NUCLEON-NUCLEON SCATTERING USING DISPERSION RELATIONS

by

KEITH STUART FELDMAN B.Sc.

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Department of Mathematics
Imperial College
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ABSTRACT

The nucleon-nucleon interaction is investigated using the dispersion theoretic formalism of Goldberger, Grisaru, MacDowell and Wong. The contributions due to the exchange of one and two pions are included. The 'N/D' method is employed and the resulting sets of integral equations are solved to a desired degree of accuracy by digital computer techniques. A partial wave approach is employed and scattering phase-shifts and coupling parameters are calculated for J (total angular momentum) =0 ... 5 , at laboratory energies up to 400 MeV. A short discussion of the deuteron is incorporated and its binding energy estimated.

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CONTENTS

CHAPTER I - INTRODUCTION

In this thesis we investigate the two nucleon interaction within the framework of dispersion theory as formulated by GOLDBERGER, GRISARU, MACDOWELL and WONG ([1]) (hereafter referred to as GGMW). As well as the well-known one-pion exchange terms (OPEC), ~~the~~ a contribution to the imaginary parts of the scattering amplitudes due to two pion exchange is included. ~~This contribution is calculated as outlined in the appendix of GGMW, viz.~~ by a consideration of the process $\pi + \pi \longrightarrow N + \bar{N}$ in Born approximation.

Although good agreement with experiment should not be expected for the low partial waves, or for laboratory energies greater than about 200 MeV (since the exchange of more massive and greater numbers of particles becomes increasingly important) we indicate below why we consider our calculation to be justified.

Numerous calculations have already been performed (e.g. by AMATI, LEADER, and VITALE ([2]) (ALV), SCOTTI and WONG ([3]) (SW) and GELANIN et. al.([4])). The calculation of Gelanin et. al. shows little agreement with experiment, and we disagree with their conclusion that a

model of the nucleon-nucleon interaction based on one and two meson exchange only is almost totally inadequate. The above calculations employ quite different methods from ours for approximating the two pion contribution. In all cases a knowledge of the amplitude for the process

$$\pi + \pi \longrightarrow N + \bar{N} \qquad \text{(I1)}$$

is required. ALV employ the dispersion relations given by Chew, Goldberger, Low and Nambu([11]) for πN scattering which involve the usual amplitudes A and B. Thus

$$\text{Re(A or B)} = \text{Born Terms} + \int \frac{\text{Im(A or B)}}{\nu' \pm \nu} d\nu' \qquad \text{(I2)}$$

(the notation is explained after equations (II4.8)of this thesis). The dispersion integral involves terms which can be calculated from a knowledge of the 33 resonance of πN scattering. Also ALV find that the best agreement with experiment is obtained when the p-wave of the process (I1) above is represented completely by the ρ-meson.

Galanin et. al. employ the work of GALANIN and GRASHIN ([4]) to calculate the process (I1). A series expansion of the corresponding amplitude is obtained and integrated directly to the desired degree of approximation in order to calculate the contribution to NN scattering.

SW consider the effect of multimeson resonances in their work. The inclusion of the ρ for example necessitates a cutoff, since there is a contribution to the $P_1(\bar{z})$ term in the partial wave expansion of (I1).

$$\bar{z} = 1 + \frac{2s}{t - 4m^2} \qquad (I\ 3)$$

and this is linearly divergent in the energy s. (t is the momentum transfer). A similar difficulty arises in the analogous Tamm-Dancoff calculation (see for example GOTTMAN and HOCHBERG (6)). SW, however, consider such particles as Regge poles and they indicate that this merely amounts to multiplying the contribution by a dying exponential factor of the form

$$\exp[c_\rho(t-m_\rho^2)\ln(-1+s/2m^2)] \qquad (I\ 4)$$

where m_ρ is the effective mass of the ρ, and c_ρ is left as an adjustable parameter. m is the nucleon mass (939 MeV). SW also find it convenient to leave some of the masses of the resonances as adjustable parameters also, since if the resonance is broad it is difficult to attach a unique exact value. The parameters are then adjusted to give the best fit with experimental data.

In our work, however, a different point of view is adopted. The only terms that are included are

those which we can compute uniquely and unambiguously. Using just this simple model, we investigate to what degree of accuracy experimental data can be reproduced.

In Chapter II we outline GGMW's formulation of the problem. Since it is obviously impossible to give a detailed account of their paper, we have confined our summary to those sections most relevant to our calculation, and have endeavoured to be as brief as is consistent with clarity.

The one particle exchange terms are well known, and we merely quote them from the paper of CHARAP, LUBKIN and SCOTTI ([7]) (CLS). In order to calculate the contribution due to the exchange of two pions, we employ the unitarity condition of CLS (equation (6.19a) of their paper), which we write symbolically as

$$\mathrm{Im}_{2\pi} \langle f | \phi | i \rangle = \int \frac{q}{4\pi} \, d\Omega_q \, \mathcal{F}^* \mathcal{F} \qquad (15)$$

q is the 3-momentum of one of the pions in the CM system for the process $\pi\,\pi \longrightarrow N\bar{N}$, and the integration is over all directions in space of either of the pions. i and f denote respectively the initial and final states of the nucleon-antinucleon pair for the process $N\bar{N} \longrightarrow N\bar{N}$,

and $\mathcal{F}$ denotes the Frazer-Fulco ([5]) helicity amplitude for the process $N\pi \longrightarrow \pi N$.

In Chapter III we discuss how the 'N/D' method of Chew and Mandelstam ([8]) is applied in conjunction with the unitarity conditions

$$\text{Im } h = \frac{p}{E} \left| h \right|^2 \tag{I6}$$

which permit us to write h in terms of a phase shift as follows

$$h = \frac{E}{p} e^{i\delta} \sin\delta \tag{I7}$$

p is the magnitude of the 3-momentum of one of the nucleons in the CM system of the process $NN \longrightarrow NN$ (channel 1), and E its energy given by

$$E^2 = p^2 + m^2 \tag{I8}$$

Assumptions about the analytic properties of the functions N and D are used in conjunction with Cauchy's integral formula. The resulting integral equations are then transformed into a form suitable for direct numerical solution by making a change of variable to reduce

the integration range from an infinite interval to the unit interval. The phase shifts and coupling parameters for each of the partial waves (J=0,5) are computed for laboratory energies $\frac{1}{4}(\frac{1}{4}) \longrightarrow 3$ pion masses. For the special case J=1, we examine the denominator function D for the coupled triplet amplitudes. We expect the determinant of D (a two by two matrix) to vanish for a value of the CM momentum $\nu = \nu_D$ related to the deuteron mass m_D as follows

$$4(\nu_D + m^2) = m_D^2 \qquad (19)$$

(see chapter III section 2)

In Chapter IV some of the computational and numerical details are discussed, and in Chapter V we display graphically our calculated phase shifts. For comparison, the corresponding results of other theoretical and experimental investigations are exhibited. In the appendix we outline a possible procedure for improving the solutions of our integral equations by a combined iteration-matrix inversion procedure.

For future reference we now summarize our input data.

m = nucleon mass 939 MeV

μ = pion mass 137 MeV

$g^2/4\pi = 14$ and g is the rationalised, renormalized coupling constant.

$a_{pp}^{I=1} = {}^1S_0$ scattering length -7.7×10^{-13} cm.

$a_{np}^{I=0} = {}^3S_1$ scattering length $+5.4 \times 10^{-13}$ cm.

for both S-waves our scattering lengths (a) and effective range (r) are related to the amplitude h and phase shift δ by

$$p \cot \delta = -1/a + \tfrac{1}{2} r p^2 \qquad \text{(II0)}$$

$$h(0) = -ma \qquad \text{(II1)}$$

CHAPTER II

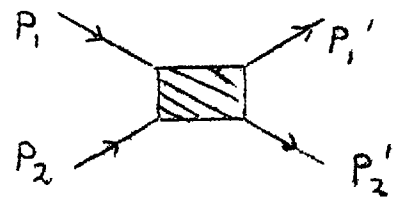
The GGMW formulation - and calculation of the absorptive parts of the scattering amplitudes.

In this chapter for the sake of completeness we have reproduced those sections of GGMW (with suitable modifications when necessary) which are relevant to our calculation.

1) Kinematics and the Mandelstam representation

N_1 and N_2 are the incoming nucleons, N_1' and N_2' the outgoing particles. The respective 4-momenta are p_1, p_2, p_1', p_2'. The process

$N_1 + N_2 \rightarrow N_1' + N_2'$ is referred to as 'channel I'



(Fig. II 1)

We define our scalar product by $A.B = \underline{A}.\underline{B} - A_o B_o$, and the Mandelstam variables by

$$\left.\begin{aligned} s &= -(P_1+P_2)^2 \\ t &= -(P_1-P_2')^2 \\ \bar{t} &= -(P_1-P_1')^2 \end{aligned}\right\} \qquad \text{(II 1.1a)}$$

Energy-momentum conservation implies $P_1+P_2 = P_1'+P_2'$ (II 1b) and the 'mass-shell' relations $P_1^2 = -m^2$ etc.; protons and neutrons are assumed to have equal mass m. Then

$$s+t+\bar{t} = 6m^2 - 2P_1\cdot P_2 + 2P_1\cdot P_2' + 2P_1\cdot P_1'$$

$$= 6m^2 + 2P_1\cdot(P_1'+P_2'-P_2) = 6m^2+2P_1\cdot P_1=4m^2 \quad \text{(II 1c)}$$

As explained by GGMW ([1]) for each isospin state (I=0,1) we require five amplitudes viz.

$J=\ell$ (total angular momentum = orbital angular momentum)

for both spin singlet and triplet states (2 amplitudes)

For $\ell=J\pm1$ three amplitudes are required

$$J+1 \longrightarrow J+1$$

$$J-1 \longrightarrow J-1$$

$$J-1 \longleftrightarrow J+1$$

The double arrow denotes two transitions, but due to time-reversal invariance these can be represented by one and the same amplitude.

The Feynman amplitude for channel I can be written as

$$\mathfrak{T}_1 = \sum_{j=1,5} \sum_{I=0,1} F_j^I \, C^j_{p1} \, \mathcal{P}_I \qquad \text{(II 1.2)}$$

The F's are functions of $s,\bar{t},t$; $\mathcal{P}_I$ (I=0,1) are the I-spin projection operators.

$$\left.\begin{aligned} \mathcal{P}_0 &= (\tilde{1} - \underline{\tau}_1 \cdot \underline{\tau}_2)/4 \\ \mathcal{P}_1 &= (3 + \underline{\tau}_1 \cdot \underline{\tau}_2)/4 \end{aligned}\right\} \qquad \text{(II 1.3)}$$

The Cpl denote five independent combinations of S,T.....etc. (the nature of the coupling involved - see below). The actual combinations are arbitrary, but the set chosen by GGMW is convenient when used in conjunction with the Pauli principle.

Thus
$$\left.\begin{aligned} \text{Cpl}^1 &= S - \tilde{S} \\ \text{Cpl}^2 &= T + \tilde{T} \\ \text{Cpl}^3 &= A - \tilde{A} \\ \text{Cpl}^4 &= V + \tilde{V} \\ \text{Cpl}^5 &= P - \tilde{P} \end{aligned}\right\} \text{(II 1.4)}$$

where

$S = \bar{u}(P_2')\bar{u}(P_2)u(P_1')u(P_1)$ SCALAR - coupling

$T = \frac{1}{2}\bar{u}(P_2')\sigma_{\mu\nu}u(P_2)\bar{u}(P_1')\sigma_{\mu\nu}u(P_1)$ TENSOR "

$A = \bar{u}(P_2')i\gamma_5\gamma_\mu u(P_2)\bar{u}(P_1')i\gamma_5\gamma_\mu u(P_1)$ AXIAL-VECTOR "II(1.5

$V = \bar{u}(P_2')\gamma_\mu u(P_2)\bar{u}(P_1')\gamma_\mu u(P_1)$ VECTOR "

$P = \bar{u}(P_2')\gamma_5 u(P_2)\bar{u}(P_1')\gamma_5 u(P_1)$ PSEUDO-SCALAR "

$\tilde{S}$ etc. are obtained by interchanging the final particles $\bar{u}(P_1') \Longleftrightarrow \bar{u}(P_2')$ and can be related to the S,T etc. by a Fierz matrix given in GGMW.

The u's are positive energy spinors normalized so that $\bar{u}u = 1$

The full amplitude changes sign when $t \leftrightarrow \bar{t}$ and so does β_0 (but not β_1). The Pauli principle thus implies

$$F_i^{\,I}(s,\bar{t},t) = (-1)^{i+I} F_i^{\,I}(s,t,\bar{t}) \qquad \text{(II 1.6)}$$

If we also consider the process (as we shall later)

$$N_1 + \bar{N}_2 \rightarrow N_1' + \bar{N}_2' \qquad \text{(Channel II)}$$

we have the amplitudes $\bar{F}_i^{\,I}$ related to the $F_i^{\,I}$ by

$$F_j(s,\bar{t},t) = \Gamma_{jk}\, B\, \bar{F}_k(\bar{t},s,t) \qquad \text{(II 1.7)}$$

We have suppressed the I-spin indices but these are implicit in the matrix notation for I-spin. B is the well known I-spin crossing matrix

$$B = \frac{1}{2}\begin{Bmatrix} -1 & 3 \\ 1 & 1 \end{Bmatrix} \qquad \text{II 1.8)}$$

and

$$\Gamma = \frac{1}{4}\begin{pmatrix} -1 & 6 & -4 & 4 & -1 \\ 1 & 2 & 0 & 0 & 1 \\ -1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 2 & -1 \\ -1 & 6 & 4 & -4 & -1 \end{pmatrix} \qquad \text{(II 1.9)}$$

We assume a Mandelstam representation (9) for the F's:

$$F_j(s,\bar{t},t) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \int_{4\mu^2}^{\infty} \frac{d\bar{t}'}{\pi} \frac{\rho_{12}^{\,j}(s',\bar{t}')}{(s'-s)(\bar{t}'-\bar{t})}$$

$$+ \int_{4m^2}^{\infty} \frac{ds'}{\pi} \int_{4\mu^2}^{\infty} \frac{dt'}{\pi} \frac{\rho_{13}^{\,j}(s',t')}{(s'-s)(t'-t)} \qquad \text{(II 1.10)}$$

$$+ \int_{4\mu^2}^{\infty} \frac{d\bar{t}'}{\pi} \int_{4\mu^2}^{\infty} \frac{dt'}{\pi} \frac{\rho_{23}^{\,j}(\bar{t}',t')}{(\bar{t}'-\bar{t})(t'-t)} + B_j(s,\bar{t},t)$$

B_j being the one meson exchange terms, μ the pion mass and m the nucleon mass.

We write a similar representation for channel II involving $\bar{F}_j(s,\bar{t},t)$. We must replace the ρ by $\bar{\rho}$ and the limits of integration will change in some cases since the roles of s and $\bar{t}$ are interchanged. (N.B. if we do not explicitly mention I-spin indices they are implied)

The Pauli principle then leads to

$$\begin{aligned} \rho_{13}^{j}(s',\bar{t}') &= (-1)^{j+I}\rho_{12}^{j}(s',\bar{t}') \\ \rho_{23}^{j}(\bar{t},t') &= (-1)^{j+I}\rho_{23}(t',\bar{t}') \end{aligned} \qquad \text{(I 1.11)}$$

Denoting by Ω the 10 x 10 matrix $\Gamma \otimes B$ we obtain

$$\begin{aligned} \rho_{12}^{j}(s,\bar{t}) &= \Omega^{j}{}_{k}\bar{\rho}_{12}^{k}(\bar{t},s) \\ \rho_{13}^{j}(s,t) &= \Omega^{j}{}_{k}\bar{\rho}_{23}^{k}(s,t) \\ \rho_{23}^{j}(\bar{t},t) &= \Omega_{jk}\bar{\rho}_{13}^{k}(\bar{t},t) \end{aligned} \qquad \text{(II 1.12)}$$

These relations enable us to define the absorptive parts of the amplitudes

$$A_j(s,t) = \int_{4\mu^2}^{\infty}\frac{dt'}{\pi}\frac{\rho_{13}^{j}(s,t')}{t'-t} + \int_{4\mu^2}^{\infty}\frac{d\bar{t}'}{\pi}\frac{\rho_{12}^{j}(s,\bar{t}')}{\bar{t}'-\bar{t}}$$

$$\bar{A}_j(\bar{t},t) = \int_{4\mu^2}^{\infty}\frac{dt'}{\pi}\frac{\bar{\rho}_{13}^{j}(\bar{t},t')}{t'-t} + \int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\bar{\rho}_{12}^{j}(\bar{t},s')}{s'-s} \qquad \text{(II 1.13)}$$

$$= \Omega_{jk}\left\{\int_{4\mu^2}^{\infty}\frac{dt'}{\pi}\frac{\rho_{23}^{k}(\bar{t},t')}{t'-t} + \int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\rho_{12}^{k}(s',\bar{t})}{s'-s}\right\}$$

These formulae will be employed when we calculate the absorptive part of the two pion contribution. (Section 5 of this chapter).

(2) Partial Wave Analysis

In order to carry out a partial wave analysis, the Jacob-Wick ([10]) helicity amplitude formalism is followed. Primes again denote the final particles λ's denote their respective spins. The differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = | \langle \lambda_1' \lambda_2' | \phi | \lambda_1 \lambda_2 \rangle |^2 \qquad \text{(II 2.1)}$$

in the C.M. system of channel I

ϕ is a matrix in spin space, and is a function of the total energy W. In the C.M. system our Mandelstam variables have the following form

$$\left.\begin{aligned} s = 4E^2 = W^2 &= 4(p^2 + m^2) \\ \bar{t} &= -2p^2 (1 + z) \\ t &= -2p^2 (1 - z) \end{aligned}\right\} \qquad \text{(II 2.2)}$$

E,p denote respectively the energy and 3-momentum of one of the nucleons, and z is the cosine of the scattering angle. Suitably modifying the formulae of Jacob and Wick, the connection between ϕ and $\mathcal{I}_1$ is found to be

$$\mathcal{I}_1 = 2\pi \frac{W}{m^2} \langle \lambda_1' \lambda_2' | \phi | \lambda_1 \lambda_2 \rangle \qquad \text{(II 2.3)}$$

Since the nucleon spin can only be $\pm\frac{1}{2}$ we attach to λ only the appropriate sign. GGMW now define

$$\phi_1 = \langle ++|\phi|++\rangle = \frac{1}{p}\sum_J (2J+1)\,\langle ++|T^J(W)|++\rangle\, d^J_{oo}(\theta)$$

$$\phi_2 = \langle ++|\phi|--\rangle = \frac{1}{p}\sum_J (2J+1)\,\langle ++|T^J(W)|--\rangle\, d^J_{oo}(\theta)$$

$$\phi_3 = \langle +-|\phi|+-\rangle = \frac{1}{p}\sum_J (2J+1)\,\langle +-|T^J(W)|+-\rangle\, d^J_{11}(\theta) \qquad \text{(II 2.4)}$$

$$\phi_4 = \langle +-|\phi|-+\rangle = \frac{1}{p}\sum_J (2J+1)\,\langle +-|T^J(W)|-+\rangle\, d^J_{-11}(\theta)$$

$$\phi_5 = \langle ++|\phi|+-\rangle = \frac{1}{p}\sum_J (2J+1)\,\langle ++|T^J(W)|+-\rangle\, d^J_{10}(\theta)$$

where $d^J_{oo}(\theta) = P_J(z)$

$$d^J_{10}(\theta) = \frac{\sqrt{J(J+1)}}{2J+1}\,\frac{P_{J+1}(z)-P_{J-1}(z)}{1-z^2} \qquad \text{(II 2.5)}$$

$$d^J_{11}(\theta) = \frac{1}{1+z}\left\{P_J(z) + \frac{(J+1)P_{J-1}(z)+JP_{J+1}(z)}{2J+1}\right\}$$

$$d^J_{-11}(\theta) = \frac{1}{1-z}\left\{-P_J(z) + \frac{(J+1)P_{J-1}(z)+JP_{J+1}(z)}{2J+1}\right\}$$

For ϕ_1 and ϕ_2 $J = 0, 1, 2 \ldots$ for ϕ_3, ϕ_4, ϕ_5 $J = 1, 2, 3 \ldots$ (since $d^{o}_{jj'}$ is only defined for $j = j' = 0$ and has the value $+1$). The usual conservation laws for strong interactions imply

(i) Parity Conservation

$$\langle \lambda_1' \lambda_2' | T^J(W) | Y_1 \lambda_2 \rangle = \langle -\lambda_1' -\lambda_2' | T^J(W) | -\lambda_1 -\lambda_2 \rangle \qquad \text{(II 2.6a)}$$

(ii) Time Reversal Invariance

$$\langle \lambda_1' \lambda_2' | T^J(W) | \lambda_1 \lambda_2 \rangle = \langle \lambda, \lambda_2 | T^J(W) | \lambda_1' \lambda_2' \rangle \qquad \text{(II 2.6b)}$$

(iii) Conservation of total spin

$$\langle \lambda_1' \lambda_2' | T^J(W) | \lambda_1 \lambda^2 \rangle = \langle \lambda_2' \lambda_1' | T^J(W) | \lambda_2 \lambda_1 \rangle \qquad \text{(II 2.6c)}$$

It is thus easily seen that we can define no more independent helicity amplitudes than those in (II 2.4) above.

The spinors u_λ can be represented explicitly as follows

$$u_{\lambda_1} = \frac{1}{N} \begin{Bmatrix} E+m \\ 2p\lambda_1 \end{Bmatrix} \chi_{\lambda_1} \qquad u_{\lambda'_1} = \frac{1}{N} \begin{Bmatrix} E+m \\ 2p\lambda_1' \end{Bmatrix} e^{-i\sigma_y \theta/2} \chi_{\lambda'_1}$$

$$u_{\lambda_2} = \frac{1}{N} \begin{Bmatrix} E+m \\ 2p\lambda_2 \end{Bmatrix} \chi_{-\lambda_2} \qquad u_{\lambda'_2} = \frac{1}{N} \begin{Bmatrix} E+m \\ 2p\lambda_2' \end{Bmatrix} e^{-i\sigma_y \theta/2} \chi_{-\lambda'_2} \qquad \text{(II 2.7)}$$

where

$N = \{2m(E+m)\}^{\frac{1}{2}}$ and χ_λ is an eigenstate of $\frac{1}{2}\sigma_z$ with eigenvalue λ.

The combinations of spinors occurring in equations (II 1.5) can now be evaluated and the ϕ_i can be obtained in terms

of the F's (GGMW equs. 4.17)

In order to find combinations of the $\langle\lambda_1'\lambda_2'|T^J(W)|\lambda_1\lambda_2\rangle$ which give rise to an uncoupled unitarity condition we begin with the unitarity condition given by Charap et al. (7)

For the process N+N $\rightarrow$ N+N (Channel I)

$$\frac{1}{2i}\left\{\langle\lambda_1'\lambda_2';p_f|\phi|\lambda_1\lambda_2p_i\rangle - \langle\lambda_1'\lambda_2';p_f|\phi^\dagger|\lambda_1\lambda_2p_i\rangle\right\}= \tag{II 2.8}$$

$$\frac{p}{4\pi}\sum_{\mu_1\mu_2}\int d\Omega_p\langle\lambda_1'\lambda_2';p_f|\phi^\dagger|\mu_1\mu_2p\rangle\langle\mu_1\mu_2p|\phi|\lambda_1\lambda_2;p_i\rangle$$

i denotes an initial and f a final state of 2 nucleons. μ_1 μ_2 are their intermediate helicities.

Let us for example choose $\lambda_1'\lambda_2'\lambda_1\lambda_2 = +\frac{1}{2}$, then performing the decomposition of (II 2.4) we find

$$\frac{1}{2i}\frac{1}{p}\sum_J(2J+1)P_J(\cos\theta_{fi})\left\{\langle++|T^J|++\rangle - \langle++|T^J|++\rangle\right\}=$$

$$\frac{1}{4\pi p}\int d\Omega_p\Big\{\sum_{JJ'}(2J+1)P_J(\cos\theta_{fI})P_{J'}(\cos\theta_{Ii})(2J+1)\ X$$

$$\Big\{\langle++|T^{+J}|++\rangle\langle++|T^{J'}|++\rangle + 2\langle++|T^{+J}|+-\rangle\langle+-|T^{J'}|++\rangle$$
$$+\langle++|T^{+J}|--\rangle\langle--|T^{J'}|++\rangle\Big\} \tag{II 2.9}$$

where we have made use of the symmetry properties of the $\langle\lambda_1'\lambda_2'|T^J|\lambda_1\lambda_2\rangle$ above; equations (II 2.6)

We then perform the angular integrations - giving rise to the factor $\frac{4\pi}{2J+1}\delta_{JJ'}$.

After equating coefficients of the single summation ensuing, we find

$$\mathrm{Im}\,\langle{+}{+}|T^J|{+}{+}\rangle = |\langle{+}{+}|T^J|{+}{+}\rangle|^2 + 2|\langle{+}{+}|T^J|{+}{-}\rangle|^2 + |\langle{+}{+}|T^J|{-}{-}\rangle|^2 \qquad \text{(II 2.10a)}$$

A similar calculation leads to

$$\mathrm{Im}\,\langle{+}{+}|T^J|{-}{-}\rangle = \langle{+}{+}|T^J|{+}{+}\rangle\langle{+}{+}|T^J|{-}{-}\rangle + 2\langle{+}{+}|T^J|{+}{-}\rangle \times \langle{+}{-}|T^J|{-}{-}\rangle + \langle{+}{+}|T^J|{-}{-}\rangle\langle{-}{-}|T^J|{-}{-}\rangle \qquad \text{(II 2.10b)}$$

Subtracting the second from the first yields

$$\mathrm{Im}\{\langle{+}{+}|T^J|{+}{+}\rangle - \langle{+}{+}|T^J|{-}{-}\rangle\} = \langle{+}{+}|T^J|{+}{+}\rangle - \langle{+}{+}|T^J|{-}{-}\rangle \times \langle{+}{+}|T^J|{+}{+}\rangle - \langle{+}{+}|T^J|{-}{-}\rangle \qquad \text{(II 2.11)}$$

We define $f_0^J = \langle{+}{+}|T^J|{+}{+}\rangle - \langle{+}{+}|T^J|{-}{-}\rangle$ SPIN SINGLET (II 2.12)

Then (II 2.11) can be written as

$$\mathrm{Im}\, f_0^J = |f_0^J|^2$$

Similarly

$$\mathrm{Im}\, f_1^J = |f_1^J|^2 \qquad \text{(II 2.12)}$$

$$\mathrm{Im}\begin{pmatrix} f_{11}^J & f_{12}^J \\ f_{12}^J & f_{22}^J \end{pmatrix} = \begin{pmatrix} f_{11}^J & f_{12}^J \\ f_{12}^J & f_{22}^J \end{pmatrix} \begin{pmatrix} f_{11}^J & f_{12}^J \\ f_{12}^J & f_{22}^J \end{pmatrix}$$

where GGMW we define

$$f_1^J = \langle{+}{-}|T^J|{+}{-}\rangle - \langle{+}{-}|T^J|{-}{+}\rangle \quad \ldots\ldots\ldots \text{UNCOUPLED TRIPLET} \quad \ell = J$$

$$\left.\begin{aligned} f_{11}^J &= \langle{+}{+}|T^J|{+}{+}\rangle + \langle{+}{+}|T^J|{-}{-}\rangle \\ f_{12}^J &= 2\langle{+}{+}|T^J|{+}{-}\rangle \\ f_{22}^J &= \langle{+}{-}|T^J|{+}{-}\rangle + \langle{+}{-}|T^J|{-}{+}\rangle \end{aligned}\right\} \text{coupled TRIPLET} \quad \ell = J \pm 1 \qquad \text{(II 2.13)}$$

Making use of (II 2.4), (II 2.5), (II 2.13), amplitudes f_i are defined by GGMW as follows

$$f_1 = E(\phi_1 - \phi_2)$$

$$f_2 = E(\phi_1 + \phi_2)$$

$$f_3 = E\left\{\frac{\phi_3}{1+z} - \frac{\phi_4}{1-z}\right\} \qquad \text{where } z = \cos\theta, \quad y = \sin\theta \qquad \text{(II 2.14)}$$

$$f_4 = E\left\{\frac{\phi_3}{1+z} + \frac{\phi_4}{1-z}\right\}$$

$$f_5 = \frac{2m\phi_5}{y}$$

Using (II 2.7) leads, as we have said, to GGMW (equ. 4.17) relating the ϕ's and F's. GGMW find it is convenient to consider linear combinations of the F's to give amplitudes G_i (also satisfying a Mandelstam representation

$$4\pi G_1 = F_1 - 4F_3 + F_3 \qquad F_1 = \tfrac{\pi}{2}[G_1 + 4G_3 + 3G_5]$$

$$4\pi G_2 = 2F_2 \qquad F_2 = \tfrac{\pi}{2}[4G_2]$$

$$4\pi G_3 = F_1 - 2F_3 - F_5 \qquad F_3 = \tfrac{\pi}{2}[-G_1 + G_5]$$

$$4\pi G_4 = 2F_4 \qquad F_4 = \tfrac{\pi}{2}[4G_4]$$

$$4\pi G_5 = F_1 + 4F_3 + F_5 \qquad F_5 = \tfrac{\pi}{2}[3G_1 - 4G_3 + G_5]$$

(II 2.15a) (II 2.15b)

Then $f_1 = E^2G_1 - zp^2G_2 + m^2G_3$

$f_2 = (E^2G_2 + m^2G_4)z - p^2G_5$

$f_3 = -p^2G_3$ (II 2.16)

$f_4 = m^2G_2 + E^2G_4$

$f_5 = -m^2(G_2 + G_4)$

It can easily be seen from these relations that the ϕ_i are not Mandelstam amplitudes since they contain the factor E which gives rise to a new branch cut in the unphysical region $E^2 < 0$ i.e. $p^2 < -m^2$.

The f_i^J can now be projected out from (II 2.4) using the orthogonality of the d^J functions

$$\int_0^1 d^j_{m\mu}(z) d^{j'}_{m\mu}(z) dz = \delta_{jj'} \frac{2}{2j+1} \qquad \text{(II 2.17)}$$

Thus

$$f_o^J = \frac{p}{4E}\int_{-1}^{1} f_1(s,z) P_J(z) dz$$

$$f_{11}^J = \frac{p}{4E}\int_{-1}^{1} f_2(s,z) P_J(z) dz$$

$$f_{12}^J = \frac{p}{4m}\int_{-1}^{1} f_5(s,z) \sqrt{\frac{J(J+1)}{2J+1}} \left\{ P_{J+1}(z) - P_{J-1}(z) \right\} dz \qquad \text{(II 2.18)}$$

$$f_{22}^{J} = \frac{p}{4E}\int_{-1}^{1}\left\{ f_3(s,z)P_J(z)+f_4(s,z)\ \frac{JP_{J+1}(z)+(J+1)P_{J-1}(z)}{2J+1}\right\}dz$$

$$f_{1}^{J} = \frac{p}{4E}\int_{-1}^{1}\left\{ f_4(s,z)P_J(z)+f_3(s,z)\ \frac{JP_{J+1}(z)+(J+1)P_{J-1}(z)}{2J+1}\right\}dz$$

These formulae differ from the corresponding ones of GGMW (equs. 4.25) by a factor of $\frac{1}{2}$. This is due to the fact that I-spin has been suppressed in their work and a factor of 2 must be introduced into the original definition of the S-matrix to account for the fact that the nucleons are identical as far as I-spin properties are concerned.*

We now define

$$h_o^J = \frac{E}{p} f_o^J ; h_{11}^J = \frac{E}{p} f_{11}^J ; h_{12}^J = \frac{E}{p} f_{12}^J ; h_{22}^J = \frac{E}{p} f_{22}^J$$

$$h_1^J = \frac{E}{p} f_1^J \text{ and } \left.\right\}_{12}^J = \frac{m}{p} f_{12}^J \qquad \text{(II 2.19)}$$

Since the nucleon-nucleon transitions must satisfy the well known relation $(-1)^{\ell+s+I} = -1$, only the following are possible

		s = 1	
	s = 0	J = ℓ	J = $\ell \pm 1$
I = 0	J odd	J even	J odd
I = 1	J even	J odd	J even

(TABLE II A)

* I am indepted to Professor D. Y. Wong and Guisseppe Marchesini for private communications on this point.

States of given orbital angular momentum ℓ can now be found from the $f_{11}^{J} = f_{12}^{J}$ and f_{22}^{J} by means of Clebsch Gordan coefficients

Thus according to Jacob and Wick ([10]) equation B.5

$$\langle JM;LS/JM;\lambda_1\lambda_2\rangle = \left\{\frac{2L+1}{2J+1}\right\}^{1/2} C(LSJ;0,\lambda)C(s_1s_2s;\lambda_1,-\lambda_2) \quad \text{(II 2.20)}$$

J,M denote respectively the total and z-component of angular momentum, L the orbital angular momentum and S the spin. As usual λ_1 and λ_2 are the helicities

$\lambda = \lambda_1 - \lambda_2$

Thus

$$\langle JM;LS| = \sum_{\lambda_1\lambda_2} \langle JM;\lambda_1,\lambda_2 | C(LSJ;0,\lambda)C(s_1s_2s;\lambda_1,-\lambda_2)\sqrt{N} \quad \text{(II 2.21)}$$

Thus

$$\langle JM;LS|T|JM;LS\rangle = \sum_{\lambda_1\lambda_2}\sum_{\lambda_1'\lambda_2'} \langle JM;\lambda_1-\lambda_2|T|JM;\lambda_1',-\lambda_2'\rangle N \quad \text{(II 2.22)}$$

$$\times\ C(LSJ;0,\lambda)C(s_1s_2s;\lambda_1,-\lambda_2)C(LSJ,0,\lambda')$$

$$\times\ C(s_1's_2's;\lambda_1',-\lambda_2')$$

where $L = J \pm 1$ and N is a normalisation factor. Similar formulae hold for the other transitions $L = J+1 \leftrightarrow J-1$. After evaluating the corresponding coefficients equations (4.26) of GGMW result.

$$f_{\ell = J-1} = \frac{1}{2J+1}\left[J f_{11}^{J} + (J+1) f_{22}^{J} + 2[J(J+1)]^{\frac{1}{2}} f_{12}^{J} \right]$$

$$f_{\ell = J+1} = \frac{1}{2J+1}\left[(J+1) f_{11}^{J} + J f_{22}^{J} - 2[J(J+1)]^{\frac{1}{2}} f_{12}^{J} \right] \qquad \text{(II 2.23a)}$$

$$f_{J-1 \leftrightarrow J+1} = \frac{1}{2J+1}\left[[J(J+1)]^{\frac{1}{2}} (f_{22}^{J} - f_{11}^{J}) - f_{12}^{J} \right]$$

It is interesting to note that these relations can be expressed as a similarity transformation as follows

$$\begin{pmatrix} f_{\ell=J-1} & f_{J-1 \leftrightarrow J+1} \\ f_{J-1 \leftrightarrow J+1} & f_{\ell=J+1} \end{pmatrix} = \begin{pmatrix} -\frac{J}{2J+1} & -\frac{J+1}{2J+1} \\ \frac{J+1}{2J+1} & -\frac{J}{2J+1} \end{pmatrix} \times$$

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix} \begin{pmatrix} -\frac{J}{2J+1} & \frac{J+1}{2J+1} \\ -\frac{J+1}{2J+1} & -\frac{J}{2J+1} \end{pmatrix} \qquad \text{(II 2.23b)}$$

The multiplying matrix is easily seen to be unimodular and orthogonal. The inverse relations are obtained trivially from these. It is also clear that the matrix of the f_{ℓ}'s will satisfy a relation identical to (II 2.12), and it is in fact from this that the coupled phase-shifts $\delta_{J\pm1,J}$ and coupling parameter ϵ_J are defined, (see Chapter III). We prefer, however, to work with the

equations for f_{11}^{J}, f_{12}^{J} and f_{22}^{J} and apply the transformations (II 2.23) only in the final stage of the calculation in order to obtain these parameters.

3) Crossing

In a later section (II 4) we calculate the absorptive part of the amplitude corresponding to $N\bar{N}$ scattering (in lowest order)—channel II. In order to find the contribution of this to NN scattering (channel I) we require the relations connecting these channels.

Let us write equations (II 2.15) as $G = \Lambda F$

then using (II 1.7) we find

$$G(s,\bar{t},t) = \Delta B \bar{G}(\bar{t},s,t) \qquad \text{(II 3.1)}$$

where $\Delta = \Lambda \Gamma \Lambda^{-1}$ (II 3.2)

and as usual B is the I-spin crossing matrix (II 1.8).

More explicitly

$$\Delta = \frac{1}{4} \begin{pmatrix} -1 & 6 & 4 & 4 & -1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 2 & -1 \\ -1 & 0 & 2 & 2 & 1 \end{pmatrix} \qquad \text{(II 3.3)}$$

Using (II 2.16) we may write

$$f(s,z) = A(s,z)\, B\, \bar{G}\,(\bar{t},s,t) \qquad \text{(II 3.4)}$$

and we find the matrix A is given by

$$\begin{pmatrix} -\frac{1}{2}p^2(1+z) & 3E^2-p^2z & 3E^2-p^2 & -E^2-p^2 & -m^2-\frac{1}{2}p^2(1+z) \\ \frac{1}{2}p^2(1+z) & -3p^2+E^2z & m^2z+2p^2 & m^2z-2p^2 & \frac{1}{2}(E^2+m^2)z+p^2 \\ -\frac{1}{2}p^2 & 0 & -p^2 & -p^2 & \frac{1}{2}p^2 \\ -\frac{1}{2}p^2 & m^2 & E^2 & E^2 & \frac{1}{2}(E^2+m^2) \\ 0 & -m^2 & -m^2 & -m^2 & -m^2 \end{pmatrix}$$

(II 3.5)

Inverting (II 2.16) and formally putting a bar on <u>all</u> the quantities involved, we obtain equations (4.33) of GGMW.

The amplitudes $\bar{G}$ are written for the order of variables $\bar{G}(\bar{t},s,t)$ - recall that $\bar{t}$ and s interchange their roles of momentum transfer and total energy.

$$\begin{aligned} \bar{t} &= 4\bar{E}^2 = 4(\bar{p}^2+m^2) \\ t &= -2\bar{p}^2(1-\bar{z}) \\ s &= -2\bar{p}^2(1+\bar{z}) \end{aligned} \qquad \text{(II 3.6)}$$

Combining (II 3.5) and (4.33) of GGMW we find

$$f(E,z) = X\bar{f}(\bar{E},\bar{z}) \qquad \text{(II 3.7)}$$

(apart from I-spin crossing), and X is given overleaf.

$$X = \frac{1}{2}$$

$$\begin{bmatrix}
1 & -1 & \frac{-s}{2\bar{p}^2} & \frac{-s}{2\bar{p}^2} & 0 \\
-1 & 1-\frac{t\bar{t}}{8p^2\bar{p}^2}+\frac{t}{2p^2} & \frac{t}{2\bar{p}^2}+\frac{s\bar{t}}{8p^2\bar{p}^2} & -\left\{\frac{t}{2\bar{p}^2}+\frac{s\bar{t}}{8p^2\bar{p}^2}\left(1+\frac{s}{2\bar{p}^2}\right)\right\} & \frac{s\bar{t}t}{8p^2\bar{p}4} \\
-\frac{2p^2}{\bar{t}} & -\frac{s-4m^2}{8\bar{p}^2} & \frac{p^2}{\bar{p}^2}\left(1-\frac{2m^2}{\bar{t}}\right) & -\frac{p^2}{\bar{p}^2}\left\{1-\frac{2m^2}{\bar{t}}\left(1+\frac{s}{2\bar{p}^2}\right)\right\} & \frac{sp^2}{2\bar{p}^4} \\
-\frac{2p^2}{\bar{t}} & -\frac{s+4m^2}{8\bar{p}^2} & -\frac{p^2}{\bar{p}^2}\left\{1+\frac{2m^2}{\bar{t}}\left(1+\frac{\bar{t}}{2p^2}\right)\right\} & \frac{p^2}{\bar{p}^2}\left\{1+\frac{2m^2}{\bar{t}}\left(1+\frac{s}{2\bar{p}^2}\right)\left(1+\frac{\bar{t}}{2p^2}\right)\right\} & -\frac{2sp^2+ts}{4\bar{p}^4} \\
0 & \frac{m^2}{\bar{p}^2} & \frac{m^2}{\bar{p}^2} & -\frac{m^2}{\bar{p}^2}\left(1+\frac{s}{2\bar{p}^2}\right) & \frac{2p^2}{\bar{p}^2}\left(1-\frac{s\bar{t}}{8p^2\bar{p}^2}\right)
\end{bmatrix}$$

(II 3.8)

4) The one and two pion exchange terms

The contribution to the G amplitudes from the exchange of a particle mass μ in the $t, \bar{t}$ channels, I-spin = I, coupling i = 1, 2, 3, 4, 5 (STAVP respectively) has been calculated by CLS (equation 5.1) The appropriate Feynmann graphs are

Fig. (II 4.1a)

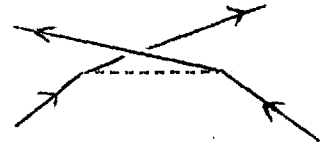


Fig (II 4.1b)

and

$$G_j^J = -\left[\frac{1}{\mu^2-t} + \frac{(-1)^{j+J}}{\mu^2-\bar{t}}\right] g^2 \left\{\delta_{J,0}(\delta_{I,0}-3\delta_{I,1}) + \delta_{J,1}(\delta_{I,0}+\delta_{I,1})\right\} A_{ji} \qquad \text{(II 4.1a)}$$

J is the spin of the exchanged particle and j refers to the coupling STAVP

$$A = \frac{1}{8\pi}\begin{pmatrix} 1 & -6 & -4 & 4 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & -2 & -1 \\ 1 & 0 & 2 & 2 & -1 \\ 1 & -6 & 4 & -4 & 1 \end{pmatrix} \qquad \text{(II 4.1b)}$$

The special case for a pion (pseudo-scalar coupling P) gives rise to the formulae given explicitly by GGMW (equ. 6.7). Since these formulae are well known we

shall not pursue this discussion any further.

We now turn to calculation of the two pion contribution. The Jacob-Wick helicity amplitudes $\mathcal{F}_{\lambda\bar{\lambda}}$ are employed and describe the process $\pi + \pi \longrightarrow N + \bar{N}$ which has differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{\bar{p}}{q} |\mathcal{F}_{\lambda\bar{\lambda}}|^2 \qquad \text{(II 4.2)}$$

where $4(\bar{p}^2 + m^2) = s = 4(q^2+\mu^2)$

$$-2\bar{p}^2(1 + \bar{z}) = \bar{t} \qquad \text{(II 4.3)}$$

$$-2\bar{p}^2(1 - \bar{z}) = t$$

s is the total energy

$\bar{p}$ the 3-momentum of a nucleon in this process - channel II

q the 3 momentum of a pion in the CM system.

s is in fact numerically equal to $\bar{t}$ of channel I .

Applying the unitarity of the S-matrix to this process, (CLS) find (their equ. 6.19)

$$\mathrm{Im}_{2\pi}\langle\lambda',\lambda';\underline{p}_f|\overline{\phi}|\lambda,\bar{\lambda};\underline{\bar{p}}_i\rangle = \frac{q}{4\pi}\int d\Omega_q \langle\underline{q}|\mathcal{F}|\lambda',\bar{\lambda}';\underline{\bar{p}}f\rangle^{*}\langle\underline{q}|\mathcal{F}|\lambda,\bar{\lambda};\underline{\bar{p}}_i\rangle$$

(II 4.4a)

$$= \frac{q}{4\pi}\int d\Omega_q \mathcal{F}^{*}_{\lambda'\lambda'}(-\theta_{2q})\mathcal{F}_{\lambda\bar{\lambda}}(\theta_{1q})\exp i[(\lambda-\bar{\lambda})\phi_{1q} + (\lambda'-\lambda')\phi_{2q}]$$

(II 4.4b)

The notation is clear - thus θ_{2q} is the angle between the final and intermediate states and θ_{1q}, for example, the angle between the initial and intermediate states. Elementary trigonometry leads to

$$z_{2q} = z_{1q}\,\bar{z} + y_{1q}\,\bar{y}\cos\phi_{1q}$$

$$e^{i\phi_{2q}} = \frac{1}{y_{2q}}\left\{\bar{y}\,z_{1q} - \bar{z}\,y_{1q}\cos\phi_{1q} + iy_{1q}\sin\phi_{iq}\right\}$$

(II 4.5)

$$z = \cos\theta \qquad y = \sin\theta$$

$$d\Omega_q = dz_{1q}\,d\phi_{1q}$$

Similar formulae to (6.1) of GGMW can be derived from these.

CLS (equs 6.7) give

$$\mathcal{F}_{++} \equiv \mathcal{F}_{\frac{1}{2},\frac{1}{2}} = \frac{mi}{8\pi\bar{E}}\left\{-\frac{\bar{p}}{m}A + B_q z_q\right\} = \mathcal{F}_{--}$$

$$\mathcal{F}_{+-} = \frac{mi}{8\pi E}\left\{\frac{\bar{E}}{m}\,Bq y_q\right\} = -\mathcal{F}_{-+} \qquad \text{(II 4.6)}$$

$A^{\pm}$, $B^{\pm}$ are the conventional amplitudes for π-N scattering

Note

$$\mathcal{F}^{(o)} = \sqrt{6}\,\mathcal{F}^{+}$$
$$\mathcal{F}(1) = 2\,\mathcal{F}^{-} \qquad \text{(II 4.7)}$$

The (o) and (1) referring to I-spin

Dispersion relations for A and B are given by Chew, Goldberger, Low and Nambu ([11]) viz. (using their notation)

$$\mathrm{Re}\ A^{(\pm)}(\nu,K^2) = \frac{\mathcal{P}}{\pi}\int_{-1-K^2/M^2}^{\infty} d\nu'\ \mathrm{Im}A^{(\pm)}(\nu',K^2)\left\{\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right\}$$

(II 4.8a)

$$\mathrm{Re}\ B^{(\pm)}(\nu,K^2) = \frac{g^2}{2M}\left\{\frac{1}{\nu_B-\nu} \mp \frac{1}{\nu_B+\nu}\right\} + \frac{\mathcal{P}}{\pi}\int_{1-K^2/M^2}^{\infty} d\nu'\ \mathrm{Im}\ B^{(\pm)}(\nu',K^2)\left\{\frac{1}{\nu'-\nu} \mp \frac{1}{\nu'+\nu}\right\}$$

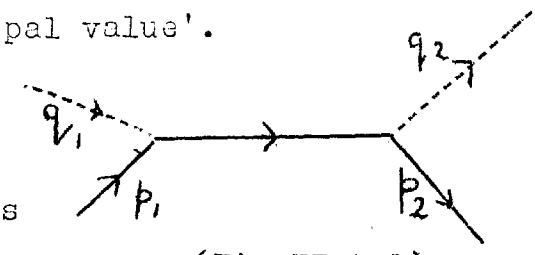
(II 4.8b)

We remark here that

$g^2/4\pi = 14$ and $\mathcal{P}$ denotes 'principal value'.

q_1 and q_2 are the 4-momenta of the incoming and outgoing pions

p_1 and p_2 are those of the nucleons



(Fig II 4.1)

$$p_1+q_1 = p_2+q_2$$

$$P = \tfrac{1}{2}(p_1+p_2) \quad Q = \tfrac{1}{2}(q_1+q_2) \quad K = \tfrac{1}{2}(q_1-q_2) \qquad \text{(II 4.9a)}$$

The pion mass is taken as unity, nucleon mass M,

$$\nu = \quad - \quad P.Q/M$$

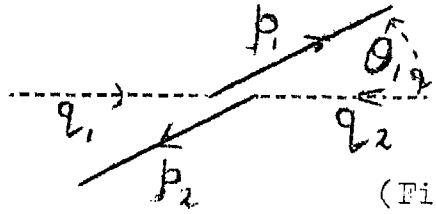
$$\nu_B = -\frac{1}{2M} - \frac{K^2}{M} \qquad \text{(II 4.9b)}$$

As pointed out by Frazor and Fulco (5) in order to go over to the process

$$\pi + \pi \longrightarrow N + \bar{N}$$

we substitute formally $p_1 \longrightarrow -p_1 \qquad q_2 \longrightarrow -q_2$

In the C.M. system for the process $\pi\pi \longrightarrow N\bar{N}$



$p_1=(E,\bar{p}) \quad q_1=(E \quad q)$

$p_2=(E-\bar{p}) \quad q_2=(E-q)$

(Fig II 4.2)

$$P = \frac{1}{2}(p_2-p_1) = (0,-\underline{p})$$
$$Q = \frac{1}{2}(q_1-q_2) = (0,-\underline{q}) \qquad \text{(II 4.10)}$$
$$K = \frac{1}{2}(q_1+q_2) = (\bar{E},0)$$

$$\text{Thus } \nu_B = -\frac{1}{2M} + \frac{\bar{E}^2}{M}$$

$$\nu = \frac{p-q}{M}$$

Hence the Born terms we require are of the form

$$\left\{ \frac{1}{-1+2\bar{E}^2-2\bar{p}q\cos\theta_1} \mp \frac{1}{-1+2\bar{E}^2+2\bar{p}q\cos\theta_1} \right\} \qquad \text{(II 4.11a)}$$

In the Born approximation scheme we are adepting we take $A=0$

$$\text{and} \quad B = \begin{Bmatrix} \sqrt{6} \\ 2 \end{Bmatrix} \frac{g^2}{2\bar{p}q} \left\{ \frac{1}{\lambda - z_1} \mp \frac{1}{\lambda + z_1} \right\} \qquad \text{(II 4.11b)}$$

The upper factor and sign refer to I=0, the lower ones to I=1

Following GGMW we have written $\lambda = \dfrac{s-2\mu^2}{4\bar{p}q} = \dfrac{s-2\mu^2}{[(s-4m^2)(s-4\mu^2)]^{\frac{1}{2}}}$

(II 4.11c)

We remark that $s = 4\bar{E}^2$ and numerically corresponds to $\bar{t}$ of channel I. Substituting back into equations (II 4.6) and (II 4.4b) we find formulae similar to those of GGMW (appendix B equation B.5)

For example

$$\begin{Bmatrix} 1/6 \\ 1/4 \end{Bmatrix} \mathrm{Im}(\bar{f}_3 + \bar{f}_4) = \frac{(g^2/4\pi)^2 q\bar{E}}{32\pi\bar{p}^2(1+\bar{z})} \int d\Omega \left\{ \sin^2\theta_1 - \frac{\sin\bar{\theta}\sin\theta_1(\cos\theta_1+\cos\theta_2)e^{-i\phi_1}}{1+\cos\bar{\theta}} \right\}$$

$$\times \left\{ \frac{1}{\lambda - z_1} \mp \frac{1}{\lambda + z_1} \right\} \left\{ \frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2} \right\} \qquad \text{(II 4.12)}$$

and similar formulae for the remaining 6 amplitudes ($\mathrm{Im}f_1^{\,0}$ and $\mathrm{Im}f_1'$ vanish) .

GGMW show that all the integrals of the type which are required in the present calculation can be obtained in terms of

$$T_1 = \frac{\lambda}{4\pi}\int d\Omega \frac{1}{\lambda - z_1} = \frac{s-2\mu^2}{[(4m^2-s)(s-4\mu^2)]^{\frac{1}{2}}} \times \tan^{-1}\frac{[(4m^2-s)(s-4\mu^2)]^{\frac{1}{2}}}{s-2\mu^2}$$

$$T_2 = \frac{\lambda^2}{4\pi}\int d\Omega \frac{1}{(\lambda-z_1)(\lambda-z_2)} = \frac{(s-2\mu^2)^2}{\left\{t(s-4\mu^2)[(s-2\mu^2)^2+\bar{t}(s-4\mu^2)]\right\}^{\frac{1}{2}}} \times \tan^{-1}\left\{\frac{t(s-4\mu^2)}{(s-2\mu^2)^2+\bar{t}(s-4\mu^2)}\right\}^{\frac{1}{2}}$$

$$T_3 = \frac{\lambda^2}{4\pi}\int d\Omega \frac{1}{(\lambda+z_1)(\lambda-z_2)} = \frac{(s-2\mu^2)^2}{\left\{\bar{t}(s-4\mu^2)[(s-2\mu^2)^2+t(s-4\mu^2)]\right\}^{\frac{1}{2}}} \tan^{-1}\left\{\frac{\bar{t}(s-4\mu^2)}{(s-2\mu^2)^2+t(s-4\mu^2)}\right\}^{\frac{1}{2}}$$

(II 4.13)

Note that terms like $\frac{\lambda^2}{4\pi}\int d\Omega \frac{1}{(\lambda-z_1)(\lambda+z_2)}$ can be obtained from T_3 by letting $\lambda \longrightarrow -\lambda$

We remark also that integrals of the form $\int d\Omega \frac{z_1 z_2}{(\lambda-z_1)(\lambda \pm z_2)}$ can easily be obtained from T_1, T_2 and T_3 by resolving

into partial fractions. Also $\int d\Omega \frac{z_2}{\lambda - z_1}$ is calculated as follows [*]

From (II 4.5)

$z_2 = z_1\bar{z} - y_1\bar{y}\cos\phi_1$, $\int d\Omega \frac{\cos\phi_1}{\lambda - z_1} = 0$ so the required integral can be evaluated.

Returning now to equation (II 4.12) the contribution from the terms involving $\sin\phi_1$ vanish, so let us consider

$$\sin^2\vartheta_1 - \frac{\sin\bar{\vartheta}\,\sin\vartheta_1(\cos\vartheta_1 + \cos\vartheta_2)\cos\phi_1}{1 + \cos\bar{\vartheta}}$$

$$= \frac{(1+\bar{z})(1-z_1^2) - z_1z_2 + z_1^2\bar{z} - z_2^2 + z_1z_2\bar{z}}{1+\bar{z}} \quad \text{(again using II 4.5)}$$

$$= 1 + z_1z_2 - \frac{(z_1+z_2)^2}{1+\bar{z}}$$

The first two terms, when combined with the usual factors $\left\{\frac{1}{\lambda - z_1} \mp \frac{1}{\lambda + z_1}\right\}\left\{\frac{1}{\lambda - z_2} \mp \frac{1}{\lambda + z_2}\right\}$ are evaluated as indicated above.

For the final term we take (for example)

$$\frac{(z_1+z_2)^2}{(\lambda - z_1)(\lambda + z_2)} = -2 + \frac{\lambda - z_1}{\lambda + z_2} + \frac{\lambda + z_2}{\lambda - z_1}$$

and so (II 4.12) can be evaluated in terms of the known functions.

[*] and because of the symmetry between z_1 and z_2, $\int \frac{z_1}{\lambda - z_2} d\Omega$ is easily obtained

We now give our version of the contributions of 2π exchange to the imaginary parts of the amplitudes $\bar{f}^{I}$.

$$\mathrm{Im}\bar{f}_1^{\,0} = \mathrm{Im}\bar{f}_1^{\,\prime} = 0$$

$$\mathrm{Im}\bar{f}_2^{\,0} = \tfrac{3}{2}\wp\, m^2 q/\bar{E}\bar{p}^2(2-4T_1+T_2+T_3)$$

$$\mathrm{Im}\bar{f}_2^{\,\prime} = \wp\, m^2 q/\bar{E}\bar{p}^2(T_2-T_3)$$

$$\mathrm{Im}(\bar{f}_3^{\,0}+\bar{f}_4^{\,0}) = \frac{3}{2}\wp\frac{q\bar{E}}{\bar{p}^2(1+\bar{z})}\left\{-2+\frac{4T_1}{1+\bar{z}}+\left(\frac{\bar{z}-3}{1+\bar{z}}+\frac{1}{\lambda^2}\right)T_2-\left(\frac{1}{\lambda^2}-1\right)T_3\right\}$$

$$\mathrm{Im}(\bar{f}_3^{\,\prime\prime}+\bar{f}_4^{\,\prime\prime}) = \wp\frac{q\bar{E}}{\bar{p}^2(1+\bar{z})}\left\{\frac{4T_1}{1+\bar{z}}+\left(\frac{\bar{z}-3}{1+\bar{z}}+\frac{1}{\lambda^2}\right)T_2+\left(\frac{1}{\lambda^2}-1\right)T_3\right\}$$

$$\mathrm{Im}(\bar{f}_3^{\,0}-\bar{f}_4^{\,0}) = \frac{3}{2}\wp\frac{q\bar{E}}{\bar{p}^2(1-\bar{z})}\left\{-2+\frac{4T_1}{1-\bar{z}}+\left(1-\frac{1}{\lambda^2}\right)T_2+\left(\frac{-\bar{z}-3}{1-\bar{z}}+\frac{1}{\lambda^2}\right)T_3\right\}$$

$$\mathrm{Im}(\bar{f}_3^{\,\prime}-\bar{f}_4^{\,\prime}) = \wp\frac{q\bar{E}}{\bar{p}^2(1-\bar{z})}\left\{-\frac{4T_1}{1-\bar{z}}+\left(1-\frac{1}{\lambda^2}\right)T_2-\left(\frac{-\bar{z}-3}{1-\bar{z}}+\frac{1}{\lambda^2}\right)T_3\right\}$$

$$\mathrm{Im}\bar{f}_5^{\,0} = \frac{3}{2}\wp\frac{m^2 q}{\bar{p}^2\bar{E}}\left\{-\frac{2\bar{z}}{1-\bar{z}^2}T_1-\frac{1}{1+z}T_2+\frac{1}{1-\bar{z}}T_3\right\}$$

$$\mathrm{Im}f_5^{\,\prime} = \wp\frac{m^2 q}{p^2\bar{E}}\left\{\frac{2T_1}{1-\bar{z}^2}-\frac{T_2}{1+\bar{z}}-\frac{T_3}{1-\bar{z}}\right\}$$

where $\left(g^2/4\pi\right)^2 = \wp$

(II 4.14)

It can be seen that these functions contain factors in their denominators which appear to vanish for some values of the parameters $\bar{p}$, $\bar{z}$. It is, however, to be remembered that before these are used they must be multiplied by the crossing matrix (II 3.8). This matrix has similar vanishing factors in the denominators of its elements. We indicate below (section 5) that the actual combinations of functions occurring are such that no infinity of this type occurs.

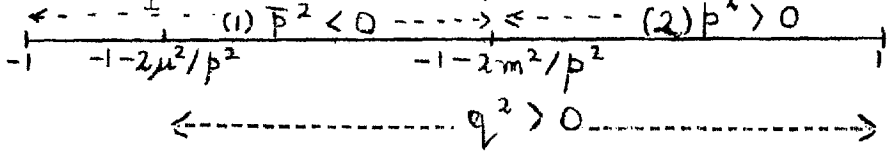
To summarize, from now on our Mandelstam variables will be as follows

✓	C.M Channel I	C.M Channel II	×
s	$4E^2=4(p^2+m^2)$	$=-2\bar{p}^2(1+\bar{z})$	$\bar{t}$
$\bar{t}$	$=-2p^2(1+z)$	$=4\bar{E}^2=4(\bar{p}^2+m^2)$	s
t	$=-2p^2(1-z)$	$=-2\bar{p}^2(1-\bar{z})$	t

Table II B

$$\text{Thus } \lambda = \frac{\bar{t}-2\mu^2}{[(\bar{t}-4m^2)(\bar{t}-4\mu^2)]^{1/2}} = \frac{\bar{t}-2\mu^2}{4\bar{p}q}$$

$$T_l = i\lambda \tan^{-1}(1/i\lambda) \qquad \text{(II 4.5)}$$



(Fig II 3)

In the region we are considering here p^2 is negative — unphysical region for channel I.

In region 1 of Fig(II 3) we have

$$T_1 = x\tan^{-1}1/x \qquad x = \frac{\bar{t}-2\mu^2}{4-\bar{p}^2 q} \qquad \text{(II 4.16)}$$

In region 2 $$= \lambda/2 \ln\left[\frac{\lambda+1}{\lambda-1}\right]$$

and $\lambda > 1$ if and only if $z > -1-2m^2/p^2-(2m^2-\mu^2)^2/-2m^2p^2$ which is always true in region 2, so T_1 is always real.

If we define

$$a = \left\{ \frac{(\bar{t}-2\mu^2)+s(\bar{t}-4\mu^2)}{t(\bar{t}-4\mu^2)} \right\}^{1/2} \qquad b = \frac{(\bar{t}-2\mu^2)^2+s(\bar{t}-4\mu^2)}{(\bar{t}-2\mu^2)^3} \qquad \text{(II4.17)}$$

$$a^2 = -1+\frac{4}{t}\left(m^2+\frac{\mu^4}{\bar{t}-4\mu^2}\right) \qquad \text{(II 4.17')}$$

$$T_2 = a/b \tan^{-1}1/a \text{ and if } a=ia'$$

$$= ia'/b \tan^{-1}(-i/a') = a'/2b \ln\left[\frac{a'+1}{a'-1}\right] \qquad \text{(II 4.18a)}$$

$$\text{Re } T_2 = a'/2b \ln\left|\frac{a'+1}{a'-1}\right| \qquad \text{(II 4.18b)}$$

The expression in (II 4.18a) can have an imaginary part in the region in which we are interested. However, as we show in the next section, only the real part need be considered (due to the fact that principal values are taken in the Mandelstam representation). Since T_2 contains the factor a/b, it tends to infinity as

$$\tilde{\Delta} = (\bar{t}-2\mu^2)^2 + s(\bar{t}-4\mu^2) \to 0$$

As is indicated in equation (II 5.7), the quantity we require is obtained by integrating the Im f's wrt. z between $-1-2\mu^2/p^2$ and $+1$. Although, as we shall show $T_2 \to \infty$ within the integration range in which we are

interested, this type of infinity causes no difficulty since it is only $1/\Delta^{1/2}$ that is involved, and the corresponding integral converges.

Substituting from table (II B) we find $T_2 \to \infty$ if

$$p^2(1+z)=(p^2+m^2-\mu^2) \pm \sqrt{(p^2+m^2)^2+2\mu^2(p^2+m^2)} \qquad \text{(II4.18c)}$$

If $p^2+m^2>0$, the value of z will be real — we recall $p^2<-\mu^2$ for a contribution to the two pion exchange terms.

$-1-2\mu^2/p^2 \leq z \leq 1$ [z in integration range of

i.e. $-2\mu^2/p^2 \leq 1+z \leq 2$ (II 5.7 below.]

or $-2\mu^2 \geq p^2(1+z) \geq 2p^2$ since p^2 is negative.

Writing $p^2+m^2=\Lambda>0$, we require

$$-2\mu^2 \geq (\Lambda-\mu^2) \pm \sqrt{\Lambda^2+2\mu^2\Lambda} \geq 2.p^2$$

$$0 \geq (\Lambda+\mu^2) \pm \sqrt{\Lambda^2+2\mu^2\Lambda} \geq 2(p^2+\mu^2)$$

and since Λ and μ^2 must be positive, this condition cannot be satisfied.

If $p^2+m^2<0$, for the square root to be real we require

$p^2+m^2+2\mu^2<0$ $(=-\Lambda$ say) and again

$$-2\mu^2 \geq (-\Lambda-3\mu^2) \pm \sqrt{(-\Lambda-2\mu^2)^2+2\mu^2(-\Lambda-2\mu^2)} \geq 2p^2 \quad \text{i.e.}$$

$$0 \geq -\Lambda-\mu^2 \pm \sqrt{\Lambda^2+2\mu^2\Lambda} \geq 2(p^2+\mu^2) \quad \text{or}$$

$$-\Lambda-\mu^2 \pm \sqrt{\Lambda^2+2\mu^2\Lambda} \geq 2(-\Lambda-m^2-\mu^2) \quad \text{or finally}$$

$$\Lambda+\mu^2+2m^2 \pm \sqrt{\Lambda^2+2\mu^2\Lambda} \geq 0$$

which is of course always satisfied, and so we can find two values of z for which $T_2 \to \infty$

In order to calculate T_3 we define analogously

$$\tilde{a}=\left\{\frac{(\bar{t}-2\mu^2)^2+t(\bar{t}-4\mu^2)}{s(\bar{t}-4\mu^2)}\right\}^{1/2} \qquad \tilde{b}=\frac{(\bar{t}-2\mu^2)^2+t(\bar{t}-4\mu^2)}{(\bar{t}-2\mu^2)^2} \qquad \text{(II4.19)}$$

alternatively $$a^2=-1+\frac{4}{s}\left[m^2+\frac{\mu^4}{\bar{t}-4m^2}\right] \qquad \text{(II4.19')}$$

and we may write $$T_3=\frac{a}{b}\tan^{-1}\frac{1}{a} \qquad \text{(II4.20)}$$

If a is complex $=ia'$ say, then

$$\text{Re } T_3=\frac{a'}{2b}\ln\left|\frac{a'+1}{a'-i}\right| \qquad \text{(II4.20')}$$

Solving the equations implied by table (IIB) and using (II 4.3) we obtain

$$\left.\begin{aligned} q^2&=-\frac{p^2(1+z)}{2}-\mu^2 \\ 1-\bar{z}&=\frac{-2p^2(1-z)}{2m^2+p^2(1+z)} \\ 1+\bar{z}&=\frac{4(p^2+m^2)}{2m^2+p^2(1+z)} \\ \bar{p}^2&=-\left[m^2+\frac{\mu^2}{2}(1+z)\right] \end{aligned}\right\} \qquad \text{(II 4.21)}$$

We now tabulate some useful properties of the variables.

FUNCTION	VANISHES	POLE
s	$p^2=-m^2$	$p^2=-\infty$
$\bar{t}$, $\bar{E}^2$	* $z=-1$, $p^2 \neq \infty$ i.e. $p^2=-m^2$	$p^2=-\infty$, $z\neq -1$
q^2	$z=-1-2\mu^2/p^2$	$p^2=-\infty$ $z\neq -1$
t	$z=1$	$p^2=-\infty$ $z\neq +1$
$1-\bar{z}$	$z=1$	$z=-1-2m^2/p^2$, $-p^2>m^2$ Includes $p^2=-\infty$, $z=-1$
$1+\bar{z}$	$p^2=-m^2$, $z\neq 1$	
$\bar{p}^2$	$z=-1-2m^2/p^2, -p^2>m^2$	$p^2=-\infty$, $z=-1$

TABLE II C

* This set of conditions does not lie within the integration range of (II 5.7) below.

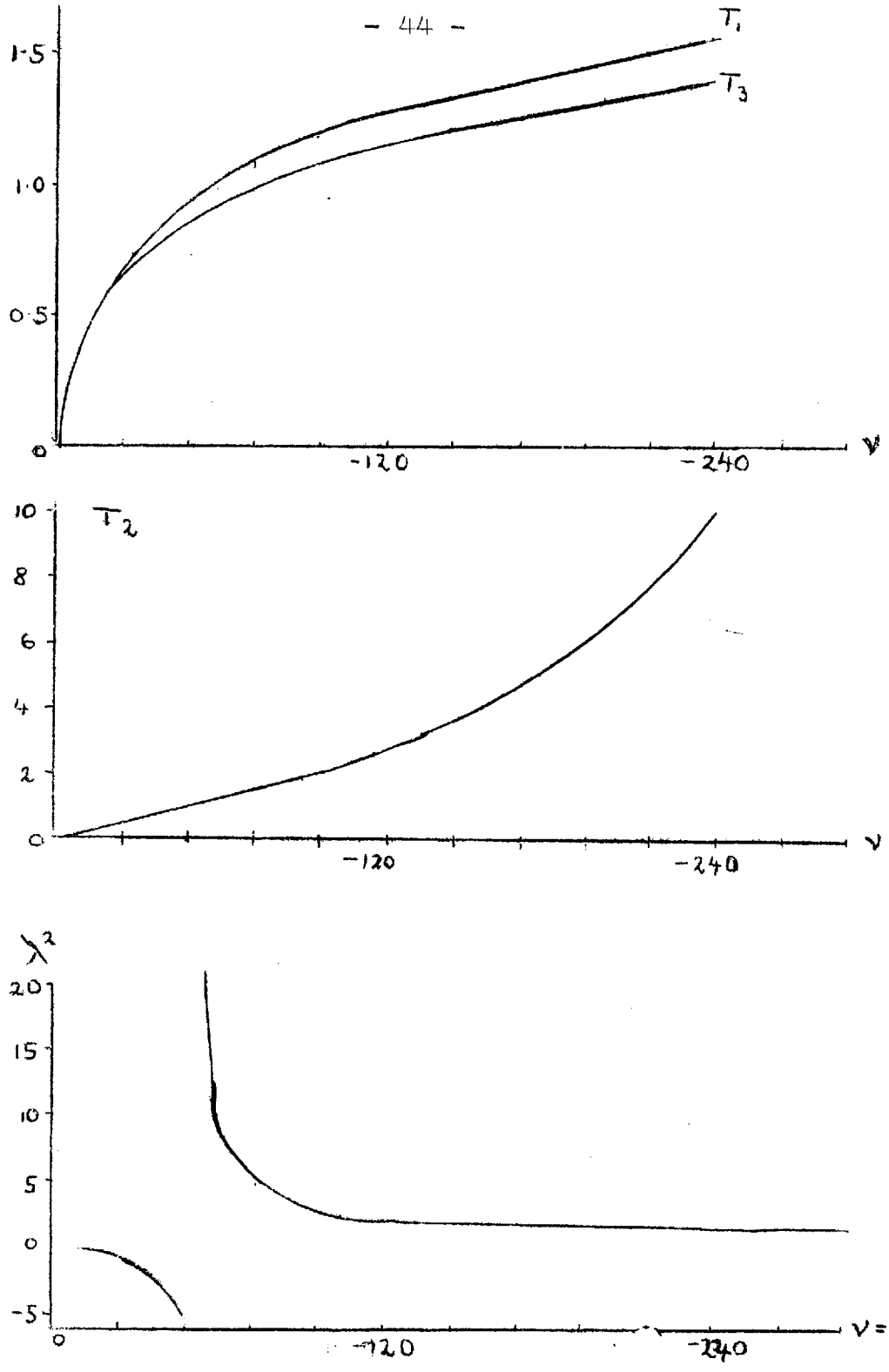
Before concluding this section we give expansions of T_1, T_2, T_3 which are valid if $\lambda^2>1$

$$T_1 = \sum_{m=0}^{\infty} \left\{\frac{2}{2\lambda^2}\right\}^m \frac{1}{2m+1} \qquad \text{(II 4.22a)}$$

$$T_2 = \sum_{m=0}^{\infty} \left\{\frac{1-\bar{z}}{2\lambda^2}\right\}^m \sum_{r=0}^{m} \frac{{}^mC_r}{2m-2r+1} \left\{\frac{1+\bar{z}}{1-\bar{z}}\right\}^r \qquad \text{(II 4.22b)}$$

$$T_3 = \sum_{m=0}^{\infty} \left\{\frac{1+\bar{z}}{2\lambda^2}\right\}^m \sum_{r=0}^{m} \frac{{}^mC_r}{2m-2r+1} \left\{\frac{1-\bar{z}}{1+\bar{z}}\right\}^r \qquad \text{(II 4.22c)}$$

The ${}^mC_r$ are the usual binomial coefficients.



(Figs. II 4.4)

T_1, T_2, T_3 and λ^2 plotted against ν ($=p^2/\mu^2=1$) for $z=0.6$

5) Application of the Mandelstam representation in order to calculate the imaginary parts of the h's.

[Some of the following can be found in section 5 of GGMW]

Let us denote equations (II 2.16) by

$$f_i = \sum_j a_{ij}(s,z) G_j(s, \bar{t}, t) \qquad \text{(II 5.1)}$$

Substituting into (II 2.18) and using (II 2.19) we obtain

$$h_\alpha{}^J(s) = \int_{-1}^{1} dz \sum_{i,j,J'} C_{\alpha i}{}^{JJ'} a_{ij}(s,z) G_j(s, \bar{t}, t) P_{J'}(z) \qquad \text{(II5.2)}$$

The C's are constant coefficients e.g. $\dfrac{[J(J+1)]^{1/2}}{2J+1}$ etc.

The Born terms are of the form $[\mu^2 + 2p^2(1 \pm z)]^{-1}$, so as z varies in the integration range of (II 5.2), this leads to a branch cut in p^2 $(=\nu)$ between $-\mu^2/4$ and $-\infty$. Since the G's have a Mandelstam representation like equation (II 1.10), we have a singularity in s from $4m^2 \longrightarrow \infty$ as $\nu = 0 \longrightarrow \infty$. Also $\bar{t}' - \bar{t}$ gives rise to a singularity since it occurs in a denominator and can vanish. Since $\nu = -2\bar{t}/(1+z)$ we have a cut in the ν-plane from $-\mu^2$ to $-\infty$ (since $\bar{t}$ has a minimum value of $4\mu^2$). Hence the denominator only vanishes when $-1 - 2\mu^2/\nu < z < 1$ ($\nu < 0$ for $\bar{t}' - \bar{t} = 0$). We can supply a similar discussion to the variable t. GGMW show that the contributions from channels II and III are in fact equivalent in virtue of the Pauli principle.

$$\text{Thus} \quad h_{\alpha}^{J}(\nu) = h_{\alpha B}^{J}(\nu) + \frac{1}{\pi}\int_{0}^{\infty} d\nu' \, \frac{\mathrm{Im}\, h_{\alpha}^{J}(\nu')}{\nu' - \nu}$$

$$+ \frac{1}{\pi}\int_{-\infty}^{-\mu^2} d\nu' \, \frac{\mathrm{Im}\left[h_{\alpha}^{J}(\nu') - h_{\alpha B}^{J}(\nu')\right]}{\nu' - \nu} \qquad \text{(II 5.3)}$$

where we have split off the Born terms $h_{B}^{J}(\nu)$. We refer to the integral between 0 and ∞ as the right hand cut (or integral), and to that between $-\infty$ and $-\mu^2$ (or if we include the Born terms $-\infty$ to $-\frac{1}{4}\mu^2$) as the left hand cut.

Using equ. (II 5.2) and the Mandelstam representation for the G's, we may obtain the discontinuity across the left hand cut using Cutkosky's rule.

$$\mathrm{Im}\left[h_{\alpha}^{J}(\nu) - h_{B}^{J}(\nu)\right] = -\sum_{i,j,J'} C_{\alpha i}^{JJ'} \int_{-1-2\mu^2/\nu}^{1} dz \; a_{ij}(s,z)$$

$$\int_{4\mu^2}^{\infty} \frac{ds'}{\pi} \frac{\rho_{12}^{j}(s',\bar{t})}{s'-s} + \mathcal{P}\int_{4\mu^2}^{\infty} \frac{dt'}{\pi} \frac{\rho_{23}^{j}(\bar{t},t')}{t'-t} P_{J'}(z) \qquad \text{(II 5.4)}$$

minus a similar term due to singularities in the t-variable.

Since the discontinuity must be real (we assume a hermitean analyticity property $h^*(\nu)=h(\nu^*)$), and $t'-t$ can vanish in the integration range, we must make a Cauchy principal value integration as indicated by $\mathcal{P}$ in the above equation. The negative sign outside the summation is the result of the way imaginary part is defined

$$\mathrm{Im}\left\{\frac{1}{\bar{t}' - \bar{t}}\right\} = \mathrm{Im}\left\{\frac{1}{\bar{t}' + 2(s/4-m^2)(1+z)+i\epsilon}\right\} = -\pi\delta(\bar{t}'-\bar{t}) \qquad \text{(II 5.5)}$$

since the $i\epsilon$ is attached to s. Comparing equs. (II 5.4) and (II 1.13) we write

$$\mathrm{Im}\left[h^J(\nu)-h_B{}^J(\nu)\right] = -2\sum_{J'} C^{JJ'} \int_{-1-2\mu^2/\nu}^{1} dz \left[a(s,z) \quad B \right.$$

$$\left. x \ \mathrm{Re}\ \bar{A}(\bar{t},t)\ P_{J'}(z)\right] \qquad \text{(II 5.6)}$$

The factor 2 is present since the contributions from channels II and III (due to singularities in $\bar{t}$,t respectively) are equal. Finally, this may be abbreviated to

$$\mathrm{Im}\left[h^J(\nu)-h_B{}^J(\nu)\right] =$$

$$-2\sum_{J'} C^{JJ'} \int_{-1-2\mu^2/\nu}^{1} dz \ X \ B \ \mathrm{Re}\left[\mathrm{Im}'\,\bar{f}(\bar{t},t)\right] P_{J'}(z) \qquad \text{(II 5.7)}$$

X is the matrix (II 3.8) and we recall that $\bar{f}$ must be considered as a (1 x 2) matrix in I-spin space.

The amplitudes Im $\bar{f}$ calculated in (II 4) can now be substituted into this expression. Only the real part of these amplitudes is required, and this tells us that T_2 and T_3 must be evaluated as indicated in equs. (II 4.18).

We now demonstrate that the $\bar{G}$ amplitudes involved have no new singularity due to the appearance of vanishing factors in the relevant denominators. On inverting the system of equations (II 2.16) and considering the amplitudes for the channel II reaction (formally placing bars over all quantities), GGMW obtain their equations (4.33)

$$\bar{G}_1 = \frac{1}{\bar{E}^2}\left[\bar{f}_1 + \frac{m^2}{\bar{p}^2}\,\bar{f}_3 - \bar{z}\,\bar{f}_4 - \bar{z}\,\frac{\bar{E}^2}{m^2}\,\bar{f}_5\right]$$

$$\bar{G}_2 = -\frac{1}{\bar{p}^2}\left[\bar{f}_4 + \frac{\bar{E}^2}{m^2}\,\bar{f}_5\right]$$

$$\bar{G}_3 = -\frac{1}{\bar{p}^2}\,\bar{f}_3 \qquad \text{(II 5.8)}$$

$$\bar{G}_4 = \frac{1}{\bar{p}^2}\left[\bar{f}_4 + \bar{f}_5\right]$$

$$\bar{G}_5 = -\frac{1}{\bar{p}^2}\left[\bar{f}_2 + \bar{z}\,\bar{f}_4 + \bar{z}\left(\frac{\bar{E}^2 + m^2}{m^2}\right)\bar{f}_5\right]$$

If the $\bar{G}$'s are to remain finite at $\bar{E}^2, \bar{p}^2 = 0$, we can see (using $\bar{E}^2 = \bar{p}^2 + m^2$) that

$\bar{f}_4 + \bar{f}_5$, $\bar{f}_3$, and $\bar{f}_2 + \bar{z}\,\bar{f}_5 = O(\bar{p}^2)$ and

$$\bar{f}_1 - \bar{f}_3 - \bar{z}\,\bar{f}_4 = O(\bar{E}^2) \qquad \text{(II 5.9)}$$

We now consider the imaginary parts of the quantities. Thus, for example, taking the I=1 component of the 2-pion contribution to the absorptive parts of our f amplitudes (equs. II 4.14), we require

$$\frac{m^2 q}{\bar{E}\,\bar{p}^2}\left\{T_2 - T_3 + \bar{z}\left(\frac{2T_1}{1-\bar{z}^2} - \frac{T_2}{1+\bar{z}} - \frac{T_3}{1-\bar{z}}\right)\right\} = O(\bar{p}^2) \qquad \text{(II 5.10)}$$

Let us consider the case $p^2 \to -m^2$, then $\bar{p}^2 \to 0$ if $\bar{z}$ remains finite. In this case $1/\lambda^2 \sim \bar{p}^2$ and we may use the expansions (II 4.22). Taking the first three terms (see also GGMW equ. B8) we have

$$\begin{aligned} T_1 &= 1 + 1/3\lambda^2 + 1/5\lambda^4 \\ T_2 &= 1 + (2+\bar{z})/3\lambda^2 + (7+6\bar{z}+2\bar{z}^2)/15\lambda^4 \\ T_3 &= 1 + (2-\bar{z})/3\lambda^2 + (7-6\bar{z}+2\bar{z}^2)/15\lambda^4 \end{aligned} \qquad \text{(II 4.22')}$$

Substituting into equs. (II 5.10) we obtain

$$\frac{m^2 q}{\bar{E}\,\bar{p}^2}\left\{\frac{2\bar{z}}{3\lambda^2} + \frac{12\bar{z}}{15\lambda^4} + \bar{z}\left[\frac{2}{1-\bar{z}^2}\left(1 + \frac{1}{3\lambda^2} + \frac{1}{5\lambda^4}\right) - \right.\right.$$

$$\left.\left.\frac{1}{1+\bar{z}}\left(1 + \frac{2+\bar{z}}{3\lambda^2} + \frac{7+6\bar{z}+2\bar{z}^2}{15\lambda^4}\right) - \frac{1}{1-\bar{z}}\left(1 + \frac{2-\bar{z}}{3\lambda^2} + \frac{7-6\bar{z}+2\bar{z}^2}{15\lambda^4}\right)\right]\right\} \qquad \text{(II 5.10')}$$

$$= \frac{m^2 q}{\bar{E} p^{-2}} \left\{ \frac{\bar{z}}{3\lambda^2(1-\bar{z}^2)} \left[2(1-\bar{z}^2) + 2-(1-\bar{z})(2+\bar{z})-(1+\bar{z})(2-\bar{z}) \right] \right.$$
$$\left. +0(1/\lambda^4) \right\}$$

$$= \frac{m^2 q}{\bar{E} p^{-2}} \quad (\text{terms involving } 1/\lambda^4 \text{ and higher powers only})$$

Since $\lambda^4 \sim 1/\bar{p}^4$, the expression (II 5.10) is $0(\bar{p}^2)$. Similar, but somewhat longer calculations along these lines can be carried out for the remaining equations of the set (II 5.9) for both I-spin amplitudes. We note $q^2 \geq 0$, so $\bar{E}$ never reaches its branch cut ($q^2 = -\mu^2$) in the integration range of (II 5.7), so the factor $1/\bar{E}$ causes no difficulty here. It is also seen that when $\bar{z} \to \pm 1$, the Im $\bar{f}$ appear to develop singularities, but again it can be seen that this is not so if we use table (II D) below. We have given the first and second derivatives of T_2 and T_3 (wrt $\bar{z}$) to enable us to employ a L'Hôspital limit when appropriate. For such a calculation it is convenient to consider λ and $\bar{z}$ as independant variables. Let us define

$$x = \left\{ \frac{1+\bar{z}-2\lambda^2}{1-\bar{z}} \right\}^{1/2} \qquad x^2 y = -\frac{2\lambda^2}{1-\bar{z}} \qquad \text{(II 5.11)}$$

then $T_2 = xy \tan^{-1} 1/x$

$$\frac{dT_2}{d\bar{z}} = \frac{y^2}{4\lambda^2} \left\{ (x-x^3)\tan^{-1} 1/x + x^2) \right\} \qquad \text{(II 5.12)}$$

$$\frac{d^2 T_2}{d\bar{z}^2} = \frac{y^3}{16\lambda^4} \left\{ (3x-2x^3+3x^5)\tan^{-1} 1/x + (3x^2-3x^4) \right\}$$

The corresponding formulae for T_3 are obtained by changing $\bar{z} \longrightarrow -\bar{z}$ everywhere (and of course prefixing a minus sign to $dT_3/d\bar{z}$).

$\bar{z}$	1	-1
T_2	$\frac{\lambda^2}{\lambda^2-1}$	T_1
T_2'	$\frac{\lambda^2}{3(\lambda^2-1)^2}$	$\frac{1}{4}\left\{T_1(1+\frac{1}{\lambda^2})-1\right\}$
T_2''	$\frac{4\lambda^2}{15(\lambda^2-1)^3}$	$\frac{1}{16\lambda^4}\left\{(3+2\lambda^2+3\lambda^4)T_1 -3\lambda^2(1+\lambda^2)\right\}$
T_3	T_1	$\frac{\lambda^2}{\lambda^2-1}$
T_3'	$-\frac{1}{4}\left\{T_1(1+\frac{1}{\lambda^2})-1\right\}$	$-\frac{\lambda^2}{3(\lambda^2-1)^2}$
T_3''	$\frac{1}{16\lambda^4}\left\{(3+2\lambda^2+3\lambda^4)T_1 -3\lambda^2(1+\lambda^2)\right\}$	$\frac{4\lambda^2}{15(\lambda^2-1)^3}$

Table (II D)

The dashes denote derivatives with respect to $\bar{z}$. Using these formulae in conjunction with equs. (II 4.14) we find

$$\mathrm{Im}(\bar{f}^{o}_{3}+\bar{f}^{o}_{4})_{\bar{z}=-1}=\mathrm{Im}(\bar{f}^{o}_{3}-\bar{f}^{o}_{4})_{\bar{z}=1}=$$

$$\frac{3\wp q\bar{E}}{2\bar{p}^{2}}\left\{\frac{1}{8}+\frac{1}{8\lambda^{2}}+\frac{1}{3(1-\lambda^{2})}-\frac{T_{1}}{8\lambda^{4}}(1-\lambda^{2})^{2}\right\}$$

$$\mathrm{Im}(\bar{f}^{1}_{3}+\bar{f}^{1}_{4})_{\bar{z}=-1}=-\mathrm{Im}(\bar{f}^{1}_{3}-\bar{f}^{1}_{4})_{\bar{z}=1}=$$

$$\frac{\wp q\bar{E}}{\bar{p}^{2}}\left\{\frac{1}{8}+\frac{1}{8\lambda^{2}}-\frac{1}{3(1-\lambda^{2})}-\frac{T_{1}}{8\lambda^{4}}(1-\lambda^{2})^{2}\right\} \qquad \text{(II 5.13)}$$

$$\mathrm{Im}\,\bar{f}^{o}_{5})_{\bar{z}=-1}=-\mathrm{Im}\,\bar{f}^{o}_{5})_{\bar{z}=1}=$$

$$\frac{-3\wp m^{2}q}{2\bar{p}^{2}\bar{E}}\left\{T_{1}\left[\tfrac{1}{2}+\tfrac{1}{4}(1+\frac{1}{\lambda^{2}})\right]-\tfrac{1}{4}-\frac{2}{2(\lambda^{2}-1)}\right\}$$

$$\mathrm{Im}\,\bar{f}^{1}_{5})_{\bar{z}=-1}=\mathrm{Im}\,\bar{f}^{1}_{5})_{\bar{z}=1}=$$

$$\frac{\wp m^{2}q}{\bar{p}^{2}\bar{E}}\left\{T_{1}\left[\tfrac{1}{2}-\tfrac{1}{4}(1+\frac{1}{\lambda^{2}})\right]+\tfrac{1}{4}-\frac{\lambda^{2}}{2(\lambda^{2}-1)}\right\}$$

As a further example, we demonstrate that when $\bar{p}=0, \bar{z}=1$ $\mathrm{Im}(\bar{f}^{1}_{4}+\bar{f}^{1}_{5})=O(\bar{p}^{2})$. Using (II 5.13) and (II 4.14), we obtain

$$\mathrm{Im}\,f^{1}_{4}=\frac{q\bar{E}}{\bar{p}^{2}}\left\{\frac{2}{3\lambda^{2}}+\frac{4}{15\lambda^{4}}\cdots\right\}$$

$$\mathrm{Im}\,\bar{f}^{1}_{5}=\frac{q\bar{E}}{\bar{p}^{2}}(1+\frac{\bar{p}^{2}}{m^{2}})^{-1}\left\{-\frac{2}{3\lambda^{2}}-\frac{8}{15\lambda^{4}}\cdots\right\} \qquad \text{(II 15.4)}$$

and the above condition follows immediately.

CHAPTER III

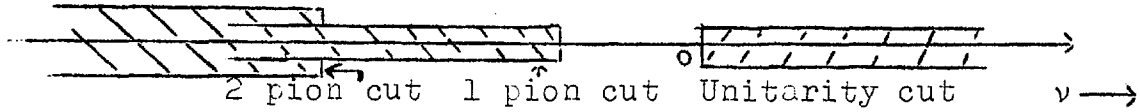
THE N/D INTEGRAL EQUATIONS

In this chapter we employ two versions of the N/D method. The first is that described by GGMw, and is extremely effective for the uncoupled amplitudes. Since the amplitude h^J_{12} has a branch cut due to the factor E (starting at $p^2=-m^2$), and this method solves the equation for D on the left hand cut, new difficulties arise due to the fact that only $\mathrm{Im}\,\eta_{12}$ is known and not $\mathrm{Im}(E\eta_{12})$ [$=\mathrm{Im}\ h_{12} \times m$], to the left of this branch cut (see chapter II, section 2 for definitions).

For the coupled amplitudes, we therefore employ the method described by Scotti and Wong ([3]), since this involves solving equations on the right hand (unitarity) cut. In the appendix we describe a method for improving the solutions of these coupled equations.

Method 1)

The amplitudes h^J have a right hand cut, and a left hand cut due to singularities of the exchange terms.



Branch points at $p^2=-\mu^2,-\mu^2/4$

(Fig.III 1)

We now take $h^J(\nu)=N^J(\nu)/D^J(\nu)$ (III 1.1)

where $N(\nu)$ has only the left hand cut and $D(\nu)$ only the right hand cut. Since all quantities involved are assumed to satisfy the hermitean analyticity property $X^*(x)=X(x^*)$, we may write

$$\begin{aligned} \text{Im } N(\nu) \underset{\text{on left hand cut}}{=} & \frac{N(\nu+i\epsilon)-N(\nu-i\epsilon)}{2i} \\ \text{Im } D(\nu) \underset{\text{on right hand cut}}{=} & \frac{D(\nu+i\epsilon)-D(\nu-i\epsilon)}{2i} \end{aligned} \qquad \text{(III 1.2)}$$

The unitarity condition may be taken as

$$\text{Im } h(\nu)=\sqrt{\frac{\nu}{\nu+m^2}}\,\left|h(\nu)\right|^2 \quad \text{or} \quad \text{Im}[1/h(\nu)]=-\sqrt{\frac{\nu}{\nu+m^2}} \qquad \text{(III 1.3)}$$

$$\text{thus} \quad \frac{\text{Im } D(\nu)}{N(\nu)} = -\sqrt{\frac{\nu}{\nu+m^2}} \quad \text{(on the right hand cut)} \qquad \text{(III 1.3')}$$

We may, therefor, write the following dispersion relation containing one subtraction, and arbitrarily normalise $D(0)=1$ —— only the ratio N/D is to be determined, so we can choose either N or D to have any finite value we please at an arbitrary point —— except of course when we have a bound state pole whose position is determined by other considerations.

$$D(\nu) = 1 - \frac{1}{\pi}\int_0^\infty \frac{\nu N(\nu')}{\nu'(\nu' - \nu)}\sqrt{\frac{\nu'}{\nu' + m^2}}\, d\nu' \qquad \text{(III 1.4)}$$

On the left hand cut Im $h(\nu)$ is given by the contributions of one and two pion exchange as derived in chapter II, and following GGMW we write

$$\text{Im } h(\nu) = \pi a(\nu) \qquad \text{(III 1.5)}$$

thus $$\frac{\text{Im } N(\nu)}{D(\nu)} = \pi a(\nu) \text{ on the left hand cut} \qquad \text{(III 1.5')}$$

Again making one subtraction and writing

$$N(0) = h(0) \qquad \text{(III 1.6)}$$

since $D(0)=1$, we have

$$N(\nu) = h(0) + \int_{-\infty}^{-1/4} \frac{\nu\, a(\nu')D(\nu')\, d\nu'}{\nu'(\nu' - \nu)} \qquad \text{(III 1.7)}$$

$h(0)$ will vanish in all cases except when we are considering S-waves, and then we supply the scattering lengths (and hence $h(0)$) from experiment. Note that we have taken the pion mass as unity in equation (III 1.7), the upper limit, in fact being given by $-\mu^2/4$ (see Fig. III 1)

Substituting (III 1.7) into (III 1.4) and interchanging the orders of integration we find

$$D(\nu)= 1 - \frac{h(0)}{\pi}\int_0^{\infty} \frac{\nu}{\nu'(\nu'-\nu)}\sqrt{\frac{\nu'}{\nu'+ m^2}}\, d\nu' - \frac{\nu}{\pi}\int_{-\infty}^{-1/4} \frac{a(\nu'')D(\nu'')d\nu''}{\nu''}\int_0^{\infty} \frac{\sqrt{\frac{\nu'}{\nu'+m^2}}\, d\nu'}{(\nu'-\nu)(\nu''-\nu')} \qquad \text{(III 1.8)}$$

In the notation of GGMW

$$D(\nu)=1+h(0)\gamma(-\nu)-\nu\int_{-\infty}^{-1/4} a(\nu')D(\nu')\,\frac{\gamma(-\nu')-\gamma(-\nu)}{\nu'-\nu}\,\frac{d\nu'}{\nu'} \qquad \text{(III 1.9)}$$

$$\text{where } \gamma(-\nu) = \frac{1}{\pi} X \ln\frac{X+1}{X-1}, X = \left\{\frac{\nu}{\nu+m^2}\right\}^{1/2} \qquad \text{(III 1.10)}$$

We now employ a device similar to that of NOYES[12] to facilitate the numerical solution of our integral equations. He makes the substitution

$$\nu' = -1/4y' \qquad \text{(III 1.11)}$$

and the integration range [$\nu'=-\infty$ to $-1/4$] becomes the unit interval. From an examination of (III 1.19) it is seen that this substitution will be most effective when $a(\nu')=O(1/\nu')$ at $-\infty$. An inspection of equations

(II 4.17-4.20) and an analysis similar to that of chapter II, section 5 indicates that, although the appearance of factors of ν' in the appropriate denominators will ensure the convergence of the integrals, due to the occurrence of logarithmic factors, a substitution of the form

$$\nu' = -1/4y'^{n} \qquad \nu = -1/4y^{n} \qquad n>1 \qquad \text{(III 1.12)}$$

will be more convenient numerically. [We recall that $\int_0 \ln x\, dx$ is more easily performed numerically if we write $x=y^n$, and the integral takes the form $\int_0 n^2 y^{n-1} dy \ln y$, and if $n>1$ the integrand vanishes at the lower limit]. If we wish to take n an integer, the most obvious choice would appear to be $n=2$, but then y would have to be imaginary in the physical region since $\nu>0$ — thus unnecessarily complicating the arithmetic. We therefore choose $n=3$. Making the substitutions

$$\nu=-1/4y^3 \qquad \nu'=-1/4y'^3 \qquad d\nu'=3/4y'^4\, dy' \qquad \text{(III 1.13)}$$

$$\alpha(\nu')=R(y) \qquad D(\nu')=D(y) \qquad \pi\gamma(-\nu)=\Gamma(y)$$

in equation (III 1.9), we find

$$D(y)=1+\frac{h(0)\Gamma(y)}{\pi}-\frac{3}{\pi}\int_0^1 R(y')D(y')\frac{\Gamma(y')-\Gamma(y)}{y'^3-y^3}dy' \quad \text{(III 1.14)}$$

If an integration formula is used which requires a knowledge of the function at the end points of the integration range, the above form of integral equation is not suitable, since the kernel is infinite when $y=y'=0$. It is, therefore, more convenient to consider the equation

$$\mathcal{D}(y)=\left\{1+\frac{h(0)\Gamma(y)}{\pi}\right\}y^2-\frac{3y^2}{\pi}\int_0^1 dy' R(y')\mathcal{D}(y')\frac{\Gamma(y')-\Gamma(y)}{y'^3-y^3} \quad \text{(III1.15)}$$

where $\mathcal{D}(y)=y^2D(y)$ (III 1.16)

$$\Gamma(y)=X\ln\frac{X+1}{X-1} \quad \text{(III 1.17)}$$

$$X=(1-4y^3m^2)^{-1/2} \quad \text{(III 1.17')}$$

At the 'diagonal points' we require $\Gamma'(y)$ and we have

$$\Gamma'(y)=6y^2m^2X^3\left[\ln\frac{X+1}{X-1}-2X/(X^2-1)\right] \quad \text{(III 1.18)}$$

It is now seen that the kernel of (III 1.15) remains finite even at the net point $y=y'=0$. In practice, Gaussian quadrature formulae were found most convenient, and the end points of the integration range are not used (see chapter IV for details).

The integral equation can be replaced by the matrix equation

$$\sum_j (\delta_{ij} - K_{ij} w_j) \mathcal{D}_j = \left[1 + \frac{h(o)\Gamma_i}{\pi}\right] y_i^2 \qquad \text{(III 1.19)}$$

where δ_{ij} is the Kronecker delta, w_j are the quadrature weighting factors, and

$$K_{ij} = -\frac{3y_i^2}{\pi} R(y_j) \frac{\Gamma(y_j) - \Gamma(y_i)}{y_j^3 - y_i^3} \qquad i \neq j$$

$$= -\frac{1}{\pi} R(y_j) \Gamma'(y_j) \qquad i = j \qquad \text{(III 1.20)}$$

Having solved this system of equations by matrix inversion, the value of $\mathcal{D}(y)$ for $y<0$ ($\nu>0$, physical region) may be obtained by a direct integration of equ. (III 1.15), provided we remember that $\Gamma(y)$ has an imaginary part in this region. Thus

$$\frac{\mathrm{Re}\,\mathcal{D}(y)}{y^2} = 1 + \frac{h(o)\tilde{\Gamma}(y)}{\pi} - \frac{3}{\pi}\int_o^1 R(y')\mathcal{D}(y') \times \left\{\frac{\Gamma(y') - \tilde{\Gamma}(y)}{y'^3 - y^3}\right\} dy' \qquad \text{(III 1.21)}$$

where $$\tilde{\Gamma}(y) = X \ln\left|\frac{X+1}{X-1}\right|$$

$$X = (1 - 4y^3 n^2)^{-1/2} \qquad \text{(III 1.22)}$$

$$\text{Also} \quad N(y)=h(0)+3\int_0^1 \frac{R(y')\nu(y')y'^2dy'}{y'^3-y^3} \qquad \text{(III 1.23)}$$

The integration of equs. (II 1.21,1.23) then gives $h(\nu)=N(\nu)/D(\nu)$ for any positive value of ν we require. The uncoupled phase-shifts are given by

$$\delta = \tan^{-1}[\text{Im } h(\nu)/\text{Re } h(\nu)] \qquad \text{(III 1.24)}$$

Also, since in the scattering length effective range approximation

$$h=E/p\ e^{i\delta}\sin\delta \text{ and } p\cot\delta=-1/a+r/2\ p^2 \qquad \text{(III 1.25)}$$

for a nucleon with 3-momentum p. Thus

$1/h = p/E\ (\cot\delta - i)$ and $\text{Re}[1/h]\sim p/m\cot\delta$ for small p (since $E\sim m$) (III 1.26)

The scattering length is given by

$$a = -h(0)\ m \qquad \text{(III 1.27)}$$

Method 2)

We again write

$$D(\nu)=1-\frac{1}{\pi}\int_0^\infty \frac{\nu N(\nu')}{\nu'(\nu'-\nu)}\sqrt{\frac{\nu'}{\nu'+m^2}}\,d\nu' \qquad \text{(III 2.1)}$$

but instead of equ. (III 1.7), which relates $N(\nu)$ to values of $D(\nu)$ on the left hand cut, we employ the exchange terms

$$h^e(\nu)= h^e(0) + \frac{\nu}{\pi}\int_{-\infty}^{-1/4} \frac{\text{Im } h(\nu')d\nu'}{\nu'(\nu'-\nu)} \qquad \text{(III 2.2)}$$

(we have performed just one subtraction). Since the full amplitude contains both left and right hand cuts, we may write

$$h(\nu)=h^e(\nu) + h^R(0) + \frac{\nu}{\pi}\int_0^\infty \frac{\text{Im } h(\nu')d\nu'}{\nu'(\nu'-\nu)} \qquad \text{(III2.3)}$$

$h^R(0)$ is, of course, the contribution to $h(0)$ from the unitarity cut. Note $h(0)=h^e(0)+h^R(0)=-a/m$ (II 2.4)

Again normalising $D(0)=1$, and observing that h^e contains the left hand cut only, we write, following SW

$$N(\nu)=h^e(\nu)D(\nu)-h^e(0)-a/m$$

$$-\frac{\nu}{\pi}\int_0^\infty \frac{h^e(\nu')\ \text{Im } D(\nu')d\nu'}{\nu'(\nu'-\nu)} \qquad \text{(III 2.5)}$$

The dispersion integral merely removes the right hand cut of the product $h^e(\nu)D(\nu)$, but leaves the singularities of the exchange terms unaffected. Taking the imaginary parts of both sides of equ. (III 2.5) gives

$$\text{Im } N(\nu)_{\overline{\text{right}}} 0 \qquad \text{(III 2.5')}$$

$$= \text{Im } h^e(\nu).\ D(\nu) \ = \text{Im } h(\nu)D(\nu)$$
left

as it should. Substituting equ. (III 2.1) into (III 2.5) leads to

$$N(\nu) = h^e(\nu) - h^e(0) - a/m + \frac{\nu}{\pi}\int_0^{\infty} d\nu' \ \frac{h^e(\nu')-h^e(\nu)}{\nu'(\nu'-\nu)}\sqrt{\frac{\nu'}{\nu'+m^2}}\ N(\nu') \qquad \text{(III 2.6)}$$

as shown by SW. This method is used for the coupled equations, so that h^e, N, and D are two by two matrices. The advantage over method 1) above, is that we can write

$$h^e_{12}(\nu)-h^e_{12(0)} = \frac{\nu}{\pi m}\int_{\infty}^{-1/4} \frac{\text{Im}[\sqrt{\nu'+m^2}\ h_{12}(\nu')]d\nu'}{\nu'(\nu'-\nu)} \qquad \text{(III 2.7)}$$

$$\text{on right hand cut} = \frac{\nu\sqrt{\nu+m^2}}{\pi m}\int_{-\infty}^{-1/4} \frac{\text{Im } h_{12}(\nu')\ d\nu'}{\nu'(\nu'-\nu)} \qquad \text{(III 2.7')}$$

by Cauchy's integral formula (if both integrals exist).

In the corresponding expression of method 1) (III 1.9)

$\alpha_{12}(\nu')=1/m\pi \quad \mathrm{Im}[E\}_{12}(\nu')]$, which is unknown, since it involves $\mathrm{Re}\}_{12}(\nu')$.

Again, for the numerical integration of equ. (III 2.6), it is convenient to make substitutions to improve the convergence behaviour at the end points of the range. $\int_0 \nu'^{-1/2} d\nu'$ converges, but the integrand is infinite at the origin. The substitution $\nu'=y^2$ eliminates this difficulty. A substitution is also made to reduce the integration range to finite limits. In equ. (III 2.1) it is convenient to subtract $N(\nu)$, so that only the combination $[N(\nu')-N(\nu)]/(\nu'-\nu)$ occurs. The residual integration can be performed exactly and is elementary.

Namely $$\frac{\nu N(\nu)}{\pi}\int_0^\infty \sqrt{\frac{\nu'}{\nu'+m^2}}\ \frac{d\nu'}{\nu'(\nu'-\nu)} \qquad \text{(III 2.8)}$$

We now make the following substitutions in equ.(III 2.6)

$$\nu=(1/y^2-1)^2\ ,\ \nu'=(1/y'^2-1)^2\ ,\ H(\nu)=h^e(\nu)-h^e(0)-a/m$$
$$N(\nu)=\mathcal{H}(y)=\mathcal{M}(y)/y\ ,\quad D(\nu)=\mathcal{D}(y)=\mathcal{Q}(y)/y \qquad \text{(III 2.9)}$$

$$\mathcal{M}(y)=yH(y)+\int_0^1 \frac{H(y')-H(y)}{y'-y}\,\mathcal{M}(y')dy'F(y',y) \qquad \text{(III 2.10a)}$$

$$\mathcal{M}(y) = y + \frac{\mathcal{M}(y)}{\pi} X \ln\left|\frac{X+1}{X-1}\right| - iX\mathcal{M}(y) - \int_0^1 \frac{\mathcal{M}(y')-y'/y\,\mathcal{M}(y)}{y'-y} F(y',y)dy' \qquad \text{(III 2.10b)}$$

where $X=\sqrt{\frac{\nu}{\nu+m^2}}$ and (III 2.10c)

$$F(y',y)=\frac{4}{\pi} y(1-y^2)^2 \frac{y'^2}{\sqrt{(1-y'^2)^2+m^2y'^4}\,(y'+y)(2y^2y'^2-y^2-y'^2)}$$

and we also have $h(y) = \mathcal{M}(y)\,\mathcal{N}(y)^{-1}$ (III 2.11)

and the square of the CM momentum is related to y by the expression in equ. (III 2.9).

The solution of the system of equations is performed in the same manner as in section 1) of this chapter. Since $\mathcal{M}$, $\mathcal{N}$ and h are two by two matrices we have indicated below, more explicitly, the actual matrices involved in the numerical solution.

$$\begin{Bmatrix} \mathcal{M}_{11}(1) & \mathcal{M}_{12}(1) \\ \vdots & \vdots \\ \mathcal{M}_{11}(n) & \mathcal{M}_{12}(n) \\ \mathcal{M}_{21}(1) & \mathcal{M}_{22}(1) \\ \vdots & \vdots \\ \mathcal{M}_{21}(n) & \mathcal{M}_{22}(n) \end{Bmatrix} = \begin{Bmatrix} y_1H_{11}(1) & y_1H_{12}(1) \\ \vdots & \vdots \\ y_nH_{11}(n) & y_nH_{12}(n) \\ y_1H_{21}(1) & y_1H_{22}(1) \\ \vdots & \vdots \\ y_nH_{21}(n) & y_nH_{22}(n) \end{Bmatrix} +$$

$$\begin{Bmatrix} K_{11}(1,1) & \cdots & K_{11}(1,n) & K_{12}(1,1) & \cdots & K_{12}(1,n) \\ \vdots & & \vdots & \vdots & & \vdots \\ K_{11}(n,1) & \cdots & K_{11}(n,n) & K_{12}(n,1) & \cdots & K_{12}(n,n) \\ K_{21}(1,1) & \cdots & K_{21}(1,n) & K_{22}(1,1) & \cdots & K_{22}(1,n) \\ \vdots & & \vdots & \vdots & & \vdots \\ K_{21}(n,1) & \cdots & K_{21}(n,n) & K_{22}(n,1) & \cdots & K_{22}(n,n) \end{Bmatrix} \times \begin{Bmatrix} \mathcal{M}_{11}(1) & \mathcal{M}_{12}(1) \\ \vdots & \vdots \\ \mathcal{M}_{11}(n) & \mathcal{M}_{12}(n) \\ \mathcal{M}_{21}(1) & \mathcal{M}_{22}(1) \\ \vdots & \vdots \\ \mathcal{M}_{21}(n) & \mathcal{M}_{22}(n) \end{Bmatrix}$$

(III 2.12)

Each element of the K matrix is composed of the factors $F(y_i,y_j)$, the quadrature weighting factors w_j, and the exchange terms

$$\frac{H(y_i)-H(y_j)}{y_i - y_j}$$

For the diagonal terms i=j so we must employ the derivatives of the exchange terms. These are calculated by a differentiation under the integral sign of equ. (III 2.2) In practice n=30, so we had to deal with a 60 x 60 K.

A direct integration of equ. (III 2.10b) then yields $\mathcal{D}$ for any value of ν in the physical region, since this only involves a knowledge of η at the grid points (known from the inversion of $(\delta - K)$ — δ being the Kronecker delta.) The solutions h_{11}^J, h_{12}^J, and h_{22}^J are then multiplied by the appropriate Clebsch-Gordan coefficients to give states of orbital angular momentum ℓ. The resulting amplitudes satisfy the unitarity condition stated in equ.(II 2.23) and can be expressed in terms of phase-shifts and coupling parameters as follows

$$
\begin{aligned}
h_{J-1,J} &= \frac{E}{2ip}\left\{\cos 2\epsilon_J \exp(2i\delta_{J-1,J})-1\right\} \\
h_{J+1,J} &= \frac{E}{2ip}\left\{\cos 2\epsilon_J \exp(2i\delta_{J+1,J})-1\right\} \qquad \text{(III 2.13)} \\
h^J &= \frac{E}{2p}\sin 2\epsilon_J \exp\left[i(\delta_{J-1,J}+\delta_{J+1,J})\right]
\end{aligned}
$$

(the first suffix referring to the orbital angular momentum). In the case of the J=1 triplet states, we also evaluate the determinant of the 2 x 2 matrix $\mathcal{D}$ at 10 points between ν=0 and -0.2. As explained by GGMW, section 7, we expect it to vanish for a value of ν such that

$$s - m_D^2 = 0 \qquad m_D \text{ is the deuteron mass.}$$

i.e. $$4(\nu + m^2) = m_D^2 \quad \text{or} \quad \nu = m_D^2/4 - m^2$$

Equ. (III 2.10b) is used for this calculation, and in chapter V we have plotted this determinant against ν.

CHAPTER IV

DETAILS OF NUMERICAL AND COMPUTATIONAL TECHNIQUES

Although some of the early development work was performed on the London University ATLAS computer, the main body of the work was carried out on the Imperial College IBM 7090/1401 system.

In this chapter we describe the function of each individual subroutine, how the various subroutines are connected, and we also give flow diagrams to indicate how the whole computation was executed.

If during the execution of routine A, transfer is made to routine B, then we say 'A calls B' ,and in general B will then 'return' to A. We have indicated our subroutine names by capital (upper case) letters.

The following subroutines are the same for both methods 1) and 2) of chapter III .

INVERT —— The standard matrix inversion subroutine obtained from the IBM Fortran SHARE library.

POLY —— Generates the first seven Legendre Polynomials.

MESONS —— Given any pair of values of ν and z (square of CM 3-momentum and cosine of scattering angle for channel I), the Mandelstam variables and related parameters for channels I and II are evaluated. (We recall that the contributions from channels II and III are identical). The actual variables calculated are s, t, $\bar{t}$, $\bar{p}^2$, $\bar{E}^2$, q^2, λ^2, T_1, T_2, T_3. The appropriate formulae of chapter II section 4 are used, and these depend on the sign of λ^2. As we have indicated previously only the real parts of T_2 and T_3 are required.

ABSORB —— This calculates the absorptive parts of the $\bar{f}$ amplitudes by employing equations (II 4.14 and II 5.13). The crossing in I-spin space and the evaluation of Im $\bar{f}_3$ and Im $\bar{f}_4$ from Im$(\bar{f}_3 \pm \bar{f}_4)$ are also carried out automatically. The subroutine

CROSS —— employs the values of the parameters calculated in MESONS to evaluate the crossing matrix X , equ. (II 3.8). ABSORB then employs this matrix in order to calculate the variables in equ. (II 5.7).

SOURCE This subroutine evaluates the contribution to Im h from the one pion exchange terms directly since it only involves the calculation of the appropriate Legendre polynomials. The two pion contribution is then found by performing the integration over the z variable [equ. (II 5.7)] of the terms calculated by the previous three subroutines. The limits of integration depend on the value of ν, and the range must be divided into three regions if $p^2+m^2+2\mu^2<0$ [see analysis following equ. (II 4.18c)]. The routines ABSORB, CROSS, and MESONS share 'common' storage, so that the variables calculated by MESONS are made available to the other routines.

Execution of Method 1)

In Fig. (IV 1) we have given a flow diagram of the subroutines. A straight line indicates a direct 'call' from one routine to another, a broken line indicates that a routine lower down the chart uses information already evaluated and stored by a routine above. A wavy line indicates that a routine lower down the diagram refers back to a routine above. We have indicated the various stages of the calculation within the main routine by the numbers 1 . . 6.

[1] Selects the net points (y_i) of equation (III 1.20) according to the Gaussian integration method. The appropriate weighting factors etc. are made available to all subroutines requiring integrations. The subroutine KNLFAC is then called and this evaluates Γ and Γ' at the net points. The factors in equ. (III 1.20) apart from the source term $R(y_j)$ are stored in a matrix 'PARTKL' since this will be the same for all partial waves.

[2] Selects the partial wave required — read from input data. The program was written so that either the one pion or both one and two pion exchange terms could be considered separately. The choice is also made at this stage.

[3] The source terms $R(y_i)$ are evaluated by calling ABSORB, CROSS, and MESONS. The integral equation is now of the form

$$[\mathcal{A}]=[1+h(0)\Gamma]y^2-[\text{Kernel}][\mathcal{A}]$$

in an obvious matrix notation. The matrix {1+Kernel} is then inverted* in section [4], matrix multiplication giving the required values of $\mathcal{A}$.
[5] chooses the laboratory energy for which the phase-shifts are required. $\mathcal{E}_{lab.}=2\nu/m$, and the value of $\mathcal{A}$ at the corresponding value of y is found using

* The numerical accuracy of the inversion routine is checked by verifying that $MM^{-1}=1$ (error $<10^{-6}$)

BAKSUB The solutions $\mathcal{D}(y)$ at the n Gaussian net points, having been evaluated by section [4] of the main routine, are placed in a common storage block, so that they are available to this subroutine also. Direct integrations are performed corresponding to equations (III 1.21, 1.23) to yield D(y) and N(y) at the required value of ν. Since D is complex in the physical region, full use is made of the machine software which facilitates complex arithmetic. The ratio N/D is then calculated and returned to the main routine where the phase shifts are calculated using equ. (III 1.24). Estimation of the numerical accuracy of the N/D method is then performed by section [6] of the main routine. Thus, performing only one subtraction in each of the integrals, we write

$$h(\nu)=h(0) + \frac{\nu}{\pi}\int_{-\infty}^{-1/4} \frac{\text{Im } h(\nu')d\nu'}{\nu'(\nu'-\nu)} + \frac{\nu}{\pi}\int_{0}^{\infty} \frac{\text{Im } h(\nu')d\nu'}{\nu'(\nu'-\nu)} \quad \text{(IV 2)}$$

Having selected the required value of ν (in practice, corresponding to a laboratory energy of n/4 pion masses — n=1, . . ,12) and evaluating $h(\nu)$ by BAKSUB the integrals in the above equation are evaluated, to give a further check of the accuracy of the method.

INTEGL This performs the left hand integration for $\nu=-\infty$ to $-1/4$. $\nu>0$ so no difficulties arise from a vanishing denominator. $\alpha(\nu')=\mathrm{Im}\ h(\nu')/\pi$ can be found for any value of ν' on the left-hand cut by calling the subroutine SOURCE. However, to save a considerable amount of computer time, we employ the same net points as in the integral equations, use the same substitution as before, and employ those values of $\alpha(\nu')$ already evaluated and placed in common storage. The value of the integral is then returned to the main routine.

INTEGR This routine performs the integration on the right hand cut. Some difficulties arise because

1) ν' can equal ν

2) $\nu'=0$ is in the integration range,

so the denominator vanishes for some points in the integration range. The second of these is easily overcome when we recall that

$$\mathrm{Im}\ h(\nu')=\sqrt{\frac{\nu'}{\nu'+m^2}}\ |\ h(\nu')\ |^2$$

in the integrand. The behaviour at the origin is thus $1/\sqrt{\nu'}$, and the integral is easily performed if we make a substitution of the form $\nu'=z^2$ (or higher power). The integral now has the form

$$\frac{\nu}{\pi}\int_0^\infty \frac{|h(\nu')|^2}{\sqrt{\{\nu'(\nu'+m^2)\}}}\,\frac{d\nu'}{\nu'-\nu} \qquad \text{(IV 3)}$$

The difficulty arising from the vanishing of $\nu'-\nu$ is overcome by rewriting equ.(IV 3) as

$$\frac{\nu}{\pi}\int_0^\infty \frac{d\nu'}{\sqrt{\{\nu'(\nu'+m^2)\}}}\,\frac{|h(\nu')|^2 - |h(\nu)|^2}{\nu' - \nu} \qquad \text{(IV 3')}$$

$$+\ \frac{\nu}{\pi}\,|h(\nu)|^2\int_0^\infty \frac{d\nu'}{\sqrt{\{\nu'(\nu'+m^2)\}(\nu'-\nu)}}$$

The final integral contributes an imaginary part equal to Im $h(\nu)$ in virtue of the unitarity condition (as, of course, it should), and a real part

$$\frac{X}{\pi}\ln\left|\frac{1-X}{1+X}\right|\,|h(\nu)|^2 = \frac{\text{Im } h(\nu)}{\pi}\ln\frac{1-X}{1+X} \qquad \text{(IV 4)}$$

and $$X=\sqrt{\frac{\nu}{\nu+m^2}}$$

Slightly different methods were employed for the actual numerical integration, but almost identical results were obtained.

1) The range was split into two regions 0 to 1 and 1 to ∞. In the first region we substitute $\nu'=z^2$, in the second region $\nu'=1/z^2$. (IV 5)

It is always an **advantage** to make both sets of limits reduce to the interval $0 \rightarrow 1$, then both integrations may be carried out simultaneously since the same net points and weighting factors are employed.

2) Alternatively, we can make the substitution

$$\nu'=(1/z^2-1)^2 \qquad \text{(IV 6)}$$

as is done in chapter III section 2. Again the integration range is reduced to the unit interval.

3) The real part of equ. (IV 3) may be written as

$$\frac{\nu}{\pi}\int_0^{\nu/2}\frac{d\nu'}{\sqrt{\nu'(\nu'+m^2)}}\;\frac{|h(\nu')|^2}{\nu'-\nu}+ \frac{\nu}{\pi}\int_{\nu/2}^{\infty}\frac{\mathrm{Im}\,h(\nu')-\mathrm{Im}\,h(\nu)}{\nu'(\nu'-\nu)}\,d\nu' \qquad \text{(IV 7)}$$

and then we need no further compensating term since

$$\mathrm{Im}\,h(\nu)\;.\mathrm{Re}\int_{\nu/2}^{\infty}\frac{d\nu'}{\nu'(\nu'-\nu)}=0 \qquad \text{(IV 8)}$$

Both integrals in equ. (IV 7) are easily evaluated numerically by employing substitutions similar to those outlined above. The integration ranges are again reduced to the unit interval for convenience. Note, however, that in all these methods we must repeatedly call BAKSUB in order to find $h(\nu')$ at the required points. We can ensure that the derivative of $h(\nu)$ is never required — one might expect it to be on examining the second integral of equ .(IV 7)— since we have only to make sure the Gaussian integration points do not coincide with ν.

The final results are returned to the main program and added to the results from INTEGL. We can then compare both sides of equ.(IV 2). Using our 30 point net (5-point Gaussian formula repeated 6 times) resulted in an error of only 3 or 4 % at worst, the actual error depending on the partial wave chosen. When only the one-pion exchange terms are considered both sides of equation (IV 2) agreed to 5 significant figures.

After section [6] has been executed, control is transferred back to section [2], and the next partial wave is selected.

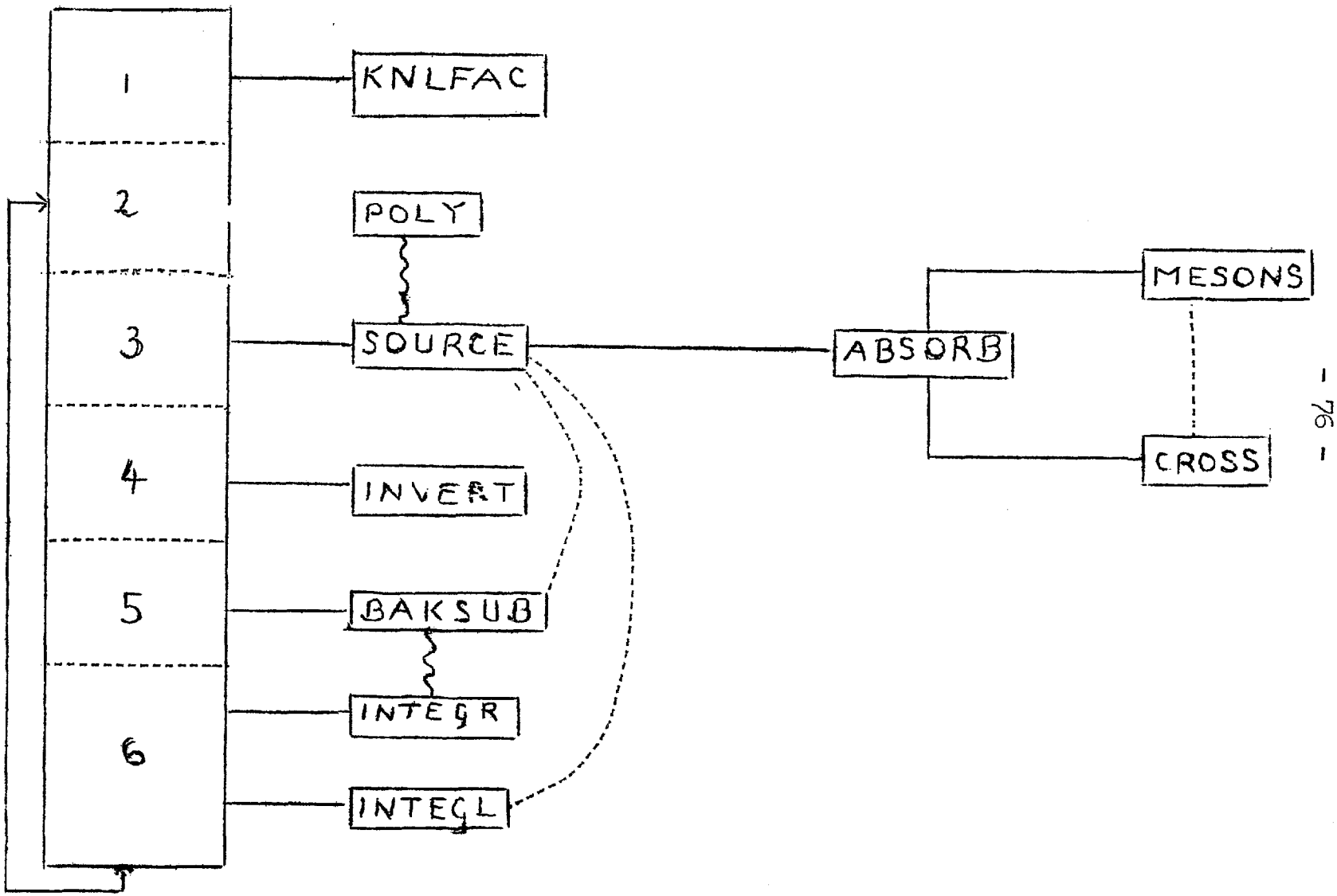

1
2
3
4
5
6
KNLFAC
POLY
SOURCE
INVERT
BAKSUB
INTEGR
INTEGL
ABSORB
MESONS
CROSS

Fig (IV 1)

For method 2) (the coupled triplet amplitudes), we have again drawn a flow diagram consisting of 9 sections. [1] performs similar functions for both methods, the only difference between the two methods at this stage being that instead of KNLFAC, the routine FACKNL is called at this point, and this evaluates the function $F(y',y)$ of equ. (III 2.10c). Sections [2] and [3] are identical for both methods, but in section [4] INTEGL is called. This performs the same functions in both methods,* i.e. evaluates the exchange terms $h^e(\nu)-h^e(0)$ ——the first integral of equ. (IV 2). All these functions are 2 x 2 matrices as we emphasized in the previous chapter. In section [5] the appropriate kernel is inverted and the numerator function is evaluated at the net points. In [6] the energy is selected and the numerator found at the corresponding point by a call to BAKN —— analogous to BAKSUB of method 1). [7] calls the routine DANDH which evaluates the denominator and the $h(\nu)$'s using equ. (III 2.10b). We recall that $h=ND^{-1}$. If J=1 the routine DENLFT is called, and this computes the determinant of the denominator function at 10 points on the left hand cut, in the region where we expect the deuteron to be located.

*In the present method $h^{e\prime}(\nu)$ is also required, and is obtained by a differentiation under the integral sign.

In section [8], the matrix h(ν) returned from DANDH is fed into TRPLET which multiplies the amplitudes by the Clebsch-Gordan coefficients of equs. (II 2.23), and evaluates the phase-shifts and coupling parameter by using equs. (III 2.13). Again full use is made of the software to handle the complex arithmetic. In [9] a checking procedure very similar to the one outlined for method 1) is performed. This again uses the routines INTEGR and INTEGL. It is a further check to note that we consider the amplitudes h_{11}, η_{12} and h_{22} in order to verify that equ. (IV 2) is satisfied. The numerical techniques required for this section are, of course, the same as for the corresponding section of method 1). Again after the execution of [9] control is transferred to [2]. We observe that the above checking procedure cannot be used for the J=1 state, since there is an extra term (residue/$(s-m_D^2)$) that must be added to the left hand side of equ. (IV 2).

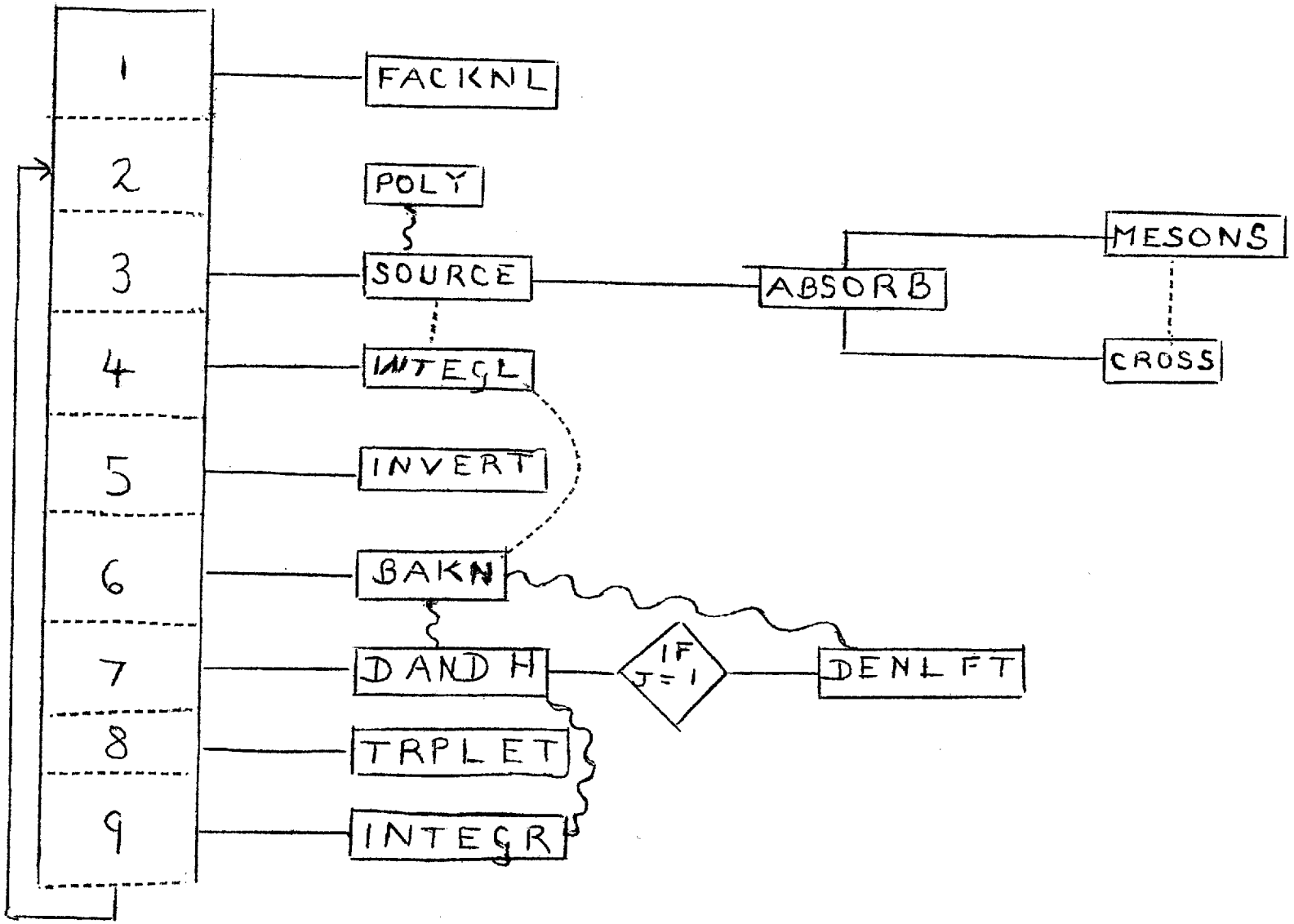

1
2
3
4
5
6
7
8
9
FACKNL
POLY
SOURCE
INTEGL
INVERT
BAKN
DAND H
TRPLET
INTEGR
IF J = 1
DENLFT
ABSORB
MESONS
CROSS

Fig (IV 1)

We now give the various Gaussian quadrature formulae employed in the calculation. 5, 8, and 16 point formulae were used for the net of the integral equations. Very satisfactory results were obtained when 5 points were used to represent the one pion contribution region ($-\frac{1}{4}$ to -1), and 25 points (5 times repeated 5-point formula) to represent the region -1 to $-\infty$. We have

$$\int_a^b f(x)dx = \sum_t w_t f(t)$$, and the numerical values of the w's and t's are given by

$t' = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$

w_t	t
.094725305	.095012510
.091301708	.281603551
.084578260	.458016778
.074797995	.617876244
.062314486	.755404408
.047579256	.865631202
.031126762	.944575023
.013576230	.989400935

16-point

w_t	t
.181341892	.183434642
.156853323	.522532410
.111190517	.796666477
.050614268	.960289856

8-point

w_t	t
.284444444	.000000000
.239314335	.538469310
.118463443	.906179846

5-point

t also takes the same values as above but with the negative sign, the same weights being attached as above.

CHAPTER V

RESULTS AND CONCLUSION

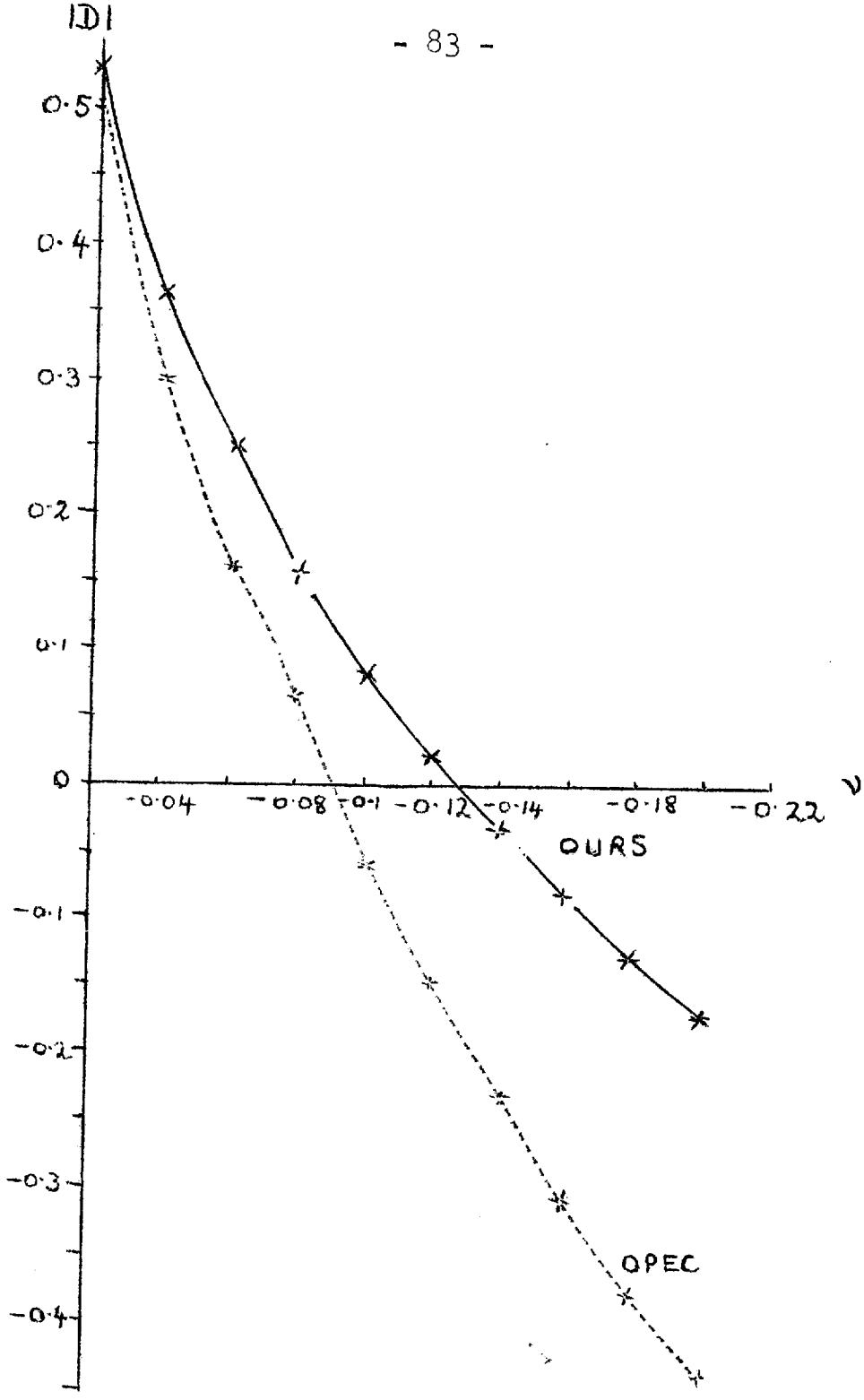
We now display graphically the energy dependence of our phase shifts. As well as our final results calculated from one and two pion exchange we give the corresponding graphs for one pion exchange only (OPEC). When the OPEC's consist of a series of crosses this indicates that the results were obtained by our own calculation only. Otherwise (for the higher angular momenta) they are taken from the paper of Amati, Leader and Vitale ([2]) (ALV), but, of course, we were able to reproduce these results also. We also display the results of those experimental groups given in ALV, and have followed their abbreviations : SMMN ([13]), YLAN 1 ([14]) and YLAN 3M ([15]). For the lower angular momenta the results of Scotti and Wong are given (SW^3) and these agree closely with those of the Yale group (YLAN 3M) or the Livermore group (private communication between STAPP and SW).

In the paper of Galanin et. al ([4]) two sets of results are given: (a) OPEC + Two pion contribution

(b) Supplementary contribution from the ρ meson.

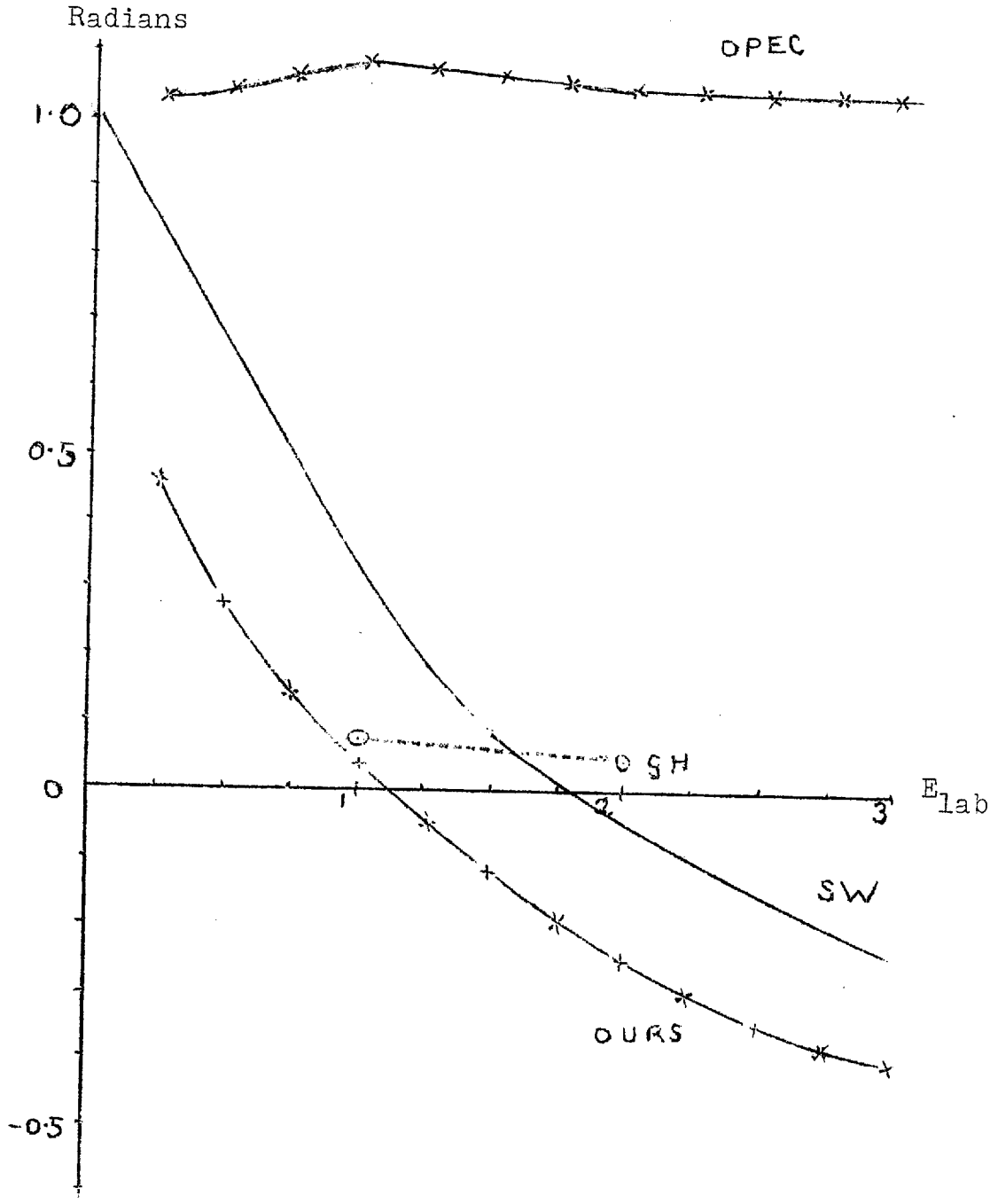
We repeat only the results of (a) since these are analogous to our calculation. For most of the I=0 phase-shifts (even for the higher angular momenta) the parameters have either the wrong sign or are too small in absolute value when compared with the experimental results. For some of these cases we have not attempted to include their (I=0) results on our graphs. The Galanin calculation is hereafter referred to as 'G'. For the uncoupled states some of the results of the one and two pion contributions calculated by the Tamm-Dancoff method are displayed — Gotsman and Hochberg ([6]) (GH). All our parameters are measured in radians, and the laboratory energy E in pion masses.

Before we give these results the deuteron bound-state (see Fig. V 1) is discussed. It is seen that the determinant of the denominator function vanishes at $\nu = -0.09$ and -0.127 for OPEC and OPEC+TPEC respectively. From the equ. $4(\nu+m^2)=m_D^2$ it follows that $\nu \sim m(m_D-2m)$ and so the binding energy (the difference between the mass of two nucleons and that of the deuteron) is just the numerical value of ν/m (pion masses). Multiplying by the pion mass 137 MeV we obtain the respective values of the deuteron's binding energy — 1.75 MeV (OPEC), and 2.45 MeV (OPEC+TPEC) compared with the experimental value of 2.2 MeV.



(Fig. V 1)

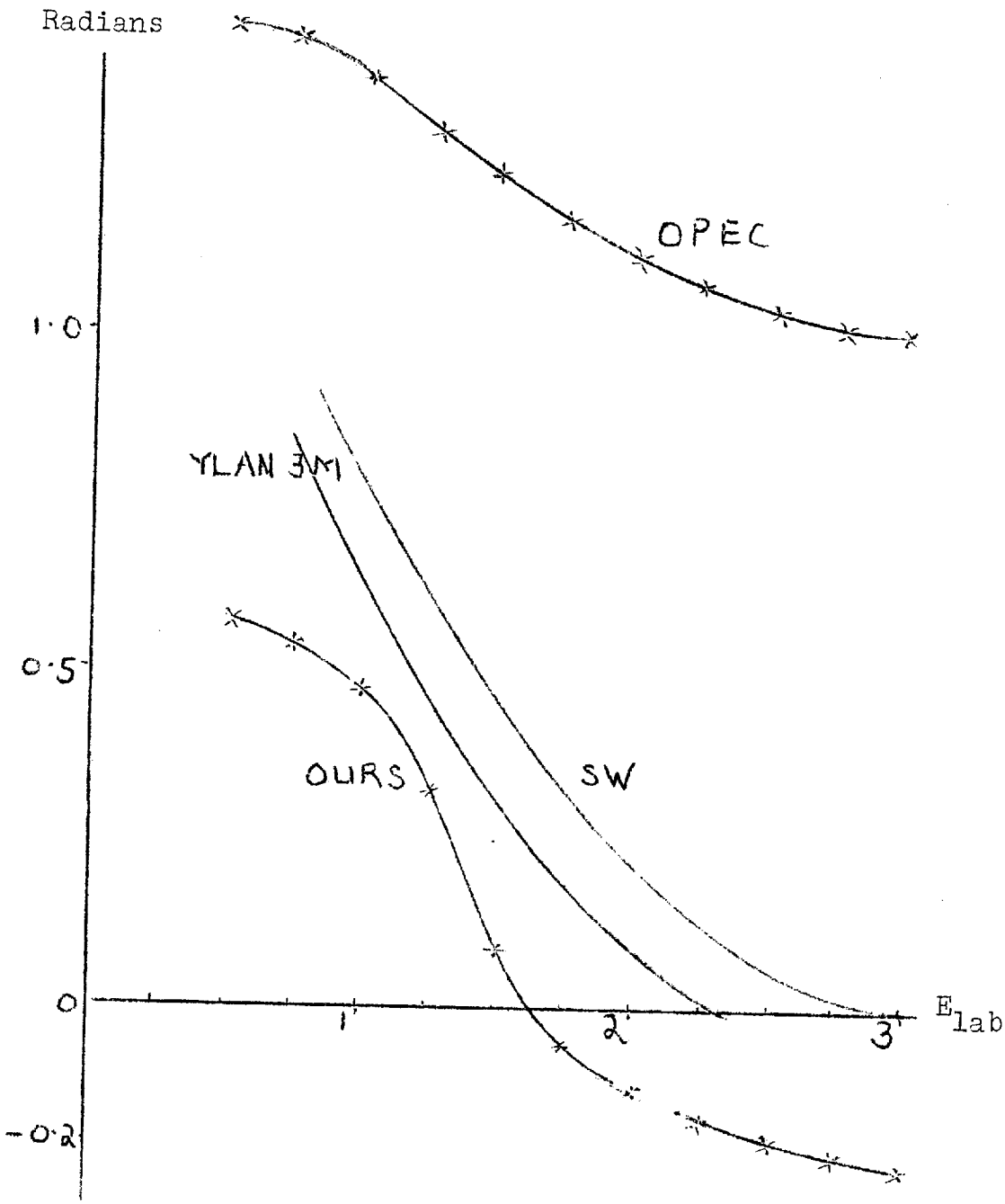
Although the $^1S_0$ and $^3S_1$ phase shifts were not calculated near $E_{lab.}=0$ we expect our graphs to have turning points and pass through the origin.



I=1

^1S_o phase-shift

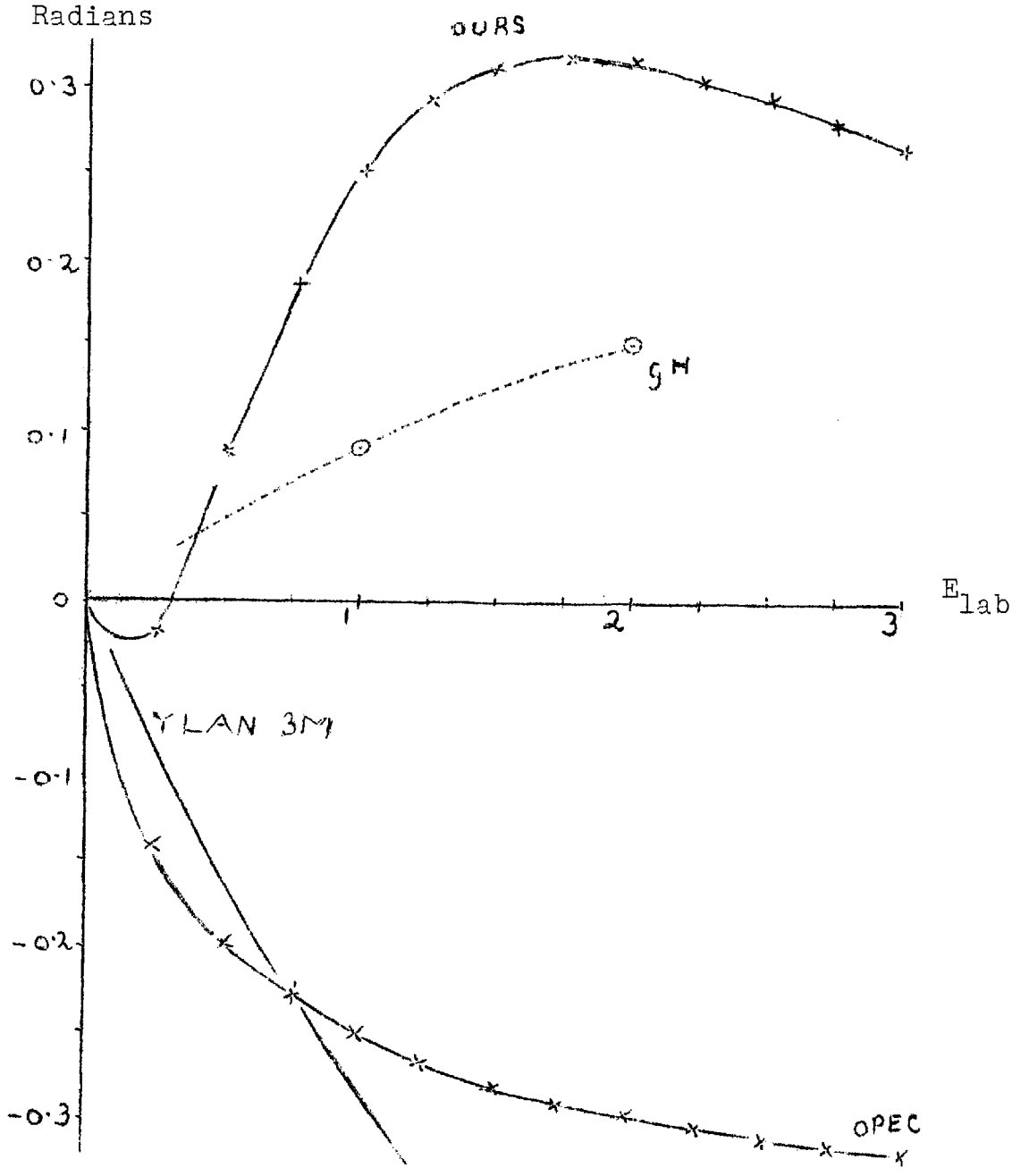
(Fig. V 2)



I=0

$^3S_1$ phase-shift

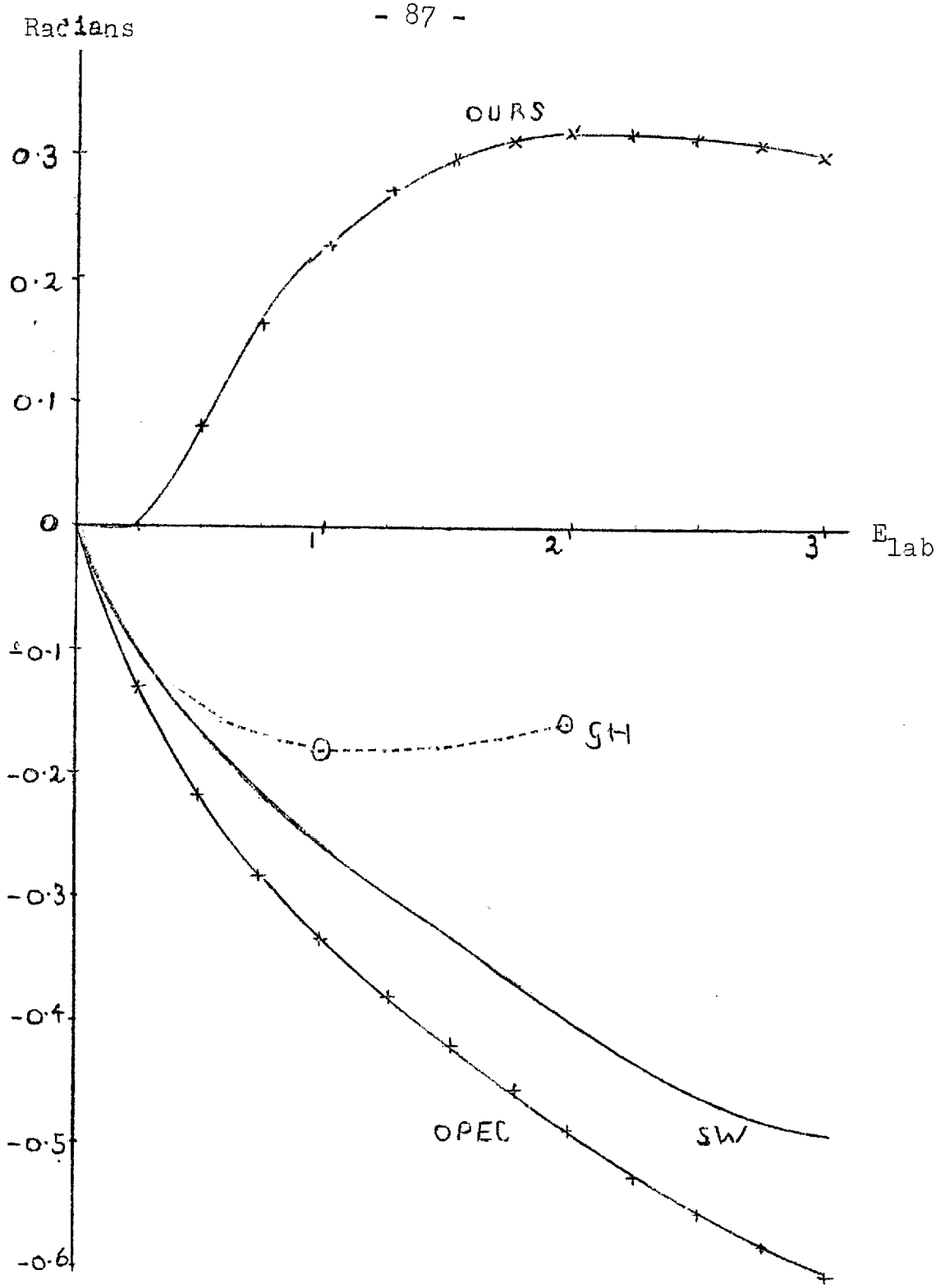
(Fig. V 3)



I=0

$^1P_1$ phase-shift

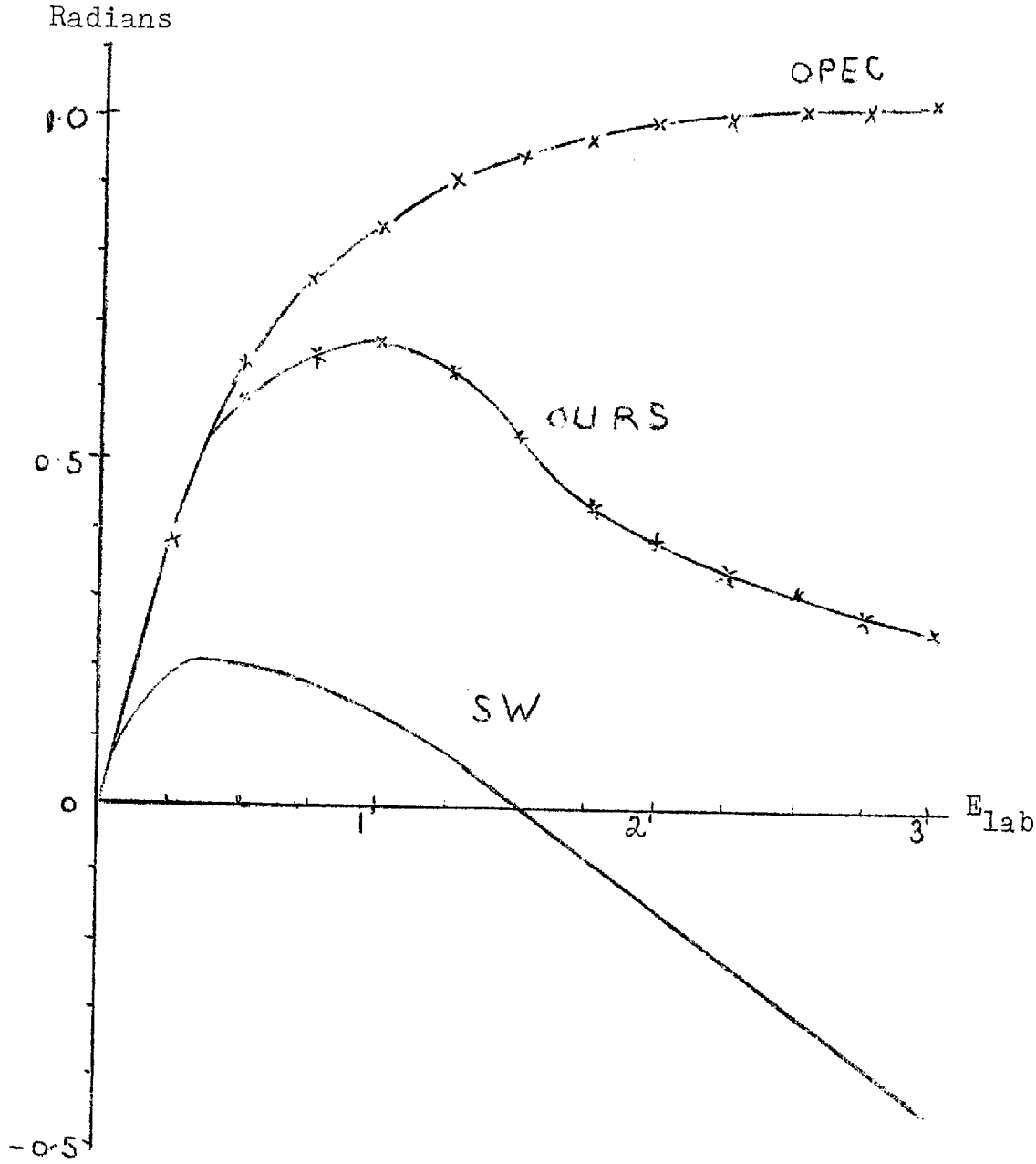
(Fig. V 4)



I=1

$^3P_1$ phase-shift

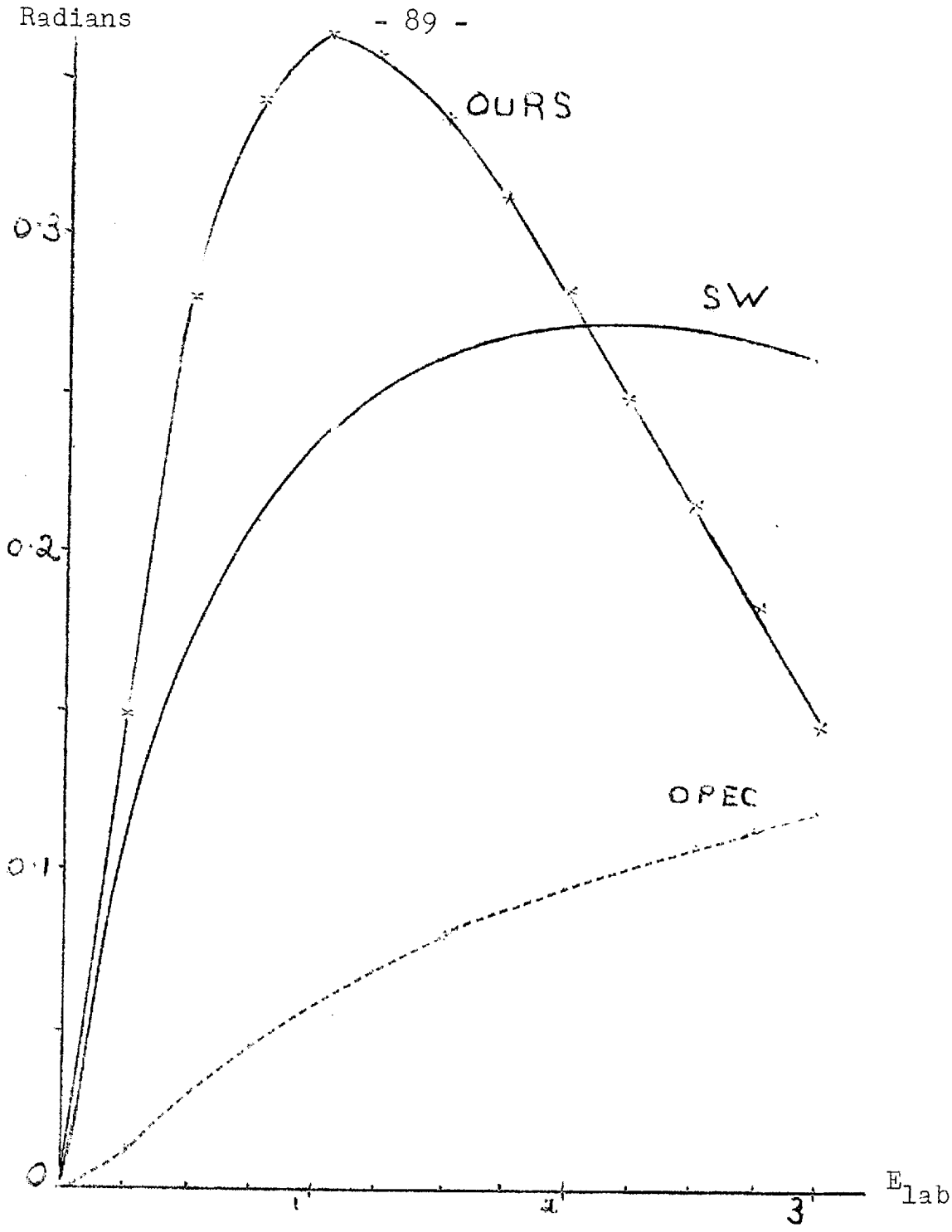
(Fig. V 5)



I=1

^3P_o phase-shift

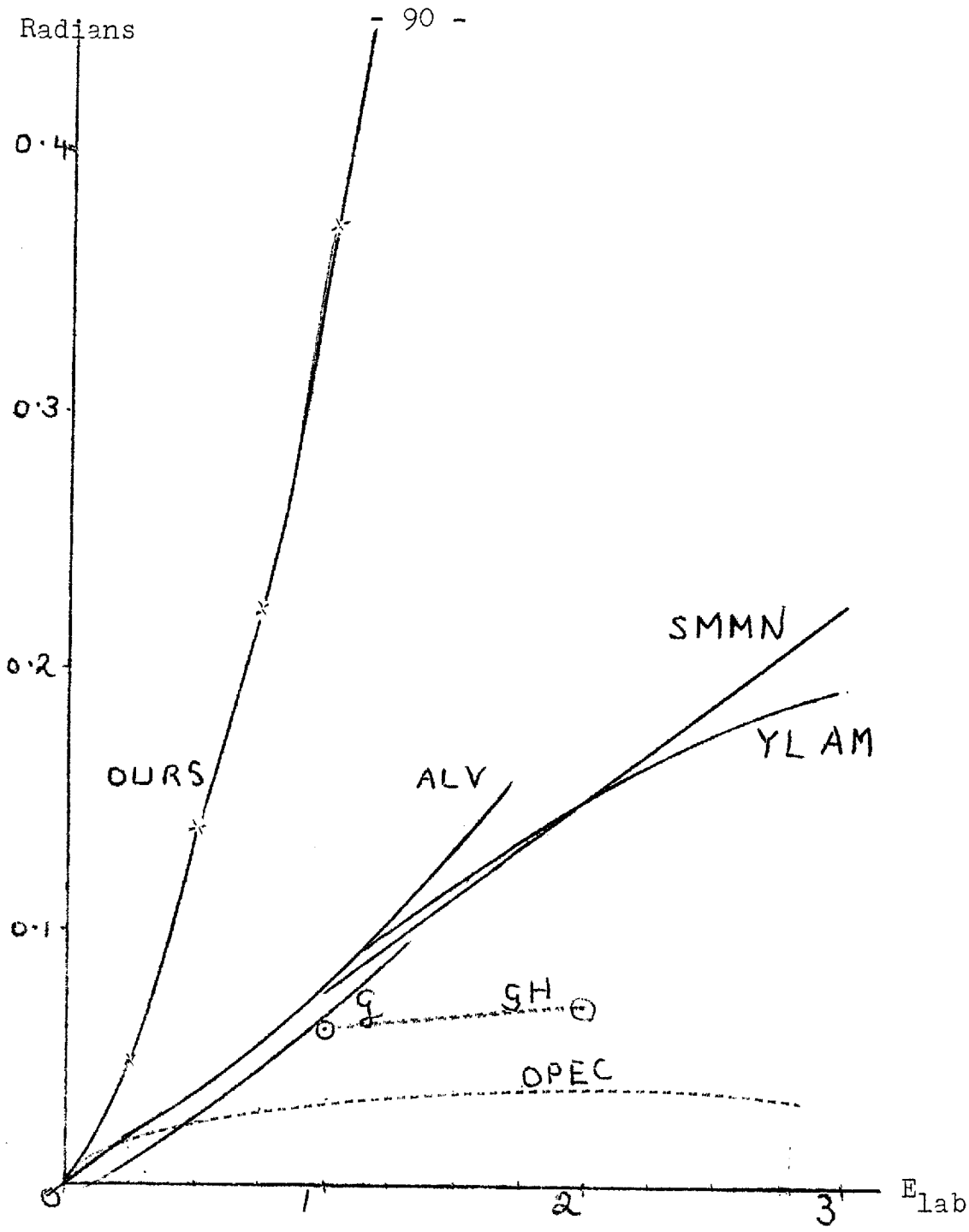
(Fig. V 6)



I=1

$^3P_2$ phase-shift

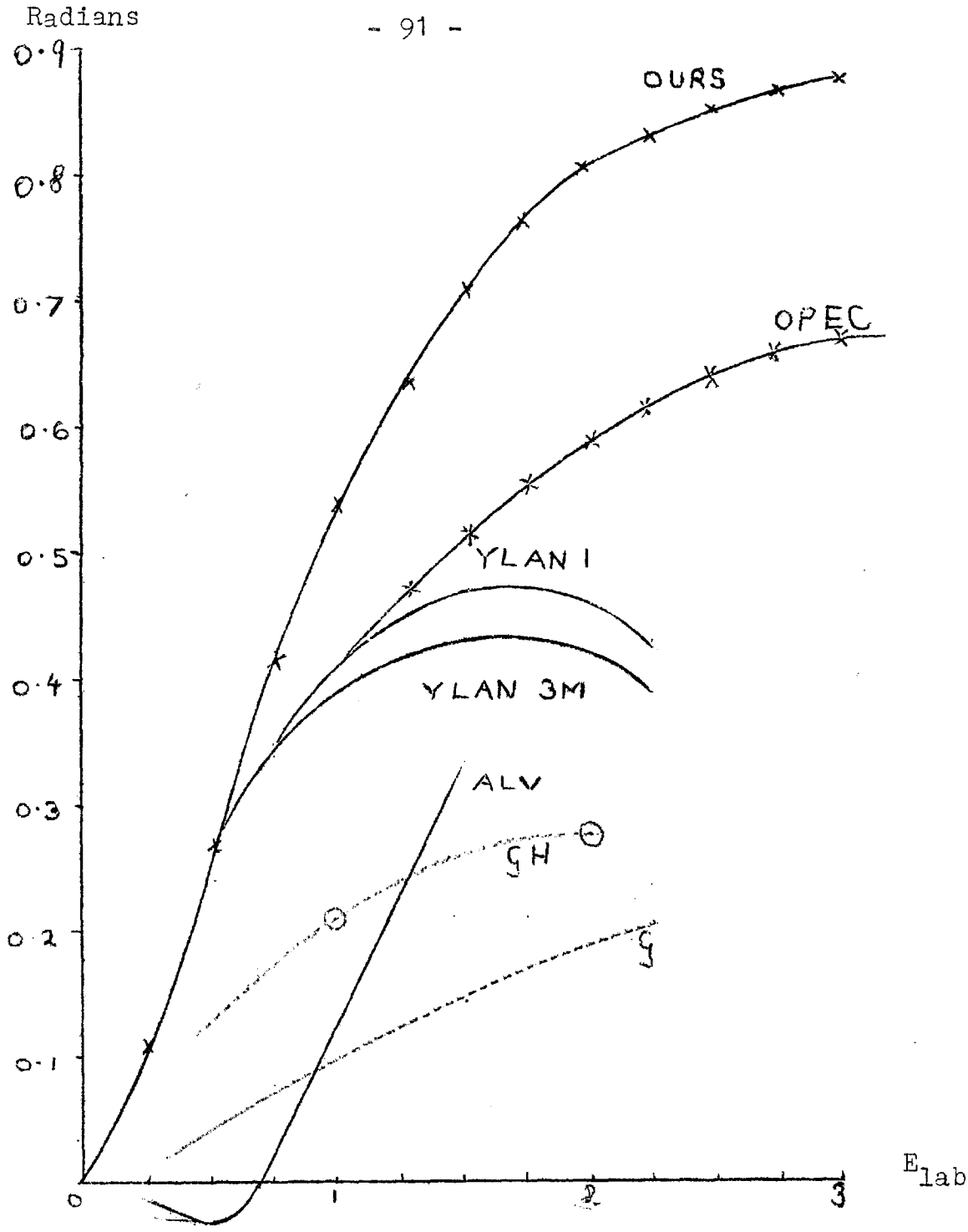
(Fig. V 7)



I=1

$^1D_2$ phase- shift

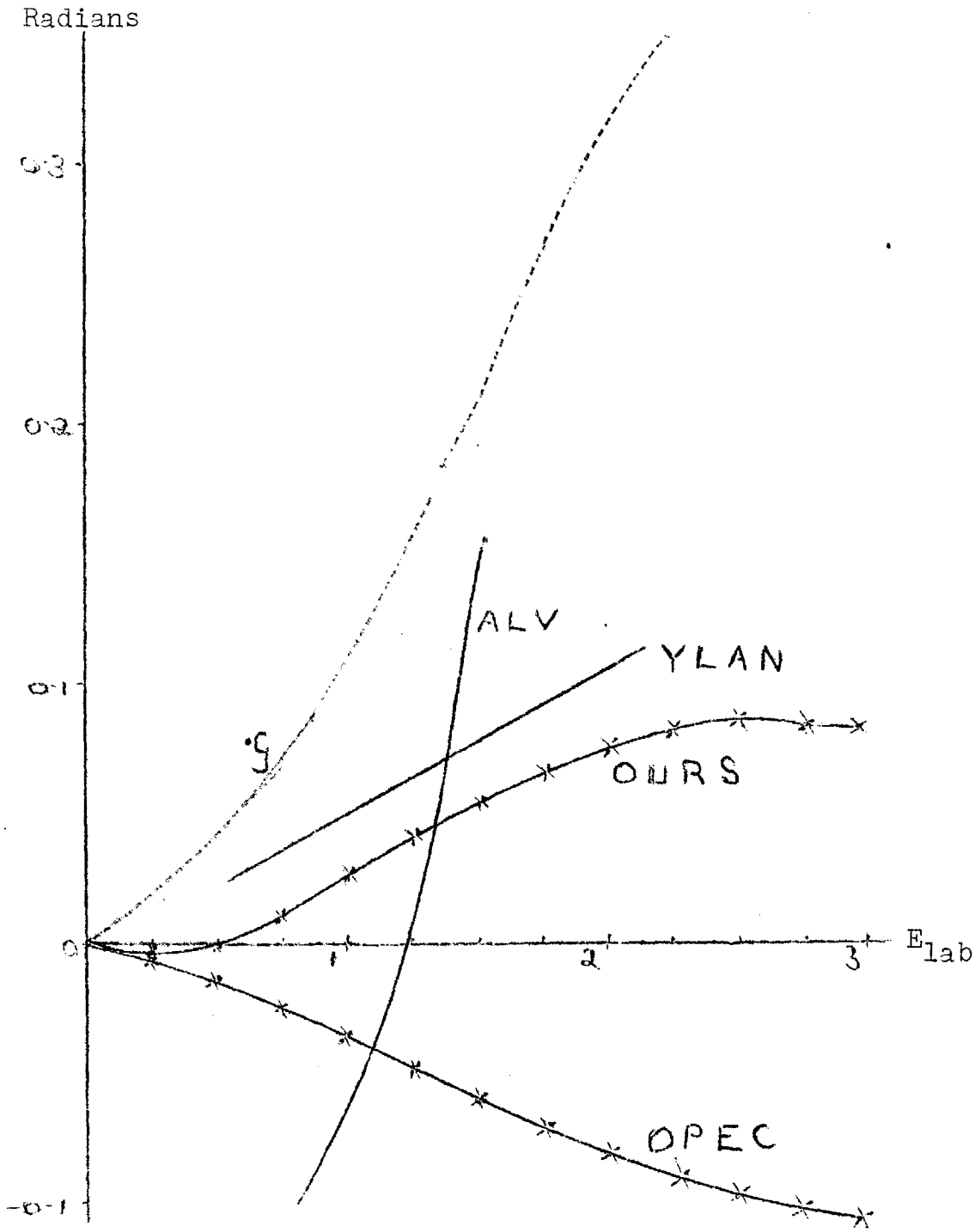
(Fig. V 8)



I=0

$^3D_2$ phase-shift

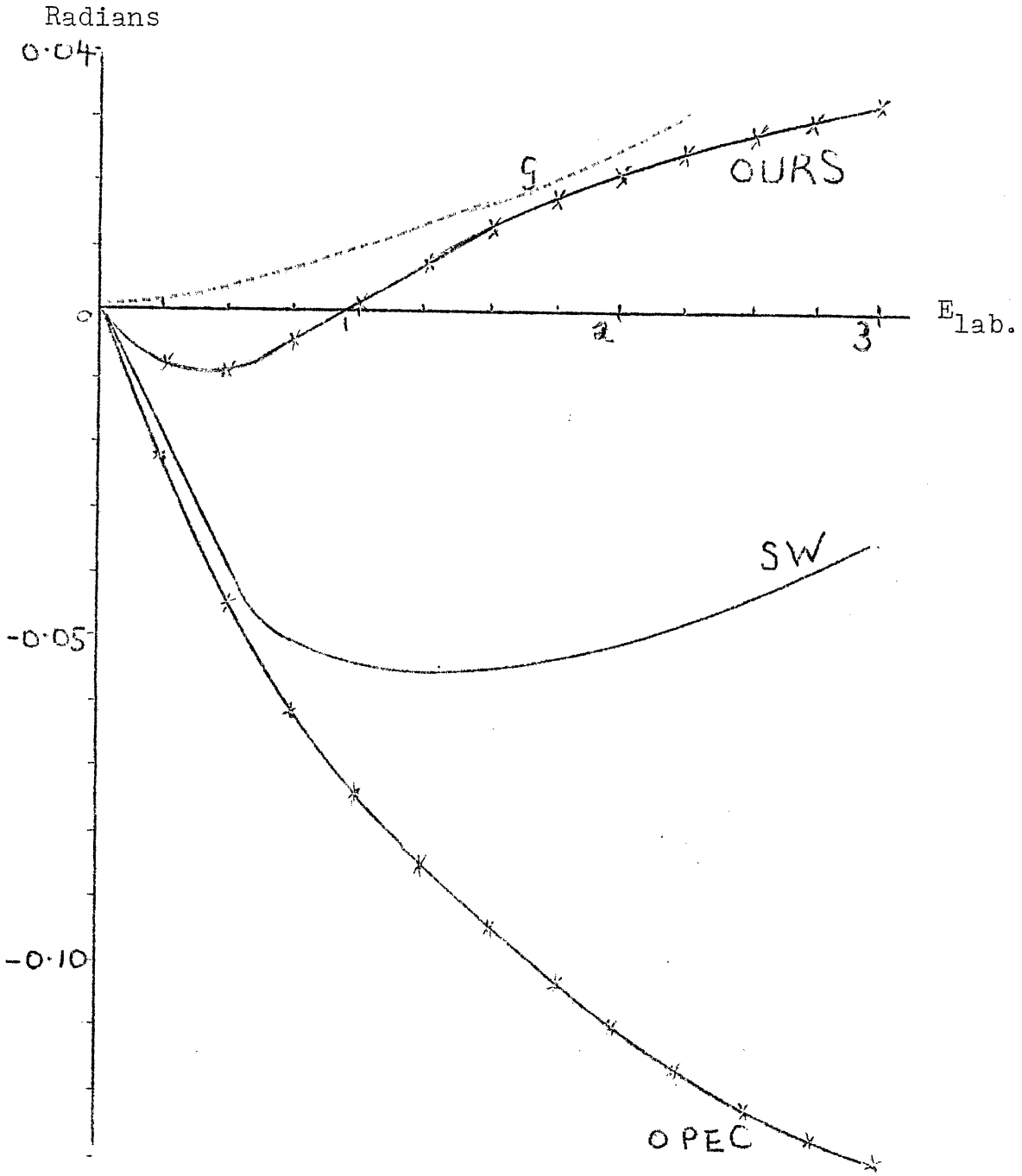
(Fig. V 9)



I=0

$^3D_3$ phase-shift

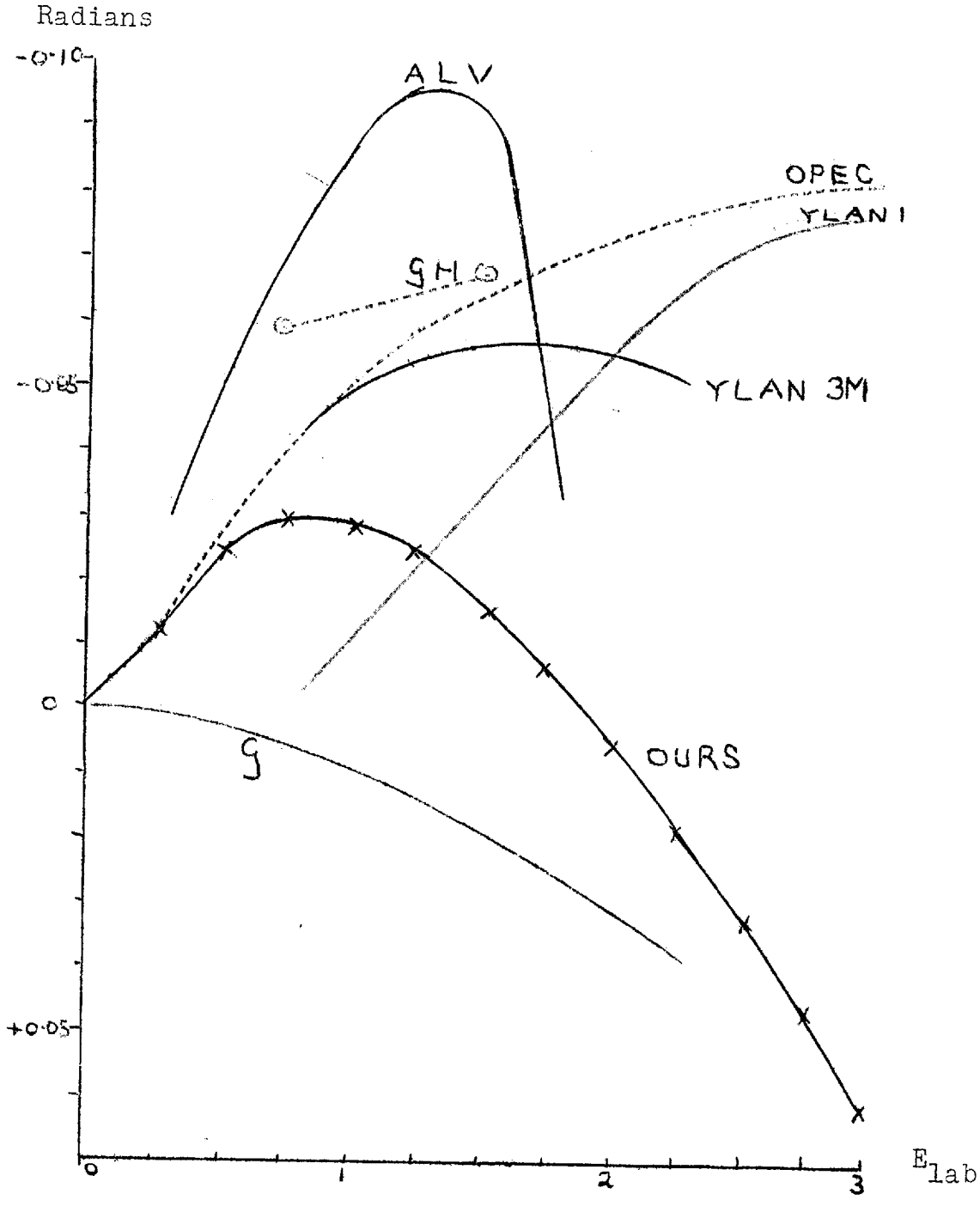
(Fig. V 10)



I=1

ϵ_2 mixing parameter

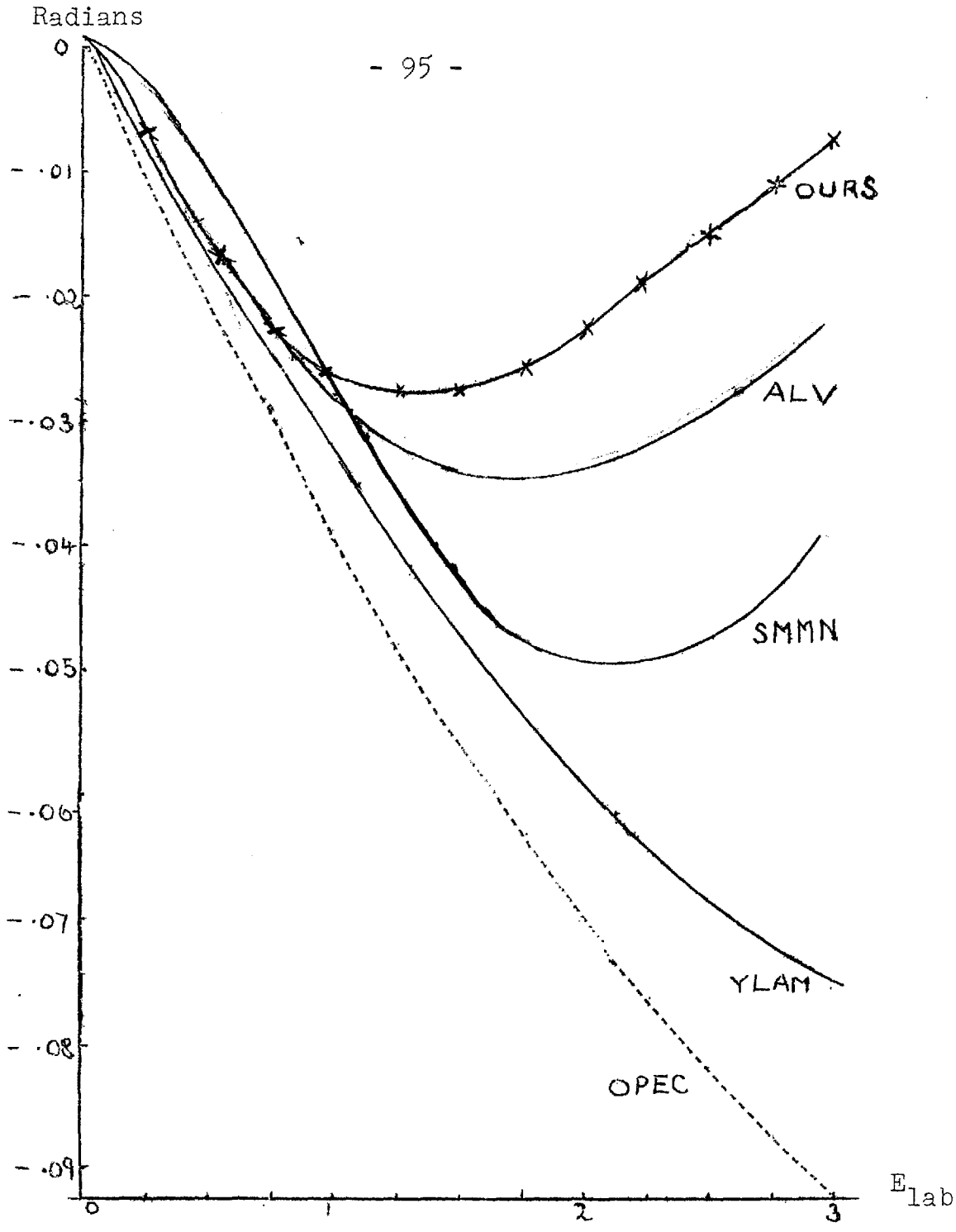
(Fig. V 11)



I=0

$^1F_3$ phase-shift

(Fig. V 12)

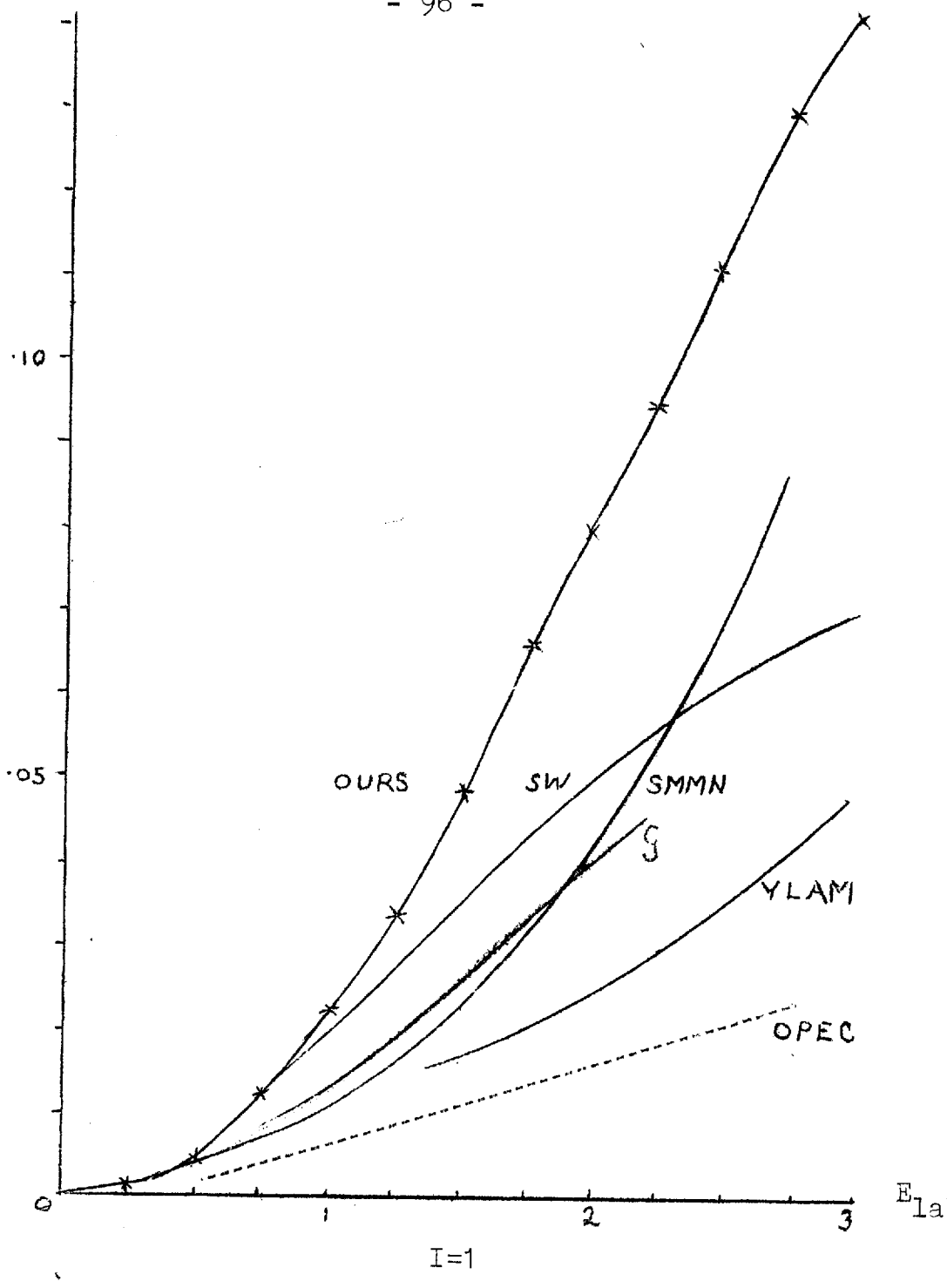


The corresponding Galanin results are .001,.007,.022,.044 radians at 40,100,200,300 MeV respectively (incorrect sign).

I=1

$^3F_3$ phase-shift

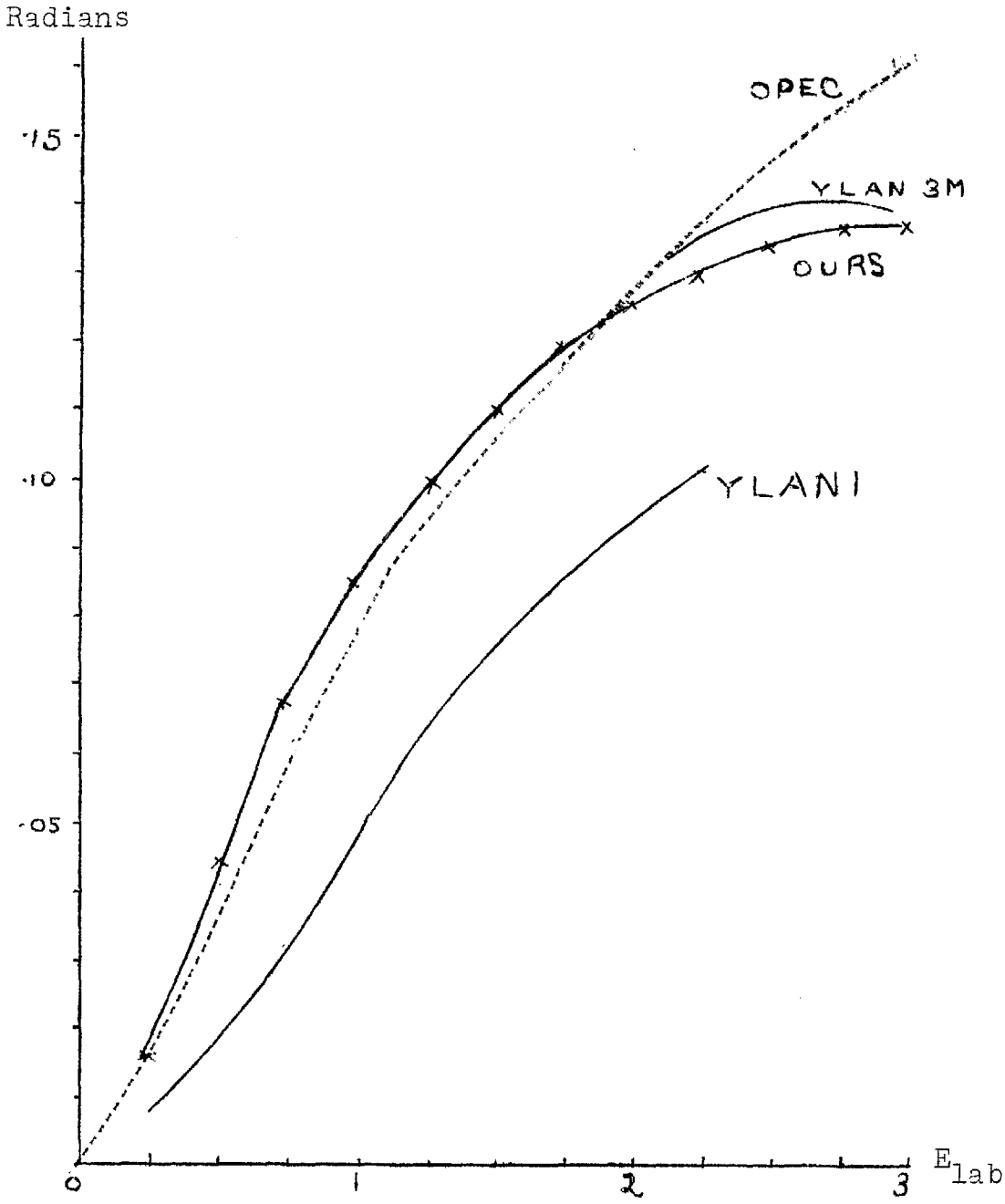
(Fig. V 13)



I=1

$^3F_4$ phase-shift

(Fig. V 14)

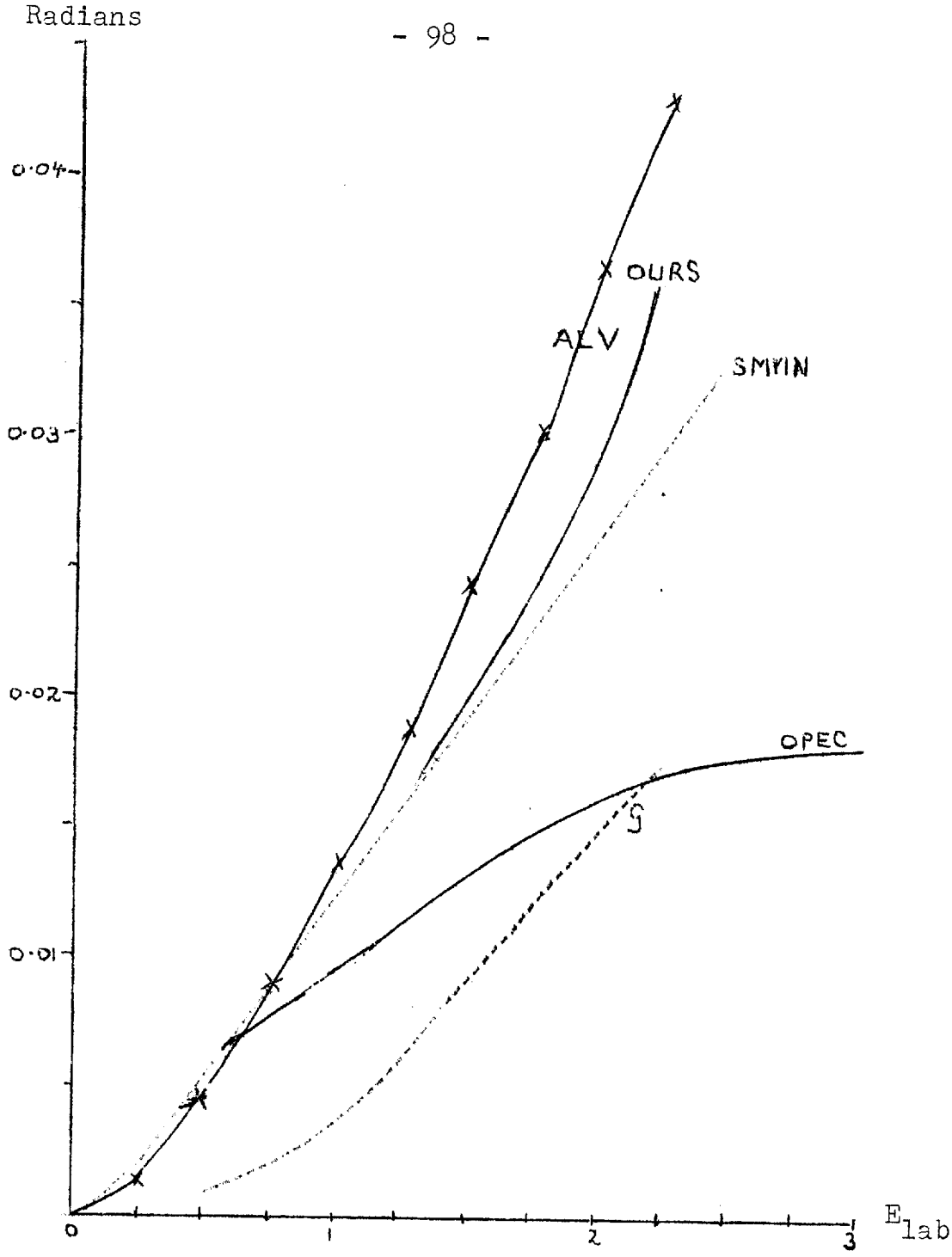


Galanin gives values of -.0005,-.007,-.0087,-.0122 radians at 40,100,200,300 MeV respectively.

I=0

ϵ_3 mixing parameter

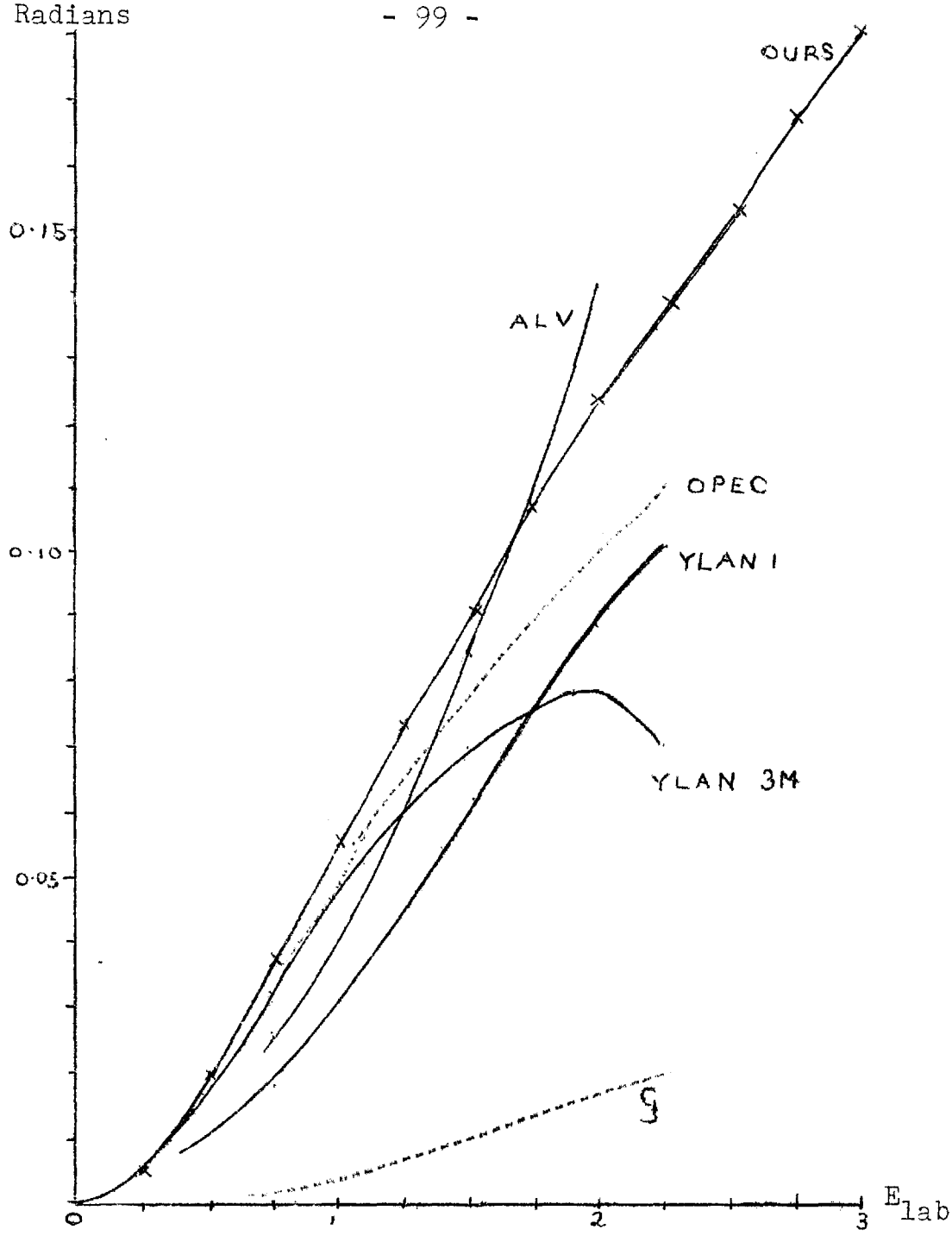
(Fig. V 15)



I=1

$^1G_4$ phase-shift

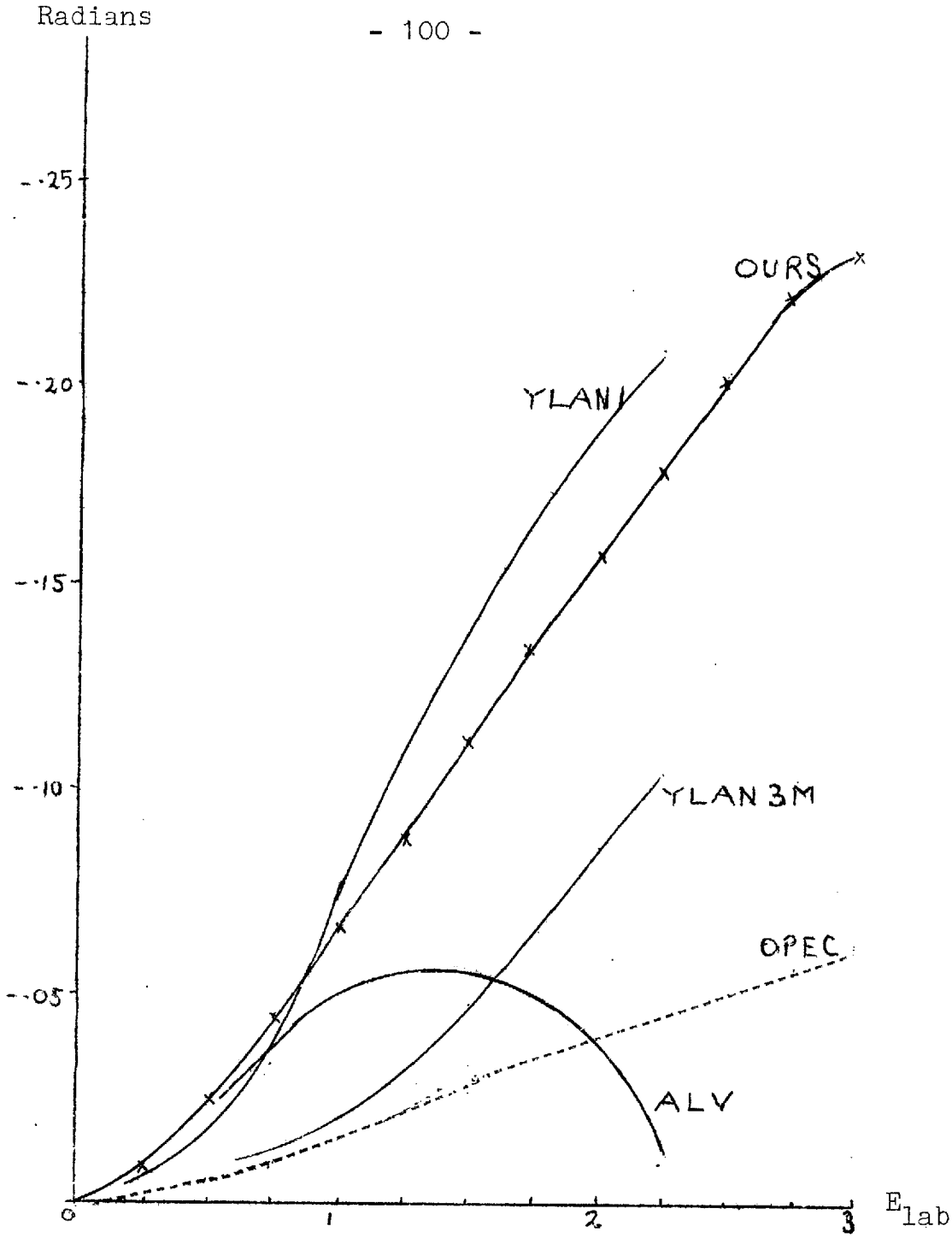
(Fig. V 16)



I=0

$^3G_4$ phase-shift

(Fig. V 17)

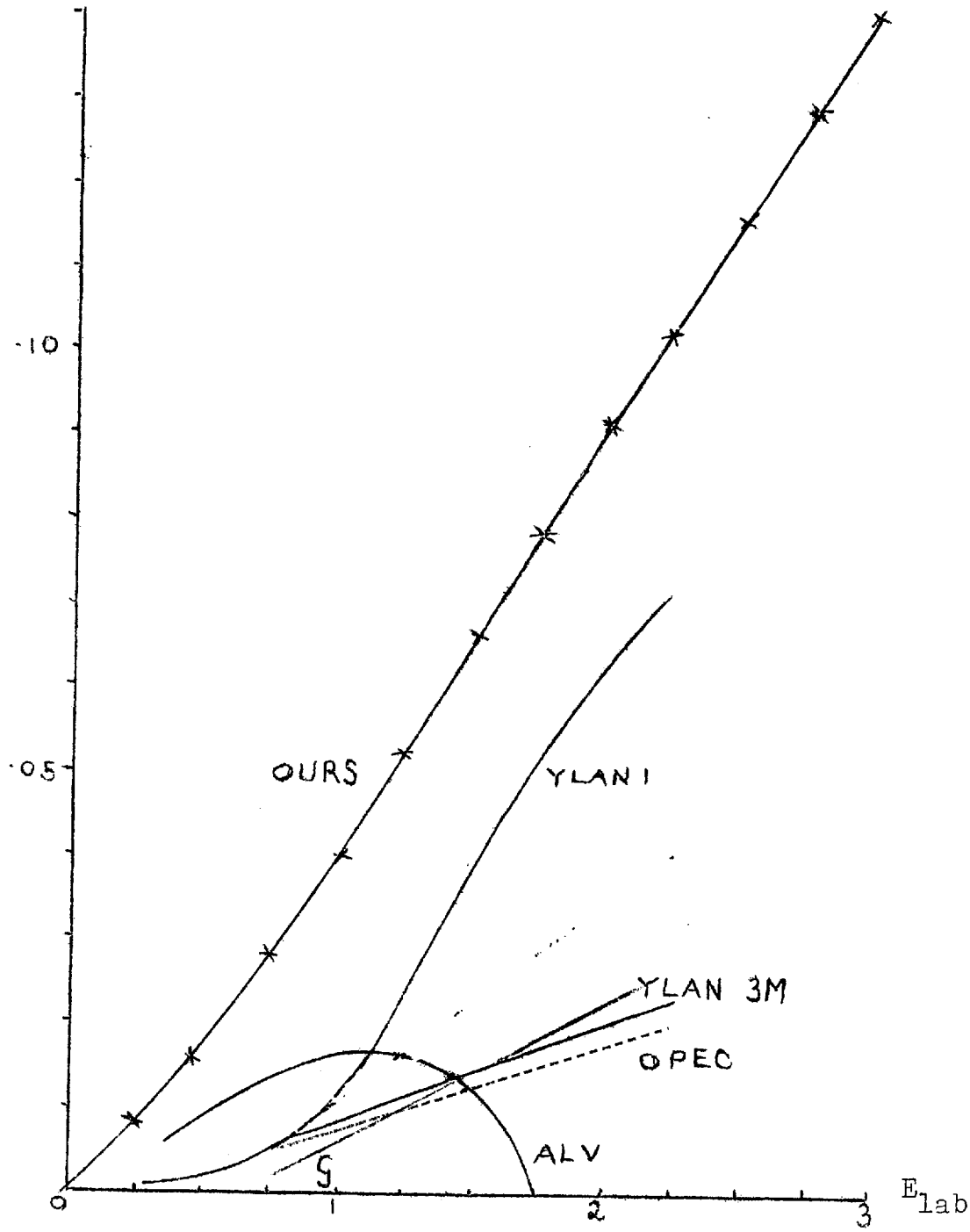


The Galanin results give phase-shifts of 10^{-4}, -7×10^{-4}, $-.0026$, $-.0014$ radians at 40, 100, 200, 300 MeV respectively.

I=0

$^3G_3$ phase-shift

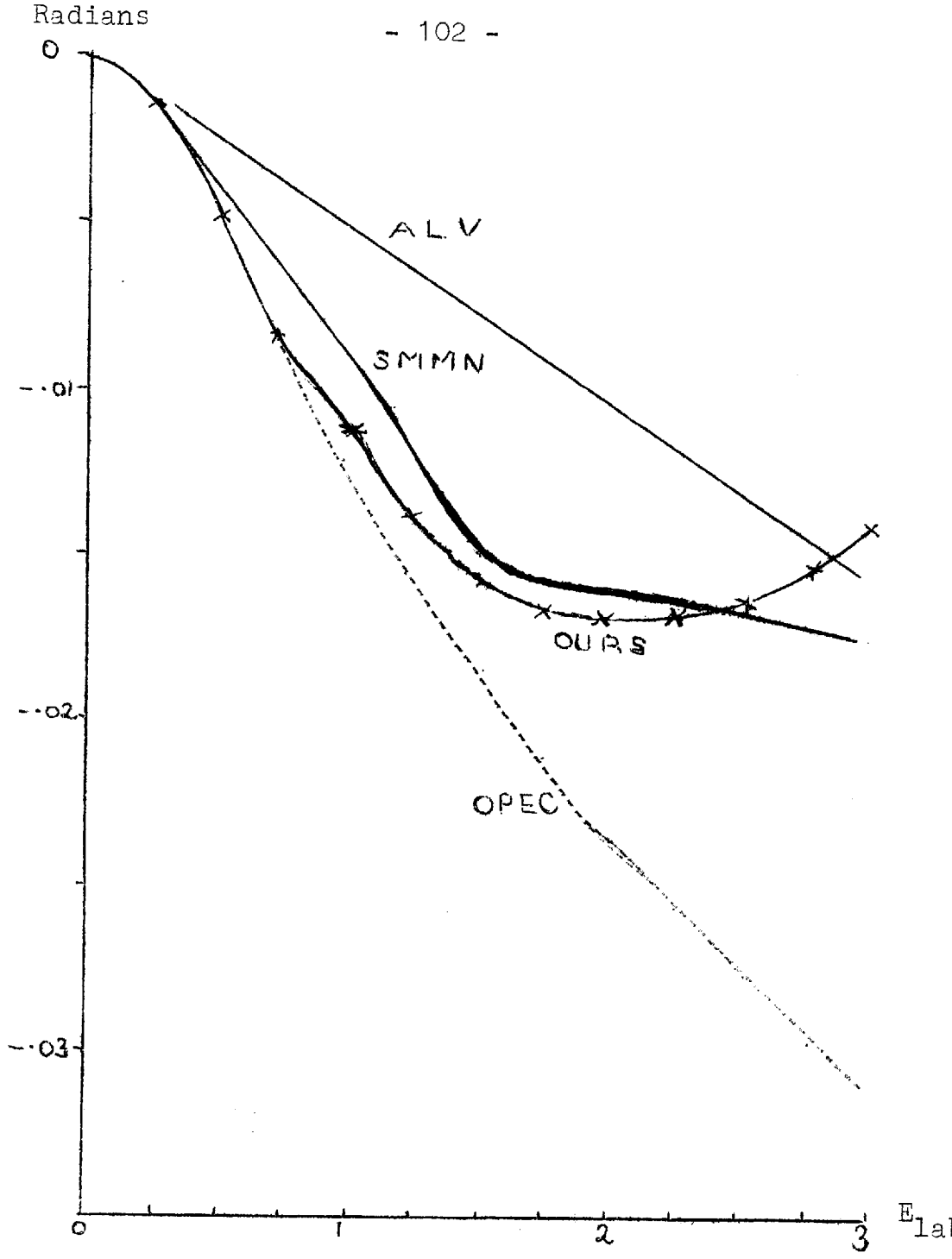
(Fig. V 18)



I=0

$^3G_5$ phase-shift

(Fig. V19)

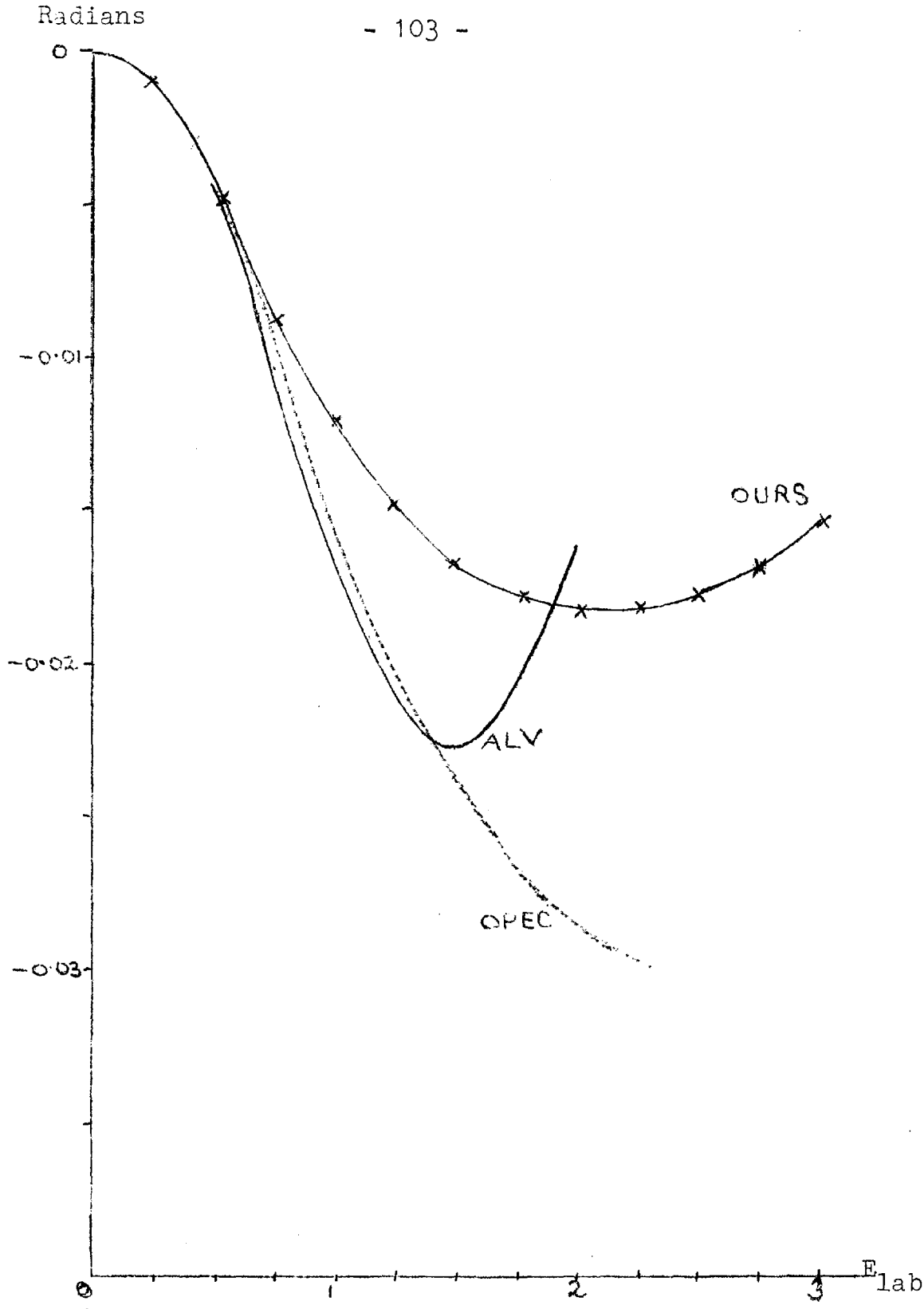


I=1

ϵ_4 mixing parameter

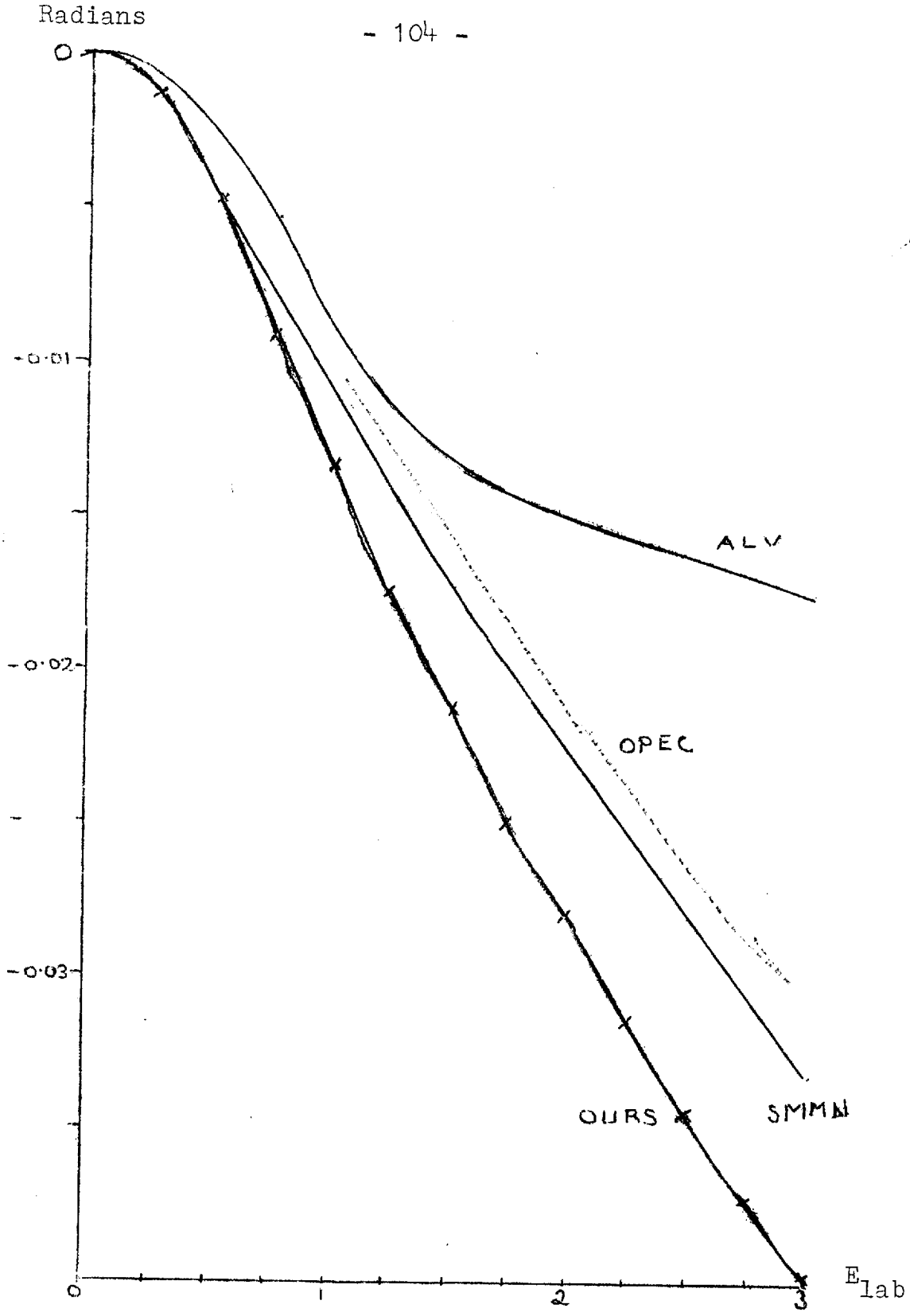
(Fig. V 20)

Galanin gives values of 5×10^{-6}, 1.2×10^{-4}, 9×10^{-4}, 2×10^{-3} radians at 40, 100, 200, 300 MeV respectively.



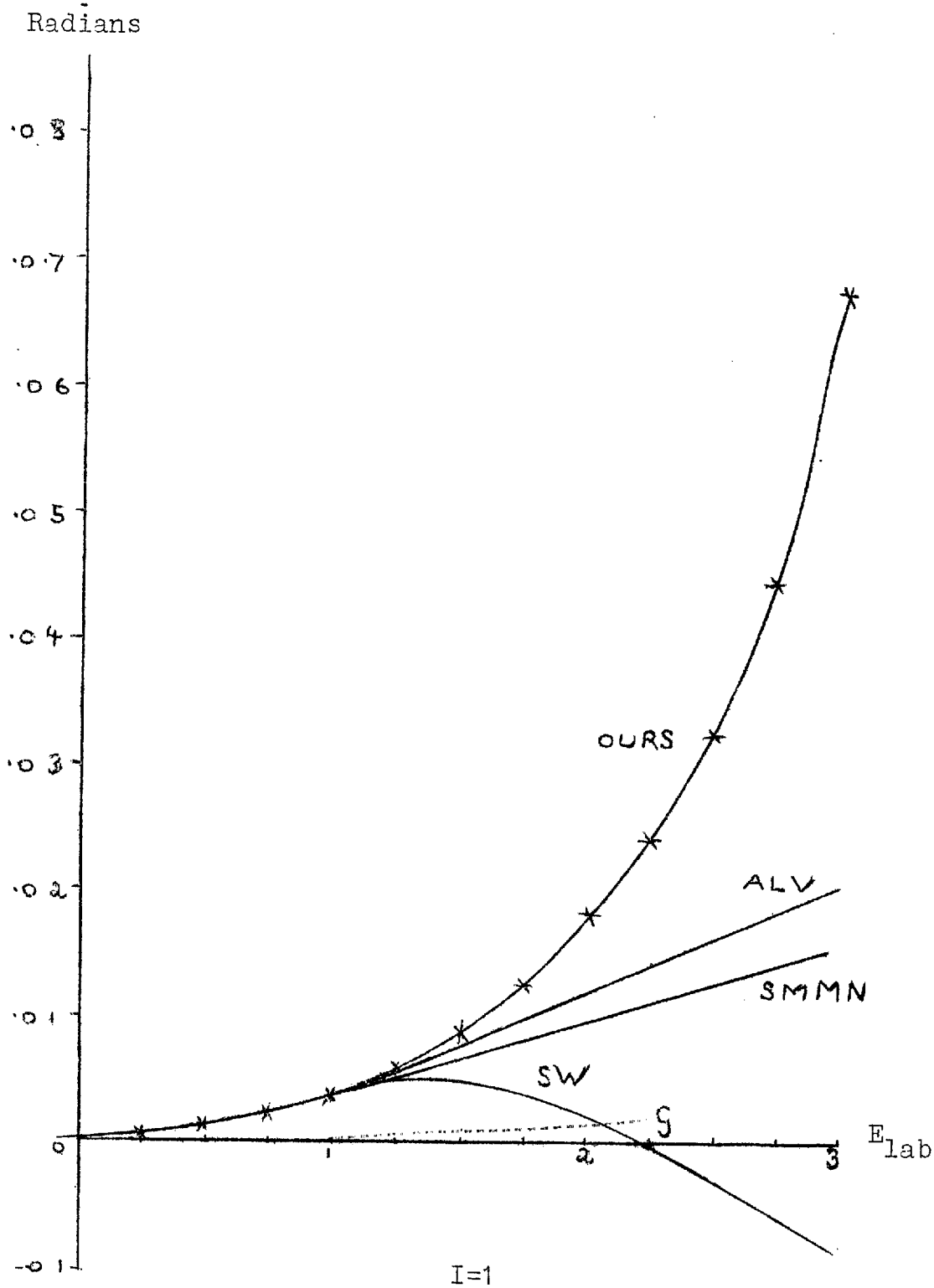
I=0, $^1H_5$ phase-shift (Fig. V 21)

Galanin gives values of 10^{-5},.0003,.0017,.0052 radians at 40,100,200,300 MeV respectively



I=1 $^3H_5$ phase-shift

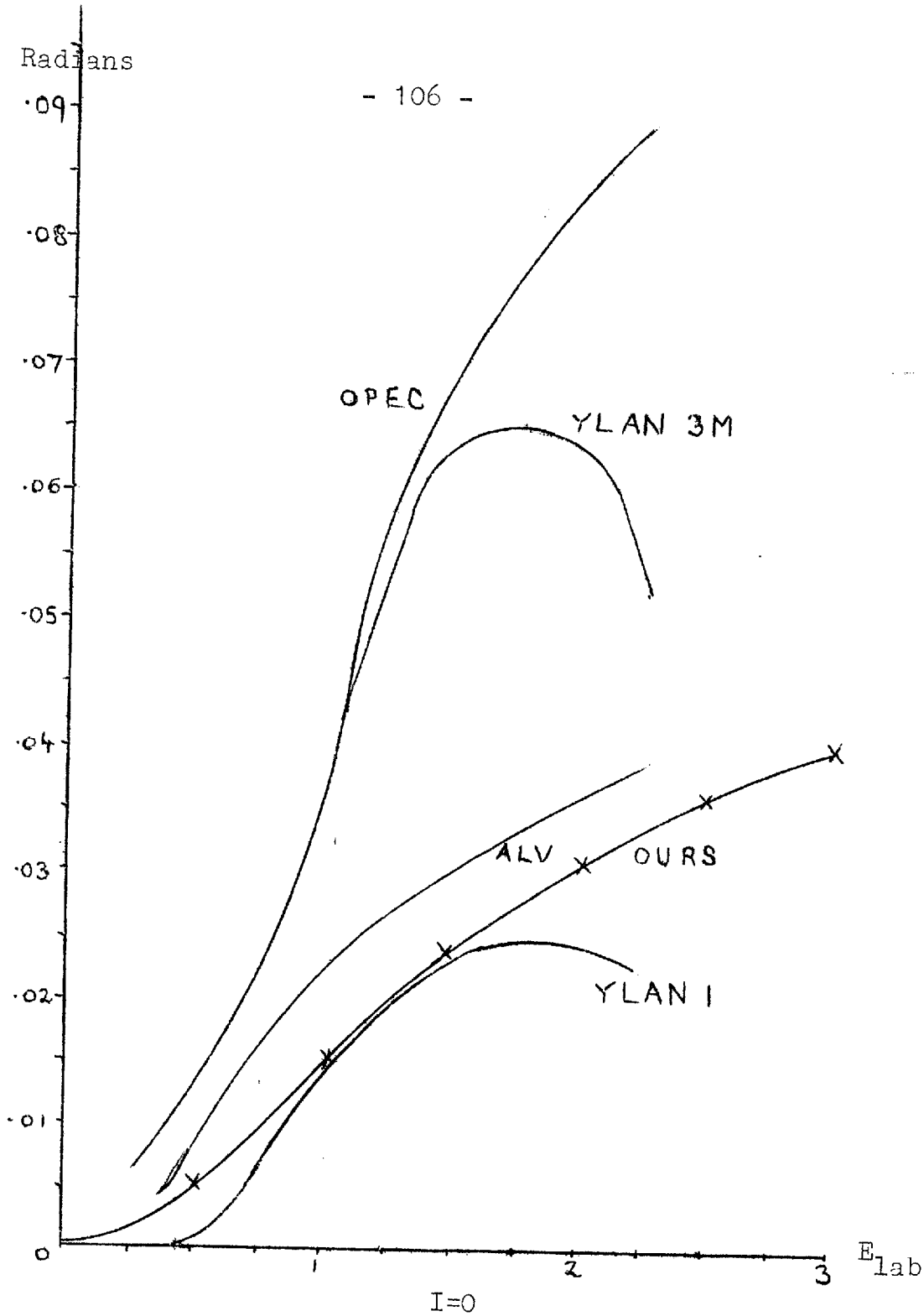
(Fig. V 22)



I=1

$^3H_4$ phase-shift

(Fig. V 23)



ε_5 mixing parameter

(Fig. V 24)

For the $^1S_0$ and $^3S_1$ phase-shifts our results follow the general shape of the SW and YLAN 3M graphs and are considerably better than the OPEC results. The numerical values of the $^1S_0$ phase-shifts are, however, rather poor. Since we have neglected the more complicated exchange processes, and would expect these to be important for the S-waves, the general tendency of these graphs towards the experimental results is rather encouraging. We suggest that this is mainly due to the fact that we have supplied the actual values of the scattering lengths from experiment.

The $^1P_1$ and $^3P_1$ phase-shifts have the wrong sign and even the OPEC results are much better. It is interesting to note that the Tamm-Dancoff method (GH) also yields $^1P_1$ phase-shifts with the wrong sign. Our $^3P_0$ and $^3P_2$ phase-shifts are beginning to show the characteristic shapes of the SW results (which closely follow experiment ([3])). Again the numerical values are rather poor but considerably better than the OPEC results. The $^1D_2$ phase-shift is much too large, and the ALV and Galanin results fit the experiments very well. For the $^3D_2$ phase-shift the OPEC terms are closer to the experiments than our results which are, however, good up to 60 MeV. The ALV, GH, and Galanin results are much too small in the whole range.

The $^3D_3$ phase-shift is considerably closer to the YLAN results than that of ALV. The Galanin values are much too large, and the OPEC values have the wrong sign. Our ϵ_2 mixing parameter (and that of Galanin) is much too large although ours has a similar shape to the SW results. The OPEC values follow the SW graph up to about 60 MeV.

Our $^1F_3$ phase-shift has the same general shape as the graphs of ALV and YLAN 3M, although our numerical values are too large. The OPEC results follow YLAN 3M up to about 100 MeV and seem to fit the YLAN 1 results better for the higher energies. The Galanin values are too large and have the wrong sign throughout the range. For the $^3F_3$ and $^3F_4$ phase-shifts the OPEC results are too small and our results too large. Again (together with ALV) we have obtained the general curvature character of the SMMN graph. For the $^3F_3$ phase-shift Galanin's results are much too small in absolute value, but for the $^3F_4$ phase-shift they fit SMMN's results quite closely. Our ϵ_3 mixing parameter, together with the OPEC results fit the YLAN 3M graph very well. Galanin's values are, however, very poor.

Our $^1G_4$ and $^3G_4$ phase-shifts are similar to those of ALV and fit in fairly well with the experimental results. From the $^3G_4$ graph we note that the OPEC results are also quite good. On the other hand, for both phase-shifts the Galanin results are much too small.

Our $^3G_3$ phase-shifts follow those of YLAN 1 quite closely, but the ALV results begin to diverge from these at about 100 MeV. In the low energy region (<100 MeV) the OPEC results follow the YLAN 3M curve. The Galanin results are again too small by a factor of about 10^{-2} (in absolute value). For the $^3G_5$ phase-shifts a similar situation occurs — our results follow the YLAN 1 graph but are a little too large. The ALV results diverge considerably from both of these and vanish at about 250 MeV. The OPEC and Galanin graphs follow the YLAN 3M phase-shifts quite closely. Our ϵ_4 mixing parameter follows the SMMN graph more closely than do the ALV results. The Galanin values are too small in absolute magnitude by a factor of about 1/10.

For the $^1H_5$ phase-shift we have a graph of similar shape to that of ALV but our values are too large. Again the Galanin results are rather small in absolute value. For the $^3H_5$ phase-shift the OPEC results are a very good approximation to the SMMN graphs, but both our calculation and that of ALV tend to produce small discrepancies ~~aggravate the behaviour~~. For the $^3H_4$ phase-shift the results of ALV, SMMN, and SW agree up to about 150 MeV and then begin to diverge, the ALV graph following SMMN most closely. The Galanin values are again too small.

For the ϵ_5 mixing parameter we have an interesting situation — the OPEC results follow the YLAN 3M graph closely up to about 150 MeV and our results follow those of YLAN 1 more closely. ALV's results are similar to ours but a little larger.

In view of the discrepancies between different experimental results, and the fact our results using one and two pion exchange tend to adhere to one set and the OPEC results to a different set for any particular graph — this adherence being inconsistent for different graphs — does not enable us to draw any absolutely firm conclusions about our results. However, except for some of the P waves our graphs have the general curvature characteristics of the experimental results and the correct sign. Whereas the OPEC results in general are smaller than the experimental values, our phase-shifts are a little too large — produce too much attraction. Evidence of this is also found in the fact that the OPEC underestimates, and our results overestimates the deuteron binding energy although our results do come closer to the experimental value. We conclude that using one and two pion exchange contributions produces a fairly adequate description of the NN interaction at least for the higher partial waves within the energy range investigated.

APPENDIX A

If accurate solutions to the coupled integral equations are required we may be forced to invert very large matrices (containing four times as many elements as those for the uncoupled equations). This can lead to computational difficulties due to both numerical and time-consuming considerations. We also have the branch cut in $h_{12}(\nu)$ which leads to the difficulties described in chapter III. It may, therefore, be useful to solve the equations by inverting a comparatively small matrix, and then to improve the solutions by a small number of iterations (one iteration may be quite adequate in some circumstances) using the method outlined below.

Our approximate solutions are h_{11}, h_{12}, and h_{22} satisfying equ. (III 2.3) on the right-hand (physical) cut. Also, on the left-hand cut

$$\begin{aligned} \text{Im}\, \eta_{12} &= \text{Im}\, \frac{m}{E} h_{12} = a_{12}(\nu)\pi \\ \text{Im}\ h_{11} &= a_{11}(\nu)\pi \\ \text{Im}\ h_{22} &= a_{22}(\nu)\pi \end{aligned} \qquad \text{(A 1)}$$

The unitarity condition can be rewritten as

$$\text{Im } h_{11} = \frac{p}{E}|h_{11}|^2 + \frac{pE}{m^2}|\eta_{12}|^2$$

$$\text{Im}\,\eta_{12} = \frac{p}{E}(h_{11}^{*}\eta_{12} + \eta_{12}^{*}h_{22}) \qquad \text{(A 2)}$$

$$\text{Im } h_{22} = \frac{p}{E}|h_{22}|^2 + \frac{pE}{m^2}|\eta_{12}|^2$$

The exact solutions we seek will satisfy equations of the form

$$h(\nu) = \frac{\nu}{\pi}\int_{-\infty}^{-1/4} \frac{\text{Im } h(\nu')d\nu'}{\nu'(\nu'-\nu)} + \frac{\nu}{\pi}\int_{0}^{\infty} \frac{\text{Im } h(\nu')d\nu'}{\nu'(\nu'-\nu)} + h(0) \qquad \text{(A 3)}$$

except for the case J=1 when we must also include the deuteron pole term $g_D^2/(4(\nu + m_D^2)$. We now define

$$\begin{aligned} \text{Re } h &= h^R \\ \text{Im } h &= h^I \end{aligned} \qquad \text{(A 4)}$$

and let us suppose that a better approximation to the solution of the system (A 3) is $h+\epsilon$, and powers of ϵ above the first may be neglected. Then using the fact that our approximate solutions will satisfy unitarity exactly*, and imposing the following conditions on the ϵ to ensure our improved solutions do so also

* This is ensured by the N/D method exactly, even if our integration formulae are very crude.

$$\epsilon_{11}^{I} = \frac{2p}{E}[\epsilon_{11}^{R} h_{11}^{R} + \epsilon_{11}^{I} h_{11}^{I}] + \frac{2pE}{n^2}[\epsilon_{12}^{R}\lambda_{12}^{R} + \epsilon_{12}^{I}\lambda_{12}^{I}]$$

$$\epsilon_{12}^{I} = \frac{p}{E}[\epsilon_{11}^{R}\lambda_{12}^{R} + \epsilon_{12}^{R} h_{11}^{R} + \epsilon_{11}^{I}\lambda_{12}^{I} + \epsilon_{12}^{I} h_{11}^{I}$$

$$+ \epsilon_{22}^{I}\lambda_{12}^{I} + \epsilon_{12}^{I} h_{22}^{I} + \epsilon_{22}^{R}\lambda_{12}^{R} + \epsilon_{12}^{R} h_{22}^{R}] \qquad (A\ 5)$$

$$\epsilon_{22}^{I} = \frac{2p}{E}[\epsilon_{22}^{R} h_{22}^{R} + \epsilon_{22}^{I} h_{22}^{I}] + \frac{2pE}{n^2}[\epsilon_{12}^{R}\lambda_{12}^{R} + \epsilon_{12}^{I}\lambda_{12}^{I}]$$

we find that the ϵ satisfy integral equations, viz.

$$\epsilon_{11}^{R}(\nu) = A_{11}(\nu) + \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{\epsilon_{11}^{I}(\nu')d\nu'}{\nu'(\nu'-\nu)}$$

$$\epsilon_{12}^{R}(\nu) = A_{12}(\nu) + \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{\epsilon_{12}^{I}(\nu')d\nu'}{\nu'(\nu'-\nu)} \qquad (A\ 6)$$

$$\epsilon_{22}^{R}(\nu) = A_{22}(\nu) + \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{\epsilon_{22}^{I}(\nu')d\nu'}{\nu'(\nu'-\nu)}$$

The functions A_{11} etc. represent the difference between the two sides of equ. (A 3) using our approximate solutions $h(\nu)$, and as usual $\mathcal{P}$ denotes principal value integration. This system of three linear simultaneous algebraic and three integral equations can easily be solved appr ximately by first taking $\epsilon_{11}^{R} = A_{11}$ etc. and then substituting into equ. (A 5) to find the imaginary parts. The next approximation to ϵ_{11}^{R} etc. is obtained

by merely integrating the right hand side of equs. (A 6). We could then expect our new solutions $h+\varepsilon$ to be considerably better approximations to the solutions of equations (A 3) than h alone.

APPENDIX B

We now summarize those misprints occuring in GGMW which we have corrected in our work.

Our equ.	(II 2.18) corresponding to GGMW equ.	(4.25)
	(II 3.5)	(4.31)
	(II 3.8)	(4.34)
	(II 4.14)	(B 9)

* * * *

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The Zilch in General Relativity.

K. S. FELDMAN
Mathematics Department, Imperial College - London

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Summary. — The conditions for the vanishing of the covariant divergence of traceless, bilinear combinations of Van der Waerden spinors are investigated for the electromagnetic field in the presence of gravitation, and the free gravitational field.

1. – Introduction.

LIPKIN (1) has recently discovered the conservation laws of Zilch for the free electromagnetic field, and these have been simplified by KIBBLE (2), using the notion of a dual tensor ${}^*F_{ab}$. The aims of the present paper are to show how the same results may be easily obtained using spinors, and how they may be extended in the presence of gravitation. We refer extensively to the work of PENROSE (3) (on the application of spinors to general relativity), and hereafter abbreviate it to P. Any equations we require from that paper will be denoted by (P 2.12 etc.). The third-order tensor $z^{ab}{}_d$ (and its six-index spinor equivalent) of KIBBLE and LIPKIN, will be called « 3-Zilch », and in Sect. **2** we investigate its properties in spinor formalism.

Section **3** is devoted to the determination of conditions for the 3-Zilch to remain (covariantly) divergence-free in a gravitational field, and in Appendix A we outline a proof that its divergences all have the same form. In Appendix B we define a 10-index spinor analogue of Zilch for a free gravitational field in empty space.

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(2) T. W. B. KIBBLE: Imperial College Preprint, ICTP/64/55, submitted to *Journ. Math. Phys.*

(3) R. PENROSE: *Ann. Phys.*, **10**, 171 (1960).

In order to obtain conservation laws from a vanishing covariant divergence a procedure explained by EDDINGTON (1) (Chapter 4) is employed. See also our conclusion.

The notation is as follows (suggested by PIRANI (5)): a, b, c ... denote tensor indices; A, B, C ... undotted spinor indices; and Z, Y, X ... dotted spinor indices. The antisymmetric spinor $\varepsilon^{AB}=\varepsilon_{AB}$ takes the value $+1$ for $AB=12$. The spinor covariant differentiation operator is ∇_{AW}, and $\Box=\nabla^{AW}\nabla_{AW}=\nabla_{AW}\nabla^{AW}$. Symmetrizing brackets are used as follows:

$$X_{(ABCD)}=\frac{1}{4!}\Big(\sum_{\text{all permutations of } ABCD} X\Big)\,.$$

2. – The 3-zilch in spinor formalism.

The spinor equivalent of the Maxwell tensor F_{ab} is given by

$$F_{AWBX}=\tfrac{1}{2}(\varphi_{AB}\varepsilon_{WX}+\varepsilon_{AB}\varphi_{WX}) \tag{P. 1.3}$$

and that of its dual ${}^*F_{ab}$ by

$${}^*F_{AWBX}=\tfrac{1}{2}i(\varepsilon_{AB}\varphi_{WX}-\varphi_{AB}\varepsilon_{WX})\,, \tag{P. 1.5}$$

where φ is symmetric. Adopting Kibble's definition (2) of 3-Zilch

$$Z^{a}{}_{cd}={}^*F^{ab}F_{bc,d}-{}^*F_{bc,d}F^{ab} \tag{2.1}$$

and replacing tensors by their spinor equivalents, partial derivatives by covariant spinor derivatives, and remembering that ∇ commutes with ε, we find that the spinor equivalent of the 3-Zilch is given by

$$Z^{AW}{}_{CYDZ}=\tfrac{1}{2}i(\bar{\varphi}^{W}{}_{Y}\nabla_{DZ}\varphi^{A}{}_{C}-\varphi^{A}{}_{C}\nabla_{DZ}\bar{\varphi}^{W}{}_{Y})\,. \tag{2.2}$$

(Cf. the density/current of, *e.g.*, the Klein-Gordon field:

$$\bar{\varphi}\delta_{\nu}\varphi-\varphi\delta\bar{\varphi}_{\nu})\,.$$

The symmetry $z^{ac}{}_{d}=z^{ca}{}_{d}$ follows immediately from the symmetry of φ; note AW replaces a, CY replaces c etc. Now in charge-free empty space, Maxwell's

(1) A. S. EDDINGTON: *The Mathematical Theory of Relativity* (Cambridge, 1960).
(5) F. A. E. PIRANI: Lectures at King's College London.

equations and the wave equations take the following forms respectively ([3]):

$$\text{(P. 3.10)} \qquad \nabla^{AW}\varphi_{AB} = 0\,, \qquad \nabla^{AW}\varphi_{WX} = 0\,,$$

$$\text{(follows (P. 3.13))} \qquad \Box\varphi_{AB} = \Psi_{ABCD}\varphi^{CD} - \tfrac{4}{3}\lambda\varphi_{AB}\,, \qquad \Box\varphi_{WX} = \bar{\Psi}_{WXYZ}\bar{\varphi}^{YZ} - \tfrac{4}{3}\lambda\bar{\varphi}_{WX}$$

where λ is the cosmological constant, and Ψ, which is totally symmetric, is part of the spinor equivalent of the conformal tensor (P and Appendix).

The vanishing of all the contractions of the 3-Zilch now follows immediately from Maxwell's equations and the fact that φ is symmetric—which implies $\varphi^{A}{}_{A} = 0$ etc.

We now discuss the various divergences of the 3-Zilch. Covariant differentiation, contraction, and a trivial cancellation, lead immediately to

$$\text{(2.3)} \qquad \nabla^{DZ}Z^{AW}{}_{CYDZ} = \tfrac{1}{2}i(\bar{\varphi}^{W}{}_{Y}\Box\varphi^{A}{}_{C} - \varphi^{A}{}_{C}\Box\bar{\varphi}^{W}{}_{Y}) = \tfrac{1}{2}i(\bar{\varphi}^{W}{}_{Y}\Psi^{A}{}_{CBD}\varphi^{BD} - \varphi^{A}{}_{C}\bar{\Psi}^{W}{}_{YXZ}\varphi^{XZ}).$$

In a flat space-time $\Box\varphi$, λ and Ψ all vanish, and so does this divergence, by inspection. The only other independent divergence is obtained by a differentiation with respect to C, Y; recall the symmetry in A, C; W, Y. We find (after altering the positions of some dummy suffices) that

$$\text{(2.4)} \qquad \nabla^{CY}Z^{YW}{}_{CYDZ} = \tfrac{1}{2}i(\bar{\varphi}^{WY}\nabla_{CY}\nabla_{DZ}\varphi^{AC} - \varphi^{AC}\nabla_{CY}\nabla_{DZ}\bar{\varphi}^{WY})\,.$$

In a flat space-time, we may alter the orders of differentiation, and then Maxwell's equations tell us that this quantity vanishes. In the Appendix we demonstrate that (2.3) and (2.4) have the same form.

3. – Conditions for the vanishing of the divergence.

From (2.3), we see that the divergence will vanish if, and only if

$$\text{(3.1)} \qquad \varphi^{WX}\Psi^{AB}{}_{CD}\varphi^{CD} = \varphi^{AB}\bar{\Psi}^{WX}{}_{YZ}\bar{\varphi}^{YZ}\,.$$

Since the complete classification of all solutions of this equation is rather involved, it suffices to demonstrate that (3.1) is satisfied for some cases of physical interest. From P. Sect. **4**, we see that $\varphi^{AB} = \tfrac{1}{2}(\theta^{A}\eta^{B} + \theta^{A}\eta^{B})$. θ and η are termed « Electromagnetic Principal Null Directions », where the null vector θ^{a} has spinor equivalent $\theta^{A}\bar{\theta}^{W}$, and similarly for η. Should $\theta = \eta = \varkappa$, say, the electromagnetic field is termed « Null », and k^{a}, with spinor equivalent $\varkappa^{A}\bar{\varkappa}^{W}$ is the direction of motion of the wave. This is the case of an electromagnetic plane-wave, and $\varphi^{AB} = \varkappa^{A}\varkappa^{B}$. In the same way the spinor Ψ may be

decomposed as follows:

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)} ,$$

α, β, γ, δ, are « Gravitational Principal Null Directions ». It is, then, easy to see that both sides of (3.1) will vanish if *a*) $\alpha = \beta = \gamma = \delta = 0$ or η (to within a scalar factor), or *b*) $\theta = \eta = \varkappa$, say, and $\varkappa$ is equal to any three of α, β, γ, δ (again to within a scalar factor).

4. – Conclusions.

It has, therefore, been demonstrated that for an electromagnetic field in a curved, charge-free, empty space-time, we may define a bilinear, trace-free combination of spinors, whose divergence vanishes in physically interesting circumstances.

We may, of course, define similar expressions for the gravitational field interacting with meson fields, neutrino fields, Dirac spinor fields etc., and it may be of interest to discover under what circumstances these have vanishing divergence.

It should be noted that the vanishing of a covariant divergence does not immediately imply a conservation law. We have first to write it in the form of the ordinary (partial derivative) divergence of a tensor-density. This will be the sum of the Zilch tensor-density and a « pseudo-tensor-density », defined from a convenient Lagrangian. For the details of this procedure see EDDINGTON [4] (Section **59**) for the case of the Material-Energy-Tensor, or any standard work on General Relativity.

* * *

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APPENDIX A

We now prove the result quoted at the end of Sect. 2. The spinor equivalent of the Riemann tensor is given by:

$$\text{(P-3.4)}' \quad R_{AWBXCYDZ} = \tfrac{1}{2}(\chi_{ABCD}\varepsilon_{WX}\varepsilon_{YZ} + \varepsilon_{CD}\varphi_{ABYZ}\varepsilon_{WX} + \varepsilon_{AB}\bar{\varphi}_{WXCD}\varepsilon_{YZ} + \varepsilon_{AB}\varepsilon_{CD}\bar{\chi}_{WXYZ}) ,$$

$$\text{(P-2.2)} \qquad \chi_{ABCD} = \Psi_{ABCD} + \lambda/3(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}) .$$

Ψ corresponds to the conformal tensor, and φ_{ABYZ} to the Einstein tensor. In the case we are considering (P-3.11), $\varphi_{ABWX} = \varphi_{BA}\bar{\varphi}_{WX}$. Using (P-2.14), and noting that any spinor may be written as the sum of products of one-index spinors, we obtain the following results:

$$\text{(A.1)} \quad \begin{cases} (\nabla_{CP}\nabla_D{}^P + \nabla_{DP}\nabla_C{}^P)\varphi^{AB} = \chi_{CD}{}^{AH}\varphi_H{}^B + \chi_{CD}{}^{BH}\varphi^A{}_H\,, \\ (\nabla_{EY}\nabla^E{}_Z + \nabla_{EZ}\nabla^E{}_Y)\varphi^{AB} = \varphi^{AH}{}_{YZ}\varphi_H{}^B + \varphi^{BH}{}_{YZ}\varphi^A{}_H, \\ (\nabla_{CP}\nabla_D{}^P + \nabla_{DP}\nabla_C{}^P)\bar{\varphi}_{WX} = \varphi_{CD}{}^{WU}\varphi_U{}^X + \varphi_{CD}{}^{XU}\bar{\varphi}_U{}^W\,, \\ (\nabla_{EY}\nabla^E{}_Z + \nabla_{EZ}\nabla^E{}_Y)\bar{\varphi}^{WX} = \bar{\chi}_{YZ}{}^{WU}\bar{\varphi}_U{}^X + \bar{\chi}_{YZ}{}^{XU}\bar{\varphi}_U{}^W\,. \end{cases}$$

These are used in conjunction with

$$\text{(P-2.12)} \quad \nabla_{AW}\nabla_{BX} - \nabla_{BX}\nabla_{AW} = \tfrac{1}{2}\varepsilon_{WX}(\nabla_{AP}\nabla_B{}^P + \nabla_{BP}\nabla_A{}^P) + \tfrac{1}{2}\varepsilon_{AB}(\nabla_{HW}\nabla^H{}_X + \nabla_{HX}\nabla^H{}_W)\,.$$

Substituting into (2.4) and using Maxwell's equations, we find

$$\text{(A.2)} \quad \begin{aligned} \nabla^{CY}Z^{AW}{}_{CYDZ} = {} & \tfrac{1}{4}i\bar{\varphi}^W{}_Z(\chi_{CD}{}^{AH}\varphi_H{}^C + \chi_{CD}{}^{CH}\varphi^A{}_H) + \tfrac{1}{4}i\bar{\varphi}^{WY}(\varphi^{AH}{}_{YZ}\varphi_{HD} + \varphi_D{}^H{}_{YZ}\varphi^A{}_H) - \\ & - \tfrac{1}{4}i\varphi^{AC}(\varphi_{CD}{}^{WU}\bar{\varphi}_{UZ} + \varphi_{CDZ}{}^U\bar{\varphi}_U{}^W) - \tfrac{1}{4}i\varphi^A{}_D(\bar{\chi}_{YZ}{}^{WU}\varphi_U{}^Y + \bar{\chi}_{YZ}{}^{YU}\bar{\varphi}_U{}^W)\,. \end{aligned}$$

Using (P-3.4), (P-3.11), it is seen that all the terms involving λ, and $\varphi^{AB}{}_{WX}$ cancel, and we are left only with Ψ terms. Note also that the symmetry of Ψ implies that $\Psi_{AC}{}^{CD}$ etc. vanishes. The final result is

$$\text{(A.3)} \quad \nabla^{CY}Z^{AW}{}_{CYDZ} = -\tfrac{1}{4}i(\bar{\varphi}^W{}_Z\Psi^A{}_{DBC}\varphi^{BC} - \varphi^A{}_D\overline{\Psi}^W{}_{ZXY}\varphi^{XY}) \quad \text{(cf. (2.3))}\,.$$

Appendix B

We now briefly discuss the problem of obtaining a Zilch tensor (or spinor) for the free gravitational field. In analogy with (2.2), we define

$$\text{(B.1)} \quad Z_{AWBXCYDZHP} = \overline{\Psi}_{WXYZ}\nabla_{HP}\Psi_{ABCD} - \Psi_{ABCD}\nabla_{HP}\overline{\Psi}_{WXYZ}\,.$$

Note that in empty space $\varphi_{ABWX} = 0$ (P-3.3), $\nabla^{DZ}\Psi_{ABCD} = 0$ (P-3.5), and $\Box\Psi_{ABCD} = 3\Psi_{(AB}{}^{EF}\Psi_{CD)EF} - 2\lambda\Psi_{ABCD}$ (P-3.8).

We see immediately that all the contractions of our 10-spinor-index Zilch (10-Zilch), vanish in virtue of the symmetry of Ψ and relation (P-3.5). By an argument similar to that in Appendix A, we see that all the divergences of this 10-Zilch have the same form, *viz.*

$$\text{(B.2)} \quad \overline{\Psi}_{WXYZ}\Box\Psi_{ABCD} - \Psi_{ABCD}\Box\overline{\Psi}_{WXYZ}\,.$$

Using (P-3.8), we see that this vanishes if, and only if

$$\Psi_{(AB}{}^{CD}\Psi_{CD)EF}\overline{\Psi}_{WXYZ} = \Psi_{ABCD}\overline{\Psi}_{(WX}{}^{UV}\Psi_{YZ)UV}\,. \tag{B.3}$$

When the Gravitational Principle Null Directions coincide in pairs, Petrov-type [22], Ψ_{ABCD} is proportional to $\Psi_{(AB}{}^{EF}\Psi_{CD)}{}^{EF}$ (see footnote ([12]), pag. 189 of P), so that (B.3) is satisfied. In the case of a plane gravitational wave, Petrov-type N, when all four principal null directions coincide, each side of (B.3) vanishes identically. It remains to show that one of those conditions is in fact necessary. Multiply both sides of (B.3) by $\alpha^A\alpha^B\alpha^C\alpha^D$, $\beta^A\beta^B\beta^C\beta^D$, etc. in turn, where, as usual, α, β, etc. are the Principal Null Directions. We obtain the following sets of conditions, to be satisfied simultaneously:

$$\alpha\cdot\beta\,\alpha\cdot\gamma\,\alpha\cdot\delta = 0\,, \qquad \beta\cdot\alpha\,\beta\cdot\gamma\,\beta\cdot\delta = 0\,, \qquad \gamma\cdot\alpha\,\gamma\cdot\beta\,\gamma\cdot\delta = 0\,, \qquad \delta\cdot\alpha\,\delta\cdot\beta\,\delta\cdot\gamma = 0$$

the only solutions of which are Petrov-types [22], N. Since the Schwarzschild solution is of type [22], this work may be of some physical interest.

RIASSUNTO (*)

Si esaminano le condizioni di annullamento della divergenza covariante delle combinazioni bilineari senza traccia degli spinori di Van der Waerden per ilc ampo elettromagnetico in presenza di gravitazione, e per il campo gravitazionale libero.

(*) *Traduzione a cura della Redazione.*

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