MIXING, SPECTRAL AND REGULARITY PROPERTIES OF FINITE AND INFINITE AUTOMORPHISMS

by

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ABSTRACT

Given any automorphism T on a measure space (X, ε, μ) there is (see K. Jacobs [8] or V.A.Rokhlin [16]) an associated unitary operator U on L^2_μ such that $Uf(x) = f(Tx)$ for all $f \in L^2_\mu$. We first define and investigate the properties of the metric invariant entropy, introduced by A.N. Kolmogorov [11] and [12], and ~~see~~ Ja. G. Sinai [18], in order to show (~~see A.N.Kolmogorov [11] and [12]~~) that there exist spectr~~metri~~cally equivalent (see V.A.Rokhlin [16]) T_1, T_2 such that the associated U_1, U_2 are not metric~~spectr~~ally equivalent (see V.A.Rokhlin [15]).

Having done this we turn to the concepts of Kolmogorov and regular automorphisms on finite measure spaces. Then following V.A.Rokhlin [15] and L.Sucheston [21] we show that both these concepts imply mixing of all degrees. Further investigation enables us to deduce that T is a Kolmogorov automorphism if, and only if, it is a regular automorphism. An immediate question then is whether or not T being a mixing of all degrees implies T is a Kolmogorov automorphism? We answer this in the negative by constructing a stationary Gaussian process which we show to be a mixing of all degrees and which cannot be a Kolmogorov automorphism since its spectrum is not a Lebesgue spectrum as is that of all Kolmogorov automorphisms see K. Jacobs [8].

In the course of the above we show that if a Gaussian process is a mixing of degree 1 then it is a mixing of all degrees.

In the last chapter we extend the notion of a Kolmogorov automorphism to σ-finite measure spaces and prove (a fact which is clear for finite measure spaces) that in this case also T is a Kolmogorov automorphism implies that T is ergodic. An open question for σ-finite measure spaces is what mixing properties does a Kolmogorov automorphism have?

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CONTENTS

1. INTRODUCTION

1·1 NOTATION

As usual we denote "is a member of" by $\in$ and the union, intersection, difference and symmetric difference of two sets by $\cup$, $\cap$, $-$, and $\triangle$ respectively. If A is a subset of B then we write $A \subseteq B$. Γ will denote the integers and Γ^+ the strictly positive integers. If X is an abstract space then by a σ-algebra α, we mean a collection of subsets of X such that:

(i) $A, B \in \alpha$ imply $A \triangle B$, $A \cap B \in \alpha$;

(ii) $A_i \in \alpha$, $i \in \Gamma^+$ implies $\bigcup_{i \in \Gamma^+} A_i \in \alpha$;

(iii) $X \in \alpha$.

We note that it is usual to define a σ-algebra to be rather more general than the above in that (iii) is replaced by

(iii)$'$ α contains a "unit"

and then (i), (ii), (iii) become necessary and sufficient conditions for α to be a σ-algebra whose unit is X. (see P.R.Halmos [5]). However, the above definition is adequate for our purpose. If α is a σ-algebra, then, since $A \cup B = (A \triangle B) \triangle (A \cap B)$, $A-B = (A \triangle B) \cap A$, we have that $A, B \in \alpha$ imply $A \cup B \in \alpha$, $A-B \in \alpha$ and since $\bigcap_{i \in \Gamma^+} A_i = X - \bigcup_{i \in \Gamma^+} (X - A_i)$ we have that $A_i \in \alpha, i \in \Gamma^+$ implies $\bigcap_{i \in \Gamma^+} A_i \in \alpha$.

Gixen X and a σ-algebra α, then by a measure on (X, α) we mean a real valued, non-negative function μ whose domain of definition is α and which satisfies:

(i) $\mu(\phi) = 0$, where here as always we use ϕ to denote the null set;

(ii) if $A_i \in \alpha$, $i \in \Gamma^+$ are such that $A_i \cap A_j = \phi$ if $i \neq j$ then

$\mu(\bigcup_{i\in\Gamma^+} A_i) = \sum_{i\in\Gamma^+}\mu(A_i)$.

We point out that we are using "measure" for what in measure theory is usually called "positive measure", (see P.R.Halmos [5]). We refer to (X, α, μ) as a measure space. If θ is a mapping of a measure space (X, α, μ) into another measure space (Y, β, λ) such that if $A \in \beta$ then $\theta^{-1}A \in \alpha$ and $\mu(\theta^{-1}A) = \lambda(A)$ then we say that θ is a homomorphism. If θ is a one-to-one mapping such that θ and θ^{-1} are homomorphism, then θ is an isomorphism and we say that (X, α, μ) and (Y, β, λ) are isomorphic. If the two measure spaces coincide then a homomorphism is called an endomorphism and an isomorphism, an automorphism. An automorphism T of the space (X, α, μ) is called isomorphic to the automorphism S of the space (Y, β, λ) if there exists an isomorphism θ of (X, α, μ) onto (Y, β, λ) such that $S = \theta T \theta^{-1}$.

An important principle of measure theory is that of neglecting sets of measure zero. In accordance with this principle, the spaces, as well as their automorphisms, need to be studied up to sets of measure sero or, as is commonly said, modulo zero (mod 0). For instance, it is not whether (X, α, μ) and (Y, β, λ) or the transformations T and S acting on them are isomorphic which is essential, but whether it is possible to make them isomorphic by subtracting some sets of measure zero from (X, α, μ) and (Y, β, λ); if the answer is positive then (X, α, μ) and (Y, β, λ) or T and S are called isomorphic modulo zero, (see V.A.Rokhlin [16]). Throughout all results are to be interpreted modulo zero. We say that a measure space (X, ε, μ) is finite and normalized, if $\mu(X) = 1$. We define a Lebesgue space to be a finite and normalized measure space which is isomorphic mod 0, to a segment of the real line with ordinary Lebesgue measure to which is attached a finite or denumerable set of points of positive measure. It turns out (see V.A.Rokhlin [14]) that all measure spaces which occur naturally in probability theory are Lebesgue spaces. Thus we shall always assume that $(X, \mathcal{E}, \mu)$ is a Lebesgue space. From now on, unless the contrary is explicitly stated, we always assume the existence of a measure space $(X, \mathcal{E}, \mu)$ which is finite and normalized and statements such as "A is a set", "α is a σ-algebra" will mean "$A \in \mathcal{E}$", α is a sub- σ-algebra of $\mathcal{E}$, i.e. α is a σ algebra such that $A \in \alpha$ implies $A \in \mathcal{E}$". Further we assume the existence of an automorphism T acting on $(X, \mathcal{E}, \mu)$. If α, β are σ-algebras such that $A \in \alpha$ implies $A \in \beta$ then we write $\alpha \leqslant \beta$. For any σ-algebras α, β we define $\alpha\beta$, $(\alpha \wedge \beta)$ to be the least, (greatest) σ-algebra containing, (contained in) α and β. If α_i, $i \in I$, where I is any index set, are σ-algebras then we define $\bigvee_{i \in I} \alpha_i$, $(\bigwedge_{i \in I} \alpha_i)$ to be the least, (greatest) σ-algebras containing, (contained in) every α_i for $i \in I$. If I is finite then $A \in \bigvee_{i \in I} \alpha_i$ if and only if $A = \bigcap_{i \in I} A_i$ with $A_i \in \alpha_i$ for each i and $A \in \bigwedge_{i \in I} \alpha_i$ if and only if $A \in \alpha_i$ for each i. However, these last statements are not true in general if I is infinite. We denote the σ-algebra whose only sets are $\emptyset$, X by ν and refer to it as the trivial algebra. Lastly we write log for $\log_2$ throughout.

Any further notation will be explained as it is introduced, and a summary of the main definitions is given at the end of this thesis.

1·2 INTRODUCTION

In this section we give a brief outline of the problems which led to the formulation and study of entropy. By a function on X we mean a mapping from X into the real line. As usual we denote,

$$\{f : f \text{ is a function on } X, \int_X |f(x)|^2 d\mu < \infty\}$$

by $\mathcal{L}^2_\mu$. Then if for arbitrary $f, g \in \mathcal{L}^2_\mu$ we put

$$(f,g) = \int_X f(x)g(x)\,d\mu$$

we have that (f,g) is an "inner product" and if

$$\|f\| = (f,f)^{\frac{1}{2}}$$

then $\|f\|$ is a "norm" and $\mathcal{L}^2_\mu$ is a Hilbert space. We now associate with T a unique transformation $U : \mathcal{L}^2_\mu \to \mathcal{L}^2_\mu$ by putting

$$Uf(x) = f(Tx), \quad f \in \mathcal{L}^2_\mu$$

If A is any set, χ_A the characteristic function of A then

$$U\chi_A(x) = \chi_A(Tx) = \chi_{T^{-1}A}$$

Moreover if A_i, $1 \leqslant i \leqslant n$ are disjoint sets a_i, $1 \leqslant i \leqslant n$ are finite real numbers and

$$f(x) = \begin{cases} a_i & \text{if } x \in A_i,\ 1 \leqslant i \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

then $$Uf(x) = \begin{cases} a_i & \text{if } x \in T^{-1}A_i,\ 1 \leqslant i \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

giving
$$\begin{aligned} \|Uf\|^2 &= \int_X |Uf(x)|^2 d\mu \\ &= \sum_{i=1}^{n} a_i^2 \mu(T^{-1}A_i) \\ &= \sum_{i=1}^{n} a_i^2 \mu(A_i) \quad \text{because} \quad (A_i) = (T \quad A_i) \\ &= \int_X |f(x)|^2 d\mu \\ &= \|f\|^2 \end{aligned}$$

Thus we see that $\mathcal{U}$ maps the "step functions" in $\mathcal{L}^2_\mu$ onto the "step functions" in $\mathcal{L}^2_\mu$ in a 1 - 1, and norm preserving manner. Further if f is as above and B_j, $1 \leq j \leq m$ are disjoint sets, b_j, $1 \leq j \leq m$ are finite real numbers and

$$g(x) = \begin{cases} b_j \text{ if } x \in B_j, & 1 \leq j \leq m \\ 0 \text{ otherwise} \end{cases}$$

$$\begin{aligned} \text{then } (\mathcal{U}f, \mathcal{U}g) &= \int_X \{\mathcal{U}f(x)\}\{\mathcal{U}g(x)\}\, d\mu \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i\, b_j\, \mu(T^{-1}A_i \cap T^{-1}B_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i\, b_j\, \mu(A_i \cap B_j) \\ &= \int_X f(x)g(x)\, d\mu \\ &= (f,g) \end{aligned}$$

Hence since "step functions" are everywhere dense in $\mathcal{L}^2_\mu$, and T maps the sets in $\mathcal{E}$ onto the sets in $\mathcal{E}$ it follows by the usual process of approximation that $\mathcal{U}$ is an automorphism on $\mathcal{L}^2_\mu$ such that $(\mathcal{U}f, \mathcal{U}g) = (f,g)$ for all $f, g \in \mathcal{L}^2_\mu$ i.e. $\mathcal{U}$ is a unitary operator (see M.H.Stone [20])

If T_1, T_2 are isomorphic automorphisms and S is the automorphism isomorphism satisfying $T_2 = ST_1S^{-1}$ and $\mathcal{U}_1, \mathcal{U}_2, V$ are the unitary operators corresponding to T_1, T_2, S then for all $f \in \mathcal{L}^2_\mu$

$$\begin{aligned} \mathcal{U}_2 f(x) &= f(T_2 x) \\ &= f(ST_1S^{-1}x) \\ &= V f(T_1 S^{-1}x) \\ &= V\mathcal{U}_1 f(S^{-1}x) \\ &= V\mathcal{U}_1 V^{-1} f(x) \end{aligned}$$

$$\text{i.e.} \qquad \mathcal{U}_2 = V\mathcal{U}_1 V^{-1}$$

Thus if T_1, T_2 are of the same metric type, $\mathcal{U}_1, \mathcal{U}_2$ are of the same spectral type. It is usual to refer to the spectral properties of U as the spectral properties of the metric type of T or simply as the spectral properties of T.

If U_1, U_2 are of the same spectral type then we say that T_1, T_2 are of the same spectral type or alternatively that they are spectrally isomorphic (see V.A.Rokhlin [16]). However, while T_1, T_2 metrically isomorphic imply T_1, T_2 spectrally isomorphic the converse is not true in general, as is shown below. We also refer to the eigenvalues, eigenfunctions, spectrum, and spectral invariants of U as the eigenvalues, eigenfunctions, spectrum and spectral invariants of T.

If $A \in \mathcal{E}$, $TA = A$ implies that either $\mu(A) = 0$ or $\mu(X-A)=0$ then we say that T is ergodic. Since $f(x) = a$, $x \in X$ implies $Uf(x) = f(Tx) = a$, $x \in X$ we see that 1 is always an eigenvalue of U and the constant functions are eigenfunctions corresponding to 1. Further if $A \in \mathcal{E}$, $TA = A$ then $U\chi_A = \chi_{T^{-1}A} = \chi_A$ giving us that χ_A is an eigenfunction. Thus we see that if the only eigenfunctions corresponding to 1 are the constant functions then T is ergodic.

If T,S are ergodic automorphisms with pure point spectrum then (see P.R.Halmos [6]) they are of the same metric type if and only if they have the same spectrum.

For other cases we call the eigenvalues and eigenfunctions, quasi-eigenvalues and quasi-eigenfunctions of the first order. Then for $n > 1$ we define a quasi-eigenvalue of order n to be a quasi-eigenfunction of order $n-1$, and if f_{n-1} is a quasi-eigenvalue of order n and $f_n \neq 0$ satisfies $U f_n = f_n f_{n-1}$ then we say that f_n is a quasi-eigenfunction of order n (see V.A.Rokhlin [16]).

If the quasi-eigenfunctions form a complete system in $\mathcal{L}^2_\Gamma$ then T has a quasi-discrete spectrum. The classification problem for ergodic automorphisms with quasi-discrete spectrum was investigated by L.M.Abramov [1] and a complete classification theory constructed for them.

If T has no eigenfunctions other than the constants then T has a purely continuous spectrum (see V.A.Rokhlin [16]). Until a few years ago it was not known whether there existed spectrally isomorphic automorphisms with purely continuous spectrum belonging to distinct metric types. In [11] and [12] A.N.Kolmogorov introduced the metric invariant, entropy, showed it was not a spectral **invariant**, and so gave a positive answer to the above question. In fact he proved a stronger result, namely, the existence of automorphisms with a denumerably multiple Lebesgue spectrum belonging to different metric types.

1·3 PREVIEW OF THE MAIN RESULTS

Having defined and investigated the entropy of an automorphism we then look at three classes of automorphisms, viz:

1. Kolmogorov automorphisms, i.e. those for which there exists a σ-algebra α such that $\alpha \leqslant T\alpha$, $\bigvee_{i\in\Gamma} T^i\alpha = \varepsilon$, $\bigwedge_{i\in\Gamma} T^i\alpha = \nu$. These were introduced by A.N.Kolmogorov see [11] under the name of quasi-regular automorphisms.

2. Regular automorphisms, i.e. those for which $\bigwedge_{i\in\Gamma} T^{-i} \bigvee_{j\in\Gamma} T^{-j}\alpha = \nu$ for all essentially finite σ-algebras α. (see L. Suchesl on [21]).

3. Automorphisms which are mixings of all degrees.

The first two will be proved equivalent later in this thesis. V.A.Rokhlin [16] and L. Sucheston [21] have shown that Kolmogorov and Regular automorphisms are mixings of all degrees. Our main result is to show that the converse is false i.e that there exist automorphisms on finite measure spaces which are mixings of all degrees, but which are neither Kolmogorov automorphisms nor Regular automorphisms. To do this we consider the Riesz product $\prod_{\nu\in\Gamma^+} (1 \quad \cos 2^{2^\nu} x) = 1 + \sum_{\nu\in\Gamma^+} \gamma_\nu \cos \nu x$ and the increasing, continuous and singular function $G(x)$ of which it is the Fourier-Stieltjes series. If we then consider $F(x) = G(t)$ where $x(t) = \frac{1}{2}\left(t + \frac{t^2}{\pi}\text{sign } t\right)$ we get that $F(x)$ is increasing, continuous and singular and that if $\varphi(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\, dF(x)$ for $n \in \Gamma$ then $\varphi(n)$ is a positive definite function and $\varphi(n) = O(n^{-\frac{1}{2}+d})$ for every $d > 0$ as $n \to \infty$. Then extending slightly the results of S.V.Fomin [4] we show that the stationary Gaussian process associated with $\varphi(n)$ is a mixing of all degrees. To complete the result we then use the results of A.N.Kolmogorov [11] and S.V.Fomin [4] to show that all Kolmogorov automorphisms have a Lebesgue spectrum whereas the stationary Gaussian process referred to above does not.

Next we look at σ-finite measure spaces and generalize the concept of a Kolmogorov automorphism. With this generalization we show that a Kolmogorov automorphism on a σ-finite measure space is ergodic.

1.4 CONVEX FUNCTIONS

Lemma 1.4 If $f(x)$ is defined for $0 \leq x \leq 1$, $f''(x)$ exists and satisfies $f''(x) \leq 0$ for $0 < x < 1$, $f(\ ) \leq \lim_{x\to 0+} f(x)$, $f(1) \leq \lim_{x\to 1-} f(x)$ then for all sequences $\{a_i\}$, $\{x_i\}$ $i \in I \subseteq \Gamma^+$ of numbers satisfying $0 \leqslant a_i$, $\sum_{i \in \Gamma^+} a_i = 1$, $0 \leqslant x_i \leqslant 1$ we have $\sum_{i \in I} a_i f(x_i) \leqslant f(\sum_{i \in I} a_i x_i)$

Proof. The existence of $f''(x)$ for $0 < x < 1$ implies (see G.H.Hardy [7] P.212) that $f'(x)$ exists for $0 < x < 1$ and that $f(x)$, $f'(x)$ are continuous for $0 < x < 1$. If $\sum_{i \in I} a_i x_i = 0$ then for each i either $a_i = 0$ or $x_i = 0$, thus

$\sum_{i \in I} a_i f(x_i) = f(0) = f(\sum_{i \in I} a_i x_i)$.

If $\sum_{i \in I} a_i x_i = 1$ then if $x_j < 1$ for some j such that $a_j \neq 0$ we have

$\sum_{i \in I} a_i x_i \leqslant \sum_{\substack{i \in I \\ i \neq j}} a_i + a_j x_i < 1$

a contradiction and so $\sum_{i \in I} a_i x_i = 1$ implies $x_i = 1$ for all i such that $a_i \neq 0$ and so

$\sum_{i \in I} a_i f(x_i) = f(1) = f(\sum_{i \in I} a_i x_i)$.

Hence if $g(x) = f(x)$, $0 < x < 1$ $g(0) = \lim_{x \to 0+} f(x)$, $g(1) = \lim_{x \to 1-} f(x)$ then $\sum_{i \in I} a_i g(x_i) \leqslant g(\sum_{i \in I} a_i x_i)$ implies $\sum_{i \in I} a_i f(x_i) \leqslant f(\sum_{i \in I} a_i x_i)$. Thus, without loss of generality, we may assume that $f(0) = \lim_{x \to 0+} f(x)$, $f(1) = \lim_{x \to 1-} f(x)$.

Further by an application of the first mean value theorem, we have that the one sided derivatives at 0, 1 are the limits of $f'(x)$ as $x \to 0, 1$ respectively.

If $x = \sum_{i \in I} a_i x_i$ then $0 \leqslant x \leqslant 1$ since $\sum_{i \in I} a_i x_i \leqslant \sum_{i \in I} a_i = 1$. Thus by the mean value theorem of the second order (G.H.Hardy [7] P.285) we have for $i \in \Gamma^+$

$$f(x_i) = f(x) + (x_i - x) f'(x) + \tfrac{1}{2}(x_i - x)^2 f''(y_i)$$

where $0 < y_i < 1$. Hence

$$f(x_i) \leqslant f(x) + (x_i - x) f'(x)$$

and so on multiplying through by a_i and adding we get

$$\sum_{i\in I} a_i f(x_i) \leqslant f(x) + (x-x)f'(x) = f(x)$$

$$\text{i.e.} \quad \sum_{i\in I} a_i f(x_i) \leqslant f(\sum_{i\in I} a_i x_i)$$

We remark that if $f''(x) < 0$ for $0 < x < 1$ then we have equality if, and only if, $x_i = x$ for all i. But this holds only if either all the x_i are equal or $a_j = 1$ for some j and consequently $a_i = 0$ for $i \neq j$.

If $f(x) = - x \log x$ for, $0 < x \leqslant 1$ and $f(0) = 0$ then $f''(x)$ exists and satisfies $f''(x) = -(\log e) / x < 0$ for $0 < x < 1$. Further $f(0) = \lim_{x \to 0+} f(x)$, $f(1) = \lim_{x \to 1-} f(x)$ and so $f(x)$ satisfies the hypotheses of lemma 1·4 and of the remark at the end.

1·5 ALGEBRAS AND PARTITIONS

If α is a σ-algebra, $A \in \alpha$ satisfies $\mu(A) \neq 0$, then we say that A is an atom of α if $B \in \alpha$, $\mu(B-A) = 0$ imply $\mu(B) = 0$ or $\mu(B) = \mu(A)$ and that A is a continuous set of α if given any $A_1 \in \alpha$ such that $A_1 \subseteq A$ and any d such that $0 < d \leqslant \mu(A_1)$ then there exists a $B \in \alpha$ with $\mu(B-A_1) = 0$ and $0 < \mu(B) \leqslant d$. If A,B are atoms of α then we say that A,B are essentially disjoint if $\mu(A \triangle B) \neq 0$ and that A,B are equivalent if $\mu(A \triangle B) = 0$. We have immediately that any σ-algebra has at most a denumerable number of essentially distinct atoms since we are assuming $\mu(X) = 1$. Clearly if A is an atom of α, and $A_1 \in \alpha$ satisfies $\mu(A_1) = 0$ then $A \cup A_1$, $A-A_1$ and $A \triangle A_1$ are atoms of α. For each atom A of a σ-algebra α we put

$$\tilde{A} = \{ B : B \text{ is an atom of } \alpha \text{ and } \mu(A \triangle B) = 0 \}$$

By the above remarks any σ-algebra α has at most a denumberable number of distinct equivalence classes $\tilde{\lambda}$. Let these be $\tilde{A}_i, i \in I$ where I is a subset of Γ^+, and let $B_i, i \in I$ be such that $B_i \in \tilde{A}_i$ for all i. Now put

$$A_1 = B_1 , \; A_i = B_i - \bigcup_{j=1}^{i-1} B_j \text{ for } i \neq 1 , \; i \in I$$

then $\mu(A_1 \triangle B_1) = 0$ and for $i \neq 1$

$$\begin{aligned} \mu(A_i \triangle B_i) &= \mu(A_i - B_i) + \mu(B_i - A_i) \\ &= 0 \end{aligned}$$

since $A_i \subseteq B_i$ and $B_i - A_i = B_i \cap \bigcup_{j=1}^{i-1} B_j = \bigcup_{j=1}^{i-1} B_i \cap B_j$; giving $A_i \in \tilde{A}_i$ for all i. Given a σ-algebra then the $\tilde{A}_i, i \in I$ are unique but the $A_i, i \in I$ depend on the choice of $B_i, i \in I$. However if for fixed α, $B_i', i \in I$ are another set of representatives of $\tilde{A}_i, i \in I$ which give rise to $A_i', i \in I$ then since $A_i, A_i' \in \tilde{A}_i$ for $i \in I$ we have $\mu(A_i \triangle A_i) = 0$ giving

$$|\mu(A_i) - \mu(A_i)| \leq \mu(A_i - A_i) + \mu(A_i - A_i) = \mu(A_i \triangle A_i) = 0$$

i.e. $\mu(A_i) = \mu(A_i)$. Thus the numbers $\mu(A_i), i \in I$ are uniquely determined.

Proposition 1·5 Given any σ-algebra α then we can find sets $A_i, i \in I$ and B such that A_i is an atom of α for each i, $A_i \cap A_j = \emptyset$ if $i \neq j$, B is a continuous set of α if $\mu(B) \neq 0$, and $X = B \cup \bigcup_{i \in I} A_i$

Proof Let $\tilde{A}_i, i \in I$ be the equivalence classes of atoms of α and the $A_i, i \in I$ chosen as above. Then A_i is an atom of α for each i and $A_i \cap A_j = \emptyset$ if $i \neq j$. If $B = X - \bigcup_{i \in I} A_i$ then it remains to prove that B is a continuous set of α, if $\mu(B) \neq 0$. If $\mu(B) \neq 0$, B_1 is any set such that $B_1 \in \alpha, B_1 \subseteq B$, d any number such that $0 < d \leq \mu(B_1)$ then either $\mu(B_1) = 0$ in which case no d exists, or $\mu(B_1) \neq 0$. If $\mu(B_1) \neq 0$

and B_1 is an atom of α then for some $i, B_1 \in A_i$ and so $\mu(B_1 \triangle A_i) = 0$ giving $\mu(B_1) = 0$ since $B_1 \subseteq B$ which is a contradiction. Thus B is not an atom of α and hence there exists a $C_1' \in \alpha$ with $\mu(C_1' - B_1) = 0$ and $0 < \mu(C_1') < \mu(B)$. If $C_1 = B_1 \cap C_1'$ then $C_1 \in \alpha$, $C_1 \subseteq B_1$ and $0 < \mu(C_1) < \mu(B_1)$. Similarly C_1 is not an atom of α and so there exists a $C_2 \in \alpha$ with $C_2 \subseteq C_1$ and $0 < \mu(C_2) < \mu(C_1)$. We now put

$$D_1 = \begin{cases} C_2 & \text{if } \mu(C_2) < \mu(B_1)/2 \\ C_1 - C_2 & \text{otherwise} \end{cases}$$

then $D_1 \in \alpha$, $D_1 \subseteq B_1$ and $0 < \mu(D_1) < \mu(B_1)/2$.

Repeating this argument a further n-1 times we find a $D_n \in \alpha$ with $D_n \subseteq D_{n-1}$ and $0 < \mu(D_n) < \mu(D_{n-1})/2$, giving $D_n \subseteq B_1$ and $0 < \mu(D_n) < \mu(B_1) \times 2^{-n}$. Since $\lim_{n\to\infty} \mu(B_1)/2^n = 0$ there exists an $m < \infty$ such that $\mu(B_1)/2^m \leq d$. If we now put $E = D_m$ then $E \in \alpha$, $\mu(E-B_1) = 0$ and $0 < \mu(E) \leq d$. Hence we conclude that B is continuous.

With the notation of the above proposition we put:

$$Z_1 = \{\alpha : \mu(B) = 0, \text{ I is finite}\}$$

$$Z_3 = \{\alpha : \mu(B) = 0\}$$

$$Z_2 = Z_3 - Z_1.$$

If $\alpha \in Z_1$, (Z_3) then we say that α is essentially finite, (denumerable)

This terminology is based on the concept of "partitions". More precisely we say that a collection of sets $P_i, i \in I$ is a partition if I is countable and $i \neq j$ implies $P_i \cap P_j = \emptyset$, and $\bigcup_{i \in I} P_i = X$. If for each $i \in I$, ϖ_i is the σ-algebra whose sets are $\emptyset, P_i, X-P_i$ and X, and if $\varpi = \bigvee_{i\in I} \varpi_i$ then we say that ϖ is the σ-algebra generated by the partition $P_i, i \in I$. We then say that a σ-algebra α is finite, (denumerable) if there exists a partition

$P_i, i \in I$ with I finite, (denumerable) such that $\alpha = \varpi$. With the notation of proposition 1·5 if $\mu(B) = 0$ it can be shown that there exists a partition $P_i, i \in I$ such that $\varpi \leq \alpha$ and if A is any set in α then there exists an $A^1 \in \varpi$ with $\mu(A \triangle A^1) = 0$. Thus ϖ, α differ only in a set of measure zero. However as we do not need the notion of a partition we shall not continue developing the connections between partitions and essentially denumerable σ- algebras.

We finish this section by considering the form the atoms and continuous set of proposition 1·5 take when we have a σ- algebra of the form $\alpha\beta$ with α, β σ- algebras.

Proposition 1·52 If α, β are σ- algebras, $A_i, i \in I$; $B_j, j \in J$ are atoms of α, β and D_1, D_2 are either null sets or continuous sets of α, β such that

$$D_1 \cup \bigcup_{i \in I} A_i = X = D_2 \cup \bigcup_{j \in J} B_j$$

then for all $i \in I, j \in J$ if $C_{ij} = A_i \cap B_j$ we have that C_{ij} is an atom of $\alpha\beta$ if $\mu(C_{ij}) \neq 0$ and if $D = D_1 \cup D_2$ then $\mu(D) = 0$ or D is a continuous set of $\alpha\beta$ and

$$X = D \cup \bigcup_{\substack{i \in I \\ j \in J}} C_{ij}$$

Proof If $\mu(D) \neq 0$ and $E \in \alpha\beta$ satisfies $E \subseteq D$ and d is such that $0 < d \leq \mu(E)$ then if $\mu(E) = 0$ no such d exists while if $\mu(E) \neq 0$ we have that either $\mu(E \cap D_1) \neq 0$ or $\mu(E \cap D_2) \neq 0$. Without loss of generality we take $\mu(E \cap D_1) \neq 0$ then $E \cap D_1 \in \alpha$ and $E \cap D_1 \subseteq D_1$ hence if $d_1 = \min\{d, \mu(E \cap D_1)\}$ then by the continuity of D_1 there exists a $E_1 \in \alpha$ with $E_1 \subseteq E$ and $0 < \mu(E_1) \leq d_1$. Now $E_1 \in \alpha\beta$ and so we have $E_1 \in \alpha\beta$, $E_1 \subseteq E$ and $0 < \mu(E_1) \leq d$ thus giving D to be a continuous set with respect to $\alpha\beta$.

It now remains to prove that C_{ij} is an atom of $\alpha\beta$ if $\mu(C_{ij}) \neq 0$. Suppose there exists a $C^1 \in \alpha\beta$ with $\mu(C^1 - C_{ij}) = 0$ and $0 < \mu(C^1) < \mu(C_{ij})$. Then $C = C^1 \cap C_{ij}$ satisfies $C \in \alpha\beta$, $\mu(C - C_{ij}) = 0$ and $0 < \mu(C) < \mu(C_{ij})$. Since $C \in \alpha\beta$ there exists $C_\alpha \in \alpha$, $C_\beta \in \beta$ such that $C = C_\alpha \cap C_\beta$. Further we can assume that $C_\alpha \subseteq A_i$ and $C_\beta \subseteq B_j$. However since A_i, B_j are atoms of α, β we must have

$$\mu(C_\alpha) = 0 \text{ or } \mu(A_i) \text{ and } \mu(C_\beta) = 0 \text{ or } \mu(B_j)$$

and since $\mu(C) \neq 0$ we must have $\mu(C_\alpha) = \mu(A_i)$, $\mu(C_\beta) = \mu(B_j)$. But this implies $\mu(C) = \mu(C_{ij})$ which is a contradiction and so we have that C_{ij} is an atom of $\alpha\beta$.

2. THE ENTROPY OF A σ-ALGEBRA

Throughout this chapter all results unless stated otherwise can be found either explicitly or implicitly in K. Jacobs [8].

2·1 THE ENTROPY OF A σ-ALGEBRA IN Z_3

If $\alpha \in Z_3$ then by the last section there exist atoms $A_i, i \in I \subseteq \Gamma^+$, (in general not unique) such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\mu(\bigcup_{i \in I} A_i) = 1$, and further the $\mu(A_i), i \in I$ are uniquely determined. Thus if we put

$$H(\alpha) = -\sum_{i \in I} \mu(A_i) \log \mu(A_i)$$

then $H(\alpha)$ is well defined for $\alpha \in Z_3$. We refer to $H(\alpha)$ as the entropy of α. Since $\mu(X) = 1$ we have $0 \leqslant H(\alpha)$ and while $H(\nu) = 0$ if $X = [0 \ \ 1]$ and $P_i = [(i-1)/n, i/n]$, $1 \leqslant i \leqslant n$ then

$$H(\varpi) = -\sum_{i=1}^{n} n^{-1} \log n^{-1} = \log n$$

where ϖ is defined as in section 1·5.

For any sets A,B we put

$$\mu(A/B) = \begin{cases} \mu(A \cap B)/\mu(B) & \text{if } \mu(B) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and using the terminology of probability theory we refer to it as the measure of A conditioned by B. If now $\alpha, \alpha' \in Z_3$ and $A_i, i \in I$, $A_i^1, i \in I^1$, are chosen as above, then we define $H(\alpha/\alpha^1)$, the entropy of α conditioned by α^1 to be,

$$-\sum_{i \in I} \sum_{j \in I^1} \mu(A_i \cap A_i^1) \log \mu(A_i / A_i^1).$$

Since all terms have the same sign we can reverse the order of summation without altering the convergence or divergence and without changing the sum in the former case.

We now introduce the following definition. If $\alpha \in Z_3$ then $A_i, i \in I$ will be an atom set of α, (in general not unique) if the $A_i, i \in I$ are chosen as at the beginning of this section. It then follows by proposition 1·52 that if $\alpha, \beta \in Z_3$ and $A_i, i \in I, B_j, j \in J$ are atom sets of α, β then $C_{ij} = A_i \cap B_j, i \in I, j \in J$ is an atom set of $\alpha\beta$ plus a number of sets of measure zero. Thus if $\gamma \in Z_3$ and $C_k, k \in K$ is an atom set of γ then

$$H(\alpha\beta) = -\sum_{i \in I} \sum_{j \in J} \mu(A_i \cap B_j) \log \mu(A_i \cap B_j)$$

$$H(\alpha\beta/\gamma) = -\sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \mu(A_i \cap B_j \cap C_k) \log \mu(A_i \cap B_j / C_k)$$

$$H(\alpha/\beta\gamma) = -\sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \mu(A_i \cap B_j \cap C_k) \log \mu(A_i / B_j \cap C_k)$$

For the remainder of this section we assume that $\alpha, \beta, \gamma, \delta \in Z_3$ and that $A_i, i \in I, B_j, j \in J, C_k, k \in K, D_l, l \in L$ are atom sets of $\alpha, \beta, \gamma, \delta$ respectively.

Proposition 2·1 1 $\qquad H(\alpha/\beta) \leq H(\alpha)$

Proof $\quad$ If $a_j = \mu(B_j)$, $x_j = \mu(A_i / B_j)$, $j \in J$ then $\sum_{j \in J} a_j = 1$, $0 \leq x_j \leq 1$, $j \in J$ giving by section 1·4 that

$$-\sum_{j \in J} a_j x_j \log x_j \leq -\left\{\sum_{j \in J} a_j x_j\right\} \log \left\{\sum_{j \in J} a_j x_j\right\}$$

i.e. $-\sum_{j \in J} \mu(B_j) \mu(A_i / B_j) \log \mu(A_i / B_j)$

$$\leq -\left\{\sum_{j \in J} \mu(B_j) \mu(A_i / B_j)\right\} \log \left\{\sum_{j \in J} \mu(B_j) \mu(A_i / B_j)\right\}$$

$$= -\mu(A_i) \log \mu(A_i)$$

Summing over i then gives the required result.

Proposition 2·12 $\quad H(\alpha\beta/\gamma) = H(\alpha/\beta\gamma) + H(\beta/\gamma)$

Proof $\quad H(\alpha\beta/\gamma) = -\sum_{i\in I} \sum_{j\in J} \sum_{k\in K} \mu(A_i \cap B_j \cap C_k) \log \mu(A_i \cap B_j / C_k)$

$$= -\sum_{i\in I} \sum_{j\in J} \sum_{k\in K} \mu(A_i \cap B_j \cap C_k) \left\{ \log \mu(A_i / B_j \cap C_k) + \log \mu(B_j / C_k) \right\}$$

$$= -\sum_{i\in I} \sum_{j\in J} \sum_{k\in K} \mu(A_i \cap B_j \cap C_k) \log \mu(A_i / B_j \cap C_k)$$

$$- \sum_{j\in J} \sum_{k\in K} \mu(B_j \cap C_k) \log \mu(B_j / C_k)$$

$$= H(\alpha/\beta\gamma) + H(\beta/\gamma)$$

Since $\alpha\beta = \beta\alpha$ we have immediately

Cor 1 $\quad H(\alpha\beta/\gamma) = H(\beta/\alpha\gamma) + H(\alpha/\gamma)$

If $\alpha \leq \beta$ then $\alpha\beta = \beta$ and this together with $H(\beta/\alpha\gamma) \geqslant 0$ gives

Cor 2 $\quad$ If $\alpha \leq \beta$ then $H(\alpha/\gamma) \leq H(\beta/\gamma)$

Now X is an atom set for ν and so

$$H(\alpha/\nu) = -\sum_{i\in I} \mu(A_i \cap X) \log \mu(A_i / X)$$

$$= -\sum_{i\in I} \mu(A_i) \log \mu(A_i)$$

$$= H(\alpha)$$

thus putting $\gamma = \nu$ in the proposition and corollary 1 gives

Corollary 3 $\quad H(\alpha\beta) = H(\alpha/\beta) + H(\beta) = H(\beta/\alpha) + H(\alpha)$

And putting $\gamma = \nu$ in corollary 2 gives

Corollary 4 $\quad$ If $\alpha \leq \beta$ then $H(\alpha) \leq H(\beta)$

Proposition 2·13 If $\beta \leqslant \gamma$ then $H(\alpha/\gamma) \leqslant H(\alpha/\beta)$

Proof For each $j \in J$ let $\varepsilon_j = \{A : A \subseteq B_j\}$, $\alpha_j = \{A : A = B_j \cap A'$ with $A' \in \alpha\}$ and $K_j = \{k : k \in K, \mu(C_k - B_j) = 0\}$ then since $\beta \leqslant \gamma$ we have $\mu(\bigcup_{k \in K_j} C_k \triangle B_j) = 0$. Thus

$$H(\alpha/\gamma) = -\sum_{i \in I} \sum_{k \in K} \mu(A_i \cap C_k) \log \mu(A_i/C_k)$$

$$= -\sum_{i \in I} \sum_{j \in J} \sum_{k \in K_j} \mu(A_i \cap C_k) \log \mu(A_i/C_k)$$

$$= -\sum_{i \in I} \sum_{j \in J} \sum_{k \in K_j} \mu(A_i \cap B_j \cap C_k) \log \mu(A_i \cap B_j/C_k)$$

If $a_k = \mu(C_k)/\mu(B_j)$, $x_k = \mu(A_i \cap B_j/C_k)$ then $\sum_{k \in K_j} a_k = 1$, $0 \leqslant x_k \leqslant 1$ and so by section 1·4

$$-\sum_{k \in K_j} a_k x_k \log x_k \leqslant -\left\{\sum_{k \in K_j} a_k x_k\right\} \log \left\{\sum_{k \in K_j} a_k x_k\right\}$$

Now $a_k x_k = \dfrac{\mu(C_k)}{\mu(B_j)} \mu(A_j \cap B_j / C_k) = \mu(A_i \cap C_k / B_j)$

and so $\sum_{k \in K_j} a_k x_k = \sum_{k \in K_j} \mu(A_i \cap C_k/B_j) = \mu(A_i/B_j)$

giving

$$H(\alpha/\gamma) \leqslant -\sum_{i \in I} \sum_{j \in J} \mu(B_j)\, \mu(A_i/B_j) \log \mu(A_i/B_j)$$

$$= -\sum_{i \in I} \sum_{j \in J} \mu(A_i \cap B_j) \log \mu(A_i/B_j)$$

$$= H(\alpha/\beta)$$

Lemma 2·14 $0 \leqslant H(\alpha)$ 1

$H(\nu) = 0$ 2

$0 \leqslant H(\alpha/\beta)$ 3

$H(\alpha/\beta) \leqslant H(\alpha)$ 4

$$H(\alpha\beta) = H(\alpha/\beta) + H(\beta) \qquad 5$$

$$= H(\beta/\alpha) + H(\alpha) \qquad 6$$

$$\leq H(\alpha) + H(\beta) \qquad 7$$

$$H(\alpha\beta/\gamma) = H(\alpha/\beta\gamma) + H(\beta/\gamma) \qquad 8$$

$$= H(\beta/\alpha\gamma) + H(\alpha/\gamma) \qquad 9$$

$$\leq H(\alpha/\gamma) + H(\beta/\gamma) \qquad 10$$

$$H(\alpha/\nu) = H(\alpha) \qquad 11$$

$$H(\alpha\beta/\beta) = H(\alpha/\beta) \qquad 12$$

$$H(\alpha/\gamma) \leq H(\alpha/\beta) + H(\beta/\gamma) \qquad 13$$

If $\alpha \leq \beta$, $\gamma \leq \delta$ then

$$H(\alpha) \leq H(\beta) \qquad 14$$

$$H(\alpha/\gamma) \leq H(\beta/\gamma) \qquad 15$$

$$H(\gamma/\beta) \leq H(\gamma/\alpha) \qquad 16$$

$$H(\alpha/\beta) = 0 \qquad 17$$

$$0 \leq H(\alpha) - H(\alpha/\gamma) \qquad 18$$

$$\leq H(\beta) - H(\beta/\gamma) \qquad 19$$

$$0 \leq H(\beta/\delta) - H(\alpha/\delta) \qquad 20$$

$$\leq H(\beta/\gamma) - H(\alpha/\gamma) \qquad 21$$

$$\leq H(\beta) - H(\alpha) \qquad 22$$

If $H(\alpha/\beta) = 0$ then there exists a σ- algebra β^* such that $\alpha \leq \beta^*$ and β, β^* differ only by sets of measure zero. 23

Thus $H(\alpha/\beta) = 0$ if, and only if $\alpha \leq \beta$ 24

Proof 1, 2, 3, 4, 5, 6, 8, 9, 11, 14, 15, 16 have already been established. 7 follows from 4, 6, and 10 from 9, 16. We now prove 13

$$
\begin{aligned}
H(\alpha/\gamma) &\leq H(\alpha\beta/\gamma) \text{ by } 15 \\
&= H(\alpha/\beta\gamma) + H(\beta/\gamma) \text{ by } 8 \\
&\leq H(\alpha/\beta) + H(\beta/\gamma) \text{ by } 16
\end{aligned}
$$

To prove 17 we note that if $\mu(A_i \cap B_j) \neq 0$

then $\mu(A_i/B_j) = 1$ giving

$$
\begin{aligned}
H(\alpha/\beta) &= -\sum_{i\in I} \sum_{j\in J} \mu(A_i \cap B_j) \log \mu(A_i/B_j) \\
&= -\sum_{i\in I} \sum_{j\in J} 0 \\
&= 0.
\end{aligned}
$$

12 now follows from 8, 17.

18 follows from 4 if $H(\alpha/\gamma) < \infty$ and is meaningless otherwise.

If $H(\beta/\gamma) = \infty$, 19 is meaningless while if $H(\beta/\gamma) < \infty$, $H(\beta) = \infty$ then 19 is true and if $H(\beta/\gamma) < \infty$, $H(\beta) < \infty$ then $H(\alpha) < \infty$ by 14 $H(\alpha/\gamma) < \infty$ by 4, and $H(\beta/\alpha) < \infty$ by 7.

Thus

$$
\begin{aligned}
H(\alpha) - H(\alpha/\gamma) &\leq H(\alpha) \;\; H(\beta/\alpha) - H(\beta/\gamma) \text{ by } 13 \\
&= H(\alpha\beta) - H(\beta/\gamma) \text{ by } 6 \\
&= H(\beta) - H(\beta/\gamma) \text{ because}
\end{aligned}
$$

20 follows from 15 if $H(\alpha/\delta) < \infty$ and is meaningless otherwise.

If $H(\alpha/\gamma) = \infty$, 21 is meaningless, while if $H(\alpha/\gamma) < \infty$, $H(\beta/\gamma) = \infty$ then 21 holds and if $H(\alpha/\gamma) < \infty$, $H(\beta/\gamma) < \infty$ then $H(\alpha/\delta) < \infty$, $H(\beta/\delta) < \infty$ by 16. Thus

$$
\begin{aligned}
H(\beta/\delta) - H(\alpha/\delta) &\leq H(\alpha\beta/\delta) - H(\alpha/\delta) \text{ by } 15 \\
&\leq H(\beta/\alpha\delta) \text{ by } 9 \\
&\leq H(\beta/\alpha\gamma) \text{ by } 16 \\
&= H(\alpha\beta/\gamma) - H(\alpha/\gamma) \text{ by } 9 \\
&= H(\beta/\gamma) - H(\alpha/\gamma) \text{ since } \alpha \leq \beta
\end{aligned}
$$

If $H(\alpha) = \infty$ 22 is meaningless, while if $H(\alpha) < \infty$, $H(\beta) = \infty$

then 22 holds, and if $H(\alpha) < \infty$, $H(\beta) < \infty$ then $H(\alpha/\gamma) < \infty$, $H(\beta/\gamma) < \infty$ by 4 and 22 follows from 19.

If there does not exist a σ- algebra $\beta^{\times}$ such that $\alpha \leq \beta^{\times}$ and $\beta, \beta^{\times}$ differ only by sets of measure zero, then there exist m, n such that $\mu(A_m \cap B_n) \neq 0$, $\mu(A_m \cap B_n) \neq \mu(B_n)$ and so

$$\begin{aligned} H(\alpha/\beta) &= - \sum_{i \in I} \sum_{j \in J} \mu(A_i \cap B_j) \log \mu(A_i/B_j) \\ &\geqslant - \mu(A_m \cap B_n) \log \mu(A_m/B_n) \\ &> 0 \end{aligned}$$

thus proving 23. 24 is a direct consequence of 17, 23.

2.2 THE ENTROPY OF A σ-ALGEBRA

For any σ- algebra α we put

$$S(\alpha) = \{ \alpha' : \alpha' \leq \alpha, \ \alpha' \in Z_3 \}$$

$$H(\alpha) = \sup_{\alpha' \in S(\alpha)} H(\alpha').$$

Since $\alpha \in S(\alpha)$ if $\alpha \in Z_3$ it follows from lemma 2.14, 14 that this definition of $H(\alpha)$ coincides with the previous one if $\alpha \in Z_3$. If β is any σ- algebra $\alpha \in Z_3$ we put

$$H(\alpha/\beta) = \inf_{\beta' \in S(\beta)} H(\alpha/\beta')$$

and note that in view of lemma 2.14, 16, this coincides with our previous definition if $\beta \in Z_3$.

Proposition 2.21 If α, β Z_3, $\alpha \leq \beta$ and γ is any σ-algebra then $H(\alpha/\gamma) \leq H(\beta/\gamma)$

Proof Let γ_i^1, γ_i^2 $i \in \Gamma^+$ be such that $\lim_{i \to \infty} H(\alpha/\gamma_i^1) = H(\alpha/\gamma)$, $\lim_{i \to \infty} H(\beta/\gamma_i^2) = H(\beta/\gamma)$ and put $\gamma_i = \gamma_i^1 \gamma_i^2$, $i \in \Gamma^+$

Then $H(\alpha/\gamma_i) \leq H(\alpha/\gamma_i^1)$, $i \in \Gamma^+$ and $H(\beta/\gamma_i) \leq H(\beta/\gamma_i^2)$, $i \in \Gamma^+$

giving $H(\alpha/\gamma) = \lim_{i\to\infty} H(\alpha/\gamma_i)$, $H(\beta/\gamma) = \lim_{i\to\infty} H(\beta/\gamma_i)$

But $\qquad H(\alpha/\gamma_i) \leq H(\beta/\gamma_i)$, $i \in \Gamma^+$ by lemma 2·14, 15.

and so

$$\begin{aligned} H(\alpha/\gamma) &= \lim_{i\to\infty} H(\alpha/\gamma_i) \\ &\leq \lim_{i\to\infty} H(\beta/\gamma_i) \\ &= H(\beta/\gamma) \end{aligned}$$

For any σ- algebras α, β we put

$$H(\alpha/\beta) = \sup_{\alpha' \in S(\alpha)} H(\alpha'/\beta)$$

and note that in view of proposition 2·21 this coincides with our previous definition if $\alpha \in Z_3$.

We now take any σ- algebras α, β and consider $S(\alpha\beta)$. Let γ^1 be any σ algebra such that $\gamma^1 \in S(\alpha\beta)$ then we have $\gamma^1 \in Z_3$. If $C_k, k \in K$ is an atom set of γ^1 then for each k we can find an $A_k \in \alpha$ and a $B_k \in \beta$ such that $C_k = A_k \cap B_k$. If $\alpha_k = \{\phi, A_k, X-A_k, X\}$, $\beta_k = \{\phi, B_k, X-B_k, X\}$ we put $\alpha^1 = \bigvee_{k\in K} \alpha_k$, $\beta^1 = \bigvee_{k\in K} \beta_k$. Clearly $\alpha^1 \in S(\alpha)$, $\beta^1 \in S(\beta)$ and $\gamma^1 \leq \alpha^1\beta^1$ except possibly on a set of measure zero. But by proposition 1·52 we have that $\alpha^1\beta^1 \in Z_3$ if $\alpha^1, \beta^1 \in Z_3$ and so we get $\alpha^1\beta^1 \in S(\alpha\beta)$. Thus we conclude that if α, β, γ are σ- algebras then

$$\begin{aligned} H(\alpha\beta) &= \sup_{\delta^1 \in S(\alpha\beta)} H(\delta^1) \\ &= \sup_{\alpha^1 \in S(\alpha), \beta^1 \in S(\beta)} H(\alpha^1\beta^1) \end{aligned}$$

by the above remarks and lemmas 2·14, 14 while

$$\begin{aligned} H(\alpha\beta/\gamma) &= \sup_{\delta^1 \in S(\alpha\beta)} H(\delta^1/\gamma) \\ &= \sup_{\alpha^1 \in S(\alpha), \beta^1 \in S(\beta)} H(\alpha^1\beta^1/\gamma) \end{aligned}$$

by the above remarks and proposition 2·21

Also if $\gamma \in Z_3$ then

$$H(\gamma/\alpha\beta) = \inf_{\delta^1 \in S(\alpha\beta)} H(\gamma/\delta^1)$$
$$= \inf_{\alpha^1 \in S(\alpha), \beta^1 \in S(\beta)} H(\gamma/\alpha^1\beta^1)$$

by the above remarks and lemma 2·14, 16.

Lemma 2·22 The results of lemma 2·14 hold for arbitrary σ-algebras $\alpha, \beta, \gamma, \delta$.

Proof 1, 2, 3 are direct consequence of lemmas 2·14 and the definitions of $H(\alpha)$, $H(\alpha/\beta)$. Now

$$H(\alpha/\beta) = \sup_{\alpha^1 \in S(\alpha)} \inf_{\beta^1 \in S(\beta)} H(\alpha^1/\beta^1)$$
$$\leq \sup_{\alpha^1 \in S(\alpha)} \inf_{\beta^1 \in S(\beta)} H(\alpha^1) \text{ by lemma 2·14, 4}$$
$$= H(\alpha)$$

giving 4. 5 and 8 are proved by the method used in proposition 2·21, bearing in mind the remarks made afterwards. 6 and 9 then follow from 5 and 8 and the fact that $\alpha\beta = \beta\alpha$. 7 is a direct consequence of 4 and 5 and 10 will follow from 8 and 16 when we have established the latter. Now

$$H(\alpha/\nu) = \sup_{\alpha^1 \in S(\alpha)} H(\alpha^1/\nu)$$
$$= \sup_{\alpha^1 \in S(\alpha)} H(\alpha^1) \text{ by lemma 2·14, 11}$$
$$= H(\alpha)$$

giving 11. 15 is proposition 2·21 and has already been established. 13 follows from 10 and 15. 14 follows from 11 and 15. To prove 16 we note that $\beta = \alpha\beta$ and so

$$\begin{aligned} H(\gamma/\beta) &= \sup_{\gamma^1\in S(\gamma)} \inf_{\beta^1\in S(\beta)} H(\gamma^1/\beta^1) \\ &= \sup_{\gamma^1\in S(\gamma)} \inf_{\beta^1\in S(\beta),\alpha^1\in S(\alpha)} H(\gamma^1/\alpha^1\beta^1) \\ &\leq \sup_{\gamma^1\in S(\gamma)} \inf_{\beta^1\in S(\beta),\alpha^1\in S(\alpha)} H(\gamma^1/\alpha^1) \\ &= H(\gamma/\alpha) \end{aligned}$$

by lemma 2·14, 16, 18, 19, 20, 21 and 22 are proved as in lemma 2·14.

To prove 17 we note that $\beta = \alpha\beta$ and so

$$\begin{aligned} H(\alpha/\beta) &= \sup_{\alpha^1\in S(\alpha)} \inf_{\beta^1\in S(\beta)} H(\alpha^1/\beta^1) \\ &= \sup_{\alpha^1\in S(\alpha)} \inf_{\beta^1\in S(\beta)} H(\alpha^1/\alpha^1\beta^1) \\ &= 0 \text{ by lemma 2·14, 17.} \end{aligned}$$

Here we use the fact that for fixed $\alpha^1 \in S(\alpha)$ we have $\beta^1 \leq \alpha^1\beta^1 \in S(\beta)$ for all $\beta^1 \in S(\beta)$. 12 follows from 8 and 17.

To prove 23 we note that $H(\alpha/\beta) = 0$ implies $H(\alpha^1/\beta) = 0$ for all $\alpha^1 \in S(\alpha)$. Given any $A \in \alpha$ we can find an $\alpha^1 \in S(\alpha)$ such that $A \in \alpha^1$. If there exists a $\beta^1 \in S(\beta)$ such that $H(\alpha^1/\beta^1) = 0$ then $\alpha \leq \beta^1 \leq \beta$ by lemma 2·14. However if $H(\alpha^1/\beta^1) \neq 0$ for all $\beta^1 \in S(\beta)$ then we choose β^1_i, $i \in \Gamma^+$ such that $\beta^1_i \in S(\beta)$ each i and $\lim_{i\to\infty} H(\alpha^1/\beta^1_i) = 0$. If $\inf_{B\in\beta} \mu(A \triangle B) = k > 0$ then there exists an atom B_i of β^1_i for each i such that

$$-\mu(B_i \cap A) \log \mu(A/B_i) \geq k^1 = k^1(k) > 0$$

and hence $\lim_{i\to\infty} H(\alpha^1/\beta^1_i) \geq k^1 \neq 0$ a contradiction.

Thus $\inf_{B\in\beta} \mu(A \triangle B) = 0$, i.e. $A \in \alpha$ implies that there exists a $B \in \beta$ such that $\mu(A \triangle B) = 0$ as required.

24 is an immediate consequence of 17 and 23.

2·3 INCREASINGLY FILTERED COLLECTIONS OF ALGEBRAS

We now introduce

$$Z = \{\alpha : \alpha \text{ is a } \sigma\text{-algebra, } H(\alpha) < \infty\}$$

Lemma 2·31 $\quad Z \subseteq Z_3$

Proof If α is a σ-algebra such that $\alpha \notin Z_3$ and Δ is any real number then it is sufficient to find a $\alpha^1 \in S(\alpha)$ with $H(\alpha^1) \geqslant \Delta$ If $\Delta \leqslant 0$ then we take $\alpha^1 = \nu$ if not then we consider the B of proposition 1·51. Since $\alpha \notin Z_3$ we have $\mu(B) \neq 0$ and hence B a continuous set of α. Further $\lim_{x \to 0+} -\mu(B) \log x = \infty$ and so we can find a real number d such that $0 < d \leqslant \mu(B)$ and $-\mu(B) \log d \geqslant \Delta$. Let Φ be the set of all sequences of disjoint sets A_i, $i \in I \subseteq \Gamma^+$ such that $A_i \in \alpha$, $A_i \subseteq B$, $0 < \mu(A_i) \leqslant d$ for all i. Since B is continuous there exists an $A^1 \in \alpha$ with $A^1 \subseteq B$ and $0 < \mu(A^1) \leqslant d$ and so Φ is non-empty. If $\beta_1, \beta_2 \in \Phi$ then we write $\beta_1 \leqslant \beta_2$ if $A \in \beta_1$ implies $A \in \beta_2$. Thus $\leqslant$ is a partial ordering of Φ. If C is a "chain" in Φ i.e. for all $\beta_1, \beta_2 \in C$ we have $\beta_1 \leqslant \beta_2$ or $\beta_2 \leqslant \beta_1$ (or both if $\beta_1 = \beta_2$) and $\beta_c = \{A:$ there exists a $\beta \in C$ with $A \in \beta\}$ then if $A_1, A_2 \in \beta_c$ there exist $\beta_1, \beta_2 \in C$ such that $A_1 \in \beta_1$, $A_2 \in \beta_2$ but $\beta_1 \leqslant \beta_2$ or $\beta_2 \leqslant \beta_1$ and so $A_1, A_2 \in \beta_2$ or $A_1, A_2 \in \beta_1$ giving in either case that A_1, A_2 are disjoint and hence that $\beta_c \in \Phi$, since $\mu(X) = 1$ implies that β_c is at most denumerable. Now $\beta \leqslant \beta_c$ for all $\beta \in C$ and so by Zorns lemma there exists a $\psi \in \Phi$ such that for all $\beta \in \Phi$ it is false that $\psi \leqslant \beta$ and $\beta \neq \psi$. Suppose $\mu(B - \bigcup_{A \in \psi} A) > 0$ then if $C = B - \bigcup_{A \in \psi} A$ we have that $C \in \alpha$, $C \subseteq B$ and so by the continuity of B there exists an $A_1 \in \alpha$, $A_1 \subseteq C$ with

$0 < \mu(A_1) \leqslant d$. Now $\psi \vee A_1 \in \Phi$, $\psi \leqslant \psi \vee A_1$ and $\psi \neq \psi \vee A_1$ giving a contradiction and so $\mu(C) = 0$. If α^1 is the σ- algebra generated by the members of ψ together with $X - \bigcup_{A \in \psi} A$ then $\alpha^1 \in S(\alpha)$ and

$$
\begin{aligned}
H(\alpha^1) &\geqslant - \sum_{A \in \psi} \mu(A) \log \mu(A) \\
&\geqslant - \sum_{A \in \psi} \mu(A) \log d \\
&= - (B) \log d \\
&\geqslant \Delta
\end{aligned}
$$

We say that a collection S of σ- algebras is increasingly filtered if given any $\alpha, \beta \in S$ then there exists a $\gamma \in S$ with $\alpha\beta \leqslant \gamma$. For any σ- algebra α we have that $S(\alpha)$ is increasingly filtered. Again if $\alpha_i, i \in I \subseteq \Gamma^+$ are σ- algebras such that $\alpha_i \leqslant \alpha_j$ if $i \leqslant j$ then $S = \{\alpha_i : i \in I\}$ is an increasingly filtered system.

Lemma 2·32 If S is an increasingly filtered collection of σ- algebras, $\alpha = \bigvee_{\beta \in S} \beta$, $\gamma \in Z_1$ and $\gamma \leqslant \alpha$ then if $C_i, 1 \leqslant i \leqslant n$ is an atom set of γ and d is any real number such that $0 < d$ there exists a $\beta \in S$ and sets $B_i \in \beta$, $1 \leqslant i \leqslant n+1$ such that $\mu(B_i \triangle C_i) < d$, $1 \leqslant i \leqslant n$; $\mu(B_{n+1}) < d$; $\mu(C_i \cap B_j) < d$ if $i \neq j$; $\mu(C_i / B_i) > 1 - d$, $1 \leqslant i \leqslant n$ and $\bigcup_{i=1}^{n+1} B_i = X$

Proof Since C_i, $1 \leqslant i \leqslant n$ is an atom set of γ we have $0 < \mu(C_i)$ for all i and so given any d such that $0 < d$ we can find a d_1 such that $0 < d_1 < d$, $d_1 < \max\{\mu(C_i), d\mu(C_i)\}$ and

$$\frac{d_1}{\mu(C_i) - d_1} < d, \quad 1 \leqslant i \leqslant n$$

For each i there exists a $\beta_i \in S$ and a $B_i^1 \in \beta_i$ such that $\mu(B_i^1 \triangle C_i) < d_1/8n^2$. Since S is increasingly filtered there exists a $\beta \in S$ such that $\beta_i \leqslant \beta$, $1 \leqslant i \leqslant n$. Let $B_i = B_i^1 - \bigcup_{j \neq i} B_j^1$, $1 \leqslant i \leqslant n$, $B_{n+1} = X - \bigcup_{i=1}^{n} B_i$. Then for each i, $B_i \in \beta$ and if $i \neq j$ then $B_i \cap B_j = \phi$. For $i \neq j$

$$\begin{aligned}\mu(B_i^1 \cap B_j^1) &\leqslant \mu(B_i^1 - C_i) + \mu(B_j^1 - C_j) \text{ since } \mu(C_i \cap C_j) = 0\\ &< d_1/8n^2 + d_1/8n^2\\ &= d_1/4n^2\end{aligned}$$

and
$$\begin{aligned}\mu(B_i^1 - B_i) &\leqslant \sum_{j \neq i} \mu(B_i^1 \cap B_j^1)\\ &< (n-1)d_1/4n^2\\ &< d_1/4n\end{aligned}$$

giving
$$\begin{aligned}\mu(C_i - B_i) &\leqslant \mu(C_i - B_i^1) + \mu(B_i^1 - B_i) \text{ since } B_i \subseteq B_i^1\\ &< d_1/8n^2 + d_1/4n\\ &< d_1/2n\end{aligned}$$

But
$$\begin{aligned}\mu(B_i - C_i) &\leqslant \mu(B_i^1 - C_i)\\ &< d_1/8n^2\end{aligned}$$

Thus
$$\begin{aligned}\mu(B_i \triangle C_i) &= \mu(B_i - C_i) + \mu(C_i - B_i)\\ &< d_1/2n + d_1/8n^2\\ &< d_1\\ &< d\end{aligned}$$

Lastly
$$\begin{aligned}\mu(B_{n+1}) &= \mu(X - \bigcup_{i=1}^{n} B_n)\\ &= 1 - \sum_{i=1}^{n} \mu(B_n)\\ &= 1 - \sum_{i=1}^{n} \mu(B_i^1 - \bigcup_{j \neq i} B_j^1)\\ &\leqslant 1 - \sum_{i=1}^{n} \{ \mu(B_i^1) - \sum_{j \neq i} \mu(B_i^1 \cap B_j^1) \}\\ &\leqslant 1 - \sum_{i=1}^{n} \{ \mu(B_i^1) - (n-1)d/4n^2 \}\\ &< 1 - \sum_{i=1}^{n} \{ \mu(C_i) - d/8n^2 - d/4n \}\end{aligned}$$

$$< 1 - 1 + nd_1/2n$$
$$= d_1/2$$
$$< d_1$$
$$< d$$

If $1 \leq i \leq n$, $1 \leq j \leq n$ $i \neq j$ we have

$$\begin{aligned} \mu(C_i \cap B_j) &\leq \mu(C_i \cap B_j^1) \\ &\leq \mu(C_i \cap C_j) + \mu(C_j \triangle B_j^1) \\ &< 0 + d_1/8n^2 \\ &< d \end{aligned}$$

If $1 \leq i \leq n$ then

$$\begin{aligned} \mu(C_i \cap B_{n+1}) &\leq \mu(B_{n+1}) \\ &< d \end{aligned}$$

If $1 \leq i \leq n$ then if $\mu(B_i) \neq 0$

$$\begin{aligned} \mu(C_i / B_i) &= \frac{\mu(C_i \cap B_i)}{\mu(B_i)} \\ &= \frac{\mu(B_i) - \mu(B_i - C_i)}{\mu(B_i)} \\ &\geq 1 - \frac{\mu(B_i \triangle C_i)}{\mu(B_i)} \\ &> 1 - \frac{d_1}{\mu(B_i)} \end{aligned}$$

Now $\mu(B_i)$ $\geq \mu(C_i) - \mu(B_i \triangle C_i)$

$$> \mu(C_i) - d_1$$

and so if $\mu(B_i) \neq 0$

$$\begin{aligned} \mu(C_i / B_i) &> 1 - \frac{d_1}{\mu(C_i) - d_1} \\ &> 1 - d \end{aligned}$$

Lemma 2·33 If S is an increasingly filtered collection of σ-algebras in Z_3, $\alpha = \bigvee_{\beta \in S} \beta$ and γ is any σ-algebra then

$$\text{(i)} \quad H(\alpha) = \sup_{\beta \in S} H(\beta)$$

$$\text{(ii)} \quad H(\alpha \mid \gamma) = \sup_{\beta \in S} H(\beta \mid \gamma)$$

and if $\gamma \in Z_3$ then

$$\text{(iii)} \quad H(\gamma \mid \alpha) = \inf_{\beta \in S} H(\gamma \mid \beta)$$

Proof Given $d > 0$ since $-x\log x$ is continuous for $0 \leqslant x \leqslant 1$ and since $\lim_{x \to 1} -\log x = 0$ there exists a d^1 such that $0 \leqslant x, y \leqslant 1$, $|x - y| \leqslant d^1$ implies $|-x\log x + y\log y| < d/n(n+1)$ and $1-d^1 \leqslant x \leqslant 1$ implies $-\log x < d/n(n+1)$.

If $\alpha^1 \in S(\alpha)$ and $C_i, i \in I$ is an atom set of α then for each $n \in \Gamma^+$ we define γ_n to be the σ-algebra generated by the $C_i, i \in I, i \leqslant n$. Then $\gamma_n \in Z_1$, $n \in \Gamma^+$ and $H(\alpha^1) = \lim_{n\to\infty} H(\gamma_n)$. we now take γ_n as the γ, and d^1 as the d of lemma 2·32.

With the notation of lemma 2·32 let β^1 be the σ-algebra generated by B_i, $1 \leqslant i \leqslant n+1$. Thus

$$\begin{aligned} |H(\gamma_n) - H(\beta^1)| &= \left| -\sum_{i=1}^{n} \mu(C_i)\log\mu(C_i) + \sum_{i=1}^{n+1} \mu(B_i)\log\mu(B_i) \right| \\ &\leqslant \sum_{i=1}^{n+1} d/n(n+1) \\ &< d \end{aligned}$$

Hence

$$\begin{aligned} H(\alpha) &= \sup_{\alpha^1 \in S(\alpha)} H(\alpha^1) \\ &= \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} H(\gamma_n) \\ &< \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} \{ H(\beta^1) + d \} \\ &\leqslant \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} \{ H(\beta) + d \} \\ &\leqslant \sup_{\alpha^1 \in S(\alpha)} \sup_{\beta \in S} \{ H(\beta) + d \} \\ &= \sup_{\beta \in S} \{ H(\beta) + d \} \end{aligned}$$

But d was arbitrary and so we have

$$H(\alpha) \leqslant \sup_{\beta \in S} H(\beta)$$

and since $S \subsetneq S(\alpha)$ we have

$$H(\alpha) \geqslant \sup_{\beta \in S} H(\beta)$$

giving (i)

Now
$$H(\gamma_n | \beta^1) = -\sum_{i=1}^{n} \sum_{j=1}^{n+1} \mu(C_i \cap B_j) \log \mu(C_i | B_j)$$
$$< n(n+1)\, d/n(n+1)$$
$$= d$$

because
$$-\mu(C_i \cap B_j) \log \mu(C_i | B_j) \leqslant -\mu(C_i \cap B_j) \log \mu(C_i \cap B_j)$$
$$\leqslant d/n(n+1) \text{ if } i \neq j$$

since $\mu(C_i \cap B_j) < d^1$ if $i \neq j$ while

$$-\mu(C_i \cap B_i) \log \mu(C_i | B_i) \leqslant -\log \mu(C_i | B_i)$$
$$< -\log(1-d^1)$$
$$< d/n(n+1)$$

Thus
$$H(\alpha | \gamma) = \sup_{\alpha^1 \in S(\alpha)} H(\alpha^1 | \gamma)$$
$$= \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} H(\gamma_n | \gamma)$$
$$\leqslant \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} H(\beta^1 \gamma_n | \gamma)$$

by lemma 2·22, 15

$$= \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} \{ H(\gamma_n | \beta^1 \gamma) + H(\beta^1 | \gamma) \}$$

by lemma 2·22, 8

$$\leqslant \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} \{ H(\gamma_n | \beta^1) + H(\beta^1 | \gamma)$$

by lemma 2·22, 16

$$< \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} \{ d + H(\beta^1 | \gamma) \}$$
$$\leqslant \sup_{\alpha^1 \in S(\alpha)} \lim_{n\to\infty} \{ d + H(\beta | \gamma) \}$$
$$\leqslant \sup_{\alpha^1 \in S(\alpha)} \sup_{\beta \in S} \{ d + H(\beta | \gamma) \}$$
$$= \sup_{\beta \in S} \{ d + H(\beta | \gamma) \}$$

But d was arbitrary and so we have

$$H(\alpha|\gamma) \leq \sup_{\beta\in S} H(\beta|\gamma)$$

and since $S \subseteq S(\alpha)$ we have

$$H(\alpha|\gamma) \geq \sup_{\beta\in S} H(\beta|\gamma)$$

giving (ii)

Again $H(\gamma|\alpha) = \inf_{\alpha^1\in S(\alpha)} H(\gamma|\alpha^1)$ for $\gamma\in Z_3$

$$= \inf_{\alpha^1\in S(\alpha)} \lim_{n\to\infty} H(\gamma|\gamma_n)$$

$$\geq \inf_{\alpha^1\in S(\alpha)} \lim_{n\to\infty} H(\gamma|\beta^1\gamma_n) \quad \text{by lemma 2·22, 16.}$$

$$= \inf_{\alpha^1\in S(\alpha)} \lim_{n\to\infty} \left\{H(\gamma\gamma_n|\beta^1) - H(\gamma_n|\beta^1)\right\}$$

by lemma 2·22, 8

$$\geq \inf_{\alpha^1\in S(\alpha)} \lim_{n\to\infty} \left\{H(\gamma|\beta^1) - d\right\}$$

by lemma 2·22, 15

$$\geq \inf_{\alpha^1\in S(\alpha)} \lim_{n\to\infty} \left\{H(\gamma|\beta) - d\right\}$$

by lemma 2·22, 16

$$\geq \inf_{\alpha^1\in S(\alpha)} \inf_{\beta\in S} \left\{H(\gamma|\beta) - d\right\}$$

$$= \inf_{\beta\in S} \left\{H(\gamma|\beta) - d\right\}$$

But d was arbitrary and so we have

$$H(\gamma|\alpha) \geq \inf_{\beta\in S} H(\gamma|\beta)$$

and since $S \subseteq S(\alpha)$ we have

$$H(\gamma|\alpha) \leq \inf_{\beta\in S} H(\gamma|\beta)$$

giving (iii)

2·4 THE METRIC SPACE (Z,ρ).

We now define a function $\rho(\alpha,\beta)$ for any pair of σ-algebras α,β by

$$\rho(\alpha,\beta) = H(\alpha|\beta) + H(\beta|\alpha)$$

Clearly by lemma 2·22

(1) $\rho(\alpha,\beta) = 0$ if, and only if $\alpha = \beta$ up to sets of measure zero

(2) $\rho(\alpha,\beta) = \rho(\beta,\alpha)$

(3) $\rho(\alpha,\gamma) \leq \rho(\alpha,\beta) + \rho(\beta,\gamma)$

i.e. ρ is a metric.

Lemma 2·41 (Z,ρ) is a complete metric space.

Proof If α^1_n, $n \in \Gamma^+$ is a Cauchy sequence in (Z,ρ), then there exists a subsequence α_n, $n \in \Gamma^+$ such that

$$\rho(\alpha_n, \alpha_{n+p}) < 2^{-n} \text{ for all } p \in \Gamma^+$$

If $\alpha = \bigwedge_{j=1}^{\infty} \bigvee_{i=j}^{\infty} \alpha_i$ then for $m > j > n$

$$H\Big(\bigvee_{i=j}^{m} \alpha_i \Big| \bigvee_{i=n}^{j-1} \alpha_i\Big) = H\Big(\alpha_j \Big| \bigvee_{i=n}^{j-1} \alpha_i\Big) + H\Big(\bigvee_{i=j+1}^{m} \alpha_i \Big| \bigvee_{i=n}^{j} \alpha_i\Big)$$

All terms are finite and so summing over j gives

$$H\Big(\bigvee_{i=n+1}^{m} \alpha_i \Big| \alpha_n\Big) = \sum_{j=n+1}^{m-1} H\Big(\alpha_j \Big| \bigvee_{i=n}^{j-1} \alpha_i\Big) + H\Big(\alpha_m \Big| \bigvee_{i=n}^{m-1} \alpha_i\Big)$$

$$= \sum_{j=n+1}^{m} H\Big(\alpha_j \Big| \bigvee_{i=n}^{j-1} \alpha_i\Big)$$

$$\leq \sum_{j=n+1}^{m} H(\alpha_j | \alpha_{j-1})$$

By lemma 2·33 letting $m \to \infty$ gives

$$H\Big(\bigvee_{i=n+1}^{\infty} \alpha_i \Big| \alpha_n\Big) = \sum_{j=n+1}^{\infty} H(\alpha_j | \alpha_{j-1})$$

and since $\alpha \leq \bigvee_{i=n+1}^{\infty} \alpha_i$ we have

$$
\begin{aligned}
H(\alpha|\alpha_n) &\leq H(\bigvee_{i=n+1}^{\infty} \alpha_i | \alpha_n) \\
&= \sum_{j=n+1}^{\infty} H(\alpha_j | \alpha_{j-1}) \\
&\leq \sum_{j=n+1}^{\infty} \rho(\alpha_j, \alpha_{j-1}) \\
&< \sum_{j=n+1}^{\infty} 2^{-(j-1)} \\
&= 2^{-(n-1)}
\end{aligned}
$$

Also $H(\alpha_n|\alpha) = \lim\limits_{j\to\infty} H(\alpha_n | \bigvee_{i=j}^{\infty} \alpha_i)$

hence there exists a $j > n$ for which

$$
\begin{aligned}
H(\alpha_n|\alpha) &\leq H(\alpha_n | \bigvee_{i=j}^{\infty} \alpha_i) + 2^{-(n-1)} \\
&\leq H(\alpha_n | \alpha_j) + 2^{-(n-1)} \\
&\leq \rho(\alpha_n, \alpha_j) + 2^{-(n-1)} \\
&< 2^{-n} + 2^{-(n-1)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
(\alpha, \alpha_n) &= H(\alpha|\alpha_n) + H(\alpha_n|\alpha) \\
&< 2^{-(n-1)} + 2^{-n} + 2^{-(n-1)} \\
&< 2^{-(n-3)}
\end{aligned}
$$

Further

$$
\begin{aligned}
H(\alpha) &\leq H(\alpha\alpha_1) \\
&= H(\alpha|\alpha_1) + H(\alpha_1) \\
&< 1 + H(\alpha_1) \\
&< \infty
\end{aligned}
$$

Hence we conclude that $\{\alpha_i\}$ and hence $\{\alpha^1_i\}$ is convergent to a σ-algebra $\alpha \in Z$.

2.5 AN ALTERNATIVE DEFINITION

We have defined the conditional entropy $H(\alpha|\beta)$ of α with respect to β by

$$
H(\alpha|\beta) = \sup_{\alpha^1 \in S(\alpha)} \inf_{\beta^1 \in S(\beta)} H(\alpha^1|\beta^1)
$$

or equivalently as

$$H(\alpha|\beta) = \lim_{\alpha^1 \in S(\alpha)} \lim_{\beta^1 \in S(\beta)} H(\alpha^1|\beta^1)$$

if we wish to make use of the theory of Moore-Smith convergence (see J.L.Kelley [10]) and such notions as 'nets', 'filters', etc. However, while K. Jacobs [8] takes an essentially equivalent definition, the Russian school proceed in a rather different manner as outlined below.

For any $\alpha, \beta \in Z_3$, if $A \in \alpha$ we define

$$\mu^A(B) = \mu(A \cap B) \text{ for } B \in \beta$$

If B_j, $j \in J$ is an atom set of β and we put

$$\mu(A|\beta)(x) = \sum_{j \in J} \chi_{B_j}(x)\, \mu(A|B_j)$$

where $\chi_{B_i}(x)$ is the characteristic function of B_j then we have

$$\mu^A(B) = \int_B \mu(A|\beta)(x)\, d\mu$$

by proposition 1·51.

We now put

$$H_\beta(\alpha, x) = -\sum_{i \in I} \chi_{A_i}(x) \log \mu(A_i|\beta)(x)$$

then $$\int_X H_\beta(\alpha,x) d\mu = -\int_X \sum_{i \in I} \chi_{A_i}(x) \log\left\{ \sum_{j \in J} \chi_{B_j}(x)\, \mu(A_i|B_j)\right\} d\mu$$

$$= -\sum_{i \in I} \sum_{j \in J} \mu(A_i \cap B_j) \log \mu(A_i|B_j)$$

$$= H(\alpha|\beta)$$

Thus we could have defined $H(\alpha|\beta)$ as an integral. We now indicate how $H_\beta(\alpha, x)$ can be defined for general β and then give an alternative definition for $H(\alpha|\beta)$ in terms of the integral of $H_\beta(\alpha, x)$.

If β is any σ-algebra then for fixed $A \in \alpha$ we define

$$\mu^A(B) = \mu(A \cap B) \quad \text{for} \quad B \in \beta$$

Now μ^A is "absolutely continuous" with respect to μ on (X, β) and so by the Radon-Nikodym theorem (see P.R.Halmos [5] P.128 theorem B) there exists a function $\mu_1(A|\beta)(x)$ on X which is measurable with respect to (X, β, μ) and such that

$$\mu^A(B) = \int_B \mu_1(A|\beta)(x) \, d\mu$$

We now put

$$H_\beta(\alpha, x) = \sum_{i \in I} \chi_{A_i}(x) \log \mu_1(A_i|\beta)(x)$$

and define

$$H_1(\alpha|\beta) = \int_X H_\beta(\alpha, x) \, d\mu$$

Since the Radon-Nikodym theorem asserts the uniqueness of $\mu_1(A|\beta)(x)$ modulo sets of measure zero, it follows that if $\beta \in Z_3$ then $\mu_1(A|\beta)(x) = \mu(A|\beta)(x)$ except possibly on a set of measure zero and consequently that $H_1(\alpha|\beta) = H(\alpha|\beta)$ in this case. Further, if $\beta_1, \beta_2 \in Z_3$ are such that $\beta_1 \leq \beta_2$ we have that $\mu(A|\beta_1)(x) \leq \mu(A|\beta_2)(x)$ giving us that $-\log \mu(A|\beta_2)(x) \leq -\log \mu(A|\beta_1)(x)$. Thus since it follows by convergence theory that for any σ- algebra

$$\mu_1(A|\beta)(x) = \lim_{\beta^1 \in S(\beta)} \mu(A|\beta^1)(x)$$

where the limit is taken in the Moore-Smith sense we have that

$$\begin{aligned} H_1(\alpha|\beta) &= \lim_{\beta^1 \in S(\beta)} H(\alpha|\beta^1) \\ &= H(\alpha|\beta) \end{aligned}$$

for $\alpha \in Z_3$ and β any σ- algebra.

3. DEFINITION AND PROPERTIES OF h(T)

3·1 THE ENTROPY OF T WITH RESPECT TO A σ-ALGEBRA

As stated in 1·1 we always assume the existence of an automorphism T on $(X, \mathcal{E}, \mu)$. If α is a σ-algebra then we put

$$T\alpha = \{ A : T^{-1}A \in \alpha \}$$
$$\alpha_T^n = \bigvee_{i=0}^{n-1} T^i\alpha, \quad n \in \Gamma^+$$
$$\alpha_T = \bigvee_{i \in \Gamma} T^i\alpha$$
$$\alpha_T^- = \bigvee_{i \in \Gamma^+} T^{-i}\alpha$$
$$\alpha_\infty^T = \bigwedge_{i \in \Gamma^+} T^{-i}\alpha$$

and if there is no danger of confusion we write $\alpha^n, \alpha^-, \alpha_\infty$ for $\alpha_T^n, \alpha_T^-, \alpha_\infty^T$.

If $\alpha, \beta \in Z_3$ and A_i, $i \in I$, B_j, $j \in J$ are atom sets of α, β then clearly T^kA_i, $i \in I$, T^kB_j, $j \in J$ are atom sets of $T^k\alpha$, $T^k\beta$ and $T^k\alpha, T^k\beta \in Z_3$ for $k \in \Gamma$. Thus since μ is measure preserving

$$\begin{aligned} H(T^k\alpha) &= -\sum_{i \in I} \mu(T^kA_i) \log \mu(T^kA_i) \\ &= -\sum_{i \in I} \mu(A_i) \log \mu(A_i) \\ &= H(\alpha) \end{aligned}$$

and
$$\begin{aligned} H(T^k\alpha \mid T^k\beta) &= -\sum_{i \in I} \sum_{j \in J} \mu(T^kA_i \cap T^kB_j) \log \mu(T^kA_i \mid T^kB_j) \\ &= -\sum_{i \in I} \sum_{j \in J} \mu(A_i \cap B_j) \log \mu(A_i \mid B_j) \\ &= H(\alpha \mid \beta) \end{aligned}$$

for $k \in \Gamma$

Lemma 3·11 If α, β are σ-algebras then $H(T^k\alpha) = H(\alpha)$ and $H(T^k\alpha \mid T^k\beta) = H(\alpha \mid \beta)$ for $k \in \Gamma$

Proof $H(T^k\alpha) = \sup_{\alpha^1 \in S(T^k\alpha)} H(\alpha^1)$ for $k \in \Gamma$

and since $S(T^k\alpha) = T^k S(\alpha)$ we have

$$
\begin{aligned}
H(T^k\alpha) &= \sup_{T^{-k}\alpha^1 \in S(\alpha)} H(\alpha^1) \\
&= \sup_{\alpha^1 \in S(\alpha)} H(T^k\alpha^1) \\
&= \sup_{\alpha^1 \in S(\alpha)} H(\alpha^1) \\
&= H(\alpha)
\end{aligned}
$$

while

$$
\begin{aligned}
H(T^k\alpha \mid T^k\beta) &= \sup_{\alpha^1 \in S(T^k\alpha)} \inf_{\beta^1 \in S(T^k\beta)} H(\alpha^1 \mid \beta^1) \\
&= \sup_{T^{-k}\alpha^1 \in S(\alpha)} \inf_{T^{-k}\beta^1 \in S(\beta)} H(\alpha^1 \mid \beta^1) \\
&= \sup_{\alpha^1 \in S(\alpha)} \inf_{\beta^1 \in S(\beta)} H(T^k\alpha^1 \mid T^k\beta^1) \\
&= \sup_{\alpha^1 \in S(\alpha)} \inf_{\beta^1 \in S(\beta)} H(\alpha^1 \mid \beta^1) \\
&= H(\alpha \mid \beta)
\end{aligned}
$$

Lemma 3·12 If α is a σ-algebra, $n, m \in \Gamma^+$ then $H(\alpha^{nm}) \leqslant mH(\alpha^n)$

Proof

$$
\begin{aligned}
H(\alpha^{nm}) &= H\left(\bigvee_{i=0}^{nm-1} T^i\alpha\right) \\
&= H\left(\bigvee_{j=0}^{m-1} T^{jn}\alpha^n\right) \\
&\leqslant \sum_{j=0}^{m-1} H(T^{jn}\alpha^n) \\
&= \sum_{j=0}^{m-1} H(\alpha^n) \\
&= m\, H(\alpha^n)
\end{aligned}
$$

For any σ-algebra α and $n \in \Gamma^+$ we have $T^{-n}\alpha^n \leqslant T^{-(n+1)}\alpha^{n+1}$ and so by lemma 2·22, 16 we have $0 \leqslant H(\alpha \mid T^{-(n+1)}\alpha^{n+1}) \leqslant H(\alpha \mid T^{-n}\alpha^n) \leqslant H(\alpha)$. Thus $H(\alpha \mid T^{-n}\alpha^n)$, $n \in \Gamma^+$ is a monotonic sequence and hence if we put

$$h_1(\alpha,T) = \lim_{n\to\infty} H(\alpha \mid T^{-n}\alpha^n)$$

then $h_1(\alpha,T)$ is well defined. Clearly $h_1(\alpha,T) \leq H(\alpha)$ and so $\alpha \in Z$ implies $h_1(\alpha,T) < \infty$. Further by lemma 2·33 we have that

$$h_1(\alpha,T) = H(\alpha \mid \alpha^-)$$

Again $H(\alpha^n \mid \alpha^-) = H(\alpha^{n-1} \mid \alpha^-) + H(T^{n-1}\alpha \mid \alpha^{n-1}\alpha^-)$

$$= H(\alpha \mid \alpha^-) + \sum_{i=1}^{n-1} H(T^i\alpha \mid \alpha^i \alpha^-)$$

$$= H(\alpha \mid \alpha^-) + \sum_{i=1}^{n-1} H(\alpha \mid T^{-i}\alpha^i \alpha^-)$$

$$= H(\alpha \mid \alpha^-) + \sum_{i=1}^{n-1} H(\alpha \mid \alpha^-)$$

$$= n\ H(\alpha \mid \alpha^-)$$

giving $h_1(\alpha,T) = \frac{1}{n} H(\alpha^n \mid \alpha^-)$, $n \in \Gamma^+$

Lemma 3·13 If $\alpha \in Z$ then $h_1(\alpha,T) = \lim_{n\to\infty} \frac{1}{n} H(\alpha^n)$

Proof $H(\alpha^n) = H(\alpha^{n-1}) + H(T^{n-1}\alpha \mid \alpha^{n-1})$

$$= H(\alpha) + \sum_{i=1}^{n-1} H(T^i\alpha \mid \alpha^i)$$

$$= H(\alpha) + \sum_{i=1}^{n-1} H(\alpha \mid T^{-i}\alpha^i)$$

Thus if $\alpha \in Z$ then the $H(\alpha \mid T^{-i}\alpha^i)$ are bounded, and since we know that $\lim_{n\to\infty} H(\alpha \mid T^{-n}\alpha^n)$ exists it follows that $\lim_{n\to\infty} \frac{1}{n} H(\alpha^n)$ exists and equals $\lim_{n\to\infty} H(\alpha \mid T^{-n}\alpha^n)$ i.e. $h_1(\alpha,T) = \lim_{n\to\infty} \frac{1}{n} H(\alpha^n)$

If $\alpha \notin Z$ then $H(\alpha^n) \geq H(\alpha) = \infty$ and so

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n) = \lim_{n\to\infty} \infty = \infty$$

Thus if we set

$$h(\alpha,T) = \lim_{n\to\infty} \frac{1}{n} H(\alpha^n)$$

then $h(\alpha,T)$ is well defined for all σ-algebras α. We call $h(\alpha,T)$ the entropy of α with respect to T, or of T with respect to α, or simply the entropy of α and T. If $\alpha \in Z$ then

$h(\alpha,T) = h_1(\alpha,T)$ but this is not true in general. To see this we consider a σ- algebra α such that $H(\alpha) = \infty$, $T\alpha \leqslant \alpha$ then $\alpha \leqslant T^{-n}\alpha^n$, $n \in \Gamma^+$ and so by lemma 2·22, 17, $H(\alpha \mid T^{-n}\alpha^n) = 0$ giving

$$\begin{aligned} h_1(\alpha,T) &= \lim_{n\to\infty} H(\alpha \mid T^{-n}\alpha^n) \\ &= \lim_{n\to\infty} 0 \\ &= 0 \end{aligned}$$

while $h(\alpha,T) = \infty$ as stated earlier. Lastly since $T^{-(n+1)}\alpha^n = \alpha^n_{T^{-1}}$ we have that

$$H(\alpha^n) = H(T^{-(n+1)}\alpha^n) = H(\alpha^n_{T-1})$$

giving that $h(\alpha,T) = h(\alpha,T^{-1})$

Lemma 3·14 If $\alpha \in Z$ and T is the identity then $h(\alpha,T) = 0$

Proof Since T is the identity $\alpha^n = \alpha$, $n \in \Gamma^+$ and so

$$\begin{aligned} h(\alpha,T) &= \lim_{n\to\infty} \frac{1}{n} H(\alpha^n) \\ &= \lim_{n\to\infty} \frac{1}{n} H(\alpha) \\ &= 0 \end{aligned}$$

Lemma 3·15 If $\alpha,\beta \in Z$ then

$$|h(\alpha,T) - h(\beta,T)| \leqslant H(\alpha \mid \beta) + H(\beta \mid \alpha) = \rho(\alpha,\beta)$$

Proof If $\alpha,\beta \in Z$ then $\alpha^n, \beta^n, \alpha^n\beta^n \in Z$ for $n \in \Gamma^+$ and so

$$\begin{aligned} |H(\alpha^n) - H(\beta^n)| &\leqslant |H(\alpha^n) - H(\alpha^n\beta^n)| + |H(\alpha^n\beta^n) - H(\beta^n)| \\ &= H(\alpha^n \mid \beta^n) + H(\beta^n \mid \alpha^n) \\ &\leqslant \sum_{i=0}^{n-1} \{H(T^i\alpha \mid \beta^n) + H(T^i\beta \mid \alpha^n)\} \quad \text{by } 2\cdot 22 \\ &\leqslant \sum_{i=0}^{n-1} \{H(T^i\alpha \mid T^i\beta) + H(T^i\beta \mid T^i\alpha)\} \quad \text{by } 2\cdot 22 \\ &= n\{H(\alpha \mid \beta) + H(\beta \mid \alpha)\} \end{aligned}$$

Giving that

$$
\begin{aligned}
|h(\alpha,T) - h(\beta,T)| &= \lim_{n\to\infty} \frac{1}{n} \,|H(\alpha^n) - H(\beta^n)| \\
&\leq \lim_{n\to\infty} \{H(\alpha|\beta) + H(\beta|\alpha)\} \\
&= H(\alpha|\beta) + H(\beta|\alpha) \\
&= \rho(\alpha,\beta)
\end{aligned}
$$

Lemma 3·16 If α, β are σ-algebras and $\alpha \leq \beta$ then $h(\alpha,T) \leq h(\beta,T)$

Proof $\alpha \leq \beta$ implies $\alpha^n \leq \beta^n$, $n \in \Gamma^+$ and so

$$
\begin{aligned}
h(\alpha,T) &= \lim_{n\to\infty} \frac{1}{n} H(\alpha^n) \\
&\leq \lim_{n\to\infty} \frac{1}{n} H(\beta^n) \\
&= h(\beta,T)
\end{aligned}
$$

Lemma 3·17 If α, β are σ-algebras, $\alpha \leq \beta_T$ then $h(\alpha,T) \leq h(\beta,T)$

Proof

$$
\begin{aligned}
H(\alpha^n) &\leq H(\alpha^n \bigvee_{i=-m}^{m+n-1} T^i\beta) \\
&= H(\bigvee_{i=-m}^{m+n-1} T^i\beta) + H(\alpha^n | \bigvee_{i=-m}^{m+n-1} T^i\beta)
\end{aligned}
$$

$$
\begin{aligned}
H(\alpha^n | \bigvee_{i=-m}^{m+n-1} T^i\beta) &\leq \sum_{j=0}^{n-1} H(T^j\alpha | \bigvee_{i=-m}^{m+n-1} T^i\beta) \\
&= \sum_{j=0}^{n-1} H(\alpha | \bigvee_{i=-m-j}^{m+n-1-j} T^i\beta)
\end{aligned}
$$

Now for $0 \leq j \leq n-1$

$$
\bigvee_{i=-m}^{m} T^i\beta \leq \bigvee_{i=-m-j}^{m+n-1-j} T^i\beta
$$

and therefore

$$
\begin{aligned}
H(\alpha^n) &\leq H(\bigvee_{i=-m}^{m+n-1} T^i\beta) + \sum_{j=0}^{n-1} H(\alpha | \bigvee_{i=-m-j}^{m+n-1-j} T^i\beta) \\
&\leq H(\bigvee_{i=-m}^{m+n-1} T^i\beta) + \sum_{j=0}^{n-1} H(\alpha | \bigvee_{i=-m}^{m} T^i\beta) \\
&= H(\beta^{2m+n}) + nH(\alpha | \bigvee_{i=-m}^{m} T^i\beta)
\end{aligned}
$$

Given $d > 0$ there exists an m such that $H(\alpha | \bigvee_{i=-m}^{m} T^i\beta) < d$ and for this m

$$\frac{1}{n} H(\alpha^n) < \left(\frac{2m+n}{n}\right) \frac{1}{2m+n} H(\beta^{2m+n}) + d$$

giving

$$h(\alpha, T) = \lim_{n\to\infty} \frac{1}{n} H(\alpha^n) \leq \lim_{n\to\infty} \left\{\left(\frac{2m+n}{n}\right) \frac{1}{2m+n} H(\beta^{2m+n}) + d\right\}$$

$$= h(\beta, T) + d$$

But d was arbitrary and so

$$h(\alpha, T) \leq h(\beta, T)$$

3·2 MORE PROPERTIES OF ENTROPY

Lemma 3·21 If $\alpha, \beta \in Z_3$ and either $\beta \leq \alpha$, $H(\alpha|\beta-) < \infty$ or $\alpha \leq \beta$, $H(\beta|\alpha-) < \infty$ then

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n|\beta-) = H(\alpha|\alpha-)$$

Proof If $\beta \leq \alpha$, $H(\alpha|\beta-) < \infty$ then we have

$$\begin{aligned} H(\alpha^n|\beta-) &= H(\alpha^{n-1}|\beta-) + H(T^{n-1}\alpha \mid \alpha^{n-1}\beta-) \\ &= H(\alpha|\beta-) + \sum_{i=1}^{n-1} H(T^i\alpha \mid \alpha^i\beta-) \\ &= H(\alpha|\beta-) + \sum_{i=1}^{n-1} H(\alpha \mid T^{-i}(\alpha^i\beta-)) \end{aligned}$$

But $H(\alpha|\beta-) < \infty$ and

$$T^{-i}(\alpha^i\beta-) = \bigvee_{j=1}^{i} T^{-j}\alpha \bigvee_{k=i+1}^{\infty} T^{-k}\beta$$

$$\to \alpha- \text{ as } i \to \infty$$

Thus $\lim_{n\to\infty} \frac{1}{n} H(\alpha^n|\beta-)$ exists and

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n|\beta-) = \lim_{n\to\infty} H(\alpha|T^{-n}(\alpha^n\beta-))$$

$$= H(\alpha|\alpha-)$$

If $\alpha \leq \beta$, $H(\beta|\alpha-) < \infty$ then we have

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n\beta^n|\beta-) = \lim_{n\to\infty} \frac{1}{n} H(\beta^n|\beta-) \text{ since } \alpha \leq \beta$$

$$= H(\beta|\beta-)$$

by the first part of the lemma since $\beta \leqslant \beta$ and

$H(\beta|\beta-) \leqslant H(\beta|\alpha-) < \infty$

Thus since

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n\beta^n|\beta-) = \lim_{n\to\infty} \frac{1}{n}\{H(\alpha^n|\beta-)+H(\beta^n|\alpha^n\beta-)\}$$

we have that the limits on the right hand side are finite and so

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n|\beta-) = \lim_{n\to\infty} \frac{1}{n} H(\alpha^n\beta^n|\beta-) - \lim_{n\to\infty} \frac{1}{n} H(\beta^n|\alpha^n\beta-)$$
$$\geqslant H(\beta|\beta-) - \lim_{n\to\infty} \frac{1}{n} H(\beta^n|\alpha^n\alpha^-)$$

since $\alpha \leqslant \beta$. Now

$$\lim_{n\to\infty} \frac{1}{n} H(\beta^n|\alpha^n\alpha^-) = \lim_{n\to\infty} \frac{1}{n} H(\alpha^n\beta^n|\alpha^-) - \lim_{n\to\infty} \frac{1}{n} H(\alpha^n|\alpha^-)$$

since $\lim_{n\to\infty} \frac{1}{n} H(\alpha^n\beta^n|\alpha^-) = \lim_{n\to\infty} \frac{1}{n} H(\beta^n|\alpha^-)$

$$= H(\beta|\beta^-) < \infty$$

and so $\lim_{n\to\infty} \frac{1}{n} H(\alpha^n|\beta^-) \geqslant H(\beta|\beta^-) - H(\beta|\beta^-) + \lim_{n\to\infty} \frac{1}{n}(\alpha^n|\alpha^-)$

$$= H(\alpha|\alpha^-)$$

since $H(\alpha|\alpha^-) \leqslant H(\beta|\alpha^-) < \infty$

Lemma 3.22 If α, β, γ are σ-algebras such that $\alpha \leqslant \beta$, $H(\beta\gamma^-|\beta^-) < \infty$ then

$$\lim_{n\to\infty} H(\alpha|\beta^- T^{-n}\gamma^-) = H(\alpha|\beta^-)$$

Proof $H(\beta^n|\beta^-\gamma^-) = H(\beta^{n-1}|\beta^-\gamma^-) + H(T^{n-1}\beta|\beta^{n-1}\beta^-\gamma^-)$

$$= H(\beta|\beta^-\gamma^-) + \sum_{i=1}^{n-1} H(T^i\beta|\beta^i\beta^-\gamma^-)$$
$$= H(\beta|\beta^-\gamma^-) + \sum_{i=1}^{n-1} H(\beta|\beta^- T^{-i}\gamma^-)$$

Now $\beta^- T^{-(i+1)}\gamma^- \leqslant \beta^- T^{-i}\gamma^-$ and so $\lim_{n\to\infty} H(\beta|\beta^- T^{-i}\gamma^-)$ exists and by the above we have

$$\lim_{n\to\infty} H(\beta|\beta^- T^{-n}\gamma^-) = \lim_{n\to\infty} \frac{1}{n} H(\beta^n|\beta^-\gamma^-)$$

since $H(\beta|\beta^-\gamma^-) \leqslant H(\beta|\beta^-) \leqslant H(\beta\gamma^-|\beta^-) < \infty$

Further $H(\beta \mid \alpha \beta^- T^{-n} \gamma^-) \leq H(\beta \mid \beta^-)$
$$\leq H(\beta\gamma \mid \beta^-)$$
$$< \infty$$

and so

$$\begin{aligned} H(\alpha \mid \beta^- T^{-n} \gamma^-) &= H(\alpha\beta \mid \beta^- T^{-n} \gamma^-) - H(\beta \mid \alpha\beta^- T^{-n} \gamma^-) \\ &= H(\beta \mid \beta^- T^{-n} \gamma^-) - H(\beta \mid \alpha\beta^- T^{-n} \gamma^-) \end{aligned}$$

But $\lim_{n\to\infty} H(\beta \mid \beta^- T^{-n} \gamma^-) = \lim_{n\to\infty} \frac{1}{n} H(\beta^n \mid \beta^- \gamma^-)$
$$= H(\beta \mid \beta^-)$$

bylemma 3·21 since $\beta \leq \beta\gamma$, $H(\beta\gamma \mid \beta^-) < \infty$ and $H(\beta \mid \alpha \beta^- T^{-n} \gamma^-) \leq H(\beta \mid \alpha\beta^-)$

giving that

$$\begin{aligned} \lim_{n\to\infty} H(\alpha \mid \beta^- T^{-n} \gamma^-) &\geq H(\beta \mid \beta^-) - H(\beta \mid \alpha\beta^-) \\ &= H(\alpha\beta \mid \beta^-) - H(\beta \mid \alpha\beta^-) \\ &= H(\alpha \mid \beta^-) \end{aligned}$$

since $H(\beta \mid \alpha\beta^-) \leq H(\beta \mid \beta^-) \leq H(\beta\gamma \mid \beta^-) < \infty$

Now $H(\alpha \mid \beta^- T^{-n} \gamma^-) \leq H(\alpha \mid \beta^-)$

giving $\lim_{n\to\infty} H(\alpha \mid \beta^- T^{-n} \gamma^-) \leq H(\alpha \mid \beta^-)$

Hence we conclude that

$$\lim_{n\to\infty} H(\alpha \mid \beta^- T^{-n} \gamma^-) = H(\alpha \mid \beta^-)$$

Lemma 3·23 If α, β are σ-algebras such that $H(\alpha\beta \mid \beta^-) < \infty$ then

$$H(\alpha\beta \mid \alpha^- \beta^-) = H(\alpha \mid \alpha^- \beta_T) + H(\beta \mid \beta^-)$$

Proof
$$\begin{aligned} H(\alpha^n \mid \alpha^- \beta^- \beta^n) &= H(\alpha^{n-1} \mid \alpha^- \beta^- \beta^n) + H(T^{n-1}\alpha \mid \alpha^{n-1} \alpha^- \beta^- \beta^n) \\ &= H(\alpha \mid \alpha^- \beta^- \beta^n) + \sum_{i=1}^{n-1} H(T^i \alpha \mid \alpha^i \alpha^- \beta^- \beta^n) \\ &= H(\alpha \mid \alpha^- \beta^- \beta^n) + \sum_{i=1}^{n-1} H(\alpha \mid T^{-i} \alpha^i \alpha^- \beta^- \beta^n) \\ &= H(\alpha \mid \alpha^- \beta^- \beta^n) + \sum_{i=1}^{n-1} H(\alpha \mid \alpha^- \beta^- \beta^{n-i}) \end{aligned}$$

But $\quad H(\alpha|\alpha^-\beta^-\beta^n) \leq H(\alpha|\beta^-)$

$$\leq H(\alpha\beta|\beta^-)$$

$$< \infty$$

while $H(\alpha|\alpha^-\beta^-\beta^n) \leq H(\alpha|\alpha^-\beta^-\beta^{n-1})$

and so

$$\lim_{n\to\infty} H(\alpha|\alpha^-\beta^-\beta^n) = H(\alpha|\alpha^-\beta_T)$$

giving that

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha^n|\alpha^-\beta^-\beta^n) = \lim_{n\to\infty} H(\alpha|\alpha^-\beta^-\beta^n)$$

$$= H(\alpha|\alpha^-\beta_T)$$

Now $H(\alpha^n\beta^n|\alpha^-\beta^-) = H(\alpha^n|\alpha^-\beta^-\beta^n) + H(\beta^n|\alpha^-\beta^-)$

and since $\alpha\beta \leq \alpha\beta$, $H(\alpha\beta|\alpha^-\beta^-) \leq H(\alpha\beta|\beta^-) < \infty$ we have by lemma 3·21 that

$$\lim_{n\to\infty} H(\alpha^n\beta^n|\alpha^-\beta^-) = H(\alpha\beta|\alpha^-\beta^-)$$

and again since $\beta \leq \alpha\beta$, $H(\alpha\beta|\beta^-) < \infty$ the same lemma gives

$$\lim_{n\to\infty} H(\beta^n|\alpha^-\beta^-) = H(\beta|\beta^-)$$

Thus we conclude that

$$H(\alpha\beta|\alpha^-\beta^-) = H(\alpha|\alpha^-\beta_T) + H(\beta|\beta^-)$$

Corollary 1 $\quad$ If α, β are σ-algebras such that $H(\alpha\beta|\beta^-) < \infty$ then

$$h_1(\alpha\beta, T) = H(\alpha|\alpha^-\beta_T) + h_1(\beta, T)$$

Corollary 2 $\quad$ If α, β Z then

$$h(\alpha\beta, T) = H(\alpha|\alpha^-\beta_T) + h(\beta, T)$$

Proof $\quad$ If $\alpha, \beta \in Z$ then $\alpha\beta \in Z$ and so $H(\alpha\beta|\beta^-) \leq H(\alpha\beta) < \infty$

The result then follows from corollary 1 and section 3·1

Lemma 3·23 is usually referred to as Pinsker's lemma, see [13] although the proof given here is based on that given by V.A.Rokhlin and Ja.G.Sinai in [17] as are the proofs of lemma 3·21 and 3·22.

3·3 THE ENTROPY OF T.

We define the entropy $h(T)$ of T by

$$h(T) = \sup_{\alpha \in Z} h(\alpha, T)$$

We have immediately that $0 \leq h(T)$ and by section 3·1 that $h(T) = h(T^{-1})$. By lemma 3·14 if T is the identity then $h(\alpha, T) = 0$ for $\alpha \in Z$ and so $h(T) = 0$.

Lemma 3·31 $\qquad h(T) = \sup_{\alpha \in Z_1} h(\alpha, T)$

Proof $\quad$ Since $Z_1 \subseteq Z$ we have $\sup_{\alpha \in Z_1} h(\alpha, T) \leq h(T)$.

If $h(T) < \infty$ then given any real number $d > 0$ there exists an $\alpha \in Z_3$ with

$$h(T) \leq h(\alpha, T) + d/2$$

Further, let A_i, $i \in I$ be an atom set of α, $A_i = \phi$ if $i \in \Gamma^+ - I$ and βn be the σ- algebra generated by $B_j = A_j$, $1 \leq j \leq n-1$ and $B_n = X - \bigcup_{j=1}^{n-1} B_j$. Thus for $i \in \Gamma^+$, $1 \leq j \leq n-1$

$$A_i \wedge B_j = \begin{cases} \phi & \text{if } i \neq j \\ A_i & \text{if } i = j \end{cases}$$

and for $i \in \Gamma^+$

$$A_i \wedge B_n = \begin{cases} \phi & \text{if } i < n \\ A_i & \text{if } n \leq i \end{cases}$$

Hence
$$\begin{aligned} H(\alpha / \beta_n) &= -\sum_{i \in \Gamma^+} \sum_{j=1}^{n} \mu(A_i \wedge B_j) \log \mu(A_i / B_j) \\ &= -\sum_{i=n}^{\infty} \mu(A_i) \log \mu(A_i / B_n) \\ &= -\sum_{i=n}^{\infty} \mu(A_i) \log \mu(A_i) + \log \mu(B_n) \sum_{i=n}^{\infty} \mu(A_i) \\ &\leq -\sum_{i=n}^{\infty} \mu(A_i) \log \mu(A_i) \end{aligned}$$

Since $\alpha \in Z$ there exists an N such that

$$- \sum_{i=N}^{\infty} \rho(A_i) \log \rho(A_i) < d/2$$

Hence since $\beta_N \leq \alpha$ we have

$$\begin{aligned} h(\alpha,T) &\leq h(\beta_N,T) + \rho(\alpha,\beta_N) \\ &= h(\beta_N,T) + H(\alpha|\beta_N) \end{aligned}$$

giving

$$\begin{aligned} h(T) &\leq h(\alpha,T) + d/2 \\ &\leq h(\beta_N,T) + d/2 + d/2 \\ &\leq \sup_{\alpha\in Z_1} h(\alpha,T) \end{aligned}$$

But d was arbitrary and so we deduce that

$$h(T) \leq \sup_{\alpha\in Z_1} h(\alpha,T)$$

If $h(T) = \infty$ then given any $\Delta > 0$ there exists an $\alpha \in Z$ with $h(\alpha,T) > \Delta$ and if the β_n, $n \in \Gamma^+$ are defined as above then there exists an N such that $h(\beta_n,T) > \Delta$ from which we deduce that $\sup_{\alpha\in Z_1} h(\alpha,T) = \infty$. This completes the proof.

Lemma 3·32 If $\alpha \in Z$, $\alpha_T = \mathcal{E}$ then $h(T) = h(\alpha,T)$

Proof For all $\beta \in Z$ we have $\beta \leq \alpha_T$ and so by $h(\beta,T) \leq h(\alpha,T)$. Thus

$$\begin{aligned} h(\alpha,T) &\leq h(T) \\ &\leq \sup_{\beta\in Z} h(\beta,T) \\ &\leq \sup_{\beta\in Z} h(\alpha,T) \\ &= h(\alpha,T) \end{aligned}$$

giving $h(T) = h(\alpha,T)$

If $\alpha \in Z$, $\alpha_T = \mathcal{E}$ then we refer to α as a generator.

Lemma 3·33 If $\alpha \in Z$, $\bigvee_{i=m}^{\infty} T^i\alpha = \mathcal{E}$, $m \in \Gamma$ then $h(T) = 0$

Proof Since $\bigvee_{i=m}^{\infty} T^i\alpha \leqslant \alpha_T$ we have that α is a generator and so

$$h(T) = h(\alpha,T)$$
$$= h(\alpha,T^{-1}) \quad \text{by } 3\cdot 1$$
$$= h_1(\alpha,T^{-1}) \quad \text{by } 3\cdot 1$$
$$= H(\alpha/\alpha_T^-) \quad \text{by } 3\cdot 1$$

But $\alpha_T^- = \bigvee_{i\in\Gamma^+} T^i\alpha = T^{-m+1} \bigvee_{i=m}^{\infty} T^i\alpha = T^{-m+1}\varepsilon = \varepsilon$

and so $\alpha \leqslant \alpha_T^-$ giving

$$h(T) = H(\alpha/\alpha_T^-) = 0$$

Lemma 3·34 If $\alpha_n \in Z$, $n \in \Gamma^+$, $\bigvee_{n\in\Gamma^+} \alpha_n = \varepsilon$ and $\alpha_n \leqslant \alpha_{n+1}$ for each n then $h(T) = \lim_{n\to\infty} h(\alpha_n,T)$

Proof If $S = \{\alpha_n, n \in \Gamma^+\}$ then S is an increasingly filtered system and since $\varepsilon = \bigvee_{\beta\in S}\beta$ it follows from lemma 2·32 that given any $\gamma \in Z_1$, if C_i, $1 \leqslant i \leqslant m$ is an atom set of γ and d^1 any real number such that $0 < d^1$ then there exists an n and sets $B_i \in \alpha_n$, $1 \leqslant i \leqslant m+1$ such that $\mu(B_i \cap C_i) < d^1$, $1 \leqslant i \leqslant n$; $\mu(B_{n+1}) < d^1$; $\mu(C_i \cap B_j) < d^1$ if $i \neq j$; $\mu(C_i | B_i) > 1-d^1$, $1 \leqslant i \leqslant n$ and $\bigcup_{i=1}^{n+1} B_i = X$.
Further

$$\mu(B_i|C_i) = \frac{\mu(B_i \cap C_i)}{\mu(C_i)}$$
$$= \frac{\mu(C_i) - \mu(C_i - B_i)}{\mu(C_i)}$$
$$\geqslant 1 - \frac{\mu(C_i \triangle B_i)}{\mu(C_i)}$$
$$\geqslant 1 - \frac{d_1}{\mu(C_i)}$$
$$\geqslant 1 - d^1 \qquad (1)$$

where the d_1 is the d_1 occurring in lemma 2·33. Hence given any

d such that $0 < d$ if we choose d^1 as in lemma 2·33 then we have $H(\gamma \mid \alpha_n^1) < d$ where α_n^1 is the σ- algebra generated by B_i, $1 \leq i \leq n+1$. By using the same method as that used in lemma 2·33 we show that (1) implies $H(\alpha_n^1 \mid \gamma) < d$. We then have

$$h(T) = \sup_{\gamma \in Z_1} h(\gamma, T) \text{ by lemma 3·31}$$

$$\leq \sup_{n \in \Gamma^+} h(\alpha_n, T) + 2d$$

by the above and lemma 3·15. But d was arbitrary and $\alpha_n \leq \alpha_{n+1}$ for all n and so using lemma 3·16 we get

$$h(T) \leq \lim_{n \to \infty} h(\alpha_n, T)$$

but trivially

$$\lim_{n \to \infty} h(\alpha_n, T) \leq h(T)$$

and so we conclude that

$$h(T) = \lim_{n \to \infty} h(\alpha_n, T)$$

Lemma 3·35 If $k \in \Gamma^+$ then $h(T^k) = kh(T)$

Proof For any α

$$h(\alpha, T^k) = \lim_{n \to \infty} \frac{1}{n} H(\alpha_{T^k}^n)$$

$$\leq \lim_{n \to \infty} \frac{1}{n} H(\alpha_T^{nk})$$

$$\leq \lim_{n \to \infty} \frac{k}{n} H(\alpha^n) \text{ by lemma 3·12}$$

$$= k\,h(T)$$

Hence

$$h(T^k) = \sup_{\alpha \in Z} h(\alpha, T^k)$$

$$\leq \sup_{\alpha \in Z} k\,h(T)$$

$$= k\,h(T)$$

If $h(T) = 0$ then $h(T^k) = 0$. If $h(T) > 0$ let d be any number satisfying $0 \leq d < h(T)$. Then there exists an $\alpha \in Z$ with

$$d < h(\alpha, T) \leq h(T).$$

Put $\beta = \alpha^k$ then

$$\frac{1}{n} H(\beta^n_{T^k}) = \frac{1}{n} H(\alpha^{nk}_T)$$

$$= k \frac{1}{nk} H(\alpha^{nk})$$

giving $h(\beta, T^k) = \lim_{n\to\infty} \frac{1}{n} H(\beta^n_{T^k})$

$$= \lim_{n\to\infty} k. \frac{1}{nk} H(\alpha^{nk})$$

$$= k\, h(\alpha, T)$$

Thus $h(T^k) = \sup_{\gamma\in Z} h(\gamma, T^k)$

$$\geqslant h(\beta, T^k)$$

$$= k\, h(\alpha, T)$$

$$> k\, d$$

But d was any number satisfying $0 \leqslant d < h(T)$ and so $h(T^k) \geqslant k\, h(T)$. The result then follows from our two inequalities.

Corollary If $k \in \Gamma$ then $h(T^k) = |k|\, h(T)$

Proof If $-k \in \Gamma^+$ then $h(T^{-k}) = -k\, h(T)$ by the lemma. Thus we have $h(T^k) = |k|\, h(T)$ if $k \in \Gamma$, $k \neq 0$. If $k = 0$ then $T^k =$ the identity and hence $h(T^k) = 0 = |k| h(T)$, in this case also.

3.4 EXISTENCE AND PROPERTIES OF CERTAIN σ-ALGEBRAS

We say that a σ-algebra is invariant with respect to T if $\alpha \leqslant T\alpha$ and that it is exhaustive with respect to T if $\alpha_T = \mathcal{E}$. If a σ-algebra is invariant and exhaustive, we say that it is a K_1-algebra with respect to T, while if α is a K_1-algebra such that $\alpha_\infty = \nu$ we say that α is a K-algebra with respect to T. Further if given T there exists a K-algebra, then we say that T is a Kolmogorov automorphism. If α is a K-algebra then $\bigwedge_{i\in\Gamma} T^i\alpha = \alpha_\infty$ and so we have that T is a Kolmogorov automorphism if, and only if, there exists a σ- algebra α such that $\alpha \leq T\alpha$, $\bigvee_{i\in\Gamma} T^i\alpha = \epsilon$, $\bigwedge_{i\in\Gamma} T^i\alpha = \nu$.

Since $\mathcal{E}$ is a K_1-algebra for all T, there always exist K_1-algebras, but as we shall see later there do not always exist K-algebras. However, if we put

$$S^* = \{\alpha : h(\alpha,T) = 0\}$$

$$\Pi(T) = \bigvee_{\alpha\in S^*} \alpha$$

then the following theorems due to V.A.Rokhlin and Ja.G.Sinai (see [17]) show that a necessary and suffieient condition for the existence of a K-algebra is $\Pi(T) = \nu$. Note that $\alpha \notin Z$ implies $h(\alpha,T) = H(\alpha) = \infty$ and hence $\alpha \notin S^*$. Thus, since (Z,ρ) is complete we have that $\alpha \leq \Pi(T)$ implies $\alpha \in Z$, and so by the corollary of lemma 3·34 that $\alpha \in S^*$ i.e. $\Pi(T) = \bigcup_{\alpha\in S^*} \alpha$.

Theorem 3·41. (1) If α is a K_1-algebra then $\Pi(T) \leq \alpha$

(2) If α is invariant and $H(T\alpha|\alpha) = h(T) < \infty$ then $\alpha_\infty \leq \Pi(T)$

Proof (1) If $\beta \leq \Pi(T)$ then $\beta \in Z$ and if $\gamma \in Z$ is such that $\gamma \leq T^m\alpha$ for some m then for all $p \in \Gamma^+$

$$H(\gamma|\gamma^-_{T^p}\,\alpha_\infty\,\beta_T) \leq H(\gamma|\alpha_\infty\beta_T) \leq H(\gamma|\alpha_\infty) \qquad (1)$$

But $T\alpha_\infty = \alpha_\infty$ and $\beta \leq \Pi(T)$, thus by section 3·1

$$h(\beta|\beta^-) = h(\beta,T) = 0$$

giving $\beta \leq \beta^-$ and hence $T\beta \leq \beta\beta^- = \beta^-$. Therefore by induction we show that $T^k\beta \leq \beta^-$, $k \in \Gamma$ i.e. $\beta_T = \beta^- = T^{pk}\eta^-$, $k \in \Gamma$

Now $H(\gamma|(\gamma\alpha_\infty)^-_{T^p}\,T^{-pi}(\bigvee_{k=1}^{p} T^{-k}\beta)^-_{T^p})$

$$= H(\gamma|\gamma^-_{T^p}\,\alpha_\infty\,T^{-pi}\bigvee_{j\in\Gamma^+} T^{-pj}\bigvee_{k=1}^{p} T^{-k}\beta)$$

$$= H(\gamma \mid \gamma^-_{T^P}\ \alpha_\infty \beta_T)$$

Thus $H(\gamma \mid \gamma^-_{T^P}\ \alpha_\infty \beta_T) = \lim_{i \to \infty} H(\gamma \mid \gamma^-_{T^P}\ \alpha_\infty \beta_T)$

$$= \lim_{i \to \infty} H(\gamma \mid (\gamma\alpha_\infty)^-_{T^P}\ T^{-Pi}(\bigvee_{k=1}^{P} T^{-k}\beta)^-_{T^P})$$

$$= H(\gamma \mid (\gamma\alpha_\infty)^-_{T^P}) \quad \text{by } 2\cdot 22$$

$$= H(\gamma \mid \gamma^-_{T^P}\ \alpha_\infty) \qquad (2)$$

While since

$$\gamma^-_{T^P} = \bigvee_{i \in \Gamma^+} T^{-Pi}\gamma \leqslant \bigvee_{i \in \Gamma^+} T^{-Pi+m}\alpha = T^{-p+m}\alpha$$

we have that $\lim_{p \to \infty} \gamma^-_{T^P}\ \alpha_\infty = \alpha_\infty$ and so

by 2·33 $\quad \lim_{p \to \infty} H(\gamma \mid \gamma^-_{T^P}\ \alpha_\infty) = H(\gamma \mid \alpha_\infty) \qquad (3)$

Thus from (1), (2), (3) we get

$$H(\gamma \mid \alpha_\infty) \leqslant H(\gamma \mid \alpha_\infty \beta_T)$$

but $\alpha_\infty \leqslant \alpha_\infty \beta_T$ and so

$$H(\gamma \mid \alpha_\infty \beta_T) \leqslant H(\gamma \mid \alpha_\infty)$$

giving $H(\gamma \mid \alpha_\infty \beta_T) = H(\gamma \mid \alpha_\infty)$

If $\delta \in Z$ and d is any real number such that $d > 0$ then there exists a $\gamma \in Z$ such that $\rho(\delta, \gamma) < d/2$ and $\gamma \in T^m\alpha$ for some m

Thus

$$
\begin{aligned}
|H(\delta \mid \alpha_\infty \beta_T) - H(\delta \mid \alpha_\infty)| \\
&\leqslant |H(\delta \mid \alpha_\infty \beta_T) - H(\delta\gamma \mid \alpha_\infty \beta_T)| \\
&\quad + |H(\delta\gamma \mid \alpha_\infty \beta_T) - H(\gamma \mid \alpha_\infty \beta_T)| \\
&\quad + |H(\gamma \mid \alpha_\infty \beta_T) - H(\gamma \mid \alpha_\infty)| \\
&\quad + |H(\gamma \mid \alpha_\infty) - H(\delta\gamma \mid \alpha_\infty)| \\
&\quad + |H(\delta\gamma \mid \alpha_\infty) - H(\delta \mid \alpha_\infty)| \\
&\leqslant H(\gamma \mid \delta\alpha_\infty \beta_T) + H(\delta \mid \gamma\alpha_\infty \beta_T) + 0 \\
&\quad + H(\delta \mid \gamma\alpha_\infty) + H(\gamma \mid \delta\alpha_\infty) \\
&\leqslant 2\rho(\delta, \gamma) \quad \text{by lemma } 2\cdot 22 \\
&< d
\end{aligned}
$$

But d was arbitrary and so

$$H(\delta|\alpha_\infty \beta_T) = H(\delta|\alpha_\infty)$$

for all $\delta \in Z$ in particular for $\delta = \beta$ giving

$$\begin{aligned} H(\beta|\alpha_\infty) &= H(\beta|\alpha_\infty \beta_T) \\ &= 0 \end{aligned}$$

Hence $\beta \leqslant \alpha_\infty$, and therefore $\pi(T) \leqslant \alpha_\infty$

Proof (2) Let $\beta \in Z$ be such that $\beta \leqslant \alpha_\infty$ and γ_p, $p \in \Gamma^+$ any sequence in (Z,p) such that $\lim_{p\to\infty} \gamma_p = \alpha$. Then by lemma 2·22

$$\begin{aligned} H(\gamma_p|\gamma_p^- \beta_T) + H(\beta|\beta^-) &= H(\gamma_p \beta|\gamma_p^- \beta^-) \\ &= H(\beta|\beta^-(\gamma_p)_T) + H(\gamma_p|\gamma_p^-) \end{aligned}$$

$$\begin{aligned} \text{giving} \quad h(\beta,T) &= H(\beta|\beta^-) \\ &= H(\beta|\beta^-(\gamma_p)_T) + H(\gamma_p|\gamma_p^-) - H(\gamma_p|\gamma_p^- \beta_T) \end{aligned}$$

$$\begin{aligned} \text{Now } \lim_{p\to\infty} H(\beta|\beta^-(\gamma_p)_T) &= H(\beta|\beta^-\alpha) \text{ by lemma 2·33} \\ &= 0 \text{ since } \beta \leqslant \alpha \end{aligned}$$

$$\begin{aligned} \text{and } \lim_{p\to\infty} H(\gamma_p|\gamma_p^-) &= H(\alpha|\alpha^-) \\ \text{while } \lim_{p\to\infty} H(\gamma_p|\gamma_p^-\beta_T) &= H(\alpha|\alpha^-\beta_T) \\ &= H(\alpha|\alpha^-) \text{ because } \beta_T \leqslant \alpha^- \end{aligned}$$

$$\begin{aligned} \text{Thus} \quad h(\beta, T) &= \lim_{p\to\infty} h(\beta,T) \\ &= 0 + H(\alpha|\alpha^-) - H(\alpha|\alpha^-) \\ &= 0 \end{aligned}$$

giving $\beta \leqslant \pi(T)$ and so completing the proof.

Corollary If $h(T) < \infty$ and α is a K_1-algebra such that $H(T\alpha|\alpha) = h(T)$ then $\alpha_\infty = \pi(T)$

Theorem 3·42 There exists a K_1-algebra α such that $\alpha_\infty = \pi(T)$ and $H(T\alpha|\alpha) = h(T)$

Proof Let β_i, $i \in \Gamma^+$ be such that $\beta_i \leq \beta_{i+1}$ for each i and $\lim_{i\to\infty} \beta_i = \varepsilon$ and n_i, $i \in \Gamma^+$ an increasing sequence of positive integers. Put $\gamma_p = \bigvee_{i=1}^{p} T^{-n_i}\beta_i$, $\gamma = \bigvee_{i\in\Gamma^+} T^{-n_i}\beta_i$ and $\alpha = \gamma^-$

Then $\alpha = \gamma^- \leq T\gamma^- = T\alpha$ showing that α is invariant.

Further

$$\begin{aligned}\varepsilon &= \bigvee_{k\in\Gamma} \beta_k \\ &\leq \bigvee_{i\in\Gamma} T^i \bigvee_{j\in\Gamma^+} T^{-j} \bigvee_{k\in\Gamma} T^{-n_k}\beta_k \\ &= \alpha_T\end{aligned}$$

and hence α is a K_1-algebra. Consider

$$\begin{aligned}F(p,q) &= H(\gamma_p|\gamma_{q-1}^-) - H(\gamma_p|\gamma_q^-),\ p,q\in\Gamma^+ \\ &= H(\gamma_p|\gamma_{q-1}^-) - H(\gamma_p|\gamma_{q-1}^- T^{-n_q}\beta_q^-)\end{aligned}$$

We put $n_1 = 1$ and assume that $n_i, 1 \leq i \leq r-1$ have been chosen such that

$$F(p,q) < \frac{1}{p}\cdot\frac{1}{2^{q-p}} \quad \text{if } p<q<r \tag{4}$$

Then by lemma 3.22 we can find an n_r such that

$$F(p,r) < \frac{1}{p}\cdot\frac{1}{2^{r-p}} \quad \text{if } p<r$$

Hence we can find n_i, $i \in \Gamma^+$ such that (3) holds for all $r \in \Gamma^+$.

If $q = p+r$ then

$$\begin{aligned}H(\gamma_p|\gamma_p^-) - H(\gamma_p|\gamma_q^-) &= \sum_{i=p+1}^{p+r} F(p,i) \\ &< \frac{1}{p}\sum_{i=1}^{r}\frac{1}{2^i} \\ &< \frac{1}{p}\end{aligned}$$

But $\lim_{r\to\infty} \gamma_{p+r}^- = \alpha$ and so we get

$$H(\gamma_p|\gamma_p^-) - H(\gamma_p|\alpha) < \frac{1}{p},\quad p\in\Gamma^+$$

i.e. $\lim_{p\to\infty} H(\gamma_p|\gamma_p^-) = \lim_{p\to\infty} H(\gamma_p|\alpha)$

Now $$\begin{aligned}\lim_{p\to\infty} H(\gamma_p|\gamma_p^-) &= H(\gamma|\gamma^-) \\ &= h(\gamma,T) \\ &= h(T) \text{ because } \gamma \text{ is a generator.}\end{aligned}$$

while $\lim_{p\to\infty} H(\gamma_p|\alpha) = H(\gamma|\alpha)$

But $\gamma \leqslant T\alpha$ and so

$$\begin{aligned} H(T\alpha|\alpha) &= H(\gamma T\alpha|\alpha) \\ &= H(\gamma|\alpha) + H(T\alpha|\gamma\alpha) \\ &= H(\gamma|\alpha) \quad \text{because } \gamma\alpha = T\alpha \end{aligned}$$

Thus
$$\begin{aligned} H(T\alpha|\alpha) &= \lim_{p\to\infty} H(\gamma_p|\alpha) \\ &= \lim_{p\to\infty} H(\gamma_p|\gamma_p^-) \\ &= h(T) \end{aligned}$$

If $\beta \in Z$, $\beta \leqslant \alpha_\infty$ then $\gamma_p^- \beta_T \leqslant \alpha$ and so by lemma 3·23

$$\begin{aligned} H(\beta|\beta^-(\gamma_p)_T) + H(\gamma_p|\gamma_p^-) &= H(\beta\gamma_p|\beta^-\gamma_p^-) \\ &= H(\gamma_p|\gamma_p^-\beta_T) + H(\beta|\beta^-) \end{aligned}$$

giving
$$\begin{aligned} h(\beta,T) &= H(\beta|\beta^-) \\ &= H(\beta|\beta^-(\gamma_p)_T) + H(\gamma_p|\gamma_p^-) - H(\gamma_p|\gamma_p^-\beta_T) \end{aligned}$$

Now $\lim_{p\to\infty} (\gamma_p)_T = \alpha_T = \varepsilon$ and so

$$\lim_{p\to\infty} H(\beta|\beta^-(\gamma_p)_T) = 0$$

and $H(\gamma_p|\gamma_p^-\beta_T) \geqslant H(\gamma_p|\alpha)$

giving
$$\begin{aligned} h(\beta,T) &= \lim_{p\to\infty} h(\beta,T) \\ &\leqslant \lim_{p\to\infty} \left\{ H(\gamma_p|\gamma_p^-) - H(\gamma_p|\alpha) \right. \\ &= 0 \end{aligned}$$

and hence $\beta \leqslant \Pi(T)$. Thus we have $\alpha_\infty \leqslant \Pi(T)$ and so by theorem 3·41 $\alpha_\infty = \Pi(T)$

4 KOLMOGOROV AND REGULAR AUTOMORPHISMS

4·1 DEFINITIONS

We say that a σ-algebra α is : invariant if $\alpha \leqslant T\alpha$; exhaustive if $\alpha_T = \varepsilon$; a K_1- algebra if $\alpha \leqslant T\alpha$ and $\alpha_T = \varepsilon$; and lastly a K-algebra if $\alpha \leqslant T\alpha$, $\alpha_T = \varepsilon$ and $\alpha_\infty = \nu$, or equivalently if $\alpha \leqslant T\alpha$, $\bigvee_{i\in\Gamma} T^i\alpha = \varepsilon$ and $\bigwedge_{i\in\Gamma} T^i\alpha = \nu$. (see K. Jacobs [8])

If there exists a K-algebra with respect to T we say that T is a Kolmogorov automorphism. (see V.A.Rokhlin [16]).

For any σ-algebra α we define the tail σ-algebra (α) of α (see L.Sucheston [21]) by

$$(\alpha) = \bigwedge_{i\in\Gamma} T^i\alpha^- = \bigwedge_{i\in\Gamma^+} T^{-i}\alpha^-$$

and say that T is regular if $(\alpha) = \nu$ for all $\alpha \in Z_1$

Lastly we say that T is a mixing of degree n if given any sets A_i, $1 \leqslant i \leqslant n+1$, $t_i \in \Gamma$, $1 \leqslant i \leqslant n+1$ then we have

$$\lim_{\Delta\to\infty} \left| \mu\left(\bigcap_{i=1}^{n+1} T^{t_i}A_i\right) - \prod_{i=1}^{n+1} \mu(A_i)\right| = 0$$

where $\Delta = \inf_{i\neq j} |t_i - t_j|$, see P.R.Halmos [6], and V.A.Rokhlin [16].

4·2 MIXING

If $t_i = t_i(n)$, $n \in \Gamma^+$, $1 \leqslant i \leqslant n+1$ are such that $\lim_{n\to\infty} \Delta_n = \infty$ where $\Delta_n = \inf_{i\neq j} |t_i(n) - t_j(n)|$ then there exists a subsequence n_m such that for all m the integers $t_i(n_m)$ are in the same order. Without loss of generality we can assume that $t_i(n_m) > t_{i+1}(n_m)$ for $1 \leqslant i \leqslant n$. Moreover since $\mu\left(\bigcap_{i=1}^{n+1} T^{t_i}A_i\right) = \mu\left(\bigcap_{i=1}^{n+1} T^{t_i - t_1}A_i\right)$ we lose no generality in assuming that $t_1(n_m) = 0$.

Thus we see that T is a mixing of degree n if given any sets A_i, $1 \leqslant i \leqslant n+1$ and any $d > 0$ there exists an $n_o > 0$ such that for all $n_i \geq n_o$, $1 \leq i \leq n$ then we have

$$\left| \mu\left(\bigcap_{i=1}^{n+1} T^{N_i} A_i\right) - \prod_{i=1}^{n+1} \mu(A_i) \right| < d$$

where $N_1 = 0$, $N_i = -\sum_{j=1}^{i-1} n_j$, $2 \leq i \leq n+1$

Another view of mixing can be obtained by considering the action of U in L^2_μ.

Theorem 4.21 T is a mixing of degree n if and only if, given f_i, $1 \leq i \leq n+1$; $t_i \in \Gamma$, $1 \leq i \leq n+1$ such that $f_i \in L^2_\mu$ each i then

$$\lim_{\Delta \to \infty} \int_X \prod_{i=1}^{n+1} U^{t_i} f_i \, d\mu . = \prod_{i=1}^{n+1} \int_X f_i \, d\mu \qquad (1)$$

where $\Delta = \inf_{i \neq j} |t_i - t_j|$

Proof If T is a mixing of degree n then given sets A_i, $1 \leq i \leq n+1$

$$\lim_{\Delta \to \infty} \mu\left(\bigcap_{i=1}^{n+1} T^{t_i} A_i\right) = \prod_{i=1}^{n+1} \mu(A_i) \qquad (2)$$

but $\mu(A_i) = \int_X \chi_{A_i} \, d\mu$ and

$$\mu\left(\bigcap_{i=1}^{n+1} T^{t_i} A_i\right) = \int_X \prod_{i=1}^{n+1} \chi_{T^{t_i} A_i} \, d\mu$$

$$= \int_X \prod_{i=1}^{n+1} U^{-t_i} \chi_{A_i} \, d\mu$$

Thus if T is a mixing of degree n (1) holds for characteristic functions. It is then obvious that (1) holds for step functions and hence by continuity for arbitrary functions in L^2_μ.

If (1) holds then given sets A_i, $1 \leq i \leq n+1$ we put $f_i = \chi_{A_i}$ for each i and get (2) thus showing T is a mixing of degree n.

Corollary 1 T is a mixing of degree n if, and only if, there exists a subset L of L^2_μ such that

(i) $\{ g : g = \sum_{i=1}^{n} a_i f_i ,\ a_i$ a real number, $f_i \in L$ each i, n finite $\}$ is everywhere dense in L^2_μ

(ii) given $f_i \in L$ $1 \leq i \leq n+1$, $t_i \in \Gamma$ $1 \leq i \leq n+1$ then

$$\lim_{\Delta \to \infty} \int_X \prod_{i=1}^{n+1} U^{t_i} f_i \, d\mu = \prod_{i=1}^{n+1} \int_X f_i \, d\mu \quad \text{where} \quad \Delta = \inf_{i \neq j} |t_i - t_j|$$

Proof If T is a mixing then we take $L = L^2_\mu$. Conversely given a subset L satisfying (i), (ii) we have that (ii) implies that

$$\lim_{\Delta \to \infty} \int_X \prod_{i=1}^{n+1} U^{t_i} f_i \, d\mu = \prod_{i=1}^{n+1} \int_X f_i \, d\mu$$

holds for all f_i belonging to the subset in (i) and hence for arbitrary $f_i \in L^2_\mu$ since integration is a continuous operation.

Corollary 2 T is a mixing of degree 1 if and only if there exists a subset L of L^2_μ such that

(i) $\{ g : g = \sum_{i=1}^{n} a_i f_i ,\ a_i$ a real number, $f_i \in L$ each i, n finite $\}$ is everywhere dense in L^2_μ

(ii) given $f, g \in L$ then

$$\lim_{t \to \infty} (U^t f, g) = \left\{ \int_X f d\mu \right\} \left\{ \int_X g d\mu \right\}$$

Proof Since $(U^t f, g) = \int_X (U^t f) \, g \, d\mu$ the result follows from corollary 1.

Lemma 4·22 T is a mixing of degree one if and only if

$$\lim_{t \to \infty} (U^t f, f) = \left\{ \int_X f \, d\mu \right\}^2, \quad f \in L^2 \quad (1)$$

Proof Given $f \in L^2_\mu$, let L_1 be the subspace of L^2_μ spanned by the constant functions together with $f, U^t f$, $t \in \Gamma^+$ and let L_2 be such that $L_1 \oplus L_2 = L^2_\mu$. If $g = U^s f$ and (1) holds then

$$\begin{aligned} \lim_{t \to \infty} (U^t f, g) &= \lim_{t \to \infty} (U^t f, U^s f) \\ &= \lim_{t \to \infty} (U^{t-s} f, f) \end{aligned}$$

$$= \left\{ \int_X f \, d\mu \right\}^2 \quad \text{by (1)}$$

$$= \left\{ \int_X f \, d\mu \right\} \left\{ \int_X g \, d\mu \right\}$$

since $\int_X g \, d\mu = \int_X U^S f \, d\mu = \int_X f \, d\mu$

Hence we get that for all $g \in L_1$ we have

$$\lim_{t \to \infty} (U^t f, g) = \left\{ \int_X f \, d\mu \right\} \left\{ \int_X g \, d\mu \right\}$$

Now if $g \in L_2$ then $(U^t f, g) = 0$, $t \in \Gamma^+$ and since L_1 contains the constant functions in particular $h(x) = 1$, $x \in X$ we have

$$\begin{aligned} 0 &= (g,h) \\ &= \int_X gh \, d\mu \\ &= \int_X g \, d\mu. \end{aligned}$$

Thus for arbitrary $g \in L^2$ we can find $g_1 \in L_1$, $g_2 \in L_2$ such that $g = g_1 + g_2$ and so

$$\begin{aligned} \lim_{t\to\infty} (U^t f, g) &= \lim_{t\to\infty} (U^t f, g_1 + g_2) \\ &= \lim_{t\to\infty} \left\{ (U^t f, g_1) + (U^t f, g_2) \right\} \\ &= \left\{ \int_X f \, d\mu \right\} \left\{ \int_X g_1 \, d\mu \right\} \\ &= \left\{ \int_X f \, d\mu \right\} \left\{ \int_X g_1 \, d\mu + \int_X g_2 \, d\mu \right\} \\ &= \left\{ \int_X f \, d\mu \right\} \left\{ \int_X g \, d\mu \right\} \end{aligned}$$

The if of the lemma then follows from theorem 4·21, corollary 2 as does the only if. (see K. Jacobs [8].)

4·3 SEQUENCES OF σ-ALGEBRAS

The main result in this section is due to J.R.Blum and D.L.Hanson (see [2]). In this section we use the term measure to mean a real valued function μ^1 defined on a σ-algebra α such that $\mu^1(\phi) = 0$ and if $A_i \in \alpha$, $i \in \Gamma^+$, $A_i \cap A_j = \phi$ for $i \neq j$ then $\mu^1(\bigcup_{i\in\Gamma^+} A_i) = \sum_{i\in\Gamma^+} \mu^1(A_i)$. However by μ we still mean a positive measure with $\mu(X) = 1$.

Lemma 4·31 If $\{\alpha_n\}$, $n \in \Gamma^+$ are a sequence of σ-algebras such that $\alpha_{n+1} \leqslant \alpha_n$ for each n, $\alpha^\infty = \bigwedge_{n\in\Gamma^+} \alpha_n$ then $\alpha^\infty = \nu$ if and only if for all $A \in \varepsilon$

$$\lim_{n\to\infty} \sup_{B\in\alpha_n} |\mu(A \cap B) - \mu(A)\mu(B)| = 0 \tag{1}$$

Proof The limit always exists since $\alpha_{n+1} \leqslant \alpha_n$, $n \in \Gamma^+$.
If (1) holds, let $A \in \alpha^\infty$ then $A \in \alpha_n$, $n \in \Gamma^+$ and so

$$0 \leqslant |\mu(A) - \mu(A)\mu(A)| \leqslant \lim_{n\to\infty} \sup_{B\in\alpha_n} |\mu(A \cap B) - \mu(A)\mu(B)| = 0$$

i.e. $\mu(A) = 0$ or 1 giving $A \in \nu$. But A was any set in α^∞ and so we deduce that $\alpha^\infty = \nu$.

If $\alpha^\infty = \nu$ and (1) is false then there exists an $A \in \varepsilon$ and a $d > 0$ such that

$$\sup_{B\in\alpha_n} |\mu(A \cap B) - \mu(A)\mu(B)| \geqslant d, \; n \in \Gamma^+ \tag{2}$$

For each n we define a measure μ_n on (X, α_n) by

$\mu_n(B) = \mu(A \cap B) - \mu(A)\mu(B)$ for $B \in \alpha_n$. We have that

$$\begin{aligned}
\mu_n(X-B) &= \mu(A \cap (X-B)) - \mu(A)\mu(X-B) \\
&= \mu(A - (A \cap B)) - \mu(A)(1 - \mu(B)) \\
&= -\mu(A \cap B) + \mu(A)\mu(B) \\
&= -\mu_n(B)
\end{aligned} \tag{3}$$

Hence $\mu_n(B) \leqslant \mu(A \cap B) \leqslant 1$

and $-\mu_n(B) = \mu(X-B) \leqslant \mu(A \cap (X-B)) \leqslant 1$

giving $|\mu_n(B)| \leqslant 1$, $B \in \alpha_n$, $n \in \Gamma^+$

If $k = \sup_{B\in\alpha_n} \mu_n(B)$ then there exists a sequence $\{B_i\}$, $i \in \Gamma^+$ with $B_i \in \alpha_n$, $\lim_{i\to\infty} \mu_n(B_i) = k$. Further if $C_n = \bigcup_{i\in\Gamma^+} B_i$ then $C_n \in \alpha_n$

because α_n is a σ-algebra and $\mu_n(C_n) \geqslant \mu_n(B_i)$ for all i, hence $\mu_n(C_n) \geqslant k$ giving $\mu_n(C_n) = k$. We note that $k \geqslant d$, by (2), (3) and the definition of k, and that for any $B \in \alpha_n$ we have $\mu_n(C_n) \geqslant \mu_n(B)$. Thus for $n \in \Gamma^+$

$$\begin{aligned} \mu_n(C_n) &\geqslant \mu_n(C_n \cup C_{n\ 1}) \\ &\geqslant \mu_n(C_{n+1}) \\ &= \mu_{n+1}(C_{n+1}) \end{aligned}$$

since $\mu_n(B) = \mu_{n+1}(B)$ if $B \in \alpha_{n+1}$

If $\mu_n(\bigcup_{i=0}^{m} C_{n+i}) \geqslant \mu_{n+m}(C_{n+m})$ for $n \in \Gamma^+$ (4)

then
$$\begin{aligned} \mu_n(\bigcup_{i=0}^{m+1} C_{n+i}) &\geqslant \mu_n(\bigcup_{i=1}^{m+1} C_{n+i}) \\ &= \mu_{n+1}(\bigcup_{i=1}^{m+1} C_{n+i}) \\ &= \mu_{n+1}(\bigcup_{i=0}^{m} C_{n+1+i}) \\ &\geqslant \mu_{n+m+1}(C_{n+m+1}) \quad \text{by (4)} \end{aligned}$$

Thus since (4) holds for $m = 1$ we have by induction that (4) holds for $m \in \Gamma^+$. Hence $\mu_n(\bigcup_{i=n}^{n+m} C_i) \geqslant \mu_{n+m}(C_{n+m}) = k \geqslant d$ for $n, m \in \Gamma^+$, letting $m \to \infty$ gives

$$\mu_n(\bigcup_{i=n}^{\infty} C_i) \geqslant d \quad \text{for } n \in \Gamma^+$$

i.e. $\mu_1(\bigcup_{i=n}^{\infty} C_i) = \mu_n(\bigcup_{i=n}^{\infty} C_i) \geqslant d$ for $n \in \Gamma^+$

and so

$$\mu_1(\bigcap_{n \in \Gamma^+} \bigcup_{i=n}^{\infty} C_i) \geqslant d \tag{5}$$

But $\bigcup_{i=n}^{\infty} C_i \in \alpha_n$, $\bigcup_{i=n+1}^{\infty} C_i \subseteq \bigcup_{i=n}^{\infty} C_i$ $n \in \Gamma^+$ and so $\bigcap_{n \in \Gamma^+} \bigcup_{i=n}^{\infty} C_i \in \alpha^\infty = \nu$ giving $\mu(\bigcap_{n \in \Gamma^+} \bigcup_{i=n}^{\infty} C_i) = 0$ or 1. In either case $\mu_1(\bigcap_{n \in \Gamma^+} \bigcup_{i=n}^{\infty} C_i) = 0$ contradicting (5). Hence we deduce that if $\alpha^\infty = \nu$ then (1) holds.

4·4 MIXING PROPERTIES OF KOLMOGOROV AUTOMORPHISMS

We are now in a position to prove a result due to A.N.Kolmogorov [11], [12] and V.A.Rokhlin [16], namely that a Kolmogorov automorphism is a mixing of all degrees. However, the proof we give is due to J.R.Blum and D.L.Hanson [2].

Theorem 4·41 If T is a Kolmogorov automorphism then it is a mixing of degree 1.

Proof Let ζ be a K-algebra, and $\alpha_n = T^{-n}\zeta$, $n \in \Gamma^+$. Then for each n, $\alpha_{n+1} \leq \alpha_n$ and $\alpha^\infty = \nu$. If $A, B \in \varepsilon$ then there exists a sequence $\{B_i\}$, $i \in \Gamma^+$ with $B_i \in T^{k_i}\zeta$ and $\mu(B \triangle B_i) < 2^{-i}$ for each i Hence for $n \in \Gamma^+$, $i \in \Gamma^+$

$$\begin{aligned}
&|\mu(A \cap T^{-(n+k_i)}B) - \mu(A \cap T^{-(n+k_i)}B_i)| \\
&\quad \leq |\mu(T^{n+k_i}A \cap B) - \mu(T^{n+k_i} A \cap B_i)| \\
&\quad \leq \mu((T^{n+k_i} A \cap B) \triangle (T^{n+k_i} A \cap B_i)) \\
&\quad \leq \mu(B \triangle B_i) \\
&\quad < 2^{-i}
\end{aligned}$$

and
$$\begin{aligned}
&|\mu(A)\mu(T^{-(n+k_i)}B_i) - \mu(A)\mu(B)| \\
&\quad = \mu(A)\,|\mu(B_i) - \mu(B)| \\
&\quad \leq \mu(A)\,\mu(B \triangle B_i) \\
&\quad < 2^{-i}
\end{aligned}$$

Given $d > 0$ choose i such that $2^{-i} < d/3$. Now $T^{-(n+k_i)} B_i \in \alpha_n$, $n \in \Gamma^+$ and so by lemma 4·31 we have

$$\lim_{n\to\infty} |\mu(A \cap T^{-(n+k_i)}B_i) - \mu(A)\mu(T^{-(n+k_i)}B)| = 0$$

Thus we can choose N_1 such that for $N \geq N_1$ and $n = N-k_i$

$$|\mu(A \cap T^{-(n+k_i)}B_i) - \mu(A)\mu(T^{-(n+k_i)}B)| < d/3$$

Hence $|\mu(A \cap T^{-N}B) - \mu(A)\,\mu(B)|$

$$\leq |\mu(A \cap T^{-(n+k_i)}B) - \mu(A \cap T^{-(n+k_i)}B_i)|$$
$$+ |\mu(A \cap T^{-(n+k_i)}B_i) - \mu(A)\,\mu(T^{-(n+k_i)}B_i)|$$
$$+ |\mu(A)\,\mu(T^{-(n+k_i)}B_i) - \mu(A)\,\mu(B)|$$
$$< d/3 + d/3 + d/3$$
$$= d$$

giving $\lim_{N\to\infty} |\mu(A \cap T^{-N}B) - \mu(A)\,\mu(B)| = 0.$

i.e. T is a mixing of degree 1.

Corollary If T is a Kolmogorov automorphism then it is a mixing of all degrees.

Proof It is sufficient to prove that given any sets $A_j, 1 \leq j \leq m, m \in \Gamma^+$ and $d > 0$ there exists a n_o such that if $n_j \geq n_o$, $N_1 = 0$, $N_{j+1} = -\sum_{k=1}^{j} n_j$, $1 \leq j \leq m-1$ then

$$|\mu(\bigcap_{j=1}^{m} T^{N_j} A_j) - \prod_{j=1}^{m} \mu(A_j)| < d.$$

We assume the result for m and prove that this implies the result for $m+1$. An appeal to the theorem for the case $m = 2$ then completes the proof.

Given A_j, $1 \leq j \leq m+1$, $d > 0$ and integers n_j, $1 \leq j \leq m$ put $N_1 = 0$, $N_{j+1} = -\sum_{k=1}^{j} n_j$, $1 \leq j \leq m$. There exist sequences $\{A_{ji}\}$, $i \in \Gamma^+$, $2 \leq j \leq m+1$ such that $A_{ji} \in T^{k_i}\zeta$, $\mu(A_j \triangle A_{ji}) < 2^{-i}$ for each i and $2 \leq j \leq m+1$. Hence for $i \in \Gamma^+$

$$|\mu(A_1 \cap T^{-n_1} \bigcap_{j=2}^{m+1} T^{N_j+n_1} A_j) - \mu(A_1 \cap T^{-n_1} \bigcap_{j=2}^{m+1} T^{N_j+n_1} A_{ji})|$$
$$\leq \mu(\bigcap_{j=2}^{m+1} T^{N_j+n_1} A_j \triangle \bigcap_{j=2}^{m+1} T^{N_j+n_1} A_{ji})$$
$$= \mu(\bigcap_{j=2}^{m+1} T^{N_j+n_1} A_j - \bigcap_{j=2}^{m+1} T^{N_j+n_1} A_{ji})$$
$$+ (\bigcap_{j=2}^{m+1} T^{N_j+n_1} A_{ji} - \bigcap_{j=2}^{m+1} T^{N_j+n_1} A_j)$$

$$\leqslant \mu\left(\bigcup_{j=2}^{m+1}\left(T^{N_j+n_1}\Lambda - T^{N_j+n_1}\Lambda_{ji}\right)\right) + \mu\left(\bigcup_{j=2}^{m+1}\left(T^{N_j+n_1}\Lambda_{ji} - T^{N_j+n_1}\Lambda_j\right)\right)$$

$$\leqslant \sum_{j=2}^{m+1} \mu(\Lambda_j \triangle \Lambda_{ji})$$

$$< m\, 2^{-i}$$

and $\left|\mu(\Lambda_1)\,\mu\left(T^{-n_1}\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji}\right) - \mu(\Lambda_1)\,\mu\left(\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_j\right)\right|$

$$\leqslant \mu(\Lambda_1)\left|\mu\left(\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji}\right) - \mu\left(\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_j\right)\right|$$

$$\leqslant \mu\left(\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji} \triangle \bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_j\right)$$

$$\leqslant \sum_{j=2}^{m+1} \mu(\Lambda_j \triangle \Lambda_{ji})$$

$$< m\, 2^{-i}$$

Choose i such that $m2^{-i} < d/4$. Now N_j+n_1 is independent of n_1 for $2 \leqslant j \leqslant m+1$ and $T^{-(n_1+k_i)}\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji} \in T^{-n_1}\zeta = \alpha_{n_1}$, $n_1 \in \Gamma^+$. Hence by lemma 4·31

$$\lim_{n_1\to\infty}\left|\mu\left(\Lambda_1 \cap T^{-(n_1+k_i)}\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji}\right) - \mu(\Lambda_1)\,\mu\left(T^{-(n_1+k_i)}\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji}\right)\right| = 0$$

uniformly in n_j, $2 \leqslant j \leqslant m$

Thus we can find an n_o^1 such that for $n_1 \geqslant n_o^1$

$$\left|\mu\left(\Lambda_1 \cap T^{-n_1}\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji}\right) - \mu(\Lambda_1)\,\mu\left(T^{-n_1}\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji}\right)\right| < d/4$$

but by our hypothesis there exists an n_o^{11} such that if $n_j \geqslant n_o^{11}$, $2 \leqslant j \leqslant m$ then

$$\left|\mu\left(\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_j\right) - \prod_{j=2}^{m+1}(\Lambda_j)\right| < d/4$$

and since $\mu(\Lambda_1) < 1$ we have

$$\left|\mu(\Lambda_1)\,\mu\left(\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_j\right) - \prod_{j=1}^{m+1}\mu(\Lambda_j)\right| < d/4$$

Thus if $n_o = \max\{n_o^1, n_o^{11}\}$ we have for $n_j \geqslant n_o$, $1 \leqslant j \leqslant m$ that

$$\left|\mu\left(\bigcap_{j=1}^{m+1} T^{N_j}\Lambda_j\right) - \prod_{j=1}^{m+1}\mu(\Lambda_j)\right|$$

$$\leqslant \left|\mu\left(\bigcap_{j=1}^{m+1} T^{N_j}\Lambda_j\right) - \mu\left(\Lambda_1 \cap T^{-n_1}\bigcap_{j=2}^{m+1} T^{N_j+n_1}\Lambda_{ji}\right)\right| +$$

$$|\mu(A_1 \cap T^{-n_1} \bigcap_{j=2}^{m+1} T^{N_j+n_1} A_{ji}) - \mu(A_1)\mu(T^{-n_1} \bigcap_{j=2}^{m+1} T^{N_j+n_1} A_{ji})|$$

$$+|\mu(A_1)\mu(T^{-n_1} \bigcap_{j=2}^{m+1} T^{N_j+n_1} A_{ji}) - \mu(A_1)\mu(\bigcap_{j=2}^{m+1} {}^{N_j+n_1} A_j)|$$

$$+|\mu(A_1)\mu(\bigcap_{j=2}^{m+1} {}^{N_j+n_1} A_j) - \prod_{j=1}^{m+1} \mu(A_j)|$$

$$< \frac{d}{4} + \frac{d}{4} + \frac{d}{4} + \frac{d}{4}$$

$$= d$$

Theorem 4·42 If T is a mixing of degree 1 then T is ergodic

Proof If $A \in \mathcal{E}$, satisfies $TA = A$ then since T is a mixing of degree 1 we have

$$\lim_{n \to \infty} |\mu(A \cap T^{-n} A) - \mu(A)\mu(A)| = 0 \qquad (1)$$

But $TA = A$ implies $A \cap T^{-n} A = A$ for $n \in \Gamma^+$ and so (1) gives

$$\mu(A) - \mu(A)\,\mu(A) = 0$$

i.e. $\mu(A) = 0, 1$

i.e. $\mu(A) = 0$ or $\mu(X-A) = 0$

Hence T is ergodic.

Corollary If T is a Kolmogorov automorphism then T is ergodic.

Proof The result is an immediate consequence of the corollary of theorem 4·41 and the theorem.

4·5 EQUIVALENCE OF KOLMOGOROV AND REGULAR AUTOMORPHISMS

This follows from the work of Ja. G. Sinai, J.R. Blum and D.L. Hanson (see [19] and [2]). We do not use the above papers, but consider expressions for $\pi(T)$.

Lemma 4·51 If $\alpha \in Z$, $\beta = \alpha^-$ then $\beta_\infty \leqslant \Pi(T)$

Proof By lemma 3·23 if $\gamma \leqslant \beta_\infty$ we have

$$H(\gamma | \gamma^-) + H(\alpha | \alpha^- \gamma_T) = H(\alpha\gamma | \alpha^- \gamma^-)$$
$$= H(\alpha | \alpha^-) + H(\gamma | \gamma^- \alpha_T) \qquad (1)$$

Now $\alpha^- \gamma_T = \alpha^-$ since $T^i \gamma \leqslant T^i \beta_\infty = \beta_\infty$ because $\beta \leqslant T\beta$ giving $H(\alpha | \alpha^- \gamma_T) = H(\alpha | \alpha^-)$. But $\gamma \leqslant \alpha_T$ and so $H(\gamma | \gamma^- \alpha_T) = 0$ giving from (1) that $H(\gamma | \gamma^-) = 0$, i.e. $\gamma \leqslant \Pi(T)$. The result then follows since γ was any σ- algebra such that $\gamma \leqslant \beta_\infty$.

Lemma 4·52 $\Pi(T) = \bigvee_{\alpha \in Z} (\alpha^-)_\infty$

Proof By the previous lemma $(\alpha^-)_\infty \leqslant \Pi(T)$ for all $\alpha \in Z$ and so $\bigvee_{\alpha \in Z} (\alpha^-)_\infty \leqslant \Pi(T)$. If $\beta \leqslant \Pi(T)$ then $0 = h(\beta, T) = H(\beta | \beta^-)$ giving $\beta \leqslant \beta^-$ and hence $T\beta \leqslant T\beta^- = \beta.\beta^- = \beta^=$ and so by induction we get $T^i \beta \leqslant \beta^-$, $i \in \Gamma^+$. Thus $\beta \leqslant T^{-i}\beta^-$, $i \in \Gamma^+$ giving $\beta \leqslant (\beta^-)_\infty$ and therefore $\Pi(T) \leqslant \bigvee_{\alpha \in Z} (\alpha^-)_\infty$.

Corollary 1 $\Pi(T) = \bigvee_{\alpha \in Z_1} (\alpha^-)_\infty$

Proof Since $Z_1 \subseteq Z$ we have $\bigvee_{\alpha \in Z_1} (\alpha^-)_\infty \leqslant \Pi(T)$. If $\beta \leqslant \Pi(T)$, $B \in \beta$ consider $\gamma = \{\phi, B, X-B, X\}$. Now $\gamma \leqslant \beta$ and so $\gamma \leqslant \Pi(T)$ giving as in the proof of the lemma that $\gamma \leqslant (\gamma^-)_\infty$ and so $B \in \gamma \leqslant \bigvee_{\alpha \in Z_1} (\alpha^1)_\infty$. But B was any set in β and so $\beta \leqslant \bigvee_{\alpha \in Z_1} (\alpha^1)_\infty$ and therefore $\pi(T) \leqslant \bigvee_{\alpha \in Z_1} (\alpha^1)_\infty$.

Corollary 2 T is a regular automorphism if and only if $\Pi(T) = \nu$

Proof We observe that $(\alpha^-)_\infty = (\alpha)$

Combining this last result with 3·4 and 4·1 we get:

Theorem 4·53 T is a Kolmogorov automorphism if, and only if, T is a regular automorphism.

4·6 SPECTRAL THEORY

Given an increasing real-valued non-negative function $F(\lambda)$ defined on $[-\pi,\pi]$ then if we put

$$F_1([x,y]) = F(y) - F(x)$$

$$F_1([x,y)) = \lim_{h\to 0+} F(y-h) - F(x)$$

$$F_1((x,y]) = F(y) - \lim_{k\to 0+} F(x+k)$$

$$F_1((x,y)) = \lim_{h\to 0+} F(y-h) - \lim_{k\to 0+} F(x+k)$$

we have that F_1 is a measure on $[-\pi,\pi]$. Conversely, if F_1 is a measure on $[-\pi,\pi]$ and we put

$$F^*(\lambda) = F_1([-\pi,\lambda])$$

then F^* is an increasing real valued function on $[-\pi,\pi]$. Moreover, if given F we construct F_1 and then F^* we have that $F = F^*$ almost everywhere. Throughout this section we shall not distinguish between an increasing real-valued function on $[-\pi,\pi]$ and the associated measure on $[-\pi,\pi]$ and the same symbol will be interpreted as both; the context making clear which interpretation is meant.

Before continuing we introduce the following notation. Given any $x \in L^2_\mu$ we denote by H_x the subspace generated by $U^n x, n \in \Gamma$ and refer to it as the cyclic subspace generated by x. For each $n \in \Gamma$ we put

$$\varphi_x(n) = (U^n x, x)$$

and note that for all n

$$\varphi_x(-n) = (U^{-n}x, x)$$
$$= (x, U^n x)$$
$$= \varphi_x(n)$$

Further, since $|\varphi_x(n)| \leqslant \varphi(0)$, $n \in \Gamma$ it follows that $\varphi_x(n)$ is a positive definite function and hence (see [3]) there exists a measure G_x on $[-\pi, \pi]$ such that

$$\varphi_x(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, d\, G_x(\lambda),\ n \in \Gamma$$

We refer to $F_x = G_x / \int_{-\pi}^{\pi} G_x(\lambda)\, d\lambda$ as the spectral type of x. Lastly we put

$$Y_x = \{ y\colon F_x \sim F_y \}$$

where $\sim$ denotes the usual equivalence relation between measures, i.e. $F_x \sim F_y$ if and only if they vanish on the same sets. The reader is referred to P.R.Halmos [5] for a discussion of the relations $\sim$ and $\leqslant$ as applied to measures. The main results we need are that if F, G, are measures, a, b, are non-zero numbers then $F \sim G$ if and only if $a F \leqslant b G$ and $F \leqslant G$ if and only if $aF \leqslant bG$. Further if F_i, $i \in \Gamma$ are finite and normalized measures, $a_i, b_i, i \in \Gamma$ are non-zero, positive real numbers such that $\sum_{i \in \Gamma} a_i, \sum_{i \in \Gamma} b_i < \infty$ then $\sum_{i \in \Gamma} a_i F_i \sim \sum_{i \in \Gamma} b_i F_i$.

Lemma 4.61 If $x \in L^2_\mu$, $y \in H_x$ then $F_y \leqslant F_x$

Proof Since $y \in H_x$ there exist constants a_k, $k \in \Gamma$ such that $y = \lim_{n \to \infty} \sum_{|k| \leqslant n} a_k U^k x$. Hence

$$\varphi y(\Lambda) = (U^n y, y)$$
$$= (\sum_{k \in \Gamma} a_k U^{n+k} x, \sum_{k \in \Gamma} a_k U^k x)$$
$$= \sum_{l \in \Gamma} b_{nl} (U^l x, x)$$

$$= \sum_{l \in \Gamma} b_{nl} \int_{-\pi}^{\pi} e^{il\lambda} \, dG_x(\lambda)$$

giving us that $F_y \leqslant F_x$

Lemma 4·62 If $x \in L^2_\Gamma$, $y \in H_x$, $F_y \not\sim F_x$ then $H_y \neq H_x$

Proof $F_y \leqslant F_x$ by lemma 4·61 and $x \in H_y$ implies $F_x \leqslant F_y$ by the same lemma, and hence $F_x \sim F_y$. This is a contradiction and so we deduce that $x \notin H_y$. But $x \in H_x$ and so we have $H_y \neq H_x$.

Lemma 4·63 If $x \in L^2_\Gamma$, $y \in H_x$, $F_y \not\sim F_x$ then there exists a $z \in H_x$ such that $H_y \perp H_2$

Proof By lemma 4·62 $H_y \neq H_x$ and hence since $H_y \subseteq H_x$ there exists a $z \in H_x$ such that $z \perp H_y$, i.e. $z \perp U^n y$, $n \in \Gamma$ and so $U^n z \perp y$ for all n giving us that $H_y \perp H_2$.

Lemma 4·64 If $x, y \in L^2_\Gamma$ are such that $H_x \perp H_y$ then $F_{x+y} = (F_x + F_y)/2$

Proof
$$\varphi_{x+y}(n) = (U^n(x\ y),\ (x\ y))$$
$$= (U^n x, x) + (U^n x, y) + (U^n y, x) + (U^n y, y)$$

but $H_x \perp H_y$ imply $(U^n x, y) = 0 = (U^n y, x)$ and so

$$\begin{aligned} \varphi_{x+y}(n) &= (U^n x, x) + (U^n y, y) \\ &= \int_{-\pi}^{\pi} e^{in\lambda} \, dG_x + \int_{-\pi}^{\pi} e^{in\lambda} \, dG_y \\ &= \int_{-\pi}^{\pi} e^{in\lambda} \, d(G_x + G_y) \end{aligned}$$

giving $G_{x+y} = G_x + G_y$ and hence $F_{x+y} = (F_x + F_y)/2$

Lemma 4·65 If $x_i, i \in I$ satisfy $x_i \in L^2_\Gamma$, $\|x_i\| = \text{constant}$ for all i, $Hx_i \perp Hx_j$ if $i \neq j$ and a_i, $i \in I$ are real numbers such that $\sum_{i \in I} |a_i| \leqslant M$ and $|a_i| \leqslant 1$ for each i then if $y = \sum_{i \in I} a_i x_i$ we have that $F_y = (\sum_{i \in I} a_i^2 F_{xi}) / \sum_{i \in I} a_i^2$.

Proof $\|\sum_{i\in I} a_i \; x_i\| \leq \sum_{i\in I} |a_i| \, \|x_i\|$
$\leq M^2$

and so y is well defined.

$$(U^n y, y) = (U^n \sum_{i\in I} a_i \, x_i, \sum_{i\in I} a_i x_i)$$
$$= \sum_{i\in I} a_i^2 \, (U^n x_i, x_i)$$
$$= \sum_{i\in I} a_i^2 \int_{-\pi}^{\pi} e^{i\lambda n} \, dG_{xi}$$

Thus since $a_i^2 \leq 1$, $M \geq (x_i, x_i) = \int_{-\pi}^{\pi} dG_{xi}$ we have that $\sum_{i\in I} a_i^2 G_{xi}$ is well defined and so $G_y = \sum_{i\in I} a_i^2 \; G_{xi}$ giving $F_i = (\sum_{i\in I} a_i^2 \, F_{xi}) / \sum_{i\in I} a_i^2$.

Lemma 4·66 If L^2_μ is separable and $x_i, i\in I$ $y_j, j\in J$ are such that $x_i \in L^2_\mu$, $y_j \in L^2_\mu$ for all i, j and

(1) $H_{xi} \perp H_{xk}$ if $i \neq k$

(2) $H_{yj} \perp H_{yl}$ if $j \neq l$

(3) $\bigoplus_{i\in I} H_{xi} = L^2_\mu = \bigoplus_{j\in J} H_{yj}$

then for all non-zero, positive real numbers a_i, $i\in I$, b_j, $j\in J$ we have $\sum_{i\in I} a_i \, F_{xi} \sim \sum_{j\in J} b_j \, F_{yj}$

Proof Since L^2_μ is separable $I, J \subseteq \Gamma$ and since F_{xi}, F_{yj} are normalized measures it follows that $\sum_{i\in I} a_i \, F_{xi}$, $\sum_{j\in J} b_j \, F_{yj}$ are well defined and finite. By (3) we have that for each $j \in J$ there exist v_{ji}, $i\in I$ such that $v_{ji} \in H_{xi}$ for each i and $y_j = \sum_{i\in I} v_{ji}$ By lemma 4·65 we have that $F_{yj} = F_{v_{ji}}$ and since by lemma 4·61 we have that $F_{v_{ji}} \leq F_{xi}$ for each i it follows that $F_{yj} \leq \sum_{i\in I} a_i F_{xi}$, $j \in J$. Thus we deduce that $\sum_{j\in J} b_j F_{yj} \leq \sum_{i\in I} a_i \, F_{xi}$. Similarly we show that $\sum_{i\in I} a_i F_{xi} \leq \sum_{j\in J} b_j F_{yj}$ and so we have $\sum_{i\in I} a_i F_{xi} \sim \sum_{j\in J} b_j \, F_{yj}$.

We now define the maximal spectral type of $\mathcal{U}$ if L^2_μ is separable to be the equivalence class of measures which contains $\sum_{i\in I} a_i F_{xi}$ where x_i, $i\in I$ are any elements of L^2_μ such that $H_{xi} \perp H_{xj}$ if $i\neq j$, $L^2_\mu = \bigoplus_{i\in I} H_{xi}$ and a_i are any non-zero, positive real numbers such that $\sum_{i\in I} a_i < \infty$. By lemma 4·66 we have immediately that the maximal spectral type is well defined and unique.

We say that $\mathcal{U}$ has a Lebesgue spectrum of multiplicity $\aleph_0$ if there exists $x_i, i\in I$ such that $x_i \in L^2_\mu$ for each i, $H_{xi} \perp H_{xj}$ if $i\neq j$, $L^2_\mu = \bigoplus_{i\in I} H_{xi}$, F_{xi} for each i is equivalent to the ordinary Lebesgue measure and I is countable finite.

If $\mathcal{U}$ satisfies all the above conditions except the last then we say that $\mathcal{U}$ has a Lebesgue spectrum of multiplicity $\theta =$ cardinal number of I.

Lemma 4·67 If there exists an orthonormal basis f_{ij}, $i\in I$, $j\in J_i$ of L^2_μ such that $J_i = \Gamma$ or Γ^+ for each i and $\mathcal{U} f_{ij} = f_{ij+1}$ for all i, j then $\mathcal{U}$ has a Lebesgue spectrum.

Proof Let $x_i = f_{ij}$ for some $j \in J_i$. Then

$$(\mathcal{U}^n x_i, x_i) = \begin{cases} \|x_i\| & \text{if } n = 0 \\ 0 & \text{if } n \in \Gamma^+ \end{cases}$$

hence we must have G_{xi} equal to a constant times the ordinary Lebesgue measure. Hence F_{xi} is equivalent to the Lebesgue measure and since f_{ij}, $i\in I$, $j\in J_i$ is an orthonormal basis we must have $L^2_\mu = \bigoplus_{i\in I} H_{xi}$, and $H_{xi} \perp H_{xk}$ if $i\neq k$. Hence we see that $\mathcal{U}$ has a Lebesgue spectrum.

Lemma 4·68 If there exists an $x \in L^2_\Gamma$ such that F_x is singular with respect to Lebesgue measure then $\mathcal{U}$ doesnot have a Lebesgue spectrum.

Proof As usual we say that two measures are singular if the only measure which is absolutely continuous with respect to both is the zero measure (see [5])

If $\mathcal{U}$ has a Lebesgue spectrum then there exist x_i, $i \in I$ such that $x_i \in L^2_\Gamma$ for each i, $Hx_i \perp Hx_j$ if $i \neq j$, $L^2_\Gamma = \bigoplus_{i \in I} Hx_i$. Hence given $x \in L^2_\Gamma$ there exist v_i, $i \in I$ such that $v_i \in Hx_i$ each i and $x = \sum_{i \in I} v_i$. By lemma 4·65 $F_x = \sum_{i \in I} F_{v_i}$ and so since $F_{v_i} \leqslant F_{x_i}$ and F_{x_i} is absolutely continuous with respect to Lebesgue measure for each i, so too is F_x. But this is a contradiction and so we deduce that $\mathcal{U}$ does not have a Lebesgue spectrum.

4·7 THE SPECTRUM OF A KOLMOGOROV AUTOMORPHISM

In this section we look at some spectral properties of T, and in particular the spectrum of T if T is a K - automorphism.

For any σ- algebra α we put $L_\alpha = \{ f: f \in L^2_\Gamma,\ f$ is measurable with respect to $(X,\alpha) \}$ If $\Lambda \in \alpha$, χ_Λ is the characteristic function of Λ, $\mathcal{U}$ as in section 1·2 then

$$\chi_\Lambda = \chi_\Lambda T = \chi_{T^{-1}\Lambda}$$

Thus we see that for any step-function $f \in L_\alpha$ we have $\mathcal{U}f \in L_{T^{-1}\alpha}$. And as usual we can approximate any f by step functions and obtain for all $f \in L^2_\Gamma$ that

$$f \in L_\alpha \text{ implies } \mathcal{U}f \in L_{T^{-1}\alpha}$$

Moreover if $\alpha \leqslant T\alpha$ then $L_{T^{-1}\alpha} \leqslant L_\alpha$ and so $\mathcal{U}L_\alpha \leqslant L_\alpha$.

We define a subspace of L of L^2_μ to be invariant if

$$\mathcal{U}L \leq L$$

and to be exhaustive if

$$\bigvee_{t\in\Gamma^+} \mathcal{U}^t L = L^2_\mu$$

Here as always we use $\bigvee_{t\in\Gamma^+} \mathcal{U}^t L$ to denote the closure of $\bigcup_{t\in\Gamma^+} \mathcal{U}^t L$.

Lastly we point out that if α is a K_1-algebra then L_α is invariant and exhaustive.

The following four lemmas are essentially proved in K. Jacobs [8].

Lemma 4·71 If L is an invariant exhaustive subspace of L^2_μ and $L_\infty = \bigcap_{t\in\Gamma^+} \mathcal{U}^t L$, L^+ satisfies $L^2_\mu = L^+ \oplus L_\infty$ then $\mathcal{U}$ has a Lebesgue spectrum in L^+ if $L^+ \neq \{0\}$

Proof Let H_1 be the subspace such that

$$L = H_1 \oplus \mathcal{U} L$$

and define $H_t = \mathcal{U}^{t-1} H_1$ for $t \in \Gamma^+$

then $\mathcal{U} H_t = H_{t+1}$ and $\mathcal{U}^{t-1} L = H_t \oplus \mathcal{U}^t L$.

Thus
$$L^2_\mu = \bigoplus_{t\in\Gamma} \mathcal{U}^{-t} L$$
$$= \bigoplus_{t\in\Gamma} H_t \oplus L_\infty$$

If $\{f_i\}$ $i\in I$ is an orthonormal basis in H_1 and $\{f_{i_t}\} = \mathcal{U}^{t-1} f_{i1}$ then $\{f_{i_t}\}$, $i \in I$, $t\in\Gamma$ is an orthonormal basis of $L^+ = \bigoplus_{t\in\Gamma} H_t$ such that $f_{i_{t+1}} = \mathcal{U} f_{i_t}$. Thus $\mathcal{U}$ has a Lebesgue spectrum in L .

Lemma 4·72 If T is ergodic, α an invariant σ-algebra and $T^{-1}\alpha \neq \alpha$ then $(X, \mathcal{E}, \mu)$ is atom free.

Proof Since $T\alpha \neq \alpha$ there exists a set $A_1 \neq \phi$ such that $A_1 \in T\alpha$, $A_1 \notin \alpha$, and for each $n \in \Gamma^+$ there exists a set $A_n \neq \phi$

such that $A_n \in T^n\alpha$, $A_n \notin T^{n-1}\alpha$ and hence $A_n \neq A_j$, $1 \leq j \leq n-1$. Let β be the σ-algebra generated by $\{A_n\}$ $n \in \Gamma^+$. Let δ be the least limit point of the $\{\mu(B)\}$, $B \in \beta$, $B \neq \phi$ and suppose $\delta > 0$. Then there exists a sequence $\{B_i\}$, $i \in \Gamma^+$ such that $B_i \neq B_j$ if $i \neq j$ and such that $\lim_{i\to\infty} \mu(B_i) = \delta$, and an N such that for $i \geqslant N$ we have $\mu(B_i) < 3\delta/2$. Consider $C_{ij} = B_i \wedge B_j$ for $i,j \geqslant N$, then we must have $C_{ij} \neq \phi$ for an infinite number of pairs i, j since otherwise the Bi, $i \geqslant$ some N_1 are disjoint and so $1 = \mu(X) \geqslant \sum_{i=N_1}^{\infty} \mu(B_i) \geqslant \sum_{i=N}^{\infty} \delta = \infty$ which is absurd. For these pairs either $0 < \mu(B_i \wedge B_j) < \delta/2$ or $0 < \mu(B_i - B_j) < \delta/2$, giving a limit point of $\{\mu(B)\}$, $B \in \beta$, $B \neq \phi$ which is $\leqslant \delta/2 < \delta$ i.e. a contradiction to $\delta > 0$. Thus we have shown that there are sets in ε with arbitrarily small measure.

For any set A with $\mu(A) > 0$ we can find a set B with $0 < \mu(B) < \mu(A)$, and since T is ergodic there is a $t \geqslant 0$ such that $\mu(T^{-t}B \wedge A) > 0$. But $T^{-t}B \wedge A \subseteq A$ and $\mu(T^{-t}B \wedge A) \leqslant \mu(B) < \mu(A)$ hence (X, ε, μ) is atom free.

Lemma 4.73 If (X, ε, μ) is atom free, α any σ-algebra and H_1 the subspace defined by $L^2_\mu = H_1 \oplus L_\alpha$ then either $H_1 = \{0\}$ or H_1 is infinite dimensional.

Proof If $H_1 \neq \{0\}$ then there exists an $f \neq 0$ such that $f \in H_1$. Further if $F = \{x: f(x) \neq 0\}$ then $F \in \alpha$ and $\mu(F) > 0$. Moreover the space

$$L = \{g\chi_F: g \in L^2\}$$

is infinite dimensional. Let,

$$L_1 = \{g\chi_F: g \in L_\alpha\}$$

and L_o be defined by

$$L = L_o \oplus L_1$$

If $g \in L_o$, $h \in L_\alpha$ and $g = g_1 \chi_F$, $g_1 \in L^2_\mu$ we have

$$\begin{aligned}(g,h) &= (g_1 \chi_F, h) \\ &= (g_1 \chi_F, h \chi_F) \\ &= 0 \quad \text{since } g_1 \chi_F \in L_o,\ h \chi_F \in L_1\end{aligned}$$

Thus $L_o \subseteq H_1$. If L_1 has finite dimension then L_o and hence H_1 is infinite dimensional. If L_1 is infinite dimensional then there exist $\{h_i\}$, $i \in \Gamma^+$ such that h_i is bounded, $h_i \in L^2_\mu$ each i and $\{h_i \chi_F\}$ are linearly independent. Since $f(x) \neq 0$ for $x \in F$ we have that $\{h_i f\}$, $i \in \Gamma^+$ are linearly independent. Also if $h \in L_\alpha$ then

$$\begin{aligned}(h_i f, h\chi_F) &= (f, h_i h) \\ &= 0 \qquad\qquad \text{for all } i \in \Gamma^+\end{aligned}$$

because $h_i h \in L_\alpha$, $f \in H_1$.

Thus $h_i f \in L_o$, $i \in \Gamma^+$ and so L_o and therefore H_1 is infinite dimensional.

Theorem 4.74 If α is a K_1 algebra, $L_\infty = \bigcap_{t \in \Gamma^+} U^t L_\alpha$ then U has a Lebesgue spectrum in L^+ where L^+ is the subspace such that $L^2_\mu = L^+ \oplus L_\infty$, if $L^+ \neq \{0\}$. If T is ergodic, $L^+ \neq \{0\}$ then U has an infinite Lebesgue spectrum.

Proof $$U L_\infty = \bigcap_{t \in \Gamma^+} U^{t+1} L_\alpha \subseteq \bigcap_{t \in \Gamma^+} U^t L_\alpha = L_\infty$$

$$\bigcup_{t \in \Gamma^+} U^{-t} L_\infty = \bigcup_{t \in \Gamma^+} U^{-t} \bigcap_{s \in \Gamma^+} U^s L_\alpha = \bigcup_{t \in \Gamma} L_\alpha$$

Thus L_∞ is invariant and exhaustive, and so by lemma 4.71 U has a Lebesgue spectrum in L^+ if $L^+ \neq \{0\}$.

Since α is a K_1 algebra either $\alpha = \varepsilon$ giving $L^+ = \{0\}$ or

$T\alpha \neq \alpha$ giving (X, ε, μ) to be atom free by lemma 4·62. The result then follows from lemma 4·63.

Corollary If T is a Kolmogorov automorphism then $\mathcal{U}$ has an infinite Lebesgue spectrum in the orthogonal complement of the subspace of constant functions.

Proof There exists a K-algebra α, and for this α, $L_{\infty} = \bigcap_{t \in \Gamma^+} \mathcal{U}^t L_{\alpha}$ = subspace of constant functions. By theorems 4·41 and 4·42 T is ergodic and the result then follows from the theorem.

This last result was first indicated by A.N.Kolmogorov in [11].

5 MIXINGS WHICH ARE NOT KOLMOGOROV AUTOMORPHISMS

5.1 RIESZ PRODUCTS

We consider the Riesz product

$$\prod_{\nu\in\Gamma^+}(1 + \alpha_\nu \cos n_\nu x) \tag{1}$$

where $n_\nu \in \Gamma^+$, $n_{\nu+1}/n_\nu \geqslant q \geqslant 3$, $0 < |\alpha_\nu| \leqslant 1$ for $\nu \in \Gamma^+$.
If $\mu_k = \sum_{\nu=1}^{k} n_\nu$, $\mu'_k = n_{k+1} - \mu_k$ for $k \in \Gamma^+$ then $\mu_k < n_k q/(q-1)$, $\mu'_k > n_{k+1}(q-2)/(q-1)$ and so $\mu'_k/\mu_k \geqslant 1$. For $k \in \Gamma^+$ we put

$$p_k(x) = 1 + \sum_{\nu=1}^{k} \gamma_\nu \cos \nu x = \prod_{i=1}^{k} (1 + \alpha_i \cos n_i x) \tag{2}$$

where the γ_ν are chosen (uniquely) to satisfy the second equality for all x. Thus we have $\gamma_\nu = 0$ if ν is not of the form $n_{i_1} \pm n_{i_2} \pm \ldots$ with $k \geqslant i_1 > i_2 > \ldots$.
Now $p_{k+1}(x) = p_k(x)\ (1 + \alpha_{k+1} \cos n_{k+1} x)$
giving that the difference $p_{k+1} - p_k$ is a polynomial whose lowest term is of rank $\mu'_k > \mu_k$. Hence the passage from p_k to p_{k+1} consists in adding to p_k a group of terms whose ranks all exceed μ_k. Letting $k \to \infty$ in (2) we obtain the series

$$1 + \sum_{\nu\in\Gamma^+} \gamma_\nu \cos \nu x \tag{3}$$

in which $\gamma_\nu = 0$ if $n \neq n_i \pm n_{i_1} \pm n_{i_2} \pm \ldots$ with $i > i_1 > i_2 > \ldots$.
The partial sums $S_n(x)$ of (3) have the property that $S_{\mu_k}(x) = p_k(x) \geqslant 0$, $k \in \Gamma^+$. Moreover if

$$G(x) = \lim_{k\to\infty} \int_0^x p_k(t)\, dt$$

then it follows (see [22]) that (3) is the Fourier-Stieltjes series for $G(x)$ which is a non-decreasing continuous function.

If we formally multiply out (1) and replace the products of cosines by linear combinations of cosines then it is easy to show that no two terms are of the same rank.

Before continuing our discussion we need a lemma and the following notation.

$$A_{n_k} = \alpha_k \cos n_k x, \; k \in \Gamma^+; \quad a_{n_i} = \begin{cases} n^{-1} \text{if } 1 \leq i \leq n, \; n \in \Gamma^+ \\ 0 \quad \text{otherwise} \end{cases}$$

$$R_{mn} = \sum_{l=n}^{\infty} a_{m_l} \qquad \tau_m(x) = \sum_{k\in\Gamma^+} A_{n_k}(x) \; R_{mn_k}$$

Lemma 5·11 If $\sum_{k\in\Gamma^+} \alpha_k^2 = \infty$ then the set of points at which

$$\tau_m^+(x) = o\left\{ \sum_{k\in\Gamma^+} \alpha_k^2 \; R_{mn_k}^2 \right\}^{\frac{1}{2}} \tag{5}$$

where $\tau_m^+(x) = \max\{0, \tau_m(x)\}$ is of measure zero.

Proof If $\Gamma_m^2 = \sum_{k\in\Gamma^+} {}_k^2 \; R_{mn_k}^2$, $E = \{x: \; (1) \text{ holds}\}$ and $|A|$ denotes the measure of any measurable set in the real line then if $|E| > 0$ and d_1 is any number such that $d_1 > 0$, there exists a set $\mathcal{E} \subseteq E$ with $|\mathcal{E}| > |E|/2$ such that $\tau_m(x)/\Gamma_m \leq d_1$ in $\mathcal{E}$ for $m > m_o = m_o(d_1)$.

By omitting the first few terms of $\sum_{k\in\Gamma^+} A_{n_k}(x)$ we may without loss of generality and without changing $\mathcal{E}$ suppose n_1 as large as we please.

Then
$$\begin{aligned} \int_{\mathcal{E}} |\tau_m(x)| \; dx &\leq \int_{\mathcal{E}} \{ |\tau_m(x) - d_1\Gamma_m| + d_1 \Gamma_m \} dx \\ &= \int_{\mathcal{E}} \{ 2d_1\Gamma_m - \tau_m(x) \} \; dx \\ &= 2d_1\Gamma_m |\mathcal{E}| - \int_{\mathcal{E}} \tau_m(x) \; dx \end{aligned}$$

but
$$\begin{aligned} \int_{\mathcal{E}} \tau_m(x) \; dx &= \int_{\mathcal{E}} \left\{ \sum_{k\in\Gamma^+} \alpha_k \cos n_k x \; R_{mn_k} \right\} dx \\ &= \sum_{k\in\Gamma^+} \pi \, a_{n_k} \alpha_k \; R_{mn_k} \end{aligned}$$

where a_n is the n^{th} Fourier coefficient of the characteristic function of the set $\mathcal{E}$.

Thus $\int_{\mathcal{E}} |\tau_m(x)| \, dx \leqslant 2d_1 \Gamma_m |\mathcal{E}| - \pi \sum_{k \in \Gamma^+} a_{n_k} \alpha_k R_{mn_k}$

$$\leqslant 2d_1 \Gamma_m |\mathcal{E}| + \pi \Gamma_m \Big\{ \sum_{k \in \Gamma^+} a_{n_k}^2 \Big\}^{\frac{1}{2}}$$

by Holder's inequality. Hence if n_1 is sufficiently large

$$\int_{\mathcal{E}} |\tau_m(x)| \, dx \leqslant d_1 \Gamma_m (2|\mathcal{E}| + \pi)$$

i.e. $\quad {}_m(x) \, dx = o(\Gamma_m)$ (6)

But by Holder's inequality

$$\int_{\mathcal{E}} \tau_m^2 (x) \, dx \leqslant \Big\{ \int_{\mathcal{E}} |\tau_m(x)| \, dx \Big\}^{2/3} \Big\{ \int_{\mathcal{E}} \tau_m^4(x) \, dx \Big\}^{1/3}$$

and A. Zygmund [22] shows that

$$\Gamma_m^2 = O\Big(\int_{\mathcal{E}} \tau_m^2 (x) \, dx \Big) \text{ for } n_1 \text{ large enough}$$

and $\int_{\mathcal{E}} \tau_m^4 (x) dx = O \ (\Gamma_m^4)$

thus giving

$$\Gamma_m = O\Big(\int |\tau_m(x)| \Big) dx \qquad (7)$$

This is a contradiction to (6) and so we conclude that $|E| = 0$

We now return to our discussion of Riesz products.

Lemma 5·12 If $\Sigma \alpha_\nu^2 = \infty$ then the function G has a derivative 0 almost everywhere.

Proof The series (3) is almost everywhere summable (C,1) to sum $G'(x)$. (see [22] vol.1 P.105). Further the series has infinitely many gaps (μ_k, μ_k') and since $\mu_k'/\mu_k > 1$ we have (see [22] vol.1, P.79) that the partial products $p_k(x)$ converge to $G'(x)$ almost everywhere. But $1 + u \leqslant e^u$ and so

$$0 \leqslant p_k(x) \leqslant \exp\Big(\sum_{\nu=1}^{k} \alpha_\nu \cos n_\nu x \Big)$$

In lemma 5·11 for fixed k we have $\lim_{m \to \infty} R_{mn} = 1$ and so $\lim_{m \to \infty} \Gamma_m^2 = \infty$. Thus applying lemma we see that $\sum_{\nu=1}^{k} \alpha_\nu \cos n_\nu x$ takes arbitrarily large negative values, as $k \to \infty$, for almost all x.

Hence $\liminf_{k\to\infty} p_k(x) = 0$, i.e. $G'(x) = 0$ almost everywhere.

Remark   We have also proved that (1) converges to 0 almost everywhere.

5·2   A PARTICULAR PRODUCT

We consider the Riesz product:

$$\prod_{\nu\in\Gamma^+}(1+\cos n_\nu x) = 1+\sum_{\nu\in\Gamma^+} \gamma_\nu \cos\nu x = \sum_{\nu\in\Gamma} \delta_\nu \; e^{i\nu x}$$

where $n_k = 2^{2^k}$ and the δ_ν, $\nu\in\Gamma$ are chosen so that the last equality is satisfied. By the last section the series is the Fourier-Stieltjes series of an increasing, continuous and singular function $G(x)$, and that $0 \leq \delta_\nu \leq 1$ for $\nu\in\Gamma$. For any $N > 3$ there is a k such that $n_{k-1} \leq N < n_k$ and hence

$$\sum_{\nu=-N}^{N} \delta_\nu \leq \sum_{\nu=-\Gamma_k}^{\Gamma_k} \delta_\nu = \prod_{\nu=1}^{k} 2 = 2^k \leq 2^2 \log n.$$

giving the δ_ν small "on the average".

Consider the mapping

$$x = x(t) = \frac{1}{2}\left(t+\frac{t^2}{\pi} \text{ sign } t\right), -\pi \leq t \leq \pi$$

of the interval $[-\pi, \pi]$ onto itself.

If $F(x) = G(t)$ then since $1/2 \leq x'(t) \leq 3/2$ we have that $F(x)$ is increasing, continuous and singular. Moreover (see [22] vol 1.P.158)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\, dF(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx(t)}\, dG(t) = \sum_{\nu\in\Gamma} \lambda_{n,\nu}\, \delta_\nu$$

and the series is absolutely convergent since $e^{-inx(t)}$ has a derivative of bounded variation and so its Fourier coefficients

$$\lambda_{n,\nu} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(nx(t)+\nu t)}\, dt$$

are $O(\nu^{-2})$.

We now leave this product in order to prove three lemmas.

Lemma 5·21 If $f(t)$ is a real valued function for $a \leqslant t \leqslant b$, $f'(t)$ is monotone and there exists a $\lambda > 0$ such that either $f'(t) \geqslant \lambda$ or $f'(t) \leqslant -\lambda$ for $a \leqslant t \leqslant b$ then $\left|\int_a^b e^{2\pi i f(t)}\, dt\right| < \lambda^{-1}$

Proof $\int_a^b e^{2\pi i f(t)} dt = \frac{1}{2\pi i} \int_a^b \frac{1}{f'(t)} \, d\, e^{2\pi i f(t)}$

and by the second mean value theorem there exists c_1, c_2 in (a,b) such that

$$\int_a^b \frac{1}{f'(t)} d\cos(2\pi f(t)) = \frac{1}{f'(a)}\int_a^{c_1} d\cos(2\pi f(t)) + \frac{1}{f'(b)}\int_{c_1}^b d\cos(2\pi f(t))$$

$$\int_a^b \frac{1}{f'(t)} d\sin(2\pi f(t)) = \frac{1}{f'(a)}\int_a^{c_2} d\sin(2\pi f(t) + \frac{1}{f'(b)}\int_{c_2}^b d\sin(2\pi f(t))$$

giving $\left|\int_a^b e^{2\pi i f(t)} dt\right| \leqslant \frac{1}{2\pi}\left\{\left|\int_a^b \frac{1}{f'(t)} d\cos(2\pi f(t))\right| + \left|\int_a^b \frac{1}{f'(t)} d\sin(2\pi f(t)\right|\right\}$

$$\leqslant \frac{1}{2\pi} \cdot \frac{2}{\lambda} \cdot 2$$

$$< \frac{1}{\lambda}$$

Lemma 5·22 If $f(t)$ is a real valued function for $a \leqslant t \leqslant b$ and there exists a $\rho > 0$ such that $f''(t) \geqslant \rho$ or $f''(t) \leqslant -\rho$ for $a \leqslant t \leqslant b$ then

$$\left|\int_a^b e^{2\pi i\, f(t)} \, dt\right| \leqslant 4\rho^{-\frac{1}{2}}$$

Proof If $f''(t) \leqslant -\rho$ we consider $-f(t)$. Hence without loss of generality we take $f''(t) \geqslant \rho$. If $f'(t) \geqslant 0$ for $a \leqslant t \leqslant b$, and $a < \gamma < b$ then $f'(t) \geqslant (\gamma - a)\rho$ for $\gamma \leqslant t \leqslant b$

Hence $\left|\int_a^b e^{2\pi i f(t)} dt\right| \leqslant \left|\int_a^\gamma e^{2\pi i (t)} dt\right| + \left|\int_\gamma^b e^{2\pi i f(t)} dt\right|$

$$\leqslant \gamma - a + \frac{1}{(\gamma - a)\rho} \quad \text{by lemma 5·21}$$

But $\gamma - a + \frac{1}{(\gamma - a)\rho}$ has a minimum value when $\gamma = a + \rho^{-\frac{1}{2}}$ and so

$$\left|\int_a^b e^{2\pi i f(t)}dt\right| \leqslant \rho^{-\frac{1}{2}}+\rho^{-\frac{1}{2}} = 2\rho^{-\frac{1}{2}}$$

If $f'(t) \leqslant 0$ for $a \leqslant t \leqslant b$ then for $a < \gamma < b$, $f'(t) \leqslant -(b-\gamma)\rho$ for $a \leqslant t \leqslant \gamma$ and result follows.

If $f'(t)$ changes sign in $a \leqslant t \leqslant b$ then we have to consider the two intervals in which $f'(t)$ is of constant sign. Thus in general

$$\left|\int_a^b e^{2\pi i f(t)}dt\right| \leqslant 2\rho^{-\frac{1}{2}}+2\rho^{-\frac{1}{2}} = 4\rho^{-\frac{1}{2}}$$

Lemma 5·23 $|\lambda_{n,\nu}| \leqslant An^{-\frac{1}{2}}, \quad \nu \in \Gamma$

$$|\lambda_{n,\nu}| \leqslant A\nu^{-2}, \quad |\nu| \geqslant 3n$$

where A is a constant.

Proof If $f(t) = nx(t) + \nu t$ then

$$f'(t) = nx'(t) + \nu$$
$$= \frac{n}{2}\left(1+\frac{2t}{\pi}\text{ sign } t\right) + \nu$$
$$f''(t) = \frac{n}{\pi}\text{ sign } t$$

Thus for t in $(0,\pi)$, $f''(t) = n\pi^{-1} > 0$, and so by lemma 5·22

$$\left|\int_0^\pi e^{-if(t)}dt\right| \leqslant 4(2n)^{-\frac{1}{2}}$$

Similarly $\left|\int_{-\pi}^0 e^{-if(t)}dt\right| \leqslant 4(2n)^{-\frac{1}{2}}$

Hence $|\lambda_{n,\nu}| \leqslant \frac{1}{2\pi}8(2n)^{-\frac{1}{2}} \leqslant \frac{8}{\pi}n^{-\frac{1}{2}}$, $\nu \in \Gamma$ $f(t)$ is an odd function and so

$$\pi\lambda_{n,\nu} = \int_0^\pi \cos f\, dt$$
$$= \int_0^\pi \frac{1}{f'}d\sin f = \int_0^\pi \frac{\sin f}{(f')^2}.f''dt$$
$$= n\pi^{-1}\int_0^\pi \frac{\sin f}{f'^3} f'\, dt$$

Further $f'(t)$ is monotone for $0 \leqslant t \leqslant \pi$ and is of constant sign if $|\nu| \geqslant 3n/2$. For $|\nu| \geqslant 3n$ we have $|f'| \geqslant |\nu|/2$ and so by the second mean value theorem

$$|\lambda_{n,\nu}| \leq n\,\pi^{-2}\,\frac{2^3}{|\nu|^3}\left|\int_0^{\pi} \sin\ f\ \ f'\ dt\right|$$

$$\leq \frac{2\ 4\,\pi^{-2}}{\nu^2}\cdot 2$$

Thus if $A = 48/\pi^2$

$$|\lambda_{n,\nu}| \leq A\,n^{-\frac{1}{2}}, \quad \nu \in \Gamma$$

$$|\lambda_{n,\nu}| \leq A\,\nu^{-2}, \quad |\nu| \geq 3n$$

Returning to our particular Riesz product we put

$$\varphi(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}dF(x),\ n \in \Gamma$$

then
$$|\varphi(n)| \leq \sum_{\nu\in\Gamma} |\lambda_{n,\nu}|\,\delta_\nu$$
$$= \sum_{|\nu|\leq 3n} |\lambda_{n,\nu}|\delta_\nu + \sum_{|\nu|>3n} |\lambda_{n,\nu}|\delta_\nu$$
$$\leq An^{-\frac{1}{2}} \sum_{|\nu|\leq 3n}\delta_\nu + A \sum_{|\nu|>3n} \nu^{-2}\,\delta_\nu$$
$$\leq An^{-\frac{1}{2}} \sum_{|\nu|\leq 3n} 1 + A \sum_{|\nu|>3n} \nu^{-2}$$
$$= O(n^{-\frac{1}{2}}\log n) + O(n^{-1})$$
$$= O(n^{-\frac{1}{2}+d}) \quad \text{for every } d>0$$

5·3 STATIONARY GAUSSIAN PROCESSES

Throughout this section we **let** R_∞ be the infinite dimensional Euclidean space whose points are of the form $u = \{u_j\}$, $j \in \Gamma$ where for each i, u_i is a real number. Further for any finite set $J \subseteq \Gamma$ we let R_J denote the finite dimensional Euclidean space whose points are of the form $u = \{u_j\}$, $j \in J$. We say that a real valued, non-negative and countably additive set function μ_J such that $\mu_J(R_J) = 1$ is a Gaussian measure on R_J if there exists a positive definite quadratic form $Q_J(x)$ such that for all $x \in R_J$ we have

$$\int_{R_J} e^{i\sum_{j\in J} x_j u_j}\, d\mu_J(u) = e^{-Q_J(x)/2}$$

As usual by a positive definite quadratic form $Q_J(x)$ we mean that $Q_J(x)$ can be written as

$$Q_J(x) = \sum_{j,k\in J} a_{jk}\, x_j\, x_k$$

and that $Q_J(x) \geqslant 0$ for all $x \in R_J$.

If P_J is the transformation on R_∞ which sends $x = \{x_j\}$, $j \in \Gamma$ into $x = \{x_j\}$, $j \in J$, μ_∞ is a real valued, non-negative and countably additive set function on R_∞, μ_J is defined by

$$\mu_J(A) = \mu_\infty(P^{-1}A)$$

for all measurable sets A in R_J then we say that μ_∞ is a Gaussian measure if for every finite set $J \subseteq \Gamma$ we have that μ_J is a Gaussian measure. If S is the transformation on R_∞ which is given by

$$Sx = y$$

where if $x = \{x_j\}$, $j \in \Gamma$, $y = \{y_j\}$, $j \in \Gamma$ we have $y_j = x_{j-1}$, $j \in \Gamma$, and if S is measure preserving with respect to a Gaussian measure μ_∞ then we refer to μ_∞ as a stationary Gaussian measure. Lastly we say that S is a stationary Gaussian process if μ_∞ is a stationary Gaussian measure. If $\varphi(n)$, $n \in \Gamma$, $-n \notin \Gamma^+$ is a real valued function such that for all finite sets $J \subseteq \Gamma$ if we put

$$Q_J(x) = \sum_{j,k\in J} \varphi(|j-k|)\, x_j x_k$$

then we have that $Q_J(x)$ is a positive definite quadratic form, we say that $\varphi(n)$ is a positive definite function. We now quote some well known results.

Theorem 5·31 A function $\varphi(n)$, $n = 0,1,2,\ldots$ is positive definite if and only if there exists a monotone non-decreasing, real-valued function $F(x)$ defined on $[-\pi, \pi]$ and such that

$$\varphi(n) = \int_{-\pi}^{\pi} e^{-inx}\, dF(x), \quad n = 0,\ 1,\ 2,\ \ldots$$

Proof See [3] P. 474. This result has already been quoted in section 4·6 and is repeated here only for convenience.

For each $n \in \Gamma$ we define a function U_n' on R_∞ by

$$U_n'(U) = U_n$$

for all $u = \{u_j\}$, $j \in \Gamma$ in R_∞. Further in expressions such as $\int_{R_\infty} U_n'(u)\, U_m'(u)\, d\mu_\infty(u)$ we omit the $'$ and simply write $\int_R U_n U_m d\mu_\infty$ With this convention we get:

Theorem 5·32 If S is a stationary Gaussian process and we put $\varphi(n) = \int_{R_\infty} U_n U_o\, d\mu_\infty$, $n = 0, 1, 2, \ldots$ then if the second moments exist $\varphi(n)$ is a uniquely determined positive function. Conversely if $\varphi(n)$ is a positive definite function then there exists a unique stationary Gaussian process such that

$$\varphi(n) = \int_{R_\infty} U_n U_o d\mu_\infty \;, \; n = 0, 1, 2, \ldots$$

Proof See [3] P. 473. From now on we always assume that S is a stationary Gaussian process and $\varphi(n)$, $n = 0, 1, 2, \ldots$ is the associated positive definite function. μ_∞ will always be the stationary Gaussian measure associated with S, J will always be a finite subset of Γ, μ_J the measure formed from μ_∞, J as previously and $Q_J(x)$ will denote $\sum_{j,k \in J} \varphi(|j-k|) x_j x_k$. We have that

$$\int_{R_J} e^{\,i \sum_{j \in J} x_j u_j}\, d\mu_J(u) = e^{-Q_J(x)/2} \tag{1}$$

expanding the left hand side gives

$$\int_{R_J} \left\{ 1 + i \sum_{j \in J} x_j u_j + \frac{1}{2!}\left(i \sum_{j \in J} x_j u_j \right)^2 + \ldots \right\} d\mu_J$$

$$= 1 + i \sum_{j \in J} x_j \int_{R_J} u_j d\mu_J - \frac{1}{2} \sum_{j,k \in J} x_j x_k \int_{R_J} u_j x_k\, d\mu_J + \ldots$$

and expanding the right hand side of (1) gives

$$1 - \frac{Q}{2}(x) + \frac{1}{2!}\left(\frac{Q(x)}{2}\right)^2 - \ldots$$

$$= 1 - \frac{1}{2} \sum_{j,k\in J} \varphi(|j-k|)\, x_j x_k + \frac{1}{8}\left(\sum_{j,k\in J} \varphi(|j-k|)\, x_j x_k\right)^2 - \ldots$$

But (1) is an identity in x and so we may compare coefficients to obtain

$$\int_{R_J} \prod_{j=1}^{k} u_{n_j}\, d\mu_J = 0 \text{ if } k \text{ is odd}$$

$$\int_{R_J} u_j\, u_k\, d\mu_J = \varphi(|j-k|)$$

$$\int_{R_J} \prod_{j=1}^{k} u_{n_j}\, d\mu_J = \sum_{p=1}^{P} \prod_{q=1}^{Q} \varphi(n_{pq}) \text{ if } k \text{ is even}$$

where $Q = k/2$, $P = (2Q)!/2^Q Q!$, $n_{pq} = n(p,q)$.

Hence we deduce that

$$\int_{R_\infty} \prod_{j=1}^{k} u_{n_j}\, d\mu_\infty = 0 \text{ if } k \text{ is odd}$$

$$\int_{R_\infty} u_j u_k\, d\mu_\infty = \varphi(|j-k|)$$

$$\int_{R_\infty} \prod_{j=1}^{k} u_{n_j}\, d\mu_\infty = \sum_{p=1}^{P} \prod_{q=1}^{Q} \varphi(n_{pq}) \text{ if } k \text{ is even}$$

since given any n_j, $1 \leq j \leq k$ we can find a J such that $n_j \in J$ for each j.

Theorem 5·33 If S is a mixing of degree one then $\lim_{n\to\infty} \varphi(n) = 0$

Proof

$$\begin{aligned}\lim_{n\to\infty} \varphi(n) &= \lim_{n\to\infty} \int_{R_\infty} u_n\, u_0\, d\mu_\infty \\ &= \lim_{n\to\infty} \int_{R_\infty} (U^n u_0)\, u_0\, d\mu_\infty \\ &= \lim_{n\to\infty} (U^n u_0, u_0) \\ &= \left\{\int_{R_\infty} u_0\, d\mu_\infty\right\}^2 \\ &= 0\end{aligned}$$

Theorem 5·34 If $\lim_{n\to\infty} \varphi(n) = 0$ then S is a mixing of all degrees.

Proof Let L_1 be the subset of L^2_ρ which consists of all functions of the form

$$\prod_{j=1}^{i} x'_{n_j}(x) = \prod_{j=1}^{i} x_{n_j}, \quad i \text{ finite}$$

If $f_l = \prod_{j=1}^{i_l} x_{l_j}$, $1 \leqslant l \leqslant k$ are any k functions in L_1 we put

$$F = F(t_1, \ldots, t_k) = \int_{R_\infty} \prod_{l=1}^{k} S^{t_l} f_l \, d\rho_\infty$$

$$G = \prod_{l=1}^{k} \int_{R_\infty} f_l \, d\rho_\infty$$

$$I = \sum_{l=1}^{k} i_l$$

If I is odd then $F = 0$ by the remarks immediately prior to theorem 5·33. However, I odd implies i_l odd for some l and consequently $\int_{R_\infty} f_l \, d\rho_\infty = 0$ for this l giving $G = 0$ and hence $F = G$, for all $t_1, \ldots, t_k$. If I is even then by the earlier part of this section we can write

$$F = \sum_{p=1}^{P} \prod_{q=1}^{Q} \varphi(n_{pq})$$

where $Q = I/2$ and $P = \dfrac{(2Q)!}{2^Q \cdot Q!}$

If i_l is odd for some l say $l = m$ then as above $G = 0$. Further we have that for each p, $1 \leqslant p \leqslant P$ there is a $q = q(p)$ such that

$$n_{pq(p)} = |r_j + t_r - (m_n + t_m)|$$

where $1 \leqslant r \leqslant k$, $r \neq m$

Thus if $\Delta = \inf\limits_{l \neq n} |t_l - t_n|$ we have that

$$\lim_{\Delta \to \infty} \varphi(n_{pq(p)}) = 0$$

and so for each p, $1 \leqslant p \leqslant P$

$$\lim_{\Delta \to \infty} \prod_{q=1}^{Q} \varphi(n_{pq}) = 0$$

giving us that

$$\lim_{\Delta \to \infty} F = 0 = G$$

If i_ℓ is even for all ℓ, and $r = t(p,q,)$, $r_j = t_j(p,q)$ $s = s(p,q)$, $s_n = s_n\ (p,q)$ are defined by

$$n_{pq} = |r_j + t_r - (s_n + t_s)|$$

then we have

$$\lim_{\Delta\to\infty} \varphi(n_{pq}) = 0 \text{ if } r \neq s$$

while if $r = s$ we have

$$\varphi(n_{pq}) = \varphi\ (|r_j - r_n\ )$$

and so

$$\lim_{\Delta\to\infty} \varphi(n_{pq}) = \varphi\ (\ r_j - r_n\ )$$

Hence $\lim\limits_{\Delta\to\infty} F = \prod\limits_{\ell=1}^{k} \sum\limits_{p=1}^{P_\ell} \prod\limits_{q=1}^{Q_\ell} \varphi(|\ell_j - \ell_n|)$

where $j = j(p,q)$, $n = n(p,q)$, $Q_\ell = i_\ell/2$ and, $P_\ell = (2Q_\ell)!/2^{Q_\ell} Q_\ell !$

giving $\lim\limits_{\Delta\to\infty} F = \prod\limits_{\ell=1}^{k} \int_{R_\infty} f_\ell \quad d\mu_\infty = G.$

Thus in all cases we have

$$\lim_{\Delta\to\infty} F = G$$

and so by theorem 4·21 corollary 1, we have that S is a mixing of all degrees.

Theorem 5·35 If S is a mixing of degree 1 then it is a mixing of all degrees.

Proof If S is a mixing of degree 1 then by theorem 5·33 $\lim\limits_{n\to\infty} \varphi(n) = 0$ and so by theorem 5·34 we have that S is a mixing of all degrees.

5·4 SPECTRAL PROPERTIES OF STATIONARY GAUSSIAN PROCESSES

Keeping the notation of the previous section we now turn to the spectral properties of S. As usual if $f \in L^2_\mu$ then we refer to the subspace spanned by $S^n f$, $n \in \Gamma$ as the cyclic subspace generated by f, and to the set function G defined on $[-\pi,\pi]$ and such that $(S^k f,f) = \int_{-\pi}^{\pi} e^{i\lambda k} dG(\lambda)$ as the spectral type of f. (cf. section 4·6). Further since the set of all finite polynomials in a finite number of the x_n''s is everywhere dense in L^2_μ it follows that the sum of all cyclic subspaces generated by a polynomial of the above form will cover L^2_μ. However, in order to find the maximal spectral type of S we need to express L^2_μ as an orthogonal sum of cyclic spaces (see section 4·6). We now proceed to find an orthogonal sequence of polynomials such that the cyclic subspaces generated by them are mutually orthogonal and their orthogonal sum equals L^2_μ. Before starting we prove a lemma which we shall need shortly.

Lemma 5·41 If F, G are integrable functions on $[-\pi,\pi]$ and defined elsewhere so as to be periodic with period 2π then

$$\left\{\int_{-\pi}^{\pi} e^{i\lambda k}\, dF(\lambda)\right\}\left\{\int_{-\pi}^{\pi} e^{iuk}\, dG(u)\right\} = \int_{-\pi}^{\pi} e^{i\lambda k}\, dH(\lambda)$$

where H is the convolution of F with G.

Proof
$$\begin{aligned}
&\left\{\int_{-\pi}^{\pi} e^{i\lambda k} dF(\lambda)\right\}\left\{\int_{-\pi}^{\pi} e^{-iuk} dG(u)\right\} \\
&= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(\lambda-u)k} dF(\lambda)\, dG(u) \\
&= \left[F(\lambda)\int_{-\pi}^{\pi} e^{i(\lambda-u)k} dG(u)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} ike^{i(\lambda-u)k} F(\lambda)\, dG(u)\, d\lambda \\
&= -\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} i\, k\, e^{ivk}\, F(\lambda)\, dG(\lambda - v)\, dv \\
&= \left[-e^{ivk}\int_{-\pi}^{\pi} F(\lambda)\, dG(\lambda-v)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} e^{ivk} \frac{d}{dv}\left\{\int_{-\pi}^{\pi} F(\lambda)\, dG(\lambda-v)\right\} \\
&= \int_{-\pi}^{\pi} e^{ivk}\, dH(\lambda)
\end{aligned}$$

where
$$\begin{aligned}
H(\lambda) &= \int_{-\pi}^{\pi} F(\lambda - u)\, dG(u) \\
&= \int_{-\pi}^{\pi} F(v)\, d\, G(\lambda - v)
\end{aligned}$$

Let $h_1 = x_0'$ and H_1 be the cyclic subspace generated by h_1. Then

$$\begin{aligned}(S^k h_1, h_1) &= \int_{R_\infty} x_k\, x_o\, d\mu \\ &= \varphi(k) \\ &= \int_{-\pi}^{\pi} e^{i\lambda k}\, dF(\lambda)\end{aligned}$$

showing that F is the spectral type of h_1.

Also $\int_{R_\infty} h_1(x)\, d\mu(x) = \int_{R_\infty} x_o\, d\mu = 0$

giving us that $H_1 \perp H_o$, where H_o is the cyclic subspace generated by the constant functions.

Before continuing, we pause to point out that since F is singular we have already by lemma 4·68 that S does not have a Lebesgue spectrum.

Returning to our investigation of the spectral type of S we have that if $h_2^{(o)} = (x_o')^2 - 1$, $H_2^{(o)}$ the cyclic subspace generated by $h_2^{(o)}$ then

$$\begin{aligned}\int_{R_\infty} h_2^{(o)}(x)\, d\mu(x) &= \int_{R_\infty} (x_o^2 - 1)\, d\mu \\ &= 0\end{aligned}$$

and $\int_{R_\infty} h_2^{(o)}(x) S^k h_1(x) d\mu(x) = \int_{R_\infty} (x_o^2\ x_k - x_k)\, d\mu$
$= 0$

giving us that $H_2^{(o)}$ is orthogonal to $H_o \oplus H_1$

$$\begin{aligned}(S^k h_2^{(o)}, h_2^{o}) &= \int_{R_\infty} (x_k^2\, x_o^2 - x_k^2 - x_o^2 + 1)\, d\mu \\ &= 2\{\varphi(k)\}^2 + \{\varphi(0)\}^2 - 2\varphi(0) + 1 \\ &= 2\{\varphi(k)\}^2 \\ &= 2\int_{-\pi}^{\pi} e^{i\lambda k}\, dF_2(\lambda)\end{aligned}$$

where $F_2(\lambda) = \int_{-\pi}^{\pi} F(\lambda - u)\, dF(u)$ by lemma 5·41.

Thus we see that F_2 is the spectral type of $h_2^{(o)}$.

Similarly if we let $h_3^{(o)} = (x_o')^3 - x_o'$, $H_3^{(o)}$ be the cyclic subspace generated by $h_3^{(o)}$ then we get $H_3^{(o)} \perp H_o \oplus H_1 \oplus H_2^{(o)}$ and

$$(S^k h_3^{(o)}, h_3^{(o)}) = 6\int_{-\pi}^{\pi} e^{i\lambda k} dF_3(\lambda)$$

where $F_3(\lambda) = \int_{-\pi}^{\pi} F_2(\lambda - u)\, dF(u)$. Thus F_3 is the spectral type of $h_3^{(o)}$.

In general we take

$$h_{2r}^{(o)} = (x_o')^{2r} + \alpha_1 (x_o)^{2(r-1)} + \cdots + \alpha_r$$

where the α_i, $1 \leqslant i \leqslant r$ are chosen so that

$$(h_{2r}^{(o)}, h_n^{(o)}) = 0 \quad \text{for} \quad 2 \leqslant n \leqslant 2r - 1$$

$$(h_{2r}^{(o)}, h_1) = 0$$

and $\int_{R_\infty} h_{2r}^{(o)}(x)\, d\mu(x) = 0$

Although we appear to have $2r$ conditions these reduce to r in view of 5·3

Similarly we take

$$h_{2r-1}^{(o)} = (x_o')^{2r-1} + \beta_1 (x_o)^{2(r-1)-1} + \cdots + \beta_{r-1} x_o'$$

where the α_i, $1 \leqslant i \leqslant r-1$ are chosen so that

$$(h_{2r-1}^{(o)}, h_n^{(o)}) = 0 \quad \text{for } 2 \leqslant n \leqslant 2r-2$$

$$(h_{2r-1}^{(o)}, h_1) = 0$$

and $\int_{R_\infty} h_{2r-1}^{(o)}(x)\, d\mu(x) = 0$

Again the same remarks as applied to the conditions the α_i's. satisfied apply here also. We then let $H_n^{(o)}$ denote the cyclic subspace generated by $h_n^{(o)}$ for $n = 2,3,4,\ldots$. Clearly we have $H_n^{(o)} \perp H_o \oplus H_1$ for all n and $H_n^{(o)} \perp H_m^{(o)}$ if $n \neq m$. Further we have

$$(S^k h_n^{(o)}, h_n^{(o)}) = p(\varphi(k))$$

where p is a polynomial of degree n and so by lemma 5·41 we have that the spectral type of $h_n^{(o)}$ is absolutely continuous with respect to $\sum_{i=1}^{n} F_i(\lambda)$ where $F_1(\lambda) = F(\lambda)$ and $F_i(\lambda) = \int_{-\pi}^{\pi} F_{i-1}(\lambda-u)dF(u)$ for $2 \leqslant i \leqslant n$.

In general $L^2_{\Gamma} \neq H_o \oplus H_1 \oplus \bigoplus_{n=2}^{\infty} H_n^{(o)}$ and so we consider $x'_o x'_n - \varphi(n)$, $n \in \Gamma^+$ and define $h_n^{(1)}$ to be the projection of $x'_o x'_n - \varphi(n)$ on the orthogonal complement of $H_o \oplus H_1 \oplus \bigoplus_{m=2} H_m^{(o)} \oplus \bigoplus_{i=o}^{n-1} H_i^{(1)}$ and $H_n^{(1)}$ to be the cyclic subspace spanned by $h_n^{(1)}$. Further

$$\begin{aligned}(S^k(x'_o x'_n - \varphi(n)),\ (x'_o x'_n - \varphi(n))) &= \int_{R_\infty} (x_k x_{n+k} x_o x_n - x_k x_{n+k}\varphi(n) - \\ &\qquad x_o x_n \varphi(n) + \{\varphi(n)\}^2)\ d\mu \\ &= \{\varphi(n)\}^2 + \{\varphi(k)\}^2 + \varphi(n+k)\ \varphi(n-k) \\ &\quad - \{\varphi(n)\}^2 - \{\varphi(n)\}^2 + \{\varphi(n)\}^2 \\ &= \{\varphi(k)\}^2 + \varphi(n+k)\ \varphi(n-k)\end{aligned}$$

and
$$\varphi(n+k)\ \varphi(n-k) = \left\{\int_{-\pi}^{\pi} e^{i\lambda(n+k)} dF(\lambda)\right\}\left\{\int_{-\pi}^{\pi} e^{iu(n-k)}\ dF(u)\right\}$$

$$= \int_{-\pi}^{\pi} e^{i\lambda k}\ dG(\lambda)$$

where $G(\lambda) = \int_{-\pi}^{\pi} e^{i(\lambda - u)n}\ F(\lambda - u)\ d\{e^{iun} F(u)\}$ by lemma 5·41. Thus we see that

$$(U^k h_n^{(1)}, h_n^{(1)}) = \int_{-\pi}^{\pi} e^{i\lambda k}\ d\Phi_2^n(\lambda)$$

where $\Phi_2^n(\lambda)$ is absolutely continuous with respect to $F_2(\lambda)$.

Again

$$\begin{aligned}(S^k(x'_o x'_n x'_m),\ x'_o x'_n x'_m) &= \int_{R_\infty} x_k\ x_{n+k}\ x_{m+k}\ \ x_o\ x_n\ x_m\ d\mu \\ &= \varphi(n)\{\varphi(m+k)\varphi(n-m) + \varphi(m+k-n)\ \varphi(m) \\ &\qquad + \varphi(k)\ \varphi(n)\} \\ &\quad + \varphi(m)\{\varphi(n+k)\varphi(n-m) + \varphi(k)\ \varphi(m) \\ &\qquad + \varphi(n+k-m)\ \varphi(n)\} \\ &\quad + \varphi(k)\{\{\varphi(n-m)\}^2 + \{\varphi(k)\}^2 \\ &\qquad + \varphi(n+k-m)\varphi(m+k-n)\}\end{aligned}$$

$$+\varphi(n-k)\{\varphi(n-m)\,\varphi(m)+\varphi(n+k)\,\varphi(k)+\varphi(n+k-m)\varphi(m+k)\}$$
$$+\varphi(m-k)\{\varphi(n-m)\,\varphi(n)+\varphi(n+k)\,\varphi(m+k-n)+\varphi(k)\varphi(m+k)\}$$

Thus if $h_n^{(2)}$ is the projection of the n^{th} term of the sequence

$$x_0'x_0'x_0',\ x_0'x_1'x_0',\ x_0'x_1'x_1',\ x_0'x_2'x_0',\ x_0'x_2'x_1',x_0'x_2'x_2',\ x_0'x_3'x_0',\ x_0'x_3'x_1',\ \ldots$$

on $\quad H_0 \oplus H_1 \oplus \bigoplus_{m=2}^{\infty} H_m^{(0)} \oplus \bigoplus_{\ell=0}^{\infty} H_\ell^{(1)} \oplus \bigoplus_{i=1}^{n-1} H_i^{(2)}$

and if $\Phi_3^{(n)}$ is the spectral type of $h_n^{(2)}$ we have that

$\Phi_3^{(n)} \leq F+F_2+F_3$ since for all $n,m\in\Gamma^+$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} F_m(\lambda-u)\ dF_n(u) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_{m-1}(\lambda-u-v_1)\ dF(v_1)\ dF_n(u)\\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} F(\lambda-u-\sum_{i=1}^{m} v_i)\ dF(v_m)\ldots dF(v_1)dF_n(u)\\ &= \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_n(\lambda-u-\sum_{i=1}^{m} v_i)\ dF(u)\ dF(v_m)\ \ldots dF(v_1)\\ &= \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(\lambda-u-\sum_{i=1}^{n+m} v_i)\ dF(u)\ dF(v_{n+m})\ldots dF(v_1)\\ &= F_{n+m+1}(\lambda)\end{aligned}$$

Thus if a_i, $i\in\Gamma^+$ satisfy $a_i>0$ each i, $\sum_{i\in\Gamma^+} a_i<\infty$ and $\psi_1(\lambda) = \sum_{j\in\Gamma^+} a_j\, F_j(\lambda)$ then by the method used earlier in this section it is easy to show that the spectral type of all polynomials in a finite number of the x_j', $j\in\Gamma$ is absolutely continuous with respect to $\psi_1(\lambda)$. Thus if we order these polynomials and consider their projections on the orthogonal complement of the $H_j^{(i)}$'s already defined we can express L_Γ^2 as the orthogonal sum of cyclic subspaces whose spectral types are all absolutely continuous with respect to $\psi_1(\lambda)$. Hence if $F_o(\lambda)$ is the spectral type of H_o, $a_o>0$ we have that $\psi(\lambda) = a_oF_o(\lambda)+\psi_1(\lambda)$ is the maximal spectral type of L_Γ^2 since $F_n(\lambda)$ is the spectral type of $H_n^{(o)}$ for $n\geq 2$ and $F_1(\lambda) = F(\lambda)$ is the spectral type of H_1.

5·5 MIXING AND REGULARITY PROPERTIES OF AN AUTOMORPHISM

We have previously defined Kolmogorov and regular automorphisms and the concept of mixing of all degrees. In section 4·5 we showed that an automorphism on a finite measure space is regular if and only if it is a Kolmogorov automorphism and in section 4·4 we showed that this condition implied mixing of all degrees. The aim of this section is to show that the converse is not true, i.e. that there exist automorphisms on finite measure spaces which are mixing of all degrees but which are not Kolmogorov automorphisms.

Theorem 5·51 There exist automorphisms of finite measure spaces which are mixing of all degrees but which are not Kolmogorov automorphisms.

Proof We consider the $F(x)$ and $\varphi(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\, dF(x)$. of section 5·2. Since $F(x)$ is monotone increasing we have by 5.31 that $\varphi(n)$ is a positive definite function and so by 5·32 there is a stationary Gaussian process S on R_∞ such that $\varphi(n)$ is the associated positive definite function. Now it was proved in 5·2 that $\varphi(n) = 0\,(n^{-\frac{1}{2}+d})$ for every $d > 0$ and so we have $\lim_{n\to\infty} \varphi(n) = 0$. Theorem 5·34 now gives us that S is a mixing of all degrees. Further using the notation and results of section 5·4 we see that the maximal spectral type of S is $\sum_{n=0}^{\infty} a_n F_n(\lambda)$ where $F_0(\lambda)$ concentrates on the eigenvalue corresponding to the constants, $F_1(\lambda)$ is singular and the $F_n(\lambda)$ for $n \geqslant 2$ are absolutely continuous. Hence we see that S cannot have a Lebesgue spectrum in the space orthogonal to the constant functions and so by 4·7 S is not a Kolmogorov-automorphism.

Corollary There exist automorphisms of finite measure spaces which are mixing of all degrees, but which are not regular.

Proof This is an immediate consequence of 4·4 and the theorem.

With the notation of 5·4 we have

$$\int_{-\pi}^{\pi} e^{in\lambda} F(\lambda)\,d\lambda = \left[\frac{e^{in\lambda}}{in} F(\lambda)\right]_{-\pi}^{\pi} + \frac{i}{n}\int_{-\pi}^{\pi} e^{in\lambda}\, dF(\lambda)$$

$$= \frac{i\,\varphi(n)}{n}$$

and so

$$F_2(\lambda) = \int_{-\pi}^{\pi} \sum_{n\in\Gamma} \frac{i\,\varphi(|n|)}{n} e^{-in(\lambda-u)} dF(u)$$

$$= i \sum_{n\in\Gamma} \frac{\varphi(|n|)}{n} e^{-in\lambda} \int_{-\pi}^{\pi} e^{inu}\, dF(u)$$

$$= i \sum_{n\in\Gamma} \frac{\{\varphi(|n|)\}^2}{n} e^{-in\lambda}$$

$$= \int \sum_{n\in\Gamma} \{\varphi(|n|)\}^2 e^{-in\lambda}\, d\lambda$$

Hence if $\varphi(n) = 0\,(n^{-\frac{1}{2}+d})$ for every $d > 0$ as $n \to \infty$ then $\{\varphi(|n|)\}^2 = 0\,(n^{-1+d})$ for every $d > 0$ as $n \to \infty$ and so we see that $F_2(\lambda)$ is the integral of a function in L^2_μ and is therefore absolutely continuous. Similarly $F_n(\lambda)$, $n \geqslant 2$ is absolutely continuous.

6 σ-FINITE MEASURE SPACES

6·1 INTRODUCTION

In this chapter we no longer require μ to satisfy $\mu(X) = 1$. Instead we assume that μ is a σ-finite, i.e. that there exist sets A_i, $i \in \Gamma^+$ such that for all i, $\mu(A_i) < \infty$ and such that $\bigcup_{i\in\Gamma^+} A_i = X$.

We say that a set A is a wandering set if for all $i, j \in \Gamma$ such that $i \neq j$ we have $T^i A \cap T^j A = \phi$. Clearly this last condition is equivalent to $A \cap T^{-i} A = \phi$, $i \in \Gamma^+$.

Proposition 6·11 If there are no wandering sets of positive measure then for all $A \in \varepsilon$ we have $\mu(TB) = \mu(B)$ where $B = \bigcup_{i\in\Gamma^+} T^{-i} A$

Proof If $C = TB - B$ then for $n \in \Gamma^+$

$$\begin{aligned} C \cap T^{-n} C &= \{ \bigcup_{i=0}^{\infty} T^{-i}A - \bigcup_{i\in\Gamma^+} T^{-i}A \} \cap \{ \bigcup_{i=n}^{\infty} T^{-i}A - \bigcup_{i=n+1}^{\infty} T^{-i}A \} \\ &= \{ A - \bigcup_{i\in\Gamma^+} T^{-i}A \} \cap \{ T^{-n} A - \bigcup_{i=n+1}^{\infty} T^{-i}A \} \\ &\subseteq \{ A - \bigcup_{i\in\Gamma^+} T^{-i}A \} \cap T^{-n}A \\ &= \phi \quad \text{since } T^{-n} A \subseteq \bigcup_{i\in\Gamma^+} T^{-i} A \end{aligned}$$

Thus C is a wandering set and so we must have $\mu(C) = 0$ and therefore $\mu(TB) = \mu(B)$ since $B \subseteq TB$.

Proposition 6·12 If A is any set such that $\mu(A) > 0$ and we put $A_i = \{ x: x \in A, T^i x \in A, T^j x \notin A, 1 \leq j \leq i-1 \}$ for $i \in \Gamma^+$ then if there are no wandering sets of positive measure, $\mu(A - \bigcup_{i\in\Gamma^+} A_i) = 0$.

Proof If $B = A - \bigcup_{i\in\Gamma^+} A_i$ then for $n \in \Gamma^+$

$$\begin{aligned} B \cap T^{-n} B &= \{ A - \bigcup_{i\in\Gamma^+} A_i \} \cap \{ T^{-n} A - \bigcup_{i\in\Gamma^+} T^{-n} A_i \} \\ &\subseteq \{ A - \bigcup_{i\in\Gamma^+} A_i \} \cap T^{-n} A \end{aligned}$$

$= \phi$

since for all i $A_i = A \cap T^{-i} A - \bigcup_{j=1}^{i-1} A_j$ giving $A \cap T^{-i} A = \bigcup_{j=1}^{i} A_j$ and hence $T^{-n} A \subseteq \bigcup_{i \in \Gamma^+} A_i$ for all $n \in \Gamma^+$. But there are no wandering sets of positive measure and so $\mu(B) = 0$ as required.

When dealing with σ-finite measure spaces, with no wandering sets of positive measure we keep the same definitions of invariant exhaustive and K_1-algebras, but we redefine a K-algebra to be a K_1-algebra α, such that $\alpha_\infty = \nu$ and such that $0 < \mu(A) < \infty$ for at least one $A \in \alpha$. Clearly this coincides with our previous definition if $\mu(X) < \infty$. If α is a K-algebra and $A \in \alpha$ satisfies $0 < \mu(A) < \infty$ then we let $A_i = T^{-i} A$, $i \in \Gamma^+$. By proposition 6·11 we have that $T \bigcup_{i \in \Gamma^+} A_i = \bigcup_{i \in \Gamma^+} A_i$ and since $\bigcup_{i \in \Gamma^+} A_i \in \alpha$ we deduce that $\bigcup_{i \in \Gamma^+} A_i \in T^j \alpha$, $j \in \Gamma$. Hence $\bigcup_{i \in \Gamma^+} A_i \in \bigwedge_{j \in \Gamma} T^j \alpha = \nu$ giving us that $\bigcup_{i \in \Gamma^+} A_i = X$ up to a set of measure zero since $0 < \mu(A)$ implies $\mu(\bigcup_{i \in \Gamma^+} A_i) \neq 0$.

6·2 INDUCED AUTOMORPHISMS

If A is any set such that $\mu(A) > 0$, then we put

$$\mathcal{E}_A = \{ B \; ; \; B \in \mathcal{E}, \; B \subseteq A \}$$

and we define a measure μ_A on $(X, \mathcal{E}_A)$ by putting

$$\mu_A(B) = \mu(B) \quad \text{for} \quad B \in \mathcal{E}_A$$

and we define S_A, by

$$S_A(x) = \{ T^i x \; : \; T^i x \in A, \; T^j x \notin A, \; 1 \leq j \leq i-1 \} \quad \text{for } x \in A.$$

Clearly by proposition 6·12 S_A is an automorphism and it is measure preserving since T is. We refer to S_A as the automorphism induced on $(A, \mathcal{E}_A)$ by T. Lastly if α is any σ-algebra we put

$\alpha_\Lambda = \{ B : \text{there exists a } C \in \alpha \text{ such that } B = \Lambda \cap C \}$

Clearly α_Λ is a σ-algebra of $(\Lambda, \mathcal{E}_\Lambda)$

Proposition 6·21 If α is an invariant σ-algebra with respect to T and $\Lambda \in \alpha$ is such that $\mu(\Lambda) > 0$ then α_Λ is an invariant σ-algebra with respect to S_Λ if there are no wandering sets of positive measure in (X, ε, μ, T)

Proof For any $B \in \alpha_\Lambda$, (and hence to α) we put $B_k = T^k\Lambda \cap B - \bigcup_{j=1}^{k-1} B_j$, $k \in \Gamma^+$ then $B_k \subseteq B$ and $S_\Lambda^{-1} B_k = T^{-k} B_k$ for all k.
If $C = B - \bigcup_{k\in\Gamma^+} B_k$ then for $n \in \Gamma^+$ we have

$$C \cap T^n C = \{ B - \bigcup_{k\in\Gamma^+} B_k \} \cap \{ T^n B - \bigcup_{k\in\Gamma^+} T^n B_k \}$$

$$\subseteq \{ B - \bigcup_{k\in\Gamma^+} B_k \} \cap T^n B$$

Now $x \in B \cap T^n B$ implies $T^{-n} x \in B \subseteq \Lambda$ i.e. $x \in T^n \Lambda$ but $x \in B$ and so $x \in T^n \Lambda \cap B$ giving $x \in \bigcup_{j=1}^{n} B_j$. Thus $C \cap T^n C = \phi$ and hence $C \cap T^{-n} C = \phi$ for all $n \in \Gamma^+$ and therefore C is a wandering set and so we have $\mu(C) = 0$. Further $B_1 = T\Lambda \cap B \in T\alpha$ and by induction we get $B_k \in T^k \alpha$ for $k \in \Gamma^+$. Hence neglecting a set of measure zero we have

$$\begin{aligned} B &= \bigcup_{k\in\Gamma^+} B_k \\ &= S_\Lambda S_\Lambda^{-1} \bigcup_{k\in\Gamma^+} B_k \\ &= S_\Lambda \bigcup_{k\in\Gamma^+} T^{-k} B_k \end{aligned}$$

But $T^{-k} B_k \in \alpha$, and $T^{-k} B_k \subseteq \Lambda$ giving $T^{-k} B_k \in \alpha_\Lambda$ for all $k \in \Gamma^+$ and hence $\bigcup_{k\in\Gamma^+} T^{-k} B_k \in \alpha_\Lambda$. Thus $B \in S_\Lambda \alpha_\Lambda$ and so we deduce that $\alpha_\Lambda \leq S_\Lambda \alpha_\Lambda$ i.e. α_Λ is invariant.

Corollary 1 $\alpha_\Lambda \leq (T\alpha)_\Lambda \leq S_\Lambda \alpha_\Lambda$

Proof Since $\alpha \leq T\alpha$ we have immediately that $\alpha_\Lambda \leq (T\alpha)_\Lambda$. If $B \in (T\alpha)_\Lambda$ then since $\Lambda \in \alpha \leq T\alpha$ we have $B \in T\alpha$. The proof of the proposition remains valid for this B and so we get $B \in S_\Lambda \alpha_\Lambda$ giving $(T\alpha)_\Lambda \leq S_\Lambda \alpha_\Lambda$

Corollary 2 If α is a K_1-algebra with respect to T then α_Λ is a K_1-algebra with respect to S_Λ.

Proof $\varepsilon = \alpha_T$ and so

$$\begin{aligned} \varepsilon_\Lambda &= (\alpha_T)_\Lambda \\ &\leq (\alpha_\Lambda)_{S_\Lambda} \quad \text{by corollary 1} \\ &\leq \varepsilon_\Lambda \end{aligned}$$

giving $(\alpha_\Lambda)_{S_\Lambda} = \varepsilon_\Lambda$

i.e. α_Λ is exhaustive. But α_Λ is invariant by the proposition and so we have that α_Λ is a K_1-algebra with respect to S_Λ.

Proposition 6.22 If α is a K_1-algebra, $\Lambda \in \alpha$, $\infty > \mu(\Lambda) > 0$ and $B \in \varepsilon_\Lambda$ is such that $S_\Lambda B = B$ then $B \in \alpha_\Lambda$

Proof α_Λ is a K_1-algebra and so for each $k \in \Gamma^+$ there exists an n_k and a $B_k \in S_\Lambda^{nk}\ \alpha_\Lambda$ such that

$$\mu(B \triangle B_k) < 2^{-k}$$

But $B = S_\Lambda B$ and so

$$\begin{aligned} \mu(B \triangle S_\Lambda^{-nk} B_k) &= \mu\{ S^{-nk}(B \triangle B_k)\} \\ &= \mu(B \triangle B_k) \\ &< 2^{-k} \end{aligned}$$

If $C_k = S^{-nk}B_k$, $k \in \Gamma^+$ then $C_k \in \alpha_\Lambda$ for each k.

Now $\mu(B - \bigcup_{k=n}^{\infty} C_k) \leq \mu(B - C_m)$ for $n \leq m$

$$\leq \mu(B \triangle C_m)$$

$$< 2^{-m}$$

Thus we get $\mu(B - \bigcup_{k=n}^{\infty} C_k) = 0$ and so $B \subseteq \bigcup_{k=n}^{\infty} C_k$ up to a set of measure zero for all $n \in \Gamma^+$ giving us that $B \subseteq \bigcap_{n\in\Gamma^+} \bigcup_{k=n}^{\infty} C_k$

But
$$\begin{aligned}\mu(\bigcap_{n\in\Gamma^+} \bigcup_{k=n}^{\infty} C_k - B) &\leq \mu(\bigcup_{k=n}^{\infty} C_k - B),\ n \in \Gamma^+ \\ &\leq \sum_{k=n}^{\infty} \mu(C_k - B) \\ &\leq \sum_{k=n}^{\infty} \mu(C_k \triangle B) \\ &< \sum_{k=n}^{\infty} 2^{-k} \\ &= 2^{1-n}\end{aligned}$$

giving $\mu(\bigcap_{n\in\Gamma^+} \bigcup_{k=n}^{\infty} C_k - B) = 0$ i.e. $\bigcap_{n\in\Gamma^+} \bigcup_{k=n}^{\infty} C_k \subseteq B$ up to a set of measure zero. Thus $B = \bigcap_{n\in\Gamma^+} \bigcup_{k=n}^{\infty} C_k$ up to a set of measure zero and hence $B \in \alpha_A$ if we neglect a set of measure zero as we are at liberty to do.

Corollary If α is a K_1-algebra, $A \in \alpha$, $\mu(A) > 0$ and $B \in \epsilon_A$ is such that $T^k B = B$ then $B \in \alpha_A$

Proof If $B_i = T^{-k} A \cap B - \bigcup_{j=1}^{i-1} B_j$ then since $T^k B = B$ we have $T^{-k} A \cap B = B$ giving us that $B = \bigcup_{i=1}^{k} B_i$ with $S_A B_i = T^i B_i$ and so $S_A B = B$. The result then follows from the proposition.

Theorem 6·23 If α is a K_1-algebra, $\beta \in Z_1$ is such that $T\beta = \beta$ then $\beta \leq \alpha$ if we neglect a set of measure zero.

Proof If $A = X$ in proposition 6·22 then since $T\beta = \beta$, $\beta \in Z_1$ implies that for each $B \in \beta$ $\mu(B) > 0$, there exists a $k \in \Gamma^+$ with $T^k B = B$ the result follows from corollary 1 of that proposition.

Proposition 6·24 If $A \in \varepsilon$, $\mu(A) > 0$, S_A is ergodic and there are no wandering sets of positive measure in (X, ε, μ) then T is ergodic in $(B, \varepsilon_B, \mu_B)$ where $B = \bigcup_{i \in \Gamma^+} T^{-i}A$.

Proof By proposition 6.11 $TB = B$ and so T is an automorphism on $(B, \varepsilon_B, \mu_B)$ and $S_B x = Tx$ for $x \in B$. If $C \in \varepsilon_B$ is such that $TC = C$ then we put

$$C_A = \{ x : x \in A, T^{-i}x \in C \text{ for some } i \in \Gamma^+ \}$$

Now $TC = C$ and so $T^{-i} x \in C$ for some $i \in \Gamma^+$ implies that $T^i x \in C$ for all $i \in \Gamma$ thus $S_A C_A = C_A$ giving $\mu(C_A) = 0$ or $\mu(A - C_A) = 0$ since S_A is ergodic. If $C^1 = A \cap C$ then we define C^k, $k \in \Gamma^+$ inductively by putting

$$C^k = T^{-k} A \cap C - \bigcup_{j=1}^{k-1} C^j$$

Thus $C^i \cap C^j = \phi$ if $i \neq j$ and $C = \bigcup_{k \in \Gamma^+} C^k$ up to a set of measure zero by 6·12. . .

Further $C_A = \bigcup_{k \in \Gamma^+} T^k C^k$ and so $\mu(C_A) = 0$ implies $\mu(T^k C^k) = 0$ for $k \in \Gamma^+$ which in turn implies $\mu(C) = 0$. While $\mu(C_A) \neq 0$ implies $\mu(A - C_A) = 0$, $A \subseteq C_A$ up to a set of measure zero. Thus if $D = B - C$ we have $TD = D$ and hence $\mu(D_A) = 0$ or $\mu(A - D_A) = 0$ as above. But $A \subseteq C_A$ implies $A \cap D_A = \phi$ and so we must have $\mu(D_A) = 0$ which implies $\mu(D) = 0$ as above. And so $\mu(B - C) = \mu(D) = 0$.

This last result is due to S.Kakutani see [9].

6·3 KOLMOGOROV AUTOMORPHISMS

Theorem 6·31 If T is a Kolmogorov-automorphism then T is ergodic.

Proof Let α be a K-algebra and A_n, $n \in \Gamma^+$ such that $A_n \in \alpha$, $0 < \mu(A_n) < \infty$ for each n and $\bigcup_{n \in \Gamma^+} A_n = X$. We write $\varepsilon_n, \mu_n, S_n, \alpha_n$ for $\varepsilon_{A_n}, \mu_{A_n}, S_{A_n}, \alpha_{A_n}$. By proposition 6·21 corollary 2 α_n is a K -algebra. Suppose that S_n is not ergodic for some $n \in \Gamma^+$. Then there exists a $B \in \varepsilon_n$ such that $0 < \mu(B) < \mu(A_n)$ and $S_n B = B$. By proposition 6·22 $B \in \alpha_n$ and hence $B \in \alpha$. Now $T \bigcup_{k \in \Gamma^+} T^{-k}B = \bigcup_{k \in \Gamma^+} T^{-k}B$ by proposition 6·11 and so $B \subseteq A_n \cap \bigcup_{k \in \Gamma^+} T^{-k}B$. If $x \in A_n \cap \bigcup_{k \in \Gamma^+} T^{-k}B$ then $x \in A_n \cap T^{-l}B$ for some $l \in \Gamma^+$ and so $x \in B$ since $S_n B = B$ giving $A_n \cap \bigcup_{k \in \Gamma^+} T^{-k}B \subseteq B$ and hence $B = A_n \cap \bigcup_{k \in \Gamma^+} T^{-k}B$. If $C = \bigcup_{n \in \Gamma^+} T^{-k}B$ then by proposition 6·11 $TC = C$. However $T^{-k}B \in T^{-k}\alpha \leqslant \alpha$, and so $C \in \alpha$ and therefore since $TC = C$ we have $C \in \alpha_\infty^- = \nu$. But $0 < \mu(B) \leqslant \mu(C)$ and therefore $\mu(X-C) = 0$ which in turn gives $A_n \cap C = A_n$ modulo zero i.e. $\mu(B) = \mu(A_n)$ a contradiction. Thus S_n is ergodic for all n. By proposition 6·24 we have that T is ergodic in $\bigcup_{i \in \Gamma^+} T^{-i}A_n$ and since $T \bigcup_{i \in \Gamma^+} T^{-i}A_n = \bigcup_{i \in \Gamma^+} T^{-i}A_n$ by proposition 6·11 we have in view of $\bigcup_{i \in \Gamma^+} T^{-i}A_n \in \alpha$, $\alpha_\infty^- = \nu$ that $\bigcup_{i \in \Gamma^+} T^{-i}A_n = X$ and hence the desired result.

Corollary If T is a Kolmogorov automorphism then T^k is ergodic for $k \in \Gamma^+$

Proof If α is a K-algebra with respect to T then $\alpha \leqslant T\alpha$ and so $\alpha \leqslant T^k\alpha$. Now $\bigvee_{i=-\infty}^{n} T^i\alpha = T^n\alpha \leqslant T^{kn}\alpha = \bigvee_{i=-\infty}^{n} T^{ki}\alpha$ for $n \in \Gamma^+$ hence letting $n \to \infty$ gives $\alpha_T \leqslant \alpha_{T^k}$ but $\alpha_T = \varepsilon$ and so $\alpha_{T^k} = \varepsilon$. Further $\bigwedge_{i=-n}^{\infty} T^{ki}\alpha = T^{-kn}\alpha \leqslant T^{-n}\alpha = \bigwedge_{i=-n}^{\infty} T^i\alpha$ for $n \in \Gamma^+$ and letting $n \to \infty$ gives $\alpha_\infty^{T^k} \leqslant \alpha_\infty^T$, but $\alpha_\infty^T = \nu$ and so $\alpha_\infty^{T^k} = \nu$. Thus α is a K-algebra with respect to T^k, $k \in \Gamma^+$ and so T^k is a Kolmogorov automorphism for $k \in \Gamma^+$. The ergodicity of T^k then follows from the theorem.

SUMMARY OF NOTATION AND DEFINITIONS

Numbers refer to the page(s) where the definition or symbol was first introduced.

atom:	A is an atom of α if $B \in \alpha$, $\mu(B-A) = 0$ imply $\mu(B) = 0$ or $\mu(B) = \mu(A)$.	16
atom set:	A_i, $i \in I \subseteq \Gamma^+$ is an atom set of $\alpha \in Z_3$ if A_i is an atom of α for each i, $A_i \wedge A_j = \phi$ whenever $i \neq j$ and $\mu(\bigcup_{i \in I} A_i) = 1$.	22.
automorphism:	8.	
continuous set:	A is a continuous set of α if given any $A_1 \in \alpha$ such that $A_1 \subseteq A$ and any d such that $0 < d \leq \mu(A_1)$ then there exists a $B \in \alpha$ with $\mu(B-A_1) = 0$ and $0 < \mu(B) \leq d$.	16.
endomorphism:	8.	
entropy:	21, 27, 28, 44, 51.	
ergodic:	T is ergodic if $TA = A$ implies $\mu(A) = 0$ or $\mu(X-A) = 0$.	12.
exhaustive:	α is exhaustive if $\alpha_T = \varepsilon$.	55,61.
	L is exhaustive if $\bigvee_{t \in \Gamma^+} U^t L = L^2_\mu$.	78.
Gaussian measure:	88, 89.	
generator:	α is a generator if $\alpha_T = \varepsilon$.	52.
Homomorphism:	8.	
increasingly filtered:	32.	
induced automorphism:	102.	
invariant:	α is invariant if $\alpha \leq T\alpha$.	55, 61.
	L is invariant if $UL \leq L$.	78.
isomorphic:	20.	

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