THE EFFECTS OF HEAT AND MASS TRANSFER PROCESSES ON SHALLOW LIQUID FLOWS

by

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ABSTRACT

Fluid flows effected by surface tension variations are considered. A discussion of the various agencies causing these variations is given together with laboratory, industrial and natural illustrations of such flows. The equations of motion, mass transfer and state for the surface phase are described and discussed. The various theoretical models produced for Bénard's cells are described and compared. Yih's model for two dimensional variable surface tension driven flow when the variation is caused by an insoluble contaminant is discussed and an analysis of a singular point in a phase plane is used to examine the possible instabilities in such a flow. By extending Yih's analysis, a model is produced for the flow in a shallow fuel layer near, but not close to, a burning wick. The solutions for the variations of surface tension and fuel depth with distance from the wick are obtained numerically. A separate solution, developed from slow viscous corner flow theory, is provided for the flow close to the wick. Neglecting the variation of depth associated with these flows, a model is produced for the effect of a heat source placed at the corner of a rectangular cell. The solutions are found using a Green's function which is obtained as a series of eigenfunctions and eigenvalues. A model produced for flow owing to a moving heat source involves the solution in a fast moving boundary layer close to the surface by formulating Yih's basic equations in a moving frame and then obtaining numerical solutions for surface tension and liquid depth, and the solution in the layer below using a Von Kármán-Pohlhausen technique. Using this Von Kármán-Pohlhausen technique again,

an analysis for the flow in an aqueous solution on contact with a hot gas is presented.

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CONTENTS

CHAPTER 1

General Introduction

The greatest influence on shallow liquid flows is probably that owing to surface tension variations: the variation of surface tension along the interface of a fluid gives rise to tangential stresses which affect the motion of the fluid. These variations can be caused by changes in the surface temperature, changes in molecular composition of the interface when more than one substance is present and also by electrical effects. Heat and mass transfer processes taking place in a shallow liquid layer have, therefore, a strong effect on the motion taking place in that layer.

The name ''Marangoni effects'' has been given to motions which are induced by variations of surface tension at the interface between two fluids, examples of which are given by Scriven and Sternling (1960) and in the review papers by Kenning (1968) and by Levich and Krylov (1969). Perhaps the easiest commonly observed effect is the ''Tears of Strong Wine'' which occurs when the walls of a glass are wetted by wine. A thin film of wine forms on the inside of the glass but the evaporation of alcohol rapidly reduces the concentration of alcohol in this film and causes the 'old' film to have a higher surface tension than that of the 'new' films just rising from the bulk of the wine. The 'old' thus retracts from the 'new' film and soon yields small droplets.

A simple experiment which demonstrates very easily motion produced by surface tension variations is the ''Camphor

Dance''. A grain of camphor placed on the surface of clean water is seen to start moving about violently. Because of the dissymmetry of the grain the amount of camphor entering the water surface varies at different points of the grain. Hence the surface tension of water is lowered to a different extent at different points on the circumference and the grain is pulled most strongly towards the point of greatest surface tension. Both the ''Camphor Dance'' and the ''Tears of Strong Wine'' are described by Bickerman (1968). The effect of surface temperature variations on a shallow liquid layer can be observed in the laboratory by placing a heat source close to the surface of a pure liquid which does not decompose under the action of heat. It will be seen that there is a net surface flow away from the source and a corresponding depression of the surface owing to the decrease in surface tension at the hot spot. The opposite effects will be observed when a liquid is locally cooled.

Marangoni effects occur naturally, on the surface of a polluted river for example. The effect is also important where a river enters the sea owing to the presence of natural solutes in river and sea water, in addition to the possible pollution of the river. A commonly observed effect is the damping of gravity waves by a thin layer of crude oil; a mathematical treatment of this effect is given by Miles (1967).

As well as being apparent in nature, Marangoni effects are of industrial importance involving heat or mass transfer across an interface eg distillation, fractional distillation and liquid-liquid extraction. Minute quantities of some substances present on the surface of some liquids can produce extremely large changes in surface tension. Experiments, involving processes such as fractional distillation, carried out in the laboratory with clean apparatus and pure materials may not therefore be applicable to a large scale industrial plant where every trace of contaminant cannot be removed.

This thesis contains a theoretical analysis of some flows driven by surface tension variations. Chapter 2 contains the derivation and a discussion of the equations and boundary conditions at the surface which will be needed, in addition to equations for flow in the bulk of the layer, for the analysis of such flows.

Chapter 3 and 4 are concerned with important work done previously on variable surface tension driven flows. Chapter 3 contains a review and discussion of the work done in connection with Benard's cells problem. Originally a classical example of buoyancy driven convection and subsequently shown to be affected by the surface tension gradient present, these cells have been the subject for many theoretical models based on buoyancy only, surface tension only and also models based on both buoyancy and surface tension. Chapter 4 considers in detail the analysis produced by Yih for two dimensional variable surface tension driven flow in a shallow liquid layer where the surface tension gradient is produced by an insoluble surfactant. In particular, the instabilities mentioned by him are considered and some of his conclusions are shown to be incorrect.

Chapters 5, 6, 7 and 8 provide theoretical analyses for variable surface tension driven flows produced by constant heat sources, the ones in Chapters 5, 6 and 7 being fixed in space. The effect of buoyancy is neglected in all four models and also in the model presented in Chapter 9. In Chapter 5 a model applicable to the problem of a shallow layer of fuel burning at a wick is presented. As the solution obtained does not satisfy the zero velocity condition at the wick a separate model, which is given in Chapter 6, is needed to describe the flow in the wedge shaped region formed by the wick and the surface. In Chapter 7 a model applicable to the case of a heat source present at the centre of a pool of fuel is produced. The only limitation placed on the depth of the liquid layer in this model is the implicit one made by assuming buoyancy effects are negligible. This depth is however assumed constant, contrary to all the other models presented in this thesis. In Chapter 8 a theoretical analysis is presented for the flow produced in a liquid layer by a constant heat source moving with constant velocity over a liquid layer.

The flow considered in Chapter 9 is that produced in a solution whose surface tension is varied both by the action of temperature and a soluble contaminant lowering the surface tension of the solvent. An experiment has been performed by Skogen where the solute in such a solution is destroyed by the action of a heat source present at the surface, thereby producing two opposing agencies affecting the surface tension. In Chapter 9 a theoretical analysis is provided for this Skogen effect.

The concluding chapter, Chapter 10, contains suggestions how the models presented here could be extended. It also contains a review of the future importance of flows affected by variable surface tension.

CHAPTER 2

Surface Phase Equations

2.1 Introduction

The concept of a surface phase has been introduced when considering two phase flows. This phase consists of the molecules of each bulk phase which are directly affected by the molecules of both bulk phases. One can suppose that these molecules constitute a two dimensional continuum with its own laws of physics. The basic results obtained from these laws are needed, mainly as boundary conditions, for theoretical models for flows in shallow liquid layers where surface phenomena are important.

2.2 Dynamical Equations

Scriven (1960) formulated the equations of motion for this phase. One must use a two dimensional set of coordinates and its associated metric tensor, $a^{\mu\lambda}$, defined by the surface. Scriven considers the interface between two fluids to be a two dimensional isotropic continuum. He also considers it to be Newtonian so that by definition the relation between stress and rate of strain is linear,

$$T^{\mu\lambda} = \rho^{\mu\lambda} + E^{\mu\lambda\alpha\beta} S_{\alpha\beta} \tag{2.1}$$

where $T^{\mu\lambda}$ is the surface stress tensor and $S_{\alpha\beta}$ the surface rate of strain tensor. Using the property of isotropy one gets after suitable rearrangement

$$T^{\mu\lambda} = \tau a^{\mu\lambda} + K a^{\mu\lambda} a^{\alpha\beta} S_{\alpha\beta} + \varepsilon(a^{\mu\lambda} a^{\lambda\beta} + a^{\mu\beta} a^{\lambda\alpha} - a^{\mu\lambda} a^{\alpha\beta}) S_{\alpha\beta}, \tag{2.2}$$

where τ the interfacial tension, K the coefficient of dilational surface viscosity, and ε the coefficient of shear viscosity are scalar quantities dependent on the temperature and molecular composition of the interface.

With the fluid at rest, the surface is in a state of uniform stress, $S_{\alpha\beta}=0$, so that

$$T^{\mu\lambda} = \tau a^{\mu\lambda} \tag{2.3}$$

τ coincides with the equilibrium interfacial tension from thermodynamic considerations. From these relations one can see clearly the analogy between surface tension in two dimensions and pressure in three dimensions.

The relation (2.2) between stress and rate of strain is then used by Scriven to obtain the equations of motion for the surface phase. These equations are formulated in a fixed Cartesian frame because in general the surface frame is non inertial. The equation balancing the forces which act in the direction $\underline{\ell}$ on a portion of surface, Σ, defined by the contour C, is

$$\iint_{\Sigma} \gamma A^i \ell_i d = \iint_{\Sigma} F^i \ell_i d + \int_{C} T^i \ell_i ds \tag{2.4}$$

where latin indices refer to the fixed frame, γ is the surface density, A^i the acceleration. F^i is the resultant of the external forces, which includes the effect of the bulk stresses at the interface. If, in the absence of any other forces, the stress tensors in the bulk phases A and B are $T_A{}^{ij}$ and $T_B{}^{ij}$, respectively,

$$F^i = (T_A{}^{ij} - T_B{}^{ij})n_j \tag{2.5}$$

where n_j is the normal to the surface Σ pointing into the bulk phase B. T^i is the expression in the Cartesian frame for $T^{\mu\lambda}m_\lambda$, m_λ being the outward normal to the contour C. Usually the terms involving γ can be neglected so that the equation of motion at the interface is just a relation between the stresses in the bulk phases at the interface and the surface stress. For a plane stationary interface in the (x,y) plane the relation

$$T_A - T_B = \frac{d\tau}{dx} + (K+\varepsilon)\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + \varepsilon\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) \qquad (2.6)$$

is obtained where (v_x,v_y) is the velocity of the liquid at the surface and T_A, T_B are the bulk stresses at the interface in the x direction, with the normal from phase A to B being in the positive z direction.

For most problems the terms in K and ε are neglible. If in addition phase B is air and phase A is a viscous incompressible fluid then as the bulk stress of the air is negligible one obtains a boundary condition for the fluid velocity (u,v,w) given by

$$\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{d\sigma}{dx} \quad , (\sigma = \gamma), \qquad (2.7)$$

which becomes

$$\mu\frac{\partial u}{\partial z} = \frac{\partial\sigma}{dx} \qquad (2.8)$$

if the mass transfer across the interface is uniform. This boundary condition will be used later in the problems of two dimensional surface tension driven flow which follow.

2.3 Equations of mass transfer

If in addition to two homogeneous fluids as phases A and B a contaminant is present, the equation of mass transfer for the contaminant in the surface phase is required. The equation given in Levich (1962) is

$$\frac{\partial \Gamma}{\partial t} + \nabla.(\Gamma u_S - D_S \nabla\Gamma) + j_n = 0 \tag{2.9}$$

where Γ is the surface concentration of contaminant, D_S is the surface diffusion coefficient, j_n is the number of molecules leaving the surface to either bulk phase/unit area, and $\underline{u}_S$ is the tangential surface velocity. The differential operators refer to the surface coordinates.

For insoluble contaminants $j_n=0$. For soluble contaminants j_n is controlled by the adsorption-desorption process and by diffusion in the bulk phase (S) in which the contaminant is soluble; normally one of the two limiting cases for which j_n is controlled completely by the slower process is considered. When adsorption or desorption is the controlling process Levich derived the equations

$$j_n = \alpha(\Gamma - \Gamma_o) \tag{2.10}$$

where Γ is the equilibrium value of Γ and α is given by,

$$\alpha = \left(\frac{\partial P}{\partial \lambda}\right)_{\Gamma = \Gamma_o} - \left(\frac{\partial Q}{\partial \lambda}\right)_{\Gamma = \Gamma_o}$$

with $P(\lambda)$ the desorption flux and $Q(\lambda)$ the adsorption flux. When diffusion is the controlling process

$$j_n = D \frac{\partial c}{\partial n} \tag{2.11}$$

where c is the concentration in the bulk phase, D the bulk mass diffusivity, and the outward normal to that bulk phase.

2.4 Equations of state

For a three dimensional continuum there is an equation of state, $f(P,\rho,T)=0$. For the two dimensional surface phase with a contaminant, concentration Γ, present there is an analogous relation, $F(\sigma,\Gamma,T)=0$, called the surface equation of state. For a pure liquid the equation of state

$$\sigma = K(T_{c}-T) \tag{2.12}$$

where K is a positive constant and T_c the critical temperature of the liquid, is in good agreement with experimental evidence.

Experimental results for the variation of surface tension, of an aqueous solution in contact with air, with concentration of solute show most solutes belong to one of three classes, see Figure 1. A good account is given by Davies and Rideal (1963) and Bickerman (1968). For the first class which contains unionized organic compounds at low concentrations, surface tension is a linear decreasing function of concentration. For the second class which contains inorganic salts at high concentrations, surface tension is a linear increasing function of concentration. In the third class are the long chain ions such as soap for which at low concentrations surface tension decreases very rapidly with concentration. Sulphuric acid for which surface tension is not a monotonic function of concentration is an example of a solute which does not belong to these categories.

The theoretical surface equations of state which are based on thermodynamics and chemical kinetics have good agreement with the experimental evidence. Gibb's adsorption which was derived from thermodynamic consideration is true in its most general form for any solution in contact with its vapour. The adsorption isotherm for a particular system which is derived from chemical kinetics is then used in conjunction with Gibb's equation to derive the equation of state for that system. For solutes of the first class the equation derived is

$$\pi = KT\Gamma \tag{2.13}$$

where $\pi=\sigma_{oT}-\sigma$, σ_{oT} being the value of σ at temperature T and $\Gamma=0$, and K is a positive constant. It is easy to see the analogy between this relation and the ideal gas law $P=RT\rho$. The relationship $\pi=KT\Gamma$ could have been obtained by regarding the surface phase as an ideal gas, each molecule having just two degrees of freedom.

Contaminants which are insoluble in a liquid generally lower the surface tension of that liquid. For systems with insoluble liquid contaminants in which the bulk phase of the contaminant is neglected the surface equation of state is generally not ideal. Because the molecules in these so called monolayers are closely packed, neither the forces of attraction between the molecules nor the area occupied by them is negligible as is assumed in the ideal theory: One of the equations of state for monolayers which considers these factors corresponds to van der Waal's equation for gases.

In the problems which follow the equation of state will be the Taylor expansion for $\sigma(T,c)$ with quadratic and higher powers neglected,

$$\sigma = \sigma(T_o,c_o) + \alpha(T_o-T)+ \beta(c-c_o) \tag{2.14}$$

where $\alpha = -\left(\frac{\partial\sigma}{\partial T}\right)_{\substack{T=T_o\\ c=c_o}}$ and $\beta = \left(\frac{\partial\sigma}{\partial c}\right)_{\substack{T=T_o\\ c=c_o}}$

The physical justification for this is seen in the preceeding review.

CHAPTER 3

3 Surface tension driven cellular convection

3.1 Introduction

It has been observed that when the temperature gradient across a shallow layer of liquid, heated from below, exceeds a critical value the layer becomes unstable and after an irregular transition period the flow becomes steady. In this steady state the layer is divided into vertical cells of polygonal cross section, the fluid rising in the centre and falling along the edges.

These cells were first observed by Bénard in 1900 when he heated a thin layer of spermaceti and cooled it above. An illustration of these Bénard Cells appears in 'Laminar Boundary Layers' (1963). A theory based on buoyancy was put forward by Rayleigh (1916) to explain this phenomenon. This basic theory has been refined by Jeffreys (1926), (1928), Low (1929), Pellow and Southwell (1940) and others.

Block (1956) repeated Bénard's experiments and observed cellular convection for temperature gradients less than the critical value predicted by the buoyancy theory. He also observed cellular convection in a layer cooled from below, which cannot be explained in terms of buoyancy. He found that when a film of silicone, which reduces the surface tension of the liquid, was placed on the surface of the layer heated from below, convection was reduced by a half. Block concluded that the Bénard cells were caused by variations of surface tension which are due to non-uniformities of temperature over the free surface. Led by the observation that the cells occur in drying paint films whether the free surface is above or below, Pearson produced a theory for Bénard's cells based on variations of surface tension at the free surface.

This chapter contains a review of the models produced for Bénard's cells.

3.2 Models assuming that the flow is driven by buoyancy.

An account of the buoyancy theory is given by Lin (1955). This section contains a summary of that work.

Consider a thin layer of fluid contained within the boundaries $y=0,d$. The equation of motion is

$$\rho \frac{\Delta \underline{u}}{\Delta t} = -\rho g\hat{j} - \nabla P + \rho\nu\nabla^2\underline{u} \qquad (3.1)$$

the equation of continuity is

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\underline{u}) = 0 \qquad (3.2)$$

the equation of heat transfer is

$$\frac{\partial T}{\partial t} + \underline{u}.\nabla T = \kappa\nabla^2 T \qquad (3.3)$$

and the equation of state is

$$\rho = \rho_0[1-\alpha(T-T_0)] \qquad (3.4)$$

where ρ_0 is the density at reference temperature T, and α is the coefficient of expansion, $\hat{j}$ the unit vector in the positive y direction. It is assumed that ν, κ, α are constants. In the unperturbed state

$$\bar{\underline{u}} = 0 \qquad (3.5)$$

$$\bar{T} = T_0 - \beta y \qquad (3.6)$$

$$\bar{\rho} = \rho_0(1+\alpha\beta y) \qquad (3.7)$$

where β is the temperature gradient T_1-T_0/d, T_1 and T_0

being the temperatures at the upper and lower plates respectively. When the system is perturbed write

$$T = \overline{T} + T' \tag{3.8}$$

$$\rho = \overline{\rho} + \rho' \tag{3.9}$$

and linearize the equations for T', u', ρ' to give two final equations for v and T', v being the velocity component in upward y direction,

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)T' = \beta v \tag{3.10}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 v = \alpha g \nabla^2 T' \tag{3.11}$$

The boundary conditions on v are

$$y=0,d \; : \; v = \frac{\partial v}{\partial y} = 0 \tag{3.12}$$

The boundary conditions on the temperature depend on the heat transfer properties of the surroundings. In Bénard's experiments the lower surface was metallic and the upper surface was air so it can be assumed in this case that the lower surface is a perfect conductor and the upper surface is a perfect insulator. The conditions are therefore

$$y=0 \; : \; T' = 0 \tag{3.13}$$

$$y=d \; : \; \frac{\partial T'}{\partial y} = 0 \tag{3.14}$$

Dimensionless variables are now introduced,

$$(\xi, \eta, \zeta) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right) , \; \tau = \frac{t\kappa}{d^2} \tag{3.15}$$

and a solution of the form

$$v = \frac{-\kappa}{d} F(\xi, \zeta) f(\eta) e^{p\tau} \tag{3.16}$$

$$T' = \beta d \, F(\xi, \zeta) g(\eta) e^{p\tau} \tag{3.17}$$

This requires

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} + \alpha^2 F = 0 \tag{3.18}$$

and a sixth order eigenvalue system is obtained:

$$[p-(D^2-\alpha^2)]\ g = -f \tag{3.19}$$

$$[pPr^{-1}-(D^2-\alpha^2)]\ [D^2-\alpha^2]f = R\alpha^2 g \tag{3.20}$$

$$\eta = 0,1\ :\ f=f'=0 \tag{3.21}$$

$$\eta = 0\ :\ g=0 \tag{3.22}$$

$$\eta = 1\ :\ g'=0 \tag{3.23}$$

where Pr is the Prandtl number and R is the Rayleigh number defined as

$$R = \frac{g\alpha\beta d^4}{\kappa} \tag{3.24}$$

By putting $p=0$, the neutral stability, (R,α), curve for the system is obtained. The curve exhibits a minimum value, R_c of R. For values of $R<R_c$ all disturbances are stable. For instability due to buoyancy $R>R_c$; in the case of the conducting boundary below and insulating above $R_c=571$ so that for Bénards cells to be buoyancy driven

$$\frac{g\alpha\beta d^4}{\kappa} > 571 \tag{3.25}$$

This buoyancy model will be compared with the surface tension driven model which now follows.

3.3 Models assuming that the flow is surface tension driven

This model assuming that these cells were induced by changes in surface temperature varying the surface tension along the surface is due to Pearson (1958). The variation of density with temperature is neglected.

Using a perturbation analysis similar to that in 3.2 the equations for v✓ and T' are

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right) \nabla^2 v = 0 \qquad (3.26)$$

and
$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)T' = \beta v \qquad (3.27)$$

The boundary conditions on v are

$$y = 0 \quad : \quad v = \frac{\partial v}{\partial y} = 0 \qquad (3.28)$$

$$y = d \quad : \quad v = 0 \qquad (3.29)$$

with a fourth condition to be obtained by equating the viscous stress at the surface to the surface tension gradient there. The relation between surface temperature and surface tension is written

$$S = S_o - \sigma T'_S \qquad (3.30)$$

where $\sigma = \left(\frac{\partial S}{\partial T}\right)_{T=T_{oS}}$ and S denotes surface quantity.

Using (2.9),

$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\partial S}{\partial x} \qquad (3.31)$$

and
$$\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{\partial S}{\partial z} \qquad (3.32)$$

which combine with (3.30) to give the fourth boundary condition on v ,

$$y = d \quad : \quad \frac{\partial^2 v}{\partial y^2} = \sigma \nabla_1{}^2 T' \quad , \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \qquad (3.33)$$

It is seen how, in the surface tension driven theory, the role of the equation of state for the surface corresponds with the role in the buoyancy theory of the bulk equation of state.

The temperature boundary conditions considered are the two extreme cases of conduction and insulation for the lower boundary,

$$y = 0 \quad : \quad T' = 0 \quad \text{or} \quad \frac{\partial T'}{\partial y} = 0 \tag{3.34}$$

and a general condition at the top,

$$-K \frac{\partial T'}{\partial y} = qT' \tag{3.35}$$

where K is the thermal conductivity of the liquid and q is the rate of change with temperature of the rate of loss of heat/unit area to its upper environment.

Introducing dimensionless variables, as before,

$$(\xi, \eta\ \zeta) = (\frac{x}{d}, \frac{y}{d}, \frac{z}{d}) \quad , \quad \tau = \frac{t\kappa}{d^2} \tag{3.36}$$

and trying for a solution with

$$v = \frac{-\kappa}{d} F(\xi,\zeta)f(\eta)e^{p\tau} \tag{3.37}$$

$$T' = \beta d\ F(\xi,\zeta)g(\eta)e^{p\tau} \tag{3.38}$$

requires

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \zeta^2} + \alpha^2 F = 0 \tag{3.39}$$

so that a sixth order eigenvalue system is obtained:

$$[p - Pr(D^2-\alpha^2)][D^2 - \alpha^2]f = 0 \tag{3.40}$$

$$[p - (D^2-\alpha^2)]g = -f \tag{3.41}$$

$$f(0)=f'(0)=0, \ f(1)=0, \ f''(1)=\alpha^2\beta g(1), \ g'(1)=Lg(1) \tag{3.42}$$

and either $g(0)=0$ or $g'(0)=0$. The constant B , the Marangoni number, is given by

$$B = \frac{\sigma\beta d^2}{\rho\nu\kappa} \tag{3.43}$$

and $$L = \frac{\alpha d}{K} \tag{3.44}$$

The neutral stability curves for this system are then obtained. For the case when the boundary below is conducting, the characteristic equation to be satisfied is

$$B = \frac{8\alpha(\alpha\cosh\alpha + L\sinh\alpha)(\alpha - \sinh\alpha\cosh\alpha)}{(\alpha^3\cosh\alpha - \sinh^2\alpha)} \tag{3.45}$$

For each value of L, the neutral stability curve given by this equation exhibits a minimum value, B_c, of B. For values of $B < B_c$ all disturbances are stable. In the case of Bénard's experiments, assuming the boundaries to be conducting below and insulating above, $B_c=80$ so that for Bénard's Cells to be surface tension driven

$$\frac{\sigma\beta d^2}{\rho\nu\kappa} > 80 \tag{3.46}$$

It is seen how the instability is governed either by the Rayleigh number or the Marangoni number according to whether it is thought to be buoyancy or surface tension driven. When the values of the Rayleigh and Marangoni numbers for Bénard's experiments are calculated it is found that although the critical Marangoni number is exceeded the critical Rayleigh number is not. The onset of instability is governed by d^4 or d^2 depending on whether it is thought to be driven by buoyancy or surface tension, respectively. For shallow flows, $d < 1$ so one would expect B_c to be exceeded rather than R_c.

Scriven and Sternling (1964) extended Pearson's theory by including the effects of surface curvature and surface viscosity. The perturbation analysis used is similar to Pearson's and the characteristic equation for neutral stability which they derive for the case of the conducting boundary below is

$$N_{MA} = \frac{8d(\alpha\cosh\alpha + N_{Nu}\sinh\alpha)[\alpha - \sinh\alpha\ \cosh\alpha + (N_{vi}/2)\alpha(\alpha^2 - \sinh^2\alpha)]}{\alpha^3\cosh\alpha - \sinh^3\alpha - N_{cr}\alpha^3\cosh\alpha} \tag{3.47}$$

where N_{MA} is the Marangoni number, N_{Nu} is the Nusselt number and corresponds with L in Pearson's theory, N_{cr} is the crispitation group which represents the effect due to curvature and is inversely proportional to the mean surface tension, and N_{vi} is the surface viscosity group which is self-explanatory. Equation (3.47) reduces to that due to Pearson's (3.45) when $N_{cr}=N_{vi}=0$. The neutral stability curves, N_{MA} against α , are then obtained for various values of N_{cr}, N_{vi}, N_{Nu}. From the curves with $N_{vi}=0$ it was seen that except in the mathematical limit $N_{cr}=0$ no critical Marangoni number exists. This model therefore predicts that the layer is always unstable to disturbances of very low wave number. However the limit $N_{cr}=0$ is a good approximation for large wave numbers and $N_{cr} < 10^{-4}$ as in the case of Benard's experiments. The stability of the layer is increased both by increasing surface tension and by increasing the surface viscosity. The increase is most marked for disturbances of low wave numbers.

Smith (1966) considered the effects of curvature on both gravity and surface tension driven waves. He found that gravity has a stabilizing effect for disturbances of small wave number and that there was a critical Marangoni number. This clarified Scriven and Sternling's analysis in which no critical Marangoni number existed. Scanlon and Segal (1967) have considered a non-linear analysis of the surface tension driven theory.

The convection cells occur when a liquid solute which lowers the surface tension of the solvent evaporates from a thin film of the solution. If the temperature, T, is interpreted as the concentration, C, of the solute, the thermal diffusion coefficient, κ , is interpreted as the diffusion coefficient, D, and the constant, q, is interpreted as the rate of loss of solute from the surface then one can use the previous analysis for the insulating boundary below to describe the cells. One must consider the gradient β to be quasi steady because a steady value for β is now impossible.

3.4 Models assuming that the flow is driven by both buoyancy and surface tension

Nield (1964) considered the effect of both buoyancy and surface tension variations. Curvature and surface viscosity were however both neglected. He used the usual linear perturbation techniques to derive a sixth order system as before. A Fourier series method is used to obtain the eigenvalue equation for the case where the lower boundary is a perfect conductor and the upper surface is subject to the more general condition. For neutral stability, values of B were computed for various values of α, R and L where α, R, B and L are defined as before. The minimum of B with respect to α was obtained by interpolation and the $(R/R_c, B/B_c)$ locus corresponding to marginal stability was plotted for each of the two cases L=0. $L=\infty$ where R_c, B_c are the values of R and B at marginal stability when B and R=0 respectively. For intermediate values of L the loci lay between the two curves.

As the critical Marangoni number decreases with increase of R, Nield deduces that the two agencies causing instability reinforce each other. He presents a table for the numerical values of B_c and a_B, the critical Marangoni number and wave number ~~and wave number~~ when R=0, and of R_c and a_r, the critical Rayleigh number and wave number when R=0, for various values of L. He noticed that the values of a_B and a_r when L=0, 1.993 and 2.086 respectively, are extremely close. Pearson quoted the value 3.5 for the wave number at the onset of instability due to buoyancy which was obtained by Jeffreys in 1926 using erroneous boundary conditions. Pearson used the difference between the predicted value and the observed value as an argument against the buoyancy mechanism which as Nield points out is no longer justified.

Assuming Bénard's cells to be driven by both buoyancy and surface tension, Cabelli and De Vahl Davies (1971) have solved the equations and boundary conditions directly by a numerical method.

CHAPTER 4

The Surface Tension Driven Flow in a Shallow Liquid Layer produced by an Insoluble Surfactant

4.1 Introduction

Levich (1962) put forward in his book a model for flow, considered to be two dimensional, in a shallow liquid layer driven by a surface tension gradient which is produced by an insoluble surface active material. Finding inconsistencies in Levich's model, Yih (1968) produced a model for such flows. In this chapter the model produced by Yih will be discussed in detail. In particular the instabilities mentioned by him will be analysed in detail and it will be shown that some of his arguments are incorrect.

If industrial liquid waste discharged into a river contains quantities of an insoluble surfactant, the flow of the river close to its surface will be affected by the presence of such material; especially as even extremely minute quantities of certain surfactants produce very large changes in the surface tension of water. Because of this, the flows considered in this chapter are of industrial importance.

4.2 Equations of Motion

Using Cartesian coordinates (x,y,z), a channel of length L is thought to connect two reservoirs, at $x=0$ where a contaminant is added, and at $x=L$. The horizontal base of the channel is at $z=0$ and the free surface of the liquid at $z=h(x)$, allowing for the variation in depth of the layer produced by the flow. The channel is assumed to be very much wider than it is long so that the motion is two dimensional, but it is sufficiently long that the edge effects at $x=0, L$ can be neglected.

The Navier Stokes equations for steady motion are

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad , \qquad (4.1)$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - g \qquad (4.2)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad , \qquad (4.3)$$

where $(u,0,w)$ are the Cartesian components of velocity. It is assumed that the assumptions for shallow water theory, $w\ll u$ and $H\ll L$, are true. If the inertial effects are negligible so that the Reynolds number, VH/ν based on a vertical height H and a vertical velocity V, is small it follows that equation (4.1) reduces to

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial z^2} \qquad . \qquad (4.4)$$

Further, if $\nu V/H^2 \ll g$ so that viscous forces in the z direction are very much smaller than gravity forces, then (4.2) reduces to

$$\frac{\partial p}{\partial z} = -\rho g \qquad . \qquad (4.5)$$

(4.4) and (4.5) are the equations used by Yih. The equation of continuity becomes

$$\frac{d}{dx}\int_0^{h(x)} u\,dz = 0 \qquad . \qquad (4.6)$$

The boundary condition at the base is simply

$$z = 0 \quad : \quad u = 0 \qquad . \qquad (4.7)$$

At the free surface the viscous stress must be equated with the surface tension gradient, so neglecting the curvature of the surface, condition (2.8),

$$z = h(x) \quad : \quad \mu \frac{\partial u}{\partial z} = \frac{d\sigma}{dx} \qquad , \qquad (4.8)$$

must be applied. Equating the normal stresses at the surface and neglecting curvature again,

$$z = h(x) \quad : \quad p = 0 \qquad . \qquad (4.9)$$

Yih assumes a form for u which satisfies (4.4), (4.7) and (4.8) given by

$$u = \frac{1}{\mu} \frac{d\sigma}{dx} z - \frac{1}{2\mu} \frac{\partial p}{\partial x} z(2h-z) \qquad . \qquad (4.10)$$

It is interesting to note the similarity between this part of the problem and lubrication theory; for an account of lubrication theory see Cameron's book (1966). Yih satisfies (4.5) and (4.9) by assuming that the pressure is hydrostatic, and given by

$$p = \rho g(h-z) \qquad . \qquad (4.11)$$

Hence the expression (4.10) for u can be rewritten

$$u = \frac{1}{\mu} \frac{d\sigma}{dx} z - \frac{1}{2\mu} \rho g \frac{dh}{dx} z(2h-z) \qquad . \qquad (4.12)$$

Substituting this expression for u in (4.6) and integrating gives the result

$$\frac{h^2}{6\mu} \left[3\frac{d\sigma}{dx} - 2 \rho g h \frac{dh}{dx} \right] = Q \qquad , \qquad (4.13)$$

where Q is the constant discharge of fluid/unit width. Using this equation the expression for the surface velocity, u_S, is

$$u_S = \frac{h}{4\mu}\left(\frac{d\sigma}{dx} + \frac{6\mu Q}{h^2}\right) \quad . \qquad (4.14)$$

From considerations of mass transport at the surface Yih derives the equation

$$u_S \sigma = D\frac{d\sigma}{dx} + q \quad . \qquad (4.15)$$

A more detailed derivation of this is now given. The surface diffusion equation (2.9) is

$$\frac{\partial \Gamma}{\partial t} + \nabla . (\Gamma u_S - D\nabla\Gamma) + j_n = 0 \quad , \qquad (4.16)$$

For steady state solutions in which the contaminant remains on the surface, $\frac{\partial\Gamma}{\partial t} = j_n = 0$. Neglecting curvature and integrating (4.16) gives the result

$$u_S \Gamma = D\frac{d\Gamma}{dx} + q' \quad , \qquad (4.17)$$

where q' is the constant discharge of surface material/unit width. It is assumed that the contaminant is added at the reservoir $x=0$, so $q' > 0$. One can assume a surface equation of state

$$\sigma - \sigma_0 = k\Gamma \qquad (4.18)$$

where σ_0 is the surface tension of the pure liquid and k is a positive or negative constant according to whether surface tension is raised or lowered by the addition of contaminant. In general insoluble contaminants lower the surface tension. Consider the following instability which would occur with an insoluble contaminant raising the surface tension. If a very small area of surface did not contain any contaminant then due to the surface tension surrounding that area being greater than that within it this area would enlarge and so such a contaminant would not therefore spread evenly over the surface. In general, therefore, $k < 0$.

If σ redefined as the relative surface tension $\sigma - \sigma_0$, the previous equations, except (4.18), remain unaltered because only derivatives of σ occur. (4.18) becomes

$$\sigma = k\Gamma \quad . \tag{4.19}$$

noting that as $k < 0$, $\sigma < 0$.

Eliminating Γ between (4.17) and (4.19) gives the result

$$u_S \sigma = D \frac{d\sigma}{dx} + q$$

where $q = kq'$. This is the equation derived by Yih, (4.15), and for the reasons given $q < 0$. Yih eliminates u_S between (4.14) and (4.15) to obtain

$$\left[\frac{h\sigma}{4\mu} - D\right] \frac{d\sigma}{dx} + \frac{3}{2} \, \frac{Q\sigma}{h} = q \quad . \tag{4.20}$$

The problem is now reduced to solving (4.13) and (4.20) together with some boundary conditions on σ and h.

4.3 Special solutions

Yih gave a solution, in complete form, for the case of zero bulk flow, $Q=0$. He also gave a solution for the case in which the surface velocity is zero; a solution which will now be shown to be contradictory.

When $u_S=0$, the expressions he obtained for σ and h^2 were

$$\sigma = \sigma_1 + \frac{\sigma_2 - \sigma_1}{L} x \quad , \tag{4.21}$$

and

$$\frac{\rho g h^2}{4} = \sigma_1 + \left(\frac{\sigma_2-\sigma_1}{L}\right) x - C_3 \quad , \qquad (4.22)$$

(σ_1,h_1) and (σ_2,h_2) being the boundary conditions in the reservoirs at $x=0$ and $x=L$, respectively. C_3 is a constant of integration. Substituting the expressions for h^2, $\frac{d\sigma}{dx}$, and $\frac{dh}{dx}$, obtained from (4.21) and (4.22), into (4.13) gives

$$\frac{4}{\rho g}\left(\sigma_1 + \frac{\sigma_2-\sigma_1}{L} x - C_3\right)\left(3\,\frac{\sigma_2-\sigma_1}{L} - 4\,\frac{(\sigma_2-\sigma_1)}{L}\right) = Q \quad . \qquad (4.23)$$

Excluding the case $\sigma_2-\sigma_1=0$ for which the problem is physically trivial, this states that a linear function of x is equal to a constant; a contradictory statement. Thus the equation of continuity, which was implicitly ignored by Yih when obtaining this solution, is not satisfied.

It is, therefore, impossible for the surface velocity to be zero. By specifying this, in addition to (4.13) and (4.20) equation (4.14) with $u_S=0$ must be satisfied. This system of 3 differential equations for only two dependent variables, σ and h, is in general insoluble.

4.4 General Solutions

For $Q \neq 0$ one can specify q, Q, L, σ_1 and h_1, and then solve numerically (4.13) and (4.20) with the boundary condition

$$x = 0 \;:\; \sigma = \sigma_1 \quad , \quad h = h_1$$

In addition one could try to vary the values of q, Q or L to fit given boundary conditions at the second reservoir. In order to consider the possible instability of solutions obtained numerically it is necessary to examine the nature of integral curves in the (σ,h), phase plane.

The phase plane equation, obtained from (4.13) and (4.20) is

$$\frac{d\sigma}{dh} = \frac{g\rho h^2[2hq-3\sigma Q]}{3h^2q-6\sigma hQ+6Q\mu D} \quad . \qquad (4.24)$$

This equation is made non-dimensional by the transformations

$$Y = \frac{3}{2}|Q|\ [6|Qq|\ \mu D]^{-1/2}\sigma \ , \ X = |q|\ [6|Qq|\ \mu D]^{-1/2}h . \qquad (4.25)$$

The non dimensional equation has two different forms according to whether Qq is positive or negative: for $Qq>0$,

$$\frac{dY}{dX} = \frac{CX^2[X-Y]}{3X^2-4XY+1} \quad , \qquad (4.26)$$

and for $Qq<0$,

$$\frac{dY}{dX} = \frac{CX^2[X+Y]}{3X^2+4XY+1} \quad , \qquad (4.27)$$

where C is a non dimensional constant,

$$C = 3g\rho \left| \left[\frac{6Q^3\mu D}{q^3}\right]\right|^{1/2} \quad .$$

As $q<0$, (4.26) must be considered when $Q<0$ and (4.27) when $Q>0$, noting that Q is positive or negative according to whether the bulk flow is in the same or the opposite direction to that of the contaminant. As h is positive and σ is negative the region of physical significance in either case is $X>0$, $Y<0$.

Equation (4.26) has two finite singular points at (1,1) and (-1,-1). The integral curves in the neighbourhood of these singular points are examined using the discussion given in the book by Davies and James (1966). With the singular point transferred to the origin and the numerator and denominator of the phase plane equation linearized so that it is of the form

$$\frac{dy}{dx} = \frac{cx+dy}{ax+by} \quad ,$$

the nature of the singular point is given by the following table:

$\Delta > 0$	$ad-bc>0$	Node ,	Stable	$a+d<0$
			Unstable	$a+d>0$
$\Delta > 0$	$ad-bc<0$	Col ,	Unstable	
$\Delta < 0$	$a + d \neq 0$	Focus,	Stable	$a+d<0$
			Unstable	$a+d>0$

where $\Delta = (a+d)^2-4(ad-bc)$.
The point (1,1) is a node or a focus according to whether $C^2-12C+4$ is positive or negative. It is stable for $C<2$, and unstable $C>2$. The point (-1,-1) is always a saddle point (Col).
The integral curves for a large value of C are sketched over the whole (X,Y) plane in Figure 3, although only those in the fourth quadrant are of physical significance. For $C<2$ a limit cycle, which necessarily contains a node or a focus, is thought to exist. It is thought further that this limit cycle will be contained within the first quadrant so that instability in the flow owing to the presence of a limit cycle is impossible (when a limit cycle is present no point inside the limit cycle can be joined by an integral curve to a point outside. This restricts the choice of possible boundary conditions). The fourth quadrant does not contain any section of the isocline $\frac{dY}{dX} = \infty$ and therefore instabilities caused by h obtaining a minimum value are also excluded from the case $Q<0$.

For $Q>0$, Equation (4.27) has no singular points in the finite plane. The integral curves, for a typical value of C , are sketched in Figure 2. Although instability due to the presence of a minimum value for h are possible.

Another flow instability to be investigated is the possibility of cusps arising in solutions; cusps being physically impossible. Writing

$$\xi = \frac{q^3}{12\mu\rho g D^2 |Qq|} x \quad , \qquad (4.28)$$

the dimensionless equations for $\frac{d\sigma}{dx}$ and $\frac{dh}{dx}$ are for $Q<0$,

$$\frac{dY}{d\xi} = \frac{C(X-Y)}{X(XY-1)} \quad , \quad \frac{dX}{d\xi} = \frac{3X^2-4XY+1}{X^3(XY-1)} \quad , \qquad (4.29)$$

and for $Q>0$,

$$\frac{dY}{d\xi} = \frac{-C(X+Y)}{X(XY-1)} \quad \frac{dX}{d\xi} = \frac{-(3X^2+4XY-1)}{X^3(XY-1)} \quad . \qquad (4.30)$$

Cusps will arise in either case where $XY-1=0$. Since this curve does not lie anywhere in the fourth quadrant, cusp type instabilities cannot arise. This contradicts Yih's discussion of cusp type instability.

4.5 Conclusion

Yih (1969) has considered the three dimensional motion of a shallow liquid layer with variable surface tension produced by an insoluble surfactant. The vertical boundaries are two parallel plates with a cylinder placed symmetrically between them. For the situation in which $\Delta\sigma >> \rho g h_0^2$, $\Delta\sigma$ being a characteristic change in σ and h_0 a vertical scale, he finds the depth and surface tension to be functionally related and shows that a simple polynomial of the depth is a harmonic function of the horizontal components x and y . When considering the flow close to the vertical boundaries, Yih postulates the existence of a velocity boundary layer with a thickness of the same order of magnitude as the depth. For the case where the angle of contact between the free surface and the boundary is $^{\pi}/2$ a solution for the velocity distribution in the layer is given.

Adler and Sowerby (1970) have also considered three dimensional flows with variable surface tension produced by an insoluble surfactant. The analysis used is also an extension of Yih's two dimensional model. Having eliminated the vertical coordinate from the set of equations to be solved by means of profiles similar to (4.10), they look for similarity solutions to the resulting pair of partial differential equations for $\sigma(x,y)$ and $h(x,y)$, x and y being the horizontal coordinates. In terms of the similarity variable $\xi(x,y)$ [If $\xi(x,y)$ is the similarity variable then σ and h assume constant values on each member of the family of curves $\xi(x,y)=\text{constant}$] the pair of ordinary differential equations obtained for $\sigma(\xi)$ and $h(\xi)$ is equivalent to (4.13) and (4.20). They consider in particular the problem of a channel flowing into a lake, with the surface contaminant flowing in either direction.

CHAPTER 5

The Surface Tension Driven Flow in a Shallow Layer of Fuel owing to a Burning Wick

5.1 Introduction

The flow when a shallow layer of liquid burns at a wick has been investigated experimentally by Burgoyne et al (1968a), the flow pattern observed consisting of an eddy close to the wick and a second region of circulation outside. A mathematical model approximating to the physical situation in this outer region is now presented.

Adler (1970) presented a model for this flow and obtained solutions by assuming that the fuel layer was subdivided into three different layers: a boundary layer near the free surface, a stagnation region in the centre of the layer and a velocity boundary layer near the base. In the model to be presented here the layer is not subdivided; the profiles assumed for the velocity and temperature being valid throughout the layer.

In the work of Burgoyne et al (1968a) it was assumed that the flow induced was buoyancy driven, whereas Glassman and Hansel (1968) assume it to be variable surface tension driven. It has been seen in Chapter 3, how in models including both buoyancy and surface tension effects for cellular convection, the surface tension effects were more dominant in very shallow layers. A similar argument justifies the neglect of buoyancy effects in the theory presented here. The variation of viscosity with temperature has also been neglected.

The liquid fuel is contained in a rectangular tray very much wider than it is long, with a plane wick immersed vertically, widthways in the liquid. The fuel is assumed to burn steadily at the wick producing a two dimensional symmetrical flow in the liquid layer. The flow produced by the flame, being surface tension driven, will be of finite extent so that if the tray is sufficiently long the effects of the edge of the tray can be ignored. No account is taken of the processes occuring in the vapour phase. It is assumed that all the evaporation takes place at the wick, outside the region of interest.

The analysis used is an adaption of Yih's for two dimensional flow with variable surface tension; a heat source being the agency producing the variation in surface tension instead of an insoluble contaminant. With suitable alterations the model here is applicable to the case where the surface tension variations are effected by a soluble contaminant.

5.2 Equations of momentum and continuity

The motion is steady and two dimensional and will be described in terms of a fixed Cartesian coordinate system Oxy. The xy plane is vertical with the base of the tray at $y=0$ and the wick at $x=0$. The surface of the liquid, which is variable, is $y=h(x)$. As the problem is symmetrical only the region $x \geq 0$ has to be investigated.

In the region away from the wick, $u \gg v$ and $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$. Because of the upward circulation current these assumptions break down close to the wick. The above are the assumptions on which Yih's analysis was based so the equations (4.4) - (4.14) are also valid here. The first order differential equations obtained from dynamical considerations only, equation (4.13), was,

$$\frac{h^2}{6\mu}\left[3\frac{d\sigma}{dx} - 2g\rho h\frac{dh}{dx}\right] = Q \qquad , \qquad (5.1)$$

$\sigma(x)$ being the surface tension and Q the discharge of fluid per unit width. The expressions obtained for the horizontal velocity, u, and u_S the surface velocity were

$$u = \frac{1}{\mu}\frac{d\sigma}{dx}y - \frac{1}{2\mu}\rho g\frac{dh}{dx}y(2h-y) \qquad , \qquad (5.2)$$

and $$u_S = \frac{1}{\mu}h\frac{d\sigma}{dx} - \frac{\rho g}{2\mu}h^2\frac{dh}{dx} \qquad . \qquad (5.3)$$

The suffix S will denote surface quantities.

5.3 Equations of heat transfer

In the shallow layer with the heat conduction in the y direction much larger than in the x direction, the convective diffusion equation becomes

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \kappa\frac{\partial^2 T}{\partial y^2}$$

The base of the layer is kept at a fixed temperature, T_1, and the heat transport from the free surface is assumed to be negligible. The condition that $T=T_1$ assumes that the base is a good conductor as in the experiments of Burgoyne and Roberts. If the model is adapted to describe the burning of a candle the temperature of the base will be the melting temperature of wax. Putting

$$\theta(x,y) = T(x,y) - T_1 \qquad ,$$

the convective diffusion equations and these two boundary conditions are written as

$$\frac{\partial}{\partial x}(u\theta) + \frac{\partial}{\partial y}(v\theta) = \kappa\,\frac{\partial^2\theta}{\partial y^2} \quad , \qquad (5.4)$$

$$y = 0 \;:\; \theta = 0 \quad , \qquad (5.5)$$

$$y = h(x) \;:\; \frac{\partial\theta}{\partial y} = 0 \quad . \qquad (5.6)$$

Integrating equation (5.4) over the depth of the layer

$$\int_0^{h(x)} \frac{\partial}{\partial x}(u\theta) + u_S\frac{dh}{dx}\,\theta_S = -\kappa\,\frac{\partial\theta}{\partial y}\bigg|_{y=0} \quad ,$$

which can be rewritten

$$\frac{\partial}{\partial x}\int_0^{h(x)} u\theta\,dy = -\kappa\,\frac{\partial\theta}{\partial y}\bigg|_{y=0} \quad . \qquad (5.7)$$

It is now necessary to choose a profile for θ compatible with the boundary conditions. The profile

$$\theta = \theta_S\,\frac{y}{h}\left(2 - \frac{y}{h}\right) \qquad (5.8)$$

satisfies the conditions (5.5) and (5.6) and at the surface $\theta=\theta_S$ as required. The expression (5.2) can be rewritten

$$u(x,y) = u_S - \frac{1}{\mu}\frac{d\sigma}{dx}(h-y) + \frac{\rho g}{2\mu}\frac{dh}{dx}(h-y)^2 \qquad (5.9)$$

Using (5.8) and (5.9)

$$\int_0^{h(x)} u\theta dy = \theta_S\int_0^{h(x)} \frac{y}{h}\left(2-\frac{y}{h}\right)\left\{u_S - \frac{1}{\mu}\frac{d\sigma}{dx}(h-y) + \frac{\rho g}{2\mu}\frac{dh}{dx}(h-y)^2\right\} dy$$

$$= h\theta_S\int_0^1 t(2-t)\left\{u_S - \frac{1}{\mu}\frac{d\sigma}{dx}h(x)(1-t) + \frac{\rho g}{2\mu}\frac{dh}{dx}h^2(1-t)^2\right\} dt$$

$$= h\theta_S\left\{\frac{2}{3}u_S - \frac{h}{4\mu}\frac{d\sigma}{dx} + \frac{\rho g}{15\mu}h^2\frac{dh}{dx}\right\} \qquad (5.10)$$

Using expression (5.10), equation (5.7) becomes

$$\frac{d}{dx}\left\{h\theta_S\left[\frac{2u_S}{3} - \frac{h}{4\mu}\frac{d\sigma}{dx} + \frac{\rho g}{15\mu}h^2\frac{dh}{dx}\right]\right\} = \frac{-2\kappa}{h}\theta_S . \qquad (5.11)$$

Substituting for u_S from equation (5.3), equation (5.11) becomes

$$\frac{d}{dx}\left\{\frac{h^2}{3\mu}\theta_S\frac{d}{dx}\left(\frac{5\sigma}{4} - \frac{2}{5}\rho g h^2\right)\right\} = \frac{-2\kappa}{h}\theta_S \qquad (5.12)$$

Assuming that surface tension decreases linearly with surface temperature according to the relation

$$\sigma - \sigma_1 = -\alpha\theta_S \quad , \alpha > 0 \qquad ,$$

σ_1 being the surface tension where the surface temperature is T_1, equation (5.12) gives the result

$$\frac{d}{dx}\left\{\frac{h^2(\sigma-\sigma_1)}{3\mu}\frac{d}{dx}\left(\frac{5}{4}\sigma - \frac{2}{5}\rho g h^2\right)\right\} = \frac{-2\kappa}{h}(\sigma-\sigma_1). \qquad (5.13)$$

The functions σ and h can be determined by solving this differential equation together with equation (5.1)

5.4 Solution

To facilitate a solution of equations (5.1) and (5.13) they are made non-dimensional by putting

$$\xi = \frac{x}{x_1} \quad , \quad H = \frac{h}{h_1} \quad , \quad S = \frac{\sigma_1-\sigma_2}{\rho g h_1} \qquad .. \qquad (5.14)$$

As the motion produced is finite, the scaling factors x_1 and h_1 arise naturally by making x_1 the distance from the wick where the ambient conditions, $T=T_1$ and $h=h_1$, prevail ~~x_1~~. The conditions at $x=x_1$ necessitate the boundary conditions

$$\xi = 1 \quad : \quad S = 0 \qquad , \qquad (5.15)$$

$$\xi = 1 \quad : \quad H = 1 \qquad . \qquad (5.16)$$

The equations (5.1) and (5.13) become, in non-dimensional form

$$\frac{d}{d\xi}\left(H^2 S \frac{d}{d\xi}\left(\frac{5}{4}S + \frac{2}{5}H^2\right)\right) = \frac{a^2 S}{H} \qquad , \qquad (5.17)$$

$$H^2\left(\frac{dS}{d\xi} + \frac{2}{3} H \frac{dH}{d\xi}\right) = b \qquad , \qquad (5.18)$$

where

$$a = \left(\frac{6\mu\kappa}{\rho g h_1{}^3}\right)^{1/2} \frac{x_1}{h_1}$$

and

$$b = \frac{\mu Q}{\rho g h_1{}^4}$$

Introduce a new variable, η , so that

$$H^2 \frac{d}{d\xi} = \frac{1}{\eta_1} \frac{d}{d\eta} \qquad , \qquad (5.19)$$

where

$$\eta_1 = \int_0^1 \frac{d\xi}{H^2} \qquad . \qquad (5.20)$$

By this definition

$$\eta = \frac{1}{\eta_1} \int_0^\xi \frac{d\xi}{H^2} \qquad , \qquad (5.21)$$

so the region of interest $0 < \xi < 1$, becomes the region $0 < \eta < 1$. The explicit relation for ξ in terms of η is

$$\xi = \frac{\int_0^\eta H^2(\eta)\, d\eta}{\int_0^1 H^2(\eta)\, d\eta} \qquad . \qquad (5.22)$$

In terms of the new variable equations (5.17), (5.18) become

$$\frac{d}{d\eta}\left(S\,\frac{d}{d\eta}\left(\frac{5}{4}S + \frac{2}{5}H^2\right)\right) = a_1{}^2 SH \quad , \qquad (5.23)$$

$$\frac{d}{d\eta}\left(S + \frac{1}{3}H^2\right) = b_1 \quad , \qquad (5.24)$$

where

$$a_1 = a\,\eta_1 \quad , \qquad (5.25)$$

$$b_1 = b\,\eta_1 \quad . \qquad (5.26)$$

The boundary conditions (5.15) and (5.16) become

$$\eta = 1 \quad : \quad S = 0 \quad , \qquad (5.27)$$

$$\eta = 1 \quad : \quad H = 1 \quad . \qquad (5.28)$$

Equation (5.24) integrates directly to give the result

$$H^2 = 1 - 3S - 3b_1(1-\eta) \quad , \qquad (5.29)$$

when the constant of integration has been determined by boundary conditions (5.28) and (5.29). Using this expression for H^2 to eliminate H from equation (5.23),

$$\frac{d}{d\eta}\left(S\,\frac{dS}{d\eta}\right) + 24b_1\,\frac{dS}{d\eta} = 20a_1{}^2 S\left[1 - 3S - 3b_1(1-\eta)\right]^{1/2} \quad . \qquad (5.30)$$

In order that equation (5.30) is satisfied when $S = 0$,

$$\frac{dS}{d\eta} = -24b_1 \quad \text{or} \quad \frac{dS}{d\eta} = 0 \quad .$$

Although $\frac{dS}{d\eta} = 0$ would give the physically realistic condition, no heat transfer across the boundary at the extent of the motion, this condition gives only the trivial solution $S(\eta) = 0$. The boundary condition to be applied is therefore

$$\eta = 1 \quad : \quad \frac{dS}{d\eta} = -24b_1 \quad . \qquad (5.31)$$

Because equation (5.30) is indeterminate at $\eta=1$ the numerical integration of (5.30) cannot be evaluated directly by means of a backward integration from $\eta=1$ so it is necessary to develop a series solution for S in the neighbourhood of $\eta=1$ in order to start the numerical integration. Suppose, therefore, that in the neighbourhood of $\eta=1$

$$S = C_1(1-\eta) + C_2(1-\eta)^2 + \dots \qquad (5.32)$$

Substituting this expression in equation (5.30), and balancing terms $O(1-\eta)^0$ yields

$$C_1^{\,2} - 24b_1C_1 = 0 \quad ,$$

the solution of which is, for reasons already stated,

$$C_1 = 24b_1 \quad .$$

Balancing terms $O(1-\eta)$ yields

$$2C_2C_1 + 2C_2C_1 = 20a_1^{\,2}C_1 \quad ,$$

so that

$$C_2 = 5a_1^{\,2}$$

Substituting for C_1 and C_2 in (5.32) gives the series expansion close to $\eta=1$ as

$$S = 24b_1(1-\eta) + 5a_1^{\,2}(1-\eta)^2 + \dots \qquad (5.33)$$

5.5 Numerical results

The functions $S(\eta)$, $H(\eta)$, $S(\xi)$, $H(\xi)$ have been evaluated numerically for different values of a and b using a CDC 6600 computer.

For fixed a_1 and b_1, the values of S and $\frac{dS}{d\eta}$ were evaluated at $\eta = 0.99$ using the series (5.33). With these values as boundary values, the function $S(\eta)$ in $0 < \eta < 0.99$ was evaluated using a Runge-Kutta method. The function $H(\eta)$ was calculated similtaneously using (5.29). The explicit relation for ξ is, using (5.22) and (5.29),

$$\xi = \frac{\eta(1-3b_1+\frac{3}{2}b_1\eta) - 3\int_0^\eta S d\eta}{1-\frac{3}{2}b_1 - 3\int_0^1 S d\eta} \tag{5.34}$$

so that the function $S(\eta)$ has been obtained throughout the range $0 < \eta < 1$ enabling $\int_0^1 S d\eta$ to be evaluated, the function $\xi(\eta)$ can be found, and hence the functions $S(\xi)$, $H(\xi)$.

It is necessary to find the values of the parameters a and b, which are groupings of the physical parameters, corresponding to values of a_1 and b_1. From the definition of a_1, (5.25), and the expression (5.29) for H^2,

$$a = \frac{a_1}{1-\frac{3}{2}b_1-3\int_0^1 S d\eta} , \tag{5.35}$$

and similarly

$$b = \frac{b_1}{1-\frac{3}{2}b_1-3\int_0^1 S d\eta} . \tag{5.36}$$

By varying the values of a_1 and b_1 the functions $S(\xi)$ and $H(\xi)$ can be obtained for different values of a and b, typical curves being shown in Figures 4, 5 and 6, those in Figure 4 being for the case when the sink is absent. It is seen that H is a monotonic increasing and S a monotonic decreasing function of X. This implies that the depth and surface tension of the liquid both increase monotonically with distance from the wick.

The expression for the non dimensional surface velocity, U_S, $U_S = \frac{\mu x_1}{\rho g h_1{}^3} u_S$, is found from (5.3) to be

$$U_S = -H(\frac{dS}{d\xi} + \frac{1}{2} H \frac{dH}{d\xi}) \quad .$$

This can be written in terms of η and S only using (5.21) and (5.29)

$$u_S = \frac{-1}{4H\eta_1}(\frac{dS}{d\eta} + 3b_1) \quad . \qquad (5.37)$$

Using this expression the surface velocity can be obtained as a function of ξ for different values of a and b ; a typical $u_S(\xi)$ curve is shown in Figure 7. From this curve it can be seen how the surface velocity, which is away from the wick throughout, decays as x increases in agreement with experimental evidence.

5.6 Discussion

On examining the numerical solutions obtained it is seen that for values of a_1 and b_1 which are not small the expression (5.29) for H^2 becomes negative at some η_2, $0 < \eta_2 < 1$. Although physically the solution obtained does not apply close to the wick where the assumptions on which this theory is based break down, it is necessary to be able to obtain the functions $H(\eta)$ and $S(\eta)$ throughout the region $0 \leq \eta \leq 1$ in order to obtain the functions $H(\xi)$, $S(\xi)$ in the region of interest. The values of a_1 and b_1 used must therefore be small.

As the value of a_1 is increased for fixed b_1, the value of $H(0)$ decreases until a value a_{1C} is reached where $H(0)$ is zero. No solution exists with this value of b_1 for $a_1 > a_{1C}$. For different values of b_1, the values of a_{1C} decrease as b_1 is increased from zero. The maximum value of b_1 for which solutions can exist is given by the case $a_{1C}=0$ and the maximum value of b_1 possible is found to be $b_1 = \frac{1}{75}$. It follows that for all solutions to the problem $0 < b_1 < \frac{1}{75}$, $0 < a_1 < a_{1C}(b_1)$; the function $a_{1C}(b_1)$ is shown in Figure 8.

From the results obtained it is seen that $\frac{\partial a}{\partial a_1}, \frac{\partial b}{\partial b_1} > 0$ and $\frac{\partial a}{\partial b_1}, \frac{\partial b}{\partial a_1} < 0$. Hence the maximum possible value for $a(a_1,b_1)$ is $a(a_{1C}(0),0)$ and the maximum value for $b(a_1,b_1)$ is $b(\frac{1}{75},0)$. By evaluating these values of a and b the restriction placed on those parameters is $0 < a < 0.220$, $0 < b < 0.0067$.

A Marangoni number for this flow is given by

$$Ma = \frac{\rho g \, h_1^{\,3}}{\mu \kappa}$$

so that the parameter a can be rewritten

$$a = \frac{\sqrt{6}}{Ma^{1/2}} \quad \frac{x_1}{h_1}$$

For flows governed by surface tension effects, the Marangoni number is large, so, because the ratio x_1/h_1 is not very large the parameter a is small. As the source strength Q is small the parameter b must also be small. It is seen that the physical constraints on a and b are compatible with the mathematical ones.

This solution with $u \neq 0$ at $x = 0$ cannot be valid close to the wick, even if the strong upward current there were ignored, because of the no slip condition at the wick. A separate theory is needed to describe the flow close to the wick.

CHAPTER 6

The Surface Tension Driven Flow in a Corner

6.1 Introduction

In the previous chapter it was stated that when a shallow liquid fuel layer burns at a wick the flow pattern divides the liquid layer into two regions. The flow in the region away from the wick was considered in the previous chapter and in this one the flow in the region close to the wick is investigated. Local solutions, valid only in the region close to the wick are obtained.

It is observed when a candle burns that close to the wick the depth of liquid increases as the distance from the wick decreases. The Reynolds number for the flow will be small and by assuming the surface to be planar close to the wick, the flow there will be slow viscous flow in an acute angled corner. Flows of this nature have been investigated by Dean and Montagnon (1941) and Moffatt (1964 a,b).

Bouyancy effects and variations of viscosity with temperature are neglected throughout.

6.2 Basic dynamical equations

To describe the two-dimensional steady flow in the corner, polar coordinates (r,θ) are introduced with the plane $\theta=0$ being the wick and $\theta=\alpha$ the surface. In the case when the sink is negligible α will be the contact angle.

The flow can be described by a stream function ψ which, as the Reynolds number is small, must satisfy the biharmonic equation

$$\nabla^4 \psi = 0 \tag{6.1}$$

Following Dean and Montagnon, and Moffatt, solutions of (6.1) can be of the form $r^n f_n(\Theta)$ provided that

$$f_n(\Theta) = A \cos n\Theta + B \sin n\Theta + C \cos(n-2)\Theta + D \sin(n-2)\Theta,$$
$$\text{for } n = 0,\ 1,\ 2.$$

$$f_o(\Theta) = A \cos 2\Theta + B \sin 2\Theta + C\Theta+D \quad . \qquad (6.3)$$

(c.f. Moffatt's expression for $f_o(\Theta)$).

$$f_1(\Theta) = A\cos\Theta + B \sin\Theta + C\ \Theta\cos\Theta + D\ \Theta\sin\Theta \qquad (6.4)$$

$$f_2(\Theta) = A \cos 2\Theta + B \sin 2\Theta + C\Theta+D \quad . \qquad (6.5)$$

Moffatt (1967a) considers the problem of slow viscous flow in a corner when one of the boundaries is rigid and on the other a constant stress τ is applied. Moffatt's analysis is extended by assuming that the stress applied at the surface, produced by the surface tension gradient there, can be expressed as,

$$\tau = \tau_o + \tau_1 r + \tau_2 r + \ .\ .\ . \quad . \qquad (6.6)$$

The velocity in polar coordinates, expressed in terms of ψ, is $\left(\frac{1}{r}\frac{\partial\psi}{\partial\Theta},\ -\frac{\partial\psi}{\partial r}\right)$. The boundary conditions at the wick are

$$\Theta=0: \quad \frac{\partial\psi}{\partial\Theta} = \frac{\partial\psi}{\partial r} = 0 \quad , \qquad (6.7a,b)$$

and the normal velocity being zero at the surface implies

$$\Theta=\alpha: \quad \frac{\partial\psi}{\partial r} = 0 \qquad (6.6)$$

Equating the viscous stress tangential to the surface with the applied stress τ gives the condition

$$\Theta=\alpha: \quad \frac{1}{r}\frac{\partial}{\partial\Theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\Theta}\right) = \frac{\tau}{\nu\rho} \quad . \qquad (6.9)$$

6.3 Solution for ψ in terms of τ when the sink is negligible

An expression for ψ is sought as a function of the unknown coefficients τ_n for the case where the sink is negligible. The expression for τ suggests the following form for ψ,

$$\psi = \frac{\tau_0 r^2}{\nu\rho} f_2(\theta) + \frac{\tau_1 r^3}{\nu\rho} f_3 \quad (\theta) + \ldots. \tag{6.10}$$

This solution will satisfy (6.1) provided the $f_n(\theta)$ are of the form (6.5) for n=2 and (6.2) for n>2. The boundary conditions (6.7a,b), (6.8) and (6.9) will be satisfied if each $f_n(\theta)$ satisfies $f_n(0) = 0$, $f_n(\alpha) = 0$, $f_n'(0) = 0$, $f_n''(\alpha) = 1$

(6.11 a,b,c,d)

For each n, these four conditions give a set of four non-homogeneous equations, determining the four arbitary constants A_n, B_n, C_n, and D_n in the expressions for $f_n(\theta)$. They are found to be;

$$\begin{aligned}
A_2 &= \frac{1}{4} \frac{(\sin 2\alpha - \alpha)}{(2\alpha\cos 2\alpha - \sin 2\alpha)} \\
B_2 &= \frac{1}{4} \frac{(1 - \cos 2\alpha)}{(2\alpha\cos 2\alpha - \sin 2\alpha)} \\
C_2 &= \frac{1}{2} \frac{(\cos 2\alpha - 1)}{(2\alpha\cos 2\alpha - \sin 2\alpha)} \\
D_2 &= \frac{(\alpha - \sin 2\alpha)}{(2\alpha\cos 2\alpha - \sin 2\alpha)} \quad ,
\end{aligned} \tag{6.12}$$

and for n>2,

$$\begin{aligned}
A_n &= \frac{1}{\Delta_n} \left[\frac{n}{(n-2)} \sin (n-2)\alpha - \sin n\alpha\right] \\
B_n &= \frac{1}{\Delta_n} \left[\cos n\alpha - \cos (n-2)\alpha\right] \\
C_n &= \frac{1}{\Delta_n} \left[\sin n\alpha - \frac{n}{(n-2)} \sin (n-2)\alpha\right] \\
D_n &= \frac{1}{\Delta_n} \frac{n}{(n-2)} \left[\cos(n-2)\alpha - \cos n\alpha\right] \quad .,
\end{aligned} \tag{6.13}$$

where $\Delta_n = \frac{4(n-1)}{(n-2)} \left[n \sin 2\alpha - 2 \cos (n-2)\alpha \sin n\alpha\right]$.(6.14)

The stream function has now been expressed in terms of α and the unknown coefficients τ_i.

6.4 Zero stress solutions

When $\tau = 0$ the boundary condition (6.11d) becomes $f_n''(\alpha) = 0$, so that the set (6.11) becomes homogeneous. In this case, there can be a non trivial solution for the constants A_n, B_n, C_n and D_n and hence a non trivial solution for ψ, if and only if $\Delta_n=0$. Because zero stress solutions for the cases n=0, 1 or 2 would not depend on this condition they were considered separately before this conclusion was reached. It was seen that $f_1(\Theta)$ and $f_2(\Theta)$ could not satisfy the homogeneous set, and the case n=0, the only one in which the physical conditions (6, 7a,b), (6.8) and (6.9) do not reduce to (6.11), would be a zero stress solution but it would also represent a sink in the flow, a case which is excluded here. $\Delta_n=0$ when

$$(n-1) \sin 2\alpha = \sin 2(n-1)\alpha \quad . \qquad (6.15)$$

This equation was investigated by Moffatt and shown to have no real solutions for $\alpha<78^{\circ}$, which is assumed to hold in this problem. The ~~least~~ positive real ~~values~~ parts of the complex roots were shown to increase from 3.84 as α decreased from 78°.

Whatever the stress may be, any zero stress solutions can be superimposed on the solution obtained. As r is small, it has been shown that the dominant zero stress solution $r^m f_n(\Theta)$ must have $m>3.84$ so it can be seen that the flow will be dominated by the first two terms in the expression for ψ already obtained. The zero stress solutions become more important as r increases since they represent the effects of the flow far from the corner region on the flow close to it.

6.5 Heat Transfer Close to the Wick

The convective diffusion equation in polar coordinates is

$$\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\theta}\right) = \kappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}\right) \quad . \qquad (6.16)$$

The wick is assumed to be an insulator so that,

$$\theta = 0: \quad \frac{\partial T}{\partial\theta} = 0 \quad . \qquad (6.17)$$

Close to the wick it is assumed that the heat flux into the liquid layer is uniform, giving a condition at the surface

$$\theta = \alpha: \quad \kappa\rho c\,\frac{1}{r}\,\frac{\partial T}{\partial\theta} = H \quad , \qquad (6.18)$$

where H is the amount of heat crossing a unit area of the surface; κ is the thermal diffusivity, ρ the density and C the specific heat of the liquid.

Because of the form for ψ, it is seen that the convective diffusion equation will admit solutions of the form $r^n\,\Theta(\theta)$ and a superpositioning of such solutions. The series

$$T = T_o + rg_1(\theta) + r^2g_2(\theta) + \ldots . \qquad (6.19)$$

gives a valid solution to the convective diffusion equation provided that the $g_n(\theta)$ satisfy the sequence of ordinary differential equations obtained by balancing powers of r when (6.19) is substituted in (6.16). The differential equations obtained by matching powers of r^{-1}, r^o and r^1 are

$$g_1'' + g_1 = 0 \quad , \qquad (6.20)$$

$$g_2'' + g_2 = 0 \quad , \qquad (6.21)$$

$$\frac{\tau_0}{\rho\nu}(f_2'g_1 - 2f_2g_1') = 9g_3 + g_3'' \quad . \qquad (6.22)$$

The boundary conditions will be satisfied provided

$$g_1'(\alpha) = \frac{H}{\rho\kappa c} \quad , \quad g_n'(\alpha) = 0 \text{ for } n \geqslant 2 \qquad (6.23)$$

and $$g_n'(0) = 0 \quad \text{for all } n. \tag{6.24}$$

The sequence of differential equations and the boundary conditions (6.23) and (6.24) can be solved directly giving the results

$$g_1(\Theta) = -\frac{H}{\kappa} \text{ cosec}\alpha \cos\Theta \quad . \tag{6.25}$$

and $$g_2(\Theta) = 0 \tag{6.26}$$

Using the expression (6.25) for $g_1(\Theta)$, (6.22) can now be solved with the boundary conditions, giving the result

$$g_3(\Theta) = \frac{\tau o H}{\rho\kappa c} \text{ cosec } \alpha \left[-A_2 \cot 3\alpha - \text{cosec } 3\alpha \left(\frac{5}{16} C_2 \sin\alpha + \frac{1}{4} \alpha\cos\alpha + \frac{A_2 \cos\alpha}{2} \right) \cos 3\Theta - \frac{A_2 \sin 3\Theta}{6} + \left(C_2\cos\Theta - \frac{1}{4} C_2 \Theta\sin\Theta + \frac{A_2\sin\Theta}{2} \right) \right] , \tag{6.27}$$

the constants A_2 and C_2 are the coefficients of $f_2(\Theta)$ which are given by (6.12).

It is seen that $g_1(\Theta)$ is independent of convective effects so that heat transfer close to the wick is mainly by conduction when the sink is neglected.

6.6 Determination of τ

The stress of τ is produced by the surface tension gradient. The surface tension will decrease linearly with increasing temperature so that the stress is related to the surface temperature gradient according to

$$\frac{dT}{dr}(r,\alpha) = -\frac{\tau}{\gamma} \quad , \tag{6.28}$$

where γ is a constant. Substituting the series expansions for $T(\alpha)$ and τ into (6.28),

$$g_1(\alpha) + 2rg_2(\alpha) + \ldots = \frac{1}{\gamma}(\tau_o + \tau_1 r + \tau_2 r^2 + \ldots) \quad , \tag{6.29}$$

so that when the functions $g_n(\theta)$ have been determined, the values of τ_n can be found by balancing powers of r in (6.29). Thus equating the constants ~~of~~ to zero,

$$\tau_0 = \frac{H\gamma}{\rho c \kappa} \cot \alpha \qquad , \qquad (6.30)$$

and equating the coefficients of r to zero,

$$\tau_1 = 0 \qquad . \qquad (6.31)$$

When these expressions for τ_i have been found, ψ and T can each be expressed as a series in powers of r with coefficients expressed in terms of known physical parameters.

6.7 Solution when a sink is present.

At a burning wick, the liquid fuel is vapourized at the surface and passes across a very thin inert gas layer, where it is supposed that heat transfer is by conduction only so that (6.18) holds, into the combustion region above where mixing with the atmosphere and combustion take place. The model assumes that all the vapourized fuel leaves the liquid at the corner.

The basic equations in §6.2 must be satisfied together with the condition representing the effect of the sink,

$$\psi_{\text{wick}} - \psi_{\text{surf}} = Q \qquad , \qquad (6.32)$$

where Q is the strength of the sink (Q>0). The angle between the surface and the wick, α, will no longer be the contact angle. Very close to the corner the typical velocity will become very large so that the Reynolds number will no longer be small. This model therefore breaks down extremely close to the corner.

Suppose that the expression for ψ in the presence of a sink is

$$\psi = f_0(\theta) + \frac{\tau_0 r^2}{\nu\rho} f_2(\theta) + \frac{\tau_1 r^3}{\rho\nu} f_3(\theta) + \ldots \quad , \qquad (6.33)$$

where $f_0(\theta)$ is of the form (6.3) and the coefficients A_n, B_n, C_n and D_n in the expressions for $f_n(\theta)$ when $n \geqslant 2$ are given by (6.12) or (6.13). It satisfies the biharmonic equation and boundary conditions (6.7b) and (6.8). The conditions (6.7a), (6.9) and (6.32) require

$$f_0'(0) = 0, \; f_0''(\alpha) = 0 \text{ and } f_0(\alpha) - f_0(0) = Q \qquad (6.34)$$

The definition of a stream function allows the constant D_0, in the expression $f_0(\theta)$, to be set zero so that these three conditions (6.34) determine the constants A_0, B_0 and C_0 giving the result

$$f_0(\theta) = \frac{Q}{(2\alpha - \tan s\alpha)} \quad (\sin 2\theta - 2\theta - \tan 2\alpha \cos 2\theta),$$

so that

$$\psi = \frac{Q}{(2\alpha - \tan 2\alpha)} \quad (\sin 2\theta - 2\theta - \tan 2\alpha \cos 2\theta) +$$
$$+ \frac{\tau_0 r^2 f_2}{\nu\rho} (\theta) + \frac{\tau_1 r^3}{\nu\rho} f_3 (\theta) + \ldots \qquad (6.35)$$

An exact solution for the Navier-Stokes equation for flow in a wedge shaped region is given in Langlois (1964). The solution which is due to Hamel shows qualitatively that flow with a sink differs from the corresponding flow with a source, a fact which, due to the neglect of inertia effects, the biharmonic equation does not predict.

When a sink is present the heat transfer equation (6.16) and the boundary conditions (6.17) and (6.18) must still be satisfied so that as in §6.5 a solution for T of the form

(6.19),

$$T = T_o + rg\ (\theta) + r^2 g_2(\theta) + \ldots\ ,$$

is sought. Substitution of this expression into (6.16) and balancing powers of r^{-1} gives

$$g_1 f_o' = g_1 + g_1'' \quad . \qquad (6.36)$$

In general, no analytic solution to this equation can be found. As the strength of the sink, Q, is small suppose that

$$g_1(\theta) = G_{10}(\theta) + QG_{11}(\theta) + Q^2 G_{12}(\theta) + \ldots \quad . \qquad (6.37)$$

Substituting (6.37) into (6.36) and equating coefficients of Q^o gives

$$G_{10}'' + G_{10} = 0$$

which has the solution satisfying the boundary conditions (6.23) and (6.24) on $g_1(\theta)$,

$$G_{10} = \frac{-H}{\kappa\rho c} \operatorname{cosec}\alpha \cos\theta \quad . \qquad (6.38)$$

Substituting (6.37) into (6.36) and equating coefficients of Q gives

$$G_{11}'' + G_{11} + \frac{2H\left[\cos 2\theta - 1 + \tan 2\alpha \sin 2\theta\right]}{\kappa\rho c\ (2\alpha - \tan 2\alpha)} = 0\ ,$$

the solution of this equation and conditions (6.23) and (6.24) being

$$G_{11}\ (\theta) = \frac{H \operatorname{cosec}^2\alpha}{4\kappa(2\alpha - \tan 2\alpha)} \Big[-7 \sin\alpha \cot\alpha \tan 2\alpha - 3 \sin 3\alpha\ +$$

$$3 \tan 2\alpha \cos 3\alpha + \tan 2\alpha \cos \alpha - 4\alpha \tan 2\alpha \sin\alpha\Big] \cos\theta$$

$$- \frac{7}{4} \frac{H \operatorname{cosec}\alpha \tan 2\alpha \sin\theta}{\kappa\ (2\alpha - \tan 2\alpha)} + \frac{H \operatorname{cosec} \alpha}{\kappa(2\alpha - \tan 2\alpha)} \times$$

$$\left[\frac{1}{4} \cos 30 + \frac{1}{4} \tan 2\alpha \sin 3\theta + \tan 2\alpha \cos\theta\right] \quad . \qquad (6.39)$$

The differential equation for $g_2(\theta)$, obtained by equating the coefficients of r to zero after (6.19) has been substituted in (6.16) is

$$2g_2f_o' = 4g_2 + g_2'' \quad . \qquad (6.40)$$

Trying for a solution

$$g_2(\theta) = G_{20}(\theta) + QG_{21}(\theta) + Q^2G_{22}(\theta) + \ldots ,$$

it can be shown that the solution of (6.40) compatible with the boundary conditions is

$$g_2(\theta) = 0 \quad .$$

The coefficients τ_i can be eliminated from the expressions for ψ and T , as in §6.6, so that they can be expressed as a series in r with coefficients in terms of known physical parameters and the unknown angle α . Each τ_i will be obtained as a series in powers of Q, those for τ_o and τ_i being

$$\tau_o = \frac{\gamma H \cot \alpha}{\rho c \kappa} - Q\gamma G_{11}(\alpha) - 2Q^2\gamma G_{12}(\alpha) + \ldots, \qquad (6.41)$$

$$\tau_1 = 0 \qquad (6.42)$$

6.8 Discussion

The corner solution must in general be matched to the solution in the outer region, obtained in Chapter 5. This problem is similar to that of laminar incompressible flow past a flat plate, where the solution valid near the leading edge has to be matched to the flow downstream. In his paper Davis (1967) uses a series truncation method on the full Navier Stokes equations in order to overcome the matching problem and such an approach might be used here.

CHAPTER 7

The Surface Tension Driven Flow in a Rectangular Cell owing to a Heat Source in a Corner

7.1 Introduction

A line heat source placed on the surface of a liquid layer contained in a tray will produce motion in the layer of a two dimensional nature which is driven by buoyancy and surface tension effects. Essentially, a problem such as this arises when a liquid is heated steadily at the centre of a pool of fuel, as in the paper of Murad et al (1970).

The theory presented here assumes that the buoyancy forces are negligible compared to the surface tension forces. If a model were developed including buoyancy effects as well as surface tension effects, a Rayleigh number would represent buoyancy whilst a Marangoni number would represent surface tension. For shallow layers the Rayleigh number has less influence on the regime that the Marangoni number since it is proportional to the depth cubed and the Marangoni number is proportional to the depth; cf chapter 3 on Cellular Convection for Bénards Cells.

The steady motion produced will be of a cellular nature and of finite extent. The motion outside the two symmetrically placed cells adjacent to the source will be negligible and is therefore neglected. It is only necessary to investigate the motion in one of these two cells, taken to be of rectangular shape. To do this, a virtual heat source which passes all its heat into this cell is thought to be placed actually inside the cell and as close as possible to the position of the real source.

This cell is assumed to be of aspect ratio 1, in order to reduce the number of parameters, but the analysis is easily adapted for other aspect ratios. In the experiments of Murad et al it was 5. In stability flows Malkus (1954a,b) has proposed a criteria: that the fluid will tend to flow in a manner which maximises heat transfer. Although the criterion has been confirmed in a proof, for small amplitudes only, given by Schulter et al (1965), it has been questioned by Foster (1969) who concluded that it was not valid at low Rayleigh numbers for two dimensional flow. For the two dimensional flow, with the Rayleigh number zero, considered here one must have reservations about assuming that the actual aspect ratio could be obtained from Malkus's criteria.

As usual variations of viscosity with temperature are neglected.

7.2 Equations of motion and heat transfer

The motion induced is thought to be two dimensional so a fixed Cartesian coordinate system $0\,\bar{x}\,\bar{y}$ is introduced. A bar will denote dimensional quantities throughout this chapter. The $\bar{x}\,\bar{y}$ plane is vertical with the base of the layer at $\bar{y} = 0$ and the surface, assumed horizontal, at $\bar{y} = L$. The origin lies in the base, vertically below the position of the real source. The virtual source is at the point $(L\delta, L[1-\delta])$, $\delta \ll 1$. The motion created in the layer by this virtual source is within the boundaries $x=0$ and $x=L$.

The Reynolds number for the motion in the cell is small so the stream function, $\bar{\psi}$, satisfies

$$\nabla^4 \bar{\psi} = 0 \quad , \qquad (7.1)$$

the biharmonic equation. By the definition of the cell, there can be no motion across any of the boundaries of the cell and as the base is rigid there will be no motion along the base of the cell. As there is no motion across the base of the cell, the base is a streamline ψ = constant; and because a stream function is unaffected by the addition of a constant, this constant may be set zero for convenience. As the Cartesian velocity components are $(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x})$ the aforementioned boundary conditions can be written

$$\bar{y} = 0 \quad : \quad \bar{\psi} = \frac{\partial\bar{\psi}}{\partial\bar{y}} = 0 \qquad , \qquad (7.2a, 7.2b)$$

$$\bar{y} = L \quad : \quad \frac{\partial\bar{\psi}}{\partial\bar{x}} = 0 \qquad , \qquad (7.3)$$

$$\bar{x} = 0, L \quad : \quad \frac{\partial\bar{\psi}}{\partial\bar{y}} = 0 \qquad . \qquad (7.4a, 7.4b)$$

The Heat Transfer equation, which holds everywhere except at the source, can be written in the form

$$\frac{\partial(T, \bar{\psi})}{\partial(\bar{x}, \bar{y})} = \kappa \nabla^2 T \qquad , \qquad (7.5)$$

T being the temperature. The base is assumed to be a pure conductor so that throughout it is at a constant temperature T_0; and at the surface the heat transfers by radiation and conduction are assumed negligible compared to the strength of the heat source. Due to the existence of the symmetrically placed cell there is no heat transfer across the boundary $\bar{x} = 0$ and as the motion outside this cell is thought to be negligible it is consistent to argue that the heat transfer across $\bar{x} = L$ is negligible. The boundary conditions on temperature are, therefore,

$$\bar{y} = 0 \quad : \quad T = T_o \qquad , \qquad (7.6)$$

$$\bar{y} = L \quad : \quad \frac{\partial T}{\partial \bar{y}} = 0 \qquad , \qquad (7.7)$$

$$\bar{x} = 0, L \quad : \quad \frac{\partial T}{\partial \bar{x}} = 0 \qquad . \qquad (7.8a, 7.8b)$$

The surface stress must be equated to the surface tension gradient so

$$\bar{y} = L \quad : \quad \mu \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = \frac{d\sigma}{dx} \qquad . \qquad (7.9)$$

If the surface tension, σ , and surface temperature are related by an equation

$$(\sigma - \sigma_o) = \lambda (T_o - T_S) \qquad , \qquad (7.10)$$

where the suffix s denotes a surface quantity, so that

$$\frac{d\sigma}{d\bar{x}} = -\lambda \frac{dT_S}{d\bar{x}} \qquad , \qquad (7.11)$$

the condition(7.9) becomes

$$\mu \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = -\lambda \frac{dT_S}{d\bar{x}} \qquad . \qquad (7.12)$$

7.3 Solution

The system of equations and boundary conditions (7.1)-(7.8) and (7.12) is made non dimensional by putting

$$\begin{aligned} \psi &= UL\bar{\psi} \quad , \\ \theta &= \frac{T - T_o}{\Delta T} \quad , \\ x &= L\bar{x} \quad , \\ y &= L\bar{y} \quad , \end{aligned} \qquad (7.13)$$

where ΔT is the temperature difference across the cell and U is a typical velocity. The equations of momentum and heat transfer become

$$\nabla^4 \psi = 0 \qquad , \qquad (7.14)$$

and
$$\frac{\partial(\theta,\psi)}{\partial(x,y)} = \frac{1}{Pe}\nabla^2\theta \qquad , \qquad (7.15)$$

where Pe, the Peclet number, is given by

$$Pe = \frac{UL}{\cancel{\nu}}\ \kappa$$

The boundary condition (7.12) becomes

$$-\frac{Pe}{Ma}\ \frac{\partial^2\psi}{\partial y^2} = \frac{\partial\theta}{\partial x} \qquad ; \qquad (7.16)$$

where Ma, the Marangoni number, is given by

$$Ma = \frac{\lambda \Delta T L}{\mu\kappa} \quad .$$

A solution of the biharmonic equations subject to the non-dimensional forms of boundary conditions (7.2b), (7.4a), (7.4b) is

$$\psi = \sin n\pi x\ [\sinh n\pi y + y(a \sinh n\pi y + b \cosh n\pi y)] \quad , \qquad (7.17)$$

where n is an integer. The parameters a and b are determined by the conditions (7.2a) and (7.3) to give

$$\psi = \sin n\pi x[\sinh n\pi y + y([n\pi \coth n\pi - 1]\sinh n\pi y - n\pi \cosh n\pi y)] \quad , \qquad (7.18)$$

as a solution of the biharmonic equation with all the boundary conditions (7.2)-(7.4). Clearly a superpositioning of such solutions is a solution so that

$$\psi = \sum_{n=1}^{\infty} A_n \sin n\pi x \left\{[\sinh n\pi y \sinh n\pi](1-y) + y \sinh n\pi(y-1)\right\} \quad , \qquad (7.19)$$

where the coefficient A_n are to be determined. For convenience this expression can be rewritten as

$$\psi = \sum_{n=1}^{\infty} A_n \sin n\pi x \, f_n(y) \quad ,$$

with $f_n(y) = (1-y) \sinh n\pi \sinh n\pi y + y \sinh n\pi(y-1)$.

The problem has been formulated in terms of two parameters; the Marangoni and Peclet numbers so clearly any solutions obtained must depend explicitly on these parameters. If the Peclet number is assumed small it is reasonable to suppose that ψ and θ can each be expressed as a series in powers of the Peclet number. From boundary condition (7.16) it is necessary that $\theta \sim Pe\psi$. Even though when the Peclet number is small, the heat transfer by convection is negligible compared to that by conduction, nevertheless, the magnitudes of the velocities in the convection currents need not be small. It is therefore clear that the expression for ψ and θ should be of the form

$$\begin{aligned} \psi &= \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \dots \quad , \\ \theta &= \varepsilon\theta_1 + \varepsilon^2\theta_2 + \dots \quad , \end{aligned} \qquad (7.20)$$

where $\varepsilon = Pe$.

In order that the expression for ψ already obtained can be written in the form of equation (7.20) each of the unknown constants A_n in that expression must be expanded as a series in ε ;

$$A_n = \sum_{r=0}^{\infty} \varepsilon^r A_{nr} \ .$$

The expression for each ψ_r is then

$$\psi_r = \sum_{n=0}^{\infty} A_{nr} \sin n\pi x \left\{(1-y)\sinh n\pi \sinh n\pi y + y \sinh n\pi(y-1)\right\} . \tag{7.21}$$

The result of making the substitutions (7.20) into the equation of heat transfer (7.15) and balancing the powers of ε is the sequence of equations

$$\nabla^2\theta_1 = 0 \quad , \tag{7.22}$$

$$\frac{\partial(\theta_1, \psi_o)}{\partial(x,y)} = \nabla^2\theta_2 \quad , \tag{7.23}$$

with the general equation for $n \geq 2$ being

$$\sum_{r=1}^{n-1} \frac{\partial(\theta_r, \psi_{n-r-1})}{\partial(x,y)} = \nabla^2\theta_n \quad . \tag{7.24}$$

Each of the functions θ_n satisfy the same set of homogeneous boundary conditions as θ , which are

$$\begin{aligned} y = 0 \quad &: \quad \theta = 0 \quad , \\ y = 1 \quad &: \quad \frac{\partial\theta}{\partial y} = 0 \quad , \\ x = 0,1 \quad &: \quad \frac{\partial\theta}{\partial x} = 0 \quad . \end{aligned} \tag{7.25}$$

With the expansions (7.20) the boundary condition (7.16) yields the condition

$$y = 1 \quad : \quad \frac{\partial^2\psi_n}{\partial y^2} = -\mathrm{Ma}\, \frac{\partial\theta_{n+1}}{\partial x} \quad , \tag{7.26}$$

for each n .

In order to find the functions θ_n the Greens Function for Laplaces equation and the given homogeneous boundary conditions (7.25) is required. This can be written as an expansion involving the eigenfunctions and eigenvalues for the corresponding solution to Poisson's equation with the same boundary conditions. A detailed derivation is given in Appendix A and the Green's function obtained is

$$G(x,y,x',y') = \frac{16}{\pi^2}\sum_{K=1}^{\infty}\sum_{m=0}^{\infty}\frac{\cos K\pi x \sin\frac{(2m+1)}{2}\pi y \cos K\pi x' \sin\frac{(2m+1)}{2}\pi y'}{4K^2+(2m+1)^2} . \tag{7.27}$$

The function θ_1 satisfies, Laplaces equation everywhere except at $(\delta, 1-\delta)$ where the source is present, and the boundary condition (7.25). Hence

$$\theta_1 = \frac{16}{\pi^2} H \sum_{K=1}^{\infty}\sum_{m=0}^{\infty}\frac{\cos K\pi\delta \sin\frac{(2m+1)}{2}\pi(1-\delta)\cos K\pi x \sin\frac{(2m+1)}{2}\pi y}{4K^2+(2m+1)^2} , \tag{7.28}$$

H being the constant non-dimensional rate of heat produced by the virtual source. It makes this expression simpler if it is rewritten;

$$\theta_1 = \sum_{K=1}^{\infty}\sum_{m=0}^{\infty} B_{Km} \sin\frac{(2m+1)}{2}\pi y \cos K\pi x , \tag{7.29}$$

B_{Km} being constants. The boundary condition (7.26) with $n = 0$ is

$$y = 1 \; : \; \frac{\partial^2 \psi_o}{\partial y^2} = -Ma \frac{\partial \theta_1}{\partial x} .$$

This condition is true if

$$\sum_{n=1}^{\infty} A_{no} f_n''(1) \sin n\pi x = -Ma \sum_{n=1}^{\infty}\left(\sum_{m=0}^{\infty} B_{nm}(-1)^m\right)\sin n\pi x .$$

Equating coefficients of $\sin n\pi x$, the boundary condition is true if, for each n,

$$A_{n0} = \frac{nMa\left(\sum_{m=0}^{\infty}(-1)^m B_{nm}\right)}{f_n''(1)} \quad . \qquad (7.30)$$

This condition determines the function ψ_o.

Making the substitutions for the now knownfunctions θ_1 and ψ_o, equation (7.23) can be written

$$\nabla^2\theta_2 = -\,\phi(x,y) \quad , \qquad (7.31)$$

where $\phi(x,y)$ is a known function. θ_2 satisfies the same set of homogeneous boundary conditions as the Greens function (7.26) for Laplaces equation. From the fundamental property of Green's functions, it follows that a solution of equation (7.31), with those boundary conditions is

$$\theta_2 = \int_0^1\int_0^1 \phi(x',y')\, G(x,y,x',y')\,dx'dy' \quad ,$$

or $$\theta_2 = -\int_0^1\int_0^1 G(x,y,x',y')\,\frac{\partial(\theta_1(x',y'),\,\psi_o(x',y'))}{\partial(x',y')}\,dx'\,dy' \quad .$$

Hence

$$\theta_2 = \sum_{K=1}^{\infty}\sum_{m=0}^{\infty} c_{Km} I_{Km} \cos K\pi x \,\sin\frac{(2m+1)}{2}\pi y \quad , \qquad (7.32)$$

where $$c_{Km} = \frac{4}{\pi}\sqrt{\frac{1}{4K^2+(2m+1)^2}} \quad ,$$

and I_{Km} =

$$-\frac{4K}{3\pi}A_{OK}c_{Km}\sum_p B_{Kp}(m-p)(m+p+1)\left[\frac{1}{K^2+(m+p+1)^2} - \frac{1}{(K^2+(p-m)^2)^2}\right]\times$$

$$\left[(-1)^{m+p+1}\left[\tfrac{1}{2}\sinh 2K\pi+1\right]+\sinh 2K\pi+\cosh K\pi\right]$$

if $K=1,3,5 \ldots$

or $I_{Km}=0$ if $K=2,4,6 \ldots$.

The boundary condition (7.26) can now be used to show that the coefficients in the expansion of ψ_1 satisfy

$$A_{n1} = \frac{M_{an}}{f_n''(1)} \sum_{m=1}^{\infty} (-1)^m c_{nm} I_{nm} \quad . \quad (7.33)$$

When the functions θ_r, ψ_{r-1} have been determined for all $r \leq n-1$, the equation (7.24) can be written in the form

$$\nabla^2\theta_n = \oint (x,y) \quad ,$$

which is the form of equation (7.31) so that the method used to find θ_2 and ψ_1 can be repeated to find θ_n and ψ_{n-1}. In this way the perturbation series for θ and ψ can be developed.

7.4 Discussion

The stream function and temperature have each been obtained as a series in powers of the Peclet number. A solution, in complete form, which is valid for small Pe and throughout the entire cell has been found.

As the Marangoni number tends to zero, which only occurs realistically when $\Delta T \to 0$, it can be seen from the solution obtained that the magnitude of the characteristic velocity U must also tend to zero thereby making the Peclet number tend to zero and hence θ tend to zero. Schematically,

$$\Delta T \to 0 \Rightarrow Ma \to 0 \Rightarrow U \to 0 \Rightarrow Pe \to 0 \Rightarrow \theta \to 0 .$$

Thus the limiting case when $Pe = 0$ in the expansions must occur simultaneously with $Ma = 0$

No direct assumption that the layer is shallow has been necessary although buoyancy effects will become more significant as the depth of the layer is increased.

CHAPTER 8

Surface Tension Driven Flow owing to a Moving Heat Source

8.1 Introduction

The problems in the three preceding chapters have all been concerned with fixed heat sources. In this chapter the motion and heat transfer processes set up in a liquid layer, over which a constant line heat source passes with uniform velocity, are considered. By supposing that the flame is a line source of heat, this model can be applied to that of flame spread over a liquid surface.

It is assumed that the motion in the layer is surface tension driven and that buoyancy effects can be neglected. Variations of viscosity with temperature, evaporation, and curvature effects are also assumed negligible.

Experiments have shown, eg Glassman and Hansel (1968), that two mechanisms control the rate of flame spread above a liquid fuel according to whether the liquid fuel temperature is above or below its closed flash point. Above the flash point the vapour phase mechanisms are controlling, below it the liquid phase mechanisms. Burgoyne and Roberts (1968c) imply that the motion in the liquid layer was buoyancy driven. The reason why the dominant driving mechanism in shallow heated liquid layers is variable surface tension rather than buoyancy has been discussed previously. Because Glassman and Hansel showed that the layer of convecting fluid away from the flame is thin, it is reasonable to suppose that the motion is surface tension driven as in the paper of Glassman and Hansel. This model is, therefore, applicable to the case of a flame propagating over a shallow liquid fuel below its flash point.

8.2 Statement of the problem

As found experimentally by Glassman and Hansel, the effect of the source on the motion of the fluid is mainly confined to a region close to the surface. The layer has, therefore, two characteristic regions which will be considered separately and then matched.

A theoretical model is considered in which a line heat source placed across the width of the tray, travels lengthways over the liquid layer in the tray, so that the motion is two dimensional. The tray is assumed to be of infinite length. The problem is formulated in the frame in which the heat source is at rest so that a steady state problem occurs. With (x,y) as Cartesian coordinates, the x axis is such that the fixed frame moves with velocity $\overset{+U}{\cancel{+u}}$ in the positive x-direction. The y axis is vertical and the origin is vertically below the heat source. The horizontal and vertical components of velocity will be u and v.

8.3 Equations of motion in the boundary layer

In the boundary layer it is assumed that the conditions, $Re \ll 1$, $u \gg v$, and $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$, for shallow water theory hold. These are the assumptions of Yih (1968), and the analysis given here is Yih's analysis adapted for a moving frame.

The Navier-Stokes equations reduce to

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \quad , \quad \frac{\partial p}{\partial y} = -\rho g \qquad (8.1a,b)$$

The equation of continuity and the boundary conditions on v are unaltered in the moving frame so that, as in Yih's paper, the equation of continuity reduces to

$$\frac{\partial}{\partial x}\int_0^{h(x)} u \, dy = 0 \quad . \qquad (8.2)$$

The boundary condition at the bottom of the layer is

$$y = 0 \; : \; u = U \quad . \qquad (8.3)$$

At the free surface the conditions are

$$\mu \frac{\partial u}{\partial y} = \frac{d\sigma}{dx} \qquad (8.4)$$

and

$$p = 0 \quad . \qquad (8.5)$$

(8.1a), (8.3) and (8.4) are satisfied by assuming

$$u = \frac{1}{\mu}\frac{d\sigma}{dx}\, y - \frac{1}{2\mu}\frac{\partial p}{\partial x}\, y(2h-y) + U \quad . \qquad (8.6)$$

The hydrostatic approximation,

$$p = \rho g(h-y) \quad , \qquad (8.7)$$

is made in order to satisfy (8.1b) and (8.5). Equations (8.2), (8.6) and (8.7) combine to give

$$\frac{h^2}{6\mu}\left[3\frac{d\sigma}{dx} - 2g\rho h\frac{dh}{dx}\right] + Uh = Q \quad , \qquad (8.8)$$

where Q is a constant of integration which represents the flux of liquid at infinity,

$$Q = Uh_{\infty} \quad . \qquad (8.9)$$

Using (8.7) and (8.6) the expression for the surface velocity, u_S , is

$$u_S = \mathbf{U} + \frac{h}{\mu}\left[\frac{d\sigma}{dx} - \frac{h\rho g}{2}\frac{dh}{dx}\right] \tag{8.10}$$

Throughout, the suffix S will be used to denote surface quantities.

These equations are comparable with the hydrodynamic equations in the paper of Sirignano and Glassman. They write the Navier-Stokes equations in non-dimensional form as

$$\mathrm{Re}\left(\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}\right) + N\frac{\partial\bar{p}}{\partial\bar{x}} = \frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\partial^2\bar{u}}{\partial\bar{y}^2} \quad , \tag{S1}$$

$$\mathrm{Re}\left(\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}\right) + N\frac{\partial\bar{p}}{\partial\bar{y}} = \frac{\partial^2\bar{v}}{\partial\bar{x}^2} + \frac{\partial^2\bar{v}}{\partial\bar{y}^2} - N \quad , \tag{S2}$$

where N is the ratio of gravity forces to surface tension forces, $N=R/F^2$ where F is the Froude Number. A variable with a bar will denote non dimensional form of that variable. Because $N \gg 1$, (S2) becomes

$$\frac{\partial\bar{p}}{\partial\bar{y}} = 1 \quad ,$$

which is the non dimensional form of (8.1b). By assuming $N \gg \mathrm{Re}$ and $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$, (S1) becomes

$$N\frac{\partial\bar{p}}{\partial\bar{x}} = \frac{\partial^2\bar{u}}{\partial\bar{x}^2} \quad .$$

which is the non dimensional form of (8.1a). Sirignano and Glassman have obtained (8.1a) by assuming $N \gg \mathrm{Re}$ instead of $R \ll 1$. All the other assumptions made here, $u \gg v$, $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$, p is hydrostatic, and the form (8.6) for u , are also used by them. Their final hydrostatic equation, which will be discussed in § 8.8, is the non-dimensional form of (8.8) when Q has been set equal to Uh_∞.

8.4 Heat transfer equations in the boundary layer

The equation of heat transfer in the boundary layer, under the assumptions $\frac{\partial}{\partial y} >> \frac{\partial}{\partial x}$ and $u >> v$ becomes

$$u \frac{\partial \overline{T}}{\partial x} + v \frac{\partial \overline{T}}{\partial y} = \kappa \frac{\partial^2 \overline{T}}{\partial y^2} , \qquad (8.11)$$

where $\overline{T}$ is the temperature. Define $T(x,y)$ by

$$T(x,y) = \overline{T}(x,y) - \overline{T}_o ,$$

$\overline{T}_o$ being the temperature far away from the heat source. Assume a profile for $T(x,y)$ of the form

$$T(x,y) = f(\frac{h-y}{\Delta(x)})\ T_S(x) , \qquad (8.12)$$

where $\Delta(x)$ is the thickness of the boundary layer. The velocity components in the boundary layer can be written

$$\begin{aligned} u &= u_S(x) + O(h-y) , \\ v &= u_S(x) \frac{dh}{dx} + O(h-y) . \end{aligned} \qquad (8.13)$$

Substitution of (8.13) and (8.12) into (8.11) gives

$$u_S f \frac{dT_S}{dx} + \frac{u_S}{\Delta} \frac{dh}{dx} f' T_S = \frac{\kappa}{\Delta^2} f'' T_S + O(h-y) , \qquad (8.14)$$

where $'$ denotes differentiation with respect to the variable $(\frac{h-y}{\Delta})$. The function f can be expressed as the Taylor Series

$$f(\frac{h-y}{\Delta}) = 1 + (\frac{h-y}{\Delta})f'(0) + (\frac{h-y}{\Delta})^2 \frac{f''(0)}{2!} + \ldots .$$

Heat passes through the free surface where the line heat source is present. The heat losses, due to conduction, evaporation and radiation at the surface, are taken to be negligible, which has been shown to be true experimentally by Burgoyne and Quinton (1968a) for the static heat source case. The above condition can be written

$$\frac{\partial\theta}{\partial y} = R\ \delta(x) \qquad ,$$

$\delta(x)$ being the Dirac delta function and κcR is equal to the constant rate of heat emitted by the source, c being the specific heat of the liquid. It follows that, except at $x = 0$,

$$f'(0) = 0 \qquad ,$$

so $$f\left(\frac{h-y}{\Delta}\right) = 1 + \left(\frac{h-y}{\Delta}\right)^2 f''(0) + \ldots \qquad .$$

Ignoring terms $O(h-y)$, equation (8.14) now becomes

$$u_S \frac{dT_S}{dx} = \frac{\kappa}{\Delta(x)^2} \quad f''(0)\ T_S \qquad . \qquad (8.15)$$

All the heat from the source must pass through the base of the boundary layer, so

$$\int_{-\infty}^{+\infty} \left.\frac{\partial T}{\partial y}\right|_{y=h-\Delta(x)} dx = R \qquad ,$$

which using (8.12) gives the result

$$-f'(1) \int_{-\infty}^{+\infty} \frac{T_S}{\Delta(x)}\ dx = R \qquad . \qquad (8.16)$$

A mean boundary layer thickness δ_1 can be defined by

$$\delta_1 = \frac{\int_{-\infty}^{+\infty} T_S \, dx}{\int_{-\infty}^{+\infty} \frac{T_S}{\Delta(x)} \, dx} \quad . \qquad (8.17)$$

It will be assumed that the boundary layer thickness $\Delta(x)$ can be replaced by its mean value δ_1. This approximation produces considerable simplification in the subsequent analysis. It is also necessary to choose the function f so that as many observables as possible are satisfied. The conditions at the surface, $f(0)=1$ and $f'(0)=0$, will be satisfied by assuming

$$f\left(\frac{h-y}{\delta_1}\right) = \cos\frac{\pi}{4\delta_1}(h-y) \quad . \qquad (8.18)$$

This function decreases monotonically with decreasing depth in agreement with observation. With these assumptions equation (8.15) becomes

$$u_S \frac{dT_S}{dx} = \frac{-\kappa}{16\delta_1^2} \pi^2 T_S \quad . \quad ,$$

which can be rewritten

$$u_S \frac{dT_S}{dx} = -\kappa \lambda^2 T_S \quad , \qquad (8.19)$$

where $\lambda = \frac{\pi}{4\delta_1}$. (8.20)

Equation (8.19) could have been obtained, as follows, by making directly the assumptions which have already been made implicitly. When terms of $O(h-y)$ are neglected in equation (8.15) it is implied that u and v take their surface values. Making this assumption directly on equation (8.11), with $T=\overline{T}(x,y)-\overline{T}_0$,

$$u_S \left[\frac{\partial T}{\partial x} + \frac{dh}{dx}\frac{\partial T}{\partial y} \right] = \kappa \frac{\partial^2 T}{\partial y^2} \quad .$$

Trying for a separable solution of the form $X(x)F(h(x)-y)$, which is assumed by equation (8.12) together with the constant boundary layer thickness assumption, the heat transfer equation becomes

$$u_S X' = \kappa F''X \quad .$$

This separates into

$$u_S X'(x) = -\lambda^2 \kappa X$$

and

$$F'' = -\lambda^2 F \quad ,$$

where λ^2 is constant. By assuming that $F(0)=1$, an assumption made in the form of the Taylor series,

$$X(x) = T_S(x) \quad ,$$

so that the separated equation for $\underline{x}$ becomes (8.19).

8.5 Solution in the boundary layer

Using the usual relation between surface tension and surface temperature,

$$\sigma-\sigma_0 = -\alpha T_S \quad , \qquad (8.21)$$

where α is a positive constant and σ_0 is the surface tension at infinity; equation (8.9) can be rewritten in terms of T_S and h as

$$\frac{-h^2}{6\mu}\,[3\alpha T_S+2g\rho h\, h'] + Uh = Q \quad . \qquad (8.22)$$

Substituting for u_S in equation (8.19), using (8.10), yields another equation for T_S and h, namely

$$[U - \frac{h}{\mu}\,\frac{d}{dx}(\alpha T_S+ \frac{1}{4}\,\rho g h^2)]\; T_S' = -\lambda^2\kappa T_S \quad . \qquad (8.23)$$

This pair of first order ordinary differential equations for T_S and h must be solved together with the boundary conditions at infinity which are,

$$x \rightarrow \pm\infty \qquad T_S \rightarrow 0 \quad , \qquad (8.24)$$

$$x \rightarrow \pm\infty \qquad h \rightarrow h_\infty \quad . \qquad (8.25)$$

The condition, $\frac{dh}{dx} = \frac{dT_S}{dx} = 0$ when $T_S=0$ and $h=h_\infty$, required by these conditions is satisfied by equations (8.22) and (8.23)

The system (8.22)-(8.25) is made non-dimensional by putting

$$\theta = \alpha\,[\frac{\lambda^4\kappa^2}{\mu^2 U^4\rho g}]\; T_S \quad , \qquad (8.26)$$

$$H = [\frac{\lambda^2\kappa\rho g}{\mu U^2}]^{1/3}\; h \quad , \qquad (8.27)$$

$$X = \frac{\lambda^2\kappa}{U}\; x \quad . \qquad (8.28)$$

It becomes

$$H - \frac{1}{2}\,H^2\,\frac{d}{dX}\,(\theta + \frac{1}{3}\,H^2) = B \quad , \qquad (8.29)$$

$$[1 - H\,\frac{d}{dX}\,(\theta + \frac{1}{4}\,H^2]\,\frac{d\theta}{dX} = -\,\theta \quad , \qquad (8.30)$$

$$X \rightarrow \pm\infty \;:\; \theta \rightarrow 0 \quad , \qquad (8.31)$$

$$X \to \pm\infty \quad : \quad H \to B \qquad , \qquad (8.32)$$

where B is the non-dimensional parameter

$$B = Q \left[\frac{\lambda^2 \rho g}{\mu U^5}\right]^{1/3} \qquad .$$

The only parameter in this non dimensional system is B . From (8.29) and (8.30) the equations for $\frac{d\theta}{dX}$ and $\frac{dH}{dX}$ are

$$\frac{d\theta}{dX} = -\frac{1}{H}\left(1-\frac{3B}{H}\right) \pm \frac{1}{H}\sqrt{\left(1-\frac{3B}{H}\right)^2 + 4H\theta} \qquad , \qquad (8.33)$$

and $$\frac{dH}{dX} = \frac{3}{2H^2}\left(3-\frac{5B}{H^2}\right) \mp \frac{3}{2H^2}\sqrt{\left(1-\frac{3B}{H}\right)^2 + 4H\theta} \qquad . \qquad (8.34)$$

From physical considerations, the temperature gradient is positive in front of the source and negative behind it. It follows that the negative root in (8.33) should be taken in the region X positive and the positive root where X is negative.

Behind the source the system to be solved is, therefore,

$$\frac{d\theta}{dX} = -\frac{1}{H}\left(1-\frac{3B}{H}\right) - \frac{1}{H}\sqrt{\left(1-\frac{3B}{H}\right)^2 + 4H\theta} \qquad , \qquad (8.35)$$

$$\frac{dH}{dX} = \frac{3}{2H^2}\left(3-\frac{5B}{H}\right) + \frac{3}{2H^2}\sqrt{\left(1-\frac{3B}{H}\right)^2 + 4H\theta} \qquad , \qquad (8.36)$$

$$X \to +\infty \quad : \quad \theta \to 0 \qquad , \qquad (8.37)$$

$$X \to +\infty \quad : \quad H \to B \qquad . \qquad (8.38)$$

Because the boundary conditions are to be applied at infinity, an analysis of the (θ,H) phase plane is necessary before a numerical solution to this system can be found.

The phase plane equation

$$\frac{dH}{d\Theta} = \frac{\frac{3}{2H^2}\left(3-\frac{3B}{H}\right) + \frac{3}{2H^2}\sqrt{\left(1-\frac{3B}{H}\right)^2 + 4H\Theta}}{-\frac{1}{H}\left(1-\frac{3B}{H}\right) + \frac{1}{H}\cdot\sqrt{\left(1-\frac{3B}{H}\right)^2 + 4H\Theta}} \quad , \quad (8.39)$$

has a singular point (0,B) which is a saddle point and corresponds to the points at infinity in the (H,X) and (Θ,X) planes. Hence the required integral curve in the phase plane is a separatrix. One separatrix is the line $H=B$ which corresponds to a trivial solution to the problem, and the other separatrix of physical significance has the equation

$$H = B - \frac{3}{2}\,\frac{B^2\Theta}{(3+B^3)} \quad , \quad (8.40)$$

close to the saddle point. Close to the source at $\Theta=\Theta^*$, $\Theta^* \ll 1$, $H(\Theta^*)$ can be evaluated using (8.40) and with the condition $\Theta=\Theta^* : H=H(\Theta^*)$, equation (8.39) can be solved numerically to give the equation of the separatrix. Assuming that the effective temperature at the source is Θ_F , the value for H at the source, H_F , can be obtained from the equation of the separatrix. Using the boundary conditions

$$X = 0 \quad : \quad \Theta = \Theta_F \qquad H = H_F \quad , \quad (8.41)$$

together with equations (8.35) and (8.36), numerical solutions for $\Theta(X)$ and $H(X)$ in the region $X \geq 0$ can be obtained for arbitrary B and Θ_F .

In the region $X \leq 0$, the positive root is to be taken in (8.33) so that the equations for $\frac{dH}{dX}$ and $\frac{d\Theta}{dX}$ are

$$\frac{d\theta}{dX} = -\frac{1}{H}\left(1-\frac{3B}{H}\right) + \frac{1}{H}\sqrt{\left(1-\frac{3B}{H}\right)^2 + 4H\theta} \qquad (8.42)$$

and $$\frac{dH}{dX} = \frac{3}{2H^2}\left(3-\frac{5B}{H}\right) - \frac{3}{2H^2}\sqrt{\left(1-\frac{3B}{H}\right)^2 + 4H\theta} \quad . \qquad (8.43)$$

The functions H and θ must be continuous at $X=0$ although their gradients will be discontinuous because of the input of heat there. The boundary condition to be applied to (8.42) and (8.43) is therefore (8.41). A solution in the region $X \leq 0$ must also satisfy the downstream conditions (8.31) and (8.32). The condition, $\theta = \frac{d\theta}{dX} = 0$ as $X \to -\infty$, is not compatible with equation (8.42) because $H_\infty = B$ and when $\theta = \frac{d\theta}{dX} = 0$, equation (8.42) requires $H > 3B$. However, $\theta(x)=0$ is a solution to (8.29) and (8.30) but it was discarded when deriving (8.33) and (8.34). When $\theta=0$, (8.30) is satisfied identically and (8.29) gives a differential equation, for H only, satisfied by

$$\tfrac{1}{3}(H^3-H_1{}^3) + \tfrac{1}{2}B(H^2-H_1{}^2) + B^2(H-H_1) + B^3\log\frac{(H-B)}{(H_1-B)} = 3(X-X_1) \quad , \qquad (8.44)$$

where $H=H_1$ at $X=X_1$. This expression satisfies the condition $H \to B$ as $X \to -\infty$. This expression (8.44) for H together with $\theta=0$ is a valid solution to the problem in the region $x \leq x_1$. If a numerical solution to equations (8.42) and (8.43) with boundary conditions (8.41), is obtained with $\theta(x_2)=0$ and $H(x_2) > 3B$; then by putting $x_1=x_2$ and $H(x_2)=H_1$ these two solutions match together at $x=x_1$ to give a solution valid throughout $x \leq 0$.

8.6 Numerical results for solution in boundary layer

The numerical results to the method given in §8.5 were obtained using the CDC 6600 at London University. The numerical integrations were performed using the trapezoidal rule.

Provided that they gave solutions with $\theta(x_2)=0$ and $H(x_2) > 3B$, solutions for $H(X)$ and $\theta(X)$, in $-\infty \leq x \leq \infty$, were obtained for arbitrary values of θ_F and B. A typical solution for $H(X)$ and $\theta(X)$ is shown in Figure 9. It is seen how the fluid is pushed up into the form of a solitary wave in front of the source and is depressed behind it. The non dimensional surface velocity $\bar{u}_S(x)$,.

$$u_S(X) = \frac{\bar{u}_S(x)}{u} = 1 - H\left[\frac{d\theta}{dX} + \frac{1}{2} H \frac{dH}{dX}\right] \qquad , \qquad (8.45)$$

was obtained for arbitrary B and θ_F when valid solutions for $H(X)$ and $\theta(X)$ had been obtained. Figure 10 shows a typical surface velocity curve. It is seen from this curve that the fluid flows away from the source throughout the region behind the source. Ahead of the source it flows away only as far as a stagnation point which occurs where $H=3B$, after which it flows towards the source throughout.

Also calculated for arbitrary values of θ_F and B was the value of A,

$$A = \int_{-\infty}^{\infty} \theta \, dX \qquad .$$

From (8.16) it follows that

$$A = \sqrt{2}\left[\frac{1}{\mu^2 \kappa \rho g}\right]^{1/3} \left(\frac{\lambda}{u}\right)^{7/3} R \qquad . \qquad (8.46)$$

Thus Λ is fixed by the heat source strength $\kappa c \cdot R$ and hence the effective source temperature is fixed by the source strength From the non dimensional solution in the boundary layer the dimensional solution can easily be deduced. It will be obtained in terms of two parameters, λ and λ/U .

8.7 Solution in the region outside the boundary layer

By the definition of the boundary layer the motion in this region produced by the source is negligible. As $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$ the heat transfer equation becomes

$$U \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} \quad , \qquad (8.47)$$

$T = \overline{T} - T_0$ as before. To make this equation non dimensional

$$\phi = \alpha \sqrt{2} \left[\frac{\lambda^{4}\ 2}{\mu^{2} u^{4} \rho g} \right] T \quad , \qquad (8.48)$$

$$Y = \left[\frac{\lambda^{2}\ \rho g}{\mu\ u^{2}} \right]^{1/3} y \quad , \qquad (8.49)$$

$$\varepsilon = \left[\frac{\rho g}{\lambda \mu u^{2}} \right]^{1/3} \quad . \qquad (8.50)$$

The boundary layer thickness being small $\lambda \gg 1$, so that $\varepsilon \ll 1$. With these definitions, the heat transfer equation becomes

$$\frac{\partial \phi}{\partial x} = \varepsilon^{2} \frac{\partial^{2} \phi}{\partial Y^{2}} \quad . \qquad (8.51)$$

The solution obtained in this stagnation region must be matched to the solution in the boundary layer above by matching the temperature at the interface and the heat flux across it.

In terms of the non dimensional variables these two conditions are

$$Y = H - \frac{\pi}{4}\varepsilon \quad : \quad \phi = \varepsilon\frac{\partial\phi}{\partial y} = \theta(x) \qquad . \qquad (8.52)$$

A Laplace transform method was used to try and solve (8.51) with the boundary condition (8.52), as shown in Appendix B , but when the Laplace transform of the solution had been found its inverse could not be obtained. This method did not, therefore, give a solution but the following Pohlhausen method does.

Using the Pohlhausen method, a profile is chosen for ϕ

$$\phi = \theta\left\{1 + \frac{1}{\varepsilon\rho(x)}\left(1 - e^{\rho\left(H - \frac{\pi}{4}\varepsilon - Y\right)}\right)\right\} \qquad , \qquad (8.53)$$

which satisfies the boundary conditions (8.52) and in which $\rho(x)$ is a parametric function. The integral of (8.51) across this layer is

$$\frac{\partial}{\partial x}\int_0^{H(x) - \frac{\pi}{4}\varepsilon} \phi \, dY - \theta\frac{dH}{dX} = \varepsilon\theta - \varepsilon^2\left.\frac{\partial\phi}{\partial Y}\right|_{Y=0} .$$

Substitution for ϕ reduces this to

$$\frac{d}{dX}\left\{\frac{\theta}{\rho}\left(H - \frac{\pi}{4}\varepsilon\right) + \frac{\theta}{\rho^2}\left(1 - e^{\rho\left(H - \frac{\pi}{4}\varepsilon\right)}\right)\right\} + \varepsilon\left(H - \frac{\pi}{4}\varepsilon\right)\frac{d\theta}{dX} = \varepsilon^2\theta\left\{1 - e^{\rho\left(H - \frac{\pi}{4}\varepsilon\right)}\right\} . \qquad (8.54)$$

Define ξ by

$$\xi = \rho\left(H - \frac{\pi}{4}\varepsilon\right) \qquad , \qquad (8.55)$$

so that (8.54) becomes

$$\frac{d}{dX}\left\{\theta\left(H - \frac{\pi}{4}\varepsilon\right)^2\left(\frac{1 + \xi - e^{\xi}}{\xi^2}\right)\right\} + \varepsilon\left(H - \frac{\pi}{4}\varepsilon\right)\frac{d\theta}{dX} = \varepsilon^2\theta\left(1 - e^{\xi}\right) \qquad . \qquad (8.56)$$

Writing $\frac{1+\ \xi-\ e^{\xi}}{2} = \varepsilon\ \eta_1+\varepsilon^2\eta_2+\ \ldots$, (8.57)

(8.56) becomes

$$\frac{d}{dX}\left\{\theta(H-\ \tfrac{\pi}{4}\varepsilon)^2(\varepsilon\,\eta_1+\varepsilon^2\eta_2+\ \ldots)\right\} + \varepsilon(H-\ \tfrac{\pi}{4}\varepsilon)\frac{d\theta}{dX} = \varepsilon^2\theta(1-e^{\xi}) \quad . \tag{8.58}$$

It follows by considering the profile (8.53) in detail that as $\varepsilon << 1$, $\ell >> 1$ and hence $-\xi >> 1$. As $-\xi >> 1$, (8.57) shows that

$$\xi \ \underline{\frown}\ \frac{1}{\varepsilon\ \ {}_1} \quad . \tag{8.59}$$

Equating coefficients of ε in (8.58) to zero one obtains the result

$$\frac{d}{dX}\ (\theta H^2\eta_1)\ +\ H\ \frac{d\theta}{dX} = 0 \quad ,$$

which when integrated determines η_1 ,

$$\eta_1 = \frac{-1}{\theta H^2}\int_{X_1}^{X} H(\frac{d\theta}{dX})\ dX \quad .$$

Using this result and (8.58),

$$\xi = \frac{-\theta H^2}{\varepsilon\displaystyle\int_{X_1}^{X} H(\frac{d\theta}{dX})dX} \quad . \tag{8.60}$$

From this expression the parameter ℓ in the profile for ϕ is easily deduced.

The integral of (8.51) over the entire range of X reduces to

$$\varepsilon A + \int_{-\infty}^{+\infty} \theta\ \frac{dHdX}{dX} = \varepsilon^2\int_{-\infty}^{\infty}\left.\frac{\partial\phi}{\partial Y}\right|_{Y=0} dX - \int_{0}^{H(X)-\ \frac{\pi}{4}\varepsilon} \phi(X_1,Y)dY \quad .$$

Substituting from (8.53) for $\emptyset$, and using the result (8.60) this becomes

$$\varepsilon A + \int_{X_1}^{\infty} \theta \frac{dH}{dX}\, dX = \varepsilon \int_{X_1}^{\infty} \theta \exp \left[\frac{-\theta H^2}{\varepsilon \int_{X_0}^{X} H(\frac{d\theta}{dX})\, dX} \right] dX \quad . \qquad (8.61)$$

This equation determines the parameter ε . It can be solved iteratively:

$$\varepsilon_1 = - \frac{1}{A} \int_{X_1}^{\infty} \theta \, (\frac{dH}{dX})\, dX \qquad ,$$

$$\varepsilon_2 = - \frac{1}{A} \int_{X_1}^{\infty} \theta(\frac{dH}{dX}) dX + \frac{\varepsilon_1}{A} \int_{X_1}^{\infty} \theta \exp \left[\frac{-\theta H^2}{\varepsilon_1 \int_{X_1}^{\infty} H(\frac{d\theta}{dX}) dX} \right] dX \qquad ,$$

$$\varepsilon_n = - \frac{1}{A} \int_{X_1}^{\infty} \theta(\frac{dH}{dX}) dX + \frac{\varepsilon_{n-1}}{A} \int_{X_1}^{\infty} \theta \exp \quad \frac{-\theta H^2}{\varepsilon_{n-1} \int_{X_1}^{\infty} H(\frac{d\theta}{dX}) dX} \quad dX$$

if the values of ε_n converge to ε. This seems likely although a proof has not been established.

A value for ε is also given by (8.50) so that this is an eigenvalue problem with λ , which is directly related to the boundary thickness, as the eigenvalue. It can be determined numerically by comparing the values of ε obtained from (8.61) and (8.51) for different values of λ . The solutions obtained in both boundary and stagnation layers can then be written in terms of the strength and speed of the source, and the known physical constants.

In the problem of a flame propagating over a liquid fuel below its closed flash point, the speed of propagation, U , will be determined by coupling this solution with the solution in the gas phase.

8.8 Discussion

This model is comparable with that of Glassman and Sirignano for flame spread over a liquid surface. It has already been stated that their model assumes the flow to be surface tension driven and the hydrodynamic equations have been compared in §8.3. They assume that the surface tension gradient is prescribed by the flame, ignoring the heat transfer effects in the liquid layer. The only differential equation they have to solve is their non-dimensional form of equation (8.8) and the solution for h is obtained in terms of this prescribed surface tension gradient. They describe the solutions which suitable surface tension gradients would give. Their expression for h is monotonic in disagreement with this model but their surface velocity has a stagnation point ahead of the flame as it does here.

CHAPTER 9

The Surface Tension Driven Flow in a Solution on Contact with a Hot Gas

9.1 Introduction

The surface tension gradients driving the flows considered in the previous problems have been produced either by a surface temperature gradient or a surface active contaminant, but not both together as in the case considered here. The contaminant is a solute which lowers the surface tension of the solvent and is destroyed by heat, so that a hot gas on the surface will act on the solution as both a heat source and a solute sink, producing opposing effects on the flow.

In a simple experiment, Skogen (1958) has demonstrated that it is possible to increase the surface tension of a solution on contact with a hot gas in spite of the decrease in tension caused by the rise in temperature. He used solutions of water and soap or detergents, but the effect can be produced with any surface active substance provided the concentration lies within certain limits. When a flame of coal gas was directed vertically downwards towards the surface of the solution, Skogen showed that under suitable conditions the following phenomena occured: (a) a rapid flow of the surface layer from all directions towards the area of contact between flame and water; (b) a surface elevation of several millimetres at this area; (c) a downward current from the elevation. The observed phenomena are consistent with a rise in surface tension directly below the flame and this was verified by measurements on the solution. Instead of a flame, a similar effect can be produced by placing an electric arc directly above the solution, but it appears that this experiment is more difficult to perform.

The model presented here is a theoretical analysis of the Skogen effect under the assumptions that the process takes place in a shallow layer under steady state conditions. A two-dimensional symmetrical flow is considered, separated by the Oy'z' - plane, in which a thin flame sheet touching the surface of the liquid is assumed to lie. The flame sheet raises the temperature of the liquid and maintains zero concentration of contaminant along the line of contact. It is assumed that the concentration of surface active agent is everywhere small and that the physical properties of the solution are independent of temperature and concentration of contaminant. Buoyancy effects have been neglected, the flow is taken to be surface tension driven with the tension gradient maintained by heat and mass transfer processes along the layer. Such an assumption is consistent with experimental observations by Skogen, since no effect was observed when the layer became covered with a thin inert film. Under the above assumptions it will be shown that the phenomena are characterised by the rates of heat and mass transfer along the layer, Prandtl, Schmidt and Reynolds numbers and the Marangoni numbers for temperature and concentration changes.

9.2 Equations of continuity and momentum

Consider a Cartesian frame of reference Ox'y'z' with z' measured vertically from the horizontal bed of the liquid, which is locally of depth $h'(x')$. Let (u',o,w') be the components of velocity in the liquid. The continuity equation is

$$\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial z'} = 0 \qquad (9.1)$$

Dimensional quantities are denoted by primed symbols, the corresponding dimensionless variables will appear without primes. The boundary conditions at the base and free surface are

$$\begin{aligned} z' &= 0 : & w' &= 0 , \\ z' &= h' : & w' &= u' \frac{dh'}{dx'} . \end{aligned} \qquad (9.2)$$

Integration of (9.1) over z' and use of boundary conditions (9.2) gives

$$\frac{d}{dx'} \int_0^{h'} u' \, dz' = 0 \qquad (9.3)$$

Since the flow is symmetrical and the volumetric flow of contaminant may be taken to be neglible, equation (9.3) gives an integration

$$\int_0^{h'} u' \, dz' = 0 \qquad (9.4)$$

For this slow two-dimensional viscous flow in a shallow layer, the equations of Yih (1968) are again appropriate. They may be written

$$\frac{\partial p'}{\partial x'} = \mu \frac{\partial^2 u'}{\partial z'^2} , \qquad (9.5)$$

$$\frac{\partial p'}{\partial z'} = - \rho g , \qquad (9.6)$$

in which p' is the pressure, μ the viscosity, ρ the density and g the gravitational acceleration.

The boundary conditions at the base and free surface may be written

$$z' = 0 \quad : \quad u' = 0 \quad , \qquad (9.7)$$

$$z' = h' \quad : \quad \mu \frac{\partial u'}{\partial z'} = \frac{d\sigma'}{dx'} \quad , \qquad (9.8)$$

$$p' = 0 \quad , \qquad (9.9)$$

where σ' is the surface tension, it being assumed that the maximum curvature of the surface is small compared to some characteristic length.

The solution for p', from (9.6) and (9.9) is

$$p' = \rho g \, (h' - z') \qquad (9.10)$$

The solution of (9.5) subject to boundary conditions (9.7), (9.8) is

$$u' = \frac{1}{\mu} \frac{d\sigma'}{dx'} z' - \frac{1}{2\mu} \frac{\partial p'}{\partial x'} z' \, (2h' - z') \qquad (9.11)$$

Substitution in the continuity equation (9.4) and integration over z' gives

$$\frac{h'^2}{2\mu} \left(\frac{d\sigma'}{dx'} - \frac{2}{3} \rho g h' \frac{dh'}{dx'} \right) = 0 \qquad (9.12)$$

Hence
$$\sigma' = \sigma_o' + \frac{1}{3} \rho g \, (h'^2 - d^2) \quad , \qquad (9.13)$$

where d is the depth and σ_o' the surface tension at $x' = 0$. Substitution of (9.13) and (9.10) in (9.11) now gives

$$u' = \frac{\rho g}{2\mu} \frac{dh'}{dx'} z'(z' - \frac{2}{3} h') \qquad (9.14)$$

The other component of velocity may be obtained by substituting in (9.1) and integrating over z'. The result is

$$w' = \frac{\rho g}{6\mu} z'^2 \left[\left(\frac{dh'}{dx'} \right)^2 + \frac{d^2 h'}{dx'^2} (h' - z') \right] \qquad (9.15)$$

Equation (9.14) shows that near the surface, $u' \lessgtr 0$, according to whether $\frac{dh'}{dx'} \gtrless 0$, and that u' changes sign when $z' = \frac{2}{3} h'$. From (9.15) it is seen that w' is everywhere small.

9.3 Equations of heat and mass transfer

In a shallow layer, with Prandtl and Schmidt numbers large, the equations of heat and mass transfer can be written

$$u' \frac{\partial T'}{\partial x'} + w' \frac{\partial T'}{\partial z'} = K \frac{\partial^2 T'}{\partial z'^2} \quad , \qquad (9.16)$$

and

$$u' \frac{\partial c'}{\partial x'} + w' \frac{\partial c'}{\partial z'} = D \frac{\partial^2 c'}{\partial z'^2} \quad , \qquad (9.17)$$

where T' is the temperature, c' the concentration of contaminant, K the thermal diffusivity and D the diffusion coefficient for mass transfer. The temperature and concentration at the surface are related to the surface tension and hence by (9.13) to the depth. Denoting surface values with a suffix S, one can take

$$\sigma' = \sigma_o' + \alpha'(T_1' - T_S') - \beta' c_S' \quad , \qquad (9.18)$$

where T_1' is the surface temperature directly below the flame where the concentration of contaminant is assumed to vanish. α' and β' are positive constants given by

$$\alpha' = -\left(\frac{\partial \sigma'}{\partial T'_S}\right)_{\substack{T_S' = T_1' \\ c_S' = 0.}} \quad , \quad \beta' = -\left(\frac{\partial \sigma'}{\partial c'}\right)_{\substack{T_S' = T_1' \\ c_S' = 0.}} \qquad (9.19)$$

9.4 Dimensionless equations

Introduce dimensionless components of velocity

$$u = \frac{u'd}{\cancel{\nu}\,[illegible]} \quad , \quad w = \frac{w'd}{\cancel{\kappa}\,[illegible]} \quad , \qquad (9.20)$$

with ~~μ~~ ν the kinematic viscosity, and dimensionless coordinates

$$x = \frac{x'}{d} \quad , \quad z = \frac{z'}{d} \quad , \quad h = \frac{h'}{d} \quad . \tag{9.21}$$

Expressions (9.14), (9.15) for the components of velocity now become

$$u = \frac{1}{2}\,\mathrm{Re}^2\,\frac{dh}{dx}\,z\,\left(z - \frac{2}{3}\,h\right) \quad , \tag{9.22}$$

and

$$w = \frac{1}{6}\cdot\mathrm{Re}^2\,z^2\left\{\left(\frac{dh}{dx}\right)^2 + \frac{d^2h}{dx^2}\,(h - z)\right\} \quad , \tag{9.23}$$

where the Reynolds number occurring in these expressions is defined by

$$\mathrm{Re} = \left(\frac{gd^3}{u}\right)^{\frac{1}{2}} . \tag{9.24}$$

Taking T'_0 and c'_0 as reference values for the temperature and concentration, introduce dimensionless variables

$$T = \frac{T' - T_0'}{T_1' - T_0'} \quad , \tag{9.25}$$

and

$$c = \frac{c'}{c'_0} \quad , \tag{9.26}$$

in terms of which the equations of heat and mass transfer become

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \frac{1}{\mathrm{Pr}}\,\frac{\partial^2 T}{\partial z^2} \quad , \tag{9.27}$$

$$u\frac{\partial c}{\partial x} + w\frac{\partial c}{\partial z} = \frac{1}{\mathrm{Sc}}\,\frac{\partial^2 c}{\partial z^2} \quad , \tag{9.28}$$

where $\mathrm{Pr} = \frac{\cancel{\partial}\,\nu}{K}$, $\mathrm{Sc} = \frac{\cancel{\gamma}\,\nu}{D}$

Eliminating the surface tension between (9.13) and (9.18) and expressing the result in dimensionless form, one obtains

$$h^2 = 1 + \alpha\,(1 - T_s) - \beta c_s \quad , \tag{9.29}$$

in which α, β are dimensionless parameters, given by

$$\alpha = \frac{3Ma_1}{Re^2 Pr} \quad , \qquad \beta = \frac{3Ma_2}{Re^2 Sc} \qquad (9.30)$$

The Marangoni numbers for temperature and concentration changes, occurring in (9.30) are defined by

$$Ma_1 = \alpha' \frac{(T_1' - T_0')\,d}{\mu\,K} \quad ,$$
$$Ma_2 = \beta' \frac{c_0' d}{\mu D} \qquad (9.31)$$

From the definitions above it follows that the surface variables satisfy boundary conditions

$$x=0:\ h=1,\ T_s=1,\ c_s=0. \qquad (9.32)$$

It is assumed that all the heat entering the surface near the flame is convected downstream. Further, since there is no downstream mass transfer across the base and free surface, the boundary conditions on temperature and concentration are

$$x>0,\ z=0,h: \qquad \frac{\partial T}{\partial z} = \frac{\partial c}{\partial z} = 0 \quad . \qquad (9.33)$$

9.5 Solution by von Kármán-Pohlhausen method

The equations of heat and mass transfer (9.27), (9.28) are now integrated over the depth of the layer. Using the equation of continuity and in view of boundary conditions (9.33) and the relationship between the velocity components at the surface, one obtains,

$$\int_0^h uT\,dz = A \quad , \qquad (9.34)$$

$$\int_0^h uc\,dz = -B \quad , \qquad (9.35)$$

where A, B are positive constants which represent the dimensionless heat and mass fluxes along the layer, respectively.

The temperature distribution in the layer will be taken in the form

$$T(x,z) = T_b(x) + \left[T_s(x) - T_b(x)\right] f_1(t) \quad , \qquad (9.36)$$

$$t = 1 - \frac{z}{h} \quad ,$$

where $T_b(x)$, $T_s(x)$ are the temperatures along the base and free surface, respectively, and $f_1(t)$ is a suitable profile function. Similarly, the distribution of contaminant will be assumed of the form

$$c(x,z) = c_b(x) + \left[c_s(x) - c_b(x)\right] f_2(t) \quad , \qquad (9.37)$$

with $c_b(x)$, $c_s(x)$ the base and surface concentrations and $f_2(t)$ a profile function. These definitions, together with the boundary conditions on temperature and concentration, show that the profile functions $f_i(t)$, i=1,2 must satisfy the following boundary values

$$\begin{aligned} f_i(0) &= 1 \quad , & f_i(1) &= 0 \quad , \\ f_i'(0) &= 0 \quad , & f_i'(1) &= 0 \quad , \end{aligned} \qquad (9.38)$$

with primes here denoting derivatives.

Expressing (9.22) in terms of $u_s(x)$, the velocity component at the surface, one obtains

$$u = u_s(1-t)(1-3t) \quad , \qquad (9.39)$$

and similarly (9.23) can be written

$$w = u_s \frac{dh}{dx} + O(t) \quad . \qquad (9.40)$$

Substituting the above functions for temperature and concentration in equations (9.34), (9.35) and using (9.39)

$$hu_s \,(T_s - T_b) = A_1 \quad , \qquad (9.41)$$

where $A = A_1 \int_0^1 (1-t)(1-3t)\, f_1(t)\, dt$,

and $hu_s(c_s - c_b) = -B_1$, (9.42)

where $B = B_1 \int_0^1 (1-t)(1-3t)\, f_2(t)\, dt$.

Since $f_i(t)$ $i=1,2$ are specified profile functions, the constants arising in (9.41), (9.42) are just the heat and mass fluxes multiplied by numerical factors. Further information about the profile functions and other functions introduced may be obtained by direct substitution in equations (9.27), (9.28). In view of (9.38)

$$f_i(t) = 1 + \frac{1}{2!}\, t^2\, f_i''(0) + \frac{1}{3!}\, t^3\, f_i'''(0) + \ldots. \tag{9.43}$$

Substituting (9.36), (9.39), (9.40) in (9.27) and equating terms without t one obtains

$$u_s \frac{dT_s}{dx} = \frac{1}{Pr}\, \frac{f''(0)}{h^2}\, (T_s - T_b) \quad . \tag{9.44}$$

A similar equation, but with a different numerical factor on the right hand side is obtained on equating terms in t. Comparing this with (9.44), one finds that

$$f_1'''(0) = -4\, f_1''(0) \tag{9.45}$$

Similarly, substituting in (9.28) and equating powers ot t

$$u_s \frac{dc_s}{dx} = \frac{1}{Sc}\, \frac{f_2''(0)}{h^2}\, (c_s - c_b) \quad , \tag{9.46}$$

where $f_2'''(0) = -4\, f_2''(0)$. (9.47)

The temperature difference between the base and free surface can be eliminated between (9.41) and (9.44). The result is

$$h^3\, u_s^{\,2}\, \frac{dT_s}{dx} = \frac{-A_2}{Pr} \quad , \tag{9.48}$$

where $A_2 = -A_1 f_1''(0)$.

Close to the surface the vertical component of the temperature gradient will be positive or negative according to whether the surface temperature is less than or greater than the base temperature. In either case $f'(t)>0$ for small t, and examination of (9.43) then shows that $f''(0)<0$. A_2 is, therefore, a positive constant related to the dimensionless heat flux along the layer. Similarly, combination of (9.42) and (9.46) gives

$$h^3 u_s^2 \frac{dc_s}{dx} = \frac{B_2}{Sc} , \qquad (9.49)$$

where $B_2 = -B_1 f_2''(0)>0$.

Equation (9.48), (9.49) can now be substituted into the differential form of the surface relation (9.29) to give an equation for h. Since by definition

$$u_s = \frac{1}{6} Re^2 h^2 \frac{dh}{dx} , \qquad (9.50)$$

one obtains

$$h^8 \left(\frac{dh}{dx}\right)^3 = \frac{54}{Re^6} \left(\frac{Ma_1}{Pr^2} A_2 - \frac{Ma_2}{Sc^2} B_2 \right) . \qquad (9.51)$$

The solution of (9.51) with boundary condition (9.32) is

$$h = (1 + Cx)^{\frac{3}{11}} , \qquad (9.52)$$

where $$C^3 = \frac{2662}{Re^6} \left(\frac{Ma_1}{Pr^2} A_2 - \frac{Ma_2 B_2}{Sc^2} \right) .$$

The free surface is thus raised or lowered below the flame according to whether $C \lessgtr 0$. The significance of this discriminant will be discussed below. The solution for the other variables of the system is now straightforward. The surface velocity component becomes

$$u_s = \frac{1}{22} \mathrm{Re}^2 \, C \, (1 + Cx)^{-2/11} \quad , \qquad (9.53)$$

and hence from (9.48) and (9.49)

$$T_s = 1 + \frac{1}{264} \frac{A_2}{\mathrm{PrRe}^4 C3} \left\{ 1 - (1+\rho x)^{6/11} \right\} \quad , \qquad (9.54)$$

$$c_s = \frac{1}{264} \frac{B_2}{\mathrm{ScRe}^4 C^3} \left\{ (1+Cx)^{6/11} - 1 \right\} \quad , \qquad (9.55)$$

on using boundary conditions (9.32). Substitution in (9.41) and (9.42) also shows that

$$T_s - T_b = \frac{22A_1}{\mathrm{Re}^2 c} (1+Cx)^{-1/11} \quad , \qquad (9.56)$$

$$c_s - c_b = \frac{-22B_1}{\mathrm{Re}^2 C} (1+Cx)^{-1/11} \quad . \qquad (9.57)$$

To describe the temperature and contaminant distributions in the layer suitable profile functions $f_i(t)$ are required. Since the boundary conditions are similar in view of (9.45) and (9.47), take

$$f_i(t) = 1 + pt^2 - \frac{4}{3} pt^3 + qt^4 \quad , \qquad (9.58)$$

with p, q parameters to be determined. Applying boundary conditions (9.38), p = -6, q = -3, so that

$$f_i(t) = (1-t)^3(1+3t) \quad . \qquad (9.59)$$

9.6 Conclusions

From the results obtained in the previous section the following conclusions can be drawn:

(i) h increases or decreases with x according to whether $C \gtrless 0$. i.e. there is a depression or rise in the liquid surface where the heat source is applied,

(ii) the surface velocity $u_s \lessgtr 0$ according to whether $C \gtrless 0$,

(iii) the surface temperature T_s decreases with x,

(iv) $T_s - T_b \gtrless 0$ according to whether $C \gtrless 0$. e.g. when $C < 0$, $u_s < 0$ and hot liquid is convected downwards below the heat source towards the base,

(v) the surface concentration c_s increases with x,

(vi) $c_s - c_b \gtrless 0$ according to whether $C \gtrless 0$. e.g. for $C > 0$, $u_s > 0$ and the concentration of contaminant near the surface tends to be lower than that at the base,

(vii) for similar profile functions of the form (9.59), the discriminant C is given by

$$C^3 = \frac{210 \times 11^3}{Re^6}\left(\frac{Ma_1}{Pr^2} A - \frac{Ma_2}{Sc^2} B\right) ,$$

in which A, B are the dimensionless heat and mass fluxes along the layer, respectively.

Reference to equation (9.13) shows that a rise in the surface below the heat source is associated with an increase in surface tension, and from conclusion (i) C is negative in this case. The discriminant above shows that this condition is satisfied when mass transfer is dominant compared to heat transfer or when relatively large changes in surface tension are associated with small changes in the concentration of contaminant ($Ma_2 >> Ma_1$). Although both these conditions are likely to be satisfied in a Skogen type experiment, more measurements are required in order to substantiate the dependence of the flow on the discriminant C.

CHAPTER 10

General Conclusion

The models produced in Chapters 5-9 could all be extended by considering the effects of buoyancy, the surface curvature and the variation of viscosity with temperature as was done by many workers for the Bénard cells problem. The effect of buoyancy is probably the most significant of the three. The models could also be improved by considering, where applicable, the effects of distributed instead of concentrated heat and mass losses.

Further analysis of variable surface tension driven flows is needed in connection with several current problems of practical importance. From the safety aspect, a problem of industrial importance is: will a spilt liquid fuel below its flash point permit a flame to spread if an ignition source is present? The layer spilt will probably be shallow and therefore the motion in the layer, which determines whether or not a flame will spread, will be driven by variable surface tension.

Heat exchanges are becoming of increasing importance in engineering, particularly in connection with nuclear reactors. In some devices, coolant passes under pressure through a tube so that the external heat flux produces a central core of vapour and a thin annular liquid film on the inside of the tube. Heat transfer takes place by conduction through the liquid film, followed by evaporation on the interface. In the phenomena of 'burnout', the liquid becomes locally unstable, the absence of the liquid film produces a sharp increase in temperature with resultant catastrophic breakdown of the tube wall. Although the phenomenon of burnout is complex, c.f. Symposium on two-phase flow dynamics (1967), surface tension effects must play a role in producing annular film breakdown. This is therefore another problem to be investigated in connection with variable surface tension driven flows.

The effect of pollution on our environment is becoming increasingly more important. The presence of surface active material in industrial waste being discharged into a river will affect the flow of the surface of that river. This could be of particular importance where the waste mixes with the river water. A report was given in Science Journal (1969) of an experiment in which an oil slick was burnt off the surface of a pool of water by placing a chemical Cab-O-Sil on the surface. This surfactant isolated the oil from the water, which acts as a heat sink, so that the oil could be ignited at the surface. Once ignited the particles of Cab-O-Sil act as a wick to keep the oil burning. Although the processes taking place in that experiment appear complex, it seems that variation in depth caused by surfactants could be used to help in the removal of crude oil from the surface of the sea. Further experimental and theoretical studies of this problem are clearly desirable.

From the evidence given it is seen that more work is desirable in connection with flows affected by variable surface tension and the associated investigations of the heat and mass transfer processes producing such variations.

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APPENDIX A

In one dimension it can be proved that for a linear differential operator L , the Greens function for the differential equation

$$L[u] = 0 ,$$

which holds everywhere in a region, $a < x < b$, except at some source point ξ, $a<\xi<b$; together with a set of homogeneous boundary conditions, is given by

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{u_n(x)u_n(\xi)}{\lambda_n}$$

where λ_n and u_n are the eigenvalues and normalised eigenfunctions for the problem

$$L\,u + \lambda u = 0$$

with the same boundary conditions. This formula relating Greens function to eigenfunctions and eigenvalues of the above eigenvalue problem is called the bilinear relation. It can be extended to 2 dimensions, an example iluustrating this extension is given in Courant and Hilbert (1953).

In Chapter 7 it is necessary to find the Greens function for

$$\nabla^2\theta = 0 \qquad \text{(A1)}$$

in the region $0<x<1$, $0<y<1$ with the boundary conditions

$$y = 0 \; : \; \theta = 0 , \qquad \text{(A2)}$$

$$y = 1 \; : \; \frac{\partial\theta}{\partial y} = 0 , \qquad \text{(A3)}$$

$$x=0,1 \; : \; \frac{\partial\theta}{\partial x} = 0 . \qquad \text{(A4a,A4b)}$$

To obtain a solution in the form suggested, it is necessary to find the normalised eigenfunctions and eigenvalues for

$$\nabla^2\Theta + \lambda^2\Theta = 0 \quad , \qquad (A5)$$

with the boundary conditions stated. Substituting a solution in the separable form

$$\Theta(x,y) = X(x)\, Y(y) \quad .$$

one obtains the equations

$$\frac{1}{X}\frac{d^2X}{dx^2} = -K'^2 \qquad (A6)$$

and

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m'^2 \quad , \qquad (A7)$$

where K' and m' are constants connected by

$$K'^2 + m'^2 = \lambda^2 \quad .$$

The solution of (A6) satisfying (A4a) and (A4b) is

$$X = A\cos K\pi x \quad ,$$

where A is constant and

$K' = K\pi$, K being an integer.

The solution of (A7) satisfying (A2) and (A3) is

$$Y = B\sin\frac{(2m+1)\pi}{2}\, y \quad ,$$

where B is constant and

$m' = \frac{(2m+1)}{2}$, m being an integer.

The required set of eigenfunctions, Θ_{km}, and corresponding eigenvalues, λ_{km}, is therefore,

$$\Theta_{Km} = A_{Km}\cos K\pi x \sin\frac{(2m+1)}{2}\pi y, \qquad (A8)$$

$$\lambda_{Km} = \frac{\pi^2}{4}\left[(2m+1)^2 + 4K^2\right] \quad , \qquad (A9)$$

where A_{Km} is constant. The eigenfunctions must now be normalised by determining the constants A_{Km}. The normality condition is

$$\int_0^1 \int_0^1 \theta^2_{Km}\, dx\, dy = 1 \qquad ,$$

which gives the result

$$A_{Km} = \frac{1}{2} \qquad .$$

The normalised eigenfunction is therefore

$$\theta_{Km} = 2 \cos K\pi x \sin \frac{(2m+1)}{2}\pi y \qquad . \qquad \text{(A10)}$$

Using the bilinear relation it now follows that the required Greens Function is

$$G(x,y,\xi,\eta) =$$

$$\sum_{K=1}^{\infty} \sum_{m=0}^{\infty} \frac{4 \cos K\pi x \sin \frac{(2m+1)}{2}\pi y \cos \xi\pi x \sin \frac{(2\eta+1)}{2}\pi y}{\frac{\pi^2}{4}\left[(2m+1)^2 + 4K^2\right]} \quad .$$

APPENDIX B

Trying to use a Laplace transformation method to solve

$$\frac{\partial \phi}{\partial x} = \varepsilon^2 \frac{\partial^2 \phi}{\partial Y^2} \quad , \qquad \text{(B1)}$$

with boundary conditions

$$Y = H - \frac{\pi\varepsilon}{4}: \quad \phi = \varepsilon\frac{\partial \phi}{\partial Y} = \theta(x) \quad , \qquad \text{(B2)}$$

and $$x \to \pm\infty: \quad \phi \to 0 \quad ; \qquad \text{(B3)}$$

rewite the system by putting

$$z = H - Y - \frac{\pi\varepsilon}{4}$$

so that $$\frac{\partial \phi}{\partial x} + \frac{dH}{dx}\frac{\partial \phi}{\partial z} = \varepsilon^2 \frac{\partial^2 \phi}{\partial z^2} \quad , \qquad \text{(B4)}$$

$$z = 0: \quad \phi = -\varepsilon\frac{\partial \phi}{\partial z} = \theta(x) \quad . \qquad \text{(B5)}$$

Suppose that $\bar{\phi}$ is the Laplace transform of the solution for ϕ,

$$\bar{\phi} = L\,\{\phi\} = \int_0^\infty e^{-sz}\,\phi(x,z)\,dz \quad .$$

Using the boundary conditions (B5), it follows that

$$L\,\left\{\frac{\partial \phi}{\partial z}\right\} = s\bar{\phi} - \theta$$

and $$L\,\left\{\frac{\partial^2 \phi}{\partial z^2}\right\} = s\{s\bar{\phi} - \theta\} + \frac{1}{\varepsilon}\theta \quad ,$$

so that the transformed equation to be solved is

$$\frac{\partial \bar{\phi}}{\partial x} + \left(s\,\frac{dH}{dx} - \varepsilon^2 s^2\right)\bar{\phi} = \left\{-\varepsilon^2 s + \varepsilon + \frac{dH}{dx}\right\}\theta \quad . \qquad \text{(B6)}$$

Because of the form of $\theta(x)$, this equation must be solved separately in the regions $x \geqslant 0$ and $x < 0$. Integrating (B6) in $x \geqslant 0$ and using the boundary condition at infinity

$$\bar{\phi} = -e^{-SH+\varepsilon^2 sx} \int_x^\infty (-\varepsilon^2 s + \varepsilon + \frac{dH}{dt} \quad \theta e^{sH-\varepsilon^2 s^2 t} \, dt. \qquad (B7)$$

Unfortunately, the inverse of this Laplace transformation is difficult to obtain so this method does not give a solution.

FIGURES

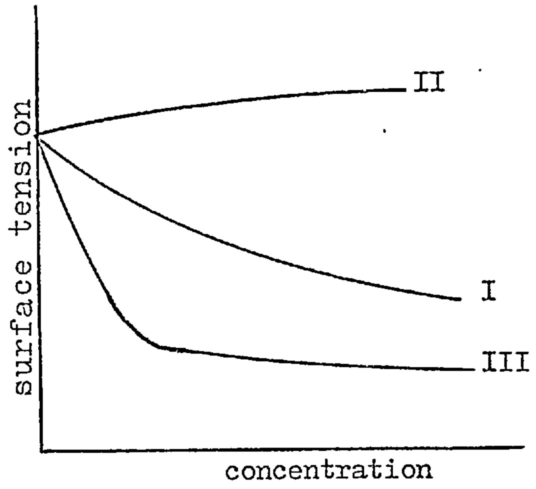

Figure 1 The three types of curves for the change of surface tension of water by solutes .

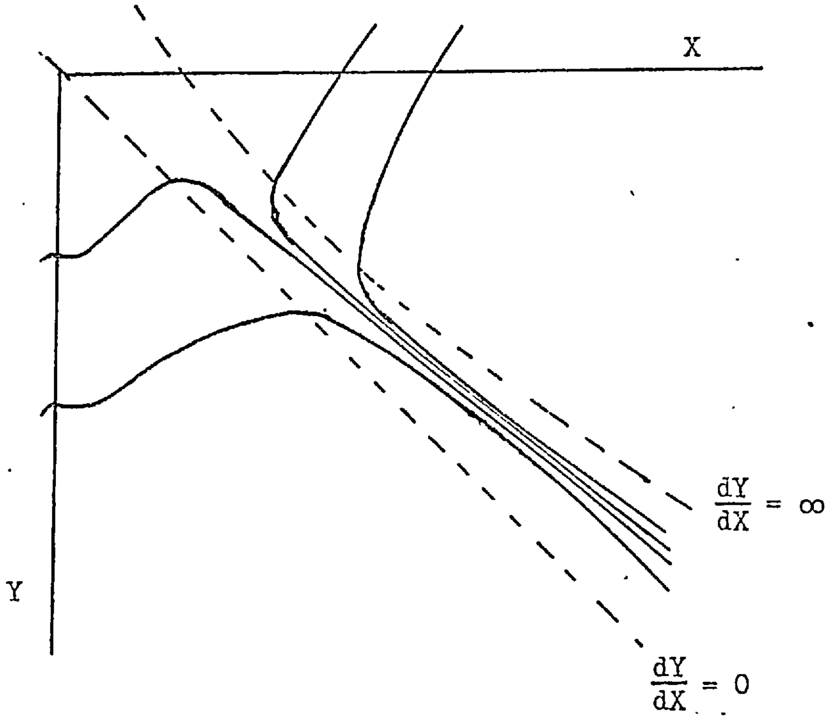

Figure 2 Sketch of integral curves in the region of physical significance for the equation

$$\frac{dY}{dX} = \frac{CX^2(X+Y)}{3X^2+4XY-1}$$

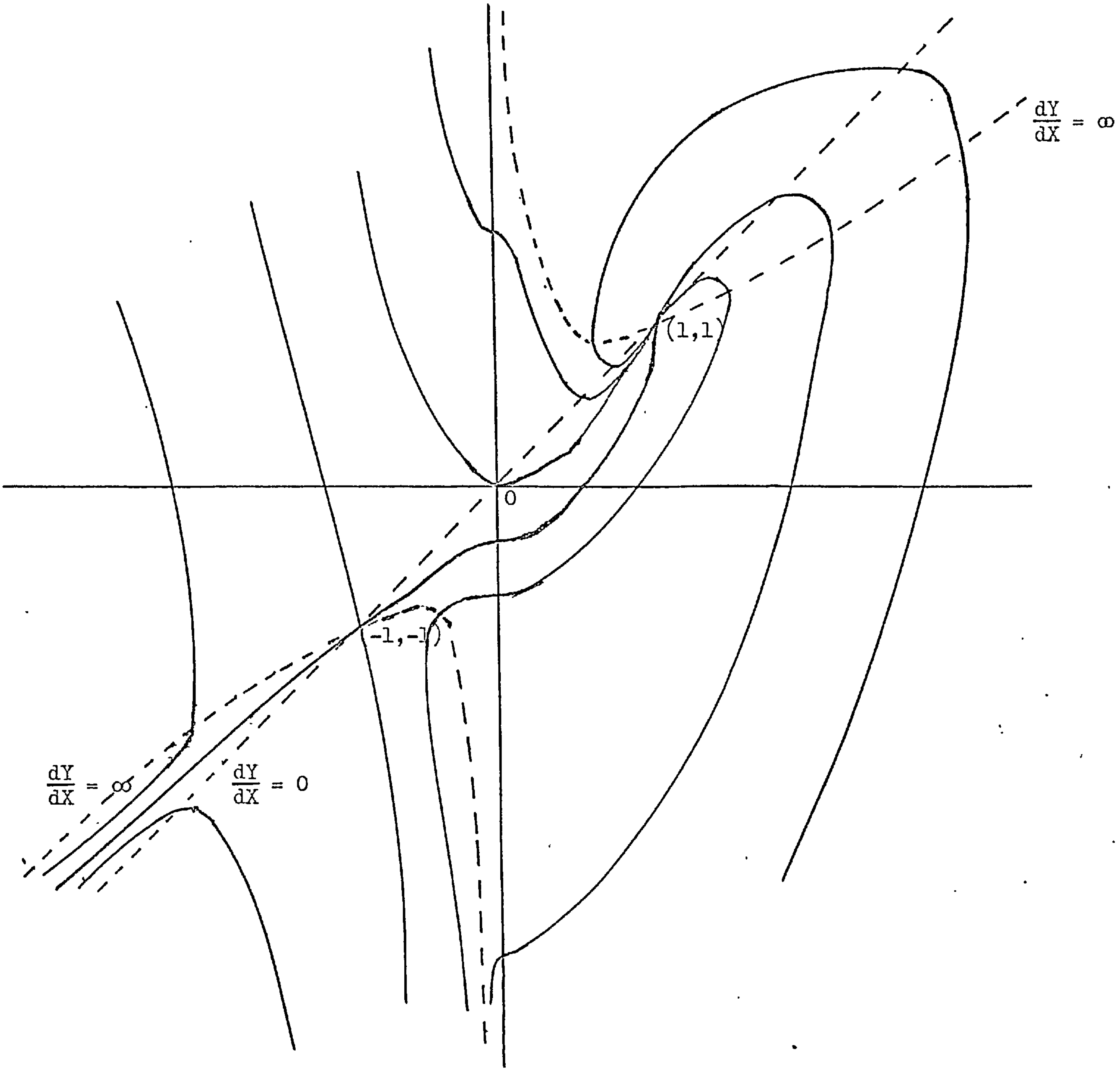

Figure 3 Sketch of the integral curves for the equation

$$\frac{dY}{dX} = \frac{CX^2(X-Y)}{3X^2-4XY+1}$$

when C is large

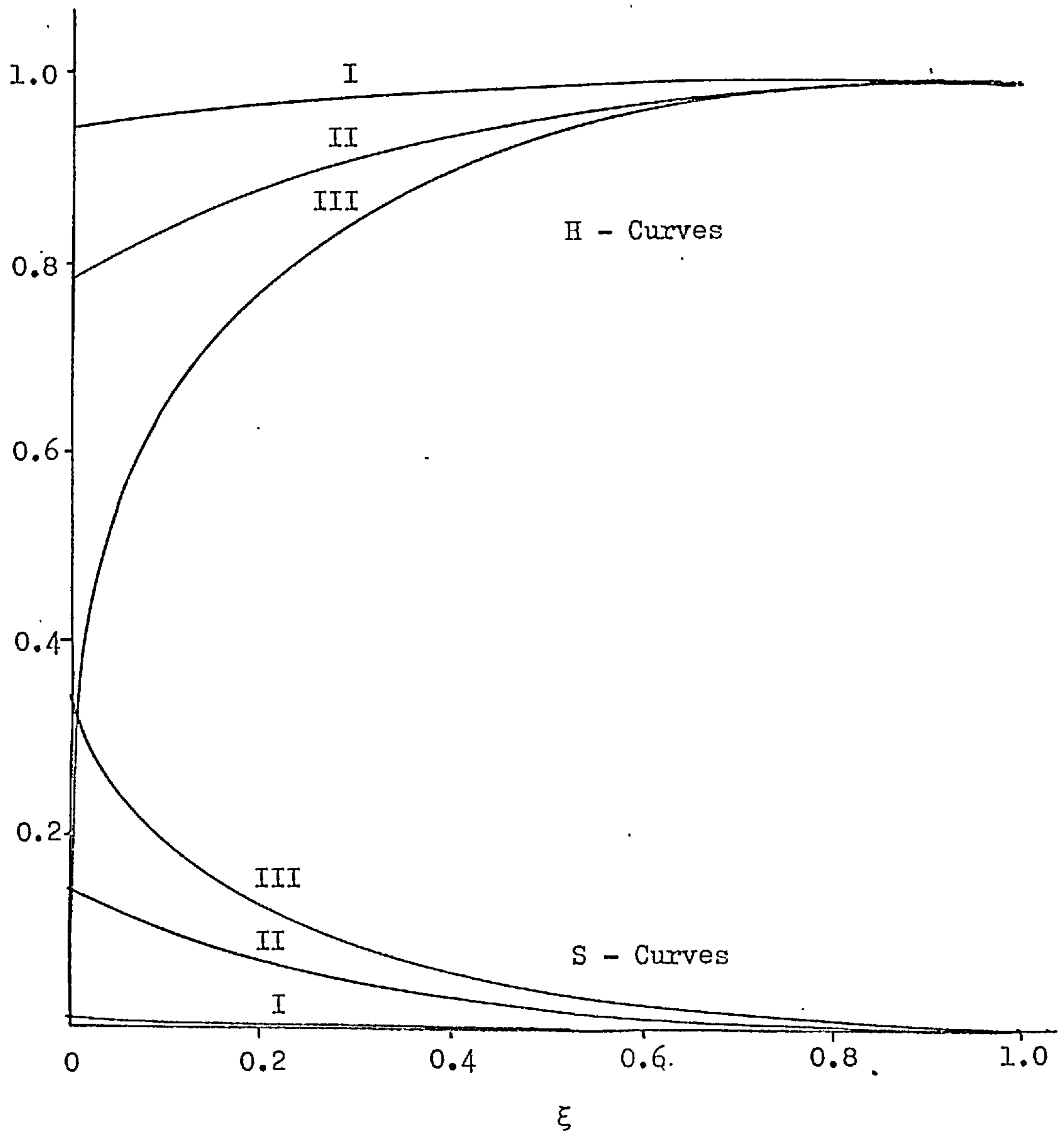

Figure 4 Curves of S and H versus for $b_1 = 0$,

I $a_1 = 0.1$, $a = 0.097$, $b = 0.0$

II $a_1 = 0.2$, $a = 0.17$, $b = 0.0$

III $a_1 = 0.346$, $a = 0.220$, $b = 0.0$

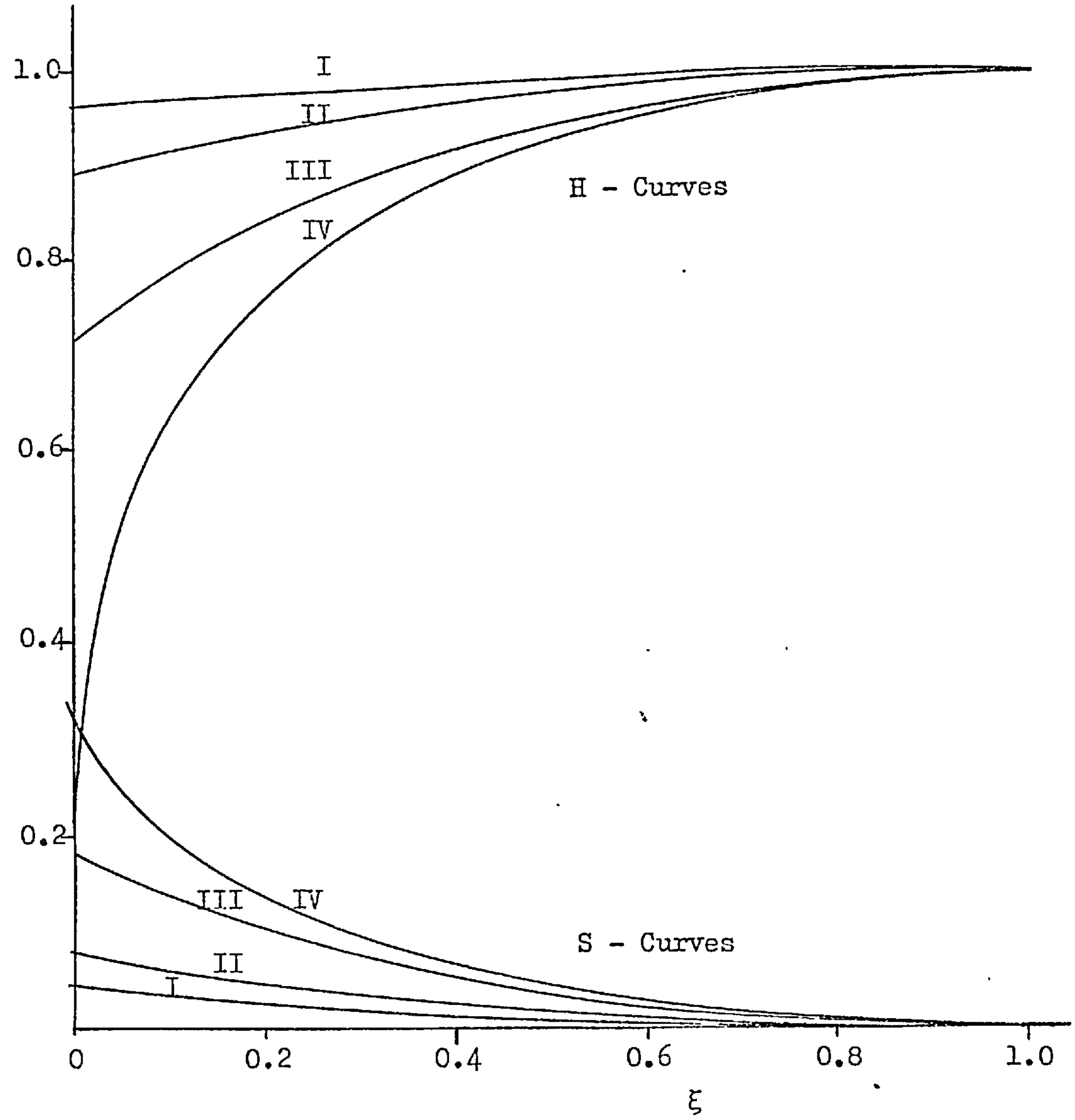

Figure 5 Curves of S and H versus for $b_1 = 0.001$,

I $a_1 = 0.0$, $a = 0.0$, $b = 0.00962$

II $a_1 = 0.1$, $a = 0.092$, $b = 0.00921$

III $a_1 = 0.2$, $a = 0.163$, $b = 0.00816$

IV $a_1 = 0.324$, $a = 0.201$, $b = 0.00622$

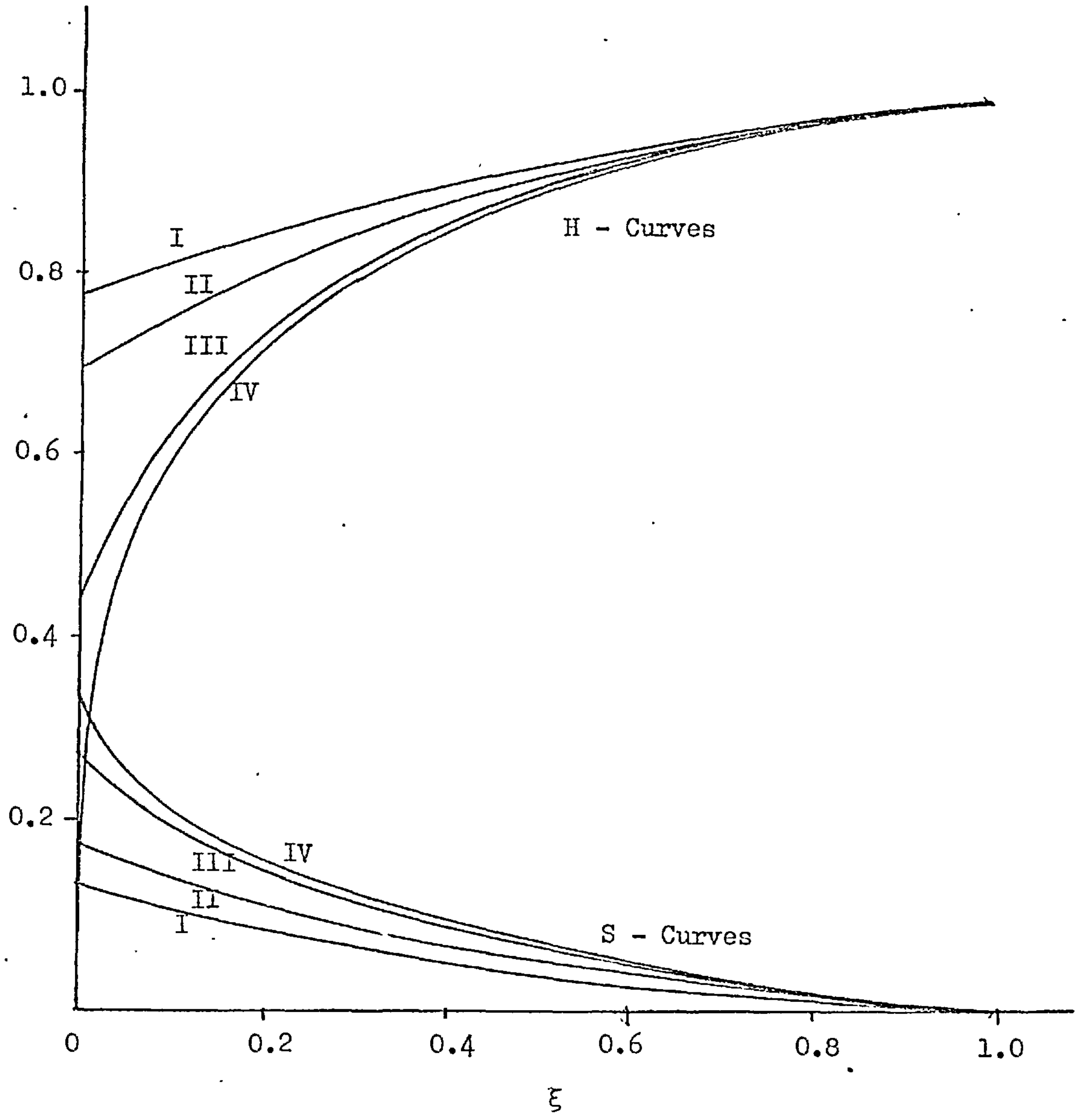

Figure 6 Curves of S and H versus for $b_1 = 0.005$,

I	$a_1 = 0.0$,	$a = 0.0$,	$b = 0.00406$
II	$a_1 = 0.1$,	$a = 0.077$,	$b = 0.00384$
III	$a_1 = 0.2$,	$a = 0.131$,	$b = 0.00328$
IV	$a_1 = 0.252$,	$a = 0.146$,	$b = 0.00290$

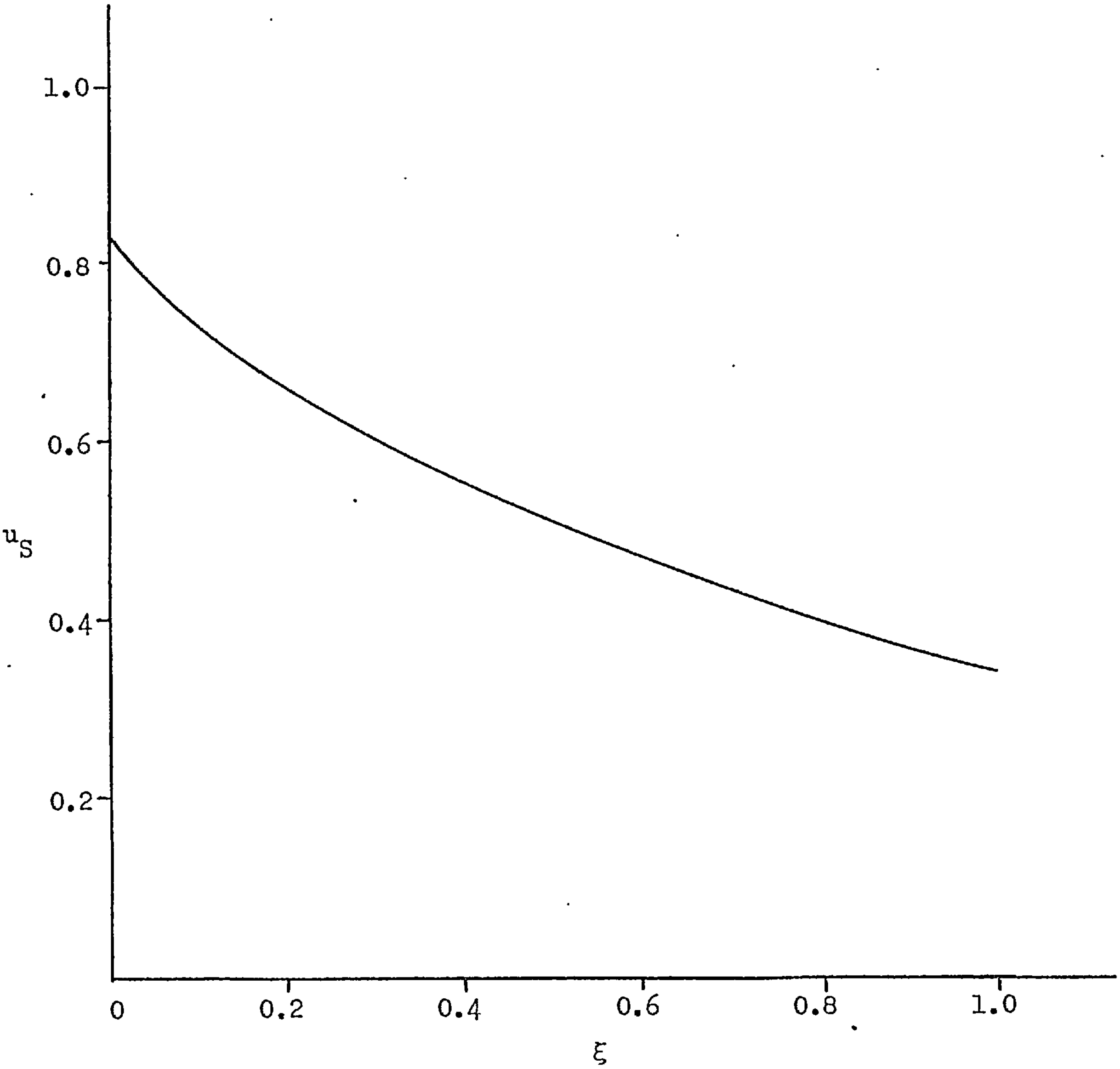

Figure 7 Typical curve for $u_S(\xi)$;

$a_1 = 0.1$, $b_1 = 0.005$

$a = 0.077$, $b = 0.00384$

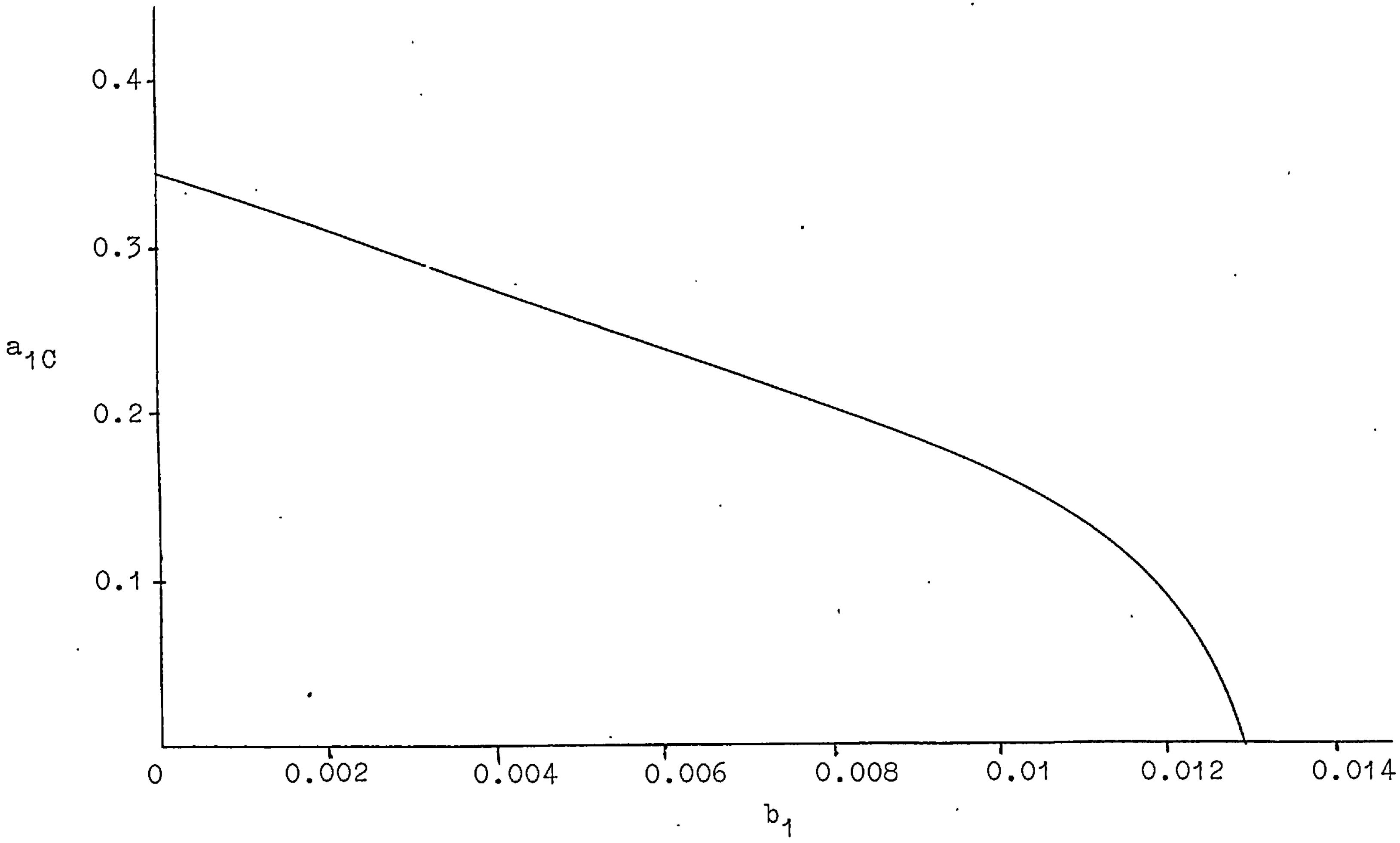

Figure 8 Curve for $a_{1C}(b_1)$

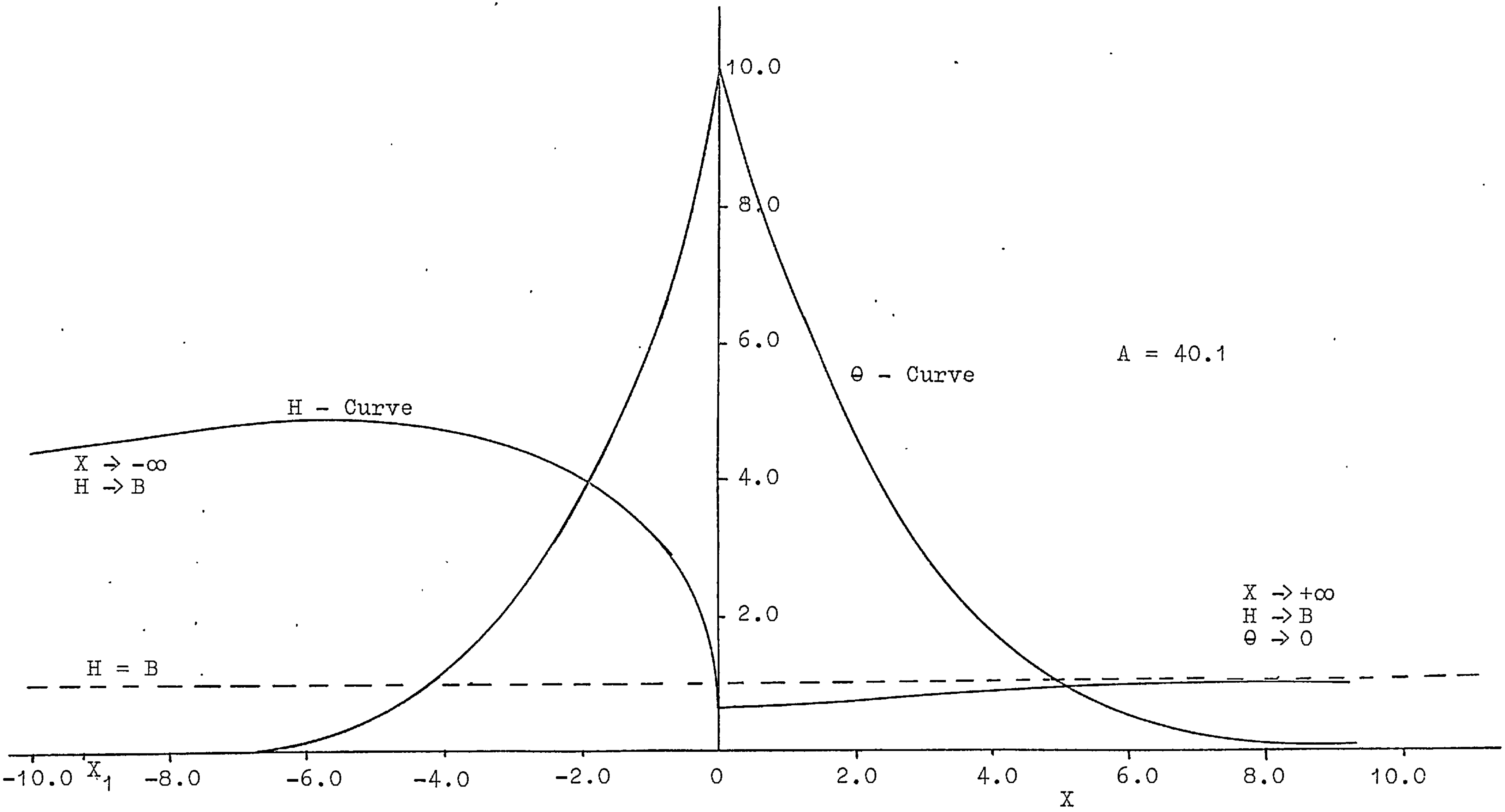

Figure 9 Curves for H and θ versus X when $\theta_F = 10.0$, B = 1.0

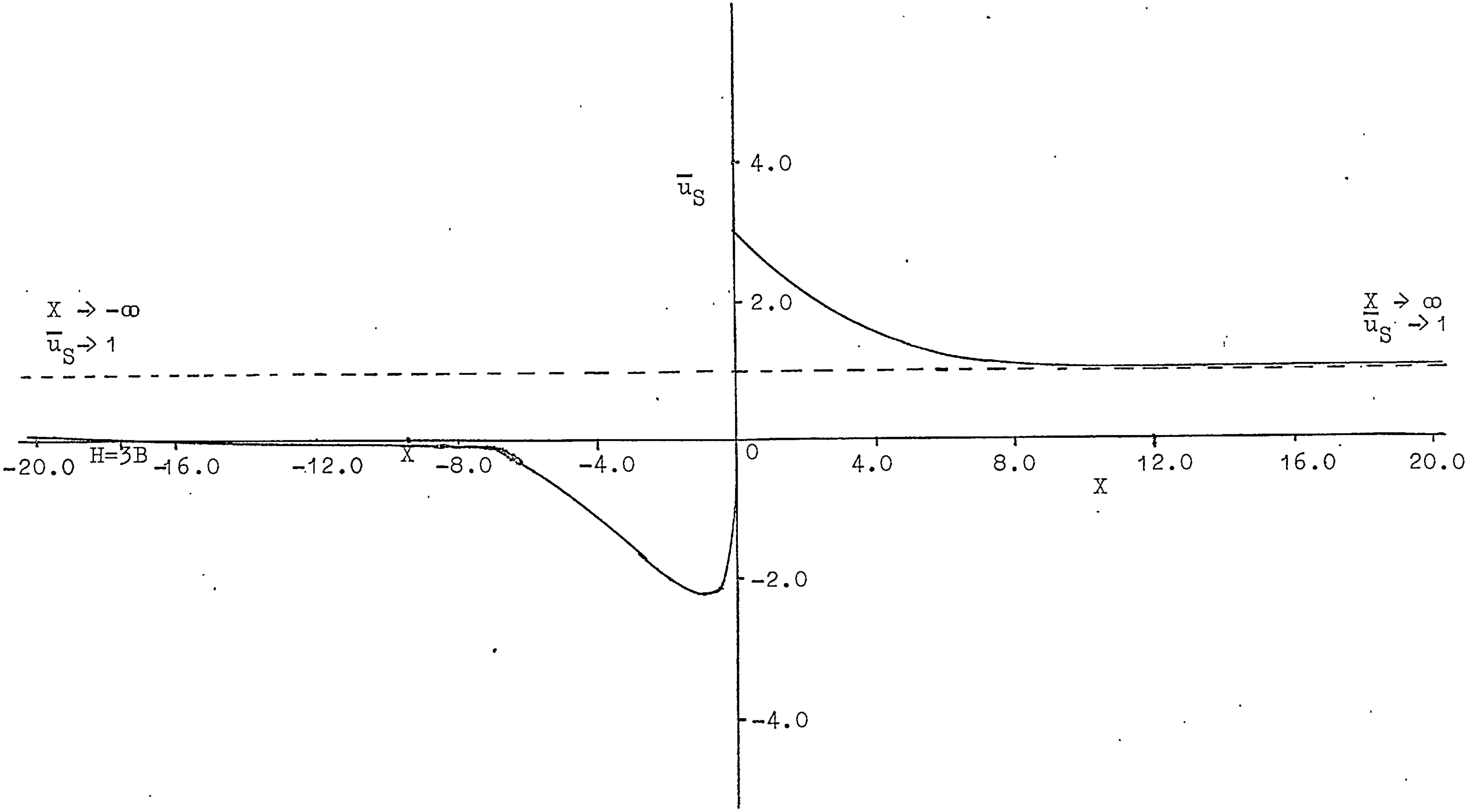

Figure 10 Curve for $\bar{u}_S(X)$ when B= 1.0 , θ_F = 10.0

CORRIGENDA