ON SOME PROBLEMS RELATED TO THE BOUNDARY OF MARKOV CHAINS

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ABSTRACT

This work is about the entrance boundary of the Markov processes in a countable state space A. The analytical definitions of J. Neveu and J.L. Doob are shown to be equivalent. The entrance boundary is an extension of A which has the following property: every process with values in A has a standard modification with values in the entrance boundary and which is right continuous and strongly Markovian.

We show that the size of the entrance boundary is the best possible but in some cases its topology is not the finest to keep the process right continuous. We attempt to metricate the finest topology by means of taboo semigroups, where the taboo sets are subsets of the entrance boundary. A solution is found in two very simple examples, which are introduced for their interesting topologies on the entrance boundary.

We investigate the relations between the entrance boundaries of the original semigroup and of a taboo semigroup. In particular, we show that for every point y of the entrance boundary, but outside A, we can find a sequence of points in A which converges to y in both entrance boundaries.

CHAPTER I

Introduction

Throughout this work we use the following notations :

N = the set of all positive integers

Q = the set of all rational numbers

R = the set of all finite real numbers

R_+ = the interval $[0, \infty)$

R_+° = the interval $(0, \infty)$

II (16) [resp, th II 2] is written to refer to relation (16) in Chapter II [resp. to theorem 2 in Chapter II]. If the roman number is not written the reference is to a relation (or a theorem) in the same Chapter.

To begin we give some basic notions on general processes. By way of simplification definitions and results will be quoted mainly from P.A.Meyer's books [1] and [2], and in this case M XI8 is written for Chapter XI, no. 8 in [1] or [2]

Let E be a set and $\mathcal{E}$ be a σ-field of subsets of E.

A transition semi-group on $(E, \mathcal{E})$ is a family of real valued functions, $p_{xB}(t)$, x in E, B in $\mathcal{E}$, t in R_+, say, such that

(i) for all $t>0$, and all B in $\mathcal{E}$, the function $p_{xB}(t)$; x in $E \to [0, 1]$ is $\mathcal{E}$ measurable

(ii) for all $t > 0$ and all x in E, the function $p_{xB}(t)$: $B \in \mathcal{E} \to [0, 1]$ is a measure on $\mathcal{E}$

(iii) For all x in E, all B in $\mathcal{E}$, all $t>0$ and all $s>0$

$$\int_E p_{xdy}(t)\, p_{yB}(s) = p_{xB}(t+s)$$

Such a semi-group is usually extended to t = 0 by setting for all x in E

$$p_{xB}(0) = \mathcal{E}_x(B)$$

Where $\mathcal{E}_x(\,.\,)$ is the atomic measure concentrated in x and of total mass 1.

The transition semi-group is said to be stochastic if all x in E and all $t>0$

$$p_{xE}(t) = 1$$

A family of measures on $(E, \mathcal{E})$ $\mu_{\cdot}(t)$, $t>0$ say is called an <u>entrance</u> relative to $p_{xB}(t)$ if for all B in $\mathcal{E}$, all $t>0$ and all $s>0$ we have

$$\int_E \mu_{dy}(t)\, p_{yB}(s) = \mu_B(t+s)$$

Let $\Omega = E^{R^{\circ}_{+}}$ and denote its elements by w.

Let $\mathcal{F}$ be the σ-field of Ω generated by the co-ordinate $X_t(w) = w_t$, $t>0$, say .

If the entrance $\mu_{\cdot}(t)$, $t>0$ relative to the stochastic semi-group $p_{xB}(t)$ is such that

$$\mu_E(t) = 1 \text{ for all } t>0$$

$\mu_{\cdot}(t)$ is called a <u>stochastic entrance.</u>

By M XII. 12 there exists in this case a probability measure P[ ] on $(\Omega, \mathcal{F})$ such that

(1) $P[X_t(w) \in B] = \mu_B(t)$

for all B in $\mathcal{E}$ and all $t>0$, and

(2) $$\begin{cases} P[X_{t_n}(w) \in B \mid X_{t_1}(w) \in B_1, X_{t_2}(w) \in B_2, \quad , X_{t_{n-1}}(w) = x] = \\ P[X_{t_n}(w) \in B \mid X_{t_{n-1}}(w) = x] = p_{xB}(t_n - t_{n-1}) \end{cases}$$

for all x in E, all $0<t_1<t_2<\ldots<t_{n-1}<t_n$ and all B, $B_1, B_2, \quad , B_{n-2}$ in $\mathcal{E}$.

The equality of elementary conditional probabilities in (2) is called the Markov property and $X_t(w)$ is then a <u>Markov process</u> with $p_{xB}(t)$ as transition function.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. If f is a $\mathcal{F}$-measurable function defined on Ω , the <u>conditional expectation</u> of f relative to $\mathcal{G}$ is a (non uniquely defined) $\mathcal{G}$-measurable function to be denoted by $E[f \mid \mathcal{G}]$ which satisfies

$$\int_G E[f \mid \mathcal{G}]\, P[dw] = \int_G f(w)\, P[dw]$$

for all G in $\mathcal{G}$.

The conditional probability of a set B in $\mathcal{F}$ is a $\mathcal{G}$-measurable function to be denoted by $P[B|\mathcal{G}]$ which satisfies

$$\int_G P[B|\mathcal{G}]\, P\,[dw] = \int_G I_B\,(w)\, P\,[dw]$$

for all G in $\mathcal{G}$, $I_B\,(w)$ being the characteristic function of B.

Denote by $\mathcal{F}_t$ (resp $_t\mathcal{F}$) the σ-field of Ω generated by $X_s\,(w)$, $0<s\leq t$ (resp. $X_s\,(w)$, $t<s$)

By M II 51, (2) is equivalent to

(3) $\qquad P[\Lambda\, M|\mathcal{F}_t] = P[\Lambda\,|\,X_t]\; P\,[M\,|\,X_t]$ a.s.

for all Λ in $\mathcal{F}_t$, all M in $_t\mathcal{F}$, all $t>0$, where the conditional expectation relative to a random variable is the one relative to the σ-field generated by this random variable.

A random variable $\zeta\,(w)$, possibly infinite, is called a stopping time relative to an increasing family of σ-field $\mathcal{G}_t$, $t>0$ if

$$[\,\zeta\,(w)\leq t]\in\mathcal{G}_t \text{ for all } t>0$$

If $\mathcal{G}_\infty$ is the σ-field generated by the union of all $\mathcal{G}_t$, the elements B in $\mathcal{G}_\infty$ such that

$$B\cap[\,\zeta\,(w)<t]\in\mathcal{G}_t \text{ for all } t>0$$

form a σ-field, denoted by $\mathcal{G}_\zeta$, it is the set of events preceding ζ.

If ζ is a finite stopping time relative to $\mathcal{F}_t$, we may associate with any w in Ω and any $s\geq 0$ the point $X_{\zeta(w)+s}\,(w)$ in E. Under certain conditions this new random variable is measurable (see e.g. M IV. 49). In this case and if for all B in $\mathcal{E}$ and all $s\geq 0$

(4) $\qquad P[\,X_{\zeta+s}\in B|\mathcal{F}_\zeta] = P\,[\,X_{\zeta+s}\in B\,|\,X_\zeta]$ a.s.

the process is said to enjoy the strong Markov property.

If $_\zeta\mathcal{F}$ is the σ-field generated by $X_{\zeta+s}$, $s\geq 0$, then by M II.51 the condition (4) is equivalent to

(5) $\qquad P[\Lambda\, M|\mathcal{F}_\zeta] = P\,[\Lambda|\,X_\zeta]\; P\,[M|X_\zeta]$ a.s.

for all Λ in $\mathcal{F}_\zeta$ and all M in $_\zeta\mathcal{F}$.

Two processes $X_t\,(w)$ and $Y_t\,(w)$, t in some interval I of R, defined on the same probability triple $(\Omega\,,\mathcal{F}\,,P)$ and with values in the same space

$(\Xi,\mathcal{E})$ are said to be standard modifications (or versions) of each other if

$$P[w \mid X_t(w) = Y_t(w)] = 1 \text{ for all } t \text{ in } I.$$

Let E be a compact metrisable space.

A process X_t, t in R_+, with values in E, is said to be separable relative to the closed sets of E, if there exists a countable set S dense in R_+ such that if C is a closed set in E and I an open interval in R_+ then the event

$$[\, w \mid X_t(w) \in C \text{ for all } t \text{ in } S \cap I] - [\, w \mid X_t(w) \in C \text{ for all } t \text{ in } I]$$

is contained in an event of probability zero.

A right continuous process is a process X_t, $t \geq 0$ with values in a topological space such that

$$P[\, w \mid X_t(w) \text{ is right continuous at all } t \geq 0] = 1$$

In this work we deal only with countable state spaces which are denoted by A. Results and definitious concerning this particular case will usually be quoted from K.L.Chung's book [3] and C th. II. 3 .3 will then be used for theorem 3 in§9 of part II in [3]. We now give some basic facts about this special case.

We consider the σ-field of all the subsets of A.

A transition semi-group on this measurable space is called a transition matrix, i.e. a set of functions $p_{ij}(t)$, i in A, j in A, and $t > 0$ such that

(6) $\quad 0 \leq p_{ij}(t) \qquad$ for all $t > 0$

(7) $\quad \sum_{k \in A} p_{ik}(t) \leq 1 \qquad$ for all $t > 0$

(8) $\quad p_{ij}(t+s) = \sum_{k \in A} p_{ik}(t)\, p_{kj}(s)$ for all $t > 0$ and all $s > 0$

We always have the additional condition of stochastic continuity, namely

(9) $\quad \lim_{s=0} p_{ij}(s) = \delta_{ij}$

Naturally we extend $p_{ij}(t)$ to $t = 0$ by setting $p_{ij}(0) = \delta_{ij}$, and $(p_{ij}(t))$ is said to be a standard transition matrix.

The conditions (6) to (9) are known to be enough to ensure the continuity in t on $[0, \infty)$ of all the functions pij (t), see e.g. C th. II. 1.3.

We will also use the equivalent matrix notation

(10) $\quad 0 \leq P(t)$

(11) $\quad P(t)\, \underline{\underline{1}} \leq \underline{\underline{1}}$ where $\underline{\underline{1}}$ is the unit vector (1,1,1,...) in the ~~Hilbert~~ Banach space of bounded sequences indexed by A

(12) $\quad P(t+s) = P(t)\, P(s)$

(13) $\quad \lim_{s \to 0} P(s) = I$ where I is the identity matrix

An <u>entrance</u> relative to (pij (t)) is a set of functions $f_i(t)$, i in A, $t > 0$ such that

(14) $\quad 0 \leq f_i(t) \qquad$ for all $t > 0$

(15) $\quad f_i(t+s) = \sum_{k \in A} f_k(t)$ pki (s) for all $t > 0$ and all $s > 0$

(16) $\quad \sup_{0 < t < \infty} \sum_{k \in A} f_k(t) < \infty$

Again in the vector matrix notations the family of vectors f (t), $t > 0$ of the ~~Hilbert~~ Banach space of converging ~~sequences~~ series indexed by A is an entrance relative to P(t) if only if

(17) $\quad 0 \leq f(t)$

(18) $\quad f(t+s) = f(t)\, P(s)$

(19) $\quad \sup_{0 < t < \infty} \| f(t) \| < \infty$

where the norm is the one in (.

Note that in the <u>stochastic</u> case (i.e. equality in (7) or (11) for all t) we have f(t) = c. Hence (19) holds whenever $\|f(s)\| < \infty$ for one $s > 0$.

The set of all entrances relative to P(t), including the trivial one, is easily seen to be a convex cone which will be denoted by F.

We recall now some definitions about cones.

A subcone $\bar{F}$ of F is said to be <u>thick</u> in F if the conditions $f \in F$, $\bar{f} \in \bar{F}$ and $f \leq \bar{f}$ (the order being the inner order in F) imply $f \in \bar{F}$.

A subcone $\overline{F}$ is a positive band of F if it is thick in F and if every nonvoid bounded above subset H contained in $\overline{F}$ has a least upper bound in $\overline{F}$.

In a cone F a point f is extremal if for any $g \leq f$ there exists an α in $[0, 1]$ such that $g = \alpha f$.

We will also use the extremality of points in a convex set, C, say.

f is said to be extremal in C if the equality $f = \alpha g + (1 - \alpha) h$, where α is in $(0, 1)$ and both g and h are elements of C implies $f = g = h$.

Now if P(t) is stochastic and if p(t) is an entrance such that

$$(20) \qquad \sum_{i \in A} p_i(t) = 1 \quad \text{for all } t > 0$$

We can apply what was recalled before about entrances relative to stochastic transition semi-groups.

Let

$$\Omega(A) = A^{R_+^\circ}$$

$$X_t^A(w(A)) = \text{the } t \text{ - co-ordinate of } w_A(A) \text{ in } \Omega(A),\ t > 0$$

$$\mathcal{F}(A) = \sigma\text{-field generated by all } X_t^A,\ t > 0$$

then there exists a probability measure $P^A[\ ]$ on $(\Omega(A), \mathcal{F}(A))$ such that

$$(21) \qquad P^A[X_t^A(w(A)) = i] = p_i(t)$$

for all i in A and all $t > 0$, and

$$(22) \qquad P^A[X_{t_n}^A = i_n \mid X_{t_1}^A = i_1, \ldots, X_{t_{n-1}}^A = i_{n-1}] = p_{i_{n-1} i_n}(t_n - t_{n-1})$$

for all $i_1, i_2, \ldots, i_n$ in A and all $0 < t_1 < t_2 < \ldots < t_n$.

If the discrete topology is used on A, then it is even possible to find a standard modification of X_t^A separable relative to the closed sets (see C th. II. 4.3). However it is not always possible to find a standard modification of this process which is right-continuous and enjoys the strong Markov property.

Therefore it is useful to find an extension of A in which such a standard modification may be obtained. Observe that the Alexandroff compactification is generally of no use in this problem (cf. C th II. 9.3 and notes following

II. 9)

Before proceeding to the description of our work, we give an account of the manner in which A is imbeded in a bigger space E in order that a process X_t^A in A may be considered as a process in E.

Let E be a topological space and $\mathcal{E}$ its Borel σ-field. Assume that the measurable space $(E; \mathcal{E})$ satisfies

(i) A is contained in E

(ii) every point of A considered as a subset of E is an element of $\mathcal{E}$.

Let

$$\Omega(E) = E^{R_+}$$

$$X_t^E\ (w(E)) = \text{the t-co-ordinate of } w\ (E) \text{ in } \Omega(E),\ t > 0$$

$$\mathcal{F}\ (E) = \sigma\text{-field generated by all } X_t^E,\ t > 0$$

We call an element Δ of $\mathcal{F}(E)$ elementary if all the factor sets are equal to the whole space E, except a finite number of co-ordinates $t_1, t_2, \dots, t_n$ say, where the corresponding factor sets are $B_1, B_2, \quad B_{tn}$ Borel sets of E. On such Δ's we may define a finitely additive function $P^E[\quad]$ by setting

(23) $$P^E[\Delta] = P^A\ [X_{t_1}^A \in B_{t_1} \cap A,\ X_{t_2}^A \in B_{t_2} \cap A, \quad \dots \quad, X_{t_n}^A \in B_{t_n} \cap A]$$

Where the set in the second bracket is measurable in ~~$\mathcal{A}$~~ $\mathcal{F}(A)$, since every subset of A is measurable ~~there~~ in A.

By Caratheodory's theorem (e.g.M II. 25), $P^E[\quad]$ can be extended in a probability measure on $(\Omega(E); \mathcal{F}(E))$, which we also denote by $P^E[\quad]$

Natuurally for the process X_t^E $t > 0$ we define

(24) $$P^E\ [\ X_{t_1}^E \in B_{t_1},\ X_{t_2}^E \in B_{t_2}, \qquad, X_{t_n}^E \in B_{t_n}\ ] = P^E[\Delta]$$

(23) and (24) give in particular

(25) $$P^E[\ X_t^E\ (w(E)) = i] = P^A[\ X_t^A\ (w(A)) = i] = p_i\ (t)$$

for all $t > 0$ and all i in A.

Similarly if $0 < \quad t_1 < t_2 < t_3 \dots < t_n$ and $B_{t_1}, B_{t_2}, \quad, B_{t_n}$ are in $\mathcal{E}$, we have the following equalities of elementary conditional probabilities

(26)

$$P^E [X^E_{t_n} \in B_{t_n} \mid X^E_{t_1} \in B_{t_1}, \qquad , X^E_{t_{n-1}} \in B_{t_{n-1}}] =$$

$$\frac{P^E [X^E_{t_n} \in B_{t_n}, X^E_{t_1} \in B_{t_1}, \qquad , X^E_{t_{n-1}} \in B_{t_{n-1}}]}{P^E [X^E_{t_1} \in B_{t_1} \qquad X^E_{t_{n-1}} \in B_{t_{n-1}}]} =$$

$$\frac{P^A [X^A_{t_n} \in B_{t_n} \cap A, X^A_{t_1} \in B_{t_1} \cap A , \ldots \qquad , X^A_{t_{n-1}} \in B_{t_{n-1}} \cap A]}{P^A [X^A_{t_1} \in B_{t_1} \cap A, \qquad X^A_{t_{n-1}} \in B_{t_{n-1}} \cap A]}$$

AS the Markov property holds for X^A_t , the last term is equal to

$$\frac{P^A [X^A_{t_n} \in B_{t_n} \cap A \quad X^A_{t_{n-1}} \in B_{t_{n-1}} \cap A]}{P^A [X^A_{t_{n-1}} \in B_{t_{n-1}} \cap A]} = \frac{P^E [X^E_{t_n} \in B_{t_n}, X^E_{t_{n-1}} \in B_{t_{n-1}}]}{P^E [X^E_{t_{n-1}} \in B_{t_{n-1}}]}$$

(27)

$$P^E [X^E_{t_n} \in B_{t_n} \mid X^E_{t_{n-1}} \in B_{t_{n-1}}]$$

The equality (26) = (27) means that X^E_t , $t > 0$ is a Markov process, moreover if we put $B_{t_{n-1}} = i$ and $B_{t_n} = j$ we can deduce that its transition semi-group is P (t).

So with every entrance p (t), $t > 0$ relative to P(t) satisfying (20) we can associate two Markov processes, one with values in A and the other with values in E. Both have P(t) as transition semi-group and p(t) as absolute distribution. From now on we shall distinguish between them simply by saying the process in A (or in E) and drop all the indexing by A (or E) of $\Omega, P, \mathcal{F}$ and so on.

To close this Chapter we give a short summary of the other Chapters.

The general content of this work is the extension of A (in the sense of (i) and (ii) in p 11 by the so called entrance boundary.

In Chapter II, § 1 we set out the analytical definition given by J.Neveu [8] the definition used by J.L.Doob in [5] is the subject of the second paragrah.

In § 3 these two definitions are shown to be equivalent as was stated by Doob in p. 237 of [5] .

Doob's proof is used in Chapter III to show that every Markov process in A has a standard modification in the entrance boundary which is right continuous and enjoys the strong Markov property.

Much of the content of these two Chapters is of course only a rearrangment of papers [5] and [8] and is introduced here for the sake of completeness.

In Chapter IV we see that the entrance boundary is the smallest extension of A on which the right continuity of almost all trajectories can be expected.

In Chapter V we are mainly concerned with the topology defined on the entrance boundary in Chapter II, Chapter V may be said to throw some darkness on the relations between the analytical and the probabilistic properties of P(t). The trivial example 1, in § 2 shows that this topology is not the best for our purposes. Then in a search for a better one we define in § 3 the taboo semi-group where the taboo set is in the extended space. In fact we try to define the best topology by adapting the techniques used in Chapter II to the taboo semi-groups. But they appear to be difficult to handle in this respect and the example 2 in § 4 is given to rule out the most obvious and general attempts in this direction.

Finally in Chapter VI we obtain some interesting analytical results about the taboo semi-groups.

Among other papers on the entrance boundary and dealing partly with countable state spaces are Ray [9], Kunita and Watanabe [6], and Williams [13].

CHAPTER II.

Analytical Definitions of the Entrance Boundary.

0 Two preliminary results.

(a) A theorem on the weak convergence of probability measures.

Let E be a topological metric space and d its metric.

Let $\mathcal{E}$ be the σ-field of its Borel sets.

Let μ and μ_n, n in N, be probability measures on $(E;\mathcal{E})$.

Let C(E) be the set of all bounded continous functions from E into R.

μ_n is said to converge weakly to μ as $n \to \infty$, $\mu_n \xrightarrow{w} \mu$, if and only if $\mu_n f \to \mu f$ as $n \to \infty$ for all f in C(E).

For every measurable function f and every real number α define the functions φ and φ_n by setting

$$\varphi(f;\alpha) = \mu(f \leq \alpha)$$

$$\varphi_n(f;\alpha) = \mu_n(f \leq \alpha) .$$

Lemma (which is a simplified version of theorem 2.1 in P.Billingsley [4])

The following statements are equivalent

(i) $\mu_n \xrightarrow{w} \mu$ as $n \to \infty$

(ii) $\mu(F) \geq \limsup_{n=\infty} \mu_n(F)$ for all sets F closed in E.

(iii) For any measurable function, which is continuous except on a set of μ - measure zero we have

$\lim_{n=\infty} \varphi_n(f;\alpha) = \varphi(f;\alpha)$

At every α where $\varphi(f,\alpha)$ is continuous .

Proof:

(i) => (ii) (which is reproduced here from [4])

Choose a closed set F

Let

$$U_\delta = \left\{ x \,\middle|\, d(x, F) < \delta \right\} \qquad \text{where} \quad 0 < \delta$$

If δ_r is a sequence of positive numbers decreasing to zero we have

$$\bigcap_{r=1}^{\infty} (U\delta_r - F) = \emptyset$$

hence

$$\liminf_{r=\infty} \mu(U\delta_r - F) = \mu(\emptyset) = 0$$

Fix $\varepsilon > 0$ and choose a δ_ε such that

$$\mu(U\delta_\varepsilon - F) < \varepsilon$$

Define f(x) as the following function

$$f(x) = \frac{d(x, E - U\delta_\varepsilon)}{d(x, E - U\delta_\varepsilon) + d(x:F)}$$

As the denominator is bounded away from 0 by δ_ε, f(x) is continous, always between 0 and 1, equal to 1 on F and to 0 on E - $U\delta_\varepsilon$.

We have then

$$\mu_n(F) \leqslant \mu_n f \qquad \text{for all n}$$

$$\lim_{n=\infty} \mu_n f = \mu f \qquad \text{by (i)}$$

$$\mu f \leqslant \mu(F) + \varepsilon$$

We can deduce that

$$\limsup_{n=\infty} \mu_n(F) \leqslant \mu(F) + \varepsilon$$

This inequality holds for every ε so that (ii) is established.

(ii) => (iii)

First note that (ii) implies

(ii)' $\mu(B) = \lim_{n=\infty} \mu_n(B)$

for every Borel set B such that its boundary (to be denoted by $\tilde{B}$) is of μ- measure zero.

Choose aB such that $\mu(\tilde{B}) = 0$

We have for all n.

$$1 - \mu_n(B^c) = \mu_n(B)$$

Hence

(1) $\liminf_{n=\infty} (1 - \mu_n(B^c)) = \liminf_{n=\infty} \mu_n(B) \leqslant \limsup_{n=\infty} \mu_n(B)$

But $B \cup \tilde{B}$ is closed so that (ii) implies

$$(2) \quad \limsup_{n=\infty} \mu_n(B) \leq \limsup_{n=\infty} \mu_n(B \cup \tilde{B}) \leq \mu(B \cup \tilde{B}) \leq$$

$$\mu(B) + \mu(\tilde{B}) = \mu(B)$$

similarly as $B^c \cup \tilde{B}$ is a closed set we have

$$(3) \quad \liminf_{n=\infty} (1 - \mu_n(B^c)) \geq \mu(B)$$

And from (1), (2) and (3) we get (ii)'

Now pick an f satisfying the hypothesis of (iii) i.e.

$$\mu(D_f) = 0 \text{ where } D_f = \{ y \mid f(x) \text{ is discontinuous at } y \}$$

Let α be a point of continuity of $\varphi(f, \alpha)$

Let $C = \{ x \mid f(x) \leq \alpha \}$

We have $\tilde{C} = \{ x \mid$ there exist two sequences y_i and z_i such that $\lim_{i=\infty} y_i = \lim_{i=\infty} z_i = x$ and $y_i \in C$ for all i in N $z_i \notin C$ for all in N $\}$

$\tilde{C}$ is contained in $\tilde{C} \cap (E - D_f) \cup D_f$

For every x in $\tilde{C} \cap (E - D_f)$ we have

$$\alpha \geq f(y_i) \longrightarrow f(x) \quad \text{as } i \to \infty$$

and

$$\alpha \leq f(z_i) \longrightarrow f(x) \quad \text{as } i \to \infty$$

so that

$$f(x) = \alpha$$

and this shows

$$\tilde{C} \cap (E - D_f) \subset \{ x \mid f(x) = \alpha \}$$

where the set on the R.H.S. is obviously equal to

$$\bigcap_{l=1}^{\infty} \{ x \mid \alpha - \tfrac{1}{l} < f(x) \leq \alpha + \tfrac{1}{l} \}$$

By choice of α we get

$$\mu\left(\bigcap_{l=1}^{\infty} \{ \cdot \} \right) = \liminf_{l=\infty} \mu\left(\left\{ x \mid \alpha - \tfrac{1}{l} < f(x) \leq \alpha + \tfrac{1}{l} \right\} \right) =$$

$$\liminf_{l=\infty} \left[\varphi(f, \alpha + \tfrac{1}{l}) - \varphi(f, \alpha - \tfrac{1}{l}) \right] = 0$$

so that

$$\mu(\tilde{C}) \leqslant \mu\left(\left\{x \mid f(x) = \alpha\right\}\right) + \mu(D_f) = 0$$

and we can apply (ii)' to C and obtain

$$\lim_{n=\infty} \varphi_n(f;\alpha) = \varphi(f;\alpha)$$

(iii) $\Longrightarrow$ (i)

Choose a bounded continuous function f. In fact as f is bounded we may even assume $0 \leqslant f \leqslant M < \infty$.

As $D_f = \emptyset$, $\mu(D_f) = 0$ and f satisfies the hypothesis of (iii). The function $\varphi(f,\alpha)$ is monotonic and hence has at most a countable number of jumps; let J be the set of those points. For every positive integer r and every $j \leqslant m(r)$ choose an a_{kj} not in J such that

$$0 = a_{ko} < a_{k1} < a_{k2} < \cdots < a_{km(r)} = M$$

and

$$\max_{0<j\leqslant m(r)} (a_{kj} - a_{kj-1}) \downarrow 0 \text{ as r tends to } \infty$$

we have

$$(4) \quad \mu f = \lim_{r=\infty} \sum_{j=0}^{m(r)-1} a_{kj}\, \mu(a_{kj} < f \leqslant a_{kj+1}) =$$

$$\lim_{r=\infty} \sum_{j=0}^{m(r)-1} a_{kj}\left[\varphi(f;a_{kj+1}) - \varphi(f;a_{kj})\right]$$

As all a_{kj} are outside J and $\mu(D_f) = 0$ we can use (iii) and the last sum becomes

$$\lim_{r=\infty} \lim_{n=\infty} \sum_{j=0}^{m(r)-1} a_{kj}\left[\varphi_n(f,a_{kj+1}) - \varphi_n(f;a_{kj})\right]$$

The positivity of all terms is ensured by the monotonicity of all $\varphi_n(f, .)$ and the sums themselves being monotonic increasing in r we can interchange the limits to obtain

$$(5) \quad \lim_{n=\infty} \lim_{r=\infty} \sum_{j=0}^{m(r)-1} a_{kj}\left[\varphi_n(f,a_{kj+1}) - \varphi_n(f;a_{kj})\right] = \lim_{n=\infty} \mu_n f$$

The equality (4) = (5) is the statement (i).

Theorem 0

Let B be a Borel set in a metric space E, consider the induced topology on B, it is also metric with the same metric (restricted to B).

If μ and μ_n are probability measures on E, all fully supported by B such that $\mu_n \xrightarrow{w} \mu$ on E, then $\mu_n \xrightarrow{w} \mu$ on B.

Conversely if $\mu_n \xrightarrow{w} \mu$ on B the measures extended to E by setting $\mu_n(E - B) = 0$ for all n tend weakly to μ extended to E in the same way.

Proof:

A set $F' \subset B$ is closed in the induced topology if and only if it is of the form $B \cap F$, where F is a closed set in E.

We have

$$\mu_n(F') = \mu_n(B \cap F) = \mu_n(F)$$
$$\mu(F') = \mu(B \cap F) = \mu(F)$$

By the lemma we have

$$\limsup_{n=\infty} \mu_n(\underline{F'}) \leq \mu(\underline{F'})$$

F/

Hence $\limsup_{n=\infty} \mu_n(F') \leq \mu(F')$ is true for all closed sets in the induced topology and by the lemma again this shows that $\mu_n f \to \mu f$ as n tends to ∞ for all bounded continuous functions defined on B.

The converse is obvious as any continuous function on E restricted to B is continuous there.

(b) A result which we shall need very often is the following theorem of Henry Scheffé in [II] . From now on we shall refer to it as Scheffé's theorem.

Scheffé's theorem:

Let $(E, \mathcal{E}, \mu)$ be a measure triple. If $f_n(x)$, n in N, is a sequence of positive $\mathcal{E}$- measurable functions defined on E such that

$$\lim_{n=\infty} f_n(x) = f(x) \qquad \text{for } \mu \text{ - almost all } x$$

and

$$\lim_{n=\infty} \int_E f_n(x)\mu(dx) = \int_E f(x)\mu(dx) < \infty$$

then

$$\lim_{n=\infty} \int_B |f_n(x) - f(x)| \, \mu(dx) = 0 \qquad \text{uniformly for all sets B in } \mathcal{E}$$

1 The entrance boundary as defined by J. Neveu.

Contrarily to Neveu in [8] we restrict our study to the stochastic case (except in the last chapter) for the following reasons:

a) it is always possible to increase A by an additional absorbing state δ and so obtain a stochastic matrix on $A \cup \{\delta\}$ (see e.g. C th. II 3.3.).

b) we want to compare the entrance boundaries as defined by Neveu and Doob in [5], but Doob works with stochastic matrices only, hence the procedure a) has already been used.

We begin by quoting two essential analytical results about entrances.

Theorem 1 (Neveu's th. 2.1.1in [7]).

For every i in A, $f_i(t)$ is continuous in t on (o, ∞), and tends to a limit $f_i(o)$, say, as t tends to 0. Morever the vector f(o) satisfies

$$f(o)\,P(t) \leq f(t) \quad \text{for all } t > 0$$

For all $\lambda > 0$, define $R_{ij}(\lambda)$ and $\hat{f}_i(\lambda)$ as the following Laplace transforms.

$$R_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt \quad \text{for all i and j in A}$$

$$\hat{f}_i(\lambda) = \int_0^\infty e^{-\lambda t} f_i(t)dt \quad \text{for all i in A}$$

Th. 2 (Neveu's proposition 1 in [8]).

The Laplace transforms $\hat{f}_i(\lambda)$ are such that

(6) $\|\hat{f}(\lambda)\| < \infty$ for all $\lambda > 0$

(7) $\hat{f}(\lambda) - \hat{f}(\mu) = (\mu - \lambda)\hat{f}(\lambda)R(\mu)$ for all λ and $\mu > 0$

(8) $\hat{f}(\lambda) \geq e^{-\lambda s}\hat{f}(\lambda)P(s)$ for all $\lambda > 0$ and all $s \geq 0$

Conversely 1°, any family of positive numbers $\hat{f}_i(\lambda)$, i in A, $\lambda > 0$ such that (6) and (7) are true is the family of Laplace transforms. of a uniquely determined entrance relative to P(t).

Converely 2° if m(i), i in A, is a sequence of positive numbers such that

(9) $$\sum_{k \in A} m(k) < \infty$$

(10) $$m(i) \geq e^{-\lambda s} \sum_{k \in A} m(k)\, p_{ki}(s) \quad \text{for one } \lambda \text{ and all } s > 0$$

then there exists a uniquely determined entrance relative to P(t), f(t), say, such that

(11) $$m(i) = \int_0^\infty e^{-\lambda t} f_i(t)\, dt \qquad \text{for all } i \text{ in } A$$

If we consider the elements in the cone F which satisfy the additional conditions:

(12) $$\sum_{k \in A} f_k(s) \leq 1 \qquad \text{for one (or all) } s > 0$$

then every convex combination of such elements satisfies the same inequality (or inequalities), and hence those entrances form a convex set, F_1, say.

Let M_1 be the set of all positive measures on A(i.e. sequences of positive numbers indexed by A) such that

(13) $$\sum_{k \in A} m(k) \leq 1$$

(14) $$e^{-s} \sum_{k \in A} m(k)\, p_{ki}(s) \leq m(i) \quad \text{for all } i \text{ in } A, \text{ and all } s \geq 0$$

We remark that M_1 is also a convex set. There is a one-to-one correspondence between M_1 and F_1.

Proof:

a) Obviously the relations I (15) and (12) imply that $\hat{f}(1)$ is an element of M_1 for all elements of F_1. Furthermore if f(t) and g(t) are two elements of F_1 such that $\hat{f}(1) = \hat{g}(1)$ then theequation (7) gives

$$\hat{f}(\lambda) = \hat{f}(1) + (1-\lambda)\, \hat{f}(1)R(\lambda)$$
$$= \hat{g}(1) + (1-\lambda)\, \hat{g}(1)R(\lambda) = \hat{g}(\lambda)$$

or

$$\hat{f}_i(t) = \hat{g}_i(\lambda) \quad \text{for all } \lambda > 0 \text{ and all i in A}$$

But by th. 1 we know that $f_i(t)$ and $g_i(t)$ are both continuous on $(0,\infty)$, hence this last equality is enough to check that

$$f_i(t) = g_i(t) \quad \text{for all i in A and all } t > 0$$

b) Conversly if $\{m(i), \text{ i in A}\}$ is in M_1, by th.2, there exists an entrance f(t) in F such that (11) holds for $\lambda = 1$. Since P(t) is stochastic we have for $t > 0$:

$$\sum_{i\in A} f_i(t) = \sum_{i\in A} f_i(t) \left[\sum_{k\in A} \int_0^\infty e^{-s} p_{ik}(s)\, ds \right]$$

$$= \int_0^\infty \sum_{i\in A} \sum_{k\in A} f_i(t)\, p_{ik}(s)\, e^{-s}\, ds$$

$$= \int_0^\infty \sum_{k\in A} f_k(t+s)\, e^{-s}\, ds = \int_0^\infty \sum_{k\in A} \sum_{j\in A} f_j(s)\, p_{jk}(t)\, e^{-s}\, ds$$

$$= \sum_{j\in A} \int_0^\infty f_j(s) e^{-s} ds \left[\sum_{k\in A} p_{jk}(t) \right] = \sum_{j\in A} m(j) \leqslant 1$$

and hence f(t) is indeed in F_1.

Now if $m_1(i)$ and $m_2(i)$, i in A are both in M_1 and such that

$$f_i^1(s) = f_i^2(s) \quad \text{for all i in A and all } s > 0$$

then

$$m_1(i) = \int_0^\infty e^{-s} f_i^1(s)\, ds = \int_0^\infty e^{-s} f_i^2(s)\, ds = m_2(i)$$

We consider the single convergence topology on M_1 and denote it by T. If m^n and m are in M_1, $m^n \to m$ in T as n tends to ∞, if and only $m^n(i) \to m(i)$ as n tends to ∞ for all i in A. If β_i, i in A, is a sequence of strictly positive numbers such that their sum over all i in A is finite, then the topology T is metrisable by setting for all m and m' in M_1:

$$d(m;m') = \sum_{i\in A} \beta_i \left| m(i) - m'(i) \right|$$

A metric space is compact if and only if every sequence of elements has a convergent subsequence.

Let $\{m^n(i), i \text{ in } A\}$ be in M_1, for all n in N, then $m^n(i)$ is in $[0, 1]$ for all i in A and all n in N; hence by the diagonal procedure we may extract a subsequence n_r such that $m^{nr}(i)$ tends to m(i) a point of $[0,1]$ as n_r tends to ∞ for every i in A.

The inequalities

$$e^{-s}\sum_{k\in A} m^{nr}(k)\, p_{ki}(s) \leqslant m^{nr}(i)$$

for all $s>0$, all i in A and all n_r and

$$\sum_{k\in A} m^{nr}(k) \leqslant 1 \qquad \text{for all } n_r$$

yield by Fatou's lemma

$$e^{-s}\sum_{k\in A} m(k)\, p_{ki}(s) \leqslant m(i)$$

for all $s>0$ and all i in A and

$$\sum_{k\in A} m(k) \leqslant 1$$

Thus the measure $\{m(i), i \text{ in } A\}$ lies in M_1, and we have m^{nr} converges to n in $\dot{T}$ as n_r tends to ∞ . This establishes that (M_1, T) is a convex set, which is a compact space for the metrisable topology of the simple convergence.

Th 3.

For every k in A, the measure $\{R_{ki}(1), i \text{ in } A\}$ is an extremal point of M_1.

Proof: First notice that I.(6); I.(7) and I.(8) ensure that $\{R_{ki}(1), i \text{ in } A\}$ is a point of M_1. Now assume

$$R_{ki}(1) = \alpha\, m(i) + (1-\alpha)\, l(i) \qquad \text{for all i in A}$$

where m and l are in M_1, and $0 < \alpha < 1$.

By th. 2 there exist two entrances g(s) and h(s) such that for all i in A

$$m(i) = \int_0^\infty e^{-s} g_i(s)\, ds = \hat{g}_i(1)$$

$$l(i) = \int_0^\infty e^{-s} h_i(s)\, ds = \hat{h}_i(1)$$

In fact their Laplace transforms exist for all $\lambda > 0$ and satisfy for all i in A.

$$(\lambda - 1)\sum_{j \in A} \hat{g}_j(1)\, R_{ji}(\lambda) = \hat{g}_i(1) - \hat{g}_i(\lambda)$$

$$(\lambda - 1)\sum_{j \in A} \hat{h}_j(1)\, R_{ji}(\lambda) = \hat{h}_i(1) - \hat{h}_i(\lambda)$$

which in turn imply for all i in A

$$R_{ki}(1) - R_{ki}(\lambda) = (\lambda - 1)\sum_{j \in A} R_{kj}(1)\, R_{ji}(\lambda) =$$

$$(\lambda - 1)\sum_{j \quad A} [\alpha \hat{g}_j(1) + (1-\alpha)\hat{h}_j(1)]\, R_{ji}(\lambda) =$$

$$\alpha \hat{g}_i(1) - \alpha \hat{g}_i(\lambda) + (1-\alpha)\hat{h}_i(1) - (1-\alpha)\hat{h}_i(\lambda) .$$

Hence $R_{ki}(\lambda) = \alpha \hat{g}_i(\lambda) + (1-\alpha)\hat{h}_i(\lambda)$ holds for all $\lambda > 0$ and all i in A, and from this we deduce the following equality

(15) $\quad p_{ki}(t) = \alpha\, g_i(t) + (1-\alpha)\, h_i(t)$ for all $t > 0$ and all i in A.

Now use th 1 to define g'(t) and h'(t) by

$$g'(t) = g(t) - g(0)\, P(t) \geqslant 0$$

$$h'(t) = h(t) - h(0)\, P(t) \geqslant 0$$

where $g_i(0) = \lim_{s=0} g_i(s)$ and $h_i(0) = \lim_{s=0} h_i(0)$

As g(t) and h(t) are both in F_1 we get $g_i(0) \leqslant 1$ and $h_i(0) \leqslant 1$ for all i in A. But if we let t decrease to 0 in (15) we get for all i in A

$$\delta_{ki} = \alpha g_i(0) + (1-\alpha)\, h_i(0)$$

And the two last facts imply

$$\delta_{ki} = g_i(0) = h_i(0) \qquad \text{for all } i \text{ in } A$$

so that(15) can be rewritten as

$$p_{ki}(t) = p_{ki}(t) + \alpha\, g_i'(t) + (1-\alpha)\, h_i'(t) \quad \text{for all } t > 0$$
$$\text{and all } i \text{ in } A.$$

As $g'(t)$ and $h'(t)$ are both positive the last equation is possible only if

$g_i'(t) = h_i'(t) = 0$ for all $t > 0$ and all i in A which implies

$$m(i) = l(i) = \int_0^\infty e^{-s}\, p_{ki}(s)\, ds \qquad \text{for all } i \text{ in } A$$

i.e. $\{R_{ki}(1),\ i \text{ in } A\}$ is extremal in the convex set M_1. Naturally with every k in A we associate the element $\{R_{ki}(1),\ i \text{ in } A\}$ of M_1 and we may write $A \subset M_1$. Define A_e as the set of all the extreme points of M_1 different from those of A and not equal to the trivial measure 0.

Definition:

The set $A + A_e$ (contained in M_1) with the topology induced by T is called the Neveu entrance boundary for P(t) and will be denoted by $(A + A_e, T)$.

By th. 2 we know that with any x in M_1 is associated a uniquely determined entrance relative to P(t). This entrance will be denoted by $p_{xi}(t)$, i in A, $t > 0$ and its Laplace transforms by $R_{xi}(\lambda)$, i in A, $\lambda > 0$. For every y in A_e the corresponding entrance is such that

$$\sum_{i \in A} \int_0^\infty e^{-s}\, p_{yi}(s)\, ds = 1$$

If this were not true, i.e. if this sum were equal to $c < 1$, then

$$(1-c)\, 0 + c \int_0^\infty e^{-s}\, \frac{1}{c}\, p_{yi}(s)\, ds \qquad \text{for all } i \text{ in } A$$

would be a non-trivial convex decomposition of y in M_1, and y would not be extremal in M_1. As we know that $p_{yi}(t)$, i in A, $t > 0$ is indeed in F_1, we can conclude

$$(16) \quad \sum_{i \in A} p_{yi}(t) = 1 \quad \text{for all } t > 0$$

Th. 4.

A is dense in A_e in the topology T.

Proof: Pick an element $\{f_i(1), \text{ in } A\}$ of M_1. The following relations are known

$$(17) \quad \hat{f}(1) \geq \lambda \hat{f}(1) R(\lambda+1) \quad \text{for all } \lambda > 0$$

$$(18) \quad \lim_{\lambda=\infty} \| (\lambda+1) \hat{f}(1) R(\lambda+1) - \hat{f}(1) \| = 0$$

$$(19) \quad \| \lambda \hat{f}(\lambda) \| = c \ (\leq 1 \text{ as we are in } M_1) \text{ for all } \lambda > 0$$

$$(20) \text{ Let } \hat{f}(1) D(\lambda) = \lambda (\hat{f}(1) - \lambda \hat{f}(1) R(\lambda+1)) \geq 0$$

$$(21) \quad [\hat{f}(1) D(\lambda)] R(1) = \lambda [\hat{f}(1) R(1) - \lambda \hat{f}(1) R(\lambda+1) R(1)] =$$

$$\lambda [\hat{f}(1) R(1) - \hat{f}(1) R(1) + \hat{f}(\lambda+1) R(1)] =$$

$$(22) \quad \hat{f}(1) - \hat{f}(\lambda+1) =$$

$$(23) \quad \lambda \hat{f}(1) R(\lambda+1)$$

By the resolvent equation, (22) is increasing as λ increases to ∞; so if we use (18), the equation (21) = (23) for all λ yields

$$(24) \quad \lim_{\lambda=\infty} \uparrow [\hat{f}(1) D(\lambda)] R(1) = \hat{f}(1)$$

Define A* as the set of measures on A which are limits of the measures generated by A, i.e. $x = \{ m(i), \text{ i in } A\}$ is in A* if and only if there exists a sequence i_n of points in A such that

$$(25) \quad m(i) = R_{xi}(1) = \lim_{n=\infty} R_{i_n i}(1) \text{ for all i in A}$$

If the topology on A* is the simple convergence one (see definintion of (M_1,T)p.21 then for reasons similar to those used for M_1, A* is a compact metric space.

(20) is $\sum_{k \in A} [\hat{f}(1)\, D\,(\lambda)]_k\, R_{ki}(1)$ for all i in A

and hence $\hat{f}(1)D\,(\lambda)$ may be considered as a measure, $g(\lambda\,;\,dx)$ say, on the Borel sets of A*, which is fully supported by A.

(24) now becomes

$$(26) \quad \hat{f}_i(1) = \lim_{\lambda=\infty} \int_{A^*} g(\lambda\,;dx)\, R_{xi}(1) \quad \text{for all i in A}$$

where $g(\lambda\,;\,A^*) = \| \lambda f(\lambda+1) \| \leqslant 1$ for all $\lambda > 0$ by (19).

The set of all measures of total mass $\leqslant 1$ on a compact set being itself compact, we may extract a sequence λ_m increasing to ∞ such that $g(\lambda_m;\,dx) \rightarrow \nu(dx)$, a measure on A*. By the very definition of A*, $R_{xi}(1)$ is continuous from A* into R; hence (26) gives

$$(27) \quad \hat{f}_i(1) = \int_{A^*} \nu(dx)\, R_{xi}(1) \quad \text{for all i in A}$$

If $\hat{f}(1)$ is extremal in M_1, we have $\| \hat{f}(1) \| = 1$, so the corresponding measure $\nu(\,.\,)$ must be of total mass equal to 1, and indeed fully supported by the points in A* such that

$$\sum_{i \in A} R_{xi}(1) = 1$$

As $\nu(\,.\,)$ is not identically zero, there exists an x_o in A* such that any neighbourhood Vx_o of x_o is of strictly positive ν-measure, We may write $\hat{f}_i(1) = \int_{Vx_o} \nu(dx)\, R_{xi}(1) + \int_{A^*-Vx_o} \nu(dx)\, R_{xi}(1)$

Now if $\nu(Vx_o) < 1$, we get

$$\hat{f}_i(1) = \nu(Vx_o) \int_{Vx_o} \frac{\nu(dx)}{\nu(Vx_o)} R_{xi}(1) + (1-\nu(Vx_o)) \int_{A^*-Vx_o} \frac{\nu(dx)}{1-\nu(Vx_o)} R_{xi}(1)$$

and the extremality of $\hat{f}(1)$ implies $\hat{f}_i(1) = \int_{Vx_o} \frac{\nu(dx)}{\nu(Vx_o)} R_{xi}(1)$

for all i in A, and all Vx_o.

Choosing as neighbourhoods Vx_o a sequence of open spheres centred in x_o and whose radii decrease to 0, the continuity of $R_{xi}(1)$ ensures that

$$\hat{f}_i(1) = R_{x_o} i(1) \quad \text{for all } i \text{ in } A$$

Now we turn back to the definition of A* to obtain a sequence i_n in A for which (25) holds, and we get

(28) $\hat{f}_i(1) = R_{x_o i}(1) = \lim_{n=\infty} Ri_n i\,(1)$ for all i in A

This completes the proof of the density of A in A_e for the topology T. For a fixed y in A+Ae and every $t>0$ the entrance $p_{yi}(t)$ generates a measure on the Borel sets of (M_1,T) in the following way. For every Borel set B define $p_{yB}(t)$ by

(29) $p_{yB}(t) = \sum_{i \in A\cap B} p_{yi}(t)$

In fact as this measure is fully supported by A we may also consider it as a measure on the Borel sets of $(A+A_e;\, T)$.

Th. 5.

These measures satisfy two interesting properties:

(i) the strong Feller property

i.e. for any bounded measurable function f defined on $(A+A_e,T)$, with values in R, we have

$\sum_{i\in A} p_{yi}(t)\, f\,(i)$ is a continuous function from

$(A+A_e,T)$ into R, for every fixed $t > 0$.

(ii) the stochastic continuity property

i.e. for every bounded continuous function f defined on $(A+A_e,T)$ with values in R, we have

$\lim_{t=0} \sum_{i\in A} p_{yi}(t)\, f\,(i) = f(y)$ for all y in $A+A_e$.

lemma (which is the proposition 2 of Neveu in [8]).

If m and m^n, n in N, are elements of M_1 such that

(30) $$\sum_{i \in A} \left| m^n(i) - m(i) \right| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

then their corresponding entrances $f(t)$ and $f^n(t)$, $t > 0$ are such that

(31) $$\lim_{n=\infty} f_i^n(t) = f_i(t) \quad \text{for all } i \text{ in } A \text{ and for all } t > 0$$

Moreover the convergence is uniform on $[a, \infty]$ for all $a > 0$.

Proof, (which is reproduced here from p.326-7 of [8]).

For any entrance $f(t)$ and any i in A we have for $0 < u < v < \infty$

$$\int_u^v e^{-t} f_i(t)dt = \int_u^\infty e^{-t} f_i(t)dt - \int_v^\infty e^{-t} f_i(t)dt =$$

$$\int_0^\infty e^{-t} e^{-u} f_i(t+u)dt - \int_0^\infty e^{-t} e^{-v} f_i(t+v)dt =$$

$$\sum_{k \in A} e^{-u} p_{ki}(u) \int_0^\infty e^{-t} f_k(t)dt - \sum_{k \in A} e^{-v} p_{ki}(v) \int_0^\infty e^{-t} f_k(t)dt$$

Hence

$$\left| \int_u^v e^{-t} f^n i(t)dt - \int_u^v e^{-t} fi(t)dt \right| =$$

$$\Big| \sum_{k \in A} [\hat{f}^n{}_k(1) - \hat{f}_k(1)]\, e^{-u} p_{ki}(u) - \sum_{k \in A} [\hat{f}^n{}_k(1) - \hat{f}_k(1)]\, e^{-v} p_{ki}(v) \Big|$$

As $e^{-s} p_{ki}(s)$ 1 for all $s > 0$, all k and all i in A the last term is bounded by

$$2 \sum_{k \in A} \left| m^n(k) - m(k) \right|$$

so that if we use (30) we get for all i in A

(32) $$\lim_{n=\infty} \int_u^v e^{-t} f^n{}_i(t)dt = \int_u^v e^{-t} f_i(t)dt$$

By I (15) we have for all n in N, all i in A, and all $0 < u < t$ ∞

$$f^n{}_i(u) p_{ii}(t-u) \leqslant f_i^n(t)$$

Hence

$$e^{-u} f_i^n(u) \int_0^{v-u} e^{-s} p_{ii}(s)ds \leqslant \int_u^v e^{-t} f_i^n(t)dt$$

If n tends to ∞ in this last inequality, (32) gives

$$e^{-u} \limsup_{n=\infty} f_i^n(u) \int_0^{v-u} e^{-s} p_{ii}(s)ds \leqslant \int_u^v e^{-t} f_i(t)dt$$

Next divide both sides by (v-u) and let v decrease to u, as $f_i(t)$ is continuous we obtain

(33)

$$\limsup_{n=\infty} f_i^n(u) \quad \leqslant \quad f_i(u)$$

By using I (15) again we have for all n in N, all i in A, and all $0 < t < v$

$$f_i^n(t) \leqslant f_i^n(v) \; [\; p_{ii}(v-t)]^{-1}$$

and hence

$$\int_u^v e^{-t} f_i^n(t)dt \leqslant \int_u^v e^{-t} f_i^n(v)[p_{ii}(v-t)]^{-1}dt$$

$$e^{-u} f_i^n(v)[\inf_{o < s \leqslant v-u} p_{ii}(s)]^{-1}(v-u)$$

If n tends to ∞ in this inequality, (32) then gives

$$\frac{1}{v-u} \int_u^v e^{-t} f_i(t)dt \leqslant e^{-u} \liminf_{n=\infty} \; f_i^n(v) \; [\inf_{o < s \leqslant v-u} p_{ii}(s)]^{-1}$$

The continuity of $f_i(t)$ ensures that when u increase to v this inequality becomes

(34) $$f_i(v) \leqslant \liminf_{n=\infty} \; f_i^n(v)$$

Since (33) and (34) hold for every strictly positive number and all i in A they are equivalent to (31).

As P(t) is stochastic, (30) yields for all $t > 0$

$$\sum_{k \in A} f_k(t) = \sum_{k \in A} m(k) = \lim_{n=\infty} \sum_{k \quad A} m^n(k) = \lim_{n=\infty} \sum_{k \in A} f_k^n(t)$$

This fact used in conjuction with (31) is enough to yield by the Scheffé's theorem

(35) $\lim_{n=\infty} \sum_{k\in A} \left| f_k^n(t) - f_k(t) \right| = 0$ for all $t>0$

If $t>0$ and $s>0$ we have for all i in A

$$\left| f_i^n(t+s) - f_i(t+s) \right| =$$

$$\left| \sum_{k\in A} f_k^n(t)\, p_{ki}(s) - \sum_{k\in A} f_k(t)\, p_{ki}(s) \right| \leqslant$$

$$\sum_{k\in A} \left| f_k^n(t) - f_k(t) \right|$$

This inequality and (35) prove that the convergence in (31) is uniform on $[t, \infty]$, for every $t>0$, and the lemma is then established.

<u>Proof of th.5 (i)</u>

Suppose y converges to y_o in $(A+A_e, T)$, i.e.

$$\lim_{y=y_o} Ryi(1) = Ry_oi(1) \quad \text{for all } i \text{ in } A$$

The additional condition (true on $A+A_e$)

$$\sum_{i\in A} R_{yi}(1) = 1 = \sum_{i\in A} R_{y_o i}(1)$$

is then enough to give by the Scheffé's theorem

$$\lim_{y=y_o} \sum_{i\in A} \left| R_{yi}(1) - R_{y_o i}(1) \right| = 0$$

Hence the le applies and it yields

(36)

$$\lim_{y=y_o} p_{yi}(t) = p_{y_o i}(t) \quad \text{for all } i \text{ in } A \text{ and all } t>0$$

But (16) holds for all points of $A+A_e$ so that if f is a bounded measurable function defined on $A+A_e$, the relation (36) and the Scheffé's theorem imply for any fixed $t>0$.

$$\lim_{y=y_o} \sum_{i\in A} p_{yi}(t) f(i) = \sum_{i\in A} p_{y_o i} f(i)$$

i.e. (i) is true.

Proof of th 5(ii)(reproduced here from p. 328.9 of Neven [8]).

Fix i in A and consider the function defined for all x in M_1, as the value of the measure x on the Borel set $\{i\}$, $x(i) = R_{xi}(1)$. The function $R_{xi}(1)$ is continuous from (M_1,T) into $[0,1]$ by the very definition of the single convergence topology.

Pick y in $A+A_e$, then as M_1 is compact the measures $p_{y\cdot}(t)$ which satisfy $p_{yM_1}(t) = p_{yA}(t) = 1$ for all $t>0$, have at least one weak limit for a suitable sequence t_n, n in N, decreasing to 0. Let such a weak limit be $\mu(\ )$; it has the property that $\mu(M_1) \leq 1$.

By definition, μ is such that in particular

$$\lim_{n=\infty} \int_{M_1} p_{ydx}(t_n)\, R_{xi}(1) = \int_{M_1} \mu(dx)\, R_{xi}(1)$$

But
$$\sum_{k\in A} p_{yk}(t_n)\, R_{ki}(1) = \sum_{k\in A} p_{yk}(t_n) \int_0^\infty e^{-s}\, p_{ki}(s)ds$$

$$= \int_0^\infty e^{-s} \sum_{k\in A} p_{yk}(t_n)p_{ki}(s)\, ds$$

$$= \int_0^\infty e^{-s} \sum_{k\in A} p_{yk}(s)p_{ki}(t_n)\, ds$$

$$= \sum_{k\in A} R_{yk}(1)\, p_{ki}(t_n)$$

and the last term tends to $R_{yi}(1)$ as t_n decreases to 0. Hence we get

$$\int_{M_1} \mu(dx)R_{xi}(1) = R_{yi}(1) \quad \text{for all i in A} \tag{37}$$

By an argument similar to the one used to show the density of A in A_e, (37) and the extremality of y in M_1, imply that $\mu(\ ) = \varepsilon_y(\ )$. As (37) is true for any sequence $\{t_n\}$, $\varepsilon_y(\ )$ is in fact the weak limit of $p_{y\cdot}(t)$ as t tends to 0.

By a theorem of Choquet (see e.g. M XI 24) we know that $0 + A + A_e$ is a G_δ - set in (M_1,T). Since $A+A_e = (M_1-\{o\}) \cap (o + A + A_e)$ it is also a G_δ - set and a Borel set of M_1.

As $\mathcal{E}_y(\ )$ and $p_{y.}(t)$ are all fully supported by $A + A_e$, th 0 yields

$$\lim_{t=0} \sum_{k \in A} p_{yk}(t) f(k) = f(y)$$

for every bounded continuous function defined on $(A + A_e, T)$,

i.e. (ii) is true.

Another remarkable consequence of the lemma is the following.

The function $p_{yi}(t)$ is not only continuous from $(A + A_e, T)$ into $[0,1]$ for a fixed $t > 0$ and a fixed i in A as established in (36), but is in fact continuous from $(A+A_e \times (0,\infty);\ T \times$ (euclidean topology)) into $[0,1]$ for every fixed i in A. This is readily concluded from the uniform convergence on any $[a,\infty)$, $a > 0$.

We now establish a useful property on the neighbourhoods of y in $(A+A_e,T)$. If V is a neighbourhood of y, then for every $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that

$$(38) \quad \sum_{i \in V \cap A} p_{yi}(t) \geqslant 1 - \epsilon \quad \text{for all } t \leqslant t_\epsilon$$

<u>Proof:</u>

By the lemma of paragraph 0 we know that if E is a metric space, the probability measures on its Borel sets μ_t converge weakly to the probability measure μ_o as t tends to 0 if and only if

$$\limsup_{t=0} \mu_t(F) \leqslant \mu_o(F) \quad \text{for all closed sets F}$$

By conplementation it yields

$$\liminf_{t=0} \mu_t(G) \geqslant \mu_o(g) \quad \text{for all open sets G}$$

Now the statement (ii) in th 5 is $p_{y.}(t) \xrightarrow{W} \mathcal{E}_y(\ .\ )$ on $A+A_e$ as t tends to 0. As every neighbourhood V of y must contain an open set containing y, $G(y)$ say, we get

$$\liminf_{t=0} p_{yV}(t) \geqslant \liminf_{t=0} p_{y\,G(y)}(t) \geqslant \mathcal{E}_y(G(y)) = 1$$

and this implies (38).

Another result which will be used later is the following theorem

Theorem 6

If y is in $M_1-(A+A_e)$, then the measure $p_{y\cdot}(t)$ does not tend weakly to $\mathcal{E}_y(\,.\,)$ as t tends to 0 on the set M_1.

Proof: The case of y = 0 in M_1 is obvious and hence we assume $y \neq 0$ in the following.

By M XI 25 and M XI 29 we know that for all y not in $A+A_e$, there exists a uniquely defined measure $\nu(\,.\,)$ on $A+A_e$ such that

$$\nu(0 + A + A_e) = 1$$

and

$$R_{yi}(1) = \int_{A+A_e} \nu(dx)\, R_{xi}(1) \quad \text{for all i in A}$$

Equivalently by the Fubini's theorem

(39) $$p_{yi}(t) = \int_{A+A_e} \nu(dx)\, p_{xi}(t) \quad \text{for all i in A and all } t > 0$$

As $\nu(\,.\,)$ is fully supported by the extreme points of M_1, we can find a point z in $A+A_e$ such that all its neighbourhoods are of strictly positive ν- measure. By choice of y, y is different of z, and so a suitable $\varepsilon > 0$ may be found such that the closed sphere centred in z with radius ε, $\overline{B}(z, \varepsilon)$ say, does not contain y.

If we let

$$f(x) = \frac{d(x; \overline{B}(z, \varepsilon))}{d(x;y) + d(y;\overline{B}(z,\varepsilon))} \qquad \text{for all x in } M_1$$

then f is a continuous function defined on M_1 which satisfies

$$f(y) = 1$$

$$0 \leq f(x) \leq 1 \qquad \text{for all x in } M_1$$

$$f(x) = 0 \qquad \text{for all x in } \overline{B}(z,\varepsilon).$$

Now by (39) and positivity we have

$$\sum_{k\in A} p_{yk}(t)f(k) = \sum_{k\in A} \int_{A+A_e} \nu(dx)p_{xk}(t)f(k) =$$

(40) $$= \int_{A+A_e} \nu(dx) \sum_{k\in A} p_{xk}(t)\, f(k)$$

The sums in (40) are bounded by 1 for all x in $A+A_e$ and they converge to f(x) as t tends to 0 by th 5(ii) and th 0. Hence we can use the Lebasgue's dominated convergence theorem to get

$$\lim_{t=0} \sum_{k \in A} p_{yk}(t)\ f\ (k) = \int_{A+A_e} \nu(dx) f(x) \leqslant$$

$$1 - \nu\,(\bar{B}(z;\epsilon)) < 1 = f(y)$$

and this proves th 6.

2. The entrance boundary as defined by J.L. Doob.

With every k in A associate the following countable set of non-negative numbers

(41) $k \to \{\lambda R_{ki}(\lambda),\ i \text{ in } A,\ \lambda \text{ in } Q_+\}$

where $Q_+ = Q \cap (0, \infty)$

(41) is a vector of the countable product space

$C = [0,1]x[0,1]x[0,1]x.$, where the unit interval is taken $A \times Q_+$ times. If we consider on C the simple convergence topology (again denoted by T) then C is a compact metrisable spade(as M_1 was)

As $p_{ki}(t)$ is continuous on $[o, \infty)$ for fixed k and i in A, its Laplace transfoem is also continuous on $(0, \infty)$ and the values $\lambda R_{ki}(\lambda)$ for all λ in Q_+ are enough to detrrmine $\lambda R_{ki}(\lambda)$ for all λ in R_+. Hence if two points k_1 and k_2 of A are such that their corresponding vectors in C are identical, then the values $R_{k_1i}(\lambda)$ and $R_{k2i}(\lambda)$ are also equal for all λ in R_+ and all i in A; but this yields

$$p_{k_1i}(t) = p_{k_2i}(t) \quad \text{for all } t > 0 \text{ and all } i \text{ in } A$$

which in turn implies $k_1 = k_2$.

Hence A may be considered as a subset of C.

Let K be the set of all the points $\xi = \{\xi i(\lambda),\ i \text{ in } A,\ \lambda \text{ in } Q_+\}$ in C such that there exists a sequence of points i_n, in A for which

(42) $\xi i(\lambda) = \lim_{n=\infty} \lambda R_{i_n i}(\lambda)$ for all λ in Q_+ and all i in A

By this very definition K is a closed set in C and therefore is a compact metric space for the induced topology.

Define K_o as those elements in K for which there exists one λ in Q_+ such that

(43) $\sum_{i \in A} \xi i(\lambda) = 1$

This property does not in fact depend on a particular λ.

Proof:

By (42) we have if μ is in Q_+

(44) $\frac{1}{\lambda} \zeta i(\lambda) = \lim_{n=\infty} Ri_n i(\lambda)$ for all i in A

and $\frac{1}{\mu} \zeta i(\mu) = \lim_{n=\infty} Ri_n i(\mu)$ for all i in A

As all in are in A we have for all i in A

$$Ri_n i(\lambda) - Ri_n i(\mu) = (\mu - \lambda) \sum_{k \in A} Ri_n k(\lambda) R_{ki}(\mu)$$

and by (43)

(45) $\lim_{n=\infty} \sum_{k \in A} R_{i_n} k(\lambda) = \frac{1}{\lambda} = \sum_{k \in A} \frac{\zeta k(\lambda)}{\lambda}$

If we take the limits as n tends to ∞ on both sides of the resolvent equation, (44), (45) and the Scheffé's theorem allow an interchange between sum and limit so that we get

(46) $\frac{1}{\lambda} \zeta i(\lambda) - \frac{1}{\mu} \zeta i(\mu) = (\mu - \lambda) \sum_{k \in A} \frac{\zeta k(\lambda)}{\lambda} R_{ki}(\mu)$

for all μ in Q_+ and all i in A.

If we sum this last relation over all i in A we find

$$\frac{1}{\lambda} \sum_{i \in A} \zeta i(\lambda) - \frac{1}{\mu} \sum_{i \in A} \zeta i(\mu) = (\mu - \lambda) \sum_{k \in A} \frac{\zeta k(\lambda)}{\lambda} \sum_{i \in A} R_{ki}(\mu)$$

Since P(t) is stochastic and using (43) we get

$$\frac{1}{\lambda} - \frac{1}{\mu} \sum_{i \in A} \zeta i(\mu) = (\mu - \lambda) \frac{1}{\lambda} \frac{1}{\mu} = \frac{1}{\lambda} - \frac{1}{\mu}$$

i.e

(47) $\sum_{i \in A} \zeta i(\mu) = 1$ for all μ in Q_+

From (42), (47) and the Scheffé's theorem we can deduce that (46) holds for all λ in Q_+ and all μ in Q_+.

As usual (42) for $\lambda = 1$ and Fatou's lemma give

$$\sum_{k \in A} \xi_k(1)\, e^{-s} p_{ki}(s) \leqslant \xi_i(1) \quad \text{for all } s > 0 \text{ and all } i \text{ in } A$$

Furthermore if ξ is in K_0 (47) holds for $\lambda = 1$ in particular ; the measure on A defined by $\{\xi_i(1),\ i \text{ in } A\}$ is then an element of M_1. Hence by th. 2 there exists an entrance relative to P(t), $p_{\xi i}(t)$, i in A, $t > 0$ and its corresponding Laplace transforms $R_{\xi i}(\ )$, i in A, μ in R_+, say, such that

$$\xi_i(1) = \int_0^\infty e^{-t}\, p_{\xi i}(t)\,dt \quad \text{for all } i \text{ in } A$$

and by stochasticity and (47)

$$(48) \quad \mu \sum_{i \in A} R_{\xi i}(\mu) = 1 \quad \text{for all } \mu \text{ in } R_+$$

We have for all i in A and all μ in R_+

$$R_{\xi i}(1) - R_{\xi i}(\mu) = (\mu - 1) \sum_{k \in A} R_{\xi k}(1) R_{ki}(\mu)$$

or

$$\xi_i(1) - R_{\xi i}(\mu) = (\mu - 1) \sum_{k \in A} \xi_k(1)\, R_{ki}(\mu)$$

If we compare this last relation to (46) for $\lambda = 1$ and μ in Q_+ we get

$$(49) \quad \mu R_{\xi i}(\mu) = \xi_i(\mu) \quad \text{for all } \mu \text{ in } Q_+ \text{ and all } i \text{ in } A$$

But (40) also gives for all i in A, all λ in Q_+ and all μ in Q_+

$$\left| \frac{1}{\lambda} \xi_i(\lambda) - \frac{1}{\mu} \xi_i(\mu) \right| \leqslant \left| (\mu - \lambda) \right| \frac{1}{\lambda\mu}$$

So that $\xi_i(\,.\,)$ has a continuous extension to all ν in R_+, satisfying for all i in A

$$\xi_i(\nu) = \lim_{\substack{\mu = \nu \\ \mu \in Q_+}} \xi_i(\mu) = \lim_{\substack{\mu = \nu \\ \mu \in Q_+}} \mu R_{\xi i}(\mu) = \nu R_{\xi i}(\nu)$$

If we use the Scheffé's theorem, these last relations and (48) are then enough to allow an interchange in (46) of summation and limit as μ tends to ν along Q_+, and this proves that (40) holds for the extended $\xi i(\,.\,)$ for all λ in R_+, all μ in R_+ and all i in A.

As in § 1 we associate with every ξ in K_o and every $t>0$ a measure on the compact space K, by setting

(50) $$p_{\xi B}(t) = \sum_{i \in B \cap A} p_{\xi i}(t) \quad \text{for all Borel sets B in } (K,T)$$

The set K_b is defined as the set of all ξ in K_o for which $p_\xi(t)$ does not tend weakly to $\mathcal{E}_\xi(\,.\,)$ as t tends to 0. K_b is called the set of branching points.

For a fixed i in A, the function $\xi i(1)$ is continuous from (K,T) into $[0,1]$; hence the sum function

$$\sum_{i \in A} \xi i(1) \quad \text{is measurable and lower semicountinuous}$$

from (K.T) into $[0,1]$. Since K_o is the inverse image of 1 by this sum function, it is a Borel set of K and indeed a G_δ - set.

Proof:

If f is a lower semicontinuous function in a metric space then by Saks p.43 in [10] the set

$$\{x \mid f(x) \leq a\}$$

is a closed set for every real number a.

We have

$$\{x \mid f(x) = a\} = \{x \mid f(x) \leq a\} \cap \bigcap_{n=1}^{\infty} \{x \mid f(x) > a - \tfrac{1}{n}\}$$

Again in a metric space every closed set is a G_δ - set (see th 84 in Sierpinski [12]). hence the set on the left hand side above is an intersection of two G_δ- sets and thus itself a G_δ- set.

If ξ is in $K_o - K_b$ the measure $p_{\xi\,.}(t)$ and $\mathcal{E}_\xi(\,.\,)$ are all fully supported by the Borel set $K_o \subset K$, so that by th.0 the weak convergence also holds on K_o only.

If k is in A, (43) obviously holds for $\{ \lambda R_{ki}(\lambda)$, i in A , λ in $Q_+ \}$, and so A is contained in K_o. Moreover as $p_{kk}(t)$ tends to 1 as t tends to 0 we get

$$p_{k\cdot}(t) \xrightarrow{w} \mathcal{E}_k(\cdot) \qquad \text{as } t \text{ tends to } 0$$

and A is in fact a subset of $K_0 - K_b$

Definition:

The set $K_o - K_b$ (contained in K)with the topology induced by T is called the Doob entrance boundary for P(t) and will be denoted by $(K_o - K_b; T)$.

3 <u>Equivalence of the Neveu and Doob entrance boundaries.</u>

In this paragraph we prove that the Neveu and Doob definitions are equivalent. First we construct two mappings (Φ and Ψ) connecting these two entrance boundaries and then we show that they form a topological isomorphism.

<u>Construction of the mapping Φ defined on $A + A_e$ with values in K_o.</u>

Choose a y in $A + A_e$. By the density of A in $A + A_e$, relative to T, (see th. 20) there exists a sequence i_n, n in N of points in A such that

(51) $R_{yi}(1) = \lim_{n=\infty} R_{i_n i}(1)$ for all i in A

On $A + A_e$ this condition is enough to check (30) and we may apply the lemma p to get

$$p_{yi}(t) = \lim_{n=\infty} p_{i_n i}(t) \quad \text{for all } t > 0 \text{ and all } i \text{ in } A$$

As all these functions are bounded by 1, the Lebesgue's theorem on dominated convergence yields for all $\lambda > 0$ and all i in A

(52) $$\int_0^\infty e^{-\lambda t} p_{yi}(t)\, dt = \lim_{n=\infty} \int_0^\infty e^{-\lambda t} p_{i_n i}(t)\, dt$$

We now define a mapping $\Phi(y)$ from y in $A + A_e$ into C by letting

(53) $[\Phi(y)]_i(\lambda) = \lim_{n=\infty} \lambda R_{i_n i}(\lambda)$

for all λ in Q_+ and all i in A.

This mapping is well defined as the value $\Phi(y)$ does not depend on a particular choice of sequence i_n. Let i_n, n in N and i_r, r in N be two sequence in A such that

$$R_{yi}(1) = \lim_{n=\infty} R_{i_n i}(1) = \lim_{r=\infty} R_{i_r i}(1) \quad \text{for all i in A}$$

The equality

$$1 = \sum_{i \in A} Ryi(1) = \sum_{i \in A} Ri_n i(1) = \sum_{i \in A} Ri_r i(1)$$

which holds for all n and r, allows us (by Scheffé's theorem) to interchange summation and limits in the following

$$\lim_{n=\infty} R_{i_n i}(\lambda) = \lim_{n=\infty} R_{i_n i}(1) + (1-\lambda)\sum_{k\in A} \lim_{n=\infty} R_{i_n k}(1)\, R_{ki}(\lambda)$$

$$= \lim_{r=\infty} R_{i_r i}(1) + (1-\lambda)\sum_{k\in A} \lim_{r=\infty} R_{i_r k}(1)\, R_{ki}(\lambda)$$

$$= \lim_{r=\infty} R_{i_r i}(\lambda)$$

for all λ in Q_+ and all i in A. And this shows that $\Phi(y)$ is uniquely defined. The topologies on $A + A_e$ and C being those of the simple convergence, the relations (51), (52) and (53) imply the continuity of Φ.

As y is in $A + A_e$, we have

$$1 = \sum_{i\in A} R_{yi}(1) = \sum_{i\in A} \lim_{n=\infty} R_{i_n} i(1) = \sum_{i\in A} [\Phi(y)]i(1)$$

hence we get the inclusion

$$\Phi(A + A_e) \subset K_o$$

If x and y are two distinct points of $A + A_e$, then $R_{xi}(1) \neq R_{yi}(1)$ for at least one i in A, but by definition of Φ (cf (51) and (53)) this yields

$$[\Phi(x)]_i(1) \neq [\Phi(y)]_i(1)$$

so that Φ is one-to-one from $A + A_e$ into K_o.

If k is a point of $A \subset A + A_e$, the special sequence $i_n = k$, for all n, may be chosen to define $\Phi(k)$, hence for all λ in Q_+ and all i in A

$$[\Phi(k)]_i(\lambda) = \lim_{n=\infty} \lambda R_{ki}(\lambda) = \lambda R_{ki}(\lambda)$$

(54) i.e. $\Phi(k) = k$ in K_o for all k in $A \subset A + A_e$.

<u>Construction of the mapping Ψ defined on K_o with values in M_1.</u>

Pick a ζ in K_o; as noted before (§ 2 p. 37) the measure $\zeta i(1)$, i in A, is an element of M_1. Define the mapping Ψ from K_o into M_1, by letting

(55) $\Psi(\zeta)_i = \zeta_i(1)$ for all i in A

$\Psi(\zeta)$, being the projection of ζ on the countable product of unit intervals indexed by i in A only, is obviously a continuous mapping relative to the simple convergence topologies on K_o and M_1. On K_o the equation (46) holds

$$\zeta_i(\lambda)\frac{1}{\lambda} - \zeta_i(1) = (1-\lambda)\sum_{k\in A} \zeta_k(1)R_{ki}(\lambda)$$

Hence the equality $\Psi(\zeta) = \Psi(\chi)$ yields

$$\zeta_i(\lambda) = \chi_i(\lambda) \quad \text{for all } \lambda \text{ in } Q_+ \text{ and all i in A}$$

so that Ψ is also one-to-one from K_o into M_1. If k is a point of $A \subset K_o$ we have

$$\Psi(k)_i = R_{ki}(1) \quad \text{for all i in A}$$

(56) i.e. $\Psi(k) = k$ in M_1 for all k in $A \subset K_o$

Each of the mappings $\Phi\cdot\Psi$ and $\Psi\cdot\Phi$ (whenever defined) is the identity.

Proof:

The relation(53) for $\lambda = 1$ in particular gives

$$[\Phi(y)]_i(1) = R_{yi}(1) \quad \text{for all i in A}$$

as by (55)

$$\Psi(\zeta)_i = \zeta_i(1) \quad \text{for all i in A}$$

we get

$$\Psi(\Phi(y))_i = R_{yi}(1) \quad \text{for all i in A}$$

Hence $\Psi\cdot\Phi$ is always defined and is equal to the identity mapping on $A + A_e$.

Let ζ be in K_o, i.e. there exists a sequence in A such that

$$\zeta_i(\lambda) = \lim_{n=\infty} \lambda R i_n i(\lambda) \quad \text{for all } \lambda \text{ in } Q_+ \text{ and all i in A}$$

By definition of Ψ (see (55)) we have

$$\Psi(\zeta)_i = \zeta_i(1) = \lim_{n=\infty} R_{i_n i}(1) \quad \text{for all i in A}$$

If $\Psi(\zeta)$ lies in $A + A_e$, then $\Phi(\Psi(\zeta))$ is defined and must satisfy

$$[\Phi(\Psi(\zeta))]_i(\lambda) = \lim_{r=\infty} \lambda R_{i_r i}(\lambda) \quad \text{for all } \lambda \text{ in } Q_+ \text{ and all i in A}$$

where i_r, r in N is a sequence in A such that

$$\lim_{r=\infty} R i_r i(1) = \Psi(\zeta)_i = \lim_{n=\infty} R i_n i(1) \quad \text{for all i in A}$$

But as we have just seen when checking the consistency of the definition of Φ, the fact that $\Psi(\zeta)$ is in $A + A_e$ is enough to ensure that $\Phi(\Psi(\zeta))$ does not depend on the sequence used and we get

$$[\Phi(\Psi(\zeta))]_i(\lambda) = \lim_{n=\infty} \lambda R i_n i(\lambda) \text{ for all } \lambda \text{ in } Q_+ \text{ and all i in A}$$

Hence $\Phi \cdot \Psi$ is equal to the identity mapping on the subset of K_o where $\Psi(\zeta)$ is in $A + A_e$.

We proceed now to prove that $(A + A_e, T)$ and $(K_0 - K_b, T)$ are topologically isomorphic by Φ and Ψ.

By th. 2 we know that with y in $A + A_e$ and $\Phi(y)$ in K_o are associated two entrances $p_{yi}(t)$ and $p_{\Phi(y)i}(t)$, say. As their Laplace transforms satisfy the resolvent equation the equality

$$Ryi(1) = [\Phi(y)]_i (1) \text{ for all i in A}$$

is then enough to get

$$Pyi(t) = P\Phi(y)i(t) \quad \text{for all } t > 0 \text{ and all i in A.}$$

Now fix y in $A + A_e$. As before we consider the measures generated by $Pyi(t)$ on A, $A + A_e$ and $A \cup \{y\}$, and the measures generated by $P\Phi(y)i(t)$ on K_o and $A \cup \{\Phi(y)\}$. Note that all these measures are fully supported by A.

By th. 5 (ii) we have

$$p_{y\,.}(t) \xrightarrow{w} \mathcal{E}_y(\,.\,) \text{ as } t \to 0 \text{ on } A + A_e$$

Note that these measures are fully supported not only by $A + A_e \subset M_1$, but also by the smaller set $A \cup \{y\}$. Since this latter set is countable it is a Borel set of $A + A_e$ and from the th. 0 we deduce that for all bounded continuous functions g defined on $A \cup \{y\}$ we have

$$\lim_{t=0} \sum_{k \in A} p_{yk}(t)\, g(k) = g(y)$$

Let f be any bounded continuous function defined on $A \cup \{\Phi(y)\} \subset K_o$. As Φ is one-to-one we may let

$$g(x) = f(\Phi(x)) \quad \text{for all } x \text{ in } A \cup \{y\}$$

and the function g is bounded continuous from $A \cup \{y\}$ into R, because Φ is continuous.

By (54) we have

$$g(k) = f(\Phi(k)) = f(k) \quad \text{for all } k \text{ in } A$$

Hence we get for all $t > 0$

$$(57) \quad \sum_{k \in A} p_{yk}(t) g(k) = \sum_{k \in A} p_{\Phi(y)k}(t)\, f(k)$$

According to the remark we have just made about weak convergence on $A \cup \{y\}$, the L.H.S. of (57) tends to g(y) as t tends to 0. But by construction we have $g(y) = f(\Phi(y))$ so that

$$(58) \quad p_{\Phi(y)\,.}(t) \xrightarrow{w} \mathcal{E}_{\Phi(y)}(\,.\,) \quad \text{as } t \to 0 \text{ on } A \cup \{\Phi(y)\}$$

All the masures in (58) are fully supported by $A \cup \{\Phi(y)\}$, whith is a Borel subset of K_o, by its mere countability and we can use th.0 to ensure this weak convergence on K_o. But the points of K_b were defindd as those in K_o which do not enjoy this property (see § 2 p 38) and it proves

$$(59) \quad \Phi(A + A_e) \subset K_o - K_b$$

For the same reasons with any $t > 0$ and ξ in $K_o - K_b$ we associate the measures $p_{\xi .}(t)$ on K_o and on $A \cup \{\xi\} \subset K_o$ and the measures $p_{\Psi(\xi).}(t)$ on M_1 and $A \cup \{\Phi(\xi)\} \subset M_1$ which satisfy

(60) $$p_{\Psi(\xi)i}(t) = p_{\xi i}(t) \quad \text{for all } i \text{ in } A$$

Let g be any bounded continuous function defined on $A \cup \{\Phi(\xi)\} \subset M_1$, As Ψ is one-to-one we may let

$$f(\xi) = g(\Psi(\xi)) \quad \text{for all } \xi \text{ in } A \cup \{\xi\} \subset K_o$$

and f is abounded continuous function defined on $A \cup \{\xi\}$ because Ψ is continuous.

By (56) we have

$$f(k) = g(\Psi(k)) = g(k) \quad \text{for all } k \text{ in } A$$

As above the weak convergence of $p_{\xi .}(t)$ to $\varepsilon_{\xi}(\;.\;)$ as t tends to 0 may be consideredas only on $A \cup \{\xi\}$ and we obtain

$$\lim_{t=0} \sum_{k \in A} p_{\Psi(\xi)k}(t)g(k) = f(\xi) = g(\Psi(\xi))$$

Again the countability of $A \cup \{\Psi(\xi)\}$ and the th. O prove this convergence on M_1, itself. And this is enough, by th.6 to check that $\Psi(\xi)$ is an ėlement of $A + A_e$, so that

(61) $\Psi(K_o - K_b) \subset A + A_e$

By the fact that $\Psi \cdot \Phi = I$ and the relations (59) and (61), we obtain

$$A + A_e = \Psi(\Phi(A+A_e)) \subset \Psi(K_o - K_b) \subset A + A_e$$

and

$$A + A_e = \Psi(K_o - K_b)$$

Similarly and using (61) to make sure that $\Phi \cdot \Psi$ is defined on $\Psi(K_o - K_b)$ and is then equal to the identity we get

$$K_o - K_b = \Phi(\Psi(K_o - K_b)) \subset \Phi(A + A_e) \subset K_o - K_b$$

and

$$K_o - K_b = \Phi(A + A_e)$$

We have now proved the following theorem

Theorem 7.

$A + A_e$ and $K_o - K_b$, both with their simple convergence topologies are topologically isomporphic by the mappings Φ and Ψ .

CHAPTER III

Markov Processes on the Entrance Boundary

Throughout this chapter the topology considered on the entrance boundary is always T we write $A+A_e$ for $(A+A_e, T)$ and $K_o - K_b$ for $(K_o - K_b, T)$.

Let $p(t)$, $t > o$ be an entrance relative to the stochastic semigroup $P(t)$ satisfying

$$(1) \qquad \sum_{i \in A} p_i(t) = 1 \qquad \text{for all } t > o$$

Then as pointed out in Chapter I the main interest of the entrance boundary is that (as stated by Doob in theorems 3.1, 4.3, 7.1 and 8.3 of [5]) a right continuous process in $K_o - K_b$ can be found such that its absolute distribution is equal to $p(t)$ for all $t > o$ and satisfying the strong Markov property with the transition semigroup extended to the entrance boundary by means of II (50).

The existence of a Markov process in $A + A_e$ with similar properties is obvious from the existence of the topological isomorphism Ψ from $K_o - K_b$ into $A + A_e$ defined in II (55).

Indeed if X_t , $t \geq o$ is a process in $K_o - K_b$ defined on the probability triple $(\Omega, \mathcal{F}, P)$ which has the properties just described, then the process Y_t, $t \geq o$ defined by letting

$$Y_t(w) = \Psi(X_t(w)) \qquad \text{for all } t \geq o, \text{ and all } w \text{ in } \Omega$$

has the same properties in $A + A_e$.

Proof:

As Ψ is a topological isomorphism $Y_t(w)$ is right continuous from t in $[o, oo]$ into $A + A_e$ for a fixed w, whenever $X_t(w)$ is right continuous. Hence Y_t is right continuous with probability one.

Next for all Borel sets B in $A + A_e$ and all $t \geq o$ we have

$$[\, w \mid Y_\rho(w) \in B \,] = [\, w \mid X_\rho(w) \in \Psi^{-1}(B) \,]$$

so that all the σ-fields generated by X_t or their corresponding Y_t are identical. A stopping time for Y_t is then also a stopping time for X_t. By II (60) for $t > o$ and trivially for $t = o$, we get for all Borel sets D in $K_o - K_b$, all ξ in $K_o - K_b$ and all $t \geq o$ the measure equality

$$p_{\Psi(\xi)\,\Psi(D)}(t) = p_{\xi D}(t)$$

These two last facts are enough to ensure that the strong Markov property which holds for X_t must also hold for Y_t (with the transition semi-group extended to $A + A_e$ as in II (29)).

As the analytical construction of $A + A_e$ needs only one auxiliary space (namely M1) instead of the two (K and K_o) used in the definition of $K_o - K_b$, $A + A_e$ seems slightly simpler than $K_o - K_b$. Thus it might be interesting to see if a proof of the existence of a right continuous strong Markov process in $A + A_e$ can be obtained faster than in $K_o - K_b$ (and not using Ψ). If we proceed along the lines of Doob some results are easier to check; unfortunately it turns out that the use of $A + A_e$ instead of $K_o - K_b$ is no real simplification.

What follows reads as a copy of Doob's proof, except that the spaces M_1 and $A + A_e$ are used rather than K, K_o, $K_o - K_b$.

As seen in Chapter I to every stochastic entrance p (t), $t > o$, relative to $P(t)$ we can associate a Markov process X_t, $t > o$ defined on a probability triple $(\Omega; \mathcal{F}; P)$ and such that

(2) $p_i(t) = P\,[\, X_t(w) = i \,]$ for all $t > o$

and

(3) $P[X_t(w) \in A] = 1$ for all $t > o$

As usual the σ-field $\mathfrak{F}$ is completed.

M_1 is an extension of A in the sense of (i) and (ii) I p 11 and X_t, $t>o$ can be considered as a process in M_1. But as $\mathfrak{F}$ is completed and M_1 is a compact metrisable space we can apply M. IV 19 to get a standard modification of X_t separable relative to the closed sets of M_1 and again de_noted by X_t.

Now fix an i in A and consider the family of random variables

(4) $$e^{-t} R_{X_t(w)\, i}(1) \quad \text{for all } t > o$$

This family forms a separable super martingale relative to the σ-fields $\mathfrak{F}_t$, $t > o$ (which are as usual those generated by X_s, $o < s \leq t$ and containing all null sets).

<u>Proof:</u>

Let $s' \leq s < t$ and choose an elementary event of $\mathfrak{F}$ of the following form

$$\Lambda = [\, w \mid X_{s'}(w) = k \,] \quad \text{for one } k \text{ in } A$$

We have

(5) $$\int_\Lambda E[e^{-t} R_{X_t i}(1) \mid \mathfrak{F}_s]\, P[dw] = \int_\Lambda e^{-t} R_{X_t\, i}(1)\; P[dw]$$

By (3) this last term is equal to

$$\sum_{j \in A} e^{-t} R_{ji}(1)\; P[\Lambda\, X_t(w) = j] =$$

$$\sum_{j \in A} e^{-t} R_{ji}(1)\; P_{kj}(t-s')\; P[\Lambda] =$$

$$= P[\Lambda]\, e^{-s'} \int_0^\infty \sum_{j \in A} p_{kj}(t-s')\, e^{-t}\, e^{s'} e^{-u}\, p_{ji}(u)\, du$$

(6) $$= P[\Lambda]\, e^{-s'} \int_{t-s'}^\infty e^{-v} p_{ki}(v)\, dv$$

Similarly we get

(7) $$\int_\Lambda e^{-s} R_{X_s i}(1)\; P[dw] =$$

(8) $$P[\Lambda]\, e^{-s'} \int_{s-s'}^\infty e^{-v}\, p_{ki}(v)\, dv$$

But the inequality (6) $\leq$ (8) yields (5) $\leq$ (7) (for all $s' \leq s < t$ and all k in A) and this completes the proof that (4) is a super martingale.

As the functions $R_{xi}(1)$, i in A, are continuous and separate the points of M_1, we find by M VI 3 that almost all sample paths have a right limit in M_1 for all $t \geq 0$, to be denoted by X_{t+}, $t \geq 0$.

The super martingale (4) is also such that

(9) $$\int_\Omega e^{-t} R_{X_t i}(1)\; P[dw] = \sum_{k \in A} e^{-t} R_{ki}(1)\, p_k(t)$$

By (1) and the Scheffé's theorem we get for all i in A

$$\lim_{t=t'} \sum_{k \in A} e^{-t} p_k(t) R_{ki}(1) = \sum_{k \in A} e^{-t'} p_k(t') R_{ki}(1)$$

i.e. $$\lim_{t=t'} E[e^{-t} R_{X_t i}(1)] = E[e^{-t'} R_{X_{t'} i}(1)]$$

By C th II 8 1 it is known that $\mathfrak{F}_t$, $t > 0$ is a right continuous family i.e.

$$\mathfrak{F}_t = \bigcap_{s>t} \mathfrak{F}_s \qquad \text{for all } t > 0$$

Hence M VI 4.3) can be used to get for all $t > o$ and all i in A

(10) $\quad R_{X_t(w)\,i}\,(1) = R_{X_{t+}(\omega)\,i}\,(1)$ a.s.

Now keep $t > o$ fixed. Since A is countable the probability that (10) holds for all i in A simultaneously is equal to one.

This implies

(11) $\quad X_t\,(w) = X_{t+}\,(w)$ in M_1 a.s.

From (2) and (11) we deduce that the absolute probability distribution of X_{t+}, $t > o$ is $p\,(t)$, $t > o$.

The Markov property (I (2)) is defined with elementary events (i.e. with a finite number of different times), since (2) and (3) hold for X_t, $t > o$ and X_{t+}, $t > o$, the process X_{t+}, $t > o$ is also a Markov process with the same transition semigroup as X_t, $t > o$. Therefore P(t) is the transition semigroup of X_t, $t > o$.

The process X_t, $t > o$ is extended to t=o by letting

(12) $\quad X_o\,(w) = \lim_{t \downarrow o} X_t\,(w)$

for all w such that this limit exists (i.e. with probability one) and choosing as X_o (w) any arbitrary value in M_1 for the other w 's.

From (9) we get for all i in A

$$\sup_{t>o} E[e^{-t} R_{X_t(w)\,i}\,(1)] = \sum_{k\in A} e^{-t} R_{ki}\,(1)\, p_k(t) \leqslant 1$$

These inequalities are enough by M VI 7 to ensure the measurability of X_o relatively to the σ-field

$$\mathcal{F}_o = \bigcap_{o<s} \mathcal{F}_s$$

The right continuity of the family of σ-fields $\mathcal{F}_t$ is then extended to t =0.

What has been obtained so far is summarised in the following theorem (which corresponds to th. 3.1 in Doob [5])

<u>Theorem 1</u>

If p (t), t>o is an entrance relative to P (t) such that (1) holds, then there exists a right continuous Markov process X_{t+} , $t \geq o$ (denoted by X_t only from now on) with values in M_1 , P (t) as transition semi group and p (t) as absolute distribution for t > o.

Now by Choquet's theorem (M XI 25 and 29) the entrance p (t) is known to be of the form

$$(13) \qquad p_i(t) = \int_{A+A_e} \mu(dx)\, p_{xi}(t) \qquad \text{for all } t>o \text{ and all } i \text{ in } A$$

Where $\mu(\,.\,)$ is a uniquely defined measure on $A+A_e$ such that $\mu(A+A_e) = 1$.

We proceed now to prove the following result

<u>Theorem 2</u>

The measure $\mu(\,.\,)$ is the absolute distribution of X_o ($=X_{o+}$ a.s)

<u>Proof:</u>

X_o being a random variable the function defined on the Borel sets B of M_1 by setting

$$P[\,X_o \in B\,] : B \longrightarrow [0,1]$$

is a probability measure on M_1.

Let f be a positive bounded continuous function defined on M_1. The integral

(14) $$\int_{M_1} f(y) \; P [X_o \in dy]$$

is by our choice of X_o (in (12)) equal to

$$\int_{M_1} f(y) \; P [X_{o+} \in dy] = \int_{\Omega} f (X_{o+}) \; P [dw] =$$

$$\int_{\Omega} \lim_{t \downarrow o} f (X_t) \; P [dw]$$

By Fatou's lemma the last term is bounded above by

$$\liminf_{t=o} \int_{\Omega} f (X_t) \; P [dw] =$$

(15) $$\liminf_{t=o} \sum_{k \in A} p_k (t) \; f (k)$$

By (13) and positivity, (15) becomes

(16) $$\liminf_{t=o} \int_{A+A_e} \mu(dx) \sum_{k \in A} p_{xk} (t) \; f (k)$$

As t decreases to 0 the sum over all k in A in (16) converges to f(x) for all x in $A+A_e$ (see th. II 5 (ii) and th II.o) and since f is bounded the Lebesgue's dominated convergence theorem can be applied to (16) which is then equal to

(17) $$\int_{A+A_e} \mu (dx) \; f (x) = \int_{M_1} \mu (dx) \; f (x)$$

The inequality (14) $\leq$ (17) which holds for all positive bounded continuous functions on M_1 implies

(18) $P[X_o \in B] \leq \mu(B)$ for all Borel sets B in M_1

But we have

$$P[X_o \in M_1] = \mu(M_1) = \mu(A+A_e) = 1$$

and so there is indeed equality in (18) and this proves th 2

<u>Corollary</u>

If we choose for a given y in A+ A_e the particular entrance $\rho_{yi}(t)$, i in A, $t > o$ then the associated process (as in th 1) starts in y almost surely.

Hence if the entrance boundary has to be an extension of A in which every process has a right continuous standard modification then all its points are actually needed.

The arguments used between (15) and (17) ~~prove in fact~~ may be used to show that

~~$p_.(t) \xrightarrow{w} p_.(e) = \int_{A+Ae} \mu(dx)\, \mathcal{E}_x(\,.\,)$ on M_1 as t tends to 0.~~

~~Hence we have~~

$$\rho_y(M_1 - (A+A_e))(t) = 0 \qquad \text{for all } t \geq o$$

Which is a result similar to lemma 8.1 in Doob [5]. Now should the strong Markov property be valid for X_t, $t \geq o$ in M_1, the proofs of lemma 8.2 and theorem 8.3 in Doob [5] could then be used to get

<u>Theorem 3</u>

The process described in theorem 1 is such that almost no sample path ever meets $(M_1 - (A+A_e))$.

<u>Proof of the strong Markov property</u>

We have to show

(19) $P[X_{\zeta+s}\in B | \mathcal{F}_\zeta] = p_{X_\zeta B}(s)$ a.s.

for every finite stopping time ζ, every $s \geq 0$ and every Borel set B in M_1. Note that by right continuity of X_t, M IV 47 and 49 the event in the L.H.S of (19) is measurable.

If $s = 0$ $p_{X_\zeta B}(0) = \mathcal{E}_{X_\zeta}(B)$ and (19) is true.

If $s > 0$, (19) is equivalent to

(20) $$E[f(X_{\zeta+s}) | \mathcal{F}_\zeta] = \sum_{k\in A} p_{X_\zeta k}(s) \quad f(k) \qquad \text{a.s.}$$

for all bounded continuous functions on M_1.

Let σ be a discrete valued stopping time for X_t. Denote by t_n, n in N, its values and by Λ_n the set of w's where $\sigma(w) = t_n$. Ω is the union of all the Λ_n, which are disjoint.

Pick a Λ in $\mathcal{F}_\sigma$, i.e.

$$\Lambda \cap [w \mid \sigma(w) \leq t] \in \mathcal{F}_t \qquad \text{for all } t \geq 0$$

If f is a bounded continuous function on M_1 we have for any $s > 0$

(21) $$\int_\Lambda E[f(X_{\sigma+s}) | \mathcal{F}_\sigma] \, P[dw] = \sum_{n=0}^{\infty} \int_{\Lambda\Lambda_n} E[f(X_{\sigma+s}) | \mathcal{F}_\sigma] P[dw]$$

But as t_n is a fixed real number one term of the sum above is equal to

(22) $$\int_{\Lambda\Lambda_n} f(X_{t_n+s}) \, P[dw]$$

Furthermore as X_{t_n+s} and X_{t_n} are both almost surely in A the last integral is equal to

$$\sum_{k\in A} f(k) \int_{\Lambda\Lambda_n} P[X_{t_n+s} = k \mid \mathcal{F}_{t_n}] P[dw] \quad =$$

$$\sum_{k \in A} f(k) \int_{\Lambda\Lambda_n} \sum_{i \in A} P[X_{t_n+s} = k \; X_{t_n} = i \mid \mathcal{F}_{t_n}] \; P[dw]$$

As $\Lambda\Lambda_n$ is in $\mathcal{F}_{t_n}$ and P (t) is the transition semigroup of the Markov process X_t, the last term becomes

$$\sum_{k \in A} f(k) \sum_{i \in A} \int_{\Lambda\Lambda_n} p_{ik}(s) \, 1_{[X_{t_n}=i]}(w) \; P[dw]$$

(23) $$= \int_{\Lambda\Lambda_n} \sum_{k \in A} p_{X_{t_n} k}(s) \; f(k) \; P[dw]$$

Now if we sum over n in N the equalities (22) =(23) we get by (21)

(24) $$\int_{\Lambda} E[f(X_{\sigma+s}) \mid \mathcal{F}_{\sigma}] \; P[dw] = \int_{\Lambda} \sum_{k \in A} p_{X_{\sigma} k}(s) \; f(k) \; P[dw]$$

As this holds for all Λ in $\mathcal{F}_{\sigma}$ it yields (20).

If τ is a finite stopping time it is always possible to construct a decreasing sequence of discrete valued stopping times σ_r converging to τ . By M IV 40 we have $\mathcal{F}_{\tau} \subseteq \mathcal{F}_{\sigma_r}$ so that (20) for σ_r implies for all r

$$E[f(X_{\sigma_r+s}) \mid \mathcal{F}_{\tau}] = E[\sum_{k \in A} p_{X_{\sigma_r} k}(s) \; f(k) \mid \mathcal{F}_{\tau}] \qquad \text{a.s.}$$

As **f** is continuous on M_1 we have

$$\lim_{r=\infty} E[f(X_{\sigma_r+s}) \mid \mathcal{F}_{\tau}] = E[f(X_{\tau+s}) \mid \mathcal{F}_{\tau}] \qquad \text{a.s.}$$

The strong Markov property would then be proved if any sequence y_m, m in N, converging to y in M_1 were such that

(25) $$\lim_{m=\infty} \sum_{k \in A} p_{y_m k}(s) \; f(k) = \sum_{k \in A} p_{yk}(s) \; f(k)$$

for all $s > 0$ and all bounded continuous functions f on M_1.

But it is not known if this is true: the nearest result to this property being the lemma Chapter I p 27 where the condition

$$(26) \quad \lim_{m=\infty} \sum_{k \in A} \left| Ry_m k\ (1) - Ryk\ (1) \right| = 0$$

is assumed.

Starting from $y_m \to y$ in M_1 i.e.

$$\lim_{m=\infty} Ry_{mi}\ (1) = Ryi\ (1) \quad \text{for all } i \text{ in } A$$

the most obvious way to get (26) is to use the Scheffé's theorem and state

$$(27) \quad \sum_{k \in A} Ry_m k\ (1) = \sum_{k \in A} Ryk\ (1) = c \qquad 0<c<1$$

As X_t, $t>0$ is in A for almost all w (see theorem 1), we are led to choose c = 1 and to introduce a set M_1 (1) defined as the part of M_1 where (27) is equal to one (and corresponding to the set K_0 of Doob). But we must now check that the sample paths remain constantly in M_1 (1) for almost all w. To do that we use lemma 4.2 and theorem 4.3 of Doob [5], and therefore no step used by him in K can be ommitted in M_1.

It should be noted that by M VI 3 the supermartingales (4) have left limits at all $t>0$ for almost all w; hence X_t has the same property in M_1. But these limits do not lie necessarily in $A+A_e$ and therefore in general X_t is not a Hunt process.

Here we give briefly an example where the left limits are not always in $A+A_e$.

Let X_t be an ascending escalator on $A = \mathbb{N}$ (see C. II. 20 ex 1) with first infinity (C II § 19) almost surely finite. From the first infinity, jump to two different absorbing states, δ_1 and δ_2 say, both with probability $\frac{1}{2}$. Then the left limits of X_t at the first infinity will be

$$\frac{1}{2} R \delta_1 \, i\,(1) + \frac{1}{2} R \delta_2 i\,(1) \qquad \text{for all i in } N \cup \{\delta_1\} \cup \{\delta_2\}$$

Which is not an extreme point of the convex set M_1.

Nevertheless X_t verifies the so called quasi left continuity which is the matter of the next theorem which will be needed once in Chapter V. It corresponds to th. 7.2 in Doob [5] and is not proved here.

Theorem 4

Let ζ_n be an increasing sequence of stopping times and ζ be its limit. If $X_{\zeta_-(w)}(w)=\lim_{s\uparrow\zeta} X_s(w)$, then

$$X_{\zeta_-(w)}(w) = X_{\zeta(w)}(w) \qquad \text{a.s. on the}$$

set of w's such that

(i) $\zeta_n(w) < \zeta(w) < \infty$ for all n

(ii) $X_{\zeta_-(w)}(w) \in A + A_e$

Finally we remark that if B is a Borel set in $A+A_e$ the random variable defined as

$$\zeta^1_B(w) = \inf\left\{t \mid t \geqslant o \quad X_t(w) \in B\right\}$$

is a stopping time relative to $\mathcal{F}_t$.

This is true by M IV 52 and 53. Both results can be used because the σ-fields $\mathcal{F}_t$ have been shown to be right continuous (see p 50 and p 52) and the right continuity of the process X_t ensures its progressive measurability (see M IV 47)

If we define another random variable ζ_B by letting

(28) $$\zeta_B(w) = \inf\left\{t \mid t > o \quad X_t(w) \in B\right\}$$

then it is also a stopping time relative to $\mathcal{F}_t$.

Proof:

For all positive integers n let

$$(28)\quad \zeta_B^n(w) = \inf \left\{ t \,\middle|\, t \geq \tfrac{1}{n} \quad X_t(w) \in B \right\}$$

$$= \inf \left\{ t \,\middle|\, t \geq 0 \quad X_t(\theta_{\frac{1}{n}}(w)) \in B \right\}$$

where as usual (see M X II 16) $\theta_{\frac{1}{n}}$ is the shift operator i.e.

$$X_s(\theta_{\frac{1}{n}}(w)) = X_{s+\frac{1}{n}}(w) \quad \text{for all } s \geq 0 \text{ and all } w \text{ in } \Omega$$

Obviously we have the equality

$$[\zeta_B \leq t] = \bigcup_{n=1}^{\infty} [\zeta_B^n \leq t - \tfrac{1}{n}]$$

By the arguments used above for ζ_B we get for all n

$$[\zeta_B^n \leq t - \tfrac{1}{n}] \quad \in \tfrac{1}{n}\mathcal{F}_t$$

For all n, $\frac{1}{n}\mathcal{F}_t$ is contained in $\mathcal{F}_t$ and this proves that ζ_B is a stopping time.

CHAPTER IV

On the Size of the Entrance Boundary

Notation: In the last 3 Chapters $A + A_e$ is used for the known completion of A instead of $K_o - K_b$, because it is easier to distinguish at first sight between the points of A the initial state space (usually noted i,j, or k) and those of A_e (usually noted x,y or z).

In Chapter III the topological space $(A+A_e; T)$ was shown to have the following property:

for every stochastic entrance p(t), $t > 0$, relative to P(t), there exists a probability triple $(\Omega, \mathcal{F}, P)$ on which there exists a Markov process X_t, $t \geq 0$, such that:

(i) all the values are in $A + A_e$

(ii) almost all sample paths are right continuous at all $t \geq 0$

(iii) the strong Markov property holds with the transition probabili defined on $A + A_e$ (II.(2 9)).

(iv) $P[X_t(w) = i] = p_i(t)$ for all $t > 0$ and i in A.

An interesting question is whether this extension of A by the entrance boundary is in some way general. The purpose of this Chapter is to show that if certain assumptions hold for a measurable space $(E, \mathcal{E})$ such that for every stochastic entrance p(t) a process X_t in E satisfying (i) to (iv) may be found, then E is at least as big as $A + A_e$, i.e. there exists an injection of $A + A_e$ into E.

Assumptions on E.

(a) Let E be a state space such that E has a Hansdorff topology which is metrisable.

(b) A is included in E, every i in A is also a point of E.

(c) If we denote by $\mathcal{E}$ the σ-field of the Borel sets in E, then there exists a transition semigroup on $(E, \mathcal{E})$ to be denoted by $p^E_{eB}(s)$ which is such that in particular if i = e in A and B = j in A.

$$p^E_{ij}(s) = p_{ij}(s) \text{ for all } s > 0$$

Whenever (a) (b) and (c) hold, hypotheses (i) and (ii) p11 are satisfied.

Hence with every stochastic entrance p (t) we can associate a probability triple and a Markov process defined on Ω and with values in E satisfying the equality

(1) $P[X_t(w) = i] = p_i(t)$

for all $t > 0$ and i in A.

(d) Suppose that for every stochastic entrance p (t) the process X_t in E has a standard modification, which is right continuous in the topology of E at all $t > 0$ and which tends to a limit in E as t tends to zero with probability one. In this case X_t is extended to $t = 0$ by letting

$$X_o(w) = \lim X_t(w)$$

for all w such that this limit is defined and choosing as $X_o(w)$ any arbitrary point in E otherwise.

The Markov process X_t, $t \geq 0$, is then right continuous at all t in $[o, \infty)$.

(e) This extended process has transition probabilities $p^E_{eB}(s)$, $s \geq 0$.

Theorem 1.

If E satisfies (a) to (e) then there exists a mapping Ξ defined on $A + A_e$ and with values in E which is one-to-one. In other words $A + A_e$ is the best extension of A with regard to the size for which a right continuous process may be found for every stochastic entrance p (t), $t > 0$.

Proof;

Pick y in $A + A_e$ and choose the entrance $p_i(t) = p_{yi}(t)$. We know that the corresponding process with values in $A + A_e$ defined in Chapter III is concentrated in y at time $t = 0$ (th. III 2). We will show that the associated process with values in E is also concentrated in one point of E at time $t = 0$.

By (d) we have $P[X_o \in E] = 1$, hence there exist some Borel sets B such that

(2) $P[X_o \in B] > 0$

For such B's the elementary conditional probabilities $P[X_t = i \mid X_o \in B]$ are defined for all $t > 0$ and i in A.

(3) As
$$P[X_t \in A] = \sum_{k \in A} p_{yk}(t) = 1 \text{ for all } t > 0$$
we have for any B satisfying (2)

(4) $$P[X_{t+s} = i \mid X_o \in B] =$$

$$\sum_{k \in A} P[X_{t+s} = i \; X_t = k \; X_o \in B] \frac{1}{P[X_o \in B]}$$

The set $[X_o \in B]$ belongs to $\mathcal{F}_t$, for all $t > 0$; therefore if we apply the Markov property this sum becomes

$$\sum_{k \in A} p_{ki}(s) \, P[X_t = k \; X_o \in B] \frac{1}{P[X_o \in B]}$$

(5) $$= \sum_{k \in A} P[X_t = k \mid X_o \in B] \, p_{ki}(s)$$

But (4) = (5) for all $t > 0$, $s \geq 0$ and all i in A means that $P[X_t = i \mid X_o \in B]$ is an entrance relative to P(t) (for all B satisfying (2))

Now suppose that X_o is not concentrated in one point of E, in this case there exists a Borel set B and its complement in E, B^c, for which

(6) $1 > P[X_o \in B] > 0$

and

(7) $1 > P[X_o \in B^c] > 0$

The equality $P[X_o \in E] = 1$ gives

$$P[X_t = i] = P[X_t = i \; X_o \in E] =$$

$$P[X_t = i \; X_o \in B] + P[X_t = i \; X_o \in B^c] =$$

$$P[X_t = i \mid X_o \in B] \, P[X_o \in B] + P[X_t = i \mid X_o \in B^c] \, P[X \; B^c]$$

and by (1) we get

$$p_{yi}(t) = P[X_t = i \mid X_o \in B] \, P[X_o \in B] + P[X_t = i \mid X_o \in B^c] \, P[X_o \in B^c]$$

for all i in A and $t > 0$.

The inequalities (6) and (7) (which imply that both $P[X_t = i \mid X_o \in B]$ and $P[X_t = i \mid X_o \in B^c]$ are entrances) together with the last relation and the extremality of $p_{yi}(t)$ (which is an analytical notion) are enough to give the equalities

(8) $$P[X_t = i \mid X_o \in B] = P[X_t = i \mid X_o \in B^c] = p_{yi}(t)$$

for all i in A and $t > 0$

From (6),(7) and (8) we obtain

$$\frac{P[X_t = i \;\; X_o \in B]}{P[X_o \in B]} = \frac{P[X_t = i \;\; X_o \in B^c]}{P[X_o \in B^c]}$$

or

(9) $P[X_t = i \;\; X_o \in B] \, P\,[X_o \in B^c] = P[X_t = i \; X_o \in B^c] \, P\,[X_o \in B]$

But $P[X_t = i \; X_o \in B^c] = P[X_t = i] - P[X_t = i \;\; X_o \in B]$

so that (9) becomes

$$P[X_t = i \; X_o \in B] \Big(P[X_o \in B^c] + P[X_o \in B] \Big) = P[X_t = i] \; P\,[X_o \in B]$$

and finally we get

(10) $\quad P[X_t = i \;\; X_o \in B] = P[X_t = i] \; P[X_o \in B]$

which holds for all Borel sets B, i in A and $t > 0$, the case for B^c being obtained by inverting the roles of B and B^c and the cases for B's such that $P[X_o \in B] = 0$ or $P[X_o \in B] = 1$ being trivially true.

Next the assumption that X_o is not concentrated in one point of E implies that there exist two distinct points e_1 and e_2 say, such that all their neighbourhoods are visited with strictly positive probability by X_o .

As $e_1 \neq e_2$, we have $d(e_1, e_2) > 0$, where d is a metric defining the Hausdorff topology on E (assumption (a)).

Choose $\varepsilon > 0$ such that

(11) $\quad \varepsilon < \dfrac{d(e_1, e_2)}{4}$

Denote by B(e,r) the open sphere in E centred in e and of radius r.

By the choice of e_1 and e_2 we have

$$P[X_o \in B(e_1, \varepsilon)] = a_1 > 0$$

and

$$P[X_o \in B(e_2, \varepsilon)] = a_2 > 0$$

Let $U_n^1 = [w \mid X_o \in B(e_1, \varepsilon), \; X_q \in B(e_1, 2\varepsilon) \text{ for all } q \text{ in } Q_t \; q < \frac{1}{n}]$

U_n^1 is in $\mathcal{F}$ for all n

and

$U_n^1 \subset U_{n+1}^1$ for all n

The continuity of X_t (w) at $t=0$ for almost all w implies the following inclusion

$$[X_0 \in B(e_1, \varepsilon)] \subset \bigcup_{n=1}^{\infty} U_n^1$$

Similarly define

$$U_n^2 = [w \mid X_0 \in B(e_2, \varepsilon) \quad X_q \quad B(e_2, 2\varepsilon) \text{ for all } q \text{ in } Q+, q \leq \tfrac{1}{n}]$$

then U_n^2 is in $\mathcal{F}$ for all n and

$$[X_0 \in B(e_2, \varepsilon)] \subset \bigcup_{n=1}^{\infty} U_n^2$$

Choose a $\delta > 0$ such that

(12) $$\frac{a_1 - \delta}{a_1} > \frac{3}{4}$$

and

(13) $$\frac{a_2 - \delta}{a_2} > \frac{3}{4}$$

By the monotonicity of U_n^1 and U_n^2, n may be chosen sufficiently large to satisfy both

(14) $P[U_n^1] \geq P[X_0 \in B(e_1, \varepsilon)] - \delta$

and

(15) $P[U_n^2] \geq P[X_0 \in B(e_2, \varepsilon)] - \delta$

We have

(16) $P[U_n^1] \leq P[X_0 \in B(e_1, \varepsilon) \text{ and } X_{\frac{1}{n}} \in B(e_1, 2\varepsilon)]$

By (3) the right hand side equals

$$P[X_0 \in B(e_1, \varepsilon) \quad X_{\frac{1}{n}} \in B(e_1, 2\varepsilon) \cap A]$$

$$= \sum_{i \in B(e_1, \varepsilon) \cap A} P[X_0 \in B(e_1, \varepsilon) \quad X_{\frac{1}{n}} = i]$$

and using (10) this sum becomes

$$\sum_{i \in B(e_1, \varepsilon) \cap A} P[X_0 \in B(e_1, \varepsilon)] \, P[X_{\frac{1}{n}} = i]$$

$$= P[X_0 \in B(e_1, \varepsilon)] \, P[X_{\frac{1}{n}} \in B(e_1, \varepsilon) \cap A]$$

Hence (16) reads as

$$P[U_n^1] \leq P[X_o \in B(e_1,\epsilon)]\; P\,[X_{\frac{1}{n}} \in B(e_1,\epsilon) \cap A]$$

and by (14) we get

$$\frac{P[X_o \in B(e_1,\epsilon)] - \delta}{P[X_o \in B(e_1,\epsilon)]} \leq P[X_{\frac{1}{n}} \in B(e_1, 2\epsilon) \cap A]$$

But our choice of δ (see (12)) implies

$$\tfrac{3}{4} \leq P[X_{\frac{1}{n}} \in B(e_1, 2\epsilon)]$$

Similary using (13) and (15) we get

$$\tfrac{3}{4} \leq P[X_{\frac{1}{n}} \in B(e_2, 2\epsilon)]$$

But the condition (11) for ϵ ensures that $B(e_1, 2\epsilon)$ and $B(e_2, 2\epsilon)$ are not overlapping, therefore the two last inequalities which are established for the same $t = 1/n$ are not possible simultaneously.

The process X_t in E corresponding to the entrance $p_{yi}(t)$ must then be concentrated in one point of Ξ at time $t = 0$.

Denote this point by $\Xi(y)$.

Ξ is the identity from A as subset of $A+A_e$ into A as subset of E.

If k is a fixed point of A the process $p_{ki}(t)$ is concentrated in one point, $\Xi(k)$, at time $t = 0$ i.e.

(17) $P[X_o = \Xi(k)] = 1$

Choose a sequence of positive numbers δ_n, n in N, such that

$$\sum_{n=1}^{\infty} \delta_n < \tfrac{1}{2}$$

As $p_{kk}(t)$ tends to one as t tends to 0, we can get a decreasing sequence of positive numbers t_n, n in N, such that t_n tends to 0, as $n \longrightarrow \infty$ and

$$1 - \delta_n \leq p_{kk}(t_n) \qquad \text{for all } n$$

But by (1) this yields

$$P[X_{t_n} = k \text{ for all } t_n] \geq 1 - \sum_{n=1} \delta_n > 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

According to the assumption (d) $X_t(w)$ tends to $X_o(w)$ as t tends to zero with probability one. Thus the last inequality shows that $X_o(w) = k$ on a set of probability at least $\frac{1}{2}$.

Taking into account (17) this is enough to give

$$\Xi(k) = k \text{ in } E \text{ for all } k \text{ in } A$$

Ξ is one-to-one from $A+A_e$ into E.

Let y_1 and y_2 be elements of $A+A_e$ such that

$$\Xi(y) = \Xi(y_2)$$

With y_1 is associated a Markov process $\bar{X}_t$ in E such that

(18) $$P[X_o = \Xi(y_1)] = 1$$

and

$$P[X_t = i] = p_{y_1 i}(t) \text{ for all } i \text{ in } A \text{ and } t > 0$$

Hence

$$P[X_t = i] = P[X_t = i \quad X_o = \Xi(y_1)] =$$

$$P[X_t = i \mid X = \Xi(y_1)]\ P\ [X \ = \Xi(y_1)]$$

By assumption (e) and (18) the last line is equal to

$$p^{E}_{\Xi(y_1)\, i}(t) \quad \text{for all } i \text{ in } A \text{ and } t > 0$$

and therefore

$$p_{y_1 i}(t) = p^{E}_{\Xi(y_1)\, i}(t) \text{ for all } i \text{ in } A \text{ and } t > 0$$

Similarly we get

$$p_{y_2 i}(t) = p^{E}_{\Xi(y_2)\, i}(t) \text{ for all } i \text{ in } A \text{ and } t > 0$$

But the equality $\Xi(y_1) = \Xi(y_2)$ then implies that $y_1 = y_2$ in $A+A_e$ i.e. Ξ is one-to-one and this completes the proof of theorem 1. Note that we do not need to assume the strong Markov property in E to prove theorem 1. On the other hand the assumptions (a) to (e) are by themselves not sufficient to show the strong Markov property. Usually some analytical assumptions are made about the transition semi-group itself, which used with the right continuity of sample paths are enough to check the strong Markov property. E.g. in Chapter III we used the strong Feller property to obtain the convergence in III. (25).

CHAPTER V

On the Topology of the Entrance Boundary

1. Introduction.

Limitation on the size of A_e.

The semigroup P(t) is assumed to have only a countable number of extremal entrances. This simplifies the notation in some cases and as in fact no general results are obtained it does not matter very much. Having made this assumption, pick a stochastic entrance p(t), $t > 0$. such that its corresponding measure on $A + A_e$ at $t = 0$ is a probability measure attaching strictly positive weight to every point of $A + A_e$.

Denote by $(\Omega, \mathcal{F}, P)$ a probability triple and by X_t, $t \geq 0$ a process in $A + A_e$ right continuous relative to T, of absolute distribution p(t) and strongly Markovian with the extended semigroup (as in Chapter III). From now on the trajectory for every w in Ω is kept fixed.

In this chapter a process is always supposed to be as described above unless otherwise stated.

In § 2 the trivial example 1 shows that there exist topologies finer than T for which the fixed trajectories which are right continuous in T are also right continuous relative to these finer topologies. Then for every process within the scope of this chapter T' will denote the finest topology on $A + A_e$ such that every right continuous trajectory in T is also right continuous in T'. To end the second paragraph we show that T' is coarser that T*, the fine topology, which is not equal to T, as was originally overlooked by Chung.

My hope was to see that T' is metrisable and to determine an equivalent metric with the help of taboo semigroups. The taboo set is to be a subset of $A + A_e$ and in paragraph 3 we define the taboo semigroups and remark that in some sence a taboo semigroup may be more discriminating than the original P(t).

In my mind the metric would have had to be of the following form: Let B_n, n in N, be a sequence of subsets in $A + A_e$ such that the taboo semigroups exist. Denote by ${}_{B_n}R(\lambda)$ the corresponding resolvents. Let α_n, n in N and β_i, i in A be two sequences of strictly positive numbers such that

$$\sum_{n=o}^{\infty} \alpha_n < \infty \qquad \text{and} \qquad \sum_{i \quad A} \beta_i < \infty$$

For all x and y in $A + A_e$ define a metric d(x;y) by letting

(1) $$d(x;y) = \sum_{n=o}^{\infty} \alpha_n \sum_{i \in A} \beta_i \left| {}_{B_n}Rxi(1) - {}_{B_n}Ryi(1) \right|$$

The main difficulty is to make sure that a given sequence of B_n's is suitable to obtain a metric (1) generating the topology T'. As the sequences $B_n = \{i_n\}$ and $B_n' = \{y_n\}$ (where i_n and y_n are enumerations of A and $A + A_e$) are very simple and general, they are interesting choices to use in (1). But both ideas are ruled out by the example 2 given in paragraph 4.

Another very general way to define a metric with the taboo semigroups is to let, for all x and y in $A + A_e$.

(2) $$d(x;y) = \sup_{B} \sum_{i \in A} \beta_i \left| {}_{B}Rxi(1) - {}_{B}Ryi(1) \right|$$

where the supremum is taken over all the subset B's for which the taboo semigroup is defined. Example 2 is again a counter example.

In fact examples 1 and 2 suggest a probabilistic characterisation for a suitable B, but it appears to be a set with a very elusive analytical definition.

2. Basic example

Example 1.

Let $A = (0,1,2,\ldots)$

Define a corresponding conservative Q matrix by

$$Q = \begin{bmatrix} 0 & 0 & 0 & & \\ 1 & -1 & 0 & & 0 \\ 2 & 0 & -2 & & \\ \vdots & 0 & & \ddots & \\ n & & & & -n \\ \vdots & & 0 & & & \ddots \end{bmatrix}$$

The resolvent is then

$$R(\lambda) = \begin{bmatrix} \frac{1}{\lambda} & 0 & & & \\ \frac{1}{\lambda+1} \cdot \frac{1}{\lambda} & \frac{1}{\lambda+1} & & 0 & \\ \vdots & 0 & \ddots & & \\ \frac{n}{\lambda+n} \cdot \frac{1}{\lambda} & & 0 & \frac{1}{\lambda+n} & \\ \vdots & & & & \ddots \end{bmatrix}$$

We have

$$R_{no}(1) = \frac{n}{n+1} \cdot \frac{1}{1} \longrightarrow 1 = R_{oo}(1) \quad \text{as } n \longrightarrow \infty$$

and

$$R_{ni}(1) \leqslant \frac{1}{n+1} \longrightarrow 0 = R_{oi}(1) \qquad \text{as } n \longrightarrow \infty$$

Hence $\{0\}$ is limit of the sequence $\{n\}$ in the topology T.

Note that A_e is void.

Proof:

By th.II 4 we know that A is dense in A_e. Take any sequence of points in A, $\{i_r\}$ say, where r is in N.

There are three possibilities

(i) i_r may be equal to the same point i of A for all sufficiently big r; then the limit of i_r is i itself as r tends to ∞

(ii) i_r may be increasing to ∞ as r increases to ∞; in this case we have just seen that the limit is $\{0\}$ in (A,T).

(iii) if i_r is different from (i) or (ii) then $\bar{i}_r$ is not convergent

in its components $R_{i_r i}(1)$.

(i), (ii) and (iii) are enough to show that A_e is void.

Description of the sample paths.

From the usual interpretations for the matrix Q(see C.th II.5.5 and p. 259) and the X_{t+} version we can deduce that all sample paths are particularly simple. Either they start in 0 and stay always in it or they start in an $i > 0$, stay there for a while and leave it by a jump to 0 in which they remain thereafter. So every trajectory is composed of a finite number (one or two) of left closed and right open intervals. The discrete topology is then such that all sample paths are right continuous and as it is the finest topology on A, it is T'. Therefore in this example T' is strictly finer than T.

An interesting feature of this example is that it is a counter-example to part of C th.II.11.4.

Let P(t) be a stochastic standard transition matrix on A and define with Chung the fine topology T_F on A. For all k in A denote by $S_k(w)$ the subset of $[o, \infty]$ on which $X_t(w)$ is equal to k.

Let i be in A and H be a subset of A. Consider the probabilities.

$$P[\bigcup_{k \in H} S_k(w) \cap (t, t+\epsilon) = 0 \text{ for some } \epsilon > o \mid X_t(w) = i]$$

and

$$P[\bigcup_{k \in H} S_k(w) \cap (t, t-\epsilon) = 0 \text{ for some } \epsilon > 0 \mid X_t(w) = i]$$

By C th II. 5. 6 for a stable i and C th II. 11. 3 for an instantaneous one, we know that these probabilities are equal and that their common value is 0 or 1.

H is called nonadjacent to i, if and only if this value is one. A fine neighbourhood of i is a complement of a nonadjacent set to i.

Finally the fine topology T_F on A is the topology generated by all fine neighbourhoods of all states in A.

Part of C th II. 11. 4 reads as follows: a base of fine neighbourhoods of i is given by the sets

$$C_i(\delta) = \left\{ k \in A \,\middle|\, \sup_{0 \leq t \leq \delta} p_{ki}(t) > 1 - \delta \right\} \qquad 0 < \delta$$

In example 1, choose $i = \{0\}$ and $H = \{k > 0\}$

Then

$$P\left[\bigcup_{k \in H} S_k(w) \cap (t, t+\varepsilon) = 0 \text{ for some } \varepsilon > 0 \,\middle|\, X_t(w) = \{0\} \right] =$$

$$P\left[\bigcup_{k>0} S_k(w) \cap (t, \infty) = 0 \,\middle|\, X_t(w) = 0 \right] = 1$$

as $\{0\}$ is absorbing.

Hence H is nonadjacent to $\{0\}$ and $\{0\}$ is a fine neighbourhood of itself.

For every $\delta > 0$ we have

$$\begin{aligned} C_0(\delta) &= \left\{ k \,\middle|\, \sup_{0 \leq t \leq \delta} p_{k0}(t) \geq 1 - \delta \right\} \\ &= \{0\} \cup \left\{ k > 0 \,\middle|\, \sup_{0 \leq t \leq \delta} (1 - e^{-kt}) \geq 1 - \delta \right\} \\ &= \{0\} \cup \left\{ k > 0 \,\middle|\, 1 - e^{-k\delta} \geq 1 - \delta \right\} \\ &= \{0\} \cup \left\{ k_\delta, k_{\delta+1}, k_{\delta+2}, \ldots \right\} \end{aligned}$$

where k_δ is the smallest positive integer such that

$$e^{-k_\delta \delta} < \delta$$

It is now clear that no $C_0(\delta)$ nor any finite intersection of them is contained in $\{0\}$. Therefore the family $C_0(\delta)$, $\delta > 0$, does not form a base of neighbourhoods of $\{0\}$ in the topology T_F.

Nevertheless the following weaker result is contained in Chung's proof:

T_F is finer than the topology generated by $C_i(\delta)$, i in A, $\delta > 0$.

Proof:

Recall the relation (10) in C p 191.

"Let i and $C_i(\delta)$ be given; then for almost every w the sample path $X_t(w)$ has the following property: if $X_t(w) = i$ then there exists $h(w) > 0$ such that

$$X_s(w) \in C_i(\delta) \cup \{\infty\} \qquad \text{for all s in } (t, t+h(w))$$

where $\{\infty\}$ is the Alexandroff additional point".

From this we can deduce that $A - C_i(\delta)$ is nonadjacent to i, and hence $C_i(\delta)$ is a fine neighbourhood of i. As this is true for every $C_i(\delta)$, this completes the proof.

Definition of the fine topology T^* on $A+A_e$

(the fine topology of Chung, T_F, was on A only)

As assumed in the introduction A_e is countable; hence every subset of $A+A_e$ is a Borel set for T and by III (28) its corresponding stopping time exists.

Let T^* be the topology generated by the following open sets (cf. Meyer[2] p. 152).

$G \subset A + A_e$ is open if and only if

(3) $P[\zeta_{G^c} > 0 \mid X_o = y] = 1 \qquad$ for all y in G

G^c is the complement of G in $A + A_e$, and by the 0 or 1 law (M XIII 14), (3) must be equal to 0 or 1; in particular as $G \cap G^c = \emptyset$ all G satisfying (3) are such that

$$P[\zeta_G = 0 \mid X_o = y] = 1 \qquad \text{for all y in G}$$

These sets from a topology because

(i) $A + A_e$ and $\emptyset$ are open

(Ii) every union of open sets is an open set

(iii) every finite intersection of open sets is an open set.

Proof: (in which open set stands for open relative to T^*)

(i) $P[\zeta_{\emptyset} = \infty \mid X_o = y] = 1 \qquad$ for all y in $A + A_e$

i.e. $A + A_e$ is an open set.

As $\emptyset$ does not contain any point of $A + A_e$, the condition (3) is meaningless for $\emptyset$ and so $\emptyset$ is an open set.

(ii) Let G_α be open sets, where α runs in some family. Then by our countability assumption their union is a Borel set relative to T, so that its complement is also one. Thus the corresponding stopping time is well defined.

By the set inclusion

$$(\cup G_\alpha)^c \subset G_\alpha^c \quad \text{for all } \alpha$$

we get

$$P[\zeta_{(\cup G_\alpha)^c} > 0 \mid X_o = y] \geqslant P[\zeta_{G_\alpha^c} > 0 \mid X_o = y] = 1$$

for all y in G_α, and hence also for all y in $\cup G_\alpha$, i.e. (3) is satisfied.

(iii) Let G_1 and G_2 be two open sets. Pick an w in

$$[\zeta_{(G_1 \cap G_2)^c} = 0]$$

and then there exists a sequence $t_n(w)$ decreasing to 0 such that

$$X_{t_n}(w) \text{ is in } (G_1 \cap G_2)^c = G_1^c \cup G_2^c$$

So at least one subsequence of $t_n(w)$ is such that $X_{t_n}(w)$ lies always in the same G_i^c and hence w is in

$$[\zeta_{G_1^c} = 0] \cup [\zeta_{G_2^c} = 0]$$

This inclusion gives the probabilistic inequality

$$P[\zeta_{(G_1 \cap G_2)^c} = 0 \mid X_o = y] \leqslant$$

$$P[\zeta_{G_1^c} = 0 \mid X_o = y] + P[\zeta_{G_2^c} = 0 \mid X_o = y]$$

and the last sum is equal to 0 for all y in $G_1 \cap G_2$, i.e.(3) holds for $G_1 \cap G_2$.

As we know (th III. 1 and th III. 3) that T is such that almost every sample path is right continuous at all t, then a sample path starting in any open set E in T will stay there for a strictly positive time with probability one, i.e.

$P[\zeta_{E^c} > 0 \mid X_o = y] = 1$ for all y in E

Hence E is an open set for T* and this shows that T* is finer than T.

Let T'' be a topology strictly finer than T*, i.e. there exists an open set for T'' which is not open in T*. Pick such a set, C say. By assumption there exists a y in C such that

$$P[\zeta_{C^c} > 0 \mid X_o = y] \neq 1$$

By the 0 - 1 law this last relation yields

$$P[\zeta_{C^c} = 0 \mid X_o = y] = 1$$

which implies that a.e. sample path starting in y leaves the open neighbourhood C at least once as soon as it leaves t = 0; hence X_+ is right discontinuous in 0 relative to T''.

So if a topology To on A + A_e is such that all the right continuous trajectories for T are also right continuous for T_o, then T_o must be coarser than T*. Therefore the finest topology with this property must also be coarser than T* and this gives

T* finer than T' finer than T

the second relation being obvious as T itself is a topology for which all right continuous trajectories in T are right continuous in T!

The next question is naturally: does T* itself keep the right continuity property of the trajectories which are right continuous for T?

Unfortunately the answer is affirmative in the obvious examples but not clear in general. We now give some reasons why it is difficult to answer.

Taking into account the countability of A_e here is a simplified version of M XV 38.

Let F be a closed set of (A + A_e, T*); then for a fixed w the set

$$\{ t \mid X_t(w) \in F \}$$

is such that every right adherent point to it is in it, except on a set of w 's of probability zero.

Now we use this result to show the separability of X_t relative to T*.

Let I be an open interval .

Let S be a countable dense subset of R_+ .

Choose an w outside the exceptional set, Ω_F say.

If w is such that

$$X_t(w) \in F \quad \text{for all } t \text{ in } I$$

then

$$X_t(w) \in F \quad \text{for all } t \text{ in the right closure of } I \cap S$$

i.e.

$$X_t(w) \in F \quad \text{for all } t \text{ in } I$$

Therefore

$$P[\ \{X_t \in F\ ,\ \forall t \in I \cap S\} - \{X_t \in F,\ \forall t \in I\}\ ] \leq$$
$$P[\Omega_F] + P[(\ \{X_t \in F,\ \forall t \in I \cap S\} - \{X_t \in F,\ \forall t \in I\}\ ) \cap (\Omega - \Omega_F)]$$
$$= 0 + P[\ \emptyset\ ] = 0$$

in other words X_t is separable with respect to the closed sets of T* and any S.

Consider the usual topology on R and define C(A + A_e, T*; R) as the set of all continuous functions defined on (A + A_e, T*) with values in R. The theorem M XV 39 reads as follows:

if f is an element of C(A + A_e, T*; R) then f(X_t(w)) is right continuous for almost all w.

Consider the following diagram:

$$t \in [\ 0, \infty\ ] \longrightarrow X_t(w) \in (A + A_e, T^*) \longrightarrow f(X_t(w)) \in R$$

If we want to deduce the right continuity of X_t(w) itself from this diagram and M XV 39 we need some additional conditions on (A + A_e, T*) e.g. :

(a) C(A + A_e, T*; R) must be good enough to define T* as its initial topology. This is known for a compact (or locally compact) space but as T* is a refinement of T, we do not know if this property (used with $(M_1; T)$ or $(K; T)$ still holds.

(b) $C(A + A_e, T^*; R)$ must be spanned by a countable set a functions f_n so that the union of Ωf_n (the exceptional sets depending on f_n in M XV 39) is of probability zero.

(a) and (b) would be enough to imply that for any w outside $\bigcup_n \Omega f_n$, $f(X_t(w))$ is right continuous from $[0, \infty]$ into R for all f, and hence $X_t(w)$ itself is right continuous from $[0, \infty]$ into $(A + A_e; T^*)$.

3 Definition of the taboo semigroups. Let B be a Borel set of $A \cup A_e$ relative to T. It is known (see III (28) that the random variable

(4) $\zeta_B(w) = \inf \left\{ t \mid t > o \; X_t(w) \in B \right\}$

is a stopping time relative to the family of σ-fields $\mathcal{F}_t$, $t \geq o$

Hence the set

$$[w \mid X_o(w) = i,\; X_s(w) \notin B,\; o < s \leq t \;\; X_t(w) = j]$$

$$= [w \mid X_o(w) = i,\; X_t(w) = j \;\; \zeta_B(w) > t]$$

is in $\mathcal{F}_t$ ($\subset \mathcal{F}$), and we can define for all i, j and $t > o$, the number $_Bp_{ij}(t)$ as the following elementary conditional probability:

(5) $P[X_t = j \;\; \zeta_B > t \mid X_o = i] = \dfrac{P[X_o = i \;\; B^t \;\; X_t = j]}{P[X_o = i]}$

Obviously we have for all i and j in A, and all $t > o$,

(6) $_Bp_{ij}(t) \leq p_{ij}(t)$

Next we check the semigroup equation for $_BP(t)$. As P (t) is stochastic we have for all $t > o$ and all $s > o$

(7) $_Bp_{ij}(t+s) = \sum_{k \in A} P[X_{t+s} = j \;\; \zeta_B > t+s \;\; X_t = k \;\; X_o = i]$

Now fix k in A, $t > o$ and $s > o$.

Let

$$\Delta_k = [X_t = k]$$

We have

$$P[X_o = i \;\; X_t = k \;\; X_{t+s} = j \;\; \zeta_B > t + s] =$$

(8) $P[X_o = i \;\; \zeta_B > t \;\; X_t = k \quad \zeta_B \circ \theta_t > s \;\; X_s \circ \theta_t = j]$

Where θt is as usual the shift operator.

As ζ_B is a stopping time the event

$$[\zeta_B \circ \theta_t > s] \text{ is in } {}_t\mathcal{F}_{t+s}$$

where ${}_t\mathcal{F}_{t+s}$ is the augmented σ-field generated by

$$X_u, \; t \leq u \leq t+s$$

Let

$$\Lambda = [X_o = i \;\; \zeta_B > t] \in \mathcal{F}_t$$

$$M = [X_s \circ \theta_t = j \;\; \zeta_B \circ \theta_t > s] \in {}_t\mathcal{F}$$

By the Markov property we get the following equality of random variables:

$$P[\Lambda M \mid X_t] = P[\Lambda \mid X_t] \, P[M \mid X_t] \qquad \text{a.s.}$$

so that (8) is equal to

(9) $$\int_{\Delta_k} P[\Lambda \mid X_t] \, P[M \mid X_t] \, P[dw]$$

By definition $P[M \mid X_t]$ is a random variable such that

$$\int_{\Delta_k} P[M \mid X_t] \, P[dw] = P[X_s \circ \theta_t = j \;\; \zeta_B \circ \theta_t > s \;\; X_t = k] =$$

$$P[X_s \circ \theta_t = j \;\; \zeta_B \cdot \theta_t > s \mid X_o \circ \theta_t = k] \, P[\Delta k]$$

By stationarity and (5) the last term becomes

$$P[X_s = j \;\; \zeta_B > s \;\; X_o = K] P[\Delta_k] = {}_B P_{kj}(s) \, P[\Delta k]$$

We have just proved that

$$P[M \mid X_t] = {}_B p_{X_t j}(s) \qquad \text{a.s.}$$

(9) is then equal to

$${}_B p_{kj}(s) \int_{\Delta_k} P[X_o = i \;\; \tau_B > t \mid X_t] \, P[dw]$$

or using (5) again

(10) $\quad {}_B p_{kj}(s) \; {}_B p_{ik}(t) \; P[X_o = i]$

If we sum over all k in A the equalities

$$(8) \quad \frac{1}{P[X_o = i]} = (10) \quad \frac{1}{P[X_o = i]}$$

(7) becomes the semigroup equation

$$(11) \quad {}_B p_{ij}(t+s) = \sum_{k \in A} {}_B p_{ik}(t) \; {}_B p_{kj}(s)$$

As τ_B is a stopping time, $[\tau_B > 0]$ is in $\mathcal{F}_o$. Hence by the 0-1 law we have for all y in $A+A_e$

(12) $\quad P[\tau_B > 0 \mid X_o = y] = 0$ or 1

By the inequality (6) we find

$$(13) \quad 0 \leq \limsup_{t=o} {}_B p_{ij}(t) < \lim_{t=o} p_{ij}(t) = 0 \text{ if } i \neq j$$

and

$$(14) \quad 0 \leq \limsup_{t=o} \sum_{k \neq i} {}_B p_{ik}(t) \leq \lim_{t=o} \sum_{k \neq i} p_{ik}(t) = 0$$

By th. III. 1 we have

$$P[\tau_B > s \mid X_o = i] = P[\tau_B > s \;\; X_s \in A \mid X_o = i]$$

Rewrite this equality as

(15) $\quad {}_Bp_{ii}(s) = P[\zeta_B > s \mid X_o = i] - \sum_{k \neq i} {}_Bp_{ik}(s)$

By monotonicity we get

(16) $\lim_{s=0} P[\zeta_B > s \mid X_o = i] = P[\zeta_B > 0 \mid X_o = i]$

The relations (14) and (16) imply that the R.H.S. of (15) has a limit as s decreases to 0.

Hence the same property holds for its L.H.S. and by (12) this limit must satisfy

(17) $\lim_{s=o} {}_Bp_{ii}(s) = P[\zeta_B > o \mid X_o = i] = 0 \text{ or } 1$

If i is such that this limit is zero, then for all j in A and all $t > o$ we have

(18) $0 \leq {}_Bp_{ij}(t) = P[\zeta_B > t \quad X_t = j \mid X_o = i] \leq$

$P[\zeta_B > o \mid X_o = i] = 0$

Let $_BA_o = \left\{ i \text{ in } A \mid {}_Bp_{ii}(t) = 0 \text{ for all } t > o \right\}$

This set is equal to

$$\left\{ i \text{ in } A \mid {}_Bp_{ij}(t) = 0 \quad \text{for all } t > o \text{ and all } j \text{ in } A \right\}$$

One inclusion is obvious conversely if i is such that ${}_Bp_{ii}(t) = 0$ for all t, then its limit as t tends to 0 is also 0 and by (17) we are in the case (18) and this shows the other inclusion.

If we let ${}_BA = A - {}_BA_o$ the relation (11) can be written as

(19) $$ {}_Bp_{ij}(t+s) = \sum_{k \in {}_BA} {}_Bp_{ik}(t) \; {}_Bp_{kj}(s)$$

for all i and j in ${}_BA$ and all $t \geq 0$ and $s \geq 0$.

If j is in $_BA_o$ then

(20) $_Bp_{ij}(t) = 0$ for all i in A and all $t \geq 0$

<u>Proof:</u> (11) gives for all $s < t$

(21) $_Bp_{ij}(t) = \sum_{k \neq j} {}_Bp_{ik}(t-s)\ {}_Bp_{kj}(s) + {}_Bp_{ij}(t-s)\ {}_Bp_{jj}(s)$

As j is in $_BA_o$ the second term in the R.H.S. is equal to 0; on the other hand the sum over all k different from j is bounded above by

$$\sum_{k \neq j} p_{ik}(t-s)\ p_{kj}(s) = p_{ij}(t) - p_{ij}(t-s)\ p_{jj}(s)$$

Which by the stochastic continuity of P(t) tends to 0 as s decreases to 0; hence if we let s tend to 0 in (21) we obtain (20). (13), (17) and (19) mean that $_BP(t)$ is a standard substochastic semigroup on $_BA$. By th. II 3.3 we can use th. II 2.3 to check the continuity for $t \geq 0$ of $_Bp_{ij}(t)$ for all i and j in $_BA$.

C //

As the initial distribution of X_t was chosen to attach strictly positive weight to every point of the countable A_e, we can let

$p_{yi}(t) = P[\zeta_B > t\ X_t = i \,/\, X_o = y]$

By the method used to get (11) and (19) we find

$$_Bp_{yi}(t+s) = \sum_{k \in {}_BA} {}_Bp_{yk}(t)\ {}_Bp_{ki}(s)$$

for all i in A and all $t > 0$ and $s \geq 0$.

Since $_Bp_{yi}(t)$ is an entrance relative to $_BP(t)$ the theorem just quoted applies to ensure the continuity in t of $_Bp_{yi}(t)$ for all i in $_BA$ (or indeed in A because the proof of (20) works also for y).

This general continuity allows us to use $_BP(t)$ or the Laplace transforms which will be denoted by $_BRij(\lambda)$.

We have just shown that

(22) $\underset{B}{R}(\lambda) \leqslant R(\lambda)$

(23) $\lim_{\lambda=\infty} \lambda\, {}_BR(\lambda) = I_{{}_BA}$ (the identity matrix on $({}_BA \times {}_BA)$)

(24) ${}_BR(\lambda) - {}_BR(\mu) = (\mu-\lambda)\, {}_BR(\lambda)\, {}_BR(\mu)$

and ${}_BR_{yi}(\lambda)$, $\lambda > o$, i in A satisfies the resolvent equation for ${}_BR(\lambda)$ for all y in $A+A_e$.

Theorem 1

For all open and all closed sets B in $(A+A_e, T)$, ${}_BP(t)$ is completely determined by P (s) and B.

Proof:

We have to show that any ${}_Bp_{ij}(t)$ is determined by $t \geq o$, i in A, j in A, P (s) and B. This result is obvious for a point i (or j) in B, because C th. II. 5.3 and the definition (5) give

$${}_Bp_{ij}(t) = 0 \quad \text{for all } t > o \text{ and i or j in } A \cap B$$

So we choose i and j outside the taboo set in the sequel. We remark also that if ${}_BP(t)$ depends only on P (s) and B for all $t > o$, the stochastic continuity will then imply the same for t= 0. Hence from now on t is a fixed strictly positive number.

Let $\Omega_c = [w \mid X_t(w)$ is right continuous (in $(A+A_e, T)$) on $[o, \infty)]$

$\Omega_l = [w \mid X_t(w)$ has a left limit (in (M_1, T)) at all t in $(o, \infty)]$

(25) We have $P[\Omega_c] = P[\Omega_l] = 1$.

Firstly we prove theorem 1 for an open set G in $(A+A_e, T)$ using the method given in C. p. 194. As the process X_t is right continuous the values $X_s(w)$ for all s in a countable subset S dense in R_+ are enough to determine the complete sample path for all w in Ω_c. Let S be enumerated in some way and let s_n be the n^{th} element in this enumeration lying in $[0, t]$.

Let $\Gamma = [w \mid X_s(w) \notin G \quad o < s \leq t \; X_t = j]$

$\Gamma_n = [w \mid X_{s_r}(w) \notin G \; r=1,2,\ldots,n \; X_t = j]$

The following inclusion is obvious

$$\Gamma \subset \bigcap_{n=1}^{\infty} \Gamma_n$$

Conversely any w in this intersection is such that

$X_{s_n}(w) \notin G$ for all n

i.e.

$X_s(w) \notin G$ for all s in $S \cap [o,t]$

As G^c is a closed set this implies for all w in $\bigcap_{n=1}^{\infty} \Gamma_n \cap \Omega_c$

$$X_u(w) = \lim_{\substack{s \downarrow u \\ s \in S}} X_s(w) \in G^c \text{ for all } u \text{ in } [o,t)$$

As j is not in G, then

$P[X_t \notin G \; X_t=j]=1$

so that we have now proved

$$(26) \quad \Gamma \doteq \bigcap_{n=1}^{\infty} \Gamma_n$$

From the definition (5) and (26) we get

$$(27) \quad {}_Bp_{ij}(t) = \lim_{n=oo} P[\Gamma_n \mid X_o = i] \;.$$

But for any fixed n theorem III 1 gives

$$P[X_{s_r} \in G^c] = P[X_{s_r} \in G^c \cap A], r = 1,2, \;\; , n$$

which in turn implies

$$(28) \quad P[\Gamma_n \mid X_o = i] = \sum_{k_1 \in G^c \cap A, \dots, k_n \in G^c \cap A} p_{ik_1}(s_1^1) \, p_{k_1 k_2}(s_2^1 - s_1^1) \dots p_{k_n j}(t - s_n^1)$$

where $\{s_r^1, r=1,2, \;\; n\}$ is the set $\{s_r, r=1,2, \;\; n\}$

reindexed to follow the natural order in R. Some obvious changes have to be made in the sum above if one s_r is equal to 0 or t, but it does not alter the fact that (27) and (28) proves theorem 1 for the open set G. Since the event Γ is independent of S, so is the limit in (27) and we can use any countable set S dense in R_+.

Next using this result for an open set we can proceed to show the same for any closed set F in $(A+A_e, T)$. As the entrance boundary is a metric space we can define the following open sets

$$G_m = \left\{ X \mid d(x; F) < \frac{1}{m} \right\} \quad m \text{ in } N$$

which are such that

$$(29) \quad F = \bigcap_{m=1}^{\infty} G_m = \bigcap_{m=1}^{\infty} \overline{G}_m$$

Where $\overline{G}_m$ denotes the closure of G_m in (A+Ae, T)

Let $\Lambda = [w / X_s \notin F \quad o < s \leq t \quad X_t = j]$

$$\Lambda_m = [w \mid X_s \notin G_m \; o < s \leq t \quad X_t = j]$$

The following inclusion is obvious

$$(30) \quad \Lambda \supset \bigcup_{m=1}^{\infty} \Lambda_m$$

Conversely if w is in $\Delta = \Omega - \bigcup_{m=1}^{\infty} \Lambda_m$ there exist $t_m(w)$, m in N, such that

$0 < t_m(w) \leq t$

and for all m

$X_{t_m(w)}(w) \in G_m$

Define ζ_m as the stopping time associated with G_m as in (4). Using the right continuity we have

$$(31) \quad X_{\zeta_m(w)}(w) \in \overline{G}_m \qquad \text{for every } w \text{ in } \Omega_c$$

The monotonicity of the sequence G_m yields for all w

$$\zeta_m(w) \leq \zeta_{m+1}(w) \leq \lim_{m=\infty} \zeta_m(w)$$

If we denote the limit above by ζ, M IV 42 ensures that ζ is also a stopping time.

Let $\Delta_1 = \bigcup_{m=1}^{\infty} [w \mid \zeta_m(w) = \zeta(w) \leq t \quad X_t(w) = j] \subset \Delta$

$\Delta_2 = \Delta - \Delta_1$

For every w in $\Delta_1 \cap \Omega_c$, there exists an m (w) in N such that (by (31))

$$X_{\zeta(w)}(w) = X_{\zeta_m(w)}(w) \in \bar{G}_m \quad \text{for all } m \geq m(w)$$

and using (29) this implies

$$(32) \quad X_{\zeta(w)}(w) \in \bigcap_{m=1}^{\infty} \bar{G}_m = F \qquad \text{for all } w \text{ in } \Delta_1 \cap \Omega_c$$

Similarly if w is in $\Delta_2 \cap \Omega_c \cap \Omega_t$ we find

$$(33) \quad X_{\zeta\_(w)}(w) \in \bigcap_{m=1}^{\infty} \bar{G}_m = F \subset A + A_e$$

As ζ is bounded by t on Δ_2, theorem III. 4 applies and we get

$$(34) \quad X_{\zeta(w)}(w) = X_{\zeta\_(w)}(w) \in F \qquad \text{for } \sout{\text{all}}\ \text{a.a. } w \text{ in } \Delta_2 \cap \Omega_c \cap \Omega_t$$

From (25), (32) and (34) we deduce

$$\Omega - \bigcup_{m=1}^{\infty} \Lambda_m = \Delta_1 \cup \Delta_2 \subset \Omega - \Lambda$$

This last relation, the definition (5) and (30) give

$$(35) \quad {}_F P_{ij}(t) = P[\Lambda \mid X_0 = i] = \lim_{m=\infty} P[\Lambda_m \mid X_0 = i] =$$

$$= \lim_{m=\infty} {}_{G_m} P_{ij}(t)$$

The sets G_m being open the last limit depends only on P (s), i, j and G_m (i.e. F). Theorem 1 is now proved. As the event Λ does not depend on a particular sequence of G_m, any sequence of open sets G_m which satisfy (29) will define ${}_F P_{ij}(t)$ by (35).

Note that theorem 1 seems also likely for a Borel set B but the proof used here does not work (even in our particular case of a countable A_e where every Borel set is a G_δ - set) Instead of (27) we get for some 9/
family of open sets G_m the less stringent relation

$$B = \bigcap_{m=1}^{\infty} G_m \subset \bigcap_{m=1}^{\infty} \overline{G_m}$$

so that theorem III 4 cannot be used between relations corresponding to (33) and (34) to show that the corresponding Λ and Λ_m satisfy $\Lambda \in \bigcup_{m=1}^{\infty} \Lambda_m$

Theorem 2

For all open and closed sets B in $(A+A_e, T)$, $p_{yi}(t)$, i in A, is completely determined by P (s), B and y.

This is proved as theorem 1 with only one change namely $p_{ik}(s)$ is to be replaced by p_{yk} (s) in (28) (which is why y is needed).

Finally we explain the meaning of the words "more discriminating than the original semigroup" used in the introduction.

In the example 1 if the subset B of A is chosen as $\bigcup_{i>0} \{i\}$ then obviously we get

(36) $_B Roo(\lambda) = \frac{1}{\lambda} = Roo(\lambda)$

(37) $_B Rok(\lambda) = 0 = Rok(\lambda)$ for all $k > o$

and

(38) $_B Rik(\lambda) = 0$ for all k if $i \geq 1$

Therefore $_B Rio(1) = 0$ does not tend to $_B Roo(1) = 1$ as i tends to ∞, the taboo semigroup relative to B introduces a refinement of T (but only locally near $\{0\}$ in A as all the other points are merged in the trivial entrance).

4. Metrication of T'

I tried to introduce the taboo semigroups in the definition of a metric for T', because it seemed an easy way to generalise the form of metric used for the definition of T(see below (1)) and also because I did not know what else, in analytical terms, could be used.

As I said in the introduction to this chapter I wanted a metric of the form

$$(1) \quad d(x; y) = \sum_{n=1}^{\infty} \alpha_n \sum_{i \in A} \beta_i \left| {}_{B_n}Rxi(1) - {}_{B_n}Ryi(1) \right| .$$

for all x and y in $A + A_e$, where the B_n's are a sequence of subsets of $A + A_e$, and α_n, n in N, β_i, i in A, are the strictly positive terms of two converging series. I also pointed out before that the use of the sequences of singletons $\{i\}$ of A or $\{y\}$ of $A + A_e$ as sequences of taboo sets is very tempting, because it requires no further knowledge.

But in the next example in which T' is indeed metrisable these two new metrics are unfortunately not equivalent to the one defining T'. We remark that if the sequence $A + A_e$ does not give the solution, the sequence A which induces a smaller metric can not work either. So we will only consider the sequence $A + A_e$.

Example 2.

Let A = (..., -2, -1, 0, 1, 2,...)

Let q_i be a sequence of stricly positive numbers (indexed by $i > 0$ only) such that $\sum_{i=1}^{\infty} \frac{1}{q_i} < \infty$

Define a conservative Q-matrix in the following way.

$q_{oo} = 0$	
$q_{ii} = q(-i)(-i) = - q_i$	$i \neq 0$
$q_{i,i-1} = q_i$	for all $i > 0$
$q_{i,(-i)-1} = q_i$	for all $i < 0$
$q_{ij} = 0$	everywhere else

By our choice of an absorbing $\{0\}$ the corresponding minimal solution is a stochastic matrix. Its terms are

$R_{oo}(\lambda) = \frac{1}{\lambda}$, $R_{oi}(\lambda) = 0$ for all $i \neq 0$

$R_{ii}(\lambda) = \frac{1}{\lambda + q_i}$ for $i \neq 0$

$R_{(-i)j}(\lambda) = \frac{1}{\lambda + q_j} \prod_{k=j+1}^{i} \frac{qk}{\lambda + qk}$ if $i > 0$, and $0 < j < i$

$R_{(-i)0}(\lambda) = \frac{1}{\lambda} \prod_{k=1}^{i} \frac{qk}{\lambda + qk}$ if $i > 0$

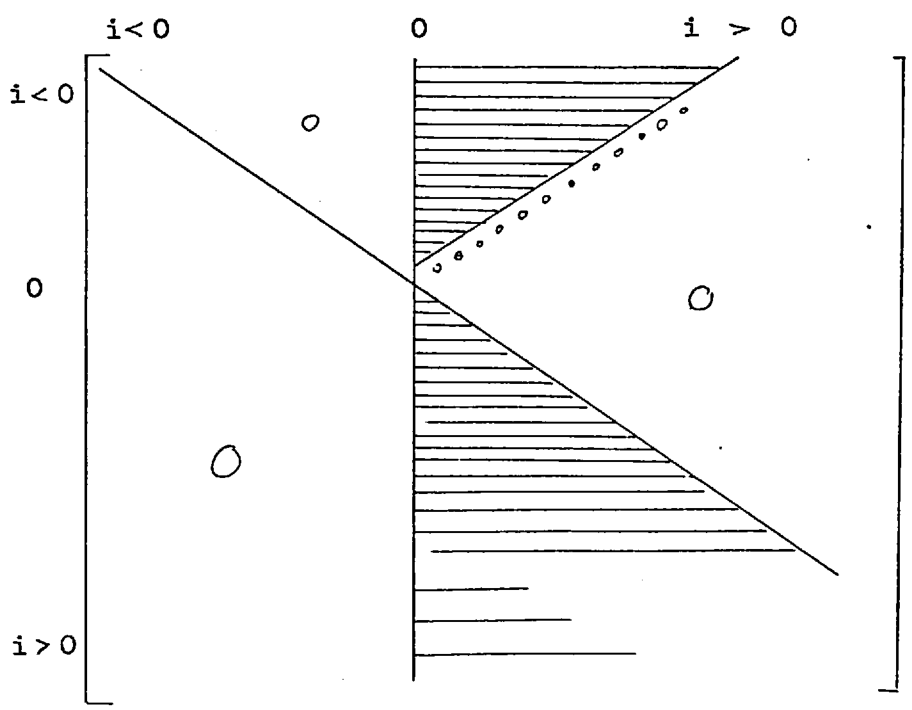
$R_{(-i)j}(\lambda) = 0$ if $i > 0$ and $j \geq i$
and $i > 0$, $j < 0$ but $j \neq -i$

$R_{ij}(\lambda) = 0$ if $i > 0$ and $j > i$ or $j < 0$

$R_{ij}(\lambda) = \frac{1}{\lambda + qj} \prod_{n=j+i}^{i} \frac{qk}{\lambda + qk}$ if $i > 0$ and $o < j < i$

$R_{io}(\lambda) = \frac{1}{\lambda} \prod_{k=1}^{i+1} \frac{qk}{\lambda + qk}$ if $i > 0$

It is helpful to draw the (A x A) - matrix to compare it later with the taboo resolvents.



Let i_n be any sequence in A such that the absolute values $|i_n|$

form an increasing sequence.

If $j > 0$ (resp $= 0$) then for all i_n such that $|i_n| > j$ we find irrespectively of its sign

$$R_{i_n j}(1) = \frac{1}{1+qj} \prod_{k=j+1}^{|i_n|} \frac{qk}{1+qk} \quad \left(\text{resp } \prod_{k=1}^{|i_n|} \frac{qk}{1+qk}\right)$$

Now if n is increasing to ∞ the R.H.S. has a decreasing limit and we get

$$(39) \quad \lim_{n=\infty} R_{i_n j}(1) = \frac{1}{1+qj} \prod_{k=j+1}^{\infty} \frac{qk}{1+qk} \left(\text{resp } \prod_{k=1}^{\infty} \frac{qk}{1+qk}\right)$$

If $j < 0$, then we have for all i_n

$$R_{i_n j}(1) \leqslant \frac{1}{1+qn}$$

Again if n is increasing to ∞ the R.H.S. decreases to 0 and we get

$$(40) \quad \lim_{n=\infty} R_{i_n j}(1) = 0 \qquad \text{for all } j < 0$$

From the density of A in A_e and the sort of arguments used after example 1 we deduce that A_e is composed of only one point, y say, defined by the R. H. S's of (39) and (40).

The interesting point about the topology T is that (-i) tends to y as i tends to $+\infty$.

Description of the sample paths.

(a) Sample paths starting in the state i of A

The usual interpretations for Q (Cth.II5.5. and p.259) and the X_+ version imply that every sample path is composed of a finite number $(|i| + 1)$ of left closed and right open intervals, the last one (spent in $\{0\}$) being ∞.

(b) Sample path starting in y

The sample path are step functions. The number of steps is countably infinite and they accumulate at $t = 0$; all the steps are spent in some $i \geqslant 0$ of A and the left closed right open interval spent in i is followed by one spent in (i-1) and so on until reaching $\{0\}$ where the sample path remains for ever. With probability one there is a finite number of steps after any strictly positive time t.

By (a) we have

$P[X_s$ has a finite number of steps after $t \mid X_o = y] =$

$\sum_{n=o}^{\infty} P[X_s$ has n steps before reaching 0 after $t \mid X_o = y] =$

$$\sum_{n=o}^{\infty} P[X_t = n \mid X_o = y] = \sum_{n=o}^{\infty} p_{yn}(t) = 1$$

For almost all trajectories starting in y, y is a right limit of points $i > 0$ of A increasing to $+\infty$.

On $A \cup \{y\}$ define a topology T_1 as follows:

every i is isolated;

(T_1) every neighbourhood Vy of y contains all positive i's bigger than some N_{Vy}; there exists one Vy not containing any negative i.

This T_1 is metrisable by (e.g)

$$(41)\quad \begin{cases} d(i,j) = 1 & \text{if i or j is negative and } i \neq j \\ d(i,j) = 1/2^i + 1/2^j & \text{if i and j are positive and } i \neq j \\ d(y,i) = 1 & \text{if } i < 0 \\ d(y,i) = 1/2^i & \text{if } i \geqslant 0 \\ d(i,j) = d(y, y) = 0 & \text{for all i in A} \end{cases}$$

By the very description of sample paths ((a) and (b)) it is clear that all the $X_t(w\ )$ which are right continuous in T are also right continuous from $[0, \infty]$ into $(A \cup \{y\}\ , T_1)$.

Moreover in any topology T^o strictly than T_1, we can find a neighbourhood of y, V^o_y say, such that (T_1) does not hold. Hence there exists a sequence $\{i_\nu\}$ of positive i's increasing to $+\infty$ but not in $V^o y$.

By the description given in (b) we have

$P[X_s$ visits $\{i_\nu\}$ for <u>one</u> s in $[o; \frac{1}{n}] \mid X_o = y] = 1$

for all n and hence

$P[X_s \text{ is in } V_y^o \text{ for } \underline{\text{all}} \text{ s in } [0; \frac{1}{n}] \mid X_o = y] = 0$

for all n. But the right continuity relative to T^o would need in particular

$P[X_s \text{ is in } V_y^o \text{ for all s in } [0; \frac{1}{n}] \mid X_o = y] \quad \uparrow \quad 1$

as n tends to $+\infty$.

Since the last two relations are contradictory, T_1 is the finest topology on $A \cup \{y\}$ for which the right continuous trajectories for T are also right continuous for T_1, i.e. $T_1 = T'$

Next we use (a) and (b) to compute the resolvents related to the taboo set $\{x\}$, a point of $A \cup \{y\}$.

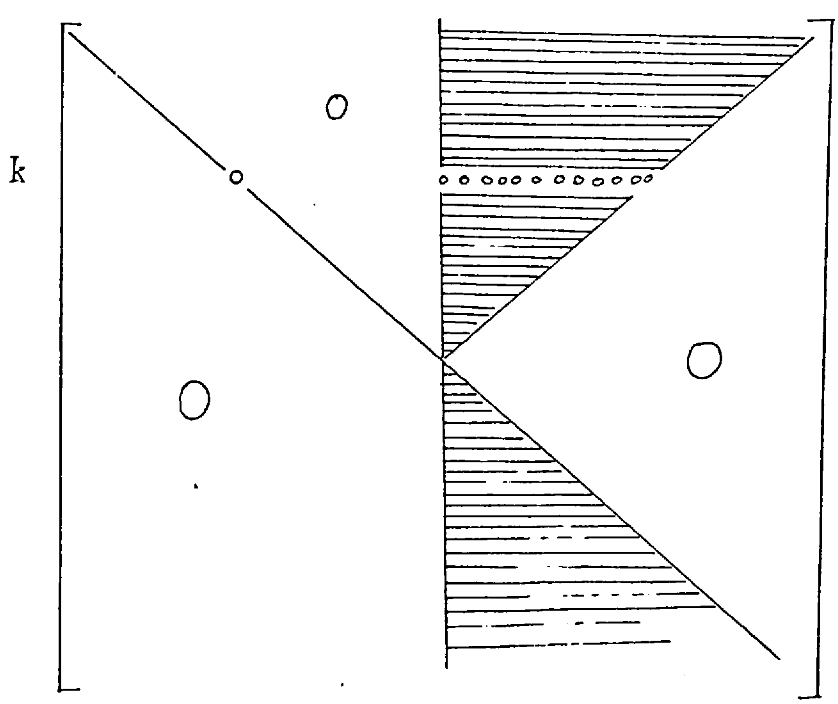
<u>Case (i)</u> $x = k < 0$; the only trajectories which ever visit k are those starting there,

${}_kR_{kj}(\lambda) = 0$ for all j in A

${}_kR_{ij}(\lambda) = R_{ij}(\lambda)$ for all $i \neq k$ and j in A

${}_kR_{yj}(\lambda) = R_{yj}(\lambda)$ for all j

${}_kR(\lambda)$ is the following matrix



<u>Case (ii)</u> $x = k \geq 0$: if $0 < i < k$ or $-k \leq i < 0$ the sample paths starting in i never meet k,

hence

$${}_kR_{ij}(\lambda) = R_{ij}(\lambda) \text{ for all } j \text{ and } 0 < i < k \text{ or } -k \leq i < 0$$

if i > k or i < -k, the sample paths descend the finite escalator between i and k and stop at k, never reaching any $j \leq k$. Hence

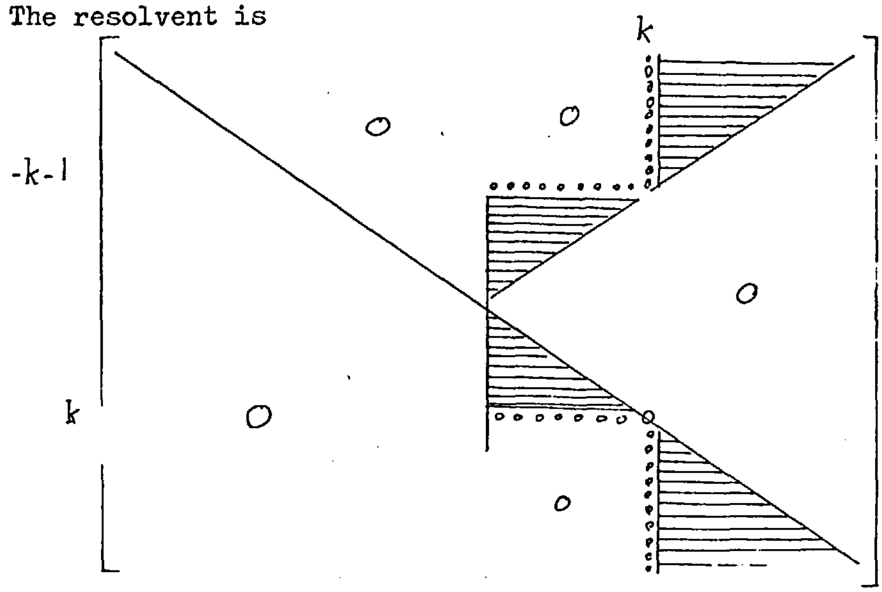
${}_kR_{ij}(\lambda) = R_{ij}(\lambda)$ if $j > k$ and $k < i$ or $i < -k$

${}_kR_{ij}(\lambda) = 0$ if $j \leq k$ and $k < i$ or $i < -k$

if i = k then ${}_kR_{kj}(\lambda) = 0$ for all j in A

$${}_kR_{yj}(\lambda) = \begin{cases} R_{yj}(\lambda), & j > k \\ 0, & j \leq k \end{cases}$$

The resolvent is



Case (iii) x = y: then almost no sample paths starting in i ever meets y, i.e. ${}_yR(\lambda) = R(\lambda)$.

Now look at what happens to these various resolvents (for $\lambda = 1$), when (- i) tends to $-\infty$ (i.e. to y in $(A \cup \{y\}; T)$).

In case (i) we get

(42) $\lim_{i=\infty} {}_kR_{(-i)j}(1) = {}_kR_{yj}(1)$ for all j in A

In case (ii) we get

(43) $\lim_{i=\infty} {}_kR_{(-i)j}(1) = {}_kR_{yj}(1)$ for all $j > k$

$= 0$ for all $j \leq k$

In case (iii) nothing is changed.

Similarly if i tends to ∞ (i.e. to y in $(A \cup \{y\}, T)$).

In case (i) we get

(44) $\lim_{i=\infty} {}_kR_{ij}(1) = {}_kR_{yj}(1)$ for all j in A

In case (ii) we get

(45) $\lim_{i=\infty} {}_kR_{ij}(1) = {}_kR_{yj}(1)$ for all $j > k$

$= 0$ for all $j \leq k$

In case (iii) nothing is changed.

Let T_2 be the topology defined by the metric (1) where the sequence of B_n is the sequence of singletons $\{x\}$ in $A \cup \{y\}$. This is equivalent to say that T_2 is defined by the simple convergence of $R_{\cdot i}(1)$, ${}_kR_{\cdot i}(1)$ and ${}_yR_{\cdot i}(1)$ for all i and k.

The equalities (42) = (44) and (43) = (45)

are then enough to show that (-i) tends to y in T_2 as well as in T ($\neq$ T'). This completes the proof that the use of $A + A_e$ in the definition of the metric (1) is not a good way to obtain T'.

As this method to define a metric is not sharp enough to refine T and obtain T', it might be interesting to disrupt T in a more brutal way; for example to define boldly for all x and z in $A + A_e$.

(2) $$d(x; z) = \sup_B \sum_{k \in A} \beta k \left| {}_BR_{xk}(1) - {}_BR_{zk}(1) \right|$$

where the sup is taken over all the subsets of $A + A_e$.

Let i tend to + ∞ (i.e. i tends to y in T) and choose $B_i = (i + 1)$.

By the case (ii) p. 91 we get for all $i > 0$.

$$d(i;y) = \sup_B \sum_{k \in A} \beta k \left| {}_BR_{ik}(1) - {}_BR_{yk}(1) \right|$$

$$\geq \sum_{k \in A} \beta k \left| {}_{(i+1)}R_{ik}(1) - {}_{(i+1)}R_{yk}(1) \right|$$

$$= \sum_{k=0}^{i} \beta k \; R_{ik}(1) + \sum_{k=i+2}^{\infty} \beta k \; R_{yk}(1)$$

$$\geq \beta_o R_{io}(1) \geq \beta_o R_{yo}(1) > 0$$

Thus y is isolated in the topology ddfined by the metric (2) and hence the sample paths starting in y are no longer right continuous, so that this method is not delicate enough to obtain T'.

This suggests that we should handle these taboo resolvents more carefully. Taking into account the description of the sample paths (given in (a) and (b) p 89) a good candidate as metric is:

(46) $$d(x;z) = \sum_{k \in A} \beta_k \left| R_{xk}(1) - R_{zk}(1) \right| + \sum_{k \in A} \theta_k \left| {}_B R_{xk}(1) - {}_B R_{zk}(1) \right|$$

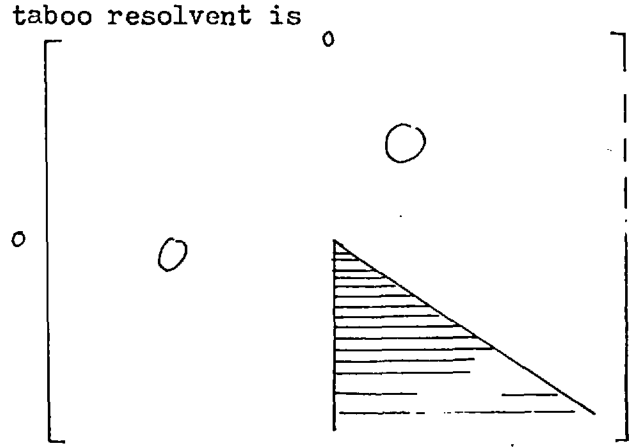
for all x and z in $A \cup \{y\}$, where B is the set of all strictly negative integers in A.

The first sum ensures that all i's are isolated (property of T). By (a) and (b) p. 89 again it is obvious that the only sample paths affected by the taboo B are those starting there, and the corresponding ${}_B R_{ij}(\lambda)$ are

(47) ${}_B R_{ij}(\lambda) = 0$ for all $\lambda > 0$, $i < 0$ and j in A

(48) ${}_B R_{ij}(\lambda) = R_{ij}(\lambda)$ for all $\lambda > 0$, $i \geq 0$ and j in A

The taboo resolvent is



By (b) we find

(49) ${}_B R_{yj}(1) = R_{yj}(1)$ for all j in A

We have by (47) and (48)

(50) $\lim_{i=\infty} {}_B R_{(-i)j}(1) = 0$ for all j in A

(51) and $\lim_{i=\infty} {}_B R_{ij}(1) = R_{yj}(1)$ for all j in A

By (48) and (49) the metric (46) is such that

$$(52)\quad \lim_{i=\infty} d(i,y) = \lim_{i=\infty} 2\sum_{k \in A} \beta_k \left| R_{ik}(1) - R_{yk}(1) \right| = 0$$

By (49) and (50) the metric (46) is such that

$$(53)\quad \lim_{i=\infty} d(-i;y) = \lim_{i=\infty} \sum_{k\in A} \beta_k \left| R_{(-i)k}(1) - R_{yk}(1) \right| + \sum_{k\in A} \beta_k R_{yk}(1)$$

$$= \sum_{k \in A} \beta_k R_{yk}(1) > 0$$

Now (52) and (53) are enough to show that the metric (46) is equivalent to (41) and defines T' (= T_1 as seen in p. 91)

If we look back at the example 1 and in particular at the description of the sample paths given in p 70., we find that the set of all stictly positive integers is a good candidate to define a metric (46) where A must be read as A of example 1. Recall (36), (37), and (38)

$${}_BR_{oo}(\lambda) = \frac{1}{\lambda}$$

$${}_BR_{ok}(\lambda) = 0 \qquad k = 0$$

$${}_BR_{ik}(\lambda) = 0 \qquad i > 0 \text{ and all } k$$

For all i and j in A define a metric d(i;j) by setting

$$d(i;j) = \sum_{k \in A} \beta_k \left| R_{ik}(1) - R_{jk}(1) \right| + \sum_{k \in A} \beta_k \left| {}_BR_{ik}(1) - {}_BR_{jk}(1) \right|$$

In the topology generated by this metric all strictly positive i 's are isolated by the first sum. Moreover for all $i > 0$ we have:

$$d(i,o) = \beta_o\left(\frac{i}{i+1} - 1\right) + \beta_i \frac{1}{i} + \beta_o \, 1 \geq \beta_o$$

so that $\{0\}$ is also isolated in this topology which is then the discrete one we were looking for as T'.

Examples 1 and 2 suggest a probabilistic definition of the kind of sets needed so that (46) is a metric for T'.

Definition :

A subset V of $A + A_e$ is called a right neighbourhood of y if and only if

(i) $P[\tau_{V^c} > 0 \mid X_0 = y] = 1$

(ii) If $V - \{y\}$ is not $\emptyset$, then every infinite sequence S of different points contained in $V - \{y\}$ has the property that

$$P[\tau_S = 0 \mid X_0 = y] = 1$$

Note that by this definition the set $\{i\}$ is a right neighbourhood of itself for a stable point but not for an instantaneous one, as in the latter case we have by C th. II. 5.4.

(54) $\quad P[\tau_{(A - \{i\})} = 0 \mid X_0 = i] = 1$

i.e. (ii) does not hold.

Another interesting point to be stressed is that contrary to D. Williams's conjecture in [13], if y is in A_e and V is a right neighbourhood of y but not a T-neighbourhood as well, then if W is a T-neighbourhood of y, the set $W - V$ is not necessarily visited by the sample paths just before hitting y.

D. Williams' conjecture is:

Let y be in $A + A_e$ and y_n be a sequence of points in $A + A_e$ such that y_n does not equal y for all n.

In this case a necessary and sufficient condition that y_n tends to y in T as n tends to ∞ is

(55) $\quad \lim_{n=\infty} \max[P[\tau_y < t \mid X_0 = y_n],\ P[\tau_{y_n} < t \mid X_0 = y]] = 1$

for all strictly positive t.

The example 2 where the escalator process starting in y (or $+\infty$) is somewhat parasited by the processes starting in the negative integers is a counter example.

As seen in (a) p. 89 any sample path starting in (-n) is a finite step function which never reaches y; hence

$$P[\tau_y = \infty \mid X_0 = -n] = 1 \qquad \text{for all } -n < 0$$

From (b) p.89 we know that a sample path starting in y, is a step function on the positive integers which is finally absorbed by $\{0\}$, and hence never visits a strictly negative n. So we have

$$P[\, \zeta_{(-n)} = \infty \mid X_o = y] = 1 \qquad \text{for all } -n < 0$$

These two equalities yield for all $-n < 0$

$$P[\, \zeta_y < t \mid X_o = -n] = P[\, \zeta_{(-n)} < t \mid X_o = y] = 0$$

Thus their maximum is zero for all $-n < 0$, and if we take the limit as $-n$ tends to $-\infty$ we get

$$\lim_{n=\infty} \max [P[\, \zeta_y < t \mid X_o = -n],\ P[\, \zeta_{(-n)} < t \mid X_o = y]] = 0$$

As $-n$ converges to y in T, this shows that Williams' conjecture does not hold for a semi-polar point y.

The next problem is to try to find an analytical characterisation of a V satisfying (i) and (ii), i.e. is it possible to define such a V by means of $R_{ij}(\lambda)$ and $R_{yj}(\lambda)$ only?

It is easy to get a necessary condition for (i).

We have

$$\sum_{k \in V^c} p_{yk}(t) = P[\, X_t \in V^c \cap A \mid X_o = y]$$

$$\leqslant P[\, \zeta_{V^c} \leqslant t \mid X_o = y]$$

$$= 1 - P[\zeta_{V^c} > t \mid X_o = y]$$

But the assumption (i) implies that the last term tends to zero as t tends to zero and we get

$$\lim_{t=o} \sum_{k \in V^c} p_{yk}(t) = 0$$

This is an insufficient condition as the case of an instantaneous point i readily shows.

For such an i we have

$$\sum_{k \in A-\{i\}} p_{ik}(t) \longrightarrow 0 \quad \text{as } t \longrightarrow 0$$

but as seen before the relation (54) is contradictory to (i).

Let $D_y = \{ i \text{ in } A \mid R_{yi}(\lambda) = 0 \}$

We have

$$P[\,\zeta_{D_y} = \infty \mid X_o = y] = 1$$

Hence if D_y is an infinite set, it cannot be contained in a right neighbourhood of y (otherwise it would contradict (ii)). But it is only incidental that in both our examples the complement of D_y in $A + A_e$ is actually a right neighbourhood of the troublesome point ($\{o\}$ in ex. 1 and $\{y\}$ in ex. 2).

It should be noted now that the existance of a right neighbourhood has not been established except in the constructed examples!

CHAPTER VI

On some analytical relations between the original semi-group and its associated taboo semi-groups.

The fact that D. Williams' conjecture is false may lead us to wonder whether analysis is a good enough tool to obtain probabilistic properties. But this does not change the likelihood of the following analytical result:

Would be theorem 1.

Let B be a subset of $A + A_e$

Let y be a point of $A + A_e$ not isolated in the topology T.

Then there exists a sequence, y_n, n in N of points of $A + A_e$ such that

(1) $y_n \neq y$ for all n

(2) $y_n \longrightarrow y$ in T as $n \longrightarrow \infty$

(3) ${}_Bp_{y_n i}(t) \longrightarrow {}_Bp_{yi}(t)$ for all $t > 0$ and i in A

This chapter is mainly concerned with the proof of a weaker result (and also with an important restriction on the choice of y). On the way some interesting related points are also investigated.

Note that the countability assumption made earlier on A_e may be relaxed, if we choose a set B such that the corresponding semigroup and entrances are well defined as in Chapter V § 3. e.g. if B is a Borel set for T.

The cone of entrances relative to ${}_BP(t)$ will be denoted by ${}_BF$.

Naturally the first thing to check is the extremality in ${}_BF$ of the probabilistically defined ${}_Bp_{yi}(t)$.

Theorem 2.

The entrance ${}_Bp_{yi}(t)$ is extremal in ${}_BF$ for all y in $A + A_e$.

To prove this result we use the th 3.2.2 of J. Neveu [7] of which we give a version adapted to our special pair of semi-groups.

For every f in F, we have

$$f(u)\,{}_BP(t+s) = f(u)\,{}_BP(t)\,{}_BP(s) \leqslant$$

$$f(u)\,P(t)\,{}_BP(s) = f(u+s)\,{}_BP(s)$$

so that $f(s)\,{}_BP(t-s)$ is monotonic decreasing as s decreases to 0, for all $t > 0$.

For all i in A and $t > 0$, define

(4) $\varphi[\,f\,]_i(t) = \lim\limits_{s=0} (f(s)\,{}_BP(t-s))_i$

$\varphi[\,f\,]$ is an element of ${}_BF$.

Proof:

$$\varphi[f]_i(t+u) = \lim_{s\downarrow 0} \sum_{k\in A} f_k(s)\,{}_Bp_{ki}(t+u-s) =$$

$$\lim_{s\downarrow 0} \sum_{k\in A} f_k(s) \sum_{j\in A} {}_Bp_{kj}(t-s)\,{}_Bp_{ji}(u) =$$

(5) $\lim\limits_{s\downarrow 0} \sum\limits_{j\in A} \left[\sum\limits_{k\in A} f_k(s)\,{}_Bp_{kj}(t-s) \right] {}_Bpji(u)$

In (5) the sums in brackets are monotonic decreasing as s tends to 0.

Therefore the relation

(5) $= \varphi[f]_i(t+u) \leqslant f_i(t+u) < \infty$

allows us to interchange the summation over j in A and the limit as s tends to 0 in (5) and we get:

$$\varphi[f]_i(t+u) = \sum_{j\in A} \lim_{s\downarrow 0} \left[\sum_{k\in A} f_k(s)\,{}_Bp_{kj}(t-s) \right] {}_Bp_{ji}(u)$$

$$= \sum_{j\in A} \varphi[f]_j(t)\,{}_Bp_{ji}(u)$$

For every ${}_Bf$ in ${}_BF$ we have

$${}_Bf(u)P(t+s) = {}_Bf(u)P(t)P(s) \geqslant$$

$${}_Bf(u)\,{}_BP(t)P(s) = {}_Bf(u+t)P(s).$$

so that ${}_Bf(s)P(t-s)$ is monotonic increasing as s decreases to 0, for all $t > 0$. For all i in A and all $t > 0$ define

$$(6) \quad \Psi[{}_Bf]_i(t) = \lim_{s \downarrow 0} ({}_Bf(s)P(t-s))_i$$

$\Psi[{}_Bf]$ is an element of F.

<u>Proof:</u>

First note that the limit in (6) is always finite.

We have for all s 0

$$\sum_{i\in A}\sum_{k\in A} {}_Bf_k(s)p_{ki}(t-s) = \sum_{k\in A} {}_Bf_k(s) \sum_{i\in A} p_{ki}(t-s) =$$

$$\sum_{k\in A} {}_Bf_k(s) \leqslant c < \infty \qquad \text{by I.(16)}$$

hence

$$\lim_{s\downarrow 0} \sum_{k\in A} {}_Bf_k(s)p_{ki}(t-s) \leqslant \limsup_{s\downarrow 0} \sum_{k\in A} {}_Bf_k(s) \leqslant c < \infty$$

Now we prove the entrance equation

$$\Psi[{}_Bf]_i(t+u) = \lim_{s\downarrow 0} \sum_{k\in A} {}_Bf_k(s)p_{ki}(t+u-s)$$

$$= \lim_{s\downarrow 0} \sum_{k\in A}\sum_{j\in A} {}_Bf_k(s)p_{kj}(t-s)p_{ji}(u)$$

As $\Psi[{}_Bf]_i(t)$ is the limit of the increasing $({}_Bf(s)P(t-s))_i$, we may interchange limit and summation in (7) to get

$$\Psi[{}_Bf]_i(t+u) = \sum_{j\in A} \lim_{s\downarrow 0}\Big(\sum_{k\in A} {}_Bf_k(s)p_{kj}(t-s)\Big)p_{ji}(u)$$

$$= \sum_{j\in A} \Psi[{}_Bf]_j(t)p_{ji}(u)$$

Neveu's theorem 3.2.2 in [7] reads as follows:

There exists a positive band $\bar{F}$ contained in F such that φ and Ψ are isomorphisms between ${}_BF$ and $\bar{F}$. Moreover φ and Ψ satisfy:

for every k in A

(8) $\varphi[p_{k\cdot}(\;)]_i(t) = {}_Bp_{ki}(t)$ or 0 for all i in A

and for every k such that ${}_Bp_{k\cdot}(\cdot)$ is not identically zero

(9) $\Psi[{}_Bp_{k\cdot}(\cdot)]_i(t) = p_{ki}(t)$ for all i in A

Proof of th 2

The extremality of the trivial entrance is clear, therefore we now choose a y in $A + A_e$ such that ${}_Bp_{yi}(t) \neq 0$ for at least one i. We have for all $s > 0$ and j in A

$$\varphi[p_{y\cdot}(\cdot)]_j(s) = \lim_{s'\downarrow 0} \sum_{k\in A} p_{yk}(s')\,{}_Bp_{kj}(s-s') \leqslant p_{yj}(s)$$

so that for all $t>0$ and all i in A

(10)
$$\begin{cases} \Psi[\varphi[p_{y\cdot}(\cdot)]]_i(t) = \lim_{s\downarrow 0} \sum_{j\in A} \varphi[p_{y\cdot}(\cdot)]_j(s)p_{ji}(t-s) \leqslant \\ \lim_{s\downarrow 0} \sum_{j\in A} p_{yj}(s)p_{ji}(t-s) = p_{yi}(t) \end{cases}$$

Let ${}_Bh$ be an element of ${}_BF$ such that

(11) ${}_Bh \leqslant \varphi[p_{y\cdot}(\cdot)]$

Since Ψ is an isomorphism (10) and (11) give

$$\Psi[{}_Bh]_i(t) \leqslant \Psi[\varphi[p_{y\cdot}(\cdot)]]_i(t) \leqslant p_{yi}(t)$$

As y is extremal in F we can find an α in $[0,1]$ such that

$$\Psi[{}_Bh]_i(t) = \alpha\, p_{yi}(t)$$

which in turn yields

$${}_Bh_i(t) = \varphi[\Psi[{}_Bh]]_i(t) = \alpha\, \varphi[p_{y\cdot}(\cdot)]_i(t)$$

As this can be done for every ${}_Bh$ in ${}_BF$ this implies the extremality of $\varphi[p_{y\cdot}(\cdot)]$ in ${}_BF$.

By V (6) we get the inequality

$$\sum_{k\in A} p_{yk}(s)\,{}_Bp_{ki}(t-s) \geqslant {}_Bp_{yi}(t)$$

and letting s decrease to 0, this yields

(12) $\varphi[p_{y\cdot}(\cdot)]_i(t) \geqslant {}_Bp_{yi}(t)$

Hence the extremality in ${}_BF$ of ${}_Bp_{yi}(t)$ itself is proved

Note that the inequality (12) is in fact an equality (as is already known for a state in A by (8). We have

$$(13)\quad \varphi\,[p_{y\,.}(\,.\,)]_i(t) = \lim_{n=\infty} \sum_{k \in A} p_{yk}(\tfrac{1}{n})\ {}_Bp_{ki}(\,t - \tfrac{1}{n}\,) =$$

$$\lim_{n=\infty} P[X_t = i \quad X_s \notin B \text{ for all } s \text{ in } [\tfrac{1}{n}\,,\,t] \mid X_o = y] .$$

But as n increases to ∞ , the events considered in the probability are decreasing and we get

$$\lim_{n=\infty} P[\bigcap_{n=1}^{\infty} \{X_t=i \quad X_s \notin B \text{ for all } s \text{ in } [\tfrac{1}{n}\,,\,t]\} \mid X_o = y] =$$

$$P[X_t=i \quad X_s \notin B \text{ for all } s \text{ in } (0,t] \mid X_o = y]$$

(14) $= {}_Bp_{yi}(t)$ by definition (see V(5))

The equality (13) = (14) gives for all $t > 0$ and i in A

$$(15)\quad \varphi\,[p_{y\,.}(\,.\,)]_i(t) = {}_Bp_{yi}(t)$$

Similarly we now extend the relation (9) to all y in A_e such that ${}_Bp_{yi}(t)$ is not identically 0.

We have

$$(16)\quad \psi\,[{}_Bp_{y\,.}(\,.\,)]_i(t) = \lim_{n=\infty} \sum_{k \in A} {}_Bp_{yk}(\tfrac{1}{n})\ p_{ki}(t-\tfrac{1}{n}) =$$

$$\lim_{n=\infty} P[\,\zeta_B > \tfrac{1}{n} \quad X_t=i \mid X_o=y]$$

As n is increasing to ∞ the events considered in the probability are increasing and we get

$$\lim_{n=\infty} P[\bigcup_{n=1}^{\infty} [\,\zeta_B > \tfrac{1}{n}\,] \quad X_t=i \mid X_o=y] =$$

$$(17)\quad P[\,\zeta_B > 0 \quad X_t=i \mid X_o=y]$$

By our choice of y there exists a k in A such that

$$0 < P[\,\zeta_B > t \quad X_t=k \mid X_o=y] \leqslant P[\,\zeta_B > 0 \mid X_o=y]$$

By the o - 1 law this implies

$P[\tau_B > 0 \mid X_o = y] = 1$

If we use this last relation in the equality (16) = (17) we get for all $t > 0$ and all i in A

$$(18) \qquad \psi[{}_Bp_{y\,.}(\,.\,)]_i(t) = p_{yi}(t)$$

As a first attempt to show would be theorem I we imitate the stochastic case (see th II. 4)

Recall that ${}_BA = \{ i \in A \mid {}_Bp_{ii}(t) \neq 0 \}$

Let ${}_BM_1$ be the convex set of all the positive measures such that

$$(19) \sum_{k \in {}_BA} {}_BM(k) \leqslant 1$$

and for all $s > 0$ and all i in ${}_BA$

$$(20) \quad e^{-s} \sum_{k \in {}_BA} {}_BM(k)\ {}_Bp_{ki}(s) \leqslant {}_BM(i)$$

By Neveu's result (our th. II. 2 which holds also for substochastic semigroups) there exists an isomorphism from ${}_BF$ onto ${}_BM$ the convex cone of positive finite measures on ${}_BA$ satisfying (20). Therefore to any extremal ray ${}_BF$ corresponds an extremal point of ${}_BM_1$. Denote by ${}_B(A + A_e)$ those extreme points which are not 0. For every y such that ${}_Bp_{yi}(t)$ is not identically 0 for all i the corresponding entrance is extremal, as an extrêmal point of ${}_BM_1$ must be of total mass equal to 1, the measures

$$(21) \frac{1}{{}_Br(y)}\ {}_BR_{yi}(1), \qquad i \text{ in } {}_BA .$$

$$(22) \text{ where } \quad {}_Br(y) = \sum_{k \in {}_BA} {}_BR_{yk}(1) > 0$$

are elements of ${}_B(A + A_e)$

If ${}_B\hat{f}(1)$ is an element of ${}_BM_1$, II (17) and II (18) hold with respect to ${}_BR(\lambda)$ and we get

(23) $\lim\limits_{\lambda=\infty} \uparrow \left[{}_B\hat{f}(1)\ {}_B D(\lambda)\right]_B R(1) = {}_B\hat{f}(1)$

where ${}_B D(\lambda)$ is the obvious equivalent of II (20)

Similarly to A* we define ${}_B A^*$ as the set of measures on ${}_B A$ which are limits of the measures of total mass equal to 1 generated by ${}_B A$, i.e. $X = \left\{ {}_B m(i),\ i \text{ in } {}_B A \right\}$ is in ${}_B A^*$ if and only if there exists a sequence i_n of points in ${}_B A$ such that

(24) ${}_B m(i) = \lim\limits_{n=\infty} \dfrac{1}{{}_B r(i_n)}\ {}_B R_{i_n i}(1)$ for all i in ${}_B A$

Once again this set ${}_B A^*$ is a compact metrisable space if the topology is the simple convergence one.

To generate a measure on ${}_B A^*$ by means of ${}_B\hat{f}(1)\,{}_B D(\lambda)$ as in th II 4 we let

${}_B g(\lambda\ ;\ k) = {}_B r(k)\ \left[{}_B\hat{f}(1)\,{}_B D(\lambda)\right]_k$ for all k in ${}_B A$

3/ so that (20) becomes

(25) $${}_B\hat{f}(1) = \lim_{\lambda=\infty} \int_{{}_B A^*} {}_B g(\lambda\ ,\ dx)\ \frac{1}{{}_B r(x)}\ {}_B R_{xi}(1)$$

Now we have

$${}_B g(\lambda\ ;\ {}_B A^*) = \sum_{k\in{}_B A} \lambda\left({}_B \hat{f}_k(1) - \lambda \sum_{j\in{}_B A} {}_B \hat{f}_j(1)\,{}_B R_{jk}(\lambda+1)\right) {}_B r(k)$$

$$= \lambda \sum_{k\in{}_B A} {}_B \hat{f}_k(\lambda+1) \sum_{i\in{}_B A} {}_B R_{ki}(1) \qquad \text{by (22)}$$

$$= \sum_{i\in{}_B A} {}_B \hat{f}_i(1) - \sum_{i\in{}_B A} {}_B \hat{f}_i(\lambda+1) \leqslant 1$$

Hence if ${}_B\hat{f}(1)$ is the extreme point of ${}_B M_1$, associated with y (by (21) and (22)) we may use the arguments of th. II 4 to show the existence of an x_o in ${}_B A^*$ such that ${}_B g(\lambda\ ,\ .\ ) \longrightarrow \varepsilon_{xo}(\ .\ )$ as $\lambda \to \infty$. By the construction of ${}_B A^*$ this is enough to get a sequence i_n in ${}_B A$ satisfying (24) and it yields for all i in ${}_B A$

(26) $\dfrac{1}{{}_B r(y)}\ {}_B R_{yi}(1) = \lim\limits_{n=\infty} \dfrac{1}{{}_B r(i_n)}\ {}_B R_{i_n i}(1)$

If we hope to deduce (1), (2) and (3) from (26) we must check that the sequence in (26) converges to y in the topology T. By lemma p.27 this amounts to showing

$$\lim_{n=\infty} p_{i_n i}(t) = p_{yi}(t) \quad \text{for all } t > 0 \text{ and all } i \text{ in } A$$

But we get only the following inequality:
every sequence i_n in ${}_BA$ for which (26) holds is such that for all i in A and t > 0

$$(27) \quad \frac{1}{{}_Br(y)} p_{yi}(t) \leq \liminf_{n=\infty} \frac{1}{{}_Br(i_n)} p_{i_n i}(t)$$

<u>Proof: of (27)</u>

By V(6) we have for all i_n, all i and all $s > 0$

$$p_{i_n i}(t) \geq \sum_{k \in {}_BA} {}_Bp_{i_n k}(s)\, p_{ki}(t-s)$$

If we divide both sides by ${}_Br(i_n)$ and take liminf as n tends to ∞, we may use Fatou's lemma to obtain

$$\liminf_{n=\infty} \frac{1}{{}_Br(i_n)} p_{i_n i}(t) \geq \sum_{k \in {}_BA} \liminf_{n=\infty} \frac{1}{{}_Br(i_n)} {}_Bp_{i_n k}(s) p_{ki}(t-s)$$

Let i_n be a sequence satisfying (26). The corresponding sums over all i in ${}_BA$ are normalised by ${}_Br(i_n)$ (see (21) and (22)). Therefore the lemma p 27 in its original form for substochastic semigroups given by Neveu implies for all k in A and all $s > 0$

$$\lim_{n=\infty} \frac{1}{{}_Br(i_n)} {}_Bp_{i_n k}(s) = \frac{1}{{}_Br(y)} {}_Bp_{yk}(s)$$

From the two last relations we deduce

$$\liminf_{n=} \frac{1}{{}_Br(i_n)} p_{i_n i}(t) \geq \sum_{k \in {}_BA} \frac{1}{{}_Br(y)} {}_Bp_{yk}(s) p_{ki}(t-s)$$

If s decreses to 0 in this inequality we get

$$\liminf_{n=\infty} \frac{1}{{}_Br(i_n)} p_{i_n i}(t) \geq \frac{1}{{}_Br(y)} \Psi[{}_Bp_{y\,\cdot}(\,\cdot\,)]_i(t)$$

which is equivalent to (27) by the equality (18).

Note that this proof does not use the fact that i_n lie in ${}_BA$ only; so if we change i_n in y_n, points of ${}_B(A + A_e)$, such that (26) holds, the same inequality (27) is satisfied.

The inequality (27) cannot be improved upon as the following example shows:

<u>Example 3.</u>

Let A be the set $\{1; 2; 3; \ldots\}$

Define the following conservative Q matrix

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ \\ n \\ \\ \end{array}
\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & & n \end{array} \\
\left[\begin{array}{cccccc}
0 & 0 & 0 & & & \\
0 & 0 & 0 & & & O \\
3 & 3.2 & -3^2 & & & \\
4 & 4.3 & 0 & -4^2 & & \\
\vdots & \vdots & \cdot & & \ddots & \\
n & n(n-1) & 0 & & & -n^2 \\
\vdots & \vdots & & O & & \ddots
\end{array}\right]\end{array}
$$

As 1 and 2 are absorbing states the minimal solution is stochastic and is equal to

$$
R(\lambda) = \left[\begin{array}{cccc}
\frac{1}{\lambda} & 0 & 0 & \\
0 & \frac{1}{\lambda} & 0 & \\
\frac{1}{\lambda}\frac{3}{\lambda+3^2} & \frac{1}{\lambda}\frac{3^2-3}{\lambda+3^2} & \frac{1}{\lambda+3^2} & \\
\vdots & \vdots & 0 & \\
\frac{1}{\lambda}\frac{n}{\lambda+n^2} & \frac{1}{\lambda}\frac{n^2-n}{\lambda+n^2} & 0 & \frac{1}{\lambda+n^2} \\
\vdots & \vdots & & \ddots
\end{array}\right]
$$

From $R(\lambda)$ we get

(28) $\lim_{n=\infty} R_{n1}(1) = \lim_{n=\infty} \frac{n}{1+n^2} = 0 = R_{21}(1)$

(29) $\lim_{n=\infty} R_{n2}(1) = \lim_{n=\infty} \frac{n^2-n}{1+n^2} = 1 = R_{22}(1)$.

(30) $\lim_{n=\infty} R_{ni}(1) \leqslant \lim_{n=\infty} \frac{1}{1+n^2} = 0 = R_{2i}(1)$, $i > 2$

i.e. n tends to $\{2\}$ in the topology T, as $n \to \infty$

Note also that for all $n > 2$

(31) $R_{11}(1) - R_{n1}(1) = 1 - \frac{n}{n^2+1} \geqslant 1 - \frac{3}{6} = \frac{1}{2}$

i.e. $\{1\}$ is isolated in T.

By the usual interpretations for Q (C th.II. 5.5. and p.259) and the X_{t+} version we can now describe the sample paths. Either they start in 1 or 2 which they never leave, or they start in $n > 2$, remain there for a while and then jump to the absorbing states 1 and 2, with respective probabilities $1/n$ and $^{n-1}/n$. Therefore if the point $\{2\}$ is chosen as taboo set the associated resolvent is equal to $R(\lambda)$ except in the second row and column which are identically 0.

As $_2\Gamma(1) = \sum_{k\neq 2} {}_2R_{1k}(1) = 1$

$$_2r(n) = \sum_{k\neq 2} {}_2R_{nk}(1) = \frac{n}{1+n^2} + \frac{1}{1+n^2} = \frac{n+1}{1+n^2}$$

the extreme points of $_2M_1$, which are given by (21) and (22) are

$$1 \sim R_{1i}(1) \quad \text{for all } i \neq 2$$

and

$$2 < n \sim \frac{n^2+1}{n+1} R_{ni}(1) \quad \text{for all } i \neq 2$$

These relations imply

(32) $\lim_{n=\infty} \frac{1}{_2r(n)} {}_2R_{n1}(1) = \lim_{n=\infty} \frac{n^2+1}{n+1} \frac{n}{n^2+1} = 1 = \frac{1}{_2r(1)} {}_2R_{11}(1)$

and if $i > 2$

$$(33)\quad \lim_{n=\infty} \frac{1}{{}_2r(n)}\ {}_2R_{ni}(1) \leq \lim_{n=\infty} \frac{n^2+1}{n+1}\ \frac{1}{n^2+1} = 0 = \frac{1}{{}_2r(1)}\ {}_2R_{1i}(1)$$ 1/

Therefore $\{n\}$ is a sequence in ${}_2A = (1,3,4,\ldots)$ such that (26) holds for $y = \{1\}$ and $B = \{2\}$.

Consider now (27) for $i = 2$,

$$0 = p_{12}(t) = \Psi\,[{}_2p_1(1)]_2\,(t)$$

$$\liminf_{n=\infty} \frac{1}{{}_2r(n)}\ p_{n2}(t) = \liminf_{n=\infty} \frac{n^2+1}{n+1}\ \frac{n-1}{n}\ (1 - e^{-n^2 t}) = \infty$$

Thus in this case (27) is a strict inequality.

From example 3 we can also deduce the interesting fact that if T and ${}_BT$ are the simple convergence topologies on $A + A_e$ and ${}_B(A + A_e)$, (as Neveu defines in [8]) then T and ${}_BT$ are wildly different.

By (28),(29),(30) and (31) we know that

$\{n\} \longrightarrow \{2\}$ in T as $n \longrightarrow \infty$

and

$\{1\}$ is isolated in T

But (32) and (33) mean

$\{n\} \longrightarrow \{1\}$ in ${}_2T$ as $n \longrightarrow \infty$

Next we prove a kind of converse to inequality (27)

If the sequence y_n converges to y in $(A + A_e, T)$, then for all i in ${}_BA$ we have

$$(34)\quad \limsup_{n=\infty}\ {}_Bp_{y_n i}(t) \leq {}_Bp_{yi}(t) \qquad \text{for all } t > 0$$

<u>Proof of (34)</u>

We have for all $s > 0$ and $t > s$, and all y_n

$$(35)\quad {}_Bp_{y_n i}(t) \leq \sum_{k \in A} p_{y_n k}(s)\ {}_Bp_{ki}(t-s)$$

Remember that the relation II (16) holds for all extreme points of M_1, i.e. we have for all $s > 0$

$$\sum_{k \in A} p_{y_n k}(s) = \sum_{k \in A} p_{yk}(s) = 1$$

This fact and the choice of a sequence y_n converging in T are enough by the Scheffé's theorem to check that the sum in (35) has a limit as n tends to ∞ , which then satisfies

$$\limsup_{n=\infty} {}_Bp_{y_n i}(t) \leq \lim_{n=\infty} \sum_{k \in A} p_{y_n k}(s)\, {}_Bp_{ki}(t-s)$$

and we get

$$(36) \quad \limsup_{n=\infty} {}_Bp_{y_n i}(t) \leq \sum_{k \in A} p_{yk}(s)\, {}_Bp_{ki}(t-s)$$

Now if we let s decrease to 0 in (36) we get

$$\limsup_{n=\infty} {}_Bp_{y_n i}(t) \leq \varphi\,[p_y(\cdot)]_i(t)$$

Finally this last inequality is equivalent to (34) by the equality (15)

As the inequality (27), (34) cannot be improved upon.

Example 1 provides a trivial counter example. We have $\{n\} \to 0$ in T (see p.69) but relations V(36) and V(38) give for the taboo set B = $\{1, 2, \ldots \}$

$$\limsup_{n=\infty} {}_Bp_{no}(t) = 0 < 1 = {}_Bp_{oo}(t)$$

Thus in this case (34) is a strict inequality.

Now look at what happens in this example under the hypothesis of would be th. 1

$\{0\}$ is not isolated in T.

Every sequence i_n such that

$i_n \neq 0$ for all n (i.e. (1))

$i_n \to 0$ in T as $n \to \infty$ (i.e. (2))

has the property that

$${}_Bp_{i_n o}(t) = 0 \not\to {}_Bp_{oo}(t) = 1 \quad \text{as } n \to \infty$$

Therefore would be th 1 is false in the case of a y in A and we must assume in its hypothesis that y is in A_e.

The results obtained so far are summarised in the following theorem.

Theorem 3.

If B is a Borel set of)A + A_e, T) the topologies T on A + A_e and ${}_BT$ on ${}_B$(A + A_e) are completely unrelated but the following analytical inequalities always hold.

For any sequence y_n converging to y in (A + A_e, T), we have for all $t > 0$ and i in A.

(34) $\limsup_{n=\infty} {}_Bp_{y_n i}(t) \leq {}_Bp_{yi}(t)$

For any sequence y_n converging to y in (${}_B$(A + A_e); ${}_BT$), we have for all $t > 0$ and all i in A

(27) $\frac{1}{{}_Br(y)} \; p_{yi}(t) \leq \liminf_{n=\infty} \frac{1}{{}_Br(y_n)} \; p_{y_n i}(t)$

Later on we shall need the inequality (34) in its Laplace transforms form. As ${}_Bp_{y_n i}(t) \leq 1$ for all y_n, i and $t \geq 0$, we can use Fatou's lemma in their respective Laplace transforms to get

$$\limsup_{n=\infty} \int_0^\infty e^{-\lambda t} \; {}_Bp_{y_n i}(t)dt \leq \int_0^\infty e^{-\lambda t} \limsup_{n=\infty} {}_Bp_{y_n i}(t)dt$$

Taking (34) into account, this shows that every sequence y_n converging to y in (A + A_e, T) is such that

(37) $\limsup_{n=\infty} {}_BR_{y_n i}(\lambda) \leq {}_BR_{yi}(\lambda)$

for all $\lambda > 0$ and all i in A.

Here is a second attempt to prove Would be th 1.

Choose y in A_e; then either ${}_Bp_{yi}(t) = 0$ for all i in A and all $t > 0$ or there is at least one i in A such that ${}_Bp_{yi}(t) > 0$ for all $t > 0$

In the first case any sequence y_n in A + A_e such that

$y_n \neq y$ for all n (i.e. (1))

and

$y_n \to y$ in T as $n \to \infty$ (i.e. (2))

satisfies for all i in A and all $t > 0$

(38) $$0 \leqslant \liminf_{n=\infty} {}_Bp_{y_n i}(t) \leqslant \limsup_{n=\infty} {}_Bp_{y_n i}(t)$$

But by th. 3 (relation (34)) the last term is bounded above by ${}_Bp_{yi}(t)$ which is equal to 0 by choice of y; hence limsup and liminf are equal in (38) and their common value is ${}_Bp_{yi}(t)$ as expected.

So only the second case remains unsolved and from now on y in A_e is such that ${}_BR_{yi}(\lambda)$ is not identically 0 for one i at least.

Lemma

Let y be in A_e

Let C be a subset of A such that there exist one i (kept fixed) in A and a $\lambda > 0$ for which

(39) $$\sup_{k \in C} {}_BR_{ki}(\lambda) < {}_BR_{yi}(\lambda)$$

Then

$$\lim_{s=0} \sum_{k \in C} p_{yk}(s) = 0$$

Proof of lemma

As the inequality (39) is strict, there exists a $\delta > 0$ such that

(40) $$\sup_{k \in C} {}_BR_{ki}(\lambda) < {}_BR_{yi}(\lambda) - \delta$$

Now let $\alpha = \limsup_{s=0} \sum_{k \in C} p_{yk}(s)$

Choose a sequence s_r of strictly positive numbers decreasing to 0 such that

(41) $$\lim_{r=\infty} \sum_{k \in C} p_{yk}(s_r) = \alpha$$

Obviously α lies in $[0,1]$

First we show that $\alpha < 1$, Assume the converse (i.e. $\alpha = 1$) and choose $\varepsilon > 0$ such that

(42) $$2\varepsilon < \delta$$

By II (16), (41) and our assumption we may find $s(\epsilon) > 0$

such that

(43) $\sum_{k \in A-C} p_{yk}(s_r) < \epsilon$ for all $s_r < s(\epsilon)$

By definition

$${}_BR_{yi}(\lambda) = \int_0^s e^{-\lambda t}\, {}_Bp_{yi}(t)dt + \int_s^\infty e^{-\lambda t}\, {}_Bp_{yi}(t)dt$$

As $e^{-\lambda t}\, {}_Bp_{yi}(t) \leq 1$ for all t, we have for all $s < \min(\epsilon, s(\epsilon))$

(44) $${}_BR_{yi}(\lambda) \leq \epsilon + \int_s^\infty e^{-\lambda t}\, {}_Bp_{yi}(t)dt$$

The integral on $[s, \infty)$ is equal to

$$\int_s^\infty e^{-\lambda t} \sum_{k \in A} {}_Bp_{yk}(s)\, {}_Bp_{ki}(t-s)dt$$

$$\leq \int_s^\infty e^{-\lambda t} \sum_{k \in A} p_{yk}(s)\, {}_Bp_{ki}(t-s)dt$$

By positivity we may interchange the summation and the integration in the last term to get

$$\sum_{k \in A} p_{yk}(s) \int_s^\infty e^{-\lambda t}\, {}_Bp_{ki}(t-s)dt$$

$$= e^{-\lambda s} \sum_{k \in A} p_{yk}(s)\, {}_BR_{ki}(\lambda)$$

$$= e^{-\lambda s} \Big(\sum_{k \in C} p_{yk}(s)\, {}_BR_{ki}(\lambda) + \sum_{k \in A-C} p_{yk}(s)\, {}_BR_{ki}(\lambda) \Big)$$

$$\leq e^{-\lambda s} \sup_{k \in C} {}_BR_{ki}(\lambda) \sum_{k \in C} p_{yk}(s) + e^{-\lambda s} \sum_{k \in A-C} p_{yk}(s)$$

Now let s decrease to 0 along s_r, the second term is bounded above by ϵ (see (43)) and the last expression remains bounded by

$$\sup_{k \in C} {}_BR_{ki}(\lambda) \limsup_{r = \infty} \sum_{k \in C} p_{yk}(s_r) + \epsilon \leq$$

$$\sup_{k \in C} {}_BR_{ki}(\lambda) + \epsilon \qquad \text{by II(16)}$$

The inequality (44) gives then

$$ {}_B R_{yi}(\lambda) \leq \sup_{k \in C} {}_B R_{ki}(\lambda) + 2\varepsilon $$

which by our choice of ε in (42) is incompatible with (40).

Hence $\alpha < 1$.

As α is < 1 we can now proceed to prove $\alpha = 0$.

Fix n in N and let B_n be the open sphere centred at y and of radius 1/n. By II (38), for every $\varepsilon > 0$ we can find $S_n(\varepsilon)$ such that

$$ (45) \quad \sum_{k \in B_n} p_{yk}(s) \geq 1 - \varepsilon \quad \text{for all } s < s_n(\varepsilon) $$

By the analytical arguments we have just used we get for all $0 < s < \min(\varepsilon, s_n(\varepsilon))$

$$ (46) \quad {}_B R_{yi}(\lambda) \leq e^{-\lambda s} \sum_{k \in B_n} p_{yk}(s)\, {}_B R_{ki}(\lambda) + 2\varepsilon $$

As B_n is contained in $(B_n - C) \cup C$ the R. H.S. of (46) is bounded above by

$$ e^{-\lambda s} \sum_{k \in B_n - C} p_{yk}(s)\, {}_B R_{ki}(\lambda) + e^{-\lambda s} \sum_{k \in C} p_{yk}(s)\, {}_B R_{ki}(\lambda) + 2\varepsilon $$

Taking into account the inequality (40) in the second sum the last expression is bounded above by

$$ (47) \quad e^{-\lambda s} \sup_{k \in B_n - C} {}_B R_{ki}(\lambda) \sum_{k \in B_n - C} p_{yk}(s) + e^{-\lambda s} \left({}_B R_{yi}(\lambda) - \delta \right) \sum_{k \in C} p_{yk}(s) + 2\varepsilon $$

So we have L.H.S (46) $\leq$ (47); next if we divide both sides by

$$ (48) \quad e^{-\lambda s} \sum_{k \in B_n - C} p_{yk}(s) $$

we may rearrange the terms to get the inequality

$$ (49) \quad {}_B R_{yi}(\lambda) \frac{\left[1 - e^{-\lambda s} \sum_{k \in C} p_{yk}(s) \right]}{e^{-\lambda s} \sum_{k \in B_n - C} p_{yk}(s)} + \frac{\delta\, e^{-\lambda s} \sum_{k \in C} p_{yk}(s)}{e^{-\lambda s} \sum_{k \in B_n - C} p_{yk}(s)} \leq \sup_{k \in B_n - C} {}_B R_{ki}(\lambda) + \frac{2\varepsilon}{e^{-\lambda s} \sum_{k \in B_n - C} p_{yk}(s)} $$

For all $s > 0$ we have

$$1 - e^{-\lambda s} \sum_{k \in C} p_{yk}(s) \geq e^{-\lambda s} \sum_{k \in B_n} p_{yk}(s) - e^{-\lambda \rho} \sum_{k \in C} p_{yk}(s)$$

Hence the coefficient of ${}_B R_{yi}(\lambda)$ in the upper side of (49) is always $\not\leq 1$.

As s decreases to 0 along the sequence s_r, (41) and (45) ensure that the denominator in (49)(which is (48)) is always bigger than $(1 - \epsilon - \alpha)$.

From the two last facts we can deduce that as s decreases to 0 along s_r (49) becomes

$${}_B R_{yi}(\lambda) + \frac{\delta \alpha}{(1 - \epsilon - \alpha)} \leq \sup_{k \in B_n - C} {}_B R_{ki}(\lambda) + \frac{2 \epsilon}{(1 - \epsilon - \alpha)}$$

Next let ϵ decrease to 0 in this inequality to get

$$(50) \quad {}_B R_{yi}(\lambda) + \frac{\delta \alpha}{1 - \alpha} \leq \sup_{k \in B_n - C} {}_B R_{ki}(\lambda)$$

To establish (50) we do not use a particular property of n, hence as II (38) holds for all n we can find a y_n in every $B_n - C$ such that

$${}_B R_{yi}(\lambda) + \frac{\delta \alpha}{1 - \alpha} - \frac{1}{n} \leq {}_B R_{y_n i}(\lambda)$$

From this we get

$${}_B R_{yi}(\lambda) + \frac{\delta \alpha}{1 - \alpha} \leq \liminf_{n = \infty} {}_B R_{y_n i}(\lambda)$$

But as y_n is in B_n for all n, y_n tends to y in T as n tends to ∞, so that we can use (37) to get

$$\limsup_{n = \infty} {}_B R_{y_n i}(\lambda) \leq {}_B R_{yi}(\lambda)$$

These two last inequalities are obviously incompatible for $\alpha > 0$.

Therefore α is equal to 0 and this completes the proof of the lemma.

Now we fix $\lambda = 1$ for convenience.

In what follows A is not only countable but actually enumerated along N (i.e. we use the order relation of N to define subsets of A;

but this order has usually no relation whatever with the topology T).

Choose a sequence of strictly positive numbers σj decreasing to 0 as j tends to ∞.

For all i in A define the following sets in A.

$$I_i(\sigma j) = \left\{ k \text{ in } A \;\middle|\; {}_BR_{ki}(1) \leqslant {}_BR_{yi}(1) - \sigma j \right\}$$

Note that $I_i(\sigma j)$ is void if either ${}_BR_{yi}(1) = 0$ or ${}_BR_{yi}(1) < \sigma j$.

By definition $I_i(\sigma j)$ satisfies (39) of the lemma; hence we get

$$(51) \quad \lim_{s=0} \sum_{k \in I_i(\sigma j)} p_{yk}(s) = 0$$

j/

$$\text{Let } A_j = A - \bigcup_{i=1}^{j} I_i(\sigma j)$$

We have the inequality

$$\sum_{k \in A_j} p_{yk}(s) + \sum_{i=1}^{j} \sum_{k \in I_i(\sigma j)} p_{yk}(s) \geqslant \sum_{k \in A} p_{yk}(s) = 1$$

The double sum above being a finite sum of sums satisfying (51), we get for all j

$$(52) \quad \lim_{s=0} \sum_{k \in A_j} p_{yk}(s) = 1$$

Now for n in N (or A) consider at the same time B_n and A_n. We have

$$(53) \sum_{k \in A_n \cap B_n} p_{yk}(s) + \sum_{k \in A-A_n} p_{yk}(s) + \sum_{k \in A-B_n} p_{yk}(s) \geqslant \sum_{k \in A} p_{yk}(s) = 1$$

If s decreases to 0, the sum over $A-A_n$ in (53) tends to 0 by (52) and so does the sum over $A - B_n$, by the known property of T-neighbourhoods (II(38)), so that we find

$$\lim_{s=0} \sum_{k \in A_n \cap B_n} p_{yk}(s) = 1$$

This is enough to ensure that $A_n \cap B_n$ is not void, and we can now choose i_n in A_n B_n for all n. Note that as A_n is a subset of A this point can be an i_n and not just a y_n, which might be in A_e.

By definitions of A_n and B_n, i_n satisfies both

(54) $\quad d(i_n;y) < 1/n \qquad$ for all n

and

(55) $\quad {}_B R_{yi}(1) - \sigma_n \leqslant {}_B R_{i_n i}(1)$

for all i in A, $1 \leqslant i \leqslant n$.

(55) yields for all i in A

$${}_B R_{yi}(1) \leqslant \liminf_{n=\infty} {}_B R_{i_n i}(1)$$

By (54), (37) can be used and gives for all i in A

$$\limsup_{n=\infty} {}_B R_{i_n i}(1) \leqslant {}_B R_{yi}(1)$$

This proves that the sequence i_n which converges to y in T is such that

(56) $\quad \lim_{n=\infty} {}_B R_{i_n i}(1) = {}_B R_{yi}(1)$ for all i in A

The convergence is now extended to all $\lambda > 0$

(a) Case of $\lambda < 1$

for all i_n we have

$${}_B R_{i_n i}(\lambda) = {}_B R_{i_n i}(1) + (1-\lambda) \sum_{k \in A} {}_B R_{i_n k}(1)\, {}_B R_{ki}(\lambda)$$

which yields by Fatou's lemma

(57) $$\liminf_{n=\infty} R_{i_n i}(\lambda) \geqslant \liminf_{n=\infty} {}_B R_{i_n i}(1) + (1-\lambda) \sum_{k \in A} \liminf_{n=\infty} {}_B R_{i_n k}(1)\, {}_B R_{ki}(\lambda)$$

But by (56) , the R.H.S of (57) is in fact equal to

$${}_B R_{yi}(1) + (1-\lambda) \sum_{k \in A} {}_B R_{yk}(1)\, {}_B R_{ki}(\lambda)$$

As we know that the resolvent equation holds for y (see p 82),

(57) can be rewritten as

$$\liminf_{n=\infty} {}_B R_{i_n i}(\lambda) \geqslant {}_B R_{yi}(\lambda) \text{ for all i in A}$$

Once again we use (37) and obtain

$$ {}_B R_{yi}(\lambda) \geq \limsup_{n=\infty} {}_B R_{i_n i}(\lambda) \text{ for all i in A} $$

so that we have for all i in A and $\lambda \leq 1$

(58) $\lim_{n=\infty} {}_B R_{i_n i}(\lambda) = {}_B R_{yi}(\lambda)$

(b) Case of $\lambda > 1$

First choose a $\mu < 1$; we have for all i_n and all i

$$ \sum_{k \in A} {}_B R_{i_n k}(1) {}_B R_{ki}(\mu) - \sum_{k \in A} {}_B R_{yk}(1) {}_B R_{ki}(\mu) = $$

(59)

$$ \frac{1}{1-\mu} ({}_B R_{i_n i}(\mu) - {}_B R_{i_n i}(1) - {}_B R_{yi}(\mu) + {}_B R_{yi}(1)) $$

By (56) and (58) the lower side of (59) tends to 0 as n tends to ∞ ; hence we get

$$ \lim_{n=\infty} \sum_{k \in A} {}_B R_{i_n k}(1) {}_B R_{ki}(\mu) = \sum_{k \in A} {}_B R_{yk}(1) {}_B R_{ki}(\mu) $$

(60)

$$ = \sum_{k \in A} \lim_{n=\infty} {}_B R_{i_n k}(1) {}_B R_{ki}(\mu) $$

As $\mu < 1 < \lambda$ we have for all k and i in A

(61) ${}_B R_{ki}(\lambda) \leq {}_B R_{ki}(\mu)$

Consider the following sum

$$ \sum_{k \in A} {}_B R_{i_n k}(1) {}_B R_{ki}(\mu) \left(\frac{{}_B R_{ki}(\lambda)}{{}_B R_{ki}(\mu)} \right) $$

By (61) the coefficients in parentheses are bounded by 1 for all k, so that (60) and the Scheffé's theorem are enough to give after obvious simplifications:

(62) $\lim_{n=\infty} \sum_{k \in A} {}_B R_{i_n k}(1) {}_B R_{ki}(\lambda) = \sum_{k \in A} {}_B R_{yk}(1) {}_B R_{ki}(\lambda)$

Write the resolvent equation for i_n in the following form:

$$ {}_B R_{i_n i}(\lambda) = {}_B R_{i_n i}(1) + (1-\lambda) \sum_{k \in A} {}_B R_{i_n k}(1)\, {}_B R_{ki}(\lambda) $$

By (58) and (62) we see that the R.H.S. has a limit as n tends to ∞ , so the L.H.S. must also have one satisfying

$$ \lim_{n=\infty} {}_B R_{i_n i}(\lambda) = {}_B R_{yi}(1) + (1-\lambda) \sum_{k \in A} {}_B R_{yk}(1)\, {}_B R_{ki}(\lambda) $$

$$ = {}_B R_{yi}(\lambda) $$

This completes the proof of the following theorem:

<u>Theorem 4</u>.

Let B be a Borel set of $(A + A_e, T)$.

Let y be a point in A_e.

Then there exists a sequence, i_n, n in N, of points of A such that

$i_n \longrightarrow y$ in T as $n \longrightarrow \infty$

and

$$ \lim_{n=\infty} {}_B R_{i_n i}(\lambda) = {}_B R_{yi}(\lambda) \quad \text{for all i in A and all } \lambda > 0 $$

This theorem is the so called "civilised" form of would be th1.(for a point in A_e).

Unfortunatelly it is not clear if the two are equivalent.

BIBLIOGRAPHY

M [1] Paul-André MEYER: Probabilités et potenti .l, Hermann, Paris (1966.) I to XI

[2] Paul-André MEYER: Processus de Markov, Springer, Berlin-Heidelberg-New York, (1967.) XII to XV

C [3] Kai Lai CHUNG: Markov Chains with Stationary Transition Probabilities, 2nd edition, Springer, Berlin-Heidelberg-New York, (1967.)

[4] Patrick BILLINGSLEY: The Invariance Principle for Dependent Random Variables, TAMS, volume 83, no. 1, p. 250,(1956.)

[5] J.L. DOOB: Compactification of the Discrete State Space of a Markov Process, Z. fur Wahrscheinlichkeitstheorie, Band 10, Heft. 3, p. 236-251, (1968.)

[6] H. KUNITA and T. WATANABE: Some Theorems Concerning Resolvents over Locally Compact Spaces, Proceedings of the fifth Berkeley symposium, volume., p. 131-164, University of California Press, (196 .)

[7] Jacques NEVEU; Lattice Methods and Submarkovian Processes, Proceedings of the fourth Berkeley Symposium, vol. II p. 347, University of California Press, (1961.)

[8] Jacques NEVEU: Sur les états d' entrée et les états fictifs d' un processus de Markov, Annales de l' Institut Henri Poincaré, vol. XVII, p.323, (1961 - 2.)

[9] D. RAY: Resolvents, Transition Functions and Strongly Markovian Processes, Ann. of Mathematics, vol. 70, pp. 43 - 78,(1959.)

[10] Stanislaw SAKS: Theory of the Integral, 2nd edition, Hafner, New York, (1937.)

[11] Henry SCHEFFÉ: A Useful Convergence Theorem for Probability Distributions, Ann, of Mathematical Statistics, vol. XVIII, p. 434 - 438, (1947.)

[12] W. SIERPINSKI: General Topology, University of Toronto Press, (1952.)

[13] David WILLIAMS: Fictitious States, Coupled Laws and Local Time, Z. fur Wahrscheinlichkeitstheorie, Band 11, Heft 4, p. 288 - 310, (1969.)