.

G.E. Gadd, 1965, Nature 206, 463.

G.E. Gadd, 1966, Nature 212, 874.

M. Gaster, 1962, J. Fluid Mech. 14, 222.

M. Gaster, 1965, Prog. Aeron. Sci. 6, 251.

M. Gaster, 1965, J. Fluid Mech. 22, 433.

M. Gaster and A. Davey, 1968, J. Fluid Mech. 32, 801.

Y. Goren, 1966, thesis, University of Liverpool.

V.A. Gorodtsov and A.I. Leonov, 1967, J. Math. Mech. 31, 310.

A.E. Green and R.S. Rivlin, 1957, Arch. Rat. Mech. Anal. 1, 1.

H.P. Greenspan and D.J. Benney, 1963, J. Fluid Mech. 15, 133.

D. Grohne, 1969, AVA Göttingen Report 69-A-30.

A.S. Gupta, 1967, J. Fluid Mech. 28, 17.

A.S. Gupta and L. Rai, 1967, Proc. Camb. Phil. Soc. 63, 527.

A.S. Gupta and L. Rai, 1968, J. Fluid Mech. 33, 87.

W. Heisenberg, 1924, Ann. Phys. Lpz. (4) 74, 577.

H.C. Hershey and J.L. Zakin, 1967, Ind. Eng. Chem. Fundamentals, 6, 381.

H. Holstein, 1950, Z. angew. Math. Mech. 30, 25.

J.W. Hoyt and A.G. Fabula, 1964, Proc. Fifth Symp. Naval Hydrodynamics, 947.

T.H. Hughes and W.H. Reid, 1965, J. Fluid Mech. 23, 715.

E.L. Ince, 1944, Ordinary Differential Equations, Dover, p210.

D.T. Jones, 1967, thesis, University of Wales.

D.T. Jones and K. Walters, 1968, A.I. Ch. E. Jl. 14, 658.

R.E. Kaplan, 1964, MIT report ASRL-TR-116-1.

H.T. Kim, S.J. Kline and W.C. Reynolds, 1971, J. Fluid Mech. 50, 133.

P.S. Klebanoff and K.D. Tidstrom, 1959, NASA TN D-195.

P.S. Klebanoff, K.D. Tidstrom and L.M. Sargent, 1962, J. Fluid Mech. 12, 1.

S.J. Kline, 1967, Fluid Mechanics of Internal Flow (ed. Gino Sovran), Elsevier 27.

L.S.G. Kovasznay, H. Komoda and B.R. Vasudeva, 1962, Proc. 1962 Heat Trans., Fluid, Mech. Inst., Stanford Univ. Press, 1.

L.H. Lee and W.C. Reynolds, 1967, Q.J. Mech. Appl. Math. 20, 1.

C.C. Lin, 1944, Proc. Nat. Acad. Sci. Washington 30, 316.

C.C. Lin, 1945, Q. Appl. Math. 3, 117, 218 and 277.

C.C. Lin, 1967, Theory of Hydrodynamic Stability, Cambridge.

A.T. Listrov, 1965, P.M.T.F. 5, 102.

R.C. Lock, 1955, Proc. Roy. Soc. A233, 105.

F.J. Lockett, 1969, Int. J. Eng. Sci. 7, 337.

F.J. Lockett, 1969, Nature 222, 937.

J.L. Lumley, 1969, Ann. Rev. of Fluid Mech. 1, 367.

D. Meksyn and J.T. Stuart, 1951, Proc. Roy. Soc. A208, 517.

E.W. Merill, K.A. Smith, H. Shin and H.S. Mickley, 1966, Trans. Soc. Rheol. 10, 335.

A.B. Metzner and M.G. Park, 1964, J. Fluid Mech. 20, 291.

K.A. Meyer and S.J. Kline, 1961, Report MD-7, Thermosciences Division, Mech. Eng. Dept., Stanford Univ.

D.H. Michael, 1964, J. Fluid Mech. 18, 19.

D.T. Mook, 1967, Univ. of Michigan Tech. Rep. 06505-2-T.

P.R. Nachtsheim, 1964, NASA TN D-2414.

W. Noll, 1958, Arch. Rat. Mech. Anal. 2, 197.

J.G. Oldroyd, 1950, Proc. Roy. Soc. A200, 523.

J.G. Oldroyd, 1958, Proc. Roy. Soc. A245, 278.

W.M. Orr, 1907, Proc. Roy. Irish Acad. A27, 9.

M.R. Osborne, 1967, SIAM J. Appl. Math. 15, 539.

C.L. Pekeris and B. Shkoller, 1967, J. Fluid Mech. 29, 31.

J. Platten and R.S. Schechter, 1970, Phys. Fluids 13, 832.

K.C. Porteous and M.M. Denn, 1971, Univ. of Delaware Water Resources Center, No. 16.

Lord Rayleigh, 1880, Scientific Papers 1, 474.

Lord Rayleigh, 1913, Scientific Papers 6, 197.

W.H. Reid, 1965, Basic Developments in Fluid Dynamics Vol. 1 (ed M. Holt) Academic Press.

O. Reynolds, 1884, Scientific Papers 2, 153.

O. Reynolds, 1895, Scientific Papers 2, 535.

W.C. Reynolds and M.C. Potter, 1967, J. Fluid Mech. 27, 465.

R.S. Rivlin, 1955, J. Rat. Mech. Anal. 4, 681.

R.S. Rivlin and J.L. Ericksen, 1955, J. Rat. Mech. Anal. 4, 323.

P.W. Runstadler, S.J. Kline and W.C. Reynolds, 1963, Report MD-8, Thermosciences Div., Mech. Eng. Dept., Stanford Univ.

P.G. Saffman and J.S. Turner, 1956, J. Fluid Mech. 1, 16.

G.S. Schubauer and H.K. Skramstad, 1943, NACA Report 909.

W.H. Schwarz and D.H. Chun, 1968, Phys. Fluids 11, 5.

R.G. Shaver, 1957, thesis, M.I.T.

A. Sommerfeld, 1908, Proc. Fourth Int. Cong. Maths., 116.

A.J.M. Spencer and R.S. Rivlin, 1960, Arch. Rat. Mech. Anal. 4, 214.

H.B. Squire, 1933, Proc. Roy. Soc. A142, 621.

J.T. Stuart, 1954, Proc. Roy. Soc. A221, 189.

J.T. Stuart, 1958, J. Fluid Mech. 4, 1.

J.T. Stuart, 1960, J. Fluid Mech. 9, 353.

J.T. Stuart, 1962, Adv. in Aeron. Sci. 3, 121.

J.T. Stuart, 1965, Appl. Mech. Rev. 18, 523.

J.T. Stuart, 1965, NPL Aero Report 1147.

J.T. Stuart, 1971, Ann. Rev. Fluid Mech. 3, 347.

L.H. Thomas, 1953, Phys. Rev. 91, 780.

W. Tollmien, 1929, Nachr. Ges. Wiss. Göttingen, Math-Phys. Klasse 22.

W. Tollmien, 1947, Z.A.M.M. 25, 33 and 27, 70.

B.A. Toms, 1949, Proc. Second Int. Cong. Rheology, 135.

C. Truesdell and W. Noll, 1965, Handbuch der Physik III/3, Springer.

M.P. Tulin, 1966, paper presented at Sixth Symp. Naval Hydrodynamics, Washington D.C.

P.S. Virk, E.W. Merrill, H.S. Mickley, K.A. Smith and E.L. Mollo-Christensen, 1967, J. Fluid Mech. 30, 305.

K. Walters, 1970, Z.A.M.P. 21, 592.

J. Watson, 1960, J. Fluid Mech. 9, 371.

J. Watson, 1962, J. Fluid Mech. 14, 211.

A. White, 1966, Hendon College of Technology Res. Bull. No. 3.

BM

APPENDIX

Expressions for the non-linear terms used in chapter 4. Use has been made of relations obtained in (4.6.1) and (4.6.13) and continuity conditions.

$$
\begin{aligned}
U_{001} = {} & -\phi\tilde{\psi}'-\tilde{\phi}\psi' + \lambda i\alpha(\tilde{\psi}\psi''-\psi\tilde{\psi}'') + \lambda(\phi\tilde{\phi}'''+\tilde{\phi}\phi''') - \lambda\alpha^{2}(\phi\tilde{\phi}'+\tilde{\phi}\phi') \\
& +i\alpha(3\lambda+2\mu)(\psi\tilde{\psi}''-\tilde{\psi}\psi''-3\phi\tilde{\phi}'') - \lambda i\alpha\tilde{\phi}\phi'' \\
& +(\lambda+\mu)(\psi'\tilde{\phi}''+\tilde{\psi}'\phi''+2\phi'\tilde{\psi}''+2\tilde{\phi}'\psi''),
\end{aligned}
$$

where suffices on all ψ and ϕ are 1,0,0.

$$W_{001} = 0.$$

$$
\begin{aligned}
U_{002} = {} & -\phi\tilde{\psi}'-\tilde{\phi}\psi'+\lambda i\alpha(\tilde{\psi}\psi''-\psi\tilde{\psi}'')+\lambda(\phi\tilde{\psi}'''+\tilde{\phi}'\psi''')-\lambda(\alpha^{2}+\gamma^{2})(\phi\tilde{\psi}'+\tilde{\phi}\psi') \\
& +\lambda i\gamma(\tilde{\chi}\psi''-\chi\tilde{\psi}'')+\lambda i\gamma(\alpha^{2}+\gamma^{2})(\chi\tilde{\psi}-\tilde{\chi}\psi) \\
& + (3\lambda+2\mu)i\alpha(\psi\tilde{\psi}''-\tilde{\psi}\psi'')+(\lambda+\mu)(\alpha^{2}+\gamma^{2})(\psi'\tilde{\phi}+\tilde{\psi}'\phi) \\
& +(\lambda+\mu)(\psi'\tilde{\phi}''+\tilde{\psi}'\phi''+2\phi'\tilde{\psi}''+2\tilde{\phi}'\psi'')+(\lambda+\mu)i\gamma(\alpha^{2}+\gamma^{2})(\psi\tilde{\chi}-\chi\tilde{\psi}) \\
& +(\lambda+\mu)i\gamma(\psi\tilde{\chi}''-\tilde{\psi}\chi''-2\chi'\tilde{\psi}'+2\tilde{\chi}'\psi') \\
& +(2\lambda+\mu)i\alpha(\phi\tilde{\phi}''-\tilde{\phi}\phi''+\chi\tilde{\chi}''-\tilde{\chi}\chi''),
\end{aligned}
$$

where suffices on all ψ,ϕ and χ are 1,1,0.

$$
\begin{aligned}
U_{003} = U_{003}(\alpha,\gamma,\psi_{1-10},\phi_{1-10},\chi_{1-10}) & = U_{002}(\alpha,-\gamma,\psi_{110},\phi_{110},-\chi_{110}) \\
& = U_{002}(\alpha,\gamma,\psi_{110},\phi_{110},\chi_{110}).
\end{aligned}
$$

Hence $U_{003} = U_{002}$.

$$
\begin{aligned}
W_{002} = {} & i\alpha(\psi\tilde{\chi}-\tilde{\psi}\chi)-\phi\tilde{\chi}'-\tilde{\phi}\chi'+\lambda i\alpha(\tilde{\psi}\chi''-\psi\tilde{\chi}'')+\lambda i\alpha(\alpha^{2}+\gamma^{2})(\psi\tilde{\chi}-\tilde{\psi}\chi) \\
& +\lambda(\phi\tilde{\chi}'''+\tilde{\phi}\chi''')-\lambda(\alpha^{2}+\gamma^{2})(\phi\tilde{\chi}'+\tilde{\phi}\chi')+\lambda i\gamma(\tilde{\chi}\chi''-\chi\tilde{\chi}'') \\
& +(\lambda+\mu)i\alpha(\alpha^{2}+\gamma^{2})(\tilde{\psi}\chi-\psi\tilde{\chi})+(\lambda+\mu)(\alpha^{2}+\gamma^{2})(\chi'\tilde{\phi}+\phi\tilde{\chi}') \\
& +(\lambda+\mu)(\chi'\tilde{\phi}''-\tilde{\chi}'\phi''+2\phi'\tilde{\chi}''+2\tilde{\phi}'\chi'') \\
& +(\lambda+\mu)\{i\alpha(\chi\tilde{\psi}''-\tilde{\chi}\psi'')+i\gamma(\chi\tilde{\chi}''-\tilde{\chi}\chi'')+2i\alpha(\chi'\tilde{\psi}'-\tilde{\chi}'\psi')\} \\
& +(2\lambda+\mu)i\gamma(\psi\tilde{\psi}''+\phi\tilde{\phi}''+\chi\tilde{\chi}''-\tilde{\psi}\psi''-\tilde{\phi}\phi''-\tilde{\chi}\chi''),
\end{aligned}
$$

where suffices on all ψ, ϕ and χ are 1,1,0.

$$\begin{aligned}W_{003} = W_{003}(\alpha,\gamma,\psi_{1-10},\phi_{1-10},\chi_{1-10}) &= W_{002}(\alpha,-\gamma,\psi_{110},\phi_{110},-\chi_{110})\\ &= -\,W_{002}(\alpha,\gamma,\psi_{110},\phi_{110},\chi_{110}).\end{aligned}$$

Hence $W_{003} = -W_{002}$

$$\begin{aligned}U_{011} = {} & -\tilde\phi_{100}\tilde\psi'_{110}-\psi'_{100}\phi_{110}+\lambda i\alpha(\tilde\psi_{100}\psi''_{110}-\tilde\psi''_{100}\psi_{110})+\lambda(\tilde\phi_{100}\psi'''_{110}+\phi_{110}\tilde\psi'''_{100})\\ & -\lambda\alpha^2(\tilde\phi_{100}\psi'_{110}+\tilde\psi'_{100}\phi_{110})-\lambda\gamma^2\tilde\phi_{100}\psi'_{110}\\ & +(\lambda+\mu)\{i\alpha(\psi_{110}\tilde\psi''_{100}-\tilde\psi_{100}\psi''_{110})+\alpha^2(\phi_{110}\tilde\psi'_{100}+\psi'_{110}\tilde\phi_{100})\\ & \qquad -\gamma^2\tilde\psi'_{100}\phi_{110}+i\alpha\gamma^2\tilde\psi_{100}\psi_{110}+\tilde\psi'_{100}\phi''_{110}+\psi'_{110}\tilde\phi''_{100}+2\tilde\phi'_{100}\psi''_{110}+2\phi'_{110}\tilde\psi''_{100}\}\\ & +(2\lambda+\mu)\{i\alpha\gamma^2\tilde\psi_{100}\psi_{110}+i\alpha\gamma^2\tilde\phi_{100}\phi_{110}\\ & \qquad +i\alpha(\psi_{110}\tilde\psi''_{100}-\tilde\psi_{100}\psi''_{110}+\phi_{110}\tilde\phi''_{100}-\tilde\phi_{100}\phi''_{110}\}.\end{aligned}$$

$$\begin{aligned}U_{012} = {} & -\phi_{100}\tilde\psi'_{1-10}-\psi'_{100}\tilde\phi_{1-10}-\lambda i\alpha(\psi_{100}\tilde\psi''_{1-10}-\psi''_{100}\tilde\psi_{1-10})+\lambda(\phi_{100}\tilde\psi'''_{1-10}+\tilde\phi_{1-10}\psi'''_{100})\\ & -\lambda\alpha^2(\phi_{100}\tilde\psi'_{1-10}+\psi'_{100}\tilde\phi_{1-10})-\lambda\gamma^2\phi_{100}\tilde\psi'_{1-10}\\ & +(\lambda+\mu)\{-i\alpha(\tilde\psi_{1-10}\psi''_{100}-\psi_{100}\tilde\psi''_{1-10})+\alpha^2(\tilde\phi_{1-10}\psi'_{100}+\tilde\psi'_{1-10}\phi_{100})-\gamma^2\psi'_{100}\tilde\phi_{1-10}\\ & \qquad -i\alpha\gamma^2\psi_{100}\tilde\psi_{1-10}+\psi'_{100}\tilde\phi''_{1-10}+\tilde\psi'_{1-10}\phi''_{100}+2\phi'_{100}\tilde\psi''_{1-10}+2\tilde\phi'_{1-10}\psi''_{100}\}\\ & +(2\lambda+\mu)\{-i\alpha\gamma^2\psi_{100}\tilde\psi_{1-10}-i\alpha\gamma^2\phi_{100}\tilde\phi_{1-10}-i\alpha(\tilde\psi_{1-10}\psi''_{100}-\psi_{100}\tilde\psi''_{1-10}\\ & \qquad +\tilde\phi_{1-10}\phi''_{100}-\phi_{100}\tilde\phi''_{1-10}\}.\end{aligned}$$

Hence $U_{012} = \tilde U_{011}$.

$$\begin{aligned}V_{011} = {} & i\alpha(\psi_{110}\tilde\phi_{100}-\tilde\psi_{100}\phi_{110})-(\tilde\phi_{100}\phi'_{110}+\phi_{110}\tilde\phi'_{100})+\lambda i\alpha(\tilde\psi_{100}\phi''_{110}-\psi_{110}\tilde\phi''_{100})\\ & +\lambda i\alpha^3\psi_{110}\tilde\phi_{100}-\lambda i\alpha(\alpha^2+\gamma^2)\tilde\psi_{100}\phi_{110}+\lambda(\tilde\phi_{100}\phi'''_{110}+\phi_{110}\tilde\phi'''_{100})-\lambda\alpha^2\phi_{110}\tilde\phi'_{100}\\ & -\lambda(\alpha^2+\gamma^2)\tilde\phi_{100}\phi'_{110}\\ & +(\lambda+\mu)\{i\alpha^3(\tilde\psi_{100}\phi_{110}-\psi_{110}\tilde\phi_{100})+i\alpha\gamma^2\psi_{110}\tilde\phi_{100}+(\alpha^2-\gamma^2)\phi_{110}\tilde\phi'_{100}+\alpha^2\tilde\phi_{100}\phi'_{110}\\ & \qquad +3(\tilde\phi'_{100}\phi''_{110}+\tilde\phi''_{100}\phi'_{110})+i\alpha(\tilde\psi''_{100}\phi_{110}-\tilde\phi_{100}\psi''_{110}+2\tilde\psi'_{100}\phi'_{110}-2\psi'_{110}\tilde\phi'_{100})\}\\ & +(2\lambda+\mu)\{(\alpha^2-\gamma^2)\psi_{110}\tilde\psi'_{100}+\alpha^2\psi'_{110}\tilde\psi_{100}-\gamma^2\phi_{110}\tilde\phi'_{100}+2\tilde\psi'_{100}\psi''_{110}+2\tilde\psi''_{100}\psi'_{110}\\ & \qquad +3\tilde\phi'_{100}\phi''_{110}+3\tilde\phi''_{100}\phi'_{110}+i\alpha(\tilde\psi'_{100}\phi'_{110}-\psi'_{110}\tilde\phi'_{100}+\tilde\psi''_{100}\phi_{110}-\psi''_{110}\tilde\phi_{100})\}.\end{aligned}$$

$$V_{012} = i\alpha(\psi_{100}\tilde{\phi}_{1-10}-\tilde{\psi}_{1-10}\phi_{100})-(\phi_{100}\tilde{\phi}'_{1-10}+\tilde{\phi}_{1-10}\phi'_{100})+\lambda i\alpha(\tilde{\psi}_{1-10}\phi''_{100}-\psi_{100}\tilde{\phi}''_{1-10})$$
$$-\lambda i\alpha^3\phi_{100}\tilde{\psi}_{1-10}+\lambda i\alpha(\alpha^2+\gamma^2)\psi_{100}\tilde{\phi}_{1-10}+\lambda(\tilde{\phi}_{100}\tilde{\phi}'''_{1-10}+\tilde{\phi}_{1-10}\phi'''_{100})-\lambda\alpha^2\tilde{\phi}_{1-10}\phi'_{100}$$
$$-\lambda(\alpha^2+\gamma^2)\phi_{100}\tilde{\phi}'_{1-10}$$
$$+(\lambda+\mu)\{i\alpha^3(\tilde{\psi}_{1-10}\phi_{100}-\psi_{100}\tilde{\phi}_{1-10})-i\alpha\gamma^2\tilde{\psi}_{1-10}\phi_{100}+(\alpha^2-\gamma^2)\tilde{\phi}_{1-10}\phi'_{100}$$
$$+\alpha^2\tilde{\phi}'_{1-10}\phi_{100}+3\phi'_{100}\tilde{\phi}''_{1-10}+3\phi''_{100}\tilde{\phi}'_{1-10}+i\alpha(\phi_{100}\tilde{\psi}''_{1-10}-\psi''_{100}\tilde{\phi}_{1-10}$$
$$+2\phi'_{100}\tilde{\psi}'_{1-10}-2\psi'_{100}\tilde{\phi}'_{1-10})\}$$
$$+(2\lambda+\mu)\{(\alpha^2-\gamma^2)\tilde{\psi}_{1-10}\psi'_{100}+\alpha^2\tilde{\psi}'_{1-10}\psi_{100}-\gamma^2\tilde{\phi}_{1-10}\phi'_{100}+2\psi'_{100}\tilde{\psi}''_{1-10}+2\psi''_{100}\tilde{\psi}'_{1-10}$$
$$+3\phi'_{100}\tilde{\phi}''_{1-10}+3\phi''_{100}\tilde{\phi}'_{1-10}+i\alpha(\tilde{\psi}'_{1-10}\phi'_{100}-\psi'_{100}\tilde{\phi}'_{1-10}+\tilde{\psi}''_{1-10}\phi_{100}$$
$$-\psi''_{100}\tilde{\phi}_{1-10})\}.$$

Hence $V_{012} = \tilde{V}_{011}$.

$$W_{011} = -i\alpha\tilde{\psi}_{100}\chi_{110}-\tilde{\phi}_{100}\chi'_{110}+\lambda i\alpha\tilde{\psi}_{100}\chi''_{110}-\lambda i\alpha(\alpha^2+\gamma^2)\tilde{\psi}_{100}\chi_{110}$$
$$+\lambda\tilde{\phi}_{100}\chi'''_{110}-\lambda(\alpha^2+\gamma^2)\tilde{\phi}_{100}\chi'_{110}$$
$$+(\lambda+\mu)\{i\alpha\chi_{110}\tilde{\psi}''_{100}+2i\alpha\chi'_{110}\tilde{\psi}'_{100}+2\tilde{\phi}'_{100}\chi''_{110}+\chi'_{110}\tilde{\phi}''_{100}\}$$
$$+(2\lambda+\mu)\{i\gamma(\tilde{\psi}'_{100}\psi'_{110}+\psi_{110}\tilde{\psi}''_{100}+2\tilde{\phi}'_{100}\phi'_{110}+\phi_{110}\tilde{\phi}''_{100})$$
$$+i\alpha^2\gamma\tilde{\psi}_{100}\psi_{110}+\alpha\gamma(\tilde{\phi}_{100}\psi'_{110}-\tilde{\psi}'_{100}\phi_{110})\}.$$

$$W_{012} = i\alpha\psi_{100}\tilde{\chi}_{1-10}-\phi_{100}\tilde{\chi}'_{1-10}-\lambda i\alpha\psi_{100}\tilde{\chi}''_{1-10}+\lambda i\alpha(\alpha^2+\gamma^2)\psi_{100}\tilde{\chi}_{1-10}$$
$$+\lambda\phi_{100}\tilde{\chi}'''_{1-10}-\lambda(\alpha^2+\gamma^2)\phi_{100}\tilde{\chi}'_{1-10}$$
$$+(\lambda+\mu)\{2\phi'_{100}\tilde{\chi}''_{1-10}+\tilde{\chi}'_{1-10}\phi''_{100}-i\alpha\tilde{\chi}_{1-10}\psi''_{100}-2i\alpha\tilde{\chi}'_{1-10}\psi'_{100}\}$$
$$+(2\lambda+\mu)\{-\alpha\gamma(\phi_{100}\tilde{\psi}'_{1-10}-\psi'_{100}\tilde{\phi}_{1-10})+i\alpha^2\gamma\psi_{100}\tilde{\psi}_{1-10}$$
$$+i\gamma(\psi'_{100}\tilde{\psi}'_{1-10}+\tilde{\psi}_{1-10}\psi''_{100}+2\phi'_{100}\tilde{\phi}'_{1-10}+\tilde{\phi}_{1-10}\phi''_{100})\}.$$

Hence $W_{012} = -\tilde{W}_{011}$.

$$U_{201} = -i\alpha\psi\psi-\phi\psi'+\lambda i\alpha\psi\psi''+\lambda\phi\psi'''$$
$$-\lambda i\alpha^3\psi\psi-\lambda\alpha^2\phi\psi'$$
$$+(\lambda+\mu)\{-3i\alpha^3\psi\psi+i\alpha\psi\psi''+2i\alpha\psi'\psi'-3\alpha^2\psi'\phi+\psi'\phi''+2\psi''\phi'\}$$
$$+(2\lambda+\mu)\{-3i\alpha^3\psi\psi+i\alpha\psi\psi''+i\alpha\psi'\psi'-2\alpha^2\psi'\phi-2i\alpha^3\phi\phi+i\alpha\phi\phi''+2i\alpha\phi'\phi'\},$$

where suffices on ψ and ϕ are 1,0,0.

$$
\begin{aligned}
V_{201} = {} & \lambda(i\alpha\psi\phi''+\phi\phi''') \\
& +(\lambda+\mu)\{-3i\alpha^3\psi\phi+i\alpha\psi''\phi+2i\alpha\psi'\phi'+3\phi'\phi''-3\alpha^2\phi\phi'\} \\
& +(2\lambda+\mu)\{i\alpha\psi''\phi+i\alpha\psi'\phi'+3\phi'\phi''-2\alpha^2\phi\phi'-3\alpha^2\psi\psi'+2\psi'\psi''\},
\end{aligned}
$$

where suffices on ψ and ϕ are 1,0,0.

$$W_{201} = 0.$$

U_{020}, V_{020} and W_{020} arise from products of functions with suffices 1,1,0 and 1, -1, 0. Equation (4.6.13) has been used to express them entirely in terms of suffices 1,1,0.

$$
\begin{aligned}
U_{020} = {} & -\phi\tilde{\psi}'-\tilde{\phi}\psi--i\gamma(\chi\tilde{\psi}-\tilde{\chi}\psi) \\
& +\lambda\{i\alpha(\tilde{\psi}\psi''-\psi\tilde{\psi}'')+\phi\tilde{\psi}'''+\tilde{\phi}\psi'''-(\alpha^2+\gamma^2)(\phi\tilde{\psi}'+\tilde{\phi}\psi') \\
& \qquad +i\gamma(\chi\tilde{\psi}''-\tilde{\chi}\psi'')+i\gamma(\alpha^2+\gamma^2)(\tilde{\chi}\psi-\chi\tilde{\psi})\} \\
& +(\lambda+\mu)\{i\alpha(\psi\tilde{\psi}''-\tilde{\psi}\psi'')+i\gamma(\tilde{\psi}\chi''-\psi\tilde{\chi}'')+\psi'\tilde{\phi}''+\tilde{\psi}'\phi'' \\
& \qquad -(\alpha^2+\gamma^2)(\psi'\tilde{\phi}+\tilde{\psi}'\phi)+i\gamma(\alpha^2+\gamma^2)(\psi\tilde{\chi}-\tilde{\psi}\chi) \\
& \qquad +2\alpha^2(\phi\tilde{\psi}'+\tilde{\phi}\psi')-2\gamma^2(\tilde{\phi}\psi'+\phi\tilde{\psi}')+2(\phi'\tilde{\psi}''+\tilde{\phi}'\psi'') \\
& \qquad +2i\gamma(\chi'\tilde{\psi}'-\tilde{\chi}'\psi')+2i\alpha^2\gamma(\chi\tilde{\psi}-\tilde{\chi}\psi)+2i\gamma^3(\tilde{\chi}\psi-\chi\tilde{\psi})\} \\
& +(2\lambda+\mu)i\alpha(\psi\tilde{\psi}''-\tilde{\psi}\psi''+\phi\tilde{\phi}''-\tilde{\phi}\phi''-\chi\tilde{\chi}''+\tilde{\chi}\chi''),
\end{aligned}
$$

where suffices on all ψ, ϕ and χ are 1,1,0.

Hence U_{020} is real.

$$
\begin{aligned}
V_{020} = {} & i\alpha(\phi\tilde{\psi}-\psi\tilde{\phi})-\phi\tilde{\phi}'-\tilde{\phi}\phi'+i\gamma(\phi\tilde{\chi}-\tilde{\phi}\chi) \\
& +\lambda\{i\alpha(\psi\tilde{\phi}''-\phi''\tilde{\psi})+i\alpha(\alpha^2+\gamma^2)(\phi\tilde{\psi}-\psi\tilde{\phi})+\phi\tilde{\phi}'''+\tilde{\phi}\phi''' \\
& \qquad -(\alpha^2+\gamma^2)(\phi\tilde{\phi}'+\tilde{\phi}\phi')+i\gamma(\chi\tilde{\phi}''-\tilde{\chi}\phi'')+i\gamma(\alpha^2+\gamma^2)(\phi\tilde{\chi}-\tilde{\phi}\chi)\} \\
& +(\lambda+\mu)\{i\alpha(\phi\tilde{\psi}''-\tilde{\phi}\psi''+2\tilde{\psi}'\phi'-2\psi'\tilde{\phi}')+i\gamma(\tilde{\phi}\chi''-\phi\tilde{\chi}''+2\chi'\tilde{\phi}'-2\tilde{\chi}'\phi') \\
& \qquad +3\phi'\tilde{\phi}''+3\phi''\tilde{\phi}'-2\gamma^2(\phi'\tilde{\phi}+\phi\tilde{\phi}')\} \\
& +(2\lambda+\mu)\{(\alpha^2-2\gamma^2)(\psi'\tilde{\psi}+\psi\tilde{\psi}')+3\gamma^2(\chi'\tilde{\chi}+\tilde{\chi}'\chi)-2\gamma^2(\phi'\tilde{\phi}+\phi\tilde{\phi}') \\
& \qquad +2(\psi'\tilde{\psi}''+\tilde{\psi}'\psi'')+3(\phi'\tilde{\phi}''+\phi''\tilde{\phi}')-2(\chi'\tilde{\chi}''+\tilde{\chi}'\chi'') \\
& \qquad +i\alpha(\phi'\tilde{\psi}'-\tilde{\phi}'\psi'+\phi\tilde{\psi}''-\tilde{\phi}\psi'')+i\gamma(\chi'\tilde{\phi}'-\tilde{\chi}'\phi'+\tilde{\phi}\chi''-\phi\tilde{\chi}'') \\
& \qquad -\alpha\gamma(\psi\tilde{\chi}'+\chi'\tilde{\psi}+\psi'\tilde{\chi}+\chi\tilde{\psi}')\},
\end{aligned}
$$

where suffices on all ψ, ϕ and χ are 1,1,0.

Hence V_{020} is real.

$$
\begin{aligned}
W_{020} = {} & -i\alpha(\chi\tilde{\psi}+\psi\tilde{\chi})+\phi\tilde{\chi}'-\phi\chi'+2i\gamma\chi\tilde{\chi} \\
& +\lambda\{i\alpha(\psi\tilde{\chi}''+\chi''\tilde{\psi})-i\alpha(\alpha^2+\gamma^2)(\chi\tilde{\psi}+\tilde{\chi}\psi)+\tilde{\phi}\chi'-\phi\tilde{\chi}' \\
& \qquad +2i\gamma(\alpha^2+\gamma^2)\chi\tilde{\chi}-i\gamma(\chi''\tilde{\chi}+\chi\tilde{\chi}'')\} \\
& +(\lambda+\mu)\{i\gamma(6\gamma^2-2\alpha^2)\chi\tilde{\chi}+i\gamma(\alpha^2-3\gamma^2)(\chi\tilde{\psi}+\tilde{\chi}\psi)+i\alpha(\chi\tilde{\psi}''+\tilde{\chi}\psi''+2\chi'\tilde{\psi}'+2\tilde{\chi}'\psi') \\
& \qquad -i\gamma(\chi\tilde{\chi}''+\tilde{\chi}\chi''+4\chi'\tilde{\chi}')+(\alpha^2-3\gamma^2)(\chi'\tilde{\phi}-\phi\tilde{\chi}') \\
& \qquad +\chi'\tilde{\phi}''-\tilde{\chi}'\phi''+2\tilde{\phi}'\chi''-2\phi'\tilde{\chi}''\} \\
& +(2\lambda+\mu)\{2i\gamma(\alpha^2-2\gamma^2)\psi\tilde{\psi}-4i\gamma^3\phi\tilde{\phi}+6i\gamma^3\chi\tilde{\chi}-2i\alpha\gamma^2(\psi\tilde{\chi}+\tilde{\psi}\chi) \\
& \qquad +i\gamma(\psi\tilde{\psi}''+\tilde{\psi}\psi''+2\psi'\tilde{\psi}'+\phi\tilde{\phi}''+\tilde{\phi}\phi''+4\phi'\tilde{\phi}'-\chi\tilde{\chi}''-\tilde{\chi}\chi''-2\chi'\tilde{\chi}') \\
& \qquad +2\alpha\gamma(\psi'\tilde{\phi}-\phi\tilde{\psi}')+2\gamma^2(\phi\tilde{\chi}'-\chi'\tilde{\phi})\},
\end{aligned}
$$

where suffices on all ψ, ϕ and χ are 1,1,0.

Hence W_{020} is pure imaginary.

$$
\begin{aligned}
U_{201} = {} & -i\alpha\psi\psi-\phi\psi'+\lambda(i\alpha\psi\psi''+\phi\psi'''-i\alpha^3\psi\psi-\alpha^2\phi\psi') \\
& +(\lambda+\mu)\{i\alpha(\psi\psi''+2\psi'\psi')+2\phi'\psi''-3\alpha^2\phi\psi' \\
& \qquad +\psi'\phi''-3i\alpha^3\psi\psi\} \\
& +(2\lambda+\mu)\{i\alpha(\psi\psi''+\phi\phi''+\psi'\psi'+2\phi'\phi')-i\alpha^3(3\psi\psi+2\phi\phi)-2\alpha^2\phi\psi'\},
\end{aligned}
$$

where suffices on all ψ and ϕ are 1,0,0.

$$
\begin{aligned}
V_{201} = {} & \lambda\{i\alpha\psi\phi''+\phi\phi'''\} \\
& +(\lambda+\mu)\{i\alpha\phi\psi''+\phi'\phi''\} \\
& +(2\lambda+\mu)\{2\psi'\psi''+3\phi'\phi''+i\alpha(\phi\psi''+\psi'\phi')-\alpha^2(3\psi\psi'-2\phi\phi')\},
\end{aligned}
$$

where suffices on all ψ and ϕ are 1,0,0.

$$W_{201} = 0.$$

$$U_{202} = -2\phi\psi'-2i\alpha\psi^2+2i\gamma\psi\chi+\lambda\{2i\alpha\psi\psi''+2\phi\psi'''-2i\gamma\chi\psi''\}$$
$$+\lambda 2(\alpha^2+\gamma^2)(i\gamma\chi\psi-\phi\psi'-i\alpha\psi\psi)$$
$$+2(\lambda+\mu)\{(3\alpha^2-\gamma^2)(i\gamma\chi\psi-i\alpha\psi\psi-\psi'\phi)+i\alpha\psi\psi''+2i\alpha\psi'\psi'+\psi'\phi''$$
$$+2\psi''\phi'-i\gamma\psi\chi''-2i\gamma\psi'\chi'\}$$
$$+2(2\lambda+\mu)\{-3i\alpha^3\psi\psi-2i\alpha^3\phi\phi+2i\alpha^3\chi\chi+2i\alpha^2\gamma\psi\chi-2\alpha^2\psi'\phi+2\alpha\gamma\phi\chi'$$
$$+i\alpha(\psi\psi''+\psi'\psi'+\phi\phi''+2\phi'\phi'-\chi\chi''-\chi'\chi')\},$$

where suffices on all ψ, ϕ and χ are 1,1,0.

$$V_{202} = -2i\alpha\psi\phi-2\phi\phi'+2i\gamma\chi\phi$$
$$+2\lambda\{i\alpha\psi\phi''-i\alpha(\alpha^2+\gamma^2)\psi\phi+\phi\phi'''-(\alpha^2+\gamma^2)\phi\phi'-i\gamma\chi\phi''+i\gamma(\alpha^2+\gamma^2)\phi\chi\}$$
$$+2(\lambda+\mu)\{(3\alpha^2-\gamma^2)(-i\alpha\psi\phi+i\gamma\phi\chi-\phi\phi')+3\phi'\phi''+i\alpha\phi\psi''$$
$$+2i\alpha\phi'\psi'-i\gamma\phi\chi''-2i\gamma\phi'\chi'\}$$
$$+(2\lambda+\mu)\{-6\alpha^2\psi\psi'-4\alpha^2\phi\phi'-2(2\alpha^2-\gamma^2)\chi\chi'+4\psi'\psi''+6\phi'\phi''-4\chi'\chi''$$
$$+2i\alpha(\psi'\phi'+\psi''\phi)-2i\alpha(\phi'\chi'+\phi\chi'')+\alpha\gamma(\psi\chi'+\chi\psi')\},$$

where suffices on all ψ, ϕ and χ are 1,1,0.

$$W_{202} = 0.$$

$$U_{210} = -2i\alpha\psi_{100}\psi_{110}-\phi_{100}\psi'_{110}-\psi'_{100}\phi_{110}$$
$$+\lambda(\phi_{100}\psi'''_{110}+\phi_{110}\psi'''_{100}-\gamma^2\phi_{100}\psi'_{110})$$
$$+(\lambda+\mu)\{i\alpha(3\psi'_{100}\psi'_{110}+\psi''_{100}\psi_{110})-4i\alpha^3\psi_{100}\psi_{110}-\gamma^2\psi'_{100}\phi_{110}$$
$$-2\alpha^2(\psi'_{110}\phi_{100}+\psi'_{100}\phi_{110})+\psi'_{100}\phi''_{110}+2\psi''_{100}\phi'_{110}+2\psi''_{110}\phi'_{100}\}$$
$$+(2\lambda+\mu)\{i\alpha(2\psi_{100}\psi''_{110}+3\psi'_{100}\psi'_{110}+\psi''_{100}\psi_{110}+\phi_{100}\phi''_{110}+4\phi'_{100}\phi'_{110}+\phi''_{100}\phi_{110})$$
$$-3\alpha^2\psi'_{100}\phi_{110}-3\alpha^2\psi'_{110}\phi_{100}-i\alpha(8\alpha^2+2\gamma^2)\psi_{100}\psi_{110}$$
$$-i\alpha(2\alpha^2+\gamma^2)\phi_{100}\phi_{110}\}.$$

$$U_{2-10} = U_{2-10}(\alpha,\gamma,\psi_{1-10},\phi_{1-10},\psi_{100},\phi_{100}) = U_{210}(\alpha,-\gamma,\psi_{110},\phi_{110},\psi_{100},\phi_{100}).$$
$$= U_{210}(\alpha,\gamma,\psi_{110},\phi_{110},\psi_{100},\phi_{100}).$$

Hence $U_{2-10} = U_{210}$.

$$
\begin{aligned}
V_{210} = & -i\alpha\psi_{100}\phi_{110}-i\alpha\psi_{110}\phi_{100}-\phi_{100}\phi'_{110}-\phi'_{100}\phi_{110} \\
& +\lambda\{i\alpha(\psi_{100}\phi''_{110}+\psi''_{110}\phi_{100})+\phi_{100}\phi'''_{110}+\phi_{110}\phi'''_{100}-i\alpha^3(\psi_{100}\phi_{110}+\psi_{110}\phi_{100}) \\
& \quad -i\alpha\gamma^2\psi_{100}\phi_{110}-\alpha^2(\phi_{100}\phi'_{110}+\phi_{110}\phi'_{100})-\gamma^2\phi_{100}\phi'_{110}\} \\
& +(\lambda+\mu)\{-3i\alpha^3(\phi_{100}\psi_{110}+\psi_{100}\phi_{110})-i\alpha\gamma^2\phi_{100}\psi_{110} \\
& \quad +i\alpha(\phi_{100}\psi''_{110}+\phi_{110}\psi''_{100}+2\phi'_{100}\psi'_{110}+2\phi'_{110}\psi'_{100}) \\
& \quad -3\alpha^2(\phi'_{100}\phi_{110}+\phi_{100}\phi'_{110})-\gamma^2\phi'_{100}\phi_{110}+3(\phi'_{100}\phi''_{110}+\phi''_{100}\phi'_{110})\} \\
& +(2\lambda+\mu)\{i\alpha(\phi_{100}\psi''_{110}+\phi_{110}\psi''_{100}+\phi'_{100}\psi'_{110}+\phi'_{110}\psi'_{100})-\gamma^2(\phi'_{100}\phi_{110}+\psi'_{100}\psi_{110}) \\
& \quad -\alpha^2(2\phi_{100}\phi'_{110}+3\psi'_{100}\psi_{110}+3\psi_{100}\psi'_{110})+3\phi'_{100}\phi''_{110}+3\phi''_{100}\phi'_{110} \\
& \quad +2\psi'_{100}\psi''_{110}+2\psi''_{100}\psi'_{110}\}.
\end{aligned}
$$

$$
\begin{aligned}
V_{2-10} = V_{2-10}(\alpha,\gamma,\psi_{1-10},\phi_{1-10},\psi_{100},\phi_{100}) & = V_{210}(\alpha,-\gamma,\psi_{110},\phi_{110},\psi_{100},\phi_{100}) \\
& = V_{210}(\alpha,\gamma,\psi_{110},\phi_{110},\psi_{100},\phi_{100}).
\end{aligned}
$$

Hence $V_{2-10} = V_{210}$.

$$
\begin{aligned}
W_{210} = & -i\alpha\psi_{100}\chi_{110}-\phi_{100}\chi'_{110}+\lambda\{i\alpha\psi_{100}\chi''_{110}+\phi_{100}\chi'''_{110}-i\alpha(\alpha^2+\gamma^2)\psi_{100}\chi_{110} \\
& \qquad -(\alpha^2+\gamma^2)\phi_{100}\chi'_{110}\} \\
& +(\lambda+\mu)\{-3i\alpha^3\psi_{100}\chi_{110}-3\alpha^2\phi_{100}\chi'_{110}+i\alpha(2\psi'_{100}\chi_{110}+\psi''_{100}\chi_{110}) \\
& \quad +2\phi'_{100}\chi''_{110}+\phi''_{100}\chi'_{110}\} \\
& +(2\lambda+\mu)\{-i\alpha^2\gamma(3\psi_{100}\psi_{110}+2\phi_{100}\phi_{110})-\alpha\gamma(\psi'_{100}\phi_{110}+\psi'_{110}\phi_{100}) \\
& \quad +i\gamma(\psi'_{100}\psi'_{110}+\psi_{110}\psi''_{110}+\phi_{110}\phi''_{100}+2\phi'_{100}\phi'_{110})\}.
\end{aligned}
$$

$$
\begin{aligned}
W_{2-10} = W_{2-10}(\alpha,\gamma,\psi_{1-10},\phi_{1-10},\chi_{1-10},\psi_{100},\phi_{100}) & = W_{210}(\alpha,-\gamma,\psi_{110},\phi_{110}, \\
& \qquad -\chi_{110},\psi_{100},\phi_{100}) \\
= -W_{210}(\alpha,\gamma,\psi_{110},\phi_{110},\chi_{110},\psi_{100},\phi_{100}). &
\end{aligned}
$$

Hence $W_{2-10} = -W_{210}$.

$$
\begin{aligned}
U_{220} = & -i\alpha\psi\psi-\phi\psi'-i\gamma\chi\psi \\
& +\lambda\{i\alpha\psi\psi''+\phi\psi''+i\gamma\chi\psi''-(\alpha^2+\gamma^2)(i\alpha\psi\psi+\phi\psi'+i\gamma\chi\psi)\} \\
& +(\lambda+\mu)\{\psi'\phi''+2\psi''\phi'+i\alpha(\psi\psi''+2\psi'\psi')+i\gamma(\psi\chi''+2\psi'\chi')
\end{aligned}
$$

$$-3(\alpha^2+\gamma^2)(i\alpha\psi\psi+\phi\psi'+i\gamma\chi\psi)\}$$
$$+(2\lambda+\mu)\{-2i\alpha(\alpha^2+\gamma^2)(\psi\psi+\phi\phi+\chi\chi)-i\alpha(\alpha^2\psi\psi+2\alpha\gamma\psi\chi+\gamma^2\chi\chi)$$
$$-2\alpha\gamma\phi'\chi-2\alpha^2\psi'\phi+i\alpha(\psi\psi''+\psi'\psi'+\phi'\phi''+2\phi'\phi'+\chi\chi''+\chi'\chi')\},$$

where suffices on all ψ, ϕ and χ are 1,1,0.

$$U_{2-20} = U_{2-20}(\alpha,\gamma,\psi_{1-10},\phi_{1-10},\chi_{1-10}) = U_{220}(\alpha,-\gamma,\psi_{110},\phi_{110},-\chi_{110})$$
$$= U_{220}(\alpha,\gamma,\psi_{110},\phi_{110},\chi_{110}).$$

Hence $U_{2-20} = U_{220}$.

$$V_{220} = -i\alpha\psi\phi-\phi\phi'-i\gamma\chi\phi+2(\lambda+\mu)(\phi'\phi''-\phi\phi''')$$
$$+2(2\lambda+\mu)\{\phi'\phi''-(\alpha^2+\gamma^2)\phi\phi'+\psi'\psi''+\chi'\chi''-(\alpha^2+\gamma^2)(\psi\psi'+\chi\chi')\},$$

where suffices on all ψ, ϕ and χ are 1,1,0.

$$V_{2-20} = V_{2-20}(\alpha,\gamma,\psi_{1-10},\phi_{1-10},\chi_{1-10}) = V_{220}(\alpha,-\gamma,\psi_{110},\phi_{110},-\chi_{110})$$
$$= V_{220}(\alpha,\gamma,\psi_{110},\phi_{110},\chi_{110}).$$

Hence $V_{2-20} = V_{220}$.

$$W_{220} = -i\alpha\psi\chi-\phi\chi'-i\gamma\chi\chi'+\lambda(\phi\chi'''-\phi'\chi'')$$
$$+(\lambda+\mu)\{\phi'\chi''-\phi''\chi'-\phi'''\chi+2(\alpha^2+\gamma^2)(\phi'\chi-\chi'\phi)\}$$
$$+(2\lambda+\mu)\{(\alpha^2+\gamma^2)(\phi'\chi-\chi'\phi)-2i\gamma(\alpha^2+\gamma^2)(\psi\psi+\phi\phi+\chi\chi)$$
$$+i\gamma(\psi\psi''+\psi'\psi'+\chi\chi''+\chi'\chi'-\phi\phi''+3\phi'\phi')\},$$

where suffices on all ψ, ϕ and χ are 1,1,0.

$$W_{2-20} = W_{2-20}(\alpha,\gamma,\psi_{1-10},\phi_{1-10},\chi_{1-10}) = W_{220}(\alpha,-\gamma,\psi_{110},\phi_{110},-\chi_{110})$$
$$= -W_{220}(\alpha,\gamma,\psi_{110},\phi_{110},\chi_{110}).$$

Hence $W_{2-20} = -W_{220}$.

$$
\begin{aligned}
U_{101} = {} & -i\alpha\psi_{201}\tilde{\psi}_{100}-\phi_{201}\tilde{\psi}'_{100}-\tilde{\phi}_{100}\psi'_{201} \\
& +\lambda\{2i\alpha\tilde{\psi}_{100}\psi''_{201}-i\alpha\psi_{201}\tilde{\psi}''_{100}-7i\alpha^{3}\tilde{\psi}_{100}\psi_{201} \\
& \qquad +\tilde{\phi}_{100}\psi'''_{201}+\phi_{201}\tilde{\psi}'''_{100}-\alpha^{2}\phi_{201}\tilde{\psi}'_{100}-4\alpha^{2}\tilde{\phi}_{100}\psi'_{201}\} \\
& +(\lambda+\mu)\{6i\alpha^{3}\tilde{\psi}_{100}\psi_{201}+i\alpha(\tilde{\psi}'_{100}\psi'_{201}+2\psi_{201}\tilde{\psi}''_{100}-\tilde{\psi}_{100}\psi''_{201}) \\
& \qquad +3\alpha^{2}\tilde{\phi}_{100}\psi'_{201}+\tilde{\psi}'_{100}\phi''_{201}+\psi'_{201}\tilde{\phi}''_{100}+2\tilde{\phi}'_{100}\psi''_{201}+2\phi'_{201}\tilde{\psi}''_{100}\} \\
& +(2\lambda+\mu)\{6i\alpha^{3}\tilde{\psi}_{100}\psi_{201}+4i\alpha^{3}\tilde{\phi}_{100}\phi_{201}-\alpha^{2}(4\tilde{\psi}'_{100}\phi_{201}+\psi'_{201}\tilde{\phi}_{100}) \\
& \qquad +2\alpha^{2}(\tilde{\phi}_{100}\psi'_{201}+\phi_{201}\tilde{\psi}'_{100}) \\
& \qquad +i\alpha(2\psi_{201}\tilde{\psi}''_{100}+\tilde{\psi}'_{100}\psi'_{201}-\tilde{\psi}_{100}\psi''_{201}+2\phi'_{201}\tilde{\phi}'_{100}+2\phi_{201}\tilde{\phi}''_{100}-\tilde{\phi}_{100}\phi''_{201})\}.
\end{aligned}
$$

$$
\begin{aligned}
V_{101} = {} & i\alpha\psi_{201}\tilde{\phi}_{100}-2i\alpha\tilde{\psi}_{100}\phi_{201}-\tilde{\phi}_{100}\phi'_{201}-\phi_{201}\tilde{\phi}'_{100} \\
& +\lambda i\alpha(2\tilde{\psi}_{100}\phi''_{201}-\psi_{201}\tilde{\phi}''_{100}+\alpha^{2}\psi_{201}\tilde{\phi}_{100}-8\alpha^{2}\tilde{\psi}_{100}\phi_{201}) \\
& +\lambda(\tilde{\phi}_{100}\phi'''_{201}+\phi_{201}\tilde{\phi}'''_{100}-\alpha^{2}\phi_{201}\tilde{\phi}'_{100}-4\alpha^{2}\tilde{\phi}_{100}\phi'_{201}) \\
& +(\lambda+\mu)\{6i\alpha^{3}\tilde{\psi}_{100}\phi_{201}+3\alpha^{2}\tilde{\phi}_{100}\phi'_{201}+i\alpha(4\tilde{\psi}'_{100}\phi'_{201}-2\psi'_{201}\tilde{\phi}'_{100}+2\phi_{201}\tilde{\psi}''_{100} \\
& \qquad\qquad -\tilde{\phi}_{100}\psi''_{201}) \\
& \qquad +3(\tilde{\phi}'_{100}\phi''_{201}+\tilde{\phi}''_{100}\phi'_{201})\} \\
& +(2\lambda+\mu)\{\alpha^{2}(3\psi'_{201}\tilde{\psi}_{100}+2\phi_{201}\tilde{\phi}'_{100}+\tilde{\phi}_{100}\phi'_{201})+2(\tilde{\psi}'_{100}\psi''_{201}+\tilde{\psi}''_{100}\psi'_{201}) \\
& \qquad +3(\tilde{\phi}'_{100}\phi''_{201}+\tilde{\phi}''_{100}\phi'_{201})+i\alpha(2\tilde{\psi}'_{100}\phi'_{201}+\tilde{\psi}''_{100}\phi_{201}-\psi'_{201}\tilde{\phi}'_{100}-\psi''_{201}\tilde{\phi}_{100})\}.
\end{aligned}
$$

$$W_{101} = 0.$$