TRANSIENT AND PERIODIC FLUID MOTIONS IN ROTATING SYSTEMS

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ABSTRACT

The problems discussed in this thesis lie in the realm of geophysical fluid mechanics and concern periodic and transient fluid motions produced by kinematic or thermal perturbations from a basic isothermal state of steady rigid rotation. We consider either a semi-infinite expanse of fluid bounded by an infinite disk or the fluid between two parallel infinite disks when, initially, the whole system is rotating with constant angular velocity.

For the semi-infinite case, the linearized initial-value problem associated with the disk performing non-torsional or torsional oscillations in its own plane is examined. Oscillatory boundary layer solutions are found except when the frequency of the imposed oscillatians is twice the angular velocity of rotation. For this resonant case, a non-oscillatory solution is obtained which penetrates through the fluid with time. When both disks are present, the corresponding linearized problems are examined and oscillatory solutions can always be found. Also, for the semi-infinite problem, oscillatory solutions always exist when a length scale is introduced in the plane of the disk through the imposed oscillations.

When two disks are present, we consider the linearized initial-value problem connected with a spacially varying temperature distribution on the lower disk. A final steady state is obtained consisting of Ekman layers on the disks together with an inviscid interior thermal-wind flow. The effect on this steady solution of assuming a one-dimensional step-function or normal distribution for the temperature variations, and also the consequence of introducing a favourable or adverse temperature gradient in the initial flow, are discussed. The effect of the non-linear convective terms is examined by seeking exact solutions of the inviscid equations.

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CHAPTER 1

INTRODUCTION

In geophysical fluid mechanics of the oceans, the atmosphere and even possibly the earth's interior, one is concerned with currents and circulations which are dominated by the Coriolis effect of the earth's rotation. Moreover, the effects of heating whether regarded as steady or periodic (night/day cycle) are profound and surely lie at the root of phenomena such as hurricanes. For these reasons, a class of problems has been considered in this thesis which is concerned with periodic and transient fluid motions produced by small perturbations either in the velocity field or the temperature field from a basic isothermal steady state of solid-body rotation.

For any fluid system which is rotating with constant angular velocity, Ω, and for which there exists typical length and velocity scales, L, V, respectively, the two dimensionless parameters

$$R_o = \frac{V}{L\Omega} , \qquad (1.1.1)$$

$$R = \frac{\Omega L^2}{\nu} , \qquad (1.1.2)$$

can be defined, where ν is the kinematic viscosity. The parameter R_o, defined by (1.1.1), is the Rossby number and represents the ratio of the convective acceleration to the Coriolis force and hence the relative importance of the non-linear terms in the equations of motion. The parameter R, defined by (1.1.2), is the Reynolds number and represents the ratio of the Coriolis force to the viscous force. In this thesis, whenever the parameters (1.1.1), (1.1.2) can be defined, we will assume that

$$R_o \ll 1 , \qquad R \gg 1 , \tag{1.1.3}$$

which implies that the convective acceleration and therefore the non-linear terms in the equations of motion, are negligible compared with the Coriolis force and that the effects of viscosity are only important in regions where there are discontinuities in the velocity profile or in the neighbourhood of confining boundaries. Therefore we are only concerned with a class of problems for which there is an interior inviscid core where the motion is dominated by the Coriolis force, viscous free shear layers in the neighbourhood of sharp or discontinuous velocity profiles, and viscous boundary layers on any confining walls where the Coriolis and viscous forces balance.

For slow ($R_o \rightarrow 0$), steady, inviscid ($R \rightarrow \infty$) flow the momentum equations reduce to a balance between the Coriolis force and the pressure gradient, which is called the geostrophic balance, and can be used to describe the steady flow in an inviscid isothermal core. This geostrophic mode possesses circulation and only exists for containers having closed contours of constant total height [16, p.43] a condition which is always satisfied for the geometries considered in this thesis. ~~While~~ for other containers, an infinite number of low frequency Rossby waves [16, p.85] arise to replace geostrophy. When temperature variations also exist, then the relevant equations for steady flow in an inviscid core become the thermal-wind equations [30] which relate the vertical shear of the horizontal velocity to the horizontal gradient of the temperature field.

Proudman [27] showed theoretically and later Taylor [37, 38, 39] confirmed experimentally that all slow, steady, inviscid motions in an incompressible rotating fluid are two dimensional in the sense that the motion is independent of the co-ordinate measured along the axis of rotation. This implies that the flow is identical in every plane normal to the axis of rotation. Then it follows that, wherever there is an obstacle present in the system, the fluid must flow around the obstacle in every plane normal to the axis of rotation and hence around the circumscribing cylinder. A particular example of this phenomenon occurs when a sphere travels with a small velocity either along or normal to the axis of rotation because, as the sphere moves along, it will carry with it the fluid inside the circumscribing cylinder which has generators parallel to the axis of rotation [34, p.146, 149].

In the neighbourhood of any confining walls or in any region in the interior where there are sharp or discontinuous velocity profiles, the effects of viscosity become significant and boundary or free shear layers are formed. These viscous layers may have dimensionless thicknesses of order $R^{-\frac{1}{2}}$, $R^{-1/3}$, or $R^{-1/4}$, depending upon their role in the flow field.

One very important boundary layer present in a rotating system is the so-called Ekman layer [16, p.30]. It should be noticed that the expression (2.3.5) for χ which is given by Greenspan [16] on page 31 is incorrect and, in fact, it should be replaced by

$$\frac{\partial \chi}{\partial z} = -\operatorname{Im}\left[\exp(-(2i)^{\frac{1}{2}}E^{\frac{1}{2}}z) + F(z,t)\right].$$

This boundary layer is formed on an infinite plane disk which bounds a semi-infinite expanse of incompressible fluid, when both the fluid and the disk are in rigid rotation with constant angular velocity, Ω, and then at some instant of time the rate of rotation of the disk is slightly altered. For the Ekman layer, the depth of penetration of vorticity is of order $(\nu/\Omega)^{\frac{1}{2}}$ or $R^{-\frac{1}{2}}$ in dimensionless variables. The ~~purpose~~ effect of the Ekman layer is to increase (decrease) the angular momentum of the fluid in the vicinity of the disk by impelling the fluid radially outwards (inwards). This fluid is replaced by an inflow (outflow) from the inviscid interior and is called the Ekman layer suction. The unsteady Ekman layer forms within a few revolutions and tends, eventually, to a steady state in which it has finite thickness and the diffusion of vorticity is balanced by the distortion of the vortex lines.

For steady flow in the neighbourhood of confining walls parallel to the axis of rotation or around discontinuities in the velocity profile, a more complicated structure arises than that found for the Ekman layer. In fact two viscous layers are formed [16, p.97]. The first layer has a depth of penetration of vorticity of order $(\nu L^2/\Omega)^{1/4}$ or $R^{-1/4}$ in dimensionless quantities and is responsible for smoothing out any abrupt changes and discontinuities that occur in the azimuthal velocity component. The second layer has a depth of penetration of vorticity of order $(\nu L/\Omega)^{1/3}$ or $R^{-1/3}$ in dimensionless quantities and is responsible for vertical mass transfer usually between two Ekman layers and for matching the required condition on a vertical wall.

The structure of vertical free shear layers was discussed by Stewartson [35] for the system consisting of incompressible fluid bounded by two concentric infinite split disks when the inner finite disks rotate with a slightly different angular velocity from the fluid and the remainder of the disks. Stewartson found that when antisymmetric boundary conditions were imposed on the inner disks only the shear layer which had a thickness of order $R^{-1/3}$ was present, because the interior flow, which is the average of the imposed boundary conditions [16, p.93], was zero and hence no adjustment in the azimuthal velocity was required. While both free shear layers were present when symmetric boundary conditions were imposed on the inner disks. The free shear layers formed when two concentric spheres rotate at slightly different speeds has been discussed by Proudman [26]. He found that the same structure existed as in the Stewartson problem except in the neighbourhood of the equator of the inner sphere. This demonstrates that the basic physical processes are almost unaltered by variations in the geometry.

In a rotating system containing incompressible fluid the Ekman boundary layers play an active role in the flow field and, in fact, control the motion in the interior inviscid core, while the vertical boundary and free shear layers, the $R^{-1/3}$ and $R^{-1/4}$ layers, have only a passive role in the flow pattern and do not influence the motion in the interior core.

We now consider the particular system consisting of two infinite plane disks with incompressible fluid between them when both the fluid and the disks are in rigid rotation with some constant angular velocity.

if the angular velocity of the two disks is altered slightly by equal and opposite amounts Ekman layers are formed on the disks. The inflow into one Ekman layer is equal to the outflow from the other Ekman layer while the angular velocity of the interior is unchanged. Alternatively, if the angular velocity of the two concentric disks is varied by identical small amounts then the final steady state that persists after the transient effects have decayed is solid-body rotation at the new angular velocity. The time dependent process, the so-called spin-up process, has been discussed by Greenspan and Howard [17].

The initial impulsive change in the angular velocity of the disks causes a Rayleigh layer [31, p.136], which penetrates through the fluid in the standard diffusive manner, to be formed on each disk. Within a few revolutions, that is in a time scale of order one, the effect of the change in rotation is felt and quasi-steady Ekman layers develop on the disks. In addition, inertial oscillations at twice the frequency of rotation and with very small amplitude arise in the fluid. If, initially, the angular velocity of the disks is increased, then, in the Ekman layers, there is a radial outflow which produces a corresponding inflow into the Ekman layer from the inviscid interior. In order to satisfy conservation of mass there must exist a radial inflow in the interior which increases the angular momentum of the fluid. Hence by this process the interior fluid is spun-up to a new state of rigid rotation in a time of order $\left\{ L/(\Omega \nu)^{\frac{1}{2}} \right\}$, or $R^{+\frac{1}{2}}$ in dimensionless variables, which will be referred to as the spin-up time and agrees with the experimental results of Wedemeyer [44]. Therefore the Ekman layers act as sinks for the low momentum fluid in the interior and this fluid is replaced by higher momentum fluid from larger radii. The inertial oscillations persist through the spin-up time and require the viscous diffusion time, which is of order R, to decay. This spin-up process also applies when, initially, the angular velocity is decreased and for any arbitrary symmetric container.

Veronis [42] has shown for two dimensional problems that the theories for an incompressible, rotating, isothermal system where the motion is driven by velocity changes on horizontal boundaries and for a stably stratified, non-rotating system where the motion is driven by temperature variations on vertical boundaries are analogous, when the direction parallel to the constraining mechanisms are equated. This analogy is no longer applicable when a third dimension is introduced because, for the rotating system, the constraint of the vorticity of the basic rotation acts equally in both horizontal directions while for the stratified system the constraint of stratification acts solely in the vertical direction and hence the third dimension introduces a degree of freedom. In a second paper Veronis [43] generalized these results to include stratification to the rotating system and rotation to the stratified system and again the analogy held for two dimensions but not for three.

The introduction of stable stratification into a rotating system tends to destroy the novel phenomena produced by the Coriolis force. The buoyancy forces inhibit vertical motion in the fluid which implies that the Ekman layer suction is impeded and hence the control exercised by the Ekman layers over the inviscid interior is lessened. Also, in the interior, there is a tendency towards horizontal flow and the Proudman-Taylor theorem no longer applies. Depending upon the relative importance of the rotation and the stratification, vortex line stretching can be rendered ineffective and the flow can be controlled by viscous diffusion in a time scale of order (L^2/ν).

Firstly, we will consider steady motions in a stably stratified rotating system for which we can define the dimensionless parameters

$$\sigma = \frac{\nu}{\kappa}, \quad \text{the Prandtl number,} \qquad (1.1.4)$$

$$H = \frac{ag\,\Delta T}{\Omega^2 L}, \quad \text{the thermal Rossby number,} \qquad (1.1.5)$$

where κ is the thermal diffusivity, g the acceleration due to gravity, a the coefficient of thermal expansion and ΔT the basic vertical temperuture difference. When $\sigma \ll R^{-1}$ or $H \ll R^{-1}$, the diffusion of heat is much more important than thermal convection and the effects of stratification are not profound. The problem can be resolved in terms of an inviscid interior core in which the flow is a solution of the thermal-wind equations and viscous boundary layers. The secondary flow present in the interior is controlled by Ekman layer suction. Hence, when the diffusion of heat dominates over thermal convection, the resultant flow is comparable to that for a homogeneous fluid except that the Proudman-Taylor theorem is no longer satisfied in the interior.

On the other hand, when $\sigma \gg 1$ or $H \gg 1$, no vertical motion is permitted and therefore all movement is confined to horizontal planes. The effects of viscosity are felt throughout the whole fluid. Hence for this particular case, there is little resemblance between the motions produced in a homogeneous and in a stratified fluid.

When $\sigma = O(1)$, $H = O(1)$, Barcilon and Pedlosky [2] showed that the solution is closely related to the case $\sigma \gg 1$. The Ekman layers are absent to first order and although they exist at lower orders, they now assume a secondary and passive role. The vertical $R^{-1/3}$,

$R^{-1/4}$ - layers no longer exist but are replaced by a new type of boundary layer which has a thickness of order $R^{-\frac{1}{2}}$. The interior core is no longer controlled by vortex line stretching but by dissipative processes.

In their second paper, Barcilon and Pedlosky [3] examined the transition from the case when the stratification is unimportant and the fluid behaves as if it were homogeneous to the case when the stratification is substantial, which they considered in their first paper [2]. From this analysis, a unified picture of the steady, linear dynamics of rotating fluids with given arbitrary stratification was obtained. They found that the parameter, σH, determines the behaviour of the fluid. When

$$\sigma H < R^{-2/3} ,$$

the fluid behaves as if it were homogeneous. Ekman layers are formed on the horizontal boundaries, Stewartson's $R^{-1/3}$, $R^{-1/4}$ - layers are present on the vertical boundaries and the motion in the interior is controlled by suction into the Ekman layers. When

$$R^{-2/3} < \sigma H < R^{-\frac{1}{2}} ,$$

that is when a weak stratification exists in the fluid, the buoyancy forces are no longer negligible. The Ekman layers still exist and control the interior motion. On the vertical boundaries, however, a triple boundary layer structure is found which consists of layers having thicknesses of order $R^{-1/4}$, $(\sigma H)^{\frac{1}{2}}$, the hydrostatic layer, $(\sigma H)^{-1/4}R^{-\frac{1}{2}}$, the buoyancy layer. When

$$\sigma H > R^{-\frac{1}{2}},$$

which corresponds to a strong stratification present in the fluid, the Ekman layers are absent to first order and the interior is controlled by viscous diffusion. On the vertical walls the $R^{-1/4}$ and $(\sigma H)^{\frac{1}{2}}$ layers combine together and penetrate through the fluid leaving a single boundary layer which has a depth of penetration of order $(\sigma H)^{-1/4}R^{-\frac{1}{2}}$. The special case of Barcilon and Pedlosky [2] discussed above lies in this last range.

For time dependent motions vortex line stretching is again present but the stratification renders this process less effective than it was for the homogeneous case. Holton [19] studied experimentally the problem of spin-up for a stratified fluid. He showed that the fluid adjusts in the spin-up time to a quasi-steady state in which the relative angular velocity is zero at the edge of the Ekman layer and increases exponentially away from the Ekman layer. The ultimate state of rigid rotation is accomplished in the viscous diffusion time. Pedlosky [25] showed theoretically that the interior was spun-up by strictly diffusive processes in a time of order (L^2/ν), but Holton and Stone [20] noticed that, in fact, there was an error in the scalings employed by Pedlosky and they suggested that the three distinct time scales

$$\Omega^{-1}, \quad (L^2/\nu\Omega)^{\frac{1}{2}}, \quad (L^2/\nu),$$

are all present in the adjustment process.

Walin [47] assumed that the spin-up process in a stratified fluid required a time scale, τ, which was large compared to the rotation time but small compared to the time taken for diffusion to penetrate through the interior of the system. Then by introducing a perturbation series in the parameter

$$\frac{1}{2\Omega\tau} = \left(\frac{\nu}{2\Omega L^2}\right)^{\frac{1}{2}},$$

where 2L is the depth of the fluid, he obtained a solution of the linearized spin-up problem which was valid in the interior, that is outside the diffusive regions in the neighbourhood of the boundaries. From this analysis, he deduced that the flow was characterised by the parameter

$$B = \frac{(Q_s g/\rho_0 L)^{\frac{1}{2}}}{2\Omega},$$

where ρ_o is a constant density and Q_s is the scale of the basic stratification, which represents the ratio of the stability frequency and the Coriolis parameter. Walin found, for the case when no lateral boundaries were present, that the effect of velocity variations on horizontal boundaries penetrates a distance $B^{-1}H$ into the fluid where H is the horizontal length scale. The time required for transient effects to decay is equal to the spin-up time based on the real penetration depth instead of the total depth of the fluid. For a closed container of radius aL, Walin deduced by considering the transport of fluid in the corner regions, that, when $B^{-1}a \ll 1$, the process is essentially the same as the spin-up of a homogeneous fluid except in a region close to the vertical boundary of thickness BL, while, when $B^{-1}a \gg 1$, the spin-up process only penetrates to a dimensional height $B^{-1}aL$.

An important phenomenon occurring in systems in rigid rotation with constant angular velocity, Ω, is the resonance effect which is experienced when oscillations are imposed on a boundary with a frequency

$$2\Omega\,(\underline{n}.\underline{k}) , \qquad (1.1.6)$$

where $\underline{k}$ is a unit vector parallel to the axis of rotation and $\underline{n}$ is a unit vector normal to the boundary. This frequency, (1.1.6), will be referred to as the resonant frequency. This resonance effect occurs in Hunt and Johns' [21] problem which is concerned with the boundary layer produced on a smooth sea bed by tidal or gravity waves, since no periodic solution of the linearized equations exists at certain critical latitudes. Hunt and Johns gave no discussion of the behaviour at these critical latitudes.

In chapter 2, the flow generated in a semi-infinite expanse of incompressible fluid bounded by an infinite plane disk is considered, when both the fluid and the disk are in rigid rotation with a constant angular velocity and, additionally, the disk performs non-torsional oscillations,

$$u + iv = ae^{int} + be^{-int}, \qquad (1.1.7)$$

in its own plane, where u,v are the cartesian velocity components in the plane of the disk relative to the rigid rotation, n the frequency and a,b complex constants [40]. Periodic solutions are first sought and it is found that a modified Stokes layer is formed on the disk for all frequencies except the resonant frequency, which is twice the angular velocity of rotation. In the latter case there is no oscillatory solution which satisfies the boundary conditions. Rott and Lewellen [48] noticed this behaviour but gave no further discussion

In order to seek a resolution of the difficulty associated with the resonant case, an initial-value problem is posed; in most cases the oscillatory solutions are reached at large times. In the resonant case, however, the flow is found to be a linear combination of a modified Rayleigh layer, which penetrates outwards perpetually from the disk in the standard diffusive manner, and a layer confined to the disk which, at large times, becomes a modified Stokes layer. The shear oscillations continue to penetrate outwards indefinitely, unless the imposed oscillations are chosen so that the velocity vector of the disk rotates with constant magnitude in the same direction as the basic rotation, but with twice its angular speed; then the Rayleigh layer is absent. On the other hand, the presence of a second disk produces, at large times, in the resonant case, a modified plane Couette flow of oscillatory amplitude superimposed on the modified Stokes layer.

For the special case $n = 0$, that is when the angular velocity of the disk is changed by a constant amount, the solution of the initial-value problem is not an analogue of the classical Rayleigh layer, that would be present in non-rotating systems, but is a steady Ekman layer.

In chapter 3, the effect of replacing the non-torsional oscillations, (1.1.7), on the disk by torsional oscillations about the axis of rotation is examined. For the problem when only one disk is present, the fluid at infinity is unaffected by boundary movements and the linearized problem is identical to the problem associated with non-torsional oscillations. The oscillatory solution, derivable for non-resonant frequencies, agrees with the solution of the linearized form of Benney's problem [5]. Benney [5] also discusses, by using the method of multiple scales, the periodic solutions of the non-linear problem for oscillations near to the resonant frequency. However, this analysis is not valid at the resonant frequency.

When a second disk is introduced parallel to and at a finite distance from the first, radial pressure gradients are required because a unique axis is defined about which torsional oscillations are performed; a new problem arises. For certain frequency ranges, this new problem is of the type associated with spin-up to solid-body rotation of a cylindrical can of liquid, when the motion is driven by secondary circulations rather than molecular diffusion. The linearized initial-value problem is considered, for the more general case, when arbitrary torsional oscillations are imposed on both disks and the Reynolds number, given by (1.1.2), where 2L is the distance between the disks, is large. This is, in fact, a generalization of the problem considered by Greenspan and Howard [17]. For the four cases $\sigma = 0$ (steady), $\sigma R^{\frac{1}{2}} \ll 1$ (low frequency), $\sigma R^{\frac{1}{2}} = O(1)$ (intermediate frequency), $\sigma R^{\frac{1}{2}} \gg 1$ (high frequency), where $\Omega\sigma$ is the frequency of the oscillations imposed on a disk, the times taken for the transient terms to decay are found firstly from the solution of the initial-value problem and then by employing the approximations used in [17], and the final states are discussed. Again, it should be noted, that the introduction of the second boundary produces final states which are always oscillatory.

Greenspan [15] examined the transient motion produced in a viscous fluid contained in a spherical shell rotating with a constant angular velocity, when an arbitrary initial state was resolved into rigid rotation. He found that there existed critical latitudes where the modal frequency of the inertial oscillations present in the fluid was equal to twice the component of the rotation vector normal to the boundary, (1.1.6). For the linearized problem, Greenspan found that, in the immediate vicinity of these critical latitudes, the boundary layer solution was composed of error functions of time while, elsewhere, the boundary layer solution consisted of a simple exponential function of time. This implies that a resonance effect is experienced. At the critical latitudes, this ~~buoyancy~~ boundary layer solution has the same structure as the solution obtained in chapters 2 and 3 when either non-torsional or torsional resonant oscillations ~~were~~ are imposed on an infinite disk which bounds a semi-infinite expanse of fluid.

When resonance occurs, these solutions are obtainable only because the time dependence ~~was~~ is retained, explicitly, in the analysis and not replaced by an assumption of periodicity. From Greenspan's results and the corresponding solutions in chapters 2 and 3 we deduce that, when resonance occurs, the acceleration balances the Coriolis and viscous forces separately and independently of each other.

For the linearized problem, Greenspan obtained a solution for the fluid motion inside the sphere by an expansion procedure in which the general inviscid solution is corrected for viscous effects and is then made uniformly valid in time through the spin-up phase. In the interior, the depth average circulation about a contour of constant cylindrical radius is extracted from the fluid by the Ekman layer suction within the spin-up time, $\{ L/(\Omega \nu)^{\frac{1}{2}} \}$. The excess circulation is not eliminated in this way but excites inviscid inertial oscillations which again decay within the spin-up time due to the influence of the boundary layers. There are small residual effects which persist until the viscous diffusion time is reached but the essential processes require a much shorter time scale. Greenspan calculated the amplitude of the modal (resonant) oscillations present inside the sphere and he found that a very good agreement existed between his theoretical results and the experimental work of Aldridge and ~~Toorme~~ Toomre [6, p.406] . From this agreement it is deduced that the non-linear convective terms are unimportant and that the linearized analysis of Greenspan gives a good description of the transient fluid motion inside the sphere.

Roberts and Stewartson [29] examined the stability of the Maclaurin spheroid, which consists of incompressible fluid in the form of a spheroid of revolution under the effect of its own gravity, for infinitesimal perturbations when the fluid was assumed to have small viscosity. Normal mode solutions of the linearized problem in oblate spheroidal co-ordinates, (Υ, Θ, ϕ), are sought. A solution which consists of an inviscid interior core surrounded by a viscous boundary layer, which has a depth of penetration of order $\nu^{\frac{1}{2}}$(or $R^{-\frac{1}{2}}$), is found except in two singular zones which occur when the frequency of the modal oscillations was

$$2\Omega \cos \psi ,$$

where the velocity vector is $(\sin \psi , \cos \psi , i)e^{i\phi}$ in the co-ordinate system, (Υ, Θ, ϕ). These critical regions require a separate analysis and their existence shows that a resonance phenomenon is present in the problem. In order to obtain a boundary layer solution for the singular zones, lateral shear is included in the analysis, that is both the co-ordinates ζ and $\mu = \cos \Theta$ are stretched although again the most rapid changes occur in the ζ - direction. From this analysis, a new boundary layer arises which has a depth of penetration of order $\nu^{2/5}$ (or $R^{-2/5}$).+ Therefore for the critical zones the boundary layer is much thicker than elsewhere on the spheroid although the effect of these eruptions on the interior is negligible compared with the influence of the $\nu^{\frac{1}{2}}$ boundary layer.

Busse [7] considered the steady, laminar motion of a viscous incompressible fluid inside a precessing spheroidal shell. He retained the non-linear convective terms in his analysis and found that these terms were important for finite amplitude motion because, in the interior, there existed a differential rotation superimposed on the constant vorticity which is the solution obtainable by linear theory. Therefore the linear

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+Greenspan [16, p.62] states that "this change would appear as a singularity in the linear theory", but, in fact, Roberts and Stewartson only consider the linear theory and hence linear theory should be replaced by normal boundary layer theory.

solution was not approached in the limit of vanishing viscosity. Firstly Busse sought a series solution which consisted of an inviscid interior with linear boundary layers and found that there existed critical circles where the boundary layer thickness tended to infinity. Again an analysis of these singular regions would produce a new boundary layer which had a depth of penetration of order $\nu^{2/5}$ (or $R^{-2/5}$). Then Busse considered the problem when the non-linear terms were retained in the boundary layer equations and found that the critical circles cause the differential rotation, in the interior, to be divergent. Busse's theoretical prediction of the steady zonal flows which tend to form a zonal jet at critical latitudes have been observed in experiments performed by Malkus [6, p.407] Busse's analysis can also be used to describe the steady fluid flow due to a tidal bulge. Busse anticipated that the differential rotation would show a smooth profile in the interior when the effects of viscosity were included.

Greenspan [46] showed that the non-linear interaction of inviscid inertial modes does not produce a resonant response in the steady geostrophic circulation. Therefore he anticipates, in agreement with Busse [7] , that the steady currents produced in a closed rotating container by oscillatory disturbances arise from a combination of viscous and non-linear effects within the boundary layers.

Stewartson and Rickard [36] investigated the free periods of oscillations in an incompressible inviscid fluid bounded by two rigid concentric spheres, a,b ($a > b$), when the whole system was rotating with angular velocity, Ω, about a common diameter of the spheres. Firstly, oscillatory solutions were sought for the linearized problem in the form of an expansion in powers of

$$\varepsilon = \frac{a - b}{a + b} ,$$

which was assumed to be small, for small disturbances from the basic state of steady rigid rotation. It was found that the solution for the pressure became singular when the frequency was (1.1.6), which defined two critical circles where the characteristic cones of the governing equations touched the inner boundary. For these critical regions an inner expansion in powers of $\varepsilon^{\frac{1}{2}}$ was developed and it was found that in order to remove the singularity in the pressure, an integrable singularity in the velocity components must be introduced on the characteristic cone which touched the inner boundary. Further integrable singularities were introduced by repeated reflections at the shell boundaries and so, even outside the critical regions, the expressions for the velocity components contained a "pathological" term of order $\varepsilon^{\frac{1}{2}}$.

Stewartson and Rickard deduced that this phenomenon applied to a large class of rotating cavities provided the characteristic cones touched the inner boundary.

In chapter 7, it is found, for an infinite disk bounding a semi-infinite expanse of incompressible fluid when the fluid and the disk are in steady rigid rotation, that an oscillatory solution always exists, when a length scale is introduced in the plane of the disk either by imposing oscillations on the disk which are sinusoidal in one of the co-ordinates in the plane of the disk, or by splitting the disk so that oscillations are imposed only on an inner finite region. For these problems, a resonance effect is still present and is shown by the fact that different oscillatory solutions exist for resonant and non-resonant frequencies.

Also in chapter 4, section 4.13, we find when the above imposed oscillations on the disk are replaced by oscillatory temperature variations on the disk, which depend upon a length scale in the plane of the disk through the membrane equation, then an oscillatory solution always exists. Again a resonance phenomenon is present because different oscillatory solutions exist for resonant and non-resonant frequencies.

Therefore, we deduce, for the cases considered in this thesis, that the introduction of a length scale into the problem either normal to or in the plane of the disk always produces an oscillatory solution. This is also true for the problems discussed by Roberts and Stewartson, [29] and Busse [7].

The remaining chapters in this thesis, namely chapters 4, 5, 6, 8, are concerned with problems in which motion is generated in a fluid by temperature variations rather than velocity variations on the confining boundaries. In particular, the development of vorticity in the atmosphere due to temperature changes on the earth's surface is considered and it is hoped that strong circulatory motions will be developed which could describe the formation of a hurricane. The time independent solutions of this problem closely resemble the solutions of the steady problems discussed by Duncan [10], Hunter [22], and Barcilan and Pedlosky [4].

Firstly, an idealized model of the atmosphere is considered which, for any given latitude, is composed of two infinite plane horizontal disks with viscous fluid between them when, initially, the fluid and the disks are in steady, isothermal rigid rotation about an axis normal to the disks.

In chapter 4, the flow generated in the fluid is examined when, from $t = 0$, a steady heating is applied to the lower disk which depends upon the co-ordinates in the plane of the disk through a function satisfying the membrane equation, while the temperature of the upper disk remains at its initial value. The linearized initial-value problem is solved for the component of vorticity perpendicular to the disk (the vertical component of vorticity) on the assumption that the Reynolds number, (1.1.2), is large. The Coriolis force and therefore the rotation must be responsible for the production of vorticity perpendicular to the disk because for the corresponding problem without rotation, the vertical vorticity is always zero. The time required for the transient effects to decay is discussed and the final steady state is shown to be composed of Ekman layers on the disks and an interior flow which is a particular solution of the thermal-wind equations.

In chapter 5, the effect on the steady vertical vorticity which persists at large times, after the transient effects have decayed, is examined, when different temperature distributions are imposed on the lower disk. Firstly, the linearized steady problem associated with an applied temperature distribution which takes the form of a one-dimensional step-function in the rotating system, is considered. This implies that the temperature on an inner finite strip of the lower plane is increased or decreased by a finite amount while elsewhere on this plane the temperature remains at its initial-value. It is found that the steady state is composed of Ekman layers on the disks, an interior flow which is a particular solution of the thermal-wind equations and free shear layers at the discontinuities, which have depths of penetration of order $R^{-1/3}$ and $R^{-1/4}$. These layers are similar to the free shear layers discussed by Stewartson [35]. Secondly, the applied temperature variations on the lower disk are assumed to be a one-dimensional "normal distribution" of the form $A \exp\{-x^2/\sigma^2\}$, where A, σ are real constants and σ is positive. No general solution has been obtained for this case but, instead, the extreme cases, σ large and σ small, are considered for time-independent flows. It is found that, to the highest order, the case σ large is equivalent to the steady problem discussed in chapter 4, while the case σ small is equivalent to the problem associated with the step-function temperature distribution discussed in chapter 5.

In chapters 4 and 5, the temperature field is assumed, initially, to be constant throughout the atmosphere. This approximation to the actual temperature field present in the atmosphere can be improved by assuming, for the initial flow, a constant adverse temperature gradient. Therefore in chapter 6, the problem considered in chapter 4 is extended to include, initially, either a favourable or an adverse temperature gradient. It is found, when a favourable temperature gradient is present, initially, that the steady solution obtained from linear theory represents the final flow present in the fluid when all the transient effects have decayed. However, when an initial adverse temperature gradient is present in the fluid, the steady solution obtained from linear theory becomes singular when the Rayleigh number

$$\mathcal{R} = \frac{g\alpha \Delta T L^3}{\kappa \nu} , \qquad (1.1.8)$$

attains a critical value, and hence only represents the final flow for values of $\mathcal{R}$ much less than this critical value. It is deduced that when the Reynolds number, R, (1.1.2), tends to infinity [redacted] the asymptotic value of the minimum critical Rayleigh number is

$$3 \left\{ \tfrac{1}{2}\pi R^2 \right\}^{2/3} \qquad (1.1.9)$$

for both free/free and rigid/rigid boundaries. A comparison between these results and the results obtained by Chandrasekhar [9] is given.

In chapters 4, 5, 6, only the linear problems have been considered and therefore, in chapter 8, the effect of the non-linear convective terms on the interior flow for the idealized model of the atmosphere is considered, when the upper disk is at infinity and the temperature of the lower disk is maintained at a constant value for all time. If

(v_r, v_Θ, v_z) are the velocity components in the non-rotating cylindrical polar co-ordinates, (r, Θ, z), then a class of inviscid solutions of the form

$$v_r = -\frac{f(r)}{r}, \qquad v_z = \frac{z}{r}\frac{d}{dr}\{f(r)\}, \qquad v_\Theta = r\Omega \quad \text{at } t = 0,$$

is found, where f(r) satisfies an ordinary non-linear differential equation. Also some exact solutions of the complete Navier-Stokes equations are derived which satisfy inviscid boundary conditions. These exact solutions represent possible interior flows which satisfy inviscid boundary conditions at the disk. However, in the neighbourhood of the disk, these interior solutions would have to be replaced by viscous boundary layer solutions which satisfy the non-slip condition at the disk and are matched with the interior solution, in order to obtain a solution of the non-linear equations valid throughout the whole fluid. No solutions for these viscous non-linear boundary layers have been obtained.

CHAPTER 2

ON STOKES AND RAYLEIGH LAYERS IN A ROTATING SYSTEM

2.1 INTRODUCTION

It is common knowledge that in fluid systems in solid-body rotation a resonance effect is found if an attempt is made to force oscillations with a frequency, n, which is twice the angular velocity of rotation. A notable example of this phenomenon occurs in unsteady Ekman layers, and it is with flows of this kind that we are concerned in this **chapter**.

Let us consider first a well-known prototype oscillatory boundary layer in the absence of rotation. This is the so-called Stokes layer ([31], page 381), in which the shear oscillations imposed by the movement of a plane decay exponentially with distance away from the surface; the characteristic distance, or depth of penetration of vorticity, is $(\nu/n)^{1/2}$, where n is the frequency and ν the kinematic viscosity. The Stokes layer is one of the simplest equilibrium boundary layers.

If the whole system, plane plus fluid, is in a state of solid-body rotation, with angular velocity Ω, the corresponding depth of penetration of the oscillations is $(\nu/|n-2\Omega|)^{1/2}$, n and Ω being assumed positive. This tends to infinity if n tends to 2Ω. We may infer, therefore, that for the idealized situation of a semi-infinite body of fluid bounded by an infinite wall, no equilibrium boundary layers exist in the limit $n \rightarrow 2\Omega$. On the other hand, if another plane is placed parallel to and at some finite distance d away from the first plane, the penetration of vorticity will be limited to this distance. Then an equilibrium oscillatory flow between the planes can be expected.

There remains the question of discovering a meaningful solution for the case (with $n \rightarrow 2\Omega$) when d is very large or infinite. This we answer by posing an initial-value problem, in which the oscillatory motion of the plane commences at, and continues from, $t = 0$. In this case a depth of penetration is found, namely $(\nu t)^{1/2}$, t being the time; this distance is the characteristic length associated with a Rayleigh (or impulsive) layer. Consequently the shear oscillations continue their penetration indefinitely, if no other boundary is present, or until some confining boundary is reached.

One further comment can be made at this stage in anticipation of the analysis. Suppose the infinite disk (or plane) performs non-torsional oscillations defined as follows: if u and v are velocity components in the plane of the disk, relative to the rigid body rotation, we specify

$$q = u + iv = ae^{int} + be^{-int}$$

at the disk surface, where a and b are complex constants and n is the frequency; the latter can be assumed positive without loss of generality. This implies that every point in the disk performs elliptic harmonic motion relative to the basic rotation. If b = O, so that the velocity vector has given magnitude and rotates with angular velocity, n, we shall see that the resonance case $n = 2\Omega$ is avoided. This movement of the disk can generate contained oscillations.

2.2 EQUATIONS OF MOTION

An infinite disk at z = O bounds a semi-infinite expanse (z > O) of fluid which is assumed to be incompressible. Both the disk and the fluid rotate as a solid body with constant angular velocity, Ω , about an axis normal to the disk, but the disk additionally performs oscillations,

$$q = u + iv = ae^{int} + be^{-int} , \qquad (2.2.1)$$

in its own plane.

Let us take cartesian axes (x,y,z) such that the z-axis is parallel to the common axis of rotation of the fluid and disk, and the x,y-axes lie in the plane of the disk and rotate with it. The Navier-Stokes equations and the continuity equation in this rotating co-ordinate system are

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} . \nabla)\underline{u} + 2\Omega \underline{k}_\wedge \underline{u} + \nabla\left(\frac{P}{\rho} - \frac{1}{2}\Omega^2(x^2 + y^2)\right) = \nu \nabla^2 \underline{u} , \qquad (2.2.2)$$

$$\text{div } \underline{u} = 0 , \qquad (2.2.3)$$

where $\underline{u}$ is the velocity vector, $\underline{k}$ a unit vector parallel to the z-axis, P the pressure and ρ the density.

Suppose $\underline{u} = (u,v,w)$ where u, v, w are the velocity components in the x, y, z directions respectively. Following knowledge of the ordinary Stokes layer, we assume that the velocity field in the boundary layer is of the form,

$$\underline{u} = (u(z,t), \ v(z,t), \ w(z,t)) . \qquad (2.2.4)$$

The boundary conditions to be satisfied are

(a) $w = 0, \quad u + iv = ae^{int} + be^{-int}$ on $z = 0$,

(b) $u, v \rightarrow 0$ as $z \rightarrow \infty$,

(c) we shall use, as required, an initial condition at $t = 0$, or an assumption of periodicity.

(2.2.5)

From (2.2.3), (2.2.4) and (2.2.5) it follows immediately that

$$w \equiv 0 .$$

Using this result with (2.2.2) and (2.2.4), we have

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\Omega v &= -\frac{\partial p^*}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} , \\ \frac{\partial v}{\partial t} + 2\Omega u &= -\frac{\partial p^*}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} , \\ 0 &= -\frac{\partial p^*}{\partial z} , \end{aligned}\right\} \qquad (2.2.6)$$

where $p^* = \frac{p}{\rho} - \frac{1}{2}\Omega^2(x^2 + y^2)$ is the effective kinematic pressure.

We assume that there is no imposed pressure gradient and then

$$\frac{\partial p^*}{\partial x} = \frac{\partial p^*}{\partial y} = 0 .$$

Hence

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\Omega v &= \nu \frac{\partial^2 u}{\partial z^2} , \\ \frac{\partial v}{\partial t} + 2\Omega u &= \nu \frac{\partial^2 v}{\partial z^2} . \end{aligned}\right\} \qquad (2.2.7)$$

We note that these equations, although exact in formulation, are also boundary layer equations, since the terms usually neglected disappear exactly. With $q = u + iv$, (2.2.7) becomes

$$\frac{\partial q}{\partial t} + 2\Omega\, iq = \nu \frac{\partial^2 q}{\partial z^2} \quad . \tag{2.2.8}$$

This equation must now be solved subject to the conditions (2.2.5), which become

$$\left.\begin{array}{ll} \text{(a)} & q = ae^{int} + be^{-int} \quad \text{on} \quad z = 0 \,, \\ \text{(b)} & q \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty \,, \\ \text{(c)} & \text{an initial condition at } t = 0 \text{ or an assumption} \\ & \text{of periodicity.} \end{array}\right\} \tag{2.2.9}$$

2.3 OSCILLATORY SOLUTIONS

As a preliminary step in this study and in order to expose certain difficulties, we consider periodic solutions which we assume to have the form

$$q = q_1 e^{int} + q_2 e^{-int} \quad . \tag{2.3.1}$$

From (2.2.8), (2.2.9) and (2.3.1) we have for $n \neq 2\Omega$ the solution

$$q = ae^{int}\, e^{-\lambda_1 z} + be^{-int}\, e^{-\lambda_2 z} \,, \tag{2.3.2}$$

where

$$\lambda_1 = \left[\frac{i(n+2\Omega)}{\nu}\right]^{1/2}, \quad \lambda_2 = \left[\frac{i(2\Omega - n)}{\nu}\right]^{1/2} . \tag{2.3.3}$$

The roots λ_1, λ_2 have positive real parts. The solution (2.3.2) satisfies the required boundary conditions, (2.2.9 (a), (b)). Therefore for $n \neq 2\Omega$ the flow can always be determined and is a well-defined boundary layer on the disk having the same structure as a Stokes layer.

For $n = 2\Omega$, however, a solution of (2.2.8) which has the form (2.3.1), is

$$q = ae^{2\Omega it}e^{-\lambda_o z} + be^{-2\Omega it} \quad , \tag{2.3.4}$$

where $$\lambda_o = \left[\frac{4\Omega i}{\nu}\right]^{1/2} . \tag{2.3.5}$$

This solution satisfies the correct boundary condition, (2.2.9(a)), on the disk and is finite but non-zero ($b \neq 0$) as z tends to infinity, contradicting (2.2.9(b)). A solution satisfying all the correct boundary conditions cannot be found except when $b = 0$.

In this resonant case of $n = 2\Omega$, which will, henceforth, be referred to as the resonant frequency, although a solution can be found satisfying the boundary conditions on the disk, it is in general impossible to satisfy the correct conditions as z tends to infinity. The oscillations are not confined to a well-defined boundary layer. It is interesting to see that if another infinite disk is introduced at $z = d$, say, at rest relative to the rotating co-ordinate system, then all the boundary conditions can be satisfied by

$$q = \frac{a \sinh\{\lambda_o(d-z)\}}{\sinh\{\lambda_o d\}} e^{2\Omega it} + \frac{b}{d}[d-z]e^{-2\Omega it}. \qquad (2.3.6)$$

The first term represents a modified Stokes layer and the second term an oscillatory plane Couette flow in which the shear is uniform in space, but has an instantaneous time-dependent amplitude. The solution, and therefore the boundary layer, extends between the two disks, in accordance with the idea that, for one disk oscillating at the resonant frequency, the boundary layer extends throughout the whole fluid.

2.4 THE INITIAL-VALUE PROBLEM

We consider both (i) the case of one disk at $z = 0$ bounding a semi-infinite body of fluid and (ii) the case of two disks at $z = 0$ and $z = d$ with fluid between them. Suppose the fluid and disk(s) are initially in steady rigid rotation with angular velocity Ω and then, from $t = 0$, the oscillations, $q = ae^{int} + be^{-int}$, are imposed on the disk, $z = 0$, relative to the initial motion. The flow will be given by (2.2.8) subject to the following conditions.

For problem (i), we have

$$\left.\begin{array}{lll} \text{(a)} & q = ae^{int} + be^{-int} & \text{on } z = 0 \text{ for } t > 0\,, \\ \text{(b)} & q \to 0 & \text{as } z \to \infty \text{ for } t > 0\,, \\ \text{(c)} & q = 0 & \text{at } t = 0 \text{ for all } z. \end{array}\right\} \qquad (2.4.1)$$

For problem (ii), we have

$$\left.\begin{array}{lll} \text{(a)} & q = ae^{int} + be^{-int} & \text{on } z = 0 \text{ for } t > 0\,, \\ \text{(b)} & q = 0 & \text{on } z = d \text{ for } t > 0\,, \\ \text{(c)} & q = 0 & \text{at } t = 0 \text{ for all } z. \end{array}\right\} \qquad (2.4.2)$$

Applying the Laplace transform

$$\bar{q} = \int_0^\infty e^{-pt}\, q\, dt\,,$$

to equation (2.2.8) together with the conditions (2.4.1) and (2.4.2), we have

$$(p + 2\,\Omega\, i)\bar{q} = \nu\,\frac{\partial^2 \bar{q}}{\partial z^2}\,, \qquad (2.4.3)$$

which must be solved subject to the following conditions.

For problem (i), we have

$$\left.\begin{array}{lll} \text{(a)} & \bar{q} = \dfrac{a}{p-in} + \dfrac{b}{p+in} & \text{on } z = 0\,, \\ \text{(b)} & \bar{q} \to 0 & \text{as } z \to \infty\,. \end{array}\right\} \qquad (2.4.4)$$

For problem (ii), we have

$$\left.\begin{array}{llll}(a) & \bar{\bar{q}} = \dfrac{a}{p-in} + \dfrac{b}{p+in} & \text{on} & z = 0 , \\ (b) & \bar{\bar{q}} = 0 & \text{on} & z = d . \end{array}\right\} \qquad (2.4.5)$$

The solution for problem (i), namely (2.4.3) subject to (2.4.4), is

$$\bar{\bar{q}} = \left(\frac{a}{p-in} + \frac{b}{p+in}\right)e^{-m_1 z} , \qquad (2.4.6)$$

where

$$m_1 = \left[\frac{p+2\Omega i}{\nu}\right]^{1/2} . \qquad (2.4.7)$$

While for problem (ii), namely (2.4.3) subject to (2.4.5), we have

$$\bar{\bar{q}} = \left(\frac{a}{p-in} + \frac{b}{p+in}\right)\frac{\sinh\{m_1(d-z)\}}{\sinh\{m_1 d\}} . \qquad (2.4.8)$$

We now need to evaluate the inverse Laplace transform,

$$q = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{\bar{q}}\, e^{pt}\, dp , \qquad (2.4.9)$$

where γ is chosen to lie to the right of all the singularities.

We consider problem (ii) first since we shall show that, in the resonant case $n = 2\Omega$, (2.3.6) can be derived, without difficulty, when

$t \rightarrow \infty$. In the case of problem (i), however, (2.3.4) which corresponds to the resonant case $n = 2\Omega$ cannot be derived for $t \rightarrow \infty$; this more complex problem will be discussed later.

Problem (ii)

We now evaluate the integral (2.4.9) when $\bar{q}$ is given by (2.4.8). The singularities of (2.4.8) play an important role in the integration and are poles situated at

$$p = \pm in ,$$

$$p = -2\Omega i - \frac{k^2 \pi^2 \nu}{d^2} \quad \text{where} \quad k = 1,2,3,\ldots .$$

We can see from the structure of (2.4.8) that, although $p = -2\Omega i$ is a branch point of m_1, it is a zero of both the numerator and denominator and hence is not a singularity of (2.4.8). The inverse integral can be evaluated by transforming the path of integration into a closed contour and then applying the calculus of residues([8], page 75). Hence we have

$$q = ae^{int}\frac{\sinh\{\lambda_1(d-z)\}}{\sinh\{\lambda_1 d\}} + be^{-int}\frac{\sinh\{\lambda_2(d-z)\}}{\sinh\{\lambda_2 d\}}$$

$$+ \sum_{k=1}^{\infty}\frac{a(-1)^k 2\pi k\nu \sin[k\pi(1-z/d)]\exp\{-t(2\Omega i+k^2\pi^2\nu/d^2)\}}{k^2\pi^2\nu + id^2(2\Omega+n)}$$

$$+ \sum_{k=1}^{\infty}\frac{b(-1)^k 2\pi k\nu \sin[k\pi(1-z/d)]\exp\{-t(2\Omega i+k^2\pi^2\nu/d^2)\}}{k^2\pi^2\nu + id^2(2\Omega-n)} . \qquad (2.4.10)$$

For $t \to \infty$ this reduces to the oscillatory form

$$q = ae^{int}\frac{\sinh\{\lambda_1(d-z)\}}{\sinh\{\lambda_1 d\}} + be^{-int}\frac{\sinh\{\lambda_2(d-z)\}}{\sinh\{\lambda_2 d\}} .$$

Then, if we allow $n \to 2\Omega$, we retrieve (2.3.6). If $n = 2\Omega$ in (2.4.10), we have

$$q = ae^{2\Omega it}\frac{\sinh\{\lambda_0(d-z)\}}{\sinh\{\lambda_0 d\}} + be^{-2\Omega it}(1-\frac{z}{d})$$

$$+ \sum_{k=1}^{\infty}(-1)^k 2\pi k\nu \sin[k\pi(1-z/d)]\exp\{-t(2\Omega i + k^2\pi^2\nu/d^2)\}\left\{\frac{a}{k^2\pi^2\nu+4\Omega id^2} + \frac{b}{k^2\pi^2\nu}\right\}$$

(2.4.11)

Again, if we let $t \to \infty$, we regain (2.3.6).

Hence the solution (2.3.6) obtained is independent of the order in which the limits $t \to \infty$, $n \to 2\Omega$ are taken. We shall see that the same is not true for the unbounded case (problem (i)), to which we now turn.

Problem (i)

We now evaluate the integral (2.4.9) when $\bar{q}$ is given by (2.4.6). The singularities of (2.4.6) are poles at $p = \pm in$ and a branch point at $p = -2\Omega i$. By writing $s = p + 2\Omega i$ we transfer the poles to $s = i(2\Omega \pm n)$ and the branch point to $s = 0$. The integral (2.4.9) becomes

$$q = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{a \exp[-z(s/\nu)^{\frac{1}{2}}] \exp[(s-2\Omega i)t] ds}{s - i(n+2\Omega)}$$

$$+ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{b \exp[-z(s/\nu)^{\frac{1}{2}}] \exp[(s-2\Omega i)t] ds}{s - i(2\Omega-n)} \; . \qquad (2.4.12)$$

We rearrange (2.4.12) to give

$$q = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{a\exp\left[-z(s/\nu)^{\frac{1}{2}}\right]\exp\left[t(s-2\Omega i)\right]}{2\left[i(n+2\Omega)\right]^{\frac{1}{2}}} \times$$

$$\times\left\{\frac{1}{s^{\frac{1}{2}}-\left[i(n+2\Omega)\right]^{\frac{1}{2}}} - \frac{1}{s^{\frac{1}{2}}+\left[i(n+2\Omega)\right]^{\frac{1}{2}}}\right\} ds$$

$$+ \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{b\exp\left[-z(s/\nu)^{\frac{1}{2}}\right]\exp\left[t(s-2\Omega i)\right]}{2\left[i(2\Omega-n)\right]^{\frac{1}{2}}} \times$$

$$\times\left\{\frac{1}{s^{\frac{1}{2}}-\left[i(2\Omega-n)\right]^{\frac{1}{2}}} - \frac{1}{s^{\frac{1}{2}}+\left[i(2\Omega-n)\right]^{\frac{1}{2}}}\right\} ds \,. \qquad (2.4.13)$$

By using the inversion formula 29.3.88 on page 1026 of the Handbook of Mathematical Functions, A.M.S.55 [1], we have that

$$q = \tfrac{1}{2}ae^{int}\left\{\exp\left\{-z\lambda_1\right\}\operatorname{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} - (it(2\Omega+n))^{\frac{1}{2}}\right]\right.$$

$$\left. + \exp\left\{z\lambda_1\right\}\operatorname{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} + (it(2\Omega+n))^{\frac{1}{2}}\right]\right\}$$

$$+ \tfrac{1}{2}be^{-int}\left\{\exp\left\{-z\lambda_2\right\}\operatorname{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} - (it(2\Omega-n))^{\frac{1}{2}}\right]\right.$$

$$\left. + \exp\left\{z\lambda_2\right\}\operatorname{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} + (it(2\Omega-n))^{\frac{1}{2}}\right]\right\}, \qquad (2.4.14)$$

where $\operatorname{erfc} X = \frac{2}{\sqrt{\pi}}\int_X^{\infty} e^{-\phi^2}\,d\phi$.

This agrees with the result obtained by using the inversion formula 805.3 of Foster and Campbell [13].

If we let $t \to \infty$ keeping $n-2\Omega$ fixed we retrieve (2.3.2), the oscillatory solution. If we now allow $n \to 2\Omega$ we have (2.3.4) which does not satisfy the boundary conditions for large z, provided $b \neq 0$.

If, on the other hand, we let $n \to 2\Omega$ keeping t fixed, we have

$$q = \tfrac{1}{2}ae^{2\Omega it}\Big\{\exp\{-z\lambda_0\}\,\mathrm{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} - (4\Omega it)^{\frac{1}{2}}\right] + \exp\{z\lambda_0\}\,\mathrm{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} + (4\Omega it)^{\frac{1}{2}}\right]\Big\} + be^{-2\Omega it}\mathrm{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}}\right]. \qquad (2.4.15)$$

Now we note that

$$\mathrm{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} - (4\Omega it)^{\frac{1}{2}}\right] \longrightarrow 2$$

and $$\mathrm{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} + (4\Omega it)^{\frac{1}{2}}\right] \longrightarrow 0$$

as $t \to \infty$. Then, if we allow $t \to \infty$ in (2.4.15), we have

$$q = ae^{2\Omega it}\exp\{-z\lambda_0\} + be^{-2\Omega it}\mathrm{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}}\right]. \qquad (2.4.16)$$

This is not an oscillatory solution. Therefore the order in which the limits are taken becomes important when $z = O(\nu t)^{\frac{1}{2}}$ and the flow in the resonant case $n = 2\Omega$ is given by (2.4.15).

2.5 UNIFORM IMPULSIVE MOTION (n = O)

A classical problem in non-rotating systems ($\Omega = O$) is the time-dependent boundary layer (the classical Rayleigh layer) associated with an impulsive velocity, U_o , of the plane ([31], page 136). This layer has the structure

$$u = U_o \operatorname{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}}\right] . \tag{2.5.1}$$

In the present case ($\Omega \neq O$, $n = O$) we can show that the corresponding initial value problem leads not to an analogue of (2.5.1), but to a steady Ekman layer. This can be done as follows.

Let us consider the solution of the initial value problem when $n = O$. In this case the imposed velocity (2.2.1) at the boundary becomes

$$q = a + b = c ,$$

where c is some complex constant.

Then from (2.4.14), we have

$$q = \tfrac{1}{2}c\left\{ e^{-z\mu} \operatorname{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} - (2\Omega it)^{\frac{1}{2}}\right] + e^{z\mu} \operatorname{erfc}\left[\tfrac{1}{2}z(\nu t)^{-\frac{1}{2}} + (2\Omega it)^{\frac{1}{2}}\right]\right\}, \tag{2.5.2}$$

where $\mu = \left[\frac{2\Omega i}{\nu}\right]^{\frac{1}{2}}$. (2.5.3)

Letting $t \to \infty$, we have

$$q = ce^{-z\mu} \qquad (2.5.4)$$

This is a steady solution.

2.6 DISCUSSION

Formula (2.3.2) justifies the remarks made in the introduction concerning the existence of a depth of penetration $\left[\nu/|n-2\Omega|\right]^{\frac{1}{2}}$, and indicates the resonant phenomenon when $n \to 2\Omega$, in the sense that no oscillatory solution satisfying the condition as $z \to \infty$ is possible (for $b \neq 0$). Moreover formula (2.3.6) shows how, in the case $n \to 2\Omega$, a secondary boundary at $z = d$ can convert that part of the flow proportional to be^{int} into an oscillatory Couette flow with a shear uniform in z. Transient effects die out like $\exp\left\{-\pi^2\nu\, t/d^2\right\}$.

The result derivable from (2.4.11) agrees exactly with the result (2.3.6). Hence the flow between the disks is not affected by the impulse after sufficient time has elapsed for the damped oscillations to become negligible, and is identical with the flow that would be present if the oscillations had been present for all times.

Still more novel is the result (2.4.15) which indicates, in the case $n \to 2\Omega$, how the b part of the flow penetrates outwards from the disk in a standard diffusive manner, indeed as a Rayleigh layer with a time-dependent velocity magnitude. Greenspan [15] hinted at this type of behaviour. Formula (2.4.16) shows, in the case $n \to 2\Omega$ and for times large enough for the impulsive effects to be negligible that, added to the Rayleigh layer, is a Stokes layer, confined to the disk and proportional to $ae^{2\Omega it}$. We have, therefore, at the resonant frequency, that for large times the disturbance $be^{-2\Omega it}$ will be felt throughout a depth proportional to $(\nu t)^{\frac{1}{2}}$, the depth of penetration of the Rayleigh layer, from the disk, while the disturbance $ae^{2\Omega it}$ is only felt in a boundary layer of thickness $(\nu/4\Omega)^{\frac{1}{2}}$ on the disk.

An oscillatory solution for the flow of the form $\underline{u} = (u(z,t), v(z,t), 0)$ can only be found, in the resonant case, if either a second boundary is present at rest relative to the rotating co-ordinate system or if the Rayleigh layer is absent.

From (2.4.15) the Rayleigh layer will be absent if $b = 0$. The imposed oscillations on the disk would then have to be $q = ae^{2\Omega it}$, from which we have

$$u = |a| \cos\left\{2\Omega\,(t-t_o)\right\} ,$$
$$v = |a| \sin\left\{2\Omega\,(t-t_o)\right\} = |a| \cos\left\{2\Omega\,(t-t_o) - \pi/2\right\} ,$$

where t_o is given by $a = |a|\exp\left\{-2\Omega\, it_o\right\}$. Thus if v lags behind u by $\pi/2$ the Rayleigh layer is absent, and the problem can trivially be solved to give

$$q = ae^{+2\Omega it}e^{-\lambda_o z} . \qquad (2.6.1)$$

For the difficulty at the resonant frequency to be avoided the angular velocity of the velocity vector for the imposed oscillations must be 2Ω, i.e. twice the angular velocity of the basic rotation of the fluid/disk combination and in the same sense.

From the formula (2.5.4) we see that, when n = O, eventually a steady state is reached, that of a well-defined boundary layer on the disk; it is in fact an Ekman layer, which has a thickness of penetration of vorticity of order $(\nu/\Omega)^{\frac{1}{2}}$.

By putting n = O in (2.3.2) we retrieve (2.5.4). We could therefore have looked for a time-independent solution of (2.2.8) satisfying the boundary conditions (2.2.9) with n = O. This is basically the problem of finding the Ekman layer structure associated with steady motions of the disk.

We now consider a co-ordinate transform for the case when there is a well-defined boundary layer confined to the disk and an inviscid region outside. This occurs when the imposed oscillations of the disk are

$$q = ae^{int} + be^{-int} \qquad \text{when } n \neq 2\Omega ,$$
$$q = ae^{int} \qquad \text{when } n = 2\Omega . \qquad (2.6.2)$$

We transform the axes such that the disk is at rest relative to the co-ordinate system rotating with angular velocity, Ω , and the fluid in the inviscid region performs oscillations given by

$$u + iv = q^* = -(ae^{int} + be^{-int}) \qquad \text{when } n \neq 2\Omega ,$$
$$q^* = -ae^{int} \qquad \text{when } n = 2\Omega . \qquad (2.6.3)$$

From (2.2.2), (2.2.3) and (2.2.4) the governing boundary layer equations are (2.2.6). We write $u + iv = q^*$ and $\frac{\partial p^*}{\partial x} + i\frac{\partial p^*}{\partial y} = P$ and then (2.2.6) becomes

$$\frac{\partial q^*}{\partial t} + 2\Omega i q^* = -P + \nu \frac{\partial^2 q^*}{\partial z^2} , \qquad (2.6.4)$$

which we must solve subject to the conditions

$$\left.\begin{array}{ll} \text{a)} & q^* = 0 \text{ on } z = 0 , \\ \text{b)} & q^* \text{ is given by (2.6.3) as } z \to \infty . \end{array}\right\} \qquad (2.6.5)$$

For the flow in the inviscid region to be consistent with the equations of motion we require

$$\frac{\partial q^*}{\partial t} + 2\Omega i q^* = -P ,$$

where q^* is given by (2.6.3).

Hence we have

$$P = i(2\Omega+n)ae^{int} + i(2\Omega-n)be^{-int} . \qquad (2.6.6)$$

In particular at the resonant frequency, $n = 2\Omega$, $\frac{\partial p^*}{\partial y}$ lags behind $\frac{\partial p^*}{\partial x}$ by $\pi/2$; similarly v lags behind u by $\pi/2$ outside the boundary layer. Hence both the velocity vector of the oscillation and the pressure gradient vector have an angular velocity, 2Ω.

The solution of (2.6.4), with P given by (2.6.6), satisfying the conditions (2.6.5) is

$$q^* = a\left\{e^{-\lambda_1 z} - 1\right\}e^{int} + be^{-int}\left\{e^{-\lambda_2 z} - 1\right\} \qquad n \neq 2\Omega ,$$

$$q^* = a\left\{e^{-\lambda_0 z} - 1\right\}e^{2\Omega it} \qquad n = 2\Omega .$$

These solutions represent modified Stokes layers and could be obtained from (2.3.2) and (2.6.1) by writing

$$q^* = q - ae^{int} - be^{-int} \qquad n \neq 2\Omega ,$$

$$q^* = q - ae^{int} \qquad n = 2\Omega .$$

CHAPTER 3

TRANSIENT MOTIONS PRODUCED BY DISKS OSCILLATING TORSIONALLY ABOUT A STATE OF RIGID ROTATION

3.1 INTRODUCTION

In Chapter 2, the fluid motions produced in a system, which was initially in isothermal steady rigid rotation, were discussed, when, from some instant of time, non-torsional oscillations defined by (2.2.1) were imposed on a confining boundary. In this chapter we will examine the effect on these problems when the non-torsional oscillations are replaced by torsional oscillations about the axis of rotation.

Firstly, we consider the flow generated in a semi-infinite expanse of incompressible fluid bounded by an infinite disk, when both the fluid and the disk are, initially, in steady solid-body rotation and, then, torsional oscillations about the common axis of rotation of the fluid and the disk, are applied at the disk. The fluid at infinity is unaffected by boundary movements and therefore no radial pressure gradients can exist. We will show that, provided it is valid to linearize the equations of motion, this problem can be reduced to the semi-infinite problem discussed in Chapter 2.

However, when a second disk is introduced parallel to and rotating with the first disk, then radial pressure gradients can exist because a unique axis is defined about which the torsional oscillations are performed; a new problem arises. Thus the specification of horizontal velocity components proportional to r requires the pressure gradient to balance the internal flow when there are two boundaries present.

When two disks are present, the linearized initial-value problem for the case when arbitrary, small amplitude, torsional oscillations are imposed on both disks is solved for large Reynolds numbers, R, defined by (1.1.2), where 2L is the distance between the disks. The parameter, $\sigma R^{\frac{1}{2}}$, where $\Omega\sigma$ is the frequency of oscillation of a disk, emerges from the analysis and, in fact, characterises the behaviour of the fluid. On the assumption of large R, the times for the transient effects to decay and the final flow are discussed for the cases $\sigma R^{\frac{1}{2}} \ll 1$ (low frequency), $\sigma R^{\frac{1}{2}} = O(1)$ (intermediate frequency) and $\sigma R^{\frac{1}{2}} \gg 1$ (high frequency). A comparison with the results of Greenspan and Howard [17] is given.

3.2 EQUATIONS OF MOTION

We consider either (i) an infinite plane disk, $z = 0$, bounding a semi-infinite expanse ($z > 0$) of incompressible fluid or (ii) two infinite parallel disks, $z = \pm L$, with incompressible fluid between them. Both the fluid and the disk(s) rotate with constant angular velocity, Ω , about an axis normal to the disk(s) and, additionally, from $t = 0$, the disk(s) perform(s) arbitrary, small amplitude, torsional oscillations about the axis of rotation.

We take cartesian axes (x,y,z) such that the z-axis is parallel to the common axis of rotation of the fluid and the disk(s) and the x,y-axes rotate with the disk(s) and lie in the disk for problem (i) and parallel to and midway between the disks for problem (ii). The Navier-Stokes equation and the continuity equation in this rotating co-ordinate system and (2.2.2), (2.2.3) respectively.

Since the flow is axisymmetric and satisfies the continuity equation (2.2.3), we assume that the velocity components have the form ([31], p.356)

$$\left.\begin{aligned} u &= \frac{\partial f(z,t)}{\partial z}x - g(z,t)y , \\ v &= g(z,t)x + \frac{\partial f(z,t)}{\partial z}y , \\ w &= -2f(z,t). \end{aligned}\right\} \qquad (3.2.1)$$

These expressions, (3.2.1), correspond, in cylindrical polar co-ordinates (r, θ, z), to velocity components

$$\underline{u} = \left(r\frac{\partial f(z,t)}{\partial z}, \quad rg(z,t), \quad -2f(z,t)\right) .$$

We assume that the imposed torsional oscillations are given by, for problem (i),

$$\left.\begin{array}{lll} \text{(a)} \quad g = \omega\,(ae^{int} + be^{-int}) & \text{on } z = 0 & \text{for } t > 0 , \\ \text{(b)} \quad g \to 0 & \text{as } z \to \infty & \text{for } t > 0 , \\ \text{(c)} \quad g = 0 & \text{at } t = 0 \text{ for all } z , & \end{array}\right\} \quad (3.2.2)$$

and for problem (ii),

$$\left.\begin{array}{lll} \text{(a)} \quad g = \omega\,(ae^{in_1t} + be^{-in_1t}) & \text{on } z = L & \text{for } t > 0 , \\ \text{(b)} \quad g = \omega\,(ce^{in_2t} + de^{-in_2t}) & \text{on } z = -L & \text{for } t > 0 , \\ \text{(c)} \quad g = 0 & \text{at } t = 0 & \text{for all } z . \end{array}\right\} \quad (3.2.3)$$

In (3.2.2) and (3.2.3), n, n_1, n_2 are the frequencies, ω is a real constant and a,b,c,d are complex constants of order one, chosen so that g is real on both disks. Hence we require $a = \tilde{b}$ and $c = \tilde{d}$, where a tilde denotes the complex conjugate. In the special case when a,b,c,d are real, this condition reduces to a = b and c = d.

We assume $\omega/\Omega \ll 1$. It then appears that we may linearize the equations; and if the resulting equations and boundary conditions have a sensible solution, we may regard this linearization as valid, at least in a pragmatic sense. (This is the case in the Greenspan and

Howard [17] spin-up problem). If, on the other hand, for some boundary conditions no solution of the linear problem can be found it may be necessary to re-examine the assumption of linearization afresh. Here we merely draw attention to this possibility but do not pursue it further.

We take the curl of equation (2.2.2), omitting the non-linear terms, and then substitute (3.2.1) to give the equations

$$\frac{\partial g}{\partial t} + 2\Omega \frac{\partial f}{\partial z} = \nu \frac{\partial^2 g}{\partial z^2}, \tag{3.2.4}$$

$$\frac{\partial^3 f}{\partial z^2 \partial t} - 2\Omega \frac{\partial g}{\partial z} = \nu \frac{\partial^4 f}{\partial z^4}. \tag{3.2.5}$$

When we integrate (3.2.5), we find

$$\frac{\partial^2 f}{\partial z \partial t} - 2\Omega g = \nu \frac{\partial^3 f}{\partial z^3} + K(t), \tag{3.2.6}$$

where K(t) is some arbitrary function of time.

When only one disk is present, that is for problem (i), the required boundary conditions are, from (3.2.2),

$$\left.\begin{array}{lll}
\text{(a)} & f = \dfrac{\partial f}{\partial z} = 0, \quad g = \omega(ae^{int} + be^{-int}) & \text{on } z = 0 \text{ for } t > 0, \\
\text{(b)} & \dfrac{\partial f}{\partial z},\ g \to 0 & \text{as } z \to \infty \text{ for } t > 0, \\
\text{(c)} & f = \dfrac{\partial f}{\partial z} = g = 0 & \text{at } t = 0 \text{ for all } z.
\end{array}\right\} \tag{3.2.7}$$

From the boundary conditions, (3.2.7), and the differential equation, (3.2.6), we infer that the function $K(t)$ must be zero at infinity in order that the functions f, g decay decently to zero (exponentially) as z tends to infinity. This implies that $K(t) \equiv 0$ and hence that there is no radial pressure gradient. If we write

$$\frac{\partial f}{\partial z} + ig = q , \qquad i\omega a = a, \qquad i\omega b = b ,$$

the equations (3.2.4) and (3.2.6) reduce to the equation (2.2.8) and the boundary conditions (3.2.7) to (2.4.1). Hence the solution to this problem has already been considered in section 2.4. The oscillatory result derivable for $t \to \infty$, when $n \neq 2\Omega$, agrees with the solution of the linearized form of Benney's problem ([5], p.336). But for $n = 2\Omega$, we have a non-oscillatory solution when $t \to \infty$.

If a second disk is introduced, that is for problem (ii), then the appropriate boundary conditions together with the differential equation (3.2.6) no longer imply that $K(t) \equiv 0$. Hence radial pressure gradients can exist and important changes may be produced in the interior by the spin-up mechanisms [17]. Therefore it is necessary to solve the more complex problem, namely (3.2.4) and (3.2.5) subject to the appropriate boundary conditions. The remainder of this chapter is concerned with a discussion of this problem.

When the dimensionless variables (starred),

$$z = Lz^*, \quad t = \Omega^{-1} t^*, \quad g = \omega g^*, \quad f = L\omega f^* , \tag{3.2.8}$$

are introduced, the equations (3.2.4) and (3.2.5) become (upon dropping the asterisks)

$$\left.\begin{aligned} \frac{\partial g}{\partial t} + 2\frac{\partial f}{\partial z} &= R^{-1}\frac{\partial^2 g}{\partial z^2} , \\ \frac{\partial^3 f}{\partial t \partial z^2} - 2\frac{\partial g}{\partial z} &= R^{-1}\frac{\partial^4 f}{\partial z^4} , \end{aligned}\right\} \tag{3.2.9}$$

where R is the Reynolds number defined by (1.1.2). The equations (3.2.9) must now be solved subject to the conditions

$$\left.\begin{aligned} &(a) \quad f = \frac{\partial f}{\partial z} = 0 , \quad g = ae^{i\sigma_1 t} + be^{-i\sigma_1 t} \quad \text{on } z = 1 \quad \text{for } t > 0 , \\ &(b) \quad f = \frac{\partial f}{\partial z} = 0 , \quad g = ce^{i\sigma_2 t} + de^{-i\sigma_2 t} \quad \text{on } z = -1 \quad \text{for } t > 0 , \\ &(c) \quad f = \frac{\partial f}{\partial z} = g = 0 \qquad\qquad \text{at } t = 0 \quad \text{for all } z , \end{aligned}\right\} \tag{3.2.10}$$

where $\sigma_1 = \dfrac{n_1}{\Omega}$ and $\sigma_2 = \dfrac{n_2}{\Omega}$.

The equations (3.2.9) are identical to the equations (3.2) and (3.3) in Greenspan and Howard's paper [17] with V and ϕ replaced by g and f respectively. Here we solve these equations with more general boundary conditions (3.2.10) than those employed by Greenspan and Howard, which, in fact, correspond to the special case

$$\sigma_1 = \sigma_2 = 0, \qquad a + b = 1, \quad c + d = 1.$$

3.3 THE SOLUTION OF THE INITIAL-VALUE PROBLEM

In the following analysis we shall confine our attention to the case when the Reynolds number, R, is large.

Applying the Laplace transform

$$\bar{h}(z,p) = \int_0^\infty e^{-pt} h(z,t)dt ,$$

to the equations (3.2.9) and the boundary conditions (3.2.10), we have

$$\left.\begin{aligned} \left(R^{-1}\frac{\partial^2}{\partial z^2} - p\right)\bar{g} - 2\frac{\partial \bar{f}}{\partial z} &= 0 , \\ \left(R^{-1}\frac{\partial^2}{\partial z^2} - p\right)\frac{\partial^2 \bar{f}}{\partial z^2} + 2\frac{\partial \bar{g}}{\partial z} &= 0 , \end{aligned}\right\} \qquad (3.3.1)$$

which must be solved subject to the conditions

$$\left.\begin{aligned} \bar{f} &= \frac{\partial \bar{f}}{\partial z} = 0 , \quad \bar{g} = \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} \quad \text{on } z = 1 , \\ \bar{f} &= \frac{\partial \bar{f}}{\partial z} = 0 , \quad \bar{g} = \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \quad \text{on } z = -1 . \end{aligned}\right\} \qquad (3.3.2)$$

The solutions to equations (3.3.1) subject to the conditions (3.3.2) are

$$\bar{g} = \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} + \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \right\}$$

$$\times \left\{ \tfrac{1}{2} + \frac{Rp}{2\Delta}\left[m_1 E(m_2)(\cosh\{m_1 z\} - \cosh m_1) + m_2 E(m_1)(\cosh\{m_2 z\} - \cosh m_2)\right] \right\}$$

$$+ \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{c}{p-i\sigma_2} - \frac{d}{p+i\sigma_2} \right\} \left\{ \frac{\sinh\{m_1 z\}}{4\sinh m_1} + \frac{\sinh\{m_2 z\}}{4\sinh m_2} \right\} ,$$

(3.3.3)

$$\bar{f} = \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} + \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \right\}$$

$$\times \left\{ \frac{Rpi}{2\Delta}\left[E(m_2)(\sinh\{m_1 z\} - z \sinh m_1) - E(m_1)(\sinh\{m_2 z\} - z\sinh m_2)\right] \right\}$$

$$+ \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{c}{p-i\sigma_2} - \frac{d}{p+i\sigma_2} \right\}$$

$$\times \left\{ \frac{i(\cosh\{m_1 z\} - \cosh m_1)}{4m_1 \sinh m_1} - \frac{i(\cosh\{m_2 z\} - \cosh m_2)}{4m_2 \sinh m_2} \right\} , \qquad (3.3.4)$$

where $m_1 = R^{\frac{1}{2}}(p + 2i)^{\frac{1}{2}}$, $m_2 = R^{\frac{1}{2}}(p-2i)^{\frac{1}{2}}$,

$$E(m) = m\cosh m - \sinh m ,$$

$$\triangle = m_1^{\,3} E(m_2)\cosh m_1 + m_2^{\,3} E(m_1)\cosh m_2 .$$

This agrees with the notation employed by Greenspan and Howard [17].

The singularities of $\bar{g}$ and $\bar{f}$ play an important role in evaluating the inverse Laplace integral. We notice that although m_1 and m_2 have branch points at $p = -2i$ and $p = 2i$ respectively, these points are not singularities of the equations (3.3.3) and (3.3.4). The only singularities of the functions $\bar{f}$ and $\bar{g}$ are simple poles situated at

$$\left.\begin{aligned} p &= \pm i\,\sigma_1 , \\ p &= \pm i\,\sigma_2 , \\ p &= \pm 2i - \frac{k^2\pi^2}{R} , \quad k = 1,2,\ldots , \end{aligned}\right\} \quad (3.3.5)$$

and at the zeros of $\triangle$ other than $p = \pm 2i$. For large R, the required zeros of $\triangle$ are given, approximately, by

$$\left.\begin{aligned} p &= -R^{-\frac{1}{2}} , \\ p &= \pm 2i - \xi_n^{\,2} R^{-1} , \end{aligned}\right\} \quad (3.3.6)$$

where ξ_n are the positive roots of $\tan\xi = \xi$ $\quad (\xi_n \simeq (n+\frac{1}{2})\pi)$.

The inverse Laplace transform,

$$h(z,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{h}(z,t)e^{pt}dp ,$$

where γ is chosen to lie to the right of all the singularities, can now be evaluated by a residue calculation to give

$$\begin{aligned} g \;=\; & ae^{i\sigma_1 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\sigma_1 A + C\right]_{p = i\sigma_1} \\ & + be^{-i\sigma_1 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\sigma_1 A + C\right]_{p = -i\sigma_1} \\ & + ce^{i\sigma_2 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\sigma_2 A - C\right]_{p = i\sigma_2} \\ & + de^{-i\sigma_2 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\sigma_2 A - C\right]_{p = -i\sigma_2} \end{aligned}$$

$$- \left\{ \tfrac{1}{2}\exp\left\{-R^{-\frac{1}{2}}t\right\}\left[1 - \cos\left\{R^{\frac{1}{2}}(1-|z|)\right\}\exp\left\{-R^{\frac{1}{2}}(1-|z|)\right\}\right] + O(R^{-\frac{1}{2}})\right\}$$

$$\times\left\{\frac{a}{1 + i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1 - i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1 + i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1 - i\sigma_2 R^{\frac{1}{2}}}\right\}$$

$$+ \frac{R^{-1}}{4i}\sum_n\left\{\left(\frac{\cos\{\xi_n z\}}{\cos\xi_n} - 1\right)\exp(-\xi_n^2 R^{-1}t) + O(R^{-\frac{1}{2}})\right\}$$

$$\times\left\{ e^{2it}(2i - \xi_n^2R^{-1})\left[\frac{a}{2i-\xi_n^2R^{-1}-i\sigma_1} + \frac{b}{2i-\xi_n^2R^{-1}+i\sigma_1}\right.\right.$$

$$\left. + \frac{c}{2i-\xi_n^2R^{-1}-i\sigma_2} + \frac{d}{2i-\xi_n^2R^{-1}+i\sigma_2}\right] - e^{-2it}(2i+\xi_n^2R^{-1})$$

$$\left.\times\left[\frac{a}{2i+\xi_n^2R^{-1}+i\sigma_1} + \frac{b}{2i+\xi_n^2R^{-1}-i\sigma_1} + \frac{c}{2i+\xi_n^2R^{-1}+i\sigma_2} + \frac{d}{2i+\xi_n^2R^{-1}-i\sigma_2}\right]\right\}$$

$$+ \sum_{k=1}^{\infty} \frac{\sin(k\pi z).(-1)^k 2\pi k \exp\{-k^2\pi^2 tR^{-1}\}}{4R} \left\{ e^{-2it}\left[\frac{a}{2i+i\sigma_1+k^2\pi^2R^{-1}}\right.\right.$$

$$\left. + \frac{b}{2i-i\sigma_1+k^2\pi^2R^{-1}} - \frac{c}{2i+i\sigma_2+k^2\pi^2R^{-1}} - \frac{d}{2i-i\sigma_2+k^2\pi^2R^{-1}}\right]$$

$$+ e^{2it}\left[\frac{a}{k^2\pi^2R^{-1}-2i+i\sigma_1} + \frac{b}{k^2\pi^2R^{-1}-2i-i\sigma_1} - \frac{c}{k^2\pi^2R^{-1}-2i+i\sigma_2}\right.$$

$$\left.\left. - \frac{d}{k^2\pi^2R^{-1}-2i-i\sigma_2}\right]\right\}, \qquad (3.3.7)$$

$$f = ae^{i\sigma_1 t}\left[D-\tfrac{1}{2}R\sigma_1 B\right]_{p=i\sigma_1} + be^{-i\sigma_1 t}\left[D+\tfrac{1}{2}R\sigma_1 B\right]_{p=-i\sigma_1}$$

$$+ ce^{i\sigma_2 t}\left[-D-\tfrac{1}{2}R\sigma_2 B\right]_{p=i\sigma_2} + de^{-i\sigma_2 t}\left[-D+\tfrac{1}{2}R\sigma_2 B\right]_{p=-i\sigma_2}$$

$$+ \left\{ \frac{R^{-\frac{1}{2}}\exp\{-R^{-\frac{1}{2}}t\}}{4} \,\mathrm{Im}\left((1-i)\left[z - \frac{z}{|z|}\exp\{-R^{\frac{1}{2}}(1+i)(1-|z|)\}\right]\right) + O(R^{-1}) \right\}$$

$$\times \left\{ \frac{a}{1 + i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1 - i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1 + i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1 - i\sigma_2 R^{\frac{1}{2}}} \right\}$$

$$+ \frac{R^{-1}}{4i} \sum_n \left\{ \left(z - \frac{\sin\{\xi_n z\}}{\sin \xi_n} \right) \exp(-\xi_n^2 R^{-1} t) + O(R^{-\frac{1}{2}}) \right\}$$

$$\times \left\{ e^{2it}(2i - \xi_n^2 R^{-1}) \left[\frac{a}{2i - i\sigma_1 - \xi_n^2 R^{-1}} + \frac{b}{2i + i\sigma_1 - \xi_n^2 R^{-1}} + \frac{c}{2i - i\sigma_2 - \xi_n^2 R^{-1}} \right. \right.$$

$$\left. + \frac{d}{2i + i\sigma_2 - \xi_n^2 R^{-1}} \right] + e^{-2it}(2i + \xi_n^2 R^{-1}) \left[\frac{a}{2i + i\sigma_1 + \xi_n^2 R^{-1}} + \frac{b}{2i - i\sigma_1 + \xi_n^2 R^{-1}} \right.$$

$$\left. + \frac{c}{2i + i\sigma_2 + \xi_n^2 R^{-1}} + \frac{d}{2i - i\sigma_2 + \xi_n^2 R^{-1}} \right]$$

$$+ \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2i}{4R} (\cos(k\pi z) - (-1)^k) \exp\left\{ -k^2 \pi^2 t R^{-1} \right\} \times$$

$$\left\{ e^{-2it} \left[\frac{a}{2i + \sigma_1 i + k^2 \pi^2 R^{-1}} + \frac{b}{2i - \sigma_1 i + k^2 \pi^2 R^{-1}} - \frac{c}{2i + i\sigma_2 + k^2 \pi^2 R^{-1}} \right. \right.$$

$$\left. - \frac{d}{2i - i\sigma_2 + k^2 \pi^2 R^{-1}} \right] - e^{2it} \left[\frac{a}{k^2 \pi^2 R^{-1} + \sigma_1 i - 2i} + \frac{b}{k^2 \pi^2 R^{-1} - \sigma_1 i - 2i} \right.$$

$$\left. \left. - \frac{c}{k^2 \pi^2 R^{-1} + i\sigma_2 - 2i} - \frac{d}{k^2 \pi^2 R^{-1} - i\sigma_2 - 2i} \right] \right\} , \qquad (3.3.8)$$

where

$$A = \frac{m_1 E(m_2)\left[\cosh\{m_1 z\} - \cosh m_1\right] + m_2 E(m_1)\left[\cosh\{m_2 z\} - \cosh m_2\right]}{\Delta},$$

$$B = \frac{E(m_2)\left[\sinh\{m_1 z\} - z\sinh m_1\right] - E(m_1)\left[\sinh\{m_2 z\} - z\sinh m_2\right]}{\Delta},$$

$$C = \frac{\sinh\{m_1 z\}}{4\sinh m_1} + \frac{\sinh\{m_2 z\}}{4\sinh m_2},$$

$$D = \frac{i(\cosh\{m_1 z\} - \cosh m_1)}{4m_1 \sinh m_1} - \frac{i(\cosh\{m_2 z\} - \cosh m_2)}{4 m_2 \sinh m_2}.$$

For each type of expression in (3.3.7) and (3.3.8) we will only be concerned with the leading term and we will ignore all terms of order $R^{-\frac{1}{2}}$ compared with these. Therefore the terms ignored in the calculation of (3.3.7) and (3.3.8) are unimportant for our purposes. It should be remarked that for $\xi_n^2 \geq R$ the above expressions do not apply since the power series (3.3.6) for these zeros of Δ would not be valid.

It is an interesting fact that

$$\frac{\sin\{\xi_n z\}}{\sin \xi_n} \quad \text{and} \quad \frac{1}{\xi_n}\left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} - 1\right),$$

form an orthonormal set in the interval (-1,1), and this information is required to satisfy the initial conditions.

We will now restrict our attention to a study of the azimuthal velocity, g, since the radial velocity, $\frac{\partial f}{\partial z}$, can be found directly from g. If we now let $t \to \infty$ in (3.3.7) we have

$$g = ae^{i\sigma_1 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\,\sigma_1 A + C\right]_{p=i\sigma_1} + be^{-i\sigma_1 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\,\sigma_1 A + C\right]_{p=-i\sigma_1}$$

$$+ ce^{i\sigma_2 t}\left[\tfrac{1}{2} + \tfrac{1}{2}ARi\,\sigma_2 - C\right]_{p=i\sigma_2} + de^{-\sigma_2 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\,\sigma_2 A - C\right]_{p=-i\sigma_2} .$$

(3.3.9)

This is an oscillatory solution which is exact for all R and is identical to the solution that would be obtained by seeking an oscillatory solution

$$g(z,t) = g_1(z)e^{i\sigma_1 t} + g_2(z)e^{-i\sigma_1 t} + g_3(z)e^{i\sigma_2 t} + g_4(z)e^{-i\sigma_2 t} ,$$

of the equation

$$(R^{-1}\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t})^2 \frac{\partial g}{\partial z} + 4\frac{\partial g}{\partial z} = 0 ,$$

derived from (3.2.9) by eliminating f, subject to the boundary conditions

$$g_1 = a, \quad g_2 = b, \quad g_3 = 0, \quad g_4 = 0 \qquad \text{on } z = 1 ,$$

$$g_1 = 0, \quad g_2 = 0, \quad g_3 = c, \quad g_4 = d \qquad \text{on } z = -1 .$$

The properties of this oscillatory solution (3.3.9) depend upon the values taken for the dimensionless frequencies, σ_1 and σ_2. A discussion of this flow will be given, for large R, when the various cases for σ_1

and σ_2 are considered.

3.4 DISCUSSION

We are now concerned with the times required for the transient terms to vanish and also the final structure of the flow when we assume that R is large.

We suppose that the spin-up time

$$T_s = R^{\frac{1}{2}} \Omega^{-1} ,$$

and the periods of the imposed oscillations

$$T_1 = \frac{2\pi}{n_1} , \qquad T_2 = \frac{2\pi}{n_2} .$$

Then

$$\sigma_1 R^{\frac{1}{2}} = 2\pi \frac{T_s}{T_1} = 2\pi . \frac{\text{spin-up time}}{\text{oscillation period for disk, } z = 1} ,$$

$$\sigma_2 R^{\frac{1}{2}} = 2\pi \frac{T_s}{T_2} = 2\pi \frac{\text{spin-up time}}{\text{oscillation period for disk, } z = -1} .$$

We will now discuss the four cases

A Steady, $\sigma_1 = \sigma_2 = 0$,

B Low frequency oscillations, $T_1, T_2 \gg T_s$,

C Intermediate frequency oscillations, $T_1 = O(T_s)$, $T_2 = O(T_s)$,

D High frequency oscillations $T_1, T_2 \ll T_s$.

A <u>The steady case</u> ($\sigma_1 = \sigma_2 = 0$)

We first consider the case when $\sigma_1 = \sigma_2 = 0$. Then the angular velocity of the disks is impulsively charged, from $t = 0$, by a+b on $z = 1$ and c+d on $z = -1$. In this case (3.3.7) reduces to

$$g = (a+b)\left[\tfrac{1}{2} + \frac{\sinh\{(2iR)^{\frac{1}{2}}z\}}{4\sinh(2iR)^{\frac{1}{2}}} + \frac{\sinh\{(-2iR)^{\frac{1}{2}}z\}}{4\sinh(-2iR)^{\frac{1}{2}}}\right]$$

$$+ (c+d)\left[\tfrac{1}{2} - \frac{\sinh\{(2iR)^{\frac{1}{2}}z\}}{4\sinh(2iR)^{\frac{1}{2}}} - \frac{\sinh\{(-2iR)^{\frac{1}{2}}z\}}{4\sinh(-2iR)^{\frac{1}{2}}}\right]$$

$$- \tfrac{1}{2}\exp\{-R^{-\frac{1}{2}}t\}\left[1 - \cos\{R^{\frac{1}{2}}(1-|z|)\}\exp\{-R^{\frac{1}{2}}(1-|z|)\}\right](a+b+c+d)$$

$$+ \frac{R^{-1}\sin(2t)}{2}\sum_n\left(\frac{\cos\{\xi_n z\}}{\cos\xi_n} - 1\right)\exp(-\xi_n^2 R^{-1}t).\left[a+b+c+d\right]$$

$$+ \sum_{k=1}^{\infty}\frac{\sin(k\pi z)(-1)^k 2\pi k}{4R}\exp\{-k^2\pi^2R^{-1}t\}(-\sin\{2t\})\left[(a+b)-(c+d)\right], \qquad (3.4.1)$$

where terms of order $R^{-\frac{1}{2}}$ compared with those written down have been omitted. By arguments identical to those employed by Greenspan and Howard [17, p.388], we find that the series

$$\frac{R^{-1}\sin\{2t\}}{2}\sum_n\left(\frac{\cos\{\xi_n z\}}{\cos\xi_n} - 1\right)\exp(-\xi_n^2R^{-1}t), \qquad (3.4.2)$$

apart from the first few terms which are of order R^{-1}, is bounded by

$$\frac{K}{2\pi} \cdot \frac{1}{t} , \tag{3.4.3}$$

where $K \simeq 1$. This bound, (3.4.3), for the amplitude of these inertial oscillations is important because, although the individual oscillations have small amplitudes, $O(R^{-1})$, we also require that collectively they should have small amplitude, for the oscillations to become negligible.

Similarly, we see that for the inertial oscillations of the form

$$\sum_{k=1}^{\infty} \frac{\sin(k\pi z)(-1)^k 2\pi k \exp\{-k^2\pi^2 R^{-1} t\}}{4R} \{-\sin(2t)\} \tag{3.4.4}$$

the amplitude of the individual oscillations is small, $O(R^{-1})$, while the collective amplitude is bounded by

$$\frac{K'}{4\pi t} , \tag{3.4.5}$$

where $K' \leq 1$.

<u>$a + b + c + d \neq 0$</u>

When $a + b + c + d \neq 0$ the term

$$-\tfrac{1}{2} \exp\{-R^{-\frac{1}{2}} t\}\left[1 - \cos\{R^{\frac{1}{2}}(1-|z|)\} \exp\{-R^{\frac{1}{2}}(1-|z|)\}\right](a+b+c+d), \tag{3.4.6}$$

requires a dimensionless time of order $R^{\frac{1}{2}}$ to decay. For this time scale,

$t = R^{\frac{1}{2}}\tau$, the inertial oscillations of the form (3.4.2) and (3.4.4) make a contribution no larger than $R^{-\frac{1}{2}}$ and are therefore negligible compared with terms of order 1. Hence all the transient effects will become negligible within a time of order $R^{\frac{1}{2}}\Omega^{-1} = T_s$, the Greenspan and Howard [17] spin-up time, to leave the final state

$$g = (a+b)\left[\frac{1}{2} + \frac{\sinh\{(2iR)^{\frac{1}{2}}z\}}{4\sinh(2iR)^{\frac{1}{2}}} + \frac{\sinh\{(-2iR)^{\frac{1}{2}}z\}}{4\sinh(-2iR)^{\frac{1}{2}}}\right]$$

$$+ (c+d)\left[\frac{1}{2} - \frac{\sinh\{(2iR)^{\frac{1}{2}}z\}}{4\sinh(2iR)^{\frac{1}{2}}} - \frac{\sinh\{(-2iR)^{\frac{1}{2}}z\}}{4\sinh(-2iR)^{\frac{1}{2}}}\right]. \qquad (3.4.7)$$

The expression (3.4.7) represents Ekman boundary layers on the disks at $z = \pm 1$, having a depth of penetration of order $(\nu/\Omega)^{\frac{1}{2}}$ (or $R^{\frac{1}{2}}$ in dimensionless quantities) and an interior flow which is a solid-body rotation having a constant angular velocity,

$$g = \tfrac{1}{2}(a + b + c + d).$$

For the special case when $a + b = c + d$, that is when the angular velocity of the two disks is impulsively changed by the same amount, then the final state reduces to solid-body rotation with

$$g = a + b ,$$

and no Ekman layers are present on the disks. If $a + b = c + d = 1$, then equation (3.4.1) reduces to equation (3.10) in Greenspan and Howard's paper [17]. Similarly the expression for f would correspond to (3.9) in [17]. It should be noticed that trivial errors occur in equations (3.9) and (3.10) in Greenspan and Howard's paper [17]. These should, in fact, read

$$\phi = \tfrac{1}{2}R^{-\frac{1}{2}}\exp(-R^{-\frac{1}{2}}t)\ \mathrm{Im}\left[(1-i)\left\{z - \frac{z}{|z|}\exp\left(-R^{\frac{1}{2}}(1+i)(1-|z|)\right)\right\}\right]$$
$$+\ R^{-1}\cos 2t \sum_n{}'\left(z - \frac{\sin\{\xi_n z\}}{\sin \xi_n}\right)\exp(-\xi_n^2 R^{-1}t)\ ,$$

$$V = 1 - \exp(-R^{-\frac{1}{2}}t)\left[1 - \cos\left\{R^{\frac{1}{2}}(1-|z|)\right\}\exp\left\{-R^{\frac{1}{2}}(1-|z|)\right\}\right]$$
$$+\ R^{-1}\sin 2t \sum_n{}'\left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} - 1\right)\exp(-\xi_n^2 R^{-1}t)\ .$$

The manner in which the final state of solid-body rotation is established has been described in Chapter 1.

<u>$a + b + c + d = 0$</u>

When $a + b + c + d = 0$ the two disks are given, initially, equal and opposite changes in angular velocity. Then, from the equation (3.4.1), the only transient effects to influence the motion are the inertial oscillations of the form (3.4.4). By formula (3.4.5), these oscillations behave in a manner which decays at the worst like t^{-1}, for $t > 0$.

If we suppose that the time taken for these inertial oscillations to decay is $t = R^{\alpha}\tau$, where $\alpha > 0$, then, by (3.4.5), the collective amplitude is no greater than $O(R^{-\alpha})$. When we take $\alpha = \frac{1}{2}$ which is equivalent to the spin-up time, T_s, these oscillations become negligible. Therefore we may assume that $\alpha < \frac{1}{2}$. Hence the inertial oscillations will be very important, initially, but will decay in a finite time which is much less than the spin-up time. The final state of the fluid consists of Ekman layers on the disks with no interior flow.

B. <u>Low frequency oscillations,</u> T_1, $T_2 \gg T_s$

We have low frequency oscillations imposed on both disks when $\sigma_1 R^{\frac{1}{2}}$, $\sigma_2 R^{\frac{1}{2}} \ll 1$, which is equivalent to T_1, $T_2 \gg T_s$. Then equation (3.3.7) reduces to

$$g = ae^{i\sigma_1 t}\left[\tfrac{1}{2} + \tfrac{1}{2}\, Ri\sigma_1 A + C\right]_{p=i\sigma_1} + be^{-i\sigma_1 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\sigma_1 A + C\right]_{p=-i\sigma_1}$$

$$+ ce^{i\sigma_2 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\sigma_2 A - C\right]_{p=i\sigma_2} + de^{-i\sigma_2 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\sigma_2 A - C\right]_{p=-i\sigma_2}$$

$$- \tfrac{1}{2}\exp\left\{-R^{-\frac{1}{2}}t\right\}\left[1 - \cos\left\{R^{\frac{1}{2}}(1-|z|)\right\}\exp\left\{-R^{\frac{1}{2}}(1-|z|)\right\}\right]\Big[(a+b+c+d)$$

$$+ i\sigma_1 R^{\frac{1}{2}}(b-a) + i\sigma_2 R^{\frac{1}{2}}(d-c)\Big]$$

$$+ \frac{R^{-1}}{2} \sum_n \left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} \right) \exp(-\xi_n^2 R^{-1} t) \Big[(a+b+c+d)\sin(2t)$$

$$+ \sigma_1 (a-b) \frac{\cos(2t)}{2i} + \sigma_2 (c-d) \frac{\cos(2t)}{2i} \Big]$$

$$+ \sum_{k=1}^{\infty} \frac{\sin(k\pi z)(-1)^k 2\pi k \exp\{-k^2\pi^2 t R^{-1}\}}{4R} \Big[\{(c+d) - (a+b)\} \sin(2t)$$

$$+ \{\sigma_1 (b-a) + \sigma_2 (c-d)\} \frac{\cos(2t)}{2i} \Big] , \qquad (3.4.8)$$

where we have ignored terms of order $R^{-\frac{1}{2}}$ compared with those retained. The assumption of low frequency oscillations implies that all transient terms involving σ_1 or σ_2 can be neglected.

<u>$a + b + c + d \neq 0$</u>

When $a + b + c + d \neq 0$ we have, by the same arguments as we have employed previously for the steady case with $a + b + c + d \neq 0$, that the transient terms decay within a time of order $R^{\frac{1}{2}} \Omega^{-1}$, the spin-up time, to leave the final state

$$g = (ae^{i\sigma_1 t} + be^{-i\sigma_1 t}) \left\{ \tfrac{1}{2} + \frac{\sinh\{(2iR)^{\frac{1}{2}} z\}}{4 \sinh(2iR)^{\frac{1}{2}}} + \frac{\sinh\{(-2iR)^{\frac{1}{2}} z\}}{4 \sinh(-2iR)^{\frac{1}{2}}} \right\}$$

$$+ (ce^{i\sigma_2 t} + de^{-i\sigma_2 t}) \left\{ \tfrac{1}{2} - \frac{\sinh\{(2iR)^{\frac{1}{2}} z\}}{4 \sinh(2iR)^{\frac{1}{2}}} - \frac{\sinh\{(-2iR)^{\frac{1}{2}} z\}}{4 \sinh(-2iR)^{\frac{1}{2}}} \right\}$$

$$+ O(R^{\frac{1}{2}} \sigma_1) + O(R^{\frac{1}{2}} \sigma_2) . \qquad (3.4.9)$$

The assumption of low frequency oscillations implies that terms of order $R^{\frac{1}{2}}\sigma_1$ or $R^{\frac{1}{2}}\sigma_2$ are negligible to a first approximation. Then (3.4.9) represents boundary layers on the disks, $z = \pm 1$, of thickness $(\nu/\Omega)^{\frac{1}{2}}$ (or $R^{\frac{1}{2}}$ in dimensionless quantities). These are, in fact, Ekman layers having time-dependent amplitudes. Added to these boundary layers is the interior flow

$$g = \tfrac{1}{2}(ae^{i\sigma_1 t} + be^{-i\sigma_1 t} + ce^{i\sigma_2 t} + de^{-i\sigma_2 t}) .$$

For the special case when identical oscillations are imposed on the two disks, the final state is a rigid rotation with

$$g = ae^{i\sigma_1 t} + be^{-i\sigma_1 t} .$$

<u>$a + b + c + d = 0$</u>

When $a + b + c + d = 0$ we have, by the arguments used in the steady case with $a + b + c + d = 0$, that the time required for the transient effects to decay is a finite time which is very much less than the spin-up time. The final state is again given by (3.4.9). For the special case when the imposed oscillations have the same frequency and amplitude but have a phase difference of π then, the final state reduces to Ekman layers on the disks with no interior flow.

Hence the cases of steady and low frequency oscillations are very closely connected and depend basically upon the spin-up mechanisms except when $a + b + c + d = 0$, the special case, when only the inertial oscillations affect the motion.

C. <u>Intermediate frequency oscillations</u>, $T_1 = O(T_s)$, $T_2 = O(T_s)$

For intermediate frequency oscillations, we have that terms involving $R^{\frac{1}{2}}\sigma_1$ and $R^{\frac{1}{2}}\sigma_2$ become comparable with one and must be retained while terms involving σ_1 or σ_2 can be ignored, since $\sigma_1 \ll 1$ and $\sigma_2 \ll 1$. Then (3.3.7) reduces to

$$g = ae^{i\sigma_1 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\sigma_1 A + C\right]_{p=i\sigma_1} + be^{-i\sigma_1 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\sigma_1 A + C\right]_{p=-i\sigma_1}$$

$$+ ce^{i\sigma_2 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\sigma_2 A - C\right]_{p=i\sigma_2} + de^{-i\sigma_2 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\sigma_2 A - C\right]_{p=-i\sigma_2}$$

$$- \tfrac{1}{2}\exp\left\{-R^{-\frac{1}{2}}t\right\}\left[1 - \cos\left\{R^{\frac{1}{2}}(1-|z|)\right\}\exp\left\{-R^{\frac{1}{2}}(1-|z|)\right\}\right]$$

$$\times\left\{\frac{a}{1 + i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1 - i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1 + i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1 - i\sigma_2 R^{\frac{1}{2}}}\right\}$$

$$+ \frac{R^{-1}}{2}\sum_n\left(\frac{\cos\{\xi_n z\}}{\cos\xi_n} - 1\right)\exp(-\xi_n^2 R^{-1}t)(a + b + c + d)\sin(2t)$$

$$+ \sum_{k=1}^{\infty} \frac{\sin(k\pi z)(-1)^k 2\pi k \exp\{-k^2\pi^2 R^{-1} t\}}{4R} \Big[(c+d) - (a+b)\Big] \sin(2t) \,, \tag{3.4.10}$$

where terms of order $R^{-\frac{1}{2}}$ compared with those written down, have been ignored. Again, by the same arguments as we applied to the steady case, when

$$\frac{a}{1 + i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1 - i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1 + i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1 - i\sigma_2 R^{\frac{1}{2}}} \neq 0 \,,$$

the transient terms decay in a time of order $R^{\frac{1}{2}}\Omega^{-1}$, the spin-up time, while for

$$\frac{a}{1 + i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1 - i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1 + i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1 - i\sigma_2 R^{\frac{1}{2}}} = 0 \,,$$

a finite time, which is always much less than the spin-up time, is required. The final state is given by

$$g = ae^{i\sigma_1 t}\Bigg[\frac{1}{2} + \frac{\sinh\{(2iR)^{\frac{1}{2}}z\}}{4\sinh(2iR)^{\frac{1}{2}}} + \frac{\sinh\{(-2iR)^{\frac{1}{2}}z\}}{4\sinh(-2iR)^{\frac{1}{2}}}$$

$$+ iR^{\frac{1}{2}}\sigma_1\Big\{\cosh(-2iR)^{\frac{1}{2}}\Big[\cosh\{(2iR)^{\frac{1}{2}}z\} - \cosh(2iR)^{\frac{1}{2}}\Big]$$

$$+ \cosh(2iR)^{\frac{1}{2}}\Big[\cosh\{(-2iR)^{\frac{1}{2}}z\} - \cosh(-2iR)^{\frac{1}{2}}\Big]\Big\}\Bigg]$$

$$\div \Big\{4i\sigma_1 R^{\frac{1}{2}}\cosh(2iR)^{\frac{1}{2}}\cosh(-2iR)^{\frac{1}{2}} - (2i)^{3/2}\sinh(-2iR)^{\frac{1}{2}}\cosh(2iR)^{\frac{1}{2}}$$

$$- (-2i)^{3/2}\cosh(-2iR)^{\frac{1}{2}}\sinh(2iR)^{\frac{1}{2}} \Big\}$$

$$+ \ 3 \text{ similar terms.} \qquad (3.4.11)$$

This represents boundary layers on both disks and an interior flow which is independent of z.

D. <u>High frequency oscillations</u>, $T_1, T_2 \ll T_s$

For high frequency oscillations we require T_1, $T_2 \ll T_s$ and therefore $R^{\frac{1}{2}}\sigma_1$, $R^{\frac{1}{2}}\sigma_2 \gg 1$. We will first consider the contribution from the term

$$-\left\{\tfrac{1}{2}\exp\left\{-R^{-\frac{1}{2}}t\right\}\left[1 - \cos\left\{R^{\frac{1}{2}}(1-|z|)\right\}\exp\left\{-R^{\frac{1}{2}}(1-|z|)\right\}\right] + O(R^{-\frac{1}{2}})\right\}$$

$$\times \left\{\frac{a}{1 + i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1 - i\sigma_1 R^{\frac{1}{2}}}\right\}, \qquad (3.4.12)$$

to the expression (3.3.7) for g, on the assumption of high frequency oscillations. When $a = \tilde{b}$, we have (3.4.12) is $O(R^{-\frac{1}{2}}\sigma_1^{-1})$ except in the special case when a and b are real when (3.4.12) is $O(R^{-1}\sigma_1^{-2})$. Similar contributions arise from the terms with a, b, σ_1 replaced by c, d, σ_2 respectively in (3.4.12). Hence the amplitude of these terms is always either $O(R^{-\frac{1}{2}}\sigma_1^{-1})$ or $O(R^{-\frac{1}{2}}\sigma_2^{-1})$ or less, and therefore they can be ignored when compared with terms of order one.

In the problem discussed in chapter 2, a resonance phenomenon was found when the frequency of oscillation was twice the angular velocity of rotation. Then, for large times, for oscillations at all frequencies except the resonant frequency, the flow was confined to boundary layers on the disks while, for oscillations at the resonant frequency, when two boundaries were present, the motion penetrated throughout the whole fluid. We will show that this type of behaviour also occurs in this problem, but first we will consider the general non-resonant case.

<u>The non-resonant case</u> $\sigma_1, \sigma_2 \neq 2$

We will first consider the case when neither σ_1 nor σ_2 is equal to two, the resonant frequency. In this case (3.3.7) becomes

$$g = ae^{i\sigma_1 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\sigma_1 A + C\right]_{p=i\sigma_1} + be^{-i\sigma_1 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\sigma_1 A + C\right]_{p=-i\sigma_1}$$

$$+ ce^{i\sigma_2 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\sigma_2 A - C\right]_{p=i\sigma_2} + de^{-i\sigma_2 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\sigma_2 A - C\right]_{p=-i\sigma_2}$$

$$+ \frac{R^{-1}}{4i}\sum_n\left(\frac{\cos\{\xi_n z\}}{\cos\xi_n} - 1\right)\exp(-\xi_n^2 R^{-1} t)\left\{2ie^{2it}\left[\frac{a}{(2-\sigma_1)i} + \frac{b}{(2+\sigma_1)i}\right.\right.$$

$$\left. + \frac{c}{(2-\sigma_2)i} + \frac{d}{(2+\sigma_2)i}\right] - 2ie^{-2it}\left[\frac{a}{(2+\sigma_1)i} + \frac{b}{(2-\sigma_1)i}\right.$$

$$\left.\left. + \frac{c}{(2+\sigma_2)i} + \frac{d}{(2-\sigma_2)i}\right]\right\}$$

$$+ \sum_{k=1}^{\infty} \frac{\sin(k\pi z)(-1)^k 2\pi k \exp\{-k^2\pi^2 R^{-1} t\}}{4R} \left\{ e^{-2it}\left[\frac{a}{(2+\sigma_1)i}\right.\right.$$

$$\left. + \frac{b}{(2-\sigma_1)i} - \frac{c}{(2+\sigma_2)i} - \frac{d}{(2-\sigma_2)i}\right] + e^{2it}\left[\frac{a}{(\sigma_1-2)i} - \frac{b}{(\sigma_1+2)i}\right.$$

$$\left.\left. + \frac{c}{(2-\sigma_2)i} + \frac{d}{(\sigma_2+2)i}\right]\right\}, \qquad (3.4.13)$$

where terms of the type (3.4.12) and also terms of order $R^{-\frac{1}{2}}$ compared with those written down have been neglected. The infinite series in (3.4.13) must be retained because, although each term has an amplitude of order R^{-1}, collectively these terms are very important at small times. Again we apply the argument used for the steady case when $a + b + c + d = 0$ to both infinite series and we find that the time required for the transient terms to decay is always very much less than the spin-up time, T_s. The flow after the transient effects have decayed is given by

$$g = \frac{ae^{i\sigma_1 t}}{2}\left\{\frac{\sinh\{R^{\frac{1}{2}}(i(\sigma_1+2))^{\frac{1}{2}}(1+z)\}}{\sinh\{2R^{\frac{1}{2}}(i(\sigma_1+2))^{\frac{1}{2}}\}} + \frac{\sinh\{R^{\frac{1}{2}}(i(\sigma_1-2))^{\frac{1}{2}}(1+z)\}}{\sinh\{2R^{\frac{1}{2}}(i(\sigma_1-2))^{\frac{1}{2}}\}}\right\}$$

$$+ \frac{be^{-i\sigma_1 t}}{2}\left\{\frac{\sinh\{R^{\frac{1}{2}}(i(2-\sigma_1))^{\frac{1}{2}}(1+z)\}}{\sinh\{2R^{\frac{1}{2}}(i(2-\sigma_1))^{\frac{1}{2}}\}} + \frac{\sinh\{R^{\frac{1}{2}}(-i(\sigma_1+2))^{\frac{1}{2}}(1+z)\}}{\sinh\{2R^{\frac{1}{2}}(-i(\sigma_1+2))^{\frac{1}{2}}\}}\right\}$$

$$+ \frac{ce^{i\sigma_2 t}}{2}\left\{\frac{\sinh\{R^{\frac{1}{2}}(i(\sigma_2+2))^{\frac{1}{2}}(1-z)\}}{\sinh\{2R^{\frac{1}{2}}(i(\sigma_2+2))^{\frac{1}{2}}\}} + \frac{\sinh\{R^{\frac{1}{2}}(i(\sigma_2-2))^{\frac{1}{2}}(1-z)\}}{\sinh\{2R^{\frac{1}{2}}(i(\sigma_2-2))^{\frac{1}{2}}\}}\right\}$$

$$+ \frac{de^{-i\sigma_2 t}}{2}\left\{\frac{\sinh\{R^{\frac{1}{2}}(i(2-\sigma_2))^{\frac{1}{2}}(1-z)\}}{\sinh\{2R^{\frac{1}{2}}(i(2-\sigma_2))^{\frac{1}{2}}\}} + \frac{\sinh\{R^{\frac{1}{2}}(-i(\sigma_2+2))^{\frac{1}{2}}(1-z)\}}{\sinh\{2R^{\frac{1}{2}}(-i(\sigma_2+2))^{\frac{1}{2}}\}}\right\}$$

$$+ \quad O(R^{-\frac{1}{2}}) \; . \tag{3.4.14}$$

We are assuming that R is large and therefore terms of $O(R^{-\frac{1}{2}})$ can be ignored. Then (3.4.14) represents boundary layers on the disks having depths of penetration of vorticity of order

$$\left(\frac{\nu}{|n_1 - 2\Omega|}\right)^{\frac{1}{2}} \quad \text{on} \quad z = 1 \, ,$$

$$\left(\frac{\nu}{|n_2 - 2\Omega|}\right)^{\frac{1}{2}} \quad \text{on} \quad z = -1 \, .$$

These well-defined boundary layers are, in fact, modified Stokes layers.

The resonant case, σ_1 or $\sigma_2 = 2$

We will assume first, without loss of generality, that for the resonant case $\sigma_1 = 2$ and $\sigma_2 \neq 2$. Then, when terms of the type (3.4.12) have been ignored (3.3.7) becomes

$$g = a e^{i\sigma_1 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\,\sigma_1 A + C\right]_{p=i\sigma_1} + b e^{-i\sigma_1 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\,\sigma_1 A + C\right]_{p=-i\sigma_1}$$

$$+ \; c e^{i\sigma_2 t}\left[\tfrac{1}{2} + \tfrac{1}{2}Ri\,\sigma_2 A - C\right]_{p=i\sigma_2} + d e^{-i\sigma_2 t}\left[\tfrac{1}{2} - \tfrac{1}{2}Ri\,\sigma_2 A - C\right]_{p=-i\sigma_2}$$

$$+ \; \frac{R^{-1}}{4i} \sum_n \left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} - 1\right)\exp(-\xi_n^2 R^{-1} t)\left\{2i\, e^{2it}\left[-\frac{aR}{\xi_n^2} + O(1)\right]\right.$$

$$\left. - \; 2i\, e^{-2it}\left[\frac{bR}{\xi_n^2} + O(1)\right]\right\}$$

$$+ \sum_{k=1}^{\infty} \frac{\sin(k\pi z)(-1)^k 2\pi k \exp(-k^2\pi^2 R^{-1} t)}{4R} \left\{ e^{-2it}\left[\frac{bR}{k^2\pi^2} + O(1)\right]\right.$$

$$\left. + e^{2it}\left[\frac{aR}{k^2\pi^2} + O(1)\right]\right\} , \qquad (3.4.15)$$

where terms of order $R^{-\frac{1}{2}}$ compared with those retained have been ignored. We will first consider the term

$$- \tfrac{1}{2}(ae^{2it} + be^{-2it}) \sum_n \left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} - 1\right) \exp(-\xi_n^2 R^{-1} t) \frac{1}{\xi_n^2} . \qquad (3.4.16)$$

Then, by the arguments used by Greenspan and Howard [17], we have that except for the first few terms the series (3.4.16) is bounded by

$$\tfrac{1}{2} \left| ae^{2it} + be^{-2it} \right| \sum_n \exp(-\xi_n^2 R^{-1} t). \frac{1}{\xi_n}$$

$$\leq \tfrac{1}{2} \left| ae^{2it} + be^{-2it} \right| \sum_n \exp(-\xi_n^2 R^{-1} t) ,$$

since $\xi_n \simeq (n + \frac{1}{2})\pi$. Similarly the series

$$\sum_{k=1}^{\infty} \frac{\sin(k\pi z)(-1)^k \exp(-k^2\pi^2 R^{-1} t)(be^{-2it} + ae^{2it})}{2\pi k} ,$$

except for the first few terms is bounded by

$$\sum_{k=1}^{\infty} \left| ae^{2it} + be^{-2it} \right| \tfrac{1}{2} \exp(-k^2\pi^2R^{-1}t) \; .$$

From these bounds we see that the transient effects certainly die out in a time of order $R\Omega^{-1} = T_d$, the viscous diffusion time. The final flow is given by

$$g = ae^{2it}\left\{\frac{\sinh\{(4iR)^{\frac{1}{2}}(1+z)\}}{2\sinh\{2(4iR)^{\frac{1}{2}}\}} + \frac{3z^2-1+2z}{8}\right\}$$

$$+ \; be^{-2it}\left\{\frac{\sinh\{(-4iR)^{\frac{1}{2}}(1+z)\}}{2\sinh\{2(-4iR)^{\frac{1}{2}}\}} + \frac{3z^2-1+2z}{8}\right\}$$

$$+ \; \frac{ce^{i\sigma_2 t}}{2}\left\{\frac{\sinh\{R^{\frac{1}{2}}(i(\sigma_2+2))^{\frac{1}{2}}(1-z)\}}{\sinh\{2R^{\frac{1}{2}}(i(\sigma_2+2))^{\frac{1}{2}}\}} + \frac{\sinh\{R^{\frac{1}{2}}(i(\sigma_2-2))^{\frac{1}{2}}(1-z)\}}{\sinh\{2R^{\frac{1}{2}}(i(\sigma_2-2))^{\frac{1}{2}}\}}\right\}$$

$$+ \; \frac{de^{-i\sigma_2 t}}{2}\left\{\frac{\sinh\{R^{\frac{1}{2}}(i(2-\sigma_2))^{\frac{1}{2}}(1-z)\}}{\sinh\{2R^{\frac{1}{2}}(i(2-\sigma_2))^{\frac{1}{2}}\}} + \frac{\sinh\{R^{\frac{1}{2}}(-i(\sigma_2+2))^{\frac{1}{2}}(1-z)\}}{\sinh\{2R^{\frac{1}{2}}(-i(\sigma_2+2))^{\frac{1}{2}}\}}\right\},$$

$$+ \; O(R^{-\frac{1}{2}}) \; . \qquad (3.4.17)$$

Since R is large, we may neglect terms of order $R^{-\frac{1}{2}}$. Hence (3.4.17) represents modified Stokes layers on $z = \pm 1$, which have depths of penetration of order

$$\left(\frac{\nu}{4\Omega}\right)^{\frac{1}{2}}, \quad \left(\frac{\nu}{|n_2 - 2\Omega|}\right)^{\frac{1}{2}} \text{ respectively ,}$$

and also a flow between the plates which is quadratic in z and has a time dependent amplitude.

When both disks oscillate at the resonant frequency, $\sigma_1 = \sigma_2 = 2$, the time required for the transient effects to decay is again T_d, the viscous diffusion time. The final state is given by

$$
\begin{aligned}
g = {} & ae^{2it}\left\{\frac{\sinh\{(4iR)^{\frac{1}{2}}(1+z)\}}{2\sinh\{2(4iR)^{\frac{1}{2}}\}} + \frac{3z^2-1+2z}{8}\right\} \\
& + be^{-2it}\left\{\frac{\sinh\{(-4iR)^{\frac{1}{2}}(1+z)\}}{2\sinh\{2(-4iR)^{\frac{1}{2}}\}} + \frac{3z^2-1+2z}{8}\right\} \\
& + ce^{2it}\left\{\frac{\sinh\{(4iR)^{\frac{1}{2}}(1-z)\}}{2\sinh\{2(4iR)^{\frac{1}{2}}\}} + \frac{3z^2-1-2z}{8}\right\} \\
& + de^{-2it}\left\{\frac{\sinh\{(-4iR)^{\frac{1}{2}}(1-z)\}}{2\sinh\{2(-4iR)^{\frac{1}{2}}\}} + \frac{3z^2-1-2z}{8}\right\} \\
& + O(R^{-\frac{1}{2}}). \qquad (3.4.18)
\end{aligned}
$$

For this case modified Stokes layers are formed on the disks, each having a depth of penetration of vorticity of order $(\nu/4\Omega)^{\frac{1}{2}}$. The interior flow is again quadratic in z with a time dependent amplitude. When both disks oscillate in the same manner that is when $a = c$, $b = d$, the terms which are linear in z cancel. When the imposed oscillations on the two disks have the same amplitude and a phase difference of π, that is $a = -c$, $b = -d$, then the interior flow reduces to a shear flow uniform in z with a time-dependent amplitude.

Interior flow

We now consider the structure of the interior flow which persists after the transient effects have vanished when both disks oscillate at the resonant frequency. From (3.4.18) the interior azimuthal velocity is given by

$$r\,g_I = r(ae^{2it} + be^{-2it})\left(\frac{3z^2 - 1 + 2z}{8}\right) + r(ce^{2it} + de^{-2it})\left(\frac{3z^2 - 1 - 2z}{8}\right) + O(R^{-\frac{1}{2}}). \qquad (3.4.19)$$

Therefore, from (3.2.9), the interior radial velocity is given by

$$r\frac{\partial f_I}{\partial z} = -ir(ae^{2it} - be^{-2it})\left(\frac{3z^2 - 1 + 2z}{8}\right) - ir(ce^{2it} - de^{-2it})\left(\frac{3z^2 - 1 - 2z}{8}\right) + O(R^{-\frac{1}{2}}), \qquad (3.4.20)$$

and the interior vertical velocity is given by

$$-2f_I = \frac{i}{4}\left[(ae^{2it} - be^{-2it})(z^3 + z^2 - z - 1) + (ce^{2it} - de^{-2it})(z^3 - z^2 - z + 1)\right] + O(R^{-\frac{1}{2}}). \qquad (3.4.21)$$

In the following analysis we will neglect terms of order $R^{-\frac{1}{2}}$ compared with one.

The particle paths for the interior motion are given by

$$\frac{dr}{r\,\frac{\partial f_I}{\partial z}} = \frac{r d\phi}{r g_I} = \frac{dz}{-2 f_I} = dt \,. \qquad (3.4.22)$$

In particular from (3.4.22), (3.4.20), (3.4.21), we find that

$$\frac{dr}{r\left[(3z^2 - 1 + 2z)|a|\sin\{2(t-t_a)\} + (3z^2 - 1 - 2z)|c|\sin\{2(t-t_c)\}\right]}$$

$$= \frac{dz}{-2\left[(z^3 + z^2 - z - 1)|a|\sin\{2(t-t_a)\} + (z^3 - z^2 - z + 1)|c|\sin\{2(t-t_c)\}\right]} \,, \qquad (3.4.23)$$

where t_a and t_c are the phases of a and c respectively. When

$$\left| t_a - t_c \right| = \tfrac{1}{2} m \pi, \quad m = 0, 1, 2, \ldots, \qquad (3.4.24)$$

the time dependence in (3.4.23) cancels and we can integrate this equation to give

$$r^2 (z^2 - 1)(z[\alpha + \beta] + [\alpha - \beta]) = \text{constant}, \qquad (3.4.25)$$

where $\alpha = |a|$ and $\beta = \pm |c|$. The particle paths lie on the steady surfaces (3.4.25) which are shown in Fig.3.1.

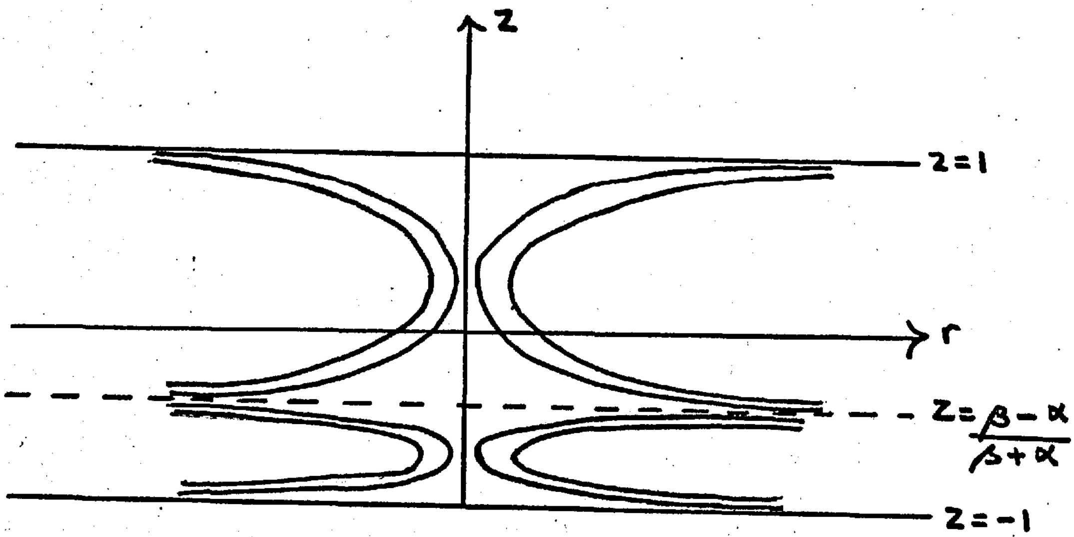


FIG.3.1:— Sketch of the steady surfaces,

$$r^2(z^2-1)(z\,[\alpha+\beta]+[\alpha-\beta]) = \text{constant}.$$

Also, from (3.4.22), when t_a and t_c are related by (3.4.24) we have

$$\frac{dz}{dt} = -\tfrac{1}{2}\sin\{2(t-t_a)\}\,.\,(z^2-1)(z\,[\alpha+\beta]+[\alpha-\beta]), \qquad (3.4.26)$$

which we can integrate by parts to give

$$\log\left\{\left(\frac{z-1}{z_o-1}\right)^{\frac{1}{\alpha}}\left(\frac{z+1}{z_o+1}\right)^{\frac{1}{\beta}}\left(\frac{[\alpha+\beta]\,z_o+[\alpha-\beta]}{[\alpha+\beta]\,z+[\alpha-\beta]}\right)^{\frac{\alpha+\beta}{\alpha\beta}}\right\}$$

$$= 2\sin t\,\sin\left[2t_a-t\right], \qquad (3.4.27)$$

where z_o is the position of a given particle at $t = 0$. This represents an oscillation which takes the value $z = z_o$ at

$$\left.\begin{aligned} t &= 0,\ \pm\pi,\ \pm 2\pi, \ldots, \\ t &= 2t_a,\ 2t_a \pm \pi,\ 2t_a \pm 2\pi, \ldots, \end{aligned}\right\} \quad (3.4.28)$$

and which is trapped either between the planes $z = 1$ and $z = \frac{\beta-\alpha}{\alpha+\beta}$ or $z = -1$ and $z = \frac{\beta-\alpha}{\alpha+\beta}$, depending upon the value of z_o.

We now consider the above results when we impose certain restrictions on α and β.

<u>(i) $\alpha = \beta$</u>

In this case the imposed oscillations are such that

$$|a| = |c| \qquad \text{and} \qquad |t_a - t_c| = 0,\ \pi,\ 2\pi,\ \ldots .$$

Then, the steady surfaces, (3.4.25), reduce to

$$r^2 z(z^2-1) = \text{constant}, \qquad (3.4.29)$$

which are shown in Fig.3.2 and (3.4.27) become

$$\log\left\{\frac{(z^2-1)z_o^{\,2}}{(z_o^2-1)z^2}\right\} = 2\alpha \sin t \sin\left[2t_a - t\right]. \qquad (3.4.30)$$

The expression (3.4.30) represents an oscillation trapped between $z = 0$ and $z = 1$ if z_o is positive or $z = 0$ and $z = -1$ if z_o is negative

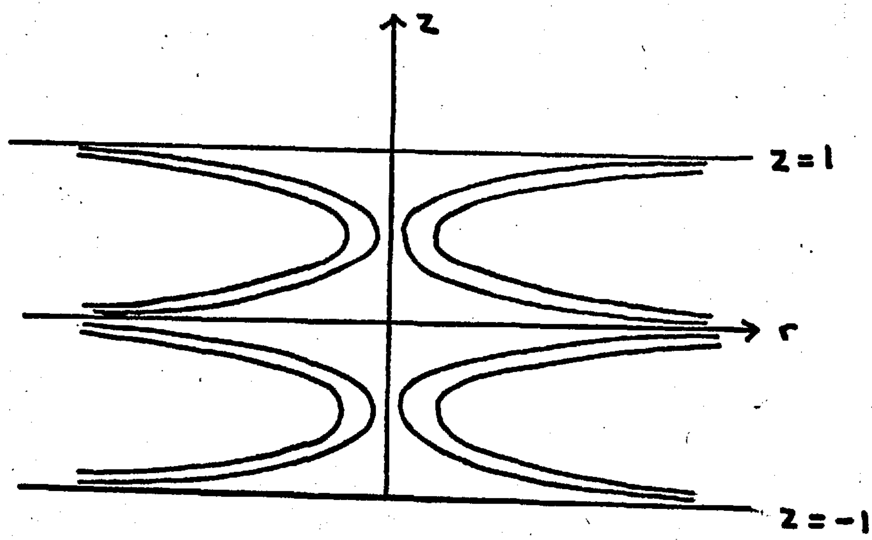


FIG.3.2: Sketch of the steady surfaces, $r^2z(z^2-1)$ = constant.

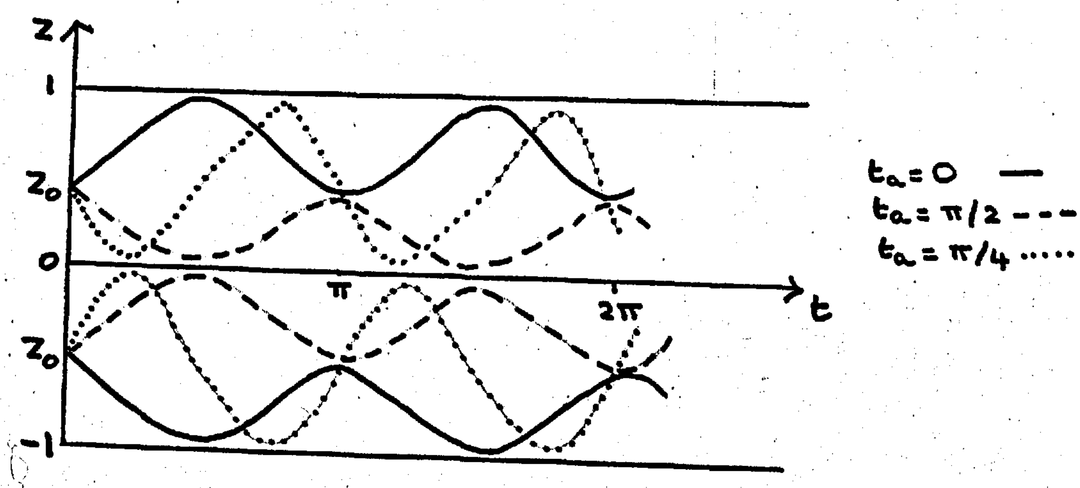


FIG. 3.3: Sketch of the oscillations (3.4.30) for $t_a = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

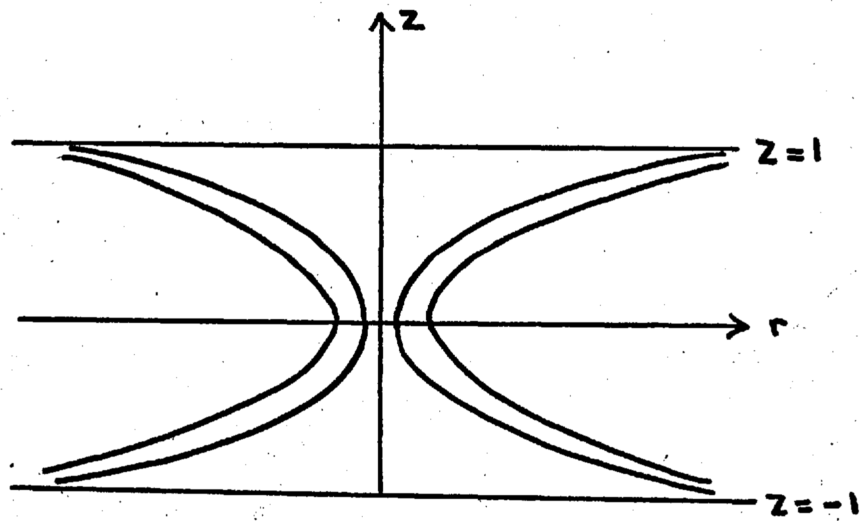


FIG.3.4: Sketch of the steady surfaces $r^2(z^2-1)$ = constant.

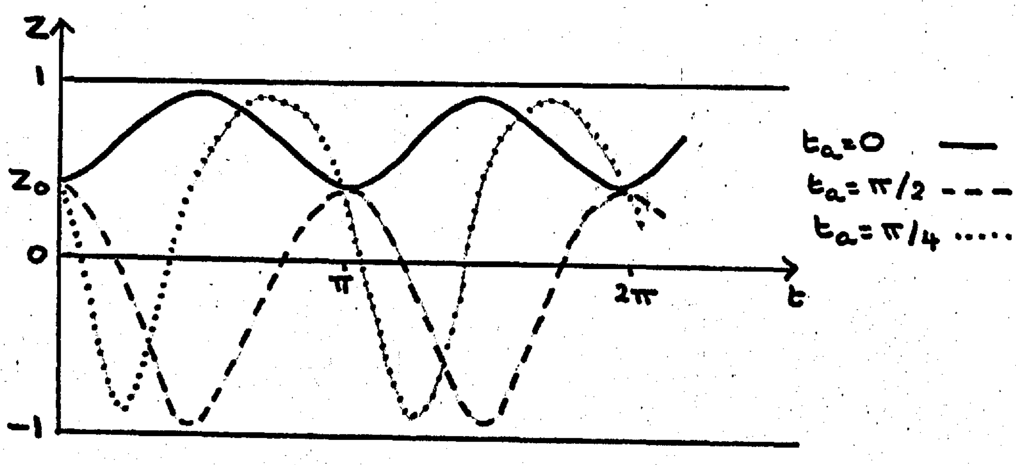


FIG.3.5: Sketch of the oscillations (3.4.32) for $t_a = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

which takes the value $z = z_o$ at the times (3.4.28). These oscillations are shown for various values of t_a in Fig.3.3.

(ii) $\alpha = -\beta$

The case $\alpha = -\beta$ corresponds to imposed oscillations which satisfy

$$|a| = |c| \quad \text{and} \quad |t_a - t_c| = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ \ldots .$$

Then the steady surfaces, (3.4.25), reduce to

$$r^2(z^2-1) = \text{constant}, \qquad (3.4.31)$$

which are shown in Fig.3.4. From (3.4.27) we have

$$\log\left\{\left(\frac{z-1}{z+1}\right)\left(\frac{z_o+1}{z_o-1}\right)\right\} = 2\alpha \sin t \sin\left[2t_a - t\right], \qquad (3.4.32)$$

which represents oscillations trapped between $z = 1$ and $z = -1$ which have the value $z = z_o$ at the times (3.4.28). These oscillations are shown for various values of t_a in Fig.3.5.

(iii) either $\alpha = 0$ or $\beta = 0$

We may suppose without loss of generality that $\beta = 0$, which corresponds to $c = d = 0$. By this assumption we derive the interior flow when either the disk at $z = -1$ is oscillating at a frequency other than the resonant frequency or is at rest. The steady surfaces, (3.4.25), reduce to

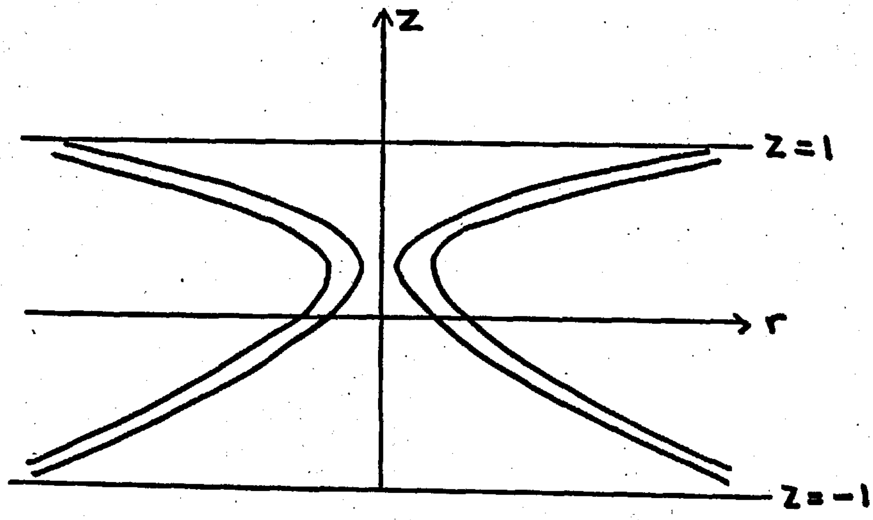


FIG.3.6: Sketch of the steady surfaces $r^2(z-1)(z+1)^2$ = constant.

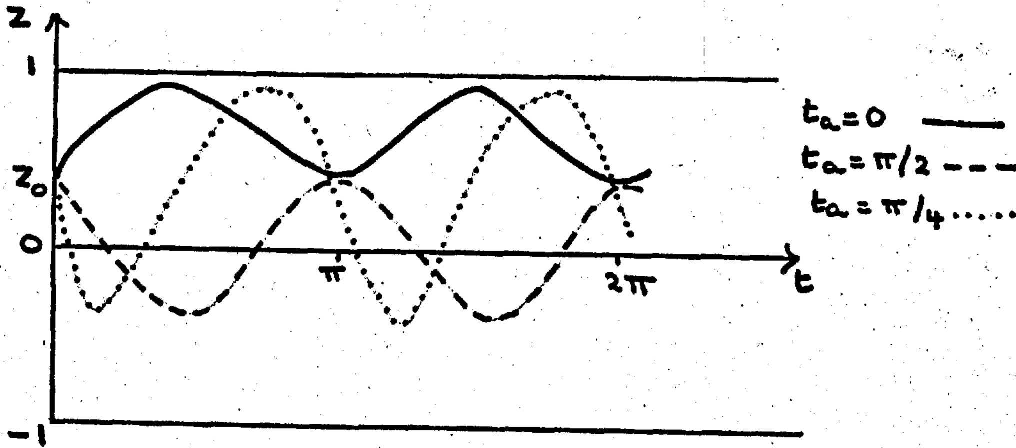


FIG.3.7: Sketch of the oscillations (3.4.34) for $t_a = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

$$r^2(z-1)(z+1)^2 = \text{constant} , \qquad (3.4.33)$$

which are shown in Fig.3.6. For this case, the oscillations are obtained by integrating (3.4.26) to give

$$\frac{2(z_o-z)}{(1+z)(1+z_o)} + \log\left\{\left(\frac{z-1}{z+1}\right)\left(\frac{z_o+1}{z_o-1}\right)\right\} = 2\alpha \sin t \sin\left[2t_a-t\right] , \qquad (3.4.34)$$

which are shown in Fig.3.7 for various values of t_a.

When a and c have phases not connected by (3.4.24), no simple discussion of this type of behaviour can be given.

E. Mixed frequencies

The cases which occur when one disk is oscillating at a high frequency and the other at a low frequency can be found by taking a combination of the previous results.

The previous results concerning the times required for transient effects to decay are combined in Table 3.1.

3.5 AN APPROXIMATE SOLUTION

We will now, by making various approximations, develop a representation of the solution, g, which gives a more qualitative description of the transient (time-dependent) state. From this representation we will confirm the remarks made in section 3.4 concerning the times required

TYPE		CONDITIONS	DECAY TIMES
STEADY	$\sigma_1 = \sigma_2 = 0$	$a+b+c+d \neq 0$	T_s
		$a+b+c+d = 0$	$\ll T_s$
LOW FREQUENCY	$\sigma_1 R^{\frac{1}{2}} \ll 1$	$a+b+c+d \neq 0$	T_s
	$\sigma_2 R^{\frac{1}{2}} \ll 1$	$a+b+c+d = 0$	$\ll T_s$
INTERMEDIATE FREQUENCY	$\sigma_1 R^{\frac{1}{2}} = O(1)$	$\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}} \neq 0$	T_s
	$\sigma_2 R^{\frac{1}{2}} = O(1)$	$\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}} = 0$	$\ll T_s$
HIGH FREQUENCY	$\sigma_1 R^{\frac{1}{2}} \gg 1$	Non-resonant ($\sigma_1, \sigma_2 \neq 2$)	$\ll T_s$
	$\sigma_2 R^{\frac{1}{2}} \gg 1$	Resonant (σ_1 or $\sigma_2 = 2$)	T_d

TABLE 3.1

for the transient effects to decay.

In section 3.4, we found that, except for oscillations at the resonant frequency (σ_1 and/or $\sigma_2 = 2$) which occurs in the high frequency range, the simple poles at

$$p = \pm i\sigma_1, \quad \pm i\sigma_2, \quad -R^{-\frac{1}{2}} \quad , \tag{3.5.1}$$

were the important singularities. The residue contribution from the simple poles

$$\begin{aligned} p &= \pm 2i - \frac{k^2\pi^2}{R}, \qquad k = 1,2,3,\dots, \\ p &= \pm 2i - \frac{\zeta_n^2}{R}, \qquad n = 1,2,3,\dots, \end{aligned} \tag{3.5.2}$$

although important initially, decayed at the worst like t^{-1}, for $t > 0$, and has a magnitude no greater than $R^{-\frac{1}{2}}$ for a time of order $R^{\frac{1}{2}}$. Therefore it seems reasonable to use Greenspan and Howard's [17] approximations to obtain an inversion integral which can be readily evaluated, retains the character and location of the important singularities, (3.5.1), and replaces the remaining poles, (3.5.2), by branch points at $p = \pm 2i$.

For the resonant case, however, the poles, (3.5.2), become very important and it would not be valid to replace them by branch points at

$p = \pm 2i$. Hence we must treat this special case by a different method.

Non-resonant oscillations

We consider oscillations at all frequencies except the resonant frequency (σ_1, $\sigma_2 \neq 2$) and we will use the three approximations employed by Greenspan and Howard [17] to obtain a representation for the solution, g, as a linear combination of an interior solution and a boundary layer solution, which is valid for large R and for $t < R$.

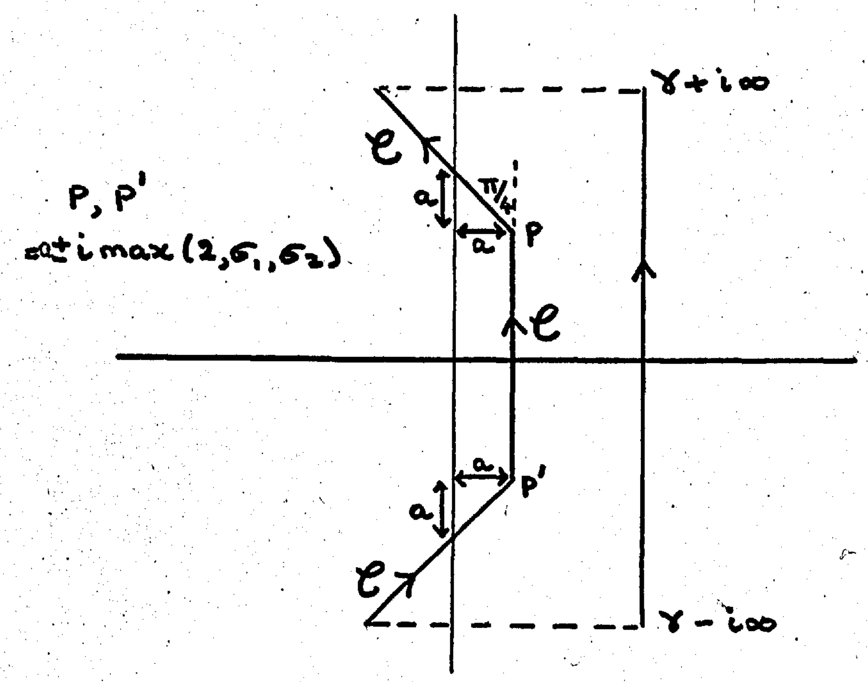


FIG.3.8: Transformed contour for the Laplace inversion integral.

We can transform the Laplace inversion contour $(\gamma - i\infty, \gamma + i\infty)$ into the contour $\mathscr{C}$ shown in Fig.3.8, where $a = R^{-1+\delta}$ for some $\delta > 0$ and P, P' are situated at $a \pm i \max(2, \sigma_1, \sigma_2)$, because all the singularities of $\bar{g}$, (3.3.3), lie to the left of $\mathscr{C}$ and the contribution from the broken lines becomes negligible as they approach infinity.

Then along $\mathscr{C}$, we have that, for $t < O(R^{1-\delta})$, $|e^{pt}|$ is bounded and tends to zero exponentially as p tends to infinity for any fixed $t > 0$, and that m_1 and m_2 have large positive real parts [17, p.389].

First approximation

In order to separate the terms in (3.3.3) responsible for the boundary layer flow from those responsible for the interior flow we make the approximations

$$\cosh(m_i z) = \tfrac{1}{2}\exp(m_i|z|), \quad \sinh(m_i z) = \tfrac{1}{2}\frac{z}{|z|}\exp(m_i|z|),$$

$$E(m_i) = \tfrac{1}{2}(m_i - 1)\exp m_i, \qquad i = 1,2, \tag{3.5.3}$$

which are valid, with exponentially small errors, along $\mathscr{C}$. When these approximations, (3.5.3), are substituted in (3.3.3) we find

$$\bar{g} = \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} + \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \right\} \frac{i(m_2 - m_1)}{D(p)}$$

$$+\left\{\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}+\frac{c}{p-i\sigma_2}+\frac{d}{p+i\sigma_2}\right\}\left\{\frac{pm_1(m_2-1)\exp\{m_1(|z|-1)\}}{2D(p)}\right.$$

$$\left.+\ \frac{pm_2(m_1-1)\exp\{m_2(|z|-1)\}}{2D(p)}\right\}$$

$$+\ \left\{\frac{a}{p-i\sigma_1}+\frac{b}{p+i\sigma_1}-\frac{c}{p-i\sigma_2}-\frac{d}{p+i\sigma_2}\right\}\frac{z}{4|z|}\ (\exp\{m_1(|z|-1)\}$$

$$+\exp\{m_2(|z|-1)\}\ ), \quad (3.5.4)$$

where

$$D(p) \;=\; R^{-1}\left[m_1^3(m_2-1)+m_2^3(m_1-1)\right]$$

$$=\; \left[m_1(m_2-1)+m_2(m_1-1)\right]p \;+\; 2i(m_2-m_1) \quad , \qquad (3.5.5)$$

is an approximation valid along $\mathcal{C}$ which satisfies the required boundary conditions at $z = \pm 1$.

In (3.5.4), the simple poles, (3.5.1), are preserved while the poles, (3.5.2), are replaced by branch cuts extending to the left from $p = \pm 2i$ and simple poles at

$$p = \pm 2i + \frac{1}{R}\ . \qquad (3.5.6)$$

These new poles, (3.5.6), give a residue contribution $O(R^{-1}\exp(tR^{-1}))$ which is small for $t < R$ but becomes important for $t > R$. The terms neglected would, in fact, nullify this growth. Hence here we must restrict our attention to $t < R$.

The first term in (3.5.4) represents the interior solution, $\bar{g}_I$, which is independent of the z-co-ordinate, while the remaining two terms in (3.5.4) represent the boundary layer solution, $\bar{g}_B$.

Second approximation

The choice of the transformed inversion contour, $\mathcal{C}$, is such that, along $\mathcal{C}$, m_1 and m_2 always have large positive real parts and therefore we may ignore the terms m_1, m_2 compared with the term $m_1 m_2$ in (3.5.4) and (3.5.5) to give

$$\bar{g}_I = \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} + \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \right\} \frac{i(m_2-m_1)}{D^*(p)}, \tag{3.5.7}$$

$$\bar{g}_B = \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} + \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \right\} \frac{pm_1m_2}{2D^*(p)} \Big[\exp\{m_1(|z|-1)\}$$
$$+ \exp\{m_2(|z|-1)\} \Big]$$
$$+ \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{c}{p-i\sigma_2} - \frac{d}{p+i\sigma_2} \right\} \frac{z}{4|z|} \Big[\exp\{m_1(|z|-1)\}$$
$$+ \exp\{m_2(|z|-1)\} \Big], \tag{3.5.8}$$

where

$$D^*(p) = p2m_1m_2 + 2i(m_2-m_1). \tag{3.5.9}$$

The expressions (3.5.8) and (3.5.9) satisfy the required boundary conditions at $z = \pm 1$, retain the simple poles, (3.5.1), and branch points, $p = \pm 2i$, and move the extra simple poles (3.5.6) to

$$p = \pm 2i + \frac{1}{4R} . \qquad (3.5.10)$$

The Laplace inversion integrals for the functions (3.5.7) and (3.5.8) cannot be evaluated in terms of known functions and therefore we must introduce another approximation.

Third approximation

The extra poles, (3.5.10), arise from the previous approximations and are not a property of the original transform function, (3.3.3). For $t < R$, they produce a negligible contribution to the transient motion of the fluid and therefore we will assume that

$$\begin{aligned} D^*(p) &= 2m_1m_2p + 2i(m_2-m_1) \\ &= 2m_1m_2(p + R^{-\frac{1}{2}})(1+B(p)) , \end{aligned} \qquad (3.5.11)$$

where

$$\begin{aligned} B(p) &= \frac{i(m_2-m_1) - R^{-\frac{1}{2}}m_1m_2}{m_1m_2(p + R^{-\frac{1}{2}})} \\ &= \frac{i\left[(p-2i)^{\frac{1}{2}} - (p+2i)^{\frac{1}{2}}\right] - (p^2+4)^{\frac{1}{2}}}{R^{\frac{1}{2}}(p^2+4)^{\frac{1}{2}}(p+R^{-\frac{1}{2}})} . \end{aligned} \qquad (3.5.12)$$

The complex function B(p) is small in magnitude along the entire contour $\mathscr{C}$ and therefore we may expand $(1 + B(p))^{-1}$, which appears in the integrand of (3.5.7) and (3.5.8) in powers of B(p) and retain only the first term in the resulting series. Then the expressions (3.5.7) and (3.5.8) become

$$\bar{g}_I = \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} + \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \right\} \frac{i\left[(p+2i)^{-\frac{1}{2}} - (p-2i)^{-\frac{1}{2}}\right]}{2R^{\frac{1}{2}}(p + R^{-\frac{1}{2}})}, \tag{3.5.13}$$

$$\bar{g}_B = \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} + \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \right\} \frac{p\left[\exp\left\{m_1(|z|-1)\right\} - \exp\left\{m_2(|z|-1)\right\}\right]}{4(p+R^{-\frac{1}{2}})}$$

$$+ \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{c}{p-i\sigma_2} - \frac{d}{p+i\sigma_2} \right\} \frac{z}{4|z|} \left[\exp\left\{m_1(|z|-1)\right\} + \exp\left\{m_2(|z|-1)\right\}\right]. \tag{3.5.14}$$

Hence by the above procedure we have retained the simple poles, (3.5.1), and have replaced all the poles in the neighbourhood of $p = \pm 2i$ by branch points at $p = \pm 2i$. It should also be noticed that the small and large p behaviour agree with that found for the original function. Therefore we may conclude that the expressions (3.5.13) and (3.5.14) are valid approximations to the original transform function along $\mathscr{C}$, for $t < R$. The above method can be expected to give a reasonably correct description for all t since we have previously shown, in the discussion, that for $t > R$ the final state is controlled by the poles $p = \pm i\sigma_1, \pm i\sigma_2$, although no rigorous justification will be given here.

The inverse Laplace transform

$$g = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{pt}\,\bar{g}\,dp\ , \qquad (3.5.15)$$

can now be evaluated in terms of known functions.

The time dependent interior motion, g_I, is given by evaluating (3.5.15) when $\bar{g}$ is given by (3.5.13). Then, from Foster and Campbell [13] No.546 , we have

$$g_I = \frac{ai\,e^{i\sigma_1 t}}{2(1+i\sigma_1 R^{\frac{1}{2}})}\left\{\frac{\operatorname{erf}\left[(2i+i\sigma_1)^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(2i+i\sigma_1)^{\frac{1}{2}}} - \frac{\operatorname{erf}\left[(i\sigma_1-2i)^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(i\sigma_1-2i)^{\frac{1}{2}}}\right\}$$

$$+ \frac{bi\,e^{-i\sigma_1 t}}{2(1-i\sigma_1 R^{\frac{1}{2}})}\left\{\frac{\operatorname{erf}\left[(2i-i\sigma_1)^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(2i-i\sigma_1)^{\frac{1}{2}}} - \frac{\operatorname{erf}\left[(-2i-i\sigma_1)^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(-2i-i\sigma_1)^{\frac{1}{2}}}\right\}$$

$$+ \frac{ci\,e^{i\sigma_2 t}}{2(1+i\sigma_2 R^{\frac{1}{2}})}\left\{\frac{\operatorname{erf}\left[(2i+i\sigma_2)^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(2i+i\sigma_2)^{\frac{1}{2}}} - \frac{\operatorname{erf}\left[(-2i+i\sigma_2)^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(i\sigma_2-2i)^{\frac{1}{2}}}\right\}$$

$$+ \frac{di\,e^{-i\sigma_2 t}}{2(1-i\sigma_2 R^{\frac{1}{2}})}\left\{\frac{\operatorname{erf}\left[(2i-i\sigma_2)^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(2i-i\sigma_2)^{\frac{1}{2}}} - \frac{\operatorname{erf}\left[(-2i-i\sigma_2)^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(-2i-i\sigma_2)^{\frac{1}{2}}}\right\}$$

$$- \exp\left\{-R^{-\frac{1}{2}}t\right\}\left\{\frac{\operatorname{erf}\left[(2i - R^{-\frac{1}{2}})^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(2i - R^{-\frac{1}{2}})^{\frac{1}{2}}} - \frac{\operatorname{erf}\left[(-2i - R^{-\frac{1}{2}})^{\frac{1}{2}}t^{\frac{1}{2}}\right]}{(-2i - R^{-\frac{1}{2}})^{\frac{1}{2}}}\right\}$$

$$\times \left[\frac{ai}{2(1+i\sigma_1 R^{\frac{1}{2}})} + \frac{bi}{2(1-i\sigma_1 R^{\frac{1}{2}})} + \frac{ci}{2(1+i\sigma_2 R^{\frac{1}{2}})} + \frac{di}{2(1-i\sigma_2 R^{\frac{1}{2}})}\right] . \qquad (3.5.16)$$

When we neglect $R^{-\frac{1}{2}}$ in the terms $(\pm 2i - R^{-\frac{1}{2}})^{\frac{1}{2}}$ and use an asymptotic expansion for the error function, (3.5.16) becomes for large times

$$g_I = \frac{ai\, e^{i\sigma_1 t}}{2(1+i\sigma_1 R^{\frac{1}{2}})}\left\{\frac{1}{(2i + i\sigma_1)^{\frac{1}{2}}} - \frac{1}{(i\sigma_1 - 2i)^{\frac{1}{2}}}\right\}$$

$$+ \frac{bi\, e^{-i\sigma_1 t}}{2(1 - i\sigma_1 R^{\frac{1}{2}})}\left\{\frac{1}{(2i - i\sigma_1)^{\frac{1}{2}}} - \frac{1}{(-i\sigma_1 - 2i)^{\frac{1}{2}}}\right\}$$

$$+ \frac{ci\, e^{i\sigma_2 t}}{2(1 + i\sigma_2 R^{\frac{1}{2}})}\left\{\frac{1}{(2i + i\sigma_2)^{\frac{1}{2}}} - \frac{1}{(i\sigma_2 - 2i)^{\frac{1}{2}}}\right\}$$

$$+ \frac{di\, e^{-i\sigma_2 t}}{2(1 - i\sigma_2 R^{\frac{1}{2}})}\left\{\frac{1}{(2i - i\sigma_2)^{\frac{1}{2}}} - \frac{1}{(-2i - i\sigma_2)^{\frac{1}{2}}}\right\}$$

$$- \frac{\exp\left\{-R^{-\frac{1}{2}}t\right\}}{2}\left\{\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1 - i\sigma_2 R^{\frac{1}{2}}}\right\}$$

$$+ O\left\{\frac{e^{2it}}{t^{\frac{1}{2}}}\left[\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}}\right]\right\} . \qquad (3.5.17)$$

For the special case with $\sigma_1 = \sigma_2 = 0$ and $a + b + c + d = 1$, (3.5.17) reduces to

$$g_I = 1 - \exp\left\{-R^{-\frac{1}{2}}t\right\} + O(t^{-\frac{1}{2}}e^{2it}) ,$$

which agrees with the result obtained for V_I by Greenspan and Howard [17] at the bottom of page 391.

The time dependent boundary layer motion, g_B, is given by evaluating (3.5.15) when $\bar{g}$ is given by (3.5.14). When we use the inversion formula No.819 in Foster and Campbell [13], we obtain

$$g_B = \frac{ae^{i\sigma_1 t}}{2}\left\{\frac{i\sigma_1 R^{\frac{1}{2}}}{4(1+i\sigma_1 R^{\frac{1}{2}})} + \frac{z}{4|z|}\right\}\left\{\exp\left\{-\zeta(2i + i\sigma_1)^{\frac{1}{2}}\right\} \times\right.$$

$$\times \operatorname{erfc}\left\{\tfrac{1}{2}\zeta t^{-\frac{1}{2}} - \left[t(2i + i\sigma_1)\right]^{\frac{1}{2}}\right\} + \exp\left\{\zeta(2i + i\sigma_1)^{\frac{1}{2}}\right\} \times$$

$$\times \operatorname{erfc}\left\{\tfrac{1}{2}\zeta t^{-\frac{1}{2}} + \left[t(2i + i\sigma_1)\right]^{\frac{1}{2}}\right\} + \exp\left\{-\zeta(-2i + i\sigma_1)^{\frac{1}{2}}\right\} \times$$

$$\times \operatorname{erfc}\left\{\tfrac{1}{2}\zeta t^{-\frac{1}{2}} - \left[t(i\sigma_1 - 2i)\right]^{\frac{1}{2}}\right\} + \exp\left\{\zeta(-2i + i\sigma_1)^{\frac{1}{2}}\right\} \times$$

$$\left.\times \operatorname{erfc}\left\{\tfrac{1}{2}\zeta t^{-\frac{1}{2}} + \left[t(i\sigma_1 - 2i)\right]^{\frac{1}{2}}\right\}\right\}$$

$$+ \frac{a\exp\left\{-R^{-\frac{1}{2}}t\right\}}{8(1 + i\sigma_1 R^{\frac{1}{2}})}\left\{\exp\left[-\zeta(2i - R^{-\frac{1}{2}})^{\frac{1}{2}}\right]\operatorname{erfc}\left\{\tfrac{1}{2}\zeta t^{-\frac{1}{2}} - \left[t(2i - R^{-\frac{1}{2}})\right]^{\frac{1}{2}}\right\}\right.$$

$$+ \exp\left\{\zeta(2i - R^{-\frac{1}{2}})^{\frac{1}{2}}\right\}\operatorname{erfc}\left\{\tfrac{1}{2}\zeta t^{-\frac{1}{2}} + \left[t(2i - R^{-\frac{1}{2}})\right]^{\frac{1}{2}}\right\}$$

$$+ \ \exp\left\{-\Upsilon(-2i - R^{-\frac{1}{2}})^{\frac{1}{2}}\right\} \operatorname{erfc}\left\{\tfrac{1}{2}\Upsilon t^{-\frac{1}{2}} - \left[t(-2i - R^{-\frac{1}{2}})\right]^{\frac{1}{2}}\right\}$$

$$+ \ \exp\left\{\Upsilon(-2i - R^{-\frac{1}{2}})^{\frac{1}{2}}\right\} \operatorname{erfc}\left\{\tfrac{1}{2}\Upsilon t^{-\frac{1}{2}} + \left[t(-2i - R^{-\frac{1}{2}})\right]^{\frac{1}{2}}\right\} \Bigg\}$$

$$+ \ 3 \text{ similar terms,} \qquad (3.5.18)$$

where $\Upsilon = R^{\frac{1}{2}}(1 - |z|)$.

Again by neglecting $R^{-\frac{1}{2}}$ in the terms $(\pm 2i - R^{-\frac{1}{2}})^{\frac{1}{2}}$ and then using the asymptotic expansion for the complementary error function, (3.5.18) becomes for large times

$$g_B = \tfrac{1}{2}ae^{i\sigma_1 t}\left\{\frac{i\sigma_1 R^{\frac{1}{2}}}{4(1+i\sigma_1 R^{\frac{1}{2}})} + \frac{z}{4|z|}\right\}\Big\{\exp\left[-\Upsilon(2i + i\sigma_1)^{\frac{1}{2}}\right]$$

$$+ \ \exp\left[-\Upsilon(-2i + i\sigma_1)^{\frac{1}{2}}\right]\Big\}$$

$$+ \ \frac{a\exp\{-R^{-\frac{1}{2}}t\}}{4(1+i\sigma_1 R^{\frac{1}{2}})}\left\{\exp\left[-\Upsilon(2i)^{\frac{1}{2}}\right] + \exp\left[-\Upsilon(-2i)^{\frac{1}{2}}\right]\right\}$$

$$+ \ O\left(t^{-\frac{1}{2}}\exp\left\{-\tfrac{1}{4}\Upsilon^2 t^{-1} + 2it\right\}\right)$$

$$+ \ 3 \text{ similar terms .} \qquad (3.5.19)$$

The sum of the expressions (3.5.16) and (3.5.18) satisfies the initial condition and tends to the required final state as t tends to infinity. We must now show that it satisfies the required boundary conditions at the disks, $z = \pm 1$.

For high frequency oscillations, $\sigma_i R^{\frac{1}{2}} >> 1$, $i = 1,2$, we have from (3.5.16) that $g_I = O\left[(\sigma_i R^{\frac{1}{2}})^{-1}\right]$, which is negligible and, from (3.5.18), that g_B obviously satisfies the required boundary conditions, namely, $g_B = ae^{i\sigma_1 t} + be^{-i\sigma_1 t}$ on $z = 1$, $g_B = ce^{i\sigma_2 t} + de^{-i\sigma_2 t}$ on $z = -1$, when terms of order $(\sigma_i R^{\frac{1}{2}})^{-1}$ are ignored.

For steady, low and intermediate frequency oscillations we have that $\sigma_1, \sigma_2 \leq O(R^{-\frac{1}{2}})$ and therefore, on $z = \pm 1$, (4.16) and (4.18) become

$$g_I = \frac{ai}{2(1+i\sigma_1 R^{\frac{1}{2}})}\left\{\frac{\mathrm{erf}(2it)^{\frac{1}{2}}}{(2i)^{\frac{1}{2}}} - \frac{\mathrm{erf}(-2it)^{\frac{1}{2}}}{(-2i)^{\frac{1}{2}}}\right\}\left[\exp\{i\sigma_1 t\} - \exp\{-R^{-\frac{1}{2}}t\}\right]$$

$+$ 3 similar terms, (3.5.20)

$$g_B = \frac{ae^{i\sigma_1 t}}{2}\left[\frac{i\sigma_1 R^{\frac{1}{2}}}{1+i\sigma_1 R^{\frac{1}{2}}} \pm 1\right] + \frac{a\exp\{-R^{-\frac{1}{2}}t\}}{2(1+i\sigma_1 R^{\frac{1}{2}})}$$

$+$ 3 similar terms. (3.5.21)

When $1 >> \max(\sigma_1 t, R^{-\frac{1}{2}}t) = R^{-\frac{1}{2}}t$, we may expand the exponential terms in (3.5.20) and (3.5.21) to give

$$g_I = \frac{ai}{2(1+i\sigma_1 R^{\frac{1}{2}})}\left[\frac{\mathrm{erf}(2it)^{\frac{1}{2}}}{(2i)^{\frac{1}{2}}} - \frac{\mathrm{erf}(-2it)^{\frac{1}{2}}}{(-2i)^{\frac{1}{2}}}\right]\left[O(R^{-\frac{1}{2}}t)\right]$$

$+$ 3 similar terms,

$$g_B = \frac{ae^{i\sigma_1 t}}{2}\left[1 \pm 1\right] + O(R^{-\frac{1}{2}}t) \qquad g_B = \frac{a}{2}\left[1 \pm 1\right] + O\left(R^{-1/2}t\right)$$

+ 3 similar terms.

These expressions show that the term involving a in $g = g_I + g_B$ is $ae^{i\sigma_1 t}$ on $z = 1$ (positive sign) and zero on $z = -1$ (negative sign), when terms of order $R^{-\frac{1}{2}}t$ have been ignored, since the error function is always less than one in magnitude. Hence, for $t \ll R^{\frac{1}{2}}$, the highest order terms satisfy the required boundary conditions. On the other hand, when $t^{\frac{1}{2}} \gg 1$, we can use the asymptotic expansions for the error functions in (3.5.20) to give

$$g_I = \frac{a}{2(1+i\sigma_1 R^{\frac{1}{2}})}\left[\exp\{i\sigma_1 t\} - \exp\{-R^{-\frac{1}{2}}t\}\right]$$

$$+ \; O\left[(t^{\frac{1}{2}}\{1+i\sigma_1 R^{\frac{1}{2}}\})^{-1}\right] . \qquad (3.5.22)$$

The expression (3.5.22) together with (3.5.21) satisfies the required boundary conditions when terms of order $t^{-\frac{1}{2}}$ are ignored. For the intermediate range, $1 \ll t \ll R^{\frac{1}{2}}$, both the exponential terms and the error functions can be expanded and again the highest order terms satisfy the boundary conditions.

Hence the highest order terms in (3.5.16) and (3.5.18), and therefore the asymptotic expressions (3.5.17) and (3.5.19), always satisfy the required boundary conditions on the disks. It is interesting to note that in agreement with Greenspan and Howard's [17] results for V the expressions (3.5.16) and (3.5.18) fail to match the boundary conditions by a small amount. Here we can ignore this discrepancy as we are only concerned with the highest order terms.

In order to determine the time required for the transient effects to vanish it is sufficient to consider the asymptotic expressions (3.5.17) and (3.5.19). For steady ($\sigma_1 = \sigma_2 = 0$), low frequency ($\sigma_1 R^{\frac{1}{2}} \ll 1$, $\sigma_2 R^{\frac{1}{2}} \ll 1$) and intermediate frequency ($\sigma_1 R^{\frac{1}{2}} = O(1)$, $\sigma_2 R^{\frac{1}{2}} = O(1)$) oscillations, (3.5.17) and (3.5.19) reduce to

$$g_I = \frac{ae^{i\sigma_1 t}}{2(1+i\sigma_1 R^{\frac{1}{2}})} + \frac{be^{-i\sigma_1 t}}{2(1-i\sigma_1 R^{\frac{1}{2}})} + \frac{ce^{i\sigma_2 t}}{2(1+i\sigma_2 R^{\frac{1}{2}})} + \frac{de^{-i\sigma_2 t}}{2(1-i\sigma_2 R^{\frac{1}{2}})}$$
$$- \tfrac{1}{2}\exp\{-R^{-\frac{1}{2}}t\}\left[\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}}\right]$$
$$+ O(t^{-\frac{1}{2}}e^{2it}), \tag{3.5.23}$$

$$g_B = \frac{1}{4}\left[\exp\{-\zeta(2i)^{\frac{1}{2}}\} + \exp\{-\zeta(-2i)^{\frac{1}{2}}\}\right] \times$$
$$\times \Bigg\{ ae^{i\sigma_1 t}\left(\frac{i\sigma_1 R^{\frac{1}{2}}}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{z}{|z|}\right) + be^{-i\sigma_1 t}\left(\frac{-i\sigma_1 R^{\frac{1}{2}}}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{z}{|z|}\right)$$
$$+ ce^{i\sigma_2 t}\left(\frac{i\sigma_2 R^{\frac{1}{2}}}{1+i\sigma_2 R^{\frac{1}{2}}} - \frac{z}{|z|}\right) + de^{-i\sigma_2 t}\left(\frac{-i\sigma_2 R^{\frac{1}{2}}}{1-i\sigma_2 R^{\frac{1}{2}}} - \frac{z}{|z|}\right)$$

$$+ \ \exp\left\{-R^{-\frac{1}{2}}t\right\}\left(\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}}\right)\Bigg\}$$

$$+ \ O\left(t^{-\frac{1}{2}} \exp\left\{-\frac{1}{4}\zeta^2 t^{-1} + 2it\right\}\right) . \qquad (3.5.24)$$

Immediately, we see that if

$$\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}} \neq 0,$$

which is equivalent to

$$a + b + c + d \neq 0 ,$$

for steady and low frequency oscillations, then a time of order the spin-up time, T_s, is required for the transient effects to vanish. For the special case when

$$\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}} = 0 ,$$

(3.5.23) and (3.5.24) show that the transient effects decay in a time which is less than T_s but large enough for $O(t^{-\frac{1}{2}})$ terms to become negligible.

For high frequency non-resonant oscillations we have from (3.5.17) that g_I will be $O(R^{-\frac{1}{2}}\sigma^{-1})$ and therefore negligible and from (3.5.19) we find that the transient effects decay in a time less than T_s but large enough for terms of order $t^{-\frac{1}{2}}\exp\left\{-\frac{1}{4}\zeta^2 t^{-1}\right\}$ to become negligible.

Hence these results agree with the predicted values given in Table 3.1, section 3.4.

<u>Resonant case (σ_1 and/or $\sigma_2 = 2$)</u>

We now consider the special case of oscillations at the resonant frequency which belongs to the high frequency range. ~~Without loss of generality~~ As a typical case we may assume that both σ_1 and σ_2 are equal to two.

We can no longer use the previous approximations because the poles near $p = \pm 2i$ control the transient behaviour and large errors occur if they are replaced by branch points at $p = \pm 2i$. However, for small times, which corresponds to large p, the poles near $p = \pm 2i$ become unimportant and we can use these approximations to give, from (3.5.18),

$$g_B = \frac{ae^{2it}}{2}\left\{\frac{1}{4} + \frac{z}{4|z|}\right\}\left\{\exp\left[-\gamma(4i)^{\frac{1}{2}}\right]\operatorname{erfc}\left[\tfrac{1}{2}\gamma t^{-\frac{1}{2}} - (4it)^{\frac{1}{2}}\right]\right.$$
$$\left. + \exp\left[\gamma(4i)^{\frac{1}{2}}\right]\operatorname{erfc}\left[\tfrac{1}{2}\gamma t^{-\frac{1}{2}} + (4it)^{\frac{1}{2}}\right] + 2\operatorname{erfc}\left[\tfrac{1}{2}\gamma t^{-\frac{1}{2}}\right]\right\}$$
$$+ O(R^{-\frac{1}{2}}) + 3 \text{ similar terms}, \qquad (3.5.25)$$

which satisfies the required boundary conditions at the disks. In order to evaluate the above expression, (3.5.25), from tables of transforms we require the fact that the integral of the approximate function, valid for large p, evaluated along the branch cut, $\mathcal{C}_1$, is a good approximation to the exact integral.

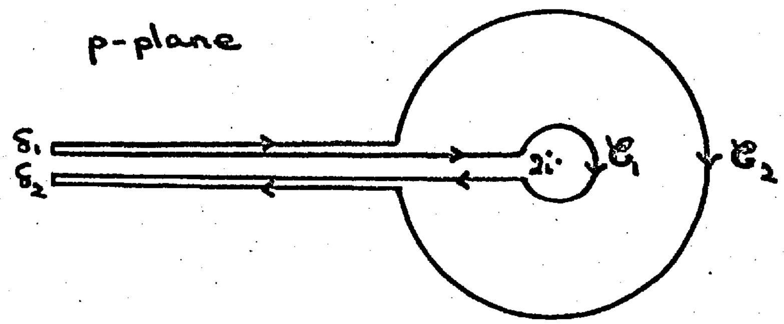


FIG.3.9: Integration contours along the branch cut which extends from p = 2i .

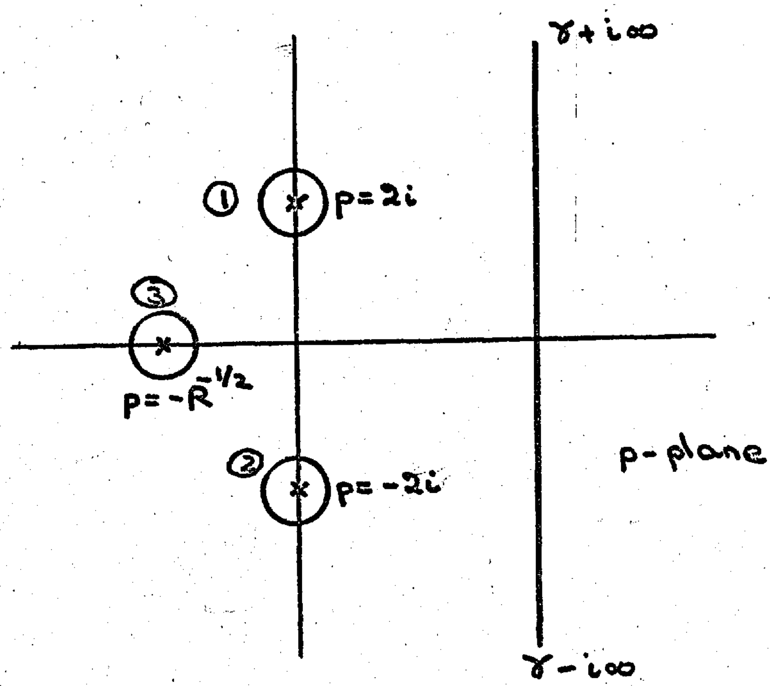


FIG.3.10: Replacement of the contour (γ - i ∞, γ + i ∞) for the inverse Laplace transform by the circles ①,②,③ enclosing the poles.

In Fig. 3.9, we have by Cauchy's theorem that the integral of the approximated function around $\mathcal{C}_1$ is equal to the integral of the approximated function around $\mathcal{C}_2$ provided δ_1 and δ_2 are so small that their contributions to the integral are negligible. Then provided the contour $\mathcal{C}_2$ is always a distance greater than R^{-1} away from $p = 2i$ the integral of the approximated function around $\mathcal{C}_2$ will be a good approximation to the exact integral. Hence the integral of the approximated function along $\mathcal{C}_1$ is a good approximation to the exact integral.

The expression (3.5.25) satisfies the initial condition and also the required boundary conditions at $z = \pm 1$. The first two terms represent well-defined boundary layers on the disks which tend to modified Stokes layers as $t \rightarrow \infty$; the required boundary layer structure. The third term is a Rayleigh layer which penetrates out from the disks like $(\nu t)^{\frac{1}{2}}$ and will, eventually, produce the final quasi-steady state in the interior.

In order to calculate the general time dependent interior flow we surround the points $p = \pm 2i$, $p = -R^{-\frac{1}{2}}$, by circles of radius $R^{-1+\delta}$, where δ is some positive number, and then use the calculus of residues. For the circle ①, in Fig.3.10, all the terms involving $m_1 \simeq (4iR)^{\frac{1}{2}}$ can be replaced by the approximations (3.5.3) while all terms involving

m_2 must be retained, since m_2 is of order one. Hence the interior flow, from (3.3.3) becomes

$$\bar{g}_I = \left\{ \frac{a}{p-2i} + \frac{b}{p+2i} + \frac{c}{p-2i} + \frac{d}{p+2i} \right\} \left\{ \frac{1}{4} + \frac{m_2(\cosh\{m_2 z\} - \cosh m_2)}{4(m_2 \cosh m_2 - \sinh m_2)} \right\}$$

$$+ \left\{ \frac{a}{p-2i} + \frac{b}{p+2i} - \frac{c}{p-2i} - \frac{d}{p+2i} \right\} \frac{\sinh\{m_2 z\}}{4 \sinh m_2} \, . \qquad (3.5.26)$$

The poles of (3.5.26) inside ① are

$$p = 2i,$$

$$p = 2i - \frac{k^2\pi^2}{R}, \qquad k = 1,2,3,\ldots,$$

$$p = 2i - \frac{\xi_n^2}{R}, \qquad n = 1,2,3,\ldots,$$

and from a residue calculation we have that the contribution to the interior flow from circle ① is

$$g_{I_1} = \frac{e^{2it}}{8}\left[a(3z^2 - 1 + 2z) + c(3z^2 - 1 - 2z)\right]$$

$$+ \sum_{k=1}^{\infty} (a-c)\exp\left\{(2i - k^2\pi^2R^{-1})t\right\}(-1)^k \frac{\sin\{k\pi z\}}{2k\pi}$$

$$- \sum_{n=1}^{\infty}\left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} - 1\right)\exp\left\{(2i - \xi_n^2 R^{-1})t\right\}\frac{(a+c)}{2\xi_n^2}$$

$$+ O(R^{-1}) \, .$$

By similar arguments we have that the contribution to the interior flow from the circles (2) and (3) are

$$g_{I_2} = \frac{e^{-2it}}{8}\left[b(3z^2 - 1 + 2z) + d(3z^2 - 1 - 2z)\right]$$

$$+ \sum_{k=1}^{\infty} (b-d)\exp\left\{(-2i - k^2\pi^2R^{-1})t\right\} (-1)^k \frac{\sin\{k\pi z\}}{2k\pi}$$

$$- \sum_{n=1}^{\infty}\left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} - 1\right)\exp\left\{(-2i - \xi_n^2R^{-1})t\right\}\frac{(b+d)}{2\xi_n^2}$$

$$+ O(R^{-1}) ,$$

$$g_{I_3} = O(R^{-\frac{1}{2}}) ,$$

respectively. From Cauchy's theorem we have that

$$g_I = g_{I_1} + g_{I_2} + g_{I_3} ,$$

and hence the time dependent interior solution is

$$g_I = (ae^{2it} + be^{-2it})\frac{(3z^2 - 1 + 2z)}{8} + (ce^{2it} + de^{-2it})\frac{(3z^2 - 1 - 2z)}{8}$$

$$+ \sum_{k=1}^{\infty} \exp\left\{-k^2\pi^2R^{-1}t\right\}(-1)^k\frac{\sin\{k\pi z\}}{2\pi k}\left[ae^{2it} + be^{-2it} - ce^{2it} - de^{-2it}\right]$$

$$+ \sum_{n=1}^{\infty} - \left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} - 1\right)\frac{\exp\left\{-\xi_n^2R^{-1}t\right\}}{2\xi_n^2}\left[ae^{2it}+be^{-2it}+ce^{2it}+de^{-2it}\right]$$

$$+ \quad O(R^{-\frac{1}{2}}) \; . \qquad (3.5.27)$$

The expression (3.5.27) can be shown to satisfy the initial condition by expanding $(3z^2 - 1)$ and z in Fourier series with respect to the orthonormal sets

$$\frac{1}{\xi_n}\left(\frac{\cos\{\xi_n z\}}{\cos \xi_n} - 1\right), \quad \sin(k\pi z) \quad ,$$

respectively, in the interval (-1,1). Also (3.5.27) together with the Stokes layer terms from (3.5.25) satisfy the boundary conditions at $z = \pm 1$. We see immediately from (3.5.27) that a time of order the viscous diffusion time, T_d, is required for the transient effects to vanish which agrees with the predicted result in Table 3.1. It should be noticed that (3.5.27) is identical to the result obtained for the interior flow from (3.3.7).

3.6 THE CONNECTION WITH THE SINGLE DISK PROBLEM

From (3.3.3) and (3.3.4) we have that

$$\frac{\partial \bar{f}}{\partial z} + i\bar{g} = \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} + \frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2} \right\} \left\{ \frac{i}{2} + \frac{Rpi}{\Delta} m_1 E(m_2)(\cosh\{m_1 z\} - \cosh m_1) \right\}$$

$$+ \left\{ \frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1} - \frac{c}{p-i\sigma_2} - \frac{d}{p+i\sigma_2} \right\} \frac{i \sinh\{m_1 z\}}{2 \sinh m_1} \quad . \qquad (3.6.1)$$

We will now consider the boundary layer solutions for small times which correspond to large p. From (3.6.1), we have, near $z = 1$,

$$\frac{\partial \bar{f}}{\partial z} + i\bar{g} = i\left(\frac{a}{p-i\sigma_1} + \frac{b}{p+i\sigma_1}\right)\exp\left\{ m_1(z-1)\right\} , \quad (3.6.2)$$

while near $z = -1$,

$$\frac{\partial \bar{f}}{\partial z} + i\bar{g} = i\left(\frac{c}{p-i\sigma_2} + \frac{d}{p+i\sigma_2}\right) \exp\left\{ m_2(-z-1)\right\} . \quad (3.6.3)$$

The expressions (3.6.2) and (3.6.3) correspond to the equation (2.4.6) with a change in the origin of z. Hence, for small times, the boundary layers behave as if only one disk is present and hence are not influenced by radial pressure gradients. The motion is given by (2.4.14) which, for small times, represents Rayleigh layers penetrating through the fluid from the disks.

3.7 CONCLUSION

In section 3.4, we found that the important parameters for the problem where

$$\sigma_1 R^{\frac{1}{2}} , \quad \sigma_2 R^{\frac{1}{2}} ,$$

which related the spin-up time, T_s, to the periods of oscillation of the imposed frequencies T_1, T_2.

We will first consider the case when the imposed oscillations are steady, low frequency or intermediate frequency. When the oscillations are such that

$$\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}} \neq 0 ,$$

then the spin-up mechanism plays a very important role and we require a time of order T_s for the transient effects to decay. The final state consists of boundary layers on the disks and an interior flow which is independent of z. The results of Greenspan and Howard [17] can be obtained as a special case. When

$$\frac{a}{1+i\sigma_1 R^{\frac{1}{2}}} + \frac{b}{1-i\sigma_1 R^{\frac{1}{2}}} + \frac{c}{1+i\sigma_2 R^{\frac{1}{2}}} + \frac{d}{1-i\sigma_2 R^{\frac{1}{2}}} = 0 ,$$

a much shorter time is required for the transient effects to vanish although the final state has the same structure.

When high frequency oscillations are imposed on the disks then the spin-up mechanism associated with the simple pole, $p = -R^{-\frac{1}{2}}$, becomes unimportant compared with viscous diffusion. A resonant phenomenon is found when either disk oscillates at a frequency which is twice the angular velocity of the basic rotation. For the case when the disks oscillate at non-resonant frequencies then a time much less than the spin-up time is required for the transient effects to vanish and the final state consists of modified Stokes layers on the disks. When a disk oscillates at the resonant frequency, the viscous diffusion time, T_d, is always required for transient effects to decay and the final state consists of modified Stokes layers on the disk and an interior flow which is quadratic in z with a time dependent amplitude. These results correspond closely to the solution for the problem discussed in Chapter 2.

CHAPTER 4

TRANSIENT AND STEADY STATE VORTICITY GENERATED BY HORIZONTAL TEMPERATURE GRADIENTS

PART 1: THE FUNDAMENTAL SOLUTION

4.1 INTRODUCTION

In the atmosphere and the oceans, the Coriolis force together with the variations in the temperature field influence the development of circulatory fluid motions and hence the vorticity which is present in the fluid. In particular in the atmosphere, strong circulatory currents can be produced by temperature variations on the earth's surface. These currents could be connected with the evolution of hurricanes and this provides the motivation for the problems that are considered in this chapter and in chapter 5.

It is assumed that the atmosphere can be represented by the idealized situation of two infinite horizontal disks with fluid between them, when the fluid and the disks are in steady isothermal rigid rotation about an axis normal to the disks. This assumption implies that, for any given latitude, it is valid to ignore the effects of the earth's curvature, to suppose that the component of the earth's rotation normal to the surface of the earth is constant, and to neglect the adverse temperature gradient which exists in the atmosphere.

When a temperature distribution is imposed on the disks, density variations in a horizontal plane can be produced in the fluid. For a non-rotating system, these density variations produce circulations in vertical planes since no hydrostatic pressure distribution can balance the horizontal variation of the buoyancy forces. Then conduction and viscous effects are, usually, only significant in the neighbourhood of the disks while, in the interior, the heat is convected by the movement of the fluid particles. For a rotating system, however, the presence of the Coriolis force has the primary effect of producing horizontal flow perpendicular to the density gradient rather than circulation in vertical planes. In particular, the idealized model of the atmosphere is axisymmetric and hence the density gradients produce an azimuthal component of velocity which cannot alter the temperature field by convective processes. Hence for a rotating system, heat can only be convected by circulation in an axial plane and this flow may be inhibited to such an extent that conduction processes predominate everywhere.

The conditions necessary for either conduction or convection processes to predominate have been discussed by Duncan [10], when an axisymmetric non-uniform temperature distribution with a minimum at the point of intersection of the disk and the axis of rotation is imposed on the upper disk and the lower disk is insulated. Duncan considered the STEADY problem when the inertial accelerations were negligible in comparison with the Coriolis acceleration ($H \ll 1$) and found that the critical parameter was,

$$\sigma H R^{\frac{1}{2}} ,$$

where σ is the Prandtl number, H the thermal Rossby number, (1.1.5), and R the Reynolds number, (1.1.2).

When $\sigma H R^{\frac{1}{2}} \ll 1$, the conduction process takes precedence throughout the fluid and the steady state consists of Ekman layers on the disks and an interior region where there is an azimuthal flow and also a lower order secondary circulation, a down-draught, which is driven by the Ekman layers. On the other hand when $\sigma H R^{\frac{1}{2}} \gg 1$, convection predominates in the interior but, in addition to the Ekman layers where conduction again prevails, there are thermal boundary layers, which have a depth of penetration of order $\sigma^{-1} H^{-1} R^{-\frac{1}{2}}$, where the conduction and convection processes balance. These thermal layers are always much thicker than the Ekman layers. In order to satisfy the physical situation, Duncan found that only the lower thermal layer could exist. For the case $\sigma HR^{\frac{1}{2}} \ll 1$, this thermal layer is much thicker than the distance between the disks, which agrees with the above result that conduction is the predominant mechanism for the transfer of heat.

Duncan also investigated the effect of replacing the temperature minimum by a temperature maximum on the upper disk and of replacing the upper disk by a stress-free surface. He found that the above results were also applicable to the case with a temperature maximum except that, when $\sigma HR^{\frac{1}{2}} \gg 1$, an exact boundary condition, say uniform temperature, was required on the lower boundary, the thermal layer was present only on the upper disk, and, in the interior, there was an up-draught and all the velocities were one order of magnitude smaller. When the upper boundary was a stress-free surface then Duncan showed that conduction always predominates over convection when the inertial effects are ignored.

Duncan, in his analysis, employed the similarity variables introduced by Von Kármán [24] but, in fact, the terms neglected vanish to first order. Hence these results apply to any general axisymmetric flow between horizontal planes when the effects of vertical boundaries can be neglected and applied temperature gradients do not change sign.

Vertical boundaries, which are at rest relative to the fluid and the disks, are introduced into the system so that their separation distance is much greater than the distance between the horizontal boundaries. The STEADY, axisymmetric régime present in the fluid when the horizontal boundaries are insulated and the vertical boundaries are maintained at different constant temperatures has been discussed by Hunter [22]. Firstly, he considered the case when conduction is the predominant mechanism for the transfer of heat and then modified this solution to include small convective effects by calculating successive corrections in the form of a power series in the parameter $\sigma HR^{\frac{1}{2}}$. This parameter is identical to the critical parameter found by Duncan [10]. Hunter's solution consisted of Ekman layers on the horizontal boundaries, a thermal layer, $O(R^{-1/3})$, on the vertical boundaries when conduction predominated, two thermal layers, $O(R^{-1/3})$, $O(R^{-1/4})$, on the vertical boundaries when convection predominated and an interior flow composed of a vertically sheared azimuthal flow and a much weaker circulation in axial planes. When the upper disk was replaced by a free surface, then only the situation when conduction is predominant is covered by Hunter's analysis and he finds that a double boundary layer structure on the vertical walls is now required.

Barcilon and Pedlosky [4] gave a discussion of the STEADY flow that would be produced in the fluid, when the upper disk is heated uniformly, the lower disk is cooled uniformly and the vertical walls are insulated. Under these conditions a stable stratification is developed in the fluid. In particular they considered the case when the fluid was rotating rapidly (Ω large) which implies that the centrifugal effects are important and that a buoyancy term is present in the radial momentum equation. The linear solution, $H \equiv O$, is composed of Ekman layers on the disks, $R^{-1/3}$ - layers on the vertical walls and an interior flow which is a solution of the thermal-wind equations. For this case conduction predominates over convection and the results are equivalent to those found by Duncan and Hunter. When $\sigma HR^{\frac{1}{2}} = O(1)$, that is when conduction and convection are of equal importance, Barcilon and Pedlosky found that the vertical boundary layers become important and influence the motion. When $\sigma HR^{\frac{1}{2}} \gg 1$, they found that "thermal conduction is important throughout the fluid" because of the influence of the side wall layers and that the Ekman layers were absent to first order.
From their results, Barcilon and Pedlosky conclude that, when a strong stratification, $\sigma HR^{\frac{1}{2}} \geq O(1)$, is present, the vertical boundaries influence the motion and the study of "flows which are unbounded laterally may

lead to solutions which are not the limit of any physically realizable experiment* .

In this chapter, we will consider the following initial-value problem which, at large times, reduces to a steady flow which is closely related to the problems discussed by Duncan, Hunter and Barcilon and Pedlosky. In particular, we will be interested in the behaviour of the component of vorticity parallel to the axis of rotation (vertical). For the idealized model of the atmosphere in the absence of any vertical walls, from some instant of time, a steady heating is applied to the lower disk. This heating depends upon the co-ordinates in the plane of the disk through a function which satisfies the membrane equation. The temperature of the upper disk is maintained at its initial constant value.

It is found that a cellular flow is developed which is not influenced by the introduction of insulating walls at the boundaries of the cells except for the additional boundary layers which must be present in order to satisfy the non-slip condition. The time required for the transient effects to decay is discussed and it is found to depend, greatly, upon the horizontal wave number defined by the membrane equation. The final steady state is composed of Ekman layers on the disks and an interior inviscid flow which is a particular solution of the thermal-wind equations.

Where the temperature of the lower disk is increased, the vertical component of vorticity for the fluid close to this disk is also increased, while further away from the disk it is decreased. The reverse is true when the temperature of the lower disk is decreased. We deduce that the rotation is responsible for the production of this vorticity perpendicular to the disks because for the corresponding problem without rotation the vertical vorticity is always zero.

In chapter 5, the steady problems associated with different temperature distributions on the lower disk, will be discussed.

As a corollary of this initial-value problem, we find, when we allow the upper disk to move to infinity and impose an oscillatory heating, dependent upon a solution of the membrane equation, on the remaining disk, a resonance effect is experienced in the sense that different oscillatory solutions are found for resonant and non-resonant frequencies. The existence of this oscillatory solution, for the resonant case, is due to the introduction of a length scale in the plane of the disk.

4.2 EQUATIONS OF MOTION

We consider a viscous fluid bounded by two infinite parallel plane horizontal disks, $z = \pm d$. Initially, both the fluid and the disks are at some constant temperature, T_o, and are in rigid-body rotation with constant angular velocity, Ω , about an axis normal to the disks. Then, from $t = 0$, a steady heating is applied to the lower disk, $z = -d$, which is dependent upon the co-ordinates in the plane of the disk, while the temperature of the upper disk, $z = d$, remains at its initial value.

In the following analysis we make two important assumptions. Firstly we apply the Boussinesq approximation [33, p.759] which supposes that the fluid is incompressible, that the density variations are small, that they depend only upon temperature variations by a linear relationship and that they can be ignored except in the buoyancy force when they are associated with gravity. Therefore, when we suppose that the density is given by

$$\rho = \rho_o(1 - a\Theta) , \qquad (4.2.1)$$

the Boussinesq approximation implies $a\Theta \ll 1$, where ρ_o is the density at the temperature T_o, Θ is the temperature variation from T_o and a is the coefficient of thermal expansion.

The temperature variations in this problem are due to the applied heating at the lower boundary and therefore the dissipation of energy and volume changes in the energy equation can always be neglected for liquids and also for gases provided

$$\left(\frac{U_1}{a_1}\right)^2 = M_1^2 \ll \frac{\Theta}{T_o} \ll \frac{1}{aT_o} , \qquad (4.2.2)$$

where U_1 is a typical velocity and a_1 is the speed of sound [31, p.126]. In this problem, the velocities will always be very small compared with the speed of sound and therefore (4.2.2) will always be satisfied since $(aT_0)^{-1}$ is greater than one for liquids and approximately one for gases.

The Boussinesq approximation also implies that the properties of the fluid namely a, ν the kinematic viscosity, κ the thermal diffusivity, are independent of temperature and therefore can be taken as constants.

Secondly, we assume that the temperature variations, Θ, and hence the velocity vector, $\underline{u}$, remain small so that it appears valid to linearize the equations of motion and the energy equation. Whether this linearization is, in fact, valid, depends upon the solution derived from the linear equations with suitable boundary conditions. If we have found a sensible solution to the problem then we assume that the linearization was valid, otherwise we must re-examine the problem. In this particular problem the solutions for the velocities from the linearized equations are found to depend upon the thermal Rossby number,

$$H = \frac{ag\,\Delta T}{d\,\Omega^2}, \qquad (4.2.3)$$

where ΔT is a typical scale for the temperature variations and g the acceleration due to gravity. This parameter, H, is always small and therefore we may assume that the linearization was valid.

We consider cartesian axes (x, y, z) such that the z-axis is parallel to the common axis of rotation of the fluid and the disks, while the x,y-axes lie in a plane parallel to and midway between the disks and rotate with constant angular velocity, Ω. The velocity components in this rotating frame of reference are given by

$$\underline{u} = (u, v, w) . \qquad (4.2.4)$$

The continuity equation and the linearized Navier-Stokes and energy equations in this rotating frame of reference are, after the application of the Boussinesq approximation,

$$\operatorname{div} \underline{u} = 0 , \qquad (4.2.5)$$

$$\frac{\partial \underline{u}}{\partial t} + 2\Omega\, \underline{k} \wedge \underline{u} + \nabla\left[\frac{p}{\rho_0} - \tfrac{1}{2}\Omega^2(x^2 + y^2)\right]$$

$$= g\alpha\Theta\underline{k} + \nu\nabla^2\underline{u} , \qquad (4.2.6)$$

$$\frac{\partial \Theta}{\partial t} = \kappa_0 \nabla^2\Theta , \qquad (4.2.7)$$

where $\underline{k}$ is a unit vector in the z-direction and $\{p - \tfrac{1}{2}\rho_0\, \Omega^2(x^2 + y^2)\}$ is the departure of the effective kinematic pressure from the hydrostatic pressure that prevails when the fluid is at rest at a uniform temperature, T_0.

We suppose that the imposed heating on the disk at $z = -d$ has the form

$$\left.\begin{aligned} \Theta &= h(x,y)\,\Phi\,\Delta T \qquad & t > 0\,, \\ \Theta &= 0 & t \leq 0\,, \end{aligned}\right\} \qquad (4.2.8)$$

where Φ is some constant and $h(x,y)$ satisfies the membrane equation,

$$\nabla_1^2\, h(x,y) + a^2 h(x,y) = 0\,, \qquad (4.2.9)$$

where $\nabla_1^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and a is the wave number in the x,y plane which can be assumed positive, without loss of generality. In particular we can choose

$$h(x,y) = \exp\{ i l_1 x + i l_2 y \}\,,$$

where $a^2 = l_1^2 + l_2^2$, which represents an imposed heating which is oscillatory in x and y.

When we write $r^2 = x^2 + y^2$, we find that a solution of the membrane equation (4.2.9) is

$$h(r) = A\, J_0(ar)\,, \qquad (4.2.10)$$

for some arbitrary constant A.

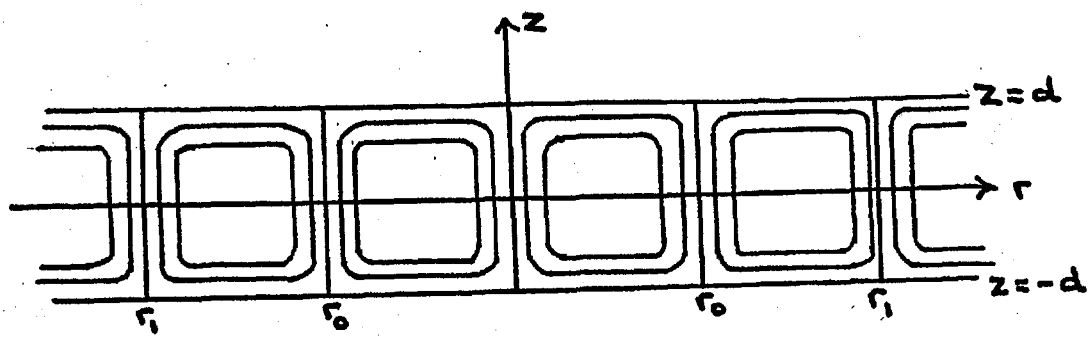


FIG.4.1: Sketch of the streamlines $\psi = -ra^{-1}AJ_1(ar)w$ where $r = r_0, r_1, \ldots$ occur at the zeros of $J_1(ar)$.

Then (4.2.10)* defines the streamlines

$$\psi = -ra^{-1}A\, J_1(ar)w \,, \tag{4.2.11}$$

which are shown, near $z = 0$, in Fig.4.1. The cell boundaries $r = 0,\ r_0,\ r_1, \ldots$ occur at the zeros of $J_1(ar)$. The radial and vertical velocity components are then given by

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial z}\,, \qquad v_z = -\frac{1}{r}\frac{\partial\psi}{\partial r}\,.$$

When the flow is two dimensional, that is independent of y, the solution of (4.2.9) and the corresponding streamfunction are

* using (4.2.18)

$$h(x) = A \cos\{ax\}$$
$$\psi = -Aa^{-1} w \sin\{ax\} \quad ,$$

where the x, z - velocity components are defined by

$$v_x = \frac{\partial \psi}{\partial z}, \qquad v_z = -\frac{\partial \psi}{\partial x}.$$

The streamlines again form the cellular pattern shown in Fig.4.1 with the cell boundaries situated at $x = \pm n\pi a^{-1}$, $n = 0,1,2,\ldots$.

The boundary and initial conditions, from (4.2.8) are

$$\left.\begin{array}{llll} \text{(i)} & \Theta = \Phi\,\triangle T\, h(x,y) & \text{on } z = -d & \text{for } t > 0, \\ \text{(ii)} & \Theta = 0 & \text{on } z = d & \text{for } t > 0, \\ \text{(iii)} & u = v = w = 0 & \text{on } z = \pm d & \text{for } t > 0, \\ \text{(iv)} & u = v = w = \Theta = 0 & \text{at } t = 0 & \text{for all } z. \end{array}\right\} \quad (4.2.12)$$

We define the dimensionless (starred) variables

$$\left.\begin{array}{l} \underline{r} = d\underline{r}^*, \quad t = \Omega^{-1} t^*, \quad \underline{u} = d\,\Omega\,\underline{u}^*, \quad \Theta = \triangle T \Theta^*, \\ a = a^* d^{-1}, \quad p\rho_0^{-1} - \tfrac{1}{2}\Omega^2(x^2 + y^2) = d^2 \Omega^2 p^*, \end{array}\right\}$$

(4.2.13)

where $\underline{r}$ is the position vector (x,y,z).

From (4.2.13), it appears that the neglected non-linear terms would be of the same order as the retained linear terms but, in fact, $\underline{u}^*$ depends upon H, the thermal Rossby number (4.2.3), which is always small and hence the linearization is valid. When we introduce (4.2.13) into the equations (4.2.5), (4.2.6), (4.2.7), we find (upon dropping the asterisks)

$$\frac{\partial \underline{u}}{\partial t} + 2\underline{k}\wedge\underline{u} + \nabla p = H\Theta\underline{k} + R^{-1}\nabla^2\underline{u}\,, \qquad (4.2.14)$$

$$\operatorname{div}\underline{u} = 0\,, \qquad (4.2.15)$$

$$\frac{\partial \Theta}{\partial t} = (\sigma R)^{-1}\nabla^2\Theta\,, \qquad (4.2.16)$$

where H is the thermal Rossby number defined by (4.2.3), $R = \frac{\Omega d^2}{\nu}$, the Reynolds number and $\sigma = \frac{\nu}{\kappa}$, the Prandtl number.

We define the component of vorticity in the z-direction (vertical) to be

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,. \qquad (4.2.17)$$

We now seek a solution of the above problem of the form

$$\left.\begin{aligned} \Theta &= h(x,y)\phi(z,t)\,, \\ w &= h(x,y)w_1(z,t)\,, \\ \zeta &= h(x,y)\,\zeta_1(z,t). \end{aligned}\right\} \qquad (4.2.18)$$

Then (4.2.18) and (4.2.15) imply that

$$u = \frac{1}{a^2}\left\{\frac{\partial h}{\partial x}\frac{\partial w_1}{\partial z} + \frac{\partial h}{\partial y}\zeta_1\right\},$$

$$v = \frac{1}{a^2}\left\{-\frac{\partial h}{\partial x}\zeta_1 + \frac{\partial h}{\partial y}\frac{\partial w_1}{\partial z}\right\}.$$

When we substitute (4.2.18) into the equations (4.2.14), (4.2.15), (4.2.16), eliminate the pressure from the momentum equations and use the continuity equation, we have

$$\frac{\partial \phi}{\partial t} = (\sigma R)^{-1}\left[D^2 - a^2\right]\phi, \qquad (4.2.19)$$

$$-\frac{\partial \zeta_1}{\partial t} + 2Dw_1 = -R^{-1}\left[D^2 - a^2\right]\zeta_1, \qquad (4.2.20)$$

$$(D^2 - a^2)\left[R^{-1}(D^2 - a^2) - \frac{\partial}{\partial t}\right]w_1 - 2D\zeta_1 = a^2 H\phi, \qquad (4.2.21)$$

where $D \equiv \frac{\partial}{\partial z}$. These equations (4.2.19), (4.2.20), (4.2.21) must be solved subject to the conditions

$$\left.\begin{array}{lllll}
\text{(i)} & \phi = \Phi & \text{on } z = -1 & \text{for} & t > 0, \\
\text{(ii)} & \phi = 0 & \text{on } z = 1 & \text{for} & t > 0, \\
\text{(iii)} & \zeta_1 = D^2\zeta_1 = w_1 = 0 & \text{on } z = \pm 1 & \text{for} & t > 0, \\
\text{(iv)} & \zeta_1 = D^2 w_1 = w_1 = \phi = 0 & \text{at } t = 0 & \text{for all } z. &
\end{array}\right\} \qquad (4.2.22)$$

The condition, (4.2.22(iv)) is correct because, initially, there was no motion in the fluid relative to the basic rotation and hence it is valid to assume that all the space differentials of the velocity components vanish at $t = 0$.

4.3 THE APPLICATION OF THE LAPLACE TRANSFORM TO THE INITIAL-VALUE PROBLEM

The initial-value problem defined by the equations (4.2.19), (4.2.20), (4.2.21) subject to the boundary conditions (4.2.22) will now be solved on the assumption that the Reynolds number, R, is large.

Applying the Laplace transform,

$$\bar{f}(z,p) = \int_0^\infty f(z,t)e^{-pt}dt ,$$

to (4.2.19), (4.2.20), (4.2.21) and (4.2.22), we have

$$p\bar{\phi} = (\sigma R)^{-1}\left[D^2 - a^2\right]\bar{\phi} , \qquad (4.3.1)$$

$$-p\bar{\zeta}_1 + 2D\bar{w}_1 = -R^{-1}\left[D^2 - a^2\right]\bar{\zeta}_1 , \qquad (4.3.2)$$

$$(D^2-a^2)\left[R^{-1}(D^2-a^2) - p\right]\bar{w}_1 - 2D\bar{\zeta}_1 = a^2H\bar{\phi} , \qquad (4.3.3)$$

which must be solved subject to the conditions

$$\left.\begin{array}{ll}(a) & \bar{\zeta}_1 = D^2\bar{\zeta}_1 = \bar{w}_1 = 0 \quad \text{on } z = \pm 1 , \\ (b) & \bar{\phi} = 0 \quad \text{on } z = 1, \quad \bar{\phi} = \dfrac{\Phi}{p} \text{ on } z = -1 .\end{array}\right\} \qquad (4.3.4)$$

The transformed energy equation (4.3.1) can immediately be solved subject to the appropriate boundary conditions, while the remaining equations (4.3.2), (4.3.3) can be combined to give a sixth-order equation in Υ_1 with a forcing term that depends upon the solution of the energy equation.

The solution of (4.3.1) which satisfies (4.3.4(b)) is

$$\bar{\phi} = \frac{\Phi \sinh\{\chi(1-z)\}}{p \sinh\{2\chi\}} , \qquad (4.3.5)$$

where $\chi^2 = Rp\sigma + a^2$. (4.3.6)

The singularities of (4.3.5) are simple poles at

$$p = 0 ,$$

$$p = -\frac{1}{R\sigma}(a^2 + \frac{m^2\pi^2}{4}) , \quad \text{where } m = 1,2,\ldots,$$

since the branch point associated with $p = -\{a^2/R\sigma\}$ is a zero of both the numerator and the denominator and is therefore not a singularity. The inverse Laplace transform,

$$f(z,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\, \bar{f}(z,p)dp , \qquad (4.3.7)$$

where γ lies to the right of all the singularities in the complex p-plane, is evaluated by transforming the path of integration into a closed contour and then applying the calculus of residues [8,p.75] to give

$$\phi = \frac{\Phi \sinh\{a(1-z)\}}{\sinh\{2a\}}$$

$$+ \sum_{m=1}^{\infty} \frac{\Phi (-1)^m m\pi \sin\left[\frac{1}{2}m\pi(1-z)\right] \exp\left\{-t(a^2 + \frac{1}{4} m^2\pi^2)/R\sigma\right\}}{2(a^2 + \frac{1}{4} m^2\pi^2)} \quad . \qquad (4.3.8)$$

Hence we see that, for the temperature perturbation (4.3.8), the transient terms decay in a dimensionless time of order $\left\{ R\sigma/(a^2 + \frac{1}{4}\pi^2)\right\}$, which is always less than the thermal diffusion time, $R\sigma\,\Omega^{-1}$, to leave

$$\phi = \frac{\Phi \sinh\{a(1-z)\}}{\sinh\{2a\}} \quad , \qquad (4.3.9)$$

which is shown in Fig.4.2.

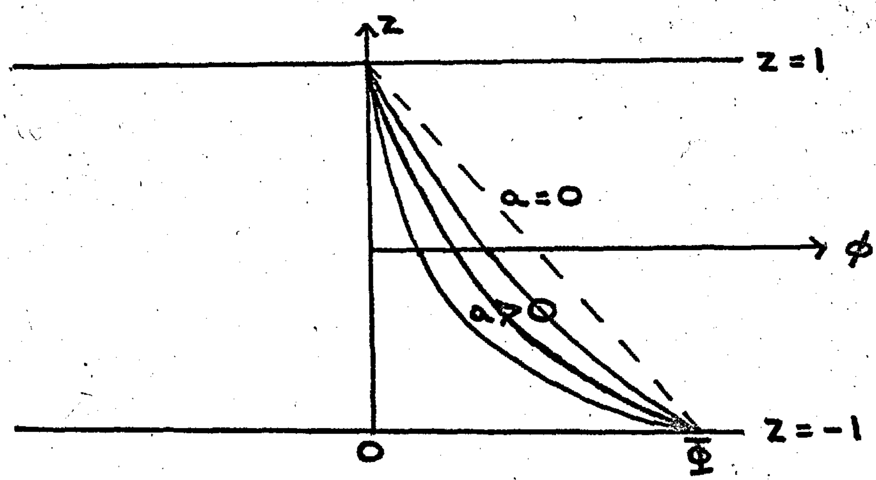


FIG.4.2: Sketch of $\phi = \dfrac{\Phi \sinh\{a(1-z)\}}{\sinh\{2a\}}$.

The decay time $\left\{R\sigma/(a^2 + \frac{1}{4}\pi^2)\right\} \longrightarrow 0$ if either $\sigma \longrightarrow 0$ which corresponds to a large thermal diffusivity as in the sun or if $a \longrightarrow \infty$ which implies a small length scale.

We now return to the equations (4.3.2) and (4.3.3) in order to find ζ_1, the z-component of vorticity. From the equations (4.3.2), (4.3.3) and (4.3.5), we have

$$(D^2 - a^2)\left[R^{-1}(D^2 - a^2) - p\right]^2 \bar{\zeta}_1 + 4D^2 \bar{\zeta}_1$$

$$= \frac{2a^2 H \Phi \chi \cosh\{\chi(1-z)\}}{p \sinh\{2\chi\}} . \qquad (4.3.10)$$

The general solution of the equation (4.3.10) consists of the complementary function together with a particular integral. Immediately we see that a particular integral of (4.3.10) is

$$\bar{\zeta}_1 = A \cosh\{\chi(1-z)\} , \qquad (4.3.11)$$

where

$$A = \frac{2a^2 H \Phi \chi}{p \sinh\{2\chi\}\left[Rp^3\sigma(\sigma-1)^2 + 4Rp\sigma + 4a^2\right]} . \qquad (4.3.12)$$

Hence it only remains to find the complementary function of (4.3.10), which we will assume to have the form $e^{\lambda z}$. This implies that

$$(\lambda^2 - a^2)\left[R^{-1}(\lambda^2 - a^2) - p\right]^2 + 4\lambda^2 = 0 . \qquad (4.3.13)$$

4.4 THE ROOTS OF THE CUBIC EQUATION

We now wish to determine the roots of the cubic equation (4.3.13) for λ^2 on the assumption of large R. Therefore we may express the roots in terms of a power series in R, namely

$$\lambda^2 = \mu_1 R + \mu_2 + \frac{\mu_3}{R} + \dots ,$$

where the highest term is taken to be O(R) in order that, in (4.3.13), there exists terms comparable in magnitude with the term involving the highest ~~derivative~~ power.

We will assume that $a \ll R^{\frac{1}{2}}$ and non zero in order that the following expansions are always valid.

From (4.3.13) we see that the leading terms in the expansions for the roots are given by the solutions of the equations

$$\lambda^6 - 2pR\lambda^4 + R^2\lambda^2(4 + p^2) = 0 , \qquad (4.4.1)$$

$$(4 + p^2)\lambda^2 - a^2p^2 = 0 , \qquad (4.4.2)$$

which are

$$\lambda^2 = R(p \pm 2i) \quad \text{or} \quad \lambda^2 = \frac{a^2p^2}{(p^2+4)} \qquad (4.4.3)$$

When $p \rightarrow \pm 2i$, we see that the last of these roots tends to infinity and therefore the above expansions cease to be valid. Hence, we must re-examine the equation (4.3.13) to obtain new expansions in the neighbourhood of $p = \pm 2i$. These roots, (4.4.3), will be called the OUTER EXPANSION and when the next terms are calculated we find

$$\lambda^2 = \frac{a^2p^2}{(4+p^2)} + \frac{32\ pa^4}{(4+p^2)^3}\ \frac{1}{R} + O\left(\frac{1}{R^2}\right), \tag{4.4.4}$$

or $$\lambda^2 = R(p \pm 2i) + \frac{a^2\left[ip \mp 3\right]}{\left[ip \mp 2\right]} + O\left(\frac{1}{R}\right). \tag{4.4.5}$$

For the special case p = O, we see from (4.4.4) that $\lambda^2 = O\left(\frac{1}{R^2}\right)$ and, from (4.3.13), we find that this root is

$$\lambda^2 = \frac{a^6}{4R^2} + O\left(\frac{1}{R^4}\right) .$$

Thus for the OUTER EXPANSIONS we define

$$\left.\begin{aligned} \lambda_1 &= \frac{ap}{(p^2+4)^{\frac{1}{2}}} + \frac{16a^3(p^2+4)^{\frac{1}{2}}}{(p^2+4)^3R} + O\left(\frac{1}{R^2}\right), \quad p \neq O, \\ \lambda_1 &= \frac{a^3}{2R} + O\left(\frac{1}{R^3}\right) \qquad\qquad\qquad\qquad, \quad p = O, \end{aligned}\right\} \tag{4.4.6}$$

$$\lambda_2 = R^{\frac{1}{2}}(p+2i)^{\frac{1}{2}} + \frac{a\left[ip-3\right](p+2i)^{-\frac{1}{2}}}{\left[ip-2\right]\ 2R^{\frac{1}{2}}} + O(R^{-3/2}), \tag{4.4.7}$$

$$\lambda_3 = R^{\frac{1}{2}}(p-2i)^{\frac{1}{2}} + \frac{a\left[ip+3\right](p-2i)^{-\frac{1}{2}}}{\left[ip+2\right]\ 2R^{\frac{1}{2}}} + O(R^{-3/2}), \tag{4.4.8}$$

where the signs of the square roots are chosen so that λ_1, λ_2, λ_3 have positive real parts.

We will next consider the roots of (4.3.13) when $p = \pm 2i$, that is when the inviscid form of the partial differential equation corresponding to (4.3.10) namely

$$D^2 \overline{\mathcal{J}}_1 \left[p^2 + 4 \right] + p^2 \nabla_1^2 \overline{\mathcal{J}}_1 = 0 ,$$

reduces to

$$\nabla_1^2 \overline{\mathcal{J}}_1 = 0 ,$$

which is elliptic. We will refer to the expansions found for these roots of (4.3.13) as the INNER EXPANSIONS. The leading terms in the INNER EXPANSIONS, for large R, are given by the solution of the equations

$$\lambda^6 - 2pR\lambda^4 = 0 , \qquad (4.4.9)$$

$$-2pR^{-1}\lambda^4 - a^2 p^2 = 0 , \qquad (4.4.10)$$

where $p = \pm 2i$. From (4.4.9) and (4.4.10) we have, for the INNER EXPANSIONS, when $p = 2i$,

$$\left.\begin{aligned} \lambda_2 &= (4iR)^{\frac{1}{2}} + O(R^{-\frac{1}{2}}) , \\ \lambda_3 &= R^{1/4} a^{\frac{1}{2}} \exp\{-\pi i/8\} + O(R^{-1/4}) , \\ \lambda_1 &= R^{1/4} a^{\frac{1}{2}} \exp\{3\pi i/8\} + O(R^{-1/4}) , \end{aligned}\right\} \qquad (4.4.11)$$

and when $p = -2i$,

$$\left.\begin{aligned} \lambda_3 &= (-4iR)^{\frac{1}{2}} + O(R^{-\frac{1}{2}}), \\ \lambda_2 &= R^{1/4}\, a^{\frac{1}{2}} \exp\{ i\pi/8 \} + O(R^{-1/4}) , \\ \lambda_1 &= R^{1/4}\, a^{\frac{1}{2}} \exp\{ -3\pi i/8 \} + O(R^{-1/4}) , \end{aligned}\right\} \qquad (4.4.12)$$

where the appropriate choices for λ_1, λ_2, λ_3 will be justified later.

Thus we find that, for $p = \pm 2i$, the depth of penetration of λ_1 is decreased while that for λ_3 or λ_2 is increased.

We now wish to consider the region in which the transition from the inner expansion to the outer expansion occurs. We suppose that

$$p = \pm 2i + \frac{k}{R^{\alpha}} ,$$

where k is a constant and α a positive real number. Then we choose $\alpha = \frac{1}{2}$ so that all the terms in (4.4.2) and (4.4.10) have the same order. The equations which yield the leading terms in the expansion for the roots of (4.3.13) are

$$\left.\begin{aligned} \lambda^6 \mp 4iR\lambda^4 &= 0 , \\ \pm \frac{4ik\lambda^2}{R^{\frac{1}{2}}} \mp \frac{4i\lambda^4}{R} + 4a^2 &= 0 . \end{aligned}\right\} \qquad (4.4.13)$$

Thus, for the TRANSITION REGION, we have that

$$\left.\begin{aligned} \lambda_2 &= (4iR)^{\frac{1}{2}} + O(R^{-\frac{1}{2}}) , \\ \lambda_3^2,\ \lambda_1^2 &= \frac{R^{\frac{1}{2}}}{2}\left\{k \pm (k^2 - 4a^2 i)^{\frac{1}{2}}\right\} + O(1) \\ \text{when} \quad p &= 2i + \frac{k}{R^{\frac{1}{2}}} , \end{aligned}\right\} \qquad (4.4.14)$$

and

$$\left.\begin{aligned} \lambda_3 &= (-4iR)^{\frac{1}{2}} + O(R^{-\frac{1}{2}}) , \\ \lambda_2^2,\ \lambda_1^2 &= \frac{R^{\frac{1}{2}}}{2}\left\{k \pm (k^2 + 4a^2 i)^{\frac{1}{2}}\right\} + O(1) \\ \text{when} \quad p &= -2i + \frac{k}{R^{\frac{1}{2}}} . \end{aligned}\right\} \qquad (4.4.15)$$

These solutions of the cubic equation (4.3.13) could have been obtained by applying Cardan's method [41, p.117] to find the general solution and then making the appropriate approximations for each region.

We have found that three different regions occur, namely

a) THE OUTER REGION when $|p \pm 2i| > O(R^{-\frac{1}{2}})$ where the roots are given by (4.4.6), (4.4.7), and (4.4.8);

b) THE INNER REGION when $|p \pm 2i| < O(R^{-\frac{1}{2}})$ where the roots are given by (4.4.11) and (4.4.12);

c) <u>THE TRANSITION REGION</u> when $\left|p \pm 2i\right| = O(R^{-\frac{1}{2}})$ where the roots are given by (4.4.14) and (4.4.15).

We can see that to the highest order if $\left|k^2\right| \gg 4a^2$ then the roots in the transition region (4.4.14), (4.4.15) tend to the roots in the outer region (4.4.6), (4.4.7), (4.4.8), while if $\left|k^2\right| \ll 4a^2$ the roots in the transition region tend to the values calculated for the inner region (4.4.11) and (4.4.12). This justifies the choice of suffixes for the roots of (4.3.13) in the inner region. It should be noticed that, when $a \equiv 0$ the roots of (4.3.13) are given by the outer expansions (4.4.3) for all values of p.

4.5 THE GENERAL SOLUTION OF THE TRANSFORMED EQUATIONS

The solution of equation (4.3.10) is

$$\begin{aligned} \bar{\mathfrak{f}}_1 = {} & A_1 \cosh\{\lambda_1 z\} + A_2 \cosh\{\lambda_2 z\} + A_3 \cosh\{\lambda_3 z\} \\ & + B_1 \sinh\{\lambda_1 z\} + B_2 \sinh\{\lambda_2 z\} + B_3 \sinh\{\lambda_3 z\} \\ & + A \cosh\{\chi(1-z)\}, \end{aligned} \tag{4.5.1}$$

where λ_1, λ_2, λ_3 are the roots of (4.3.13) discussed above, A is given by (4.3.12) and A_1, A_2, A_3, B_1, B_2, B_3 are constants which are determined by the boundary conditions (4.3.4(a)).

The boundary conditions (4.3.4(a)) yield the following two systems of equations.

$$\left.\begin{aligned}
&A_1 \cosh \lambda_1 + A_2 \cosh \lambda_2 + A_3 \cosh \lambda_3 + \tfrac{1}{2}A(\cosh\{2\chi\} + 1) = 0 \,, \\
&\lambda_1^2 A_1 \cosh \lambda_1 + \lambda_2^2 A_2 \cosh \lambda_2 + \lambda_3^2 A_3 \cosh \lambda_3 + \tfrac{1}{2}A\chi^2(\cosh\{2\chi\} + 1) = 0 \,, \\
&A_1\left[\frac{(a^2 + Rp)}{\lambda_1} - \lambda_1\right]\sinh \lambda_1 + A_2\left[\frac{(a^2 + Rp)}{\lambda_2} - \lambda_2\right]\sinh \lambda_2 \\
&+ A_3\left[\frac{(a^2 + Rp)}{\lambda_3} - \lambda_3\right]\sinh \lambda_3 + \tfrac{1}{2}A \sinh\{2\chi\}\left[\frac{(a^2 + Rp)}{\chi} - \chi\right] = 0 .
\end{aligned}\right\}$$

(4.5.2)

$$\left.\begin{aligned}
&B_1 \sinh \lambda_1 + B_2 \sinh \lambda_2 + B_3 \sinh \lambda_3 + \tfrac{1}{2}A(1 - \cosh\{2\chi\}) = 0 \,, \\
&\lambda_1^2 B_1 \sinh \lambda_1 + \lambda_2^2 B_2 \sinh \lambda_2 + \lambda_3^2 B_3 \sinh \lambda_3 + \tfrac{1}{2}A\chi^2(1 - \cosh\{2\chi\}) = 0 \,, \\
&B_1\left[\frac{(a^2 + Rp)}{\lambda_1} - \lambda_1\right]\cosh \lambda_1 + B_2\left[\frac{(a^2 + Rp)}{\lambda_2} - \lambda_2\right]\cosh \lambda_2 \\
&+ B_3\left[\frac{(a^2 + Rp)}{\lambda_3} - \lambda_3\right]\cosh \lambda_3 + \tfrac{1}{2}A \sinh\{2\chi\}\left[\chi - \frac{(a^2 + Rp)}{\chi}\right] = 0 .
\end{aligned}\right\}$$

(4.5.3)

It should be noticed that when $\overline{w}_1$ is found by substituting (4.5.1) in equation (4.3.2) another arbitrary constant arises which must be identically zero in order to satisfy the equation (4.3.3).

The above systems of equations (4.5.2) and (4.5.3) can be solved to give

$$B_1 = \frac{A}{2}\Big\{Rp(1-\sigma)\frac{\sinh\{2\chi\}}{\chi}(\lambda_2^2-\lambda_3^2)\sinh\lambda_2\sinh\lambda_3$$

$$+(\chi^2-\lambda_3^2)(1-\cosh\{2\chi\})\left[\frac{(a^2+Rp)}{\lambda_2}-\lambda_2\right]\cosh\lambda_2\sinh\lambda_3$$

$$+(\lambda_2^2-\chi^2)(1-\cosh\{2\chi\})\left[\frac{(a^2+Rp)}{\lambda_3}-\lambda_3\right]\cosh\lambda_3\sinh\lambda_2\Big\}$$

$$\div\Big\{\left[\frac{(a^2+Rp)}{\lambda_1}-\lambda_1\right]\cosh\lambda_1\sinh\lambda_2\sinh\lambda_3(\lambda_2^2-\lambda_3^2)$$

$$+(\lambda_3^2-\lambda_1^2)\left[\frac{(a^2+Rp)}{\lambda_2}-\lambda_2\right]\sinh\lambda_1\cosh\lambda_2\sinh\lambda_3$$

$$+(\lambda_1^2-\lambda_2^2)\left[\frac{(a^2+Rp)}{\lambda_3}-\lambda_3\right]\sinh\lambda_1\sinh\lambda_2\cosh\lambda_3\Big\}, \qquad (4.5.4)$$

$$B_2 = \frac{1}{(\lambda_2^2-\lambda_3^2)\sinh\lambda_2}\Big\{(\lambda_3^2-\chi^2)\frac{A}{2}(1-\cosh\{2\chi\})+(\lambda_3^2-\lambda_1^2)B_1\sinh\lambda_1\Big\} \qquad (4.5.5)$$

$$B_3 = \frac{1}{(\lambda_3^2-\lambda_2^2)\sinh\lambda_3}\Big\{(\lambda_2^2-\chi^2)\frac{A}{2}(1-\cosh\{2\chi\})+(\lambda_2^2-\lambda_1^2)B_1\sinh\lambda_1\Big\} \qquad (4.5.6)$$

$$A_1 = \frac{A}{2}\Big\{-Rp(1-\sigma)\frac{\sinh\{2\chi\}}{\chi}(\lambda_2^2-\lambda_3^2)\cosh\lambda_2\cosh\lambda_3$$

$$+(\chi^2-\lambda_3^2)(1+\cosh\{2\chi\})\sinh\lambda_2\cosh\lambda_3\left[\frac{(a^2+Rp)}{\lambda_2}-\lambda_2\right]$$

$$+(\lambda_2^2-\chi^2)(1+\cosh\{2\chi\})\sinh\lambda_3\cosh\lambda_2\left[\frac{(a^2+Rp)}{\lambda_3}-\lambda_3\right]\Big\}$$

$$\div \Bigg\{ \left[\frac{(a^2 + Rp)}{\lambda_1} - \lambda_1 \right] (\lambda_2^2 - \lambda_3^2) \cosh \lambda_2 \cosh \lambda_3 \sinh \lambda_1$$

$$+ (\lambda_3^2 - \lambda_1^2) \cosh \lambda_1 \sinh \lambda_2 \cosh \lambda_3 \left[\frac{(a^2 + Rp)}{\lambda_2} - \lambda_2 \right]$$

$$+ (\lambda_1^2 - \lambda_2^2) \cosh \lambda_1 \cosh \lambda_2 \sinh \lambda_3 \left[\frac{(a^2 + Rp)}{\lambda_3} - \lambda_3 \right] \Bigg\} , \tag{4.5.7}$$

$$A_2 = \frac{1}{(\lambda_2^2 - \lambda_3^2)\cosh \lambda_2} \left\{ (\lambda_3^2 - \chi^2) \frac{A}{2} (1 + \cosh \{2\chi\}) + (\lambda_3^2 - \lambda_1^2) A_1 \cosh \lambda_1 \right\} , \tag{4.5.8}$$

$$A_3 = \frac{1}{(\lambda_3^2 - \lambda_2^2)\cosh \lambda_3} \left\{ (\lambda_2^2 - \chi^2) \frac{A}{2} (1 + \cosh \{2\chi\}) + (\lambda_2^2 - \lambda_1^2) A_1 \cosh \lambda_1 \right\} . \tag{4.5.9}$$

Then, from (4.5.4), we see that $B_1 \sinh \{\lambda_1 z\}$ has no branch points, since for small λ_1 (or λ_2, or λ_3) both the numerator and the denominator have λ_1 (or λ_2, or λ_3) as a common factor leaving only even powers of λ_1 (or λ_2, or λ_3) in the expression $B_1 \sinh \{\lambda_1 z\}$. Similarly we find, from (4.5.5), (4.5.6), (4.5.7), (4.5.8), (4.5.9), that $B_2 \sinh \{\lambda_2 z\}$, $B_3 \sinh \{\lambda_3 z\}$, $A_1 \cosh \{\lambda_1 z\}$, $A_2 \cosh \{\lambda_2 z\}$, $A_3 \cosh \{\lambda_3 z\}$ have no branch points. Hence we see that $\overline{\mathcal{T}}_1$ given by (4.5.1), as a function of p, has no branch points.

4.6 NORMAL MODE METHOD

Before we evaluate the inverse Laplace transform (4.3.7) we will consider the results obtained by applying a normal mode argument to the above problem. This will determine the frequencies, but not the amplitudes, for all possible modes of oscillation. The solution (4.3.8) for the temperature variation, ϕ, shows that after a time of order $\{R\,\sigma\,\Omega^{-1}/(a^2 + \frac{1}{4}\pi^2)\}$ ϕ attains a steady value, ϕ_s. Thus we will seek a solution of the form

$$\mathcal{S}_1(z,t) = \mathcal{S}_{1s}(z) + e^{pt}f(z)\,, \qquad (4.6.1)$$

of the equations (4.2.20) and (4.2.21) when ϕ is replaced by ϕ_s, subject to the boundary conditions

$$\mathcal{S}_1 = D^2\mathcal{S}_1 = w_1 = 0 \quad \text{on} \quad z = \pm 1\,. \qquad (4.6.2)$$

This problem is equivalent to solving the equations

$$R^{-2}\left[D^2 - a^2\right]^3 \mathcal{S}_{1s} + 4D^2\mathcal{S}_{1s} = -2a^2 H D\phi_s\,, \qquad (4.6.3)$$

$$\left[D^2 - a^2\right]\left[R^{-1}\left[D^2 - a^2\right] - p\right]^2 f + 4D^2 f = 0\,, \qquad (4.6.4)$$

subject to the homogeneous boundary conditions obtained from (4.6.2).

The equation (4.6.3) serves to determine the steady solution, which will be discussed later, while (4.6.4) produces either amplifying or decaying modes. A solution of (4.6.4) is (4.5.1) with A identically zero and the boundary conditions yield the two systems of equations (4.5.2) and (4.5.3) with A equated to zero. For a non-trivial solution we require either

$$\begin{vmatrix} \cosh\lambda_1 & \cosh\lambda_2 & \cosh\lambda_3 \\ \lambda_1^2\cosh\lambda_1 & \lambda_2^2\cosh\lambda_2 & \lambda_3^2\cosh\lambda_3 \\ \left[\frac{(a^2+Rp)}{\lambda_1}-\lambda_1\right]\sinh\lambda_1 & \left[\frac{(a^2+Rp)}{\lambda_2}-\lambda_2\right]\sinh\lambda_2 & \left[\frac{(a^2+Rp)}{\lambda_3}-\lambda_3\right]\sinh\lambda_3 \end{vmatrix} = 0, \quad (4.6.5)$$

or

$$\begin{vmatrix} \sinh\lambda_1 & \sinh\lambda_2 & \sinh\lambda_3 \\ \lambda_1^2\sinh\lambda_1 & \lambda_2^2\sinh\lambda_2 & \lambda_3^2\sinh\lambda_3 \\ \left[\frac{(a^2+Rp)}{\lambda_1}-\lambda_1\right]\cosh\lambda_1 & \left[\frac{(a^2+Rp)}{\lambda_2}-\lambda_2\right]\cosh\lambda_2 & \left[\frac{(a^2+Rp)}{\lambda_3}-\lambda_3\right]\cosh\lambda_3 \end{vmatrix} = 0 \quad (4.6.6)$$

We find that (4.6.5) reduces to

$$(\lambda_2^2-\lambda_3^2)\left[\frac{(a^2+Rp)}{\lambda_1}-\lambda_1\right]\sinh\lambda_1\cosh\lambda_2\cosh\lambda_3$$

$$+\ (\lambda_3^2-\lambda_1^2)\left[\frac{(a^2+Rp)}{\lambda_2}-\lambda_2\right]\cosh\lambda_1\sinh\lambda_2\cosh\lambda_3$$

$$+ \; (\lambda_1^2 - \lambda_2^2)\left[\frac{(a^2 + Rp)}{\lambda_3} - \lambda_3\right]\cosh\lambda_1 \cosh\lambda_2 \sinh\lambda_3 = 0 \,, \qquad (4.6.7)$$

while (4.6.6) becomes

$$(\lambda_2^2 - \lambda_3^2)\left[\frac{(a^2 + Rp)}{\lambda_1} - \lambda_1\right]\cosh\lambda_1 \sinh\lambda_2 \sinh\lambda_3$$

$$+ \; (\lambda_3^2 - \lambda_1^2)\left[\frac{(a^2 + Rp)}{\lambda_2} - \lambda_2\right]\sinh\lambda_1 \cosh\lambda_2 \sinh\lambda_3$$

$$+ \; (\lambda_1^2 - \lambda_2^2)\left[\frac{(a^2 + Rp)}{\lambda_3} - \lambda_3\right]\sinh\lambda_1 \sinh\lambda_2 \cosh\lambda_3 = 0. \qquad (4.6.8)$$

We now wish to evaluate the zeros of (4.6.7) and (4.6.8) on the assumption that R is large. If we seek a zero of the form $p = O(R^{\beta})$ where $\beta \geq 0$ and use the outer expansions for λ_1, λ_2, λ_3 then the highest order term in (4.6.7) is

$$(\lambda_2^2 - \lambda_3^2)\,\frac{Rp}{\lambda_1} \sinh\lambda_1 \cosh\lambda_2 \cosh\lambda_3 \,, \qquad (4.6.9)$$

and in (4.6.8) is

$$(\lambda_2^2 - \lambda_3^2)\,\frac{Rp}{\lambda_1} \cosh\lambda_1 \sinh\lambda_2 \sinh\lambda_3 \,. \qquad (4.6.10)$$

For these highest order terms to be zero we require $\beta = 0$ and for (4.6.9) either $p = 0$ or $\sinh\lambda_1 = 0$ or $\cosh\lambda_2 = 0$ or $\cosh\lambda_3 = 0$, while for (4.6.10) either $p = 0$ or $\cosh\lambda_1 = 0$ or $\sinh\lambda_3 = 0$ or $\sinh\lambda_2 = 0$.

Firstly, for the zero at $p = 0$ in (4.6.9), the next term in the power series in R for this root of (4.6.7) is

$$p = -\frac{1}{R^{\frac{1}{2}}} + O(R^{-1}) . \qquad (4.6.11)$$

This represents a steadily decaying mode which will be referred to as the spin-up mode because any contribution to the flow, found from this term, will decay within a time of order $R^{\frac{1}{2}} \Omega^{-1}$, the spin-up time.

For the zero at $p = 0$ in (4.6.10), we find that the root of (4.6.8) associated with this zero is

$$p = -\frac{a^2}{R} + O(R^{-9/2}) . \qquad (4.6.12)$$

This represents a steadily decaying mode which vanishes within a time of order $\{R \Omega^{-1}/a^2\}$.

For $\sinh \lambda_1 = 0$, we require that

$$\lambda_1 = \pm i m_1 \pi , \qquad m_1 = 1,2,3,\ldots ,$$

which becomes upon using the outer expansions

$$p = \pm \frac{2 i m_1 \pi}{(a^2 + m_1^2 \pi^2)^{\frac{1}{2}}} .$$

This term in the power series for p represents a pure oscillation and further terms must be determined in order to discover whether this oscillation is, in fact, amplified or damped.

When the next term in the power series is calculated, we find that the roots of (4.6.7) are situated at

$$p = \pm \frac{2im_1\pi}{(a^2 + m_1^2\pi^2)^{\frac{1}{2}}} + \frac{2a^2\left[-\left(\frac{\gamma_1^2}{\beta_1} + \frac{\beta_1^2}{\gamma_1}\right) \pm i\left(\frac{\gamma_1^2}{\beta_1} - \frac{\beta_1^2}{\gamma_1}\right)\right]}{R^{\frac{1}{2}}\quad(a^2 + m_1^2\pi^2)}, \quad (4.6.13)$$

where

$$\gamma_1 = \left\{1 + (1 + a^2/m_1^2\pi^2)^{-\frac{1}{2}}\right\}^{\frac{1}{2}},$$

$$\beta_1 = \left\{1 - (1 + a^2/m_1^2\pi^2)^{-\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

These roots (4.6.13) represent two infinite series of decaying oscillations which vanish within a time of order

$$\frac{R^{\frac{1}{2}}\,\Omega^{-1}(a^2 + m_1^2\pi^2)}{a^2\left[\frac{\gamma_1^2}{\beta_1} + \frac{\beta_1^2}{\gamma_1}\right]}.$$

This time increases like $\left\{R^{\frac{1}{2}}m_1\pi\,\Omega^{-1}/a\right\}$ as a becomes small or m_1 becomes large.

Similarly, for $\cosh\lambda_1 = 0$, we find that the roots of (4.6.8) are situated at

$$p = \pm \frac{(2m_2 + 1)i\pi}{(a^2 + \frac{1}{4}\pi^2(2m_2+1)^2)^{\frac{1}{2}}} + \frac{2a^2\left[-(\frac{\gamma_2^2}{\beta_2} + \frac{\beta_2^2}{\gamma_2}) \pm i(\frac{\gamma_2^2}{\beta_2} - \frac{\beta_2^2}{\gamma_2})\right]}{R^{\frac{1}{2}}\ (a^2 + \frac{1}{4}\pi^2(2m_2 + 1)^2)} , \tag{4.6.14}$$

where

$$m_2 = 0, 1, 2, \ldots,$$

$$\gamma_2 = \left\{1 + (1 + 4a^2/(2m_2 + 1)^2\pi^2)^{-\frac{1}{2}}\right\}^{\frac{1}{2}} ,$$

$$\beta_2 = \left\{1 - (1 + 4a^2/\pi^2(2m_2 + 1)^2)^{-\frac{1}{2}}\right\}^{\frac{1}{2}} .$$

These roots (4.6.14) represent another two series of decaying oscillations which vanish within a time of order

$$\frac{R^{\frac{1}{2}}\ \Omega^{-1}(a^2 + \frac{1}{4}\pi^2(2m_2 + 1)^2)}{a^2\left[\frac{\gamma_2^2}{\beta_2} + \frac{\beta_2^2}{\gamma_2}\right]} .$$

When $a \longrightarrow 0$ or m_1 and $m_2 \longrightarrow \infty$, in (4.6.13) and (4.6.14) we find that $p \longrightarrow \pm 2i$. Hence, when

$$\beta_1^2 \text{ or } \beta_2^2 \leq O(R^{-\frac{1}{2}}) , \tag{4.6.15}$$

the outer expansions for the λ's cease to be valid and we must reconsider the analysis using the expansions for either the transition or the inner region. Hence we have that (4.6.13) and (4.6.14) are roots of the equations (4.6.7) and (4.6.8) provided we choose m_1 and m_2 so that (4.6.15) is not satisfied. We will assume that M_1, M_2 are the maximum values of m_1, m_2 respectively which render (4.6.15) false.

Also for sinh λ_2, sinh λ_3, cosh λ_2, cosh λ_3 to be zero we require that p is near to $\pm 2i$ and again we must revise the above analysis for the highest order terms using the expansions for either the transition region or the inner region. Then for $p = 2i + O(R^{-\frac{1}{2}})$, the dominant terms being equated to zero is equivalent to

$$\lambda_1^{-1} \tanh \lambda_1 = \lambda_3^{-1} \tanh \lambda_3 \quad \text{for (4.6.7)}, \qquad (4.6.16)$$

$$\lambda_1 \tanh \lambda_1 = \lambda_3 \tanh \lambda_3 \quad \text{for (4.6.8)}. \qquad (4.6.17)$$

We now wish to determine the solutions of (4.6.16) and (4.6.17) which satisfy the additional condition

$$\lambda_1^2 \lambda_3^2 = a^2 iR , \qquad (4.6.18)$$

which is obtained from (4.4.10) or (4.4.13). If we assume that $\lambda_1(\lambda_3)$ is real or purely imaginary then, from (4.6.16) and (4.6.17), we require $\lambda_3^{-1} \tanh \lambda_3$ ($\lambda_1^{-1} \tanh \lambda_1$) and $\lambda_3 \tanh \lambda_3$ ($\lambda_1 \tanh \lambda_1$) to be real, which implies that $\lambda_3(\lambda_1)$ must be real or purely imaginary and hence contradicts the condition (4.6.18). Therefore the only solutions of (4.6.16) and (4.6.17) which satisfy (4.6.18) occur when λ_1 and λ_3 are complex numbers with positive real part and are given by

$$\lambda_1 = \lambda_3 ,$$

provided $a > O(R^{-\frac{1}{2}})$. It should be noticed that λ_1, λ_3 are very small when $a \leq O(R^{-\frac{1}{2}})$ and other solutions of (4.6.16) can exist. The special case $a = 0$ will be considered later.

The particular case that arises when $\lambda_1 = \lambda_3$ will now be considered. We see, from (4.4.14), that $\lambda_1 = \lambda_3$ when

$$k^2 = 4a^2 i \qquad \text{and} \qquad p = 2i \pm \frac{(4a^2 i)^{\frac{1}{2}}}{R^{\frac{1}{2}}} . \qquad (4.6.19)$$

The complementary function used in (4.5.1) is not applicable when $\lambda_1 = \lambda_3$ because this would introduce only four arbitrary constants. Instead, for this specific value of p, (4.6.19), we replace (4.5.1) by

$$\begin{aligned} \overline{\zeta}_1 &= C_1 \cosh\{\lambda_1 z\} + C_2 \cosh\{\lambda_2 z\} + zC_3 \cosh\{\lambda_1 z\} \\ &+ D_1 \sinh\{\lambda_1 z\} + D_2 \sinh\{\lambda_2 z\} + zD_3 \sinh\{\lambda_1 z\} \qquad (4.6.20) \\ &+ A \cosh\{\chi(1-z)\} , \end{aligned}$$

where C_1, C_2, C_3, D_1, D_2, D_3 are arbitrary constants which are determined by the boundary conditions (4.3.4(a)), A is given by (4.3.12) and

$$\lambda_2 = (4iR)^{\frac{1}{2}}, \quad \lambda_1 = R^{1/4} a^{\frac{1}{2}} \exp\{i\pi/8\} \quad \text{or} \quad R^{1/4} a^{\frac{1}{2}} \exp\{-3\pi i/8\}, \qquad (4.6.21)$$

depending on whether the positive or negative sign is chosen in (4.6.19).

When we apply the normal mode argument to (4.6.20), we find that the determinants (4.6.5) and (4.6.6) are replaced by

$$\begin{vmatrix} \cosh\lambda_1 & \cosh\lambda_2 & \sinh\lambda_1 \\ \lambda_1^2\cosh\lambda_1 & \lambda_2^2\cosh\lambda_2 & 2\lambda_1\cosh\lambda_1 + \lambda_1^2\sinh\lambda_1 \\ \left[\frac{(a^2+Rp)}{\lambda_1} - \lambda_1\right]\sinh\lambda_1 & \left[\frac{(a^2+Rp)}{\lambda_2} - \lambda_2\right]\sinh\lambda_2 & \left\{\sinh\lambda_1\left[-1-\frac{(a^2+Rp)}{\lambda_1^2}\right] + \cosh\lambda_1\left[\frac{(a^2+Rp)}{\lambda_1} - \lambda_1\right]\right\} \end{vmatrix} = 0, \qquad (4.6.22)$$

$$\begin{vmatrix} \sinh\lambda_1 & \sinh\lambda_2 & \cosh\lambda_1 \\ \lambda_1^2\sinh\lambda_1 & \lambda_2^2\sinh\lambda_2 & \lambda_1^2\cosh\lambda_1 + 2\lambda_1\sinh\lambda_1 \\ \left[\frac{(a^2+Rp)}{\lambda_1} - \lambda_1\right]\cosh\lambda_1 & \left[\frac{(a^2+Rp)}{\lambda_2} - \lambda_2\right]\cosh\lambda_2 & \left\{\cosh\lambda_1\left[-1-\frac{(a^2+Rp)}{\lambda_1^2}\right] + \sinh\lambda_1\left[\frac{(a^2+Rp)}{\lambda_1} - \lambda_1\right]\right\} \end{vmatrix} = 0. \qquad (4.6.23)$$

We find that (4.6.22) and (4.6.23) reduce to

$$\cosh \lambda_2 \left\{ \left[1 + \frac{(a^2+Rp)}{\lambda_1^2}\right] \frac{\sinh\{2\lambda_1\}}{2} (\lambda_1^2 - \lambda_2^2) + \left[\frac{(a^2+Rp)}{\lambda_1} - \lambda_1\right] (\lambda_2^2 - \lambda_1^2) \right.$$

$$\left. + \lambda_1 \sinh\{2\lambda_1\} \left[\frac{(a^2+Rp)}{\lambda_1} - \lambda_1\right] \right\} + \sinh \lambda_2 \left[\frac{(a^2+Rp)}{\lambda_2} - \lambda_2\right] (-2\lambda_1 \cosh^2\lambda_1)$$

$$= 0 , \qquad (4.6.24)$$

$$\sinh \lambda_2 \left\{ \left[1 + \frac{(a^2+Rp)}{\lambda_1^2}\right] \frac{\sinh\{2\lambda_1\}}{2} (\lambda_1^2 - \lambda_2^2) + \left[\frac{(a^2+Rp)}{\lambda_1} - \lambda_1\right] (\lambda_1^2 - \lambda_2^2) \right.$$

$$\left. + \left[\frac{(a^2+Rp)}{\lambda_1} - \lambda_1\right] \lambda_1 \sinh\{2\lambda_1\} \right\} + \cosh \lambda_2 \left[\frac{(a^2+Rp)}{\lambda_2} - \lambda_2\right] (-2\lambda_1 \sinh^2\lambda_1)$$

$$= 0 . \qquad (4.6.25)$$

When we use (4.6.21) we find that the highest order terms in (4.6.24) and (4.6.25) are

$$(\cosh \lambda_2)\lambda_2^2 \left\{ -\frac{Rp}{\lambda_1^2} \frac{\sinh\{2\lambda_1\}}{2} + \frac{Rp}{\lambda_1} \right\}, \qquad (4.6.26)$$

$$(\sinh \lambda_2)\lambda_2^2 \left\{ -\frac{Rp}{\lambda_1^2} \frac{\sinh\{2\lambda_1\}}{2} - \frac{Rp}{\lambda_1} \right\}, \qquad (4.6.27)$$

respectively. For the expressions (4.6.26) and (4.6.27) to be zero we require

$$\sinh 2\lambda_1 = \pm 2\lambda_1 . \qquad (4.6.28)$$

When we write $2\lambda_1 = X + iY$ we find from (4.6.21) and (4.6.28) that

$$X > 0, \quad \frac{Y}{X} = \tan\frac{\pi}{8} \text{ or } \tan(-\frac{3\pi}{8}) , \qquad (4.6.29)$$

$$\sinh X \cos Y = \pm X , \qquad (4.6.30)$$

$$\cosh X \sin Y = \pm Y . \qquad (4.6.31)$$

Hence we have three equations (4.6.29), (4.6.30), (4.6.31), for the two unknown quantities X and Y and therefore, in general, there is no solution.

We will now demonstrate that the only solution of (4.6.28) is $a \equiv 0$ when (4.6.29) is not applicable. We will now assume $a \neq 0$.

If we write

$$2\lambda_1 = iZ = i(x + iy) ,$$

where x,y are real, then (4.6.28) and (4.6.29) become

$$\sin Z = \pm Z , \qquad (4.6.32)$$

$$\frac{x}{y} = -\tan(\frac{\pi}{8}) \text{ or } -\tan(-3\pi/8)$$
$$= -.4142 \text{ or } 2.4142 . \qquad (4.6.33)$$

Then Hillman and Salzer [18] have tabulated the first ten roots of $\sin Z = Z$, and Robbins and Smith [28] have tabulated the first ten roots of $\sin Z = -Z$. From these tabulated results, which are shown in

TABLE 4.1

n	x	y	x/y
1	7.4977	2.7687	2.7078
2	13.9000	3.3522	4.1466
3	20.2385	3.7168	5.4450
4	26.5546	3.9831	6.6681
5	32.8597	4.1933	7.8379
6	39.1588	4.3668	8.9682
7	45.4541	4.5146	10.067
8	51.7468	4.6434	11.146
9	58.0377	4.7575	12.198
10	64.3272	4.8599	13.238

(a) The first ten roots of $\sin Z = Z$ as given by Hillman and Salzer to four decimal places.

n	x	y	x/y
0	4.2124	2.2507	1.8711
1	10.7125	3.1032	3.4514
2	17.0734	3.5511	4.8073
3	23.3984	3.8588	6.0632
4	29.7081	4.0937	7.2579
5	36.0099	4.2838	8.4062
6	42.3068	4.4435	9.5212
7	48.6007	4.5811	10.610
8	54.8924	4.7021	11.673
9	61.1826	4.8100	12.721

(b) The first ten roots of $\sin Z = -Z$ as given by Robbins and Smith to four decimal places.

Table 4.1, we can see that the ratio (4.6.33) is never satisfied and hence for $X \leq 4.8$, which is the range covered by the tabulated results, we have no roots of (4.6.32) which satisfy (4.6.33).

Then for $X > 4.8$ we can replace sinh X and cosh X in (4.6.30) and (4.6.31) by $\frac{1}{2}e^X$ to give

$$e^X \cos Y = \pm 2X , \qquad (4.6.34)$$

$$e^X \sin Y = \pm 2Y . \qquad (4.6.35)$$

Then we see immediately from (4.6.29), (4.6.34) and (4.6.35) that

$$\tan Y = \frac{Y}{X} = \tan \frac{\pi}{8} \text{ or } \tan(-\frac{3\pi}{8}),$$

and hence

$$\left.\begin{aligned} Y &= \frac{\pi}{8} + n\pi , \\ \text{or} \quad Y &= -\frac{3\pi}{8} + n\pi , \end{aligned}\right\} \qquad (4.6.36)$$

where $\quad n = 0, \pm 1, \pm 2, \ldots \quad .$

When we substitute (4.6.36) into the equations (4.6.29) and (4.6.34) we find

$$X = \frac{\frac{\pi}{8} + n\pi}{\tan \frac{\pi}{8}} \quad \text{and} \quad e^X = \pm \frac{2X(-1)^n}{\cos \frac{\pi}{8}}, \qquad (4.6.37)$$

or

$$X = \frac{-\frac{3\pi}{8} + n\pi}{\tan(-\frac{3\pi}{8})} \quad \text{and} \quad e^X = \frac{\pm 2X(-1)^n}{\cos\frac{3\pi}{8}} \ . \qquad (4.6.38)$$

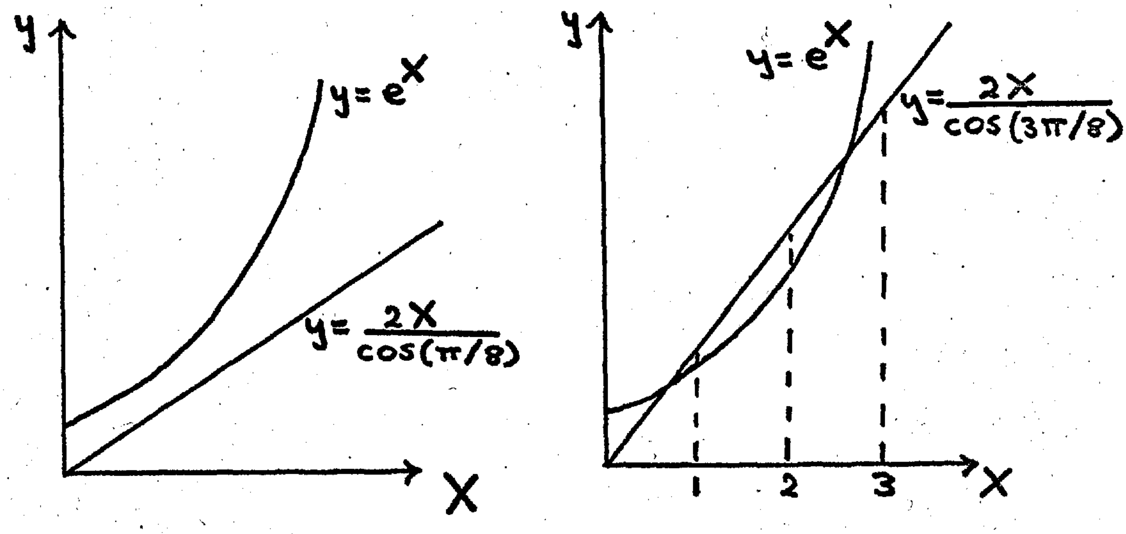


FIG.4.3: Sketch to show the position of the roots of

$$\text{(a)} \quad e^X = \frac{2X}{\cos\frac{\pi}{8}} \ , \qquad \text{(b)} \quad = e^X = \frac{2X}{\cos\frac{3\pi}{8}} \ .$$

Hence we see, from Fig.4.3(a), that (4.6.37) is never satisfied, while, from Fig.4.3(b), that (4.6.38) is not satisfied for $X \geq 3$. Therefore for $X > 4.8$ there are no roots of (4.6.28) which satisfy (4.6.21).

Hence there are no normal modes produced when $p = 2i \pm \frac{(4a^2 i)^{\frac{1}{2}}}{R^{\frac{1}{2}}}$, $a \neq 0$. Similarly we can show that no normal modes are produced when $p = -2i \pm \frac{(-4a^2 i)^{\frac{1}{2}}}{R^{\frac{1}{2}}}$, $a \neq 0$, which occurs when $\lambda_1 = \lambda_2$.

Thus from the above analysis we find that apart from the steady solution all the possible modes decay with time and consist of two steadily decaying modes (4.6.11) and (4.6.12) and four series of damped oscillations (4.6.13) and (4.6.14), provided $a > O(R^{-\frac{1}{2}})$. It should be noticed that none of these modes involve the Prandtl number σ. For all other values of p, f(z) is identically zero.

4.7 THE SPECIAL CASE $a \equiv 0$

When a is identically zero we see from (4.3.13) and (4.3.12) that

$$\left.\begin{aligned} &\lambda_1 = 0, \quad \lambda_2 = R^{\frac{1}{2}}(p+2i)^{\frac{1}{2}}, \quad \lambda_3 = R^{\frac{1}{2}}(p-2i)^{\frac{1}{2}}, \\ &A \equiv 0. \end{aligned}\right\} \quad (4.7.1)$$

Then we must replace the solution (4.5.1) by

$$\bar{\mathcal{P}}_1 = A_1 + A_2 \cosh\{\lambda_2 z\} + A_3 \cosh\{\lambda_3 z\}$$

$$+ B_1 z + B_2 \sinh\{\lambda_2 z\} + B_3 \sinh\{\lambda_3 z\}, \quad (4.7.2)$$

and the determinants (4.6.5) and (4.6.6) by

$$\begin{vmatrix} 1 & \cosh\lambda_2 & \cosh\lambda_3 \\ 0 & \lambda_2^2\cosh\lambda_2 & \lambda_3^2\cosh\lambda_3 \\ Rp & \left[\frac{Rp}{\lambda_2}-\lambda_2\right]\sinh\lambda_2 & \left[\frac{Rp}{\lambda_3}-\lambda_3\right]\sinh\lambda_3 \end{vmatrix} = 0, \quad (4.7.3)$$

$$\begin{vmatrix} 1 & \sinh\lambda_2 & \sinh\lambda_3 \\ 0 & \lambda_2^2\sinh\lambda_2 & \lambda_3^2\sinh\lambda_3 \\ (\frac{Rp}{2}-1) & \left[\frac{Rp}{\lambda_2}-\lambda_2\right]\cosh\lambda_2 & \left[\frac{Rp}{\lambda_3}-\lambda_3\right]\cosh\lambda_3 \end{vmatrix} = 0. \quad (4.7.4)$$

These determinants (4.7.3) and (4.7.4) may be written

$$\lambda_2^2\left[\frac{Rp}{\lambda_3}-\lambda_3\right]\sinh\lambda_3\cosh\lambda_2 - \lambda_3^2\left[\frac{Rp}{\lambda_2}-\lambda_2\right]\sinh\lambda_2\cosh\lambda_3$$
$$+ (\lambda_3^2-\lambda_2^2)Rp\cosh\lambda_2\cosh\lambda_3 = 0, \quad (4.7.5)$$

$$\lambda_2^2\left[\frac{Rp}{\lambda_3}-\lambda_3\right]\cosh\lambda_3\sinh\lambda_2 - \lambda_3^2\left[\frac{Rp}{\lambda_2}-\lambda_2\right]\cosh\lambda_2\sinh\lambda_3$$
$$+ (\lambda_3^2-\lambda_2^2)(\frac{Rp}{2}-1)\sinh\lambda_2\sinh\lambda_3 = 0. \quad (4.7.6)$$

When $p = O(R^{\beta})$, where $\beta \geq 0$, the dominant terms in (4.7.5) and

(4.7.6) are

$$\left.\begin{aligned}&(\lambda_3^2 - \lambda_2^2)Rp \cosh \lambda_2 \cosh \lambda_3 ,\\ &(\lambda_3^2 - \lambda_2^2)\frac{Rp}{2} \sinh \lambda_2 \sinh \lambda_3 ,\end{aligned}\right\} \qquad (4.7.7)$$

respectively, which implies $\beta = 0$ and $p = 0$ or $\cosh \lambda_2 = 0$ or $\sinh \lambda_2 = 0$ or $\cosh \lambda_3 = 0$ or $\sinh \lambda_3 = 0$ for (4.7.7) to vanish.

Firstly, for the zero of (4.7.7) at $p = 0$, the next term in the power series in R for this root of (4.7.5) is again (4.6.11), and of (4.7.6) is

$$p = -\frac{2}{R^{\frac{1}{2}}} + O(R^{-1}) . \qquad (4.7.8)$$

For the roots in the neighbourhood of $p = 2i$, we have that the dominant terms in (4.7.5) and (4.7.6) being equated to zero is equivalent to

$$\lambda_3^{-1} \tanh \lambda_3 = 1 , \qquad (4.7.9)$$

$$\lambda_3 \tanh \lambda_3 = 1 , \qquad (4.7.10)$$

respectively. The equation (4.7.9) can be satisfied by $\lambda_3 = 0$, but, for this case, we must replace (4.7.2) by

$$\Upsilon = A_1 + B_1 z + A_3 z^2 + B_3 z^3 + A_2 \cosh \{\lambda_2 z\} + B_2 \sinh \{\lambda_2 z\}, \quad (4.7.11)$$

and then the new determinants calculated from (4.7.11), corresponding to (4.7.3) and (4.7.4), are always found to be non-zero. Hence there is no normal mode solution when $\lambda_3 = 0$.

Similar results apply in the neighbourhood of $p = -2i$ with λ_3 replaced by λ_2.

Hence from the terms involving A_1, A_2, A_3 we have a steadily decaying mode (4.6.11) and two infinite series of decaying oscillations,

$$p = \pm 2i - \frac{\xi_n^2}{R} ,$$

where ξ_n are the positive roots of $\xi = \tan \xi$. These results are equivalent to the poles (3.7) and (3.8) found by Greenspan and Howard [17] and could have been obtained by letting a tend to zero in the previous analysis. While, from the terms involving B_1, B_2, B_3 we have the steadily decaying mode (4.7.8) and also two series of decaying oscillations,

$$p = \pm 2i - \frac{\eta_n^2}{R} ,$$

where η_n are the positive roots of $\eta \tan \eta = -1$. This case cannot be obtained by letting a tend to zero in the previous analysis.

When $a \equiv 0$ the steady solutions found from (4.6.3) and (4.3.9) are

$$\mathcal{Y}_{1s} \equiv 0 \quad \text{and} \quad \phi_s = \frac{\Phi(1-z)}{2}$$

Hence, when a uniform temperature is applied to the disk $z = -d$, the steady solution is a stable uniform temperature gradient.

4.8 THE LOCATION OF THE SINGULARITIES OF $\overline{\mathcal{Y}}$ REQUIRED FOR THE EVALUATION OF THE INVERSE LAPLACE TRANSFORM

We now return to the evaluation of the inverse Laplace transform (4.3.7) on the assumption of large R when $\overline{\mathcal{Y}}_1$ is given by equation (4.5.1). The singularities of $\overline{\mathcal{Y}}_1$ play an important role in the evaluation of the inverse integral and hence we begin by locating them. We see that, since A, (4.3.12), appears in all the coefficients A_1, A_2, A_3, B_1, B_2, B_3, $\overline{\mathcal{Y}}_1$ has simple poles at

$$p = 0, \tag{4.8.1}$$

$$p = \frac{1}{R\sigma}\left(-a^2 - \frac{m^2\pi^2}{4}\right), \quad m = 1,2,3,\dots . \tag{4.8.2}$$

The poles of A given by the roots of

$$Rp^3\sigma(\sigma-1)^2 + 4Rp\sigma + 4a^2 = 0, \tag{4.8.3}$$

are, in fact, regular points of $\overline{\mathcal{Y}}_1$ because this equation (4.8.3) for p is equivalent to χ being a root of (4.3.13). Hence χ must be equal to either λ_1 or λ_2 or λ_3 when $\sigma \neq 0$. There are no roots of (4.8.3) when $\sigma = 0$. If we suppose that $\chi = \lambda_i$, $i = 1,2,3$, then the particular integral (4.3.11) must be replaced by

$$\overline{\mathcal{F}}_1 = A^* z \sinh\{\lambda_i(1-z)\} , \qquad (4.8.4)$$

where $$A^* = \frac{2a^2 H \Phi}{p \sinh(2\lambda_i)\left\{-6R^{-2}(\lambda_i^2-a^2)^2 + 8pR^{-1}(\lambda_i^2-a^2) - 2(p^2+4)\right\}} . \qquad (4.8.5)$$

It can easily be shown that the denominator of (4.8.5) is non-zero when $\chi = \lambda_1$, $p = (-a^2/R\sigma)$; $\chi = \lambda_2$, $p = \{2i/(\sigma-1)\}$, $\sigma \neq 1$; $\chi = \lambda_3$, $p = \{-2i/(\sigma-1)\}$, $\sigma \neq 1$. When $\sigma = 1$ we see, from (4.8.3), that $p = (-a^2/R)$, which corresponds to $\chi = \lambda_1$, is the only root and hence $\chi = \lambda_2$ and $\chi = \lambda_3$ are not applicable. Hence we find, when $\chi = \lambda_i$, that A^* is always regular and therefore there is no residue contribution from the roots of (4.8.3).

The other singularities occur at the zeros of the denominator of A_1, (4.5.7), and B_1, (4.5.4), which are identical to (4.6.7) and (4.6.8), and have already been calculated. Also it seems that for B_2, (4.5.5), there are poles at $\sinh \lambda_2 = 0$ but these values for p also render the numerator zero and hence they are regular points. A similar argument applies for A_2, (4.5.8), B_2, (4.5.5), A_3, (4.5.9), B_3, (4.5.6).

To summarize, therefore, we have a pole at p = O, (4.8.1), which produces the steady solution, two steadily decaying modes, (4.6.11), (4.6.12), an infinite set of steadily decaying modes, (4.8.2) and four series of decaying oscillations (4.6.13) and (4.6.14), provided $a > O(R^{-\frac{1}{2}})$.

The inversion integral can now be evaluated using the calculus of residues [8, p.75] but before doing this we will consider the steady solution that persists after all the transient effects have decayed.

4.9 THE STEADY SOLUTION

The steady solution for the z-component of vorticity, ζ_{1s}, arises from the simple pole at p = O, (4.8.1), in equation (4.5.1) for $\overline{\zeta}_1$ or alternatively from the equation (4.6.3). When p = O, we have, from (4.4.6), (4.4.7) and (4.4.8), that

$$\lambda_1 = \frac{a^3}{2R}, \quad \lambda_2 = (2iR)^{\frac{1}{2}}, \quad \lambda_3 = (-2iR)^{\frac{1}{2}}, \qquad (4.9.1)$$

when only the highest order terms are retained. When $\lambda_1, \lambda_2, \lambda_3$ are given by (4.9.1) and $\chi = a$, we see that the residue at the pole, p = O, gives

$$\zeta_{1s} = \frac{Ha\,\Phi}{2\sinh\{2a\}}\left\{\cosh\{a(1-z)\} - \frac{(1-\cosh\{2a\})}{2}\left[\frac{a}{2R^{\frac{1}{2}}}\sinh(a^3 z/2R)\right.\right.$$

$$\left.\left. + \sinh\{(2iR)^{\frac{1}{2}}z\}\exp\{-(2iR)^{\frac{1}{2}}\} + \sinh\{(-2iR)^{\frac{1}{2}}z\}\exp\{-(-2iR)^{\frac{1}{2}}\}\right]\right.$$

$$+ \frac{(1+\cosh\{2a\})}{2}\left[-\cosh(a^3z/2R) - \frac{a^2}{R^{\frac{1}{2}}}\cosh\{(2iR)^{\frac{1}{2}}z\}\exp\{-(2iR)^{\frac{1}{2}}\}\right.$$

$$\left.\left. - \frac{a^2}{R^{\frac{1}{2}}}\cosh\{(-2iR)^{\frac{1}{2}}z\}\exp\{-(-2iR)^{\frac{1}{2}}\}\right]\right\} , \qquad (4.9.2)$$

where terms of order $R^{-\frac{1}{2}}$ compared with those written down have been neglected $a > O(R^{-\frac{1}{2}})$ and $a \ll R^{1/4}$. It should be noticed that (4.9.2) satisfies all the boundary conditions to the highest order. If only the highest order terms in (4.9.2) are kept then

$$\Upsilon_{1s} = \frac{Ha\,\bar{\Phi}}{2\sinh\{2a\}}\left\{\cosh\{a(1-z)\} - \frac{(1+\cosh\{2a\})}{2}\right\}$$

$$- \frac{Ha\,\Phi\,(1-\cosh\{2a\})}{4\sinh\{2a\}}\left[\sinh\{(2iR)^{\frac{1}{2}}z\}\exp\{-(2iR)^{\frac{1}{2}}\}\right.$$

$$\left. + \sinh\{(-2iR)^{\frac{1}{2}}z\}\exp\{-(-2iR)^{\frac{1}{2}}\}\right]. \qquad (4.9.3)$$

Then the actual steady vertical component of vorticity is from (4.2.18)

$$\Upsilon_s = h(x,y)\,\Upsilon_{1s} .$$

The second term in (4.9.3) represents Ekman layers on both disks, $z = \pm 1$, which have a depth of penetration of vorticity of order $(\nu/\Omega)^{\frac{1}{2}}$ while the first term represents the distribution of vorticity in the interior and is shown in Fig.4.4.

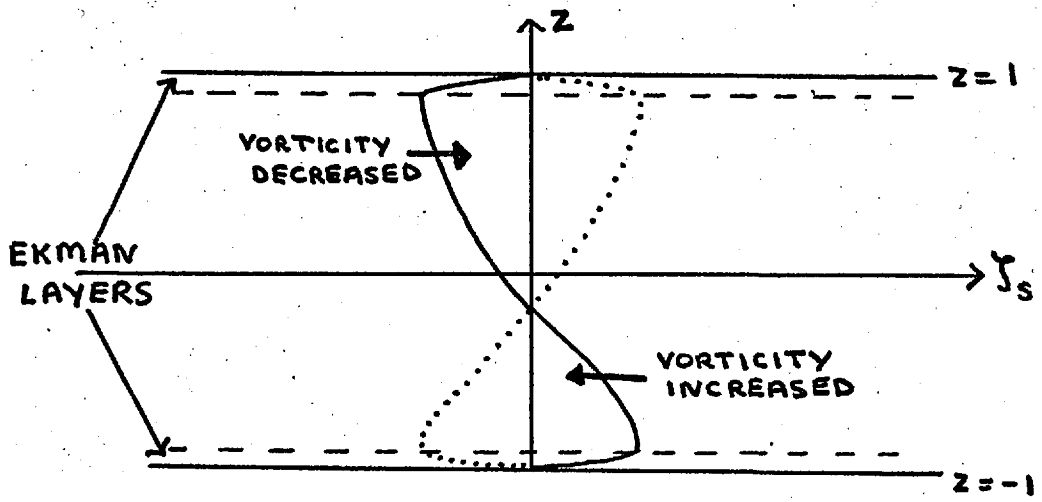


FIG.4.4: Sketch of the steady z-component of vorticity, ζ_s, where ——— represents $\Phi\, h(x,y) > 0$ and —······ represents $\Phi\, h(x,y) < 0$.

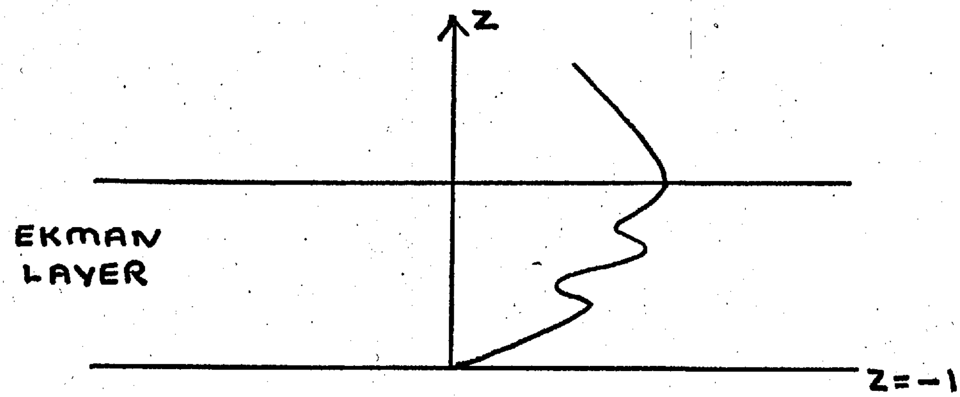


FIG.4.5: Sketch of the steady z-component of vorticity, ζ_{1s}, in the boundary layer on the disk, z = -1.

The steady z-component of vorticity will vanish in the interior when

$$\cosh\{a(1-z)\} = \cosh^2 a . \qquad (4.9.4)$$

When $a \rightarrow \infty$ and $a \rightarrow 0$ the zeros of (4.9.4) occur at

$$z = -1 + a^{-1} \log 2 ,$$

$$z = 1 - \sqrt{2} ,$$

respectively. Hence, in the interior, the zero of Υ_s is always on the negative z-axis and tends from $1 - \sqrt{2}$ to $-1 + a^{-1}\log 2$ as a increases.

We now wish to examine in more detail the flow in the boundary layers on the disks, $z = \pm 1$. The equation (4.9.3) may be written in the form

$$\Upsilon_{1s} = \frac{Ha\,\Phi}{2\sinh(2a)}\Bigg\{\cosh\left[a(1-z)\right] - \frac{(1+\cosh\{2a\})}{2}$$

$$+ \frac{(1-\cosh\{2a\})}{2}\Big[\exp\{-R^{\frac{1}{2}}(1+z)\}\cos\{R^{\frac{1}{2}}(1+z)\}$$

$$- \exp\{-R^{\frac{1}{2}}(1-z)\}\cos\{R^{\frac{1}{2}}(1-z)\}\Big]\Bigg\} . \qquad (4.9.5)$$

We will now concentrate on the boundary layer near the disk, $z = -1$. When we write

$$z = -1 + \epsilon ,$$

where ε is a small real number, we find (4.9.5) becomes

$$\Upsilon_{1s} = \frac{Ha\,\Phi\,\varepsilon}{2\sinh\{2a\}}\left[(\cosh\{2a\}-1)\frac{R^{\frac{1}{2}}}{2} - a\sinh\{2a\}\right] ,$$

which is always positive because we have assumed $a \ll R^{\frac{1}{2}}$ in order that the calculated roots of equation (4.3.13) are always valid. Hence in the Ekman layer, on $z = -1$, the steady z-component of vorticity is always positive.

From (4.9.5) we find that

$$\frac{\partial \Upsilon_{1s}}{\partial z} = \frac{Ha\,\Phi}{2\sinh\{2a\}}\Bigg\{-a\sinh\{a(1-z)\}$$

$$+ \frac{(1-\cosh\{2a\})}{2}(-R^{\frac{1}{2}})\Big[\exp\{-R^{\frac{1}{2}}(1+z)\}\left(\cos\{R^{\frac{1}{2}}(1+z)\} + \sin\{R^{\frac{1}{2}}(1+z)\}\right)$$

$$+ \exp\{-R^{\frac{1}{2}}(1-z)\}\left(\cos\{R^{\frac{1}{2}}(1-z)\} + \sin\{R^{\frac{1}{2}}(1-z)\}\right)\Big]\Bigg\} ,$$

which becomes near $z = -1$

$$\frac{\partial \Upsilon_{1s}}{\partial z} = \frac{Ha\,\Phi}{2\sinh\{2a\}}\Bigg\{-a\sinh\{a(1-z)\}$$

$$- \frac{R^{\frac{1}{2}}}{\sqrt{2}}(1-\cosh\{2a\})\exp\{-R^{\frac{1}{2}}(1+z)\}\cos\left[R^{\frac{1}{2}}(1+z) - \frac{\pi}{4}\right]\Bigg\} .$$

Hence the gradient of the steady z-component of vorticity changes sign with $\cos\left[R^{\frac{1}{2}}(1+z) - \frac{\pi}{4}\right]$ and the flow in the boundary layer near $z = -1$ is shown in Fig.4.5.

A similar argument gives the flow in the boundary layer on $z = +1$.

We will now examine the previous steady flow on the assumption that a is large but is always very much less than $R^{1/4}$ so that the expression (4.9.2) is valid. Also we choose Φ so that $a\Phi$ is of order one. Then, from (4.3.9) and (4.9.3), we find

$$\phi_s = \Phi \exp\{-a(z+1)\},$$

$$\psi_{1s} = \frac{Ha\Phi}{2}\left[\exp\{-a(1+z)\} - \tfrac{1}{2}\right]$$

$$+ \frac{Ha\Phi}{4}\left[\sinh\{(2iR)^{\frac{1}{2}}z\}\exp\{-(2iR)^{\frac{1}{2}}\}\right.$$

$$\left.+ \sinh\{(-2iR)^{\frac{1}{2}}z\}\exp\{-(-2iR)^{\frac{1}{2}}\}\right].$$

Hence for large a, but $a \ll R^{1/4}$, we have a second boundary layer on the disk, $z = -1$, having a depth of penetration of order a^{-1}, and a constant interior flow.

From (4.9.3), we find that the steady z-component of velocity is

$$w_{1s} = \frac{Ha\Phi}{4\sinh\{2a\}} \cdot \frac{1}{R}\left\{-\sinh^2 a\left[(2iR)^{\frac{1}{2}}\cosh\{(2iR)^{\frac{1}{2}}z\}\exp\{-(2iR)^{\frac{1}{2}}\}\right.\right.$$

$$\left.\left.+ (-2iR)^{\frac{1}{2}}\cosh\{(-2iR)^{\frac{1}{2}}z\}\exp\{-(-2iR)^{\frac{1}{2}}\}\right] + \sinh^2(a)R^{\frac{1}{2}}\right\} + O(R^{-1}). \quad (4.9.6)$$

The first two terms in (4.9.6) represent the Ekman layers on the disks while the last term shows that the z-component of velocity is constant in the interior.

The interior flow described above is often referred to as the thermal wind [30,p.504] because, as we will now show, it is a particular solution of the equations

$$\left.\begin{aligned} -2v_I + \frac{\partial p_I}{\partial x} &= 0, && \text{(a)}\\ 2u_I + \frac{\partial p_I}{\partial y} &= 0, && \text{(b)}\\ \frac{\partial p_I}{\partial z} &= H\Theta_I, && \text{(c)}\\ 0 &= \nabla^2\Theta_I, && \text{(d)}\\ \frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y} + \frac{\partial w_I}{\partial z} &= 0, && \text{(e)} \end{aligned}\right\} \qquad (4.9.7)$$

where the subscript I denotes the solution for the interior. The equations (4.9.7(a), (b)) represent a geostrophic balance, that is a balance between the pressure gradients and the Coriolis force. The solution of (4.9.7(d)) which satisfies the boundary conditions on the two disks is

$$\Theta_I = h(x,y)\,\Phi\,\frac{\sinh\{a(1-z)\}}{\sinh\{2a\}},$$

which agrees with the solution (4.3.9). Then we have, from (4.9.7(c)),

$$p_I = -\frac{H}{a} h(x,y) \Phi \frac{\cosh\{a(1-z)\}}{\sinh\{2a\}} + g(x,y),$$

where g(x,y) is an arbitrary function of x and y. From (4.9.7(a),(b)), the general solution

$$(\zeta_s)_I = \frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}$$

$$= \tfrac{1}{2}aH\Phi h(x,y) \frac{\cosh\{a(1-z)\}}{\sinh\{2a\}} + \tfrac{1}{2}(g_{xx} + g_{yy}) .$$

If we now suppose that g satisfies the membrane equation (4.2.9) and then take

$$g = \frac{a\Phi H}{2\sinh\{2a\}} (\cosh\{2a\} + 1) h(x,y) . \frac{1}{a^2} ,$$

$(\zeta_s)_I$ reduces to the first term in equation, (4.9.3). Hence the interior flow is a particular solution of the equations (4.9.7).

Alternatively, from (4.9.7(a),(b)), we find that

$$(\zeta_s)_I = \tfrac{1}{2}\nabla_1^2 p_I ,$$

and, from (4.9.7(c),(d)), that

$$\frac{\partial}{\partial z}(\nabla_1^2 p_I) = H\nabla_1^2 \theta_I = -H\frac{\partial^2\theta_I}{\partial z^2} .$$

Hence we find that

$$\nabla_1^2 p_I = -H\frac{\partial\Theta_I}{\partial z} + G(x,y) ,$$

where G(x,y) is an arbitrary function of x and y and

$$(\zeta_s)_I = -\tfrac{1}{2}H\frac{\partial\Theta_I}{\partial z} + \tfrac{1}{2}G(x,y). \qquad (4.9.8)$$

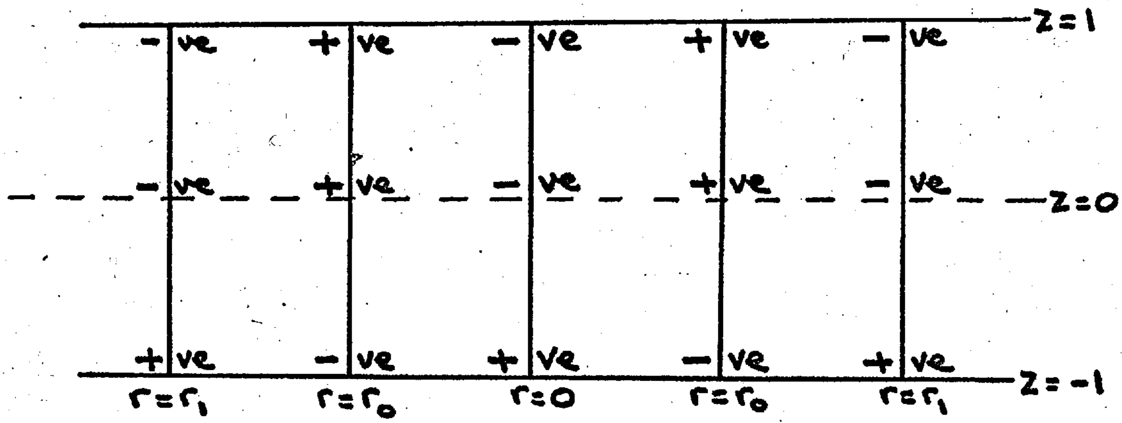
Therefore from (4.9.8) we see that the contribution to the interior vorticity produced by the inviscid terms is always positive provided $\Phi\, h(x,y) > 0$. The term involving G(x,y) cannot be determined by inviscid considerations alone and, in fact, comes from the boundary layers.

The horizontal components of vorticity, in the interior, are

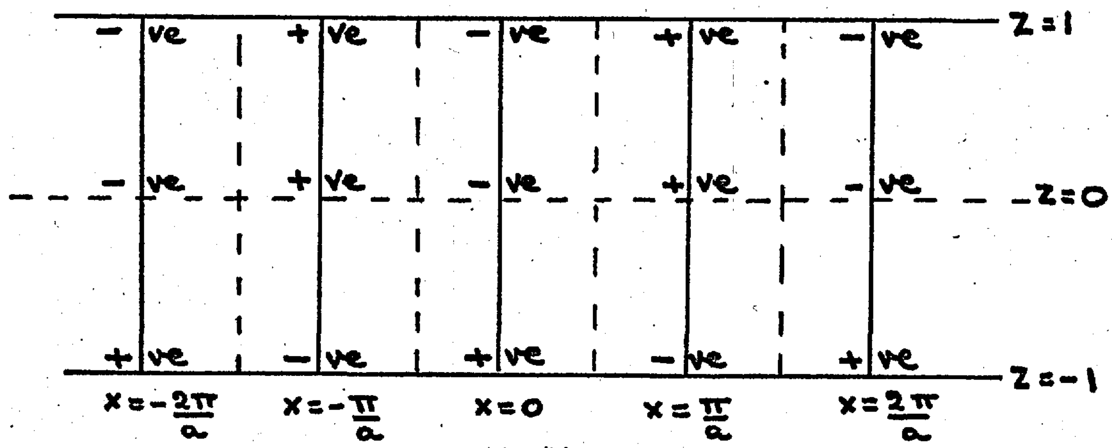
$$\eta_I = \frac{\partial u_I}{\partial z} - \frac{\partial w_I}{\partial x} = -\tfrac{1}{2}H\frac{\partial\Theta_I}{\partial y} ,$$

$$\xi_I = \frac{\partial w_I}{\partial y} - \frac{\partial v_I}{\partial z} = -\tfrac{1}{2}H\frac{\partial\Theta_I}{\partial x} .$$

Hence the steady solution consists of Ekman layers on both disks having thicknesses of order $(\nu/\Omega)^{\frac{1}{2}}$ and an interior flow which is a special solution of the thermal-wind relationships. These thermal-wind relationships relate, for variable density, the vertical shear of the horizontal velocities to the horizontal gradient of the temperature field and are also such that the horizontal temperature gradient and the shear



4.6(a)



(4.6(b)

FIG.4.6: The sign of the interior steady vertical vorticity $(\zeta_s)_I$ at the cell boundaries when (a) $h(x,y) = J_1(ar)$, (b) $h(x,y) = \cos a\,x$ and $\Phi > 0$. The lines --- represent $(\zeta_s)_I \equiv 0$.

of the horizontal velocity vector are orthogonal. It is also interesting to note that the interior flow does not satisfy the inviscid boundary conditions at the disk.

The sign of the interior steady vertical vorticity

$$(\zeta_s)_I = h(x,y)(\zeta_{1s})_I \;,$$

is shown at the cell boundaries for given z and $\Phi > 0$ in Fig.4.6(a),(b), when

$$h(x,y) = J_1(ar) \;,$$

and $$h(x,y) = \cos ax \;,$$

respectively.

Hence if the temperature of the lower disk is increased from $t = 0$, then positive vorticity is produced in the region near the lower disk and negative vorticity further away, while the reverse is true if the temperature of the lower disk is initially decreased. The region between the disks is divided into cells and a consideration of a volume integral for the vorticity over a large area shows that over the whole region no nett vorticity is produced.

4.10 THE INVERSE LAPLACE TRANSFORM

The residues calculated at the poles discussed in section 4.8 give, for large R and $R^{1/4} \gg a > O(R^{-\frac{1}{2}})$,

$$\Upsilon_1 = \frac{Ha\Phi}{2\sinh\{2a\}}\left\{\cosh\{a(1-z)\} - \frac{(1+\cosh\{2a\})}{2}\right.$$

$$- \frac{(1-\cosh\{2a\})}{2}\left[\sinh\{(2iR)^{\frac{1}{2}}z\}\exp\{-(2iR)^{\frac{1}{2}}\}\right.$$

$$\left.\left. + \sinh\{(-2iR)^{\frac{1}{2}}z\}\exp\{-(-2iR)^{\frac{1}{2}}\}\right]\right\}$$

$$+ O(R^{-\frac{1}{2}}) \qquad (4.10.1)$$

$$+ \sum_{m=1}^{\infty}\frac{a^2H\Phi(-1)^m\exp\{-t(a^2+\tfrac{1}{4}m^2\pi^2)/R\sigma\}}{4a^2+m^2\pi^2}\left\{-(1+(-1)^m)\right.$$

$$+ 2\cos\{\tfrac{1}{2}m\pi(1-z)\} + (1-(-1)^m)\left[\sinh\{(2iR)^{\frac{1}{2}}z\}\exp\{-(2iR)^{\frac{1}{2}}\}\right.$$

$$\left.\left. - \sinh\{(-2iR)^{\frac{1}{2}}z\}\exp\{-(-2iR)^{\frac{1}{2}}\}\right]\right\} + O(R^{-\frac{1}{2}}) \qquad (4.10.2)$$

$$+ \frac{\exp\{-tR^{-\frac{1}{2}}\}}{R^{1/4}}\left[\frac{a^2H\Phi(1+\cos\{2R^{1/4}\sigma^{-\frac{1}{2}}\})}{4\sigma^{\frac{1}{2}}\sin\{2R^{1/4}\sigma^{-\frac{1}{2}}\}}\right]\left\{-\cosh\left(-\frac{az}{2R^{\frac{1}{2}}}\right)\right.$$

$$\left. + \cosh\{(2iR)^{\frac{1}{2}}z\}\exp\{-(2iR)^{\frac{1}{2}}\} + \cosh\{(-2iR)^{\frac{1}{2}}z\}\exp\{-(-2iR)^{\frac{1}{2}}\}\right\}$$

$$+ O(R^{-\frac{1}{2}}) \qquad (4.10.3)$$

$$+ \frac{\exp\{-a^2 t/R\}}{R^{3/2}} \left[- \frac{H \overline{\Phi}\, a^2 (1-\cosh\{2a(1-\sigma)^{\frac{1}{2}}\})}{8(1-\sigma)^{\frac{1}{2}} \sinh\{2a(1-\sigma)^{\frac{1}{2}}\}} \right] \Bigg\{ - \frac{a^3 z}{2}$$

$$+ \frac{a^3}{8} \Big[\sinh\{(2iR)^{\frac{1}{2}} z\} \exp\{-(2iR)^{\frac{1}{2}}\} + \sinh\{(-2iR)^{\frac{1}{2}} z\} \exp\{-(-2iR)^{\frac{1}{2}}\} \Big] \Bigg\}$$

$$+ O(R^{-2}) \qquad (4.10.4)$$

$$+ \sum_{m_1=1}^{M_1} \exp\{t(\nu_1 + X_1 R^{-\frac{1}{2}})\} \frac{A_1^*}{R} \Big[\cos(m_1 \pi z)$$

$$+ (-1)^{m_1} \frac{(\nu_1 - 2i)}{2i} \cosh\{R^{\frac{1}{2}}(\nu_1 + 2i)^{\frac{1}{2}} z\} \exp\{-R^{\frac{1}{2}}(\nu_1 + 2i)^{\frac{1}{2}}\}$$

$$- (-1)^{m_1} \frac{(\nu_1 + 2i)}{2i} \cosh\{R^{\frac{1}{2}}(\nu_1 - 2i)^{\frac{1}{2}} z\} \exp\{-R^{\frac{1}{2}}(\nu_1 - 2i)^{\frac{1}{2}}\} \Big]$$

$$+ O(R^{-3/2}) \qquad (4.10.5)$$

$$+ \sum_{m_1=1}^{M_1} \exp\{t(\overline{\nu}_1 + \overline{X}_1 R^{-\frac{1}{2}})\} \frac{B_1^*}{R} \Big[\cos(m_1 \pi z)$$

$$+ (-1)^{m_1} \frac{(\overline{\nu}_1 - 2i)}{2i} \cosh\{R^{\frac{1}{2}}(\overline{\nu}_1 + 2i)^{\frac{1}{2}} z\} \exp\{-R^{\frac{1}{2}}(\overline{\nu}_1 + 2i)^{\frac{1}{2}}\}$$

$$- (-1)^{m_1} \frac{(\overline{\nu}_1 + 2i)}{2i} \cosh\{R^{\frac{1}{2}}(\overline{\nu}_1 - 2i)^{\frac{1}{2}} z\} \exp\{-R^{\frac{1}{2}}(\overline{\nu}_1 - 2i)^{\frac{1}{2}}\} \Big]$$

$$+ O(R^{-3/2}) \qquad (4.10.6)$$

$$+ \sum_{m_2=0}^{M_2} \exp\left\{t\left(\nu_2+X_2R^{-\frac{1}{2}}\right)\right\} \frac{A_2^*}{R} \left[i \sin\left\{\frac{\pi z}{2}(2m_2+1)\right\}\right.$$

$$+ (-1)^{m_2} \frac{(\nu_2-2i)}{2} \sinh\left\{R^{\frac{1}{2}}(\nu_2+2i)^{\frac{1}{2}}z\right\} \exp\left\{-R^{\frac{1}{2}}(\nu_2+2i)^{\frac{1}{2}}\right\}$$

$$\left. - (-1)^{m_2} \frac{(\nu_2+2i)}{2} \sinh\left\{R^{\frac{1}{2}}(\nu_2-2i)^{\frac{1}{2}}z\right\} \exp\left\{-R^{\frac{1}{2}}(\nu_2-2i)^{\frac{1}{2}}\right\}\right]$$

$$+ O(R^{-3/2}) \qquad (4.10.7)$$

$$+ \sum_{m_2=0}^{M_2} \exp\left\{t\left(\bar{\nu}_2+\bar{X}_2R^{-\frac{1}{2}}\right)\right\} \frac{B_2^*}{R} \left[i \sin\left\{\frac{\pi z}{2}(2m_2+1)\right\}\right.$$

$$+ (-1)^{m_2} \frac{(\bar{\nu}_2-2i)}{2} \sinh\left\{R^{\frac{1}{2}}(\bar{\nu}_2+2i)^{\frac{1}{2}}z\right\} \exp\left\{-R^{\frac{1}{2}}(\bar{\nu}_2+2i)^{\frac{1}{2}}\right\}$$

$$\left. - (-1)^{m_2} \frac{(\bar{\nu}_2+2i)}{2} \sinh\left\{R^{\frac{1}{2}}(\bar{\nu}_2-2i)^{\frac{1}{2}}z\right\} \exp\left\{-R^{\frac{1}{2}}(\bar{\nu}_2-2i)^{\frac{1}{2}}\right\}\right]$$

$$+ O(R^{-3/2}) , \qquad (4.10.8)$$

where the poles (4.6.13) and (4.6.14) are written

$$p = \nu_1 + \frac{X_1}{R^{\frac{1}{2}}} , \qquad p = \nu_2 + \frac{X_2}{R^{\frac{1}{2}}} ,$$

and the bar denotes the complex conjugate. Also

$$A_1^* = \frac{2a^2 H \left[\nu_1 \sigma\right]^{\frac{1}{2}} \Phi}{\sigma 2 \nu_1 \sinh\left\{2(R\nu_1\sigma)^{\frac{1}{2}}\right\}\left[4\nu_1 + \nu_1^3(\sigma-1)^2\right]}$$

$$\times \Bigg\{ -\frac{4\nu_1 i(1-\sigma)}{(\nu_1\sigma)^{\frac{1}{2}}} \sinh\left\{2(R\nu_1\sigma)^{\frac{1}{2}}\right\}$$

$$+ \left[2i + \nu_1(\sigma-1)\right]\left[\frac{\nu_1}{(\nu_1+2i)^{\frac{1}{2}}} - (\nu_1+2i)^{\frac{1}{2}}\right]\left(1+\cosh\left\{2(R\nu_1\sigma)^{\frac{1}{2}}\right\}\right)$$

$$+ \left[2i + \nu_1(1-\sigma)\right]\left[\frac{\nu_1}{(\nu_1-2i)^{\frac{1}{2}}} - (\nu_1-2i)^{\frac{1}{2}}\right]\left(1+\cosh\left\{2(R\nu_1\sigma)^{\frac{1}{2}}\right\}\right)\Bigg\}$$

$$\div \Bigg[\frac{(a^2+m_1^2\pi^2)^{3/2}}{2a^2} \cdot \frac{4\nu_1(-1)^{m_1}}{m_1\pi}\left\{1 + \frac{X_1}{2(\nu_1+2i)^{\frac{1}{2}}} + \frac{X_1}{2(\nu_1-2i)^{\frac{1}{2}}}\right\}$$

$$+ \frac{\nu_1(-1)^{m_1}}{2}\left\{\left(\frac{\nu_1-2i}{\nu_1+2i}\right) - \left(\frac{\nu_1+2i}{\nu_1-2i}\right) - \left(\frac{\nu_1+2i}{\nu_1-2i}\right)^{\frac{1}{2}} + \left(\frac{\nu_1-2i}{\nu_1+2i}\right)^{\frac{1}{2}}\right\}$$

$$+ 2i(-1)^{m_1}\Bigg] ,$$

A_2^* is identical to A_1^* with m_1, $(-1)^{m_1}$, X_1, ν_1, $\sinh\{2(R\sigma\nu_1)^{\frac{1}{2}}\}$, $(1+\cosh\{2(R\sigma\nu_1)^{\frac{1}{2}}\})$ replaced by $\frac{1}{2}(2m_2+1)$, $i(-1)^{m_2}$, X_2, ν_2, $-\sinh\{2(R\sigma\nu_2)^{\frac{1}{2}}\}$, $(1-\cosh\{2(R\sigma\nu_2)^{\frac{1}{2}}\})$, respectively and B^*_1 and B^*_2 are identical to A_1^* and A_2^* when ν_1, ν_2, X_1, X_2 are replaced by their complex conjugates. It should be noticed that A_1^*, A_2^*, B_1^*, B_2^* are constants of order one.

We see that in the above expression for $\mathcal{J}_1$, the first term (4.10.1) is the steady solution which has already been discussed in section 4.9. The second term (4.10.2) is an infinite series of steadily decaying modes which have an amplitude of order one and require a time of order $\{\Omega^{-1} R\sigma/(a^2 + \frac{1}{4}\pi^2)\}$, which is always less than or equal to the thermal diffusion time, $R\sigma\Omega^{-1}$, to decay. The contribution, from this term (4.10.2), to the vorticity in the interior is

$$\sum_{m=1}^{\infty} \Phi\, h(x,y) \frac{a^2 H(-1)^m}{4a^2 + m^2\pi^2} \exp\left\{-t(a^2 + \tfrac{1}{4}m^2\pi^2)/R\sigma\right\} \quad \times$$

$$\times \left[-(1 + (-1)^m) + 2\cos\left\{\frac{m\pi}{2}(1-z)\right\}\right] \quad , \qquad (4.10.9)$$

which represents an oscillation in z with an amplitude that decreases as m increases for any given time. Hence the modes associated with the smallest values of m characterise the behaviour of (4.10.9).

From Fig.4.7, which shows the variation of the modes m = 1,2,3,4,5 of (4.10.9) with z, for fixed time, when $\Phi\, h(x,y) > 0$, we see that the even modes always make a negative contribution to the interior flow while the odd modes produce positive vorticity near z = −1 and negative vorticity near z = 1. Hence the term (4.10.2) represents negative vorticity in the

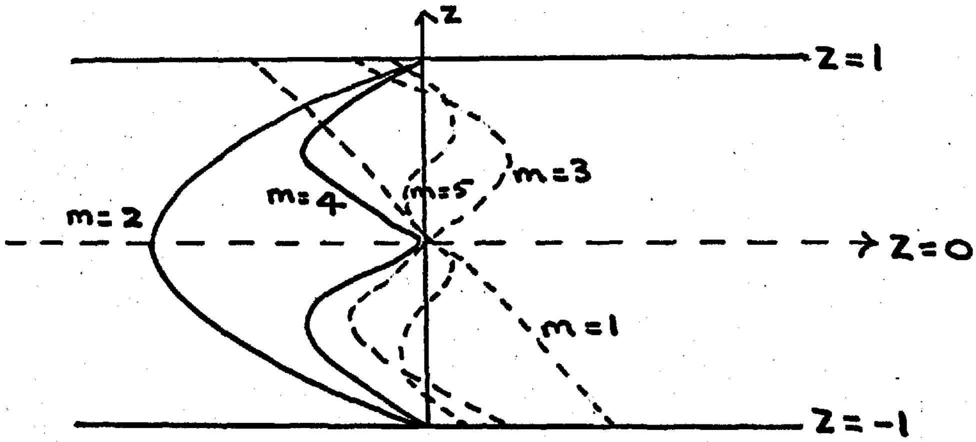


FIG.4.7: The variation of the modes $m = 1,2,3,4,5$ of (4.10.9) with z, for fixed time, when $\Phi\, h(x,y) > 0$.

interior except in the neighbourhood of the disk $z = -1$. On the other hand, when $\Phi\, h(x,y) < 0$, the term (4.10.2) represents positive vorticity in the interior except in the neighbourhood of the disk $z = -1$. From (4.10.2), we see that the only contributions to the vorticity in the boundary layer arises from the odd modes.

The term (4.10.3) represents a steadily decaying mode which has an amplitude of order $R^{-1/4} \cot (R^{1/4} \sigma^{-\frac{1}{2}})$ and requires the spin-up time, $\Omega^{-1} R^{\frac{1}{2}}$, to decay. The contribution to the interior vorticity from this term (4.10.3) is

$$\Phi\, h(x,y)\exp\left\{-tR^{-\frac{1}{2}}\right\} \frac{a^2 H \cot\left[R^{1/4}\sigma^{-\frac{1}{2}}\right]}{4R^{1/4}\sigma^{-\frac{1}{2}}} \left\{-\cosh\left(-\frac{az}{2R^{\frac{1}{2}}}\right)\right\}. \qquad (4.10.10)$$

Since R is assumed to be large, $\cosh\{-az/2R^{\frac{1}{2}}\}$ may be approximated by one and hence we see that at any instant of time the vorticity (4.10.10) is independent of z and takes a constant value for any prescribed x and y. Therefore the term (4.10.10) represents constant vorticity in the interior for any fixed x and y with the sign of this vorticity given by the sign of

$$-\Phi\, h(x,y) \cot\left[R^{1/4}\sigma^{-\frac{1}{2}}\right].$$

The term (4.10.4) is a steadily decaying mode which has an amplitude of order

$$\frac{\tanh\left[a(1-\sigma)^{\frac{1}{2}}\right]}{R^{3/2}(1-\sigma)^{\frac{1}{2}}},$$

and decays in a time of order $\left\{\Omega^{-1}R/a^2\right\}$. When $\sigma \leq 1$, this term (4.10.4) can be neglected on account of its small amplitude. From (4.10.4) the contribution to the vorticity in the interior is

$$-\Phi\, Hh(x,y)a^5 z \exp\left\{-ta^2/R\right\} \frac{\tanh\left[a(1-\sigma)^{\frac{1}{2}}\right]}{16R^{3/2}(1-\sigma)^{\frac{1}{2}}}, \qquad (4.10.11)$$

which, for any fixed time, has a sign given by

$$\operatorname{sgn}\left[-\oint h(x,y)z.\frac{\tanh\left[a(1-\sigma)^{\frac{1}{2}}\right]}{(1-\sigma)^{\frac{1}{2}}}\right].$$

The remaining terms (4.10.5), (4.10.6), (4.10.7) and (4.10.8) represent four series of damped oscillations with each oscillation having an amplitude of order R^{-1} and decaying in a time of order

$$\frac{R^{\frac{1}{2}}\Omega^{-1}(a^2+m_1^2\pi^2)}{a^2\left\{\frac{\gamma_1^2}{\beta_1}+\frac{\beta_1^2}{\gamma_1}\right\}}, \qquad \text{for (4.10.5) and (4.10.6)},$$

and

$$\frac{R^{\frac{1}{2}}\Omega^{-1}(a^2+\frac{1}{4}\pi^2(2m_2+1)^2)}{a^2\left\{\frac{\gamma_2^2}{\beta_2}+\frac{\beta_2^2}{\gamma_2}\right\}}, \quad \text{for} \quad \text{(4.10.7) and (4.10.8).}$$

Although, for these oscillations, the individual amplitudes are small, they may combine to give a non-negligible contribution and therefore must be retained. From (4.10.5), the contribution to the interior vorticity behaves like

$$\sum_{m_1=1}^{M_1}\mathcal{A}_{m_1}(-1)^{m_1}\,h(x,y)\cos(m_1\pi z)\;, \qquad (4.10.12)$$

where $\mathcal{A}_{m_1}$ is independent of z, dependent upon t and is of order R^{-1}. The sign of each mode in (4.10.12) is given by

$$\mathrm{sgn}(\mathcal{A}_{m_1} h(x,y)) = \mathrm{sgn}((-1)^{m_1} A_1^* h(x,y)) ,$$

at $z = \pm 1$ and by sgn $(A_1^* h(x,y))$ at $z = 0$. Hence each mode is symmetrical about the axis $z = 0$. A similar result applies for (4.10.6). From (4.10.7), the contribution to the interior vorticity behaves like

$$\sum_{m_2=0}^{M_2} \mathcal{A}_{m_2} (-1)^{m_2} h(x,y) \sin\left[(2m_2+1)\frac{\pi z}{2}\right] , \qquad (4.10.13)$$

where $\mathcal{A}_{m_2}$ is independent of z, dependent upon t and is of order R^{-1}. The sign of each mode in (4.10.13) is given by $\mathrm{sgn}\left[\pm A_2^*(-1)^{m_2} h(x,y)\right]$ on $z = \pm 1$ and (4.10.13) is identically zero on $z = 0$. Therefore each mode is antisymmetrical about the axis $z = 0$. A similar result holds for (4.10.8).

From the above results it follows that all the transient effects will become negligible in a time of order

$$\max\left[R^{\frac{1}{2}} \Omega^{-1}, \ \frac{R\sigma\Omega^{-1}}{(a^2 + \frac{1}{4}\pi^2)} , \ \frac{\Omega^{-1} R}{a^2} , \ \frac{R^{\frac{1}{2}} \Omega^{-1}(a^2 + m_1^2\pi^2)}{a^2\left\{\frac{\gamma_1^2}{\beta_1} + \frac{\beta_1^2}{\gamma_1}\right\}} \right. ,$$

$$\left.\frac{R^{\frac{1}{2}}\,\mathcal{S}\zeta^{-1}\left(a^2+\frac{1}{4}\pi^2(2m_2+1)^2\right)}{a^2\left\{\frac{\gamma_2^2}{\beta_2}+\frac{\beta_2^2}{\gamma_2}\right\}}\quad\right] ,$$

which depends upon the value chosen for a, to leave, as a final state, the steady solution (4.10.1).

4.11 VERTICAL BOUNDARIES

We will now examine the effect of introducing vertical walls at the zeros of $J_1(ar)$, that is at the cellular boundaries. Then from (4.2.10) and (4.3.9), we find that the steady temperature distribution is given by

$$\Theta = A\, J_o(ar).\,\Phi\,\frac{\sinh\{a(1-z)\}}{\sinh\{2a\}} , \qquad (4.11.1)$$

which implies that, if a vertical boundary is situated at $r = r_i$, where $J_1(ar_i) = 0$, the imposed temperature distribution on this boundary must be

$$\Theta = A\, J_o(ar_i)\,\Phi\,\frac{\sinh\{a(1-z)\}}{\sinh\{2a\}} , \qquad (4.11.2)$$

in order that the inviscid flow in the interior is not affected by the introduction of the vertical wall. Also, from (4.11.1), we find that

$\frac{\partial \Theta}{\partial r}$ vanishes at the cellular boundaries, which implies that the introduction of insulating walls at the boundaries of the cells will not affect the interior inviscid flow.

The velocity components for the inviscid interior calculated from (4.2.11) are

$$v_z = A\ J_o(ar)w,$$

$$v_r = -Aa^{-1} J_1(ar) \frac{\partial w}{\partial z} \quad ,$$

and therefore we find that the radial component of velocity will always vanish on the vertical boundaries.

When the effects of viscosity are included, we would expect shear layers, which have a depth of penetration of order $R^{-1/3}$ and $R^{-1/4}$, to exist on the vertical boundaries, although no justification of this statement will be given here.

Hence, if insulating vertical boundaries are introduced at the zeros of $J_1(ar)$ with a temperature distribution (4.11.2), then the interior flow will remain unchanged and is a solution of the thermal-wind equations.

4.12 NO BASIC ROTATION ($\Omega \equiv 0$)

We consider the effect on the previous problem when there is no initial basic rotation, that is when the angular velocity of rotation, Ω, is identically zero. For this special case, we cannot use the dimensionless variables (4.2.13). Instead we define the new dimensionless (starred) variables

$$\Theta = \Delta T \Theta^*, \quad t = \frac{d^2}{\kappa} t^*, \quad \underline{r} = d\underline{r}^*, \quad \underline{u} = \frac{\kappa}{d}\underline{u}^*,$$

$$\frac{p}{\rho_0} = \frac{\kappa^2}{d^2} p^*, \quad a = \frac{a^*}{d}. \qquad (4.12.1)$$

When these new variables (4.12.1) are introduced into the equations (4.2.5), (4.2.6), (4.2.7), with $\Omega \equiv 0$, we find (upon dropping the asterisks) that

$$\left.\begin{aligned} \operatorname{div} \underline{u} &= 0, \\ \frac{\partial \underline{u}}{\partial t} + \nabla p &= \sigma \mathcal{R}\, \Theta\, \underline{k} + \sigma \nabla^2 \underline{u}, \\ \frac{\partial \Theta}{\partial t} &= \nabla^2 \Theta, \end{aligned}\right\} \qquad (4.12.2)$$

where $\mathcal{R} = \dfrac{g\alpha\, \Delta T d^3}{\kappa \nu}$ is the Rayleigh number for the flow.

When we assume (4.2.9), (4.2.18) and eliminate the pressure, we find that (4.12.2) becomes

$$\frac{\partial \phi}{\partial t} = (D^2 - a^2)\phi \, , \qquad (4.12.3)$$

$$\frac{\partial \zeta_1}{\partial t} = \sigma (D^2 - a^2) \zeta_1 \, , \qquad (4.12.4)$$

$$(D^2 - a^2)\left[(D^2 - a^2) - \sigma^{-1}\frac{\partial}{\partial t}\right] w_1 = a^2 \mathcal{R} \phi \, . \qquad (4.12.5)$$

We see that the equation (4.12.3) is of the same structure as the equation (4.2.19) and can immediately be solved to give the temperature distribution throughout the fluid. The equation (4.12.4) is independent of the equations (4.12.3) and (4.12.5) and determines the z-component of vorticity, ζ_1. The equation (4.12.5) is a fourth order equation with a forcing term dependent upon the solution of (4.12.3) and determines the z-component of velocity, w_1.

Hence we have, for this special case, that ζ_1 and w_1 are solutions of the equations (4.12.4) and (4.12.5) respectively while, in the previous problem, ζ_1 or w_1 is the solution of the sixth order equation obtained from the equations (4.2.20) and (4.2.21). The equation (4.12.4) is the diffusion equation which must have the solution

$$\zeta_1 \equiv 0 \, , \qquad (4.12.6)$$

in order to satisfy the boundary conditions

$$\zeta_1 = D^2 \zeta_1 = 0 \quad \text{on } z = \pm 1, \text{ for all time.}$$

Then (4.12.6) shows that there is no vorticity introduced into the flow in the z-direction. Hence in the previous analysis when $\Omega \neq 0$, it was the basic rotation that was responsible for the introduction of vorticity in the z-direction.

When we apply the Laplace transform to (4.12.3), (4.12.5), we find that the solutions of the resulting equations which satisfy the boundary conditions (4.3.4) are

$$\bar{\phi} = \frac{\Phi \sinh\{\chi_1(1-z)\}}{p \sinh\{2\chi_1\}} , \qquad (4.12.7)$$

$$\bar{w}_1 = C\left\{\sinh\{\chi_1(1-z)\} - \sinh\{a(1-z)\}\frac{\sinh\{2\chi_1\}}{\sinh\{2a\}}\right\}$$

$$+ C_1\left\{\sinh\{\mu z\} - \sinh\{az\}\frac{\sinh \mu}{\sinh a}\right\} \qquad (4.12.8)$$

$$+ D_1\left\{\cosh\{\mu z\} - \cosh\{az\}\frac{\cosh \mu}{\cosh a}\right\} ,$$

where

$$\chi_1 = (a^2 + p)^{\frac{1}{2}} , \qquad \mu = (a^2 + p\sigma^{-1})^{\frac{1}{2}} , \qquad (4.12.9)$$

$$C = \frac{a^2 R \Phi}{\sinh\{2\chi_1\}\, p^3(1-\sigma^{-1})} , \qquad (4.12.10)$$

$$C_1 = \frac{c}{2}\left[\frac{-\chi_1(\sinh a)(1+\cosh\{2\chi_1\}) + a\cosh a \sinh\{2\chi_1\}}{a\sinh\mu\cosh a - \mu\cosh\mu\sinh a}\right], \quad (4.12.11)$$

$$D_1 = \frac{c}{2}\left[\frac{\chi_1(\cosh a)(1-\cosh\{2\chi_1\}) + a\sinh a \sinh\{2\chi_1\}}{\mu\sinh\mu\cosh a - a\sinh a\cosh\mu}\right]. \quad (4.12.12)$$

The singularities of (4.12.7), (4.12.8), are required in order to evaluate the inverse Laplace transform and we will now locate them. The expression (4.12.7) has simple poles at

$$\left.\begin{aligned} p &= 0, \\ p &= -a^2 - \tfrac{1}{4}\pi^2 m^2, \quad m = 1,2,3,\dots . \end{aligned}\right\} \quad (4.12.13)$$

The branch point associated with $p = -a^2$ is a zero of both the numerator and the denominator of (4.12.7) and hence is not a singularity. The expression (4.12.8) has simple poles at (4.12.13) and at the roots of

$$a\sinh\mu\cosh a = \mu\cosh\mu\sinh a, \quad (4.12.14)$$

$$\mu\sinh\mu\cosh a = a\sinh a\cosh\mu, \quad (4.12.15)$$

$\mu \neq 0$, provided the numerators of C_1 and D_1 remain non-zero.

It appears, from (4.12.10), that the expression (4.12.8) has a triple pole at $p = 0$ but, in fact, $p = 0$ is a double zero of the numerator in (4.12.8) and, therefore, we only retain a simple pole at $p = 0$. The branch points associated with $p = -a^2$ and $p = -a^2\sigma$ render the numerator and the denominator of (4.12.8) zero and hence are not singularities.

From (4.12.14) and (4.12.15), we see that, if μ is real and non-zero, the only root occurs when $\mu = a$, but this value also renders the numerators of C_1 and D_1 zero and hence is not a singularity of (4.12.8). If we assume that μ is purely imaginary, that is $\mu = i\mu_1$ where μ_1 is real and non-zero, then (4.12.14) and (4.12.15) become

$$a \cosh a \sin \mu_1 = \mu_1 \cos \mu_1 \sinh a \,, \qquad (4.12.16)$$

$$-\mu_1 \sin \mu_1 \cosh a = \cos \mu_1 \, a \sinh a \,. \qquad (4.12.17)$$

The roots of (4.12.16) and (4.12.17) are shown in Fig.4.8 and Fig.4.9 respectively and we will assume that they are situated at

$$\left.\begin{aligned} \mu &= \pm i x_j \qquad j = 1,2,3,\ldots \,, \\ \mu &= \pm i y_j \qquad j = 1,2,3,\ldots \,, \end{aligned}\right\} \qquad (4.12.18)$$

respectively, where x_j, y_j are positive real numbers. The actual position of these roots depends upon the value taken for a and is unimportant for the following discussion. If we assume that μ is complex, that is $\mu = x + iy$ where x,y are real and $x \neq 0$, $y \neq 0$, then we can rewrite (4.12.14) and (4.12.15) as

$$a \coth a = \mu \coth \mu \,, \qquad a \tanh a = \mu \tanh \mu \,, \qquad (4.12.19)$$

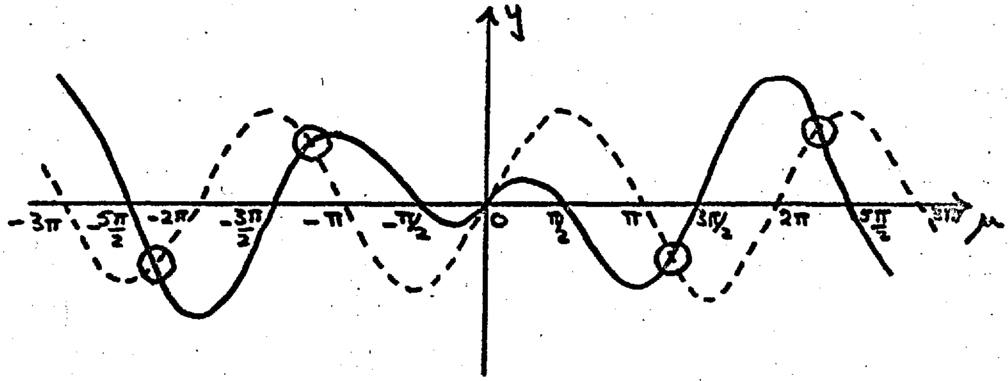


FIG.4.8: Sketch of ---- $y = a \coth a \sin \mu_1$,

—— $y = \mu_1 \cos \mu_1$,

where O denotes the roots of $a \coth a \sin \mu_1 = \mu_1 \cos \mu_1$, $\mu_1 \neq 0$.

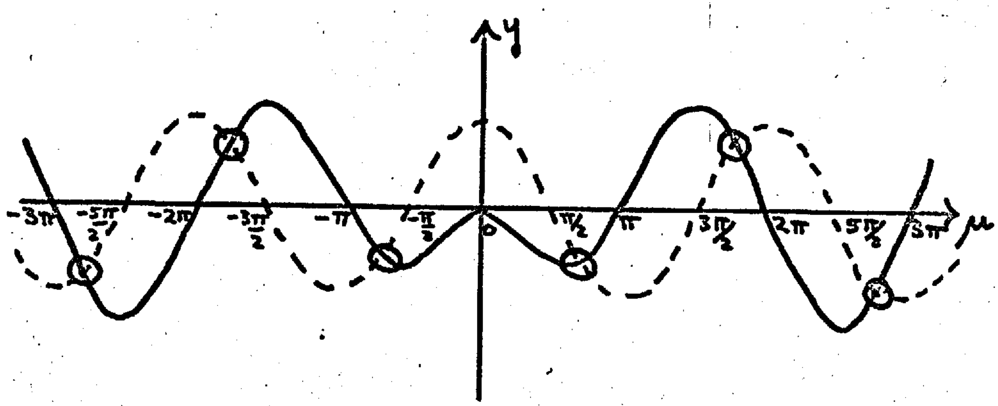


FIG.4.9: Sketch of ---- $y = a \tanh a \cos \mu_1$,

—— $y = -\mu_1 \sin \mu_1$,

where O denotes the roots of $a \tanh a \cos \mu_1 = -\mu_1 \sin \mu_1$, $\mu_1 \neq 0$.

and we require

$$y \sinh\{2x\} = \pm x \sin\{2y\}, \qquad (4.12.20)$$

in order that the imaginary parts of (4.12.19) should vanish. The only solution of (4.12.20) is $x = y = 0$ which is not a singularity of (4.12.8). Hence there are no complex roots of (4.12.14) and (4.12.15).

Therefore the only roots of (4.12.14) and (4.12.15) which are singularities of (4.12.8) are purely imaginary and we assume that these are situated at (4.12.18).

We now calculate the inverse Laplace transform (4.3.7) for (4.12.7) and (4.12.8). From (4.12.7), we have that

$$\phi = \frac{\Phi \sinh\{a(1-z)\}}{\sinh\{2a\}} + \sum_{m=1}^{\infty} \frac{\Phi (-1)^m m\pi \sin\left[\frac{m\pi}{2}(1-z)\right] \exp\left[-t(a^2 + \frac{1}{4} m^2\pi^2)\right]}{2(a^2 + \frac{1}{4} m^2\pi^2)} . \qquad (4.12.21)$$

This solution (4.12.21) corresponds to (4.3.8) and consists of a steady term identical to (4.3.9) and an infinite series of steadily decaying modes which vanish within a dimensionless time of order $(a^2 + \frac{1}{4} m^2\pi^2)^{-1}$.

For the z-component of velocity, w_1, the steady solution can be found either by calculating the residue contribution from the simple pole at $p = 0$ in (4.12.8) or from the steady terms in (4.12.5). When the inverse Laplace transform is calculated for (4.12.8), we find

$$w_1 = \frac{\mathcal{R}_0 \Phi}{8}\Bigg[(z^2-1)\frac{\sinh\{a(1-z)\}}{\sinh\{2a\}} - \frac{\cosh a}{a+\sinh a \cosh a}\Big\{z \sinh\{az\}$$

$$- \cosh\{az\}\frac{\sinh a}{\cosh a}\Big\} + \frac{\sinh a}{\cosh a \sinh a - a}\Big\{z\cosh\{az\} - \sinh\{az\}\frac{\cosh a}{\sinh a}\Big\}\Bigg]$$

$$+ \sum_{m=1}^{\infty} \frac{a^2 \Phi \mathcal{R}_0 m\pi (-1)^m \exp\left[-t(a^2+\frac{1}{4}m^2\pi^2)\right]}{2(a^2+\frac{1}{4}m^2\pi^2)^3(1-\sigma^{-1})}\Bigg[\sin\Big\{\frac{m\pi}{2}(1-z)\Big\}$$

$$+ \tfrac{1}{2}\Big\{\sinh(\mu^* z) - \sinh(az)\frac{\sinh\mu^*}{\sinh a}\Big\}\Big\{\frac{-\frac{1}{2}m\pi(\sinh a)(1+(-1)^m)}{a\sinh\mu^*\cosh a - \mu^*\sinh a\cosh\mu^*}\Big\}$$

$$+ \tfrac{1}{2}\Big\{\cosh(\mu^* z) - \cosh(az)\frac{\cosh\mu^*}{\cosh a}\Big\}\Big\{\frac{\frac{1}{2}m\pi(\cosh a)(1-(-1)^m)}{\mu^*\sinh\mu^*\cosh a - a\sinh a\cosh\mu^*}\Big\}\Bigg]$$

$$+ \sum_j \frac{a^2 \Phi \mathcal{R}_0 2x_j \exp\left[-t\sigma(a^2+x_j^2)\right]}{\sinh\{2\chi^*\}(a^2+x_j^2)^3\sigma^2(1-\sigma^{-1})}\Bigg[\Big\{\sin(x_j z) - \sinh(az)\frac{\sin x_j}{\sinh a}\Big\}$$

$$\times \Big\{\frac{-\chi^*(\sinh a)(1+\cosh\{2\chi^*\}) + a\cosh a \sinh\{2\chi^*\}}{a\cos x_j \cosh a - \cos x_j \sinh a + x_j \sin x_j \sinh a}\Big\}\Bigg]$$

$$+ \sum_j \frac{-a^2 \Phi \mathcal{R}_0 2y_j \exp\left[-t\sigma(a^2+y_j^2)\right]}{\sinh\{2\chi^+\}(a^2+y_j^2)^3\sigma^2(1-\sigma^{-1})}\Bigg[\Big\{\cos(y_j z) - \cosh(az)\frac{\cos y_j}{\cosh a}\Big\}$$

$$\times \Big\{\frac{\chi^+(\cosh a)(1-\cosh\{2\chi^+\}) + a\sinh a\sinh\{2\chi^+\}}{y_j\cos y_j\cosh a + \sin y_j \cosh a - a\sin y_j \sinh a}\Big\}\Bigg], \qquad (4.12.22)$$

where $\mu^* = (a^2 - \sigma^{-1}(a^2 + \frac{1}{4}m^2\pi^2))^{\frac{1}{2}}$,

$\chi^* = \left[a^2 - \sigma(a^2 + x_j^2)\right]^{\frac{1}{2}}$,

$\chi^+ = \left[a^2 - \sigma(a^2 + y_j^2)\right]^{\frac{1}{2}}$.

The first term in (4.12.22) is the steady term, which becomes, at $z = 0$,

$$w_1 = \frac{R_o \Phi}{8} \frac{\sinh a}{\sinh\{2a\}} \frac{(\sinh\{2a\} - 2a)}{(2a + \sinh\{2a\})} . \qquad (4.12.23)$$

Hence we see, from (4.2.18) and (4.12.23), that, at $z = 0$, the steady vertical velocity is positive when $\Phi h(x,y) > 0$ and negative when $\Phi h(x,y) < 0$. The second, third and fourth terms in (4.12.22) represent infinite series of steadily decaying modes which vanish in a dimensionless time of order

$$(a^2 + \tfrac{1}{4}m^2\pi^2)^{-1}, \quad \left[\sigma(a^2 + x_j^2)\right]^{-1}, \quad \left[\sigma(a^2 + y_j^2)\right]^{-1},$$

respectively. Therefore after a time of order

$$\max\left[(a^2 + \tfrac{1}{4}m^2\pi^2)^{-1}, \quad \sigma(a^2 + x_j^2)^{-1}, \quad \sigma(a^2 + y_j^2)^{-1}\right],$$

all the transient effects will have decayed and (4.12.22) will reduce to the steady term.

The solution obtained for this problem is exact unlike the previous results involving rotation, when a large parameter, the Reynolds number, was introduced and small terms were neglected.

4.13 A RESONANCE EFFECT IN OSCILLATORY HEATING

In chapters 2 and 3 we considered the problem of either an infinite plane disk bounding a semi-infinite expanse of incompressible fluid or two infinite disks with incompressible fluid between them, when the fluid and the disk(s) were in rigid rotation and additionally, one of the disk(s) was performing torsional or non-torsional oscillations in its own plane. A resonance effect was found when the frequency of oscillation was twice the angular frequency of the basic rotation. This effect became apparent when only one disk was present because no oscillatory solution satisfying all the boundary conditions could be found, while when two disks were present an interior flow depending on z was developed; provided a linearization was valid when torsional oscillations were imposed.

We now wish to examine whether or not a similar effect will occur when we replace the imposedoscillations by some imposed oscillatory heating,

$$\Theta = h(x,y) \, \Delta T \, \Phi \, e^{int} , \qquad (4.13.1)$$

where n is the frequency which can be assumed positive without loss of generality.

When we have two infinite plane disks, $z = \pm d$, with fluid between them, then, by using the previous notation and analysis, we find, when the heating is applied to $z = -d$, that there always exists a solution for the z-component of vorticity of the form

$$\begin{aligned} \zeta = \; & A_1 \cosh\{\lambda_1 z\} + A_2 \cosh\{\lambda_2 z\} + A_3 \cosh\{\lambda_3 z\} \\ + \; & B_1 \sinh\{\lambda_1 z\} + B_2 \sinh\{\lambda_2 z\} + B_3 \sinh\{\lambda_3 z\} \qquad (4.13.2) \\ + \; & A \cosh\{\chi(1-z)\} \; , \end{aligned}$$

where A_1, A_2, A_3, B_1, B_2, B_3 are constants determined by the boundary conditions, χ is given by (4.3.6), A is given by (4.3.12) and the λ's are the roots of (4.3.13) with $p = \frac{in}{\Omega}$. A similar result applies when the disk $z = d$ is heated.

On the other hand, when we have one infinite plane disk, $z = 0$, bounding a semi-infinite expanse of fluid, we have no geometric length scale, d, from which to form the Reynolds number, R. Instead we use a^{-1} as a length scale and define a new Reynolds number

$$R_o = \frac{\Omega}{\nu a^2} , \qquad (4.13.3)$$

which we will assume to be large. Then we can always find a solution for the z-component of vorticity of the form

$$\zeta = A_1 e^{-\lambda_1 z} + A_2 e^{-\lambda_2 z} + A_3 e^{-\lambda_3 z} + A^* e^{-\chi z}, \qquad (4.13.4)$$

where A_1, A_2, A_3 are constants determined by the boundary conditions on the disk, χ is given by (4.3.6), $A^* = A \sinh\{2\chi\}$ where A is given by (4.3.12) and the λ's are the roots of (4.3.13) with $p = \frac{in}{\Omega}$, $a = 1$, $R = R_o$.

We see, from the inviscid form of the partial differential equation corresponding to (4.3.10) with $p = in/\Omega$, namely

$$D^2\zeta\left[4 - n^2/\Omega^2\right] - \frac{n^2}{\Omega^2}\nabla_1^2\zeta = 0,$$

that we can divide the frequency range into the two regions

(i) Elliptic $n \geq 2\Omega$,

(ii) Hyperbolic $n < 2\Omega$.

From the results of section 4.4, we see that we can further divide the frequency range into the ELLIPTIC OUTER REGION when $\frac{n}{\Omega} - 2 > O(R_o^{-\frac{1}{2}})$, the HYPERBOLIC OUTER REGION when $2 - \frac{n}{\Omega} > O(R_o^{-\frac{1}{2}})$, the INNER REGION when $\left|\frac{n}{\Omega} - 2\right| < O(R_o^{-\frac{1}{2}})$ and the TRANSITION REGION when $\left|\frac{n}{\Omega} - 2\right| = O(R_o^{-\frac{1}{2}})$.

REGION	The thickness of the boundary layers associated with λ_1	λ_2	λ_3
ELLIPTIC OUTER	$\frac{(n^2 - 4\Omega^2)^{\frac{1}{2}}}{n}$	$\frac{R_o^{-\frac{1}{2}}}{(\frac{n}{\Omega} + 2)^{\frac{1}{2}}}$	$\frac{R_o^{-\frac{1}{2}}}{(\frac{n}{\Omega} - 2)^{\frac{1}{2}}}$
INNER	$R_o^{-1/4}$	$R_o^{-\frac{1}{2}}$	$R_o^{-1/4}$
HYPERBOLIC OUTER	$(4 - \frac{n^2}{\Omega^2})^{5/2} R_o$	$\frac{R_o^{-\frac{1}{2}}}{(\frac{n}{\Omega} + 2)^{\frac{1}{2}}}$	$\frac{R_o^{-\frac{1}{2}}}{(- \frac{n}{\Omega} + 2)^{\frac{1}{2}}}$

(a) Dimensionless

REGION	The thickness of the boundary layers associated with λ_1	λ_2	λ_3
ELLIPTIC OUTER	$\frac{(n^2 - 4\Omega^2)^{\frac{1}{2}}}{na}$	$(\frac{\nu}{n + 2\Omega})^{\frac{1}{2}}$	$(\frac{\nu}{n - 2\Omega})^{\frac{1}{2}}$
INNER	$(\frac{\nu}{\Omega a^2})^{1/4}$	$(\frac{\nu}{\Omega})^{\frac{1}{2}}$	$(\frac{\nu}{\Omega a^2})^{1/4}$
HYPERBOLIC OUTER	$\frac{(4\Omega^2 - n^2)^{5/2}}{\Omega^4 \nu a^3}$	$(\frac{\nu}{n + 2\Omega})^{\frac{1}{2}}$	$(\frac{\nu}{2\Omega - n})^{\frac{1}{2}}$

(b) Dimensional

TABLE 4.2: The (a) dimensionless, (b) dimensional thickness of the boundary layers associated with λ_1, λ_2, λ_3, for the case when one disk bounds a semi-infinite expanse of fluid.

When only one disk is present, the dimensionless and dimensional thicknesses of the boundary layers corresponding to λ_1, λ_2, λ_3 are found from (4.13.4) and are shown in Table 4.2. The boundary layer thicknesses associated with λ_1 for the outer elliptic and hyperbolic regions differ because, for the elliptic outer region, it is the first term in (4.4.6) that gives the decay distance while, for the hyperbolic outer region, the first term in (4.4.6) is purely imaginary and it is the second term that yields the decay distance.

For the elliptic and hyperbolic outer regions, we see that the boundary layer having the deepest penetration is that associated with λ_1. This layer penetrates a depth of $O(a^{-1})$ for the elliptic outer region and $O(\Omega/\nu a^3)$, which is much greater than $O(a^{-1})$, for the hyperbolic outer region. Hence we see that the influence of the heating is felt through a much greater depth of fluid for the hyperbolic outer region than for the elliptic outer region. The boundary layer thicknesses associated with λ_2 and λ_3 are identical for both the elliptic and hyperbolic outer regions.

For the inner region, the boundary layers having the greatest penetration are those associated with λ_1 and λ_3 and have a thickness of $O(\nu/\Omega a^2)^{1/4}$, which is always much thinner than the boundary layer associated with λ_1, for both the elliptic and hyperbolic outer regions. The thickness of the boundary layer associated with λ_2 is the same order for the inner region and the elliptic and hyperbolic outer regions. Hence we see that as $n \rightarrow 2\Omega$ the thickness of the boundary layer associated with λ_1 decreases while the thickness of the boundary layer associated with λ_3 increases and the thickness of the boundary layer associated with λ_2 remains unchanged.

The transition from the inner region to either the elliptic or hyperbolic outer region occurs when

$$\left| \frac{n}{\Omega} - 2 \right| = O(R_o^{-\frac{1}{2}}) .$$

For the boundary layer associated with λ_2, the depth of penetration is again $O(\nu/\Omega)^{\frac{1}{2}}$, while for the boundary layers associated with λ_1 and λ_3 the depths of penetration are

$$O\left\{\left(\frac{\nu}{\Omega a^2}\right)^{1/4}\left[\frac{2}{k \mp (k^2-4i)^{\frac{1}{2}}}\right]^{\frac{1}{2}}\right\} , \text{ when } \frac{in}{\Omega} = 2i + \frac{k}{R_o^{\frac{1}{2}}} ,$$

provided $\left\{k \pm (k^2-4i)^{\frac{1}{2}}\right\}^{\frac{1}{2}}$ is not purely imaginary. When $\left\{k \pm (k^2-4i)^{\frac{1}{2}}\right\}^{\frac{1}{2}}$ is purely imaginary then the next term in the series expansions (4.4.14) determines the boundary layer thickness.

Also, from (4.13.4), we see that there is a fourth boundary layer having a depth of penetration of order $(\nu/n\sigma)^{\frac{1}{2}}$ which is always much thinner than the boundary layer associated with λ_1.

For the special case when $a \equiv 0$, the solutions of (4.2.7), (4.2.5), (4.2.6) are

$$\Theta = \Phi \exp\{-z(in/\kappa_0)^{\frac{1}{2}}\} \quad , \quad \Upsilon = w = 0, \qquad (4.13.5)$$

since the x,y-momentum equations and the continuity equation are sufficient to determine Υ and w and contain no forcing term involving Θ. The solutions, (4.13.5), agree with the results that would be obtained by letting a tend to zero in (4.13.4) since the particular integral would vanish and hence $A_1 \equiv A_2 \equiv A_3 \equiv 0$ in order to satisfy the boundary conditions on the disk.

Hence, there is a resonance effect present in the sense that a different solution exists in the neighbourhood of $n = 2\Omega$. Then provided that the Reynolds number, R_o, remains large, we have shown that the introduction of the horizontal length scale, a^{-1}, guarantees that an oscillatory solution always exists. When only one disk is present this oscillatory solution consists of well-defined boundary layers confined to the disk.

Also, by the same arguments as we employed above, we can show that oscillatory solutions, which satisfy imposed boundary conditions, can always be found, for both the cases when one and two disks are present, when, instead of imposing an oscillatory heating on the disk, the disk is made to oscillate in the z-direction with a velocity,

$$w = \varepsilon h(x,y)e^{int} .$$

We must assume that the amplitude, ε, is very small so that we may consider the disk to be at $z = 0$ for all time, while it is, in fact, moving in a wave-like manner. Again, we have introduced a new length scale, a^{-1}, into the problem.

CHAPTER 5

STEADY STATE VORTICITY GENERATED BY HORIZONTAL TEMPERATURE GRADIENTS

PART 2: SOME OTHER SOLUTIONS

5.1 INTRODUCTION

In this chapter we will examine the effect on the steady vertical vorticity discussed in section 4.9, when the additional heating imposed on the lower disk,

$$\Theta = \Delta T \, \Phi \, h(x,y) ,$$

is replaced by either

$$\begin{aligned} \Theta &= \Delta T \, \Phi \qquad |x| \leq da , \\ &= 0 \qquad\quad\; |x| > da , \end{aligned} \tag{5.1.1}$$

or

$$\Theta = \Delta T \, \Phi \, \exp\{-x^2/\sigma^2\} , \tag{5.1.2}$$

where a, σ are positive real constants. Hence we are assuming an imposed temperature distribution which depends solely on one of the co-ordinates in the plane of the disks and takes the form of either a step-function, (5.1.1), where a non-zero temperature is present only on a finite strip $|x| \leq da$ or a normal distribution, (5.1.2).

When the imposed temperature distribution is the step-function (5.1.1), the steady solution is composed of an inviscid interior flow which satisfies the thermal-wind equations, Ekman layers on the disks, which have a depth of penetration of order $R^{-\frac{1}{2}}$ and free shear layers at the discontinuities, which are similar to the layers found by Stewartson [35] and have thicknesses of order $R^{-1/3}$ and $R^{-1/4}$. When x is considered constant, the variation of the interior vertical vorticity takes the same form as in section 4.9.

When the applied temperature distribution is the normal distribution (5.1.2), a solution has only been obtained for the two extreme cases, σ large and σ small. For both these cases, the solution consists of Ekman layers on the disks and an interior flow which is a particular solution of the thermal-wind equations. It is found, for large σ, that this problem reduces to the two dimensional problem associated with an applied cosine temperature distribution, with a small wave number, a, which has been discussed in section 4.9, provided

$$\frac{1}{\sigma^2} = \frac{a^2}{2}, \qquad (5.1.3)$$

and terms of order σ^{-4} are negligible. On the other hand, when σ is small, this problem is identical to the problem connected with (5.1.1) when a tends to zero, provided

$$\sigma = \frac{2a}{\sqrt{\pi}} \quad , \qquad (5.1.4)$$

and terms of order σ^3 are negligible. It should be noticed that in formula, (5.1.3), a is a wavenumber, while in formula, (5.1.4), a is a length scale and therefore the two formulae are unrelated.

5.2 EQUATIONS OF MOTION

We consider two infinite plane disks, $z = \pm d$, bounding a viscous fluid, when the fluid and the disks are rotating with constant angular velocity, Ω , and, additionally, a heating of the form

$$\theta = \Delta T \, \Theta(x), \qquad (5.2.1)$$

where $\Delta T \, \Theta(x)$ is defined by (5.1.1) or (5.1.2), is applied to the disk $z = -d$.

Again we assume that it is valid to apply the Boussinesq approximation and to linearize the basic equations. Also, since the required boundary conditions

$$\left.\begin{array}{ll} \text{(a)} & u = v = w = 0 \qquad \text{on } z = \pm d, \\ \text{(b)} & \theta = 0 \ \text{ on } \ z = d, \quad \theta = \Delta T \, \Theta(x) \ \text{ on } z = -d \ , \\ \text{(c)} & \theta, v \text{ tend to zero exponentially as } |x| \rightarrow \infty \ , \end{array}\right\}$$

(5.2.2)

are independent of y, we may assume that the flow is independent of y. Hence the basic equations (4.2.5), (4.2.6), (4.2.7) become, for time-independent motions,

$$\left.\begin{aligned}
-2\,\Omega v &= -\frac{\partial}{\partial x}\left[\frac{p}{\rho_0} - \tfrac{1}{2}\,\Omega^2(x^2+y^2)\right] + \nu\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right] u\,, \\
2\,\Omega u &= \nu\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right] v\,, \\
0 &= -\frac{\partial}{\partial z}\left[\frac{p}{\rho_0} - \tfrac{1}{2}\,\Omega^2(x^2+y^2)\right] + g\alpha\Theta + \nu\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right] w\,, \\
&\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\Theta = 0\,, \\
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,.
\end{aligned}\right\} \tag{5.2.3}$$

When we introduce the dimensionless variables (starred)

$$\left.\begin{aligned}
&z = dz^*, \quad x = dx^*, \quad (u,v,w) = \Omega d(u^*,v^*,w^*)\,, \\
&\frac{p}{\rho_0} - \tfrac{1}{2}\,\Omega^2(x^2+y^2) = d^2\,\Omega^2 p^*, \quad \Theta = \Delta T\Theta^*\,,
\end{aligned}\right\} \tag{5.2.4}$$

into the equations (5.2.3) and the boundary conditions (5.2.2), we find, when we eliminate u, w, p from the resulting equations and drop the asterisks,

$$R^{-2}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right]^3 v + 4\frac{\partial^2 v}{\partial z^2} = 2H\frac{\partial^2 \Theta}{\partial x\partial z} , \tag{5.2.5}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\Theta = 0 , \tag{5.2.6}$$

where R, H are again the Reynolds number, $\Omega d^2/\nu$, and the thermal Rossby number, (4.2.3), respectively. We will again assume that the Reynolds number, R, is large. The equations (5.2.5) and (5.2.6) must now be solved subject to the conditions that

$$\left.\begin{array}{lll} \text{(a)} & u = v = w = 0 & \text{on } z = \pm 1 , \\ \text{(b)} & \Theta = 0 \ \text{ on } z = 1, \quad \Theta = \textcircled{H}(x) & \text{on } z = -1 , \\ \text{(c)} & \Theta,\ v \text{ tend to zero exponentially as } |x| \longrightarrow \infty , & \end{array}\right\} \tag{5.2.7}$$

Then the Fourier transform

$$\bar{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f\, dx , \tag{5.2.8}$$

$$f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\, \bar{f}\, ds , \tag{5.2.9}$$

is applied to the equations (5.2.5), (5.2.6) and the boundary conditions (5.2.7) and we find that we must solve the equations

$$R^{-2}\left[D^2 - s^2\right]^3 \bar{v} + 4D^2\bar{v} = -2HisD\bar{\Theta} , \qquad (5.2.10)$$

$$\left[D^2 - s^2\right]\bar{\Theta} = 0 , \qquad (5.2.11)$$

where $D \equiv \frac{\partial}{\partial z}$, subject to the conditions

$$\left.\begin{array}{ll} \text{(a)} \quad \bar{u} = \bar{v} = \bar{w} = 0 & \text{on } z = \pm 1 , \\ \text{(b)} \quad \bar{\Theta} = 0 \quad \text{on } z = 1, & \bar{\Theta} = \overline{\Theta}(s) \quad \text{on } z = -1 . \end{array}\right\} \qquad (5.2.12)$$

When $\Delta T\,\Theta(x)$ is defined by (5.1.1) and (5.1.2) we find that

$$\overline{\Theta}(s) = \Phi\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin\{as\}}{s} , \qquad (5.2.13)$$

$$\overline{\Theta}(s) = \frac{\Phi}{\sqrt{2}}\,\sigma \exp\{-s^2\sigma^2/4\} , \qquad (5.2.14)$$

respectively.

The solution of the equation (5.2.11) which satisfies the required boundary condition (5.2.12(b)) is

$$\bar{\Theta} = \overline{\Theta}(s)\frac{\sinh\{s(1-z)\}}{\sinh\{2s\}} , \qquad (5.2.15)$$

which, when we apply the inverse Fourier transform (5.2.9), becomes

$$\Theta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx}\, \overline{\Theta}(s)\, \frac{\sinh\{s(1-z)\}}{\sinh\{2s\}}\, ds . \qquad (5.2.16)$$

When $\bar{\Theta}$ is given by (5.2.15), we see that a particular integral of (5.2.10) is

$$\frac{2Hi\,\overline{\Theta}(s)\,\cosh\{s(1-z)\}}{4\sinh\{2s\}} \,. \qquad (5.2.17)$$

If we seek a complementary function of (5.2.10) of the form $e^{\lambda z}$, we find that

$$R^{-2}(\lambda^2-s^2)^3 + 4\lambda^2 = 0 \,,$$

which, provided $s \ll R^{\frac{1}{2}}$, has the roots $\lambda = \pm\,\lambda_i$, $i = 1,2,3$, where

$$\left.\begin{aligned} \lambda_1 &= \frac{s^3}{2R} + O\left(\frac{s^7}{R^3}\right), \\ \lambda_2 &= (2iR)^{\frac{1}{2}} + O(R^{-\frac{1}{2}}s^2), \\ \lambda_3 &= (-2iR)^{\frac{1}{2}} + O(R^{-\frac{1}{2}}s^2), \end{aligned}\right\} \qquad (5.2.18)$$

and the signs of the square roots are chosen so that the λ's have positive real part. Then, for $s \ll R^{\frac{1}{2}}$, the solution of (5.2.10) which satisfies the boundary conditions (5.2.12(a)) to the highest order is

$$\bar{v} = -\frac{Hi\,\overline{\Theta}(s)}{4}\left\{\frac{-\tanh s}{\sinh\lambda_1 + 2R^{\frac{1}{2}}s^{-1}\cosh\lambda_1}\left[\sinh\{\lambda_1 z\}\right.\right.$$

$$\left. + 2R^{\frac{1}{2}}s^{-1}\cosh\lambda_1(\sinh\{\lambda_2 z\}\,e^{-\lambda_2} + \sinh\{\lambda_3 z\}\,e^{-\lambda_3})\right]$$

$$+ \frac{\coth s}{\cosh \lambda_1 + 2R^{\frac{1}{2}} s^{-1} \sinh \lambda_1} \left[\cosh\{\lambda_1 z\} + 2R^{\frac{1}{2}} s^{-1} \sinh \lambda_1 (\cosh\{\lambda_2 z\} e^{-\lambda_2}\right.$$

$$\left. + \cosh\{\lambda_3 z\} e^{-\lambda_3}) \right] - \frac{2 \cosh\{s(1-z)\}}{\sinh\{2s\}} \Bigg\} . \qquad (5.2.19)$$

When we restrict our attention to the range $s \ll R^{1/4}$, the dominant terms in (5.2.19) are

$$\bar{v} = - \frac{Hi\,\overline{\Theta}(s)}{4} \Bigg\{ -(\tanh s)(\sinh\{\lambda_2 z\} e^{-\lambda_2} + \sinh\{\lambda_3 z\} e^{-\lambda_3})$$

$$+ \coth s - \frac{2 \cosh\{s(1-z)\}}{\sinh(2s)} \Bigg\} , \qquad (5.2.20)$$

since, for the range of interest, $\lambda_1 \ll 1$. In the equation (5.2.20) the first term, which we will refer to as $\bar{v}_B$, determines the boundary layer motion while the remaining two terms, $\bar{v}_I$, determine the motion in the interior. When we apply the inverse Fourier transform (5.2.9) to (5.2.20), we find

$$v = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \Bigg\{ - \frac{Hi\,\overline{\Theta}(s)}{4} \Bigg[- \tanh s\,(\sinh\{\lambda_2 z\} e^{-\lambda_2}$$

$$+ \sinh\{\lambda_3 z\} e^{-\lambda_3}) + \coth s - \frac{2 \cosh\{s(1-z)\}}{\sinh\{2s\}} \Bigg] \Bigg\} ds , \qquad (5.2.21)$$

which gives a valid representation of the y-component of velocity provided the contribution to the integral is negligible for s outside the range $s \ll R^{1/4}$. From the expression (5.2.21), we can calculate the z-component of vorticity

$$\zeta = \frac{\partial v}{\partial x} , \qquad (5.2.22)$$

in order to obtain a comparison with the results obtained in chapter 4, section 4.9.

We will now consider the general solutions (5.2.16) and (5.2.21) when $\Theta(s)$ takes the particular forms (5.2.13) and (5.2.14).

5.3 A STEP-FUNCTION TEMPERATURE DISTRIBUTION

When the imposed temperature distribution on the lower disk, $z = -d$, is the step-function (5.1.1), the temperature distribution (5.2.16) and the y-component of velocity (5.2.21) in the fluid become, when $\overline{\Theta}(s)$ is given by (5.2.13) ,

$$\theta = \frac{2\Phi}{\pi} \int_0^\infty \frac{\sin\{as\}}{s} \, \frac{\sinh\{s(1-z)\}}{\sinh\{2s\}} \cos\{sx\} \, ds , \qquad (5.3.1)$$

$$v = -\frac{H\Phi}{2\pi} \int_0^\infty \Big[-(\tanh s)(\sinh\{\lambda_2 z\} e^{-\lambda_2} + \sinh\{\lambda_3 z\} e^{-\lambda_3})$$

$$+ \coth s - \frac{2\cosh\{s(1-z)\}}{\sinh\{2s\}} \Big] \sin\{sx\} \cdot \frac{\sin\{as\}}{s} \, ds. \qquad (5.3.2)$$

Firstly, we will evaluate the integral (5.3.1), for the temperature distribution, which can be rewritten as

$$\Theta = \frac{\Phi}{\pi} \int_{0}^{\infty} \frac{\sinh\{s(1-z)\}}{\sinh\{2s\}} \left[\int_{-a}^{x} \cos\{(a+x)s\}\, dx - \int_{+a}^{x} \cos\{(a-x)s\}\, dx \right] ds . \qquad (5.3.3)$$

When $z \neq -1$, the integrals in (5.3.3) converge for large s and therefore we may interchange the order of integration to give

$$\Theta = \frac{\Phi}{\pi} \left[\int_{-a}^{x} \int_{0}^{\infty} \frac{\sinh\{s(1-z)\}}{\sinh\{2s\}} \cos\{(a+x)s\}\, ds\, dx - \int_{+a}^{x} \int_{0}^{\infty} \frac{\sinh\{s(1-z)\}}{\sinh\{2s\}} \cos\{(a-x)s\}\, ds\, dx \right] . \qquad (5.3.4)$$

When we apply the results given by Edwards [12, p.274], (5.3.4) becomes

$$\Theta = \frac{\Phi}{4} \left[\int_{-a}^{x} \frac{\sin\left[\frac{1}{2}\pi(1-z)\right]}{\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi(a+x)\right]}\, dx - \int_{+a}^{x} \frac{\sin\left[\frac{1}{2}\pi(1-z)\right]}{\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi(a-x)\right]}\, dx \right]$$

$$= \frac{\Phi}{\pi}\left[\tan^{-1}\left\{\frac{\exp\left[\frac{1}{2}\pi(a+x)\right] + \cos\left[\frac{1}{2}\pi(1-z)\right]}{\sin\left[\frac{1}{2}\pi(1-z)\right]}\right\} - \tan^{-1}\left\{\frac{\exp\left[\frac{1}{2}\pi(x-a)\right] + \cos\left[\frac{1}{2}\pi(1-z)\right]}{\sin\left[\frac{1}{2}\pi(1-z)\right]}\right\}\right], \qquad (5.3.5)$$

provided $z \neq -1$, when we use Dwight [11] page 38, formula 160.01. The expression (5.3.5) represents the temperature variation in the fluid and we require the additional condition

$$\Theta = \Phi \quad |x| \leq a, \qquad \Theta = 0 \quad |x| > a \quad \text{on } z = -1,$$

in order to give a solution valid for all z.

We will now return to the evaluation of the integral (5.3.2) for the y-component of velocity, v, from which the z-component of vorticity, Υ, may be calculated by using the relation (5.2.22). The expression (5.3.2) will yield a valid representation for v since the integrand is oscillatory and negligibly small, $O(R^{-1/4})$ when $s \geq R^{1/4}$. Also the integral (5.3.2) is convergent, since for large s the integrand behaves like

$$\frac{\sin\{sx\}\sin\{as\}}{s},$$

and hence the actual value of the upper limit is unimportant and we can replace $\int_0^\infty$ by

$$\lim_{n\to\infty}\int_0^{\frac{(2n+1)\pi}{2(a+x)}}, \text{ except in the neighbourhood of } x = -a,$$

$$\text{or} \quad \lim_{n\to\infty} \int_0^{\frac{(2n+1)\pi}{2(a-x)}} \quad , \text{ except in the neighbourhood of } x = a \,. \tag{5.3.6}$$

Since the boundary conditions are discontinuous at $x = \pm a$, we expect free shear layers to exist parallel to the axis at these points and hence the above approximations, (5.3.6), will be valid outside the shear layers.

Then, from (5.3.2), we have

$$v = -\frac{H\Phi}{2\pi} \int_0^\infty \Big[(-\tanh s (\sinh \{\lambda_2 z\} e^{-\lambda_2} + \sinh \{\lambda_3 z\} e^{-\lambda_3})$$

$$+ \coth s - \frac{2 \cosh \{s(1-z)\}}{\sinh \{2s\}} \Big] \sinh \{as\} \int_0^x \cos \{sx\}\, dx\, ds \,,$$

$$= -\frac{H\Phi}{4\pi} \int_0^\infty \Big[-(\tanh s (\sinh \{\lambda_2 z\} e^{-\lambda_2} + \sinh \{\lambda_3 z\} e^{-\lambda_3})$$

$$+ \coth s - \frac{2 \cosh \{s(1-z)\}}{\sinh \{2s\}} \Big] \Big[\int_0^x \sin \{(a+x)s\}\, dx$$

$$+ \int_0^x \sin \{(a-x)s\}\, dx \Big] ds \,.$$

When we interchange the order of integration and use the approximations (5.3.6), we find that

$$v = -\frac{H\Phi}{4\pi}\int_0^x \left\{ \lim_{n\to\infty}\int_0^{\frac{(2n+1)\pi}{2(a+x)}} \left[(-\tanh s)(\sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}) + \coth s - \frac{2\cosh\{s(1-z)\}}{\sinh\{2s\}} \right] \sin\{(a+x)s\}\, ds \right.$$

$$+ \lim_{n\to\infty}\int_0^{\frac{(2n+1)\pi}{2(a-x)}} \left[(-\tanh s)(\sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}) \right.$$

$$\left.\left. + \coth s - \frac{2\cosh\{s(1-z)\}}{\sinh\{2s\}} \right] \sin\{(a-x)s\}\, ds \right\} dx.$$

Then the first integration can be performed by employing exactly the same method as used by Edwards [12] page 274 § 1105, to give

$$v = \frac{H\Phi}{8}\int_0^x \left\{ \left[\frac{1}{\sinh[\frac{1}{2}\pi(a+x)]} + \frac{1}{\sinh[\frac{1}{2}\pi(a-x)]} \right] (\sinh\{\lambda_2 z\}\, e^{-\lambda_2} \right.$$

$$+ \sinh\{\lambda_3 z\}\, e^{-\lambda_3}) + \left[\frac{\sinh[\frac{1}{2}\pi(a+x)]}{\cos[\frac{1}{2}\pi(1-z)] + \cosh[\frac{1}{2}\pi(a+x)]} - \frac{1}{\tanh[\frac{1}{2}\pi(a+x)]} \right]$$

$$\left. + \left[\frac{\sinh[\frac{1}{2}\pi(a-x)]}{\cos[\frac{1}{2}\pi(1-z)] + \cosh[\frac{1}{2}\pi(a-x)]} - \frac{1}{\tanh[\frac{1}{2}\pi(a-x)]} \right] \right\} dx . \qquad (5.3.7)$$

It should be noticed that the method used by Edwards [12] is not valid when p = q (in his notation) because $\int_0^\infty \sin\{mx\}\,dx$ does not converge whereas Edwards assumed this integral to be equal to (1/m). If, however, we alter the upper limit to $[(2n+1)\pi/2m]$, where n is large, then the method is valid. Hence the results given by Edwards for when p = q, page 277 § 1106, are only valid if the upper limit is defined in this special way, namely

$$\int_0^\infty = \lim_{n\to\infty} \int_0^{\frac{(2n+1)\pi}{2m}} .$$

Then, from (5.3.7), we find that

$$v = \frac{H\bar{\Phi}}{8\pi}\left[\left(\sinh\{\lambda_2 z\}\,e^{-\lambda_2} + \sinh\{\lambda_3 z\}\,e^{-\lambda_3}\right) \times \right.$$

$$\left.\times \left\{-\log\left[\frac{\cosh[\frac{1}{2}\pi(a+x)] + 1}{\cosh[\frac{1}{2}\pi(a+x)] - 1}\right] + \log\left[\frac{\cosh[\frac{1}{2}\pi(a-x)] + 1}{\cosh[\frac{1}{2}\pi(a-x)] - 1}\right]\right\}\right]$$

$$+ \frac{H\bar{\Phi}}{4\pi}\left\{\log\left[\frac{\cos[\frac{1}{2}\pi(1-z)] + \cosh[\frac{1}{2}\pi(a+x)]}{\sinh[\frac{1}{2}\pi(a+x)]}\right]\right.$$

$$\left. - \log\left[\frac{\cos[\frac{1}{2}\pi(1-z)] + \cosh[\frac{1}{2}\pi(a-x)]}{\sinh[\frac{1}{2}\pi(a-x)]}\right]\right\}$$

$$+ O(R^{-1}). \qquad (5.3.8)$$

When we substitute the solution (5.3.8) into the equation (5.2.5) with Θ given by (5.3.5), we find that the error involved for the first term, which represents the boundary layer, is of order R^{-1} and the error involved in the second term, which represents the interior flow, is of order R^{-2}. Then, since (5.3.8) satisfies the required boundary conditions for the y-component of velocity and since we have assumed that R is large, the solution (5.3.8) gives a valid representation of the flow involving a negligibly small error, $O(R^{-1})$. Therefore we may deduce that the above approximations were valid.

When the z-component of vorticity is calculated from (5.3.8) by using (5.2.22) we find that

$$\zeta = \frac{H\Phi}{8}\Bigg[\Big\{\sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}\Big\}\Bigg\{\frac{1}{\sinh\left[\frac{1}{2}\pi(a+x)\right]} + \frac{1}{\sinh\left[\frac{1}{2}\pi(a-x)\right]}\Bigg\}\Bigg]$$

$$+ \frac{H\Phi}{8}\Bigg[\Bigg\{\frac{\sinh\left[\frac{1}{2}\pi(a+x)\right]}{\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi(a+x)\right]} - \frac{1}{\tanh\left[\frac{1}{2}\pi(a+x)\right]}\Bigg\}$$

$$+ \Bigg\{\frac{\sinh\left[\frac{1}{2}\pi(a-x)\right]}{\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi(a-x)\right]} - \frac{1}{\tanh\left[\frac{1}{2}\pi(a-x)\right]}\Bigg\}\Bigg] + O(R^{-1}). \qquad (5.3.9)$$

The expression (5.3.9) satisfies the required boundary condition

$$\zeta = 0 \quad \text{on} \quad z = \pm 1,$$

with a negligibly small error, $O(R^{-1})$. The first term in (5.3.9), ζ_B, represents Ekman layers on the disks, $z = \pm 1$, which have a depth of penetration of order $R^{-\frac{1}{2}}$. The remaining term, ζ_I, represents the interior flow which, as in section 4.9, is a particular solution of the thermal wind equation,

$$2 \frac{\partial \zeta_I}{\partial z} = H \frac{\partial^2 \theta}{\partial x^2} .$$

It should be emphasized that the solution (5.3.9) is only valid outside the shear layers which occur around $x = \pm a$. The presence of these free shear layers is demonstrated by the singularities of (5.3.9) at $x = \pm a$. Before we examine the behaviour of the fluid in the shear layers we will discuss the equation (5.3.9).

The sign of ζ_I, the steady, interior, vertical vorticity, at $z = 0$, ± 1, when $\Phi > 0$, is shown in Fig.5.1. The opposite signs occur when $\Phi < 0$. When we compare Fig.5.1 with Fig.4.6(b), we see that, near the axis of rotation, ζ_I behaves in the same manner for both temperature distributions. Also, from (5.3.9), for any x, we see that ζ behaves like ζ_s in Fig.4.4 and the value of z such that $\zeta_I \equiv 0$ always occurs on the negative axis.

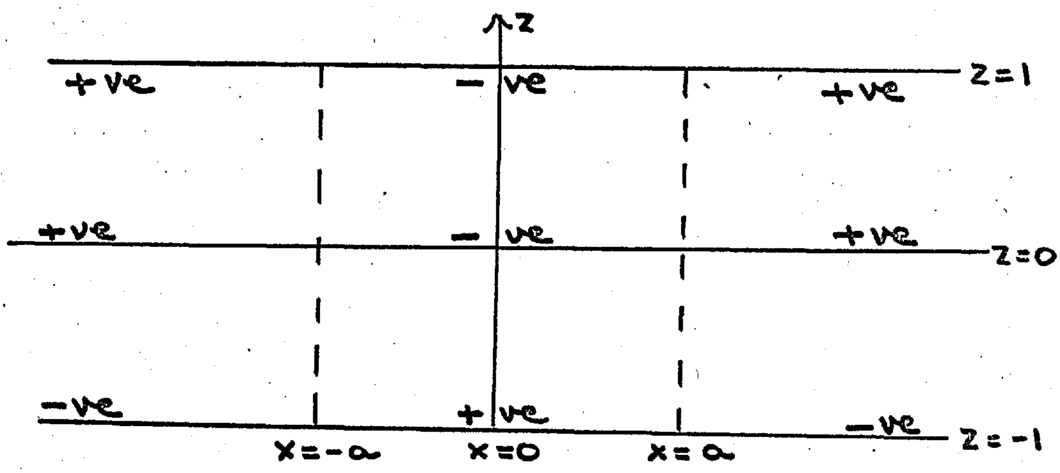


FIG. 5.1: The sign of the steady interior vertical vorticity, ζ_I, when $\Phi > 0$, for z = 0, 1, -1.

We will now return to the equation (5.2.19) in order to calculate the y-component of velocity in the shear layers by the method employed by Stewartson [35]. We will consider only the part of the shear layers which lies outside the Ekman layers, that is where $|z \pm 1| > O(R^{-\frac{1}{2}})$. Then, from (5.2.19) with $\overline{\Theta}(s)$ given by (5.2.13), we find that the y-component of velocity for the shear layers in the interior is

$$v_I = -\frac{H\Phi}{4\pi}\int_0^\infty \left[\cos\{(a-x)s\} - \cos\{(a+x)s\}\right] \times$$

$$\times\left\{\frac{-\tanh s \sinh\{\lambda_1 z\}}{s\sinh\lambda_1 + 2R^{\frac{1}{2}}\cosh\lambda_1} + \frac{\coth s\cosh\{\lambda_1 z\}}{s\cosh\lambda_1 + 2R^{\frac{1}{2}}\sinh\lambda_1} - \right.$$

$$\left. - \frac{2\cosh\{s(1-z)\}}{s\sinh\{2s\}}\right\} ds. \qquad (5.3.10)$$

We will now restrict our attention to the shear layer in the neighbourhood of $x = a$. Then from (5.3.10), we see that the term involving $\cos\{(a+x)s\}$ can be evaluated by retaining only the dominant terms for $s \ll R^{1/4}$ and by employing the same method as for the flow outside the shear layers, to give

$$\frac{H\Phi}{4\pi}\log\left\{\frac{\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi(a+x)\right]}{\sinh\left[\frac{1}{2}\pi(a+x)\right]}\right\} .$$

This agrees with the appropriate term in (5.3.8), namely the term for the interior which is non-singular in the neighbourhood of $x = a$. The remaining term in (5.3.10), which we will refer to as I, may be written

$$I = -\frac{H\Phi}{8\pi}\,\mathrm{Re}\int_{-\infty}^{\infty} e^{i|a-x|s}\left[-\frac{\tanh s\sinh\{\lambda_1 z\}}{s\sinh\lambda_1 + 2R^{\frac{1}{2}}\cosh\lambda_1}\right.$$

$$+ \frac{\coth s \cosh\{\lambda_1 z\}}{s \cosh \lambda_1 + 2R^{\frac{1}{2}}\sinh \lambda_1} - \frac{2 \cosh\{s(1-z)\}}{s \sinh\{2s\}} \Bigg] ds \,. \qquad (5.3.11)$$

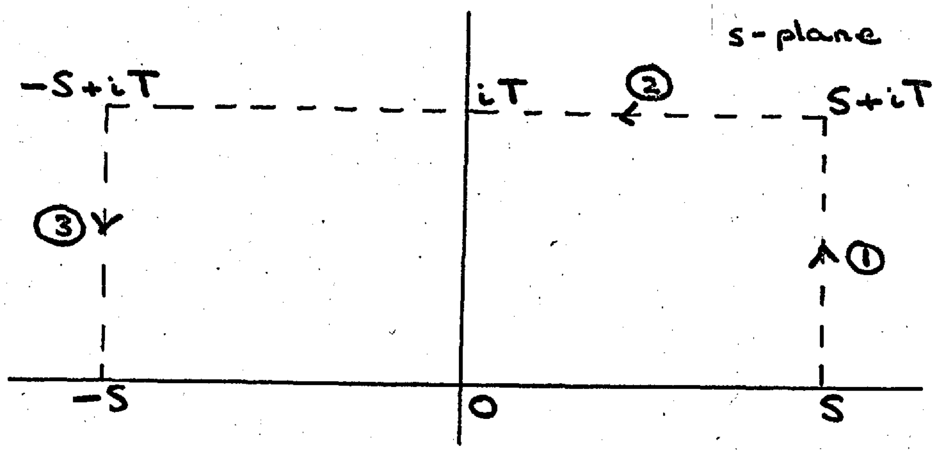


FIG.5.2: Path of integration in the complex s-plane for the integral (5.3.11).

We now wish to show that the contribution to the integral (5.3.11) from the contour which is shown in Fig.5.2 by a broken line, is negligibly small. Along the line ②, which is defined by

$$s = s_1 + iT \;, \qquad -S \leq s_1 \leq S \;,$$

we find, from (5.3.11), after some detailed arguments, that

$$|I| < \frac{H\Phi}{8\pi} e^{-T|a-x|} \, 2S \left\{ \left| \frac{1}{(\cos T)(T\sin[T^3/2R] - 2R^{\frac{1}{2}})} \right| \right.$$

$$+ \left| \frac{1}{(\sin T)(T\cos[T^3/2R] - 2R^{\frac{1}{2}})} \right| + \left| \frac{2}{T\sin(2T)} \right| \Bigg\} , \qquad (5.3.12)$$

which, for $S < T$, tends to zero as T tends to infinity with S fixed. Along the line (1), which is defined by

$$s = S + is_2 , \qquad 0 \leq s_2 \leq T ,$$

we find, from (5.3.11), after some detailed arguments, that

$$|\mathcal{I}| < \frac{H\Phi}{8\pi} \frac{T}{S} \left| \exp\{\lambda_1^*(|z|-1)\} [1 - \text{sgn } z] - 2\exp\{-S(1+z)\} \right| , \qquad (5.3.13)$$

where λ_1^* is the minimum value of λ_1 along this contour and will always have a large positive real part. This contribution, (5.3.13), to the integral vanishes as S tends to infinity. Similarly we can show that the contribution to the integral from the line (3) also vanishes as S tends to infinity.

Therefore we have shown that the contribution to the integral (5.3.11) vanishes around the large contour (broken line) in Fig.5.2 . The integral, $\mathcal{I}$, defined by (5.3.11), is equal to the sum of the residues calculated at the poles of the integrand which are situated in the upper half-plane. The poles of the integrand of (5.3.11) in the upper half-plane are

$$\left.\begin{array}{lll} i\nu_1 , & (\frac{\sqrt{3}}{2}+\frac{i}{2})\nu_1 , & (-\frac{\sqrt{3}}{2}+\frac{i}{2})\nu_1 , \\ i\nu_2 , & (\frac{\sqrt{3}}{2}+\frac{i}{2})\nu_2 , & (-\frac{\sqrt{3}}{2}+\frac{i}{2})\nu_2 , \\ iR^{1/4} , & & \\ \frac{i(2m-1)\pi}{2} , & im\pi , & \frac{im\pi}{2} , \end{array}\right\} \qquad (5.3.14)$$

where $\nu_1 = \left[(2n-1)\pi R\right]^{1/3}$, $\nu_2 = \left[2n\pi R\right]^{1/3}$,

$n = 1,2,3,\ldots$,

$m = 1,2,3,\ldots$.

It should be noticed that the double zero of the denominator in (5.3.11) at s=0 is also a double zero of the numerator and hence is not a singularity of the integrand of (5.3.11).

When I is found by calculating the residues at the poles (5.3.14), we find, from (5.3.10), that, near x = a and outside the Ekman layer,

$$v_I = \frac{H\Phi}{4\pi}\log\left[\frac{\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi(a+x)\right]}{\sinh\left[\frac{1}{2}\pi(a+x)\right]}\right]$$

$$+\frac{H\Phi}{12R^{1/6}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\left[(2n-1)\pi z/2\right]}{\left[(2n-1)\pi\right]^{2/3}}\left\{\tan\nu_1\exp\{-\nu_1|x-a|\}\right.$$

$$-\frac{2\exp\{-\frac{1}{2}\nu_1|x-a|\}}{\cosh\{\sqrt{3}\,\nu_1\} + \cos\nu_1}\left[\sin\nu_1\cos(\frac{\sqrt{3}}{2}\nu_1|x-a| - \frac{\pi}{3})\right.$$

$$- \sinh \{\sqrt{3}\nu_1\} \cos(\frac{\sqrt{3}}{2} \nu_1|x-a| + \frac{\pi}{6}) \Big] \Big\}$$

$$+ \frac{H\bar{\Phi}}{12R^{1/6}} \sum_{n=1}^{\infty} \frac{\cos[n\pi z] (-1)^n}{[2n\pi]^{2/3}} \Big\{ \cot \nu_2 \exp\{-\nu_2|x-a|\}$$

$$+ \frac{2\exp\{-\frac{1}{2}\nu_2|x-a|\}}{\cosh\{\sqrt{3}\nu_2\} - \cos\nu_2} \Big[- \sin\nu_2 \cos(\frac{\sqrt{3}}{2}\nu_2|x-a| - \frac{\pi}{3})$$

$$- \sinh\{\sqrt{3}\nu_2\} \cos(\frac{\sqrt{3}}{2} \nu_2|x-a| + \frac{\pi}{6}) \Big] \Big\}$$

$$+ \frac{H\bar{\Phi}}{8} \cot R^{1/4} \exp\{-R^{1/4}|x-a|\}$$

$$- \frac{H\bar{\Phi}}{4} \Big\{ - \sum_{m=1}^{\infty} \frac{\exp\{-(2m-1)|x-a|\pi/2\} \sin[(2m-1)^3\pi^3 z/16R]}{\frac{1}{2}\pi(2m-1)\sin[(2m-1)^3\pi^3/16R] + 2R^{\frac{1}{2}}\cos[(2m-1)^3\pi^3/16R]}$$

$$+ \sum_{m=1}^{\infty} \frac{\exp\{-m\pi|x-a|\} \cos[m^3\pi^3 z/2R]}{m\pi\cos[m^3\pi^3/2R] - 2R^{\frac{1}{2}}\sin[m^3\pi^3/2R]}$$

$$- \sum_{m=1}^{\infty} \frac{2(-1)^m}{m\pi} \cos[m\pi(1-z)/2] \exp[-\tfrac{1}{2}m\pi|x-a|] \Big\}. \qquad (5.3.15)$$

Then, from (5.2.22) and (5.3.15), we find that for the shear layer around $x = a$, outside the Ekman layers, the z-component of vorticity is

$$\zeta_I = \frac{H\bar{\Phi}}{8} \Big\{ \frac{\sinh[\frac{1}{2}\pi(a+x)]}{\cos[\frac{1}{2}\pi(1-z)] + \cosh[\frac{1}{2}\pi(a+x)]} - \frac{1}{\tanh[\frac{1}{2}\pi(a+x)]} \Big\}$$

$$+\frac{H\Phi R^{1/6}}{12}\,\mathrm{sgn}(x-a)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\left[(2n-1)\pi z/2\right]}{\left[(2n-1)\pi\right]^{1/3}}\left\{-\tan\nu_1\exp\left\{-\nu_1|x-a|\right\}\right.$$

$$+\frac{2\exp\left\{-\frac{1}{2}\nu_1|x-a|\right\}}{\cosh\left\{\sqrt{3}\,\nu_1\right\}+\cos\nu_1}\left[-\sinh\left\{\sqrt{3}\,\nu_1\right\}\cos\left(\frac{\sqrt{3}}{2}\nu_1|x-a|-\frac{\pi}{6}\right)\right.$$

$$\left.\left.-\sin\nu_1\cos\left(\frac{\sqrt{3}}{2}\nu_1|x-a|+\frac{\pi}{3}\right)\right]\right\}$$

$$+\frac{H\Phi R^{1/6}}{12}\,\mathrm{sgn}(x-a)\sum_{n=1}^{\infty}\frac{(-1)^{n}\cos\left[n\pi z\right]}{\left[2n\pi\right]^{1/3}}\left\{-\cot\nu_2\exp\left\{-\nu_2|x-a|\right\}\right.$$

$$+\frac{2\exp\left\{-\frac{1}{2}\nu_2|x-a|\right\}}{\cosh\left\{\sqrt{3}\,\nu_2\right\}-\cos\nu_2}\left[-\sin\nu_2\cos\left(\frac{\sqrt{3}}{2}\nu_2|x-a|+\frac{\pi}{3}\right)\right.$$

$$\left.\left.+\sinh\left\{\sqrt{3}\,\nu_2\right\}\cos\left(\frac{\sqrt{3}}{2}\nu_2|x-a|-\frac{\pi}{6}\right)\right]\right\}$$

$$-\frac{H\Phi}{8}R^{1/4}\,\mathrm{sgn}(x-a)\cot R^{1/4}\exp\left\{-R^{1/4}|x-a|\right\}$$

$$-\frac{H\Phi}{4}\,\mathrm{sgn}(x-a)\left[\sum_{m=1}^{\infty}\frac{\exp\left\{-(2m-1)|x-a|\,\pi/2\right\}\sin\left[(2m-1)^3\pi^3 z/16R\right]}{\sin\left[(2m-1)^3\pi^3/16R\right]+4R^{\frac{1}{2}}\pi^{-1}(2m-1)^{-1}\cos\left[(2m-1)^3\pi^3/16R\right]}\right.$$

$$-\sum_{m=1}^{\infty}\frac{\exp\left\{-m\pi|x-a|\right\}\cos\left\{m^3\pi^3 z/2R\right\}}{\cos\left[m^3\pi^3/2R\right]-2R^{\frac{1}{2}}(m\pi)^{-1}\sin\left[m^3\pi^3/2R\right]}$$

$$\left.+\sum_{m=1}^{\infty}\exp\left\{-m\pi|x-a|/2\right\}\cos\left\{m\pi(1-z)/2\right\}(-1)^m\right]. \qquad (5.3.16)$$

Then, from the second and third terms in (5.3.15) and (5.3.16), we see that there is a free shear layer around x=a which has a depth of penetration of order $R^{-1/3}$. This layer permits vertical mass flow from one Ekman layer to the other. From the fourth terms in (5.3.15) and (5.3.16), we see that there is a second free shear layer which has a depth of penetration of order $R^{-1/4}$ and hence always contains the $R^{-1/3}$-layer. This second free shear layer is responsible for smoothing out the discontinuities in the y-component of velocity which were introduced by the boundary conditions. These free shear layers correspond to the layers found by Stewartson [35].

From (5.3.16), we see that the contribution to the z-component of vorticity from the shear layer terms is of order $R^{1/4}$ cot $R^{1/4}$ from the $R^{-1/4}$-layer and the sum of the two infinite series of terms which individually are of order $R^{1/6}$, from the $R^{-1/3}$-layer. The remaining terms in (5.3.16) are the first term which corresponds exactly to the interior term in (5.3.9) and is regular at $x = a$ and the last term which consists of three infinite series.

At the edge of the shear layers, which occurs when

$$|x - a| = O(R^{1/4}) ,$$

we require the solution (5.3.16) to match the interior terms in (5.3.9).

When we write

$$x - a = \frac{A}{R^{1/4}} ,$$

where A is sufficiently large for exp(-A) to become negligible, the solution (5.3.9) for the interior reduces to

$$\mathcal{Y}_I = \frac{H\bar{\Phi}}{8}\left\{\frac{\sinh\{\pi a\}}{\cos\left[\tfrac{1}{2}\pi(1-z)\right] + \cosh\{\pi a\}} - \frac{1}{\tanh\{\pi a\}}\right\}$$

$$+ \frac{H\bar{\Phi}}{8}\;\frac{2R^{1/4}}{A\pi} , \qquad (5.3.17)$$

when $z \neq -1$, and (5.3.16) reduces to

$$\mathcal{Y}_I = \frac{H\bar{\Phi}}{8}\left\{\frac{\sinh\{\pi a\}}{\cos\left[\tfrac{1}{2}\pi(1-z)\right] + \cosh\{\pi a\}} - \frac{1}{\tanh\{\pi a\}}\right\}$$

$$- \frac{H\bar{\Phi}}{4}\left[\sum_{m=1}^{\infty}\frac{\exp\{-(2m-1)\pi A/2R^{1/4}\}\sin\{(2m-1)^3\pi^3 z/16R\}}{\sin\{(2m-1)^3\pi^3/16R\} + 4R^{\frac{1}{2}}\pi^{-1}(2m-1)^{-1}\cos\{(2m-1)^3\pi^3/16R\}}\right.$$

$$- \sum_{m=1}^{\infty}\frac{\exp\{-m\pi A/R^{1/4}\}\cos\{m^3\pi^3 z/2R\}}{\cos\{m^3\pi^3/2R\} - 2R^{\frac{1}{2}}(m\pi)^{-1}\sin\{m^3\pi^3/2R\}}$$

$$\left. + \sum_{m=1}^{\infty}\exp\{-m\pi A/2R^{1/4}\}\cos\{m\pi(1-z)/2\}(-1)^m\right] . \qquad (5.3.18)$$

The first term in (5.3.17) is identical to the first term in (5.3.18) and hence we must now show that the sum of the three infinite series in (5.3.18) corresponds to the second term in (5.3.17).

Since there is a negligible contribution from the terms in the infinite series in (5.3.18) when $m \geq O(R^{1/4})$, we may simplify (5.3.18) to give

$$\psi_I = \frac{H\Phi}{8}\left\{\frac{\sinh\{\pi a\}}{\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\{\pi a\}} - \frac{1}{\tanh\{\pi a\}}\right\}$$

$$- \frac{H\Phi}{4}\left[\sum_{m=1}^{M} \exp\{-(2m-1)A\,\pi/2R^{1/4}\}\frac{(2m-1)^4\pi^4 z}{64R^{3/2}}\right.$$

$$- \sum_{m=1}^{M} \frac{\exp\{-m\pi A/R^{1/4}\}}{1 - m^2\pi^2R^{-\frac{1}{2}}}$$

$$\left. + \sum_{m=1}^{M} \exp\{-m\pi A/2R^{1/4}\}\cos\{m\pi(1-z)/2\}(-1)^m\right], \qquad (5.3.19)$$

where $M = \dfrac{\varepsilon R^{1/4}}{\pi}$ with the choice $0 < \varepsilon < 1$.

In (5.3.19) the first series makes a contribution which is always less than $O(R^{-1/4})$ and hence may be neglected. Also, when $z \neq -1$, which is always satisfied for the part of the shear layer which lies outside the Ekman layers, the third series in (5.3.19) is always oscillatory and has a sum of order one since it may be written

$$\sum_{m=1}^{M} \exp\{-m\pi A/2R^{1/4}\} \cos\{m\pi(3-z)/2\}$$

$$= \text{Re} \sum_{m=1}^{M} \exp\left\{-\frac{m\pi A}{2R^{1/4}} + \frac{im\pi}{2}(3-z)\right\}$$

$$= \text{Re} \left\{ \frac{\exp\{i\pi(3-z)/2\}}{1-\exp\{i\pi(3-z)/2\}} \right\}$$

$$= -\tfrac{1}{2} \;,$$

when only the highest order terms are retained. Therefore the third series makes no contribution of order $R^{1/4}$.

Therefore we wish to show that

$$\frac{H\Phi}{4} \sum_{m=1}^{M} \frac{\exp\{-m\pi A/R^{1/4}\}}{1-m^2\pi^2R^{-\frac{1}{2}}} \;, \qquad (5.3.20)$$

corresponds to the second term in (5.3.17), when only the highest order terms are considered. Since the largest contribution to (5.3.20) arises when m is small, we may assume that it is valid to write the series as

$$\frac{H\Phi}{4} \sum_{m=1}^{M} (1 + m^2\pi^2R^{-\frac{1}{2}} + m^4\pi^4R^{-1} + \ldots)\exp\{-m\pi A/R^{1/4}\} \;,$$

and when we sum each term independently over m, this series becomes

$$\frac{H\Phi}{4}\left(1 + \frac{\alpha^2D^2}{A^2} + \frac{\alpha^4D^4}{A^4} + \ldots\right)\left(\frac{1}{e^{\alpha}-1}\right) \;,$$

where $\alpha = \pi A R^{-1/4}$, $D \equiv \frac{\partial}{\partial \alpha}$ and $\exp\{-\alpha M\}$ is assumed negligible. A better form of this result is

$$\frac{H\bar{\Phi}}{4}\left(\frac{R^{1/4}}{\pi A}\right)\left\{1 + \frac{2!}{A^2} + \frac{4!}{A^4} + \dots + \frac{(2n)!}{A^{2n}} + \dots\right\}, \tag{5.3.21}$$

where only the terms $O(R^{1/4})$ are retained and also $\exp(-A)$ and $\exp(-\varepsilon A)$ are assumed negligible.

In order to show that the series (5.3.21) is a valid asymptotic series for the series (5.3.20), it is necessary to estimate the error which is introduced when we equate (5.3.20) and (5.3.21). Firstly, we notice that the function

$$Y = \frac{e^{-\alpha x}}{1 - x^2\alpha^2 A^{-2}},$$

is always a strictly decreasing function of x provided

$$A(1 - \varepsilon^2) > 2\varepsilon . \tag{5.3.22}$$

Then, provided A, ε are chosen so that (5.3.22) is satisfied, the sum (5.3.20) will always be bounded above by

$$\frac{H\bar{\Phi}}{4}\int_0^{M-1} \frac{e^{-\alpha y}}{1 - y^2\alpha^2 A^{-2}}\, dy , \tag{5.3.23}$$

and below by

$$\frac{H\Phi}{4} \int_{1}^{M} \frac{e^{-\alpha y}}{1 - y^2\alpha^2 A^{-2}} \, dy \; . \qquad (5.3.24)$$

Then the integral (5.3.23) may be written as

$$\frac{H\Phi}{8} \left[\int_{0}^{M-1} \frac{e^{-\alpha y}}{1 + \alpha y A^{-1}} \, dy + \int_{0}^{M-1} \frac{e^{-\alpha y}}{1 - \alpha y A^{-1}} \, dy \right]$$

$$= \frac{H\Phi}{8} \cdot \frac{A}{\alpha} \left[\frac{1}{A} - \frac{1}{A^2} + \frac{2}{A^3} + \ldots\ldots + \frac{(-1)^{n-1}(n-1)!}{A^n} + R_n \right]$$

$$+ \frac{H\Phi}{8} \cdot \frac{A}{\alpha} \left[\frac{1}{A} + \frac{1}{A^2} + \frac{2}{A^3} + \ldots\ldots + \frac{(n-1)!}{A^n} + S_n \right] ,$$

where $R_n < \dfrac{n!}{A^{n+1}}$, $\quad S_n < \dfrac{n!}{A^{n+1}(1-\varepsilon)^{n+1}}$.

Hence when the integral (5.3.23) is assumed equal to the series (5.3.21), the error involved will always be less than the last term retained provided

$$(2n + 1) < A \, (1 - \varepsilon)^{2n+2} \; . \qquad (5.3.25)$$

Similarly, we find that the integral (5.3.24) is always greater than

$$\frac{H\Phi}{4} \; \frac{A}{\alpha} \left\{ \frac{1}{A} + \frac{2}{A^3} + \ldots\ldots + \frac{(2n)!}{A^{2n+1}} \right\} ,$$

when only the highest order terms are retained and we use the fact that $\alpha \ll 1$ and $A = O(1)$.

Hence we see that it is valid to assume that the series (5.3.20) can be expressed in the form (5.3.21) with an error always less than the last term retained, provided (5.3.22) and (5.3.25) are satisfied. In fact, the condition (5.3.25), with $n > 0$, is sufficient, since this condition automatically implies that (5.3.22) is satisfied. Hence we deduce that (5.3.20) is equal to

$$\frac{H\Phi}{4}\left(\frac{R^{1/4}}{\pi A}\right), \qquad (5.3.26)$$

with an error always less than $(H\Phi R^{1/4}/\pi A^3)$ provided

$$3 < A(1-\varepsilon)^4 . \qquad (5.3.27)$$

The ratio of the error to (5.3.26) is $(4/A^2)$ which is always small when A satisfies (5.3.27). Therefore we see that the sum of the series (5.3.18) agrees with the second term in (5.3.17) to the highest order.

Hence we have shown that the interior solution in (5.3.9) and the interior shear layer solution (5.3.16) match at the edge of the free shear layers, that is when

$$|x - a| = O(R^{-1/4}).$$

The z-component of vorticity for the shear layers around $x = -a$ could be found by employing the same arguments as we used above and will, in fact, be given by (5.3.16) with

$$a + x, \qquad |a - x|, \qquad \mathrm{sgn}(x-a) \, ,$$

replaced by

$$a - x, \qquad |a + x|, \qquad \mathrm{sgn}(x+a) \, ,$$

respectively.

The solution for the part of the shear layers which lies inside the Ekman layers could be found by an analysis similar to that employed above but no discussion of this problem will be given here.

5.4 A NORMAL TEMPERATURE DISTRIBUTION

We will now consider the second special case which occurs when a temperature distribution of the form (5.1.2) is applied at the disk $z = -d$. For this particular case $\overline{\Theta}(s)$ is given by (5.2.14) and the equations (5.2.16) and (5.2.21) become

$$\Theta = \frac{\Phi \sigma}{\sqrt{\pi}} \int_0^\infty \exp\{-s^2\sigma^2/4\} \, . \, \frac{\sinh[s(1-z)]}{\sinh[2s]} \cos\{sx\} \, ds, \qquad (5.4.1)$$

$$v = -\frac{\Phi \sigma H}{4\sqrt{\pi}} \int_0^\infty \exp\{-s^2\sigma^2/4\} \left[-(\tanh s)(\sinh\{\lambda_2 z\} e^{-\lambda_2}\right.$$

$$+ \ \sinh\{\lambda_3 z\}\, e^{-\lambda_3}) + \coth s - \frac{2\cosh\{s(1-z)\}}{\sinh\{2s\}} \Bigg] \sin\{sx\}\, ds. \qquad (5.4.2)$$

The expression (5.4.2) will give a valid description of the y-component of velocity provided there is a negligibly small contribution from the integral outside the range $s \ll R^{1/4}$. This will always be true provided

$$s\sigma \gg O(1), \qquad (5.4.3)$$

outside the range $s \ll R^{1/4}$, which implies that

$$R^{1/4}\sigma \gg O(1)$$

must always be satisfied. The condition (5.4.3) can always be satisfied for σ large and provides a lower bound on σ for the case σ small.

No discussion of the integrals (5.4.1) and (5.4.2) has been obtained for general σ. Instead we will consider the two extreme cases, σ large and σ small.

σ large

When we write

$$\tfrac{1}{2}s^2 = \rho,$$

(5.4.1) and (5.4.2) become

$$e = \frac{\Phi\,\sigma}{\sqrt{\pi}} \int_0^\infty \exp\{-\tfrac{1}{2}\rho\sigma^2\} \frac{\sinh\{(2\rho)^{\frac{1}{2}}(1-z)\}}{\sinh\{2(2\rho)^{\frac{1}{2}}\}} \cos\{(2\rho)^{\frac{1}{2}}x\} \frac{d\rho}{(2\rho)^{\frac{1}{2}}},$$

(5.4.4)

$$v = -\frac{\Phi \sigma H}{4\sqrt{\pi}} \int_0^\infty \exp\{-\tfrac{1}{2}\rho\sigma^2\} \Big[-\tanh\{(2\rho)^{\frac{1}{2}}\} (\sinh\{\lambda_2 z\}\, e^{-\lambda_2}$$

$$+ \sinh\{\lambda_3 z\}\, e^{-\lambda_3}) + \coth\{(2\rho)^{\frac{1}{2}}\} - \frac{2\cosh\{(2\rho)^{\frac{1}{2}}(1-z)\}}{\sinh\{2(2\rho)^{\frac{1}{2}}\}} \qquad \times$$

$$\times \; \sin\{(2\rho)^{\frac{1}{2}} x\} \, \frac{d\rho}{(2\rho)^{\frac{1}{2}}} . \tag{5.4.5}$$

Since, for large ρ, the integrands in (5.4.4) and (5.4.5) behave like $\exp\{-\frac{1}{2}\rho\sigma^2\}$, we see that the major contribution to the integrals (5.4.4) and (5.4.5) arises from ρ small. Hence we may use WATSON'S LEMMA [23, p.501-3] to obtain the following asymptotic formulae.

$$\theta = \frac{\Phi\,(1-z)}{2} \Big\{ 1 + \frac{1}{\sigma^2}\Big[\frac{1}{3}z^2 - \frac{2}{3}z - x^2 - 1\Big]$$
$$+ \frac{3}{\sigma^4}\Big[\frac{(1-z)^4}{30} + \frac{x^4}{6} + \frac{56}{45} - \frac{4(1-z)^2}{9} + \frac{4x^2}{3} - \frac{x^2(1-z)^2}{3}\Big]\Big\}$$
$$+ O(\sigma^{-6}) , \tag{5.4.6}$$

$$v = \frac{H\Phi}{2}\Big[\frac{x}{\sigma^2} - \frac{3}{\sigma^4}\Big(\frac{2x}{3} + \frac{x^3}{3}\Big)\Big]\Big[\sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}\Big]$$
$$- \frac{H\Phi}{4}\Big[\frac{x}{\sigma^2}(2-(1-z)^2) - \frac{3}{\sigma^4}\Big\{\frac{x^3}{3}(2-(1-z)^2)$$
$$+ \frac{x}{3}(4 + \tfrac{1}{2}(1-z)^4 - 4(1-z)^2)\Big\}\Big] + O(\sigma^{-6}) , \tag{5.4.7}$$

where terms of order R^{-1} have been neglected. The expressions (5.4.6) and (5.4.7) have been obtained for fixed x and large σ and therefore these expressions are not expected to give a valid description of the flow when x tends to infinity.

Then, from (5.2.22) and (5.4.7), we find that the z-component of vorticity is given by

$$\mathcal{S} = \frac{H\Phi}{2}\left[\frac{1}{\sigma^2} - \frac{3}{\sigma^4}\left(\frac{2}{3} + x^2\right)\right]\left[\sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}\right]$$

$$- \frac{H\Phi}{4}\left[\frac{(2-(1-z)^2)}{\sigma^2} - \frac{3}{\sigma^4}\left\{x^2(2-(1-z)^2) + \frac{1}{3}(4 + \tfrac{1}{2}(1-z)^4 - 4(1-z)^2)\right\}\right] + O(\sigma^{-6}) . \qquad (5.4.8)$$

Then the first terms in (5.4.7) and (5.4.8) represent Ekman layers on the disks, $z = \pm 1$, which have a depth of penetration of order $R^{-\frac{1}{2}}$, while the second terms give the interior flow which again satisfies the thermal-wind equation

$$H \frac{\partial\Theta}{\partial x} = \frac{2\partial v_I}{\partial z} ,$$

when Θ is given by (5.4.6).

When we retain only the highest order terms in (5.4.6) and (5.4.8), we find

$$\Theta = \frac{\Phi\,(1-z)}{2} , \qquad (5.4.9)$$

$$\zeta = \frac{H\Phi}{2}\cdot\frac{1}{\sigma^2}\left[\sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}\right] - \frac{H\Phi}{4}\left[\frac{2-(1-z)^2}{\sigma^2}\right] . \qquad (5.4.10)$$

The results (5.4.9) and (5.4.10) are independent of x and agree with the results (4.3.9) and (4.9.3), in chapter 4, when we allow a in h(x) to tend to zero with x fixed and assume that

$$\frac{a^2}{2} = \frac{1}{\sigma^2} . \qquad (5.4.11)$$

The temperature distribution (5.4.9) and the distribution of the z-component of vorticity (5.4.10) are shown in Fig.4.2 and Fig.4.4 respectively. The vorticity (5.4.10) vanishes in the interior when

$$z = 1 - \sqrt{2} .$$

If we expand the results (4.3.9) and (4.9.3) for small a and assume a two dimensional flow, that is $h(x,y) = \cos\{ax\}$, we find

$$\left.\begin{aligned} \Theta &= \frac{\Phi\,(1-z)}{2}\left\{1 + \frac{a^2}{2}\left[\frac{(1-z)^2}{3} - \frac{4}{3} - x^2\right]\right\} + O(a^4) , \\ \zeta_s &= \frac{H\Phi a^2}{4}\left\{\frac{(1-z)^2}{2} - 1 + \sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}\right\} \\ &\qquad + O(a^4) . \end{aligned}\right\} \qquad (5.4.12)$$

The results (5.4.12) agree with the expressions (5.4.6) and (5.4.8) when terms of order σ^{-4} are neglected, provided the condition (5.4.11) is satisfied. If, however, the terms of order a^4 in (5.4.12) were calculated explicitly, no agreement would be found, since when we expand $\Phi \cos\{ax\}$, $\Phi \exp\{-x^2/\sigma^2\}$ for small a and large σ, we find that

$$\Phi \cos\{ax\} = \Phi\left\{1 - \frac{a^2x^2}{2} + \frac{a^4x^4}{4!} + \dots\right\} ,$$

$$\Phi \exp\{-x^2/\sigma^2\} = \Phi\left\{1 - \frac{x^2}{\sigma^2} + \frac{x^4}{2\sigma^4} + \dots\right\} ,$$

and when (5.4.11) is assumed valid only the first two terms in the above series are identical.

Hence, when σ is large, the problem of applying a normal temperature distribution to $z = -1$ is equivalent to applying a sinusoidal heating, provided terms of order σ^{-4} are negligible.

σ small

We will now consider the remaining extreme case, that is σ small but

$$\sigma \gg O(R^{-1/4}),$$

from (5.4.3) in order that the expression (5.4.2) for v is always small. If we write $s\sigma = t$, then the integrals (5.4.1) and (5.4.2) become

$$\Theta = \frac{\Phi\sigma}{\sqrt{\pi}} \int_0^\infty \exp\left\{-\frac{1}{4}t^2\right\} \frac{\sinh\{(1-z)t/\sigma\}}{\sinh\{2t/\sigma\}} \cos\{xt/\sigma\}\, d\left(\frac{t}{\sigma}\right), \tag{5.4.13}$$

$$v = -\frac{\Phi\sigma H}{4\sqrt{\pi}} \int_0^\infty \exp\left\{-\frac{1}{4}t^2\right\} \sin\{xt/\sigma\} \left[\coth\left(\frac{t}{\sigma}\right) - \frac{2\cosh\{(1-z)t/\sigma\}}{\sinh\{2t/\sigma\}}\right.$$

$$\left. - \tanh\left(\frac{t}{\sigma}\right)\left\{\sinh\{\lambda_2 z\}e^{-\lambda_2} + \sinh\{\lambda_3 z\}e^{-\lambda_3}\right\}\right] d\left(\frac{t}{\sigma}\right). \tag{5.4.14}$$

When $z \neq \pm 1$, the terms involving

$$\cos\{xt/\sigma\}\frac{\sinh\{(1-z)t/\sigma\}}{\sinh\{2t/\sigma\}}, \quad \sin\{xt/\sigma\}\frac{\cosh\{(1-z)t/\sigma\}}{\sinh\{2t/\sigma\}},$$

make a negligibly small contribution when $t > O(\sigma)$ and hence we may assume, for these terms, that the major contribution to the integral arises from the neighbourhood of $t = 0$. Therefore we may expand $\exp\{-t^2/4\}$ for small t to give

$$\exp\{-t^2/4\} = 1 - \frac{t^2}{4} + \ldots .$$

When we introduce the results given by Edwards [12, § 1105, p.274] we find that (5.4.13) and (5.4.14) become

$$\Theta = \frac{\Phi\sigma\sqrt{\pi}}{4} \frac{\sin\left[\frac{1}{2}\pi(1-z)\right]}{\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi x\right]} + O(\sigma^3), \tag{5.4.15}$$

$$v = -\frac{\Phi\sigma H}{4\sqrt{\pi}} \int_0^\infty \exp\{-\tfrac{1}{4}s^2\sigma^2\}\Big[-(\tanh s)(\sinh\{\lambda_2 z\}\, e^{-\lambda_2}$$

$$+ \sinh\{\lambda_3 z\}\, e^{-\lambda_3}) + \coth s\Big]\sin\{sx\}\, ds$$

$$+ \frac{\Phi\sigma H\sqrt{\pi}}{8} \; \frac{\sinh[\tfrac{1}{2}\pi x]}{\cos[\tfrac{1}{2}\pi(1-z)] + \cosh[\tfrac{1}{2}\pi x]} + O(\sigma^3) . \qquad (5.4.16)$$

It now remains to evaluate

$$\int_0^\infty \exp\{-\tfrac{1}{4}s^2\sigma^2\}\sin\{sx\}\coth s\, ds ,$$

and $\int_0^\infty \exp\{-\tfrac{1}{4}s^2\sigma^2\}\sin\{sx\}\tanh s\, ds$.

Since $\exp\{-\frac{1}{4}s^2\sigma^2\}$ may be approximated by one when $s < O(\sigma^{-1})$ and $\tanh s$, $\coth s$ may be approximated by one when $s > 10$, we choose n so that

$$10 < \frac{(2n+1)\pi}{2x} < O(\sigma^{-1}) , \qquad (5.4.17)$$

which can always be satisfied except in the neighbourhood of $x = 0$. Then away from $x = 0$, we have that

$$\int_0^\infty \exp\{-\tfrac{1}{4}\sigma^2 s^2\}\sin\{sx\}\coth s\, ds$$

$$= \int_0^{(\frac{2n+1}{2x})\pi} \sin\{sx\} \coth s \, ds + \int_{(\frac{2n+1}{2x})\pi}^{\infty} \exp\{-\tfrac{1}{4}s^2\sigma^2\} \sin\{sx\} ds .$$

$$= \int_0^{(\frac{2n+1}{2x})\pi} \sin\{sx\} \coth s \, ds + O(\sigma^2) . \qquad (5.4.18)$$

Then the integral on the right hand side of equation (5.4.18) can now be evaluated by the method employed by Edwards [12 p.274 ∮1105], namely

$$\int_0^{(\frac{2n+1}{2x})\pi} \sin\{sx\} \coth s \, ds$$

$$= \int_0^{(\frac{2n+1}{2x})\pi} \sum_{r=0}^{\infty} \left[\exp\{-2s(r+1)\} + \exp\{-2rs\}\right] \sin\{sx\} \, ds$$

$$= \sum_{r=0}^{\infty} \int_0^{(\frac{2n+1}{2x})\pi} \left[\exp\{-2s(r+1)\} + \exp\{-2rs\}\right] \sin\{sx\} \, ds$$

$$= \sum_{r=0}^{\infty} \left[\frac{x}{(2r+2)^2 + x^2} + \frac{x}{(2r)^2 + x^2}\right]$$

$$= \frac{\pi}{2} \coth\left[\tfrac{1}{2}\pi x\right] . \qquad (5.4.19)$$

Hence, from (5.4.18) and (5.4.19), we deduce

$$\int_0^\infty \exp\left\{-\tfrac{1}{4} s^2\sigma^2\right\} \sin\{sx\} \coth s \, ds = \frac{\pi}{2} \coth\left[\tfrac{1}{2}\pi x\right] + O(\sigma^2) . \tag{5.4.20}$$

By a similar argument we can also derive

$$\int_0^\infty \exp\left\{-\tfrac{1}{4} s^2\sigma^2\right\} \sin\{sx\} \tanh s \, ds = \frac{\pi}{2} \operatorname{cosech}\left[\tfrac{1}{2}\pi x\right] + O(\sigma^2) . \tag{5.4.21}$$

It should be noticed that the integrals 861.65, 861.66 given by Dwight [11] and the corresponding results in Edwards [12] are only valid in the sense

$$\int_0^\infty = \lim_{n\to\infty} \int_0^{\left(\frac{2n+1}{2x}\right)\pi} .$$

Hence, from (5.4.16), (5.4.20) and (5.4.21), we find that

$$v = \frac{\Phi\sigma H\sqrt{\pi}}{8}\left\{\frac{1}{\sinh\left[\tfrac{1}{2}\pi x\right]}\left(\sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}\right) + \left[\frac{\sinh\left[\tfrac{1}{2}\pi x\right]}{\cos\left[\tfrac{1}{2}\pi(1-z)\right] + \cosh\left[\tfrac{1}{2}\pi x\right]} - \frac{\cosh\left[\tfrac{1}{2}\pi x\right]}{\sinh\left[\tfrac{1}{2}\pi x\right]}\right]\right\} + O(\sigma^3) , \tag{5.4.22}$$

where terms of order R^{-1} are assumed negligible compared with those retained. Also, by applying the above arguments to (5.4.13) and (5.4.14) when $z = \pm 1$, we find that the results (5.4.15) and (5.4.22) are valid for all z and x not in the neighbourhood of $x = 0$.

The first term in (5.4.22) again represents Ekman layers on the disks, $z = \pm 1$, which have a depth of penetration of order $R^{-\frac{1}{2}}$, while the second term gives the interior flow which again satisfies the thermal-wind equation,

$$2 \frac{\partial v_I}{\partial z} = H \frac{\partial \Theta}{\partial x} ,$$

when Θ is given by (5.4.15). The singularities that appear in (5.4.22) when $x = 0$, show that free shear layers are to be anticipated around this point.

When the z-component of vorticity is calculated from (5.2.22), we find that

$$\Upsilon = \frac{\Phi \sigma H \pi^{3/2}}{16} \left\{ - \frac{\cosh\left[\frac{1}{2}\pi x\right]}{\sinh^2\left[\frac{1}{2}\pi x\right]} \left(\sinh\{\lambda_2 z\}\, e^{-\lambda_2} + \sinh\{\lambda_3 z\}\, e^{-\lambda_3}\right) \right.$$

$$\left. + \frac{\cos\left[\frac{1}{2}\pi(1-z)\right] \cosh\left[\frac{1}{2}\pi x\right] + 1}{\left(\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi x\right]\right)^2} + \frac{1}{\sinh^2\left[\frac{1}{2}\pi x\right]} \right\} + O(\sigma^3). \quad (5.4.23)$$

The presence of the singularity at $x = 0$ implies that free shear layers exist in this region and therefore we should anticipate a connection between this case and the problem in section 5.3 with a tending to zero.

When we allow a to tend to zero in (5.3.9) and retain only the highest order terms, we have that

$$\varphi = \frac{H\,\Phi\,\pi a}{8}\left\{-\frac{\cosh\left[\frac{1}{2}\pi x\right]}{\sinh^2\left[\frac{1}{2}\pi x\right]}\left(\sinh\{\lambda_2 z\}\,e^{-\lambda_2} + \sinh\{\lambda_3 z\}\,e^{-\lambda_3}\right)\right.$$

$$\left. + \frac{\cos\left[\frac{1}{2}\pi(1-z)\right]\cosh\left[\frac{1}{2}\pi x\right] + 1}{\left(\cos\left[\frac{1}{2}\pi(1-z)\right] + \cosh\left[\frac{1}{2}\pi x\right]\right)^2} + \frac{1}{\sinh^2\left[\frac{1}{2}\pi x\right]}\right\}\,, \qquad (5.4.24)$$

which agrees with the result (5.4.23), provided

$$\frac{\sigma\sqrt{\pi}}{2} = a\,. \qquad (5.4.25)$$

Hence we see that, for x away from $x = 0$, the case when a normal temperature distribution with small σ is applied to the lower disk, $z = -1$, is equivalent, to the highest order, to the case when a non-zero temperature is imposed only on the finite strip $|x| < \frac{1}{2}\sigma\sqrt{\pi}$ on the disk $z = -1$.

CHAPTER 6

THE EFFECT OF AN INITIAL TEMPERATURE GRADIENT ON THE DEVELOPMENT OF VORTICITY

6.1 INTRODUCTION

In the problem considered in Chapter 4, the effect of rotation and steady heating on the development of vorticity normal to the confining boundaries was found, when the initial temperature throughout the fluid was assumed constant. This first approximation to the actual temperature field present in the atmosphere can be improved by assuming, for the initial flow, a constant adverse temperature gradient,

$$T = T_o - \beta z , \qquad (6.1.1)$$

where T_o, β are constants and β is positive.

In this chapter we will examine the effect on the vertical vorticity when, initially, either an adverse ($\beta > 0$) or a favourable ($\beta < 0$) temperature gradient is present in the fluid. When there is an initial favourable temperature gradient, a valid expression for the steady vertical vorticity can always be found and will represent the final steady state which exists in the fluid after all the transient effects have decayed. On the other hand, when an adverse temperature gradient exists, initially, the solution obtained for the steady vertical vorticity will only represent the final flow for values of the Rayleigh number,

$$\mathcal{R} = \frac{\alpha g \beta d^4}{k \nu}, \qquad (6.1.2)$$

much less than a critical value $\mathcal{R}_c$ which depends on the horizontal wave-number, a, defined by the membrane equation, (4.2.9). Any discussion of the behaviour of the fluid in the neighbourhood of this critical Rayleigh number, $\mathcal{R}_c$, would require the inclusion of the non-linear terms in the analysis and will not be considered.

As the Reynolds number,

$$R = \frac{\Omega d^2}{\nu}, \qquad (6.1.3)$$

tends to infinity, the asymptotic behaviour of the minimum critical Rayleigh number is found to be

$$3 \left\{ \tfrac{1}{2}\pi R^2 \right\}^{2/3} .$$

This result agrees exactly with the expression for the asymptotic behaviour of the minimum critical Rayleigh number for free/free boundaries calculated by Chandrasekhar [9]. We therefore deduce, contrary to the assertions made by Chandrasekhar, that the type of boundary does not influence the asymptotic behaviour of the minimum critical Rayleigh number. A comparison with the numerical results of Chandrasekhar is given, from which we deduce that the range of Reynolds numbers considered by Chandrasekhar does not go to high enough values for the asymptotic formula to be applicable.

6.2 EQUATIONS OF MOTION

We will consider the effect on the z-component of vorticity, when we replace the assumption that, initially, the temperature field is a constant, T_o, by the initial condition (6.1.1) where $\beta > 0$, $\beta < 0$ correspond to adverse and favourable temperature gradients respectively. Then, after the application of the Boussinesq approximation, the continuity equation and the linearized Navier-Stokes equations are again (4.2.5), (4.2.6) while the linearized energy equation becomes

$$\frac{\partial \Theta}{\partial t} - \beta w = K_o \nabla^2 \Theta . \qquad (6.2.1)$$

When we introduce the dimensionless variables (starred), (4.2.13), with the typical temperature scale ΔT replaced by βd, substitute (4.2.18), eliminate the pressure and use the continuity equation, we find that (upon dropping the asterisks) ,

$$\frac{\partial \phi}{\partial t} - w_1 = (\sigma R)^{-1}(D^2 - a^2)\phi , \qquad (6.2.2)$$

$$\left[\frac{\partial}{\partial t} - R^{-1}(D^2 - a^2)\right] \Upsilon_1 - 2Dw_1 = 0 , \qquad (6.2.3)$$

$$\left[\frac{\partial}{\partial t} - R^{-1}(D^2 - a^2)\right](D^2 - a^2)w_1 + 2D\Upsilon_1 = -H^* a^2 \phi , \qquad (6.2.4)$$

where $H^* = \dfrac{g\alpha\beta}{\Omega^2}$. (6.2.5)

The equations (6.2.2), (6.2.3), (6.2.4) have the same structure as the equations (4.2.19), (4.2.20), (4.2.21) except for the convective term which appears in (6.2.2). This additional term couples the equation (6.2.2) with the equations (6.2.3) and (6.2.4) to yield an eighth order partial differential equation for either ϕ or w_1 or Υ_1. Hence we see that the introduction of an initial temperature gradient yields a more complicated problem and that the solution for the temperature field is no longer trivial.

From the equations (6.2.2), (6.2.3), (6.2.4) we find that

$$\left[(D^2-a^2) - R\frac{\partial}{\partial t}\right]^2 \left[(D^2-a^2) - \sigma R\frac{\partial}{\partial t}\right](D^2-a^2)\Upsilon_1$$

$$+ 4R^2\left[(D^2-a^2) - R\sigma\frac{\partial}{\partial t}\right]D^2\Upsilon_1 = -a^2\mathcal{R}\left[(D^2-a^2) - R\frac{\partial}{\partial t}\right]\Upsilon_1 , \qquad (6.2.6)$$

where $\mathcal{R}$ is the Rayleigh number, (6.1.2).

Now, from (6.2.5) and (6.1.2), we have that

$$\mathcal{R} = H^* R^2 \sigma .$$

Also, from (6.2.5), we have $H^* \ll 1$, which is a necessary condition for the linearization of the Navier-Stokes and energy equations to be valid. Then, if we assume that $\sigma = O(1)$, we find that

$$\mathcal{R} \ll R^2 , \qquad (6.2.7)$$

The solution of the equation (6.2.6) must now be solved subject to the boundary conditions (4.2.22). No discussion of the general time dependent solution of (6.2.6) will be given but instead we will consider the steady problem.

6.3 THE STEADY SOLUTION

The time independent solution of the equation (6.2.6) is equivalent to the solution of the equation

$$(D^2-a^2)\left[(D^2-a^2)^3 + 4R^2D^2 + a^2\mathcal{R}\right]\Upsilon_1 = 0 . \qquad (6.3.1)$$

If we seek a solution of (6.3.1) of the form $e^{\mu z}$, then we find that

$$(\mu^2-a^2)\left[(\mu^2-a^2)^3 + 4R^2\mu^2 + a^2\mathcal{R}\right] = 0 , \qquad (6.3.2)$$

which, provided $a \ll R^{\frac{1}{2}}$, has the solutions

$$\mu = \pm \mu_i , \qquad i = 1,2,3,4,$$

where

$$\left.\begin{aligned} \mu_1 &= a , \\ \mu_2 &= (2iR)^{\frac{1}{2}} + O(R^{-\frac{1}{2}}) , \\ \mu_3 &= (-2iR)^{\frac{1}{2}} + O(R^{-\frac{1}{2}}) , \\ \mu_4 &= \frac{(a^6 - a^2 \mathcal{R}_o)^{\frac{1}{2}}}{2R} + O(R^{-3}) \quad . \end{aligned}\right\} \qquad (6.3.3)$$

The signs of the square roots in (6.3.3) are chosen such that the μ's have positive real parts and μ_4 remains on the same branch when $a^2 \mathcal{R}_o > a^6$.

Then we find that the terms in the solution of (6.3.1) which satisfy the boundary conditions, (4.2.22), to the highest order are

$$\mathcal{J}_1 = - Ha^2 \Phi \left\{ \frac{\sinh \{az\}}{\sinh a} \left[- \sinh \mu_4 + (\mu_4 - \frac{a^2}{\mu_4}) \frac{\cosh \mu_4}{R^{\frac{1}{2}}} \right] \right.$$

$$\left. - (\mu_4 - \frac{a^2}{\mu_4}) \frac{\cosh \mu_4}{R^{\frac{1}{2}}} \left[\sinh \{\mu_2 z\} e^{-\mu_2} + \sinh \{\mu_3 z\} e^{-\mu_3} \right] + \sinh \{\mu_4 z\} \right\}$$

$$\div \left\{ 4a \coth a \left[-\sinh \mu_4 + (\mu_4 - \frac{a^2}{\mu_4}) \frac{\cosh \mu_4}{R^{\frac{1}{2}}} \right] + \right.$$

$$\left. + \cosh \mu_4 \{ R^{-2} \mu_4^{-1} (\mu_4^2 - a^2)^3 + 4 \mu_4 \} \right\}$$

$$+ Ha^2 \Phi \left\{ \frac{\cosh\{az\}}{\cosh a} \left[-\cosh\mu_4 + \left(\mu_4 - \frac{a^2}{\mu_4}\right) \frac{\sinh\mu_4}{R^{\frac{1}{2}}} \right] \right.$$

$$- \left(\mu_4 - \frac{a^2}{\mu_4}\right) \frac{\sinh\mu_4}{R^{\frac{1}{2}}} \left[\cosh\{\mu_2 z\}\, e^{-\mu_2} + \cosh\{\mu_3 z\}\, e^{-\mu_3} \right]$$

$$\left. + \cosh\{\mu_4 z\} \right\}$$

$$\div \left\{ 4a \tanh a \left[-\cosh\mu_4 + \left(\mu_4 - \frac{a^2}{\mu_4}\right) \frac{\sinh\mu_4}{R^{\frac{1}{2}}} \right] \right.$$

$$\left. + \sinh\mu_4 \left\{ R^{-2} \mu_4^{-1} (\mu_4^2 - a^2)^3 + 4\mu_4 \right\} \right\}. \qquad (6.3.4)$$

This solution (6.3.4) is meaningful provided

$$\left[-\sinh\mu_4 + \left(\mu_4 - \frac{a^2}{\mu_4}\right) \frac{\cosh\mu_4}{R^{\frac{1}{2}}} \right] 4a \coth a$$

$$+ \cosh\mu_4 \left\{ R^{-2} \mu_4^{-1} (\mu_4^2 - a^2)^3 + 4\mu_4 \right\}, \qquad (6.3.5)$$

$$\left[-\cosh\mu_4 + \left(\mu_4 - \frac{a^2}{\mu_4}\right) \frac{\sinh\mu_4}{R^{\frac{1}{2}}} \right] 4a \tanh a$$

$$+ \sinh\mu_4 \left\{ R^{-2} \mu_4^{-1} (\mu_4^2 - a^2)^3 + 4\mu_4 \right\}, \qquad (6.3.6)$$

are non-zero. The smallest value of $\mathcal{R}_0$, for a given a, which renders (6.3.5) or (6.3.6) zero defines a critical region where the transition from the stable (subcritical) region, with values of $\mathcal{R}_0$ less than this predicted value of $\mathcal{R}_c$ for the given a, to the unstable (supercritical) region occurs.

For the subcritical region, the solution (6.3.4) gives a valid representation of the flow at large times, provided the amplitude remains small enough to justify the linearization, since, in this region, any small perturbations will decay with time.

For the supercritical region, however, unstable modes must be added to the steady solution (6.3.4) in order to give a complete description of the flow at large times.

For the critical region, the linear solution (6.3.4) becomes infinite and the non-linear terms must be included in the analysis.

Hence the steady solution (6.3.4) gives a valid description of the final state, which occurs at large times, only in the stable region and provided the amplitude remains small. Therefore the region of validity of the solution (6.3.4) is restricted to the region which is well sub-critical.

The position of the zeros of (6.3.5) and (6.3.6) and hence the critical Rayleigh number depends upon the value chosen for a and we will now consider the special cases, $a = O(1)$, $a = O(R^{1/4})$, $a = O(R^{1/3})$.

<u>$a = O(1)$</u>

When $a = O(1)$, we see, from (6.3.3), that $\mu_4 \ll 1$ and we find that the dominant terms in (6.3.5) and (6.3.6) are

$$\left(-\mu_4 - \frac{a^2}{\mu_4 R^{\frac{1}{2}}}\right)4a \coth a + 4\mu_4 \,, \qquad (6.3.7)$$

$$- 4a \tanh a \quad , \qquad (6.3.8)$$

respectively. Then, provided $a \neq 0$, we see immediately that (6.3.8) is never zero and (6.3.7) is zero only if

$$\mu_4^2 = -\frac{a^3 \coth a}{R^{\frac{1}{2}}(a \coth a - 1)} \quad ,$$

which corresponds to a critical Rayleigh number

$$\mathcal{R}_c = \frac{4a \coth a \, R^{3/2}}{(a \coth a - 1)} \quad , \qquad (6.3.9)$$

when only the highest order term is retained.

Therefore, when a is of order one and non-zero, the critical Rayleigh number is given by (6.3.9) and is always positive. This implies that, when a favourable temperature gradient ($\beta < 0$) is present, (6.3.4) gives a valid representation of the flow at large times.

$\underline{a = O(R^{1/4})}$

When we assume $a = R^{1/4} a_1$, where a_1 is of order one, we find that $\mu_4 \ll R^{1/4}$ and the dominant terms in (6.3.5) and (6.3.6) are

$$4a_1 R^{1/4}\left\{-\sinh\mu_4 - \frac{a_1^2}{\mu_4}\cosh\mu_4\right\} + 4\mu_4\cosh\mu_4\,, \qquad (6.3.10)$$

$$4a_1 R^{1/4}\left\{-\cosh\mu_4 - \frac{a_1^2}{\mu_4}\sinh\mu_4\right\} + 4\mu_4\sinh\mu_4\,. \qquad (6.3.11)$$

When $\mathcal{R}_o \leq O(R)$ we have, from (6.3.3), that $\mu_4 = O(R^{-1/4})$ which implies that the highest order terms in (6.3.10) and (6.3.11) are non-zero. When $\mathcal{R}_o > O(R)$ and $\beta < 0$ then, from (6.3.3), we see that the highest order term in the expansion for μ_4 is real and $\mu_4 \ll R^{1/4}$ which implies that (6.3.10) and (6.3.11) are always non-zero. However, when $\mathcal{R}_o > O(R)$ and $\beta > 0$, we find, from (6.3.3), that the highest order term in the expansion for μ_4 is purely imaginary. Hence, when we retain only the highest order term in this expansion for μ_4, we can write

$$\mu_4 = i\,\nu_4\,,$$

where ν_4 is real, $\nu_4 = O(R^{-3/4}\mathcal{R}_o^{\frac{1}{2}})$ and $\nu_4 \ll R^{1/4}$. Then the expressions (6.3.10) and (6.3.11) reduce to

$$\left.\begin{aligned} &4a_1 R^{1/4}\left\{-\sin\nu_4 + \frac{a_1^2}{\nu_4}\cos\nu_4\right\} + 4\nu_4\cos\nu_4\,, \\ &4a_1 R^{1/4}\left\{-\cos\nu_4 - \frac{a_1^2}{\nu_4}\sin\nu_4\right\} + 4\nu_4\sin\nu_4\,. \end{aligned}\right\} \qquad (6.3.12)$$

For the highest order terms in (6.3.12) to vanish we require $\nu_4 = O(1)$ and hence the critical Rayleigh number

$$\mathcal{R}_c = O(R^{3/2}) .$$

Therefore, when $a = O(R^{1/4})$, the critical Rayleigh number is of order $R^{3/2}$ and positive. Hence, when $\beta < 0$, (6.3.4) always describes the flow which will exist in the fluid, at large times.

$\underline{a = O(R^{1/3})}$

When we assume that $a = R^{1/3} a_2$, where a_2 is of order one, we find that the dominant terms in (6.3.5) and (6.3.6) are

$$4a_2 R^{1/3} \left\{ - \sinh \mu_4 - \frac{a_2^2 R^{1/6}}{\mu_4} \cosh \mu_4 \right\} + 4 \mu_4 \cosh \mu_4 , \qquad (6.3.13)$$

$$4a_2 R^{1/3} \left\{ - \cosh \mu_4 - \frac{a_2^2 R^{1/6}}{\mu_4} \sinh \mu_4 \right\} + 4 \mu_4 \sinh \mu_4 . \qquad (6.3.14)$$

When $\mathcal{R} \le O(R^{4/3})$ we find, from (6.3.3), that $\mu_4 = O(1)$ which implies that the dominant terms in (6.3.13) and (6.3.14) are

$$- \frac{4a_2^3 R^{\frac{1}{2}}}{\mu_4} \cosh \mu_4 , \qquad (6.3.15)$$

$$- \frac{4a_2^3 R^{\frac{1}{2}}}{\mu_4} \sinh \mu_4 , \qquad (6.3.16)$$

respectively.

These expressions, (6.3.15) and (6.3.16), are zero when

$$\mu_4^2 = -\frac{(2m-1)^2\pi^2}{4}\ , \quad \mu_4^2 = -m^2\pi^2\ ,$$

$$m = 1,2,3,\ldots\ ,$$

respectively. Hence the first zero occurs when

$$\mu_4^2 = -\frac{\pi^2}{4} = \frac{a^6 - \mathcal{R}a^2}{4R^2}\ ,$$

which implies that

$$\mathcal{R}_c = \left(\frac{\pi^2 R^2 + a^6}{a^2}\right)\ , \text{ when } a = O(R^{1/3}). \qquad (6.3.17)$$

The equation (6.3.17) can never be satisfied when a favourable temperature gradient ($\beta < O$) is applied and defines the critical Rayleigh number when an adverse temperature gradient ($\beta > O$) is present. Since we define $\mathcal{R}_c$ to be the smallest value of $\mathcal{R}$, for any given a, which renders (6.3.5) or (6.3.6) zero, we see that any roots of (6.3.13) and (6.3.14) that exist for $\mathcal{R} > O(R^{4/3})$ will not influence the critical Rayleigh number, $\mathcal{R}_c$, given by (6.3.17), when $\beta > O$.

When $\beta < O$ and $\mathcal{R} > O(R^{4/3})$, we see, from (6.3.3), that μ_4 is always real and $1 \ll \mu_4 \ll R^{1/3}$ which implies that (6.3.13) and

(6.3.14) are never zero.

Therefore, when $a = O(R^{1/3})$, the critical Rayleigh number is given by (6.3.17) which is always positive. When $\beta < 0$, the solution, (6.3.4), is always non-singular and hence describes the motion in the fluid, at large times.

The minimum value of (6.3.17), as a function of a, occurs when

$$\left.\begin{aligned} (a_c)_{min} &= (\tfrac{1}{2}\pi^2 R^2)^{1/6} \quad , \\ (\mathcal{R}_c)_{min} &= 3\left\{\tfrac{1}{2}\pi^2 R^2\right\}^{2/3} \quad . \end{aligned}\right\} \qquad (6.3.18)$$

Hence the formula (6.3.18) gives the asymptotic behaviour of the minimum critical Rayleigh number and the minimum critical wavenumber for very large Reynolds numbers, R, in the range $a = O(R^{1/3})$.

In order to obtain a comparison between the above results and the results given by Chandrasekhar [9], we must first notice that the length scale, d, used in the above analysis is half the distance between the disks while the length scale, d, employed by Chandrasekhar is the total distance between the disks. Therefore, we find that the Taylor number, T, used by Chandrasekhar, [9, p.90] is equal to $64R^2$, the minimum critical wavenumber, a_c, used by Chandrasekhar corresponds to $2(a_c)_{min}$ in the above analysis and the minimum critical Rayleigh number, $\mathcal{R}_c$, used by

Chandrasekhar corresponds to $16(\mathcal{R}_c)_{min}$ in the above analysis.

Then, we find that the asymptotic forms (6.3.18) may be written

$$\left.\begin{aligned} 2(a_c)_{min} &= (\tfrac{1}{2}\pi^2 T)^{1/6} , \\ 16(\mathcal{R}_c)_{min} &= 3(\tfrac{1}{2}\pi^2 T)^{2/3} , \end{aligned}\right\} \qquad (6.3.19)$$

for large T, which are identical to the asymptotic laws found by Chandrasekhar [9] page 95 formula 133, for the case when both bounding surfaces were free.

Chandrasekhar, on page 104, states, for the case when both the bounding surfaces are rigid, that "it appears that the same power laws hold" as for the case when the bounding surfaces were free, that is

$$2(a_c)_{min} \longrightarrow \text{constant } T^{1/6} ,$$

$$16(\mathcal{R}_c)_{min} \longrightarrow \text{constant } T^{2/3} ,$$

for large T, "though the constants of proportionality seem to depend slightly, but definitely, on the boundary conditions". Whereas, from the above analysis, we have shown that not only the power laws but also the constants of proportionality are identical for the cases of free-free and rigid-rigid bounding surfaces. Therefore we may deduce that the minimum critical Rayleigh number for the case of rigid boundaries can be found from inviscid considerations alone.

From (6.3.4), when $a = O(R^{1/3})$, we see that the steady solution, Υ_1, consists of two boundary layers which have depths of penetration of order $R^{-\frac{1}{2}}$ and a^{-1} and an interior solution. Hence, since we have shown that the minimum critical Rayleigh number can be found solely from inviscid considerations, which corresponds to the interior solution, we can deduce that the boundary layers play only a passive role in the determination of the critical Rayleigh number.

This fact can also be seen from the equation (6.3.4) because, when we assume $a = O(R^{1/3})$ and retain only the highest order terms in the denominators namely (6.3.15) and (6.3.16), we find that

$$\Upsilon_1 = -\frac{H\Phi}{4a}(a^2-\mu_4^2)\left\{\frac{\sinh\{az\}}{\sinh a} - (\sinh\{\mu_2 z\}e^{-\mu_2} + \sinh\{\mu_3 z\}e^{-\mu_3}) - \frac{\cosh\{az\}}{\cosh a} + (\cosh\{\mu_2 z\}e^{-\mu_2} + \cosh\{\mu_3 z\}e^{-\mu_3})\right\}$$
$$+\frac{H\Phi\mu_4 R^{\frac{1}{2}}}{4a\cosh\mu_4}\left\{\frac{\sinh\{az\}}{\sinh a}(-\sinh\mu_4) + \sinh\{\mu_4 z\}\right\}$$
$$-\frac{H\Phi\mu_4 R^{\frac{1}{2}}}{4a\sinh\mu_4}\left\{\frac{\cosh\{az\}}{\cosh a}(-\cosh\mu_4) + \cosh\{\mu_4 z\}\right\}. \qquad (6.3.20)$$

Then from (6.3.20), we see that, in the region which is well subcritical, the first expression gives the highest order contribution and consists entirely of boundary layer terms which are always regular. While, as the critical region is approached, the remaining terms become increasingly more important and, in fact, determine the critical Rayleigh number. However, the structure of the second and third terms in (6.3.20) are such that the critical Rayleigh number may be determined from the interior solution alone, which agrees with the above deductions.

From the above results, we deduce that the critical Rayleigh number, (6.3.17), could have been calculated from

$$4R^2D^2w_1 = -(\mathcal{R}a^2 - a^6)w_1 , \qquad (6.3.21)$$

instead of the steady equation obtained by eliminating, Υ_1, ϕ from (6.2.2), (6.2.3) and (6.2.4), by seeking a solution of the form

$$w_1 = A_m \cos\left[\frac{(2m-1)\pi z}{2}\right] + B_m \sin(m\pi z) ,$$

where A_m, B_m are constants and m is an integer. Hence, when we compare the above result with the result given by Chandrasekhar [9] p.104, we deduce that (6.3.21) determines the critical Rayleigh number for both free/free and rigid/rigid boundaries.

From the tables 6.1 and 6.2, we see that the results calculated from the asymptotic formula (6.3.19) are always high compared with the results given by Chandrasekhar [9] in Table VIII page 102. This discrepancy may be accounted for by the fact that, in the present analysis, we have

T	$2(a_c)_{min}$	$16(\mathcal{R}_c)_{min}$
10^5	8.89	1.873×10^4
10^6	13.05	8.696×10^4
10^8	28.11	1.873×10^6
10^{10}	60.56	4.037×10^7

TABLE 6.1: The values of the minimum critical wavenumber and Rayleigh number calculated from the asymptotic forms (6.3.19).

T	a_c	R_c
10^5	7.20	1.672×10^4
10^6	10.80	7.113×10^4
10^8	24.5	1.531×10^6
10^{10}	55.5	3.457×10^7

TABLE 6.2: The values of the minimum critical wavenumber and Rayleigh number given by Chandrasekhar page 102 Table VIII.

T	$2(a_c)_{min}$	$16(\mathcal{R}_c)_{min}$
10^5	6.061	1.078×10^4
10^6	9.622	5.650×10^4
10^8	23.08	1.426×10^6
10^{10}	53.176	3.381×10^7

TABLE 6.3: The values of the minimum critical wavenumber and Rayleigh number calculated from the first two terms in the series (6.3.22).

assumed terms of order $R^{-1/6}$ are negligible compared with the terms retained, while for the values of T used in table 6.1, $R^{1/6}$ is of order one. Hence the asymptotic forms (6.3.19) are only valid for values of T much larger than 10^{10}.

When the first correction term is calculated by retaining

$$4a_2R^{1/3}\left(-\sinh\mu_4 - \frac{a_2^2R^{1/6}}{\mu_4}\cosh\mu_4\right) ,$$

$$4a_2R^{1/3}\left(-\cosh\mu_4 - \frac{a_2^2R^{1/6}}{\mu_4}\sinh\mu_4\right) ,$$

for the dominant terms in (6.3.13) and (6.3.14), we find that

$$\left.\begin{aligned} (a_c)_{min} &= (\tfrac{1}{2}\pi^2R^2)^{1/6} - R^{\frac{1}{2}}(\tfrac{1}{2}\pi^2R^2)^{-1/6} + O(1) , \\ (\mathcal{R}_c)_{min} &= 3(\tfrac{1}{2}\pi^2R^2)^{2/3} - 4R^{\frac{1}{2}}(\tfrac{1}{2}\pi^2R^2)^{1/3} + O(R) . \end{aligned}\right\} \qquad (6.3.22)$$

When we compare the results calculated from (6.3.22), which are given in Table 6.3, with the results given by Chandrasekhar, which are shown in Table 6.2, we find a better agreement for the cases $T = 10^8$ and $T = 10^{10}$, although this time the calculated results are always low.

Hence it appears that, for the values of T used by Chandrasekhar, the asymptotic formula (6.3.19) is not valid. Instead $(a_c)_{min}$ and $(\mathcal{R}_c)_{min}$ must be expressed in the series (6.3.22) and provided sufficient terms were calculated, we would expect agreement with the results given by Chandrasekhar.

CHAPTER 7

THE INFLUENCE OF NON-UNIFORM CONDITIONS ON A PLANE BOUNDARY

7.1 INTRODUCTION

In chapters 2 and 3, we considered the flow generated in a semi-infinite expanse of incompressible fluid bounded by an infinite disk when both the fluid and the disk were in steady rigid rotation and, additionally, from some instant of time, non-torsional oscillations, (2.2.1), or torsional oscillations were imposed on the disk. It was found that no oscillatory solutions of the linearized equations which satisfied all the required boundary conditions existed when the frequency of the imposed oscillations was twice the angular velocity of the basic rotation. If, however, a second disk was introduced parallel to and at a finite distance away from the first disk, then an oscillatory solution could always be found. The presence of this second disk introduced a length scale, namely the distance between the disks, into the problem.

In chapter 4, section 4.13, it was found that an oscillatory solution which satisfied the required boundary conditions always existed when the imposed oscillations of the disk were replaced by an oscillatory heating of the form (4.2.8). The introduction of a length scale in the plane of the disk through the membrane equation, provided, at the resonant frequency, a second boundary layer thickness and therefore an oscillatory solution which, however, differed from the oscillatory solution obtained for non-resonant frequencies.

Roberts and Stewartson [29] and Busse [7] considered problems in which a length scale can be defined and they found that an oscillatory solution for the boundary layers always existed. This boundary layer had a depth of penetration of order $R^{-2/5}$ at the critical latitudes and $R^{-\frac{1}{2}}$ elsewhere, and therefore a resonance effect was present in the sense that different oscillatory solutions existed for critical and non-critical latitudes.

In this chapter, by considering two specific examples, we will determine whether or not an oscillatory solution always exists when a length scale in the plane of the disk is introduced into the problem through the imposed oscillations on the boundary.

7.2 EQUATIONS OF MOTION

We consider an infinite disk, $z = 0$, bounding a semi-infinite expanse, $z > 0$, of incompressible fluid when both the fluid and the disk are in solid-body rotation with constant angular velocity, Ω .

The cartesian axes (x,y,z) are taken so that the z-axis is perpendicular to the disk and parallel to the common axis of rotation of the fluid/disk combination and the x,y-axes lie in the plane of the disk and rotate with it. The velocity vector in this rotating co-ordinate system is $\underline{u}=(u,v,w)$.

We assume that the imposed oscillations on the disk are in the y-direction and depend only on x and t, which implies that

$$u = w = 0, \qquad v = F(x)e^{int} \quad \text{at} \quad z = 0, \tag{7.2.1}$$

where n is the frequency and F(x) is some function of x which will be prescribed later.

We will suppose that the velocity of the fluid is always small so that it is valid to linearize the equations of motion. Also since the imposed oscillations are independent of y, we will assume that the motion of the fluid is independent of y. Then the equations of motion (2.2.2) and the continuity equation (2.2.3) become

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\Omega v &= -\frac{\partial}{\partial x}\left(\frac{p}{\rho} - \tfrac{1}{2}\Omega^2(x^2+y^2)\right) + \nu\nabla^2 u, \\ \frac{\partial v}{\partial t} + 2\Omega u &= \nu\nabla^2 v, \\ \frac{\partial w}{\partial t} &= -\frac{\partial}{\partial z}\left(\frac{p}{\rho} - \tfrac{1}{2}\Omega^2(x^2+y^2)\right) + \nu\nabla^2 w, \end{aligned}\right\} \tag{7.2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 , \tag{7.2.3}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$.

From (7.2.3), we define a streamfunction, ψ , by

$$u = \frac{\partial \psi}{\partial z} , \qquad w = - \frac{\partial \psi}{\partial x} , \tag{7.2.4}$$

and then the y-component of vorticity is

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \nabla^2 \psi . \tag{7.2.5}$$

When the pressure is eliminated from the equations, (7.2.2), and the streamfunction, (7.2.4), is introduced, we find that

$$\left.\begin{aligned} &(\frac{\partial}{\partial t} - \nu \nabla^2)\nabla^2 \psi - 2\Omega \frac{\partial v}{\partial z} = 0 , \\ &(\frac{\partial}{\partial t} - \nu \nabla^2) v + 2\Omega \frac{\partial \psi}{\partial z} = 0 , \end{aligned}\right\} \tag{7.2.6}$$

which we must solve subject to the conditions

(a) $\psi = \frac{\partial \psi}{\partial z} = 0 , \quad v = F(x)e^{int} \quad$ on $z = 0$,

(b) $\psi , \frac{\partial \psi}{\partial z} , v \longrightarrow 0 \quad$ as $z \longrightarrow \infty$, (7.2.7)

(c) an assumption of periodicity in t.

We will now solve the problems that arise when we assume that F(x) is given by

(i) a sinusoidal function,

or (ii) $F(x) = \text{constant}$ for $|x| < a$

and $F(x) = 0$ for $|x| > a$.

7.3 SINUSOIDAL X-DEPENDENCE

We suppose that the imposed oscillations, (7.2.1), take the form

$$u = w = 0\,, \qquad v = \varepsilon\, \alpha^{-1} \Omega\, e^{int} \sin(\alpha x + \phi) \quad \text{at } z = 0\,, \tag{7.3.1}$$

where n is the frequency, α the wavenumber, ϕ the phase angle and ε a constant. In order to ensure the validity of the linearization, we require that $\varepsilon \ll 1$.

When we introduce the dimensionless quantities (starred),

$$x = \alpha^{-1} x^*, \quad z = \alpha^{-1} z^*, \quad v = \varepsilon\, \alpha^{-1} \Omega\, v^*, \quad \psi = \varepsilon \Omega \alpha^{-2} \psi^*\,,$$
$$t = \Omega^{-1} t^*\,, \tag{7.3.2}$$

into the equations, (7.2.6), we have (upon dropping the asterisks)

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - R^{-1} \nabla^2\right) \nabla^2 \psi - 2\,\frac{\partial v}{\partial z} &= 0\,, \\ \left(\frac{\partial}{\partial t} - R^{-1} \nabla^2\right) v + 2\,\frac{\partial \psi}{\partial z} &= 0\,, \end{aligned}\right\} \tag{7.3.3}$$

where $R = (\Omega/\alpha^2\nu)$ is the Reynolds number, (1.1.2), with a length scale equal to α^{-1}. The equations, (7.3.3), must now be solved subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(a)} & \psi = \frac{\partial \psi}{\partial z} = 0, \quad v = e^{i\sigma t}\sin(x+\phi) \quad \text{on } z = 0, \\ \text{(b)} & \psi, \frac{\partial \psi}{\partial z}, v \rightarrow 0 \quad \text{as } z \rightarrow \infty, \\ \text{(c)} & \text{an assumption of periodicity in } t, \end{array}\right\} \quad (7.3.4)$$

where $\sigma = (n/\Omega)$.

In the following analysis we will assume that the Reynolds number, R, is large. When we substitute a solution of the form

$$v = v_1(z)e^{i\sigma t}\sin(x+\phi),$$
$$\psi = \psi_1(z)e^{i\sigma t}\cos(x+\phi),$$

into (7.3.3) and eliminate $\psi_1(z)$, we find

$$(D^2-1)\left[i\sigma - R^{-1}(D^2-1)\right]^2 v_1 + 4D^2 v_1 = 0, \qquad (7.3.5)$$

where $D \equiv \frac{\partial}{\partial z}$, which must be solved subject to the conditions

$$\left.\begin{array}{ll} \text{(a)} & \psi_1 = \frac{\partial \psi_1}{\partial z} = 0, \quad v_1 = 1 \quad \text{on } z = 0, \\ \text{(b)} & \psi_1, \frac{\partial \psi_1}{\partial z}, \quad v_1 \rightarrow 0 \quad \text{as } z \rightarrow \infty. \end{array}\right\} \quad (7.3.6)$$

When we seek a solution of (7.3.5) of the form $\exp\{-\lambda z\}$, we find that

$$(\lambda^2-1)\left[i\sigma - R^{-1}(\lambda^2-1)\right]^2 + 4\lambda^2 = 0 . \qquad (7.3.7)$$

The equation (7.3.7) is identical to the equation (4.3.13) in chapter 4 with $a = 1$ and $p = i\sigma$. The roots of (7.3.7) have been discussed in section 4.4 and a summary of the highest order terms in the power series for the roots, λ, which have positive real part, are shown in Table 7.1.

For the outer range, the solution is

$$v = \tfrac{1}{2}e^{i\sigma t}\sin(x+\phi)\left\{\exp\left[-R^{\frac{1}{2}}(i\sigma+2i)^{\frac{1}{2}}z\right] + \exp\left[-R^{\frac{1}{2}}(i\sigma-2i)^{\frac{1}{2}}z\right]\right\} + O(R^{-\frac{1}{2}}) , \qquad (7.3.8)$$

because the terms involving λ_1 produce a contribution of order $R^{-\frac{1}{2}}$. Hence, for frequencies of oscillation away from the resonant frequency, the solution, (7.3.8), represents two boundary layers, confined to the disk, having depths of penetration of vorticity of order

$$\left(\frac{\nu}{|n \pm 2\Omega|}\right)^{\frac{1}{2}} .$$

These are, in fact, modified Stokes layers with an x-dependent amplitude.

RANGE OF FREQUENCY	λ_1	λ_2	λ_3
OUTER $\lvert \sigma \pm 2 \rvert > O(R^{-\frac{1}{2}})$	$\sigma-(\sigma^2-4)^{-\frac{1}{2}}$ $+ 16R^{-1}(4-\sigma^2)^{-5/2}$	$R^{\frac{1}{2}}(i\sigma+2i)^{\frac{1}{2}}$	$R^{\frac{1}{2}}(i\sigma-2i)^{\frac{1}{2}}$
INNER $\lvert\sigma-2\rvert < O(R^{-\frac{1}{2}})$ $\lvert\sigma+2\rvert < O(R^{-\frac{1}{2}})$	$R^{1/4}\exp\{3\pi i/8\}$ $R^{1/4}\exp\{-3\pi i/8\}$	$(4iR)^{\frac{1}{2}}$ $R^{1/4}\exp\{\pi i/8\}$	$R^{1/4}\exp\{-\pi i/8\}$ $(-4iR)^{\frac{1}{2}}$
TRANSITION $\sigma=2+\frac{\mu}{R^{\frac{1}{2}}}$	$R^{1/4}\left[\frac{i\mu}{2} - i(\frac{\mu^2}{4}+i)^{\frac{1}{2}}\right]^{\frac{1}{2}}$	$(4iR)^{\frac{1}{2}}$	$R^{1/4}\left[\frac{i\mu}{2} + i(\frac{\mu^2}{4}+i)^{\frac{1}{2}}\right]^{\frac{1}{2}}$
$\sigma=-2+\frac{\mu}{R^{\frac{1}{2}}}$ μ real $\mu=O(1)$	$R^{1/4}\left[\frac{i\mu}{2} - i(\frac{\mu^2}{4}-i)^{\frac{1}{2}}\right]^{\frac{1}{2}}$	$R^{1/4}\left[\frac{i\mu}{2} + i(\frac{\mu^2}{4}-i)^{\frac{1}{2}}\right]^{\frac{1}{2}}$	$(-4iR)^{\frac{1}{2}}$

TABLE 7.1

We see that as n approaches the resonant frequency the thickness of one of these boundary layers increases rapidly.

The solution for the inner and transition ranges in the neighbourhood of $\sigma = 2$ is

$$v = \frac{2}{(\sigma + 2)} e^{i\sigma t} \sin(x + \phi) \left\{ \frac{\lambda_1}{\lambda_1 - \lambda_3} e^{-\lambda_1 z} + \frac{\sigma}{2} e^{-\lambda_2 z} - \frac{\lambda_3}{\lambda_1 - \lambda_3} e^{-\lambda_3 z} \right\} + O(R^{-1/4}) , \qquad (7.3.9)$$

and in the neighbourhood of $\sigma = -2$ is

$$v = \frac{2}{(2-\sigma)} e^{i\sigma t} \sin(x + \phi) \left\{ \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_1 z} - \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 z} - \frac{\sigma}{2} e^{-\lambda_3 z} \right\} + O(R^{-1/4}) . \qquad (7.3.10)$$

At the resonant frequency, which corresponds to the inner range, the solutions (7.3.9) and (7.3.10) represent boundary layers, confined to the disks, having depths of penetration of vorticity of order

$$\left(\frac{\nu}{4\Omega}\right)^{\frac{1}{2}} \quad \text{and} \quad \left(\frac{\nu}{a^2 \Omega}\right)^{1/4} . \qquad (7.3.11)$$

The boundary layer of thickness $(\nu/4\Omega)^{\frac{1}{2}}$ is again a modified Stokes layer while the other boundary layer depends on the imposed length scale, a^{-1}. For the transition range the boundary layer thicknesses are again of order (7.3.11). The ratio of these boundary layer thicknesses, (7.3.11), is

$$\frac{(\nu/a^2\Omega)^{1/4}}{(\nu/4\Omega)^{\frac{1}{2}}} = 2R^{1/4} > 1 ,$$

since $R \gg 1$. Hence the boundary layer dependent on the length scale, a^{-1}, is always much thicker than the modified Stokes layer.

Therefore we see that the introduction of a length scale, a^{-1}, in the plane of the disk provides a second boundary layer thickness near the resonant frequency and an oscillatory solution always exists. When we allow a to tend to zero, the second boundary layer thickness in (7.3.11) tends to infinity and hence, near the resonant frequency, no oscillatory solution can be found which satisfies all the required boundary conditions. This special case ($a \rightarrow 0$) corresponds to the problem considered in chapter 2.

7.4 SPLIT DISK

We assume that the disk is split at $|x| = a$ and that the imposed oscillations, (7.2.1), take the form

$$\left.\begin{array}{lll} u = w = 0, & v = \varepsilon a \Omega e^{int} & |x| < a , \\ & v = 0 & |x| > a , \end{array}\right\} \quad (7.4.1)$$

where n is the frequency and ε a constant. We again suppose that $\varepsilon \ll 1$ in order to ensure that the linearization is valid.

When we introduce the dimensionless (starred) variables

$$x = ax^*, \quad z = az^*, \quad v = \varepsilon a \Omega v^*, \quad \psi = \varepsilon a^2 \Omega \psi^*, \quad t = \Omega^{-1} t^*, \tag{7.4.2}$$

into the equations (7.2.6), we have (upon dropping the asterisks)

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - R_o^{-1}\nabla^2\right)\nabla^2\psi - 2\frac{\partial v}{\partial z} &= 0, \\ \left(\frac{\partial}{\partial t} - R_o^{-1}\nabla^2\right)v + 2\frac{\partial \psi}{\partial z} &= 0, \end{aligned}\right\} \tag{7.4.3}$$

where $R_o = (\Omega a^2/\nu)$ is the Reynolds number, (1.1.2), with a length scale equal to a. We will assume that this Reynolds number, R_o is large.

The required boundary conditions are

$$\left.\begin{aligned} &\text{(a)} \quad \psi = \frac{\partial \psi}{\partial z} = 0, \quad v = e^{i\sigma t} \; |x| < 1, \quad v = 0 \; |x| > 1, \text{ on } z = 0, \\ &\text{(b)} \quad \psi, \frac{\partial \psi}{\partial z}, \quad v \longrightarrow 0 \quad \text{as } z \longrightarrow \infty, \\ &\text{(c)} \quad \text{an assumption of periodicity in } t, \\ &\text{(d)} \quad \psi, v \text{ tend to zero exponentially as } |x| \longrightarrow \infty, \end{aligned}\right\} \tag{7.4.4}$$

where $\sigma = n/\Omega$.

When we substitute

$$v = v_2(z,x)e^{i\sigma t}, \qquad \psi = \psi_2(z,x)e^{i\sigma t},$$

into the equations, (7.4.3), apply the Fourier transform

$$\bar{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f \, dx , \quad f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \bar{f} \, d\alpha , \qquad (7.4.5)$$

and eliminate $\overline{\psi_2}$, we have that

$$(D^2-\alpha^2)\left[i\sigma - R_o^{-1}(D^2-\alpha^2)\right]^2 \bar{v}_2 + 4D^2 \bar{v}_2 = 0 , \qquad (7.4.6)$$

where $D \equiv \frac{\partial}{\partial z}$. This equation, (7.4.6), must now be solved subject to the conditions

$$\left.\begin{aligned} &(a) \quad \overline{\psi_2} = \frac{\partial \overline{\psi_2}}{\partial z} = 0 , \quad \bar{v}_2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin \alpha}{\alpha} \quad \text{on } z = 0 , \\ &(b) \quad \overline{\psi_2} , \frac{\partial \overline{\psi_2}}{\partial z} , \bar{v}_2 \longrightarrow 0 \quad \text{as } z \longrightarrow \infty . \end{aligned}\right\} \qquad (7.4.7)$$

When we seek a solution of (7.4.6) of the form $\exp\{-\lambda z\}$, we find that

$$(\lambda^2 - \alpha^2)\left[i\sigma - R_o^{-1}(\lambda^2 - \alpha^2)\right]^2 + 4\lambda^2 = 0 , \qquad (7.4.8)$$

which is identical to the equation (4.3.13) in chapter 4, when we write

RANGE OF FREQUENCY	λ_1	λ_2	λ_3
OUTER $\lvert\sigma\pm 2\rvert > O(R^{-\frac{1}{2}})$	$\lvert\alpha\rvert\sigma(\sigma^2-4)^{-\frac{1}{2}}$ $+16R_o^{-1}\lvert\alpha\rvert^3(4-\sigma^2)^{-5/2}$	$R_o^{\frac{1}{2}}(i\sigma+2i)^{\frac{1}{2}}$	$R_o^{\frac{1}{2}}(i\sigma-2i)^{\frac{1}{2}}$
INNER $\lvert\sigma-2\rvert < O(R^{-\frac{1}{2}})$	$\lvert\alpha\rvert^{\frac{1}{2}}R_o^{1/4}\exp\{3\pi i/8\}$	$(4iR_o)^{\frac{1}{2}}$	$R_o^{1/4}\lvert\alpha\rvert^{\frac{1}{2}}\exp\{-\pi i/8\}$
$\lvert\sigma+2\rvert < O(R^{-\frac{1}{2}})$	$\lvert\alpha\rvert^{\frac{1}{2}}R_o^{1/4}\exp\{-3\pi i/8\}$	$R_o^{1/4}\lvert\alpha\rvert^{\frac{1}{2}}\exp\{\pi i/8\}$	$(-4iR_o)^{\frac{1}{2}}$
TRANSITION $\sigma=2+\frac{\mu}{R^{\frac{1}{2}}}$	$R_o^{1/4}\left[\frac{i\mu}{2}-i(\frac{\mu^2}{4}+\alpha^2 i)^{\frac{1}{2}}\right]^{\frac{1}{2}}$	$(4iR_o)^{\frac{1}{2}}$	$R_o^{1/4}\left[\frac{i\mu}{2}+i(\frac{\mu^2}{4}+\alpha^2 i)^{\frac{1}{2}}\right]^{\frac{1}{2}}$
$\sigma=-2+\frac{\mu}{R^{\frac{1}{2}}}$ μ real $\mu = O(1)$	$R_o^{1/4}\left[\frac{i\mu}{2}-i(\frac{\mu^2}{4}-\alpha^2 i)^{\frac{1}{2}}\right]^{\frac{1}{2}}$	$R_o^{1/4}\left[\frac{i\mu}{2}+i(\frac{\mu^2}{4}-\alpha^2 i)^{\frac{1}{2}}\right]^{\frac{1}{2}}$	$(-4iR_o)^{\frac{1}{2}}$

TABLE 7.2

$i\sigma = p$, $\alpha = a$, $R_o = R$. The roots of (7.4.8) have been discussed in section 4.4 and a summary of the highest order terms in the power series for the roots, λ, which have positive real part, for the case $|\alpha| \ll R_o^{\frac{1}{2}}$, are shown in Table 7.2.

The restriction $|\alpha| \ll R_o^{\frac{1}{2}}$ appears valid because if $|\alpha| = O(R_o^{\frac{1}{2}})$, the boundary condition, (7.4.7(a)), becomes

$$\overline{\psi}_2 = \frac{\partial \overline{\psi}_2}{\partial z} = 0, \quad \bar{v}_2 = O(R_o^{-\frac{1}{2}}) \quad \text{on } z = 0 ,$$

and hence, since $R_o \gg 1$, there will be a negligible contribution to the inversion integral.

In the outer range, that is for frequencies of oscillation away from the resonant frequency, we have

$$\bar{v}_2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin\alpha}{\alpha} \left[\tfrac{1}{2}\exp\left\{-R_o^{\frac{1}{2}}(i\sigma + 2i)^{\frac{1}{2}}z\right\} + \tfrac{1}{2}\exp\left\{-R_o^{\frac{1}{2}}(i\sigma - 2i)^{\frac{1}{2}}z\right\}\right] + O(|\alpha|\, R_o^{-\frac{1}{2}}), \qquad (7.4.9)$$

because the terms involving λ_1 only produce a contribution of order $|\alpha|\, R_o^{-\frac{1}{2}}$. Since we are assuming $|\alpha| \ll R_o^{\frac{1}{2}}$, we can neglect terms of order $|\alpha|\, R_o^{-\frac{1}{2}}$. Hence, when we apply the Fourier inversion integral, (7.4.5), we have

$$v = \tfrac{1}{2}e^{i\sigma t}\left[\exp\left\{-R_o^{\frac{1}{2}}(i\sigma + 2i)^{\frac{1}{2}}z\right\} + \exp\left\{-R_o^{\frac{1}{2}}(i\sigma - 2i)^{\frac{1}{2}}z\right\}\right] \qquad \text{for } |x| < 1 ,$$

$$v = \frac{1}{4} e^{i\sigma t} \left[\exp\left\{ -R_o^{\frac{1}{2}} (i\sigma + 2i)^{\frac{1}{2}} z \right\} + \exp\left\{ -R_o^{\frac{1}{2}} (i\sigma - 2i)^{\frac{1}{2}} z \right\} \right]$$

for $|x| = 1$,

$$v = 0 \qquad \text{for } |x| > 1 , \qquad (7.4.10)$$

which satisfies all the boundary conditions to order one.

The expression, (7.4.10), represents two modified Stokes layers in the region $|x| < 1$ which have depths of penetration of vorticity of order $(\nu / |n \pm 2\Omega|)^{\frac{1}{2}}$, no flow in the region $|x| > 1$ and shear layers in the neighbourhood of $|x| = 1$.

The solution in the neighbourhood of the resonant frequency, that is for the transition and inner ranges, is

$$v = \frac{2e^{i\sigma t}}{\pi(\sigma + 2)} \int_{-\infty}^{\infty} \left\{ \frac{\lambda_1}{\lambda_1 - \lambda_3} e^{-\lambda_1 z} + \frac{\sigma}{2} e^{-\lambda_2 z} - \frac{\lambda_3}{\lambda_1 - \lambda_3} e^{-\lambda_3 z} \right\} \frac{\sin \alpha}{\alpha} e^{-i\alpha x} d\alpha ,$$

(7.4.11)

for σ in the neighbourhood of 2 and

$$v = \frac{2e^{i\sigma t}}{\pi(\sigma + 2)} \int_{-\infty}^{\infty} \left\{ \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_1 z} - \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 z} - \frac{\sigma}{2} e^{-\lambda_3 z} \right\} \frac{\sin \alpha}{\alpha} e^{-i\alpha x} d\alpha ,$$

(7.4.12)

for σ in the neighbourhood of -2, when terms of order $|\alpha|^{\frac{1}{2}} R_o^{1/4}$ have been ignored. The second term in (7.4.11) and the third term in (7.4.12) can immediately be integrated to give

$$\frac{\sigma}{\sigma+2}\, e^{i\sigma t} \exp\left\{-(4iR_o)^{\frac{1}{2}} z\right\} \quad \text{and} \quad \frac{\sigma}{\sigma-2}\, e^{i\sigma t} \exp\left\{-(4iR_o)^{\frac{1}{2}} z\right\},$$

respectively on $|x| < 1$ and zero elsewhere. Therefore these terms represent modified Stokes layers on the disk having a depth of penetration of order $(\nu/4\Omega)^{\frac{1}{2}}$. The remaining terms in (7.4.11) and (7.4.12) are of the form

$$A \int_{-\infty}^{\infty} \exp\left\{-|\alpha|^{\frac{1}{2}} z\, R_o^{1/4} c\right\} \left[\frac{\sin \alpha}{\alpha}\, e^{-i\alpha x}\right] d\alpha\,, \qquad (7.4.13)$$

where A, c are complex constants of order one and c has positive real part. We have been unable to evaluate the integral (7.4.13) exactly but instead when $zR_o^{1/4} \gg 1$, we see that the dominant contribution arises from the neighbourhood of $\alpha = 0$. Then by expanding $\alpha^{-1} \sin \alpha e^{-i\alpha x}$ in a Taylor series about $\alpha = 0$ we find that to the highest order (7.4.13) behaves like $(R_o^{\frac{1}{2}} z^2)^{-1}$. Hence these remaining terms represent a solution which has an algebraic decay for $z \gg R_o^{-1/4}$ and which produces a negligibly small contribution, $O(R_o^{-\frac{1}{2}})$, when $z = O(1)$.

Therefore the expressions (7.4.11) and (7.4.12) satisfy all the required boundary conditions to the highest order and make a non-negligible contribution to the flow only within a distance $z = O(1)$ from the disk.

Hence the introduction of a length scale, a, in the plane of the disk implies that an oscillatory solution exists for all possible frequencies of oscillation. The case $a \longrightarrow \infty$ corresponds to the problem considered in chapter 2 and the above analysis is not valid because the condition, (7.4.4(d)), cannot be satisfied.

7.5 CONCLUSIONS

For the two special cases considered in sections 7.3 and 7.4, we find that, even when oscillations are imposed at the resonant frequency, an oscillatory solution which satisfies the required boundary conditions always exists and takes the form of well-defined boundary layers confined to the disk. For the non-resonant case, these boundary layers are again modified Stokes layers which have a depth of penetration of order $(\nu / | n \pm 2\Omega | \quad)^{\frac{1}{2}}$, while for oscillations of a frequency within a neighbourhood of radius $R^{-\frac{1}{2}}$ (or $R_o^{-\frac{1}{2}}$) of the resonant frequency, a new solution exists which consists of one modified Stokes layer and a second, much thicker, boundary layer on the disk. The depth of penetration of this second layer depends upon the length scale which has been introduced into the problem by the imposed oscillations and tends to infinity as this length scale tends to infinity. This agrees with the results obtained for the semi-infinite problem in chapters 2 and 3.

Hence, for these two special cases, the introduction of a length scale in the plane of the disk provides a second boundary layer thickness for resonant oscillations and therefore an oscillatory solution satisfying all the boundary conditions always exists. However, a resonance effect is still present since different oscillatory solutions are found for resonant and non-resonant oscillations.

7.6 SINUSOIDAL STRESS APPLIED AT THE SURFACE OF A SEMI-INFINITE OCEAN

We can use the results of section 7.3 to discuss the more realistic problem of the flow generated in a semi-infinite ocean ($z < 0$) with a free surface ($z = 0$) which always remains planar, when a stress

$$\left.\begin{aligned} \frac{\partial v}{\partial z} &= \varepsilon \Omega \, e^{int} \sin(\alpha x + \phi) \, , \\ \frac{\partial u}{\partial z} &= 0 \, , \end{aligned}\right\} \qquad (7.6.1)$$

is imposed at the surface.

For the outer range, that is for oscillations at frequencies away from the resonant frequency, we have

$$v = \frac{1}{2}\left[\frac{1}{R^{\frac{1}{2}}(i\sigma+2i)^{\frac{1}{2}}}\exp\left\{R^{\frac{1}{2}}(i\sigma+2i)^{\frac{1}{2}}z\right\} + \frac{1}{R^{\frac{1}{2}}(i\sigma-2i)^{\frac{1}{2}}}\exp\left\{R^{\frac{1}{2}}(i\sigma-2i)^{\frac{1}{2}}z\right\}\right]\sin(x+\phi)e^{i\sigma t} + O(R^{-1}), \qquad (7.6.2)$$

in the dimensionless variables defined by (7.3.2). This solution, (7.6.2), represents stress boundary layers attached to the surface which penetrate downwards through distance of order $(\nu/|n \pm 2\Omega|)^{\frac{1}{2}}$.

For frequencies of oscillation at or near the resonant frequency, that is for the inner or transition ranges, the solution in the dimensionless variables, (7.3.2), is

$$v = \frac{2}{\sigma+2}\left\{\frac{\sigma}{2}\frac{1}{\lambda_2}e^{\lambda_2 z} + \frac{\lambda_1}{(\lambda_1^2-\lambda_3^2)}e^{\lambda_1 z} - \frac{\lambda_3}{(\lambda_1^2-\lambda_3^2)}e^{\lambda_3 z}\right\}\sin(x+\phi)e^{i\sigma t} + O(R^{-3/4}), \qquad (7.6.3)$$

near $\sigma = 2$ and

$$v = \frac{2}{2-\sigma}\left\{-\frac{\sigma}{2}\frac{1}{\lambda_3}e^{\lambda_3 z} + \frac{\lambda_1}{(\lambda_1^2-\lambda_2^2)}e^{\lambda_1 z} - \frac{\lambda_2}{(\lambda_1^2-\lambda_2^2)}e^{\lambda_2 z}\right\}\sin(x+\phi)e^{i\sigma t} + O(R^{-3/4}), \qquad (7.6.4)$$

near $\sigma = -2$, where the λ's are given in Table 7.1. The expressions (7.6.3) and (7.6.4) represent layers having thicknesses

$$(\nu/4\Omega)^{\frac{1}{2}} \quad \text{and} \quad (\nu/\alpha^2\Omega)^{1/4},$$

attached to the surface.

Hence the effect of a sinusoidal surface stress, which introduces a horizontal length scale, α^{-1}, into the problem is confined to the upper fluid and never penetrates a distance more than

$$(\nu/|n-2\Omega|)^{\frac{1}{2}}, \qquad (\nu/\alpha^2\Omega)^{1/4},$$

for non-resonant and resonant oscillations respectively.

CHAPTER 8

SOME EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

8.1 INTRODUCTION

In chapters 4, 5 and 6, the vertical vorticity produced in a fluid by the Coriolis force together with temperature variations has been discussed for the case when the non-linear convective terms were assumed negligible. In this chapter we will investigate the effect of these non-linear convective terms on the development of the vertical component of vorticity in a rotating fluid system which is heated from below. Exact solutions of the inviscid Navier-Stokes equations, the continuity equation and the inviscid energy equation, in non-rotating cylindrical polar co-ordinates (r, Θ, z), are sought for the flow in a semi-infinite expanse of fluid bounded by an infinite plane disk, $z = 0$, when, initially, the fluid and the disk are in steady, isothermal rigid rotation. If (v_r, v_Θ, v_z) are the velocity components in this co-ordinate system, then a class of solutions of the form

$$v_r = -\frac{f(r)}{r}, \qquad v_z = \frac{z}{r}\frac{\partial}{\partial r}\left[f(r)\right], \qquad v_\Theta = r\Omega \quad \text{at } t = 0,$$

is found, where f(r) satisfies an ordinary non-linear differential equation. In particular we are interested in any solutions which are non-singular and which exhibit a growth or decay, with time, of the vertical component of vorticity near the axis of rotation and could therefore describe the formation of a hurricane.

Also some exact solutions of the complete Navier-Stokes equations are derived which satisfy the inviscid boundary condition at the disk.

These solutions represent possible interior flows which satisfy the inviscid boundary condition at the disk. In order to obtain a solution of the non-linear equations valid throughout the whole fluid, we see that, in the neighbourhood of the disk, these interior solutions must be replaced by viscous boundary layers which satisfy the non-slip condition at the disk and also match the interior flow. No possible solutions for these viscous non-linear boundary layers have, as yet, been obtained.

8.2 EQUATIONS OF MOTION

We consider an infinite plane horizontal disk, $z = 0$, bounding a semi-infinite expanse of fluid, $z > 0$, when, initially, the fluid and the disk are in solid-body rotation with constant angular velocity, Ω , about an axis normal to the disk. Also we assume that the disk is maintained at a constant temperature, T_o, for all time.

We take a cylindrical polar co-ordinate system (r,θ,z) such that the z-axis is parallel to the common axis of rotation of the disk and the fluid and the r,θ-axes lie in the plane of the disk and are at rest. The velocity components for this co-ordinate system are assumed to be (v_r, v_θ, v_z).

Three basic assumptions are now made. Namely, we assume that

(i) the motion is axisymmetric and is therefore independent of θ,

(ii) it is always valid to apply the Boussinesq approximation [33, p.759] and therefore we may suppose that the density

$$\rho = \rho_o(1 - aT),$$

where ρ_o is the density at T_o, a the coefficient of thermal expansion and T is the variation in temperature from T_o,

(iii) any temperature variations are independent of time, and dissipation and volume changes can be ignored.

Then the Navier-Stokes equations, the continuity equation and the energy equation become

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho_o}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 v_r - \frac{v_r}{r^2}\right],$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = \nu\left[\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right],$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + g\alpha T + \nu \nabla^2 v_z ,$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0 ,$$

$$v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} = \kappa \nabla^2 T , \qquad (8.2.1)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, p is the departure of the effective kinematic pressure from the hydrostatic pressure, which prevails when the fluid is at rest at a uniform temperature, T_0 ; g is the acceleration due to gravity and κ is the thermal diffusivity.

We now wish to solve the above equations (8.2.1) when we specify certain conditions at the disk, z = 0. However, no restrictions will be placed on the behaviour of the fluid at infinity. Firstly, we will consider some inviscid solutions.

8.3 INVISCID SOLUTIONS

We will now seek solutions of the equations, (8.2.1), when the terms representing viscous and thermal diffusion are ignored, namely the equations

$$
\left.
\begin{aligned}
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} &= - \frac{1}{\rho_0} \frac{\partial p}{\partial r} \,, \\
\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} &= 0 \,, \\
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} &= - \frac{1}{\rho_0} \frac{\partial p}{\partial z} + g\alpha T \,, \\
\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} &= 0 \,, \\
v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} &= 0 \,.
\end{aligned}
\right\} \qquad (8.3.1)
$$

The solution of these equations, (8.3.1), corresponds to a possible interior solution when the kinematic viscosity, ν , and the thermal diffusivity, κ , are suitably small. The order of the equations (8.3.1) is less than the order of the equations (8.2.1) and, therefore, in order to satisfy the non-slip condition at the disk, a boundary layer solution must be added to the following interior solutions.

We now wish to solve the equations, (8.3.1), subject to the conditions

$$
\left.
\begin{array}{llll}
\text{(a)} & v_z = 0 & \text{on } z = 0 & \text{for all } r, t, \\
\text{(b)} & v_\theta = \Omega r & \text{at } t = 0 & \text{for all } z, \\
\text{(c)} & T = 0 & \text{on } z = 0 & \text{for all } r, t.
\end{array}
\right\} \qquad (8.3.2)
$$

Also, whenever possible, we would like to obtain a solution which also satisfies the additional conditions

$$\left.\begin{array}{ll}\text{(a)} & v_r \longrightarrow 0 \text{ as } r \longrightarrow \infty \quad \text{for all } t, z, \\ \text{(b)} & v_r,\ v_z \text{ are always regular.}\end{array}\right\} \tag{8.3.3}$$

We now CHOOSE

$$v_r = -\frac{f(r)}{r}, \qquad v_z = \frac{zf'(r)}{r}, \tag{8.3.4}$$

where $f(r)$ is some arbitrary function of r which will be determined later and prime represents differentiation with respect to r. This choice, (8.3.4), for v_r and v_z ensures that the continuity equation and the inviscid boundary condition, (8.3.2(a)), are satisfied.

When we substitute (8.3.4) into the equations (8.3.1), we find that

$$\frac{f}{r}\frac{d}{dr}\left(\frac{f}{r}\right) - \frac{v_\theta^2}{r} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r}, \tag{8.3.5}$$

$$\frac{\partial v_\theta}{\partial t} - \frac{f}{r}\frac{\partial v_\theta}{\partial r} + \frac{zf'}{r}\frac{\partial v_\theta}{\partial z} - \frac{fv_\theta}{r^2} = 0, \tag{8.3.6}$$

$$-\frac{fz}{r}\frac{d}{dr}\left(\frac{f'}{r}\right) + z\left(\frac{f'}{r}\right)^2 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + gaT, \tag{8.3.7}$$

$$-\frac{f}{r}\frac{\partial T}{\partial r} + \frac{zf'}{r}\frac{\partial T}{\partial z} = 0. \tag{8.3.8}$$

In order to obtain the general solution for the equation (8.3.6), we must consider the characteristic equations [14],

$$\frac{dt}{1} = \frac{dr}{(-f/r)} = \frac{dz}{(zf'/r)} = \frac{dv_\theta}{(fv_\theta/r^2)} ,$$

which have solutions

$$zf(r) = a, \qquad rv_\theta = b , \qquad t + \int^r \frac{r}{f(r)} . dr = c ,$$

where a, b, c are arbitrary constants. Then the general solution of (8.3.6) is

$$rv_\theta = F\left\{ zf(r) , \qquad t + \int^r \frac{r}{f(r)} dr \right\} , \qquad (8.3.9)$$

where F is some arbitrary function. In order that (8.3.9) satisfies the initial condition, (8.3.2(b)), we require

$$r^2 \Omega = F\left\{ zf(r), \qquad \int^r \frac{r}{f(r)} dr \right\} . \qquad (8.3.10)$$

Similarly, for the equation (8.3.8), we must first consider the characteristic equations

$$\frac{dr}{(-f/r)} = \frac{dz}{(zf'/r)} = \frac{dT}{0} ,$$

from which we find that the general solution is

$$T = G(zf(r)) , \qquad (8.3.11)$$

where G is some arbitrary function.

When the pressure is eliminated from the equations (8.3.5) and (8.3.7), we find that

$$\frac{\partial}{\partial z}\left(\frac{v_\Theta^2}{r}\right) + z\frac{d}{dr}\left\{-\frac{f}{r}\frac{d}{dr}\left(\frac{f'}{r}\right) + \left(\frac{f'}{r}\right)^2\right\} = g\alpha\frac{\partial T}{\partial r} . \qquad (8.3.12)$$

When (8.3.12) is integrated with respect to z, we find that

$$\frac{v_\Theta^2}{r} - H(r,t) = g\alpha\int^z \frac{\partial T}{\partial r}.dz - \tfrac{1}{2}z^2\frac{d}{dr}\left\{-\frac{f}{r}\frac{d}{dr}\left(\frac{f'}{r}\right) + \left(\frac{f'}{r}\right)^2\right\} , \qquad (8.3.13)$$

where H(r,t) is some arbitrary function. At $t = 0$, $v_\Theta = \Omega r$ and the left hand side of (8.3.13) is a function of r only. Therefore, since the right hand side of (8.3.13) is independent of time and a function of z and r, we require that

$$g\alpha\int^z \frac{\partial T}{\partial r}.dz - \tfrac{1}{2}z^2\frac{d}{dr}\left\{-\frac{f}{r}\frac{d}{dr}\left(\frac{f'}{r}\right) + \left(\frac{f'}{r}\right)^2\right\} \equiv 0 , \qquad (8.3.14)$$

so that (8.3.13) is always satisfied. For both the terms in (8.3.14) to have the same z-dependence we require, from (8.3.11), that

$$T = \gamma z f(r) , \qquad (8.3.15)$$

which satisfies the condition (8.3.2(c)) for any constant, γ.

When we substitute (8.3.15) into the equations (8.3.12) and (8.3.14), we find that

$$\left.\begin{aligned} \frac{\partial}{\partial z}\left(\frac{v_\Theta^2}{r}\right) + z\left[\frac{d}{dr}\left\{-\frac{f}{r}\frac{d}{dr}\left(\frac{f'}{r}\right) + \left(\frac{f'}{r}\right)^2\right\} - ga\gamma f'\right] &= 0 , \\ \gamma g a f' - \frac{d}{dr}\left\{-\frac{f}{r}\frac{d}{dr}\left(\frac{f'}{r}\right) + \left(\frac{f'}{r}\right)^2\right\} &= 0 . \end{aligned}\right\} \qquad (8.3.16)$$

From (8.3.16), we see immediately that v_Θ is independent of z and hence, from (8.3.9) and (8.3.10), we have that

$$\left.\begin{aligned} r v_\Theta &= F\left\{t + \int^r \frac{r}{f(r)} . dr\right\} , \\ \text{where } r^2 \Omega &= F\left\{\int^r \frac{r}{f(r)} . dr\right\} . \end{aligned}\right\} \qquad (8.3.17)$$

Therefore we have found a whole class of solutions of the equations, (8.3.1), subject to the conditions, (8.3.2), of the form

$$v_r = -\frac{f(r)}{r} , \qquad v_z = \frac{zf'(r)}{r} , \qquad T = \gamma z f(r) ,$$

$$r v_\Theta = F\left\{t + \int^r \frac{r}{f(r)} dr\right\} \quad \text{where} \quad \Omega r^2 = F\left\{\int^r \frac{r}{f(r)} . dr\right\} , \qquad (8.3.18)$$

where f(r) must be chosen to satisfy

$$\left[\frac{f'(r)}{r}\right]^2 - \frac{f(r)}{r}\frac{d}{dr}\left(\frac{f'(r)}{r}\right) - ga\gamma f(r) = K , \qquad (8.3.19)$$

where K is a constant defined by $K = -\frac{1}{z\rho_o}\frac{\partial p}{\partial z}$.

We will now consider some special cases of the above class of solutions by assuming different values for γ and K. The most interesting case occurs when $\gamma \neq 0$, $K \neq 0$, which is discussed in Section 8.5, case 2. However, before considering this case, we will discuss some problems that arise when γ and/or K are identically zero.

8.4 INCOMPRESSIBLE SOLUTIONS

We will first consider some solutions when the fluid is incompressible, that is when

$$\rho \equiv \rho_o \quad \text{and} \quad T \equiv 0 .$$

This is equivalent to assuming that $\gamma \equiv 0$. Then the equation (8.3.19) becomes

$$-\frac{f(r)}{r}\frac{d}{dr}\left(\frac{f'(r)}{r}\right) + \left(\frac{f'(r)}{r}\right)^2 = K . \qquad (8.4.1)$$

Case 1: $\gamma \equiv 0$, $K \equiv 0$

For this case, the equation (8.4.1) reduces to

$$\frac{r}{f'}\frac{d}{dr}\left(\frac{f'}{r}\right) = \frac{f'}{f} ,$$

which can immediately be integrated to give

$$f' = 2Arf , \qquad (8.4.2)$$

where A is a constant. This equation (8.4.2) can again be integrated to give

$$f(r) = B \exp\{Ar^2\} , \qquad (8.4.3)$$

where B is another arbitrary constant.

Hence for the special case when the fluid is incompressible and $K \equiv 0$, all the solutions of (8.3.19) are of the form (8.4.3) and, from (8.3.18), we find immediately that

$$v_r = -\frac{B}{r}\exp\{Ar^2\} , \quad v_z = 2ABz \exp\{Ar^2\} , \quad T \equiv 0 , \quad (8.4.4)$$

$$rv_\theta = F\left\{t - \frac{1}{2AB}\exp\{-Ar^2\}\right\} , \text{ where } r^2\Omega = F\left\{-\frac{1}{2AB}\exp\{-Ar^2\}\right\} .$$

(8.4.5)

If we assume that

$$X = -\frac{1}{2AB}\exp\{-Ar^2\} ,$$

then $-\frac{\Omega}{A}\log(-2ABX) = \Omega r^2$,

and hence, from (8.4.5), we have that

$$rv_\theta = -\frac{\Omega}{A}\log\left[-2ABt + \exp\{-Ar^2\}\right] . \qquad (8.4.6)$$

In order that (8.4.4) and (8.4.6) are the solution of a physical problem, we will restrict A,B to being real numbers, which implies that the velocity components must always be real.

We define a streamfunction, ψ, by

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad (8.4.7)$$

and the components of vorticity, $(\omega_r, \omega_\theta, \omega_z)$, by

$$\omega_r = -\frac{\partial v_\theta}{\partial z}, \quad \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \quad \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta). \qquad (8.4.8)$$

Then, from (8.4.4) and (8.4.6), we find that

$$\psi = -Bz \exp\{Ar^2\}, \quad \omega_r \equiv 0, \quad \omega_\theta = -4A^2Brz \exp\{Ar^2\},$$

$$\omega_z = \frac{2\Omega}{(1 - 2ABt \exp\{Ar^2\})}. \qquad (8.4.9)$$

The solution for v_r in (8.4.4) is always singular at $r = 0$, which corresponds to a sink or a source on the z-axis depending whether B is positive or negative and, therefore, the extra condition, (8.3.3(b)), is never satisfied. The remaining condition, (8.3.3(a)), is satisfied only if A is negative.

For the case $A > 0$ and $B > 0$, which implies a sink on the z-axis, we see, from (8.4.6) and (8.4.9), that rv_θ and ω_z have a singularity at

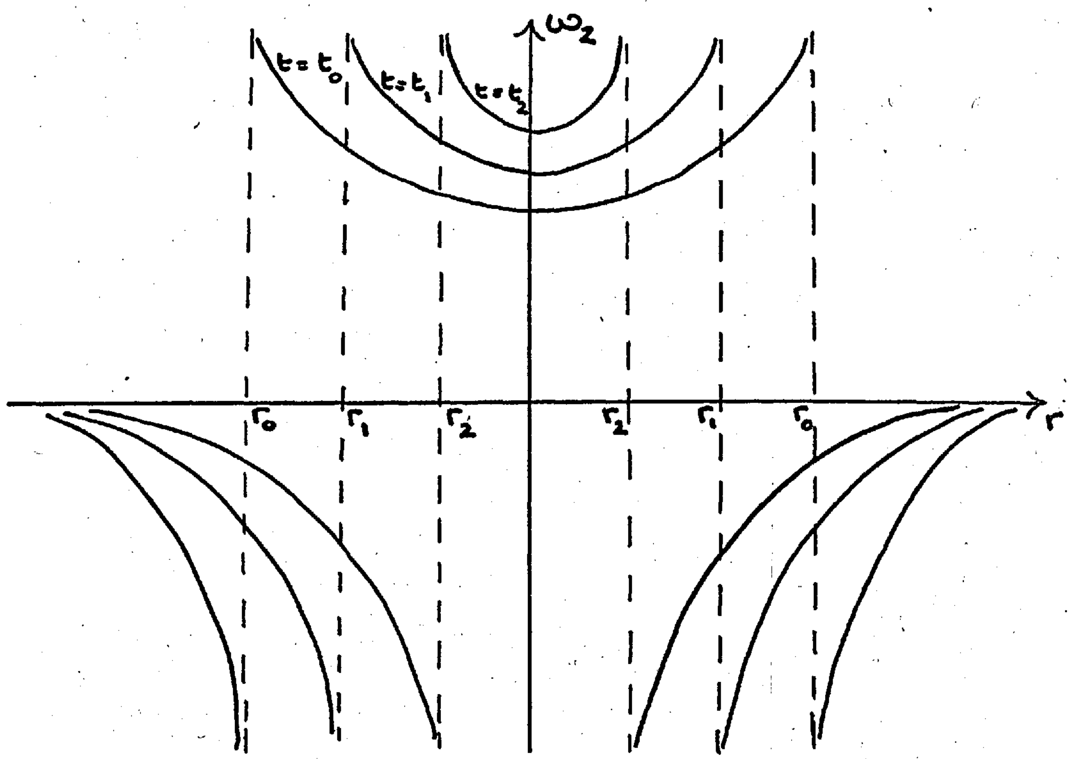
$$t = \frac{1}{2AB} \exp\{-Ar^2\} \quad , \qquad (8.4.10)$$

which moves towards the origin, $r = 0$, as t tends to infinity. When $A > 0$, $B < 0$, which corresponds to a source on the z-axis, we find, from (8.4.6) and (8.4.9), that rv_θ and ω_z are always regular. For the case $A < 0$, $B < 0$, which implies a source on the z-axis, rv_θ and ω_z have a singularity when t is given by (8.4.10), which moves away from the z-axis as t increases. When $A < 0$, $B > 0$, which corresponds to a sink on the z-axis, rv_θ and ω_z are always regular. The variations of ω_z with r, for fixed t, for these four cases are shown in Fig.8.1.

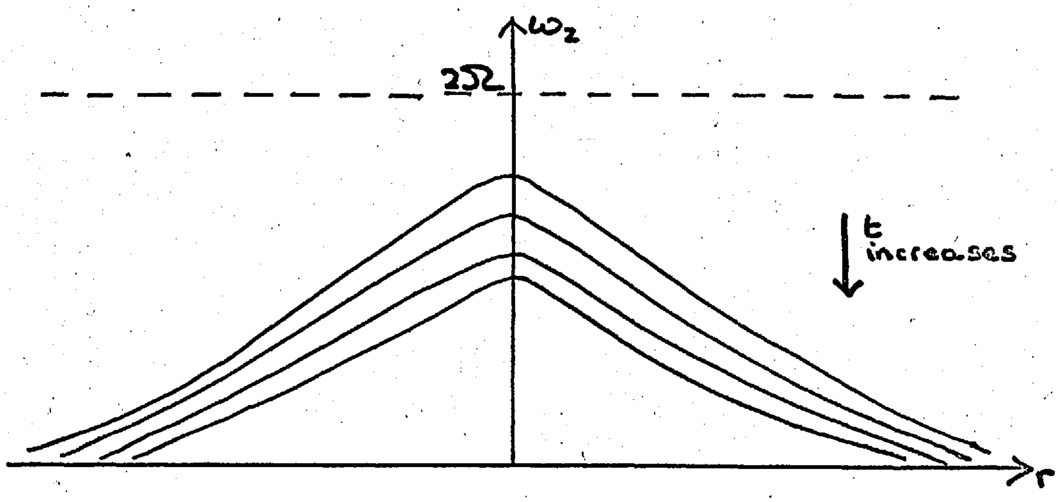
When $A \equiv 0$, we have, from (8.4.4), (8.4.6) and (8.4.9)

$$\left.\begin{array}{l} v_r = -\frac{B}{r} , \quad v_z = 0 , \quad T \equiv 0, \quad rv_\theta = \Omega r^2 , \\ \psi = -Bz, \quad \omega_r = \omega_\theta = 0 , \quad \omega_z = 2\Omega \quad . \end{array}\right\} \qquad (8.4.11)$$

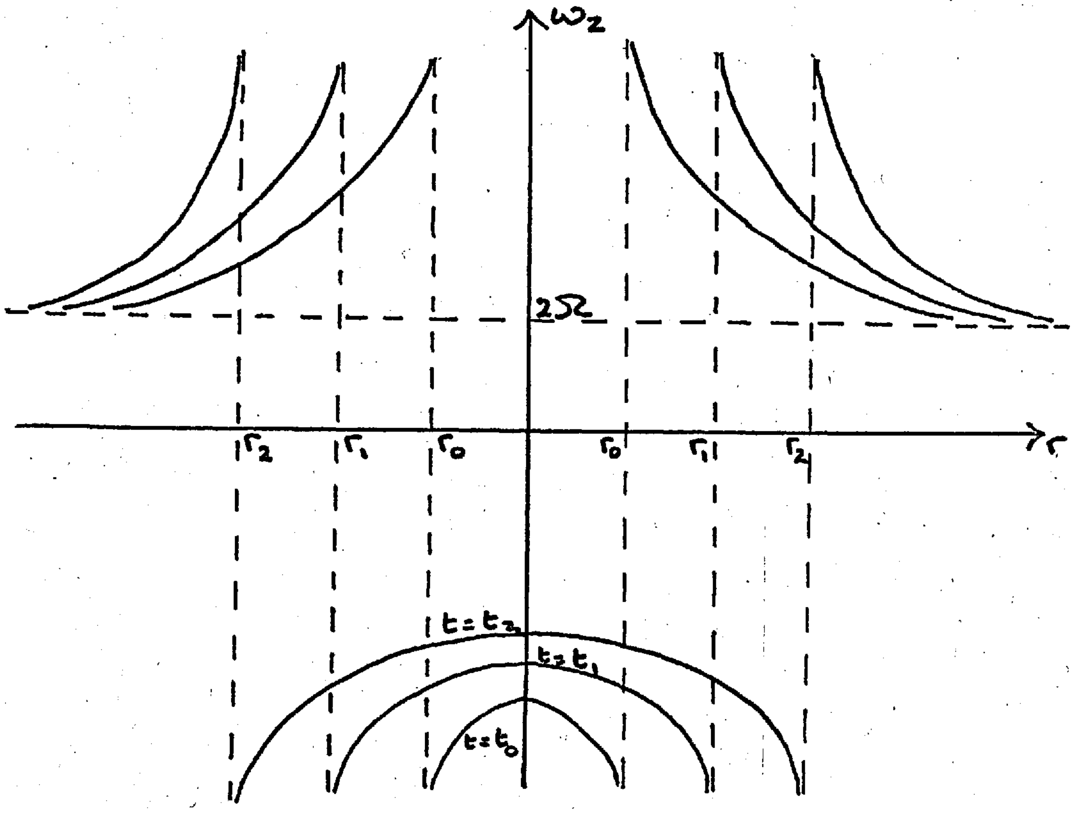
This is a very special case with a sink, $B > 0$, or a source, $B < 0$, on the z-axis, with the streamlines given by z = constant and with the z-component of vorticity and the azimuthal velocity unchanged from their initial values.


ω_z
$t=t_0$
$t=t_1$
$t=t_2$
r_0
r_1
r_2
r_2
r_1
r_0
r

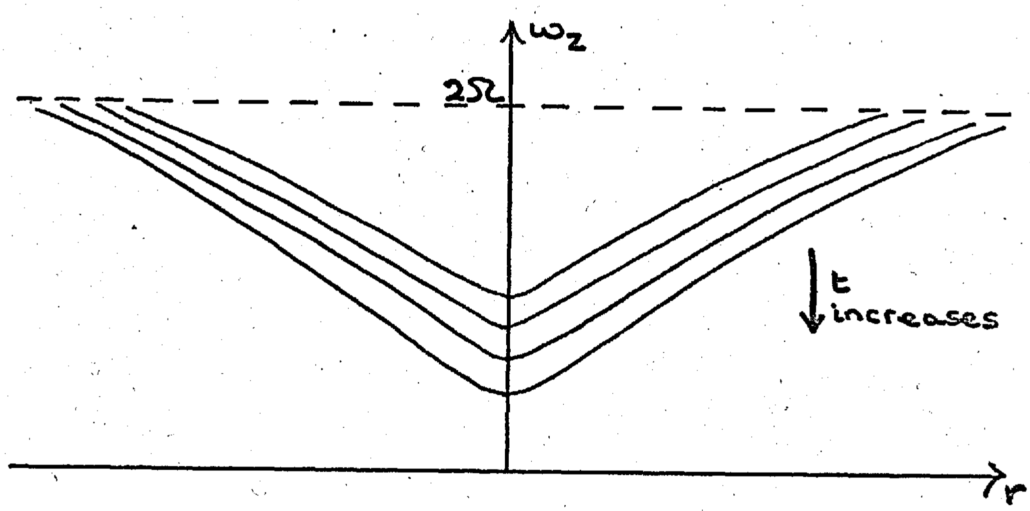
8.1(a)


ω_z
2Ω
t increases
r

8.1(b)


ω_z
2Ω
r_2
r_1
r_0
r_0
r_1
r_2
r
$t=t_2$
$t=t_1$
$t=t_0$

8.1(c)


ω_z
2Ω
t
increases
r

8.1(d)

FIG.8.1: Sketch of the vertical vorticity,

$$\omega_z = \frac{2\Omega}{(1-2ABt\ \exp\{Ar^2\})} ,$$

when

(a) $A > 0$, $B > 0$, $t_i = (2AB)^{-1}\exp\{-Ar_i^2\}$, $i = 0,1,2$ where $t_0 < t_1 < t_2$,

(b) $A > 0$, $B < 0$, for fixed t,

(c) $A < 0$, $B < 0$, $t_i = (2AB)^{-1}\exp\{-Ar_i^2\}$, $i = 0,1,2$, where $t_0 < t_1 < t_2$,

(d) $A < 0$, $B > 0$, for fixed t.

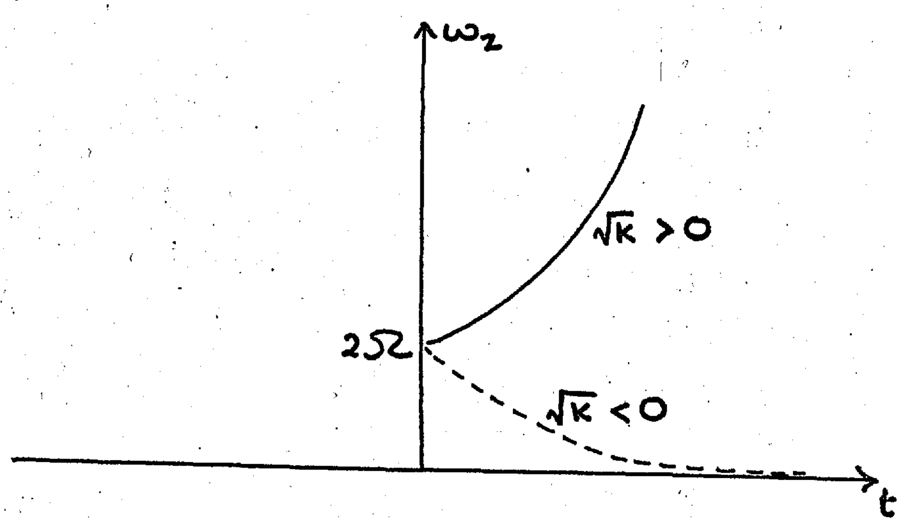


FIG.8.2: Sketch of the vertical vorticity $\omega_z = 2\Omega \exp\{t\sqrt{K}\}$.

When $B \equiv 0$, we see that the initial state of solid-body rotation is maintained.

<u>Case 2: $\gamma \equiv 0,\ K \neq 0$</u>

We must now consider the solutions of the equation (8.4.1), which becomes

$$-f\frac{d^2f}{dx^2} + \left(\frac{df}{dx}\right)^2 = \frac{1}{4}K \ , \qquad (8.4.12)$$

when we write $r^2 = x$.

A solution of (8.4.12) is

$$f = \frac{xK^{\frac{1}{2}}}{2} \ . \qquad (8.4.13)$$

When we substitute (8.4.13) into (8.3.18), we find that

$$v_r = -\frac{rK^{\frac{1}{2}}}{2} \ , \quad v_z = z\,K^{\frac{1}{2}} \ , \quad T \equiv 0 \ , \qquad (8.4.14)$$

$$rv_\theta = F\left\{t + 2K^{-\frac{1}{2}}\log r\right\} \quad \text{where} \quad r^2\Omega = F\left\{2K^{-\frac{1}{2}}\log r\right\},$$

and hence

$$rv_\theta = \Omega\, r^2 \exp\left\{t\sqrt{K}\right\} \ . \qquad (8.4.15)$$

We will assume that K is a positive real number in order that the velocity components are the solution of a physical problem. Then, from (8.4.7), (8.4.8), (8.4.14) and (8.4.15), we find that

$$\psi = -\frac{r^2 z K^{\frac{1}{2}}}{2}, \quad \omega_r \equiv \omega_\theta \equiv 0, \quad \omega_z = 2\Omega \exp\{t\sqrt{K}\}. \tag{8.4.16}$$

The above solution, (8.4.14), always satisfies the additional boundary condition, (8.3.3(b)), but never the condition, (8.3.3(a)). From (8.4.14), (8.4.15) and (8.4.16), we see that all the functions are regular and that the only vorticity produced in the flow is in the z-direction. This axial component of vorticity increases or decreases with time depending upon the sign chosen for $\sqrt{K}$ and is independent of r and z. The variation of ω_z with time is shown in Fig.8.2.

A second solution of (8.4.12) is

$$f = \frac{xK^{\frac{1}{2}}}{2} + A, \tag{8.4.17}$$

where A is an arbitrary constant. Then, from (8.4.17), (8.3.18), (8.4.7) and (8.4.8), we find

$$\left.\begin{aligned} v_r &= -\left(\frac{K^{\frac{1}{2}}r}{2} + \frac{A}{r}\right), \qquad v_z = K^{\frac{1}{2}}z, \qquad T \equiv 0, \\ rv_\theta &= \Omega r^2 \exp\{t\sqrt{K}\} + 2\Omega AK^{-\frac{1}{2}}(\exp\{t\sqrt{K}\} - 1), \\ \psi &= -z\left(\frac{K^{\frac{1}{2}}r^2}{2} + A\right), \qquad \omega_r \equiv \omega_\theta \equiv 0, \\ \omega_z &= 2\Omega \exp\{t\sqrt{K}\}. \end{aligned}\right\} \tag{8.4.18}$$

We will again assume that K is a positive real number and that A is real. The effect of introducing this extra constant is that neither of the additional conditions, (8.3.3(a)) and (8.3.3(b)), are satisfied, while the vorticity generated remains unchanged. Also we have introduced a source or a sink on the z-axis depending whether A is negative or positive. The axial component of vorticity, ω_z, is again shown in Fig.8.2.

8.5 VARIABLE DENSITY SOLUTIONS

We will now assume that $\gamma \neq 0$ in the equation (8.3.19).

Case 1: $\gamma \neq 0, \quad K \equiv 0$

When we assume that $K \equiv 0$ and write $r^2 = x$, the equation (8.3.19) becomes

$$-f\frac{d^2f}{dx^2} + \left(\frac{df}{dx}\right)^2 - \frac{ga\gamma f}{4} = 0 . \qquad (8.5.1)$$

A solution of this equation, (8.5.1), is

$$f = \frac{ga\gamma}{8}x^2 . \qquad (8.5.2)$$

Then, from (8.5.2) and (8.3.18), we have

$$v_r = -\frac{ga\gamma}{8}r^3 , \qquad v_z = \frac{ga\gamma}{2}zr^2 , \qquad T = \frac{ga\gamma^2}{8}zr^4 ,$$

$$rv_\theta = \frac{4\Omega r^2}{4 - ga\gamma tr^2} . \qquad (8.5.3)$$

Then, from (8.4.7), (8.4.8) and (8.5.3), we find that

$$\psi = -\frac{ga\gamma}{8} r^4 z, \quad \omega_r \equiv 0, \quad \omega_\theta = -zrga\gamma,$$

$$\omega_z = \frac{32}{(ga\gamma t r^2 - 4)^2}. \tag{8.5.4}$$

The solution (8.5.3) always satisfies the additional boundary condition, (8.3.3(b)), but not the condition, (8.3.3(a)). When $\gamma > 0$, the functions rv_θ and ω_z have a singularity at

$$t = \frac{4}{ga\gamma r^2}, \tag{8.5.5}$$

which moves towards the z-axis as time increases. On the other hand, when $\gamma < 0$, the functions rv_θ and ω_z are always regular. It should be noticed that the temperature variation, T, is always positive. The variation of ω_z with r for fixed z is shown in Fig.8.3.

Case 2: $\gamma \neq 0, \ K \neq 0$

When both γ and K are non-zero there exists a solution of (8.3.19) of the form

$$f = C(1 - \exp\{-kr^2\}), \tag{8.5.6}$$

where $K = -ga\gamma C$ and $\gamma = -\frac{4Ck^2}{ag}$, which is negative if $C > 0$ and positive if $C < 0$, for real k. Moreover, γC, which appears in T below,

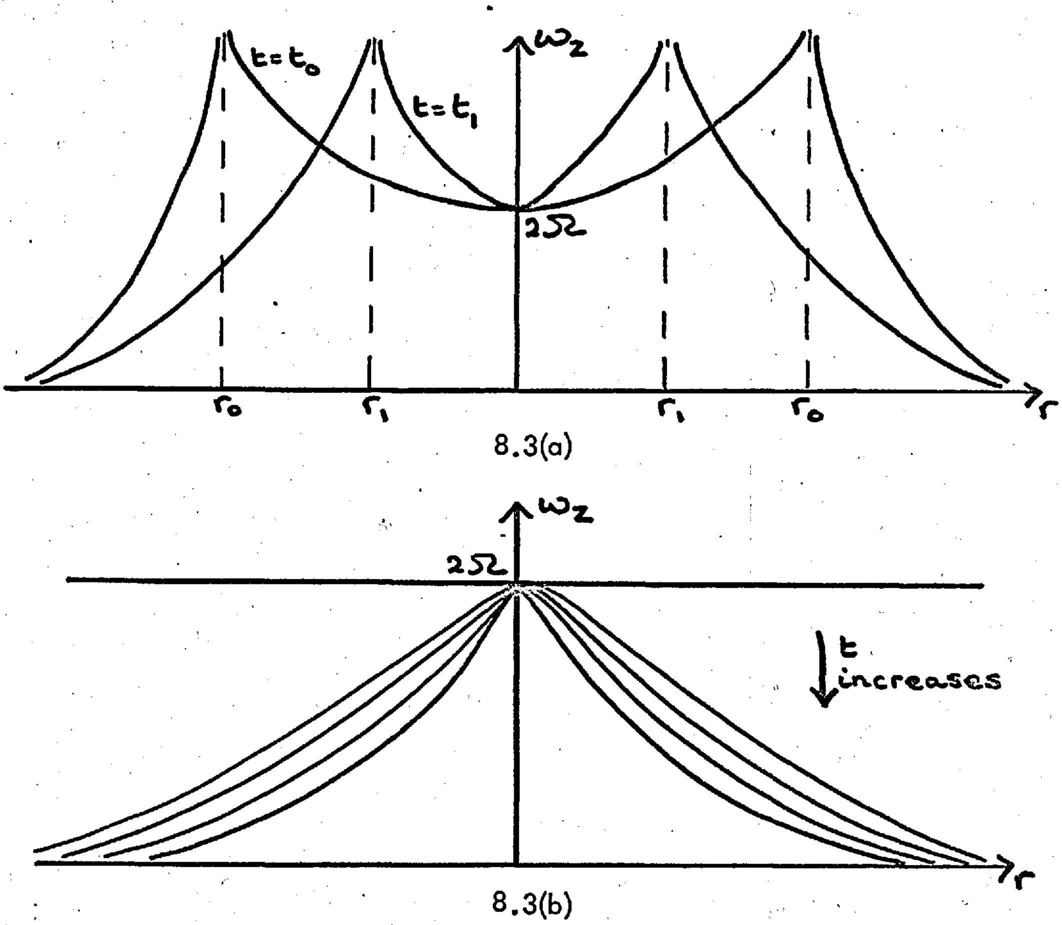


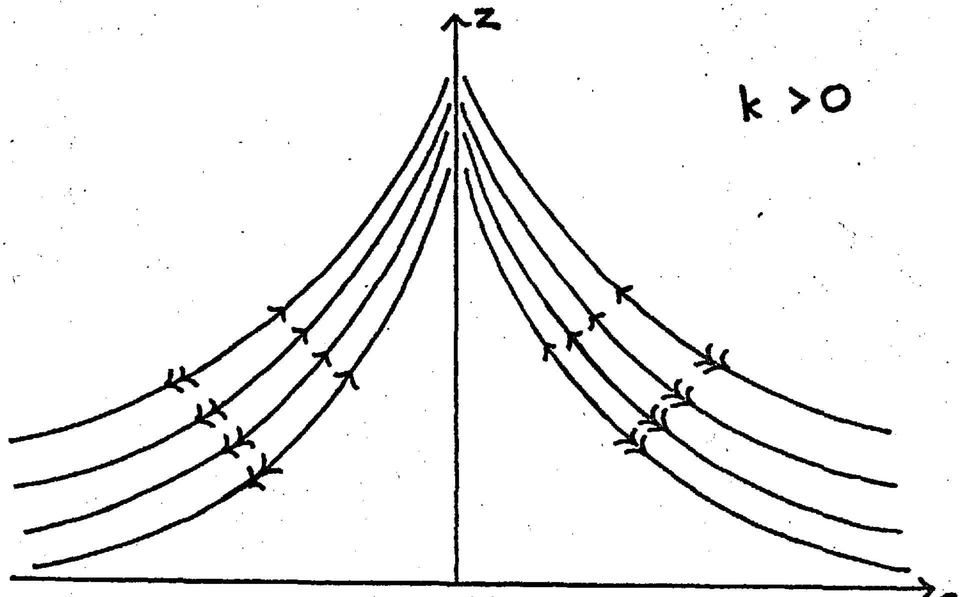
FIG.8.3: Sketch of the vertical vorticity,

$$\omega_z = \frac{32\,\Omega}{(gat\gamma r^2 - 4)^2}$$

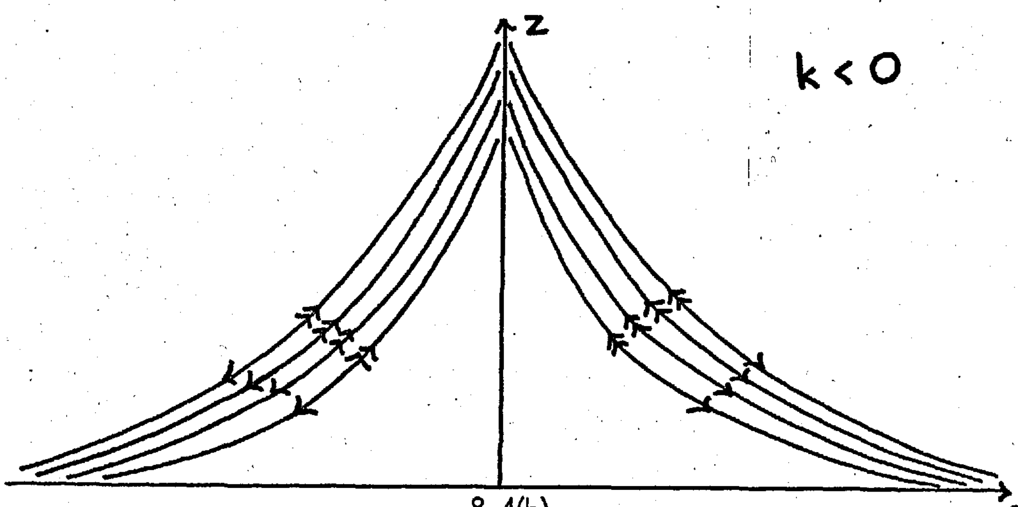
when

(a) $\gamma > 0$, $t_i = 4(ga\gamma r_i^2)^{-1}$, $i = 1,2$, and $t_o < t_1$,

(b) $\gamma < 0$ for fixed t.



8.4(a)



8.4(b)

FIG.8.4: Sketch of the streamlines

$$\psi = -C\left[1-\exp\{-kr^2\}\right] z \; ,$$

when (a) $k > O$, (b) $k < O$, where ↠ $C < O$ and → $C > O$.

is always negative, while K is always positive.

Then, with the value (8.5.6) for f, (8.3.18) becomes

$$\left.\begin{aligned} v_r &= -\frac{C\left[1 - \exp\{-kr^2\}\right]}{r}, \qquad v_z = 2Ckz \exp\{-kr^2\}, \\ T &= \gamma C\left[1 - \exp\{-kr^2\}\right] z, \\ rv_\Theta &= \frac{\Omega}{k} \log\left[\exp\{2kCt\}(\exp\{kr^2\}-1) + 1\right]. \end{aligned}\right\} \quad (8.5.7)$$

We will assume that γ, K and therefore k, C are real numbers in order that the above solution (8.5.7) represents some physical problem.

Then, from (8.4.7), (8.4.8) and (8.5.7), we find that

$$\left.\begin{aligned} \psi &= -C\left[1-\exp\{-kr^2\}\right] z, \qquad \omega_r \equiv 0, \\ \omega_\Theta &= 4Ck^2 zr \exp\{-kr^2\}, \\ \omega_z &= \frac{2\Omega \exp\{kr^2\}}{\exp\{kr^2\} - 1 + \exp\{-2kCt\}}. \end{aligned}\right\} \quad (8.5.8)$$

The streamlines are shown in Fig.8.4.

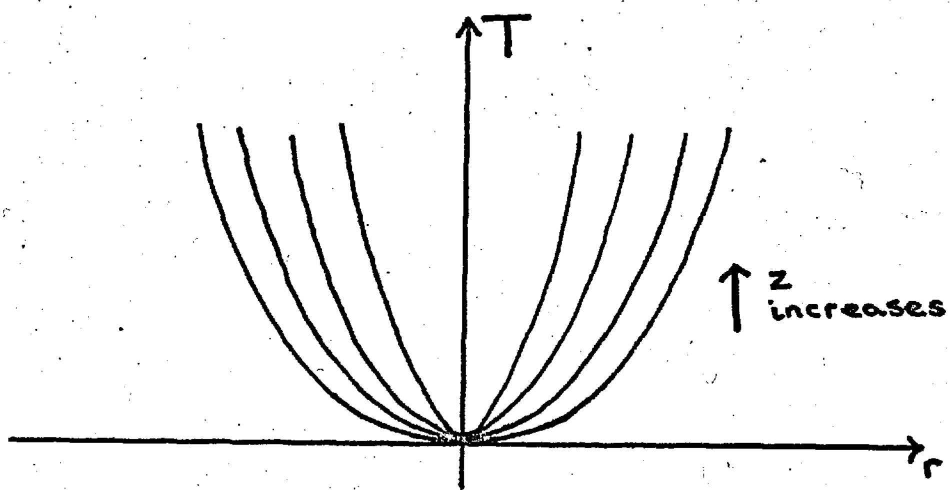
When $k < 0$, we see that the solution, (8.5.7), always satisfies the additional condition, (8.3.3(b)), but never the condition, (8.3.3(a)), and that rv_Θ and ω_z have a singularity at

$$t = -\frac{1}{2kC} \log\left[1 - \exp\{kr^2\}\right],$$

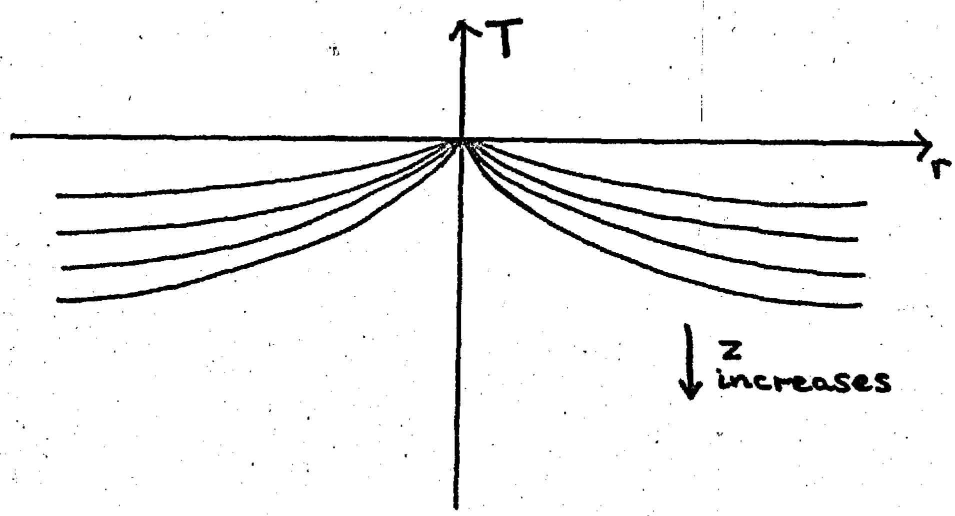
when $C < 0$, which moves towards the z-axis as time increases, but are always regular when $C > 0$. The variation of T and ω_z with r for fixed z,t are shown in Fig.8.5(a) and Fig.8.6(a),(b), respectively.

When $k > 0$, we have the most interesting case since the solution, (8.5.7), satisfies both the additional conditions, (8.3.3). For this case, all the functions (8.5.7) and (8.5.8) are regular and the fluid at a large distance away from the axis of rotation ($r \longrightarrow \infty$) has a non-zero velocity component in the azimuthal direction only. These properties demonstrate that this particular solution is the most useful solution that has been discussed. The variations of T and ω_z with r for fixed z,t are shown in Fig.8.5(b) and Fig.8.6(c),(d), respectively.

Hence, when $k > 0$, the temperature distribution, T, is independent of time and, from Fig.8.5(b), we see that at any given radius the temperature decreases and therefore the density increases as z increases. Therefore there exists lighter fluid in the neighbourhood of the disk with heavier fluid above it. This situation produces convection currents which alter the vertical vorticity present in the fluid. Hence we see, from Fig.8.6, that, when $k > 0$, the vertical vorticity in the neighbourhood of the axis of rotation (the z-axis) is increased or decreased with time while the vertical vorticity at infinity is unchanged from its initial value.

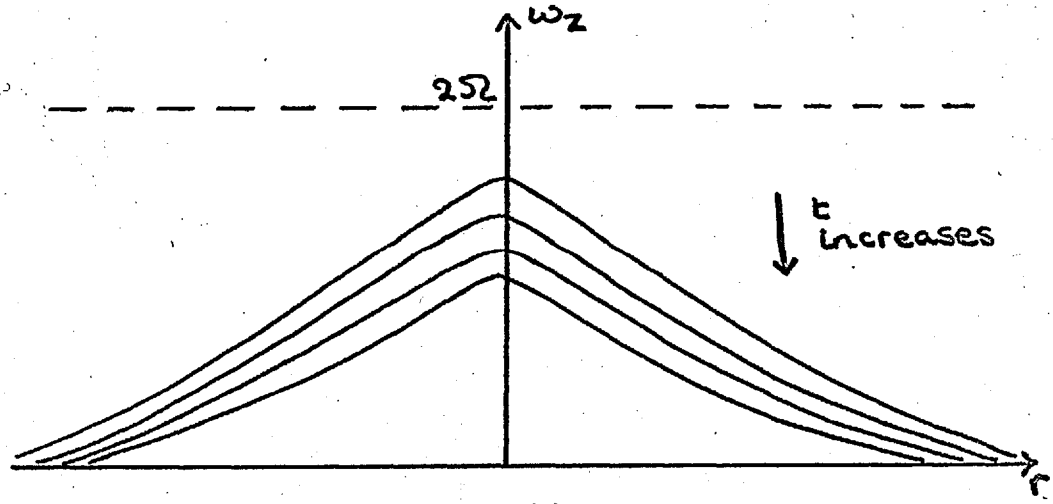


8.5(a)



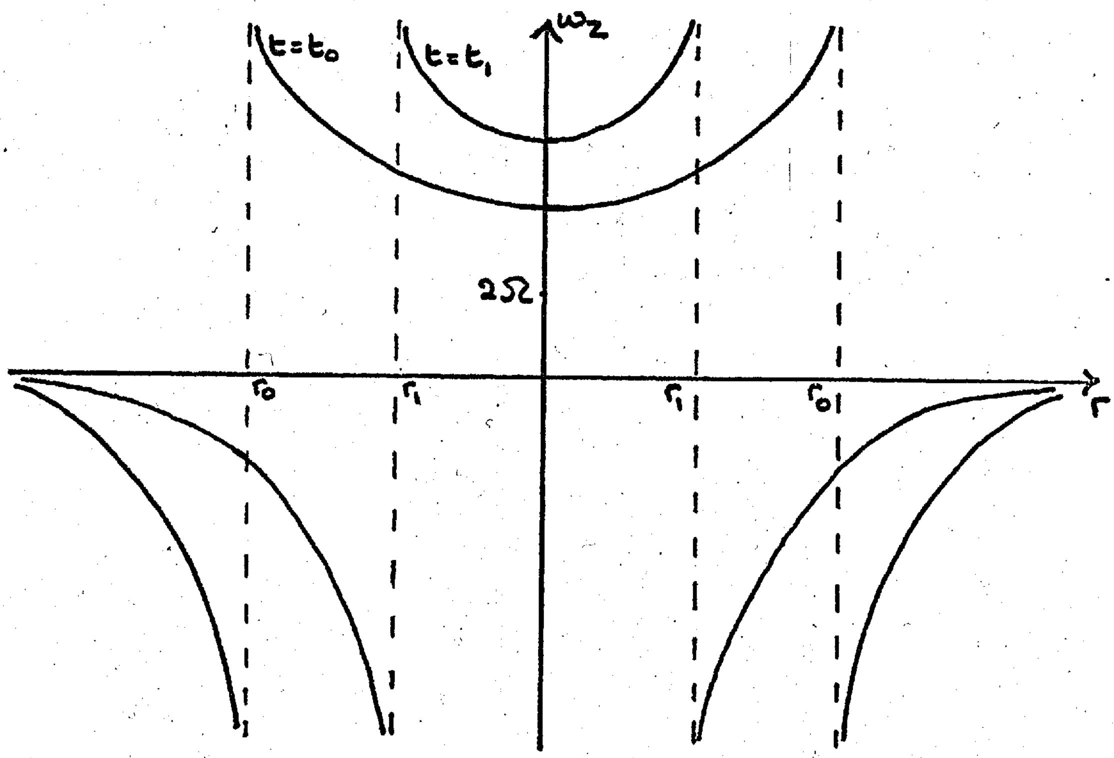
8.5(b)

FIG.8.5: Sketch of the temperature variation $T = \gamma C \left[1 - \exp\{-kr^2\}\right] z$, for fixed z, when (a) $k < 0$, (b) $k > 0$.



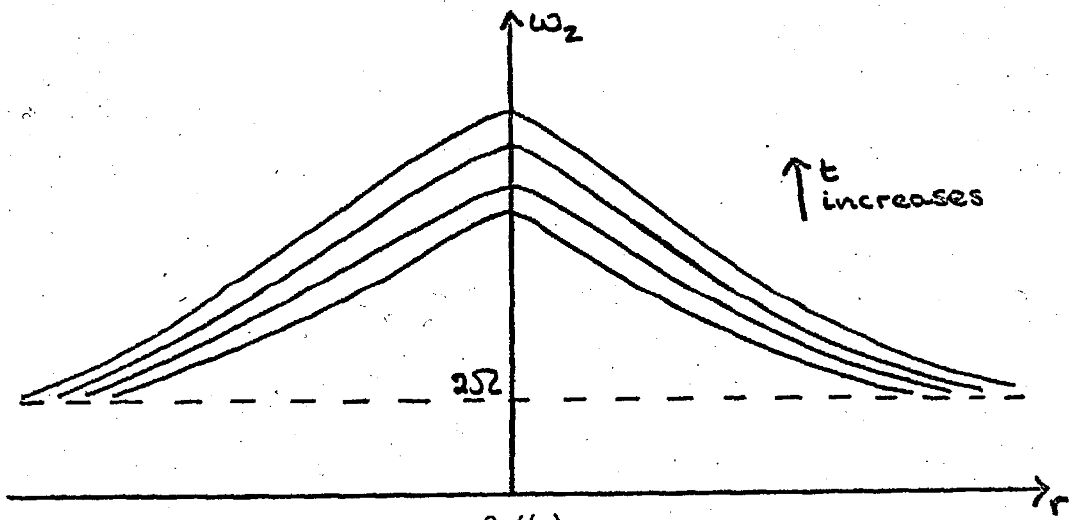
ω_z
2Ω
t increases
r

8.6(a)

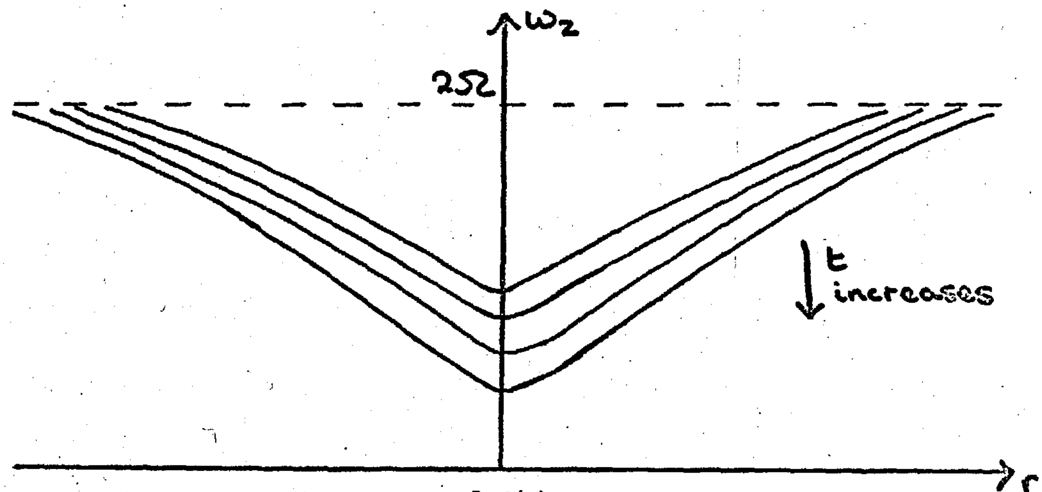


$t = t_0$
$t = t_1$
ω_z
2Ω
r_0
r_1
r_1
r_0
r

8.6(b)



8.6(c)



8.6(d)

FIG.8.6: Sketch of the vertical vorticity,

$$\omega_z = 2\Omega \exp\{kr^2\} \left[\exp\{kr^2\} - 1 + \exp\{-2kCt\}\right]^{-1}$$

when (a) $k < 0, \quad C > 0,$ for fixed t

(b) $k < 0, \quad C < 0,$ and $t_i = -(2kC)^{-1} \log \left[1 - \exp\{kr_i^2\}\right]$

$i = 1, 2,$ where $t_o < t_1$,

(c) $k > 0, \quad C > 0,$ for fixed t,

(d) $k > 0, \quad C < 0,$ for fixed t .

8.6 SOME SOLUTIONS INCLUDING VISCOSITY

We will now derive some solutions of the full Navier-Stokes equations, (8.2.1), when the temperature, and therefore the density, of the fluid remain constant, which satisfy the condition that the vertical velocity vanishes on the boundary, $z = 0$. We will not impose the initial condition that the fluid and the disk were, initially, in solid-body rotation with constant angular velocity, Ω . However, instead of specifying an initial condition, we see what possibilities the solutions allow. These solutions represent possible interior solutions to which boundary layers must be added in order to satisfy the non-slip condition at the boundary.

We assume that

$$v_r = -Ar \; , \qquad v_z = 2Az \; , \tag{8.6.1}$$

where A is a real constant. This choice, (8.6.1), satisfies the continuity equation and also the required boundary condition at $z = 0$ for the interior solution. From (8.6.1) and (8.4.7), we find that the streamlines are

$$\psi = - Ar^2 z \; .$$

When we substitute (8.6.1) into (8.2.1) with $T \equiv 0$, we find that

$$A^2 r - \frac{v_\theta^2}{r} = - \frac{1}{\rho_0} \frac{\partial p}{\partial r} \; , \tag{8.6.2}$$

$$\frac{\partial v_\theta}{\partial t} - Ar\frac{\partial v_\theta}{\partial r} + 2Az\frac{\partial v_\theta}{\partial z} - Av_\theta = \nu\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}\right]v_\theta ,$$

(8.6.3)

$$4A^2 z = -\frac{1}{\rho_o}\frac{\partial p}{\partial z} . \qquad (8.6.4)$$

The equations (8.6.2) and (8.6.4) are identical to the inviscid equations (8.3.5) and (8.3.7) with $f = Ar^2$, $T \equiv 0$, because the viscous stress terms are identically zero. When we eliminate the pressure between the equations (8.6.2) and (8.6.4), we find that

$$\frac{\partial}{\partial z}\left(\frac{v_\theta^2}{r}\right) = 0 .$$

Hence v_θ is independent of z and the equation (8.6.3) reduces to

$$\frac{\partial v_\theta}{\partial t} - A\frac{\partial}{\partial r}(rv_\theta) = \nu\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) . \qquad (8.6.5)$$

When we write

$$rv_\theta = U , \qquad r^2 = x ,$$

the equation (8.6.5) becomes

$$\frac{\partial U}{\partial t} = x\frac{\partial}{\partial x}\left[2AU + 4\nu\frac{\partial U}{\partial x}\right] . \qquad (8.6.6)$$

Before we discuss the general solution of equation (8.6.6), we will consider three special cases.

Case 1: Extension of Burger's solution

We will now discuss the solution of (8.6.6) which was derived by Rott [32, p.408]. We seek a solution of the form

$$U = K\left[1 - \exp\{-AxF(t)/2\nu\}\right] , \qquad (8.6.7)$$

where K is a real constant and F(t) a function of time. When we substitute (8.6.7) into the equation (8.6.6), we find

$$\tfrac{1}{2}F'(t) = A\left[F(t) - F^2(t)\right] , \qquad (8.6.8)$$

where prime represents differentiation with respect to t. The equation, (8.6.8), can be integrated to give

$$(1 + B\exp\{-2At\})^{-1} = F(t) , \qquad (8.6.9)$$

where B is a constant of integration.

Hence we have, from (8.6.9) and (8.6.7), that

$$U = K\left[1 - \exp\{-Ax/2\nu(1 + B\exp\{-2At\})\}\right] ,$$

or alternatively

$$rv_\Theta = K\left[1 - \exp\{-Ar^2/2\nu(1+B\exp\{-2At\})\}\right] . \qquad (8.6.10)$$

Hence we see, from (8.6.10), that v_Θ satisfies the following conditions.

(I) $v_\Theta \longrightarrow 0$ as $r \longrightarrow \infty$, for all t such that $A(1+B\exp\{-2At\}) \geq 0$,

(ii) $v_\Theta = 0$ on $r = 0$ for all t except when $1 + B\exp\{-2At\} = 0$,

(iii) When $1 + B \exp\{-2At\} = 0$, then $v_\theta = K/r$ which represents a potential vortex,

(iv) $rv_\theta = K\left[1 - \exp\{-Ar^2/2\nu(1+B)\}\right]$ at $t = 0$,

(v) near the z-axis, at $t = 0$,

$$rv_\theta = \frac{AKr^2}{2\nu(1+B)} = \Omega r^2 \text{, provided } \Omega = \frac{AK}{2\nu(1+B)} \text{ and}$$

$B \neq -1$,

(vi) when $B = -1$, the singularity discussed in (iii) occurs at $t = 0$.

The vorticity components are

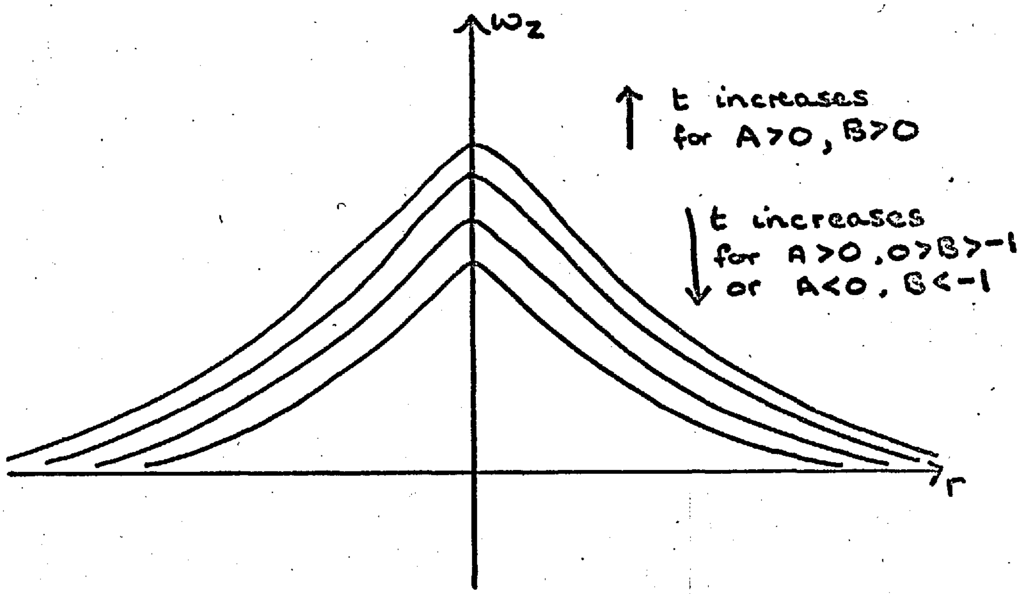
$$\omega_r \equiv \omega_\theta = 0, \quad \omega_z = \frac{KA \exp\{-Ar^2/2\nu(1+B\exp\{-2At\})\}}{\nu(1+B\exp\{-2At\})}. \tag{8.6.11}$$

When $A > 0$, $B \leq -1$ or $A < 0$, $-1 \leq B < 0$, we see, from (8.6.10) and (8.6.11), that ω_z and rv_θ have a singularity at

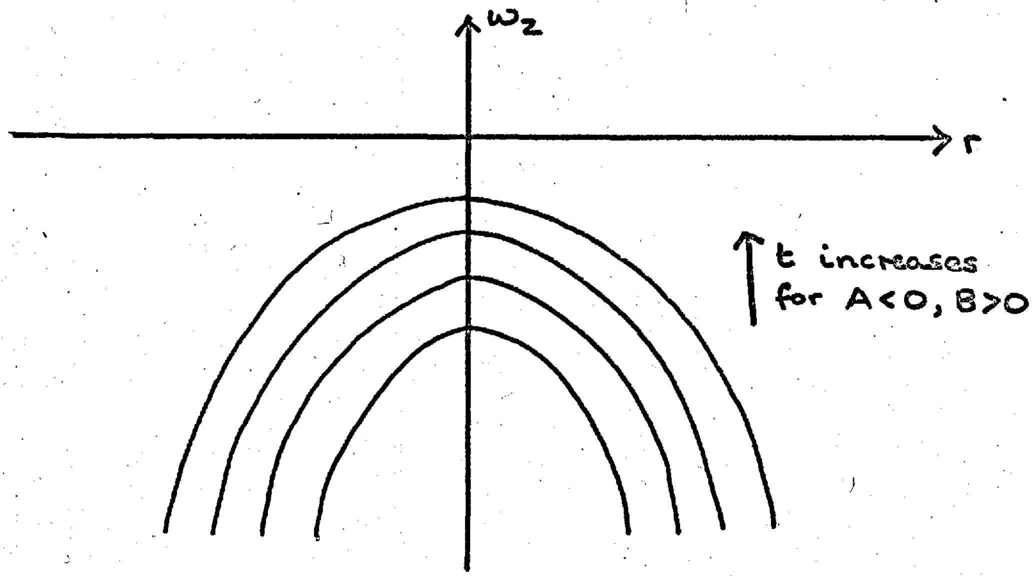
$$t = \frac{1}{2A}\log(-B).$$

For the remaining ranges $A > 0$, $B > -1$, $A < 0$, $B > 0$ and $A < 0$, $B < -1$, we see that ω_z and rv_θ are always regular. The variation of ω_z with r for fixed t is shown in Fig.8.7 for $K > 0$, which we may assume without loss of generality.

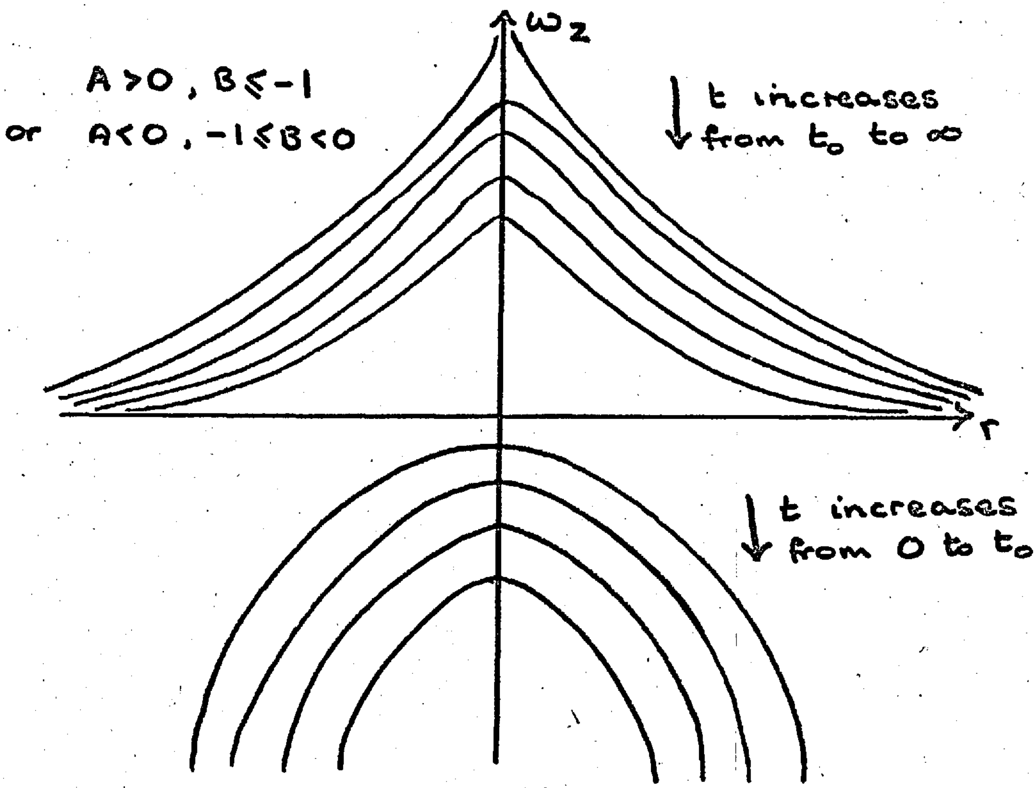
When we allow $t \to -\infty$ in (8.6.10) and (8.6.11) for $A > 0$, we find that $rv_\theta \to 0$ and $\omega_z \to 0$. Therefore when $A > 0$, the


ωz
t increases
for A>0, B>0
t increases
for A>0, 0>B>-1
or A<0, B<-1
r

8.7(a)


ωz
r
t increases
for A<0, B>0

8.7(b)



8.7(c)

FIG.8.7: Sketch of the vertical vorticity,

$$\omega_z = \frac{KA \exp\{-Ar^2/2\nu(1+B \exp\{-2At\})}{\nu(1+B\exp\{-2At\})}$$

for fixed t, when

(a) $A > O$, $B > O$; $A > O$, $O > B > -1$; $A < O$, $B < -1$,

(b) $A < O$, $B < O$,

(c) $A > O$, $B \leq -1$; $A < O$, $-1 \leq B < O$ where $t_o = \frac{1}{2A}\log(-B)$.

vertical vorticity, (8.6.11), is identically zero when $t \longrightarrow -\infty$ and becomes

$$\omega_z = \frac{KA}{\nu} \exp\{-Ar^2/2\nu\},$$

when $t \longrightarrow \infty$. Hence, at large times, a concentration of vertical vorticity is developed in the neighbourhood of the axis of rotation while at large radii the vertical vorticity is unchanged from its initial value ($t \rightarrow -\infty$).

Case 2

When we seek a solution of (8.6.6) of the form

$$U = \Omega\, x \exp\{\lambda t\}, \qquad (8.6.12)$$

we find that $\lambda = 2A$ and that the viscous stress terms are identically zero. This solution, (8.6.12), is identical to (8.4.15). Hence the inviscid solution discussed in section 8.4 when f is given by (8.4.13), is also a solution of the full Navier-Stokes equations because the viscous stress terms are all identically zero.

Case 3

When we seek a separable solution of (8.6.6) of the form

$$U = x \exp\{-\lambda x\} . G(t), \qquad (8.6.13)$$

we find that the function $G(t)$ must satisfy the equation

$$G'(t) = (2A - 8\nu\lambda)G(t) + xG(t).\{-2A\lambda + 4\nu\lambda^2\} . \quad (8.6.14)$$

From (8.6.14), we see that

$$2\nu\lambda = A \quad \text{and} \quad \frac{G'(t)}{G(t)} = -4\nu\lambda . \quad (8.6.15)$$

When we integrate (8.6.15), we find that

$$G(t) = B \exp\{-2At\} , \quad (8.6.16)$$

where B is an arbitrary constant. Hence, from (8.6.13) and (8.6.16), we have that

$$rv_\theta = Br^2 \exp\{-2At - (Ar^2/2\nu)\} . \quad (8.6.17)$$

The vorticity components are

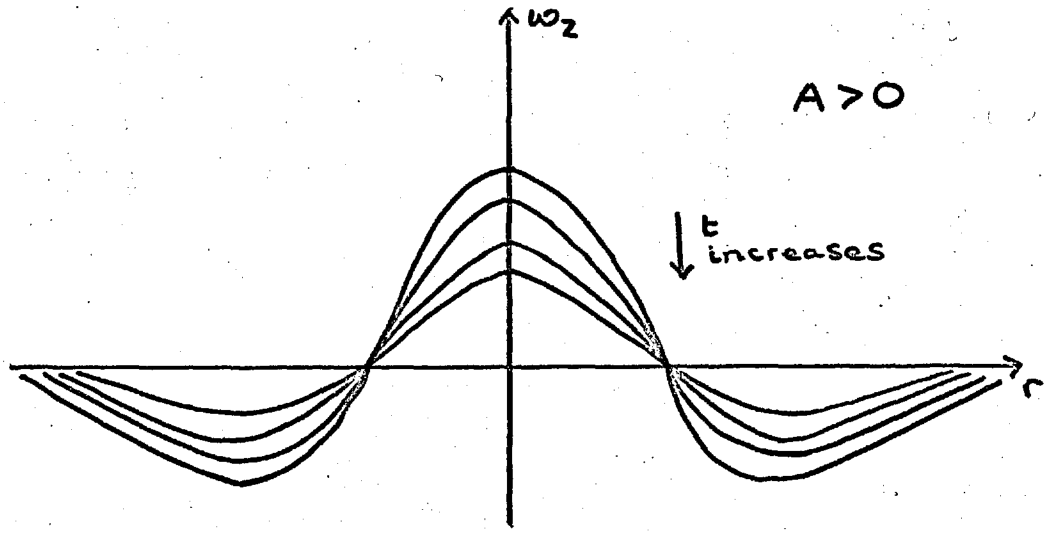
$$\omega_r \equiv \omega_\theta \equiv 0 , \quad \omega_z = 2B\left[1 - \frac{Ar^2}{2\nu}\right] \exp\{-2At - (Ar^2/2\nu)\} . \quad (8.6.18)$$

The solution, (8.6.17), satisfies the following conditions

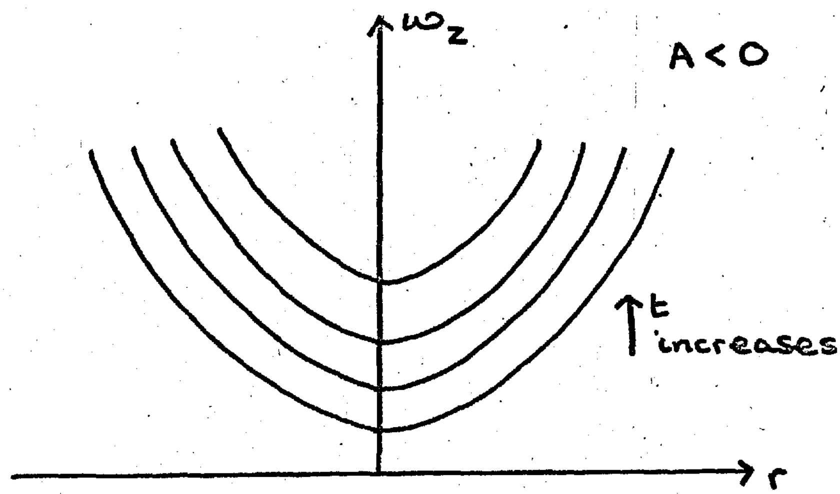
(i) $v_\theta \longrightarrow 0$ as $r \longrightarrow \infty$ when $A > 0$, for all t,

(ii) $v_\theta = 0$ on $r = 0$ for all t,

(iii) $v_\theta = Br \exp\{-Ar^2/2\nu\}$ when $t = 0$, which becomes Ωr near the z-axis provided that $B \equiv \Omega$.



8.8(a)



8.8(b)

FIG.8.8: Sketch of the vertical vorticity,

$$\omega_z = 2B(1-Ar^2/2\nu)\exp\{-A(r^2 + 4\nu t)/2\nu\}$$

when (a) $A > O$, (b) $A < O$, for $B > O$ and for fixed values of t.

The variation of ω_z with r for fixed time is shown in Fig.8.8 for $B > 0$, which we may assume without loss of generality.

THE GENERAL SOLUTION

We now return to a consideration of the general solution of (8.6.6). When we apply the Laplace transform,

$$\overline{U} = \int_0^\infty U\, e^{-pt} dt ,$$

to (8.6.6) and use the fact that

$$\int_0^\infty \frac{\partial U}{\partial t} e^{-pt} dt = U_o + p\overline{U} ,$$

where U_o is the value of $-rv_\theta$ at $t = 0$, we find that

$$p\overline{U} + U_o = x\frac{\partial}{\partial x}\left\{2A\overline{U} + 4\nu\frac{\partial\overline{U}}{\partial x}\right\} . \qquad (8.6.19)$$

If we write $U = uv$, where u,v are some functions of r and t, the equation (8.6.19) becomes

$$U_o = -puv + x\left\{2A(uv' + u'v) + 4\nu(uv'' + 2u'v' + u''v)\right\} , \qquad (8.6.20)$$

where prime represents differentiation with respect to x. In order that the terms involving v' vanish we require that

$$u = \exp\{-Ax/4\nu\} \quad . \tag{8.6.21}$$

When we substitute (8.6.21) into (8.6.20), we find that

$$v'' + v\left[-\frac{p}{4\nu x} - \frac{A^2}{16\nu^2}\right] = \frac{U_o}{4\nu u x} \quad . \tag{8.6.22}$$

If we write $\eta = (Ax/2\nu)$, (8.6.22) becomes

$$v_{\eta\eta} + v\left[-\frac{1}{4} - \frac{p}{2A\eta}\right] = \frac{U_o \exp\{\frac{1}{2}\eta\}}{2A\eta} \quad , \tag{8.6.23}$$

where U_o is, in general, a function of η .

For the special case when U_o is zero, we have

$$v_{\eta\eta} + v\left[-\frac{1}{4} - \frac{p}{2A\eta}\right] = 0 \quad , \tag{8.6.24}$$

which is Whittaker's equation [45, p.337] . This equation, (8.6.24), has solutions

$$v = \exp\{-Ax/4\nu\} \; U_{k,m}(x,p) \; ,$$

where $\qquad k = (-p/2A), \quad m = \frac{1}{2}$.

Hence the general solution of (8.6.19) for the case when $U_o \equiv 0$ is

$$\overline{U} = \exp\{-Ax/2\nu\} \; U_{k,m}(x,p) \; . \tag{8.6.25}$$

These solutions are, in general, very complicated and we will not pursue the discussion of this general solution.

We see, from (8.6.12) and (8.6.17) that for the special cases 2 and 3, U_o is $-\Omega x$ and $-Bx \exp\{-Ax/2\nu\}$ respectively, which are non-zero. We will now show that these special cases can be derived from the solutions of the equation (8.6.23).

Firstly, we seek a solution of (8.6.23), with $U_o = -\Omega x$, of the form $C\eta \exp\{\frac{1}{2}\eta\}$, where C is a constant. Then we find that

$$C = \frac{2\nu\Omega}{A\{p - 2A\}},$$

and hence

$$\overline{U} = \frac{2\nu\Omega\eta}{A\{p - 2A\}},$$

which yields (8.6.12) when the inversion integral is evaluated.

Secondly, we seek a solution of (8.6.23), with $\overline{U}_o = -Bx \exp\{-Ax/2\nu\}$, of the form $C\eta \exp\{-\frac{1}{2}\eta\}$, where C is a constant. Then we find that

$$C = \frac{2B\nu}{A\{p+2A\}},$$

and hence

$$\overline{U} = \frac{2B\nu\eta \exp\{-\eta\}}{A\{p + 2A\}},$$

which yields (8.6.17) when the inversion integral is evaluated. Hence we have shown that the special cases 2 and 3 can easily be derived from the general solution.

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