ON THE SETS WHERE A SUBHARMONIC FUNCTION

IS LARGE

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INTRODUCTION

It is a famous theorem of Iversen that if f(z) be a non-constant meromorphic transcendental function in the plane which assumes a value 'a' a finite number of times there, then there exists a path Γ tending to ∞, such that $f(z) \longrightarrow a$, as $z \longrightarrow \infty$ along Γ. The value a is called asymptotic value and Γ is called an asymptotic path. If we take $a = \infty$, we obtain the theorem that every integral function tends to infinity along some path. We investigate whether an analogue of this holds for a general subharmonic function u(z) in the plane. We answer in the affirmative by proving Theorem 1.2. The entire Chapter I is devoted to our proof of this theorem.

In our investigations we were inevitably led to consider the sets $\left\{ z \middle| u(z) > K \right\}$ and $\left\{ z \middle| u(z) \geq K \right\}$. It appeared that there may exist components of the set $\left\{ z \middle| u(z) \geq K \right\}$ on which $u(z) \equiv K$. We show that this is possible by constructing examples in Chapter II. Also such components may be non-countably infinite in number. The number of components is related to the order of the function u(z) in the case of integral functions. This is also discussed in Chapter II. In Chapter III we consider how quickly u(z) must tend to $+\infty$ along an asymptotic path Γ.

Chapter IV is devoted entirely to subharmonic functions in space. There is a marked difference in the behaviour of subharmonic functions in the plane and in space. A function non-constant and subharmonic in the plane cannot be bounded above. However, there exist subharmonic functions in space which are bounded above. An asymptotic path has to lie finally in a set in which a subharmonic function is large. The exact analogues of growth theorems like the Wiman-Heins theorem and the Milloux-Schmidt inequality are not valid in space. We first prove Theorem 4.2 which may be considered as an analogue in space of the Milloux-Schmidt inequality. We also prove Theorem 4.6 which is a space analogue of a theorem of Hayman on the infimum of u(P) on radial segments going outward from the origin. With the help of these theorems we are able to show the existence of an asymptotic path when (i) u(P) is a continuous subharmonic function, (ii) u(P) is a general subharmonic function which is bounded above in space. If u(P) is a general subharmonic which is not bounded in space, we are able to show the existence of a continuum on which $u(P) \longrightarrow +\infty$ as P tends to infinity on the continuum. The problem of constructing a path in this case still remains open.

Also in space we have investigated the relationship between the number of components of the set $\{P \mid u(P) > C\}$ and the order of the function u(P).

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CONTENTS

CHAPTER I

AN EXTENSION OF IVERSEN'S THEOREM

1.1 It was shown by Iversen (1) that if a non-constant meromorphic function $f(z)$ has a Picard value a, then there exists a path Γ tending to ∞ in the z-plane such that,

$$f(z) \longrightarrow a, \text{ as } z \longrightarrow \infty \text{ along } \Gamma .$$

If we take $a = \infty$, we obtain that for every non-constant integral function, there exists a path Γ tending to ∞, such that

$$|f(z)| \longrightarrow \infty, \text{ as } z \longrightarrow \infty \text{ along } \Gamma .$$

It is natural to ask whether the analogue, that a non-constant subharmonic function $u(z)$ in the plane tends to $+\infty$ along a path Γ, still holds. We answer this in an affirmative by proving Theorem 1.2.

The arguments of the proof for the modulus of an integral function can be carried over to the case of continuous subharmonic functions. The proof is given below for the sake of completeness. Also the steps in the proof are clear in this simple case and become somewhat more complex in the case of a general subharmonic function.

Theorem 1.1. If $u(z)$ is a continuous non-constant subharmonic function in the z-plane, then there exists a path Γ tending to ∞ on which $u(z)$ tends to $+\infty$.

Since $u(z)$ is continuous, the set $G(K)$ of points for which $u(z)$ is greater than K is open. Consequently $G(K)$ consists of a sequence of domains (say) $G^{(\nu)}(K)$. $(\nu = 1,2,\ldots .)$ We now prove that (a) Every component $G^{(\nu)}(K)$ must extend to infinity.

(b) $u(z)$ is unbounded in each $G^{(\nu)}(K)$.

We note that on account of continuity, $u(z) = K$ on the boundary of $G^{(\nu)}(K)$. Therefore if $G^{(\nu)}(K)$ did not extend to infinity, we would have

$$u(z) \leq K, \text{ for } z \in G^{(\nu)}(K) \text{ by the Maximum-principle.}$$

This contradicts the definition of $G^{(\nu)}(K)$ and hence (a) follows.

To prove (b), we make use of the following theorem which is the subharmonic form of the Phragmen-Lindelöf principle. A proof of the theorem in the form below is given by Heins (Heins (1), p.76).

'Given u subharmonic in a domain D of the closed plane. Suppose that E is a countable subset of the frontier of D. Suppose that $\operatorname{Sup} u < +\infty$ and that there exists a real number M such that

$$\limsup_{z \to \xi} u(z) \leq M, \qquad \xi \in \mathcal{F}D - E.$$

Suppose that there exists at least one non-exceptional boundary point, then $u(z) \leq M$ in D.'

Now suppose that (b) is false and $u(z)$ is bounded in some $G^{(\nu)}(K)$. Also on all the finite boundary points of $G^{(\nu)}(K)$, we have $u(z) = K$. Thus the hypotheses of the above theorem are satisfied and we have
$u(z) \leq K$ in $G^{(\nu)}(K)$.
This contradicts the definition of $G^{(\nu)}(K)$, hence the assumption that $u(z)$ is bounded in $G^{(\nu)}(K)$ is false.

Let G_K be a domain in which $u(z) > K$. Since $u(z)$ is unbounded in G_1, we can choose z_1 and z_2 in G_1 such that: $u(z_1) > 1$, and $u(z_2) > 2$. We join z_1 to z_2 by a continuous curve (say) γ_1 lying in G_1. This is possible since G_K, being a domain, is arcwise connected. Also z_2 lies in G_2. We choose z_3 in G_2 such that $u(z_3) > 3$. Join z_2 to z_3 by a path γ_2 lying in G_2.
Again z_3 lies in G_3 and with a step by step argument we can find a path $\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \ldots\ldots$ such that

$$u(z) \longrightarrow +\infty, \quad \text{as } z \longrightarrow \infty \text{ on } \Gamma .$$

Thus in each domain $G^{(\nu)}(K)$, there exists a path Γ tending to ∞, such that

$$u(z) \longrightarrow +\infty, \quad \text{as } z \longrightarrow \infty \text{ on } \Gamma .$$

<u>Theorem 1.2.</u> <u>The conclusion of theorem 1.1 still holds for a general subharmonic function.</u>

1.2 We observe that in the proof of Theorem 1.1, the set $\{z|u(z) > K\}$ plays an essential role. This set may be of considerable complexity in the case of a general subharmonic function. Therefore the assertions (a) and (b) and the subsequent construction of the path do not follow so easily.

We prove first the analogue of the assertion (a), that any component of the set $\{z|u(z) > K\}$ or $\{z|u(z) \geq K\}$ goes to the boundary i.e. in the case of the z-plane extends to infinity.

<u>Lemma 1.1.</u> <u>If $u(z)$ is subharmonic in a disc $|z| \leq r$, all the components of the sets $\{z|u(z) > K\}$ or $\{z|u(z) \geq K\}$ go to the boundary $|z| = r$.</u>

Let $f_n(z)$ be a decreasing sequence of continuous subharmonic functions with limit $u(z)$. By considering $f_n + \frac{1}{n}$ instead of f_n, we can assume $f_n(z)$ to be a strictly decreasing sequence of continuous subharmonic functions.

Suppose that the set $\{z|u(z) \geq K\}$ is not void.

Let z_o be a point inside $|z| < r$, such that

$$u(z_o) \geq K .$$

Then $f_n(z_o) > K$ for each n.

Let $G_n(K)$ be the component of the set $\{z|f_n(z) > K\}$ which contains z_o. By the assertion (a) in the proof of Theorem 1.1, $G_n(K)$ goes to the boundary $|z| = r$ for each n.

Hence $\overline{G_n(K)}$ i.e. the closure of $G_n(K)$ also goes to the boundary $|z| = r$.

And $f_n(z) \geq K$ for $z \in \overline{G_n(K)}$ for each n.

Set $B_n(K) = \overline{G_n(K)} \cap \{|z| = r\}$.

Then $B_n(K)$ is not void for any n.

Also $B_n(K)$ is a compact set and it contracts as n increases.

Since the intersection of a decreasing sequence of compact sets is a compact set, we claim that

$$B(K) = \bigcap_{n=1}^{\infty} B_n(K) \text{ is not void .}$$

Also B(K) is a part of $O(K) = \bigcap_{n=1}^{\infty} \overline{G_n(K)}$.

Now O(K) is an intersection of a decreasing sequence of continua. And therefore O(K) is a point or a continuum. Since O(K) contains z_o, inside $|z| < r$ and B(K) on $|z| = r$, obviously O(K) is a continuum containing z_o and extending to the boundary $|z| = r$. And on O(K), $u(z) \geq K$.

Since a subharmonic function is upper-semi-continuous, the set $\{z | u(z) \geq K\}$ is closed.

The components of a bounded closed set are points or continua. Thus by what we have shown above, through every point inside $|z| < r$, on which $u(z) \geq K$, there passes a component of $\{z \mid u(z) \geq K\}$ stretching to the boundary $|z| = r$.

Now we deduce a similar result about the components of the set $\{z \mid u(z) > K\}$.

Let z_o be a point such that $u(z_o) > K$.

Since $u(z_o) > K$, $u(z_o) = K + \delta$, where $\delta > o$.

Hence there is a continuum containing z_o going to the boundary on which $u(z) \geq K + \delta$.

This continuum obviously lies in the component of $u(z) > K$ containing z_o, and hence this component goes to the boundary $|z| = r$.

Since z_o is any point with the property $u(z) > K$, the same result holds for all z with the property $u(z) > K$. This completes the proof of Lemma 1.1.

It follows immediately from Lemma 1.1 that for functions subharmonic in the whole plane, any component of the sets $\{z \mid u(z) > K\}$ or $\{z \mid u(z) \geq K\}$ extends to infinity.

1.3. In the next few sections we prove the analogue of the assertion (b) for functions subharmonic in the whole plane.

We make use of the following lemma, which is the subharmonic form of the Milloux-Schmidt inequality (e.g. Nevanlinna (1), p.94). A proof of the lemma in the form below is given by Brelot (Brelot (1)).

Lemma 1.2. Suppose that $\omega(z)$ is subharmonic in $|z| < 1$ and satisfies $\omega(z) < 1$ there. Suppose also that

$$\inf_{|z| = r} \omega(z) \leq o \quad , \qquad o < r < 1 \tag{3.1}$$

then

$$\sup_{|z| = r} \omega(z) \leq \frac{4}{\pi} \tan^{-1} \sqrt{r} \quad . \quad o < r < 1 \tag{3.2}$$

We deduce from Lemma 1.2 that if $u(z_o) > K$, there exists a circle centre z_o, such that $u(z) \geq K$ on that circle. We shall call this assertion (c).

The assertion (c) is obvious if z_o is an interior point of a component of the set $\left\{ z \middle| u(z) \geq K \right\}$.

Therefore we have only to consider the case when z_o is a boundary point of C(K), the component of $\left\{ z \middle| u(z) \geq K \right\}$.

Then either for a small δ, on every circle $c(z_o, r)$ with centre z_o, radius r, such that $o < r < \delta$, we have

$$\inf_{z \in c(z_o, r)} u(z) < K \ ,$$

or for some δ, $\inf_{z \in c(z_o, \delta)} u(z) \geq K.$

If we can show that the first alternative does not hold, then the assertion (c) is proved. We show below that the first alternative implies the hypotheses of the Lemma 1.2, and consequently forces a restriction on sup u(z) on these circles $c(z_o, r)$, and this together with $u(z_o) > K$, gives a contradiction.

Let $u(z_o) = K + \epsilon$, where $o < \epsilon \leq \frac{1}{2}$.

For if not consider $v(z) = K + \dfrac{u(z) - K}{2\epsilon}$.

We assume without loss of generality that $z_o = O$.

Define $g(z) = u(Rz) - u(o) + \epsilon$,

where R is the radius of a circle with centre the origin such that

$u(z) < u(o) + 1 - \epsilon$ inside this circle.

(Since $\epsilon \leq \frac{1}{2}$, by the upper-semi-continuity of u(z), such a circle exists.)

Then $g(z) < 1$ in $|z| < 1$,

and also $\inf\limits_{|z| = r} g(z) \leq o$, $o < r < 1$ from the first alternative.

Thus the hypotheses of the Lemma 1.2 are satisfied and we have

$$\sup_{|z| = r} g(z) \leq \frac{4}{\pi} \tan^{-1}\sqrt{r} , \qquad o < r < 1 .$$

This gives for the original function,

$$\sup_{|z| = rR} u(z) \leq u(o) - \epsilon + \frac{4}{\pi} \tan^{-1}\sqrt{rR} , \qquad o < r < 1 .$$

And this gives a contradiction by the Maximum-principle if r is sufficiently small.

By similar arguments it also follows that if $u(z_o) > K$, then there exists a circle $c(z_o, \delta)$ such that $u(z) > K$ on $c(z_o, \delta)$.

We now prove a lemma which will help us to prove that u(z) is unbounded in each component of $\left\{ z \mid u(z) > K \right\}$.

Lemma 1.3. Suppose that u(z) is subharmonic in the plane, C(K) is a component of the set $\left\{ z \mid u(z) \geq K \right\}$, and define

$$v(z) = \begin{cases} u(z) & \text{for } z \in C(K) , \\ K & \text{outside } C(K) . \end{cases}$$

Then v(z) is subharmonic in the plane.

Since the complement of C(K) is an open set and v(z) is constant in it, it is subharmonic there. Also v(z) is equal to u(z) in C(K) and so is subharmonic at the interior points of C(K).

Thus we need to consider only the boundary points of C(K).

Let ζ be such a boundary point. We show first that v(z) is upper semi-continuous at $z = \zeta$.

Since u(z) is upper-semi-continuous at $z = \zeta$, these exists a neighbourhood $\delta(\epsilon)$ such that

$$u(z) - u(\zeta) < \epsilon , \qquad \text{for } |z - \zeta| < \delta(\epsilon) . \qquad (3.3)$$

We now show that (3.3) is also satisfied in the same neighbourhood by v instead of u.

Since $v(z) = u(z)$ for $z \in C(K)$, we note that (3.3) is obviously satisfied for the points z in the neighbourhood $|z - \zeta| < \delta(\epsilon)$ belonging to C(K).

For $z \notin C(K)$, $v(z) = K$, and $v(\zeta) \geq K$.

Therefore, again (3.3) is satisfied by v for the remaining points of the neighbourhood $\delta(\epsilon)$.

Thus v(z) is upper-semi-continuous at the boundary points of C(K).

We show next that the mean value inequality is satisfied by v for such points ζ.

If ζ is a boundary point and $v(\zeta) = K$, then the mean value inequality is obviously satisfied as $v(z) \geq K$ in the entire plane.

However, if $v(\zeta) > K$, then $u(\zeta) > K$ and we have by the assertion (c) which was proved in this section that there exists a circle $c(\zeta, \delta_1)$, centre ζ, radius δ_1 such that $u(z) \geq K$ on $c(\zeta, \delta_1)$.
Then the circle $c(\zeta, \delta_1)$ meets the component C(K) because its centre ζ belongs to C(K) and C(K) is connected and extends to infinity.
Thus C(K) contains this circle and so no other component of the set $\{z | u(z) \geq K\}$ meets the circle $c(\zeta, \delta_1)$.

Now if we choose $\delta \leq \delta_1$, we observe that in a neighbourhood centre ζ , radius δ, (i) the set of z at which u(z) is greater than or equal to K, is the same set for which v(z) = u(z); (ii) the set of z for which v(z) is different from u(z) has the property u(z) < K and v(z) = K.

Since the mean value inequality is satisfied by u(z) and by (i) and (ii) the mean value of v(z) is not less than that for u(z), we see that the mean value inequality is also satisfied by v(z).

This shows that v(z) is subharmonic at the boundary points of C(K) and thus completes the proof of Lemma 1.3.

We note that by the arguments similar to those of Lemma 1.3 and the fact that if $u(z_o) > K$, there exists a circle $c(z_o, \delta)$ such that $u(z) > K$ on $c(z_o, \delta)$, we can prove that the modified function $v_1(z)$ is subharmonic in the plane when it is defined as follows:

$$v_1(z) = u(z) \quad \text{for } z \in G(K), \text{ a component of } \{z \mid u(z) > K\},$$
$$= K \quad \text{for } z \text{ outside } G(K).$$

1.4 Lemmas 1.2 and 1.3 enable us to prove the analogue of the assertion (b) that u(z) is unbounded in each component of $\{z \mid u(z) > K\}$ and in those components of $\{z \mid u(z) \geq K\}$, in which u(z) > K somewhere.

Let C(K) be a component of the set $\{z \mid u(z) \geq K\}$.

First suppose that there is only one component C(K).

By considering $u(z) - K$ instead of $u(z)$ we may suppose $K = 0$. Then either $C(K)$ contains the whole plane so that $u(z)$ is constant or unbounded in $C(K)$.

Otherwise $u(z)$ is non-constant and hence unbounded in the plane and so unbounded in $C(K)$.

The unboundedness of such a non-constant $u(z)$ in the plane follows from the fact that the maximum modulus $B(r) = \sup\limits_{|z| = r} u(z)$ is positive for some $|z| = r_o$. Also $B(r)$ is a convex increasing function of $\log r$ (Rado (1) p.18).

Hence $B(r) \longrightarrow \infty$, as $r \longrightarrow \infty$.

Since $u(z)$ attains the value $B(r)$ on $|z| = r$ for some z, $u(z)$ is unbounded as $|z| \longrightarrow \infty$.

Next suppose that the set where $u(z) \geq K$ has at least two components $C_1(K)$ and $C_2(K)$.

By Lemma 1.1 if $|z| = r_o$ meets both $C_1(K)$ and $C_2(K)$, then $|z| = r$ meets both $C_1(K)$ and $C_2(K)$ for all $r \geq r_o$.

Let $A(r) = \inf\limits_{|z| = r} u(z)$.

So we have $A(r) \leq K$ for $r \geq r_o$.

If we replace $u(z)$ in $|z| \leq r_o$ by its Poisson Integral, we have $u(z)$

harmonic and bounded in $|z| \leq r_o$, and u(z) is unchanged for $|z| \geq r_o$.

To show the unboundedness of u(z) in $C_1(K)$ we first form v(z) as follows

$$\begin{aligned} v(z) &= u(z) \qquad \text{for } z \in C_1(K) , \\ &= K \qquad \text{outside } C_1(K) . \end{aligned}$$

Then by Lemma 1.3, v(z) is subharmonic in the plane.

$$\text{Set} \quad \omega(z) = \frac{v(Rz) - K}{B(R) - K} \qquad \text{for } R > r_o .$$

Then we have $\omega(z) < 1$ for $|z| < 1$, and

$$\inf_{|z| = \rho} \omega(z) \leq 0 \quad \text{for } 0 < \rho < 1 .$$

Hence $\omega(z)$ satisfies the hypotheses of the Lemma 1.2 and we have from (3.2),

$$\sup_{|z| = \rho} \omega(z) = \sup \frac{v(Rz) - K}{B(R) - K} \leq \frac{4}{\pi} \tan^{-1} \sqrt{\rho} .$$

Put $R\rho = r$.

Then for $0 < r < R$,

$$\begin{aligned} B(r) - K &\leq (B(R) - K) \frac{4}{\pi} \tan^{-1} \sqrt{\frac{r}{R}} . \\ &\leq \frac{4}{\pi} (B(R) - K) \sqrt{\frac{r}{R}} . \end{aligned}$$

Therefore $$B(R) - K \geq \frac{\pi}{4} (B(r) - K) \sqrt{\frac{R}{r}} . \tag{4.1}$$

If $u(z) > K$ somewhere on the component, then the right-hand side of (4.1) is positive, and as $R \longrightarrow \infty$, we see that $B(R)$ also tends to infinity.

Since $v(z)$ attains $B(R)$ somewhere on $|z| = R$, $u(z)$ attains the same values on $|z| = R$ and $C_1(K)$, and hence $u(z)$ is unbounded in $C_1(K)$.

This completes the proof that $u(z)$ is unbounded in each component $C(K)$ in which $u(z)$ is greater than K [somewhere]. It can be shown by the same arguments that $u(z)$ is unbounded in each component of the set $\{z | u(z) > K\}$.

1.5 We now show that if there are a finite number of the components of the set $\{z | u(z) \geq K\}$, then $u(z) > K$ somewhere on each component. Consequently by Section 1.4, $u(z)$ is unbounded in each such component.

Suppose that the set $\{z | u(z) \geq K\}$ has a finite number of components. Then in the finite plane (say $|z| \leq R$), each component being a closed set is at a positive distance δ from all the other components.

Suppose that z_o is a boundary point of a component $C^{(\nu)}(K)$ and that $u(z_o) = K$. For if $u(z_o) > K$, there is nothing to prove. Then it follows contrapositively from the Maximum-principle that if $u(z_o) = K$, then either there exist points in $|z - z_o| < \frac{\delta}{2}$ such that $u(z) > K$ or $u(z) \equiv K$ inside $|z - z_o| < \frac{\delta}{2}$.

Since we suppose z_o to be a boundary point of $C^{(\nu)}(K)$, we cannot have $u(z) \equiv K$ inside $|z - z_o| < \frac{\delta}{2}$. Also since $C^{(\nu)}(K)$ and z_o are at a distance at least δ from the other components of $\{z \mid u(z) \geq K\}$, the only points in $|z - z_0| < \frac{\delta}{2}$ on which $u(z) > K$, lie in $C^{(\nu)}(K)$. This is true for each of the components $C^{(\nu)}(K)$, if they are finite in number.

Thus $u(z) > K$ somewhere on each $C^{(\nu)}(K)$.

On the other hand if there are an infinite number of components of the set $\{z \mid u(z) \geq K\}$, there may be components on which $u(z) \equiv K$. This will be illustrated by an example later on. It will be shown further that there can be non-countably many such components. However, in any neighbourhood of such a component, there are infinitely many components of the set $\{z \mid u(z) \geq K\}$ on which $u(z) > K$ somewhere, and on which $u(z)$ is consequently unbounded.

1.5 We have shown so far that for every non-constant subharmonic function,

(i) the components of the sets $\{z \mid u(z) > K\}$, and $\{z \mid u(z) \geq K\}$ extend to infinity.

(ii) $u(z)$ is unbounded in each component of $\{z \mid u(z) > K\}$ and in those components of $\{z \mid u(z) \geq K\}$ in which $u(z)$ is not identically equal to K.

These are analogues of the assertions (a) and (b) of Theorem 1.1. The existence of an asymptotic path cannot follow easily as the general subharmonic functions are not subject to any smoothness conditions.
However by the arguments of Theorem 1.1, we prove:

Lemma 1.4. There exists a continuum Γ going to ∞ through each point z_o such that $u(z) \longrightarrow +\infty$ as $z \longrightarrow \infty$ on Γ.

Let C(1) be a component of the set $\{z \mid u(z) \geq 1\}$ with strict inequality somewhere. We have shown that such a C(1) exists, and extends to infinity, and u(z) is unbounded on each such C(1).
Suppose $z_1 \in C(1)$ and $u(z_1) > 1$. Find $z_2 \in C(1)$, such that $u(z_2) > 2$. There exists a continuum γ_1 joining z_1 to z_2 and lying in C(1). Now z_2 lies in C(2) which is a sub-continuum of C(1).

In C(2), find a point z_3 such $u(z_3) > 3$, and join z_2 to z_3 by a continuum γ_2 lying in C(2). Continuing in this way after n-steps we have $u(z_{n-1}) > n-1$, $u(z_n) > n$ and $u(z) \geq n-1$ for z on γ_{n-1} joining z_{n-1} and z_n.

Thus there exists a continuum $\Gamma = \gamma_1 + \gamma_2 + \ldots\ldots$ going to infinity through each point z_o such that

$$u(z) \longrightarrow +\infty, \quad \text{as } z \longrightarrow \infty \text{ on } \Gamma .$$

1.7 We now complete the proof of the Theorem 1.2, by proving:

Lemma 1.5. Suppose that $u(z)$ is subharmonic in a neighbourhood N of the continuum γ_K and that $u(z) \geq K$ for $z \in \gamma_K$. Let z_1 and z_2 be two points on γ_K. Then there exists a polygonal path joining z_1 to z_2 in N such that $u(z) > K - 1$ on this path.

In order to prove this, we shall need a theorem of Hayman (1). It is necessary to introduce his notation.

$$u^+(z) = \max \{u(z), 0\} \quad , \quad u^-(z) = -\min \{u(z), 0\} \quad .$$

$$T(r,u) = \frac{1}{2\pi} \int_0^{2\pi} u^+(re^{i\Theta})d\Theta \; .$$

$$u_1(re^{i\Theta}) = \sup_{0 \leq t \leq r} u^-(te^{i\Theta}) \; .$$

$$\psi(t) = \frac{(1-t)\log(1 + \frac{2\pi\sqrt{t}}{1-t})}{\pi\sqrt{t}\,\log\frac{1}{t}} \quad .$$

We now state the theorem of Hayman (1, Th.4, p.193), which we shall make use of.

Lemma 1.6. If $u(z)$ is subharmonic in $|z| \leq R$, then for $0 < r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} u_1(re^{i\Theta})d\Theta \le \left[1 + \psi\left(\frac{r}{R}\right)\right]\left\{T(R,u) - u(o)\right\} .$$

It follows from this (as remarked by Hayman, p.194) that if $u(z)$ is also non-positive in $|z| \le R$, so that $T(R,u) = O$, we have

$$u(o) \ge \frac{1}{2\pi}\int_0^{2\pi} -u_1(re^{i\Theta})d\Theta \ge \left[1 + \psi\left(\frac{r}{R}\right)\right] u(o) . \quad (7.1)$$

Now suppose that on a set of angular measure α of Θ ,

$$-u_1(re^{i\Theta}) = \inf_{o\le t\le r} u(te^{i\Theta}) \le u(o) - c, \text{ where } c > o .$$

Also on the complementary set of measure $2\pi - \alpha$,

$$-u_1(re^{i\Theta}) \le u(o) .$$

From the right hand inequality of (7.1), we have

$$\frac{1}{2\pi}\left[\alpha\left\{u(o) - c\right\} + (2\pi - \alpha)u(o)\right] \ge \left[1 + \psi\left(\frac{r}{R}\right)\right] u(o) .$$

$$\frac{1}{2\pi}\left[2\pi\, u(o) - \alpha c\right] \ge \left[1 + \psi\left(\frac{r}{R}\right)\right] u(o) .$$

$$u(o) - \frac{\alpha c}{2\pi} \ge u(o) + u(o)\,\psi\left(\frac{r}{R}\right) .$$

$$-\frac{\alpha c}{2\pi} \ge u(o)\,\psi\left(\frac{r}{R}\right) .$$

$$\alpha \le \frac{2\pi\,|u(o)|}{c}\,\psi\left(\frac{r}{R}\right) . \quad (7.2)$$

The above lemma thus gives us the quantitative estimate that for a non-positive subharmonic function in $|z| \leq R$, the radial segments going outward from the origin and having length r, on which

$\inf u(z) \leq u(o) - c$, have at most

angular measure $\dfrac{2\pi|u(o)|\ \psi(\frac{r}{R})}{c}$.

We shall be using this result in the proof of Lemma 1.5.

<u>Proof of Lemma 1.5</u>

Since u(z) is subharmonic in a neighbourhood N of the continuum γ_K, there exists a number R such that u(z) is subharmonic for all z distant not more than 2R from γ_K. Since γ_K is a continuum, we can find a finite chain of points $z_1 = z_1^{(o)}, z_1^{(1)}, \ldots\ldots z_1^{(n)} = z_2$, on γ_K such that the maximum distance between two consecutive points $z_1^{(k)}$ and $z_1^{(k+1)}$ $(k = o, 1, \ldots, n-1)$ is less than or equal to R. Therefore for the proof of Lemma 1.5, it is sufficient to show that there exists a polygonal path joining z_1 to z_2 in N on which $u(z) > K - 1$, when $|z_1 - z_2| \leq R$. |K-1

Take M(R) = Max u(z) taken over all points z distant not more than R from γ_K.

Define $v(z) = u(z_1 + z) - M(R)$.

Then $v(z)$ is subharmonic and non-positive in $|z| \leq R$ and thus satisfies the hypotheses of $u(z)$ in Lemma 1.6.

We take $c = 1$ in (7.2) and note that α, the angular measure of radial segments of length r on which the infimum $v(z) \leq v(o) - 1$ is at most

$$2\pi \, |v(o)| \, \psi(\tfrac{r}{R}) = 2\pi \, |u(z_1) - M(R)| \, \psi(\tfrac{r}{R}) .$$

If we take any $z \in \gamma_K$ as origin instead of z_1, with the same values of r and R, we get

$$\alpha \leq 2\pi \, |u(z) - M(R)| \, \psi(\tfrac{r}{R}) .$$

Thus for every point z on γ_K, the angular measure α of the radial segments of length r on which the infimum differs from $u(z)$ by an amount greater than or equal to one is at most,

$$\alpha \leq 2\pi \, |u(z) - M(R)| \, \psi(\tfrac{r}{R}) . \qquad (7.3)$$

Since $M(R)$ is bounded for a fixed R and $u(z) \geq K$ on γ_K, we have

$$|u(z) - M(R)| \leq M(R) - K .$$

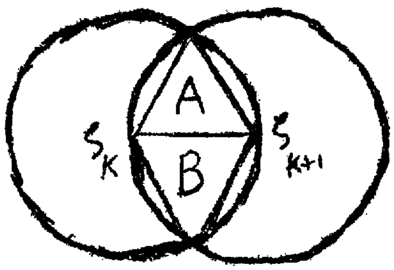
Set $M(R) - K = T$.

Then from (7.3) , $$\alpha \leq 2\pi T \, \psi(\tfrac{r}{R}) . \qquad (7.4)$$

Since $\psi(\frac{r}{R}) \longrightarrow o$ as $\frac{r}{R} \longrightarrow o$, we take δ so small that $\psi(\frac{\delta}{R}) < \frac{1}{12T}$.

Then from (7.4), $\alpha < \frac{\pi}{6}$. (7.5)

Since γ_K is a continuum, we can find a finite chain of points $z_1 = \zeta_o, \zeta_1, \zeta_2, \ldots \zeta_n = z_2$, on γ_K such that the maximum distance between two consecutive points ζ_K and ζ_{K+1} (K = 0,1,...n-1) is less than or equal to δ.



By elementary Plane Geometry, two circles of radii equal to (or greater than) the distance between their centres intersect and the angular measure of the radial segments going out from origin, on each side of the line joining their centres is equal to (or greater than) $\frac{\pi}{3}$.

We also note from the diagram that in the triangles A and B, each segment going out from the centre of one circle meets all segments going out from the centre of the other circle in that triangle.

By (7.3) and (7.5), we can find radial segments of length δ from each of ζ_K and ζ_{K+1} and making angles between 0 and $\frac{\pi}{3}$, with this line such that

$$u(z) > K - 1 \text{ on these radial segments.}$$

And by the above consideration of elementary geometry, these radial segments meet.

Thus it is possible to go from ζ_K to ζ_{K+1} and hence from z_1 to z_2 along a polygonal path on which $u(z) > K - 1$.

This shows that there exists a polygonal path with properties similar to those of the continuum in Lemma 1.4, and hence completes the proof of Theorem 1.2.

1.8 In the proof of theorem 1.2, we discussed some properties of the components of the sets $\left\{ z \mid u(z) \geq K \right\}$ and $\left\{ z \mid u(z) > K \right\}$. We also raised the question whether a function $u(z)$ can be identically constant in a component of $\left\{ z \mid u(z) \geq K \right\}$ without being identically constant in the plane. In the next chapter we give examples of such functions and discuss other properties of components of the sets $\left\{ z \mid u(z) \geq K \right\}$ and $\left\{ z \mid u(z) > K \right\}$.

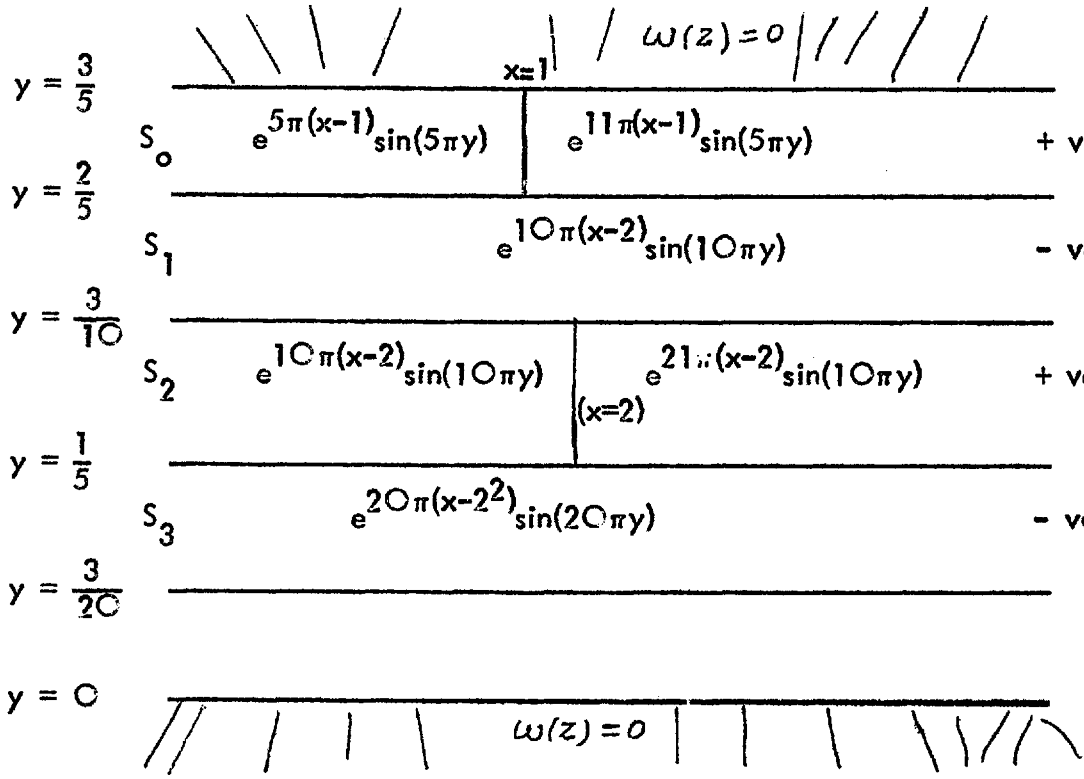
CHAPTER II

SOME EXAMPLES

2.1 It was remarked in Section 1.5, that if there are an infinite number of the components of the set $\{z \mid u(z) \geq K\}$, there may be a component of the set on which $u(z) \equiv K$.

Example 1

We illustrate this by constructing in the next four sections a function $\omega(z)$ which is identically zero in the lower half plane and this is isolated from the other components of the set $\{z \mid \omega(z) \geq 0\}$.



We form strips $S_o, S_1, S_2, \ldots$ parallel to the real axis starting from $y = \frac{3}{5}$ in the following pattern.

$$S_o = \left\{ \frac{2}{5} < y < \frac{3}{5} \quad , \quad -\infty < x < \infty \right\} .$$

$$S_1 = \left\{ \frac{3}{10} < y < \frac{2}{5} \quad , \quad -\infty < x < \infty \right\} .$$

$$S_2 = \left\{ \frac{1}{5} < y < \frac{3}{10} \quad , \quad -\infty < x < \infty \right\} .$$

$$S_3 = \left\{ \frac{3}{20} < y < \frac{1}{5} \quad , \quad -\infty < x < \infty \right\} .$$

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And thus generally,

$$S_{2n} = \left\{ \frac{2}{5.2^n} < y < \frac{3}{5.2^n} \quad , \quad -\infty < x < \infty \right\} . \qquad (1.1)$$

$$n = o, 1, 2 \ldots$$

$$S_{2n+1} = \left\{ \frac{3}{5.2^{n+1}} < y < \frac{2}{5.2^n} \quad , \quad -\infty < x < \infty \right\} . \qquad (1.2)$$

$$n = o, 1, 2 \ldots$$

We define
$$\omega(z) = \begin{cases} \omega_K(z) & \text{for } z \in S_K , \\ 0 & \text{for } z \text{ outside all the } S_K . \end{cases}$$

For $n = o, 1, 2 \ldots$, we define

$$\omega_{2n}(z) = \begin{cases} e^{5.2^n \pi (x-2^n)} \sin(5.2^n \pi y), & \text{for } x \leq 2^n \\ e^{(5.2^{n+1}+1)\pi(x-2^n)} \sin(5.2^n \pi y) & \text{for } x > 2^n . \end{cases} \qquad (1.3)$$

$$\omega_{2n+1}(z) = e^{5.2^{n+1}\pi(x-2^{n+1})}\sin(5.2^{n+1}\pi y) . \qquad (1.4)$$

We note that the width of these strips is gradually decreased and they are so defined as to lie in the upper half plane. $\omega(z)$ is identically zero in the lower half plane and on the real axis. $\omega(z)$ is also identically zero on the part of the upper half plane above $y = \frac{3}{5}$.

We also note that in the strips starting from $y = \frac{3}{5}$, $\omega(z)$ is continuous, alternately positive and negative and vanishes on the boundary. This is observed as follows.

$\omega(z)$ is always positive in S_{2n} because $\sin(5.2^n\pi y)$ is positive as y in S_{2n} varies from $\frac{2}{5.2^n}$ to $\frac{3}{5.2^n}$. There are two different continuous functions in S_{2n} for $x > 2^n$ and $x < 2^n$ but each function tends to $\sin(5.2^n\pi y)$ as x tends to 2^n. Hence $\omega(z)$ is continuous and positive inside the strip S_{2n} and vanishes on its boundary.

Also $\omega(z)$ is always negative in S_{2n+1}, because $\sin(5.2^{n+1}\pi y)$ is negative as y in S_{2n+1} varies from $\frac{3}{5.2^{n+1}}$ to $\frac{2}{5.2^n}$, and vanishes on the boundary. Also $\omega(z)$ is a harmonic function $\omega_{2n+1}(z)$ inside S_{2n+1} and therefore $\omega(z)$ is continuous and negative inside S_{2n+1} and vanishes on its boundary.

Thus $\omega(z)$ is continuous in any two adjoining strips, and identically constant in the part of the upper half-plane above $y = \frac{3}{5}$, and in the lower half-plane.

In order to show that $\omega(z)$ is continuous in the z-plane, it remains to show that $\omega(z)$ is continuous on the real axis.

This is so because the functions in the strips approach zero in the finite part of the plane as the strips approach the real axis. More precisely for any x_o on the real axis choose n such that $x_o \leq 2^n$. Then for all z in a neighbourhood of radius $\delta < \frac{2}{5.2^n}$, round this x_o, it follows from (1.1) and (1.2) that $z \in S_k$ where $k \geq 2n + 1$.

From (1.3) and (1.4), we have in this neighbourhood,

$$|\omega(z)| < e^{5.2^k \pi(x-2^k)} , \qquad k \geq n+1 ,$$

where $x < x_o + 1, \qquad x - 2^k < x_o + 1 - 2.2^n \leq 1 - 2^n \leq -1$ if $n \geq 1$.

$$\text{i.e.} \qquad |\omega(z)| < e^{-5.2^{n+1}\pi} . \qquad (1.5)$$

As $\omega(z)$ is identically zero on the real axis and in the lower half-plane, the continuity on any finite point of the real axis follows from (1.5).

2.2 So far we have shown the continuity of $\omega(z)$. Now we show that the mean value inequality is satisfied and hence that the function is subharmonic.

We first observe that if $h(z) = e^{ax}\sin by$, then the Laplacian

$$\Delta h = (a^2 - b^2)h(z).$$

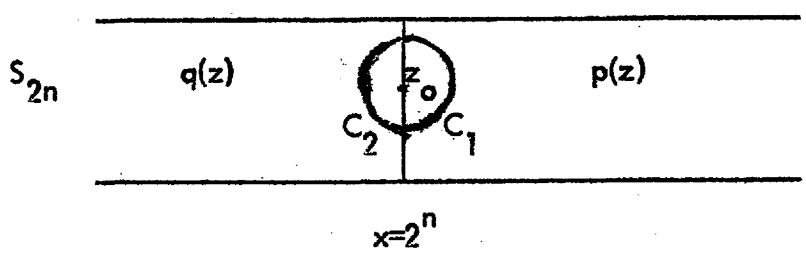
Thus h(z) is subharmonic if,

$$a \geq b, \text{ when } h(z) \geq 0, \qquad \text{and}$$

$$a \leq b, \text{ when } h(z) < 0 .$$

Our $\omega(z)$ is of the form h(z) with different constants in different parts of the strips. We note that in all S_{2n} where $\omega(z)$ is positive we have $a = b$ for $x \leq 2^n$ and $a > b$ for $x > 2^n$. Also in all S_{2n+1} where $\omega(z)$ is negative, the coefficients of x and y i.e. a and b are the same. Thus $\omega(z)$ is subharmonic in S_{2n+1}, and also in the two halves of S_{2n} on the right and the left of the line $x = 2^n$.

In order to show that $\omega(z)$ is subharmonic on this line we note that we showed in Section 2.1 that $\omega(z)$ is continuous and positive on this line.



For any point z_o on this line, let C_r be a circular neighbourhood of this point which lies in S_{2n}. Let C_1 and C_2 be the halves of the boundary of C_r in the two halves of the strip S_{2n} on the right and the left respectively.

Let $\omega(z) = p(z)$ for $z \in C_1$ and $\omega(z) = q(z)$ for $z \in C_2$,

where $$p(z) = e^{(5.2^{n+1}+1)\pi(x-2^n)}\sin(5.2^n\pi y),$$

$$q(z) = e^{5.2^n\pi(x-2^n)}\sin(5.2^n\pi y) .$$

Then $$\frac{1}{2\pi}\int_{C_1+C_2} \omega(z_o + r e^{i\theta})d\theta = \frac{1}{2\pi}\int_{C_1} p(z_o + re^{i\theta})d\theta$$

$$+ \frac{1}{2\pi}\int_{C_2} q(z_o + re^{i\theta})d\theta .$$

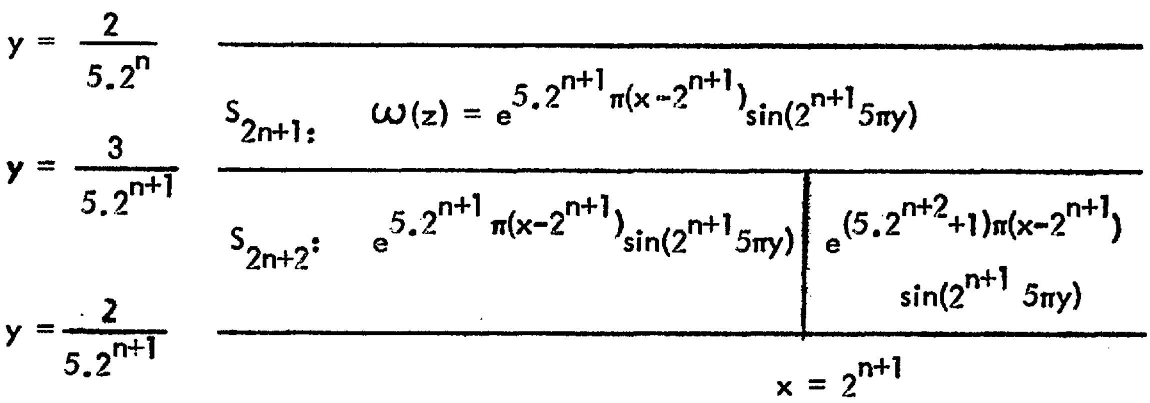
Since $q(z) \leq p(z)$ on C_1, we have

$$\frac{1}{2\pi}\int_{C_1+C_2} \omega(z_o + re^{i\theta})d\theta \geq \frac{1}{2\pi}\int_{C_1+C_2} q(z_o + re^{i\theta})d\theta = q(z_o) = \omega(z_o) ,$$

since $q(z)$ is harmonic.

Thus we have shown that $\omega(z)$ is subharmonic inside the strips. Since a constant is trivially subharmonic, it only remains to show that $\omega(z)$ is subharmonic on the boundaries of the strips and on the real axis. We propose to do this in the following sections.

2.3 There are two types of boundaries of these strips and we discuss them separately. The first is the boundary between S_{2n+1} and S_{2n+2} and this is the line $y = \frac{3}{5.2^{n+1}}$.



It is clear from the above diagram that $\omega(z)$ is subharmonic on $y = \frac{3}{5.2^{n+1}}$ for $x < 2^{n+1}$ as for this part, $\omega(z)$ is the same function in the two strips S_{2n+1} and S_{2n+2} and is harmonic inside them. Actually therefore the integral mean ~~for the~~ points on the line $y = \frac{3}{5.2^{n+1}}$ to the left of $x = 2^{n+1}$, and the value of $\omega(z)$ at these points are both equal to zero.

For the points on the line $y = \frac{3}{5.2^{n+1}}$ to the right of $x = 2^{n+1}$, $\omega(z)$ is again zero on the line. Moreover the function defined in the strip S_{2n+2} for $x \geq 2^{n+1}$ is positive and is not less than the harmonic extension of $\omega(z)$ from $x < 2^{n+1}$. Thus the integral mean is not less than that for this harmonic extension. And so the integral mean of $\omega(z)$ is positive on a circle with centre $x_o + i\frac{3}{5.2^{n+1}}$ and small positive radius if $x_o \geq 2^{n+1}$.

Hence the mean value inequality is satisfied and consequently $\omega(z)$ is subharmonic on the line $y = \frac{3}{5.2^{n+1}}$.

The second type of the boundary is the line between S_{2n} and S_{2n+1}, and this is the line $y = \frac{2}{5.2^n}$.

$\omega(z)$ is zero on the boundary line and positive in S_{2n} and negative in S_{2n+1}.

$$y = \frac{2}{5.2^n} \qquad \frac{S_{2n}:\ e^{5.2^n\pi(x-2^n)}\sin(5.2^n\pi y) \;\Big|\; e^{(5.2^{n+1}+1)\pi(x-2^n)}\sin(5.2^n\pi y)}{S_{2n+1}:\ e^{5.2^{n+1}\pi(x-2^{n+1})}\sin(5.2^{n+1}\pi y)} \qquad \begin{matrix}+\text{ ve}\\ -\text{ ve}\end{matrix}$$

By the law of the mean for every x in S_{2n}, we have

$$\omega(x,y+h) - \omega(x,y) = h\frac{\partial\omega}{\partial y}(x, y+\Theta_1 h)\ , \qquad (3.1)$$

for $0 < \Theta_1 < 1$.

Similarly in S_{2n+1},

$$\omega(x,y-h) - \omega(x,y) = -h\frac{\partial\omega}{\partial y}(x,y-\Theta_2 h)\ , \qquad (3.2)$$

for $0 < \Theta_2 < 1$.

Since $\omega(x,y) = 0$ on the line $y = \dfrac{2}{5.2^n}$, we have

for $x \leq 2^n$, $y = \dfrac{2}{5.2^n}$, from (3.1) and (3.2) ,

$$\omega(x,y+h) + \omega(x,y-h) = h\Big[5.2^n\pi e^{5.2^n\pi(x-2^n)}\cos\left\{5.2^n\pi(y+\Theta_1 h)\right\}$$

$$- 5.2^{n+1}\pi e^{5.2^{n+1}\pi(x-2^{n+1})}\cos\left\{5.2^{n+1}\pi(y-\Theta_2 h)\right\}\Big] .$$

$$\omega(x,y+h) + \omega(x,y-h) = 5.2^n\pi e^{5.2^n\pi(x-2^n)} .h$$

$$\times \left[\cos\left\{5.2^n\pi(y+\Theta_1 h)\right\} - 2e^{5.2^n\pi(x-3.2^n)}\cos\left\{5.2^{n+1}\pi(y-\Theta_2 h)\right\}\right]. \tag{3.3}$$

The maximum value of the second term in the second factor on the right hand side of the above expression (3.3) is at most $2e^{-5.2^{2n+1}\pi} \leq 2e^{-10\pi}$. Also $\cos(5.2^n\pi y)$ is 1 for $y = \dfrac{2}{5.2^n}$ and is greater than $\frac{1}{2}$ sufficiently near this value of y. So it is possible to choose h_1 such that for all $h < h_1$, and all $\Theta_1 < 1$, we have $\cos\left\{5.2^n\pi(y + \Theta_1 h)\right\} > 2e^{-10\pi}$. Thus we have from (3.3), that when $x \leq 2^n$,

$$\omega(x,y+h) + \omega(x,y-h) > 0 , \tag{3.4}$$

for all $h < h_1$.

By the same method we find that for

$$x > 2^n, \quad y = \frac{2}{5.2^n}, \text{ we have :}$$

$$\omega(x,y+h) + \omega(x,y-h) = h\Big[5.2^n\pi e^{(5.2^{n+1}+1)\pi(x-2^n)}\cos\left\{5.2^n\pi(y+\Theta_1 h)\right\}$$

$$- 5.2^{n+1}\pi e^{5.2^{n+1}\pi(x-2^{n+1})}\cos\left\{5.2^{n+1}\pi(y-\Theta_2 h)\right\}\Big],$$

$$= h.5.2^n\pi e^{5.2^{n+1}\pi(x-2^n)}\Big[e^{\pi(x-2^n)}\cos\left\{5.2^n(y+\Theta_1 h)\right\} - 2e^{-5.2^{2n+1}\pi}$$

$$\times \cos\left\{5.2^{n+1}\pi(y-\Theta_2 h)\right\}\Big]. \qquad (3.5)$$

Again the maximum value of the second term in the second factor on the right hand side of (3.5) is at most $2e^{-10\pi}$ as before. The first term is at least $\cos\ 5.2^n\pi(y+\Theta_1 h)$ for $x \geq 2^n$.

As before it is possible to choose h_2 such that for all $h < h_2$ and all $\Theta_i < 1$, the first term is greater than $\frac{1}{2}$ and so greater than the second term of (3.5).

Hence (3.4) holds also for $x > 2^n$.

Thus if $z_o = x_o + i\dfrac{2}{5.2^n}$, and $r < \min(h_1,h_2)$, we have from (3.4), that

$$\frac{1}{2\pi} \int_0^{2\pi} \omega(z_o + re^{i\Theta})d\Theta > 0 . \qquad (3.6)$$

Since $\omega(z)$ is zero and continuous on the line $y = \frac{2}{5.2^n}$, it follows from (3.6), that $\omega(z)$ is subharmonic on the line.

We have shown so far that $\omega(z)$ is subharmonic on the boundary lines between any two adjoining strips. It remains to show that $\omega(z)$ is subharmonic on the two extremal boundary lines namely the upper boundary of the first strip S_o, which is the line $y = \frac{3}{5}$, and the real axis.

It is easy to see that $\omega(z)$ is subharmonic on $y = \frac{3}{5}$. We recall that $\omega(z)$ is zero in the upper half plane above the line $y = \frac{3}{5}$, and is also zero on the line $y = \frac{3}{5}$. In the strip S_o, $\omega(z)$ is positive and continuous. Thus for any $z_o = x_o + i\,\frac{3}{5}$ and small radius r, we have

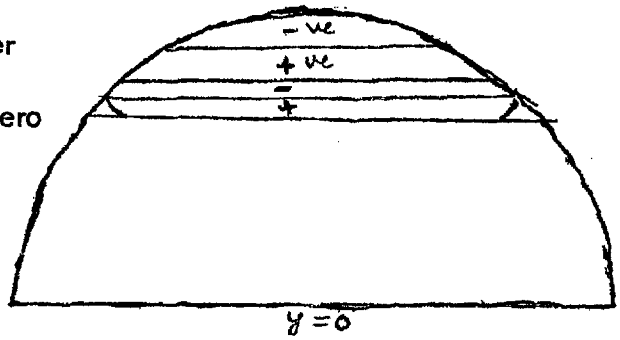
$$\frac{1}{2\pi} \int_0^{2\pi} \omega(z_o + re^{i\Theta})d\Theta > 0 ,$$

and from this it follows that $\omega(z)$ is subharmonic on the line $y = \frac{3}{5}$.

The subharmonicity on the real axis will be shown in the next section.

2.4 It was shown in the last paragraph of Section 2.1 that $\omega(z)$ is continuous on the real axis, and it remains to show that the mean value inequality is satisfied. For this purpose we compute the areal mean $\frac{1}{\pi r^2} \iint \omega(z)\, dx\, dy$ taken over the disc $|z - x_o| < r$.

If $x < 2^n$, we take $r < \frac{2}{5.2^n}$ and compute the areal mean of $\omega(z)$ in the disc with centre x_o and radius r.



The areal mean is zero for the lower semicircle as $\omega(z)$ is identically zero in the lower half plane.

We consider the portions of the strips S_{2k}, S_{2k+1}, etc., inside the upper semicircle and the contributions from them towards the areal mean. We observe the way the strips are formed and note that if the upper semicircle contains a portion D_n of S_{2n-1} (in which $\omega(z)$ is negative), then it also contains the reflection $\bar{D}_n$ of D_n in the line $y = \frac{3}{5.2^n}$.

Thus the total contribution to the areal mean from the pair of strips is always non-negative, since $\omega(z) \geq o$ in S_{2n}, and

$$\frac{1}{\pi r^2} \iint_{D_n} \omega(z)\, dx\, dy = - \frac{1}{\pi r^2} \iint_{\bar{D}_n} \omega(z)\, dx\, dy .$$

Thus the areal mean is positive for the upper semicircle and hence positive for the whole disc.

Since $\omega(z)$ is zero on the real axis, the mean value inequality is satisfied and the function $\omega(z)$ is subharmonic on the real axis.

We have thus shown that the function $\omega(z)$ of our Example 1 is continuous and subharmonic in the whole plane. The components of $\{z \mid \omega(z) > 0\}$ are precisely the (open) strips S_{2n}. On the other hand the components of $\{z \mid \omega(z) \geq 0\}$ are the closed strips S_{2n} together with the half planes $y \leq 0$, $y \geq \frac{3}{5}$ and on these half planes $\omega(z) \equiv 0$.

Also the lower half plane is isolated from all other components of the set $\{z \mid \omega(z) \geq 0\}$. This shows that components of $\{z \mid \omega(z) \geq K\}$ can exist on which $\omega(z) \equiv K$ even though $\omega(z)$ is not identically constant in the plane.

2.5 We note that it follows from Lemma 1.6 and subsequent deduction (7.2) that if $u(z_o) > K$, then the intersection of the set $\{z \mid u(z) > K\}$ with a small neighbourhood of z_o has a positive area. (5.1).

We also have the assertion (c) of Section 1.3, that if $u(z) > K$ at a point z_o, there exists a circle $c(z_o, \delta)$ round this point such that $u(z) > K$ on this circle. Since the components of $\{z \mid u(z) > K\}$ extend to infinity by Lemma 1.1, the component of $\{z \mid u(z) > K\}$ which contains z_o, also contains the circle $c(z_o, \delta)$ and has the positive area referred to in (5.1) inside this circle. Also there is no other component inside this circle. Thus each component of the set $\{z \mid u(z) > K\}$ has positive area. And also the components of $\{z \mid u(z) \geq K\}$, on which $u(z) > K$ somewhere on the component, have positive area.

Consequently such components are at most countable in number.

Thus we have:

Theorem 2.1. The components of the set $\{z \mid u(z) > K\}$ and the components of $\{z \mid u(z) \geq K\}$ on which $u(z) \not\equiv K$, each have positive area and so their total number is at most countable.

We also note that in Example 1, there are an infinite number of such components in any small neighbourhood of a point on the real axis.

Thus the components of the set $\{z \mid u(z) \geq K\}$ in which $u(z) \not\equiv K$, can be at most countable in number and need not be locally finite.

However, the components of $\{z \mid u(z) \geq K\}$ on which $u(z)$ is identically constant may have zero area. We show in the next example that the number of such components on which $u(z)$ is identically constant may be non-countable.

2.6 Example 2

We now construct a subharmonic function $u(z)$ for which the set

$\{z | u(z) \geq o\}$ has a number of components having the power of the continuum on each of which u(z) is identically zero.

We first define $\omega_o(z) = \omega(z)$ in the entire finite z-plane except the strip $\left\{\frac{3}{5} < y < 1, -\infty < x < \infty\right\}$, where $\omega(z)$ is the function defined in Example 1. In the strip $\left\{\frac{3}{5} < y < 1, -\infty < x < \infty\right\}$, we define

$$\omega_o(x,y) = \omega(x,1-y) .$$

The function $\omega_o(z)$ is thus defined in the whole plane and is symmetric about the line $y = \frac{1}{2}$. Also $\omega_o(z)$ is identically zero for $y \leq O$ and $y \geq 1$. Again $\omega_o(z)$ is subharmonic for $y < \frac{3}{5}$ and so in the whole plane by symmetry.

Finally we note that the half planes $y \leq o$ and $y \geq 1$ are components of the set $\{z | \omega_o(z) \geq O\}$.

It can be easily proved (e.g. Talpur (1), Th.1.7, p.19) that if u(z) is subharmonic in D and $z = f(\zeta)$ maps $\Delta(1,1)$ conformally into D, then $u\{f(\zeta)\}$ is subharmonic in Δ .

Let $z = f_{1,1}(\zeta) = 3\zeta - i$.

Then $f_{1,1}(\zeta)$ maps the infinite strip $\left\{\frac{1}{3} < \operatorname{Im} \zeta < \frac{2}{3}\right\}$ into the strip $\{O < \operatorname{Im} z < 1\}$.

Thus $\omega_{1,1}(z) = \omega_o\{f_{1,1}(\zeta)\}$ is subharmonic in the ζ-plane and is identically zero in the half planes $\operatorname{Im}\zeta \geq \frac{2}{3}$ and $\operatorname{Im}\zeta \leq \frac{1}{3}$.

Similarly $z = f_{2,1}(\zeta) = 3^2\zeta - i$ and

$z = f_{2,2} = 3^2\zeta - 7i$ map the strips $\left\{\frac{1}{9} < \operatorname{Im}\zeta < \frac{2}{9}\right\}$ and $\left\{\frac{7}{9} < \operatorname{Im}\zeta < \frac{8}{9}\right\}$ into the strip $\{0 < \operatorname{Im} z < 1\}$.

We also define $\omega_{2,1}(\zeta) = \omega_o\{f_{2,1}(\zeta)\}$ and

$\omega_{2,2}(\zeta) = \omega_o\{f_{2,2}(\zeta)\}$, and note that $\omega_{2,1}(\zeta)$ and $\omega_{2,2}(\zeta)$ are subharmonic in the ζ-plane.

We note that at the m^{th} step, there are 2^{m-1} complementary intervals to Cantor's ternary set. Let (a_k, b_k) be any complementary interval to Cantor's ternary set. Then $z = f_{m,k} = \frac{1}{b_k - a_k}(\zeta - ia_k)$ maps the strip $\{a_k < \operatorname{Im}\zeta < b_k\}$ into the strip $\{0 < \operatorname{Im} z < 1\}$.

We thus define m^{th} group of 2^{m-1} functions $f_{m,1}, f_{m,2}, \ldots, f_{m,2^{m-1}}$, so as to map the 2^{m-1} strips of the m^{th} group into the strip $\{0 < \operatorname{Im} z < 1\}$.

Set $$\omega_{m,k}(\zeta) = \omega_o\{f_{m,k}(\zeta)\}, \qquad (6.1)$$

where $m = 1$ to ∞ and $k = 1$ to 2^{m-1}.

Finally we define $\omega(\zeta) = \sum_{m=1}^{\infty} \sum_{k=1}^{2^{m-1}} \epsilon_{m,k}\, \omega_{m,k}(\zeta),$ (6.2)

$$\text{where } 0 < \epsilon_{m,k} \sup_{|\zeta| < m} |\omega_{m,k}(\zeta)| \leq \frac{1}{4^m} .$$

The series on the right hand side of (6.2) is clearly uniformly convergent for all ζ in the finite ζ-plane.

Since $\epsilon_{m,k} > 0$, $\epsilon_{m,k}\, \omega_{m,k}(\zeta)$ is subharmonic and $\omega(\zeta)$ is thus defined as a uniformly convergent infinite sum of functions subharmonic in the finite plane. Thus $\omega(\zeta)$ is subharmonic in the finite ζ-plane. (e.g. Rado (1) p.20).

We note that at interior points of the strips formed by the middle thirds, only one function $\omega_{m,k}$ is not identically zero. The function $\omega(\zeta)$ is equal to zero on the boundary of this strip. And these boundary lines do not belong to any component of the set $\{\zeta \mid \omega_{m,k}(\zeta) > 0\}$, because by construction as in Example 1, these boundary lines are isolated from other components of $\omega_{m,k}$. Also these boundary lines do not belong to any component of $\{\zeta \mid \omega_{m,k}(\zeta) > 0\}$ for different values of m and k. Therefore these boundary lines are components of

$$\{\zeta \mid \omega(\zeta) \geq 0\} .$$

Now we consider the function $\omega(\zeta)$ on the complementary set of these closed strips. This set consists of lines parallel to the x-axis through the points of the Cantor ternary set on the y-axis, and the half planes $\operatorname{Im} \zeta \leq 0$ and $\operatorname{Im} \zeta \geq 1$.

It follows from the definition that $\omega(\zeta)$ is equal to zero on this complementary set as each $\omega_{m,k}$ is zero on this set. Also these lines and sets are isolated from components of $\{\zeta | \omega(\zeta) > 0\}$.

Finally we note that the components of $\{\zeta | \omega(\zeta) \geq 0\}$ include all the lines $\operatorname{Im} \zeta = c$ where c is a number in Cantor's ternary set such that $0 < c < 1$, and also the half planes $\operatorname{Im} \zeta \leq 0$, $\operatorname{Im} \zeta \geq 1$. On all these components $\omega(\zeta) \equiv 0$.

In addition there are a countable set of strips of the type $\eta_\nu < \operatorname{Im} \zeta < \eta'_\nu$ on which $\omega(\zeta) > 0$. And the closures of these strips are the remaining components of $\omega(\zeta) \geq 0$.

2.7 In this section we discuss the relationship between the number of components $G(K)$ of the set $\{z | u(z) > K\}$ and the order of $u(z)$.

We recall the definition of the order K and the lower order λ of a subharmonic function.

Let $$B(r) = \max_{|z| = r} u(z) .$$

$$K = \limsup_{r \to \infty} \frac{\log B(r)}{\log r} .$$

$$\lambda = \liminf_{r \to \infty} \frac{\log B(r)}{\log r} .$$

We have shown earlier that $u(z)$ is unbounded in each such component $G(K)$. Therefore if $K' > K$, it follows that each component $G(K)$ of the set $\left\{z \mid u(z) > K\right\}$ contains at least one component $G(K')$ of the set $\left\{z \mid u(z) > K'\right\}$.

Hence the number $n(K)$ of these components $G(K)$ is a positive non-decreasing function of K which may be $+\infty$ for some and hence for all sufficiently large values of K.

We also observe that if the component $C(K)$ of the set $\left\{z \mid u(z) \geq K\right\}$ contains precisely one component $C(K + \delta)$ of the set $\left\{z \mid u(z) \geq K + \delta\right\}$ where $\delta > 0$, then $C(K)$ contains precisely one component $G(K)$ of the set $\left\{z \mid u(z) > K\right\}$.

In the case of integral functions, it was shown by Hayman (Hayman (2)) that if the lower order λ is finite, the number $n(K)$ of the components $G(K)$ of the set $\left\{z \mid u(z) > K\right\}$ is finite.

He showed that $\lambda \geq \frac{1}{2}N$, where

$$N = \lim_{K \to \infty} n(K).$$

This is a consequence of the Denjoy-Ahlfors theorem.

The situation for general subharmonic functions is different as the components $G(K)$ of the set $\left\{z \mid u(z) > K\right\}$ may not be domains.

The result however follows from the following theorem of the Denjoy-Ahlfors type proved by Heins (Heins (2), Th.5.1, p.74).

<u>Lemma 2.1</u> <u>Let $u_1,\ldots u_n$ denote $(n \geq 2)$ non-constant non-negative subharmonic functions in the finite plane which satisfy that min</u> $(u_j, u_k) = 0$ for $j \neq k$.

$$\text{Let} \qquad q(r) = \left\{ \int_0^{2\pi} \sum_{k=1}^{n} \left[u_k(re^{i\Theta}) \right]^2 d\Theta \right\}^{\frac{1}{2}} ,$$

$$\text{then} \qquad \liminf_{r \to \infty} r^{-\frac{n}{2}} q(r) > 0 \qquad (7.1)$$

We now prove a consequence of this theorem about the components $G(K)$ of a general subharmonic function.

Suppose that $u(z)$ is subharmonic in the z-plane and let $N = \lim_{K \to \infty} n(K)$ where as before $n(K)$ is the number of the components $G(K)$ of the set $\left\{ z \,|\, u(z) > K \right\}$.

Take n finite and such that $n \leq N$.

Choose n components $G_1(K)$, $G_2(K), \ldots . G_n(K)$.

Consider the function $u(z) - K$.

It is positive inside these n components.

We now define $u_1, u_2, \ldots u_n$ as follows:

$$u_\nu(z) = u(z) - K \quad \text{in } G_\nu(K),$$
$$= 0 \quad \text{outside} \quad G_\nu(K)$$
$$\text{for} \quad \nu = 1, 2, \ldots n .$$

Then, as observed at the end of Section 1.3, it can be easily seen that by the arguments similar to those of Lemma 1.3, it follows that the functions $u_1, u_2, \ldots u_n$ are subharmonic in the whole plane.

The functions $u_1, u_2, \ldots u_n$ are also non-negative in the entire plane.

Also, it follows from the construction of these functions that if one of $u_\nu(z)$ is non-zero, at a point, the others are necessarily zero. Thus the condition $\min \{u_k(z), u_j(z)\} = 0,\ k \neq j$ is satisfied.

Hence the functions $u_1, \ldots u_\nu, \ldots u_n$ satisfy the hypotheses of Lemma 2.1.

Therefore

$$\underline{\lim}_{r \to \infty} r^{-\frac{n}{2}} q(r) > 0 .$$

Since B(r), the maximum modulus for the original function u(z) is obviously greater than or equal to q(r), we have

$$\underline{\lim}_{r \to \infty} r^{-\frac{n}{2}} B(r) > 0.$$

Hence by the definition of λ, we have

$$\lambda \geq \frac{n}{2} \quad .$$

If $N = \infty$, we may take n arbitrarily large and hence $\lambda = \infty$.

Therefore we have the following conclusion:

If u(z) has a finite lower order λ, the number of the components G(K) is at most $\max\left\{2\lambda, 1\right\}$. In particular an infinite number of components G(K) is only possible in the case of a function of infinite lower order.

CHAPTER III

SOME FURTHER RESULTS

3.1 In this chapter we consider how quickly does u(z) tend to $+\infty$ along an asymptotic path.

If u(z) is a non-constant subharmonic function in the plane and as before $B(r) = \max_{|z|=r} u(z)$, it follows from the Phragmen-Lindelöf Principle that

$$\frac{B(r)}{\log r} \longrightarrow a \quad \text{where} \quad o < a \leq \infty \quad .$$

If a is finite, the problem has been settled by Hayman (Hayman (3)). He has shown that 'If u(z) is subharmonic and not constant in the plane and

$$B(r) = O(\log r) \qquad \text{as } r \longrightarrow \infty ,$$

then $u(re^{i\theta}) = B(r) + o(1)$, uniformly as $re^{i\theta} \longrightarrow \infty$ outside a set of circles subtending angles at the origin, whose sum is finite.'

He has also shown in the same paper that if

$$B(r) = O(\log r)^2 \text{ as } r \longrightarrow \infty ,$$

then $u(re^{i\theta}) \sim B(r)$ as $r \longrightarrow \infty$ for almost every fixed θ. The relation also holds uniformly in Θ as $r \longrightarrow \infty$ outside a set of finite logarithmic measure.

However, even when f(z) is a rapidly growing integral function, such general results are not easy to prove.

There is an unpublished result of Boas, that if f(z) is an integral function which is not a polynomial, then there exists a path Γ_∞ such that for every n,

$$\left|\frac{f(z)}{z^n}\right| \longrightarrow \infty, \quad \text{as } z \longrightarrow \infty \text{ along } \quad \Gamma_\infty .$$

This, of course, only shows that on the asymptotic path Γ_∞, $|f(z)| \longrightarrow \infty$ more rapidly than every polynomial. We shall prove an analogue of this result of Boas for a general subharmonic function for which $B(r) \neq O(\log r)$.

3.2

Theorem 3.1. If u(z) is a non-constant subharmonic function such that $B(r) \neq O(\log r)$, then there exists a path Γ such that

$$\frac{u(z)}{\log |z|} \longrightarrow +\infty, \text{ as } z \longrightarrow \infty \text{ on } \Gamma .$$

The function $u_n(z) = u(z) - n \log |z|$ must be unbounded in the plane for every finite n, because otherwise $B(r) = O(\log r)$ and this contradicts the hypothesis.

Consider the components of the set $\{z \mid u(z) > K\}$ in $|z| > 1$. If there is only one such component then the function $u_n(z)$ is unbounded in this component for every finite n. If there are more than one of such components, then it follows from the Wiman-Heins theorem (Heins (3)) that B(r) grows at least like $r^{\frac{1}{2}}$ in each such component. Consequently $u_n(z)$ is unbounded in each component for finite n.

Also $u_n(z)$ is subharmonic in the plane except for the origin.

It follows from Lemma 1.1, that if $u_n(z)$ is subharmonic in a disc $|z - z_o| \leq R$, all the components of the set $\{z \mid u_n(z) > K\}$ go to the boundary $|z - z_o| = R$.

Consider a component G(n) of the set $|z| > 1$ on which $u_n(z) = u(z) - n \log |z| > B(1) + n$. This is a component of the set on which

$$u(z) > n + n \log |z| + B(1) .$$

Such a component exists and stretches to infinity and $u_m(z)$ is unbounded in it for every finite m. Therefore G(n) contains G(n+1) i.e. a component (in $|z| > 1$) on which $u(z) - (n+1) \log |z| > B(1) + n+1$. (2.1)

Suppose that z_1 is a point in $|z| > 1$ such that $u(z_1) - \log |z_1| > B(1) + 1$.

Then $z_1 \in G(1)$.

Also by (2.1), G(1) contains a component G(2).

Let $z_2 \in G(2)$, then $u(z_2) - 2 \log |z_2| > B(1) + 2$.

Since z_1 and z_2 both belong to G(1), there exists a continuum γ_1 joining z_1 to z_2 and lying in the closure of G(1).

Also as before by (2.1), G(2) contains a component G(3).

Let $z_3 \in G(3)$, then

$$u(z_3) - 3 \log |z_3| > B(1) + 3 .$$

Since z_2 and z_3 both belong to G(2), there exists a continuum γ_2 joining z_2 to z_3 and lying in closure of G(2).

Continuing in this way we have after n steps,

$$u(z_n) - n \log |z_n| > B(1) + n \qquad \text{and}$$

$$u(z) - (n-1) \log |z| \geq B(1) + (n-1) \text{ on } \gamma_{n-1}$$

joining the points z_{n-1} and z_n.

Also from the construction it is obvious that $u(z_n) > n + n \log |z_n| + B(1)$. Thus z_n will tend to infinity with n, as u(z) is bounded near $|z| = 1$.

Thus we have a continuum $\Gamma = \gamma_1 + \gamma_2 + \dots$ that goes to infinity through all of these G(n) and on this continuum

$$\frac{u(z)}{\log |z|} \longrightarrow +\infty , \qquad \text{as } z \longrightarrow \infty .$$

By applying Lemma 1.5 to $u_n(z)$, it follows that there exists a polygonal path with similar properties.

3.3 We now discuss whether we can improve the conclusion of Theorem 3.1 by restricting somewhat the growth of the function. We consider subharmonic functions of order less than $\frac{1}{2}$.

Let u(z) be a non-constant function subharmonic in the finite z-plane and as before A(r) and B(r) denote inf u(z) and max u(z) on $|z| = r$ respectively. Let K and λ denote the order and the lower order of u(z) as defined in Section 2.7.

We now introduce some notation regarding measure and density of sets.

Given any set E on the part $r > 1$ of the positive r-axis we define the measure mE of E and the logarithmic measure lm E of E by

$$mE = \int_E dr, \qquad lm\ E = \int_E \frac{dr}{r} . \tag{3.1}$$

Let E(r) denote the part of E in the interval $[1, r]$, and we define the upper and lower densities of E, by the equations:

$$\overline{\text{dens}}\ E = \overline{\lim_{r\to\infty}} \frac{mE(r)}{r-1}, \quad \underline{\text{dens}}\ E = \underline{\lim_{r\to\infty}} \frac{mE(r)}{r-1} . \tag{3.2}$$

We also define the upper and lower logarithmic densities $\overline{\text{log dens}}\ E$, and $\underline{\text{log dens}}\ E$, by

$$\overline{\text{log dens}}\ E = \varlimsup_{r\to\infty} \frac{lm\ E(r)}{\log r}, \quad \underline{\text{log dens}}\ E = \varliminf_{r\to\infty} \frac{lm\ E(r)}{\log r} \tag{3.3}$$

It can be easily proved (e.g. Barry (1) p.447) that

$$0 \leq \underline{\text{dens}}\ E \leq \underline{\text{log dens}}\ E \leq \overline{\text{log dens}}\ E \leq \overline{\text{dens}}\ E \leq 1$$

and $\quad 0 \leq lm\ E \leq mE$.

In 1927, Besicovitch proved that if $u(z) = \log|f(z)|$ where $f(z)$ is an integral function of order K, $0 \leq K < 1$, and $K < \alpha < 1$, then

$$\overline{\text{dens}} \left\{ r \,\middle|\, A(r) > \cos \pi\alpha\ B(r) \right\} \geq 1 - \frac{K}{\alpha} . \tag{3.4}$$

He also gave an example of an integral function for which the lower density of the set in (3.4) is zero.

In 1952, Huber extended (3.4) to the class of general subharmonic functions of order K, $(0 \leq K < 1)$.

In 1963, (3.4) was strengthened further by Barry, who proved the following theorem in Barry (2).

<u>Lemma 3.1.</u> <u>Let $u(z)$ be a non-constant function subharmonic in the finite z-plane and of order K,</u> $0 \leq K < 1$, <u>and let</u> $K < \alpha < 1$. <u>Then</u>

$$\underline{\text{log dens}}\; E\left\{ r \,\middle|\, A(r) > \cos \pi\alpha B(r) \right\} \geq 1 - \frac{K}{\alpha} \qquad (3.5)$$

Thus if the order K of $u(z)$ is less than $\frac{1}{2}$, we choose α between K and $\frac{1}{2}$. Then by (3.5), on a set of positive lower logarithmic density c we have,

$$A(r) > \cos \pi\alpha\, B(r) \qquad (3.6)$$

3.4 We now proove the following lemma which we will use to prove a theorem about lower growth on suitable paths.

<u>Lemma 3.2.</u> <u>Suppose that a set E has positive lower logarithmic density at least c. Then there exist</u> r_o <u>such that for all</u> $r > r_o$, <u>there are points belonging to E in</u> $(r, r^{\frac{1+\epsilon}{c}})$, <u>where</u> ϵ <u>is any positive number.</u>

The set E has lower logarithmic density at least c. We assume without loss in generality that $0 < \epsilon < 1$.

There exists r_o such that for all $r > r_o$, the logarithmic measure of $E(r)$ is at least $\frac{1}{1+\epsilon^2} c \log r$.

$$\operatorname{Im} E(r^{\frac{1+\epsilon}{c}}) > \frac{1}{1+\epsilon^2} c \log r^{\frac{1+\epsilon}{c}} = \frac{1}{1+\epsilon^2} . c . \frac{1+\epsilon}{c} \log r$$

$$= \frac{1+\epsilon}{1+\epsilon^2} \log r$$

Now Im E(r) is at most log r.

The logarithmic measure of the part of E in the interval $(r, r^{\frac{1+\epsilon}{c}})$ is at least

$$\left(\frac{1+\epsilon}{1+\epsilon^2} \log r - \log r\right) = \left(\frac{1+\epsilon}{1+\epsilon^2} - 1\right) \log r .$$

On simplification this is equal to $\dfrac{\epsilon(1-\epsilon)}{1+\epsilon^2} \log r > 0$.

There exist points belonging to E in $(r, r^{\frac{1+\epsilon}{c}})$.

This completes the proof of Lemma 3.2.

We now prove a result about the lower growth which is a consequence of Lemma 3.1.

<u>Theorem 3.2.</u> <u>Let u(z) be a non-constant function subharmonic in the finite plane of order</u> K, $0 \leq K < \frac{1}{2}$, <u>and let</u> $K < \alpha < \frac{1}{2}$. <u>Then on a suitable asymptotic path</u>,

$$u(z) > \cos \pi\alpha B \left[r^{(1 - \frac{K}{\alpha})/(1+\epsilon)} \right] , \qquad (4.1)$$

<u>where</u> $|z| = r$.

Since $u(z)$ is of order less than a half there exists a sequence r_n of r, tending to ∞, such that,

$$A(r_n) > \cos \pi\alpha \; B(r_n). \tag{4.2}$$

Since $A(r_n) = \inf_{|z|=r_n} u(z)$, we have that for all z on $|z| = r_n$,

$$u(z) > \cos \pi\alpha . \; B(r_n). \tag{4.3}$$

We now use Lemmas 3.1 and 3.2 about the density of the set of r_n' which satisfy (4.2).

If r is sufficiently large we deduce from Lemmas 3.1 and 3.2 that the set E of (3.5) has points in the interval $(r, r^{\frac{1+\epsilon}{c}})$, where c is the lower logarithmic density in (3.5).

Also $c \geq 1 - \frac{K}{\alpha}$.

If the sequence is suitably chosen, then for large r, the interval $(r, r^{\frac{1+\epsilon}{1-K/\alpha}})$ contains members of the sequence in (4.2).

Thus from the above sequence we can choose r_n satisfying

$$(r_n^{\frac{1-K/\alpha}{1+\epsilon}}) < r_{n-1} \; .$$

Also it follows from (4.2) and (4.3) that for z on these r_n,

$$u(z) > \cos \pi\alpha . \; B(r_n) \; .$$

The set on which $u(z) - A(r_1) \geq 0$ contains $|z| = r_1$ and by Lemma 1.1, stretches to infinity. We choose r_2, as above and it follows from our earlier results that there is a continuum extending from $|z| = r_1$ to $|z| = r_2$, on which

$$u(z) \geq A(r_1) > \cos \pi\alpha \; B(r_1) \; .$$

Thus we can join the points z_1 and z_2 where $u(z_1) = B(r_1)$ and $u(z_2) = B(r_2)$ respectively by a continuum on which $u(z) > \cos \pi\alpha \; B(r_1)$.

Continuing in this way we have a continuum extending to infinity on which,

$$u(z) > \cos \pi\alpha . B(r^{\frac{1-K/\alpha}{1+\epsilon}}) , \quad \text{where } |z| = r .$$

By Lemma 1.5, there exists a polygonal path on which

$$u(z) > \cos \pi\alpha . B(r^{\frac{(1-K/\alpha)}{(1+\epsilon)}}) - 1 , \quad \text{where } |z| = r .$$

3.5 We note that in the case of functions of order zero, α can be chosen arbitrarily near zero. Thus both $\cos \pi\alpha$ and $\frac{1}{1+\epsilon}$ can be near one. Therefore we have that for functions of zero order we can find suitable paths on which ,

$$u(z) > (1- \epsilon)B(r^{1-\epsilon}), \quad \text{where } r = |z| .$$

CHAPTER IV

SUBHARMONIC FUNCTIONS IN SPACE

4.1 Introduction

The definition of a subharmonic function in the plane can be carried over to Euclidean space of three or more dimensions. Let u(P) be a function in a domain D such that $-\infty \leq u < +\infty$. Then it is subharmonic if it satisfies the following three conditions:

(i) u is not identically equal to $-\infty$ in D.

(ii) u is upper-semi-continuous in D.

(iii) u(P) is less than or equal to $A_u(P,\delta)$, the mean of u on any spherical surface of centre P and radius δ for all sufficiently small δ, depending on P.

We introduce the following notation.

$D(P,r) = \{Q \mid PQ < r\}$, an open ball centre P, radius r;

$C(P,r) = \{Q \mid PQ \leq r\}$, a closed ball centre P, radius r;

$D_r = D(O,r)$, $C_r = C(O,r)$;

$S(P,r) = \{Q \mid PQ = r\}$, $S_r = S(O,r)$, the spherical surface.

Thus the condition (iii) implies that,

$$u(P) \leq A_u(P,\delta) = \frac{1}{4\pi\delta^2} \int_{S(P,\delta)} u(Q)\, dS_Q \quad , \qquad (1.1)$$

where dS_Q is the element of area around Q on $S(P,\delta)$.

Also, if $a_u(P,\delta)$ denotes the volume average of u over an open ball $D(P,\delta)$, it can be easily seen that

$$a_u(P,\delta) = \frac{3}{\delta^3} \int_0^\delta A_u(P,r)\, r^2 dr \ . \qquad (1.2)$$

Also in the condition (iii), the mean may be considered on a spherical volume instead of a spherical surface.

It follows easily from the above definition that

1. If u_1 and u_2 are subharmonic, then $\max\left[u_1, u_2\right]$ and $u_1 + u_2$ are subharmonic.

2. If u is subharmonic so is Ku for any constant $K > 0$.

As in the plane case we deduce the following version of the Maximum principle from the above definition.

<u>Theorem 4.1.</u> <u>Suppose that V(P) is harmonic in a bounded domain D, of space and continuous in the closure of D, and that u(P) is subharmonic</u>

in D and upper-semi-continuous in the closure of D and $u(P) \leq v(P)$ on ∂D, the boundary of D. Then

$u(P) < v(P)$ in D or $u(P) \equiv v(P)$ in D.

Let $\omega(P) = u(P) - v(P)$.

Then $\omega(P)$ is subharmonic in D and upper-semi-continuous in the closure of D and $\omega(P) \leq 0$ on ∂D.

Let $M = \sup_{P \in D} \omega(P)$.

Suppose that $P_1, \ldots, P_n, \ldots$ is a sequence of points in D such that $\omega(P_n) \longrightarrow M$.

Then a subsequence of $\{P_n\}$ say P_{n_q} converges to a point P_o belonging to the closure of D.

Suppose that $P_o \in \partial D$. Then for large q, by upper-semi-continuity, we deduce that

$$\omega(P_{n_q}) < \omega(P_o) + \epsilon \quad .$$

Consequently $M \leq \omega(P_o) + \epsilon \leq 0$.

Therefore $M < 0$ as required unless $M = 0$ and in this case $\omega(P) = 0$ for some point in $\overline{D}$. 2

If $P_o \in D$, then again by upper-semi-continuity, $\omega(P_o) \geq \limsup_{q \to \infty} \omega(P_{n_q}) =$

$= M$.

Since by definition $\omega(P) \leq M$ for $P \in D$, we have $\omega(P_o) = M$.

We now show that if this happens, we have

$$\omega(P) \equiv M \quad \text{for} \quad P \in D.$$

The set $F = \{P \mid P \in D, \; \omega(P) = M\}$ is closed by the upper-semi-continuity.

If F is not the whole domain D, there exists a point $R \in D \setminus F$.

Let $a(T)$ be a path joining P_o to R in D. Since P_o belongs to a closed subset F of D and R is outside F, the path $a(T)$ must have an extremal point on the set F when it leave F and enters $D \setminus F$.

Let this point be T, and $T \in F$.

For arbitrary small δ,

$$M = \omega(T) \leq \frac{1}{4\pi\delta^2} \int_{S(T,\delta)} \omega(Q) \, dS_Q .$$

Thus
$$\frac{1}{4\pi\delta^2} \int_{S(T,\delta)} \{\omega(Q) - \omega(T)\} \, dS_Q \geq 0 . \qquad (1.3)$$

If $\omega(Q) = M - 2\epsilon < M$ for some $Q = Q_o$, then by the upper-semi-continuity $\omega(Q) < M - \epsilon$ in a neighbourhood of Q_o of area $\delta^2 \eta$ say. Thus the integral in (1.3) is at most $-\frac{1}{4\pi}\eta\epsilon < 0$.

This gives a contradiction. Therefore for all Q on $S(T,\delta)$ we have $\omega(Q) = M$. But this surface meets the path joining P_0 to R.

Thus the point of intersection belongs to F and contradicts the definition of T.

Since $\omega(P) \leq 0$ on ∂D, we must have $M \leq 0$.

Thus $\omega(P) \leq 0$ in all cases and equality holds if and only if $\omega(P) \equiv 0$ in D. This proves the theorem.

4.2 Thus we observe that the definition is the same for two or more dimensions and some properties like the Maximum-principle and others also hold both in the case of the plane and space. However, in some ways the behaviour of subharmonic functions in space is quite different from that in the plane. For example in the plane we have an analogue of Liouville's theorem that a function which is subharmonic and bounded above in the entire finite plane is constant. But in space we have non-constant subharmonic functions which are bounded above.

For example, consider

$$u(P) = u(x,y,z) = \begin{cases} \dfrac{-1}{\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{\frac{1}{2}}} & \text{for } (x,y,z) \neq (x_0,y_0,z_0), \\ -\infty & \text{for } (x,y,z) = x_0,y_0,z_0). \end{cases}$$

Then u(P) is bounded above in space.

u(P) is also upper-semi-continuous at (x_o, y_o, z_o) and continuous elsewhere. Also the Laplacian of u(P) is zero except at (x_o, y_o, z_o). This implies the condition (iii) of the definition of a subharmonic function. [e.g. Kellogg (1), p.316]. Hence u(P) is subharmonic in the entire space.

Thus we see that some properties which hold in the plane may not hold in space. In general we cannot expect an asymptotic path on which a subharmonic function u(P) tends to + ∞. The natural extension of Iversen's theorem would be to show the existence of a path on which u(P) tends to M where M is the upper bound of u(P) in space.

We are able to do this in the case of a continuous subharmonic function, and in the case of a general subharmonic function when M is finite.

As the analogues in space of some theorems in the plane are not valid, it is not possible to get results as strong as those in the plane. The question of finding an asymptotic path on which a general subharmonic function tends to + ∞, is still open.

4.3 In this section we state and prove some lemmas which will be useful to us in further investigations.

Lemma 4.1 If u(P) is a subharmonic function in a complete neighbourhood of the closed ball C(Q,r), then all the components of the sets

$\{P | u(P) \geq K\}$ or $\{P | u(P) > K\}$ in $C(Q,r)$ go to the boundary $S(Q,r)$.

This is an analogue of Lemma 1.1 and can be proved by the arguments similar to those given for the plane in Chapter I.

Lemma 4.2 Suppose that $u(P)$ is subharmonic in the neighbourhood of the closure $\overline{\Omega}$ of a bounded domain Ω. Let F_o be a component of $\{P | u(P) \geq 0\}$ in $\overline{\Omega}$ and define

$$v(P) = \begin{cases} u(P), & \text{for } P \in F_o, \\ 0, & \text{for } P \text{ outside } F_o. \end{cases}$$

Then $v(P)$ is subharmonic in Ω.

This lemma is the space analogue of Lemma 1.3 in the plane. In the plane we made use of the Milloux-Schmidt inequality. We now give a proof which applies in K-dimensions and does not need the application of the Milloux-Schmidt inequality.

Let u_n be a sequence of continuous functions decreasing to u in $\overline{\Omega}$ and subharmonic in a neighbourhood of $\overline{\Omega}$. Let P_o be a point of F_o and let F_n be the component of $\{P | u_n(P) \geq 0\}$ in $\overline{\Omega}$ which contains P_o.

Then $F_{n+1} \subset F_n$ and the F_n are continua.

Let $F = \bigcap_{n=1}^{\infty} F_n$.

We show that $F = F_o$.

We remark that F_o is a component of a bounded closed set $\left[\bar{\Omega} \cap \{P | u(P) \geq 0\}\right]$ and is therefore a point or a continuum. If it is a point, it lies on the boundary of Ω by Lemma 4.1. In that case v(P) is identically zero in Ω.

We therefore consider the case when F_o is a continuum.

We note that $u_n \geq u \geq 0$ in F_o, and F_o is a continuum in $\bar{\Omega}$. Thus F_n contains F_o.

Since this is true for every n, $F_o \subset F$. (3.1)

Conversely, $u_n \geq 0$ in F_n for every n.

$$u_n \geq 0 \text{ in } F$$

Thus $u \geq 0$ in F, and since F is a continuum which contains

$$P_o, \qquad F \subset F_o. \tag{3.2}$$

From (3.1) and (3.2), we have $F = F_o$.

Now set $V_n(P) = \begin{cases} u_n(P), & \text{for } P \in F_n, \\ 0, & \text{for } P \text{ outside } F_n. \end{cases}$

Then it is evident from the continuity of $u_n(P)$, that $v_n(P)$ is continuous and subharmonic in Ω.

Also we note that $v_n(P)$ decreases with n.

For if $P \in F_{n+1}$, then $P \in F_n$, and so

$$v_n(P) - v_{n+1}(P) = u_n(P) - u_{n+1}(P) \geq 0.$$

And if P is outside F_{n+1}, then $v_{n+1}(P) = 0 \leq v_n(P)$.

Since the limit of a decreasing sequence of subharmonic functions is a subharmonic function, (Rado (1), p.14) we have that $v(P) = \lim_{n\to\infty} v_n(P)$ is subharmonic in Ω.

Now if $P \in F_o$, then $P \in F_n$ for every n, and $v(P) = \lim_{n\to\infty} v_n(P) =$

$= \lim_{n\to\infty} u_n(P) = u(P)$.

If P is outside F_o, then P is outside F_n for large n, and $v_n(P) = 0$ for $n > n_o(P)$.

Therefore $v(P) = 0$.

This shows that the limit function is the original function defined in the lemma and what is required is proved.

Lemma 4.3 If $u(P)$ is subharmonic in a domain D and E is a compact subset of D, then there exists a sequence $v_\delta(P)$ of functions defined for all points in D distant not more than 3δ from the boundary of D such that

(i) $v_\delta(P)$ is subharmonic in E,

(ii) $v_\delta(P) \in C^\infty$,

(iii) $v_\delta(P)$ increases with increasing δ,

(iv) $v_\delta(P) \longrightarrow u(P)$ as $\delta \to 0$.

Consider $v(P) = \int u(Q)K(P,Q)dQ$, where dQ is a volume element and $K(P,Q) \in C^\infty$.

Assume that $P = (x,y,z)$, $Q = (\xi, \eta, \zeta)$ and $K(P,Q)$ can be written in the form $K(x-\xi, y-\eta, z-\zeta)$, and $K = 0$ when $\left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{\frac{1}{2}} > \delta$.

We shall set

$$K_\delta(x-\xi, y-\eta, z-\zeta) = C(\delta) \exp\left\{ - \left[1 - \frac{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}}{\delta^2}\right]^{-1}\right\}$$

$$\text{for} \quad \left\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right\}^{\frac{1}{2}} < \delta,$$

$$K_\delta(x-\xi, y-\eta, z-\zeta) = 0, \qquad (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \geq \delta,$$

where $C(\delta)$ is a constant $\frac{C}{\delta^3}$, and C is defined so that $4\pi C \int_0^1 e^{-\frac{1}{1-s^2}} s^2 ds = 1.$

Suppose that $P \in E$ and that $u(P)$ is subharmonic in a 3δ-neighbourhood of E. Then $u(P)$ has a finite integral in a δ-neighbourhood E' of E and

$$|v(P)| \leq M \text{ in } E'.$$

Also for $P \in E$, $u(Q)\,K(P,Q) = 0$ except when Q is at a distance not more than δ from E.

We note that the partial derivatives of v can be obtained by formal differentiation under the integral

$$v(x,y,z) = \iiint_D u(\xi,\eta,\zeta)\,K(x-\xi, y-\eta, z-\zeta)\,d\xi\,d\eta\,d\zeta \;.$$

Thus $v(x+h, y,z) - v(x,y,z) =$

$$= \iiint u(\xi,\eta,\zeta)\left\{K(x+h-\xi, y-\eta, z-\zeta) - K(x-\xi, y-\eta, z-\zeta)\right\} d\xi\,d\eta\,d\zeta,$$

$$= h\iiint u(\xi,\eta,\zeta)\left\{\frac{\partial K}{\partial x}(x-\xi+\theta h, y-\eta, z-\zeta)\right.\quad d\xi\,d\eta\,d\zeta,$$

where $0 < \theta < 1$,

and $\theta = \theta(x,y,z,\xi,\eta,\zeta)$.

Also $\dfrac{\partial K}{\partial x}$ is continuous and therefore uniformly continuous in a sphere of radius δ and so in the whole space.

Thus given $\epsilon > 0$, $\exists$ $\delta' > 0$ such that for all $|h'| < \delta'$ we have,

$$\left| \frac{\partial K}{\partial x}(x-\xi, y-\eta, z-\zeta) - \frac{\partial K}{\partial x}(x+h'-\xi, y-\eta, z-\zeta) \right| < \epsilon \text{ , for all } (x,y,z).$$

$$\text{Thus } \left| \frac{v(x+h,y,z) - v(x,y,z)}{h} - \iiint \frac{\partial K}{\partial x}(x-\xi, y-\eta, z-\zeta) u(\xi,\eta,\zeta) d\xi d\eta d\zeta \right|$$

$$< \epsilon \left| \iiint_{E'} u(\xi,\eta,\zeta) d\xi \, d\eta \, d\zeta \right| \leq M\epsilon .$$

Therefore $\frac{\partial v}{\partial x}$ exists. Also like $\frac{\partial K}{\partial x}$, it is uniformly continuous in space.

Similarly it follows that $V \in C^{\infty}$.

Now we show that v(P) is subharmonic in E.

$$v(x+h_1, y+h_2, z+h_3) = \iiint u(\xi,\eta,\zeta) K(x+h_1-\xi, y+h_2-\eta, z+h_3-\zeta) d\xi \, d\eta \, d\zeta$$

$$= \iiint u(\xi'+h_1, \eta'+h_2, \zeta'+h_3) K(x-\xi', y-\eta', z-\zeta') d\xi' d\eta' d\zeta' ,$$

$$= \iiint u(\xi+h_1, \eta+h_2, \zeta+h_3) K(x-\xi, y-\eta, z-\zeta) d\xi \, d\eta \, d\zeta . \quad (3.3)$$

We now operate on both sides of (3.3) by $\frac{1}{4\pi r^2} \iint dS_r$ where dS_r is the area element of the spherical surface radius r,

$$A_v(P,r) = \frac{1}{4\pi r^2} \iint dS_r \iiint u(\xi+h_1, \eta+h_2, \zeta+h_3) K(x-\xi, y-\eta, z-\zeta) d\xi \, d\eta \, d\zeta .$$

We change the order of integration on the right hand side as the integrand is bounded above.

$$A_v(P,r) = \iiint K(x-\xi, y-\eta, z-\zeta)d\xi\, d\eta\, d\zeta \,.\, \frac{1}{4\pi r^2}\iint u(\xi+h_1, \eta+h_2, \zeta+h_3)dS_r \,.$$

We subtract v(P) from both the sides and recall that

$$v(P) = \iiint u(\xi, \eta, \zeta)K(x-\xi, y-\eta, z-\zeta)d\xi d\eta d\zeta \,.$$

Thus $A_v(P,r) - v(P) = \iiint K(x-\xi, y-\eta, z-\zeta)d\xi\, d\eta\, d\zeta$

$$\times \left[\frac{1}{4\pi r^2}\iint u(\xi+h_1, \eta+h_2, \zeta+h_3) - u(\xi, \eta, \zeta)dS_r\right] \,.$$

The integrand on the right hand side is positive as u(P) is subharmonic in E' and $K \geq O$.

Therefore $A_v(P,r) \geq v(P)$ and v(P) is subharmonic in E.

Finally, we note that if we take P as the origin we have in spherical polar form

$$v_\delta(P) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\delta} C(\delta)e^{\frac{-1}{1-r^2/\delta^2}} u(r,\Theta,\phi)r^2\sin\Theta\, d\Theta d\phi dr \,.$$

$$= \int_0^{\delta} C(\delta)\, e^{-\frac{1}{1-r^2/\delta^2}} Au(P,r).4\pi.r^2 dr \,.$$

Since we take $C(\delta) = \frac{C}{\delta^3}$, we have

$$v_\delta(P) = \frac{4\pi C}{\delta^3} \int_0^\delta e^{-\frac{1}{1-r^2/\delta^2}} Au(P,r) r^2 dr$$

Put $r = s\delta$

$$\text{Thus } v_\delta(P) = 4\pi C \int_0^1 e^{-\frac{1}{1-s^2}} Au(P,s\delta)\, s^2 ds \qquad (3.4)$$

It can be shown by arguments analogous to those in the plane (e.g. Th.1.8, Talpur (1)) that $Au(P,r)$ is an increasing function of r.

Take $\delta_1 < \delta_2$.

$$\text{Then } v_{\delta_2}(P) - v_{\delta_1}(P) = 4\pi C \int_0^1 e^{-\frac{1}{1-s^2}} \left[Au(P, s\delta_2) - Au(P, s\delta_1) \right] s^2 ds$$

$$\geq 0 .$$

Therefore $v_\delta(P)$ is an increasing function of δ.

Also as $\delta \longrightarrow 0$, $v_\delta(P) \longrightarrow 4\pi C\, u(P) \int_0^1 e^{-\frac{1}{1-s^2}} s^2 ds$.

As the constant C was defined so that $4\pi C \int_0^1 e^{-\frac{1}{1-s^2}} s^2 ds = 1$,

we have that $v_\delta(P) \longrightarrow u(P)$ as $\delta \longrightarrow 0$.

Thus by choosing a sequence of δ tending to O, we obtain $V_\delta(P) \in C^\infty$, subharmonic in E, and decreasing to u(P) in E as $\delta \longrightarrow O$.

4.4 It was remarked in Section 4.2 that a function subharmonic in space, may be bounded above in space. We note that the theorems about the growth of subharmonic functions do not have the exact analogues in space. For example in the plane we have the Wiman-Heins theorem which can be stated as:

'<u>If u(P) is a non-negative subharmonic function in the entire plane and if</u> $A = \{P | u(P) = O\}$, $S_r = \{P | o\overline{P} = r\}$.

$$M(r) = \max_{P \in S_r} u(P)$$

<u>and if the intersection</u> $A \wedge S_r$ <u>is not null for all sufficiently large r, then either</u> $u \equiv O$, or $\lim_{r \to \infty} r^{-\frac{1}{2}} M(r) > O$.'

Huber(1) has studied the extension of this theorem to space. He has shown that by making suitably general assumptions about the set A, one can make some assertions about the growth of u(P) in space.

Following Huber, if G is a set in space, we define the solid angle $\mu(r)$ subtended by the set $G \wedge S_r$ at the origin as the Lebesgue area of $G \wedge S_r$ divided by r^2.

As before, let A denote the set $\{P \mid u(P) = 0\}$.

Huber has proved:

'*Let u(P) be a non-negative subharmonic function in space. Suppose that for all sufficiently large values of r, the solid angle $\mu(r)$ subtended by the set $A \wedge S_r$ at the origin is not less than a positive number μ_o. Then there exists a positive number β depending on μ_o, such that either* $u \equiv 0$ or $\lim\limits_{r \to \infty} r^{-\beta} M(r) > 0$.'

Since an asymptotic path has to lie inside a component of $\{P \mid u(P) > 0\}$, we have to show that (i) there is at least one component of $\{P \mid u(P) > 0\}$ in which an unbounded function u(P) is unbounded, (ii) there is at least one component in which u(P) has the same upper bound as it has in space.

However, we cannot use the above theorem directly as we cannot say in general that such a μ_o exists.

We recall that in the plane we showed that u(P) is unbounded or identically constant in a component of $\{P \mid u(P) \geq 0\}$ by applying the Milloux-Schmidt inequality.

We shall devote the next few sections to prove the following theorem which may be considered as an analogue of the Milloux-Schmidt inequality.

Theorem 4.2. Suppose that $u(P)$ is subharmonic in D_R $(0 < R \leq \infty)$, and that $u > 0$ on a set D, meeting each spherical surface S_r in a set $D(r)$ subtending a solid angle at most $\mu(r) < 4\pi$, at the origin for $r_o < r < R$.

Let $$m^2(r) = \frac{1}{4\pi r^2} \int_{D(r)} u^2 \, dS_r \quad ,$$

where dS_r denotes an element of area on S_r.

Then for $r_o < r < R$, we have

$$m^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} \exp\left\{ 2 \int_{r_o}^{\eta} a(\xi) d \log \xi \right\} d\eta \; ,$$

where $a(\xi) \geq \sqrt{\left[\frac{\pi j}{\mu(\xi)}^2 - \frac{j^2 - 1}{4}\right]} - \frac{1}{2}$, and $j = 2.4048.....$ is the first positive zero for the Bessel function J_o.

4.5 Our proof of Theorem 4.2 is long, and Sections 4.5 to 4.8 will be occupied with it.

The method of measuring the growth of analytic functions (in the plane) subject to certain boundary conditions in terms of a quadratic integral norm was first introduced by Carleman (1). Later Dinghas (1) showed that the method could be extended to the smooth functions in n-dimensions. However, the general subharmonic functions are not subject to any smoothness conditions. In the plane, Heins (Heins (2)) obtained the correct lower estimate of growth for general subharmonic functions by replacing the differential inequality of Carleman by a convexity condition. Huber (1) used a more direct method of approximating a general subharmonic function by smooth functions.

We first prove a definite inequality about the growth of smooth subharmonic functions and later as in Lemma 4.3, approximate a general subharmonic function by smooth ones.

Suppose that $P_o \in D$, let $u(P_o) \geq 2\epsilon > 0$.

Consider $v(P) = u_\epsilon(P) = u(P) - \epsilon$ ·

Then the set $v(P) \geq 0$ is the set $u(P) \geq \epsilon$ which is closed subset of $u > 0$. Therefore the solid angle $\overline{\mu}(r)$ subtended at the origin by $\{P / v(P) \geq 0\} \cap S_r$ is not more than $\mu(r)$.

Thus $\overline{\mu}(r) \leq \mu(r) < 4\pi$ for $r_o < r < R$.

Also $v(P)$ is negative in an open set Ω, say, and the solid angle $(4\pi - \overline{\mu}(r))$ subtended at the origin by $\Omega \cap S_r$ satisfies the inequality $4\pi - \overline{\mu}(r) \geq 4\pi - \mu(r) > 0$ for $r_o < r < R$. Thus the set Ω extends to R and S_r meets Ω for all $r > r_o$.

Let $r_o, r_1, r_2, \ldots, r_n, \ldots \longrightarrow R$ be a sequence of values of r. Let A_n be a compact set lying inside $\Omega \cap D_{r_n}$ and approximating $\Omega \cap D_{r_n}$. We assume further that A_n is the union of a finite number of closed spheres. It is thus possible to construct a monotonic sequence of compact sets converging to Ω such that

$$A_1 \leq A_2 \leq \ldots\ldots\ldots \uparrow \Omega \ .$$

Let the distance between the sets A_n and the boundary of Ω be denoted by δ_n.

We define $v_n(P)$ as $v_\delta(P)$ of Lemma 4.3 where $\delta = \frac{\delta_n}{6}$ and instead of u(P), we have as integrand

$$v^+(P) = \mathrm{Max}\left[v(P), 0\right] = \mathrm{Max}\left[u(P), \epsilon\right] - \epsilon \ .$$

$$\text{i.e.} \qquad v_n(P) = v_{\frac{\delta_n}{6}}(P) = \int_{C_R} v^+(Q)K(P,Q)dQ \ ,$$

defined over the closed ball C_{r_n}.

Clearly the $v_n(P)$ are C^∞ functions which are zero in A_n and approximate the function $v^+(P)$ from above.

We define

$$m^2(r) = \frac{1}{4\pi r^2} \iint_{S_r} (v^+)^2 dS_r \ , \qquad (5.1)$$

$$m_n^2(r) = \frac{1}{4\pi r^2} \iint_{S_r} v_n^2 \, dS_r \ .$$

We note that as $n \longrightarrow \infty$, $\delta_n \longrightarrow 0$ and by Lemma 4.3, $v_n \downarrow v^+$ and $m_n(r) \downarrow m(r)$. (5.2)

Let $D_n(r)$ be the intersection of S_r with the complement of A_n. Let $\mu_n(r)$ be the solid angle subtended at the origin by $D_n(r)$.

We note that as $n \uparrow \infty$, $\mu_n(r) \downarrow \bar{\mu}(r)$. (5.3)

We define $\lambda_n(r)$ for the domain $D_n(r)$ as follows:

$$\lambda_n(r) = \inf \frac{r^2 \iint_{D_n(r)} \left| \text{gradient}_{S_r} f \right|^2 dS_r}{\iint_{D_n(r)} f^2 \, dS_r} \ , \tag{5.4}$$

where f ranges over all four times continuously differentiable functions which vanish continuously on the boundary of $D_n(r)$, and are not identically zero in $D_n(r)$. In (5.4), $\text{gradient}_{S_r} f$ is considered on the spherical surface S_r of D_r. It is the tangential component on S_r of the three-dimensional gradient of f. e.g. If $\underline{t}_1$, $\underline{t}_2$, $\underline{t}_3$ be the unit vectors at an arbitrary point (r,θ,ϕ) pointing in the directions of the respective coordinate lines, then

$$\text{gradient } f = \frac{\partial f}{\partial r}\,\underline{t}_1 + \frac{1}{r}\,\frac{\partial f}{\partial \Theta}\,\underline{t}_2 + \frac{1}{r \sin \Theta}\,\frac{\partial f}{\partial \phi}\,\underline{t}_3\,, \qquad (5.5)$$

$$= f_r\,\underline{t}_1 + g\,\underline{t}_2 + h\,\underline{t}_3 \qquad (5.6)$$

Now in (5.6), $f_r\,\underline{t}_1$ is the component of the gradient perpendicular to D_r, while $g\,\underline{t}_2 + h\,\underline{t}_3$ is the component of the gradient on the spherical surface S_r. Clearly from (5.4), $\lambda_n(r)$ is a non-decreasing function of n, as $n \uparrow \infty$.

We define $\alpha_n(r)$ to be the positive solution of

$$x(x+1) = \lambda_n(r)\,. \qquad (5.7)$$

If $D_n(r) = S_r$, we define $\alpha_n(r) = 0$.

Clearly from (5.7), $\alpha_n(r)$ increases as $\lambda_n(r)$ increases.

$$\text{Thus} \quad \alpha_n(r) \uparrow \alpha(r) \quad \text{as} \quad n \longrightarrow \infty\,. \qquad (5.8)$$

With the above notation, following Huber, we prove the following analogue of a differential inequality due to Carleman, namely:

$$r\,\frac{d}{dr}\left\{\log\left[r\,\frac{d}{dr}(m_n^{\,2}(r))\right]\right\} \geq 2\alpha_n(r) + 1\,. \qquad (5.9)$$

Let G_n be the complement of A_n with respect to D_{r_n}. G_n can be considered to be a regular domain for the purposes of Green's formula and we note that

$$G_n \wedge S_r = D_n(r).$$

Since v_n vanishes identically in A_n, we have

$$m_n^2(r) = \frac{1}{4\pi r^2} \iint_{S_r} v_n^2 dS_r = \frac{1}{4\pi r^2} \iint_{G_n \wedge S_r} v_n^2 dS_r . \quad (5.10)$$

Suppose that $r < r_n$.

We have by differentiation,

$$m_n(r)\, m_n'(r) = \frac{1}{4\pi r^2} \iint_{G_n \wedge S_r} v_n \frac{\partial v_n}{\partial r}\, dS_r , \quad (5.11)$$

We recall that by Green's formula

$$\iiint (u_x v_x + u_y v_y + u_z v_z) dG + \iiint v \,\Delta u \, dG = \iint v \frac{\partial u}{\partial \nu}\, dS .$$

From (5.11), we have

$$m_n(r)\, m_n'(r) = \frac{1}{4\pi r^2} \iiint_{G_n \wedge D_r} (|\text{gradient } v_n|^2 + v_n \Delta v_n) dG_n ,$$

where gradient v_n and Δv_n are the three-dimensional gradient and the Laplacian of v_n.

$$\text{Therefore} \quad (r^2 m_n(r) m_n'(r)) = \frac{1}{4\pi} \iiint_{G_n \wedge D_r} (|\text{gradient } v_n|^2 + v_n \Delta v_n) dG_n. \qquad (5.12)$$

Differentiating (5.12) again we have,

$$(r^2 m_n(r) m_n'(r))' = \frac{1}{4\pi} \iint_{G_n \wedge S_r} (|\text{gradient } v_n|^2 + v_n \Delta v_n) dS_r \, .$$

$$\text{Thus} \quad (r^2 m_n(r) m_n'(r))' \geq \frac{1}{4\pi} \iint_{G_n \wedge S_r} |\text{gradient } v_n|^2 \, dS_r \, , \qquad (5.13)$$

$$\text{since } v_n \Delta v_n \geq 0 \text{ in } G_n \wedge S_r \, .$$

As observed in (5.5) and (5.6), we have

$$|\text{gradient } v_n|^2 = \left(\frac{\partial v_n}{\partial r}\right)^2 + |\text{gradient}_{S_r} v_n|^2 \, , \qquad (5.14)$$

where $\text{gradient}_{S_r} v_n$ is the tangential component of the gradient of v_n on S_r .

Also from (5.4) and (5.7), we have,

$$\frac{r^2 \iint_{D_n(r)} \left|\operatorname{gradient}_{S_r} v_n\right|^2 dS_r}{\iint_{D_n(r)} v_n^2\, dS_r} \geq \alpha_n(r)\,(\alpha_n(r)+1) \quad .$$

Therefore
$$\iint_{D_n(r)} \left|\operatorname{gradient}_{S_r} v_n\right|^2 dS_r \geq \frac{\alpha_n(r)\big(\alpha_n(r)+1\big)}{r^2} \iint_{D_n(r)} v_n^2\, dS_r \quad .$$

$$= \alpha_n(r)\big(\alpha_n(r)+1\big).4\pi\, m_n(r)^2 \quad . \qquad (5.15)$$

From (5.13), (5.14) and (5.15), we have

$$(r^2 m_n(r) m_n'(r))' \geq \frac{1}{4\pi} \iint_{G_n \wedge S_r} \left(\frac{\partial v_n}{\partial S_r}\right)^2 dS_r + \alpha(\alpha+1) m_n^2(r) \ , \qquad (5.16)$$

where for the sake of brevity in (5.16), we suppress n, r from $\alpha_n(r)$.

We estimate the first integral on the right hand side of (5.16), by means of Schwarz's inequality as follows:

$$(m_n(r) m_n'(r))^2 = \left(\frac{1}{4\pi r^2} \iint_{G_n \wedge S_r} v_n \frac{\partial v_n}{\partial r}\, dS_r\right)^2 \quad .$$

$$\leq \left(\frac{1}{4\pi r^2} \iint_{G_n \wedge S_r} v_n^2 dS_r\right)\left(\frac{1}{4\pi r^2} \iint_{G_n \wedge S_r} \left(\frac{\partial v_n}{\partial r}\right)^2 dS_r\right) ,$$

$$\leq \; m_n^{\,2}(r) . \frac{1}{4\pi r^2} \iint\limits_{G_n \cap S_r} \left(\frac{\partial v_n}{\partial r}\right)^2 dS_r \; .$$

Thus $$\frac{1}{4\pi} \iint\limits_{G_n \cap S_r} \left(\frac{\partial v_n}{\partial r}\right)^2 dS_r \geq r^2 \left\{ m_n'(r) \right\}^2 . \qquad (5.17)$$

Substituting (5.17) in (5.16), we have

$$(r^2 m_n(r) m_n'(r))' \;\; \geq \;\; r^2 (m_n'(r))^2 + \alpha(\alpha+1) m_n^{\,2}(r) \; .$$

Hence $$r^2(m_n'(r))^2 + r^2 m_n(r) m_n''(r) + 2r m_n(r) m_n'(r) \geq r^2 (m_n'(r))^2 + \alpha(\alpha+1) m_n^{\,2}(r) \; .$$

$$r^2 m_n(r) m_n''(r) + 2r m_n(r) m_n'(r) \geq \alpha(\alpha+1) m_n^{\,2}(r) \; . \qquad (5.18)$$

Since $m_n(r) > 0$, we can divide (5.18) by $m_n(r)$, and have,

$$r^2 m_n''(r) + 2r m_n'(r) > \alpha(\alpha+1) m_n(r) \; . \qquad (5.19)$$

We now wish to simplify (5.19) and to put it in a suitable form to be integrated later.

We make the transformation:

$$t = \log r \; , \qquad \omega = \log m_n^{\,2}(r) \qquad (5.20)$$

$$\frac{d\omega}{dt} = \omega' = \frac{d\omega}{dr} . \frac{dr}{dt} = 2 . \frac{m_n'(r)}{m_n(r)} . r \; .$$

Also $\omega'' = \frac{d\omega'}{dt} = 2\,\frac{m_n(r)\left\{rm''_n(r) + m_n'(r)\right\} - r(m'_n(r))^2}{m_n^2(r)} \cdot r$;

Thus $\omega'' = 2\left(\frac{m''_n(r)}{m_n(r)}\, r^2 + r\,\frac{m'_n(r)}{m_n(r)} - \frac{r^2(m'_n(r))^2}{m_n^2(r)}\right)$.

Again, dividing (5.19) by $m_n(r)$ we have

$$r^2\,\frac{m''_n(r)}{m_n(r)} + 2r\,\frac{m'_n(r)}{m_n(r)} \geq \alpha(\alpha+1) \;.$$

i.e. $$\frac{1}{2}\,\omega'' + \frac{1}{2}\,\omega' + \frac{1}{4}\,\omega'^2 \geq \alpha(\alpha+1) \qquad (5.21)$$

$$2\,\omega'' + 2\,\omega' + \omega'^2 \geq 4\alpha(\alpha+1)$$

$$2\,\omega'' + (1 + \omega')^2 \geq (2\alpha + 1)^2 \;.$$

$$\left(1 + \omega' + \frac{\omega''}{1 + \omega'}\right)^2 \geq (2\alpha + 1)^2 + \frac{\omega''^2}{(1 + \omega')^2} \geq (2\alpha + 1)^2 \;.$$

Either $1 + \omega' + \frac{\omega''}{1 + \omega'} \geq 2\alpha + 1$, or $1 + \omega' + \frac{\omega''}{1 + \omega'} \leq -(2\alpha+1)$.

(5.22)

Now $$\omega'' + (1 + \omega')^2 = \omega'' + 1 + 2\,\omega' + \omega'^2$$

$$= \omega'' + \omega' + \tfrac{1}{2}\,\omega'^2 + 1 + \omega' + \tfrac{1}{2}\,\omega'^2$$

$$\geq 2\alpha(\alpha+1) + (1 + \tfrac{1}{2}\,\omega')^2 > 0 \text{ by } (5.21) \;.$$

Since $1 + \omega' > 0$, $2\alpha + 1 > 0$, we have $\frac{\omega''}{1+\omega'} + 1 + \omega' > 0$.

This shows that in (5.22), only the alternative $1 + \omega' + \frac{\omega''}{1+\omega'} \geq 2\alpha + 1$,

is possible.

Therefore we have

$$\omega' + 1 + \frac{d}{dt}\left\{\log\left(1 + \omega'\right)\right\} \geq 2\alpha + 1 .$$

$$\frac{d}{dt}\left\{\omega + t\right\} + \frac{d}{dt}\left\{\log\left(1 + \omega'\right)\right\} \geq 2\alpha + 1 .$$

$$\frac{d}{dt}\log\left\{e^{\omega + t}(1 + \omega')\right\} \geq 2\alpha + 1 .$$

$$\frac{d}{dt}\log\left\{\frac{d}{dt}(e^{\omega + t})\right\} \geq 2\alpha + 1 .$$

From (5.20), $e^{\omega + t} = r m_n^2(r)$ and since $\frac{d}{dt} = r\frac{d}{dr}$, we have

$$r\frac{d}{dr}\left\{\log\left[r\frac{d}{dr}\left(r\, m_n^2(r)\right)\right]\right\} \geq 2a_n(r) + 1 . \qquad (5.23)$$

which is the required inequality.

We now integrate this inequality twice to get an inequality for the growth of $m_n(r)$. So far we have shown that (5.23) holds for $r_o < r < r_n$, $n = 1, 2, \ldots$.

Thus $\frac{d}{dr}\left\{\log\left[r\frac{d}{dr}\left(r\, m_n^2(r)\right)\right]\right\} \geq \frac{2a_n(r)}{r} + \frac{1}{r}$.

Integrating both sides w.r.t.r, we have

$$\left\{\log\ r\frac{d}{dr}(r\ m_n^2(r))\right\}_{r_o}^{\eta} \geq \int_{r_o}^{\eta} 2\alpha_n(\xi)\ d\ \log\xi + \log\frac{\eta}{r_o},$$

$$\log\frac{\left[\eta\frac{d}{d\eta}(\eta\, m_n^2(\eta))\right]}{r_o(rm_n^2(r))'_{r=r_o}} \geq \log\frac{\eta}{r_o} + \int_{r_o}^{\eta} 2\alpha_n(\xi)\ d\ \log\xi ,$$

$$\frac{\eta\frac{d}{d\eta}(\eta\, m_n^2(\eta))}{r_o(rm_n^2(r))'_{r=r_o}} \geq \frac{\eta}{r_o}\exp\left\{2\int_{r_o}^{\eta}\alpha_n(\xi)\ d\ \log\xi\right\},$$

Thus $\frac{d}{d\eta}(\eta\, m_n^2(\eta)) \geq (r\ m_n^2(r))'_{r=r_o}\exp\left\{2\int_{r_o}^{\eta}\alpha_n(\xi)\ d\ \log\xi\right\}.$

Integrating both sides w.r.t. η we have,

$$\left[\eta\ m_n^2(\eta)\right]_{r_o}^{r} \geq (rm_n^2(r))'_{r=r_o}\int_{r_o}^{r}\exp\left\{2\int_{r_o}^{\eta}\alpha_n(\xi)\ d\ \log\xi\right\} d\eta .$$

$$rm_n^2(r) - r_o m_n^2(r_o) \geq (rm_n^2(r))'_{r=r_o}\int_{r_o}^{r}\exp\left\{2\int_{r_o}^{\eta}\alpha_n(\xi)\ d\ \log\xi\right\} d\eta ,$$

where $(rm_n^2(r))'_{r=r_o} = r_o(m_n^2(r))'_{r=r_o} + m_n^2(r_o) \geq m^2(r_o) > 0$.

Thus $$m_n^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} \exp\left\{ 2 \int_{r_o}^{\eta} \alpha_n(\xi)\, d \log \xi \right\} d\eta \; . \qquad (5.25)$$

Thus we have the inequality (5.25) for the growth of $m_n(r)$. We now obtain a lower bound for $\alpha_n(r)$ in terms of the solid angle subtended at the origin by the domains $D_n(r)$. We shall need two Lemmas for this which will be proved in the next two sections.

4.6

Lemma 4.4 Suppose that D is a domain on a spherical surface S_r, whose boundary consists of a finite number of polygons whose sides are circular arcs. The Rayleigh's quotient λ of the domain D is defined as

$$\inf \frac{\iint_D \left| \text{gradient}_{S_r} f \right|^2 dS_r}{\iint_D f^2 dS_r} , \qquad \text{where}$$

f ranges over all four times continuously differentiable functions in D which vanish continuously on the boundary ∂D of D, and are not identically zero in D. Then λ is the lowest eigenvalue for the eigenvalue problem:

$$\Delta f + \lambda f = 0 \text{ in } D$$
$$f = 0 \text{ on } \partial D \qquad (6.1)$$

where Δ is the spherical part of the Laplace operator of S_r .

This result is well-known and is frequently used in Applied Mathematics. The result has been proved rigorously by Garabedian (1) when D is a plane domain. Following Garabedian we shall give a proof when D is a domain on the spherical surface S_r.

We show that (i) the equation (6.1) has a discrete spectrum of positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \longrightarrow +\infty$ and a corresponding set of eigenfunctions. (ii) the lowest eigenvalue λ_1 is equal to the Rayleigh's quotient λ as defined above for the domain.

We show first from the form of the equation (6.1) that the eigenvalues are all positive. An eigenfunction which is a solution of (6.1) vanishes on ∂D, and so must have in D either a positive maximum or a negative minimum. Since Δf would be non-positive and non-negative respectively at the maxima and minima, the negative values of λ would contradict (6.1).

Also $\lambda = 0$ is not an eigenvalue of (6.1), because in that case $\Delta f = 0$ in D, $f = 0$ on ∂D, and thus the eigenfunction would be identically zero in D. Thus the eigenvalues are all positive.

We note that D is a (multiply connected) domain whose boundary consists of circular arcs and the Green's function G(P,Q) for Laplace's equation exists in D. Also the solution u of Dirichlet problem for $\Delta u + \lambda u = 0$ in D, can be represented in the form

$$u - \lambda \int_D G u \, dS = U ,$$

where U stands for the known harmonic function in D which assumes the boundary values prescribed for u. (Garabedian (1) p.342, Courant and Hilbert, Vol.2, p.262).

Conversely we have that if G(P,Q) is the Green's function for a bounded domain D then for piecewise differentiable f, the expression

$$v = \int_D G(P,Q) \, f(Q) \, dS_Q ,$$

represents a solution of the Poisson equation $\Delta v = -f$, continuous in $D + \Gamma$ and vanishing on Γ. (Courant and Hilbert, Vol.2, p.263, P.D.E. Duff, p.159).

Thus we see that the equation (6.1) is completely equivalent to the homogeneous integral equation,

$$u(P) = \lambda \int_D G(P,Q)\, u(Q)\, dS_Q \tag{6.2}$$

Since (6.2) is an integral equation with real symmetric kernel, we use the following well known properties of integral equations.

(a) A symmetric kernel always has at least one eigenvalue.

(b) The eigenvalues of a real symmetric kernel are real.

(c) The eigenfunctions corresponding to distinct eigenvalues of (6.2) are orthogonal.

These properties are proved in standard treatises e.g. Garabedian ((1), p.370-371), Tricomi ((1), p.102), in case of integral equations of the form (6.2) in the plane. However, as remarked in Tricomi ((1), p.153), this theory can be easily extended to integral equations of the form (6.2) where $P \equiv (x_1, x_2, \ldots x_n)$, and $Q \equiv (y_1, y_2, \ldots y_n)$ are two points of a fixed n-dimensional manifold E_n whose volume element around Q is designated by dS_Q.

These results make it possible to find an expression for the symmetric kernel in terms of eigenvalues and eigenfunctions.

Let λ_1 be such an eigenvalue whose existence is asserted in (a) for (6.2). Let $u_1(P)$ be its corresponding eigenfunction which is normalized so that its square integral over D is unity.

$$\int_D u_1(Q)^2 \, dS_Q = 1 \tag{6.3}$$

We now form $G_1(P,Q) = G(P,Q) - \dfrac{u_1(P)\, u_1(Q)}{\lambda_1}$. (6.4)

Then $G_1(P,Q)$ is again a real symmetric kernel. If $G_1(P,Q) \not\equiv 0$, it has an eigenvalue λ_2 and an eigenfunction $u_2(P)$.

$$\int_D u_1(Q)u_2(Q)dS_Q = \int_D u_1(Q)\, dS_Q \left\{ \lambda_2 \int_D G_1(Q,P)u_2(P)dS_P \right\} .$$

By (6.4), the right-hand side is equal to $\lambda_2 \int_D u_2(P)dS_P \int_D \left\{ u_1(Q)G(P,Q) - \right.$

$$\left. - \frac{u_1^2(Q)u_1(P)}{\lambda_1} \right\} dS_Q .$$

And by (6.2) and (6.3), this is

$$\lambda_2 \int u_2(P)\, dS_P \left\{ \frac{u_1(P)}{\lambda_1} - \frac{u_1(P)}{\lambda_1} \right\} = 0 \tag{6.5}$$

Thus $u_1(Q)$ and $u_2(Q)$ are orthogonal and so must be distinct.

We also have $u_2(P) = \lambda_2 \int_D G_1(P,Q)u_2(Q)\, dS_Q$.

$$u_2(P) = \lambda_2 \int_D G(P,Q)u_2(Q)dS_Q - \frac{\lambda_2 u_1(P)}{\lambda_1} \int_D u_1(Q)u_2(Q)dS_Q$$

And by (6.5), $$u_2(P) = \lambda_2 \int_D G(P,Q)u_2(Q)dS_Q . \qquad (6.6)$$

Hence λ_2 and $u_2(Q)$ are an eigenvalue and an eigenfunction of the original kernel G(P,Q).

This process can be repeated and we obtain

$$G(P,Q) = \sum_{i=1}^{n} \frac{u_i(P)\, u_i(Q)}{\lambda_i} + G_n(P,Q) , \qquad (6.7)$$

where $G_n(P,Q)$ is a 'remainder' after n steps.

In the present case the remainder cannot vanish after a finite number of steps. If it did, we should have

$$G(P,Q) = \sum_{i=1}^{m} \frac{u_i(P)u_i(Q)}{\lambda_i} , \qquad (6.8)$$

where $u_i(Q)$ are orthonormal functions.

The formula (6.8) is valid for all P and Q as the right-hand side of (6.8) is continuous. But if $P \longrightarrow Q$ we have as is well known $G(P,Q) \longrightarrow \infty$, and the formula (6.8) would not be correct.

It follows that the number of eigenfunctions u_n must be infinite. We show now that to no single eigenvalue can correspond more than a finite number of linearly independent eigenfunctions. We then conclude that the number of eigenvalues is infinite and that the set of eigenvalues can only accumulate at infinity.

We now suppose that $\sum_{i=1}^{\infty} \frac{u_i(P)u_i(Q)}{\lambda_i}$ denotes an expansion (as above) of G(P,Q) corresponding to an orthonormal system.

Then we have $u_i(P) = \lambda_i \int_D G(P,Q)\, u_i(Q) dS_Q$ (6.9)

$$\int_D u_i(Q)u_j(Q)dS_Q = \delta_{ij} \qquad (6.10)$$

Obviously,

$$0 \leq \int_D \left[G(P,Q) - \sum_{i=1}^{m} \frac{u_i(P)u_i(Q)}{\lambda_i}\right]^2 dS_Q \;,$$

$$= \int_D G^2(P,Q)dS_Q - \sum_{i=1}^{m} \frac{u_i(P)^2}{\lambda_i^2} \;.$$

$$\sum_{i=1}^{m} \frac{u_i(P)^2}{\lambda_i^2} \leq \int_D G^2(P,Q)dS_Q = A \text{ (say)} . \qquad (6.11)$$

Finally, integrating both sides with respect to S_P we deduce that

$$\sum_{i=1}^{m} \int_D \frac{u_i^2(P)}{\lambda_i^2}\, dS_P \leq \int_D \int_D G^2(P,Q)\, dS_Q\, dS_P \; .$$

$$\sum_{i=1}^{m} \frac{1}{\lambda_i^2} \leq M = \int_D \int_D G^2(P,Q)\, dS_Q\, dS_P \; .$$

Thus under the assumption that $\int_D G^2(P,Q)\, dS_Q$ is uniformly bounded for $P \in D$, we have

$$\sum_{i=1}^{m} \frac{1}{\lambda_i^2} \leq M \; . \tag{6.12}$$

We now show that this condition is satisfied.

We recall that G_D is the Green's function of a multiply connected domain D on the surface S_r of a sphere with respect to Δ , the spherical part of the (three dimensional) Laplace operator. Thus Δ is the second Beltrami-Laplace operator on S_r. If u and v are the curvilinear coordinates Δ can be written as

$$\Delta f = \frac{1}{\sqrt{EG - F^2}} \left\{ \frac{\partial}{\partial u} \left(\frac{Gf_u - Ff_v}{\sqrt{EG - F^2}} \right) + \frac{\partial}{\partial v} \left(\frac{Ef_v - Ff_u}{\sqrt{EG - F^2}} \right) \right\} \; ,$$

where E, F, G denote as usual the fundamental quantities of the first order for the surface S_r given by $ds^2 = Edu^2 + 2F\,du\,dv + Gdv^2$ where s is arc length.

Thus Δ is an analogue of the Laplace operator for the plane and the Green's function G_D has all the properties of the Green's function for plane domains.

Since the boundary of our domain D consists of circular arcs, the Green's function is known to exist. Also it is known that $G_D(P,Q)$ increases with expanding D.

Suppose that C_o is a circular domain in the complement of D with respect to S_r. Let D_1 denote the complement of C_o with respect to S_r. Then $D \subset D_1$ and $G_D(P,Q) < G_{D_1}(P,Q)$.

Also the Green's function is known to be positive.

Therefore

$$\int_D G_D{}^2(P,Q)dS_Q \leq \int_D G_{D_1}^2(P,Q)dS_Q \leq \int_{D_1} G_{D_1}^2(P,Q)dS_Q \; .$$

It is evident that if we take the centre of the circle C_o as the North Pole (o,o,r) then the stereographic projection maps the domain D_1 onto a circular disc of radius R say in the z-plane. This map is conformal and since Laplace's equation is invariant under conformal mapping, the Green's function corresponds to the Green's function in the transformed circular domain.

Also dS_Q the element of area on S_r near a point Q (on S_r) is less than the corresponding element of area dA on the z-plane since $dS_Q = \frac{dA}{(1+|z|^2)^2}$.

Therefore $\int_{D_1} G^2_{D_1}(P,Q)dS_Q$ is uniformly bounded if the corresponding integral in the z-plane is uniformly bounded.

The green's function for the circular disc of radius R in the z-plane is $\log\left|\frac{R^2 - z\bar{\zeta}}{R(z-\zeta)}\right|$. Thus the corresponding integral in the z-plane is

$$\int_C \left(\log\left|\frac{R^2 - z\bar{\zeta}}{R(z-\zeta)}\right|\right)^2 dA_z$$

where dA_z is an element of area near z.

The integrand is bounded above by $\left(\log\left|\frac{2R}{z-\zeta}\right|\right)^2$ and hence the integral is dominated by

$$\int_{|z-\zeta|<2R} \left(\log\left|\frac{2R}{z-\zeta}\right|\right)^2 dA_z \quad ,$$

which works out to $2\pi R^2$. Consequently the corresponding integral $\int_{D_1} G^2_{D_1}(P,Q)dS_Q$ on the spherical surface S_r is uniformly bounded.

Hence $\int_D G^2_D(P,Q)dS_Q$ is uniformly bounded.

Setting first $\lambda = \lambda_1 = \lambda_2 = \ldots = \lambda_m$, we deduce from (6.12) that $m \leq \lambda^2 M$ from which it follows that no eigenvalue can have more than finite multiplicity. Also if $\lambda_i \leq B$, we have $m \leq B^2 M$, so that there are only a finite number of eigenvalues in an interval $[O, B]$. And it follows easily from (6.12) that $\lim_{m \to \infty} \lambda_m = \infty$.

Thus the eigenvalues can only accumulate at infinity. Hence there exists a least positive eigenvalue λ_1 for the equations (6.2).

We now use the notation of the symbolic product $G \circ u = \int_D G(P,Q)\, u(Q) dS_Q$. (6.13)

We also represent the iterated kernel $G^{(n)}$ in the form $G^{(n)} = G \circ G \circ \ldots\ldots \circ G$, where there are n factors on the right. The integration is carried out with respect to the argument adjacent to the symbol o.

For example, the iterated kernel

$$G^{(2)}(P_1, P_2) = G_o G = \int_D G(P_1, Q) G(Q, P_2) dS_Q$$

is a function of P_1 and P_2.

We now state the Hilbert-Schmidt theorem which will be needed later.

HILBERT-SCHMIDT THEOREM. 'Suppose that a function can be expressed in the form $f = G \circ h$ where h is some square integrable function and G is a symmetric kernel which satisfies the inequality

$$\int_D G^2(P,Q)\,dS_Q < A \quad \text{for all } P \in D .$$

Then f has an absolutely and uniformly convergent representation

$$f(P) = \sum_{i=1}^{\infty} (u_i \circ f)u_i(P) = \sum_{i=1}^{\infty} \frac{(u_i \circ h)}{\lambda_i} u_i(P) ,$$

in terms of the eigenfunctions u_i of G.' (Garabedian (1), p.383, Tricomi (1), p.110).

Thus the Hilbert-Schmidt theorem gives us that all the iterated kernels $G^{(n)}(P,Q)$, $n \geq 2$ can be represented by the absolutely and uniformly convergent series

$$G^{(n)}(P,Q) = \sum_{i=1}^{\infty} \frac{u_i(P)u_i(Q)}{\lambda_i^n} \qquad (6.14)$$

It can be shown easily (e.g. Garabedian (1) p.385) that if f is twice continuously differentiable function in D which reduces to zero on the boundary ∂D, then f can be represented in the form

$f = G \circ g$, where g is given by the Poisson equation $g = - \Delta f$.

However, functions f of Lemma 4.4 are four times continuously differentiable and therefore have the representation $f = G \circ g$, where $g = - \Delta f$ is again twice continuously differentiable. Therefore the functions f have a representation

$$f = G \circ G \circ h \tag{6.15}$$

in terms of some square integrable function h.

Setting $a_i = u_i \circ h$, it follows from the Hilbert-Schmidt theorem that f has a uniformly convergent series expansion

$$f = \sum_{i=1}^{\infty} \frac{a_i u_i}{\lambda_i^2} \tag{6.16}$$

in terms of eigenfunctions u_i.

The Laplacian Δf can be obtained directly by applying the operator Δ to (6.16) and using Poisson's equation

$$\Delta f = - G \circ h = - \sum_{i=1}^{\infty} \frac{a_i u_i}{\lambda_i} \tag{6.17}$$

Also since $f = 0$ on ∂D, and Δf being uniformly continuous is bounded, Green's theorem gives

$$\int_D \left| \operatorname{gradient}_{S_r} f \right|^2 dS = - \int_D f \Delta f \, dS \; . \qquad (6.18)$$

The substitution of the representations (6.16) and (6.17) in (6.18) and the fact that the system u_i is orthonormal gives us the following identities.

$$\int_D \left| \operatorname{gradient}_{S_r} f \right|^2 dS = \sum_{i=1}^{\infty} \frac{a_i^2}{\lambda_i^3} \; ,$$

and $$\int_D f^2 \, dS = \sum_{i=1}^{\infty} \frac{a_i^2}{\lambda_i^4} \; .$$

Setting $C_i = \dfrac{a_i}{\lambda_i^2} = u_i \circ f$ we find for the Rayleigh's quotient, the development,

$$\frac{\int_D \left| \operatorname{grad}_{S_r} f \right|^2 dS}{\int_D f^2 \, dS} = \frac{\sum_{i=1}^{\infty} \lambda_i C_i^2}{\sum_{i=1}^{\infty} C_i^2} \; . \qquad (6.19)$$

Thus it follows from (6.19) that the lowest eigenvalue λ_1 is the smallest value which the Rayleigh's quotient can assume. This proves Lemma 4.4.

We further remark that it also follows from (6.19) that the first eigenfunction u_1 is one choice for the minimizing function as it gives us the representation (6.15) with $h = \lambda_1^2 u_1$.

4.7 We now wish to obtain a lower bound on the product $\lambda_1 r^2$, where λ_1 is the first eigenvalue for a general domain D on the spherical surface S_r. This follows from an inequality of Peetre (1) which he used to obtain a generalization of Coursant's nodal domain theorem.

Lemma 4.5 Suppose that D is a finitely connected domain with analytic boundary on a spherical surface S_r. Suppose that $\mu(r)$ is the solid angle subtended at the origin by D, and λ_1 is the first eigenvalue of the equation (6.1) for the domain D.

Then
$$\lambda_1 r^2 > \frac{\pi j^2}{\mu(r)}\left(1 - \frac{\mu(r)}{4\pi}\right), \tag{7.1}$$

where $j = 2.4048 \ldots\ldots$ is the first positive zero of the Bessel function J_o.

Let A denote the area of D and L denote the length of its boundary; Then if D is simply connected we have by the isoperimetric inequality (e.g. Hayman (4), p.152) on the sphere of radius r,

$$L^2 \geq 4\pi A\left(1 - \frac{A}{4\pi r^2}\right). \tag{7.2}$$

We show that (7.2) also holds when D is multiply connected. Let D have complementary domains D_ν with areas A_ν , and the lengths of boundaries L_ν , Let $\sum A_\nu = A' = 4\pi r^2 - A, \quad \sum L_\nu = L$.

Then $$L^2 = (\sum L_\nu)^2 \geq \sum L_\nu^2 \geq \sum 4\pi A_\nu - \sum \frac{A_\nu^2}{r^2} .$$

Therefore $$L^2 \geq 4\pi A' - \frac{(\sum A_\nu)^2}{r^2} = 4\pi A' - \frac{A'^2}{r^2}$$

$$= 4\pi A'\left(1 - \frac{A'}{4\pi r^2}\right)$$

$$= 4\pi A\left(1 - \frac{A}{4\pi r^2}\right) .$$

Thus (7.2) holds for all D.

Let u_1 be the first eigenfunction of (6.1). We recall that u_1 is a solution of an elliptic partial differential equation and is real analytic in D (Garabedian (1), p.196). We show first that u_1 does not change its sign in D.

We prove first that if an eigenfunction u vanishes at a point P_o in D, then u must necessarily assume both positive and negative values in D. Suppose this is not true. Then in a neighbourhood $N(P_o)$ we have, let us say, $u(P) \geq 0$ for $P \in N(P_o)$.
From (6.1), we then have $\Delta u \leq 0$ for $P \in N(P_o)$.
Hence u(P) is superharmonic in $N(P_o)$ without the 'super-mean' property

at P_o. This gives a contradiction. Similarly we get a contradiction if $u(P) \leq 0$ for $P \in N(P_o)$.

Thus $u(P)$ must assume both negative and positive values in $N(P_o)$. Thus the set where u is different from zero has two subdomains.

The points at which an eigenfunction vanishes are called nodes. These nodes divide the domain D into subdomains. There is a theorem of Courant (n-dimensions) that if the eigenfunctions of (6.1) are ordered according to increasing eigenvalues, then the nodes of the n^{th} eigenfunction u_n divide the domain into no more than n subdomains (Courant and Hilbert, Vol.(1), p.452).

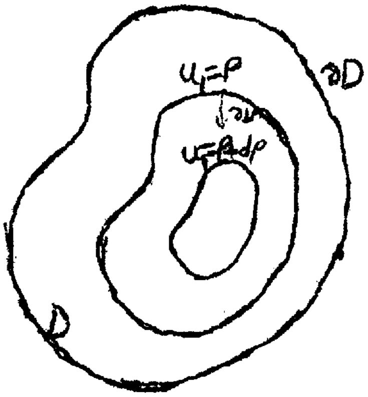
It is an immediate consequence of this theorem that if u_1 is an eigenfunction belonging to the first eigenvalue, then u_1 vanishes only on the boundary of D and nowhere in D.

Therefore by what we have shown, u_1 does not change its sign in D.

Suppose that u_1 is positive in D.

Let $\Omega(\rho)$ denote the subdomain of D where $u_1 > \rho$.

For $0 < \rho < \max u_1$, the boundary $\partial\Omega(\rho)$ of $\Omega(\rho)$ is composed of the set of level curves $u_1 = \rho$ of the eigenfunction u_1 and is therefore piecewise analytic.



Denote by s the arc length and by ν the inner normal along the level curves.

The element of area in D = ds dν $= \dfrac{ds.d\rho}{\dfrac{\partial u_1}{\partial \nu}}$.

(7.3)

Let us write $A(\rho) = \displaystyle\int_{\Omega(\rho)} dA$, $L(\rho) = \displaystyle\int_{\partial\Omega(\rho)} ds$. (7.4)

$$D(\rho) = \int_{\Omega(\rho)} |\text{gradient } u_1|^2 \, dA \,, \quad H(\rho) = \int_{\Omega(\rho)} u_1^{\,2} dA \qquad (7.5)$$

Then $A(\rho + d\rho) - A(\rho) = dA = - A'(\rho)d\rho$.

Also from (7.3), $dA = \displaystyle\int_{\partial\Omega(\rho)} ds.d\nu = d\rho \int_{\partial\Omega(\rho)} |\text{gradient } u|^{-1} \, ds.$

Similarly it can be shown that,

$$|D'(\rho)| = -D'(\rho) = \int_{\partial\Omega(\rho)} |\text{gradient } u_1| \, ds \,.$$

According to Schwarz's inequality we can write,

$$L^2(\rho) = \left(\int_{\partial\Omega(\rho)} ds \right)^2 \leq \int_{\partial\Omega(\rho)} |\text{gradient } u_1| \, ds \int_{\partial\Omega(\rho)} |\text{gradient } u_1|^{-1} \, ds.$$

$$L^2(\rho) \leq |D'(\rho)| \; |A'(\rho)| \,. \qquad (7.6)$$

By the isoperimetric inequality for these domains, we have,

$$L^2(\rho) \geq 4\pi A(\rho)\left(1 - \frac{A(\rho)}{4\pi r^2}\right). \qquad (7.7)$$

From (7.6) and (7.7), we have

$$|D'(\rho)| \geq 4\pi \frac{A(\rho)}{|A'(\rho)|} \left(1 - \frac{A(\rho)}{4\pi r^2}\right).$$

$$> 4\pi \frac{A(\rho)}{|A'(\rho)|} \left(1 - \frac{A}{4\pi r^2}\right). \qquad (7.8)$$

We now apply a process of symmetrization replacing the domains $\Omega(\rho)$ by the concentric circles $\tilde{\Omega}(\rho)$ with the same areas in the Euclidean plane.

We replace the function u_1 by a function $\tilde{u}_1$ (in the plane) which is equal to ρ on the $\partial\tilde{\Omega}(\rho)$. Thus the domain of definition of $\tilde{u}_1$ is a circle $\partial\tilde{\Omega}(o)$ (in the plane) whose area A is equal to that of the domain D on the sphere.

Thus clearly $\tilde{A}(\rho) = A(\rho)$ and $\tilde{A}'(\rho) = A'(\rho)$.

Hence from (7.8) we have,

$$|D'(\rho)| > 4\pi \frac{\tilde{A}(\rho)}{|\tilde{A}'(\rho)|} \left(1 - \frac{A}{4\pi r^2}\right) \qquad (7.9)$$

We note that in the case of the circular domains $\tilde{\Omega}(\rho)$ in the plane for the symmetrized function $\tilde{u}_1$, $\frac{\partial \tilde{u}_1}{\partial \nu}$ is constant and we get equality in (7.6).

$$\text{i.e.} \qquad \tilde{L}^2(\rho) = |\tilde{D}'(\rho)| \, |\tilde{A}'(\rho)| \, .$$

Also for the circular domains we have,

$$4\pi \tilde{A}(\rho) = \tilde{L}^2(\rho) \, .$$

$$\text{Therefore} \quad |\tilde{D}'(\rho)| = 4\pi \frac{\tilde{A}(\rho)}{|\tilde{A}'(\rho)|} \tag{7.10}$$

From (7.9) and (7.10) we get,

$$|D'(\rho)| > |\tilde{D}'(\rho)| \, (1 - \frac{A}{4\pi r^2}) \, .$$

Integration over the interval $o < \rho < \max u_1$ finally yields,

$$D > \tilde{D} \, (1 - \frac{A}{4\pi r^2}).$$

Moreover $\tilde{H}(\rho) = H(\rho)$ and $\tilde{H} = H$.

$$\text{Therefore} \quad \frac{D}{H} > \frac{\tilde{D}}{\tilde{H}}(1 - \frac{A}{4\pi r^2}) \, . \tag{7.11}$$

Thus $\lambda_1 = \frac{D}{H}$ is bounded below by $\frac{\tilde{D}}{\tilde{H}} \, (1 - \frac{A}{4\pi r^2})$, where $\tilde{D}$,

$\tilde{H}$ are defined in the plane for a circle.

By the result of Faber-Krahm the first eigenvalue is a minimum for a circle and is equal to $\frac{\pi j^2}{A}$. (Garabedian (1) p.413).

Thus we have from (7.11), that $\lambda_1 > \frac{\pi j^2}{A}\left(1 - \frac{A}{4\pi r^2}\right)$.

Since $A = \mu(r).r^2$, we have the required inequality

$$\lambda_1 r^2 > \frac{\pi j^2}{\mu(r)}\left(1 - \frac{\mu(r)}{4\pi}\right).$$

This completes the proof of Lemma 4.5 and we are now in a position to complete the proof of the theorem 4.2. We do this in the next section.

4.8 Since α is defined to be the positive solution of $x(x + 1) = \lambda$, where $\lambda = \lambda_1 r^2$ we have

$$\alpha = \frac{-1 + \sqrt{4\lambda + 1}}{2}.$$

And from (7.1), we have

$$\alpha > \tfrac{1}{2}\sqrt{\left[\frac{4\pi j^2}{\mu(r)}\left(1 - \frac{\mu(r)}{4\pi}\right) + 1\right]} - \tfrac{1}{2}.$$

$$\text{i.e.} \qquad \alpha > \sqrt{\left[\frac{\pi j^2}{\mu(r)} - \frac{j^2 - 1}{4}\right]} - \tfrac{1}{2}, \qquad (8.1)$$

for a domain D which subtends solid angle $\mu(r)$ at the origin.

Suppose that $D_n(r)$, the intersection of S_r with the complement of A_n is connected and has no isolated boundary points. Then the boundary of $D_n(r)$ consists of finite-sided polygons whose sides are arcs of circles formed by intersection of spheres in A_n with S_r. Also the function v_n is C^{∞} on $D_n(r)$ and its boundary. Thus the hypotheses of Lemmas 4.4 and 4.5 are satisfied and we have from (8.1),

$$a_n(r) > \sqrt{\left[\frac{\pi j^2}{\nu_n(r)} - \frac{j^2 - 1}{4}\right]} - \tfrac{1}{2}$$

whereas before $\nu_n(r)$ is solid angle subtended at the origin by $D_n(r)$.

We note as in (5.3) that as $n \uparrow \infty$, $\nu_n(r) \downarrow \bar{\mu}(r)$.

Therefore as $n \uparrow \infty$, $a_n(r) \uparrow \alpha(r)$ where

$$\alpha(r) \geq \sqrt{\left[\frac{\pi j^2}{\bar{\mu}(r)} - \frac{j^2 - 1}{4}\right]} - \tfrac{1}{2} .$$

Since $\bar{\mu}(r) \leq \mu(r)$, we have as $n \uparrow \infty$, $a_n(r) \uparrow \alpha(r)$,

$$\text{where} \quad \alpha(r) > \sqrt{\left[\frac{\pi j^2}{\mu(r)} - \frac{j^2 - 1}{4}\right]} - \tfrac{1}{2} \tag{8.2}$$

We recall from (5.25), that

$$m_n^2(r) \geq \frac{m^2(r)}{r} \int_{r_o}^{r} \exp\left\{ 2 \int_{r_o}^{\eta} a_n(\xi)\, d \log \xi \right\} d\eta .$$

As $n \longrightarrow \infty$, $m_n^2(r) \downarrow m^2(r)$ and $a_n(\xi) \uparrow a(\xi)$, which is given by (8.2).

$$m^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} \exp\left\{ 2 \int_{r_o}^{\eta} a(\xi) d \log \xi \right\} d\eta .$$

This proves the theorem with the difference that $m^2(r)$ is not the one defined for u in the original statement, but for $u_\varepsilon = \max\left[u, \varepsilon\right] - \varepsilon$ for a positive ε .

Letting $\varepsilon \longrightarrow 0$, we get that the result holds when $m^2(r)$ is defined for u.

In the case when $D_n(r)$ is not connected, $D_n(r)$ consists of a finite number K of domain $\Delta_{n,1}(r)$, $\Delta_{n,2}(r), \dots \Delta_{n,K}(r)$.

And let $\lambda_{n,1}, \lambda_{n,2}, \dots \lambda_{n,K}$ be the corresponding smallest eigenvalues for these domains. Then the Rayleigh's quotient for $D_n(r)$ is the least of these K eigenvalues. And it follows that (8.2) holds also in this case and the theorem follows.

We further remark that the Rayleigh's quotient is not affected by an isolated boundary point. Also $D_n(r)$ can only have a finite number of isolated boundary points as A_n is a finite union of closed spheres.

Hence (8.2) and the conclusion of the theorem holds in all cases.

4.9 We now prove the following theorem which will be needed for construction of an asymptotic path for a subharmonic function. With the previous notation, we prove,

Theorem 4.3 'Suppose that $u(P)$ is subharmonic in space. Let F_R be a component in C_R of the set $\{P | u(P) \geq 0\}$ containing a fixed point P_o such that $u(P_o) > 0$. Suppose that $u > 0$ on a set G_R in F_R. Let $a_R(\xi)$ refer to $G(\xi)$ (i.e. $G_R \wedge S_\xi$) and $a(\xi) = \lim_{R \to \infty} a_R(\xi)$.

Let $M_R = \sup_{P \in G_R} u(P)$,

Then either $M_R \longrightarrow +\infty$ as $R \longrightarrow \infty$ or (9.1)

$$\int_{r_o}^{\infty} a(\xi)\, d \log \xi < +\infty . \qquad (9.2)$$

Suppose that $R < +\infty$.

u is subharmonic in a neighbourhood of C_R.

By Lemma 4.1, all the components of $\{P | u(P) \geq 0\}$ extend to S_R.

We now define $v(P) = \begin{cases} u(P) & \text{for } P \in F_R \\ 0 & \text{elsewhere.} \end{cases}$

By Lemma 4.2, $v(P)$ is subharmonic in D_R.

Also v is positive in the set G_R.

Let $$m^2(r) = m_v^2(r) = \frac{1}{4\pi r^2} \int_{G(r)} v^2 dS_r = \frac{1}{4\pi r^2} \int_{G(r)} u^2 dS_r ,$$

as before.

Since v^2 is subharmonic $m^2(r)$ is an increasing function of r. It is non-negative.

Also $m(r)$ can be zero only if $v(P) \equiv 0$ in D_r by the Poisson-Jensen formula. Therefore $m^2(r)$ is positive for some $r = r_o$, and hence for $r_o < r < R$. We note that by Lemma 4.1, G_R meets S_r for $r_o < r < R$.

By applying Theorem 4.2 to v, we have

$$m^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} \exp\left\{ 2 \int_{r_o}^{\eta} \alpha_R(\xi)\, d \log\xi \right\} d\eta . \qquad (9.3)$$

We note that as R increases, two or more components, say F_{R_1} and F'_{R_1} of $\{p \mid u(P) \geq 0\}$ may meet in C_{R_2} for $R_2 > R_1$. Thus if $R_2 > R_1$, $G(\xi)$ for G_{R_1} is a subset of $G(\xi)$ for G_{R_2}. And consequently $\alpha_{R_1}(\xi)$ is not less than $\alpha_{R_2}(\xi)$.

Therefore if $\alpha(\xi) = \lim_{R\to\infty} \alpha_R(\xi)$, we have

$$\alpha(\xi) \leq \alpha_R(\xi) \quad \text{for all } R .$$

Thus we have $m^2(r) \geq \dfrac{m^2(r_o)}{r} \displaystyle\int_{r_o}^{r} \exp\left\{ 2 \int_{r_o}^{\eta} \alpha(\xi)\, d \log \xi \right\} d\eta.$ (9.4)

Now either $\displaystyle\int_{r_o}^{\infty} \alpha(\xi)\, d \log\xi < +\infty$

or

$$\int_{r_o}^{\eta} \alpha(\xi)\, d \log\xi = g_1(\eta) \quad \text{where } g_1(\eta) \longrightarrow +\infty \text{ as}$$

$\eta \longrightarrow \infty$.

If the second alternative holds, we have

$$m^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} \exp\left\{ 2g_1(\eta) \right\} d\eta .$$

For $r > 2r_o$ we deduce that

$$\frac{1}{r} \int_{r_o}^{r} \exp\left\{ 2g_1(\eta) \right\} d\eta \geq \frac{1}{r} \int_{\frac{1}{2}r}^{r} \exp\left\{ 2g_1(\tfrac{1}{2}r) \right\} d\eta .$$

$$\geq \tfrac{1}{2} \exp \left\{ 2g_1(\tfrac{1}{2}r) \right\} .$$

Thus $\displaystyle \frac{1}{r} \int_{r_o}^{r} \exp \left\{ 2g_1(\eta) \right\} d\eta \longrightarrow \infty$ as $r \longrightarrow \infty$.

Since $m^2(r_o) > o$, and obviously $M_R \geq m_r$, we have that the second alternative implies that

$$M_R \longrightarrow +\infty \quad \text{as} \quad R \longrightarrow \infty .$$

This completes the proof of Theorem 4.3. In the next section we study the consequences of theorems 4.2 and 4.3.

4.10 Suppose that F is a component of $\left\{ p \mid u \geq o \right\}$ in which $u > o$ somewhere. Let G be the subset of F in which $u > o$, and $G(\xi)$, and $\alpha(\xi)$ be the intersection and its corresponding α respectively as defined previously for spherical surfaces S_ξ .

Now if $\displaystyle \int_{r_o}^{\infty} \alpha(\xi)\, d \log \xi < +\infty$,

then it follows that $\alpha(\xi) < \epsilon$ except for a set of finite logarithmic measure on the r-axis. We recall that if the solid angle $\mu(\xi)$ subtended by $G(\xi)$ at the origin is not more than $4\pi - \epsilon_1$, we have

$$a(\xi) \geq \sqrt{\left[\frac{\pi i^2}{4\pi - \epsilon_1} - \frac{i^2-1}{4}\right]} - \tfrac{1}{2} ,$$

$$= \sqrt{\left[\frac{1}{4} + \frac{\epsilon_1 i^2}{4(4\pi - \epsilon_1)}\right]} - \tfrac{1}{2} ,$$

$$= f(\epsilon_1) > 0 .$$

This shows that if $\int_{r_0}^{\infty} a(\xi)\, d \log \xi < +\infty$, the solid angle $\mu(\xi)$ subtended by $G(\xi)$, at the origin is greater than $4\pi - \epsilon_1$, outside a set of finite logarithmic measure.

If a component has this property then its complement and so every subset of the complement has the property that its intersection $G(\xi)$ subtends an angle not more than ϵ_1 on a set of density one on the r-axis.

Hence for the components having this property which we shall call the 'smallness' property,

$$\int_{r_0}^{\infty} a(\xi)\, d \log \xi \quad \text{is unbounded.}$$

Consequently $M_R \longrightarrow +\infty$ as $R \longrightarrow \infty$ if $m^2(r_0) > 0$. Thus it follows that if P_1 and P_2 are distinct points where $u > 0$, then either F_R is finally identically the same for P_1 and P_2 or $M_R \longrightarrow +\infty$ for the component corresponding to at least one of these two points.

We shall say that the set $\{P|u(P) \geq O\}$ or $\{P|u(P) > O\}$ has only one component if any fixed points in space where $u \geq O$ or $u > 0$ belong to the same component of $\{P|u(P) \geq O\}$ or $\{P|u(P)>0\}$ in C_R for all sufficiently large R. We note that if u is subharmonic and bounded in space, then the set $\{P|u(P) \geq O\}$ or $\{P|u(P)>0\}$ can have only one component in space in this sense.

4.11 Now we are in a position to prove the following theorem which is the analogue of Theorem 1.1 in the continuous case.

Theorem 4.4 Suppose that u(P) is a continuous non-constant subharmonic function in space. Then if u(P) is unbounded in space, there exists a path Γ tending to ∞ on which u(P) tends to $+\infty$; and if u(P) is bounded above in space, there exists a path Γ tending to ∞ on which u(P) tends to M where M is the upper bound of u(P) in space.

Since u(P) is a continuous subharmonic function, the set $\{P|u(P) > O\}$ is open and consists of at most a sequence of domains G_ν, $\nu = 1,2,\ldots$. Also $u(P) = O$ on the boundary of each G_ν .
Therefore by the Maximum-principle each G_ν extends to infinity.
Suppose that u(P) is bounded above in space and that M is the upper bound of u(P) in space.

We claim that there exists a point P_1 such that $u(P_1) > M - \frac{1}{2}$.

For if it did not exist, the upper bound of $u(P)$ would be $M - \frac{1}{2}$.

It follows from theorem 4.3 that in this case there is only one component of the set $\left\{P \middle| u(P) > M - \frac{1}{2}\right\}$ in space, namely the one containing P_1.

In this component we choose P_2 such that $u(P_2) > M - \frac{1}{2^2}$.

Join P_1 to P_2 by a continuous curve γ_1 lying in the domain $\left\{P \middle| u(P) > M - \frac{1}{2}\right\}$.

Now P_2 lies in $\left\{P \middle| u(P) > M - \frac{1}{2^2}\right\}$.

Also this set has only one component, namely the one containing P_2.

Again choose P_3 in $\left\{P \middle| u(P) > M - \frac{1}{2^2}\right\}$ such that $u(P_3) > M - \frac{1}{2^3}$.

As before join P_2 to P_3 by a continuous curve γ_2 lying in the domain $\left\{P \middle| u(P) > M - \frac{1}{2^2}\right\}$.

Thus by a step by step argument we get a path $\Gamma = \gamma_1 + \gamma_2 + \ldots$ such that $u(P_n) > M - \frac{1}{2^n}$ and $u(P) > M - \frac{1}{2^{n-1}}$ on γ_n joining P_{n-1} to P_n.

Thus we have an asymptotic path Γ , such that $u(P) \longrightarrow M$, as $P \longrightarrow \infty$ on Γ .

We now consider the case when $u(P)$ is unbounded in space.

We remark that if $u(P)$ is a continuous subharmonic function, it can be shown easily that the modified function $v(P)$ defined as below is also subharmonic.

$$v(P) = \begin{cases} u(P) & \text{in a component of } \{P \mid u(P) > 0\}, \\ 0 & \text{elsewhere}. \end{cases}$$

Consequently the conclusions of theorem 4.3 hold for a continuous subharmonic function when G_R is a component of $\{P \mid u(P) > 0\}$ in C_R instead of being the set where u is positive in a component F_R of $\{P \mid u(P) \geq 0\}$ inside C_R.

Suppose that P_o is a point such that $u(P_o) > 0$. Let G_R be the component of $\{P \mid u(P) > 0\}$ in C_R containing P_o. Then by Theorem 4.3, either

$$M_R \longrightarrow +\infty, \quad \text{as } R \longrightarrow \infty;$$

or

$$\int_{r_o}^{\infty} a(\xi)\, d\log\xi < +\infty. \tag{11.1}$$

We noted in section 4.10 that if the second alternative holds, then every other component of $\{P \mid u(P) > 0\}$ in space has the 'smallness' property.

Suppose that the set $\{P \mid u(P) > 0\}$ has more than one component.

Then either u is unbounded in every component G of $\left\{P \middle| u(P) > K\right\}$ for any $K (> O)$ in the sense that

$$M_R = \sup_{P \in G \wedge C_R} u(P) \longrightarrow +\infty \text{ as } R \longrightarrow \infty ,$$

or there exists a component G' of $\left\{P \middle| u(P) > K\right\}$ for some K in which u is bounded.

Since u is unbounded, there exists another component G'' of $\left\{P \middle| u(P) > K\right\}$. This component and any subset of it has the 'smallness' property. Therefore u is unbounded in G'' and in every component of $\left\{P \middle| u(P) > K'\right\}$ lying in G'' where $K' > K$.

As the set $\left\{P \middle| u(P) > O\right\}$ has more than one component, we can choose a component G(o) containing a point P_o such that u(P) is unbounded in G(o) and in every component of $\left\{P \middle| u(P) > K\right\}$ (for any $K > O$) lying in G(o).

Let $G_R(o)$ be the component of $\left\{P \middle| u(P) > O\right\}$ containing the point P_o.

Then $\quad M_R(o) \longrightarrow +\infty , \quad$ as $\quad R \longrightarrow \infty ,$

$$\text{where} \quad M_R(o) = \sup_{P \in G_R(o)} u(P) .$$

Choose R_1 such that $M_{R_1}(o) > 1$.

There exist points in $G_{R_1}(o)$ on S_{R_1} on which $u(P) > 1$. By continuity

$u(P) > 1$ in a neighbourhood of such points. Thus we can find a point P_1 in a domain $G_{R_1}(o) \setminus S_{R_1}$ such that $u(P_1) > 1$.

We joint P_o to P_1 by a continuous curve γ_1 lying in a domain $G_{R_1}(o) \setminus S_{R_1}$.

Let $G_R(1)$ be the component of $\{P | u(P) > 1\}$ inside C_R containing P_1.

Then this component of $\{P | u(P) > 1\}$ also has unboundedness property, i.e.

$$M_R(1) \longrightarrow + \infty \ , \quad \text{as } R \longrightarrow \infty \ .$$

Choose R_2 such that $M_{R_2}(1) > 2$.

As before find P_2 in $G_{R_2}(1) \setminus S_{R_2}$ such that $u(P_2) > 2$.

We again join P_1 to P_2 by a continuous curve γ_2 lying in $G_{R_2} \setminus S_{R_2}$.

Continuing in this way by a step by step argument we get a path Γ = $\gamma_1 + \gamma_2 + \dots + \gamma_n + \dots$ passing through a sequence of points $\{P_n\}$ such that $u(P_n) > n$ and $u(P) > n-1$ on γ_n joining P_{n-1} to P_n.

Thus on path Γ we have that

$$u(P) \longrightarrow + \infty \ , \quad \text{as } P \longrightarrow \infty \ .$$

We note that in the proof we assumed that the set $\{P | u(P) > O\}$ hcs more than one component. If the set $\{P | u(P) > K\}$ has only one component for all ~~integral~~ real values of K, then $u(P)$ is unbounded in that component and the asymptotic path can be easily constructed by the classical argument. If for some K, the set $\{P \mid u(P) > K\}$ has more than one component, then we choose a component G which has the property that u is unbounded in G and in every component of $\{P \mid u(P) > K'\}$ lying in G when $K' > K$. By the above method we can construct an asymptotic path Γ in G such that $u(P) \longrightarrow +\infty$ as $P \longrightarrow \infty$ on Γ.

This completes the proof of Theorem 4.4.

4.12 In the case of a general subharmonic function we are able to show the existence of a continuum Γ on which u(P) tends to the upper bound M in space. In the case when M is finite, we shall show in the next few sections that the continuum can be approximated by a polygonal path.

Theorem 4.5 If u(P) is a subharmonic function in space, there exists a continuum Γ such that $u(P) \longrightarrow M$, as $P \longrightarrow \infty$ on Γ, where M is the upper bound of u(P) in space.

Suppose that M is finite.

Choose a point P_1 such that $u(P_1) > M - \frac{1}{2}$.

Such a point exists for if it did not exist, $M - \frac{1}{2}$ would be the upper bound of u(P) in space.

Choose a point P_2 such that $u(P_2) > M - \frac{1}{2^2}$.

Since u(P) is bounded in space, it follows from Theorem 4.3 that the set $\left\{P \middle| u(P) \geq M - \frac{1}{2}\right\}$ or $\left\{P \middle| u(P) > M - \frac{1}{2}\right\}$ has only one component in space.

In other words for sufficiently large values of R, the points P_1 and P_2 would belong to the same component F_R of $\left\{P \middle| u(P) \geq M - \frac{1}{2}\right\}$ in C_R.

For if it were not so for any finite value of R, the set $\left\{P \middle| u(P) > M - \frac{1}{2}\right\}$

would have two components and by theorem 4.3, this would contradict the hypothesis that M is finite.

Suppose that R_1 has this property for P_1 and P_2. Then the component $F_{R_1}(1)$ of $\left\{P \middle| u(P) \geq M - \frac{1}{2}\right\}$ in C_{R_1} containing P_1 and P_2 extends to the boundary S_{R_1} by Lemma 4.1.

And we can join P_1 to P_2 by a subcontinuum γ_1 of $F_{R_1}(1)$.

We now choose P_3 such that $u(P_3) > M - \frac{1}{2^3}$.

Similarly choose R_2 so large that P_2 and P_3 belong to the same component $F_{R_2}(2)$ of $\left\{P \middle| u(P) \geq M - \frac{1}{2^2}\right\}$ in C_{R_2}. Again we join P_2 to P_3 by a subcontinuum γ_2 of $F_{R_2}(2)$ in C_{R_2} and we have that $u(P) \geq M - \frac{1}{2^2}$ for $P \in \gamma_2$.

Continuing in this way we have $\Gamma = \gamma_1 + \gamma_2 + \dots\dots$ passing through a sequence of points $\{P_n\}$ such that $u(P_n) > M - \frac{1}{2^n}$ and $u(P) > M - \frac{1}{2^{n-1}}$ for $P \in \gamma_n$ joining P_{n-1} and P_n.

Thus there exists a continuum Γ on which

$$u(P) \longrightarrow M, \quad \text{as } P \longrightarrow \infty .$$

We now consider the case when $u(P)$ is unbounded in space.

Then either u is unbounded in every component of $\left\{ P \middle| u(P) \geq K \right\}$ for any K (> O) in the sense that

$$M_R \longrightarrow +\infty \text{ as } R \longrightarrow \infty .$$

Or there exists a component F' of $\left\{ P \middle| u(P) \geq K \right\}$ for some K in which u is bounded.

Since u is unbounded, there exists another component F'' of $\left\{ P \middle| u(P) \geq K \right\}$.

This component and any subset of it has the 'smallness' property. Therefore u is unbounded in F'' and in every component of $\left\{ P \middle| u(P) \geq K' \right\}$ lying in F'' where K' > K.

Suppose that $\left\{ P \middle| u(P) \geq O \right\}$ has more than one component then we can choose a component F(o) containing a point P_o with the property that u is unbounded in F(o) and in every component of $\left\{ P \middle| u(P) \geq K \right\}$ (for any K > O) lying in F(o).

Let $F_R(o)$ be the component of $\left\{ P \middle| u(P) \geq O \right\}$ inside C_R containing the point P_o.

Then $M_R(o) \longrightarrow +\infty$ as $R \longrightarrow \infty$.

Choose R_1 such that $M_{R_1}(o) > 1$.

Choose a point P_1 in $F_{R_1}(o)$ such that $u(P_2) > 1$.

Join P_o to P_1 by a subcontinuum γ_1 of $F_{R_1}(o)$.

Let $F_R(1)$ be the component of $\{P | u(P) \geq 1\}$ inside C_R containing P_1. Then this component $F_R(1)$ also has the unboundedness property. Choose R_2 such that $M_{R_2}(1) > 2$.

Let P_2 be a point in $F_{R_2}(1)$ such that $u(P_2) > 2$.

Again as before join P_1 to P_2 by a sub-continuum γ_2 of $F_{R_2}(1)$.

Continuing in this way we have a continuum $\Gamma = \gamma_1 + \gamma_2 + \dots$ passing through a sequence of points $\{P_n\}$ such that $u(P_n) > n$ and $u(P) > n-1$ for $P \in \gamma_{n-1}$ joining P_{n-1} to P_n.

Thus there exists a continuum Γ on which $u(P) \longrightarrow +\infty$ as $P \longrightarrow \infty$ on Γ.

We note that we assumed that the set $\{P | u(P) \geq 0\}$ has more than one component. If the set $\{P | u(P) \geq K\}$ has one component in C_R for all real K and R, then the same argument as for finite M gives us the required continuum. If at some stage the set splits into more than one component, it follows from the above argument that the required continuum exists in at least one component.

This completes the proof of Theorem 4.5.

4.13. Now we prove an analogue of Hayman's theorem (Lemma 1.6) in the plane about the minimum of u(P) on rays going out from the origin.

We will usually denote by P a point inside D_R and Q a point on the surface S_R. Let $K_R(P,Q)$ denote the Poisson Kernel of D_R so that if v(P) is harmonic in D_R, continuous in C_R, we have

$$v(P) = \iint_{S_R} K_R(P,Q)\, v(Q)\, dS_Q \ , \qquad (13.1)$$

where dS_Q is the area element of Q on S_R .

Let $G_R(P,T)$ denote the Green's function of D_R. If P and T are both in D_R, this is given by

$$G_R(P,T) = \frac{1}{PT} - \frac{R}{r}\frac{1}{P'T} \qquad (13.2)$$

where OP = r, and P' lies on OP extended and $OP' = \frac{R^2}{r}$.

Since the Green's function is symmetric $G_R(P,T) = G_R(T,P)$ and $G_R(P,T)$

can also be written as $\frac{1}{PT} - \frac{R}{r_1}\frac{1}{T'P}$ where $OT = r_1$, (13.2a)

and T' as before is the inverse of T.

We first prove an analogue of the Poisson-Jensen formula.

Lemma 4.6. Suppose that $\omega(P)$ is subharmonic in space. For every $R > O$, there exists a unique non-negative distribution $\mu(e)$ defined for all Borel measurable sets e in space and finite on compact sets, such that for all $P \in D_R$, we have

$$\omega(P) = \iint_{Q \in S_R} \omega(Q) K_R(P,Q) dS_Q - \int_{T \in D_R} G_R(P,T)\, d\mu(e_T). \qquad (13.3)$$

We recall the Fundamental theorem of F. Riesz in space (Evans (1), p.237).

If $\omega(M)$ is subharmonic in a bounded domain D, and Ω is a domain contained with its boundary in D, then $\omega(M)$ may be written in the form

$$\omega(M) = v(M) - u(M) \quad \text{for} \quad M \in \Omega ,$$

where $u(M) = \int_{T \in \Omega} \frac{1}{MT}\, d\mu(e_T)$ i.e. it is a potential of a distribution of positive mass on Ω , finite in total amount, and v(M) is harmonic in Ω'.

$\omega(P)$ is subharmonic in $C_{R+\epsilon}$ for some $\epsilon > 0$.

Choose R' such that $R < R' < R + \epsilon$. Then the Riesz mass in $D_{R'}$ is finite.

Also for all points Q on S_R, we have by Riesz's theorem,

$$\omega(Q) = v(Q) - \int_{T \in D_{R'}} \frac{1}{QT} \, d\mu(e_T) .$$

By using (13.1), we have

$$\iint_{S_R} K_R(P,Q)\omega(Q)dS_Q = v(P) - \iint_{S_R} K_R(P,Q)dS_Q \left\{ \int_{T \in D_{R'}} \frac{1}{QT} \, d\mu(e_T) \right\} \tag{13.4}$$

We invert the order of integration in the 2nd term on the right-hand side of (13.4). This is justified since the integrand is positive.

Thus
$$\iint_{S_R} K_R(P,Q)\omega(Q)dS_Q = v(P) - \int_{T \in D_{R'}} d\mu(e_T) \iint_{S_R} K_R(P,Q) \frac{1}{QT} \, dS_Q. \tag{13.5}$$

Let OT = r_1 and as before let T' be a point on OT extended such that $OT' = \frac{R^2}{r_1}$.

Suppose that Q and T belong to C_R .

Then $\frac{1}{QT}$ and $\frac{1}{QT'}$ are harmonic functions of Q , and the former has a singularity inside C_R .

And we have

$$\iint_{S_R} K_R(P,Q)\frac{1}{QT}\,dS_Q = \iint_{\substack{S_R \\ \text{and } T\in D_{R'} \\ \text{but } T\notin C_R}} K_R(P,Q)\frac{1}{QT}\,dS_Q + \iint_{\substack{S_R \\ T\in D_R}} K_R(P,Q)\frac{1}{QT}\,dS_Q + \iint_{\substack{S_R \\ T\in S_R}} K_R(P,Q)\frac{1}{QT}\,dS_Q$$

By (13.1), the first integral on the right hand side is equal to $\frac{1}{PT}$.

It is well known that if T and T' are inverse points with respect to a sphere and Q a point on the surface of the sphere, then $\frac{QT}{T'Q} = \frac{r_1}{R}$ when r_1 = QT and R is the radius of the sphere.

If T is in D_R, then T' is outside C_R and in this case,

$$\iint_{T\in D_R} K_R(P,Q)\frac{1}{QT}\,dS_Q = \iint_{\substack{S_R \\ T\in D_R}} K_R(P,Q)\frac{R}{r_1}\,\frac{1}{T'Q}\,dS_Q = \frac{R}{r_1}\,\frac{1}{T'P} .$$

We note that $\iint_{S_R} K_R(P,Q)\,\frac{1}{QT}\,dS_Q = \frac{1}{PT}$, when T is outside the closed ball C_R. Also the integral is equal to $\frac{R}{r_1}\,\frac{1}{T'P}$ when T is inside the open hall D_R. Also the integral is a continuous function of T.

Also on the surface S_R, T is its own inverse and $\frac{R}{r_1}\,\frac{1}{T'P} = \frac{1}{PT}$. Therefore the value of the integral for T on the surface is the common value $\frac{1}{PT} = \frac{R}{r_1}\,\frac{1}{T'P}$, when $T' = T$ and $R = r_1$.

From (13.5), we have

$$\iint_{S_R} K_R(P,Q)\,\omega(Q)\,dS_Q = v(P) - \int_{T\in D_{R'}\setminus D_R} \frac{1}{PT}\,d\mu(e_T) - \int_{T\in D_R} \frac{R}{r_1}\,\frac{1}{T'P}\,d\mu(e_T) \tag{13.6}$$

We note from (13.2) and (13.2a) that

$$r_1 T'P = rP'T.$$

From (13.6), we have

$$\iint_{S_R} K_R(P,Q)\omega(Q)\,dS_Q = v(P) - \int_{T\in D_{R'}\setminus D_R} \frac{1}{PT}\,d\mu(e_T) - \int_{T\in D_R} \frac{R}{r}\,\frac{1}{P'T}\,d\mu(e_T).$$

$$= v(P) - \int_{T \in D_{R'}} \frac{1}{PT}\, d\mu(e_T) + \int_{T \in D_R} \left(\frac{1}{PT} - \frac{R}{r}\frac{1}{P'T}\right) d\mu(e_T) \,.$$

$$= v(P) - u(P) + \int_{T \in D_R} G_R(P,T)\, d\mu(e_T) \,.$$

$$= \omega(P) + \int_{T \in D_R} G_R(P,T)\, d\mu(e_T) \,,$$

which is the required result.

4.14 We now introduce some more notation and then prove an analogue of Lemma 1.6 in space.

$$\omega^+(P) = \text{Max}\left[0, \omega(P)\right] \quad , \qquad \omega^-(P) = -\,\text{Min}\left[0, \omega(P)\right] \quad ,$$

$$\omega_1(r) = \sup_{0 \le t \le r} \omega^-(t, \theta, \phi) \text{ for fixed } \theta \text{ and } \phi \,.$$

We define $\psi(t) = \dfrac{\sqrt{t}}{1 - \sqrt{t}}$.

<u>Theorem 4.6</u> <u>Suppose that $\omega(P)$ is subharmonic in a neighbourhood of a closed ball</u> C_R , <u>then with the above notation</u>

$$\frac{1}{4\pi r^2} \iint_{S_r} \omega_1(r)\, dS_r < \left[\frac{\pi}{4} + \frac{1}{2} + \psi\left(\frac{r}{R}\right)\right] \frac{1}{4\pi R^2} \iint_{S_R} \left[\omega^+(Q) - \omega(O)\right] dS_Q \tag{14.1}$$

Following Hayman [1, Lemma 1, p.186], we estimate first the integral means for the suprema of the Poisson kernel and the Green's function on the radial segments.

Suppose that P lies on the radius joining the North Pole (O,O,R) and the Origin. Let $L^{POT} = \Theta$ and $OT = r_\mu$, $OP = r$.

Let

$$K_R(P,Q) = K(R,r,\Theta) = \frac{R(R^2 - r^2)}{(R^2 + r^2 - 2rR\cos\Theta)^{3/2}} \quad (14.2)$$

Let

$$k(R,r,\Theta) = \sup_{0 \le t \le r} \frac{R(R^2 - t^2)}{(R^2 + t^2 - 2tR\cos\Theta)^{3/2}} \quad (14.3)$$

We recall that the Green's function for D_R is $G_R(P,T) = \frac{1}{P_T} - \frac{R}{r}\frac{1}{P'_T}$.

We write $G_R(P,T) = G(R,r,r_\mu,\Theta)$.

Let $g(R,r,r_\mu,\Theta) = \sup\limits_{0 \le t \le r} G(R,t,r_\mu,\Theta)$ (14.4)

Lemma 4.7 With the above notation, we have

(i) $$\frac{1}{4\pi r^2}\iint_{S_r} k(R,r,\Theta)\, dS_r < 1 + \tfrac{1}{2}\log\frac{R+r}{R-r}. \quad (14.5)$$

(ii) $$\frac{1}{4\pi r^2}\iint_{S_r} g(R,r,r_\mu,\Theta)\, dS_r < \left(\frac{\pi}{4} + \frac{1}{2}\right)\frac{1}{r_\mu} - \frac{1}{R} + \frac{r^2 r_\mu^{\,2}}{4R^5} \quad (14.6)$$

Let $\quad I = \dfrac{1}{4\pi r^2} \iint\limits_{S_r} k(R,r,\Theta) dS_r$.

Then $\quad I = \dfrac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} k(R,r,\Theta) \sin \Theta \, d\Theta \, d\phi = \frac{1}{2} \int_0^{\pi} k(R,r,\Theta) \sin \Theta \, d\Theta$. (14.7)

If in (14.3), we substitute $t = t_1 R$, we have

$$k(R,r,\Theta) = \sup_{0 \le t_1 \le r_1 = \frac{r}{R}} \frac{(1 - t_1^2)}{(1 + t_1^2 - 2t_1 \cos \Theta)^{3/2}} .$$

We note that for $\frac{\pi}{2} \le \Theta \le \pi$, $K(R,r,\Theta)$ is a decreasing function of r and therefore,

$$k(R,r,\Theta) = 1 , \qquad \text{for } \frac{\pi}{2} \le \Theta \le \pi .$$

$$\frac{1 - t_1^2}{(1 + t_1^2 - 2t_1 \cos \Theta)^{3/2}} = \frac{1 - t_1^2}{(1 + t_1^2 - 2t_1 \cos \Theta)} \cdot \frac{1}{(1 + t_1^2 - 2t_1 \cos \Theta)^{\frac{1}{2}}} .$$

For $0 < \Theta < \frac{\pi}{2}$, we note that $\dfrac{1 - t_1^2}{1 + t_1^2 - 2t_1 \cos \Theta}$ increases from 1 to $\dfrac{1}{\sin \Theta}$ as t_1 increases from 0 to $\dfrac{\cos \Theta}{1 + \sin \Theta}$, and then decreases again. Also $\dfrac{1}{(1 + t_1^2 - 2t_1 \cos \Theta)^{\frac{1}{2}}}$ increases from 1 to $\dfrac{1}{\sin \Theta}$

as t_1 increases from 0 to $\cos\theta$, and then decreases again.

Since $\cos\theta > \dfrac{\cos\theta}{1+\sin\theta}$, we note that $\dfrac{1-t_1^2}{(1+t_1^2-2t_1\cos\theta)^{3/2}}$

is an increasing function of t_1 for θ between 0 and θ_o, where θ_o is given by

$$\frac{\cos\theta_o}{1+\sin\theta_o} = r_1 . \qquad (14.8)$$

Also in the range $\theta_o \leq \theta < \frac{\pi}{2}$,

$$k(R,r,\theta) \leq \sup_{o\leq t_1\leq r_1} \frac{(1-t_1^2)}{(1+t_1^2-2t_1\cos\theta)} \sup_{o\leq t_1\leq r_1} \frac{1}{\sqrt{1+t_1^2-2t_1\cos\theta}} .$$

$$\leq \frac{1}{\sin^2\theta} .$$

$$\text{Therefore} \quad k(R,r,\theta) \leq \begin{cases} K(R,r,\theta) & \text{for } o < \theta < \theta_o , \\ \dfrac{1}{\sin^2\theta} & \theta_o < \theta < \dfrac{\pi}{2} , \\ 1 & \dfrac{\pi}{2} < \theta < \pi . \end{cases}$$

We deduce from (14.8), that $\cos\theta_o = \dfrac{2r_1}{1+r_1^2}$ and $\tan\dfrac{\theta_o}{2} = \dfrac{1-r_1}{1+r_1}$. (14.9)

Thus
$$I \le \frac{1}{2}\left\{ \int_0^{\theta_0} \frac{(1 - r_1^2)\,\mathrm{Sin}\,\theta\, d\theta}{(1 + r_1^2 - 2r_1 \cos\theta)^{3/2}} + \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta}{\sin^2\theta}\, d\theta + \right.$$

$$\left. + \int_{\frac{\pi}{2}}^{\pi} \sin\theta\, d\theta \right\} .$$

$$\le \frac{1}{2}\left\{ \left[\frac{-(1 - r_1^2)}{r_1(1 + r_1^2 - 2r_1 \cos\theta)^{\frac{1}{2}}} \right]_0^{\theta_0} + \left[\log \tan\frac{\theta}{2} \right]_{\theta_0}^{\frac{\pi}{2}} - \right.$$

$$\left. - \left[\cos\theta \right]_{\frac{\pi}{2}}^{\pi} \right\} .$$

Substituting the values of θ from (14.9), we get

$$I \le \frac{1}{2}\left\{ \left[\frac{-(1 - r_1^2)}{r_1\left(1 + r_1^2 - \frac{4r_1^2}{1 + r_1^2}\right)^{\frac{1}{2}}} + \frac{1 - r_1^2}{r_1(1 - r_1)} \right] + \log\frac{1 + r_1}{1 - r_1} + 1 \right\} ,$$

$$= \frac{1}{2}\left\{ \frac{-(1 - r_1^2)\sqrt{1 + r_1^2}}{r_1(1 - r_1^2)} + \frac{1 + r_1}{r_1} + \log\frac{1 + r_1}{1 - r_1} + 1 \right\} ,$$

$$= \tfrac{1}{2} \left\{ 2 + \log \frac{1 + r_1}{1 - r_1} + \frac{1 - \sqrt{1 + r_1^2}}{r_1} \right\} .$$

Thus $I \leq 1 + \tfrac{1}{2} \log \dfrac{1 + r_1}{1 - r_1} + \tfrac{1}{2} \dfrac{1 - \sqrt{1 + r_1^2}}{r_1}$.

Substituting $r_1 = \dfrac{r}{R}$, we have

$$I \leq 1 + \tfrac{1}{2} \log \frac{R + r}{R - r} + \tfrac{1}{2} \frac{R - \sqrt{R^2 + r^2}}{r} .$$

Since the third term on the right hand side is negative, we have,

$$I < 1 + \tfrac{1}{2} \log \frac{R + r}{R - r} .$$

(ii) We observe that in our notation (14.4),

$$G(R, r, r_\mu, \theta) = \frac{1}{\sqrt{r^2 + r_\mu^2 - 2r\, r_\mu \cos \theta}} - \frac{R}{r} \frac{1}{\sqrt{\frac{R^4}{r^2} + r_\mu^2 - \frac{2R^2}{r} r_\mu \cos \theta}}$$

$$= \frac{1}{\sqrt{r^2 + r_\mu^2 - 2r\, r_\mu \cos \theta}} - \frac{R}{\sqrt{R^4 + r^2 r_\mu^2 - 2R^2 r\, r_\mu \cos \theta}}$$

If we substitute $r = \rho R$ and $r_\mu = \rho_\mu R$, then we have

$$G(R,r,r_\mu\ ,\Theta) = \frac{1}{R}\left[\frac{1}{\sqrt{\rho^2+\rho_\mu{}^2-2\rho\,\rho_\mu\ \cos\Theta}} - \frac{1}{\sqrt{1+\rho^2\rho_\mu{}^2-2\rho\,\rho_\mu\ \cos\Theta}}\right] \qquad (14.10)$$

We wish to estimate $I_2 = \frac{1}{4\pi r^2}\iint_{S_r} g(R,r,r_\mu\ ,\Theta)\ dS_r$.

Then $I_2 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} g(R,r,r_\mu\ ,\Theta)\ \sin\Theta\ d\Theta\ d\phi = \frac{1}{2}\int_0^{\pi} g(R,r,r_\mu\ ,\Theta)\sin\Theta d\Theta$.

We write

$$I_2 = \frac{1}{2}\int_0^{\pi/2} g(R,r,r_\mu\ ,\Theta)\sin\Theta d\Theta + \int_{\pi/2}^{\pi} g(R,r,r_\mu\ ,\Theta)\ \sin\Theta\ d\Theta\ .$$

$$I_2 = I_3 + I_4\ (\text{say})\ . \qquad (14.11)$$

I_4 can be easily calculated as follows.

From (14.10), $G(R,r,r_\mu\ ,\Theta) =$

$$= \frac{1}{R}\left[\frac{(1+\rho^2\rho_\mu{}^2-2\rho\rho_\mu\ \cos\Theta)-(\rho^2+\rho_\mu^2\ -2\rho\rho_\mu\ \cos\Theta)}{D}\right]$$

where $D = \sqrt{\rho^2+\rho_\mu^2 - 2\rho\rho_\mu \cos\Theta}\,\sqrt{1+\rho^2\rho_\mu^2 - 2\rho\rho_\mu \cos\Theta}\left[\sqrt{\rho^2+\rho_\mu^2 - 2\rho\rho_\mu \cos\Theta} + \sqrt{\rho^2\rho_\mu^2 + 1 - 2\rho\rho_\mu \cos\Theta}\right]$.

Therefore $$G(R,r,r_\mu,\Theta) = \frac{1}{R}\,\frac{(1-\rho^2)(1-\rho_\mu^2)}{D} \quad . \qquad (14.12)$$

For $\frac{\pi}{2} < \Theta < \pi$, the expressions under the radical sign are increasing functions of ρ (and hence of r), and since in (14.12), the numerator also decreases as ρ increases, we have that in this range $G(R,r,r_\mu,\Theta)$ is a decreasing function of r and hence its maximum is at $r = 0$ and is equal to $\frac{1}{R}(\frac{1}{\rho_\mu} - 1)$.

Thus $$I_4 = \tfrac{1}{2}\,\frac{1}{R}(\frac{1}{\rho_\mu} - 1)\int_{\pi/2}^{\pi} \sin\Theta\, d\Theta = \frac{1}{2R}(\frac{1}{\rho_\mu} - 1) =$$

$$= \tfrac{1}{2}(\frac{1}{r_\mu} - \frac{1}{R}) . \qquad (14.13)$$

For the evaluation of I_3, we consider the two terms of (14.10) separately.

Obviously $\frac{1}{2}\int_0^{\pi/2} g(R,r,r_\mu,\Theta)\sin\Theta\, d\Theta \leq$

$$\leq \frac{1}{2R}\int_0^{\pi/2} \sup_{0\leq t\leq\rho} \frac{\sin\Theta\, d\Theta}{\sqrt{t^2+\rho_\mu^2 - 2t\rho_\mu \cos\Theta}} - I_6$$ where

$$I_6 = \frac{1}{2R} \int_0^{\pi/2} \inf_{0 \leq t \leq \rho} \frac{\sin \Theta \, d\Theta}{\sqrt{1 + t^2 \rho_\mu^{\,2} - 2t\rho_\mu \cos \Theta}}$$

$$I_3 \leq I_5 - I_6 \text{ (say)} . \tag{14.14}$$

We first estimate I_5 .

For $0 < \Theta < \frac{\pi}{2}$, if $\rho > \rho_\mu$, $\sup\limits_{0 \leq t \leq \rho} \dfrac{1}{\sqrt{t^2 + \rho_\mu^{\,2} - 2t\rho_\mu \cos \Theta}}$

is $\dfrac{1}{\rho_\mu \sin \Theta}$.

However, if $\rho < \rho_\mu$, then $\dfrac{1}{\sqrt{t^2 + \rho_\mu^{\,2} - 2t\rho_\mu \cos \Theta}}$ increases from

$\dfrac{1}{\rho_\mu}$ to $\dfrac{1}{\rho_\mu \sin \Theta}$ as t increases from 0 to $\rho_\mu \cos \Theta$, and then decreases again.

This case can be easily dealt with later on as the Riesz masses would be entirely outside the sphere radius ρ and the Green's function would be harmonic inside this sphere.

In any case,

$$I_5 \leq \frac{1}{2R} \int_0^{\pi/2} \frac{\sin \Theta}{\rho_\mu \sin \Theta} d\Theta = \frac{\pi}{4R\rho_\mu} = \frac{\pi}{4r_\mu} , \tag{14.15}$$

In order to evaluate I_6, we note that for $o < \theta < \frac{\pi}{2}$,

$$\frac{1}{\sqrt{1 + t^2 \rho_\mu^{\ 2} - 2t\rho_\mu \ \cos\theta}}$$ increases as t increases from O to $\frac{\cos\theta}{\rho_\mu}$

and then decreases again. Thus the minimum value in the interval $[O, \rho]$ is attained when $t = O$ or $t = \rho$, and is 1 or $\frac{1}{\sqrt{1 + \rho^2\rho_\mu^{\ 2} - 2\rho\rho_\mu \ \cos\theta}}$ respectively.

Also $\frac{1}{\sqrt{1 + \rho^2\rho_\mu^{\ 2} - 2\rho\rho_\mu \ \cos\theta}} < 1$ if $\rho^2\rho_\mu^{\ 2} > 2\rho\rho_\mu \ \cos\theta$

i.e. if $\rho > \frac{2\cos\theta}{\rho_\mu}$

Therefore
$$I_6 = \frac{1}{2R}\int_o^{\cos^{-1}\frac{\rho\rho_\mu}{2}} 1.\text{Sin}\,\theta.d\theta + \frac{1}{2R}\int_{\cos^{-1}\frac{\rho\rho_\mu}{2}}^{\pi/2} \frac{\sin\theta\, d\theta}{\sqrt{1 + \rho^2\rho_\mu^{\ 2} - 2\rho\rho_\mu \ \cos\theta}} .$$

$$I_6 = \frac{1}{2R}\left\{1 - \frac{\rho\rho_\mu}{2} + \frac{1}{\rho\rho_\mu}\left[(1 + \rho^2\rho_\mu^{\ 2} - 2\rho\rho_\mu \ \cos\theta)^{\frac{1}{2}}\right]_{\cos^{-1}\frac{\rho\rho_\mu}{2}}^{\pi/2}\right\} .$$

$$I_6 = \frac{1}{2R}\left\{1 - \frac{\rho\rho_\mu}{2} + \frac{1}{\rho\rho_\mu}\left[\sqrt{1 + \rho^2\rho_\mu^{\ 2}} - 1\right]\right\} .$$

$$I_6 = \frac{1}{2R}\left[1 - \frac{\rho\rho_\mu}{2} + \frac{\rho^2\rho_\mu^{\,2}}{\rho\rho_\mu\left[\sqrt{1+\rho^2\rho_\mu^{\,2}}+1\right]}\right] .$$

$$= \frac{1}{2R}\left[1 + \rho\rho_\mu\left(\frac{1}{\sqrt{1+\rho^2\rho_\mu^{\,2}}+1} - \tfrac{1}{2}\right)\right]$$

Thus $I_6 > \frac{1}{2R}\left[1 + \rho\rho_\mu\left(\frac{1}{2+\rho\rho_\mu} - \tfrac{1}{2}\right)\right] = \frac{1}{2R}\left[1 + \rho\rho_\mu\left(\frac{2-2-\rho\rho_\mu}{2(2+\rho\rho_\mu)}\right)\right]$.

$$I_6 > \frac{1}{2R}\left[1 - \frac{\rho^2\rho_\mu^{\,2}}{2(2+\rho\rho_\mu)}\right] > \frac{1}{2R}\left[1 - \frac{\rho^2\rho_\mu^{\,2}}{4}\right] .$$

Now resubstituting $\rho = \frac{r}{R}$, $\rho_\mu = \frac{r_\mu}{R}$, we have

$$I_6 > \frac{1}{2R} - \frac{r^2 r_\mu^{\,2}}{4R5} \qquad (14.16)$$

From (14.14), (14.15) and (14.16), we have

$$I_3 < \frac{\pi}{4}\ \frac{'}{r_\mu} - \frac{1}{2R} + \frac{r^2 r_\mu^{\,2}}{4R^5} . \qquad (14.17)$$

And from (14.11), (14.13) and (14.17) we have

$$I_2 < \frac{1}{2}\left(\frac{1}{r_\mu} - \frac{1}{R}\right) + \frac{\pi}{4}\,\frac{1}{r_\mu} - \frac{1}{2R} + \frac{r^2 r_\mu^{\,2}}{4R^5} .$$

Thus $\quad I_2 < (\frac{1}{2} + \frac{\pi}{4})\frac{1}{r_\mu} - \frac{1}{R} + \frac{r^2 r_\mu^2}{4R^5}$,

which is the required result in (ii).

4.15. In this section we complete the proof of theorem 4.6. We showed in Lemma 4.7(ii) that

$$I_2 = \frac{1}{4\pi r^2} \iint_{S_r} g(R, r, r_\mu, \Theta) dS_r < (\frac{\pi}{4} + \frac{1}{2})\frac{1}{r_\mu} - \frac{1}{R} + \frac{r^2 r_\mu^2}{4R^5} .$$

Now suppose that $r_\mu \leq R_1 = \sqrt{rR}$.

Then $\quad I_2 < (\frac{\pi}{4} + \frac{1}{2})(\frac{1}{r_\mu} - \frac{1}{R}) + (\frac{\pi}{4} - \frac{1}{2})\frac{1}{R} + \frac{r^2 r_\mu^2}{4R^5}$.

$$I_2 < (\frac{1}{r_\mu} - \frac{1}{R}) \left[\frac{\pi}{4} + \frac{1}{2} + \frac{R r_\mu}{R - r_\mu} \left\{ (\frac{\pi}{4} - \frac{1}{2})\frac{1}{R} + \frac{r^2 r_\mu^2}{4R^5} \right\} \right] .$$

Now using the fact that $r_\mu \leq \sqrt{rR}$, we deduce that

$$I_2 < (\frac{1}{r_\mu} - \frac{1}{R}) \left[\frac{\pi}{4} + \frac{1}{2} + \frac{R\sqrt{rR}}{R - \sqrt{rR}} \left\{ (\frac{\pi}{4} - \frac{1}{2})\frac{1}{R} + \frac{r^3}{4R^4} \right\} \right] .$$

$$= (\frac{1}{r_\mu} - \frac{1}{R}) \left[\frac{\pi}{4} + \frac{1}{2} + e(\frac{r}{R}) \right] \text{, where}$$

$$e(t) = \frac{\sqrt{t}}{1-\sqrt{t}} \left\{ \frac{\pi}{4} - \frac{1}{2} + \frac{t^3}{4} \right\}.$$

We note that $e(t) < \dfrac{\sqrt{t}}{1-\sqrt{t}} \left\{ \dfrac{\pi}{4} - \dfrac{1}{2} + \dfrac{1}{4} \right\} < \dfrac{\sqrt{t}}{1-\sqrt{t}}$.

$$I_2 < \left(\frac{1}{r_\mu} - \frac{1}{R}\right) \left[\frac{\pi}{4} + \frac{1}{2} + \psi(t) \right] \text{, where } \psi(t) = \frac{\sqrt{t}}{1-\sqrt{t}}. \tag{15.1}$$

If $r_\mu > \sqrt{rR}$, we have a better estimate for I_2.

In this case the Green's function is harmonic inside a sphere of radius $\sqrt{rR} = R_1$ and the Riesz masses vanish inside D_{R_1}.

From the Poisson-Jensen formula (13.3) we have

$$G(R,r,r_\mu,\Theta) \le \frac{1}{4\pi R_1^{\,2}} \iint_{Q\in S_{R_1}} K_{R_1}(P,Q) G_R(Q,T)\, dS_Q .$$

$$g(R,r,r_\mu,\Theta) \le \frac{1}{4\pi R_1^{\,2}} \iint_{S_{R_1}} k_{R_1}(P,Q) G_R(Q,T)\, dS_Q .$$

Therefore $\dfrac{1}{4\pi r^2} \displaystyle\iint_{S_r} g(R,r,r_\mu,\Theta) dS_r \le \dfrac{1}{4\pi r^2} \displaystyle\iint_{S_r} dS_r \dfrac{1}{4\pi R_1 2}$

$$\times \iint_{S_{R_1}} k_{R_1}(P,Q) G_R(Q,T) dS_Q .$$

Inverting the order of integration on the right hand side we have,

$$I_2 \leq \frac{1}{4\pi r^2} \iint_{S_r} k_{R_1}(P,Q)dS_r\left(\frac{1}{4\pi R_1^2} \iint_{S_{R_1}} G_R(Q,T)dS_Q\right) .$$

From the harmonicity it follows that the average on the spherical surface S_{R_1} is equal to the value at the centre, i.e. $\left(\frac{1}{r_\mu} - \frac{1}{R}\right)$.

Thus by Lemma 4.7(i), we have

$$I_2 < \left(1 + \tfrac{1}{2}\log\frac{R_1 + r}{R_1 - r}\right)\left(\frac{1}{r_\mu} - \frac{1}{R}\right) . \qquad (15.2)$$

We now show that in any case (15.1) holds.

i.e. $\left(\frac{1}{r_\mu} - \frac{1}{R}\right)\left[\frac{\pi}{4} + \frac{1}{2} + \psi\left(\frac{r}{R}\right)\right] > \left(1 + \frac{1}{2}\log\frac{R_1 + r}{R_1 - r}\right)\left(\frac{1}{r_\mu} - \frac{1}{R}\right)$.

We have to show that

$$\frac{\pi}{4} + \frac{1}{2} - 1 + \psi\left(\frac{r}{R}\right) - \frac{1}{2}\log\frac{R_1 + r}{R_1 - r} > 0 .$$

We have $\quad \psi(t) = \dfrac{\sqrt{t}}{1 - \sqrt{t}}$,

Similarly $\log\dfrac{R_1 + r}{R_1 - r} = \log\dfrac{\sqrt{rR} + r}{\sqrt{rR} - r} = \log\dfrac{1 + \sqrt{t}}{1 - \sqrt{t}}$.

Since $\log x < \frac{1}{2}(x - \frac{1}{x})$, when $x > 1$, we have

$$\frac{1}{2}\log\frac{1+\sqrt{t}}{1-\sqrt{t}} < \frac{1}{4}\left[\frac{1+\sqrt{t}}{1-\sqrt{t}} - \frac{1-\sqrt{t}}{1+\sqrt{t}}\right] = \frac{\sqrt{t}}{1-t} .$$

Since t lies between O and 1, $\sqrt{t} > t$

$$1 - \sqrt{t} < 1 - t ,$$

and $$\frac{\sqrt{t}}{1-\sqrt{t}} > \frac{\sqrt{t}}{1-t} > \frac{1}{2}\log\frac{1+\sqrt{t}}{1-\sqrt{t}} .$$

$$\psi(t) > \frac{1}{2}\log\frac{1+\sqrt{t}}{1-\sqrt{t}}$$

And we see that clearly (15.1) holds in all cases.

<u>Completion of the proof of theorem 4.7</u>

We have from (13.3) the following representation for a subharmonic function $\omega(P)$ when $P \in D_R$.

$$\omega(P) = \frac{1}{4\pi R^2}\iint_{S_R}\omega(Q)K_R(P,Q)dS_Q - \int_{T\in D_R} G_R(P,T)d\mu(e_T) .$$

If P is the origin, we have

$$\omega(O) = \frac{1}{4\pi R^2}\iint_{S_R}\omega(Q)dS_Q - \int_{T\in D_R}\left(\frac{1}{OT} - \frac{1}{R}\right)d\mu(e_T) .$$

With the notation introduced in Section 14, the above formulae can be written as follows.

We use the fact that $\omega = \omega^+ - \omega^-$. Then

$$\frac{1}{4\pi R^2}\iint_{S_R}\omega^-(Q)dS_Q + \int_{T\in D_R}\left(\frac{1}{CT} - \frac{1}{R}\right)d\mu(e_T) = \frac{1}{4\pi R^2}$$

$$\times \iint_{S_R}\left[\omega^+(Q) - \omega(O)\right] dS_Q \cdot \tag{15.3}$$

$$\omega^-(P) = \frac{1}{4\pi R^2}\iint_{S_R} K_R(P,Q)\ \omega^-(Q)dS_Q + \int_{T\in D_R} G_R(P,T)d\mu(e_T) -$$

$$-\left\{\frac{1}{4\pi R^2}\iint_{S_R} K_R(P,Q)\ \omega^+(Q)dS_Q - \omega^+(P)\right\}. \tag{15.4}$$

Since $\omega^+(P)$ is subharmonic, the last term on the right hand side of (15.4) is positive.

$$\omega^-(P) \le \frac{1}{4\pi R^2}\iint_{S_R} K_R(P,Q)\ \omega^-(Q)dS_Q + \int_{T\in D_R} G_R(P,T)d\mu(e_T) \tag{15.5}$$

We recall that $\omega_1(r) = \sup\limits_{0\le t\le r} \omega^-(t,\theta,\phi)$ for fixed θ, ϕ.

Thus $\omega_1(r) \leq \frac{1}{4\pi R^2} \iint_{S_R} k_R(P,Q) \; \bar{\omega}(Q) dS_Q + \int_{T \in D_R} g(R,r,OT,\Theta) d\mu(\Theta_T)$

$$(15.6)$$

We now operate on both sides of (15.6) by

$$\frac{1}{4\pi r^2} \iint_{S_r} dS_r \quad .$$

We invert the order of integration on the right hand side which is justified since all the integrands are positive.

Therefore $\frac{1}{4\pi r^2} \iint_{S_r} \omega_1(r) dS_r \leq \frac{1}{4\pi R^2} \iint_{S_R} \left\{ \frac{1}{4\pi r^2} \iint_{S_r} k(R,r,\Theta) dS_r \right\} \bar{\omega}(Q) dS_Q$

$$+ \int_{T \in D_R} \left\{ \frac{1}{4\pi r^2} \iint_{S_r} g(R,r,OT,\Theta) dS_r \right\} d\mu(\Theta_T) \quad .$$

And by (14.5) and (15.1) we deduce that the right hand side is at most

$$\frac{1}{4\pi R^2} \left(1 + \tfrac{1}{2} \log \frac{R+r}{R-r}\right) \iint_{S_R} \bar{\omega}(Q) dS_Q + \left[\frac{\pi}{4} + \tfrac{1}{2} + \psi\left(\frac{r}{R}\right) \right]$$

$$\times \int_{T \in D_R} \left(\frac{1}{OT} - \frac{1}{R}\right) d\mu(\Theta_T) \quad .$$

Thus from (15.3), we have

$$\frac{1}{4\pi r^2}\iint_{S_r}\omega_1(r)dS_r < \left[\frac{\pi}{4}+\frac{1}{2}+\psi\left(\frac{r}{R}\right)\right]\frac{1}{4\pi R^2}\iint_{S_R}\left[\omega^+(Q)-\omega(O)\right]dS_Q$$

$$+\left[1+\frac{1}{2}\log\frac{R+r}{R-r}-\frac{\pi}{4}-\frac{1}{2}-\psi\left(\frac{r}{R}\right)\right]\frac{1}{4\pi R^2}\iint_{S_R}\omega^-(Q)dS_Q \; .$$

Therefore $$\frac{1}{4\pi r^2}\iint_{S_r}\omega_1(r)dS_r < \left[\frac{\pi}{4}+\frac{1}{2}+\psi\left(\frac{r}{R}\right)\right]\frac{1}{4\pi R^2}\iint_{S_R}\left[\omega^+(Q)-\omega(O)\right]dS_Q \; ,$$

which is the required result.

4.16 We note that if $\omega(P)$ is non-positive in C_R, and since $-\omega_1(r) = \inf\limits_{o<t<r}\omega(t,\theta,\phi)$ for fixed θ,ϕ, it follows that $\omega^+(Q) = 0$ and consequently we have from Theorem 4.7 that

$$\frac{1}{4\pi r^2}\iint_{S_r}\inf_{o<t<r}\omega(t,\theta,\phi)dS_r > \left[\frac{\pi}{4}+\frac{1}{2}+\psi\left(\frac{r}{R}\right)\right]\omega(O) \qquad (16.1)$$

We now prove

<u>Theorem 4.8</u> <u>If $\omega(P)$ is non-positive and subharmonic in the whole space, then</u>

$$\frac{1}{4\pi r^2}\iint_{S_r} \inf_{o<t<\infty} \omega(t,\Theta,\phi)dS_r \geq (\frac{\pi}{4} + \frac{1}{2})\omega(O), \qquad (16.2)$$

As $\omega(P)$ is non-positive in the whole space, we can let $R \longrightarrow +\infty$ in (16.1) and note that $\psi(\frac{r}{R}) \longrightarrow o$ as $R \longrightarrow +\infty$.

Thus we have from (16.1),

$$\frac{1}{4\pi r^2}\iint_{S_r} \inf_{o<t<r} \omega(t,\Theta,\phi)dS_r \geq \left[\frac{\pi}{4} + \frac{1}{2}\right]\omega(O) .$$

This holds for all r. If we have a sequence of r_n tending to infinity we note that

$$\frac{1}{4\pi r^2}\iint_{S_r} \inf_{o<t<r_n} \omega(t,\Theta,\phi)dS_r$$

is a decreasing sequence of functions.

Hence by Fatou's theorem,

$$\frac{1}{4\pi r^2}\iint_{S_r} \inf_{o<t<\infty} \omega(t,\Theta,\phi)dS_r \geq \left[\frac{\pi}{4} + \frac{1}{2}\right]\omega(O) .$$

We show below by a simple example that the constant $\left[\frac{\pi}{4} + \frac{1}{2}\right]$ is the best possible.

Example.

$$\omega(x,y,z) = -\left[x^2 + y^2 + (z-1)^2\right]^{-\frac{1}{2}} \quad \text{for } (x,y,z) \neq (0,0,1)$$

$$= -\infty \quad \text{for } (0,0,1) .$$

Then we note that $\omega(0,0,0) = -1$ and $\omega(x,y,z) < 0$ for the entire space.

In polar coordinates $\omega(r,\theta,\phi) = \dfrac{-1}{\sqrt{r^2 - 2r\cos\theta + 1}}$.

If $\frac{\pi}{2} < \theta < \pi$, clearly $\inf\limits_{0<r<\infty} \omega(r,\theta,\phi) = -1$ for fixed θ, ϕ .

In $0 < \theta < \frac{\pi}{2}$ $\inf\limits_{\cos\theta < r < \infty} \omega(r,\theta,\phi) = \dfrac{-1}{\sin\theta}$

Also $\inf\limits_{t \leq \cos\theta} \omega(t,\theta,\phi)$ is a decreasing function of t and equal $\dfrac{-1}{\sin\theta}$ when $t = \cos\theta$.

Since we are concerned with large values of r, we consider $r > \cos\theta$.

$$\text{Thus } \frac{1}{4\pi r^2} \iint\limits_{S_r} \inf\omega \, dS_r = \frac{1}{4\pi r^2} \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \left\{\inf\omega\right\} d\theta \, d\phi$$

$$= \frac{1}{2} \int_0^{\pi/2} -\sin\theta \, d\theta + \frac{1}{2} \int_{\pi/2}^{\pi} \frac{-1}{\sin\theta} \sin\theta \, d\theta$$

$$= \frac{1}{2}\left[0-1\right] + \frac{1}{2}\left[-\frac{\pi}{2}\right] = -\left[\frac{\pi}{4}+\frac{1}{2}\right] .$$

Since $\omega(0,0,0) = -1$, we have

$$\frac{1}{4\pi r^2} \iint_{S_r} \inf \omega \, dS_r = \left[\frac{\pi}{4}+\frac{1}{2}\right]\omega(0,0,0) .$$

Thus the above inequality is sharp.

A simple consequence of the sharp inequality is:

Theorem 4.8A: Suppose that $u(P)$ is a subharmonic function in space which is bounded above. Then on almost all straight lines through a given point, u is bounded below.

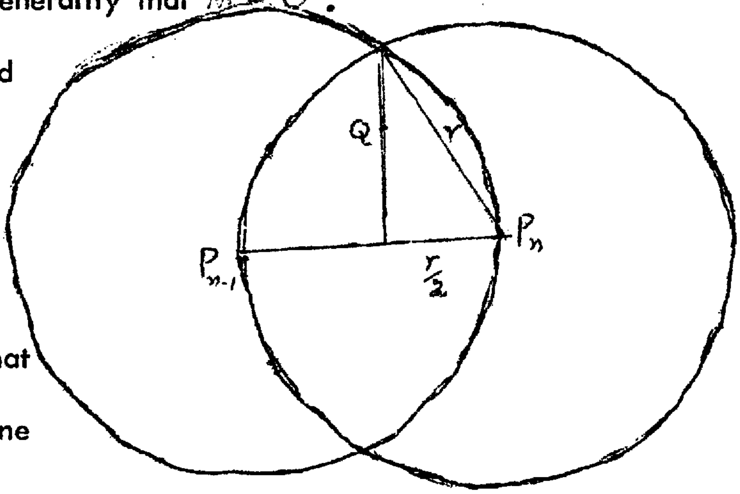
4.17 We now prove a theorem about general subharmonic functions which are bounded.

Theorem 4.9 Suppose that $\omega(P)$ is subharmonic in space and that $\omega(P)$ is bounded above in space by M. Then there exists a polygonal path such that $\omega(P) \longrightarrow M$ as $P \longrightarrow \infty$ on Γ.

It was shown in Theorem 4.5 that there exists a continuum $\Gamma = \gamma_1 + \gamma_2 + \ldots\ldots$ passing through a sequence of points $\{P_n\}$ such that $\omega(P_n) > M - \frac{1}{2^n}$, and $\omega(P) > M - \frac{1}{2^{n-1}}$ on γ_{n-1} joining P_{n-1} to P_n.

We now show that there exists a polygonal path with similar properties. It is sufficient to show that we can join two points P_n and P_{n-1} by a suitable polygonal path.

We assume without loss of generality that $M = O$.



Two spheres centres P_{n-1} and P_n and radii equal to $\overline{P_{n-1}P_n} = r$ intersect. It can be easily verified from elementary solid geometry that the area of the surface of one insidethe other is πr^2.

Since $\omega(P)$ is non positive in space, we have from (16.1), the following inequality about the average of infima on radial segments through P_{n-1}.

Take R so large that $\Psi(\frac{r}{R}) < \frac{1}{16}$. Then,

$$\frac{1}{4\pi r^2} \iint_{S_r} \inf_{o \leq t \leq r} \omega(t,\theta,\phi) dS_r > \left[\frac{\pi}{4} + \frac{1}{2} + \frac{1}{16} \right] \omega(P_{n-1}) \qquad (17.1)$$

From (17.1) we obtain a lower bound for the infimum on a portion ΔS_r of S_r (namely the part inside the other sphere which has area πr^2) instead of the whole S_r.

$$\frac{1}{4\pi r^2}\iint_{\Delta S_r} \inf_{\substack{o\le t\le r}} \omega(t,\Theta,\phi)dS_r + \frac{1}{4\pi r^2}\cdot 3\pi r^2 \cdot \omega(P_{n-1}) > (\frac{\pi}{4}+\frac{9}{16})\,\omega(P_{n-1})$$

Thus $\dfrac{1}{4\pi r^2}\displaystyle\iint_{\Delta S_r} \inf \omega(t,\Theta,\phi)dS_r > \left[\frac{\pi}{4}-\frac{3}{16}\right]\omega(P_{n-1})$.

Since the area of ΔS_r is πr^2, we note that

$$\frac{1}{\pi r^2}\iint_{\Delta S_r} \inf \omega(t,\Theta,\phi)dS_r > \left[\pi - \frac{3}{4}\right]\omega(P_{n-1}) \qquad (17.2)$$

We note that the infima on the radial segments going up to the disc through the circle of intersection of the two spheres would be not less than the infima on the radial segments going right up to the surface S_r. With each point Q in the disc A going through the circle of intersection of the two spheres we associate the coordinates Θ, ϕ, taking for Θ the angle $QP_{n-1}P_n = QP_nP_{n-1}$ and for ϕ the angle which the plane $QP_{n-1}P_n$ makes with a fixed plane through $P_{n-1}P_n$. Then if $\omega_1(\Theta,\phi)$ and $\omega_2(\Theta,\phi)$ denote the infima of ω on the rays $P_{n-1}Q$ and P_nQ respectively we deduce from (17.2) that

$$\frac{1}{\pi}\int_0^{\pi/3} d\Theta \int_0^{2\pi} \omega_1(\Theta,\phi)\sin\Theta\, d\phi > (\pi - \tfrac{3}{4})\,\omega(P_{n-1}) \;,$$

and similarly

$$\frac{1}{\pi}\int_0^{\pi/3} d\theta \int_0^{2\pi} \omega_2(\theta,\phi)\sin\theta\, d\phi > (\pi - \tfrac{3}{4})\,\omega(P_n) ,$$

so that

$$\frac{1}{\pi}\int_0^{\pi/3}\int_0^{2\pi}\left[\omega_1(\theta,\phi) + \omega_2(\theta,\phi)\right]\sin\theta\, d\theta d\phi > (\pi - \tfrac{3}{4})\left[\omega(P_{n-1}) + \omega(P_n)\right] \tag{17.3}$$

Since the minimum of ω on the broken line $P_{n-1}QP_n$ is $\inf(\omega_1, \omega_2) \geq \omega_1 + \omega_2$, we deduce that on at least one such line $P_{n-1}QP_n$ we have

$$\omega(P) \geq (\pi - \tfrac{3}{4})\left[\omega(P_{n-1}) + \omega(P_n)\right]$$
$$> (\pi - \tfrac{3}{4})(\tfrac{-3}{2^n}) > -2^{3-n} .$$

Thus if $\omega(P_{n-1}) > M - 2^{1-n}$ and $\omega(P_n) > M - 2^{-n}$, we have shown that there exists a broken line $P_{n-1}QP_n$ such that $\omega(P) > M - 2^{3-n}$ for P on $P_{n-1}QP_n$.

This completes the proof of theorem 4.9.

4.18 In this section we study the relation between the number of components of the set $\left\{P \middle| u(P) \geq K\right\}$ in which $u(P) > K$, and the lower order of $u(P)$. In case of a continuous subharmonic function our results hold for the number of components of the set $\left\{P \middle| u(P) > K\right\}$. Dinghas in Dinghas (2) has estimated growth of similar classes of functions satisfying $\Delta u \geq cu$ for given $c \geq o$ in Euclidean space of n-dimensions. Dinghas has also obtained in Dinghas (3) a lower estimate for ratios of functionals of certain classes of C" non-negative functions. These functionals are α^{th} powers $(\alpha > 1)$ of the norm of the gradient and of the function. Dinghas has then applied this estimate to obtain theorems of the Wiman type and of the Denjoy-Carleman type for harmonic functions in E^n.

In the case $n = 3$, we have Peetre's inequality $\lambda_1 \geq \frac{i^2 \pi}{A}\left(1 - \frac{A}{4\pi r^2}\right)$ which gives us a better lower estimate for λ_1. We study growth of functions having n components with the help of above inequality.

Suppose that the set $\left\{P \middle| u(P) \geq O\right\}$ has two components F_i.

$$\text{Define} \quad \omega^{(i)}(P) = \begin{cases} u(P) & \text{for } P \in F_i \\ O & \text{for } P \text{ outside } F_i . \end{cases}$$

Let $m^{(i)^2}(r) = \frac{1}{4\pi r^2} \iint_{S_r} \omega^{(i)^2}(P)\, dS_r$

We also define $q^2(r) = \frac{1}{4\pi r^2} \iint_{S_r} \sum_{i=1}^{2} \left[\omega^{(i)}(P)\right]^2 dS_r$,

and note that $q^2(r) = \sum_{i=1}^{2} m^{(i)^2}(r)$.

As before $\alpha^{(i)}(\xi)$ for each component is defined as the limit of increasing sequence of $\alpha_n^{(i)}(\xi)$ corresponding to the spherical domains in which the intersections of the component with S_r lies.

Then by summation on (9.4), we have

$$q^2(r) = \frac{m^2(r_o)}{r} \int_{r_o}^{r} \sum_{i=1}^{2} \exp\left[2 \int_{r_o}^{\eta} \alpha^{(i)}(\xi) d \log\xi\right] d\eta .$$

By Arithmetic-Geometric mean, we have

$$q^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} 2 \exp\left\{ \int_{r_o}^{\eta} \sum_{i=1}^{2} \alpha^{(i)}(\xi)\, d \log\xi \right\} d\eta . \quad (18.2)$$

We recall that

$$\alpha(\xi) \geq \sqrt{\left[\frac{\pi j^2}{\mu(\xi)} - \frac{j^2 - 1}{4}\right]} - \frac{1}{2} , \quad (18.3)$$

where $j = 2.4048...$

We now verify easily that $\sqrt{\left[\frac{\pi j^2}{\mu(r)} - \frac{j^2-1}{4}\right]} - \frac{1}{2}$ is a convex function of $\mu(r)$ at least when $\mu(r) \leq \frac{18\pi}{5}$.

It is sufficient to show that the second derivative is positive.

Let $\phi(x) = \sqrt{\left[\frac{a}{x} - b\right]} - \frac{1}{2}$, where $x < \frac{a}{b}$.

$$\phi'(x) = \frac{-a}{2x^2\sqrt{\frac{a}{x} - b}} .$$

$$\phi''(x) = \frac{a}{2}\left[\frac{2x\sqrt{\frac{a}{x} - b} - \frac{a}{2\sqrt{\frac{a}{x} - b}}}{x^2\left(\frac{a}{x} - b\right)}\right]$$

And $\phi''(x) > o$ if $x\sqrt{\frac{a}{x} - b} > \frac{a}{4\sqrt{\frac{a}{x} - b}}$

i.e. $4a - 4bx > a$

or $x < \frac{3a}{4b}$, i.e. $\mu(r) < \frac{3\pi j^2}{j^2 - 1}$

If $\mu(r) \leq \frac{18}{5}\pi$, this is obviously satisfied.

First suppose that for both the components $\mu(r) \leq \frac{18}{5}\pi$, Then $\alpha^{(1)}(\xi) + \alpha^{(2)}(\xi) \geq \left[\phi(\mu^{(1)}(\xi)) + \phi(\mu^{(2)}(\xi)) \right]$

$$\geq 2\phi\left(\frac{\mu^{(1)}(\xi) + \mu^{(2)}(\xi)}{2}\right) .$$

Since $\dfrac{\mu^{(1)}(\xi) + \mu^{(2)}(\xi)}{2} \leq 2\pi$, we deduce from (18.3) that

$$\alpha(2\pi) \geq \frac{4}{5} .$$

Therefore in this case $\sum \alpha^{(i)}(\xi) \geq \dfrac{8}{5}$. (18.4)

If for any r, $\mu^{(i)}(r)$ for one component is greater than $\frac{18\pi}{5}$, and we are unable to use the convexity relation we have that

$$\sum \alpha^{(i)} \geq \max \alpha^{(i)}(\xi) \geq \alpha(\tfrac{2}{5}\pi)$$

because the other component subtends an angle at most $\frac{2}{5}\pi$ for the same value of r.

It is easily verified from (18.3) that

$$\alpha(\tfrac{2}{5}\pi) > 2 .$$

Hence when the set $\{P | u(P) \geq K\}$ has two components in which $u(P) > K$, (18.4) holds for all $\xi \geq r_o$.

Substituting this value of $\sum \alpha^{(i)}(\xi)$ in (18.2) and integrating, we have,

$$q^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} 2 \exp\left\{ \frac{8}{5} \log \xi \right\}_{r_o}^{\eta} d\eta$$

and finally $q^2(r) \geq K^2 r^{8/5}$ where K^2 is a constant depending on $m^2(r_o)$. Therefore $M(r) = \max_{P \in G_r} u(P) \geq K r^{4/5}$ where $K > 0$. Hence if a function has order less than $\frac{4}{5}$, it cannot have more than one component.

We now study what this method gives us if the set $\{ P | u(P) \geq K \}$ has n components in which $u(P) > K$.

As before, if the solid angle subtended by the intersection of each component (with S_ξ) at the origin is less than $\frac{18\pi}{5}$, we have from the convexity property that

$$\sum_{i}^{n} \alpha^{(i)}(\xi) \geq \phi(\mu^{(1)}(\xi)) + \phi(\mu^{(2)}(\xi)) + \dots \phi(\mu^{(n)}(\xi))$$

$$\geq n \left(\phi\left(\frac{\mu^{(1)}(\xi) + \dots \mu^{(n)}(\xi)}{n}\right) \right)$$

$$\geq n\,\phi\left(\frac{4\pi}{n}\right). \tag{18.5}$$

It can be easily verified from (18.3) that

$$\phi\left(\frac{4\pi}{n}\right) > \frac{j}{2}\sqrt{n-1} - \frac{1}{2}.$$

Thus $$\sum_{i=1}^{n} \alpha^{(i)}(\xi) > \frac{n}{2}\left[j\sqrt{n-1} - 1\right] \qquad (18.6)$$

Also if one of the components has an intersection with S_ξ which subtends at the origin a solid angle greater than $\frac{18\pi}{5}$, then all other components subtend at the origin, solid angles whose sum is at most $\frac{2\pi}{5}$. Thus the convexity property holds for these n - 1 components and we have

$$\sum_{i=1}^{n-1} \alpha^{(i)}(\xi) \geq (n-1)\,\phi\left(\frac{2\pi}{5(n-1)}\right).$$

Also it can be verified from (18.3) that

$$\phi\left(\frac{2\pi}{5(n-1)}\right) \geq \frac{1}{2}\left(j\sqrt{10n-11} - 1\right)$$

Thus in this case we have

$$\sum_{i=1}^{n-1} \alpha^{(i)}(\xi) \geq \frac{n-1}{2}\left(j\sqrt{10n-11} - 1\right) \qquad (18.7)$$

Since $\frac{n-1}{2}\left(j\sqrt{10n-11} - 1\right) > \frac{n}{2}\left[j\sqrt{n-1} - 1\right]$ for $n \geq 2$, we have that (18.6) holds in all cases.

Again by summation on (9.4) for the modified function formed for each of the components we have

$$q^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} \sum_{i=1}^{n} \exp\left[2 \int_{r_o}^{\eta} \alpha^{(i)}(\xi)\, d \log\xi\right] d\eta .$$

By Arithmetic-Geometric mean, we have

$$q^2(r) \geq \frac{m^2(r_o)}{r} \int_{r_o}^{r} n \exp\left[\frac{2}{n} \int_{r_o}^{\eta} \sum_{i=1}^{n} \alpha^{(i)}(\xi)\, d \log\xi\right] d\eta .$$

From (18.6), we have $\frac{2}{n} \sum_{i=1}^{n} \alpha^{(i)}(\xi) > i\sqrt{n-1} \;-1$.

Then on integration as before we have

$q^2(r) \geq K^2 r^{i\sqrt{n-1}\;-1}$, where K is a positive constant depending on $m(r_o)$.

Hence $M(r) > K r^{\frac{i}{2}\sqrt{n-1}\;-\frac{1}{2}}$.

Therefore a function of lower order less than $\frac{i}{2}\sqrt{n-1} \;- \frac{1}{2}$ cannot have n components.

In particular an infinite number of components is possible only in the case of a function of infinite lower order.

4.19 We now consider the case when the function $u(P)$ is unbounded in space but is bounded in a component of $\{P \mid u(P) \geq 0\}$ or $\{P \mid u(P) > 0\}$. Clearly for this component by Theorem 4.3,

$$\int_{r_o}^{\infty} \alpha(\xi)\, d \log \xi < +\infty .$$

Hence its complement has the smallness property.

Thus every other component of $\{P \mid u(P) \geq 0\}$ or $\{P \mid u(P) > 0\}$ has the property that given any $\epsilon > 0$, the solid angle $\mu(r)$ subtended by the intersection of that component with S_r, at the origin is less than ϵ except on a set of finite logarithmic measure on the r-axis.

If $\mu(r) \leq 2\pi$, it follows from (18.3) that

$$\alpha(r) \geq \frac{\sqrt{2}}{3}\ j\sqrt{\pi}\ \frac{1}{\sqrt{\mu(r)}} .$$

Thus in the above case we have $\alpha(r) \geq K$ outside a set finite logarithmic measure.

$$\int_{r_o}^{\eta} \alpha(\xi)\, d \log \xi > K \log \frac{\eta}{r_o} \quad \text{for} \quad \eta > r_o .$$

And we have $m(r) > m(r_o)\, r^K$ where K is as large as we please.

Thus such a component is only possible in the case of a function of infinite lower order.

4.20 It was shown in Section 4.18 that if u(P) is a subharmonic function of lower order less than $\frac{1}{2}\sqrt{n-1} - \frac{1}{2}$, then the set $\{P \mid u(P) \geq K\}$ cannot have n or more components for any real K.

We now construct a function for which the set $\{P \mid u(P) \geq K\}$ has N components and study its growth.

It was shown by Deny and Lelong (1) that the functions of minimal growth in a cone Ω are harmonic functions of the type $r^{\rho_1} h_1(\Theta,\phi)$ where $h_1(\Theta,\phi)$ is an eigenfunction in the domain of intersection $\Omega(\xi)$ of the cone with spherical surface $S(\xi)$, corresponding to the lowest positive eigenvalue λ_1 of the equation

$$Lh + \lambda h = 0 \quad \text{in} \quad \Omega(\xi)$$

and h vanishing continuously on $\partial\Omega(\xi)$ where L is the spherical part of the Laplace operator.

Also ρ_1 is determined from λ_1 by the equation $\rho_1(\rho_1 + 1) = \lambda_1$.

Let Ω be a right circular cone consisting of points of the form rP where $0 < r < \infty$ and P is a point inside a circular domain C_1 on the unit sphere. We define a subharmonic function $v_1(P)$ as that of the minimal growth investigated by Deny and Lelong for P inside the cone, and equal to zero elsewhere.

We form N such circular domains of equal area on the surface of the unit sphere and form functions $v_2(P), \ldots v_N(P)$ as before.

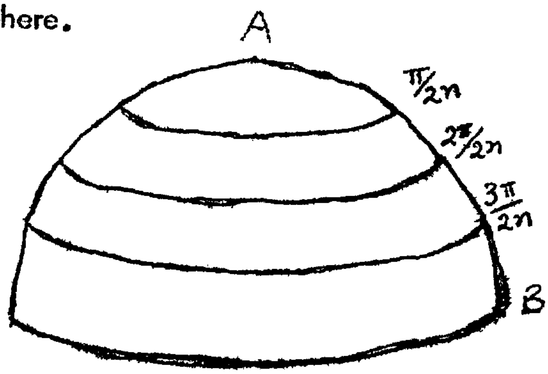
Then $v(P) = v_1(P) + \ldots + v_N(P)$ is subharmonic and the set $\left\{ P \mid v(P) \geq \epsilon \right\}$ has N components if $\epsilon > 0$.

We first give a construction for obtaining such non-overlapping circular domains on the surface of the unit sphere.

The circumference of the great circle is 2π.

Therefore $AB = \frac{\pi}{2}$.

We divide AB in n equal parts.



The spherical cap of centre A and with base the small circle $\Theta = \frac{\pi}{2n}$ gives one circular domain. We now construct other domains equal in area to this domain. We draw small circles $\Theta = \frac{2\pi}{2n}, \frac{3\pi}{2n}, \ldots\ldots$

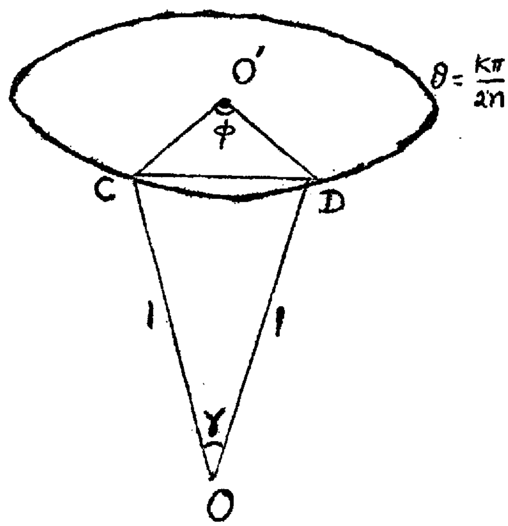
We take as centre a point on $\Theta = \frac{2\pi}{2n}$, with radius of great circle distance equal to $\frac{\pi}{2n}$, we draw a circle. This circle will lie in the zone between $\Theta = \frac{\pi}{2n}$ and $\frac{3\pi}{2n}$. We now take centres on $\Theta = \frac{2\pi}{2n}$

such that these circular domains do not overlap. Take n odd. Then we form these circular domain with centres on $\Theta = \frac{2\pi}{2n}, \frac{4\pi}{2n}, \ldots \frac{n-1\,\pi}{2n}$. Thus we have formed circular domains of equal area in the Northern hemisphere. We now want to determine the exact number of points which can be distributed round the circle $\Theta = \frac{K\pi}{2n}$ at a spherical distance equal to (or greater than) $\frac{\pi}{n}$ (when K is even).

We first determine the circumferential distance on $\Theta = \frac{K\pi}{2n}$ between the two points whose great circle distance is equal to $\frac{\pi}{n}$.

Consider two points C and D on the circle $\Theta = \frac{K\pi}{2n}$ whose great circle distance is $\frac{\pi}{n}$.

Let O' be the centre of the circle $\Theta = \frac{K\pi}{2n}$



Then $O'C = \sin\frac{K\pi}{2n} = O'D$

$$OC = OD = 1$$

$$\gamma = \frac{\pi}{n}.$$

The chordal distance $CD = 2\sin\frac{\gamma}{2} = 2\sin\Theta\sin\frac{\phi}{2}$.

$$\sin\frac{\phi}{2} = \frac{\sin\frac{\pi}{2n}}{\sin\frac{K\pi}{2n}}$$

$$\phi = 2 \sin^{-1}\left\{\frac{\sin\frac{\pi}{2n}}{\sin\frac{K\pi}{2n}}\right\} .$$

For $0 < \theta < \frac{\pi}{2}$ we have $\frac{2\theta}{\pi} < \sin\theta < \theta$.

Therefore $$\frac{\sin\frac{\pi}{2n}}{\sin\frac{K\pi}{2n}} < \frac{\pi}{2n} \times \frac{\pi}{2} \times \frac{2n}{K\pi} = \frac{\pi}{2K} .$$

$$\phi < 2 \sin^{-1}\frac{\pi}{2K} .$$

The exact number of points which can be distributed round the circle $\theta = \frac{K\pi}{2n}$ at spherical distance not less than $\frac{\pi}{n}$ is

$$\left[\frac{2\pi}{\phi}\right] \geq \left[\frac{\pi}{\sin^{-1}\frac{\pi}{2K}}\right] \geq \left[\frac{\pi}{\pi^2/4K}\right] \geq \left[\frac{4K}{\pi}\right] \geq K$$

Thus there are at least K circular domains in the zone between

$$\theta = \frac{(K-1)\pi}{2n} \quad \text{and} \quad \frac{(K+1)\pi}{2n} .$$

On a hemisphere K takes even values from 2 to n-1.

Thus the total number of circles between the zones is not less than

$$2\sum_{K=1}^{\frac{n-1}{2}} K = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) = \frac{(n^2-1)}{4} .$$

Thus the total number of circles in the Northern hemisphere is at least equal to

$$\frac{(n^2 - 1)}{4} + 1$$

Thus numbers of circles on the unit sphere is at least aqual to $\frac{(n^2 - 1)}{2} + 2$.

Let $\quad N = \frac{1}{2}(n^2 - 1) + 2$,

so that $\quad n = \sqrt{2N - 3}$

We now want an upper bound for ρ_1.

Since ρ_1 increases with λ_1, we find an upper bound for λ_1.

We recall that λ_1 is the lowest estimate of the Rayleigh's quotient.

Thus if we have any test function satisfying the boundary conditions, we get an upper bound on λ_1. Clearly cos $n\Theta$ is such a function for the spherical cap with centre North Pole and base $\Theta = \frac{\pi}{2n}$.

$$\left| \text{gradient of } \cos n\Theta \right|^2 = n^2 \sin^2 n\Theta \ .$$

We have to evaluate

$$\frac{\int_0^{2\pi} d\phi \int_0^{\pi/2n} n^2 \sin^2 n\theta \sin\theta \, d\theta}{\int_0^{2\pi} d\phi \int_0^{\pi/2n} \cos^2 n\theta \sin\theta \, d\theta} = \frac{n^2 \int_0^{\pi/2n} \sin^2 n\theta \sin\theta \, d\theta}{\int_0^{\pi/2n} \cos^2 n\theta \sin\theta \, d\theta}$$

The Numerator is $n^2 \int_0^{\pi/2n} \sin^2 n\theta \sin\theta \, d\theta$

$$< n^4 \int_0^{\pi/2n} \theta^3 \, d\theta = n^4 \left[\frac{\theta^4}{4}\right]_0^{\pi/2n} = \frac{\pi^4}{64} \quad .$$

The Denominator is $\int_0^{\pi/2n} \cos^2 n\theta \sin\theta \, d\theta$

$$= \frac{1}{n} \int_0^{\pi/2} \cos^2 \phi \sin\frac{\phi}{n} \, d\phi$$

Since $\frac{\sin\theta}{\theta}$ decreases as θ increases, we have $\frac{\sin\frac{\phi}{n}}{\frac{\phi}{n}} > \frac{\sin\phi}{\phi}$.

Therefore $\sin\frac{\phi}{n} > \frac{1}{n} \sin\phi$.

Thus the Denominator is at least $\frac{1}{n^2}\int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi$

$$= \frac{1}{n^2}\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/2} = \frac{1}{3n^2}$$

Hence $$\frac{n^2\int_0^{\pi/2n} \sin^2 n\Theta \sin\Theta \, d\Theta}{\int_0^{\pi/2n} \cos^2 n\Theta \sin\Theta \, d\Theta} < \frac{\pi^4}{64}.\quad 3n^2 < \frac{100}{64}.\quad 3n^2 = \frac{75n^2}{16}.$$

Thus we have $\lambda_1 < \frac{75n^2}{16}$.

Since $\rho_1(\rho_1 + 1) = \lambda_1$ we have

$$\rho_1 < \sqrt{\left[\frac{75n^2}{16} + \frac{1}{4}\right]} - \frac{1}{2}$$

$$\rho_1 < \sqrt{\left(\frac{75n^2 + 4}{16}\right)} - \frac{1}{2}$$

Since $n^2 = 2N - 3$, we have

$$\rho_1 \leq \sqrt{\left\{\frac{75(2N - 3) + 4}{16}\right\}} - \frac{1}{2} \text{ .}$$

$$\text{i.e.}\quad \rho_1 < \sqrt{\left(\frac{150N - 221}{16}\right)} - \frac{1}{2} < \frac{\sqrt{150}}{4}\sqrt{N-1} - \frac{1}{2}$$

$$< (3.1)\sqrt{N-1} - \frac{1}{2}$$

Thus we have shown that there exists a function for which the set $\{P | u(P) > K\}$ or $\{P | u(P) \geq K\}$ has N components and the order of the function is less than $(3.1)\sqrt{N-1} - \frac{1}{2}$, in case when $N = \frac{1}{2}(n^2 - 1) + 2$ and n is odd. We recall that our lowest estimate of the order for the growth of such function was $\frac{j}{2}\sqrt{N-1} - \frac{1}{2}$ where $j = 2.4048\ldots\ldots$

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