ON - LINE ESTIMATION

OF

STEAM BOILER PLANT DYNAMICS

by

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ABSTRACT.

The thesis is concerned with estimating, on-line, structural parameters of a linear first order system in the presence of a correlated nonstationary disturbance. The first order system represents a dynamic relationship between two quantities called the input and the output, and may be a part of a large plant. The disturbance represents then the coupling effect of the rest of the plant on the system under consideration. It is assumed to affect the output of the system only.

The problem of on-line estimation is viewed as a part of a larger procedure of on-line control of the overall plant, and the estimates obtained are assumed to be required as additional parameters necessary to control the plant. Therefore short computational times and moderate demands on the storage capacity of the process control computer are envisaged. Thus, only linear system is considered and relatively small series of sampled values of the input and output are assumed to be stored.

First ,the available digital techniques of identification are critically reviewed in Chapters 1 to 4. The techniques

are divided into the non-parametric ones, dealing only with system responses to given inputs, and parametric ones, involving the determination of structural parameters , or, the parameters of the governing differential or difference equations.

The current problem cannot be solved by any of these methods, and a new approach, described in Chapter 5 and Appendices C and D, is developed. The approach consists in representing the disturbance and the input as a non-stationary stochastic process, the model of which can be identified from an analysis of the mean square value of the input and output. The parameters of the combined model are estimated by an iterative procedure based on the Least Squares Method. A series of hypothetical outputs is calculated from the assumed model and an assumed set of parameter values. The deviations of these outputs from the actual outputs are called quasi-residuals. The method aims at obtaining a set of parameter values which result in the covariance matrix of the quasi-residuals being as close to the diagonal matrix as possible. Chapter 6 describes the application of this method to estimation of boiler dynamics.

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CONTENTS.

Introduction.

The work described in this thesis was carried out in support of a project dealing with the control of a power station boiler and associated with the Automatic Control Research Project of the Central Electricity Generating Board. One of the modes of control considered required obtaining, on-line, estimates of certain parameters of dynamic relations and using these estimates for control purposes. This requirement has led to the development of a new technique which enables first order dynamic relationships, as well as nonstationary processes encountered in the boiler operation to be identified on-line.

The problem of identification consists in the determination of the causal relationships, assumed to exist between variables, from observations of the variables over a period of time. This involves finding a form of the relationship and estimating the values of its parameters in such a way that the observations are best explained in the sense of some accepted performance criterion.

The problem first appeared in the control systems literature in connection with the design of controllers for physical systems in which the characteristics of signals, with which the controller has to cope, change in time in a random fashion (Laning and Battin,1956; Newton,Gould and Kaiser,1957) The introduction later of the analogue and digital computers opened new possibilities in the field of adaptive control systems. In such systems the controller, analogue or digital, is automatically adjusted to maintain the desirable performance in the presence of random fluctuations of process parameters (Mishkin and Braun,1960) and, therefore, the proces identification must be carried out automatically during the normal system operation.

The approach to the identification problem has been influenced by the developments in the theories of statistical estimation and communication. The former was established as a mathematical technique at the beginning of the last century with the work of Legendre and Gauss on least squares estimation (Plackett,1949; Rosenbrock,1965). The estimation techniques, gradually developed owing to many contributions, especially those due to K.Pearson and R.A.Fisher, were concerned ,up to about 1940, mainly with the classical problem of determining the best estimates of distribution parameters on the basis of a selection of samples taken from a given population.

Independently of this development, communication engineers were investigating the effect of noise, perturbing the transmitted signals, on the intelligibility of the signals.

They wished to formulate theories and synthesize equipment which could effectively detect the presence or absence of signal. This resulted in the introduction of filters which estimated the power frequency spectrum of the desired signal.

The pioneering work of Kolmogorov (1941) and Wiener(1949) showed that these problems could be incorporated in the framework of modern statistics if proper extensions were made from the classical discrete statements to those applicable in stochastic processes. The essence of Wiener's contributions is, firstly, the demonstration that the estimation theory can be applied to synthesize an electrical filter providing the best separation of signal and noise, and, secondly, the treatment of signals and noise as stochastic processes.

Following Wiener's work, considerable body of literature discusses both the analysis of nonlinear systems (Wiener,1958; Zadeh,1953) or their identification by means of finite expansions in terms of orthogonal functions (Lubbock,1960; Lubbock and Barker,1963; Simpson,1964; Barker and Hawley,1966). However, although nonlinear systems use the information about the input and output in a more efficient way than do the linear systems (Lubbock,1960), most of the literature dealing with the identification problem is concerned with linear systems. This, no doubt, arises from the fact that, first of all, various proposed theories can be comparatively easily formulated and validated when linear relationships hold. Secondly, many moderately nonlinear relationships can be linearized in the range of variations of interest so that the linear theory can be assumed to hold approximately and be applied (Pugachev, 1963). Finally, for practical data reduction systems, the prediction precision is only one of several factors to be considered in the choice of an estimator. The prediction speed and computer capability are at least equally important considerations, and sometimes it may be desirable to exchange the simplicity of a computer program for prediction precision (Deutch, 1965). In fact, a relatively fast computation of estimates is of primary importance in the present application (in view of the processes drifting in time) and, therefore, the application of only linear systems theory is of concern in this thesis.

The first approach to the determination of the dynamical characteristics of a linear system was to use Wiener's theory of optimum filtering. The variables in the system under consideration were regarded as statistically fluctuating time series which constitutes a sample from an ensemble of series representing the underlying stochastic process. The parameter values obtained from the solution of the Wiener-Hopf equation are then considered to constitute the best approximation to the underlying statistical parameters , by means of which variance of the estimates of the parameter values may be calculated.

Now the actual solutions of the problem may be divided into two classes. The first class comprises solutions obtained from the knowledge of the response of the process to external stimuli, and the estimated parameters are then the values of impulse response or the frequency response. The solutions of the second class stem from the knowledge of the physical nature of the process and the laws which govern it, and the parameters to be estimated are coefficients of the differentia (or difference) equations describing the physical behaviour of the process.

In the early days the techniques employed involved the solutions of the first class only, using the time domain approach, frequency domain approach, and expansions in terms of orthogonal functions. While the application of the latter technique to digital computation has been recently reported (Simpson,1964), this approach is essentially oriented towards analogue computation (Kitamori,1960; Braun et al.,1960; Dooge,1965) and it is only mentioned here for completeness.

As regards the techniques of the first class, the pattern seems to have been set by Goodman and Reswick(1956) and Goodman(1955) who presented a way of obtaining the impulse response of a dynamical system from normal operating records by means of a delay - line synthesizer. The introduction later of a digital computer enabled Levin(1960),Woodrow(1959) and Rosenberg and Shen(1963) to apply the fundamentals of mathematical statistics by formulating the same problem in a matrix form, and solving it by using the least squares method.

The alternative approach involves obtaining the frequency response function of a dynamical system from the consideration of power spectra of input and output, and their crossspectrum. While excellent exposition of the theoretical aspects of the technique can be found in the literature comparatively early (James et al.,1949; Laning and Battin,1956), its actual application was made possible only later owing to the pioneering work of N.R.Goodman(1957) on the estimation aspects of the technique. These were later discussed by Goodman and his associates (1961), Bendat(1960), Jenkins (1963a,1963b) and Enochson(1964). The application of the technique to estimation of system dynamics under closed loop control was dealt with by Westcott(1960) and Florentin (1959)

The actual computational aspects of this approach when using a digital computer were discussed fairly recently by Fleming and Michael(1965).

Kalman appears to be the first to seek the solutions of the second class. He formulated (1958) the response of a dynamical system in terms of a pulse transfer function and obtained the estimates of its coefficients by using the weighted least squares method. Later he reformulated the Wiener filter by using the concept of state(1960,1963) and showed that the solution of the optimal filter can be characterized by a set of differential equations. Under the influence of Kalman's contributions the state space description of dynamical systems has been almost universally accepted in control engineering (e.g.Zadeh and Desoer,1963). and the approach to the identification problem has been reformulated in many works as that consisting in the determination of the coefficients of the state transition matrix of the system under consideration. Thus, for example, Kopp and Orford(1963) enlarge the state space to include the structural parameters and use the perturbation theory and the Kalman filter for state estimation, while Mayne(1963) shows that the problem of nonstationary estimation of the coefficients of the state transition matrix can be formulated in terms of an equivalent

Kalman filter if the state of the system is known completely at every instant of time. On the other hand, Lee(1964), after formulating the estimation problem of a single-input single-output system in the state space form, points out an inefficiency in estimating the coefficients of transition matrix of such a system. He, therefore, transforms the state equations into equivalent difference equations and estimates the coefficients of the latter by the least squares method.

The common feature in all the above mentioned techniques is the synthesis of an optimum relationship between two sets of values, called "the input" and "the output", on the assumption that the latter are contaminated with white noise representing, for example, inaccuracies in measurement. However, measurement errors are not the only type of disturbance affecting the measurements. For example, the disturbances can enter the system as inputs which are not measurable, or as disturbances generated inside the system. Disturbances of this type are always present in a practical problem and their effect should be acknowledged in the design of the controller for the system under consideration. If only linear models are considered then the disturbances can always be transformed so as to appear as an effective disturbance entering the output. If such an effective disturbance is assumed to constitute a stationary random process with rational spectral density, then it can be represented as an output of a linear filter driven by white noise. This approach was adopted by Åström and Bohlin(1965a, 1965b). They represented both the system dynamics and the output disturbance in a form of a pulse transfer function, used the canonical transformation to obtain the state equations and obtained the maximum likelihood estimates of the coefficients by employing the Newton-Raphson algorithm.

The inclusion of the model of the disturbance in the system description is a considerable improvement on the previously mentioned techniques as it allows for a better design of the process controller. Even this approach,however, may be open to criticism on the ground that it assumes the disturbance to be stationary. Indeed, it has been pointed out by Box and Jenkins(1963) that a control system derived on a stationary assumption might be quite useless in the face of actual nonstationarity, and that it is often because the uncontrolled process may be highly nonstationary that the control is required.

Suppose that the single input single output system under consideration forms a small part of a large interconnected system (such a situation might, for example,arise, when one wishes to study a dynamic relationship between steam temperature and steam flowrate in a boiler, under normal operating conditions). In such circumstances the model of the disturbance should take into account the influence, on the output,of other processes coupled to it through the internal dynamics. If such processes are time-varying,the disturbance model representing their effect should be either time invariant and updated continuously or, which is considered to be more satisfactory and elegant, they should model the nonstationary behaviour of the disturbing process.

A disturbance model of this type was suggested by Box and Jenkins(1962, 1963) as part of their method of treating the problem of adaptive control systems. The model, discussed in Chapter 3,can be thought of as an unstable digital filter and is a generalization of the method of representing accumulated processes (Whittle,1963). Box and Jenkins suggest a method of identifying the structure of such a model from observations of the process. If such a model, however, is included in the description of an "open loop" dynamic system, there is, apparently, no way of EASILY identifying the structure of the model. If one is faced with identificatio of the system dynamics as well, it seems that a formidable identification problem arises.

The problem to be solved by the writer was to estimate the parameters of system dynamics in the presence of an unknown correlated nonstationary disturbance. As the estimatio procedure was to be carried out on-line , the nonstationary character of the disturbance was to be identified automaticall by the computer, and the parameters of the disturbace model could then be estimated jointly with the parameters of the system dynamics. Since none of the available techniques was suitable for solving this problem a novel approach, for which originality is claimed, has been developed. The approach involves a new method of representation of nonstationary processes and a new technique of parameter estimation.The approach is characterized by the following features;

a) A nonstationary stochastic process is represented as an output of a linear filter with time-varying coefficients the filter being such that the mean square value of its output is a polynomial in time and the degree of the polynomial is associated with a definite structure of the filter; thus, when the degree of the polynomial is known, the structure of the filter is also known;

b) the input to the system under consideration, as well as the disturbance;representing the coupling effect of the rest of the system and assumed to affect the system output only,are both represented by a filter of the type (a);

c) sequences of sample mean square values of the input and of the output are calculated for increasing sample lengths up to the maximum length of the series stored in the computer;

d) small sample averages of the sequences near the beginning and end of each sequence are calculated; these indicate the relative magnitudes of the mean square values, as well as the trend of the series;

e) sequences are successively differenced until the small sample averages are less than some prescribed fraction of the values calculated originally at (d); since the n-th difference of an n-th degree polynomial is zero, this stage determines the degree of the polynomials representing the mean square values of the input and output and, therefore, identifies the structure of the disturbance;

f) a set of parameter values for the combined system dynamics-disturbance model is assumed and, using the actual values of inputs, hypothetical values of the outputs, corresponding to this assumed set of parameter values, are calculated;

g) a series of differences between these outputs and the corresponding actual outputs is calculated; these differences are called in the thesis "quasi-residuals" and are thought of as made up of two contributions; the white random process assumed to excite the disturbance filter and the effect of deviations of the assumed parameter values from the true values of parameters;

h) the estimates of the parameters will be close to their true values if the "quasi-residuals" exhibit the characteristics of the white noise; this can only happen if the contribution of the parameter deviations to the magnitude of the quasi-residuals is small compared with the effect of the white noise; since the white noise is characterized by a diagonal covariance matrix, the convergence to the proper parameter values is achieved when the sum of squares of the quasi-residuals is minimized in such a way, that, at the same time, the covariance matrix of the quasi-residuals, is being reduced, as far as possible, to a diagonal form.

The organization of the text is as follows. In Chapters 1 to 4 a critical review of the currently available techniques is given(in addition, Appendix A gives a short discussion of the theory of estimation). The novel technique is developed in Chapter 5 and Appendices C and D. Chapter 6 discusses the application of the technique to estimation of boiler dynamics. Finally, Appendix B gives a general method, developed by the writer, of obtaining a difference equation of a general linear system with rational transfer function when the system is subject to an input smoothed by a hold circuit.

SYMBOLS AND CONVENTIONS:

Many symbols have been used in the thesis to denote differen quantities. The symbols are defined wherever they occur and, therefore, it is not proposed to list here all the symbols with all the meanings attached to them in various sections of the thesis. On the other hand, certain conventions have been kept throughout the thesis, and, to denote the quantities listed in the left hand column below the corresponding symbols listed in the right hand column have been used.

Quantity	Symbol
backward difference operator	∇
backward shift operator	B
complex conjugate of x(t)	$\bar{x}(t)$ OR $x^*(t)$
correlation of lag L of the series X(t)	$\rho_x(L)$
covariance of lag L of the series X(t)	$\gamma_x(L)$
continuous-time funtion x	$x(t)$
discrete-time function x	x_t
ensemble average, or expected value of X(t)	$E\langle x(t)\rangle$
estimate of X(t)	$\overset{*}{X}(t)$ OR $\hat{X}(t)$
exponential function of a parameter k	e^k OR $exp(k)$

Quantity	Symbol
operator shifting by 90 degrees	j or i
Laplace Transform of X(t)	$\mathcal{L}(X(t)) = X(s)$
n x m matrix with elements $a_{ij}(t)$	$\underline{A}(t)$ or $(a_{ij}(t))$
set of values $X_i(t)$ for varying i	$\{x_i(t)\}$
small increment in value of X	ΔX
time parameter	t
time derivative of X(t)	$pX(t)$ or $\dot{X}(t)$ or $\frac{dX(t)}{dt}$
transfer function	$H(s)$
transpose of a matrix $\underline{A}(t)$	$\underline{A}^T(t)$
transpose of a vector $\underline{X}(t)$	$\underline{x}^T(t)$
vector with elements $x_1(t),..x_n(t)$	$\underline{X}(t)$
weighting function	$h(t)$
Z-transfer function	$H(z)$
Z-transform of X(t)	$\mathcal{Z}(X(t)) = X(z)$

CHAPTER 1.

GENERAL CONSIDERATIONS IN THE PROBLEM OF IDENTIFICATION OF A LINEAR SYSTEM.

1.1. Introduction.

In a recent paper Tsypkin(1966) distinguishes three consecutive periods in the control theory: a deterministic period, a stochastic period and an adaptive period.

In the deterministic period the knowledge of the equations describing the behaviour of the system to be controlled, as well as that of the external inputs and disturbances was assumed. This knowledge allowed the use of classical analytical techniques for the solution of various control problems.

The stochastic period is characterized by a more realistic approach to control problems. In this approach the equations of the system to be controlled were still assumed to be known; however, the disturbances , and sometimes the system parameter were regarded as being probabilistic in nature. The mathematical techniques developed in this period were based on the use of random functions with statistical characteristics known in advance.

The characteristic feature of the current adaptive period follows from applications of automatic control to systems whose properties change with time and may not be known in advance. This feature is the use of information about the past of the controlled process, or"plant", for making current decisions.

Suppose that it is required to control a plant with incompletely defined dynamic characteristics. If the latter do not change very rapidly, a suitable controller may be required to compute, or identify, the characteristics of the plant while the system is in normal operation. The controller must then makes a decision concerning the way in which the available parameters of the system should be adjusted so as to improve the operation with respect to a defined performance index. Finally, certain signals or parameters must undergo a modification to accomplish the result. A control system, accomplishing the three functions of identification,decision and modification may be defined as an adaptive control system (Bellman et al,1966).

The identification problem, forming the subject of the present thesis, is concerned with the determination of a mathematical relationship which describes the input-output behaviour of an unknown system. The importance of the problem of identification was illustrated at a symposium organized recently in Prague by the International Federation of

Automatic Control, at which no fewer than 65 papers were presented.As observed by Godfrey and Hammond(1967), the symposium brought out a very wide range of techniques available for identification, coupled with an almost complete lack of any logical method for choosing the best one for a particular application, and with very few practical applications to industrial plants. It is hoped that the present work will not merit a comment of this type.

Reviewing the existing methods of identification Eykhoff (1966,1967) divided them into two broad classes,namely the techniques using "explicit mathematical relations" (or, open-loop techniques), and those using "model adjustment" (or closed-loop, or implicit techniques). In the first class he included the techniques which use a mathematical expression explicitly providing numerical values of parameters estimates in terms of known a priori knowledge and measured variables. The techniques of this class are least squares estimation, Markov estimation, maximum likelihood estimation and minimum risk estimation. They yield solutions which

a) are available after a finite number of elementary operation

b) require considerable memory,

c) are not available in an approximate form as an intermediate result;

The techniques of the second class employ some kind of model of the system. The parameters of this model are adjusted in such a way that the model characteristics approach the characteristics of the system in some preassigned sense. The techniques of this class depend on the minimization of the gradient, with respect to the unknown parameters, of the error between the output of the system and that of the model. The solutions obtained are

(a) available after (in principle) an infinite number of elementary operations,

(b) available in an approximate form as an intermediate result,

(c) found by a self-correcting procedure.

The approach adopted in these techniques is "closed-loop" with respect to the system performance. That is to say, the system performance is monitored and the parameters are adjusted to minimize a performance index.

Numerous papers have been written on the subject of model-reference adaptive systems incorporating the identification technique of the second class (e.g. Roberts,1962; Donaldson and Leondes,1963). These papers seem to imply that the main application of these techniques is in closed loop control systems in which the parameters of the controlled system exhibi wide variations due to changes in environment . In such applications the characteristics of suitable compensating networks are required to alter as the controlled system's parameters change. Such a procedure is usually effected throug the use of the method of steepest descents. In this method the gradient of the error, with respect to the unknown parameters, between the controlled system output and the system model output, is made proportional to the time rate of change of the respective parameters. This enables the adjustment of the compensating network to be mechanised, witho the necessity to use numerical values of any of the parameters involved.

This thesis is concerned with situations in which digital computers are employed to control processes and, therefore, analogue techniques are not relevant here. On the other hand, the last few years have witnessed the development of another approach to identification. The approach involves the use of discrete-time (sampled-data) model of the controlled plant, the parameters of the model being estimated by employing a suitable hill-climbing techniques. This is, essentially, a model reference approach which, however, results in numerical values of parameters, and which thus does not really fit in

Eykhoff's classification.

Assuming that all analogue techniques are excluded, a more appropriate way of discussing the existing digital techniques is to divide them into two classes as follows:

a) the techniques not involving any structural parameters and relationships between input and output of the controlle plant, and yielding a number of numerical values of plant response;

b) the "model reference"techniques depending on an assumed form of differential or difference equation which relates the plant input and the plant output; the techniques yield numerical values of the coefficients of the equation.

This classification is adopted in discussing existing techniques in the following chapters.

1.2. The two alternative formulations of the process identification problem

An on-line control of a process involves normally a prediction, over a suitable interval of time, of the process behaviou and taking an appropriate compensating action in accordance with some specified control policy. The former objective is achieved through the use of a mathematical model, usually in the form of a set of differential equations which expres the dynamic behaviour of the process.

Rigorous analysis of the dynamics of a typical industrial process is extremely difficult, if not impossible. The reason is that processes are usually very complex and contain numerous variables which are unwieldy to manipulate. Large number of variables, nonlinearities and uncertainties in certain physical phenomena, all contribute to the complexity of the problem, and a solution of a set of equations may well be as difficult to obtain as the synthesis of the actual mathematical description. To facilitate a solution of the problem certain simplifying assumptions may be made as long as the solutions resulting from such simplifications can still be regarded as describing the character of the dynamic behaviour correctly. (It may be possible, for example, to formulate some semi-empirical or approximate expressions for phenomena which are too complex to admit of an exact mathematical description).

The starting point of analysis is usually the formulation of mass-transfer balance equation, momentum equation and energy-balance equation. These equations are usually complicated (for example, Navier-Stokes flow equations(Davis, 1962)) and involve, in general, partial differentiation as well as nonlinearities. The introduction of allowable simplifications yields a set of differential equations describing, possibly approximately, the dynamic behaviour of the process under consideration.

Under the influence of control theory it has been customary to regard such a set of differential equations as describing a dynamical system and to formulate the equations so as to relate a set of system inputs

$$\{u_i(t)\}, \ (i = 1, 2, \ldots m)$$

and a set of system outputs $\{y_j(t)\}, \ (j = 1, 2, \ldots, l)$

If the system parameters vary slowly as compared with the time necessary for the identification of the system, only small variations about the steady operating levels can be c considered and the process behaviour can then be approximately described by that of a linear dynamical system.

A single-input single-output system of interest in this thesis may be described by a differential equation of the type

$$[a_n(t)p^n + a_{n-1}(t)p^{n-1} + \cdots + a_0(t)]\,y(t)$$
$$= [b_n(t)p^n + b_{n-1}(t)p^{n-1} + \cdots + b_0(t)]\,u(t) \qquad (1.1)$$

in which some of the coefficients $b_i(t)$ may be equal to zero.

An orthodox approach, discussed in the older literature on control systems theory is to represent the response characteristics of a linear system either in terms of an impulse response in the time domain, or frequency function in the frequency domain (alternatively, transfer function in the complex frequency domain).

The impulse response $h(t,\tau)$ is defined by

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)\,u(\tau)\,d\tau \tag{1.2a}$$

$$h(t,\tau) = M_\tau^* \, K(t,\tau) \tag{1.2b}$$

where M_τ^* is the adjoint operator of the right-hand side of the equation (1.1) and $K(t,\tau)$ is the Green's function for the left-hand side of this equation (Miller,1955).

Both, the variable-coefficient system (1.1) and the constant-coefficient system

$$(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_0)\,y(t)$$

$$= (b_n p^n + b_{n-1} p^{n-1} + \cdots + b_0)\,u(t) \tag{1.3}$$

can be characterized in terms of the transfer or frequency function(Miller,1955; Laning and Battin,1956). However, such a description of only constant coefficient system (1.3) has found application in the problem of identification.

If the weighting function of such a system is denoted by h(t) then the transfer function H(s) is given by the Laplace transform of h(t),

$$H(s) = \int_0^{\infty} h(t)\,e^{-st}\,dt \tag{1.4a}$$

or, alternatively by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \tag{1.4b}$$

where Y(s) and U(s) denote Laplace transform of the output and the input, respectively.

Similarly, the frequency function H(jw) is defined as the Fourier transform of the impulse response

$$H(j\omega) = \int_0^\infty h(\tau) e^{-j\omega\tau} d\tau \tag{1.5a}$$

or,alternatively by

$$H(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{b_n (j\omega)^n + b_{n-1} (j\omega)^{n-1} + \cdots + b_0}{a_n (j\omega)^n + a_{n-1} (j\omega)^{n-1} + \cdots + a_0} \tag{1.5b}$$

in which U(jw) is the Fourier transform of the input and Y(jw) is the Fourier transform of th output.

The method of obtaining the weighting function, and, therefore, the frequency response or transfer function,is well known (Miller,1955; Laning and Battin,1956). However, the converse problem of identification of the dynamical equation of a system from its impulse response presents formidable difficulties. In practical applications, therefore, one often identifies only the system response from the given values of input and output. This is achieved by estimating values of the impulse response at a number of time instants, or values of the frequency response at a chosen number of frequencies. The corresponding methods of identification do not allow the structural relationship between the input and output to be determined and will be referred to in the thesis as the "non-parametric methods of system identification".

A modern approach to the identification problem involves the determination of the second relationship between input and output of the system under consideration and, therefore, the techniques associated with this second approach are called in the thesis "parametric techniques of system identification". This approach can take two different forms. The first of these employs the notion of state x(t) (Zadeh and Desoer,1963), originated by Kalman(1960,1963a) and almost universally accepted in modern control theory. A linear system is characterized by means of dynamical state equations (Kalman,,1963b; Zadeh and Desoer,1963)

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)u(t) \tag{1.6a}$$

$$y(t) = \underline{C}(t)\underline{x}(t) + \underline{D}(t)u(t) \tag{1.6b}$$

the solution of which gives an explicit expression for the state $\underline{x}(t)$ as

$$\underline{x}(t) = \underline{\Phi}(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t} \underline{\Phi}(t,\tau)\underline{B}(\tau)u(\tau)\,d\tau \tag{1.7}$$

where $\underline{x}(t)$ is an n-vector, u(t) and y(t) are scalars and

$\underline{A}(t)$, $\underline{B}(t)$, $\underline{C}(t)$ and $\underline{D}(t)$ are matrices.

When the system (1.6) is controlled by a digital computer, the system output is sampled and the control is effected at discrete intervals of time. Continuous-time analysis may still, however, be used if the sampling interval is small compared to the significant time constants of the system. This, indeed, is the case in the technique of system identification described by Kopp and Orford(1963). The technique involves enlarging the state space to include the structural parameters as well as the assumption of the form of a differential equation which governs the dynamic behaviour of the system under consideration. Certain assumptions are also made about statistical characteristics of noise contaminating the data and differential equations with random forcing functions describing the parameter variations are adjoined to the system of differential equations describing the process. A linear regression technique is then used to derive a recursive relationship for the updated estimates of the state variables as a function of the last estimates and new measurement data.

An advantage of the continuous time description is that it allows to predict the system behaviour not only at the sampling instants but also between them. In certain control applications,however, it may be perfectly adequate to predict the system behaviour at the sampling instants only.. If such a relaxation of requirements is possible, the linear system is defined at discrete time instants and is referred to as a discrete time system . If such a system is time invariant, it may be defined by a linear difference equation of the form

$$\alpha_n y(t-n) + \alpha_{n-1} y(t-n+1) + \dots + \alpha_0 y(t)$$
$$= \beta_n u(t-n) + \beta_{n-1} y(t-n+1) + \dots + \beta_0 u(t) \qquad (1.8)$$

and its response may be described either by an impulse response in the form

$$h_d(t) = \sum_{l=1}^{N} h_l \delta(t - l\Delta T) \qquad (1.9)$$

which is equal to sampled values of the continuous time impulse response and is called the weighting sequence.; or by the pulse transfer function (Hurewicz,1949;Barker,1952) given by

$$G(z) = \frac{\beta_n z^n + \beta_{n-1} z^{n-1} + \dots + \beta_0}{\alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_0} \qquad (1.10)$$

As far as the writer is aware, there exist only two techniques which allow the system (1.8) to be identified parameterwise. (the method described by Kalman(1958) deals with noise-free measurements and is not considered to be realistic enough in practical applications). In the first of these, due to Åström and Bohlin(1965a,1965b) expressions for error

$$\varepsilon_j = y_j - y_j^* \tag{1.11}$$

between the actual output y_j and the predicted output y_j^* at time $j\Delta T$ are formed as functions of unknown parameters α_i, β_i and input x_j, using discrete-time state space description. On the assumption of the error being Gaussian, the likelihood equations are formed and solved by using the Newton-Raphson algorithm.

The starting point of the second technique, due to Box and Jenkins(1963,1967a,1967b) is also the formulation of the difference equation (1.8). Corresponding to the degree n of the equation, an expression, valid between the sampling instants, for the response of the system (1.8) to a step or ramp input is derived in the form

$$V_j = \int_{j-1}^{j} h(t-\tau)\, u(\tau)\, d\tau \tag{1.12}$$

where the weighting function h(t) is expressed as a function of parameters of the difference equation (1.8).

An expression for error

$$\varepsilon_k = y_k - y_k^* \tag{1.13}$$

between the actual output y_k and the predicted output y_k*·
at time $k\Delta T$

$$y_k^* = \sum_{j=1}^{k-1} v_j x_{k-1-j} \tag{1.14}$$

is formed and, assuming the error (1.13) to be normally distributed, the likelihood equations are obtained and solved by a nonlinear estimation technique.

The above mentioned techniques are discussed in some detail in Chapters 2 and 4.

CHAPTER 2.

DIGITAL NON-PARAMETRIC METHODS OF IDENTIFICATION OF A LINEAR SYSTEM

2.1. Introduction.

As discussed in Chapter 1, the response of a linear time-invariant system (1.3) may be expressed in terms of its impulse response h(t)

$$y(t) = \int_0^\infty h(\tau)u(t-\tau)\,d\tau \qquad (2.1)$$

When estimating the response of such a system from records of input u(t) and output y(t) under normal operating conditions, it has been customary to formulate the problem in either of the following two ways(Westcott,1960; Woodrow,1959);

a) either. one requires to determine an impulse response function h(t) which, when operating on the recorded values of the input process u(t), most nearly approximates the recorded values of the output process y(t). The error of approximation, $\varepsilon(t)$, is then minimized in some suitable, usually least squares, sense;

b) or, one assumes that the fluctuations about the mean operating points, of the processes u(t) and y(t), constitute stationary time series the recorded values of which represent one of many possible realizations of the processes. The output time series is then regarded as being the sum of two time series. One of these series is generated by operating on the linear system, having impulse response function h(t), with the input series $\{u(t)\}$. The other series $\{\varepsilon(t)\}$ is generated by internal disturbances not correlated with the input process $\{u(t)\}$ In most cases the series $\{\varepsilon(t)\}$ is assumed to be normally distributed.

Whichever viewpoint is taken, an ideal relationship between the measured quantities representing the process is written

$$y(t) = \int_0^\infty h(\tau)\,u(t-\tau)\,d\tau + \varepsilon(t) \qquad (.2.2)$$

on the assumption that the quantities can be observed and recorded over infinitely long time.. In any practical situation, however, estimates are obtained from records of finite duration. An estimation procedure must, therefore, provide an assessment of the validity of such results.

Only the second approach is discussed here. This involves multiplying equation (2.1) by $u(t-\tau)$ and ensemble averaging to obtain the well known Wiener-Hopf equation (Laning and Battin, 1956)

$$\gamma_{uy}(\tau) = \int_0^\infty h(r)\,\gamma_{uu}(\tau - r)\,dr \qquad (2.3)$$

where,

$$\gamma_{uy}(\tau) = E\langle u(t-\tau)y(t)\rangle \qquad ((2.4)$$

is the crosscovariance function of the input and output, and

$$\gamma_{uu}(\tau) = E\langle u(t-\tau)u(t)\rangle \qquad (2.5)$$

is the autocovariance function of the input.

Taking Fourier transforms of equation (2.3) one obtains

$$\sigma_u \sigma_y\, g_{uy}(\omega) = H(\omega)\,\sigma_u^2\, g_{uu}(\omega) \qquad (2.6)$$

where

σ_u^2	is the variance of the input
σ_y^2	is the variance of the output
$g_{uy}(w)$	is the normalized cross-spectral density function of the input and output,
$H(w)$	is the frequency response function of the system.

Thus one obtains (Jenkins,1963),

$$H(\omega) = \frac{\sigma_y}{\sigma_u}\,\frac{g_{uy}(\omega)}{g_{uu}(\omega)} \qquad (2.7)$$

Given a finite length of record of the input u(t) and output y(t) the linear system under consideration may be identified

a) either in the time domain by using an approximation to equation (2.3) and solving it for a finite number of heights h_j of the impulse response h(t);

b) or in the frequency domain, by using equation (2.7) to estimate a finite number of heights of the frequency response function $H(w)$. These techniques are discussed below.

2.2. Identification of the impulse response

Equation (2.3) involves infinite limits and, therefore, its solution for $h(t)$ cannot be obtained from a finite length of record of the input and output.

Suppose, however, that the input $u(t)$ and output $y(t)$ are sampled at intervals ΔT of time and are available in the form of a series of values

$$\{u_i\}, (i = 0,1,2,\ldots,M+K)$$
$$\{y_j\}, (j=K,K+1,\ldots,M)$$

Suppose also that, correspondingly, the impulse response $h(t)$ to be estimated is restricted to have a finite memory time $\Delta T \cdot T_0$ and to be defined by

$$h(t) = \sum_{j=0}^{K} h_j \, \delta\left(t - j\frac{T_0}{K}\right), \quad 0 \leq t \leq T_0 \Delta T \qquad (2.8)$$
$$= 0 \qquad \text{otherwise}$$

Then the finite number $(K+1)$ of the values h_j of the impulse response may be obtained by using an approximation to the Wiener Hopf equation (2.7), first suggested by Goodman and

Reswick (1955,1956) and given by

$$C_{uy}(r) = \sum_{j=0}^{K} h_j \, C_{uu}(r-j) \tag{2.9}$$

where,

$$C_{uu}(r) = \frac{1}{M+1} \sum_{i=0}^{M-r} u_i u_{i+r} \tag{2.10}$$

is the sample autocovariance function of the input, and

$$C_{uy}(r) = \frac{1}{M+1} \sum_{i=0}^{M-r} u_i y_{i+r} \tag{2.11}$$

is the sample cross-covariance of the input and the output. (The formulae (2.10) and (2.11) are based on the assumption that both the input and the output are zero-mean stationary time series).

The (K+1) values h_j of the impulse response may be obtained from equation (2.9) by substituting (K+1) values of the covariances $C_{uy}(r)$ and $C_{uu}(r)$ corresponding to $r=0,1,\ldots,K$, and solving the resultant set of (K+1) equations. Since the autocovariance function is an even function of its argument, i.e. since

$$C_{uu}(r) = C_{uu}(-r) \tag{2.12}$$

the set of equations may be written in a matrix form as

$$\begin{bmatrix} C_{uy}(0) \\ C_{uy}(1) \\ C_{uy}(2) \\ \vdots \\ C_{uy}(K) \end{bmatrix} = \begin{bmatrix} C_{uu}(0) & C_{uu}(1) & C_{uu}(2) & \cdots & C_{uu}(K) \\ C_{uu}(1) & C_{uu}(0) & C_{uu}(1) & \cdots & C_{uu}(K-1) \\ C_{uu}(2) & C_{uu}(1) & C_{uu}(0) & \cdots & C_{uu}(K-2) \\ \vdots & \vdots & \vdots & & \vdots \\ C_{uu}(K) & C_{uu}(K-1) & & \cdots & C_{uu}(0) \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ \vdots \\ W_K \end{bmatrix} \tag{2.13a}$$

or, symbolically,

$$\underline{C}_{uy} = \underline{C}_{uu} \cdot \underline{h} \tag{2.13b}$$

The solution for $\underline{h}$ is then

$$\underline{h} = \underline{C}_{uu}^{-1} \cdot \underline{C}_{uy} \tag{2.14}$$

The equations (2.13),(2.14) resulting from the approximation (2.9) to the Wiener Hopf equation, can also be given a different interpretation. When the input and output are given in the form of a series of discrete values $\{u_i\}$ and $\{y_j\}$, and the restriction (2.8) on the impulse response is imposed, the ideal relationship (2.2) may be approximated by

$$y_i = \sum_{j=0}^{K} h_j u_{i-j} + \varepsilon_i \qquad i = K, K+1, \ldots, K+1 \tag{2.15}$$

where $\{\varepsilon_i\}$ are uncorrelated identically distributed random variables having zero mean and variance σ^2.

Equations (2.15) may be put in a matrix form (Woodrow,1959; Levin,1960; Kerr,1965) as

$$\underline{y} = \underline{U} \cdot \underline{H} + \underline{\varepsilon} \qquad (2.16)$$

or,explicitly,

$$\begin{bmatrix} y_K \\ y_{K+1} \\ \vdots \\ \vdots \\ y_{K+M} \end{bmatrix} = \begin{bmatrix} u_K & u_{K-1} \ldots\ldots & u_0 \\ u_{K+1} & u_K \ldots\ldots & u_1 \\ \vdots & & \\ \vdots & & \\ u_{K+M} & u_{K+M-1} \ldots\ldots & u_M \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ \vdots \\ h_K \end{bmatrix} + \begin{bmatrix} \varepsilon_K \\ \varepsilon_{K+1} \\ \vdots \\ \vdots \\ \varepsilon_{K+M} \end{bmatrix} \qquad (2.17)$$

Equations (2.16) and (2.17) represent a linear system and may be solved by the least squares method, discussed in Appendix A. The least squares estimates $\underline{H}^*$ of $\underline{H}$ are given by

$$\underline{H}^* = \left(\underline{U}^T\underline{U}\right)^{-1}\underline{U}^T\underline{y} \qquad (2.18)$$

After performing the matrix multiplications and dividing throug by (M+1) the equations (2.18) become identical with equation (2.14). It is thus seen that the set of equations (2.13) resulting from an approximation to the Wiener Hopf Equation, can be interpreted as the normal equations of Least Squares Estimation (Plackett,1960).

A great disadvantage of this approach is that, to identify a weighting function reasonably well, K must be large. However, as the number of terms in the equations is increased, the process of matrix inversion becomes disproportionately difficult, and this may create problems when on-line system identification for control purposes is required.

2.3. Identification of Frequency response.

An explicit relation

$$H(\omega) = \frac{\sigma_y}{\sigma_u} \frac{g_{uy}(\omega)}{g_{uu}(\omega)} \tag{2.7}$$

obtained by Fourier transforming the Wiener Hopf equation (2.3), expresses the frequency response function H(w) of a linear system in terms of the spectral density $g_{uu}(w)$ of the input, and cross-spectral density $g_{uy}(w)$ of the input and the output.

The autocovariance function $\gamma_{uu}(\tau)$ is an even function of the lag τ and, therefore, the spectral density $g_{uu}(w)$ is expressed as its cosine transform

$$\sigma_u^2 g_{uu}(\omega) = \frac{2}{\pi}\int_0^\infty \gamma_{uu}(\tau)\cos\omega\tau \, d\tau \tag{2.19}$$

However, the cross-covariance function is not an even function of the lag τ. For this reason it is usual to introduce two auxiliary functions (Jenkins,1963)

$$\sigma_u \sigma_y \alpha_{uy}(\tau) = \frac{1}{2}\left[\gamma_{uy}(\tau) + \gamma_{uy}(-\tau)\right] \qquad (2.20a)$$

$$\sigma_u \sigma_y \beta_{uy}(\tau) = \frac{1}{2}\left[\gamma_{uy}(\tau) - \gamma_{uy}(-\tau)\right] \qquad (2.20b)$$

and to define, in terms of these functions, two components of the cross-spectral density, the in-phase component $C_{uy}(w)$ called the co-spectral density, and the quadrature component, $Q_{uy}(w)$ called the quadrature spectral density (Goodman,1957) as

$$C_{uy}(\omega) = \frac{2}{\pi}\int_0^\infty \alpha_{uy}(\tau)\cos\omega\tau \, d\tau \qquad (2.21a)$$

$$Q_{uy}(\omega) = \frac{2}{\pi}\int_0^\infty \beta_{uy}(\tau)\sin\omega\tau \, d\tau \qquad (2.21b)$$

The cross-spectral density $g_{uy}(w)$ is then expressed as

$$g_{uy}(\omega) = C_{uy}(\omega) - j\,Q_{uy}(\omega) \qquad (2.22)$$

It is convenient to express the frequency response function in terms of the gain $G(w)$ and phase shift $\varnothing(w)$ as

$$H(w) = G(w).\exp(-j\varnothing(w)) \qquad (2.23)$$

in which

$$G(\omega) = \frac{\sigma_y}{\sigma_u}\,\frac{\sqrt{C^2_{uy}(\omega) + Q^2_{uy}(\omega)}}{g_{uu}(\omega)} \qquad (2.24a)$$

and

$$\phi(\omega) = \tan^{-1}\left(\frac{Q_{uy}(\omega)}{C_{uy}(\omega)}\right) \tag{2.24b}$$

In practical situations the output y(t) is contaminated by noise and, if the level of the latter is high as compared with the output, it may not be possible to obtain ~~any~~ USEFUL estimates of the gain G(w) and phase Ø(w). It is essential, therefore, to have some measure indicating to what extent the output y(t) is dependent on the input u(t). Such a measure is provided by the <u>coherency</u> (Wiener,1949; Goodman,1957) defined by

$$W_{uy}^2(\omega) = \frac{|g_{uy}(\omega)|^2}{g_{uu}(\omega)\, g_{yy}(\omega)} = \frac{C_{uy}^2(\omega) + Q_{uy}^2(\omega)}{g_{uu}(\omega)\, g_{yy}(\omega)} \tag{2.25}$$

$$|W_{uy}^2(\omega)| \leq 1 \tag{2.26}$$

The coherency is a measure of correlation between u(t) and y(t) at a frequency w_j; it tends to unity when the noise is negligible and tends to zero when the spectral density $g_{\varepsilon\varepsilon}(w)$ of the noise $\varepsilon(t)$ is large compared with the spectral density $G^2(w).g_{uu}(w)$ of the input referred to the output.

The problem of identification of the frequency response appears to have been first studied by N.R.Goodman (1957)

The approach was later followed by Goodman and his associates(1961), and also discussed by Jenkins (1963a,1963b) Bendat (1960) and Enochson(1964). The approach involves essentially a judicious choice of the amount of data required (this may or may not be a critical requirement) and a critical examination of the behaviour and of the sampling properties of the estimates $\hat{G}(w)$ of gain, phase $\hat{\emptyset}$ and coherency $\hat{W}(w)$, as influenced by the length N of series, maximum lag or truncation point M and level of coherency $W(w)$. All the three quantities, gain, phase and coherency are functions of the various spectral density functions and the problem is thus seen to be that of estimation of spectral density function. It can be shown (Jenkins,1961,1963; Parzen,1964a,1964b) that the estimates of the four spectral densities $g_{uu}(w)$, $g_{yy}(w)$, $C_{uy}(w)$ and $Q_{uy}(w)$ are respectively given by

$$\hat{g}_N^{uu}(\omega_j) = \frac{\Delta T}{\hat{\sigma}_u^2 \pi}\left[C_0^{uu} + 2\sum_{s=1}^{M-1} k\left(\frac{s}{M}\right) C_s^{uu} \cos\omega_j s\right] \qquad (2.27a)$$

$$\hat{g}_N^{yy}(\omega_j) = \frac{\Delta T}{\hat{\sigma}_u^2 \pi}\left[C_0^{yy} + 2\sum_{s=1}^{M-1} k\left(\frac{s}{M}\right) C_s^{yy} \cos\omega_j s\right] \qquad (2.27b)$$

$$\hat{C}_N^{yy}(\omega_j) = \frac{\Delta T}{\hat{\sigma}_u \hat{\sigma}_y \pi}\left[\hat{\alpha}_0^{uy} + 2\sum_{s=1}^{M-1} k\left(\frac{s}{M}\right) \hat{\alpha}_s^{uy} \cos\omega_j s\right] \qquad (2.27c)$$

$$Q_N^{uy}(\omega_j) = \frac{2\Delta T}{\hat{\sigma}_u \hat{\sigma}_y \pi}\left[\sum_{s=1}^{M-1} k\left(\frac{s}{M}\right) \hat{b}_s^{uy} \sin\omega_j s\right] \qquad (2.27d)$$

where

$\hat{g}_N(w_j)$ is the spectral density, $\hat{C}_N(w_j)$ is the in-phase and $\hat{Q}_N(w_j)$ is the quadrature cross-spectral density.
Using the estimates defined by equations (2.27), the expressio for the estimates of gain $\hat{G}_N(w_j)$, phase $\hat{\phi}_N(w_j)$ and coherency $\hat{W}^2_N(w_j)$ are obtained in the form

$$\hat{G}_N(\omega_j) = \frac{\hat{\sigma}_y}{\hat{\sigma}_u} \sqrt{\frac{\left(\hat{C}_N^{uy}(\omega_j)\right)^2 + \left(\hat{Q}_N^{uy}(\omega_j)\right)^2}{\hat{g}_N^{uu}(\omega_j)}} \qquad (2.28)$$

$$\hat{\phi}_N(\omega_j) = \tan^{-1}\left(\frac{\hat{Q}_N^{uy}(\omega_j)}{\hat{C}_N^{uy}(\omega_j)}\right) \qquad (2.29)$$

$$\left(\hat{W}_N^{uy}(\omega_j)\right)^2 = \frac{\left(\hat{C}_N^{uy}(\omega_j)\right)^2 + \left(\hat{Q}_N^{uy}(\omega_j)\right)^2}{\hat{g}_N^{uu}(\omega_j)\;\hat{g}_N^{yy}(\omega_j)} \qquad (2.30)$$

Approximate confidence limits for the above quantities are obtained from the covariance matrix of the estimates which is derived by approximating infinite-sample properties of multinormal series by those of a sample of a finite length. The confidence limits depend on the scale on which the given estimated quantity is measured and, ideally, a scale should be chosen such that the limits are independent of the quantity being estimated.

Thus, if the estimates of the spectral density $\hat{g}_N(w_j)$

are measured on a logarithmic scale, the confidence intervals for $\log(\hat{g}_N(w))$ are

$$\left[\pm z_\alpha \sqrt{\frac{\ell M}{N}} \right] \tag{2.31}$$

where z_α is the upper $\frac{\alpha}{2}$% limit of the normal distribution and ℓ depends on the type of window (Jenkins,1961; Parzen, 1964a,1964b).

Logarithmic scale is also convenient for measuring the gain. Corresponding to estimates $\log(\hat{G}_N(w))$, the approximate confidence limits are (Jenkins,1963a,1963b)

$$\left[\log\left(\frac{f_1 \hat{G}}{\chi^2_{1-\alpha}}\right), \log\left(\frac{f_1 \hat{G}}{\chi^2_\alpha}\right) \right] \tag{2.32}$$

where $\chi^2_{1-\alpha}, \chi^2_\alpha$ are, respectively, the lower and upper α per cent point of χ^2 distribution with f_1 degrees of freedom defined by

$$f_1 = \frac{4N}{\ell M\left[\frac{1}{W^2(\omega)} - 1\right]} \tag{2.33}$$

The estimates $(\tan\hat{\phi}(w))$ of $\tan\phi(w)$ can be taken to be approximately normally distributed about $\tan\phi(w)$ with variance (Jenkins,1963a,1963b)

$$\operatorname{Var} \tan(\hat{\phi}(\omega)) = \frac{1}{2} \frac{\ell M}{N} \sec^4 \phi \left[\frac{1}{W^2(\omega)} - 1\right] \tag{3.34}$$

It follows that, for the confidence coefficient $(1-\alpha)$, the approximate confidence limits for $\tan\phi$ are given by

$$\beta_{1,2} = \exp\left[\tan \hat{\phi}(\omega) \pm Z_\alpha \sec^2\phi \sqrt{\frac{1}{2}\ \frac{\ell M}{N}\left[\frac{1}{W^2(\omega)} - 1\right]}\right] \tag{2.35}$$

where Z_α is the upper $\frac{\alpha}{2}$ percent limit of the normal distribution. As observed by Jenkins(1963a,1963b) a natural scale of measurement of the phase is ϕ/so that the confidence intervals for $\hat{\phi}(w)$ are

$$\tan^{-1}\left\{\left[\tan \hat{\phi}(\omega) \pm Z_\alpha \sec^2\phi \sqrt{\frac{1}{2}\ \frac{\ell M}{N}\left[\frac{1}{W^2(\omega)} - 1\right]}\right]\right\} \tag{2.36}$$

Approximate confidence limits for the coherency have been suggested by Enochson and Goodman(1965). They applied the Fisher z-transformation (Cramér,1946) to the exact distribution of sample coherency obtained by Goodman(1957). If

$$\begin{aligned} z &= \tanh^{-1} \hat{W}_N^{uy}(\omega) \\ &= \frac{1}{2} \log \frac{1 + \hat{W}_N^{uy}(\omega)}{1 - \hat{W}_N^{uy}(\omega)} \end{aligned} \tag{2.37}$$

then it is shown by Enochson and Goodman that

$$\tanh\left(z - \frac{1}{2(f_2-1)} - Z_\alpha \frac{1}{\sqrt{2(f_2-1)}}\right) \leq W^{uy}(\omega)$$

$$\leq \tanh\left(z - \frac{1}{2(f_2-1)} + Z_\alpha \frac{1}{\sqrt{2(f_e-1)}}\right) \tag{2.38}$$

where z_{α} is the upper $\frac{\alpha}{2}$ per cent limit of the normal distribution and f_2 is the number of degrees of freedom associated with estimation of the cross-spectral density and is given by

$$f_2 = \frac{N}{ML} \tag{2.39}$$

2.4. General Remarks.

The two "non-parametric" techniques discussed in this Chapter, suffer from a disadvantage in that they require a large number of estimates to characterize the system response sufficiently accurately. Moreover, long computational times required for identification of the system response are involved in both techniques, and especially the second one. An alternative, more advantageous approach is to characterize the system response by means of a finite number of parameters. The techniques associated with this approach involve characterization of the dynamic system in the discrete time and the representation of the effect of a disturbance. These techniques are discussed in Chapter 4.

CHAPTER 3.

PARAMERTIC MODELLING OF DISCRETE-TIME STOCHASTC PROCESSES

3.1 Introduction.

The Chapter discusses representation of discrete-time stochastic processes by means of finite-parameter models. Some general definitions relating to stochastic processes are given first. Modelling of stationary processes by means of autoregressive,moving average and mixed autoregressive-moving average schemes is then discussed. Finally, a brief review of modelling techniques applicable to nonstationary processes is given.

3.2. Some fundamental ideas and definitions relating to stochastic processes.

The theory of stochastic processes is generally defined (Parzen,1961) as the" dynamic" part of the probability theory in which one studies a collection of random variables, called a "stochastic process", from the point of view of their interdependence and limiting behaviour.

Central to the definition of a stochastic process is the notion of a random variable. This may be defined(Doob,1953) simply as a measurable function. More formally, a real function X, defined on a space Ω of points w is called a random variable if there is a probability measure defined on w sets, and if, for every real number λ, the inequality

$$X(\omega) \leq \lambda$$

delimits an w-set whose probability is defined, that is a measurable set (Doob,1953)

A stochastic process can now be defined (Parzen,1961) as a collection $\{X(t,w),\ t \in T,\ w \in \Omega\}$ of random variables indexed by a parameter t varying in an index set T, with w varying over the probability space Ω; when $T = \{0, \pm 1, \pm 2, \ldots\}$ the stochastic process is said to be a discrete parameter process.

A stochastic process is thus a process which is developing in time in a manner controlled by probabilistic laws, and the functions X(t,w) depend on two arguments, the time t and a random event w. If the time instant t is fixed, say $t = t_o$, then $X(t_o, w)$ is a random variable. If, on the other hand, $w = w_o$ is a fixed random event, then $X(t, w_o)$ is considered to be a possible observation, or a realization, of the stochastic process, and is called a time series (Parzen,1961)

An important role in the theory of stochastic processes is played by their first and second moments, assuming that the latter exist. For a stochastic process $\{X(t),\ t \in T\}$

these are defined by

(a) the mean value function

$$m(t) = E \langle X(t) \rangle \ , t \in T \qquad (3.1a)$$

(b) the covariance kernel

$$K(s,t) = Cov\left[X(s), X(t)\right], \quad s,t \in T \qquad (3.1b)$$

In general,however, a stochastic process may be described by the joint probability distribution function of the n random variables $X(t_1),\ldots X(t_n)$ for all integers n and n points $t_1,\ldots,t_n$ in T. Thus,

$$F_{X(t_1)\ldots X(t_n)}(x_1,\ldots x_n)$$

$$= P\left[X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\right] \qquad (3.2)$$

Among stochastic processes with finite second moments stationary processes are important for practical applications. Such processes arise when the random mechanism producing the process does not change with time. This situation is often met with in technology and physical sciences, and is often assumed to hold approximately in other fields, such as economics, if T is not too large, and if any systematic component is isolated in an appropriate way. Such processes are classified as being either strictly or weakly stationary. These are defined as follows (Grenander and Rosenblatt,1957; Parzen,1962):

(a) A stochastic process $\{X(t), t \in T\}$ is said to be strictly stationary (or stationary in the strict sense) if for any integer n and any h in T, the n-dimensional vector $[X(t_1), \ldots X(t_n)]^T$ has the same joint probability distribution function as the n-dimensional vector $[X(t_1+h), \ldots, X(t_n+h)]^T$. In other words, the simultaneous distributions depend only upon the time parameter differences (t_1-t_2), (t_1-t_3), ..., (t_1-t_n)

(b) A stochastic process $\{X(t), t \in T\}$ is said to be weakly stationary (or covariance stationary, or stationary in the wide sense) if it possesses finite second moments and if its covariance kernel $K(s,t)$ is a function only of the absolute difference $|s-t|$ in the sense that there exists a function $R(v)$ such that for all s and t in T,

$$K(s,t) = R(s-t) \tag{3.3}$$

$R(v)$ is called the covariance function of the weakly stationary process $\{X(t), t \in T\}$

3.3 Spectral representation of weakly stationary discrete parameter stochastic processes.

An important result proved by Wold (Parzen, 1961) is that the covariance function $R(v)$ of a discrete parameter weakly stationary stochastic process may be expressed in the form

$$R(v) = \int_{-\pi}^{\pi} e^{jv\omega} \, dF(\omega) \qquad v = 0, \pm 1, \ldots \tag{3.4}$$

where the function F(w), called the spectral distribution function, is bounded and nondecreasing. This function may be uniquely written as the sum of three components $F_d(w)$, $F_{sc}(w)$ and $F_{ac}(w)$ such that

(a) the function $F_{sc}(w)$ is a singular and continuous function;

(b) the function $F_d(w)$ is purely discontinuous, increases only at the discontinuity points and is defined by

$$F_d(\omega) = \sum_{\omega_j \leq \omega} \Delta F(\omega_j) \tag{3.5a}$$

$\{\omega_j\}$ being the discontinuity points of F(w) and

$$\Delta F(\omega) = F(\omega + 0) - F(\omega - 0) \tag{3.5b}$$

$$\Delta F(\omega) > 0 \tag{3.5c}$$

there being only a finite number of points of positive spectral mass in any finite interval on the real line;

(c) the function $F_{ac}(w)$ is absolutely continuous and is the integral of a non-negative integrable function f(w) called the spectral density function; the latter function is continuous except at a finite number of points where it has finite left-hand and right-hand limits.

In time series studies it is usually assumed (Parzen,1961) that the singular component F_{sc} (w) is absent and that, therefore, the spectral distribution function F(w) may be represented in the form

$$F(\omega) = \sum_{\omega' \leq \omega} \Delta F(\omega') + \int_{-\infty}^{\omega} f(\omega')d\omega' \tag{3.6}$$

In terms of the spectral distribution function F(w), one can characterize various representations of a stationary process $\{X(t), t \in T\}$ One of the most important representations employs the notion of a process $\{y(t), t \in T\}$ with orthogonal increments. This is defined (Doob,1953) as the process such that

$$E < |y(t) - y(s)|^2 > < \infty \tag{3.7a}$$

and, whenever the parameter values satisfy the inequality $s_1 < t_1 \leq s_2 < t_2$, the increments $y_{t_1} - y_{s_1}$ and $y_{t_2} - y_{s_2}$ are orthogonal to each other, i.e.

$$E < (y_{t_2} - y_{s_2})\overline{(y_{t_1} - y_{s_1})} > = 0 \tag{3.7b}$$

Corresponding to such a process a monotone non-decreasing function can be defined to satisfy

$$E < |y(t) - y(s)|^2 > = \underset{s<t}{F(t) - F(s)} \tag{3.7c}$$

or, symbolically,

$$E < |dy(t)|^2 > = dF(t) \tag{3.7d}$$

since the difference F(t)-F(s) depends only on (t-s) if the process y(t) is covariance stationary.

In terms of the process with orthogonal increments, a weakly stationary process $\{X(t), t \in T\}$ has the so-called spectral representation (Doob, 1953; Grenander and Rosenblatt, 1957)

$$X(t) = \int_{-\pi}^{\pi} e^{j\omega t} \, dy(\omega) \tag{3.8}$$

where

$$E\langle |dy(\omega)|^2 \rangle = dF(\omega) \tag{3.9}$$

As far as modelling of stochastic processes is concerned, an important case arises when the spectral jump function (3.5b) vanishes for all w and the stochastic process is characterized by the so-called absolutely continuous spectral distribution function.

It can then be shown (Doob, 1953) that, if g(w) is a function measurable with respect to the class of Borel-measurable sets in an n-dimensional space, such that

$$|g(\omega)|^2 = \frac{dF(\omega)}{d\omega} \tag{3.10}$$

then there exists a $\tilde{y}(\omega)$ process with orthogonal increments which satisfies

$$x(t) = \int_{-\pi}^{\pi} e^{jt\omega} g(\omega) \, d\tilde{y}(\omega) \tag{3.11a}$$

$$E\langle |d\tilde{y}(\omega)|^2 \rangle = d\omega \tag{3.11b}$$

Moreover, if $\frac{dF(\omega)}{d\omega}$ never vanishes,

$$\tilde{y}(\omega) = \int_{-\pi}^{\omega} \frac{1}{g(\mu)}\, dy(\mu) \tag{3.12}$$

The relations (3.11) and (3.12) play an important role in parametric models of stochastic processes as discussed in the following sections.

3.4. Parametric representation of discrete parameter weakly stationary processes.

As discussed in the preceding section, a stochastic process $\{X(t), t \in T\}$ may be defined as a family of random variables indexed by a parameter t which belongs to a linear index set T. It was also observed in the preceding section that a set of observations $\{X(t), t \in T\}$, arranged chronologically, and called a time series, is regarded as one of many possible realizations of the stochastic process.

The statistical theory of time series analysis attempts to infer, from an observed sample, the probability law of the underlying stochastic process. This is effected by postulating a stochastic model which is completely specified except for the values of certain parameters. The parameters are then estimated on the basis of the observed sample so that the complete model may be used.

(a) either to understand the mechanism generating the process

(b) or to predict the future behaviour of the time series.

The important and extensively used models of weakly stationary stochastic processes are the moving average scheme and the autoregressive scheme. Both schemes were discovered , in a finite parameter form, in 1920's. It was not until late 1930's, however ,that they were shown by Wold to constitute special cases of stationary stochastic processes possessing absolutely continuous spectral distribution function (Parzen,1961;Doob,1953). In particular, it can be shown (Doob,1953; Grenander and Rosenblatt,1957; Whittle,1963) — that if (and only if) a stochastic process $\{X(t), t \in T\}$ possesses an absolutely continuous spectral distribution function F(w), then the process can be represented as a process of <u>moving averages</u> defined by

$$X(t) = \sum_{j=-\infty}^{\infty} c_j \xi_{t-j} \tag{3.13}$$

with the condition

$$\sum_{j=-\infty}^{\infty} |c_j|^2 < \infty \tag{3.14}$$

The $\{\xi_j\}$ are mutually orthogonal random variables with mean zero and variance σ^2. They are defined, in terms of a process $\tilde{y}(w)$ with orthogonal increments (3.11) by

$$\xi_m = \int_{-\pi}^{\pi} e^{i\omega m} \, d\tilde{y}(\omega) \tag{3.15}$$

$$E\langle \zeta_s \zeta_t \rangle = \sigma^2 \delta_{s,t} = \sigma^2 \int_{-\pi}^{\pi} e^{i(s-t)\omega} \frac{d\omega}{2\pi} \tag{3.16}$$

and have an absolutely continuous spectrum with constant spectral density.

$$f_\zeta(\omega) = \frac{\sigma^2}{2\pi} \tag{3.17}$$

The spectral density $f_x(w)$ of the process $\{X(t)\}$ is then given by

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \left| \sum_{j=-\infty}^{\infty} c_j e^{i\omega j} \right|^2 \tag{3.18}$$

As observed by Whittle(1963) ,there are infinitely many functions $\phi(\omega)$ satisfying

$$\begin{aligned} f_x(\omega) &= |\phi(\omega)|^2 \\ &= \frac{\sigma^2}{2\pi} \left| \sum_{j=-\infty}^{\infty} c_j e^{i\omega j} \right|^2 \end{aligned} \tag{3.19}$$

and, therefore, infinitely many representations of the form (3.13). However, for a process in time, there is only one physically meaningful moving average representation, namely, the one not involving future values of the ζ_t so that

$$X(t) = \sum_{j=0}^{\infty} b_j \zeta_{t-j} \tag{3.20}$$

The condition for the existence of such a representation is (Whittle,1963)

$$\int_{-\pi}^{\pi} \log f(\omega)\, d\omega > -\infty \qquad (3.21)$$

The relation (3.20) represents a stationary process $\{X(t)\}$ in terms of past values on an orthogonal random process $\{\zeta_t\}$. It is possible, however, to represent the process $\{X(t)\}$ also as an autoregressive process, or, as a linear function of its past values plus a random shock

$$X(t) = \sum_{j=1}^{\infty} a_j X(t-j) + \zeta_t \qquad (3.22)$$

where the orthogonal random process $\{\zeta_t\}$ is defined by (3.15) and (3.16). The relationship between the coefficients $\{b_j\}$ and $\{a_j\}$ of equations (3.20) and (3.22) can be easily deduced (Box and Jenkins, 1966) by introducing a backward shift operator B defined by

$$B V_t = V_{t-1} \qquad (3.23a)$$

$$B^K V_t = V_{t-K} \qquad (3.23b)$$

Employing the operator B in equations (3.20) and (3.22) one obtains,

$$X_t = \left(\sum_{i=0}^{\infty} b_i B^i\right) \zeta_t \qquad (3.20a)$$

and

$$\zeta_t = \left[1 - \sum_{j=1}^{\infty} a_j B^j\right] X_t \qquad (3.22a)$$

Substituting (3.22a) into (3.20a) (Box and Jenkins,1966; Whittle,1963) one obtains

$$\left(\sum_{i=0}^{\infty} b_i B^i\right)\left(1-\sum_{j=1}^{\infty} a_j B^j\right)=1 \quad .$$

This relation may be used to derive the coefficients $\{a_j\}$ from the coefficients $\{b_i\}$ and vice versa. This, however, can only be achieved if certain invertibility conditions are satisfied, namely (Whittle,1963; Box and Jenkins,1966)

(a) the autoregressive process (3.22) may be inverted into the one-sided moving average process (3.20) if

(i) the expression $\left(1-\sum_{j=1}^{\infty} a_j B^j\right)$ is analytic in

$$|B|<1 \tag{3.24a}$$

(ii) the coefficients $\{a_j\}$ in (3.22) satisfy

$$\sum_{j=1}^{\infty} a_j^2 < \infty \tag{3.24b}$$

(b) the one-sided moving average process (3.20) may be inverted into the autoregressive process (3.22) if

(i) the expression $\sum_{j=0}^{\infty} b_j B^j$ is analytic in

$$|B|<1 \tag{3.25a}$$

(ii) the coefficients b_j in (3.20) satisfy

$$\sum_{j=0}^{\infty} b_j^2 < \infty \tag{3.25b}$$

Approaches to time series analysis based on finite parameter versions of the representations (3.20) and (3.22) were pioneered by Slutsky and Yule in 1920's (Parzen,1961). The former is credited with discovering a finite moving average scheme, which, for some integer m, is defined by

$$x_t = \sum_{i=0}^{m} b_i \zeta_{t-i} \qquad (3.26)$$

The researches of Yule, on the other hand, led to the notion of a finite autoregressive scheme, which, for some integer n, is defined by

$$x_t = \sum_{j=1}^{n} a_j x_{t-j} + \zeta_t \qquad (3.27)$$

The latter scheme may also be interpreted as a stochastic difference equation of order n (Mann and Wald,1943; Grenander and Rosenblatt,1957).

An excellent exposition of characteristic features of finite autoregressive and moving average schemes is given by Box and Jenkins (1966). For the purpose of the present discussion it will be sufficient to note that

(a) the finite moving average scheme exhibits properties of disturbed periodicity; its autocorrelation function vanishes for lags greater than m where m is the order of the scheme.

(b) the autocorrelation function of an autoregressive process satisfies the same difference equation as the process itself.

(c) the autoregressive process of order 1, defined by

$$x_t = a_1 x_{t-1} + \xi_t \tag{3.28}$$

relates the present value x_t of the process to its past through only one past value x_{t-1} and is generally called a Markov process.

(d) the autoregressive process of order 2 defined by

$$x_t = a_1 x_{t-1} + a_2 x_{t-2} + \xi_t \tag{3.29}$$

may exhibit a pseudo-periodic behaviour if the roots of the characteristic equation

$$1 - a_1 B - a_2 B^2 = 0 \tag{3.30}$$

are complex.

(e) as observed by Box and Jenkins(1966), there exists a duality between autoregressive and finite moving average processes. As a result of this duality,

1. the parameters of the autoregressive process are not required to satisfy any conditions to ensure invertibility; however, for stationarity, the roots of the characteristic equation

$$1 - \sum_{j=1}^{n} a_j B^j = 0 \tag{3.31}$$

must lie outside the unit circle.

2. the parameters of the moving average process are not required to satisfy any conditions for the stationarity however, for the invertibility of the moving average process the roots of the characteristic equation

$$\sum_{i=0}^{m} b_i B^i = 0 \tag{3.32}$$

must lie outside the unit circle.

An extension of the representations (3.26) and (3.27) is provided by an important subclass of processes with absolutely continuous spectral distribution function. These are processes whose spectral density function can be represented as rational function of exp(iw) in the form

$$f(\omega) = \sigma^2 \left| \frac{\sum_{j=0}^{m} b_j e^{i\omega j}}{\sum_{j=0}^{n} a_j e^{i\omega j}} \right|^2 \tag{3.33}$$

where both polynomials in (exp(iw)) are assumed to have all zeros strictly within the unit circle, and no common roots.

As shown by Doob(1953) such processes may be represented in the form

$$x_t = \int_{-\pi}^{\pi} e^{i\omega t} \left| \frac{\sum_{j=0}^{m} b_j e^{i\omega j}}{\sum_{j=0}^{n} a_j e^{i\omega j}} \right|^2 d\tilde{y}(\omega) \tag{3.34}$$

where $\tilde{y}(w)$ is a process with orthogonal increments.

Alternatively, the process x_t described by (3.34) may be represented by a stochastic difference equation

$$\left(\sum_{j=0}^{n} (a_j B^j) \right) x_t = \left(\sum_{j=0}^{m} b_j B^j \right) \xi_t \tag{3.35}$$

where the orthogonal random variables $\{\xi_t\}$ are defined by (3.15) and (3.16)

The representation (3.35) is referred to as the process of mixed type (Whittle,1963), or a mixed moving average-autoregressive process (Box and Jenkins,1966).

As discussed in the Appendix B, the difference equation of the type (3.35) may also be interpreted as that describing the output $\{x_t\}$ of a linear filter with a pulse transfer function

$$H(z) = \frac{\sum_{j=0}^{m} b_j z^{-j}}{\sum_{j=0}^{n} a_j z^{-j}} \qquad (3.36)$$

the input to the filter being the white noise sequence
In particular, if the sequence $\{\zeta\}$ is characterized by a Gaussian distribution, the process $\{x_t\}$ is said to be a Gauss-Markov process.

The idea that a rational spectral density may be associated with the output of a linear filter excited by a white noise was employed in a pioneering "shaping filter" method of Bode and Shannon(1950).They were concerned with the problem of prediction of a signal contaminated by noise. Since the only way in which the signal and noise entered their objective function to be minimized was through the power spectra, Bode and Shannon argued that the only statistics that are needed to solve the problem of prediction are the power spectrum of signal and noise. They suggested, therefore, representing a process $\{x_t, t \in T\}$ whose spectrum $P(w)=\sigma^2 f(w)$ is known, as the output of a linear filter with gain

$$Y(w) = \sqrt{P(w)} \tag{3.37}$$

and minimum phase

$$B(\omega_0) = -\frac{\omega_0}{\pi}\int_0^{\infty} \frac{\log P(\omega) - \log P(\omega_0)}{\omega^2 - \omega_0^2} \tag{3.37a}$$

A somewhat similar method was also described by Zadeh and Ragazzini(1950).

As is well known, two different types of signal may lead to the same spectrum and to the same optimum prediction filter The above representation is, therefore, not unique, and while it is suitable for prediction, it is not necessarily so for the modelling of the behaviour of processes. It has been demonstrated by Box and Jenkins(1966), however, that a covariance structure can uniquely determine a model, provided that the model is of a stationary-invertible type (3.35) in which the current value of the process x_t is expressed in terms of only the previous history.

It should be added that the comparatively recent approach to modelling of stochastic processes from discrete-time data does ~~no~~ involve the use of a model of the type (3.35), (or its state space equivalent), the parameters of the model being estimated by employing either linear regression (Kalman,1963) or least squares technique(Box and Jenkins,1962, 1963,1966,1967) or maximum likelihood method (Åström and Bohlin,1965a,1965b).

3.5. Modelling of discrete parameter nonstationary processes.

The preceding section dealt with the representation of stationary processes whose statistical characteristics do not change with time. Stationarity, however, seems to exist only as an ideal to which most physical processes do not conform, and, as a result, this feature may have to be acknowledged in the model of a given process.

When the assumption of stationarity is dropped, one is left with scarcely any restriction on one's model. For this reason, it is all the more difficult to specify the model, or even to specify some of the statistical characteristics of the variates. In consequence, the methods of modelling nonstationary processes have tended to be more or less empirical (Whittle, 1963).

An orthodox approach to the problem was based on Wold's theorem (Whittle, 1963) that any stationary process $\{x_t, t \in T\}$ can be uniquely represented as the sum of two mutually uncorrelated processes

$$x_t = m_t + \eta_t \tag{3.38}$$

in which

$$m_t = E\langle x_t \rangle \tag{3.39}$$

is deterministic and called the mean value function, and η_t represents a stationary random process with finite second moments and is called the fluctuation function (Parzen, 1961).

The relation (3.38) has been used for a long time to explain a nonstationary behaviour of certain processes by representing the function m_t as an appropriate function of time. Thus, m_t has been represented, by various researchers, as

(a) systematic oscillation (Parzen,1961)

$$m_t = \sum_{j=1}^{q} A_j \cos(\omega_j t + \phi_j) \tag{3.40}$$

in which the amplitudes A_j, the angular frequencies w_j and the phases ϕ_j are constants, some of which are given, and the rest are unknown and have to be estimated;

(b) polynomial trend (Zadeh and Ragazzini,1950)

$$m_t = \sum_{j=0}^{p} a_j t^j \tag{3.41}$$

in which the degree p of the polynomial is assumed and the coefficients a_j have to be estimated;

(c) sum of orthogonal polynomials (Whittle,1963; Thrall and Bendat,1965)

$$m_t = \sum_{j=0}^{p} a_j P_j(t) \tag{3.42}$$

where, for a sample $x_1, \ldots, x_N$

$$\sum_{t=1}^{N} P_j(t)\, P_k(t) = \delta_{jk} \tag{3.43a}$$

$$a_k = \sum_{t=1}^{N} m_t\, P_k(t) \tag{3.43b}$$

This type of model was used, for example, by

McCarty et al(1962,1963) for analyzing data contained in RF backscatter for information concerned with physical phenomena of missile flight. The many theorems which are presented and proved in the above reports seem, however, to be unnecessarily complicated and they could be presented in a more simple manner.

As discussed in the preceding section, the stationarity of the autoregressive, moving average and mixed autoregressive moving average models was ensured by requiring that all the zeros of the appropriate polynomials in the backward shift operator B should lie outside the unit circle. It seems reasonable, therefore, to expect that when these requirements are relaxed, a nonstationary behaviour of the corresponding models will result. (Whittle, 1963).

The earliest example of this approach (Whittle, 1963) is provided by the model of an <u>accumulated process</u> defined by the relation

$$x_t = \sum_{j=0}^{p} a_j x_{t-j} + \zeta_t \tag{3.44}$$

in which the zeros of the polynomial

$$A(B) = \sum_{j=0}^{p} a_j B^j \tag{3.45}$$

approach the unit circle and the $\{\zeta_t\}$ are orthogonal random variables, such as those characterized by the relations (3.15) and (3.16).

A more realistic approach to the representation of a process containing trends is to employ (Whittle,1965) a generalization of the mixed autoregressive-moving average model

$$\left(\sum_{j=0}^{n} a_j B^j \right) x_t = \left(\sum_{j=0}^{m} b_j B^j \right) \xi_t . \tag{3.46}$$

in which the polynomial

$$A(B) = \sum_{j=0}^{n} a_j B^j \tag{3.47}$$

has zeros on or inside the unit circle. It has been observed by Whittle(1965) that the mechanism generating this process is itself constant, although the process is evolutive, and, moreover, the nonstationarity of the process is evident not merely in its mean but also in all its moments as, indeed, is the case with observed evolutive series.

A model of the type (3.46) has found an important application in a prediction method based on an exponentially weighted moving average techniques (Whittle,1965; Otterman, 1960). In this approach no explicit model of the process is given, but the predictors are assumed to obey a low order model of the type (3.46). The coefficients in the relation are determined, partly by requiring that the predictor be exact for certain sequences (such as polynomial sequences) and partly by empirical search for values which seem to yield good predictors.

It has been observed by Box and Jenkins (1966) that, while a model of the type (3.44) (and, by virtue of the above remarks, also the model (3.46), may be of value in representing an explosive or evolutionary behaviour of processes such as bacterial growth, it is not suitable for representing many physical processes met with in practical applications. The reason is that, while, in general, the local behaviour of the latter appears to be independent of the current value x_t of the process $\{x_t\}$, the local behaviour of the evolutionary series does depend on the current value x_t of the series. In particular, the solution of the difference equation of the type (3.46) consists of a deterministic and a stochastic component. In the nonstationary case, the deterministic component builds up and dominates the stochastic component, the behaviour being essentially the same whether or not the "moving average" terms are introduced in this equation (Box and Jenkins, 1966).

Box and Jenkins (1966) show, therefore, that, for the representation of processes which are nonstationary, but nevertheless exhibit homogeneity, a model of the type

$$\left(\sum_{j=0}^{n} a_j B^j \right) \nabla^d x_t = \left(\sum_{j=0}^{m} b_j B^j \right) \zeta_t \quad (3.48)$$

should be used. Here

$$\nabla x_t = x_t - x_{t-1} \quad (3.49a)$$

$$= (1-B) x_t \quad (3.49b)$$

denotes a backward difference.

The polynomial A(z) given by (3.47) may have some of its zeros inside the unit circle, but the introduction of the difference operator ∇ ensures that the effective polynomial in B on the left hand side of the equation (3.48) has some zeros outside and some inside the unit circle. As a result, processes containing trends, but not being explosive, can be generated.

The process characterized by (3.48) has been introduced by Box and Jenkins earlier(1962,1963a,1963b) but without the above interpretation and in a different form as follows.

Replacing the operator B on the right hand side of (3.48) by the backward difference operator . one obtains,

$$A(B)\nabla^d x_t = (\lambda_{m-d}\nabla^{m-1} + \dots + \lambda_0 \nabla^{d-1} + \dots + \lambda_{d-1})\zeta_{t-1} + \nabla^d \zeta_t \qquad (3.50)$$

Employing now a summation operator S defined by (Box and Jenkins,1962)

$$S x_t = \sum_{j=0}^{\infty} x_{t-j} \qquad (3.51a)$$

$$S^2 x_t = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} x_{t-j-k} \qquad (3.51b)$$

etc., and summing (3.50) d times, one obtains,

$$A(B)x_t = P_{d-1}(t) + (\lambda_{m-d}\nabla^{m-d-1} + \dots + \lambda_0 S + \dots + \lambda_{d-1} S^d)\zeta_{t-1} + \zeta_t \qquad (3.52)$$

The process characterized by (3.52) has been called by Box and Jenkins(1966) an integrated autoregressive-moving average process of order (n,d,m). The first term $P_{d-1}(t)$ is the complementary function of the difference equation (3.50) and is a polynomial in t of degree (d-1), with coefficients depending on the starting values of the series.

The model introduced by Box and Jenkins earlier (1962,1963a, 1963b) in a rather empirical fashion, represents a particular case of (3.52) with A(B)=1 and $P_{d-1}(t)=0$.

3.6. Concluding remarks.

In this Chapter an attempt has been made to review critically the existing techniques,known to the writer,of parametric description of stationary and nonstationary processes (for this reason, the representations in terms of expansions (e.g. Karhuen,1947) are outside the scope of the review) None of the techniques was suitable for application to the problem of on-line identification on which the writer was working. This has led to the development of another description of nonstationary processes,discussed in Chapter 5 and Appendix C.

CHAPTER 4.

DIGITAL TECHNIQUES OF PARAMETRIC IDENTIFICATION OF A LINEAR SYSTEM.

4.1. Introduction.

Chapter 2 dealt with identification techniques yielding estimates of the discrete values of the impulse response or frequency response. It was pointed out that a great disadvantage of these techniques is the large number of parameters required to represent a response adequately. An alternative approach is to characterize a system by a differential or difference equation and to identify the system by determining the order of such an equation, and estimating its parameters. The techniques identifying a system in such a fashion, and involving the use of regression analysis , are reviewed in the following sections.

4.2. System identification as a Kalman filtering problem.

4.2.1. General.

When the coefficients of a differential or difference equation are known, the state of the system (Zadeh and Desoer 1963) may be estimated by using the well-established filtering techniques (Kalman,1960,1961,1963a). It is possible, however, to employ the state space approach in the identification problem in which such coefficients are unknown and are treated as parameters to be estimated. The parameters may then be considered as part of the state vector and estimated jointly with the state variables proper.

Such an approach has been adopted, for example, by Kopp and Orford (1963). The approach may be briefly summarized as follows.

(a) observable output and effective input are regarded as being contaminated by noise with given statistical characteristics;

(b) parameters of the differential (or difference) equation describing the system are considered as additional state variables and incorporated in an enlarged state vector;

(c) assumptions are made about the fashion in which these parameters are supposed to vary; from these assumptions a set of constraint equations is obtained and adjoined to the state equations of the system;

(d) the resulting non-linear differential equations are perturbed about the current estimate of the enlarged state vector;

(e) the problem of estimating the linearized state vector is formulated as a solution of the Kalman filtering proble

4.2.2. Statement of the problem and assumptions.

Consider a general single-input single-output system characterized by a differential equation

$$(a_n(t)p^n + a_{n-1}(t)p^{n-1} + \ldots\ldots + a_o(t))y(t)$$

$$= (b_n(t)p^n + b_{n-1}(t)p^{n-1} + \ldots\ldots + b_o(t))\,u(t) \qquad (4.1)$$

in which $a_n(t) \neq 0$ and some of the coefficients $b_j(t)$ may be equal to zero.

In any practical situation the input and output of the system will be contaminated by noise which, in the technique being described, is assumed to be additive. Consequently, the state equations of the system (4.1) can be written in the form

$$\dot{\underline{x}}(t) = \underline{F}(t).\underline{x}(t) + \underline{G}(t).\underline{u}(t) + \underline{\eta}(t) \qquad (4.2a)$$

$$z(t) = \underline{H}(t).\underline{x}(t) + v(t) \qquad (4.2b)$$

where

$\underline{x}(t)$ is an n x 1 vector,

$\underline{G}(t)$ is an n x 1 matrix,

$\underline{F}(t)$ is an n x n matrix

$\underline{H}(t)$ is a 1 x n matrix

The disturbing noises $\{\eta(t)\}$ and $\{v(t)\}$ are assumed to have known characteristics as follows:

(a) $\{v(t)\}$ is a normally distributed white noise with zero mean and variance $\sigma_v^2(t)\,\delta(t-\tau)$;

(b) $\{\eta(t)\}$ is a zero mean normally distributed variable which is a white noise modulated by a known function $S(u(t))$ of the noise-free input, that is

$$\eta(t) = S(u(t)).w_o(t) \tag{4.3}$$

where $w_o(t)$ is white noise with zero mean and variance $N_{w_o}(t)\delta(t-\tau)$;

The manner in which the parameters $\{f_{ij}(t)\}$ and $\{g_i(t)\}$ of the matrices $\underline{F}(t)$ and $\underline{G}(t)$, respectively, are supposed to vary, is presented by the constraint relations in the form

$$\dot{\underline{F}}(t) = \underline{\Theta}(t) \tag{4.4a}$$

$$\dot{\underline{G}}(t) = \underline{\Delta}(t) \tag{4.4b}$$

The coefficients $\Theta_{ij}(t)$ and $\phi_i(t)$ of the matrices $\underline{\Theta}(t)$ and $\underline{\Delta}(t)$, respectively, are supposed to vary continuously in a random manner so that

$$\Theta_{ij}(t) = \Theta^*_{ij}(t)f_{ij}(t) + w_{ij}(t) \tag{4.5a}$$

$$\phi_i(t) = \Theta^*_i(t)g_i(t) + \tilde{w}_i(t) \tag{4.5b}$$

where the parameters $\{\Theta^*_{ij}(t)\}$ and $\{\phi^*_i(t)\}$ are assumed to be given, and $\{w_{ij}(t)\}$ and $\{w_i(t)\}$ are zero mean gaussian noises with known variances $\sigma^2_{w_{ij}}\delta(t-\tau)$ and $\sigma^2_{\tilde{w}_i}(t)\delta(t-\tau)$, respectively.

4.2.3. Method of solution.

Let an enlarged state vector $\underline{q}(t)$, composed of the elements of the state vector $\underline{x}(t)$ and the elements of matrices $\underline{F}(t)$ and $\underline{G}(t)$ be defined as

$$\underline{q}^T(t) = (x_1(t),\ldots,x_n(t);f_{11}(t),\ldots,f_{nn}(t);\; g_1(t),\ldots,g_n(t)\ ) \tag{4.6}$$

Similarly, let an enlarged disturbance vector $\underline{\lambda}(t)$, composed of the elements of the noise vector $\underline{\mu}(t)$ and random components of the matrices $\underline{\theta}(t)$ and $\underline{\Delta}(t)$ be defined by

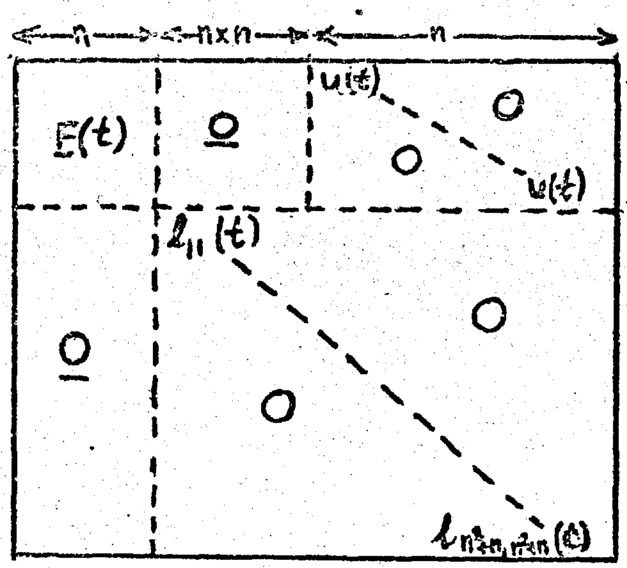
$$\underline{\lambda}(t)^T = (\mu_1(t),\ldots,\mu_n(t);\ w_{11}(t),\ldots,w_{nn}(t);\; \tilde{w}_1(t),\ldots,\tilde{w}_n(t)\ ) \tag{4.7}$$

Then, when the constraint equations (4.4) are adjoined to the state equations (4.2), the enlarged nonlinear state equations are written in the form

$$\dot{\underline{q}}(t) = \underline{L}(t)\underline{q}(t) + \underline{\lambda}(t) \tag{4.8}$$

where

$\underline{L}(t) =$



(4.9)

In the above matrix the diagonal terms $\ell_{j,j}(t)$ correspond to the asterisked parameters of equation (4.5).

For a discrete-time analysis, of interest in this thesis, it is assumed that the input and the output are sampled every ΔT seconds, at the end of each interval, and that the input $u(t)$ is constant over any interval, changing in a stepwise manner between the intervals.**

Adopting, for a variable $\xi(t)$, the notation,

$$\xi(n) = \xi(n\Delta T) \tag{4.10a}$$

$$\left.\begin{aligned}\hat{\xi}(t/n) &= \text{estimate of } \xi_t \text{ given observations} \\ &\quad \text{up to } n\Delta T \quad n\Delta T \leq t < (n+1)\Delta T\end{aligned}\right\} \tag{4.10b}$$

$$\left.\begin{aligned}\tilde{\xi}(t/n) &= \xi(t) - \hat{\xi}(t/n) \\ &= \text{estimation error}\end{aligned}\right\} \tag{4.10c}$$

$$n\Delta T \leq t < (n+1)\Delta T$$

the problem of estimation is formulated as a recursive problem of estimating the current state vector $\underline{q}(n+1)$, given the conditional estimate $\hat{\underline{q}}(n/n)$ based on observations up to the time $n\Delta T$, and the current observation of output, $z(n+1)$. The procedure is carried out in the following steps:

(a) the estimates $\hat{\underline{q}}(t/n)$ of the state vector $\underline{q}(t)$ between the sampling intervals $n\Delta T$ and $(n+1)\Delta T$, given the observation up to the time $n\Delta T$ are governed by the equations

---

** This is the case of a zero order hold discussed in Appendix B.

$$\underline{q}(t/n) = \underline{L}(t/n)\, \underline{q}(t/n) \tag{4.11}$$

where

$$\underline{L}(t/n) = \left[\begin{array}{c|c|c} \hat{F}(t|n) & 0 & \begin{matrix} u(t) & & \\ & \ddots & \\ & & u(t) \end{matrix} \\ \hline 0 & \multicolumn{2}{c}{\begin{matrix} l_{n+1,n+1}(t|n) & & 0 \\ & \ddots & \\ 0 & & l_{3n,3n}(t|n) \end{matrix}} \end{array}\right] \tag{4.12}$$

(b) the differential equations governing the estimation errors

$$\tilde{\underline{q}}(t/n) = \underline{q}(t) - \hat{\underline{q}}(t/n) \tag{4.13}$$

are obtained by perturbing equations (4.8) about the current estimate of the variables. Thus, subtracting equations (4.11) from (4.8) and neglecting second order quantities, the linearized perturbation equations are

$$\dot{\tilde{\underline{q}}}(t/n) = \underline{L}_1(t/n)\tilde{\underline{q}}(t/n) + \underline{\lambda}_1(t) \tag{4.14}$$

where

$$\underline{\lambda}_1^T(t) = (\, \mu_{11}^*(t/n), \ldots, \mu_n^*(t/n); w_{11}(t), \ldots, w_{nn}(t); \tilde{w}_1(t), \ldots, \tilde{w}_n(t)\,) \tag{4.15a}$$

$$u_l^*(t/n) = \hat{g}_l(t/n).S(u(t))w_o(t) \tag{4.15b}$$

The matrix $\underline{L}_1(t/n)$, the components of which are the estimates of the components of the enlarged state vector $\underline{q}(t/n)$ at time t given data to time $n\Delta T$, is given by

$$\underline{L}_1(t/n) = \left[\begin{array}{c|ccc|ccc} \hat{F}(t|n) & \hat{\underline{x}}^T(t|n) & & & u(t) & & \\ & & \ddots & & & \ddots & \\ & & & \hat{\underline{x}}^T(t|n) & & & u(t) \\ \hline & \ell_{1,1}(t) & & & & & \\ \underline{0} & & \ddots & & & 0 & \\ & & 0 & & \ddots & & \\ & & & & & & \ell_{n^2-n,\,n^2+n}(t) \end{array}\right] \qquad (4.16)$$

(c) It is now assumed that the disturbance vector $\underline{\lambda}_1(t)$ is constant in the interval $n\Delta T \leq t < (n+1)\Delta T$ and that its components are zero mean gaussian variables with covariance matrix $\underline{Q}(t)\underline{\delta}(t-\tau)$

$$E\left\langle \underline{\lambda}_1(t)\,\underline{\lambda}_1^T(\tau)\right\rangle = \underline{Q}(t)\underline{\delta}(t-\tau) \qquad (4.17a)$$

$$E\left\langle \underline{\lambda}_1(t)\right\rangle = \underline{0} \qquad (4.17b)$$

It is also assumed that the estimation error $\tilde{\underline{q}}(n/n)$ at the sampling instants $n\Delta T$ is distributed multinormally with zero mean and covariance matrix $\underline{P}(n/n)$, i.e.

$$E\left\langle \tilde{\underline{q}}(n/n)\,\tilde{\underline{q}}^T(n/n)\right\rangle = \underline{P}(n/n) \qquad (4.18a)$$

$$E\left\langle \tilde{\underline{q}}(n/n)\right\rangle = \underline{0} \qquad (4.18b)$$

(d) With the assumptions as above, the solution of equation (4.14) is given by

$$\underline{\tilde{q}}(n+1/n) = \underline{\emptyset}(n+1,n)\ \underline{\tilde{q}}(n/n) + \underline{\Gamma}(n+1,n)\ \underline{\lambda}_1(n) \qquad (4.19a)$$

where $\underline{\emptyset}(n+1,n)$ is the transition matrix of equations (4.14) and

$$\underline{\Gamma}(n+1,n) = \int_{n\Delta T}^{(n+1)\Delta T} \underline{\emptyset}((n+1)\Delta T, \tau)\, d\tau \qquad (4.19b)$$

(e) The solution of equations (4.11) yields the estimate $\underline{\hat{q}}(n+1/n)$ of the enlarged state vector $\underline{q}(t)$ at the time $(n+1)\Delta T$, given the observations up to the time $n\Delta T$. The problem is now to obtain the estimates of this vector at time $(n+1)\Delta T$, when the new observation $z(n+1)$ is availabl In other words, it is required to estimate

$$\underline{\hat{q}}(n+1/n+1) = \underline{\hat{q}}(n+1/n) + \underline{\tilde{q}}(n+1/n+1) \qquad (4.20a)$$

An estimate $\underline{\hat{\tilde{q}}}(n+1/n+1)$ of the error $\underline{\tilde{q}}(n+1/n+1)$ given data to $(n+1)\Delta T$ is made by linear regression, i.e.

$$\underline{\hat{\tilde{q}}}(n+1/n+1) = K(n+1)\ \tilde{z}(n+1) \qquad (4.20b)$$

where

K(n+1) is determined by minimizing the diagonal elements of the covariance matrix $\underline{P}(n+1/n+1)$ of the error $\underline{\tilde{q}}(n+1/n+1)$

$\tilde{z}(n+1)$ denotes the difference between the actual new data received and the estimate of the data conditioned on the previous sampling interval

$$\tilde{z}(n+1) = z(n+1) - \hat{z}(n+1/n) \qquad (4.20c)$$

Let a $1 \times (n + n^2 + n)$ matrix $\underline{M}$ be defined as

$$\underline{M} = (\underbrace{\underline{H}}_{n}, \underbrace{0, \ldots\ldots, 0}_{n^2+n}) \qquad (4.21)$$

Then,

$$\tilde{z}(n+1) = \underline{M}\,\underline{\tilde{q}}(n+1/n) + v(n+1) \qquad (4.22)$$

The expression for the optimal estimates $\underline{\hat{q}}(n+1/n+1)$ of the enlarged state vector $\underline{q}(n+1)$ at time $(n+1)\Delta T$ given the data up to $(n+1)\Delta T$, is obtained from (4.20a),(4.20b) and (4.22) as

$$\underline{\hat{q}}(n+1/n+1) = \underline{\hat{q}}(n+1/n) + \underline{\hat{\tilde{q}}}(n+1/n+1) \qquad (4.23a)$$

or,

$$\underline{\hat{q}}(n+1/n+1) = \underline{\hat{q}}(n+1/n) + K(n+1)\,(\,\underline{M}\,\underline{\tilde{q}}(n+1/n) + v(n+1)\,) \qquad (4.23b)$$

This equation is seen to represent a Kalman filter with the gain K(n+1) given by

$$K(n+1) = \underline{P}(n+1/n)\,\underline{M}^T\,(\,\underline{M}\,\underline{P}(n+1/n)\,\underline{M}^T + \sigma_v^2(n+1)\,)^{-1} \qquad (4.24)$$

and governed by the nonlinear Ricatti-type differential equation, called the variance equation and given by

$$\underline{P}(n+1/n) = \underline{\emptyset}(n+1,n)\,\underline{P}(n/n)\,\underline{\emptyset}(n+1,n) + \Gamma(n+1,n)\,\underline{Q}(n)\,\Gamma^T(n+1,n) \qquad (4.25a)$$

where the relation between $\underline{P}(n+1/n)$ and $\underline{P}(n+1/n+1)$ is

$$\underline{P}(n+1/n+1) = (\,\underline{I} - K(n+1)\,\underline{M}\,)\,\underline{P}(n+1/n) \qquad (4.25b)$$

$\underline{I}$ being the unit matrix.

The estimates $\underline{q}(t)$ are obtained in an iterative fashion as follows.

ances generated inside the system. Such disturbances are better represented as stationary random processes with rational spectral densities.

As discussed in Chapter 3, a process of this type may be modelled by an output of a linear dynamical system excited by white noise. In the discrete-time description, such a linear system is characterized by a pulse transfer function (see Appendix B),

$$H(z^{-1}) = \frac{N(z^{-1})}{D(z^{-1})} \tag{4.26}$$

where $N(z^{-1})$ and $D(z^{-1})$ are polynomials in the variable z^{-1}, have no common factors, and have zeros within the unit circle. A stationary discrete-time random process $e(t)$ may thus be represented by

$$e(t) = \frac{N_1(z^{-1})}{D_1(z^{-1})} \cdot w(t) \qquad t = \ldots, -\Delta T, 0, \Delta T, \ldots \tag{4.27}$$

where $\{w(t)\}$ is a zero-mean gaussian white noise sequence.

By virtue of the superposition principle, characterizing linear dynamical systems, any disturbance affecting the input can be transformed so as to appear at the output. Such transformed disturbances may be combined with any other disturbances affecting the output, into an equivalent output disturbance $e(t)$ of the form(4.27).

By virtue of the superposition property the effects of the disturbance $e(t)$ and of the input $u(t)$ on the output $y(t)$ can be considered separately. The dynamical system itself is also characterized by a pulse transfer function

$$H_2(z^{-1}) = \frac{N_2(z^{-1})}{D_2(z^{-1})} \tag{4.28}$$

and the output $y(t)$ may be written in the form

$$y(t) = \frac{N_2(z^{-1})}{D_2(z^{-1})} u(t) + \frac{N_1(z^{-1})}{D_1(z^{-1})} w(t) + k \tag{4.29}$$

where k is a constant and all the zeros od $D_2(z^{-1})$ and $D_1(z^{-1})$ are strictly within the unit circle.

If

$$D_2(z^{-1}).D_1(z^{-1}) = D(z^{-1}) \tag{4.30a}$$

$$N_2(z^{-1}).D_1(z^{-1}) = N_3(z^{-1}) \tag{4.30b}$$

$$N_1(z^{-1}).D_2(z^{-1}) = N_4(z^{-1}) \tag{4.30c}$$

then the relation (4.29) may be written (Åström and Bohlin,1965a,1965b)

$$y(t) = \frac{N_3(z^{-1})}{D(z^{-1})} u(t) + \frac{N_4(z^{-1})}{D(z^{-1})} w(t) + k \tag{4.31}$$

or, explicitly,

(4.32)

$$y(t) = \frac{\beta_n' z^{-n} + \beta_{n-1}' z^{-(n-1)} + .. \beta_0'}{\alpha_n' z^{-n} + \alpha_{n-1}' z^{-(n-1)} + .. + \alpha_0'} u(t)$$

$$+ \frac{\gamma_n' z^{-n} + \gamma_{n-1}' z^{-(n-1)} + ... + \gamma_0}{\alpha_n' z^{-n} + \alpha_{n-1}' z^{-(n-1)} + ... + \alpha_0'} w(t) + k \tag{4.32}$$

Some redundancy in this model is removed by a reparameterization as follows

$$y(t) = \frac{\beta_n z^{-n} + \beta_{n-1} z^{-(n-1)} + ... + \beta_0}{\alpha_n z^{-n} + \alpha_{n-1} z^{-(n-1)} + .. + 1} u(t)$$

$$+ \lambda \frac{\gamma_n z^{-n} + \gamma_{n-1} z^{-(n-1)} + ... + 1}{\alpha_n z^{-n} + \alpha_{n-1} z^{-(n-1)} + .. + 1} w(t) + k . \tag{4.33}$$

The model strúcture is the general representation of a finite-dimensional, completely controllable, completely observable single-input single-output system with arbitrary disturbances in terms of stationary,gaussian random process with rational spectral density.

It is clear from equations (4.29)-(4.33) that the dynamics represented by the polynomial $D(z^{-1})$ is partly due to the system dynamics and partly due to the representation of disturbances. An investigation of the common factors of $N_3(z^{-1})$ $N_4(z^{-1})$ and $D(z^{-1})$ will separate one from the other. Should there be no such common factors, every state of the system (4.33) would be controllable either from u(t) or from w(t).

The equation (4.33) contains (4n+3) parameters: n coefficients α_i , (n+1) coefficients β_i , n coefficients γ_i ,n initial conditions and k. The identification problem is, for an assumed value of n, to find estimates of these parameters from the given observations of the input u(t) and output y(t), $t = \Delta T, 2\Delta T, \ldots, N\Delta T$

In the approach of Åström and Bohlin, the parameters are estimated by using the method of maximum likelihood estimation (see Appendix A), the essence of the approach being an efficient algorithm for minimizing the logarithm of the likelihood function. The approach is briefly sketched below.

First, an expression for the probability density function of the observations y(t) ($t = \Delta T, 2\Delta T, \ldots, N\Delta T$) is obtained as a function of the inputs u(t) ($t = \Delta T, \ldots, N\Delta T$) and the parameters. For this purpose the variables w(t) of equation (4.33) are replaced by new variables*

$$\varepsilon(t) = c_0 w(t) \tag{4.34}$$

and expressed as a function of observations. The problem is then formulated in terms of state equations as follows.

$$\underline{x}(t+1) = \underline{\Phi}\, \underline{x}(t) + \underline{\Gamma} u(t) + \underline{\Delta}\left[y(t) - k\right] \tag{4.35a}$$

$$\varepsilon(t) = x_n(t) - b_0 u(t) + y(t) - k \tag{4.35b}$$

where $\underline{x}(t)$ is an n-dimensional state vector, the variables ε are independent normal with zero mean and variance c_0^2 and the matrices $\underline{\Phi}$, $\underline{\Gamma}$, and $\underline{\Delta}$ are defined by

* {w(t)} HAVE UNIT VARIANCE

$$\underline{\Phi} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\gamma_n \\ 1 & 0 & \cdots & 0 & -\gamma_{n-1} \\ 0 & 1 & & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ 0 & 0 & & 1 & -\gamma_1 \end{bmatrix} \tag{4.36a}$$

$$\underline{\Gamma} = \begin{bmatrix} -\beta_n + \beta_0 \gamma_n \\ -\beta_{n-1} + \beta_0 \gamma_{n-1} \\ \vdots \\ -\beta_1 + \beta_0 \gamma_1 \end{bmatrix} \tag{4.36b}$$

$$\underline{A} = \begin{vmatrix} \alpha_n - \gamma_n \\ \alpha_{n-1} - \gamma_{n-1} \\ \vdots \\ \alpha_1 - \gamma_1 \end{vmatrix} \tag{4.36c}$$

If L is the logarithm of the likelihood function,

$$L = -\left(\frac{1}{2c_0^2}\sum_{t=1}^{N}\varepsilon_t^2 + N\log c_0 + \frac{N}{2}\log 2\pi\right) \qquad (4.37)$$

then the maximum likelihood estimates of the 4n+3 parameters are obtained as those estimates which result in the minimum of the function (4.37). To simplify the notation, all the parameters are denoted by $\{\theta_i\}$ where

$$\begin{aligned}
\theta_i &= \gamma_i & i&=1,..,n\\
\theta_{n+i} &= -\beta_i+\beta_0\gamma_i & i&=1,..n\\
\theta_{2n+i} &= \alpha_i-\gamma_i & i&=1,..n\\
\theta_{3n+i} &= x_i(t) & i&=1,..n\\
\theta_{4n+1} &= b_0\\
\theta_{4n+2} &= k\\
\theta_{4n+3} &= c_0 = \lambda.
\end{aligned} \qquad (4.38)$$

The function L is minimized in two stages: first, a minimum of the function

$$V(\underline{\theta}) = \frac{1}{2}\sum_{t=1}^{N}\varepsilon_t^2 \qquad (4.39)$$

is obtained, where

$$\underline{\theta}^T = (\theta_1, \theta_2, \ldots, \theta_{4n+2}) \qquad (4.40)$$

Then the estimate of θ_{4n+3} is obtained from

$$\hat{\theta}_{4n+3}^2 = \hat{c}_0^2 = \frac{1}{N}\min_{\underline{\theta}}\sum_{t=1}^{N}\varepsilon_t^2 \qquad (4.41)$$

The log-likelihood function L has continuous partial derivatives of all orders and the minimum is finite, though not necessarily unique.. For this reason the technique chosen for the minimization of the function (4.39) and (4.41) is a gradient technique enabling fast convergence to be obtained through the use of th Newton-Raphson algorithm (Deutsch, 1965). The maximum likelihood estimates are thus obtained from

$$\underline{\theta}^{(k+1)} = \underline{\theta}^{(k)} - [\underline{V}_{\theta\theta}^{-1} (\underline{\theta}^{(k)}) \underline{V}_{\theta} (\underline{\theta}^{(k)})] \tag{4.42}$$

where $\underline{V}_{\theta}$ is a vector with components

$$v_{\theta_i} = \frac{\partial \underline{V}(\underline{\theta})}{\partial \theta_i}, \quad i = 1, 2, \ldots 4n+2 \tag{4.43a}$$

$$\frac{\partial V(\underline{\theta})}{\partial \theta_i} = \sum_{t=1}^{N} \varepsilon(t) \frac{\partial \varepsilon(t)}{\partial \theta_i} \tag{4.43b}$$

$\underline{V}_{\theta\theta}$ is a matrix of second partial derivatives with elements given by

$$v_{\theta\theta ij} = \frac{\partial^2 \underline{V}(\underline{\theta})}{\partial \theta_i \partial \theta_j} \quad i,j = 1, 2, \ldots 4n+2 \tag{4.44a}$$

$$\frac{\partial^2 \underline{V}(\theta)}{\partial \theta_i \partial \theta_j} = \sum_{t=1}^{N} \frac{\partial \varepsilon(t)}{\partial \theta_i} \cdot \frac{\partial \varepsilon_t}{\partial \theta_j} + \sum_{t=1}^{N} \varepsilon(t) \frac{\partial^2 \varepsilon(t)}{\partial \theta_i \partial \theta_j} \tag{4.44b}$$

and the derivatives of $\varepsilon(t)$ are obtained from the relations (4.35) in terms of the partial derivatives of the state vector $\underline{x}(t)$ with respect to the parameters θ.

Åström and Bohlin (1965a) show how to calculate these partial derivatives so as to achieve shortest possible computation times. They prove also that their estimates possess all the desirable large sample properties.(see Appendix A). The details are, however, thought to be beyond the scope of this review and the interested reader is referred to the valuable report quoted above.

4.4 Identification of a linear system subject to a nonstationary correlated disturbance.

The identification technique discussed in the preceding section is really applicable under stationary conditions, when the statistical characteristics of the disturbance do not change with time. When this is not the case, the model may be cyclically updated, as suggested by Åström and Bohlin. A more sophisticated way of dealing with such a situation is, however, to acknowledge the nonstationary character of the disturbance and to allow for it by including a suitable model in the input-output relation of the system under consideration.

This approach was adopted by Box and Jenkins. Their technique was first described in 1962 and 1963 and was originally devised to deal with closed-loop control systems. The technique has been greatly extended and consolidated during the last two years , and published as a series of Lancaster University Technical Reports (Box and Jenkins,1966,1967). While it is comparatively easy to review the earlier work, it is rather difficult to give justice,within the framework of a small section,to the above mentioned reports,constituting effectively a preprint of a projected book. Therefore, only the more important aspects of the "Box and Jenkins" approach can be highlighted here.

The essence of the early "closed loop" phase of the approach (Box and Jenkins,1962,1963) is the design of a controller to control an industrial process subjected to a nonstationary disturbance. The design is carried out in two stages. First,the process dynamics and the character of the disturbance are identified. Then, a control law is derived such that the variation of the input signal matches, as closely as possible,the variations of the output of the process due to the disturbance. Only the identification part of the procedure is reviewed below.

The characteristics of the process and the disturbance are derived from the results of two tests. In one test, no control is exercised over the process and the process is allowed to drift under the influence of the output disturbance. In the second test known adjustments are made to the process and the resultant variations in the output, due to the combined effect of control and disturbance, are noted.

The nonstationary disturbance is represented as an "integrated autoregressive-moving average" stochastic process V(t), discussed in Chapter 3 and defined by

$$V(t+1) = [\gamma_{(\ell-1)} \nabla^{\ell} + \ldots + \gamma_{-1} + \gamma_0 S + \ldots + \gamma_m S^{m+1}] e_t + e_{t+1} \qquad (4.45)$$

where e(t) is a zero-mean gaussian white noise with variance σ_v^2, γ_j are constants, and ∇ and S denote, respectively, a backward difference operator

$$\nabla e_t = e_t - e_{t-1} \qquad (4.46a)$$

and a summation operator

$$S e_t = \sum_{j=0}^{\infty} e_{t-j} \qquad (4.46b)$$

The structure of the disturbance model (4.45) is obtained from the results of the first test, as the drifting output of the process under consideration represents the disturbance itself. On differencing the model (m+1) times one obtains a moving average process of order (ℓ+m+1)

$$\nabla^{m+1} V_{t+1} = e_{t+1} + \sum_{j=0}^{\ell+m} d_j e_{t-j} \qquad (4.47)$$

The characteristic feature of such a process is that all the autocorrelations of lag greater than (ℓ+m+1) are zero.

The approximate constants m and ℓ of the most appropriate model (4.45) of a given disturbance are thus obtained as follows. First, the series of readings is differenced until it appears stationary. Then sample autocorrelation function of the resultant series is obtained and the lag is estimated at which the correlations appear insignificant. When the structure of the model (4.45) and, therefore, the numbers of the parameters involved, has thus been determined, and an approximate structure of the system dynamics is known, the parameters of the overall input-output relation may be estimated from the results of the second test. To this end the dynamic characteristics of the process are expressed by a difference equation (see Appendix B)

$$c_m u_k + c_{m-1} u_{k-1} + \dots + c_0 u_{k-m}$$

$$= d_m y_k + d_{m-1} y_{k-1} + \dots + d_0 y_{k-m} \qquad (4.48)$$

or, introducing a backward shift operator B defined by

$$B^j x_t = x_{t-j} \qquad (4.49)$$

the relation (4.48) is written

$$y_t = \frac{c_m B^0 + c_{m-1} B + \dots + c_0 B^m}{d_m B^0 + d_{m-1} B + \dots + d_0 B^m} u_t. \qquad (4.50)$$

The expression for the output z(t) due to the combined effect of the input u(t) and the disturbances v(t) is then written as

$$z_{t+1} = y_{t+1} + v_{t+1} \tag{4.51}$$

or, explicitly,

$$z_{t+1} = \frac{c_m B^0 + \dots + c_0 B^m}{d_m B^0 + \dots + d_0 B^m} u_{t+1}$$
$$+ \left[\gamma_{\ell-1} (1-B)^{\ell-1} + \dots + \gamma_0 (1-B)^{-1} + \dots + \gamma_m (1-B)^{-(m+1)} \right] e_t + e_{t+1} \tag{4.52}$$

where

$$\nabla = 1 - B \tag{4.53a}$$
$$S = \nabla^{-1} = (1-B)^{-1} \tag{4.53b}$$

The parameters of the model (4.52) are then obtained as maximum likelihood estimates as follows. First, the errors

$$\underline{\varepsilon}\,(\underline{c}, \underline{d}, \underline{\gamma} \mid \underline{u}, \underline{z})$$

are calculated recursively as a function of the parameter vectors $\underline{c}$, $\underline{d}$, $\underline{\gamma}$ conditioned on the vector of observations $\underline{u}, \underline{z}$

$$\varepsilon_{j+1} = z_{j+1} - \left(\frac{c_m B^0 + \dots + c_0 B^m}{d_m B^0 + \dots + d_0 B^m} \right) u_{j+1}$$
$$- \left[\gamma_{\ell+1} (1-B)^{\ell} + \dots + \gamma_m (1-B)^{-(m+1)} \right] e_j \tag{4.54a}$$

where

$$\underline{\varepsilon}^T = \varepsilon_1, \varepsilon_2 \dots, \varepsilon_N \tag{4.54b}$$
$$\underline{u}^T = u_1, u_2 \dots, u_N \tag{4.54c}$$
$$\underline{z}^T = z_1, z_2 \dots z_N \tag{4.54d}$$

Then the sum of squares

$$S(\underline{c}, \underline{d}, \underline{\gamma}) = \underline{\varepsilon}^T \underline{\varepsilon} \tag{4.55}$$

proportional to the log-likelihood function, is calculated and minimized by using a standard nonlinear estimation program(Booth and Peterson,1960). The minimization yields the maximum likelihood estimates of the parameters.

It should be observed that the identification of the structure of the nonstationary disturbance is SIMPLIFIED ~~made possible only~~ by the closed loop nature of the problem,However, in the "open-loop" systems the effect of the disturbance on the output cannot be EASILY separated from that of the input, and the "closed-loop" Box and Jenkins approach cannot be employed.

Box and Jenkins have evidently realized this and in their later work(1966,1967) they modified their approach to include the open loop systems also. This latest approach to the identification of open loop systems can be summarized as follows.

The technique consists of three parts: identification of the structure of the dynamic model, estimation of parameter in a tentatively entertained model, and diagnostic checks of the adequacy of the model. The technique applies now only under stationary conditions. Therefore, a preliminary check on the stationarity of the input and output is first made, for example,by obtaining and examining a sample cross-correlation function of the input and of the output. If necessary, the input and output series are differenced until they appear approximately stationary.

At this stage a preliminary identification procedure is carried out in order to obtain a rough idea of the probable structure of the dynamic model. For this purpose the output y_t is represented in the form

$$y_t = \frac{N(B)}{D(B)} B^b u_t + e_t \tag{4.56}$$

or, alternatively,

$$y_t = V(B) u_t + e_t \tag{4.57a}$$

$$V(B) = \sum_{j=0}^{\infty} v_j B^j \tag{4.57b}$$

In these relations N(B) and D(B) denote polynomials in the backward shift operator B, v_j are heights of the impulse response at instants $j\Delta T$ when the input is passed through a zero-order hold, B^b corresponds to a transport lag of b sampling periods, and $\{e_t\}$ is the noise sequence uncorrelated with the input, the effect of the noise being assumed to be small compared with the variation of the input u_t.

The preliminary procedure is based on the fact that, if the input to a linear system is in the form of a white noise, the crosscorrelation function of the input and of the output is identical with the impulse response of the system.

An approximation to a white noise input is obtained by fitting to the actual input series a mixed autoregressive-moving average model

$$(1 - \phi_1 B - \dots - \phi_p B^p) u_t = (1 - \theta_1 B - \dots - \theta_q B^q) \alpha_t \tag{4.58a}$$

or,

$$\phi_u(B)u_t = \theta_u(B)\alpha_t \tag{4.58b}$$

where $\{\alpha_t\}$ is a white noise sequence with variance σ_α^2, the estimate $\hat{\sigma}_\alpha^2$ of which is obtained from the sum of squares of residual errors.

The white noise input

$$\alpha_t = \frac{\phi_u(B)}{\theta_u(B)} u_t \tag{4.59}$$

is then crosscorrelated with the transformed output

$$z_t = \frac{\phi_u(B)}{\theta_u(B)} y_t \tag{4.60a}$$

or,

$$z_t = V(B)\alpha_t + \frac{\phi_u(B)}{\theta_u(B)} e_t \tag{4.60b}$$

and the heights v_j of the impulse response are obtained from the sample cross-covariance function

$$\hat{\gamma}_{\alpha z}(j) = \hat{\sigma}_\alpha^2 v_j \tag{4.61}$$

where

$$\gamma_{\alpha z}(j) = E\langle \alpha_t z_{t+j} \rangle \tag{4.62a}$$

and

$$\hat{\gamma}_{\alpha z}(j) = \frac{1}{N} \sum_{t=1}^{N-j} \alpha_t z_{t+j} \tag{4.62b}$$

The identification is effected by plotting the impulse response thus obtained and selecting a model whose theoretical impulse response most closely resembles the plotted response.

When an indication of the probable structure of the dynamic model has just been obtained, a model of the input-output relation is tentatively postulated and its parameters are estimated by using the least squares method. At this stage Box and Jenkins make a difference between a "linearized" model and a "nonlinear" model. In the writer's opinion, Box and Jenkins are not consistent either in their notation or convention and, as a result, it is not very easy to ascertain what do they actually mean by a "linearized" model.

As the writer understands it, the "linearized" model is the one in which the residuals in

$$D(B)y_t = N(B)B^b u_t + \varepsilon_t \tag{4.63}$$

are represented by an autoregressive-moving average model

$$(1-\phi_1 B - \dots - \phi_p B^p)\varepsilon_t = (1-\theta_1 B - \dots - \theta_q B^q)a_t \tag{4.64a}$$

or,

$$\phi(B)\,\varepsilon_t = \theta(B)\,a_t \tag{4.64b}$$

where $\{a_t\}$ is a white noise sequence. Thus, an input-output relation corresponding to a "linearized model is

$$D(B)\,y_t = N(B)\,B^b u_t + \frac{\theta(B)}{\phi(B)}\,a_t \tag{4.65}$$

In the "nonlinear " model, on the other hand, it is not the residual error but the noise e_t at the output in

$$y_t = \frac{N(B)}{D(B)}\,B^b u_t + e_t \tag{4.66}$$

that is represented by an autoregressive-moving average model

$$(1-\phi_1 B-\cdots-\phi_p B^p)e_t=(1-\theta_1 B-\cdots-\theta_q B^q)a_t \tag{4.67a}$$

or,

$$\phi(B)e_t=\theta(B)a_t \tag{4.67b}$$

As a result the input-output relation is

$$y_t=\frac{N(B)}{D(B)}B^b u_t+\frac{\theta(B)}{\phi(B)}a_t \tag{4.68}$$

~~is now nonlinear in the parameters.~~

The parameters of the linearized model (4.65) are estimated in the following way:

(a) First, a simple model $\frac{\theta(B)}{\phi(B)}$ is postulated and some set of values is adopted as starting values of the parameters $\frac{\theta_o(B)}{\phi_o(B)}$ of the model; often $\theta_o(B)=\phi_o(B)=1$

(b) Using these assumed values the relation (4.65) is written

$$D(B)\cdot\left[\frac{\phi_o(B)}{\theta_o(B)}y_t\right]=N(B)B^b\left[\frac{\phi_o(B)}{\theta_o(B)}\right]u_t+a_t. \tag{4.69a}$$

or,

$$D(B)y_t^{(o)}=N(B)B^b u_t^{(o)}+a_t \tag{4.69b}$$

where

$$y_t^{(o)}=\frac{\phi_o(B)}{\theta_o(B)}y_t \tag{4.69c}$$

$$u_t^{(o)}=\frac{\phi_o(B)}{\theta_o(B)}u_t \tag{4.69d}$$

(c) Estimates $D^{(o)}(B)$ and $N^{(o)}(B)$ of $D(B)$ and $N(B)$ in the relation (4.69b) are obtained by ordinary linear least squares method, for a suitable set of integer values of b.

(d) Using the estimates $D^{(0)}(B)$, $N^{(0)}(B)$, and $b^{(0)}$, residual errors $\varepsilon_t^{(0)}$ are computed from

$$\varepsilon_t^{(0)} = D^{(0)}(B)\, y_t^{(0)} - N^{(0)}(B) B^{b^{(0)}} u_t^{(0)} \qquad (4.70)$$

(e) from the study of the autocorrelation function of the residuals a more suitable model

$$\varepsilon_t^{(0)} = \frac{\theta^{(1)}(B)}{\phi^{(1)}(B)} \qquad (4.71)$$

may be inferred.

(f) if the indicated noise structure is sufficiently simple and depends only on one or two additional parameters, new values $y_t^{(1)}$, $u_t^{(1)}$ may be generated from (4.69c), (4.69d) for a grid of values of noise parameters, and the values of parameters resulting in the smallest attainable sum of squares of residuals are finally chosen.

A different approach is suggested for estimating the parameters of the "nonlinear" model

$$y_t = \frac{N(B)}{D(B)} B^b u_t + \frac{\theta(B)}{\phi(B)} a_t \qquad (4.68)$$

where

$$N(B) = \omega_0 - \omega_1 B - \cdots - \omega_\nu B^\nu \qquad (4.72a)$$

$$D(B) = 1 - \delta_1 B - \cdots - \delta_u B^u \qquad (4.72b)$$

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \tag{4.72c}$$
$$\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q \tag{4.72d}$$

In the relation (4.68) the dynamic model

$$\frac{N(B)}{D(B)} B^b$$

is known approximately from the preliminary identification procedure, but the structure of the noise model $\frac{\theta(B)}{\phi(B)}$ may not be known initially, and may have to be obtained through a series of progressive improvements on a previously assumed structure, the improvements being indicated by the structure of the autocorrelation function of residual errors.

The suggested approach involves linearizing the expression

$$a_t = \frac{\phi(B)}{\theta(B)} y_t - \frac{\phi(B)}{\theta(B)} \cdot \frac{N(B)}{D(B)} u_{t-b-1} \tag{4.73}$$

about current guessed values of parameters

$$\underline{\beta}_i^T = [\phi_{1i} \ldots \phi_{pi};\ \theta_{1i} \ldots \theta_{qi};\ \delta_{1i} \ldots \delta_{ui};\ \omega_{0i} \ldots \omega_{vi}] \tag{4.74}$$

and using linear least squares technique to estimate the parameters of the linearized model.

Specifically, if

$$a_t^{(i)} = \frac{\phi_i(B)}{\theta_i(B)} y_t - \frac{\phi_i(B)}{\theta_i(B)} \times \frac{N_i(B)}{D_i(B)} u_{t-b-1} \tag{4.75}$$

corresponds to the parameter vector (4.74), then linearizing the expression (4.73) about (4.75) yields

$$a_t = a_t^{(i)} + \sum_{j=1}^{p} (\phi_j - \phi_{ji}) \left[\frac{\partial a_t}{\partial \phi_j}\right]_{\beta_i} + \sum_{k=1}^{q} (\theta_k - \theta_{ki}) \left[\frac{\partial a_t}{\partial \theta_k}\right]_{\beta_i}$$

$$+ \sum_{\ell=1}^{u} (\delta_\ell - \delta_{\ell i}) \left[\frac{\partial a_t}{\partial \delta_\ell}\right]_{\beta_i}$$

$$+ \sum_{m=1}^{v} (\omega_m - \omega_{mi}) \left[\frac{\partial a_t}{\partial \omega_m}\right]_{\beta_i} \qquad (4.76)$$

The adjustments $(\phi_j - \phi_{ji})$, $(\theta_k - \theta_{ki})$, $(\delta_\ell - \delta_{\ell i})$ and $(\omega_m - \omega_{mi})$ are estimated by regressing $a_t^{(i)}$ on to the negative of the derivatives $\left(\frac{\partial a_t}{\partial \phi_j}\right)$, $\left(\frac{\partial a_t}{\partial \theta_k}\right)$, $\left(\frac{\partial a_t}{\partial \delta_\ell}\right)$, $\left(\frac{\partial a_t}{\partial \omega_m}\right)$

respectively, and the cycle is repeated until convergence occurs.

Box and Jenkins show that the linearized model to be fitted by linear least squares is approximately given by

$$a_t^{(i)} \approx -\phi(B)\left[\frac{a_t^{(i)}}{\phi_i(B)}\right] + \theta(B)\left[\frac{a_t^{(i)}}{\theta_i(B)}\right]$$

$$-D(B)\left[\frac{u_{t-b}^{(i)}}{D_i(B)}\right] + N(B)\left[\frac{u_{t-b}^{(i)}}{N_i(B)}\right] \qquad (4.77)$$

Thus, at the i-th iteration, the estimates $\phi_i(B)$, $\theta_i(B)$, $D_i(B)$ and $N_i(B)$ are obtained from the fitted regression

$$a_t^{(i-1)} = -\phi_i(B)\left[\frac{a_t^{(i-1)}}{\phi_{i-1}(B)}\right] + \theta_i(B)\left[\frac{a_t^{(i-1)}}{\theta_{i-1}(B)}\right]$$

$$-D_i(B)\left[\frac{u_{t-b}^{(i-1)}}{D_{i-1}(B)}\right] + N_i(B)\left[\frac{u_{t-b}^{(i-1)}}{N_{i-1}(B)}\right] \qquad (4.78)$$

If b is also to be estimated the iterative procedure is run for convergence for a series of values of b in the likely range, and that value resulting in a minimum sum of squares is selected.

After the estimation procedure has been completed, diagnostic checks are to be made to ascertain the validity of the identified model.

If the true model (4.68) is written as

$$y_t = V(B)u_t + \psi(B)a_t \tag{4.79}$$

and the identified model as

$$y_t = V_0(B)u_t + \psi_0(B)\hat{a}_t \tag{4.80}$$

then the errors resulting from a wrong selection of the model are given by

$$\hat{a}_t = \frac{V(B)-V_0(B)}{\psi_0(B)}u_t + \frac{\psi(B)}{\psi_0(B)}a_t \tag{4.81}$$

An indication of the validity of the model (4.80) may be obtained from an examination of the autocorrelation function of the errors a_t and of the crosscorrelation function of the errors a_t and input u_t.

In particular,

(a) If the dynamic model is correct and the noise model is incorrect, then a_t will not be cross-correlated with the input u_t but the autocorrelation function of a_t will not appear to correspond to that of white noise;

(b) If the noise model is correct but the dynamic model is incorrect, the errors a_t will be both autocorrelated and cross-correlated with the input u_t.

4.5. General Remarks.

The parametric techniques discussed in this Chapter have been deliberately arranged in the order in which , in the writer's opinion, the degree of complexity of representing the effects of the disturbing noise increases. The first technique seems to be least attractive, at least in the a application to single-input single output systems, in that, in addition to several assumptions to be made about the disturbances, it involves the estimation of state variables as well as the parameters of the system model. The remaining two techniques, while differing in details, have nevertheless some common factors. One of these is the minimization of the sum of squares of errors between the actual and the "predicted" output. In the case of a nonlinear relationship between parameters, the sum of squares function may possess multiple minima-the difficulty recognized by Åström and Bohlin and also discussed in the theory of the nonlinear program employed by Box and Jenkins (Booth and Peterson,1960). The other common feature of the two techniques is the modelling of stationary disturbances, because the mixed autoregressive-moving average scheme employed by Box and Jenkins can also be regarded as a pulsed filter excited by white noise.

The nonstationary model of the disturbance, employed by

Box and Jenkins in the solution of the"closed loop" problem, cannot be interpreted in this fashion. Bohlin(1966) considers this model in the form

$$V(t)=\left[1+\left(\gamma_{-l}\nabla^{l-1}+\dots+\gamma_0\nabla^{-1}+\dots+\gamma_m\nabla^{-(m+1)}\right)z^{-1}\right]a(t) \qquad (4,83)$$

and claims, therefore, that Box and Jenkins model represents a special case of their model. It is rather difficult to accept this point of view because the stationary model of Åström and Bohlin should rather be regarded as a subclass of nonstationary models, and not the other way round.

Neither of the described techniques permit the identification of system dynamics to be performed automatically in the presence of a nonstationary disturbance, The development of such a technique is decribed in the next Chapter.

CHAPTER 5.

ON-LINE ESTIMATION OF PARAMETERS OF A SINGLE-INPUT SINGLE-OUTPUT FIRST ORDER SYSTEM IN THE PRESENCE OF A NONSTATIONARY CORRELATED DISTURBANCE.

5.1. Introduction.

The work discussed in Chapters 5 and 6 has been carried out within the framework of the Automatic Control Research Project of the Central Electricity Generating Board. It was associated, in particular, with one aspect of the project, namely with the on-line control of a power station boiler.

A method of on-line control of a boiler (Berger,1967; Moran and Berger,1967; Moran,Berger and Xirokostas,1968) required the knowledge of certain quantities , the values of some of which had to be either assumed or estimated on-line. It was the writer's task to investigate the feasibility of on-line estimation of one such parameter, on the assumption that its estimate was to be used as a control parameter in the main control program.

A preliminary analysis carried out by the writer showed that such estimates are nonlinear functions of several quantities. It appeared, however, that it might be feasible, at least on paper, to obtain such estimates in real time by linearizing the relationships about the mean operating point and providing means to learn what the operating point is

As a result of such a procedure, the required quantity was expressible in terms of a small number of first order linear time-invariant differential equations. Thus, the problem of estimating the quantity in question was shown to be reducible to that of on-line estimation of single-input single-output linear time-invariant dynamical systems subject to disturbances representing other boiler quantities coupled to the systems under consideration. The problem appeared, therefore, to be solvable by employing one of the methods reviewed in Chapters 2-4, provided that the quantities used in the estimation could be regarded as stationary stochastic processe

The supporting tests carried out at Croydon B and Northfleet Generating Stations, and their results, are discussed in Chapter 6. It is shown there that (as, perhaps, was to be expected) the statistical characteristics of the various quantities of interest, like, for example, steam flowrate, boiler pressure or steam temperature, vary with time. This indicates that these quantities should properly be regarded as nonstationary stochastic processes.
The problem became thus that of estimating parameters of single-input single-output first order dynamical systems subject to a nonstationary disturbance. Since, however, the estimates were to be used as control parameters in an overall plant control problem, two further requirements were added, namely

(a) the input and output quantities were not to be examined by a human operator and, therefore, the character of the disturbance had to be learnt adaptively by the estimation procedure;

(b) since the estimation had to be effected in real time, and the estimation procedure was to be only a subroutine of a bigger program, only a limited storage capacity could be expected to be allocated to the estimation procedure. Therefore, relatively small-sample analysis was envisaged and the various large sample attributes of the estimates, discussed in the Appendix A, did not appear to be very relevant, the main objective being as good a fit to the recorded data as possible, and reasonably short computation times.

None of the methods reviewed in Chapters 2-4 could be made to satisfy all these requirements and, therefore, a new technique has been developed by the writer. The technique includes a novel method of modelling nonstationary processes and a new method of parameter estimation, not assuming any of the large sample properties which estimates are usually required to have. The technique and examples of its application are described in the remainder of this Chapter.

5.2. Characterization of a nonstationary process in terms of its mean square value.

It has been observed in Chapter 3 that, in view of a difficulty in specifying statistical characteristics of the variates, the methods of modelling nonstationary processes have tended to be more or less empirical. Various such methods, currently available, were reviewed in Chapter 3. In particular, it was observed that the model due to Box and Jenkins (1966) seems to describe the nonstationary behaviour of many physical processes, met with in practical applications, reasonably well. If $P_{d-1}(t)$ is the polynomial in time of degree (d-1) (5.1a)

$$\nabla \eta_t = \eta_t - \eta_{t-1} \tag{5.1b}$$

is the backward difference operator, S is a summation operator defined by

$$S\eta_t = \sum_{j=0}^{\infty} \eta_{t-j} \tag{5.1c}$$

$$S^2\eta_t = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \eta_{t-j-k} \tag{5.1d}$$

etc., and $\{\eta_t\}$ denotes a zero-mean white noise process characterized by

$$E\langle \eta_t \, \eta_{t'} \rangle = \sigma^2 \delta_{tt'} \tag{5.1e}$$

the Box and Jenkins model is written as

$$z_t = P_{d-1}(t) + [\lambda_{m-d} \nabla^{m-d-1} + \dots + \lambda_0 S + \dots + \lambda_{d-1} S^d] \eta_{t-1} + \eta_t \tag{5.2}$$

The problem to be solved by the writer was the estimation of parameters of a single-input single output first order system from a series of values of input $\{u_t\}$ and output $\{y_t\}$ sampled by a suitable scanner and stored in the computer. The input u_t was expected to be nonstationary, and the output y_t was to include the effect of an unknown nonstationary disturbance η_t "buried" in the output and representing the coupling of the rest of the plant with the system u-y. The observable output y_t is then given by

$$y_t = y_t' + \eta_t \tag{5.3}$$

where y_t' is the noise-free output due to the input u_t. The problem is illustrated in Fig,5.1.

Formulated in the above fashion, the problem bears some similarity to the Box and Jenkins approach, except that the input and output values are to be "seen" only by the computer. The estimation program was, therefore, required

(a) to identify the structure of the nonstationary disturbance contaminating the output y_t;

(b) to estimate the parameters of the disturbance jointly with the parameters (gain and time constant) of the first order dynamic system under consideration.

It is obvious that, when human judgement is not allowed, in examination of the input and output values, as well as the processed results, the model of the type (5.2) is not very suitable for automatic identification of the structure of the disturbance from a series of input and output readings. For this reason, another approach, utilizing some other characteristic of a nonstationary process must be sought.

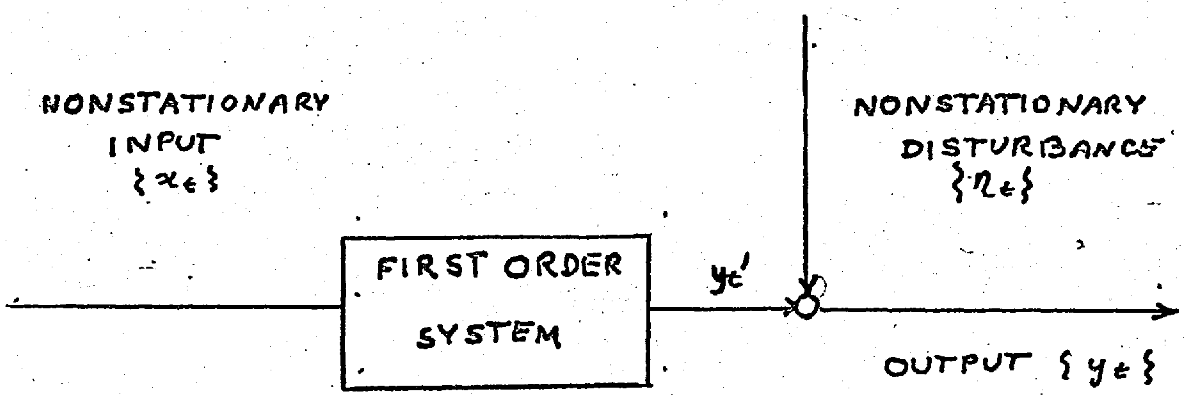


FIG. 5.1

ILLUSTRATING THE NATURE OF

THE ESTIMATION PROBLEM TO BE SOLVED

The various models, discussed in Chapter 3, represent a nonstationary stochastic process by a relation of the form

$$x_t = m_t + v_t \tag{5.4}$$

where

(a) either the mean value function m_t is assumed to be zero and v_t is a nonstationary stochastic process,

(b) or, a model is fitted to represent a deterministic trend m_t and a stationary process v_t.

The first approach represents the nonstationarity by a time-varying variance of the process; in the second approach, on the other hand, the nonstationary behaviour of the process x_t is represented by the mean value function and trend m_t as a function of time.

Now, since

$$(\text{mean})^2 + (\text{variance}) = (\text{mean square value}) \tag{5.5}$$

it seems reasonable to argue that either approach can be regarded as a particular case of fitting a model to represent a time-varying behaviour of the mean square value of the process under consideration. It has, in fact, been observed by Thrall(1964), Thrall and Bendat(1965) and Piersol(1965) that in certain applications , like analysis of mechanical vibrations data, it is the mean square value which is the significant parameter.

If a Box and Jenkins model is employed to represent a disturbance η_t in equation (5.3) or Fig.5.1, then there is no means of identifying its structure (short of reducing the model (5.3) to a stationary model and then applying a more or less trial and error procedure as indicated by Box and Jenkins(1966)). However, the model seems to represent the nonstationary behaviour of many physical processes reasonably well. Thus, what is required in the present investigation, is such a representation of a nonstationary process which would be at least as good as the representation of Box and Jenkins, and yet render itself to easy identification from the analysis of the input and output data.

The mean square value of the process (3.52) is given by

$$E\langle x_t^2\rangle = E\langle \lambda^2_{m-d}(\nabla^{d-m-1}\xi_{t-1})^2\rangle + \dots E\langle \lambda_0^2 (S\xi_{t-1})^2\rangle$$

$$+ \dots + E\langle \lambda^2_{d-1}\cdot(S^d\xi_{t-1})^2\rangle + \dots$$

$$+ 2\lambda_{d-1}\lambda_{d-2}\, E\langle (S^{d-1}\xi_{t-1})(S^d\xi_{t-1})\rangle \tag{5.5}$$

Now the stochastic process $\{\xi_t\}$ in (3.52) is assumed to be a zero mean white noise, that is

$$E\langle \xi_t\rangle = 0 \tag{5.6a}$$

$$E\langle \xi_t\,\xi_{t'}\rangle = \sigma^2\delta_{t't} \tag{5.6b}$$

Therefore,

$$E\langle(\nabla^n\xi_{t-1})^2\rangle = E\langle\Big(\sum_{j=0}^{n}(-1)^j\binom{n}{j}\xi_{t-j}\Big)^2\rangle$$

$$= \sigma^2\sum_{j=0}^{n}\binom{n}{j}^2 \tag{5.7}$$

Also, the first three summations are

$$S\xi_{t-1} = \sum_{i=0}^{t-2} \xi_{t-1-i} = \xi_{t-1} + \xi_{t-2} + \cdots + \xi_1 \tag{5.8}$$

$$S^2\xi_{t-1} = \sum_{j=0}^{t-2} \sum_{i=0}^{t-2-j} \xi_{t-1-j-i} \tag{5.9}$$

$$= \sum_{k=1}^{t-1} (t-k)\xi_k \tag{5.10}$$

$$S^3\xi_{t-1} = \sum_{k=0}^{t-2} \sum_{j=0}^{t-2-k} \sum_{i=0}^{t-2-k-j} \xi_{t-1-k-j-i} \tag{5.11}$$

$$= \sum_{j=0}^{t-1} \sum_{k=1}^{t-1} (t-k-j)\xi_k \tag{5.12}$$

Thus,

$$E\langle (S\xi_{t-1})^2 \rangle = (t-1)\sigma^2 \tag{5.13}$$

$$E\langle (S^2\xi_{t-1})^2 \rangle = \sigma^2 \sum_{k=1}^{t-1} (t^2 - 2k + k^2) \tag{5.14}$$

$$E\langle (S^3\xi_{t-1})^2 \rangle = \sigma^2 \left\{ \sum_{j=0}^{t-1} \sum_{k=1}^{t-1} (t^2 + k^2 + j^2 - 2tk - 2tj + 2kj) \right\} \tag{5.15}$$

The last twoee sums and, indeed, any higher order expressions, can be easily evaluated by using a technique of summating the factorial function (Miller,1960). However, it is already apparent that the mean square value of the nonstationary stochastic process represented by the Box and Jenkins model (3.52) may be expressed in the form

$$E\langle x_t^2 \rangle = a_0 t + a_1 t + a_2 t^2 + \cdots + a_{d+1} t^{d+1} \tag{5.16}$$

A characteristic feature of a time polynomial is that the n-th difference of the n-th degree polynomial is zero. This feature makes it very suitable for identification purposes. The problem studied was to devise a learning technique which could enable a nonstationary disturbance to be identified, and then to be taken into account during the estimation of the system dynamics. Now although the polynomial (5.16) can be identified in an automatic fashion relatively easily, its coefficients are complicated functions of the coefficients of the model (3.52). Therefore, the Box and Jenkins representation is of little use in the current problem and a new approach to modelling nonstationary processes has been developed by the writer . The approach, discussed in the Appendix C, is to associate the polynomial expression of the type (5.16) with a definite structure of a linear filter with time-varying coefficients, excited by white noise. It is shown in Appendix C that if a nonstationary process $\{\eta_t\}$ is represented in the form

$$\eta_t = \eta_0 + \sum_{j=1}^{t-1} v_j(t)\,\xi_{t-j} + \xi_t \tag{5.17}$$

where $\{\xi_t\}$ is a white noise process, then the mean square value of the process, $E\langle\eta_t^2\rangle$, ~~is~~ CAN BE MADE a polynomial in time, the degree of the polynomial being associated with a definite structure of the weights $v_j(t)$. Therefore, the identification of the degree of the polynomial, describing the time variation of the mean square value, of the process implies also the identification of the structure of the filter, representing the process. This solves the first part of the problem.

The structure of the weights $v_j(t)$

$$v_j(t) = K\,\frac{(t-j)^k j^l}{t^m} \tag{5.18}$$

was determined in a semi-empirical fashion from considerations of conditions which have to be satisfied by the impulsive response of a linear system characterized by a general linear differential equation with time-varying coefficients(Miller, 1955). Trial and error procedure was then employed to arrive at a structure of the weights resulting in the required form of the mean square value.

As proved in the Appendix C, the processes whose mean square value is a polynomial in time, are characterized by the following difference equations:

(a) for the first order polynomial,

$$\eta_t' - \left(\frac{t-1}{t}\right)^{3/2}\eta_{t-1}' = \left[G\,\frac{t-1}{t^{3/2}} - \left(\frac{t-1}{t}\right)^{3/2}\right]\zeta_{t-1} + \zeta_t \tag{5.19}$$

(b) for the second degree polynomial,

$$\eta_t' - \frac{t-1}{t}\eta_{t-1}' = (G-1)\,\frac{t-1}{t}\,\zeta_{t-1} + \zeta_t \tag{5.20}$$

(c) for the cubic polynomial

$$\eta_t' - \left(\frac{t-1}{t}\right)^{1/2}\eta_{t-1}' = \left[G\,\frac{t-1}{t^{1/2}} - \left(\frac{t-1}{t}\right)^{\frac{1}{2}}\right]\zeta_{t-1} + \zeta_t \tag{5.21}$$

(d) for the quartic polynomial,

$$\begin{aligned}\eta_t' - 2\,\frac{t-1}{t}\eta_{t-1}' + \frac{t-2}{t}\eta_{t-2}' \\ = \zeta_t + \left[G\,\frac{t-1}{t} - 2\,\frac{t-1}{t}\right]\zeta_{t-1} + (G-1)\,\frac{t-2}{t}\,\zeta_{t-2}\end{aligned} \tag{5.22}$$

The development of this approach represents the original contribution of the writer. The writer wishes, however, to acknowledge some similarity between the weights (5.16) and weighting functions for white noise described by Blackman(1965) and based on the Bode and Shannon(1950) approac

In Blackman's method the weighting function $W_r(\tau)$ is given by

$$W_r(\tau) = \frac{K_r}{T}\left[\frac{\tau}{T}\left(1-\frac{\tau}{T}\right)\right]^r \quad \text{in } 0 \leq \tau \leq T \tag{5.23}$$

where

$$K_r = \frac{(2r+1)!}{(r!)^2} \tag{5.24}$$

An approximate estimate of the degree of the polynomial assumed to characterize the time variation of the mean square value of $\{\eta_t\}$ may be found as follows.

Let

$$y_t = \frac{1}{t}\sum_{t=1}^{t} \eta_t^2 \tag{5.25}$$

where t is greater than some number A (e.g.50), thus ensuring that the variability of the estimates (5.25) is not large. Then the series $\{y_t\}$ is an estimate of

$$E\langle \eta_t^2 \rangle$$

for t =A,A+1,.....,N., where N is the length of sample available for analysis.

Let also $\{y_t^{(i)}\}$ denote a series formed by differencing the series $\{y_t\}$ i times, that is

$$\{y_t^{(1)}\} = \{y_t - y_{t-1}\} = \{\nabla y_t\} \tag{5.26}$$

$$\{y_t^{(2)}\} = \{y_t - 2y_{t-1} + y_{t-2}\} = \{\nabla^2 y_t\} \tag{5.27}$$

and so on.

Then,

$$\hat{y}_t^{(i)} = \frac{1}{D-C}\sum_{t=C}^{D} y_t^{(i)} \tag{5.28}$$

is an estimate, averaged over a small number (D-C) of samples, of the magnitude of the i-th difference of the mean square value series $\{\eta_t^2\}$.

The identification procedure involves then the following steps:

(a) Obtain a series of estimates of mean square values for increasing sample lengths, starting from a minimum length A.

(b) Obtain an estimate of the order of magnitude of the estimates near the beginning and near the end of the series; the relative magnitudes will indicate the increasing or decreasing trend of the series;

(c) difference the series of estimates (b);

(d) Keep repeating the steps (b) and (c) until the small sample averages near the beginning and end of the series are small fraction (say 5 per cent) of the corresponding original estimates of the mean square value. The number of differencings required to arrive at this stage will be equal to the degree of the polynomial representing the time variation of the mean square value. The identification of the model is now complete.

5.3. Representation of a first order system.

The quantities recorded during the tests described in Chapter 6 were obtained by sampling outputs of transducrs every 10 seconds in the Croydon test and every 15 seconds in the Northfleet tests. Most of the results indicate that any two consecutive readings differ only in the third (least significant) digit. For this reason, the use of a zero order hold for reconstructing the sampled functions appears to be justified.

It is shown in the Appendix B that, if a continuous-time first order system is described by the transfer function

$$H(s) = \frac{K}{s+\alpha} \tag{5.29}$$

then the difference equation corresponding to a discrete-time version of the system (5.29) with a zero-order hold is

$$y_t' - \phi y_{t-1}' = g(1-\phi)u_{t-1} \tag{5.30}$$

where

$$g = \frac{K}{\alpha} \tag{5.31}$$

is the gain of the system and

$$\phi = e^{-\frac{\Delta T}{L\Delta T}} \tag{5.32}$$

ΔT being the sampling interval, and $L\Delta T$ being the time constant of the system.

The difference equation (5.30) was employed by the writer to estimate the parameters ϕ and g by means of the estimation procedure to be described.

5.4. The overall input-output relation.

In the development below it will be assumed that the disturbance η_t is represented by the second order model (5.20). Using the backward difference operator ∇ defined by

$$\nabla v_t = v_t - v_{t-1} \qquad (5.33)$$

the difference equations (5.30) and (5.20) may be respectively written as

$$\left(1 + \frac{\phi}{1-\phi}\nabla\right) y_t' = g\, u_{t-1} \qquad (5.34)$$

and

$$\left(\frac{1}{t} - \frac{t-1}{t}\nabla\right)\eta_t' = \zeta_t - (1-G)\frac{t-1}{t}\zeta_{t-1} \qquad (5.35)$$

where

$$\eta_t' = \eta_t - \eta_0 \qquad (5.36)$$

η_0 being the starting value.

Using (5.34)-(5.36) the overall input-output relation is written as

$$y_t = \frac{g\, u_{t-1}}{1 + \frac{\phi}{1-\phi}\nabla} + \frac{\zeta_t - (1-G)\frac{t-1}{t}\zeta_{t-1}}{\frac{1}{t} - \frac{t-1}{t}\nabla} + \eta_2 \qquad t = 2, 3, \ldots, N \qquad (5.37)$$

where the starting value of the disturbance is at t=2.

5.5 Equation of estimation.

In the relation (5.37) the starting value η_0 of the disturbance is not known and, therefore, must be estimated together with the system parameters in one form or another. An approach suggested by the writer is to regard the quantity

$$y_2 - \eta_2$$

as an initial state of the system represented by (5.34) and to form a dual of the Kopp and Orford method (Chapter 4) by treating this quantity as an additional parameter and adjoining it to the system parameters.

Let then
$$\frac{\eta_2}{y_2} = \delta \tag{5.38a}$$

$$0 < \delta < 1 \tag{5.38b}$$

Then, (5.37) may be written as

$$y_t = \frac{g\, u_{t-1}}{1 + \frac{\phi}{1-\phi}\nabla} + \frac{\xi_t - (1-G)\frac{t-1}{t}\xi_{t-1}}{\frac{1}{t} - \frac{t-1}{t}\nabla} + \delta y_2 \qquad t = 2, 3, \ldots, N \tag{5.39}$$

The relation (5.39) may be written in the form

$$y_t = f(\underline{\theta}, \underline{u}, \underline{y}, \underline{\xi}) + \xi_t \tag{5.40a}$$

$$\underline{\theta}^T = (g, \phi, G, \delta) \tag{5.40b}$$

which expresses the output as a function of past values of the input, output and the white noise process $\{\xi_t\}$, and true values of the parameter vector $\underline{\theta}$.

When the parameter vector assumes values $\underline{\theta}^{\circ}$ which differ from the true values $\underline{\theta}$, the calculation of the right hand side of equation (5.39) yields "predicted" values of output y_t^* which differ from the actually observed values y_t. The differences between the actual and the predicted values

$$\varepsilon_t = y_t - y_t^* \tag{5.41}$$

combining the effect of the white noise process $\{\xi_t\}$ and the effect of the parameter deviations

$$\underline{\theta} - \underline{\theta}^{\circ}$$

have been called by the writer the "quasi-residuals", They can be recursively calculated from

$$\begin{aligned} \varepsilon_t = {} & \left[(1-\phi^{\circ}) + \nabla\phi^{\circ}\right]\left[\frac{1}{t} - \frac{t-1}{t}\nabla\right]\left[y_t - \delta^{\circ} y_2\right] \\ & + \left[\phi^{\circ} + \frac{t-1}{t}(1-G^{\circ})\right]\left[\varepsilon_{t-1} + \frac{\phi^{\circ}}{1-\phi^{\circ}}(1-G^{\circ})\frac{t-2}{t-1}\varepsilon_{t-2}\right. \\ & - g\left[\frac{1}{t} - \frac{t-1}{t}\nabla\right]u_{t-1} \end{aligned} \tag{5.42}$$

which relation can be easily obtained from equation (5.39). It is shown in the Appendix D that estimates of the parameter vector $\hat{\underline{\theta}}$ may be obtained by minimizing the sum of squares of the quasi-residuals in such a way, that, at the same time, the covariance matrix of the quasi-residuals is also reduced. This is discussed in the following section.

5.6. The method of parameter estimation.

The parameters are to be estimated subject to the following constraints:

(a) $$0 < \phi < 1 \tag{5.43a}$$

(b) $$0 < \delta < 1 \tag{5.43b}$$

(c) $$0 < G < 1 \tag{5.43c}$$

(d) in addition, by considering the initial values, one obtain from (5.37),

$$y_2 = \frac{g u_1}{1 + \frac{\phi}{1-\phi}\nabla} + \eta_2 \tag{5.44a}$$

$$= \frac{g u_1}{1 + \frac{\phi}{1-\phi}\nabla} + \delta y_1 \tag{5.44b}$$

from which,

$$\frac{y_2}{1-\phi} - \frac{\phi}{1-\phi} y_1 = g u_1 + y_1 \delta \tag{5.44c}$$,

This yields a constraint condition on the gain:

$$g \not> \frac{y_2}{1-\phi} - \frac{\phi}{1-\phi} y_1 - y_1 \delta \tag{5.43d}$$

The object of a suitable minimization routine was to minimize the sum of squares of the quasi-residuals, subjects to the constraints (5.43), so as to reduce the covariance matrix of the quasi-residuals to a diagonal matrix as far as possibl

When the investigations started, only one technique of constrained optimization, due to Rosenbrock(1960) was available. The technique is a variant of the well known steepest descent method and involves:

(a) working in n orthogonal directions when a function of n parameters is being minimized;

(b) moving along a direction of steepest descent, rotating the direction after a complete cycle of adjustments of the n parameters;

(c) representing the l constraints on the parameters in the form of l functions each of which is zero if the associated parameter is outside the permitted range, it is equal to unity when the associated parameter is within the permitted range, and varies parabolically from zero to unity in a narrow boundary region, the width of which is directly related to the accuracy obtainable with a given computer word length. The product of l constraint functions and the sum of squares is then the effective function to be minimized.

The method was tried in many simulation studies. It was found, however, that the rotation of axes at the end of each stage made in many cases the convergence to the proper minimum impossible, and the program tended to converge onto the nearest local minimum.

By monitoring the variance and covariances of lag one and two, of the quasi-residuals it was observed that, whenever the program converged on to a wrong minimum, the correlations and covariances of the quasi-residuals were either increasing, or decreasing only by negligible amount. However, the convergence on to the proper minimum could only be obtained if the starting values of parameters were near to the true values, in which case the convergence to the true minimum was accompanied by a rapid decrease of the correlations and covariances of the quasi-residuals.

These observations have led to abandoning the Rosenbrock method and developing a new method as follows.

(a) corresponding to n parameters, n orthogonal directions are chosen; these remain fixed throughout the minimization procedure,which corresponds to the adjustment of one parameter at a time;

(b) the method of allowing for constraints on parameters is the same as that in the Rosenbrock's technique; the boundary region was assumed to be 10^{-4}.(allowable parameter range) , as suggested by Rosenbrock;

(c) Let a "success" be defined to mean that the "new" value of the function, resulting from a change in a parameter, is smaller than or equal to the "old" value, prior to the change, and, at the same time, both the product of the l constraint functions is not zero and the first three covariances of the quasi-residuals are decreased;

Similarly, let a "failure" be defined to mean that either the new value of the function being minimized is greater than the old value , or that the product of the 1 constrain functions is equal to zero, or that the first three covariances do not decrease after the change.

Then the minimization procedure developed consists in a cyclic adjustment of the parameters in such a way as to achieve as many "successes" as possible. The adjustments are effected as follows:

(a) at the beginning of each cycle the first change to be applied to the parameter θ_i is $w\theta_i$ where w=0.02;

(b) in the case of a failure, the direction of the change is reversed; if this, too, results in a failure, the value of the parameter is restored to the original value;

(c) after each success, every succeeding change applied to a given parameter, is equal to the preceding change times two;

(d) the convergence to the proper minimum value of the minimized function is achieved if attention is paid to the rate at which the decrease in variance is increasing; In particular, let the value of the variance of the quasi-residuals $\gamma_\varepsilon(0)$ after n successes be denoted by γ_n . Then it is shown in the Appendix D that the convergence to the proper minimum is ensured if the quantity

$$\gamma_{n-1}^{(3)} = \gamma_{n-3} - 3\gamma_{n-2} + 3\gamma_{n-1} - \gamma_n$$

does not become negative;

(e) the procedure is terminated when the changes in the parameters do not result in a significant change of the minimized sum of squares, and the correlations of lag 1 and 2 of the quasi-residuals do not exceed the theoretical standard deviation $1/\sqrt{N}$ of white noise determined from the sample of size N.

5.7. Confidence regions for the parameters.

5.7.1. General.

At the end of the estimation procedure one wishes usually to obtain a rough idea of the precision with which the estimates have been obtained. This may be obtained from the consideration of confidence regions, the theory of which based on the work of Booth and Peterson(1960), Rosenbrock (1962), Rosenbrock and Storey(1965) and Deutch(1965), is discussed in the following subsections.

A method of deriving eigenvalues and eigenvectors of the correlation matrix of the estimates, employed by the writer, is described in subsection 5.7.3.

5.7.2. Confidence regions for the estimates of parameters.

The well-established theory of confidence regions(Booth and Peterson,1960; Rosenbrock,1962; Rosenbrock and Storey,1965;

Deutsch, 1965) is based on the assumption that estimated values $\hat{\underline{\theta}}$ of the parameter vector are very near to their true values $\underline{\theta}$. Therefore, when the parameters are perturbed about their optimum estimated values $\hat{\underline{\theta}}$, only first order changes in the resultant predicted outputs need be considered. Under these circumstances, the small changes

$$\delta\underline{\theta} = \underline{\theta} - \hat{\underline{\theta}}$$

are linear functions of the observation errors, and if the latter are assumed to be gaussian, the small changes $\delta\underline{\theta}$ in the parameters are also normally distributed with the covariance matrix (Booth and Peterson, 1960)

$$\underline{M} = (\underline{D}^T \cdot \underline{D})^{-1} \tag{5.45}$$

where $\underline{D}$ denotes the matrix of partial difference quotients

$$\underline{D} = \begin{pmatrix} \frac{\Delta y_1}{\Delta\theta_1} & \cdots & \frac{\Delta y_1}{\Delta\theta_p} \\ \cdots & \cdots & \cdots \\ \frac{\Delta y_n}{\Delta\theta_1} & \cdots & \frac{\Delta y_n}{\Delta\theta_p} \end{pmatrix} \tag{5.46}$$

and in which Δy_i denotes a change in the i-th predicted output value due to a small change $\Delta\theta_j$ in the parameter θ_j. With the above assumptions the sum of squares S corresponding to the perturbed values of parameters in the neighbourhood of the minimum sum of squares S_m defines a contour in the p-dimensional parameter space, defined by

$$S = S_m \left(1 + \frac{p}{N-p} F_{p,N-p}(\alpha)\right) \tag{5.47}$$

where

$S = S(\underline{\theta}, \underline{u}, \underline{y})$ = contour sum of squares,

S_m = minimum sum of squares,

N = number of observations,

p = number of parameters,

α = significance level

$F_{p,N-p}(\alpha)$ = Fisher's distribution with p and (N-p) degrees of freedom, corresponding to the significance level α.

For $S(\underline{\theta}, \underline{u}, \underline{y})$ equal to a constant, there is an associated contour of values of $\underline{\theta}$. Assuming that the deviations in the predicted outputs, resulting from the small changes of parameters are normally distributed, the contour $S(\underline{\theta}, \underline{u}, \underline{y})$ defines a likelihood contour for the estimated parameters. If the parameters are linearly related to the dependent variable, the set of parameters $\underline{\theta}$ for which $S(\underline{\theta}, \underline{u}, \underline{y})$ is a constant, is a p-dimensional ellipsoid in the space of p parameters.

It has been observed by Booth and Peterson(1960) that inferences regarding the estimates can be drawn from the consideration of the ellipsoid in the space of normalized parameters

$$\underline{\gamma} = \Delta^{-\frac{1}{2}} \underline{\theta} \tag{5.48}$$

where

$$\Delta = \begin{pmatrix} \Delta_1 & & 0 \\ & \ddots & \\ 0 & & \Delta_p \end{pmatrix} \tag{5.49}$$

and $\Delta_j^{\frac{1}{2}}$ is equal to the square root of the j-th diagonal element of the covariance matrix $(\underline{D}^T\underline{D})^{-1}$.

In the normalized parameter space, the likelihood contours are given by

$$\underline{\gamma}^T \Delta^{\frac{1}{2}} \underline{D}^T \underline{D} \Delta^{\frac{1}{2}} \underline{\gamma} = (S - S_m) \tag{5.50}$$

The semi-axes of the ellipsoid defined by the quadratic form (5.50), will have lengths equal to the square roots of the eigenvalues, and the orientation of the axes will be governed by the eigenvectors.

The eigenvalues and eigenvectors of the correlation matrix may yield useful information. For, example, if the correlation matrix were an identity matrix, this would imply that the estimates of the parameters are uncorrelated. On the other hand, if the correlation matrix contained off-diagonal elements, its eigenvalues might differ by several orders of magnitude. The inference would then be that a certain linear combination of parameters has been determined with a smaller variance than some others. The linear combination determined with greater precision would be given by the eigenvector associated with the smallest eigenvalue.

5.7.3. Calculation of eigenvalues and eigenvectors of the correlation matrix of parameter estimates.

After the optimum values of the parameters have been obtained, the eigenvalues and eigenvectors of the correlation matrix are obtained as indicated below.

First, the matrix $\underline{D}$ of partial difference quotients defined b

$$\underline{D} = \begin{pmatrix} \frac{\Delta y_1^*}{\Delta G} & \frac{\Delta y_1^*}{\Delta g} & \frac{\Delta y_1^*}{\Delta \phi} & \frac{\Delta y_1^*}{\Delta \sigma} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\Delta y_N^*}{\Delta G} & \frac{\Delta y_N^*}{\Delta g} & \frac{\Delta y_N^*}{\Delta \phi} & \frac{\Delta y_N^*}{\Delta \sigma} \end{pmatrix} \tag{5.50}$$

was obtained, the perturbations $\Delta\theta = \frac{\hat{\theta}}{100}$

having been employed (In the actual application of the method, N was of the order of 200-300; in view of the restrictions in the computer, the multiplication of two matrices 300 x 4 each presented some interesting programming difficulties)

The elements in the j-th row of the covariance matrix

$$\underline{M} = (\underline{D}^T\underline{D})^{-1} \tag{5.51}$$

were then divided by the j-th diagonal element to yield the correlation matrix

$$\underline{R} = \Delta^{-\frac{1}{2}}\, \underline{M}\, \Delta^{-\frac{1}{2}} \tag{5.52}$$

The eigenvectors and eigenvalues of this matrix were determine by using the Jacobi method(Ralston,1965), particularly suitable for application to matrices whose off-diagonal elements are small as compared with the diagonal elements. The method consists in determining, in an iterative fashion, a sequence $\{\underline{S}_k\}$ of orthogonal matrices with the property

$$\lim_{k\to\infty} \underline{S}_1 \underline{S}_2 \ldots \underline{S}_k = \underline{Q} \qquad (5.53)$$

and

$$\underline{Q}^T \underline{R} \underline{Q} = \underline{\Lambda} \qquad (5.54)$$

$\underline{\Lambda}$ being the diagonal matrix of the eigenvalues of the matrix $\underline{R}$, and $\underline{Q}$ being the matrix of eigenvectors.

Let a matrix $\underline{T}_k$ be defined by

$$\underline{T}_k = \underline{S}_k^T \underline{S}_{k-1}^T \ldots \underline{S}_1^T \underline{R} \underline{S}_1 \underline{S}_2 \ldots \underline{S}_k , \underline{T}_0 = \underline{R} ; \qquad (5.55)$$

Then the Jacobi method consists in choosing S_k in such a way that if $t_{pq}^{(k-1)}$ is the largest off-diagonal non-zero element of the matrix $\underline{T}_{k-1}$, the off diagonal term $t_{pq}^{(k)}$ of the matrix $\underline{T}_k$ is zero.

The elements of the "plane rotation matrix" S_k are defined by

$$\left.\begin{aligned} s_{pp}^{(k)} &= s_{qq}^{(k)} = \cos\theta_k \\ s_{pq}^{(k)} &= - s_{qp}^{(k)} = \sin\theta_k \end{aligned}\right\} \quad \left.\begin{aligned} s_{ii}^{(k)} &= 1, \ i \neq p \text{ or } q, \\ &= 0 \text{ otherwise} \end{aligned}\right\} \qquad (5.56)$$

The required conditions are obtained if the angle θ_k is chosen from the relation

$$\tan 2\theta_k = \frac{-t_{pq}^{(k-1)}}{\frac{1}{2}\left(t_{pp}^{(k-1)} - t_{qq}^{(k-1)}\right)} \qquad (5.57)$$

The procedure is terminated when the ratio of the sum of squares of the diagonal elements in two consecutive iterations is less than some prescribed value; (the value 10^{-11} was actually used by the writer).

5.8. Summary of the estimation method developed.

The method described above was programmed in Extended Mercury Autocode which involves one instruction per line. The program is rather bulky and, if reproduced here, it would increase the volume of the thesis by some 30 pages. It is believed that anyone wishing to pursue the line of development indicated here would employ a higher level language. For this reason, the method is summarized in steps in such a way that it can be easily coded in any higher level language.

1. Define as a success such a change in the value of the parameter θ_i that it results in
   a) smaller or equal sum of squares of quasi-residuals,
   b) smaller variance, covariance of lag 1 and covariance of lag 2, the latter covariance being smaller than the covariance of lag 1;
   c) stability limits being satisfied;
2. Define as a failure a parameter change resulting in any one of the above conditions not being satisfied;

3. Define a trial to mean a change in the parameter resulting in either a success or a failure, and define a stage to mean a cycle of trials on each parameter in turn.
4. At the beginning of a stage, start from the first parameter θ_1. The initial change to be applied to the current value θ_1^k of the parameter at the k-th stage is $w\theta_i^k$ where w=0.02.
5. In the case of a success, the next change to be applied to the parameter is equal to the preceding change times two
6. In the case of a failure the change is applied in the opposite direction. If this results in a success, proceed as in step 5; if the result is the failure, reset the parameter to the value it had before the change and start adjusting parameter θ_{i+1} if $i+1 < p$, or the parameter θ_1 if $i+1 > p$, where p= the number of parameters involved;
7. In the event of there occurring more than one success, monitor the rate at which the decrease in variance increases; stop adjusting the parameter if this rate starts decreasing;
8. If no more progress is obtained with the parameter θ_i, start adjusting the parameter θ_{i+1} if $i+1 < p$, or the parameter θ_1 if $i+1 > p$, where p is the number of parameters
9. Stop adjustments if two successive stage results differ by less than some prescribed value; when this occurs,

the correlations of the qusi-residuals of lag 1 and 2 should b of the order of $\pm 1/\sqrt{N}$ where N is the sample size; also, the covariance of lag 2 should be smaller than the covariance of lag 1.

10. Calculate the eigenvalues of the correlation matrix by the method of the preceding section, and check that the ratio of the largest to the smallest eigenvalue is not very different from unity;

11. In the case of bad estimation indicated either by lack of convergence or the ratio of the eigenvalues being very large, assume a different set od starting values and start again from step 4.

5.9. Examples.

As an illustration of the method two examples are given. Both examples involve input derived from test recordings of boiler pressure at Croydon (and, therefore, very realistic and nonstationary) being applied to a first order system with a zero order hold. In the first example the gain of the system is 13.00, the time constant is 7.61 sampling intervals and the describing difference equation is

$$y_t' - 0.8769y_{t-1}' = 13(1 - 0.8769)u_{t-1}$$

In the second example the gain of the system is 15.00, the time constant is 12.5 sampling intervals and the difference equation is

$$y_t' - 0.923y_{t-1}' = 15(1 - 0.923)u_{t-1}$$

In both examples the disturbance is generated recursively by using the model (5.20) with G=40, and a pseudo-random number generator the statistical characteristics of which, corresponding to sample sizes used in the examples, are shown in Fig. 5.2.

In the first example η_2 =1103, y'_2=6405 and y_2 =7509; This gives δ = 0.147.

In the second example η_2 = 1104, y_2' =7392, and y_2=8496. Thus, δ = 0.13

Figs.5.3-5.5 show the beginning and end of the estimation procedure relating to Example 1. Figs. 5.6-5.8 show similar results relating to example 2.

It is seen that, owing to different starting values in the two examples the number of iterations required to reach the optimum values is different (81 iterations in Example 1 and 109 iterations in example 2). The results show that the technique yields correct estimates and that minimization of the sum of squares ("SUM ETA") is accompanied by minimization of the first three covariances of the quasi-residuals. In both examples the eigenvalues of the correlation matrix are of the same order of magnitude which, according to the established theory, confirms that the estimation procedure is satisfactory.

N=200

MEAN= 5.144842, -1 MEAN SQUARE= 3.477219, -1 VARIANCE= 8.302797, -2 SLOPE=-3.248129, -4

COVARIANCES

0	8.267630, -2	1	-7.586431, -3	2	7.868612, -3	3	-4.221518, -3
4	2.976648, -3	5	6.769399, -3	6	-3.268628, -4	7	-2.750668, -4
8	-4.299299, -4	9	-3.045656, -3	10	-1.269508, -3		

CORRELATIONS

0	1.0000	1	-9.1761, -2	2	9.5174, -2	3	-5.1061, -2
4	3.6004, -2	5	8.1878, -2	6	-3.9535, -3	7	-3.3270, -3
8	-5.2002, -3	9	-3.6838, -2	10	-1.5355, -2		

N=300

MEAN= 5.182759, -1 MEAN SQUARE= 3.461495, -1 VARIANCE= 7.753968, -2 SLOPE=-3.783213, -5

COVARIANCES

0	7.752895, -2	1	-2.931937, -3	2	6.282523, -3	3	-2.898459, -3
4	3.067033, -3	5	4.441502, -3	6	-1.602609, -4	7	-2.288412, -3
8	1.621801, -3	9	-4.048786, -3	10	-2.577515, -3		

CORRELATIONS

0	1.0000	1	-3.7817, -2	2	8.1035, -2	3	-3.7386, -2
4	3.9560, -2	5	5.7288, -2	6	-2.0671, -3	7	-2.9517, -2
8	2.0919, -2	9	-5.2223, -2	10	-3.3246, -2		

FIG. 5.2.

CHARACTERISTICS OF THE RANDOM NUMBER GENERATOR

```
SINGLE EXPONENTIAL DYNAMICS, FIRST ORDER DISTURBANCE

INPUT PARAMETER VALUES
SYSTEM DYNAMICS    GAIN= 1.3000,   1      TIME CONSTANT= 8.7690,  -1      7.612533

DISTURBANCE    SLOPE= 4.0000,   1

     (G)              (g)              (φ)              (δ)
 GUESSES
  1.000000,  -1      5.000000          6.500000,  -1     1.100000,  -1
 A'=  1       M'=   0       S'=  0                                        START
 FIRST MEAN= 6.421720,   2

 MEAN SQUARE= 7.55356879,   5   COV.(0)= 3.415506,   5   COV.(1)= 3.339972,   5   COV.(2)= 3.187420,   5
 COR.(1)= 9.778849,  -1   COR.(2)= 9.332206,  -1   SUM ETA= 2.267033,   8

 Y'= 9.778849,  -1      Z'= 9.332206,  -1
 FIRST SIX RECONSTRUCTED RANDOM NUMBERS
  0.000000,   0    2.344031,   3    3.364356,   3    3.636827,   3    3.612630,   3    3.492320,   3

 STAGE RESULTS
 F'= 3.92187738,   5        M'=  81
  9.120456,  -1      1.288200,   1      8.791076,  -1      1.408994,  -1        T= 7.7611
 MEAN= 2.213705814574,   1     COR.(1)= 4.031416,  -3    COR.(2)= 4.321957,  -2          FINISH
 COV.(0)= 8.153047393192,   2     COV.(1)= 3.286832216152          COV.(2)=-3.523712128415,   1

 MEAN SQUARE= 1.30676549,   3

 RATE OF CHANGE OF VARIANCE    PRECEDING= 7.852735887479,   1     CURRENT= 2.381471315937,   2

 Z= 1.088842           R=  0
 T'=  4

 OPTIMUM VALUES OF PARAMETERS
  9.120456,  -1      1.288200,   1      8.791076,  -1      1.408994,  -1
 SYSTEM DYNAMICS    TIME CONSTANT= 7.761083           GAIN= 1.288200,   1
 DISTURBANCE    ORDINATE= 1.408994,  -1      SLOPE= 9.120456,  -1
```

FIG. 5.3
SHOWING START AND END OF ESTIMATION PROCEDURE. EXAMPLE I.

INVERTED (COVARIANCE) MATRIX

1.036808, -12	1.180641, -15	-8.253816, -16	-1.098017, -15
1.180641, -15	8.412495, -16	-5.649618, -17	-3.522977, -16
-8.253816, -16	-5.649618, -17	7.289305, -16	-3.021883, -16
-1.068017, -15	-3.522977, -16	-3.021883, -16	9.337450, -16

DIAGONAL MATRIX OF NORMALIZING SIGMAS

9.820888, 5	0.000000, 0	0.000000, 0	0.000000, 0
0.000000, 0	3.447764, 7	0.000000, 0	0.000000, 0
0.000000, 0	0.000000, 0	3.703880, 7	0.000000, 0
0.000000, 0	0.000000, 0	0.000000, 0	3.272547, 7

CORRELATION MATRIX

1.000000	3.997663, -2	-3.002358, -2	-3.528951, -2
3.997663, -2	1.000000	-7.214623, -2	-3.974964, -1
-3.002358, -2	-7.214623, -2	1.000000	-3.662860, -1
-3.528951, -2	-3.974964, -1	-3.662860, -1	1.000000

FIG. 5.4.

CORRELATION MATRIX. EXAMPLE 1.

ITERATION

9.821818, -1	2.350442, -19	1.542377, -10	0.000000, 0	EIGENVALUES
2.350442, -19	1.117653	0.000000, 0	1.479013, -14	
1.542377, -10	0.000000, 0	7.443111, -1	-6.821526, -19	
0.000000, 0	1.479013, -14	-6.821526, -19	1.155854	

GAMMA ONE= 5.453125, -11

CURRENT GREATEST ELEMENT= 1.542377, -10 S= 1 T= 3

COS THETA= 1.0000 SIN THETA=-2.0722, -10

CURRENT S-MATRIX

1.000000	0.000000, 0	-2.072221, -10	0.000000, 0
0.000000, 0	1.000000	0.000000, 0	0.000000, 0
2.072221, -10	0.000000, 0	1.000000	0.000000, 0
0.000000, 0	0.000000, 0	0.000000, 0	1.000000

CURRENT MATRIX OF EIGENVECTORS

9.920495, -1	7.999195, -2	9.303894, -2	2.798034, -2
1.661579, -2	3.646139, -1	-6.814333, -1	6.343731, -1
-1.247468, -1	6.850514, -1	6.489463, -1	3.066131, -1
-6.978551, -5	-6.255898, -1	3.253631, -1	7.090672, -1

FIG. 5.5.

MATRIX OF EIGENVALUES AND EIGENVECTORS. EXAMPLE I.

SINGLE EXPONENTIAL DYNAMICS, FIRST ORDER DISTURBANCE

FIG. 5.6
SHOWING START AND END OF ESTIMATION PROCEDURE
EXAMPLE 2.

```
INPUT PARAMETER VALUES
SYSTEM DYNAMICS    GAIN= 1.5000,   1      TIME CONSTANT= 9.2310,  -1       1.249723,   1

DISTURBANCE    SLOPE= 4.0000,   1
```

(G) (g) (ϕ) (δ)

```
GUESSES
 1.000000,  -1     5.000000          5.000000,  -1     1.015000,  -1
A'=  1     M'=   0      S'=  0
FIRST MEAN= 7.405322,   2

MEAN SQUARE= 1.04521621,   6   COV.(0)= 4.948269,   5   COV.(1)= 4.862744,   5   COV.(2)= 4.666476,   5
COR.(1)= 9.827160,  -1   COR.(2)= 9.430521,  -1   SUM ETA= 3.136772,   8

Y'= 9.827160,  -1     Z'= 9.430521,  -1
FIRST SIX RECONSTRUCTED RANDOM NUMBERS
 0.000000,   0    2.174341,   3    3.567770,   3    4.152340,   3    4.291896,   3    4.245873,   3
```

START

```
STAGE RESULTS
F'= 3.11488724,   5     M'= 109
 9.861037,  -1     1.468548,   1     9.295341,  -1     1.210259,  -1        T= 1.3685,   1
MEAN= 2.276950124083,   1     COR.(1)= 5.388031,  -2     COR.(2)= 3.064891,  -2
COV.(0)= 5.177788889923,   2     COV.(1)=-2.789808821674,   1     COV.(2)=-1.586935861894,   1

MEAN SQUARE= 1.03775877,   3

RATE OF CHANGE OF VARIANCE    PRECEDING= 6.600087038983,   1     CURRENT= 7.894253275252,   1

Z= 1.022522          R=  0
T'=  7

OPTIMUM VALUES OF PARAMETERS
 9.861037,  -1     1.468548,   1     9.295341,  -1     1.210259,  -1
SYSTEM DYNAMICS    TIME CONSTANT= 1.368517,   1      GAIN= 1.468548,   1
DISTURBANCE    ORDINATE= 1.210259,  -1     SLOPE= 9.861037,  -1
```

FINISH

INVERTED (COVARIANCE) MATRIX

9.315646, -13	8.550022, -16	-5.607506, -16	-8.131704, -16
8.550022, -16	4.751862, -16	-2.611119, -17	-1.956719, -16
-5.607506, -16	-2.611119, -17	4.129940, -16	-1.719135, -16
-8.131704, -16	-1.956719, -16	-1.719135, -16	5.260066, -16

DIAGONAL MATRIX OF NORMALIZING SIGMAS

1.036081, 6	0.000000, 0	0.000000, 0	0.000000, 0
0.000000, 0	4.587416, 7	0.000000, 0	0.000000, 0
0.000000, 0	0.000000, 0	4.920714, 7	0.000000, 0
0.000000, 0	0.000000, 0	0.000000, 0	4.360180, 7

CORRELATION MATRIX

1.000000	4.063767, -2	-2.858850, -2	-3.673495, -2
4.063767, -2	1.000000	-5.894172, -2	-3.913821, -1
-2.858850, -2	-5.894172, -2	1.000000	-3.688439, -1
-3.673495, -2	-3.913821, -1	-3.688439, -1	1.000000

FIG. 5.7.

CORRELATION MATRIX. EXAMPLE 2.

ITERATION

1.008071	-9.808161, -15	0.000000, 0	6.931291, -12	EIGENVALUES
-9.808160, -15	9.943996, -1	-2.055418, -21	0.000000, 0	
0.000000, 0	-2.055418, -21	1.211999	-3.592622, -14	
6.931291, -12	0.000000, 0	-3.592622, -14	7.855295, -1	

GAMMA ONE= 2.450617, -12

CURRENT GREATEST ELEMENT= 6.931291, -12 S= 1 T= 4

COS THETA= 1.0000 SIN THETA=-8.8237, -12

CURRENT S-MATRIX

1.000000	0.000000, 0	0.000000, 0	-8.823717, -12
0.000000, 0	1.000000	0.000000, 0	0.000000, 0
0.000000, 0	0.000000, 0	1.000000	0.000000, 0
8.823717, -12	0.000000, 0	0.000000, 0	1.000000

CURRENT MATRIX OF EIGENVECTORS

9.972878, -1	-6.095532, -2	-3.402247, -2	2.332230, -2
3.011596, -2	-1.057273, -1	4.729288, -1	-8.742157, -1
6.581480, -3	-1.839508, -1	8.549381, -1	4.849738, -1
6.683354, -2	9.753298, -1	2.103844, -1	-1.840974, -3

FIG. 5.8.

MATRIX OF EIGENVALUES AND EIGENVECTORS. EXAMPLE 2.

5.10. Conclusions.

In this Chapter a novel method of estimation of parameters of a first order system has been described. The method consist in identifying a nonstationary disturbance, assumed to contami ate the output only, from the input and output readings; the parameters of the combined model are then estimated by minimizing the sum of squares of the quasi-residuals in such a way that, at the same time, their covariance matrix is made to approach the diagonal matrix.

The examples illustrate the technique and show it to be quite satisfactory. The drawback of the method, discussed in the Appendix D, is that the starting value of the parameter δ must be reasonably close to its true value if convergence to the minimum is to be obtained.

CHAPTER 6.

APPLICATION OF THE METHOD TO ESTIMATION OF BOILER DYNAMICS.

6.1. Introduction

This Chapter describes briefly the tests carried out in support of this project, analysis of test results, and application of the method, described in Chapter 5, to estimation of boiler dynamics.

6.2. Description of plant.

The tests on a boiler-turbogenerator unit were carried out at Croydon 'B' and Northfleet Generating Stations of the Central Electricity Generating Board. The full description of a boiler and its operation would merit more than one chapter and is, therefore, outside the scope of the thesis. It might be helpful, however, to give a simple picture of processes occurring in a typical boiler, and factors governing the boiler response (Moran et al., 1968).

In a typical boiler, new coal is fed to mills where it is ground and dried by hot primary air. It is then carried by the air stream to burners where it ignites, the remaining air required for combustion being supplied as secondary air. Steam is generated in the waterwalls of the furnace, mainly by radiation and superheated to the final required temperatur in the superheater which may be both radiative and convective Heat is recovered from the hot gases by an economiser, which heats the feedwater, and the air heater which heats both the primary and secondary air. The coal flow into the furnace is, at least transiently, mainly controlled by varying the primary air flow. This varies the pickup of coal in the mills,drawing on the ground coal stored therein. The raw coal feed is then adjusted to maintain the storage. The response in heat release to primary air-flow changes is rapid, being typically for large drum-type boilers a dead lag of about 5-10 sec., corresponding to the transport time from the mills to the burners.

The response of steam generation to heat release , manifest as pressure or steam flow changes,is approximately a single lag, typically of about 5 minutes,dependent on the thermal inertia of the boiler. The response of the temperature of the steam leaving the superheater to heat absorption is slower still,and is more complex,but the simplest approximation is a single lag of about 10 min.

The boilers at Croydon B Generating Station are Simon-Carw tri-drum with twin natural circulation and pulverized fuel firing. They supply steam at 625 psi and 875 deg .F at the boiler stop valve, each boiler being rated at 320,000 lbs/hour at maximum continuous rating, equivalent to approximately 35 MW generated. Each pair of boilers supply one of the four main steam receivers which are interconnected Except for automatic control of drum water level, no other automatic control systems are provided.

The Northfleet Generating Station is provided with six Foster Wheeler boilers having each evaporative capacity of 860,000 lbs/hour, at a pressure of 1600 lb/sq in and temp. of 543 deg.C. at the superheater outlet. Each boiler is singl drum, natural circulation and has a water tube radiant furnace radiant superheater, primary and secondary convection superheater, reheater and economizer. Each boiler is associated with a separate turbine. Automatic control of drum pressure and steam outlet temperature is provided. The control of drum pressure is effected by varying the flow of mixture of pulverized coal and hot air by means of dampers. Steam temperature is controlled by varying the moisture of steam from the drum.

6.3. Description of the tests.

The tests at Croydon 'B', carried out in March 1965, consisted in running the boiler at low output (20 MW) with governor valve locked, and recording drum pressure, megawatts output and final steam temperature. There was no provision for directly measuring the steam flow , and the tests were regarded as necessary to obtain some rough idea about the behaviour of the processes. The various transducers had been installed previously in connection with the boiler optimization project (Moran et al.,1968). The analogue outputs of the transducers were scanned every 10 seconds and recorded on a 5-hole tape by means of the equipment shown in Fig.6.1. Two tests, of 6 hours duration each, were carried out . However, as the moist coal blocked one of the mills during one tests, the results of only one test could be used for the analysis.

The tests carried out at Northfleet in October 1965, comprised recording, at full output (120 MW) of boiler pressures, steam temperature, throttle valve movement, and movement of the coal feed damper when it was in operation. Two six hour tests were carried out with manual control of the dampers, and two six-hour tests with automatic control of the dampers. Automatic temperature controllers were out of action during the tests.

The various transducers had been installed before for dynamic boiler trials (Williams and Dart,1967). The analogue outputs of these transducers were sampled every 15 seconds

FIG. 6.1
DATA LOGGING EQUPMENT
AT CROYDON 'B' POWER STATION

FIG. 6.2.
DATA LOGGING EQUIPMENT
AT NORTHFLEET POWER STATION

and recorded on a 5-hole tape by means of the equipment shown in Fig.6.2.

The discussion of the transducers and data logging equipment is considered to be beyond the scope of this Chapter.

6.4. Data processing.

The test results were punched on a five-hole tape in a special C.E.G.B. code. As a result, a data translation procedure had to be developed to enable the analysis to be made on the London University Atlas computer.

It is not intended to go into details of the translation program which, in theory, should form a simple "look-up" table.It turned out, however, that the data logging equipment shown in Figs, 6.1. and 6.2 , although expensive, was not entirely free from errors. As a result, what should have been a simple program, it became an elaborate procedure, adaptively learning such possible faults as missing of a scan, and correcting the translated data in a proper manner. Fig.6.3 illustrates a new fault found by the translation program, and Fig.6.4. shows that the program has completed translation after having recognized a type of fault.

```
DSC43EP1, RUDZINSKI NORTHFLEET TEST 3 NOR011 AUGUST66
DATE 04.08.66
TIME 11.08.42
SERIAL NUMBER      9521554
START OF THE DATA

CHANNEL SEPARATION  I'=  54
CHANNEL SEPARATION  I'=  69
CHANNEL SEPARATION  I'=  77
CHANNEL SEPARATION  I'=  78
CHANNEL SEPARATION  I'=  89
CHANNEL SEPARATION  I'=  92
CHANNEL SEPARATION  I'=  98
CHANNEL SEPARATION  I'=  99
CHANNEL SEPARATION  I'= 102
CHANNEL SEPARATION  I'= 105
CHANNEL SEPARATION  I'= 124
CHANNEL SEPARATION  I'= 131
CHANNEL SEPARATION  I'= 148
CHANNEL SEPARATION  I'= 148
CHANNEL SEPARATION  I'= 154
CHANNEL SEPARATION  I'= 158
CHANNEL SEPARATION  I'= 165
CHANNEL SEPARATION  I'= 166
CHANNEL SEPARATION  I'= 226
CHANNEL SEPARATION  I'= 312

FAULTY SCANNING
Q=  3     I'= 1447
DATA TAPES FAULTY
```


FIG. 6.3

ILLUSTRATING FAULT IN TAPE NOT ALLOWED FOR IN PROGRAM

```
00.03.06 /          02.08.66   11.33.00
OUTPUT   0
DSC43EP1, RUDZINSKI NORTHFLEET TEST 1 NOR010 JULY66
```



```
EMA 16 DEC 1965

  5: HEADING FOR ROUTINE 1
  6: START OF CHAPTER 0
 253: START OF ROUTINE 1
 282: END OF ROUTINE
PROGRAMME ENTERED

START OF THE DATA

FIRST CHARACTER VALUE0 (61)   I'=  225

FIRST CHARACTER VALUE0 (61)   I'=  334

FIRST CHARACTER VALUE0 (61)   I'=  343

FIRST CHARACTER VALUE0 (61)   I'=  346

FIRST CHARACTER VALUE0 (61)   I'=  515

FIRST CHARACTER VALUE0 (61)   I'=  533

FIRST CHARACTER VALUE0 (61)   I'=  553

FIRST CHARACTER VALUE0 (61)   I'=  586

FIRST CHARACTER VALUE0 (61)   I'=  650

FIRST CHARACTER VALUE0 (61)   I'=  715

FIRST CHARACTER VALUE0 (61)   I'=  751

FIRST CHARACTER VALUE0 (61)   I'=  878

FIRST CHARACTER VALUE0 (61)   I'= 1020

FIRST CHARACTER VALUE0 (61)   I'= 1053

FIRST CHARACTER VALUE0 (61)   I'= 1185

NUMBER OF READINGS= 1256
DATA TAPES IN ORDER
```

FIG. 6.4

ILLUSTRATING FAULT IN TAPE
TAKEN INTO ACCOUNT BY PROGRAM

6.5. Scaling of the recorded data.

All quantities were recorded as three-digit integers in th range 0-999. The scaling factors to be applied to the recordings θ were as follows.

a) Croydon Test.

Drum pressure $\theta \times \frac{250}{1000} + 550$ p.s.i

Power output $\theta \times \frac{10}{1000}$ MW

Steam temperature. $\theta \times \frac{312}{1000} + 700$ °F

b) Northfleet Test

Steam flowrate $83.2\sqrt{0.0456(\theta-207)+17}$ Klb/hr

Drum pressure $0.284(\theta-212)+1620$ p.s.i.

Steam temperature $0.398(\theta-216.5)+980$°F

6.6. Analysis of the data.

The analysis of the data comprised

a) obtaining sample correlation functions, by using the formulae quoted in the thesis,

b) investigation of the behaviour of mean square values, as discussed in Chapter 5.

Figs.6.5-6.6. show sample correlation functions of drum pressure and final steam temperature., relating to Croydon test, and calculated for each quarter (1 hour recording) of the total recorded data.

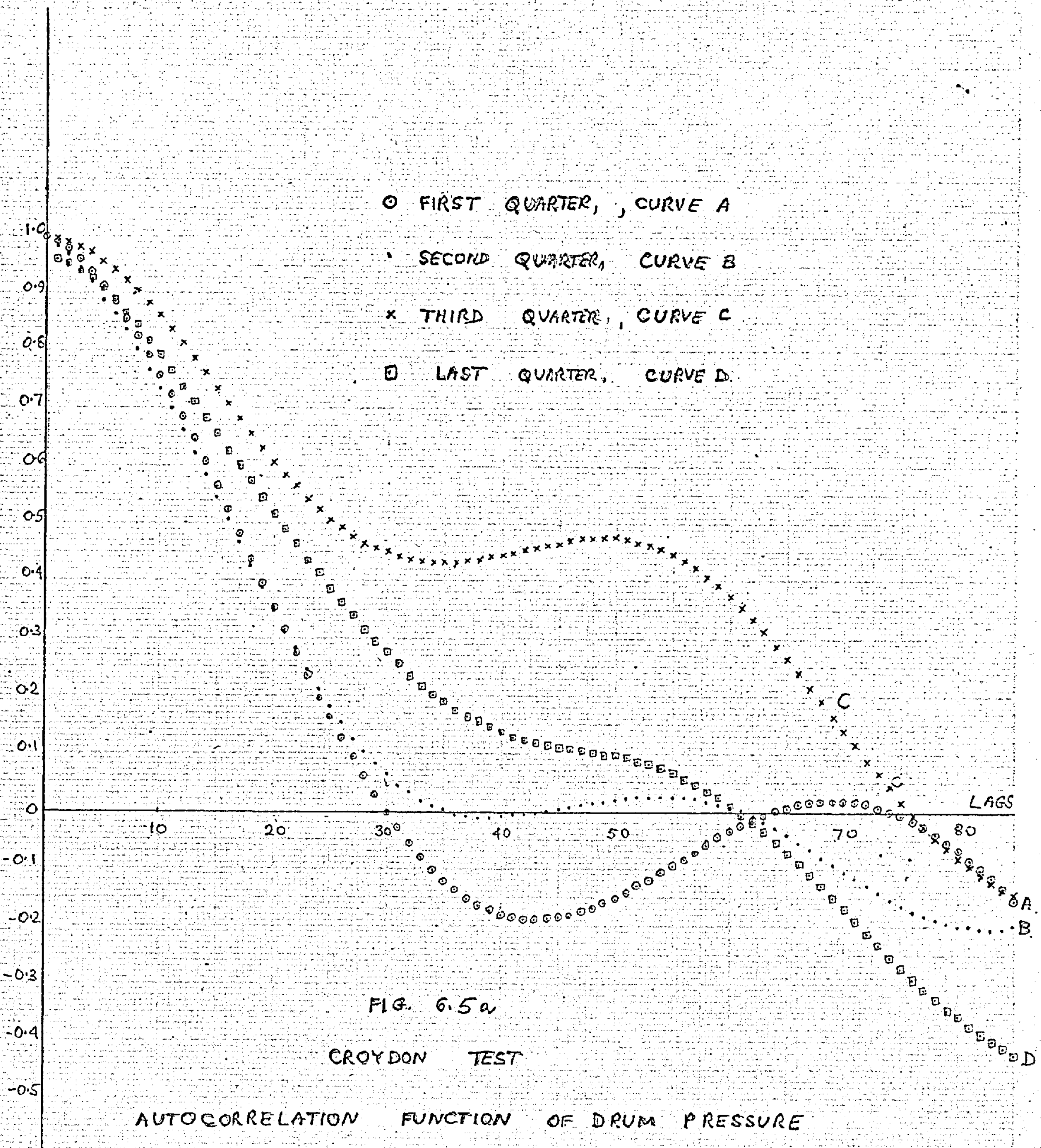


FIG. 6.5a

CROYDON TEST

AUTOCORRELATION FUNCTION OF DRUM PRESSURE

SAMPLING INTERVAL 10 SECONDS

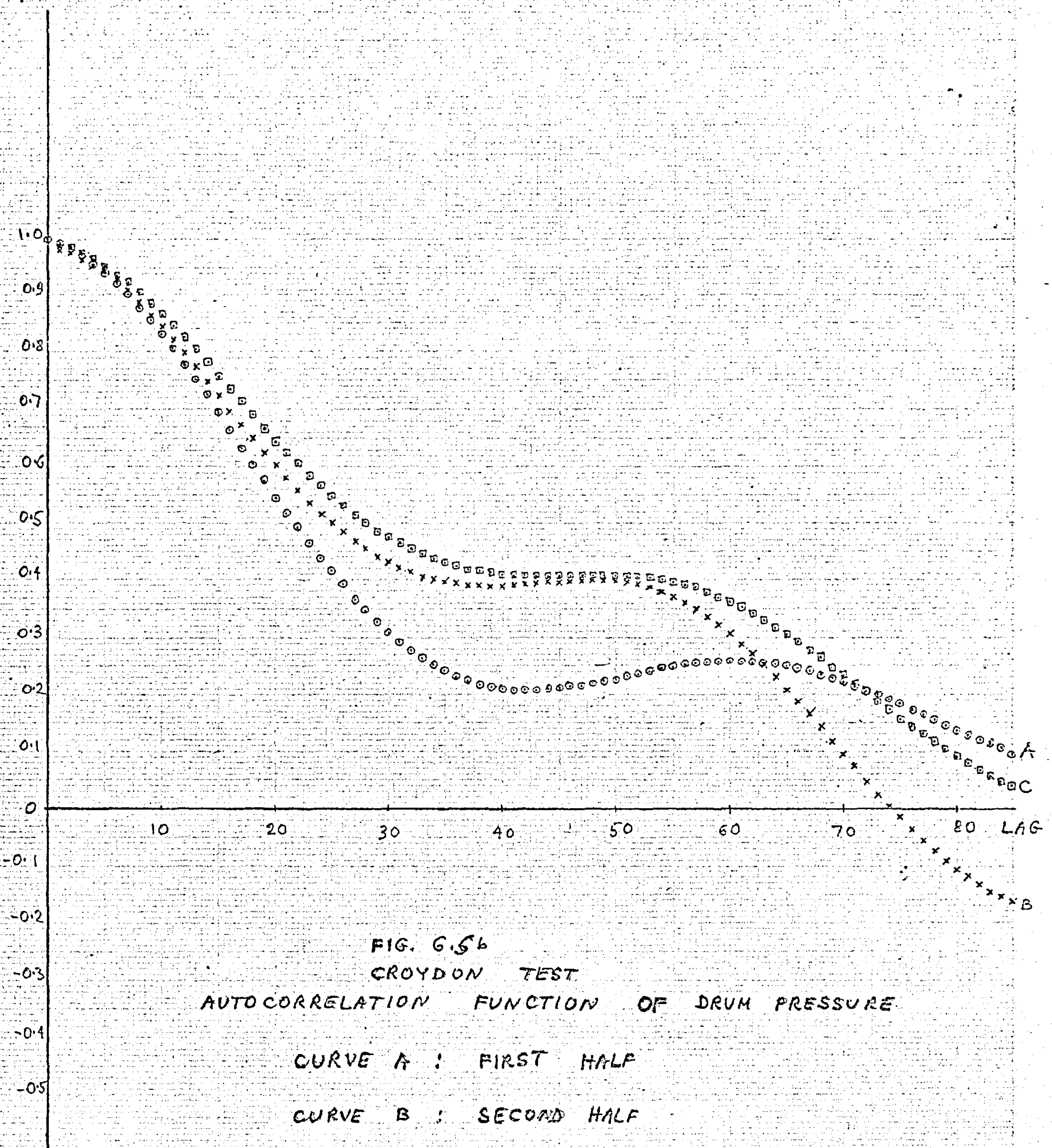


FIG. 6.5b
CROYDON TEST
AUTOCORRELATION FUNCTION OF DRUM PRESSURE

CURVE A : FIRST HALF

CURVE B : SECOND HALF

CURVE C : TOTAL SERIES (1800 READINGS)

SAMPLING INTERVAL : 10 SECONDS

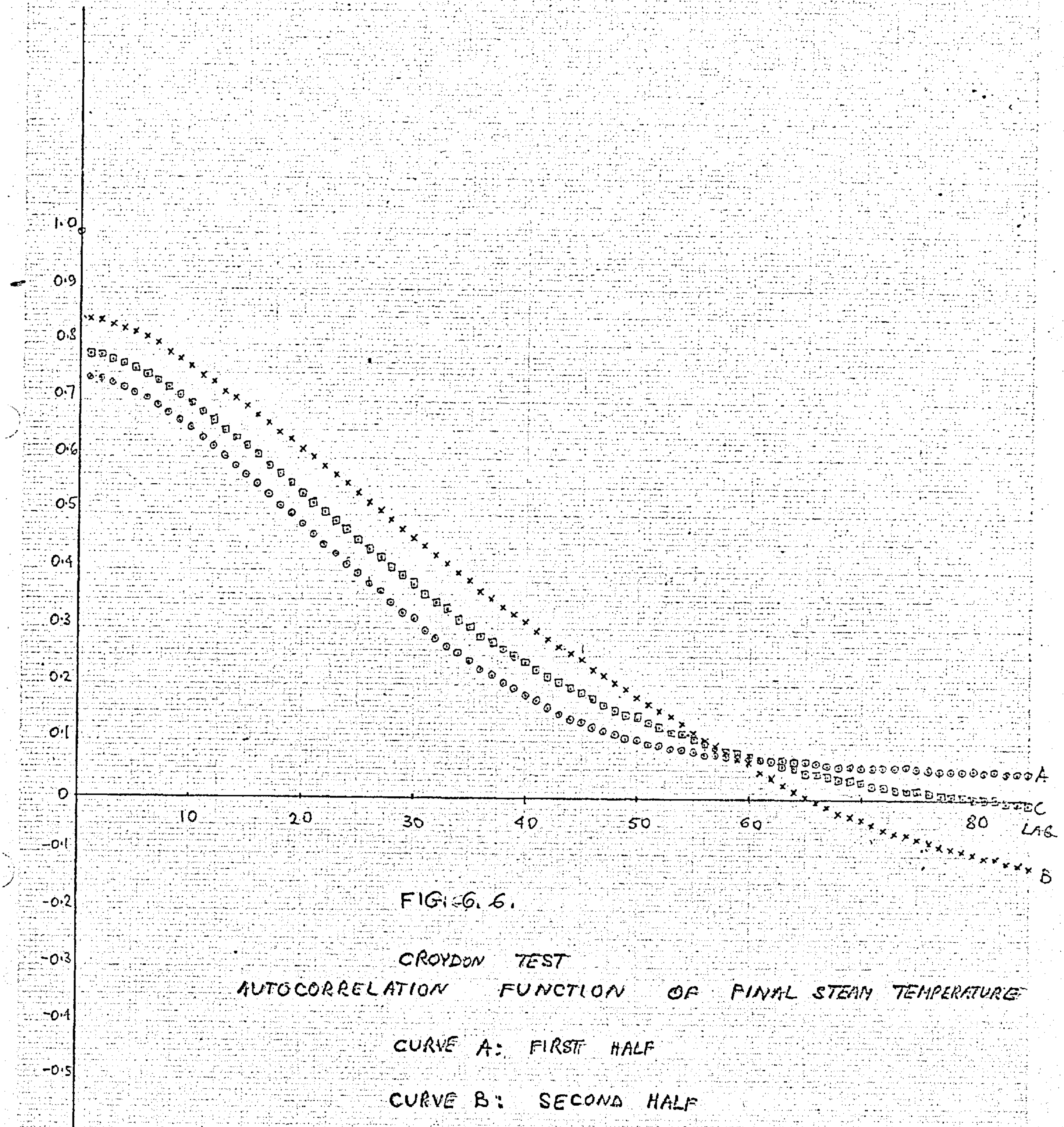


FIG. 6. 6.

CROYDON TEST

AUTOCORRELATION FUNCTION OF FINAL STEAM TEMPERATURE

CURVE A: FIRST HALF

CURVE B: SECOND HALF

CURVE C: TOTAL SERIES (1900 READINGS)

SAMPLING INTERVAL: 10 SECONDS

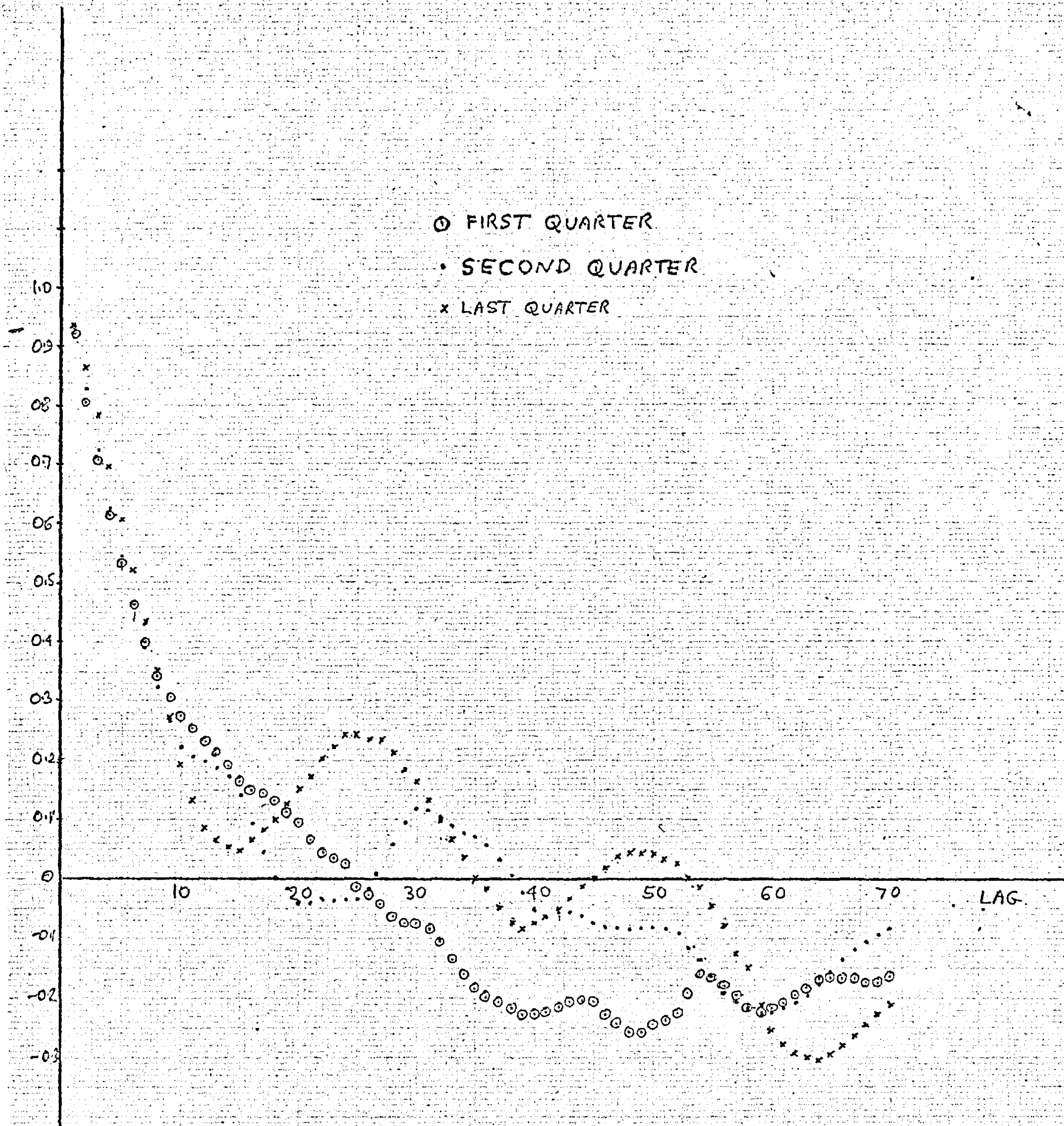


FIG. 6.7.

AUTOCORRELATION FUNCTION OF STEAM FLOWRATE

TEST 1.

SAMPLING INTERVAL 15 SECS.

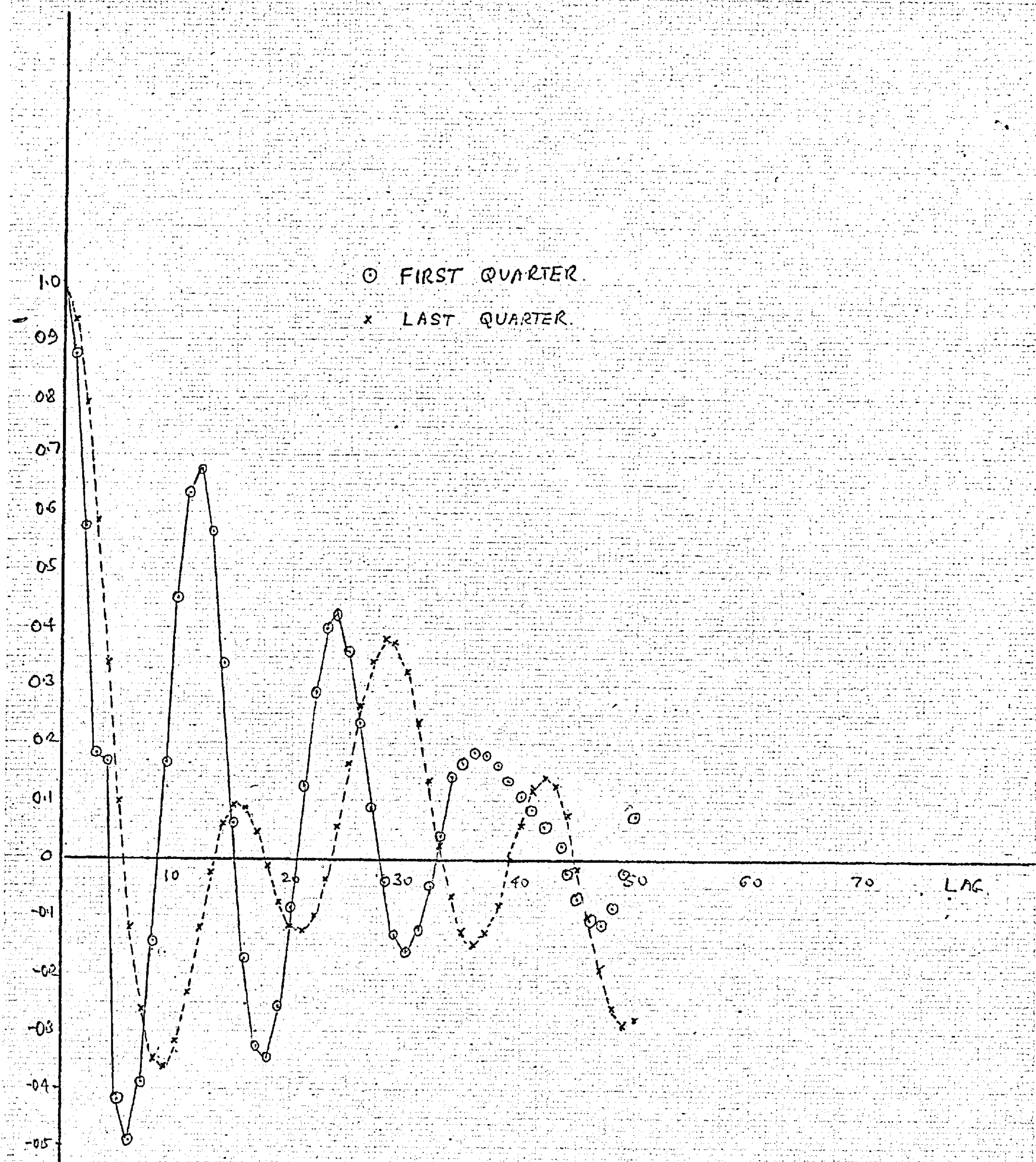


FIG. 6.8

AUTOCORRELATION FUNCTION OF FINAL STEAM TEMPERATURE. TEST 4.

SAMPLING INTERVAL 15 SEC.

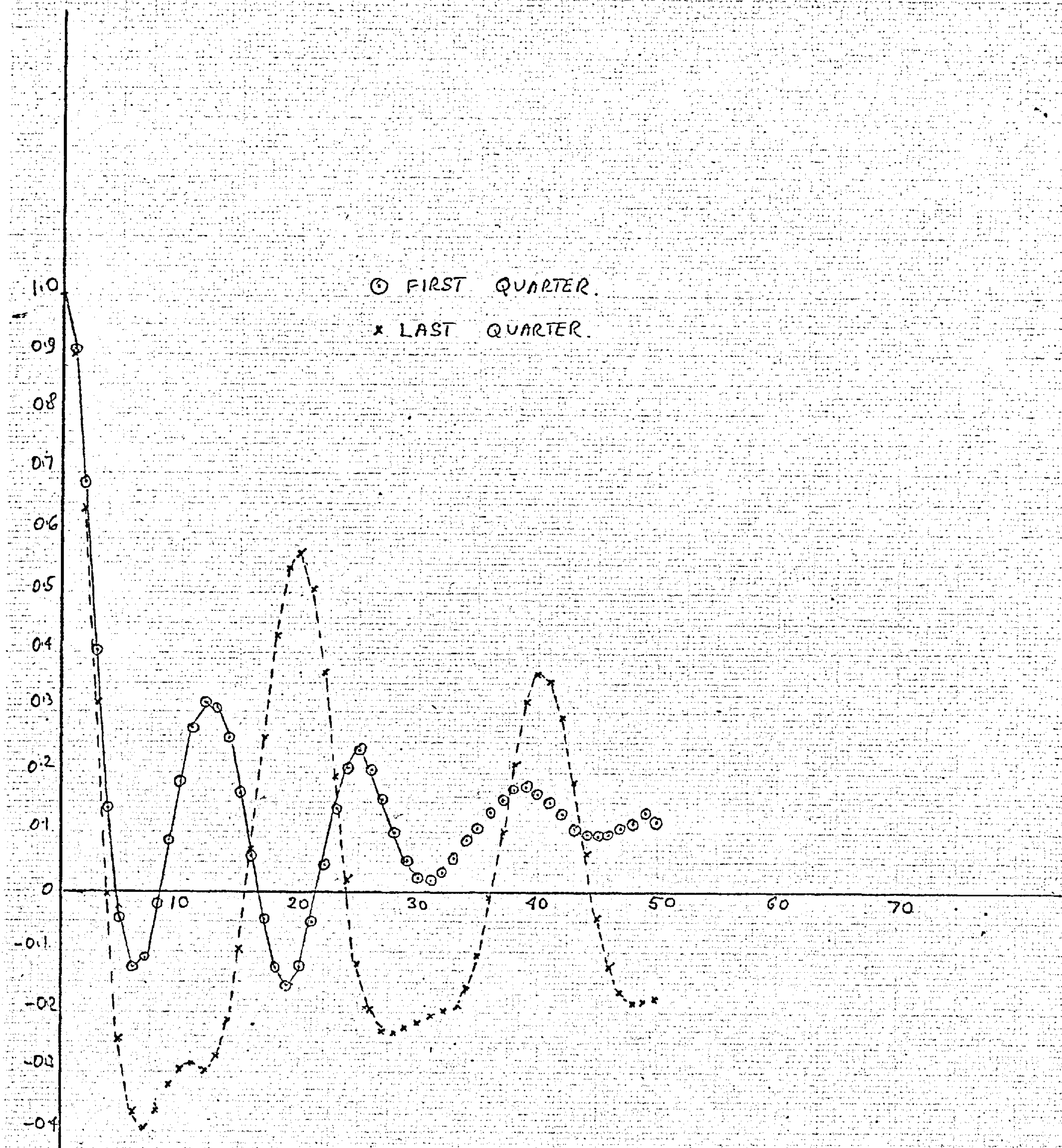


FIG. 6.9.

AUTO CORRELATION FUNCTION OF DRUM PRESSURE

TEST 3

SAMPLING INTERVAL 15 SECS.

Figs.6.7,6.8 and 6.9 show, respectively, sample correlation functions of steam flowrate, final steam temperature and drum pressure, calculated for various quarters (90 minutes records) of the total recorded data.

The plots show a nonstationary behaviour of the processes. Analysis of mean square value shows that typical values of mean square and its second difference are as follows:

a) temperature (Test 3)

mean square value 6.33×10^4

second difference 6.63×10^2

b) drum pressure (Test 4)

mean square value 1.56×10^5

second difference 1.72×10^3

c) steam flowrate (Test 2)

mean square value 3.22×10^5

second difference 3.41

The results suggest that the processes can be represented by a second order model developed in Appendix C.

6.7. Estimation of steam flow-drum pressure and steam flow steam temperature dynamics.

Analysis of boiler equations (Evans and Fry, 1964) shows that both these relationships can be represented by a first order lag with negative gain. They represent a second order effects of the variations in steam flowrate about mean operating point, the temperature and pressure being established by heat release.

Attempt has been made to estimate these small negative variations by supposing that the main process, i.e. relation between heat release and temperature, and between heat release and drum pressure, acts as a big disturbance (about 95% of the output) opposing the negative relation between steam flow and temperature, and between steam flow and drum pressure, respectively.
The input-output relation of Chapter 5 was used and each of the two dynamic relationships was estimated, using this model, from the test results of two tests.
Figs. 6.10-6.13 illustrate the estimation procedure of steam flow to drum pressure dynamics. Figs.6.14-6.17 illustrate the estimation of steam flow to steam temperature dynamics.

The results seem to indicate that

a) for the steam flow to drum pressure dynamics the gain is of the order of 8×10^{-4} and the time constant is of the order of $20 \times 15 = 300$ seconds.

```
00.00.27  /        12.04.67   03.15.28
OUTPUT    0
DSC22EB1, RUDZINSKI STEAM FLOW/DRUM PRESSURE T2 NOR542A(5) MARCH 67

    (G)              (g)              (φ)              (δ)
GUESSES
 8.000000,  -1    -1.000000,  -2     5.000000,  -1     9.600000,  -1
A'=  1        M'=   0        S'=  0                                        START
FIRST MEAN= 9.917033,   1

MEAN SQUARE= 7.18880524,   4   COV.(0)= 6.201829,   4   COV.(1)= 5.588789,   4   COV.(2)= 4.617712,   4
COR.(1)= 9.011518,  -1   COR.(2)= 7.445726,  -1   SUM ETA= 2.875604,   7

 STAGE RESULTS
 F'= 2.34280418,   5      M'=  90
  9.860544,  -1     -8.692880,  -4      9.453797,  -1      9.987840,  -1        T= 1.7804,   1     FINISH
 MEAN= 5.534645511317              COR.(1)= 3.461770,  -2      COR.(2)= 7.177395,  -2
 COV.(0)= 5.549911931986,   2      COV.(1)= 1.921251951379,   1      COV.(2)= 3.983391070656,   1

 MEAN SCUARE= 5.85640734,   2

 RATE OF CHANGE OF VARIANCE    PRECEDING= 2.170266693821,   1      CURRENT= 0.000000000000,   0

 Z= 1.000000             R=  0
 T'=  7

 OPTIMUM VALUES OF PARAMETERS
  9.860544,  -1     -8.692880,  -4      9.453797,  -1      9.987840,  -1
 SYSTEM DYNAMICS    TIME CONSTANT= 1.780352,   1      GAIN=-8.692880,  -4
 DISTURBANCE   ORDINATE= 9.987840,  -1     SLOPE= 9.860544,  -1
```

FIG. 6·10

ESTIMATION OF STEAM FLOW TO DRUM PRESSURE DYNAMICS

TEST2.

INVERTED (COVARIANCE) MATRIX

2.312086, -10	-4.897563, -12	-6.847243, -12	-4.725403, -12
-4.897563, -12	2.761013, -12	-1.037585, -12	-1.004540, -12
-6.847243, -12	-1.037585, -12	2.996360, -12	-5.753067, -13
-4.725403, -12	-1.004540, -12	-5.753067, -13	3.267209, -12

DIAGONAL MATRIX OF NORMALIZING SIGMAS

6.576548, 4	0.000000, 0	0.000000, 0	0.000000, 0
0.000000, 0	6.018188, 5	0.000000, 0	0.000000, 0
0.000000, 0	0.000000, 0	5.777008, 5	0.000000, 0
0.000000, 0	0.000000, 0	0.000000, 0	5.532374, 5

CORRELATION MATRIX

1.000000	-1.938402, -1	-2.601457, -1	-1.719287, -1
-1.938402, -1	1.000000	-3.607384, -1	-3.344603, -1
-2.601457, -1	-3.607384, -1	1.000000	-1.838713, -1
-1.719287, -1	-3.344603, -1	-1.838713, -1	1.000000

CURRENT MATRIX OF EIGENVECTORS

1.000000	1.124969, -4	1.511622, -4	9.989855, -5
-1.125479, -4	1.000000	2.095346, -4	1.942722, -4
-1.511493, -4	-2.095723, -4	1.000000	1.068641, -4
-9.986054, -5	-1.942610, -4	-1.069199, -4	1.000000

ITERATION

9.998434, -1	0.000000, 0	7.242331, -18	2.138208, -18	EIGENVALUES
0.000000, 0	9.997625, -1	1.546143, -29	-6.877070, -18	
7.242331, -18	1.546143, -29	1.000191	0.000000, 0	
2.138208, -18	-6.877070, -18	0.000000, 0	1.000204	

FIG. 6.11.
ESTIMATION OF STEAM FLOW TO DRUM PRESSURE DYNAMICS
TEST 2.

```
00.04.64  /          09.04.67    03.56.00
OUTPUT    0
DSC2?EB1, RUDZINSKI STEAM FLOW/DRUM PRESSURE T3 NOR543A(1) MARCH 67
```

(G) (g) (φ) (δ) 171

```
GUESSES
 8.000000,  -1    -1.000000,  -2    5.000000,  -1    9.600000,  -1
A'= 1      M'=  0      S'= 0
FIRST MEAN= 1.031565,   2
```

START

```
MEAN SQUARE= 7.90746231,   4   COV.(0)= 6.838274,   4   COV.(1)= 6.157643,   4   COV.(2)= 5.087624,   4
COR.(1)= 9.004674,  -1   COR.(2)= 7.439924,  -1   SUM ETA= 3.162987,   7


STAGE RESULTS
F'= 2.33752063,   5     M'=  94
 9.860544,  -1    -8.692880,  -4    9.582804,  -1    9.987840,  -1     T= 2.3466,   1
MEAN= 5.748088887907          COR.(1)= 7.767915,  -2      COR.(2)= 9.983663,  -2
COV.(0)= 5.510777620147,   2    COV.(1)= 4.280725366392,   1    COV.(2)= 5.501774495947,   1
```

FINISH

```
MEAN SQUARE= 5.84361662,   2

RATE OF CHANGE OF VARIANCE    PRECEDING= 6.598830188814,   1      CURRENT= 2.980232238781,  -8

Z= 1.000000          R=  0
T'=  7

OPTIMUM VALUES OF PARAMETERS
 9.860544,  -1    -8.692880,  -4    9.582804,  -1    9.987840,  -1
SYSTEM DYNAMICS     TIME CONSTANT= 2.346602,   1      GAIN=-8.692880,  -4
DISTURBANCE   ORDINATE= 9.987840,  -1     SLOPE= 9.860544,  -1
```

FIG. 6. 12.

ESTIMATION OF STEAM FLOW TO DRUM PRESSURE DYNAMICS
TEST 3.

INVERTED (COVARIANCE) MATRIX

2.089435, -10	-4.287859, -12	-6.213936, -12	-4.156194, -12
-4.287859, -12	2.551960, -12	-9.702686, -13	-9.514557, -13
-6.213936, -12	-9.702686, -13	2.784901, -12	-5.217420, -13
-4.156194, -12	-9.514557, -13	-5.217420, -13	3.007818, -12

DIAGONAL MATRIX OF NORMALIZING SIGMAS

6.918080, 4	0.000000, 0	0.000000, 0	0.000000, 0
0.000000, 0	6.259837, 5	0.000000, 0	0.000000, 0
0.000000, 0	0.000000, 0	5.992322, 5	0.000000, 0
0.000000, 0	0.000000, 0	0.000000, 0	5.765995, 5

CORRELATION MATRIX

1.000000	-1.856903, -1	-2.576010, -1	-1.657890, -1
-1.856903, -1	1.000000	-3.639571, -1	-3.434202, -1
-2.576010, -1	-3.639571, -1	1.000000	-1.802707, -1
-1.657890, -1	-3.434202, -1	-1.802707, -1	1.000000

CURRENT MATRIX OF EIGENVECTORS

8.484203, -1	3.624187, -1	-3.840710, -1	-3.640360, -2
-4.190105, -1	4.379536, -1	-5.653657, -1	5.594537, -1
-5.380619, -2	7.815820, -1	6.209912, -1	-2.458623, -2
3.189328, -1	-2.568637, -1	3.836940, -1	8.276967, -1

ITERATION

1.045299	0.000000, 0	-4.127667, -23	-7.278016, -15	EIGENVALUES
0.000000, 0	1.129540	-3.728128, -14	1.271671, -22	
-4.127667, -23	-3.728128, -14	6.535135, -1	0.000000, 0	
-7.278016, -15	1.271671, -22	0.000000, 0	1.171647	

FIG. 6.13.

ESTIMATION OF STEAM FLOW TO DRUM PRESSURE DYNAMICS

TEST3.

```
00.00.54 /          23.04.67. 10.20.12
OUTPUT   0
QSC22EB1, RUDZINSKI STEAM FLOW/TEMPERATURE T4 NOR534A(5) MARCH 67
```


(G) (g) (φ) (δ)


```
GUESSES
 8.000000,  -1    -1.000000,  -2     6.000000,  -1     9.000000,  -1
A'=  1      M'=   0      S'=  0
FIRST MEAN= 1.053770,   2

MEAN SQUARE= 9.19492415,   4   COV.(0)= 8.080398,   4   COV.(1)= 6.827251,   4   COV.(2)= 5.418149,   4
COR.(1)= 8.449151,  -1   COR.(2)= 6.705300,  -1   SUM ETA= 3.678046,   7
```

START

```
STAGE RESULTS
F'= 9.44193991,   4      M'= 112
 9.860544,  -1    -3.841040,  -4     9.261731,  -1     9.737993,  -1        T= 1.3039,   1
MEAN= 2.634759570893           COR.(1)= 4.920410,  -2     COR.(2)= 3.536004,  -3
COV.(0)= 2.290798996443,   2     COV.(1)=-1.127167124519,   1     COV.(2)=-8.100273810909,  -1
```

FINISH

```
MEAN SQUARE= 2.36047246,   2

RATE OF CHANGE OF VARIANCE    PRECEDING= 4.136272594347,  -1      CURRENT= 3.454622766017

Z= 1.007506          R=  0
T'=  8

OPTIMUM VALUES OF PARAMETERS
 9.860544,  -1    -3.841040,  -4     9.261731,  -1     9.737993,  -1
SYSTEM DYNAMICS    TIME CONSTANT= 1.303881,   1     GAIN=-3.841040,  -4
DISTURBANCE   ORDINATE= 9.737993,  -1     SLOPE= 9.860544,  -1
```

FIG. 6.14.

ESTIMATION OF STEAM FLOW TO STEAM TEMPERATURE DYNAMICS

TEST 4.

INVERTED (COVARIANCE) MATRIX

1.155949, -9	-3.005763, -11	-3.989525, -11	-2.855841, -11
-3.005763, -11	1.345482, -11	-4.718159, -12	-4.578499, -12
-3.989525, -11	-4.718159, -12	1.477575, -11	-2.406489, -12
-2.855841, -11	-4.578499, -12	-2.406489, -12	1.569536, -11

DIAGONAL MATRIX OF NORMALIZING SIGMAS

2.941242, 4	0.000000, 0	0.000000, 0	0.000000, 0
0.000000, 0	2.726221, 5	0.000000, 0	0.000000, 0
0.000000, 0	0.000000, 0	2.601508, 5	0.000000, 0
0.000000, 0	0.000000, 0	0.000000, 0	2.524146, 5

CORRELATION MATRIX

1.000000	-2.410164, -1	-3.052650, -1	-2.120211, -1
-2.410164, -1	1.000000	-3.346254, -1	-3.150639, -1
-3.052650, -1	-3.346254, -1	1.000000	-1.580242, -1
-2.120211, -1	-3.150639, -1	-1.580242, -1	1.000000

CURRENT MATRIX OF EIGENVECTORS

5.428217, -1	-4.755153, -1	4.269107, -1	-5.449560, -1
7.153387, -1	4.086491, -1	-5.606806, -1	-8.326926, -2
-2.372877, -1	6.947024, -1	2.944596, -1	-6.118633, -1
3.705802, -1	3.525333, -1	6.455061, -1	5.671970, -1

ITERATION

1.744607	0.000000, 0	1.275154, -13	6.696890, -18	EIGENVALUES
0.000000, 0	7.955345, -1	1.037970, -16	-1.444121, -26	
1.275154, -13	1.037970, -16	8.465618, -1	-4.055343, -25	
6.696890, -18	-1.444121, -26	-4.055343, -25	6.132963, -1	

FIG. 6.15.

ESTIMATION OF STEAM FLOW TO STEAM TEMPERATURE DYNAMICS

TEST 4.

```
00.02.40  /           12.04.67   05.02.37
OUTPUT    0
DSC22EE1, RUDZINSKI STEAM FLOW/TEMPERATURE T3 NOR533A(4) MARCH 67

     (G)               (g)               (φ)              (δ)

GUESSES
 8.000000,  -1    -1.000000,  -2      6.000000,  -1      9.000000,  -1
A'=  1      M'=   0       S'=  0
FIRST MEAN= 1.062175,   2                                                         (START)

MEAN SQUARE= 9.49735330,   4   COV.(0)= 8.368655,   4   COV.(1)= 7.050390,   4   COV.(2)= 5.592658,   4
COR.(1)= 8.429795,  -1   COR.(2)= 6.686859,  -1   SUM ETA= 3.798942,   7

STAGE RESULTS
F'= 8.82625624,    4      M'=  99                                                  FINISH
 9.860544,  -1    -3.764219,  -4      9.399807,  -1      9.741889,  -1      T= 1.6156,   1
MEAN= 2.640907763750            COR.(1)= 4.029480,  -2     COR.(2)= 3.293044,  -4
COV.(0)= 2.136159219677,   2     COV.(1)=-8.607611231855          COV.(2)=-7.034467057776,  -2

MEAN SQUARE= 2.20648568,   2

RATE OF CHANGE OF VARIANCE    PRECEDING= 2.467473523705,   1      CURRENT= 2.925607306894

Z= 1.005683             R=  0
T'=  6

OPTIMUM VALUES OF PARAMETERS
 9.860544,  -1    -3.764219,  -4      9.399807,  -1      9.741889,  -1
SYSTEM DYNAMICS    TIME CONSTANT= 1.615615,   1      GAIN=-3.764219,  -4
DISTURBANCE    ORDINATE= 9.741889,  -1     SLOPE= 9.860544,  -1
```

FIG. 6.16

ESTIMATION OF STEAM FLOW TO STEAM TEMPERATURE DYNAMICS TEST3.

INVERTED (COVARIANCE) MATRIX

1.143475, -9	-2.985362, -11	-3.967433, -11	-2.770982, -11
-2.985362, -11	1.365089, -11	-4.724625, -12	-4.725290, -12
-3.967433, -11	-4.724625, -12	1.486773, -11	-2.460040, -12
-2.770982, -11	-4.725290, -12	-2.460040, -12	1.583182, -11

DIAGONAL MATRIX OF NORMALIZING SIGMAS

2.957241, 4	0.000000, 0	0.000000, 0	0.000000, 0
0.000000, 0	2.706572, 5	0.000000, 0	0.000000, 0
0.000000, 0	0.000000, 0	2.593448, 5	0.000000, 0
0.000000, 0	0.000000, 0	0.000000, 0	2.513244, 5

CORRELATION MATRIX

1.000000	-2.389479, -1	-3.042803, -1	-2.059468, -1
-2.389479, -1	1.000000	-3.316382, -1	-3.214272, -1
-3.042803, -1	-3.316382, -1	1.000000	-1.603446, -1
-2.059468, -1	-3.214272, -1	-1.603446, -1	1.000000

CURRENT MATRIX OF EIGENVECTORS

5.439566, -1	2.413713, -1	-3.789404, -1	-7.086997, -1
-7.655964, -4	7.963932, -1	6.025779, -1	-5.154664, -2
-1.196068, -1	-5.045709, -1	6.159822, -1	-5.930163, -1
8.305449, -1	-2.300128, -1	3.374462, -1	3.787076, -1

ITERATION

1.545572	-3.792809, -13	2.609171, -22	0.000000, 0
-3.792809, -13	6.883168, -1	0.000000, 0	-1.543513, -14
2.609171, -22	0.000000, 0	1.039759	3.801131, -15
0.000000, 0	-1.543513, -14	3.801131, -15	7.263518, -1

EIGENVALUES

FIG. 6.17.
ESTIMATION OF STEAM FLOW TO STEAM TEMPERATURE DYNAMICS
TEST 3.

b) for the steam flow to steam temperature dynamics

the gain is of the order of 4×10^{-4}

and the time constant is of the order of.

$15 \times 15 = 225$ seconds.

These results are in agreement with the corresponding results obtained by Williams and Dart(1967) during dynamic boiler trials.

7. CONCLUSIONS.

The results of tests described in Chapter 6 confirm a highly nonstationary behaviour of the processes encountered in boile plant operation. It is shown that they can be represented by nonstationary models developed in the thesis.
The nonstationary estimation procedure can be applied to the particular examples of boiler dynamics considered only because of the negative relationships relating steam flow and temperature or pressure, which makes it possible to differentiate between the characteristics of the big disturbance and small dynamic relationship.

The instability relating to the parameter expressing the initial state of the system dynamics, showed itself when starting values of this parameter, very different from the true value , were assumed. It was then necessary to increase gradually the starting value until progress in the iterations could be obtained. This is undoubtedly , a big drawback of the method presented.

The method was discussed with relation to a single lag, only, because this was to be the ultimate application of the method. Simulated studies with double exponential dynamics have, however, been made and the estimation procedure was shown to be satisfactory.

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Appendix A.

Estimation.

A.1. Introduction.

In simplest terms, the problem of estimation can be formulated as follows. Given a sample of n observations, $x_1, \ldots, x_n$, taken at random from a parent population, and assuming as a working hypothesis that the population is distributed in a form which is completely determinate except for values of some parameters $\underset{\sim}{\theta} = \{ \theta_i \}, i = 1, 2, \ldots, p$ it is required to determine, with the aid of the observations, numbers which can be taken as the values of the parameters $\underset{\sim}{\theta}$, or a range of numbers which can be taken to include these values.

Since observations are random variables, any function of the observations alone, called a statistic, is also a random variable. Therefore, if a statistic is used to estimate the parameters $\underset{\sim}{\theta}$, the estimated values may on occasion differ considerably from the true values of $\underset{\sim}{\theta}$. Therefore, a method of estimation, or an estimator, is regarded as generating a distribution of particular values, or estimates, and the merit of such a method are judged by the properties of this sampling distribution. The required properties are consistency, unbiassedness, minimum variance, efficiency and sufficiency.

The discussion of these properties and of the estimators, given below, is based on Kendall and Stuart(1961), Cramér (1946), Plackett(1960) and Deutsch (1965).

A.2. Desired properties of estimators.

A.2.1. Consistency.

This property requires that the estimator should give more and more accurate values of estimates as the number of observations in the sample increases. In other words, the variance of the sampling distribution of the estimator should decrease, and the central value of the distribution should tend to the true value θ as the sample size increases.

Stated more formally, an estimator t_n computed from a sample of n values, is said to be a consistent estimator of θ if, for any positive ε and η, however small, there is some N such that the probability P that $|t_n - \theta| < \varepsilon$ is given by

$$P\{|t_n - \theta| < \varepsilon\} > 1 - \eta \;, \quad n > N \qquad \text{(A.1)}$$

A.2.2. Unbiassedness.

This criterion requires that the central value of the sampling distribution should tend to the true value θ for all sample sizes, not merely large. In other words, an estimator t_n is unbiassed if

$$E\langle t_n \rangle = \theta \qquad \text{(A.2.)}$$

It should be noted that consistent estimators are not necessarily unbiassed.

A.2.3. Minimum variance.

If there exist more than one unbiassed consistent estimators of parameters, this further criterion chooses among them the one with the smallest sampling variance.

The variance of an estimator $\underline{t}$ of a function $\tau(\underline{\theta})$ $[\underline{\theta} = \{\theta_1, \dots, \theta_p\}]$ is related to the likelihood function L through the well known Cramér-Rao inequality

$$var[\underline{t}] \geq \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial\tau}{\partial\theta_i} \frac{\partial\tau}{\partial\theta_j} I_{ij}^{-1} \qquad (A.3)$$

where the matrix I to be inverted is given by

$$\{I_{ij}\} = \{ E < \frac{1}{L} \cdot \frac{\partial L}{\partial\theta_i} \cdot \frac{\partial L}{\partial\theta_j} > \} \qquad (A.4)$$

and the Likelihood Function L of sample of n independent observations is defined as the joint frequency function of the observations

$$L(x_1, \dots, x_n | \underline{\theta}) = f(x_1 | \underline{\theta}) \dots\dots f(x_n | \underline{\theta}) \qquad (A.5)$$

assuming the existence of the first two derivatives of L with respect to $\underline{\theta}$ for all θ, as well as the independence of the range of variation of x of the θ.

The smallest possible variance attainable by an estimator, corresponding to the equality sign in (A.3), is called the minimum variance bound. The estimator which attains this variance is then referred to as the Minimum Variance Bound Estimator.

It may be shown (Kendall and Stuart, 1961) that if a minimum variance bound estimator exists it is always unique (irrespective of whether any bound is attained) and that the minimum variance bound is attained when

$$\frac{\partial \log L}{\partial \underline{\theta}} = A(\underline{\theta}) \cdot [\underline{t} - \tau(\underline{\theta})] \tag{A.6}$$

where $A(\underline{\theta})$ is independent of observations.

If the relation (A.6) is not satisfied, then the best attainable variance may be greater than the minimum variance bound. The estimator of $\tau(\underline{\theta})$ which, under these conditions, has uniformly in $\{\underline{\theta}\}$ smaller variance than any other estimator, is the called a Minimum Variance Estimator.

A.2.4. Efficiency.

The criterion of efficiency is concerned with large sample properties of estimators. Since most of the estimators are asymptotically normally distributed in virtue of the Central

Limit Theorem (Cramér,1946), the large sample distribution of an estimator depends only on its mean and variance. However, as a consistent estimate is asymtotically unbiassed, it is the variance of the asymptotically normal distribution which discriminates between consistent estimators of the same parametric function.

An estimator which in large samples attains minimum variance is called efficient. The efficiency of any other estimator, relative to the efficient estimator, is defined as the reciprocal of the ratio of sample numbers required to give the estimators equal sampling variances. The criterion of e efficiency chooses an estimator with greater efficiency, other properties being equal.

A.2.5. Sufficiency.

An estimator $\underline{t}$ of $\underline{\theta}$ $(\underline{\theta} = \{\theta_1, .., \theta_p\})$ is said to be a jointly sufficient statistic for $\underline{\theta}$ if the Likelihood function L of the observations can be represented as a product of two factors, one of which is a function of the observations alone, i.e. if

$$L(x_1, \ldots, x_n \mid \underline{\theta}) = g(\underline{t} \mid \underline{\theta}) . k(x_1, \ldots, x_n) \qquad \text{(A.7.)}$$

where

$\underline{t} = \{t_1, \ldots, t_s\}$ is the vector of estimators,

$\underline{\theta} = \{\theta_1, \theta_2, \ldots, \theta_p\}$ is the vector of parameters.

Under these conditions the estimator t contains all the information in the sample, and the sufficient statistic is unique. A point to observe, however, is that, whereas individual sufficiency of the components of the vector t implies joint sufficiency of the estimator t, the converse is not necessarily true.

It can be shown that a Minimum Variance Bound Estimator can only exist if there is a sufficient statistic. In general, irrespective of the attainability of any variance bound, the minimum variance unbiassed estimator of $\tau(\theta)$ is always a function of the sufficient statistic, if one exists.

It can be shown also that the class of distributions in which sufficient statistics exist for the parameters $\underline{\theta}$ belongs to the exponential family of distributions defined by

$$f(x|\underline{\theta}) = \exp\{A(\underline{\theta})B(x) + C(x) + D(\underline{\theta})\} \tag{A.8}$$

A.3. Maximum Likelihood Estimation.

As stated in the preceding section, the Likelihood Function L of n independent observations from the same distribution is defined as the joint probability of the observations regarded as a function of the set of parameters $\underline{\theta} = \{\theta_1, \ldots, \theta_p\}$

$$L(x|\underline{\theta}) = f(x_1|\underline{\theta}) \cdot f(x_2|\underline{\theta}) \ldots\ldots f(x_n|\underline{\theta}) \tag{A.9}$$

The maximum likelihood estimation of the set of parameters $\underline{\theta}$ consists in choosing that set $\underline{\hat{\theta}} = (\hat{\theta}_1, \ldots, \hat{\theta}_p)$ of the admissible values of the parameters $\underline{\theta}$ which makes the likelihood function an absolute maximum. It is usually simpler to employ the logarithm of the likelihood function rather than the function itself. If the range of the frequency function $f(x|\underline{\theta})$ does not depend on the set of parameters $\underline{\theta}$, and if the set of the parameters $\underline{\theta}$ may take any set of values in the p-dimensional space, then the logarithm of the function and the function will have the maxima together. Under these conditions the local turning point will be given by the roots of the set of equations

$$\frac{\partial}{\partial \theta_i} \log L(x|\underline{\theta}) = 0 \qquad i = 1, 2, \ldots, n \tag{A.10}$$

A sufficient condition that any of these stationary values be a local maximum is that the matrix

$$\left(\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j} \right) \quad \begin{matrix} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \end{matrix} \tag{A.11}$$

be negative definite.

The solutions of the equations (A.10) are the set of p maximum likelihood estimates.

If there exists a set of s statistics $t_1, \ldots, t_s$ which are jointly sufficient for the parameters $\underline{\theta} = \{\theta_1, \ldots, \theta_p\}$, the set of maximum likelihood estimators $\underline{\hat{\theta}} = \{\hat{\theta}_1, \ldots, \hat{\theta}_p\}$ will be a function of the sufficient statistics. If this is the case, the likelihood function can be factorized into two factors, one of which is independent of the set of parameters $\underline{\theta}=\{\theta_1,\ldots,\theta_p\}$, i.e.

$$L\ (x\mid\underline{\theta})=g(\underline{t}\mid\underline{\theta})\,h(x) \tag{A.12}$$

where

$$\underline{t}=\{t_1,\ldots,t_s\}$$

If the regularity conditions mentioned in Section A.2.3. are satisfied then it may be shown (Kendall and Stuart,1961) that the likelihood equations have a unique solution if s=p, and that this solution is a maximum of the likelihood function

Under these conditions the most general form of distribution (A.8), admitting a set of p jointly sufficient statistics, results in the logarithm of the likelihood function of the form

$$\log L=\sum_{j=1}^{p}A_j(\theta)\sum_{i=1}^{n}B_j(x_i)+\sum_{i=1}^{n}C(x_i)+n\,D(\underline{\theta}) \tag{A.13}$$

The solutions $\underline{\hat{\theta}}=\{\hat{\theta}_1,\ldots,\hat{\theta}_p\}$ of the corresponding likelihood equations

$$\frac{\partial\log L}{\partial\theta_r}=\sum_{j=1}^{p}\frac{\partial A_j(\underline{\theta})}{\partial\theta_r}\sum_{i=1}^{n}B_j(x_i)+n\frac{\partial D(\underline{\theta})}{\partial\theta_r}=0 \qquad r=1,2,\ldots,p \tag{A.14}$$

is a maximum if the matrix

$$\left(\frac{\partial^2 \log L}{\partial\theta_r \partial\theta_s}\right)_{\hat{\theta}} = \sum_{j=1}^{p} \left(\frac{\partial^2 A_j(\underline{\theta})}{\partial\theta_r \partial\theta_s}\right)_{\hat{\theta}} \sum_{i=1}^{n} B_j(x_i)$$

$$+ n\left(\frac{\partial^2 D(\underline{\theta})}{\partial\theta_r \partial\theta_s}\right)_{\hat{\theta}} \qquad \text{(A.15)}$$

is negative definite.

If there is not necessarily a set of p sufficient statistics for the p parameters, the likelihood function no longer has a unique maximum value and the joint maximum likelihood estimators $\hat{\underline{\theta}} = \{\hat{\theta}_1, \ldots, \hat{\theta}_p\}$ are chosen such that

$$L(x|\hat{\underline{\theta}}) \geq L(x|\underline{\theta}) \qquad \text{(A.16)}$$

Such estimators are, under very broad conditions, consistent and converge in probability, as a set, to the true set of parameter values $\underline{\theta}_0$

If the range of the frequency function $f(x|\underline{\theta})$ does not depend on the set of parameters $\underline{\theta}$, the estimators are asymtotically efficient and tend to a multivariate normal distribution with a covariance matrix whose inverse is given (Kendall and Stuart,1961) by

$$(V_{rs}^{-1}) = -E\left\langle\left(\frac{\partial^2 \log L}{\partial\theta_r \partial\theta_s}\right)\right\rangle = E\left\langle\left(\frac{\partial \log L}{\partial\theta_r}\frac{\partial \log L}{\partial\theta_s}\right)\right\rangle \qquad \text{(A.17)}$$

$$r = 1, 2, \ldots, p$$
$$s = 1, 2, \ldots, p$$

An important result proved by Kendall and Stuart (1961) is that the determinant $|D|$ of the covariance matrix D of any set of estimators, which is called the generalized variance, cannot be less than

$$\frac{1}{|V^{-1}|}$$

in value asymptotically. Since, asymptotically, for the maximum likelihood estimators,

$$|D| = |V| = 1/|V^{-1}| \qquad \text{(A.18)}$$

it follows that the maximum likelihood estimators minimize the generalized variance in large samples.

As regards the bias, the maximum likelihood estimators are in general, biassed, although the bias will tend to zero for very large samples, if the estimators have finite mean value.

A.4. Least Squares Estimation.

A.4.1. The principle of Least Squares.

The least squares method of estimation has been known for quite a long time, as it appears to have originated from Gauss (Plackett,1949; Rosenbrock,1965). The method in the present form has been mainly employed in situations in which observations are distributed with constant variance about (possibly differing) mean values, which are linear functions in a finite number of unknown parameters, and in which the observations are uncorrelated in pairs.

The situation is then described as the linear model

$$\underline{y} = X\underline{\theta} + \underline{\varepsilon} \tag{A.19}$$

where $\underline{\theta}$ is a (p x 1) vector of parameters,

$\underline{y}$ is an (n x 1) vector of observations,

X is an (n x p) matrix of known coefficients,

$\underline{\varepsilon}$ is an (n x 1) vector of error random variables whose mean and covariance matrix are respectively given by

$$E\langle \underline{\varepsilon} \rangle = \underline{\theta} \tag{A.20}$$

and,

$$V(\underline{\varepsilon}) = E\langle \underline{\varepsilon}\,\underline{\varepsilon}^T \rangle = \sigma^2 I \tag{A.21}$$

The Least Squares method selects simultaneously those values of which minimize the scalar sum of squares

$$S = (\underline{y} - X\underline{\theta})^T(\underline{y} - X\underline{\theta}) \tag{A.22}$$

for variation in the components of $\underline{\theta}$.

The solution of equation (A.22), resulting in the computation of the least squares estimators $\hat{\underline{\theta}}$, as well as the computation of the covariance matrix of the estimators, involves the inversion of the matrix X, and thus depends on the rank of the latter. For this reason, there exist two forms of the solution, one corresponding to the rank of X being equal to the number of the parameters, p, and the other applying when the rank is smaller than p. (Plackett, 1960). Only the former case is of interest in this thesis and is discussed below.

A.4.2. Least squares estimation when the rank of matrix X is equal to the number p of parameters.

In this case the matrix $X^T X$ is invertible. Differentiating (A.22) with respect to $\underline{\theta}$, and equating to 0, yields the least squares estimator in the form

$$\hat{\underline{\theta}} = (X^T X)^{-1} X^T \underline{y} \tag{A.23}$$

The estimator is unbiassed, for from (A.19) and (A.23) we have,

$$\hat{\underline{\theta}} = \underline{\theta} + (X^T X)^{-1} X^T \underline{\varepsilon} \tag{A.24}$$

and the expected value of this expression is equal to $\underline{\theta}$.

Also, the covariance matrix of the estimators is

$$\begin{aligned} V(\hat{\underline{\theta}}) &= E\langle (\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})^T \rangle \\ &= \sigma^2 (X^T X)^{-1} \end{aligned} \tag{A.25}$$

If $\underline{t}$ is any vector of estimators, linear in the observations $\underline{y}$, i.e. of the form,

$$\underline{t} = T \cdot \underline{y} \tag{A.26}$$

and if $\underline{t}$ is unbiassed for a set of linear functions of the parameters, say, $C\underline{\theta}$, i.e. if

$$E \langle \underline{t} \rangle = C \cdot \underline{\theta} \tag{A.27}$$

then it may be shown (Kendall and Stuart, 1961) that

$$\underline{t} = C \cdot \hat{\underline{\theta}} \tag{A.28}$$

AND

$$V(\underline{t}) = \sigma^2 C (X^T X)^{-1} C^T \tag{A.29}$$

These results state that the least squares method yields minimum variance linear estimators of any set of linear functions of the parameters $\underline{\theta}$. It can be shown (Kendall and Stuart, 1961) that the least squares estimator $\hat{\underline{\theta}}$ minimizes the value of the generalized variance for linear estimators of $\underline{\theta}$ always, and not only asymptotically as is the case with the maximum likelihood estimator.

By considering the set of residuals in the least squares estimation,

$$y - X\hat{\theta} = \{I_n - X(X^T X)^{-1} X^T\} \underline{\varepsilon} \tag{A.30}$$

and observing that the matrix $(y - X\hat{\theta})^T (y - X\hat{\theta})$ is idempotent, and that trace $\{X(X^T X)^{-1} X^T\}$ = trace $\{X^T X (X^T X)^{-1}\}$, it can be shown that (Plackett, 1960)

$$E\langle (\underline{y} - X\hat{\underline{\theta}})^T(\underline{y} - X\hat{\underline{\theta}})\} = \sigma^2(n-p) \qquad (A.31)$$

It follows from the above that an unbiassed estimator s^2 of the variance σ^2 is the sum of the squared residuals divided by (the number of observations minus the number of parameters estimated) , i.e.

$$s^2 = \frac{1}{n-p} \cdot (\underline{y} - X\hat{\theta})^T(\underline{y} - X\hat{\underline{\theta}}) \qquad (A.32)$$

It should be noted that, as long as it is not required to test hypotheses concerning the parameters, no assumptions about the forms of distribution of errors are necessary for obtaining the least squares estimates.

In the above discuss ion the only restriction placed on the random errors $\{\varepsilon_i\}$ is that they be uncorrelated. If, in addition, the errors are normally distributed, then they are also independent. Under these conditions the quentity

$$\frac{(n-p)s^2}{\sigma^2}$$

is a chi-square variate with (n-p) degrees of freedom (since an idempotent quadratic form in independent standarized normal variates is a chi-square variate with degrees of freedom given by the rank of the quadratic form).

Now,

$$\underline{y}^T\underline{y} \equiv \underline{y} - (X\hat{\underline{\theta}})^T(\underline{y} - X\hat{\underline{\theta}}) + (X\hat{\underline{\theta}})^T(X\hat{\underline{\theta}}) \qquad (A.33)$$

and,

$$(X\hat{\underline{\theta}})^T(X\hat{\underline{\theta}}) = \hat{\underline{\theta}}^T X^T X \hat{\underline{\theta}} = \underline{y}^T X (X^T X)^{-1} X^T \underline{y}$$

or,

$$(X\hat{\underline{\theta}})^T(X\hat{\underline{\theta}}) = (\underline{\varepsilon}^T + \underline{\theta}^T X^T) X (X^T X)^{-1} X^T (X\underline{\theta} + \underline{\varepsilon}) \qquad (A.34)$$

which for $\underline{\theta} = \underline{0}$ gives,

$$\hat{\underline{\theta}}^T X^T X \hat{\underline{\theta}} = \underline{\varepsilon}^T X (X^T X)^{-1} X^T \underline{\varepsilon} . \qquad (A.35)$$

From (A.35), (A.33) and (A.30) we have then,

$$\underline{\varepsilon}^T \underline{\varepsilon} = \underline{\varepsilon}^T \{ I_n - X (X^T X)^{-1} X^T \} \underline{\varepsilon}$$

$$+ \underline{\varepsilon}^T \{ X (X^T X)^{-1} X^T \} \underline{\varepsilon} \qquad (A.36)6$$

The rank of the first matrix in the curly brackets is (n-p) and that of the second matrix is p. The ranks of the matrices add up to the rank n of the matrix $\underline{\varepsilon}^T \underline{\varepsilon}$. Applying Cochran's theorem ** we have then the result that the two quadratic forms in equation (A.36) are independently distributed like chi-square with (n-p) and p degrees of freedom.

** Cochran's Theorem (Lindgren,1962) states:
Let $U_1, .. U_r$ be independent and normally distributed with zero means and unit standard deviations.
Let $\sum_{i=1}^{r} U_i^2 = Q_1 + Q_2 + .. + Q_s$. ,where each Q_i is a sum of square of linear combinations of $U_1, ... U_r$, with t_i degrees of freedo
Then, if $t_1 + ... + t_s = r$,, the quantities $Q_1, ... Q_s$ are independent chi square variates with $t_1, .. t_s$ degrees of freedom.

Appendix B.

Linear Discrete-Time Systems.

B.1. Introduction.

The Appendix discusses linear discrete-time systems derived from linear differential systems by means of periodic sampling. A review of the well established theory of Z transforms (Hurewicz,1947; Barker,1952; Tou,1959) is first given as an introduction. The Z-transform technique is then used to derive difference equations describing linear discrete time systems which are suitable for their identification. The Appendix uses certainm results of the recent work by Box and Jenkins (1963; 1966;1967) but the formulae developed are general and include those derived by Box and Jenkins as special cases.

B.2. Definition of a Linear Discrete-Time System.

Most physical systems are continuous by nature and their dynamic behaviour can, therefore, be represented by that of continuous-time differential systems, as discussed in Chapters 1 and 2. If, however, such a process is controlled by a digital computer, the discreteness, specific to the digital computer control, is brought about by the periodic sampling of the variables.

Suppose that a linear differential system characterized by the impulse response h(t) and described by a relation such as

$$(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_0)\, y(t)$$

$$= (b_m p^m + b_{m-1} p^{m-1} + \cdots + b_0)\, u(t) \qquad \text{(B.1)}$$

is subject to a digital computer control and that, therefore, its continuous input u(t) and output y(t) are sampled every ΔT seconds. Then the resulting input sequence $\{u_t\}$ and the output sequence $\{y_t\}$

$$\{u_t\} = u(k\Delta T + 0) \qquad \text{(B.2a)}$$

$$\{y_t\} = y(k\Delta T + 0). \qquad \text{(B.2b)}$$

$$k = 1, 2, \ldots$$

describes a linear system in which the variables can change only at discrete instants of time (sampling instants). Such a system is referred to as a <u>discrete-time system</u> or a <u>sampled-data system.</u>

B.3. Two alternative characterizations of a linear discrete-time system.

As discussed in Chapter 1, a linear continuous-time system can be characterized either by its impulse response in the time domain, or by a transfer function in the frequency domain. The dynamic response of a discrete-time system can also be formulated in two domains as follows.

In the time domain, the sampled output time function $y(k\Delta T)$ is related to the sampled input time function $u(k\Delta T)$ by a so-called convolution summation

$$y(k\Delta T) = \sum_{j=0}^{\infty} h(j\Delta T)\, u(k\Delta T - j\Delta T) \qquad k = 1, 2, \ldots \tag{B.3}$$

in which the sampled values of the impulse response $h(t)$,

$$h(n\Delta T) = h(t)\big|_{t=n\Delta T} \tag{B.4}$$

are referred to as the <u>weighting sequence</u> of the discrete-time system.

With the notation

$$h_n = h(n\Delta T) \tag{B.5}$$

the relation (B.3), characterizing the linear discrete-time system in the time domain is written

$$y_k = \sum_{j=0}^{\infty} h_j\, u_{k-j} \tag{B.6}$$

Let the sampled time functions, corresponding to the continuous time functions $u(t)$, $y(t)$ and $h(t)$ be denoted $u^*(t)$, $y^*(t)$ and $h^*(t)$, respectively. Then, bearing in mind that for physical systems both $u(t)$ and $y(t)$ are zero for $t < 0$, one can write the expressions for the sampled input, output and the impulse response functions in the form

$$u^*(t) = \sum_{n=0}^{\infty} u(n\Delta T)\,\delta(t - n\Delta T) \qquad \text{(B.7a)}$$

$$y^*(t) = \sum_{n=0}^{\infty} y(n\Delta T)\,\delta(t - n\Delta T) \qquad \text{(B.7b)}$$

$$h^*(t) = \sum_{n=0}^{\infty} h(n\Delta T)\,\delta(t - n\Delta T) \qquad \text{(B.7c)}$$

The Laplace transforms of these expressions are

$$U^*(s) = \sum_{n=0}^{\infty} u(n\Delta T)\,e^{-n\Delta T s} \qquad \text{(B.8a)}$$

$$Y^*(s) = \sum_{n=0}^{\infty} y(n\Delta T)\,e^{-n\Delta T s} \qquad \text{(B.8b)}$$

$$H^*(s) = \sum_{n=0}^{\infty} h(n\Delta T)\,e^{-n\Delta T s} \qquad \text{(B.8c)}$$

If a complex variable z, defined by the relation

$$z = \exp(\Delta Ts) \qquad \text{(B.9)}$$

is substituted into (B.8), the resulting relations

$$U(z) = \sum_{n=0}^{\infty} u(n\Delta T)\,z^{-n} = \sum_{n=0}^{\infty} u_n z^{-n} \qquad \text{(B.10a)}$$

$$Y(z) = \sum_{n=0}^{\infty} y(n\Delta T)\,z^{-n} = \sum_{n=0}^{\infty} y_n z^{-n} \qquad \text{(B.10b)}$$

$$H(z) = \sum_{n=0}^{\infty} h(n\Delta T)\,z^{-n} = \sum_{n=0}^{\infty} h_n z^{-n} \qquad \text{(B.10c)}$$

are referred to, respectively, as the z transform of the input, z transform of the output and the z transform of the impulse response.

The relation

$$Y(z) = H(z).U(z) \qquad \text{(B.11)}$$

between the three z transforms is derived by Laplace transforming the expression (B.3) and substituting in it (B.8) and (B.9). The relation applies to the output signal $y(t)$ at the sampling instants only.

The ratio

$$H(z) = \frac{Y(z)}{U(z)} \qquad (B.12)$$

of the z transforms of the input and of the output is identical with the z transform of the sampled impulse response and is referred to as the <u>z-transfer function</u> or <u>the pulse transfer function</u> of the system.

It has been observed by Hurewicz (1947) that the expression for a rational transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_o}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_o} \qquad (B.13)$$

can be decomposed into a finite number of simple partial fractions

$$H(s) = \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{K_{ij}}{(s+\alpha_i)^j} \qquad (B.14)$$

the non-zero numbers α_i and K_{ij} being not necessarily real. However, many physical transfer functions are characterized by a multiple pole at s=0 (Tou,1959). In general, therefore, the denominator of the transfer function H(s) can be factorize in the form

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_o = s^r (s+\alpha_i)^t \quad i=1,2,..,k. \tag{B.15}$$

Thus, in general, a transfer function H(s) can be decomposed into partial fractions as follows

$$H(s) = \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{K_{ij}}{(s+\alpha_i)^j} + \sum_{j=1}^{r} \frac{L_j}{s^j} \tag{B.16}$$

Noting that a component of the form

$$\frac{K_{ij}}{s+\alpha_i} \tag{B.17}$$

corresponds to a weighting function

$$h(t) = K_{ij} \exp(-\alpha_i t) \tag{B.18}$$

and a component of the form

$$\frac{L_j}{s^j} \tag{B.19}$$

corresponds to a weighting function

$$h(t) = \frac{L_j}{(j-1)!} t^{j-1} \tag{B.20}$$

one obtains the z transfer function of the general expansion (B.16) in the form

$$H(z) = \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{K_{ij}}{(j-1)!} (-1)^{j-1} \frac{\partial^{j-1}}{\partial \alpha_i^{j-1}} \left(\frac{1}{1 - z^{-1} e^{-\alpha_i \Delta T}} \right) + \sum_{j=1}^{r} (-1)^{j-1} \frac{L_j}{(j-1)!} \lim_{\alpha \to 0} \frac{\partial^{j-1}}{\partial \alpha^{j-1}} \left(\frac{1}{1 - z^{-1} e^{-\alpha \Delta T}} \right) \tag{B.21}$$

The pulse transfer function H(z) is thus a rational function of the complex variable z, the poles of which are at $z_K = e^{-\alpha_K \Delta T}$

$$z_k = \exp(-\alpha_k \Delta T)$$

or at

$$z_k = 1.$$

It follows, therefore, that the pulse transfer function of the system (B.1) can be obtained in a closed form as

$$H(z) = \frac{c_m + c_{m-1} z^{-1} + \dots + c_o z^{-m}}{d_m + d_{m-1} z^{-1} + \dots + d_o z^{-m}} \tag{B.22}$$

where some of the coefficients c_j may be equal to zero.

The weighting sequence (B.5) and the pulse transfer functio (B.22) provide two alternative characterizations of a linear discrete-time system.

B.4. Linear discrete-time systems with transport lag.

The discussion in the preceding section refers to a dynamical system in which the response to an applied input is instantaneous. Some physical systems, however, are characterized by a so-called transport lag, or dead time during which the system yields no response to the applied input. In the analysis and synthesis of such systems it is usual to represent the effect of a transport lag λ by that of a separate element having the transfer function

$$H_D'(s) = \exp(-\lambda s) \tag{B.23}$$

and relating the input $y(t)$ and output $y_D(t)$ by

$$y_D(t) = y(t-\lambda) \tag{B.24}$$

If the transfer functions of linear dynamical systems without the transport lag and with the transport lag are respectively denoted by $H(s)$ and $H_D(s)$ then

$$H_D(s) = H(s).\exp(-\lambda s) \tag{B.25}$$

and the outputs of the two systems are related by (B.24)

The pulse transfer function of a linear discrete-time system with transport lag is derived below. The time-domain description of such a system ,however,depends on the type of input, or, rather, on its behaviour between the sampling instants. This question is discussed within the framework of the identification problem in the next section.

In general, the transport lag λ is not an integral multiple of the sampling interval ΔT and can be written

$$\lambda = (n+m)\,\Delta T \tag{B.26}$$

where n is an integer and m is a positive number, smaller than unity **.

The pulse transfer function of a linear discrete-time syste with a transport lag λ defined by (B.26) is obtained by means of a so-called modified z transform. The transform is a function of the parameter m and for a time function X(t) is defined by

$$X(z,m) = z^{-n} \sum_{k=0}^{\infty} x(k\Delta T - m\Delta T) \tag{B.27}$$

The modified z transforms corresponding to the weighting i functions (B.18) and (B.20) are respectively given by

$$H'_D(m,z) = z^{-n} K'_{ij} \frac{e^{\alpha_i m \Delta T}}{1 - z^{-1} e^{-\alpha_i \Delta T}} \tag{B.28}$$

and

$$H''_D(m,z) = z^{-n} \frac{L_j'}{(j-1)!} (-1)^{j-1} \lim_{\alpha \to 0} \frac{\partial^{j-1}}{\partial \alpha^{j-1}} \left(\frac{e^{\alpha m \Delta T}}{1 - z^{-1} e^{-\alpha_i \Delta T}} \right) \tag{B.29}$$

Hence a general expression for a pulse transfer function of a linear dynamical system with a transport lag

$$\lambda = (n+m)\Delta T$$

is given by

** According to the convention adopted in literature (Barker, 1952; Tou, 1959) the fractional delay is $(1-m)\Delta T$ and $\lambda = [(n-1) + (1-m)]\Delta T$. The convention adopted in this thesis facilitates the treatment of the next section and enables the results of Box and Jenkins to be included as a special case.

$$H_D(z) = z^{-n} \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{K_{ij}'}{(j-1)!} (-1)^{j-1} \frac{\partial^{j-1}}{\partial \alpha_i^{j-1}} \left(\frac{e^{\alpha_i m \Delta T}}{1 - z^{-1} e^{-\alpha_i \Delta T}} \right)$$

$$+ z^{-n} \sum_{j=1}^{r} (-1)^{j-1} \frac{L_j'}{(j-1)!} \lim_{\alpha \to 0} \frac{\partial^{j-1}}{\partial \alpha^{j-1}} \left(\frac{e^{\alpha m \Delta T}}{1 - z^{-1} e^{-\alpha \Delta T}} \right)$$

(B.30)

This expression may also be written in a closed form similar to (B.22).

B.5. The problem of identification of linear discrete-time systems.

The aim of the analysis and synthesis of sampled-data and digital control systems is to assess the stability of the system and to obtain the output time function resulting from the application of known input time function to a known linear system. The output response at the sampling instants is obtained by first evaluating the overall z transform of the input and of the system and then inverting this transform by any of the recommended standard procedures (e.g. Tou,1959).

Converse requirements have to be satisfied ,however, when considering the problem of system identification. This problem is concerned with situations in which the order and the coeffi ients of an unknown transfer function of the system (B.1) .

are to be determined froma series of known values of both, the input $\{u_i\}$ and output $\{y_i\}$ ($i= \Delta T, 2\Delta T, \ldots, N\Delta T$) In a practical situation both sets of values will include the effect of various kinds of disturbances, and this effect must be allowed for or made insignificant. In the following it is assumed that the disturbances are allowed for and that, therefore, the input and the output values can be considered to be noise-free.

Since the input and output readings are available in the time domain, the most convenient characterization of the system to be identified is also in the time domain, in terms of difference equations rather than in terms of the z transforms. The required difference equations can be formulated in two different ways as follows.

The first formulation relates (m+1) values of samples $\{u_j\}$ of the input and $\{y_i\}$ of the output, and can be easily derived from the z transform equation (Zadeh and Desoer, 1963)

$$Y(z) = H(z).U(z) \qquad (B.11)$$

which relates the z transform U(z) of the input, Y(z) of the output and the pulse transfer function H(z). If expressions (B.10b), (B.10a) and (B.22) are substituted in (B.11), one obtains

$$(c_m + c_{m-1}z^{-1} + \ldots + c_0 z^{-m})(u_0 + u_1 z^{-1} + \ldots + u_K z^{-K} + \ldots)$$
$$= (d_m + d_{m-1}z^{-1} + \ldots + d_0 z^{-m})(y_0 + y_1 z^{-1} + \ldots + y_K z^{-K} + \ldots) \qquad (B.31)$$

This equality has to be satisfied for every power of z. Therefore, equating coefficients of, say, z^{-k}, one obtains a difference equation

$$c_m u_k + c_m u_{k-1} + \cdots + c_0 u_{k-m}$$

$$= d_m y_k + d_{m-1} y_{k-1} + \cdots + d_0 y_{k-m} \quad \text{(B.32)}$$

A set of such equations, corresponding to k=1,2,...N, and an assumed value of m, can then be solved as discussed in Chapters 2,3,and 4.

The second formulation, employed by Box and Jenkins(1963, 1967a,1967b) is obtained by approximating to a convolution integral

$$y(t) = \int_0^\infty h(\tau)\, u(t-\tau)\, d\tau \quad \text{(B.33)}$$

by means of an infinite sum of definite integrals

$$y(t) \approx \sum_{j=1}^{\infty} \left(\int_{(j-1)\Delta T}^{j\Delta T} h(\tau)\, u(t-\tau)\, d\tau \right) \quad \text{(B.34)}$$

The latter formulation is derived as follows.

The input series $\{u_i\}$, $(i = \Delta T, 2\Delta T, \ldots, N\Delta T)$ may correspond either to a genuinely discrete sequence of values, or to sampled values of continuous time function. In the former case, pertaining to sampled-data control systems the sampling process introduces high frequency complementary components into the actuating signals. These unwanted components are usually removed by a smoothing device, called a holding or a clamping circuit.. In the latter case, the sampled values of continuous output correspond to continuous input, only sampled values of which are available; in such a case the continuous input can be reconstructed to a required degree of approximation by means of mathematical interpolators (Cruickshank, 1961). It is thus seen that, whether the input is genuinely discrete, or continuous and sampled, the discrete sequence of input values is converted into a piece-wise continuous time function. For this reason no difference will be made in the treatment of these two cases. (such a difference is made, however, by Box and Jenkins, as will be discussed later).

The extrapolated time function between the consecutive sampling instants $n\Delta T$ and $(n+1)\Delta T$ depends upon its values at the preceding sampling instants $n\Delta T$, $(n-1)\Delta T$, $(n-2)\Delta T$... and can be generally described by a power series expansion in the interval $t=n\Delta T$ and $t=(n+1)\Delta T$. If $y(t)$ is the output time function and $y_n(t)$ is the output between sampling instant $n\Delta T$ and $(n+1)\Delta T$, then, in general,

$$y_n(t) \simeq y(n\Delta T) + y^{(1)}(n\Delta T)(t-n\Delta T) + \dots + \frac{y^{(k)}(n\Delta T)}{k!}(t-n\Delta T)^k \quad \text{(B.35)}$$

In the above relation the approximated value of the k-th derivative $y^{(k)}(n\Delta T)$ at $t=n\Delta T$ is obtained from

$$y^{(k)}(n\Delta T) = \frac{1}{(\Delta T)^k}\left\{y(n\Delta T) - k\,y[(n-1)\Delta T] + \dots + (-1)^k y[(n-k)\Delta T]\right\} \qquad (B.36)$$

Because of a high cost and the constructional complexity involved in the high order holding devices, and a large amount of shift introduced by them, the most common holding devices used in the sampled data control systems are

a) <u>the zero order hold circuit</u> resulting in the interpolatio by means of

$$y_n(t) = y(n\Delta T) \quad , \quad n\Delta T < t \leq (n+1)\Delta T \qquad (B.37)$$

and having the transfer function

$$H_{h_0}(s) = \frac{1-e^{-\Delta T s}}{s} \qquad (B.38),$$

and

b) <u>the first order hold circuit</u> interpolating by means of

$$y_n(t) = y(n\Delta T) + y'(n\Delta T)(t - n\Delta T) \qquad (B.39).$$

and having the transfer function

$$H_{h_1}(s) = \left(\frac{1+\Delta T s}{\Delta T}\right)\left(\frac{1-e^{-\Delta T s}}{s}\right)^2 \qquad (B.40)$$

Mathematical interpolators employed in digital control systems can be realized by a computer program and higher order interpolation can be achieved, It has been pointed out by Cruickshank(1961), however, that little gain in the accuracy of reproducing a function is obtained in engineering calculations by employing orders of interpolators higher than the second. For the above reasons the following discussion ism limited to zero and first order interpolators only.

When hold circuits or interpolators are assumed to be present, the appearance of the infinite sum (B.34) is greatly simplified and difference equations similar to (B.32) can be derived. Box and Jenkins (1967a) obtain such difference equations, which relate to a first and second order linear dynamical system, through direct integration of the differential equation of the system. This approach, however,requires the knowledge of the relevant differential equation and lacks the generality and elegance of the expression (B.21) obtained before. Moreover, it involves expressions which are quite complicated even for a relatively simple second order system. In order to preserve both the simplicity and generality of the results presented so far, an alternative approach, utilizing the z transform theory ,is adopted below. With this approach, the pulse transfer function of a linear system subject to a piece-wise continuous input is obtained first and then used to derive a difference equation in a way in which the equation (B.32) has been obtained. This is discussed in the following two sections.

B.6. The case of a zero order hold or interpolator

In this case, (referred to by Box and Jenkins as the case of a stepped input), illustrated in Fig.B.1., the input is constant over any sampling interval

$$u(t) = u_\ell \qquad \ell\Delta T < t \leq (\ell+1)\Delta T \tag{B.41}$$

and the convolution integral may be approximated by

$$y_n = \sum_{\ell=1}^{\infty} \left(\int_{(\ell-1)\Delta T}^{\ell\Delta T} h(\xi)\, d\xi \right) u_{n-\ell} \qquad n = 1, 2, \ldots \tag{B.42}$$

Adopting the notation (Box and Jenkins, 1967a)

$$V_\ell = \int_{(\ell-1)\Delta T}^{\ell\Delta T} h(\xi)\, d\xi \tag{B.43}$$

the response of a linear system is written

$$y_n = \sum_{\ell=1}^{\infty} V_\ell\, u_{n-\ell} \qquad n = 1, 2, \ldots \tag{B.44}$$

Now, referring to equations (B.16), (B.18), (B.20) and (B.21),

$$\begin{aligned} V_\ell &= \int_{(\ell-1)\Delta T}^{\ell\Delta T} K_{ij}\, e^{-\alpha_i t}\, dt \\ &= \frac{K_{ij}}{\alpha_i}\left(1 - e^{-\alpha_i \Delta T}\right) e^{-\alpha_i(\ell-1)\Delta T} \end{aligned} \tag{B.45}$$

A) NO DELAY

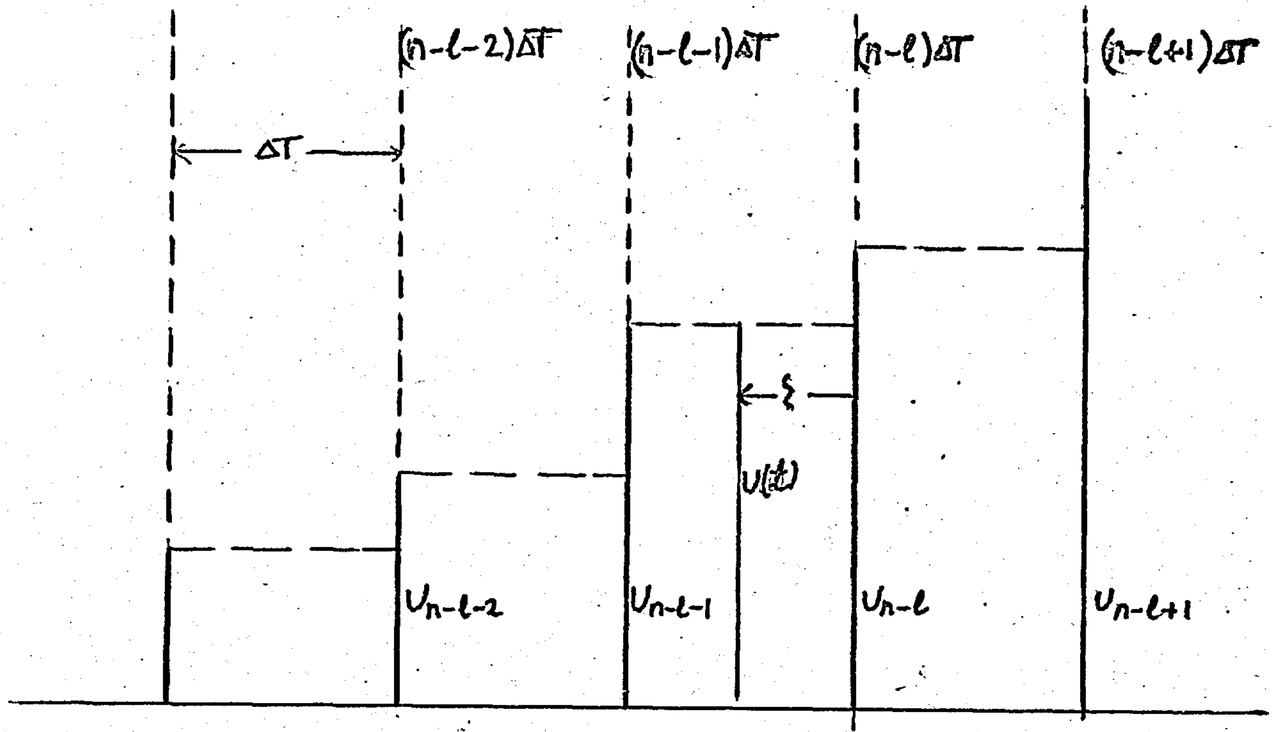
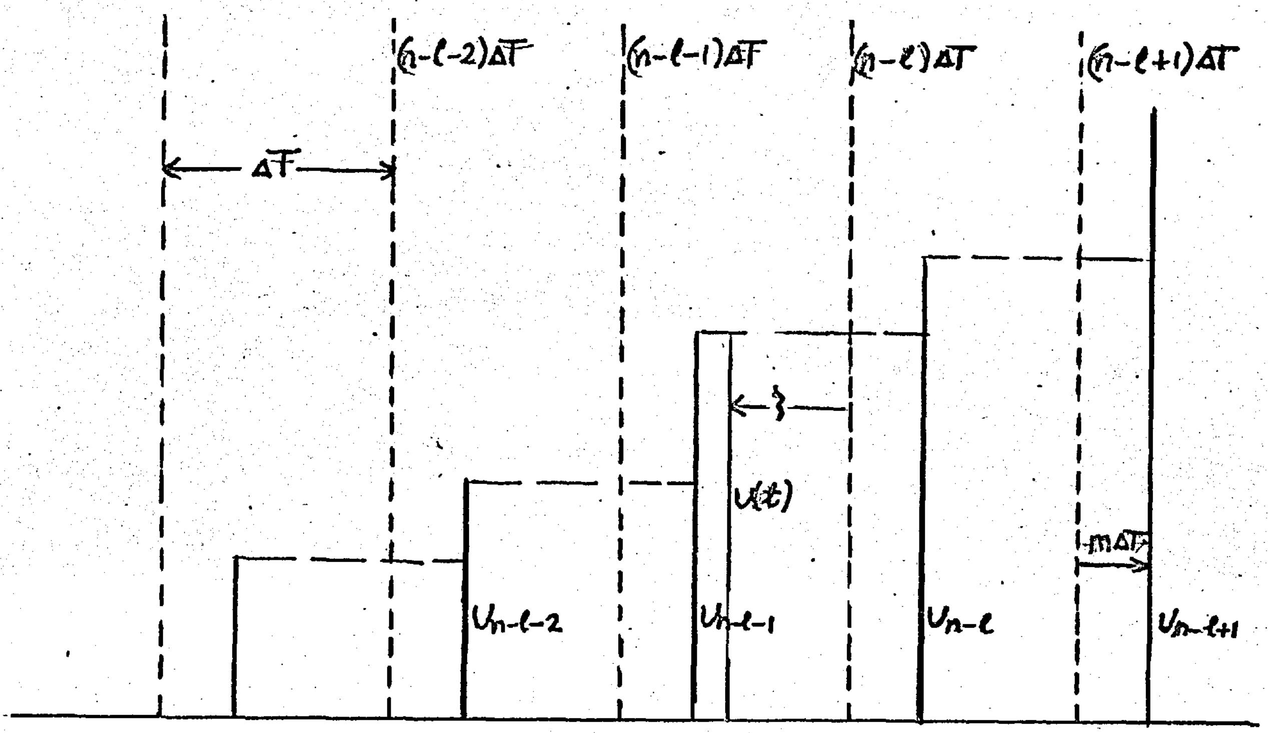


FIG. B.1. ZERO ORDER INTERPOLATION

B) DELAY $m\Delta T$, $0 < m < 1$



and

$$\mathcal{Z}(V_l) = V(z)$$
$$= \frac{K_{ij}}{\alpha_i}\left(1 - e^{-\alpha_i \Delta T}\right) z^{-1} \times \frac{1}{1 - e^{-\alpha_i \Delta T} z^{-1}} \qquad (B.46)$$

Also,

$$V_l' = \int_{(l-1)\Delta T}^{l \Delta T} \frac{L_j t^{j-1}}{(j-1)!} dt = \frac{L_j}{j!}\left[(l\Delta T)^j - ((l-1)\Delta T)^j\right]$$
$$= \lim_{a \to 0} \left[\frac{L_j}{j!}(-1)^j \frac{\partial^j}{\partial a^j}\left(e^{-a l \Delta T} - e^{-a(l-1)\Delta T}\right)\right] \qquad (B.47)$$

or,

$$V_l' = \lim_{a \to 0} \left[\frac{L_j}{j!}(-1)^{j+1} \frac{\partial^j}{\partial a^j}\left(1 - e^{-a\Delta T}\right) e^{-a(l-1)\Delta T}\right] \qquad (B.48)$$

Hence,

$$\mathcal{Z}(V_l') = V'(z)$$
$$= \frac{L_j}{j!}(-1)^{j+1} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} \cdot \frac{z^{-1}\left(1 - e^{-a\Delta T}\right)}{1 - z^{-1} e^{-a\Delta T}} \qquad (B.49)$$

An expression for the pulse transfer function of a general linear system with a zero order hold or interpolator is obtained if in the relation (B.21) components corresponding to z transforms of the weighting functions are replaced by the expressions (B.48) and (B.49), corresponding to <u>integrals</u> of the weighting functions. One obtains thus,

$$H^{o}(z)=\sum_{i=1}^{k}\sum_{j=1}^{t}\frac{K_{ij}}{(j-1)!}(-1)^{j-1}\frac{\partial^{j-1}}{\partial\alpha_i^{j-1}}\left(\frac{1}{\alpha_i}\;\frac{z^{-1}(1-e^{-\alpha_i\Delta T})}{1-z^{-1}e^{-\alpha_i\Delta T}}\right)$$
$$+\sum_{j=1}^{r}(-1)^{j+1}\frac{L_j}{j!}\lim_{a\to 0}\frac{\partial^{j}}{\partial a^{j}}\left(\frac{z^{-1}(1-e^{-a\Delta T})}{1-z^{-1}e^{-a\Delta T}}\right) \qquad ((B.50)$$

A similar procedure is adopted in deriving an expression similar to (B.50) but corresponding to the case when the linea system under consideration includes the effect of the transpor lag

$$\lambda = (n+m)\Delta T \qquad (B.26)$$

That is to say, expressions in brackets in (B.30) , which correspond to weighting functions , have to be replaced by corresponding expressions relating to <u>integrals</u> of the weighting functions, This time, however, the input is not constant over the whole interval $[(\ell-1)\Delta T, \ell\Delta T]$ but only over its sections, namely $[(\ell-1)\Delta T, (1-m)\Delta T]$ and $[(\ell-m)\Delta T, \ell\Delta T]$. In order to retain the simplicity of the expressions derived previously, a backward shift operator B defined by (Box and Jenkins,1963,1966)

$$B^{k}x_t = x_{t-k} \qquad (B.51)$$

will be introduced.

When the input u(t) is delayed by $m\Delta T$ $(0<m<1)$, the corresponding convolution integral may be approximated by

$$y(n\Delta T) = \sum_{\ell=1}^{\infty}\Big[\Big(\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} h(\tau)\,d\tau\Big)u_{n-\ell} + \Big(\int_{(\ell-m)\Delta T}^{\ell\Delta T} h(\tau)\,d\tau\Big)u_{n-\ell-1}\Big] \qquad (B.52)$$

If

$$h(t) = \exp(-\alpha_i t) \qquad (B.53)$$

then, noting that

$$u_{n-l-1} = Bu_{n-l} \qquad (B.54)$$

the relation (B.52) may be written,

$$y(n\Delta T) = \sum_{\ell=1}^{\infty}\Big[\Big(\int_{(\ell-1)\Delta T}^{(\ell-1+1-m)\Delta T} e^{-\alpha_i\tau}\,d\tau\Big) + \Big(\int_{(\ell-1+1-m)\Delta T}^{\ell\Delta T} e^{-\alpha_i\tau}\,d\tau\Big)B\Big]u_{n-\ell} \qquad (B.55)$$

This is seen to be of the form of the relation (B.42) with

$$v_\ell = \Big(\int_{(\ell-1)\Delta T}^{(\ell-1+1-m)\Delta T} e^{-\alpha_i\tau}\,d\tau\Big) + \Big(\int_{(\ell-1+1-m)\Delta T}^{\ell\Delta T} e^{-\alpha_i\tau}\,d\tau\Big)B \qquad (B.56)$$

Performing the integrations one obtains,

$$v_\ell = \frac{1}{\alpha_i}\Big[\big(1-e^{-\alpha_i(1-m)\Delta T}\big)e^{-\alpha_i(\ell-1)\Delta T} + \big(e^{-\alpha_i(1-m)\Delta T} - e^{-\alpha_i\Delta T}\big)e^{-\alpha_i(\ell-1)\Delta T}B \qquad (B.57)$$

Since

$$Bx_t = x_{t-1} \qquad (B.58)$$

and

$$\mathcal{Z}(x_{\ell-1}) = z^{-1}\mathcal{Z}(x_\ell) \tag{B.59}$$

then, taking the z transform of (B.57) one obtains,

$$V(z) = \frac{1}{\alpha_i}\,\frac{\left(1-e^{-\alpha_i(1-m)\Delta T}\right)z^{-1} + \left(e^{-\alpha_i(1-m)\Delta T} - e^{-\alpha_i \Delta T}\right)z^{-2}}{1 - z^{-1}e^{-\alpha_i \Delta T}} \tag{B.60}$$

Similarly, if

$$h(t) = \frac{t^{j-1}}{(j-1)!} \tag{B.61}$$

then

$$V_\ell' = \left(\int_{(\ell-1)\Delta T}^{(\ell-1+1-m)\Delta T} \frac{t^{j-1}}{(j-1)!}\,dt\right) + \left(\int_{j-1+1-m}^{\ell\Delta T} \frac{t^{j-1}}{(j-1)!}\,dt\right)B \tag{B.62}$$

Integrating, one obtains,

$$V_\ell' = \frac{1}{j!}\left\{\left[(\ell-1+1-m)\Delta T\right]^j - \left[(\ell-1)\Delta T\right]^j\right\}$$

$$- \frac{1}{j!}\left\{\left[\ell\Delta T\right]^j - \left[(\ell-1+1-m)\Delta T\right]^j\right\}B \tag{B.63}$$

This can be easily put in the form

$$V_\ell' = \frac{(-1)^{j+1}}{j!}\lim_{a\to 0}\frac{\partial^j}{\partial a^j}\left\{\left(1-e^{-a(1-m)\Delta T}\right)e^{-a(\ell-1)\Delta T}\right.$$

$$\left. + \left(e^{-a(1-m)\Delta T} - e^{-a\Delta T}\right)e^{-a(\ell-1)\Delta T}B\right\} \tag{B.64}$$

The z transform of this relation is

$$V'(z) = \frac{(-1)^{j+1}}{j!} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} \left\{ \frac{(1 - e^{-a(1-m)\Delta T})z^{-1}}{1 - z^{-1} e^{-a\Delta T}} + \frac{(e^{-a(1-m)\Delta T} - e^{-a\Delta T})z^{-2}}{1 - z^{-1} e^{-a\Delta T}} \right\} \tag{B.65}$$

Using (B.60), and (B.65) in (B.30) one obtains an expression for a z transform of a general linear dynamical system subject to a stepped input (or a zero-order hold) and including the effect of the transport lag (B.26). in the form

$$\begin{aligned} H'(z) = {} & z^{-n} \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{K_{ij}}{(j-1)!} (-1)^{j-1} \frac{\partial^{j-1}}{\partial \alpha_i^{j-1}} \frac{1}{\alpha_i} \left\{ \frac{(1 - e^{-\alpha_i(1-m)\Delta T})z^{-1}}{1 - z^{-1} e^{-\alpha_i \Delta T}} + \frac{(e^{-\alpha_i(1-m)\Delta T} - e^{-\alpha_i \Delta T})z^{-2}}{1 - z^{-1} e^{-\alpha_i \Delta T}} \right\} \\ & + z^{-n} \sum_{j=1}^{r} (-1)^{j+1} \frac{L_j}{j!} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} \left\{ \frac{(1 - e^{-a(1-m)\Delta T})z^{-1}}{1 - z^{-1} e^{-a\Delta T}} + \frac{(e^{-a(1-m)\Delta T} - e^{-a\Delta T})z^{-2}}{1 - z^{-1} e^{-a\Delta T}} \right\} \end{aligned} \tag{B.66}$$

B.7. The case of a first order hold circuit or interpolator.

When a first order holding device is used, the input $u(t)$ between the sampling instants $k\Delta T$ and $(k+1)\Delta T$ is extrapolated using two previous samples, u_k and u_{k-1}.

From (B.35) and (B.36) the law of extrapolation is

$$u(t) = u_k + \frac{u_k - u_{k-1}}{\Delta T}(t - k\Delta T) \qquad u_k < u(t) \leq u_{k+1} \tag{B.67}$$

In this case the convolution integral is approximated by (see Fig.B.2)

$$y(n\Delta T) = \sum_{\ell=1}^{\infty} \left\{ \int_{(\ell-1)\Delta T}^{\ell\Delta T} h(\xi)\left[u_{n-\ell} + \frac{u_{n-\ell} - u_{n-\ell-1}}{\Delta T}(\Delta T - \xi)\right] d\xi \right\} \tag{B.68}$$

If the backward shift operator is employed, this can be written,

$$y(n\Delta T) = \sum_{\ell=1}^{\infty} \left\{ \left(\int_{(\ell-1)\Delta T}^{\ell\Delta T} h(\xi)\, d\xi \right)(2-B) - \frac{1-B}{\Delta T}\left(\int_{(\ell-1)\Delta T}^{(\ell\Delta T)} \xi h(\xi) d\xi \right) \right\} u_{n-\ell} \tag{B.69}$$

Hence,

$$v_\ell = 2\int_{(\ell-1)\Delta T}^{\ell\Delta T} h(\xi)\, d\xi - \frac{1}{\Delta T}\int_{(\ell-1)\Delta T}^{\ell\Delta T} \xi h(\xi) d\xi - B\left[\int_{(\ell-1)\Delta T}^{\ell\Delta T} h(\xi)\, d\xi - \frac{1}{\Delta T}\int_{(\ell-1)\Delta T}^{\ell\Delta T} \xi h(\xi) d\xi \right] \tag{B.70}$$

The variable ξ in the second and fourth integral varies linearl only between o and ΔT over any interval $[k\Delta T, (k+1)\Delta T]$

A) NO DELAY

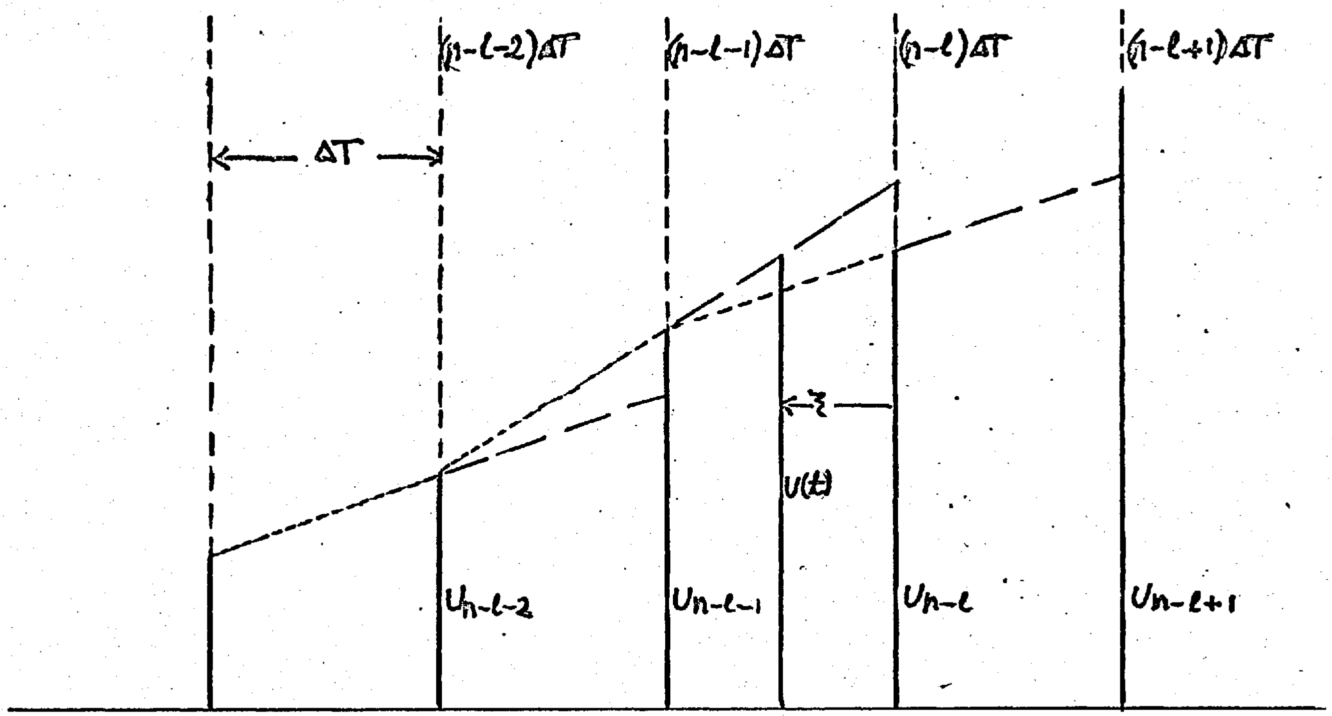
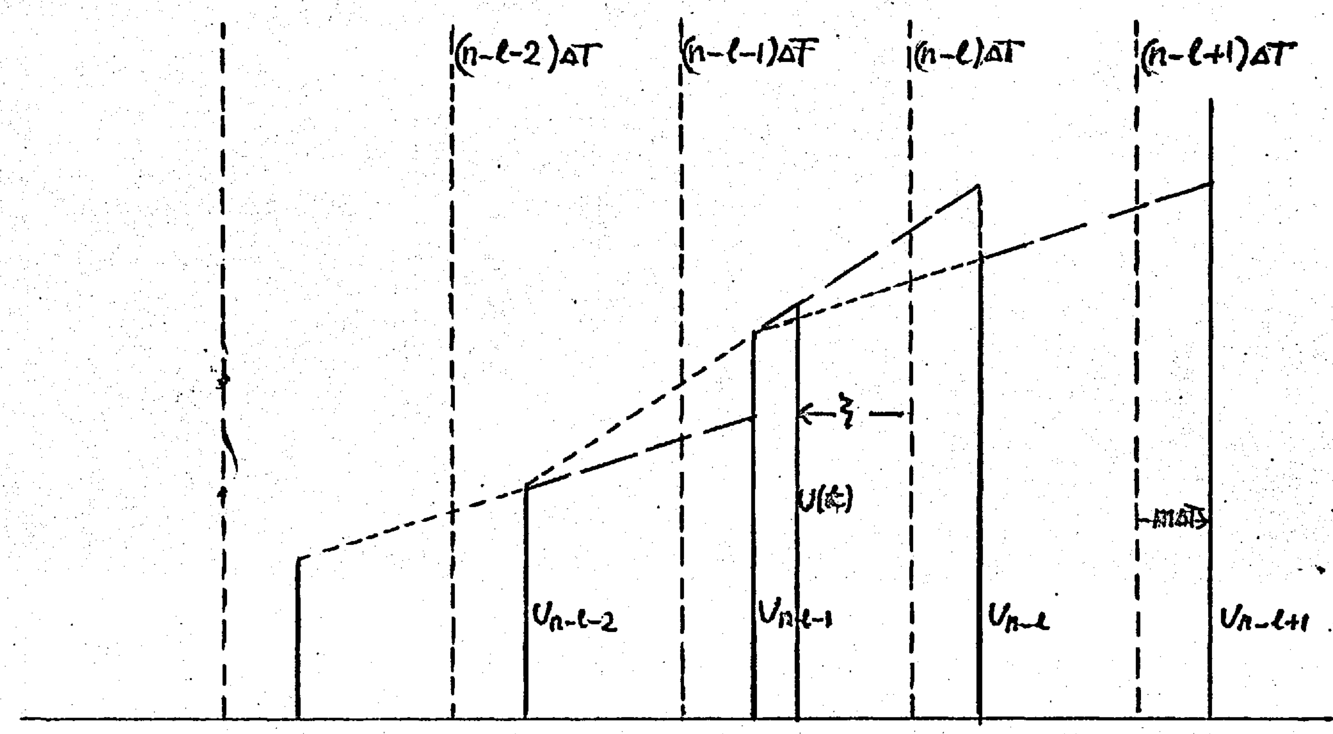


FIG. B.2. FIRST ORDER INTERPOLATION

B) DELAY $m\Delta T$, $0 < m < 1$



Hence (B.70) may be written,

$$V_l = 2\int_{(l-1)\Delta T}^{l\Delta T} h(\zeta)\,d\zeta - \frac{1}{\Delta T}\int_{-\Delta T}^{0} \zeta h(\zeta + l\Delta T)\,d\zeta$$

$$-B\Big[\int_{(l-1)\Delta T}^{l\Delta T} h(\zeta)\,d\zeta - \frac{1}{\Delta T}\int_{-\Delta T}^{0} \zeta h(\zeta + l\Delta T)\,d\zeta \qquad \text{(B.71)}$$

If

$$h(t) = K_{ij}\exp(-\alpha_i t) \qquad \text{(B.18)}$$

then,

$$V_l = K_{ij}\Big\{2\int_{(l-1)\Delta T}^{l\Delta T} e^{-\alpha_i \zeta}\,d\zeta - \frac{1}{\Delta T}\Big(\int_{-\Delta T}^{0} \zeta e^{-\alpha_i \zeta}\,d\zeta\Big)e^{-\alpha_i l\Delta T}$$

$$-B\Big[\int_{(l-1)\Delta T}^{l\Delta T} e^{-\alpha_i \zeta}\,d\zeta - \frac{1}{\Delta T}\Big(\int_{-\Delta T}^{0} \zeta e^{-\alpha_i \zeta}\,d\zeta\Big)e^{-\alpha_i l\Delta T}\Big]\Big\} \qquad \text{(B.72)}$$

Performing the integrations one obtains

$$V_l = \frac{K_{ij}}{\alpha_i}\Big\{2\big(1-e^{-\alpha_i\Delta T}\big)e^{-\alpha_i(l-1)\Delta T} + \frac{1}{\Delta T}\Big[\Delta T e^{-\alpha_i\Delta T} + \frac{1}{\alpha_i}\big(1-e^{-\alpha_i\Delta T}\big)\Big]e^{-\alpha_i l\Delta}$$

$$-B\Big[\big(1-e^{-\alpha_i\Delta T}\big)e^{-\alpha_i(l-1)\Delta T} + \frac{1}{\Delta T}\Big(\Delta T e^{-\alpha_i\Delta T} + \frac{1}{\alpha_i}\big(1-e^{-\alpha_i\Delta T}\big)\Big)e^{-\alpha_i\Delta T l}\Big]\Big\} \qquad \text{(B.73)}$$

The z transform of this relation is

$$V(z) = \frac{K_{ij}}{\alpha_i} \Big\{ 2\left(1-e^{-\alpha_i \Delta T}\right) \frac{z^{-1}}{1-z^{-1}e^{-\alpha_i \Delta T}}$$

$$+ \left(e^{-\alpha_i \Delta T} + \frac{1}{\Delta T \alpha_i}\left(1-e^{-\alpha_i \Delta T}\right)\right) \frac{1}{1-z^{-1}e^{-\alpha_i \Delta T}}$$

$$+ z^{-1}\Big[\left(1-e^{-\alpha_i \Delta T}\right) \frac{z^{-1}}{1-z^{-1}e^{-\alpha_i \Delta T}} + \left(e^{-\alpha_i \Delta T} + \frac{1}{\Delta T \alpha_i}\left(1-e^{-\alpha_i \Delta T}\right)\right) \times \frac{1}{1-z^{-1}e^{-\alpha_i \Delta T}}\Big] \tag{B.74}$$

or,

$$V(z) = K_{ij} V'(z) \tag{B.75}$$

where

$$V'(z) = \frac{1}{\alpha_i} \frac{1}{1-z^{-1}e^{-\alpha_i \Delta T}} \Big\{ \Big[\left(1 - \frac{1}{\Delta T \alpha_i}\right) e^{-\alpha_i \Delta T} + \frac{1}{\Delta T \alpha_i} \Big]$$

$$+ \Big[\left(2 + \frac{1}{\Delta T \alpha_i}\right) - \left(1 + \frac{1}{\Delta T \alpha_i}\right) e^{-\alpha_i \Delta T} \Big] z^{-1}$$

$$+ \Big[\left(1 - e^{-\alpha_i \Delta T}\right) z^{-2} \Big] \Big\} \tag{B.76}$$

If

$$h(t) = \frac{L_j}{(j-1)!} \tag{B.20}$$

then,

$$v_\ell = \frac{L_j}{(j-1)!} \Big\{ 2 \int_{(\ell-1)\Delta T}^{\ell \Delta T} \zeta^{j-1} d\zeta - \frac{1}{\Delta T} \int_{-\Delta T}^{0} \zeta \cdot (\zeta + \ell \Delta T)^{j-1} d\zeta$$

$$- B \Big[\int_{(\ell-1)\Delta T}^{\ell \Delta T} \zeta^{j-1} d\zeta - \frac{1}{\Delta T} \int_{-\Delta T}^{0} \zeta \cdot (\zeta + \ell \Delta T)^{j-1} d\zeta \Big] \Big\} \tag{B.77}$$

Performing the integrations,

$$v_l = \frac{L_j}{(j-1)!}\left\{2\left[\frac{\zeta^j}{j}\right]_{(l-1)\Delta T}^{l\Delta T} - \frac{1}{\Delta T}\left[\zeta\cdot\frac{(\zeta+l\Delta T)^j}{j}\right]_{-\Delta T}^{0} - \int_{-\Delta T}^{0}(\zeta+l\Delta T)^j\,d\zeta\right]$$
$$-B\left[\left[\frac{\zeta^j}{j}\right]_{(l-1)\Delta T}^{l\Delta T} - \frac{1}{\Delta T}\left[\zeta\cdot\frac{(\zeta+l\Delta T)^j}{j}\right]_{-\Delta T}^{0} - \int_{-\Delta T}^{0}(\zeta+l\Delta T)^j\,d\zeta\right]\right\} \qquad (B.78)$$

or,

$$v_l = \frac{L_j}{j!}\left\{2\left((l\Delta T)^j - ((l-1)\Delta T)^j\right) - \frac{1}{\Delta T}\left(\Delta T((l-1)\Delta T)^j\right) - \int_{-\Delta T}^{0}(\zeta+l\Delta T)^j d\zeta\right.$$
$$-B\left[(l\Delta T)^j - ((l-1)\Delta T)^j - \frac{1}{\Delta T}\left(\Delta T((l-1)\Delta T)^j\right.\right.$$
$$\left.\left.\left. - \int_{-\Delta T}^{0}(\zeta+l\Delta T)^j d\zeta\right)\right]\right\} \qquad (B.79)$$

On carrying out the remaining integrations and tidying up this becomes,

$$v_l = (-1)^{j+1}\frac{L_j}{j!}\lim_{a\to 0}\frac{\partial^j}{\partial a^j}\left\{\left[2(1-e^{-a\Delta T})e^{-a(l-1)\Delta T}\right.\right.$$
$$\left. + \left(e^{-a\Delta T} + \frac{1}{\Delta T a}(1-e^{-a\Delta T})\right)e^{-al\Delta T}\right]$$
$$-B\left[(1-e^{-a\Delta T})e^{-a(l-1)\Delta T} + \left(e^{-a\Delta T}\right.\right.$$
$$\left.\left.\left. + \frac{1}{\Delta T a}(1-e^{-a\Delta T})\right)e^{-al\Delta T}\right]\right\} \qquad (B.80)$$

The expression in the curly brackets is of the same form as that in (B.73). Hence, by inspection, the z transform of (B.80 is

$$V(z) = (-1)^{j+1} \frac{L_j}{j!} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} V''(z) \tag{B.81}$$

where

$$V''(z) = \frac{1}{1 - z^{-1} e^{-a\Delta T}} \Big\{ \Big[\Big(1 - \frac{1}{\Delta T a}\Big) e^{-a\Delta T} + \frac{1}{\Delta T a} \Big] + \Big[\Big(2 + \frac{1}{\Delta T a}\Big) - \Big(1 + \frac{1}{\Delta T a}\Big) e^{-a\Delta T} \Big] z^{-1} + \Big[\big(1 - e^{-a\Delta T}\big) \Big] z^{-2} \Big\} \tag{B.82}$$

The pulse transfer function H''(z) of a general linear system with a first order hold is obtained in a way similar to that in which (B.50) has been obtained. Thus, finally,

$$H''(z) = \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{K_{ij}}{(j-1)!} (-1)^{j-1} \frac{\partial^{j-1}}{\partial \alpha_i^{j-1}} V'(z) + \sum_{j=1}^{r} (-1)^{j+1} \frac{L_j}{j!} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} V''(z) \tag{B.83}$$

where V'(z) and V''(z) are given by (B.76) and (B.82) respectively.

A similar expression corresponding to the transport lag given by (B.26) is obtained, as in the case of a zero order hold, by first deriving a pulse transfer function of a system with a fractional delay $m\Delta T$, and then multiplying the result by z^{-n}. From Fig.B.2b it can be seen that the convolution integral is now approximated by

$$y(n\Delta T) = \sum_{\ell=1}^{\infty} \left\{ \int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} h(\zeta)\left[u_{n-\ell} + \frac{u_{n-\ell}-u_{n-\ell-1}}{\Delta T}\left((1-m)\Delta T-\zeta\right)\right] d\zeta \right.$$
$$\left. + \int_{(\ell-m)\Delta T}^{\ell\Delta T} h(\zeta)\left[u_{n-\ell-1} + \frac{u_{n-\ell-1}-u_{n-\ell-2}}{\Delta T}(\Delta T-\zeta)\right] d\zeta \right\}$$

(B.84)

Using the backward shift operator B this can be written

$$y(n\Delta T) = \sum_{\ell=1}^{\infty} \left\{ (2-m-(1-m)B)\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} h(\zeta)\, d\zeta - \frac{1-B}{\Delta T}\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} \zeta\, h(\zeta)\, d\zeta \right.$$
$$+ (2B-B^2)\int_{(\ell-m)\Delta T}^{\ell\Delta T} h(\zeta)\, d\zeta$$
$$\left. - \frac{B-B^2}{\Delta T}\int_{(\ell-m)\Delta T}^{\ell\Delta T} \zeta\, h(\zeta)\, d\zeta \right\} u_{n-\ell} \qquad \text{(B.85)}$$

Noting that ζ varies from 0 to $(1-m)\Delta T$ over the interval $[(\ell-1)\Delta T, (1-m)\Delta T]$, and from 0 to $m\Delta T$ over the interval $[(\ell-m)\Delta T, \ell\Delta T]$, (B.85) may be written,

$$y(n\Delta T) = \sum_{\ell=1}^{\infty} \left\{ (2-m-(1-m)B)\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} h(\zeta)\, d\zeta \right.$$
$$- \frac{1-B}{\Delta T}\int_{0}^{(1-m)\Delta T} \zeta\, h(\zeta+(\ell-1)\Delta T)\, d\zeta$$
$$\left. + (2B-B^2)\int_{(\ell-m)\Delta T}^{\ell\Delta T} h(\zeta)\, d\zeta - \frac{B-B^2}{\Delta T}\int_{0}^{m\Delta T} \zeta\, h(\zeta+(\ell-m)\Delta T)\, d\zeta \right\} u_{n-\ell}$$

(B.86)

Hence,

$$V_l = (2-m-(1-m)B\int_{(l-1)\Delta T}^{(l-1+1-m)\Delta T} h(\zeta)d\zeta - \frac{1-B}{\Delta T}\int_0^{(1-m)\Delta T} \zeta h(\zeta+(l-1)\Delta T)d\zeta$$

$$+ (2B-B^2)\int_{(l-1+1-m)\Delta T}^{l\Delta T} h(\zeta)d\zeta$$

$$- \frac{B-B^2}{\Delta T}\int_0^{m\Delta T} \zeta h(\zeta+(l-m)\Delta T)d\zeta \qquad \text{(B.87)}$$

For h(t) defined by (B.18),

$$V_l = K_{ij}\Big\{(2-m-(1-m)B)\int_{(l-1)\Delta T}^{(l-1+1-m)\Delta T} e^{-\alpha_i\zeta} d\zeta$$

$$- \frac{1-B}{\Delta T} e^{-\alpha_i(l-1)\Delta T}\int_0^{(1-m)\Delta T} \zeta e^{-\alpha_i\zeta} d\zeta$$

$$+(2B-B^2)\int_{(l-1+1-m)\Delta T}^{l\Delta T} e^{-\alpha_i\zeta} d\zeta$$

$$- \frac{B-B^2}{\Delta T} e^{-\alpha_i(l-m)\Delta T}\int_0^{m\Delta T} \zeta e^{-\alpha_i\zeta} d\zeta \Big\} \qquad \text{(B.88)}$$

Performing the integrations one obtains,

$$V_l = \frac{K_{ij}}{\alpha_i} e^{-\alpha_i(l-1)\Delta T}\Big\{(2-m-(1-m)B)(1-e^{-\alpha_i(1-m)\Delta T})$$

$$+ \frac{1-B}{\Delta T}\Big[(1-m)\Delta T e^{-\alpha_i(1-m)\Delta T} - \frac{1}{\alpha_i}(1-e^{-\alpha_i(1-m)\Delta T})\Big]$$

$$+ (2B-B^2)(e^{-\alpha_i(1-m)\Delta T} - e^{-\alpha_i\Delta T})$$

$$+ \frac{B-B^2}{\Delta T}\Big[m e^{-\alpha_i\Delta T} - \frac{1}{\alpha_i}(e^{-\alpha_i(1-m)\Delta T} - e^{-\alpha_i\Delta T})\Big]\Big\}$$

(B.89)

Collecting corresponding terms,

$$V_l = \frac{K_{ij}}{\alpha_i} e^{-\alpha_i(l-1)\Delta T}\Big\{\Big[(2-m)(1-e^{-\alpha_i(1-m)\Delta T}) + (1-m)e^{-\alpha_i(1-m)\Delta T} - \frac{1}{\Delta T\alpha_i}(1-e^{-\alpha_i(1-m)\Delta T})\Big]$$
$$+\Big[-(1-m)(1-e^{-\alpha_i(1-m)\Delta T}) - (1-m)e^{-\alpha_i(1-m)\Delta T} + \frac{1}{\Delta T\alpha_i}(1-e^{-\alpha_i(1-m)\Delta T})$$
$$+2e^{-\alpha_i(1-m)\Delta T} - 2e^{-\alpha_i\Delta T} + m e^{-\alpha_i\Delta T} - \frac{1}{\Delta T\alpha_i}e^{-\alpha_i(1-m)\Delta T} + \frac{1}{\Delta T\alpha_i}e^{-\alpha_i\Delta T}\Big]B$$
$$-\Big[e^{-\alpha_i(1-m)\Delta T} - e^{-\alpha_i\Delta T} + m e^{-\alpha_i\Delta T} - \frac{1}{\Delta T\alpha_i}e^{-\alpha_i(1-m)\Delta T} + \frac{1}{\Delta T\alpha_i}e^{-\alpha_i\Delta T}\Big]B^2\Big\} \tag{B.90}$$

Simplifying,

$$V_l = \frac{K_{ij}}{\alpha_i} e^{-\alpha_i(l-1)\Delta T}\Big\{\Big[\Big(2-m-\frac{1}{\Delta T\alpha_i}\Big) + \Big(\frac{1}{\Delta T\alpha_i}-1\Big)e^{-\alpha_i(1-m)\Delta T}\Big]$$
$$+\Big[\Big(\frac{1}{\Delta T\alpha_i}-1+m\Big) + \Big(2-\frac{2}{\Delta T\alpha_i}\Big)e^{-\alpha_i(1-m)\Delta T} + \Big(\frac{1}{\Delta T\alpha_i}+m-2\Big)e^{-\alpha_i\Delta T}\Big]B$$
$$-\Big[\Big(1-\frac{1}{\Delta T\alpha_i}\Big)e^{-\alpha_i(1-m)\Delta T} + \Big(\frac{1}{\Delta T\alpha_i}-1+m\Big)e^{-\alpha_i\Delta T}\Big]B^2\Big\} \tag{B.91}$$

Taking z transform of (B.91) one obtains,

$$V(z)\ K_{ij}V'''(z) \tag{B.92}$$

where

$$V'''(z) = \frac{1}{\alpha_i}\frac{1}{1-z^{-1}e^{-\alpha_i\Delta T}}\Big\{\Big[\Big(2-m-\frac{1}{\Delta T\alpha_i}\Big) + \Big(\frac{1}{\Delta T\alpha_i}-1\Big)e^{-\alpha_i(1-m)\Delta T}\Big]z^{-1}$$
$$+\Big[\Big(\frac{1}{\Delta T\alpha_i}-1+m\Big) + \Big(2-\frac{2}{\Delta T\alpha_i}\Big)e^{-\alpha_i(1-m)\Delta T} + \Big(\frac{1}{\Delta T\alpha_i}+m-2\Big)e^{-\alpha_i\Delta T}\Big]z^{-2}$$
$$-\Big[\Big(1-\frac{1}{\Delta T\alpha_i}\Big)e^{-\alpha_i(1-m)\Delta T} + \Big(\frac{1}{\Delta T\alpha_i}-1+m\Big)e^{-\alpha_i\Delta T}\Big]z^{-3}\Big\} \tag{B.93}$$

For h(t) defined by (B.20),

$$V_\ell = \frac{L_j}{(j-1)!}\left\{(2-m-(1-m)B)\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T}\zeta^{j-1}d\zeta\right.$$
$$-\frac{1-B}{\Delta T}\int_0^{(1-m)\Delta T}\zeta\cdot(\zeta+(\ell-1)\Delta T)^{j-1}d\zeta$$
$$+(2B-B^2)\int_{(\ell-m)\Delta T}^{\ell\Delta T}\zeta^{j-1}d\zeta$$
$$\left.-\frac{B-B^2}{\Delta T}\int_0^{m\Delta T}\zeta\cdot(\zeta+(\ell-m)\Delta T)^{j-1}d\zeta\right\} \tag{B.94}$$

Performing the integrations,

$$V_\ell = \frac{L_j}{j!}\left\{(2-m+(1-m)B)\left[((\ell-m)\Delta T)^j-((\ell-1)\Delta T)^j\right]\right.$$
$$-\frac{1-B}{\Delta T}\left[(1-m)\Delta T\cdot((1-m)\Delta T+(\ell-1)\Delta T)^j-\int_0^{(1-m)\Delta T}(\zeta+(\ell-1)\Delta T)^j d\zeta\right]$$
$$+(2B-B^2)\left[(\ell\Delta T)^j-((\ell-m)\Delta T)^j\right]$$
$$\left.-\frac{(B-B^2)}{\Delta T}\left[m\Delta T\cdot(m\Delta T+(\ell-m)\Delta T)^j-\int_0^{m\Delta T}(\zeta+(\ell-m)\Delta T)^j d\zeta\right]\right\} \tag{B.95}$$

This can be written in the form,

$$V_\ell = (-1)^{j+1}\frac{L_j}{j!}\lim_{a\to 0}\frac{\partial^j}{\partial a^j}e^{-a(\ell-1)\Delta T}\left\{(2-m-(1-m)B)(1-e^{-a(1-m)\Delta T})\right.$$
$$+\frac{1-B}{\Delta T}\left[(1-m)\Delta T e^{-a(1-m)\Delta T}-\frac{1}{a}(1-e^{-a(1-m)\Delta T})\right]$$
$$\left.+(2B-B^2)(e^{-a(1-m)\Delta T}-e^{-a\Delta T})+\frac{B-B^2}{\Delta T}\left[m\Delta T e^{-a\Delta T}-\frac{1}{a}(e^{-a(1-m)\Delta T}-e^{-a\Delta T})\right]\right\} \tag{B.96}$$

The expression in the curly brackets is seen to be of the sam form as that in the relation (B.89). Hence the pulse transfer function can be written by inspection as

$$V(z) = (-1)^{j+1}\frac{L_j}{j!}\lim_{a\to 0}\frac{\partial^j}{\partial a^j}V^{iv}(z) \tag{B.97}$$

where

$$V^{iv}(z)=\frac{1}{1-z^{-1}e^{-a\Delta T}}\left\{\left[\left(2-m-\frac{1}{\Delta Ta}\right)+\left(\frac{1}{\Delta Ta}-1\right)e^{-a(1-m)\Delta T}\right]z^{-1}\right.$$
$$+\left[\left(\frac{1}{\Delta Ta}-1+m\right)+\left(2-\frac{2}{\Delta Ta}\right)e^{-a(1-m)\Delta T}+\left(\frac{1}{\Delta Ta}+m-2\right)e^{-a\Delta T}\right]z^{-2}$$
$$\left.-\left[\left(1-\frac{1}{\Delta Ta}\right)e^{-a(1-m)\Delta T}+\left(\frac{1}{\Delta Ta}-1+m\right)e^{-a\Delta T}\right]z^{-3}\right\} \tag{B.98}$$

The pulse transfer function of a general linear system, inclu ing the effect of the transport lag, and the effect of a firs order hold is, finally,

$$H'''(z)=z^{-n}\sum_{i=1}^{k}\sum_{j=1}^{t}\frac{K_{ij}}{(j-1)!}(-1)^{j-1}\frac{\partial^{j-1}}{\partial\alpha_i^{j-1}}V'''(z)$$
$$+z^{-n}\sum_{j=1}^{r}(-1)^{j+1}\frac{L_j}{j!}\lim_{a\to 0}\frac{\partial^j}{\partial a^j}V^{iv}(z) \tag{B.99}$$

where $V'''(z)$ and $V^{iv}(z)$ are given by (B.93) and (B.98) respectively.

B.8. The case of a first order interpolation according to Box and Jenkins (1967a)

This case, discussed by Box and Jenkins(1967a,1967b), and illustrated in Fig.B.3, differs from the first order hold of the preceding section in that the values of the input function u(t) between the sampling instants $k\Delta T$ and $(k+1)\Delta T$ are obtained from

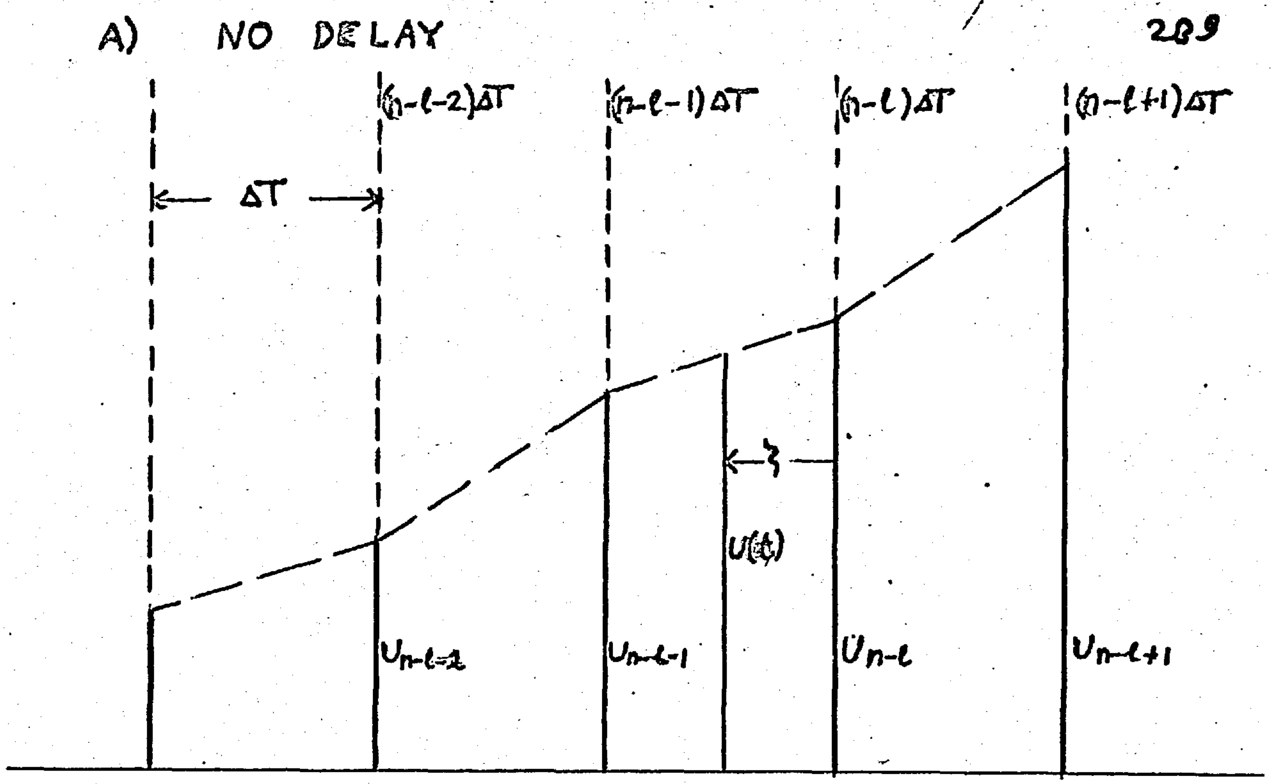
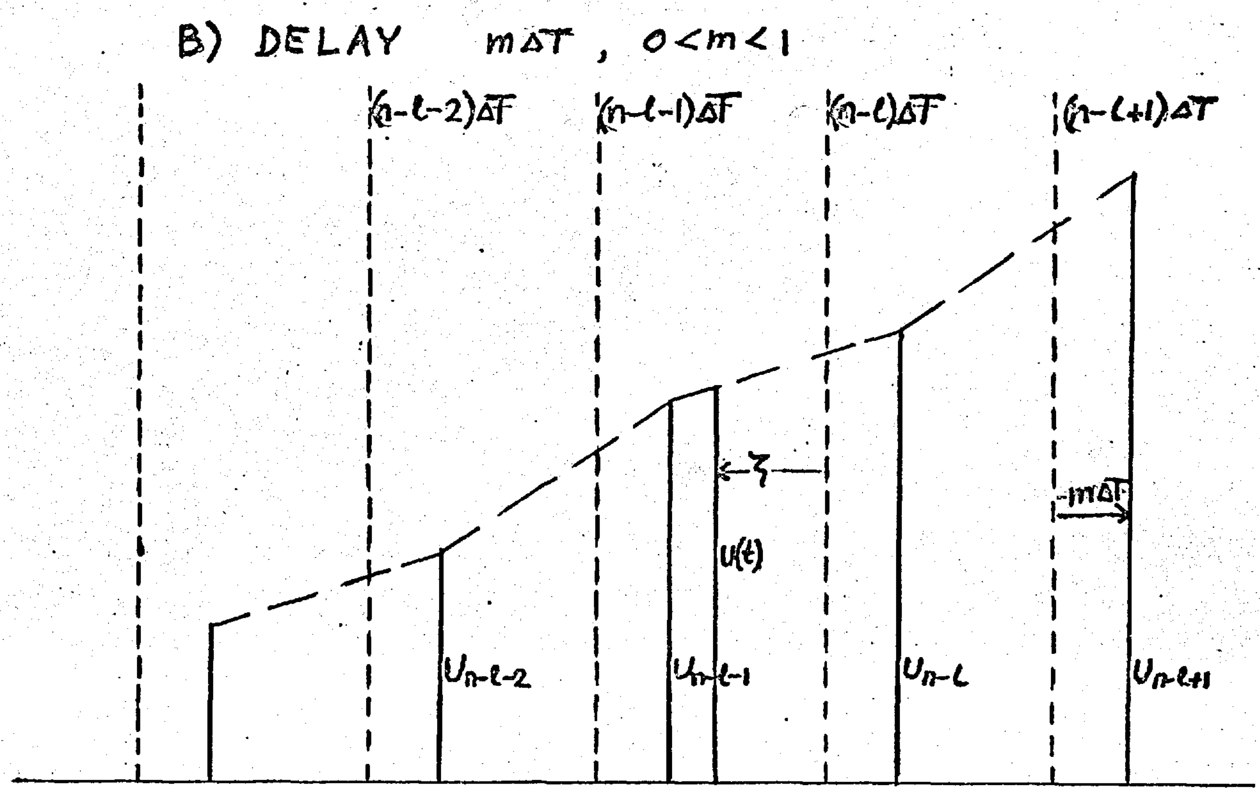


FIG. B.3. LINEAR INTERPOLATION



$$u(t) = u_K + \frac{u_{K+1} - u_K}{\Delta T} t \qquad u_K < u(t) \leq u_{K+1} \tag{B.100}$$

That is to say, <u>current</u> values u_k and u_{k+1} are used for interpolation, a complete record of sampled values being assumed to be available for analysis.

Box and Jenkins call this the case of "identification" or sampling a continuous function, in contradistinction to the "control" problem in which the control commands are given only at discrete instants of time. Box and Jenkins argue that "the most sensible" way of approximating an input function is by means of the interpolation (B.100) in the "identificatio problem, and by means of (B.41) (zero order hold) in the " "control" problem. This may be so in the control of batch processes, but not necessarily so in the control of many physical processes,where, as discussed before, higher order interpolation may be employed.

It is suggested here that this case be regarded as just another case of the first order interpolation which may be employed when a complete record of input and output is given. This case is discussed below for completeness.

Corresponding to the law of interpolation (B.100), the convolution integral is approximated by

$$y(n\Delta T) = \sum_{l=1}^{\infty} \left\{ \int_{(l-1)\Delta T}^{l\Delta T} h(\zeta) \left[u_{n-l+1} - \frac{u_{n-l+1} - u_{n-l}}{\Delta T} \zeta \right] d\zeta \right\} \tag{B.101}$$

Employing the backward shift operator B

$$y(n\Delta T)=\sum_{l=1}^{\infty}\left\{B^{-1}\int_{(l-1)\Delta T}^{l\Delta T}h(\zeta)\,d\zeta-\frac{B^{-1}-1}{\Delta T}\int_{(l-1)\Delta T}^{l\Delta T}\zeta\,h(\zeta)\,d\zeta\right\}u_{n-l} \tag{B.102}$$

Hence,

$$v_l=B^{-1}\int_{(l-1)\Delta T}^{l\Delta T}h(\zeta)\,d\zeta-\frac{B^{-1}-1}{\Delta T}\int_{(l-1)\Delta T}^{l\Delta T}\zeta\,h(\zeta)\,d\zeta \tag{B.103}$$

Since the variable ζ in the second integral varies only between 0 and ΔT,

$$v_l=B^{-1}\int_{(l-1)\Delta T}^{l\Delta T}h(\zeta)\,d\zeta-\frac{B^{-1}-1}{\Delta T}\int_{0}^{\Delta T}\zeta\,h\big(\zeta+(l-1)\Delta T\big)\,d\zeta \tag{B.104}$$

For h(t) defined by (B.18),

$$v_l=K_{ij}\left\{B^{-1}\int_{(l-1)\Delta T}^{l\Delta T}e^{-\alpha_i\zeta}\,d\zeta-\frac{B^{-1}-1}{\Delta T}\left(\int_{0}^{\Delta T}\zeta\,e^{-\alpha_i\zeta}\,d\zeta\right)e^{-\alpha_i(l-1)\Delta T}\right\} \tag{B.105}$$

Carrying out the integrations,

$$v_l=\frac{K_{ij}}{\alpha_i}\,e^{-\alpha_i(l-1)\Delta T}\left\{B^{-1}\left(1-e^{-\alpha_i\Delta T}\right)+\frac{B^{-1}-1}{\Delta T}\left[\Delta T\,e^{-\alpha_i\Delta T}-\frac{1}{\alpha_i}\left(1-e^{-\alpha_i\Delta T}\right)\right]\right\} \tag{B.106}$$

When corresponding terms are collected, one obtains

$$v_l=\frac{K_{ij}}{\alpha_i}\,e^{-\alpha_i(l-1)\Delta T}\left\{\frac{1}{\alpha_i}\left(1-e^{-\alpha_i\Delta T}\right)-e^{-\alpha_i\Delta T}+\left[e^{-\alpha_i\Delta T}+\left(1-e^{-\alpha_i\Delta T}\right)\left(1-\frac{1}{\Delta T\alpha_i}\right)\right]B^{-1}\right\} \tag{B.107}$$

Since

$$B^{-1} x_t = x_{t+1} \tag{B.108a}$$

$$\mathcal{Z}\left(B^{-1} x_t\right) = z \mathcal{Z}(x_t) \tag{B.108b}$$

the z transform of (B.107) is

$$V(z) = K_{ij} \; V^v(z) \tag{B.109}$$

where

$$V^v(z) = \frac{1}{\alpha_i} \frac{1}{1 - z^{-1} e^{-\alpha_i \Delta T}} \left\{ \left[\frac{1}{\alpha_i}\left(1 - e^{-\alpha_i \Delta T}\right) - e^{-\alpha_i \Delta T} \right] z^{-1} + \left[e^{-\alpha_i \Delta T} + \left(1 - e^{-\alpha_i \Delta T}\right)\left(1 - \frac{1}{\Delta T \alpha_i}\right) \right] \right\} \tag{B.110}$$

For the weighting function h(t) defined by (B.20),

$$V_l = \frac{L_j}{(j-1)!} \left\{ B^{-1} \int_{(l-1)\Delta T}^{l \Delta T} \zeta^{j-1} d\zeta - \frac{B^{-1} - 1}{\Delta T} \int_0^{\Delta T} \zeta \cdot \left(\zeta + (l-1)\Delta T \right)^{j-1} d\zeta \right. \tag{B.111}$$

Performing the integrations one obtains,

$$V_l = \frac{L_j}{j!} \left\{ B^{-1} \left[(l \Delta T)^j - ((l-1)\Delta T)^j \right] - \frac{B^{-1} - 1}{\Delta T} \left[\Delta T \left(\Delta T + (l-1)\Delta T \right)^j - \int_0^{\Delta T} \left(\zeta + ((l-1)\Delta T \right)^j d\zeta \right] \right\} \tag{B.112}$$

This can be written as

(B.113)

$$V_l = \frac{L_j}{j!}(-1)^{j+1} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} e^{-a(l-1)\Delta T} \Big\{ B^{-1}(1-e^{-a\Delta T})$$

$$+ \frac{B^{-1}-1}{\Delta T}\left(\Delta T e^{-a\Delta T} - \int_0^{\Delta T} e^{-a\zeta} d\zeta\right)\Big\} \tag{B.113}$$

or,

$$V_l = \frac{L_j}{j!}(-1)^{j+1} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} e^{-a(l-1)\Delta T} \Big\{ B^{-1}(1-e^{-a\Delta T})$$

$$- \frac{B^{-1}-1}{\Delta T}\left[\Delta T e^{-a\Delta T} - \frac{1}{a}(1-e^{-a\Delta T})\right]\Big\}. \tag{B.114}$$

This relation is of the same form as (B.106) and, therefore, the corresponding z transform may be written by inspection as

$$V(z) = \frac{L_j}{j!}(-1)^{j+1} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} V^{vi}(z) \tag{B.115}$$

where

$$V^{vi}(z) = \frac{1}{1-z^{-1}e^{-a\Delta T}} \Big\{ \left[\frac{1}{a}(1-e^{-a\Delta T}) - e^{-a\Delta T}\right] z^{-1}$$

$$+ \left[e^{-a\Delta T} + (1-e^{-a\Delta T})\left(1-\frac{1}{\Delta T a}\right)\right]\Big\} \tag{B.116}$$

Hence the pulse transfer function of a general linear system using Box and Jenkins interpolation (B.100) is

$$H^{iv}(z) = \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{K_{ij}}{(j-1)!}(-1)^{j+1} \frac{\partial^{j-1}}{\partial \alpha_i^{j-1}} V^{v}(z)$$

$$+ \sum_{j=1}^{r} (-1)^{j+1} \frac{L_j}{j!} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} V^{vi}(z) \tag{B.117}$$

where $V^{v}(z)$ and $V^{vi}(z)$ are given by (B.110) and (B.116), respectively.

When the system under consideration includes a transport lag (B.26), the corresponding pulse transfer function is obtained, as before, by first deriving a pulse transfer function of a system with fractional transport lag $m\Delta T$, and then multiplying the result by z^{-n}.

Referring to Fig.B.3b, the convolution integral is now approximated by

$$y(n\Delta T) = \sum_{\ell=1}^{\infty} \left\{ \int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} h(\zeta)\left[u_{n-\ell} + \frac{u_{n-\ell-1}-u_{n-\ell}}{\Delta T}((1-m)\Delta T-\zeta)\right]d\zeta \right.$$
$$\left. + \int_{(\ell-m)\Delta T}^{\ell\Delta T} h(\zeta)\,d\zeta\left[u_{n-\ell-1} + \frac{u_{n-\ell-1+1}-u_{n-\ell-1}}{\Delta T}(\Delta T-\zeta)\right]d\zeta \right\} \qquad \text{(B.118)}$$

Using the backward shift operator B, this can be written as

$$y(n\Delta T) = \sum_{\ell=1}^{\infty} (1-m)B^{-1}\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} h(\zeta)\,d\zeta - \frac{B^{-1}-1}{\Delta T}\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} \zeta h(\zeta)\,d\zeta$$
$$+ \int_{(\ell-m)\Delta T}^{\ell\Delta T} h(\zeta)\,d\zeta - \frac{1-B}{\Delta T}\int_{(\ell-m)\Delta T}^{\ell\Delta T} \zeta h(\zeta)\,d\zeta \qquad \text{(B.119)}$$

Since the variable ζ varies between 0 and $(1-m)\Delta T$ over the interval $[(\ell-1)\Delta T, (\ell-m)\Delta T]$, and between 0 and $m\Delta T$ over the interval $[(\ell-m)\Delta T, \ell\Delta T]$, (B.119) may be written,

$$y(n\Delta T) = \sum_{\ell=1}^{\infty} (1-m)B^{-1}\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T} h(\zeta)\,d\zeta$$
$$- \frac{B^{-1}-1}{\Delta T}\int_{0}^{(1-m)\Delta T} \zeta\cdot h(\zeta+(\ell-1)\Delta T)\,d\zeta$$
$$+ \int_{(\ell-m)\Delta T}^{\ell\Delta T} h(\zeta)\,d\zeta - \frac{1-B}{\Delta T}\int_{0}^{m\Delta T} \zeta\cdot h(\zeta+(\ell-m)\Delta T)\,d\zeta\Big\}\, u_{n-\ell} \qquad \text{(B.120)}$$

Hence,

$$V_l=(1-m)B^{-1}\int_{(l-1)\Delta T}^{(l-m)\Delta T} h(\zeta)\,d\zeta-\frac{B^{-1}-1}{\Delta T}\int_0^{(1-m)\Delta T}\zeta\,h(\zeta+(l-1)\Delta T)\,d\zeta$$
$$+\int_{(l-m)\Delta T}^{l\Delta T} h(\zeta)\,d\zeta-\frac{1-B}{\Delta T}\int_0^{m\Delta T}\zeta\,h(\zeta+(l-m)\Delta T)\,d\zeta$$

(B.121)

For h(t) defined by (B.18),

$$V_l=K_{ij}\left\{(1-m)B^{-1}\int_{(l-1)\Delta T}^{(l-1+1-m)\Delta T}e^{-\alpha_i\zeta}\,d\zeta-\frac{B^{-1}-1}{\Delta T}\left(\int_0^{(1-m)\Delta T}\zeta e^{-\alpha_i\zeta}\right)e^{-\alpha_i(l-1)\Delta}\right.$$
$$\left.+\int_{(l-1+1-m)\Delta T}^{l\Delta T}e^{-\alpha_i\zeta}\,d\zeta-\frac{1-B}{\Delta T}\left(\int_0^{m\Delta T}\zeta e^{-\alpha_i\zeta}\,d\zeta_i\right)e^{-\alpha_i(l-m)}\right\}$$

(B.122)

Performing the integrations one obtains,

$$V_l=\frac{K_{ij}}{\alpha_i}e^{-\alpha_i(l-1)\Delta T}\left\{(1-m)B^{-1}\left(1-e^{-\alpha_i(1-m)\Delta T}\right)\right.$$
$$+\frac{B^{-1}-1}{\Delta T}\left[(1-m)\Delta T e^{-\alpha_i(1-m)\Delta T}-\frac{1}{\alpha_i}\left(1-e^{-\alpha_i(1-m)\Delta T}\right)\right]$$
$$\left.+\left(e^{-\alpha_i(1-m)\Delta T}-e^{-\alpha_i\Delta T}\right)+\frac{1-B}{\Delta T}\left[m\Delta T e^{-\alpha_i\Delta T}-\frac{1}{\alpha_i}\left(e^{-\alpha_i(1-m)\Delta T}-e^{-\alpha_i\Delta T}\right)\right]\right\}$$

(B.123)

Collecting corresponding terms,

$$V_l=\frac{K_{ij}}{\alpha_i}e^{-\alpha_i(l-1)\Delta T}\left\{\left[\left[1-m-\frac{1}{\Delta T\alpha_i}\right]+\frac{1}{\Delta T\alpha_i}e^{-\alpha_i(1-m)\Delta T}\right]B^{-1}\right.$$
$$+\left[\frac{1}{\Delta T\alpha_i}+\left(m-\frac{2}{\Delta T\alpha_i}\right)e^{-\alpha_i(1-m)\Delta T}+\left(m-1+\frac{1}{\Delta T\alpha_i}\right)e^{-\alpha_i\Delta T}\right]$$
$$\left.-\left[\left(m+\frac{1}{\Delta T\alpha_i}\right)e^{-\alpha_i\Delta T}-\frac{1}{\Delta T\alpha_i}e^{-\alpha_i(1-m)\Delta T}\right]B\right\}$$

(B.124)

The z transform of (B.124) is

$$V(z) = K_{ij}\, V^{VII}(z) \tag{B.125}$$

where

$$V^{VII}(z) = \frac{1}{\alpha_i}\,\frac{1}{1-z^{-1}e^{-\alpha_i\Delta T}}\left\{\left[\left(1-m-\frac{1}{\Delta T\alpha_i}\right)+\frac{1}{\Delta T\alpha_i}\,e^{-\alpha_i(1-m)\Delta T}\right]\right.$$
$$+\left[\frac{1}{\Delta T\alpha_i}+\left(m-\frac{2}{\Delta T\alpha_i}\right)e^{-\alpha_i(1-m)\Delta T}+\left(m-1+\frac{1}{\Delta T\alpha_i}\right)e^{-\alpha_i\Delta T}\right]z^{-1}$$
$$\left.-\left[\left(m+\frac{1}{\Delta T\alpha_i}\right)e^{-\alpha_i\Delta T}-\frac{1}{\Delta T\alpha_i}\,e^{-\alpha_i(1-m)\Delta T}\right]z^{-2}\right\} \tag{B.126}$$

For h(t) defined by (B.20),

$$v_\ell = \frac{L_j}{(j-1)!}\left\{(1-m)B^{-1}\int_{(\ell-1)\Delta T}^{(\ell-m)\Delta T}\zeta^{j-1}\,d\zeta-\frac{B^{-1}-1}{\Delta T}\int_0^{(1-m)\Delta T}\zeta\cdot(\zeta+(\ell-1)\Delta T)^{j-1}\,d\zeta\right.$$
$$\left.+\int_{(\ell-m)\Delta T}^{\ell\Delta T}\zeta^{j-1}\,d\zeta-\frac{1-B}{\Delta T}\int_0^{m\Delta T}\zeta\cdot(\zeta+(\ell-m)\Delta T)^{j-1}\,d\zeta\right\} \tag{B.127}$$

Performing the integrations one obtains,

$$v_\ell = \frac{L_j}{j!}\left\{(1-m)B^{-1}\left[((\ell-m)\Delta T)^j-((\ell-1)\Delta T)^j\right]\right.$$
$$-\frac{B^{-1}-1}{\Delta T}\left[(1-m)\Delta T((1-m)\Delta T+(\ell-1)\Delta T)^j-\int_0^{(1-m)\Delta T}(\zeta+(\ell-1)\Delta T)^j\,d\zeta\right.$$
$$+\left[(\ell\Delta T)^j-((\ell-m)\Delta T)^j\right]-\frac{1-B}{\Delta T}\left[m\Delta T(m\Delta T+(\ell-m)\Delta T)^j\right.$$
$$\left.\left.-\int_0^{m\Delta T}(\zeta+(\ell-m)\Delta T)^j\,d\zeta\right]\right\} \tag{B.128}$$

This can also be written as

$$V_L = \frac{L_j}{j!}(-1)^{j+1} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} e^{-a(\ell-1)\Delta} \Big\{ (1-m)B^{-1}\left(1-e^{-a(1-m)\Delta T}\right)$$
$$+ \frac{B^{-1}-1}{\Delta T}\left[(1-m)\Delta T e^{-a(1-m)\Delta T} - \frac{1}{a}\left(1-e^{-a(1-m)\Delta T}\right)\right]$$
$$+ \left[e^{-a(1-m)\Delta T} - e^{-a\Delta T}\right] +$$
$$+ \frac{1-B}{\Delta T}\left[m\Delta T e^{-a\Delta T} - \frac{1}{a}\left(e^{-a(1-m)\Delta T} - e^{-a\Delta T}\right)\right]\Big\}$$

(B.129)

The above expression is of similar form to (B.123) and the z transform of (B.129) can, therefore, be written by inspection as

$$V(z) = \frac{L_j}{j!}(-1)^{j+1} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} V^{viii}(z) \tag{B.130}$$

where

$$V^{viii}(z) = \frac{1}{1-z^{-1}e^{-a\Delta T}} \Big\{ \left[(1-m) - \frac{1}{\Delta T a} + \frac{1}{\Delta T a} e^{-a(1-m)\Delta T}\right]$$
$$+ \left[\frac{1}{\Delta T a} + \left(m - \frac{2}{\Delta T a}\right)e^{-a(1-m)\Delta T} + \left(m-1+\frac{1}{\Delta T a}\right)e^{-a\Delta T}\right]z^{-1}$$
$$- \left[\left(m+\frac{1}{\Delta T a}\right)e^{-a\Delta T} - \frac{1}{\Delta T a}e^{-a(1-m)\Delta T}\right]z^{-2}\Big\}$$

(B.131)

Finally, the pulse transfer function of a general linear dynamical system with transport lag and subject to an input with the interpolation law (B.100) is

$$H^{v}(z) = z^{-n} \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{K_{ij}}{(j-1)!}(-1)^{j-1} \frac{\partial^{j-1}}{\partial \alpha_i^{j-1}} V^{vii}(z)$$
$$z^{-n} \sum_{j=1}^{r} (-1)^{j+1} \frac{L_j}{j!} \lim_{a \to 0} \frac{\partial^j}{\partial a^j} V^{viii}(z) \tag{B.132}$$

where $V^{vii}(z)$ and $V^{viii}(z)$ are respectively given by (B.126) and (B.131).

B.9. Examples of difference equations describing first and second order systems with interpolated input.

The relations (B.50),(B.66),(B.83),(B.99),(B.117) and (B.132) give expressions for a z transfer function with or without transport lag, corresponding to three methods of interpolating the input function u(t). The method of derivation of these equations is superior to that used by Box and Jenkins(1967a,1967b) and holds for <u>any</u> linear system with a rational transfer function. Once the transfer function H(s) is specified, it is factorized according to (B.16) or (B.25) and the corresponding z transfer function can then be obtained directly by using one of the relations developed in this Appendix.

The z transfer function thus obtained can be put in the form

$$H(z)=\frac{c_m+c_{m-1}z^{-1}+\dots+c_0z^{-m}}{d_m+d_{m-1}z^{-1}+\dots+d_0z^{-m}} \tag{B.133}$$

The corresponding difference equation can be derived from the relation (B.11) as described in Section B.5. That is to say, substituting (B.10b),(B.10a) in (B.11) and using (B.133), one obtains,

$$(c_m + c_{m-1}z^{-1} + \dots + c_0 z^{-m})(u_0 + u_1 z^{-1} + \dots + u_K z^{-K} + \dots)$$

$$= (d_m + d_{m-1}z^{-1} + \dots + d_0 z^{-m})(y_0 + y_1 z^{-1} + \dots + y_K z^{-K} + \dots) \tag{B.134}$$

This relation holds for any power of z. Therefore, equating coefficients of, say, z^{-k}, one obtains the required difference equation

$$c_m u_K + c_m u_{K-1} + \dots + c_0 u_{K-m} =$$

$$= d_m y_K + d_{m-1} y_{K-1} + \dots + d_0 y_{K-m} \tag{B.135}$$

This approach will now be adopted to derive difference equations describing first and second order systems. The object of these examples is, first, to illustrate how the formulae, developed in the Appendix, and holding for a <u>general</u> linear system, can be used to obtain difference equations corresponding to some <u>specified</u> transfer functions. Secondly, it is required to obtain the formulae used in the thesis and compar them with those obtained elsewhere. Only zero order interpola ion will be considered here.

<u>a) First order system without transport lag and with transport lag</u>

Consider first a simple first order system given by

$$H(s) = \frac{K}{s+\alpha} \tag{B.136}$$

Using the formula (B.50),

(B.137)

Employing the "parsimonious parameterization" of Box and Jenkins,

$$g = \frac{K}{\alpha} \tag{B.138a}$$

$$\phi = e^{-\alpha \Delta T} \tag{B.138b}$$

Then

$$H^{\circ}(z) = \frac{g(1-\phi)z^{-1}}{1-\phi z^{-1}} \tag{B.140}$$

Using (B.134), the difference equation is obtained as

$$y_k - \phi y_{k-1} = g(1-\phi)u_{k-1} \tag{B.141}$$

which agrees with the relation given by Box and Jenkins(1967a)

If the system (B.136) includes transport lag (B.26), then, using (B.66) one obtains,

$$H'(z) = z^{-n}\frac{K}{\alpha}\left\{\frac{(1-e^{-\alpha(1-m)\Delta T})z^{-1}}{1-z^{-1}e^{-\alpha\Delta T}} + \frac{(e^{-\alpha(1-m)\Delta T} - e^{-\alpha\Delta T})z^{-2}}{1-z^{-1}e^{-\alpha\Delta T}}\right. \tag{B.142}$$

Employing (B.138) this becomes,

$$H'(z) = z^{-n} g\left[\frac{(1-\phi^{1-m})z^{-1}}{1-\phi z^{-1}} + \frac{(\phi^{1-m}-\phi)z^{-2}}{1-\phi z^{-1}}\right. \tag{B.143}$$

Using (B.134) the corresponding difference equation is obtained as

$$y_k - \phi y_{k-1} = g\left[(1-\phi^{1-m})u_{k-1-n} + (\phi^{1-m} - \phi)u_{k-2-n}\right] \tag{B.144}$$

which again agrees with Box and Jenkins (1967a).

b) Second order system

Consider, finally, a second order system described by a transfer function

$$H(s) = \frac{K}{(s+\alpha_1)(s+\alpha_2)} \tag{B.145}$$

where and are real and not equal.

Splitting H(s) into partial fractions one obtains

$$H(s) = \frac{K}{\alpha_2-\alpha_1}\,\frac{1}{s+\alpha_2} - \frac{K}{\alpha_2-\alpha_1}\,\frac{1}{s+\alpha_1} \tag{B.146}$$

Using (B.50) one obtains,

$$H^{o}(z) = \frac{K}{\alpha_2-\alpha_1}\left[\frac{1}{\alpha_2}\,\frac{(1-e^{-\alpha_2\Delta T})z^{-1}}{1-e^{-\alpha_2\Delta T}z^{-1}} - \frac{1}{\alpha_1}\,\frac{(1-e^{-\alpha_1\Delta T})z^{-1}}{1-e^{-\alpha_1\Delta T}z^{-1}}\right] \tag{B.147}$$

Let

$$\phi_1 = e^{-\alpha_1\Delta T} \tag{B.148a}$$

$$\phi_2 = e^{-\alpha_2\Delta T} \tag{B.148b}$$

$$g = \frac{K}{\alpha_1\alpha_2} \tag{B.148c}$$

Then, after some calculations, one obtains,

$$H^{\circ}(z) = \frac{g}{\frac{1}{\alpha_1} - \frac{1}{\alpha_2}} \left\{ \frac{\frac{1}{\alpha_1}(1-\phi_1) - \frac{1}{\alpha_2}(1-\phi_2)}{1-(\phi_1+\phi_2)z^{-1} + \phi_1 \phi_2 z^{-2}} z^{-1} \right.$$

$$\left. + \frac{\frac{1}{\alpha_2}\phi_1(1-\phi_2) - \frac{1}{\alpha_1}\phi_2(1-\phi_1)}{1-(\phi_1+\phi_2)z^{-1} + \phi_1 \phi_2 z^{-2}} \right\} \tag{B.149}$$

The corresponding difference equation is easily obtained from (B.149) and is

$$y_K - (\phi_1+\phi_2)y_{K-1} + \phi_1 \phi_2 y_{K-2} =$$

$$= \frac{g}{\frac{1}{\alpha_1} - \frac{1}{\alpha_2}} \left\{ \left[\frac{1}{\alpha_1}(1-\phi_1) - \frac{1}{\alpha_2}(1-\phi_2) \right] u_{K-1} \right.$$

$$\left. + \left[\frac{1}{\alpha_2}\phi_1(1-\phi_2) - \frac{1}{\alpha_1}\phi_2(1-\phi_1) \right] u_{K-2} \right\} \tag{B.150}$$

If

$$\frac{1}{\alpha_1} = T_1 \tag{B.151a}$$

$$\frac{1}{\alpha_2} = T_2 \tag{B.151b}$$

then, finally,

$$y_K - (\phi_1+\phi_2)y_{K-1} + \phi_1 \phi_2 y_{K-2} =$$

$$= \frac{g}{T_1 - T_2} \left\{ \left[T_1(1-\phi_1) - T_2(1-\phi_2) \right] u_{K-1} \right.$$

$$\left. + \left[T_2 \phi_1(1-\phi_2) - T_1 \phi_2(1-\phi_1) \right] u_{K-2} \right\} \tag{B.152}$$

Thus, the equation involves gain and only two additional parameters T_1 and T_2, or, alternatively, ϕ_1 and ϕ_2, as indeed should be the case for the second order system.

The relation (B.152) differs in appearance from the corresponding relation derived by Box and Jenkins(1967a) by direct integration of a second order differential equation. In order to reduce (B.152) to this form, let

$$\delta_1 = \phi_1 + \phi_2 \tag{B.153a}$$

$$\delta_2 = -\phi_1 \phi_2 \tag{B.153b}$$

$$\nu = \frac{\frac{1}{\alpha_2}\phi_1(1-\phi_2) - \frac{1}{\alpha_1}\phi_2(1-\phi_1)}{[1-(\phi_1+\phi_2)+\phi_1\phi_2]\left[\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right]} \tag{B.154}$$

Substituting (B.153), (B.154) into (B.149), one obtains,

$$H_r^o(z) = g\,(1-\delta_1-\delta_2)\,\frac{(1-\nu)z^{-1}+\nu z^{-2}}{1-\delta_1 z^{-1}-\delta_2 z^{-2}} \tag{B.155}$$

The corresponding difference equation

$$y_k - \delta_1 y_{k-1} - \delta_2 y_{k-2} = $$
$$= g\,(1-\delta_1-\delta_2)\left[(1-\nu)u_{k-1} + \nu u_{k-2}\right] \tag{B.156}$$

is now identical with the relation given by Box and Jenkins.

Appendix C.

Representation of a nonstationary stochastic process as an output of a time-varying linear filter excited by white noise the mean square value of the process being a polynomial function of time.

C.1.Introduction

In this Appendix a novel method of representation of a nonstationary stochastic process is developed. The process is represented as an output of a linear filter with time varying coefficients, excited by white noise. The mean square value of the output of the filter is a polynomial in time such that the degree of the polynomial is a property of the filter structure.

The terms of a white noise process $\{\zeta_t\}$ with mean m_ζ and variance σ^2 defined by

$$E < \zeta_t > = m_\zeta \tag{C.1}$$

$$E < (\zeta_t - m_\zeta)(\zeta_{t'} - m_\zeta) = \sigma^2 \delta(t-t') \tag{C.2}$$

the nonstationary stochastic process $\{\eta_t\}$ is given by

$$\eta_t = \eta_0 + \sum_{j=1}^{t-1} V_j(t)\, \zeta_{t-j} + \zeta_t \tag{C.3}$$

where η_0 denotes the starting value.

The weighting functions $v_j(t)$ are of the form

$$v_j(t) = K\frac{(t-j)^k j^l}{t^m} \tag{C.4}$$

where K is the gain factor.

The weighting functions are determined in a semi-empirical fashion by first considering the conditions which have to be satisfied by an impulsive response of a linear time-varying filter characterized by a linear time-varying differential equation (Miller,1955). The weighting functions $v_j(t)$ resulting in the mean square value of the output of the corresponding filter being a polynomial in time of a prescribed degree were then determined by trial and error.

Difference equations characterizing the various filters are developed from the relations (C.3) using the variables defined by

$$\eta_t' = \eta_t - \eta_0 \tag{C.5}$$

If these relations are used for modelling a given process $\{\eta_t\}$ then stability constraints are placed on the gain factor with the result that the effective gain factor G in the difference equations is smaller than the gain factor K in the relations (C.3) , while the magnitude of the process $\{\xi_0\}$

estimated is correspondingly increased.

C.2. Representation of a first order process whose mean square value varies linearly with time.

This case corresponds to a process with a constant mean (zero or non-zero) and whose variance varies linearly with time}.

Setting in (C.4) k=1, l=0, m=3/2, one obtains,

$$\eta_t = \eta_0 + \sum_{j=1}^{t-1} K \frac{t-j}{t^{3/2}} \zeta_{t-j} + \zeta_t \qquad (C.6)$$

Now,

$$E\langle \eta_t^2 \rangle = \eta_0^2 + E\langle \Big[\sum_{j=1}^{t-1} K \frac{t-j}{t^{3/2}} \zeta_{t-j}\Big]^2 \rangle + E\langle \zeta_t^2 \rangle \qquad (C.7)$$

$$= \eta_0^2 + (\sigma^2 + m_\zeta^2)\Big[1 + \Big(\sum_{j=1}^{t-1} K \frac{t-j}{t^{3/2}}\Big)^2\Big] \qquad (C.\ 8)$$

Also

$$\sum_{j=1}^{t-1} K \frac{t-j}{t^{3/2}} = \frac{K}{t^{3/2}}\Big[t(t-1) - \frac{t(t-1)}{2}\Big] = \frac{1}{2} K \frac{t(t-1)}{t^{3/2}} \qquad (C.9)$$

Hence

$$E\langle \eta_t^2 \rangle = \eta_0^2 + (\sigma^2 + m_\zeta^2)\Big[1 + \frac{1}{4} \frac{K^2 t^2 (t-1)^2}{t^3}\Big] \qquad (C.10)$$

$$= \eta_0^2 + (\sigma^2 + m_\zeta^2)\Big[1 + \frac{K^2}{4}\Big[\Big(\frac{1}{t} - 2\Big) + t\Big]\Big] \qquad (C.11)$$

The effect of the term (1/t) decreases as t increases.

For $t > 10$ the relation (C.11) may be taken to represent a linear variation in the mean square value fairly accurately.

From (C.5) and (C.6) one obtains,

$$t^{3/2}\eta_t' = \sum_{j=2}^{t-1} K(t-j)\xi_{t-j} + K(t-j) + t^{3/2}\xi_t \tag{C.12}$$

and

$$(t-1)^{3/2}\eta_{t-1}' = \sum_{j=1}^{t-2} K(t-1-j)\xi_{t-1-j} + (t-1)^{3/2}\xi_{t-1} \tag{C.13}$$

Substituting in (C.13)

$$k=j+1 \tag{C.14}$$

eliminating the summation between (C.12) and (C.13), and repla ing K by G one obtains the difference equation

$$\eta_t' - \left(\frac{t-1}{t}\right)^{3/2}\eta_{t-1}' =$$

$$= \left[G\,\frac{t-1}{t^{3/2}} - \left(\frac{t-1}{t}\right)^{3/2}\right]\xi_{t-1} + \xi_t \tag{C.15}$$

The stability condition for the parameter G is obtained by writing (C.15) in the form

$$\xi_t = \frac{\left[1 - \left(\frac{t-1}{t}\right)^{3/2} B\right]\eta_t'}{1 - \left[\left(\frac{t-1}{t}\right)^{3/2} - G\,\frac{t-1}{t^{3/2}}\right]B} \tag{C.16}$$

and noting that the process of estimating recursively $\{\xi_t\}$ from $\{\eta_t\}$ is stable provided that the root of the equation

$$1-\left[\left(\frac{t-1}{t}\right)^{3/2}-G\,\frac{t-1}{t^{3/2}}\right]x=0 \qquad \text{(C.17)}$$

lies outside the unit circle.

This yields the condition

$$\left|\left(\frac{t-1}{t}\right)^{3/2}-G\,\frac{t-1}{t^{3/2}}\right|<1 \qquad \text{(C.18)}$$

from which

$$|G|<1 \qquad \text{(C.19)}$$

follows.

C.3. Representation of a second order process whose mean square value is a parabolic function of time.

This case corresponds to a process with linearly varying mean and/or parabolically varying variance.

Setting in (C.4) k=1, l=o, m=1, one obtains,

$$\eta_t=\eta_0+\sum_{j=1}^{t-1}K\,\frac{t-j}{t}\,\xi_{t-j}+\xi_t \qquad \text{(C.20)}$$

$$E\langle\eta_t^2\rangle=\eta_0^2+E\left\langle\left(\sum_{j=1}^{t-1}K\frac{t-j}{t}\,\xi_{t-j}\right)^2\right\rangle+E\langle\xi_t^2\rangle \qquad \text{(C.21)}$$

$$=\eta_0^2+(\sigma^2+m_\xi^2)\left[1+\left(\sum_{j=1}^{t-1}K\,\frac{t-j}{t}\right)^2\right] \qquad \text{(C.22)}$$

Now,

$$\sum_{j=1}^{t-1} K\frac{t-j}{t} = \frac{K}{t}\left\{t(t-1)-\frac{1}{2}(t-1)t\right\} = \frac{1}{2}Kt(t-1) \qquad (C.23)$$

Hence

$$E\langle \eta_t^2 \rangle = \eta_0^2 + (\sigma^2 + m_\zeta^2)\left[1 + \frac{K^2}{4}(t^2-2t+1)\right] \qquad (C.24)$$

which confirms a parabolic variation of the mean square value of the process represented by (C.20).

From (C.5) and (C.20) one obtains,

$$t\eta_t' = \sum_{j=2}^{t-1} K(t-j)\zeta_{t-j} + K(t-1)\zeta_{t-1} + t\zeta_t \qquad (C.25)$$

$$(t-1)\eta_{t-1}' = \sum_{j=1}^{t-2} K(t-1-j)\zeta_{t-j-1} + (t-1)\zeta_{t-1} \qquad (C.26)$$

Using again (C.14) in (C.26), eliminating the summation term between (C.25) and (C.26), and replacing K by G , one obtains the difference equation corresponding to (C.20) as

$$\eta_t' - \frac{t-1}{t}\eta_{t-1}' = (G-1)\frac{t-1}{t}\zeta_{t-1} + \zeta_t \qquad (C.27)$$

By using the method of the Section C.3 the stability condition is easily shown to be given again by (C.19).

C.4. Representation of a third order process whose mean square value is a cubic function of time.

Substituting in (C.4) k=1, j=0, m=$\frac{1}{2}$, one obtains,

$$\eta_t = \eta_0 + \sum_{j=1}^{t-1} K \frac{t-j}{t^{1/2}} \zeta_{t-j} + \zeta_t \qquad (C.28)$$

$$E\langle \eta_t^2 \rangle = \eta_0^2 + E\langle (\sum_{j=1}^{t-1} K \frac{t-j}{t^{1/2}} \zeta_{t-j})^2 \rangle + E\langle \zeta_t^2 \rangle \qquad (C.29)$$

$$= \eta_0^2 + (\sigma^2 + m_\zeta^2)\{1 + (\sum_{j=1}^{t-1} K \frac{t-j}{t^{1/2}})^2\} \qquad (C.30)$$

Since

$$\sum_{j=1}^{t-1} K \frac{t-j}{t^{1/2}} = \frac{K}{t^{1/2}} \{t(t-1) - \frac{t(t-1)}{2}\} = \frac{1}{2} \frac{Kt(t-1)}{t^{1/2}} \qquad (C.31)$$

$$E\langle \eta_t^2 \rangle = \eta_0^2 + (\sigma^2 + m_\zeta^2)\{1 + \frac{K^2}{4} \frac{t^2(t-1)}{t}\} \qquad (C.32)$$

$$= \eta_0^2 + (\sigma^2 + m_\zeta^2)[1 + \frac{K^2}{4} t(t-1)^2] \qquad (C.33)$$

which shows that the mean square value of the process (C.28) is a cubic function of time.

From (C.5) and (C.28),

$$t^{1/2} \eta_t' = \sum_{j=2}^{t-1} K(t-j)\zeta_{t-j} + K(t-1)\zeta_{t-1} + \zeta_t \qquad (C.34)$$

$$(t-1)^{1/2} \eta_t' = \sum_{j=1}^{t-2} K(t-1-j)\zeta_{t-1-j} + (t-1)^{1/2} \zeta_{t-1} \qquad (C.35)$$

Again, using (C.14) in (C.35), eliminating the summation term between (C.34) and (C.35), and replacing K by G, one obtains the difference equation corresponding to the process (C.28) in the form

$$\eta_t' - \left(\frac{t-1}{t}\right)^{1/2}\eta_t' = \left[G\,\frac{t-1}{t^{1/2}} - \left(\frac{t-1}{t}\right)^{1/2} \right] \zeta_{t-1} + \zeta_t \tag{C.36}$$

The stability condition for G is again given by (C.19) as can be shown by using the method of Section C.2.

C.5. Representation of a fourth order process whose mean square value is a quartic function of time.

Substituting in (C.4) k=1, j=1, m=1, one obtains,

$$\eta_t = \eta_0 + \sum_{j=1}^{t-1} K\,\frac{(t-j)j}{t}\,\zeta_{t-j} + \zeta_t \tag{C.37}$$

$$E\langle \eta_t^2 \rangle = \eta_0^{\,2} + E\left\langle \left(\sum_{j=1}^{t-1} K\,\frac{(t-j)j}{t}\,\zeta_{t-j} \right)^2 \right\rangle + E\langle \zeta_t^2 \rangle \tag{C.38}$$

$$= \eta_0^{\,2} + (\sigma^2 + m_\zeta^{\,2}) \left\{ 1 + \left(\sum_{j=1}^{t-1} K\,\frac{(t-j)(j)}{t} \right)^2 \right\} \tag{C.39}$$

Now,

$$\frac{K}{t} \sum_{j=1}^{t-1} (tj - j^2) = \frac{K}{t} \left\{ t \cdot \frac{t(t-1)}{2} - \frac{2(t-1)^3 + 3(t-1)^2 + (t-1)}{6} \right\}$$

$$= \frac{K(t-1)}{t} \left\{ \frac{3t^2 - 2(t-1)^2 - 3(t-1) - 1}{6} \right\}$$

$$= \frac{K(t-1)}{t}\;\frac{3t^2 - 2(t^2 - 2t + 1) - 3(t-1) - 1}{6} \tag{C.40}$$

or,

$$\sum_{j=1}^{t-1} K\,\frac{(t-j)j}{t} = \frac{K(t-1)}{t} \cdot \frac{t^2 - t + 1}{6} \tag{C.41}$$

Thus,

$$\left(\sum_{j=1}^{t-1}\frac{K(t-j)j}{t}\right)^2 = \frac{K^2}{36}\times\frac{t^2-2t+1}{t^2}(t^2-t+1)^2$$

$$= \frac{K^2}{36}\left\{\left[\left(\frac{1}{t^2}-\frac{1}{t}\right)+1\right](t^2-t+1)^2\right. \tag{C.42}$$

The term $(1/t^2 - 1/t)$ decreases rapidly as t increases; it is less than 0.02 when t=50. Therefore, (C.42) can be taken to be a good representation of the stochastic process whose mean square value is a quartic function of time, if length of series considered is greater than 50 terms.

The difference equation corresponding to the process (C,37) is, this time, a little more difficult to obtain because of the nonlinear nature of the weights $v_j(t)$. The equation is obtained as follows.

From (C.37),

$$\eta_t' = \sum_{j=3}^{t-1} K\frac{(t-j)j}{t}\zeta_{t-j} + K\frac{t-2}{t}\zeta_{t-2} + K\frac{t-1}{t}\zeta_{t-1} + \zeta_t \tag{C.43}$$

$$\eta_{t-1}' = \sum_{j=2}^{t-2} K\frac{(t-1-j)j}{t-1}\zeta_{t-1-j} + K\frac{t-2}{t-1}\zeta_{t-2} + \zeta_{t-1} \tag{C.44}$$

and

$$\eta_{t-2}' = \sum_{j=1}^{t-3} K\frac{(t-2-j)j}{t-2}\zeta_{t-2-j} + \zeta_{t-2} \tag{C.45}$$

Substitute

$$k=j+1 \tag{C.46}$$

in (C.44) and

$$l=j+2 \tag{C.47}$$

in (C.45)

Then (C.43),(C.44) and (C.45) become

$$\eta_t' = \sum_{j=3}^{t-1} K\frac{(t-j)j}{t}\zeta_{t-j} + K\frac{t-2}{t}\zeta_{t-2} + K\frac{t-1}{t}\zeta_{t-1} + \zeta_t \tag{C.48}$$

$$\eta_{t-1}' = \sum_{k=3}^{t-1} K\frac{(t-k)k}{t-1}\zeta_{t-k} - \sum_{k=3}^{t-1} K\frac{t-k}{t-1}\zeta_{t-k} + K\frac{t-2}{t-1}\zeta_{t-2} + \zeta_{t-1} \tag{C.49}$$

$$\eta_{t-2}' = \sum_{l=3}^{t-1} K\frac{(t-l)l}{t-2}\zeta_{t-l} - 2\sum_{l=3}^{t-1} K\frac{t-l}{t-2}\zeta_{t-l} + \zeta_{t-2} \tag{C.50}$$

These can be written as

$$t\eta_t' = \sum_{j=3}^{t-1} K(t-j)j\zeta_{t-j} + K(t-2)\zeta_{t-2} + K(t-1)\zeta_{t-1} + t\zeta_t \tag{C.51}$$

$$(t-1)\eta_{t-1}' = \sum_{j=3}^{t-1} K(t-j)j\zeta_{t-j} - \sum_{j=3}^{t-1} K(t-j)\zeta_{t-j} + K(t-2)\zeta_{t-2} + (t-1)\zeta_{t-1} \tag{C.52}$$

$$(t-2)\eta_{t-2}' = \sum_{j=3}^{t-1} K(t-j)j\zeta_{t-j} - 2\sum_{j=3}^{t-1} K(t-j)\zeta_{t-j} + (t-2)\zeta_{t-2} \tag{C.53}$$

From (C.51) and (C.52)

$$t\eta_t' - (t-1)\eta_{t-1}' = \sum_{j=3}^{t-1} K(t-j)\zeta_{t-j} + (K-1)(t-2)\zeta_{t-2} + (K-1)(t-1)\zeta_{t-1} + t\zeta_t \tag{C.54}$$

and from (C.51) and (C.53)

$$t\eta_t' - (t-2)\eta_{t-2}' = 2\sum_{j=3}^{t-1} K(t-j)\zeta_{t-j} + (K-1)(t-2)\zeta_{t-2} + K(t-1)\zeta_{t-1} + t\zeta_t \tag{C.55}$$

Hence, from (C.54) and (C.55), replacing K by G , one obtains finally,

$$\eta_t' - 2\left(\frac{t-1}{t}\right)\eta_{t-1}' + \left(\frac{t-2}{t}\right)\eta_{t-2}'$$
$$= \zeta_t + \left[G\left(\frac{t-1}{t}\right) - 2\left(\frac{t-1}{t}\right)\right]\zeta_{t-1} + (G-1)\left(\frac{t-2}{t}\right)\zeta_{t-2} \tag{C.56}$$

This equation may be written in the form

$$\zeta_t = \frac{[(t-2)B^2 - 2(t-1)B + t]\,\eta_t'}{(G-1)(t-2)B^2 + [G(t-1) - 2(t-1)]B + t} \tag{C.57}$$

The estimation procedure will be stable if the roots of the equation

$$0 = (G-1)(t-2)x^2 + [G(t-1) - 2(t-1)]x + t \tag{C.58}$$

lie outside the unit circle.

Now the representation (C.42) is valid for relatively large t. Hence under these conditions

$$t \approx t-1 \approx t-2$$

and the stability condition (C.58) is equivalent to the condition that the roots of

$$0 = (G-1)x^2 + [G-2]x + 1 \tag{C.59}$$

lie outside the unit circle

Since the product of the roots is 1/(G-1), the stability condition is again given by (C.19).

C.6. Representation of higher order processes.

The models of Sections C.2-C.5 have been treated in some detail in order to illustrate the theory and to provide a comparison with the stochastic models developed by Box and Jenkins (1963,1962,1966). It is thought that the four models discussed illustrate the theory adequately and that they are sufficient for the purpose of this thesis. Higher order models may,however, be easily developed from the weighting function (C.4) by using a suitable combination of the exponents k,l, and m.

Appendix D.

Estimation of parameters of a first order system in the presence of a disturbance.

D.1. Introduction.

It is shown in the Appendix B that a first order system with gain K and the time constant T may be described by a difference equation

$$y'_t - \phi y_{t-1} = g(1 - \phi)x_{t-1} \qquad (D.1)$$

where y'_t denotes the output, x_t is the input and

$$g=KT \qquad (D.2)$$

is the effective gain, and

$$\phi = \exp(-\Delta T/T) \qquad (D.3)$$

If, for a given sampling rate, the time constant T is expressed as a multiple of the sampling interval ΔT, say $T=L\Delta T$, then

$$\phi = \exp(-1/L) \qquad (D.4)$$

The relation (D.1) corresponds to the representation of the relation between the output y_t and the input x_t in the form

$$y_t' = \sum_{j=0}^{\infty} [g(1-\phi)\phi^j]x_{t-1-j} \qquad (D.5)$$

Any physically obseved quantity is usually subject to experimental errors. If such errors can be assumed to be characterized by a zero mean white noise process, then the noisy output y_t can be represented as

$$y_t = y'_t + \varepsilon_t^o \tag{D.6}$$

or, explicitly,

$$y_t = \sum_{j=0}^{\infty} [g(1-\phi)\phi^j] x_{t-1-j} + \varepsilon_t^o \tag{D.7}$$

Given a series of values of the input x_t and of the output y_t various estimation procedures, using least squares or maximum likelihood method, (see Appendix A), assume a set of parameter values, use it to calculate "predicted" values of output y_t^*, and obtain estimates of the parameters by minimizing the sum of squares of deviations of the predicted outputs y_t^* from the corresponding actual outputs y_t. Procedures of this kind involve a tacit assumption that a relation of the form (D.6) holds throughout the minimization process. This assumption is probably justified when the length N of the series used for the estimation is very large. However, in small samples, the effect of correlated errors generated by the differences between the true and the assumed values of parameters may become significant. The differences between the actual and the predicted values of the output have been called by the writer the "quasi-residuals" in order to describe their correlated character. It is shown in this Appendix that correct estimates can be obtained when the sum of squares of the quasi-residuals is minimized in such a way that their first three covariances are minimized and approximate the covariances of white noise. Section D.2 discusses the estimation procedure when (D.6) holds, that is, when the output readings are subject to white noise. In Section D.3 the output readings are assumed to include the effect of a nonstationary disturbance discussed in the Appendix C.

Since the relation (D.5) is used to derive only the first three covariances, it is approximated by its first three terms. Employing the formulation used earlier by Box and Jenkins (1963), let

$$\beta = 1 - \phi \qquad \text{(D.8)}$$

then the noise-free output y_t is represented by

$$y_t' = g\beta\left[x_{t-1} + (1-\beta)x_{t-2} + (1-\beta)^2 x_{t-3}\right] \qquad \text{(D.9)}$$

<u>D.2. Output readings subject to a white noise disturbance.</u>

Under these conditions, from (D.6) and (D.9) we have

$$y_t = g\beta\left[x_{t-1} + (1-\beta)x_{t-2} + (1-\beta)^2 x_{t-3}\right] + \varepsilon_t^{\circ} \qquad \text{(D.10)}$$

Suppose that, at the start , the values of the gain and of the exponential factor (D.8) are assumed to be g_o and β_o, when the corresponding true but unknown values are g and β. Then the deviations g^* and β^* of the parameters g and β from their true values are given by

$$g^* = g - g_0 \tag{D.11}$$

$$\beta^* = \beta - \beta_0 \tag{D.12}$$

Using the relations (D.11) and (D.12) in (D.10),

$$y_t = (g - g^*)(\beta - \beta^*)\left[x_{t-1} + (1-\beta+\beta^*)x_{t-2} + (1-\beta+\beta^*)^2 x_{t-3}\right] + \varepsilon_t^0 \tag{D.13}$$

This can be written

$$\begin{aligned} y_t = {} & g\beta\left[x_{t-1} + (1-\beta)x_{t-2} + (1-\beta)^2 x_{t-3}\right] \\ & + g\beta\beta^* x_{t-2} + \left[2g\beta\beta^*(1-\beta) + g\beta\beta^{*2}\right]x_{t-3} \\ & + (g^*\beta^* - g^*\beta - g\beta^*)\times\left[x_{t-1} + (1-\beta)x_{t-2} + \beta^* x_{t-2} + (1-\beta)^2 x_{t-3}\right. \\ & \left. + \left\{2\beta^*(1-\beta) + \beta^{*2}\right\}x_{t-3}\right] + \varepsilon_t^0 \end{aligned} \tag{D.14}$$

Let the total effect of the white noise disturbance ε_t^0 and the errors caused by assuming wrong values of the parameters, be represented by a "quasi-residual" error ε_t given by

$$\begin{aligned} \varepsilon_t = {} & g\beta\beta^* x_{t-2} + \left[2g\beta\beta^*(1-\beta) + g\beta\beta^{*2}\right]x_{t-3} \\ & + (g^*\beta^* - g^*\beta - g\beta^*)\times\left\{x_{t-1} + (1-\beta)x_{t-2} + \beta^* x_{t-2} + (1-\beta)^2 x_{t-3}\right. \\ & \left. + \left[2\beta^*(1-\beta) + \beta^{*2}\right]x_{t-3}\right\} + \varepsilon_t^0 \end{aligned} \tag{D.15}$$

Then (D.14) can be written

$$y_t = y_t' + \varepsilon_t \tag{D.16}$$

where y_t is given by (D.9)

To simplify calculations let

$$A = g^*\beta^* - g^*\beta - g\beta^* \tag{D.17}$$

$$B = g\beta\beta^* + (1-\beta+\beta^*)(g^*\beta^* - g^*\beta - g\beta^*) \tag{D.18}$$

$$C = g\beta\beta^{*2} + 2g\beta(1-\beta)\beta^*$$
$$+ (1-\beta+\beta^*)^2(g^*\beta^* - g^*\beta - g\beta^*) \tag{D.19}$$

Then (D.15) may be written as

$$\varepsilon_t = Ax_{t-1} + B_{t-2} + Cx_{t-3} + \varepsilon_t^0 \tag{D.20}$$
$$t = 4, 5, \ldots, N$$

In the following, N is assumed to be sufficiently large for the difference between N,(N-1),(N-2),(N-3) and (N-4) to be negligible.

Let the sample mean of the input and of the quasi-residuals be respectively given by

$$m_x = \frac{1}{N}\sum_{t=1}^{N-3} x_t \tag{D.21}$$

and

$$m_\varepsilon = \frac{1}{N}\sum_{t=4}^{N} \varepsilon_t$$
$$= (A+B+C)\,m_x\,. \tag{D.22}$$

Also let the deviations of the input values from their mean value and the deviations of the quasi-residuals from their mean value be ,respectively,

$$\tilde{x}_t = x_t - m_x \tag{D.23}$$

$$\tilde{\varepsilon}_t = \varepsilon_t - m_\varepsilon \tag{D.24}$$

Then from (D.20), (D.21), (D.22), (D.23) and (D.24),

$$\tilde{\varepsilon}_t = A\tilde{x}_{t-1} + B\tilde{x}_{t-2} + C\tilde{x}_{t-3} + \varepsilon_t^{\circ} \tag{D.25}$$

$$t = 4, 5, \ldots, N.$$

Let sample covariances of the input, quasi-residuals and the disturbance be respectively defined by

$$\gamma_x(k) = \frac{1}{N}\sum_{t=1}^{N-1-k} \tilde{x}_t\,\tilde{x}_{t+k} \tag{D.26}$$

$$\gamma_\varepsilon(k) = \frac{1}{N}\sum_{t=4}^{N-k} \tilde{\varepsilon}_t\,\tilde{\varepsilon}_{t+k} \tag{D.27}$$

$$\gamma_{\varepsilon^\circ}(k) = \frac{1}{N}\sum_{t=4}^{N-k} \varepsilon_t^{\circ}\,\varepsilon_{t+k}^{\circ} \tag{D.28}$$

Let also the sum of squares of the quasi-residuals be given by

$$S = \sum_{t=4}^{N} \varepsilon_t^2 \tag{D.29}$$

Then,

$$\gamma_\varepsilon(0) = (AB + BC)[\gamma_x(1) + \gamma_x(1)] + AC[\gamma_x(2) + \gamma_x(2)]$$
$$+ (A^2 + B^2 + C^2)\gamma_x(0) + \gamma_{\varepsilon^\circ}(0) \tag{D.30}$$

$$\gamma_\varepsilon(1) = (AB + BC)[\gamma_x(0) + \gamma_x(2)] + AC[\gamma_x(1) + \gamma_x(3)]$$
$$+ (A^2 + B^2 + C^2)\gamma_x(1) + \gamma_{\varepsilon^\circ}(1) \tag{D.31}$$

$$\gamma_\varepsilon(2) = (AB + BC)[\gamma_x(0) + \gamma_x(3)] + AC[\gamma_x(0) + \gamma_x(4)]$$
$$+ (A^2 + B^2 + C^2)\gamma_x(2) + \gamma_{\varepsilon^\circ}(2) \tag{D.32}$$

Using (D.17) – (D.19) one obtains

$$A^2+B^2+C^2=$$

$$= \beta^{*3}[-6g^2\beta-2g^2] + \beta^{*3}g^*[4g-16g\beta] + \beta^{*3}g^{*2}[24\beta-14]$$

$$+\beta^{*2}[2g^2-4g^2\beta] + \beta^{*2}g^*[-4g] + \beta^{*2}g^{*2}[2+2\beta]$$

$$+\beta^*g^*[4g\beta] + \beta^*g^{*2}[-4\beta] \qquad \text{(D.33)}$$

$$AB+BC=$$

$$= \beta^{*3}[4g^2-8g^2\beta] + \beta^{*3}g^*[8g\beta-8g] + \beta^{*3}g^{*2}[4-12\beta]$$

$$+\beta^{*2}[g^2-3g^2\beta] + \beta^{*2}g^*[12g\beta-2g] + \beta^{*2}g^{*2}[1-3\beta]$$

$$+\beta^*g^*[2g\beta] + \beta^*g^{*2}[-2\beta] \qquad \text{(D.34)}$$

$$AC=$$

$$= \beta^{*3}[3g^2\beta-2g^2] + \beta^{*3}g^*[7g\beta-4g] + \beta^{*3}g^{*2}[2-4\beta]$$

$$+\beta^{*2}[g^2-4g^2\beta] + \beta^{*2}g^*[10g\beta-2g] + \beta^{*2}g^{*2}[2\beta-3]$$

$$+\beta^*g^*[2g\beta] + \beta^*g^{*2}[-2\beta] \qquad \text{(D.35)}$$

Let

$$x = g^*/g \qquad \text{(D.36)}$$

$$y_1 = \beta^*/\beta \qquad \text{(D.37)}$$

and

$$A_0^2 + B_0^2 + C_0^2 = \frac{1}{g^2\beta^3}(A^2+B^2+C^2) \tag{D.38}$$

$$A_0 B_0 + C_0 B_0 = \frac{1}{g^2\beta^3}(AB+CB) \tag{D.39}$$

$$A_0 C_0 = \frac{1}{g^2\beta^3}(AC) \tag{D.40}$$

$$A_0^2+B_0^2+C_0^2 =$$

$$= y_1^3(-6\beta-2) + y_1^3 x(4-16\beta) + y_1^3 x^2(24\beta-14)$$
$$+ y_1^2\left(\frac{2}{\beta}-4\right) + y_1^2 x\left(-\frac{4}{\beta}\right) + y_1^2 x^2\left(\frac{2}{\beta}+2\right)$$
$$+ y_1 x\left(\frac{4}{\beta}\right) + y_1 x^2\left(-\frac{4}{\beta}\right) \tag{D.41}$$

$$A_0 B_0 + B_0 C_0 =$$

$$= y_1^3(4-8\beta) + y_1^3 x(8\beta-8) + y_1^3 x^2(4-12\beta)$$
$$+ y_1^2\left(\frac{1}{\beta}-3\right) + y_1^2 x\left(12-\frac{2}{\beta}\right) + y_1^2 x^2\left(\frac{1}{\beta}-3\right)$$
$$+ y_1 x\left(\frac{2}{\beta}\right) + y_1 x^2\left(-\frac{2}{\beta}\right) \tag{D.42}$$

$$A_0 C_0 =$$

$$= y_1^3(3\beta-2) + y_1^3 x(7\beta-4) + y_1^3 x^2(2-4\beta) + y_1^2\left(\frac{1}{\beta}-4\right)$$
$$+ y_1^2 x\left(10-\frac{2}{\beta}\right) + y_1^2 x^2\left(2-\frac{3}{\beta}\right) + y_1 x\left(\frac{2}{\beta}\right) + y_1 x^2\left(-\frac{2}{\beta}\right) \tag{D.43}$$

Define

$$\gamma_\varepsilon^*(k) = \frac{\gamma_\varepsilon(k) - \gamma_{\varepsilon 0}(k)}{g^2\beta^3\gamma_x(0)} \tag{D.44}$$

and the autocorrelation function

$$\rho_z(k) = \frac{\gamma_z(k)}{\gamma_z(0)} \tag{D.45}$$

Then,

$$\gamma_\varepsilon^*(0) = 2(A_0B_0 + B_0C_0)\rho_x(1) + 2A_0C_0\rho_x(2) + (A_0^2 + B_0^2 + C_0^2) \tag{D.46}$$

$$\gamma_\varepsilon^*(1) = (A_0B_0 + B_0C_0)[1 + \rho_x(2)] + A_0C_0[\rho_x(1) + \rho_x(3)] + (A_0^2 + B_0^2 + C_0^2)\rho_x(1) \tag{D.47}$$

$$\gamma_\varepsilon^*(2) = (A_0B_0 + B_0C_0)[1 + \rho_x(3)] + A_0C_0[1 + \rho_x(4)] + (A_0^2 + B_0^2 + C_0^2)\rho_x(2) \tag{D.48}$$

Also let

$$S^{*2} = \frac{1}{g^2\beta^3}\left[\frac{1}{N}S^2 - \gamma_{\varepsilon 0}(0)\right] \tag{D.49}$$

or, explicitly,

$$S^{*2} = \gamma_\varepsilon^*(0)\cdot\gamma_x(0) + [(A_0^2 + B_0^2 + C_0^2) + 2(A_0B_0 + B_0C_0) + 2A_0C_0]\,m_x^2 \tag{D.50}$$

or,

$$S^{*2} = 2(A_0 B_0 + B_0 C_0)\gamma_x(0) + 2A_0 B_0 \gamma_x(2)$$

$$+ (A_0^2 + B_0^2 + C_0^2)\gamma_x(0)$$

$$+ \left[(A_0^2 + B_0^2 + C_0^2) + 2(A_0 B_0 + B_0 C_0) + \right.$$

$$\left. + 2A_0 C_0 \right] m_x^2 \qquad \text{(D.51)}.$$

S^{*2} is the mean square value of the quasi-residuals normalized so that it is a function of the time constant only. The covariances (D.46) – (D.48) are likewise normalized. It is seen from the above relations that the values $x=y=0$ make the absolute minimum of S^{*2} equal to that of either covariance. However, for $x\neq0$ and/or $y\neq0$ the minimum of S^{*2} is not necessarily the same as the minimum of the covariances.

Now, according to the well established theory (Anderson, 1942; Koopmans, 1942; Dixon, 1944) the distribution of the serial correlation coefficient of lag 1 of a white noise sequence is approximately normal with mean $-1/(N-1)$ and variance $(N-2)/(N-1)^2$ when the sample size N is large. Thus, at best, the estimated value of

$$\rho_{\varepsilon^\circ}(1) = \frac{\gamma_{\varepsilon^\circ}(1)}{\gamma_{\varepsilon^\circ}(0)} \qquad \text{(D.52)}$$

which, according to the definition of white noise, is theoretically zero, will lie in the range

$$\left(-\frac{1}{\sqrt{N}} \, , \, \frac{1}{\sqrt{N}} \right)$$

Since the expressions in (A_o, B_o, C_o) are the same in all the three covariances eq. (D.46), (D.47) and (D.48), and the input autocorrelations $\rho_x(k)$ are of the same order of magnitude, the asymptotic value of can be achieved only if the expressions on the right hand side of (D.46), (D.47) and (D.48) are of the same order of magnitude as covariances $\gamma_{\varepsilon 0}(k)$ of the disturbing noise. It follows, therefore, that the absolute minimum of the sum of squares of the quasi-residuals may be approached by monitoring and minimizing at the same time the covariances of the quasi-residuals.

It will be shown below that this aim may be achieved if attention is paid to the rate at which the variance of the quasi-residuals is being decreased.

The expression (D.46) requires the knowledge of the first four autocorrelations of the input.. If the input process is not stationary then any assumed set of four autocorrelations would only apply to one particular sample, the sets of values varying from sample to sample. For the purpose of the present exposition it will not greatly matter, therefore, if instead, the first two autocorrelations are assumed to be equal. Table D.1. shows the results of the tests discussed in the next chapter. It is seen that the variability of

TABLE D.I.

CORRELATION FUNCTION $\rho_x(k)$ OF STEAM FLOW

TEST	N	PART OF SERIES	$\rho_x(1)$	$\rho_x(2)$	$\rho_x(3)$	$\rho_x(4)$
1	340	1ST QUARTER	0·921	0·810	0·710	0·616
	340	2ND QUARTER	0·930	0·829	0·725	0·628
	340	3RD QUARTER	0·919	0·805	0·689	0·582
	340	4TH QUARTER	0·936	0·864	0·783	0·695
2	324	1ST QUARTER	0·973	0·929	0·874	0·817
	324	2ND QUARTER	0·964	0·919	0·870	0·818
	324	3RD QUARTER	0·947	0·885	0·824	0·759
	324	4TH QUARTER	0·973	0·944	0·909	0·870
3	361	1ST QUARTER	0·925	0·878	0·836	0·798
	361	2ND QUARTER	0·969	0·947	0·942	0·905
	361	3RD QUARTER	0·944	0·894	0·840	0·802
	361	4TH QUARTER	0·958	0·911	0·864	0·825
4	349	1ST QUARTER	0·931	0·864	0·796	0·737
	349	2ND QUARTER	0·945	0·880	0·806	0·752
	349	3RD QUARTER	0·971	0·940	0·905	0·876
	349	4TH QUARTER	0·953	0·905	0·847	0·796

the autocorrelation is such that the approximation

$$\rho_x(1) \approx \rho_x(2) \approx 1$$

resulting in a great simplification of the expressions appears to be reasonable.

With this approximation,

$$\gamma_\varepsilon^*(0) = 2\left[A_o^2 + B_o^2 + C_o^2 + A_o C_o + A_o B_o + B_o C_o\right] \quad (D.53)$$

Now, since it is the parameter $\phi = 1-\beta$ which is to be estimated, it is convenient to introduce at this stage the following change.

From (D.12) and (D.8),

$$\beta^* = \beta - \beta_o = (1-\phi) - (1-\phi_o) \quad (D.54)$$

If

$$\phi^* = \phi - \phi_o \quad (D.55a)$$

and

$$y = \phi^*/\phi \quad (D.55b)$$

then

$$y = -\frac{\beta}{1-\beta}\, y_1 \quad (D.55c)$$

Using (D.55c), (D.52) and (D.42)-(D.44),

$$\gamma_\varepsilon^* = a_1 y^3 + a_2 y^3 x + a_3 y^3 x^2 + a_4 y^2$$

$$+ a_5 y^2 x + a_6 y^2 x + a_7 y x + a_8 y x^2 \quad (D.56)$$

or, explicitly,

$$\gamma_\varepsilon^*(0) = y^3\left[\frac{22\beta^4}{(1-\beta)^3}\right] + y^3x\left[\frac{(2\beta+32)\beta^3}{(1-\beta)^3}\right]$$
$$+ y^3x^2\left[\frac{(16-16\beta)\beta^3}{(1-\beta)^3}\right] + y^2\left[\frac{(8-22\beta)\beta}{(1-\beta)^2}\right]$$
$$+ y^2x\left[\frac{(44\beta-16)\beta}{(1-\beta)^2}\right] + y^2x^2\left[\frac{2\beta^2}{(1-\beta)^2}\right]$$
$$+ yx\left[-\frac{16}{1-\beta}\right] + yx^2\left[\frac{16}{1-\beta}\right] \qquad \text{(D.57)}$$

Table D.2 shows the values of the coefficients a_i calculated for four values of system time constant in the expected range of variation.

The hill climbing procedure, developed by the writer and discussed in Chapter 5, involves minimizing the sum of squares function along n orthogonal directions, corresponding to the n parameters. The n directions are fixed throughout the procedure. At the start of each stage the initial parameter change was obtained as the product of a fixed fraction w (the value w=0.02 was actually employed by the writer) and the currently available estimate of the parameter. After a "success" (that is, the parameter change such that the new value of the parameter satisfies all the constraints and decreases the sum of squares and the covariances of the quasi-residuals) the new change to be applied to the parameter is equal to the preceding change multiplied by two.

TABLE D.2.

VARIATION OF COEFFICIENTS a_i IN (D.56) WITH SYSTEM TIME CONSTANT $L=-1/\mathrm{LOG}(1-\beta)$

$1-\beta$ (L)	0·87 (16·5)	0·90 (21·9)	0·92 (27·6)	0·95 (44·9)
a_1	0·0095	0·0030	0·0012	0·0002
a_2	0·108	0·044	0·021	0·005
a_3	0·046	0·020	0·010	0·002
a_4	0·882	0·716	0·590	0·382
a_5	-1·76	-1·43	-1·18	-0·76
a_6	0·067	0·037	0·023	0·008
a_7	-18·40	-17·80	-17·40	-16·80
a_8	18·40	17·80	17·40	16·80

Thus, if at the start of a stage, the currently available estimates of the gain and the exponential factor are denoted by g and $\emptyset$,respectively, then the initial changes to be applied to these estimates, expressed as fractions of the parameters, are

$$w(g-g^*)/g = w(1-x) \tag{D.58a}$$

$$w(\emptyset-\emptyset^*)/\emptyset = w(1-y) \tag{D.58b}$$

Also, after n successes, the total change applied to the parameter a is

$$\frac{w(a-a^*)}{a}\ (1 + 2 + \ldots + 2(n-1)) = \frac{w(a-a^*)}{a}(2^n-1) \tag{D.59}$$

Let the value of the variance $\gamma_\varepsilon(0)$ after n successful steps be denoted by γ_n . Then, for the decreasing variance, we must have,

$$\gamma_n < \gamma_{n-1} \tag{D.60}$$

Let

$$\gamma^{(1)}_{n-1} = \gamma_{n-1} - \gamma_n \tag{D.61a}$$

$$\gamma^{(1)}_{n-2} = \gamma_{n-2} - \gamma_{n-1} \tag{D.61b}$$

$$\gamma^{(1)}_{n-3} = \gamma_{n-3} - \gamma_{n-2} \tag{D.61c}$$

We want the increases in the decrease of variance to keep on increasing, so that

$$\gamma^{(1)}_{n-2} > \gamma^{(1)}_{n-1}$$

Let

$$\gamma^{(2)}_{n-1} = \gamma^{(1)}_{n-1} - \gamma^{(1)}_{n-2} \qquad \text{(D.62a)}$$

$$\gamma^{(2)}_{n-2} = \gamma^{(1)}_{n-2} - \gamma^{(1)}_{n-3} \qquad \text{(D.62b)}$$

$$\gamma^{(3)}_{n-1} = \gamma^{(2)}_{n-1} - \gamma^{(2)}_{n-2} \qquad \text{(D.62c)}$$

We want to investigate what happens when the rate, at which the variance decreases, itself decreases, that is, when $\gamma^{(3)}_{n-1}$ given by (D.62c) becomes negative.

Now, from (D.60)÷(D.62), we have

$$\begin{aligned}
\gamma^{(3)}_{n-1} &= \gamma^{(2)}_{n-1} - \gamma^{(2)}_{n-2} \\
&= \left(\gamma^{(1)}_{n-1} - \gamma^{(1)}_{n-2}\right) - \left(\gamma^{(1)}_{n-2} - \gamma^{(1)}_{n-3}\right) \\
&= \gamma^{(1)}_{n-1} - 2\gamma^{(1)}_{n-2} + \gamma^{(1)}_{n-3} \\
&= (\gamma_{n-1} - \gamma_n) - 2(\gamma_{n-2} - \gamma_{n-1}) + (\gamma_{n-3} - \gamma_{n-2}) \\
&= \gamma_{n-3} - 3\gamma_{n-2} + 3\gamma_{n-1} - \gamma_n
\end{aligned} \qquad \text{(D.63)}$$

Now, the procedure adjusts one parameter at a time. Therefore, the variance relation may be written either as a function of the variable x,

$$\begin{aligned}
\gamma''_{\varepsilon}(0) = (a_1 y^3 + a_4 y^2) + x(y^3 a_2 + a_6 y^2 + a_8 y) \\
+ x^2(a_3 y^3 + a_5 y^2 + a_7 y)
\end{aligned} \qquad \text{(D.64)}$$

or as a function of the variable y

$$\gamma_{\varepsilon}^{*}(0) = y(a_7 x^2 + a_8 x) + y^2(a_4 + a_5 x^2 + a_6 x) + y^3(a_2 x + a_3 x^2) \tag{D.65}$$

Consider first the fractional parameter x.

Using (D.64), (D.63), (D.62), (D.61) and (D.59),

$$\gamma_{n-1}^{(3)} = 2^{n-3}\omega(1-x)\Big\{(a_2 y^3 + a_6 y^2 + a_8 y) + (a_3 y^3 + a_5 y^2 + a_7 y)\big[2x + 2\omega(1-x) - \omega(1-x)\times 27\times 2^{n-3}\big]\Big\} \tag{D.66}$$

This expression becomes negative when

$$2x + 2\omega(1-x) < \omega(1-x)\times 27\times 2^{n-4} \tag{D.67}$$

This can be written as

$$x + \omega(1-x) < (1+\tfrac{9}{16})\times 2^{n}\omega(1-x) \tag{D.68}$$

or,

$$x - (2^n - 1)\omega(1-x) < \tfrac{9}{16}\times 2^{n}\omega(1-x) \tag{D.69}$$

From (D.68) and (D.59) it follows that the rate at which the variance is decreasing begins to decrease when the remaining deviation of the current estimate of the parameter from its true value,

$$x - (2^n - 1)\omega(1-x) \tag{D.70}$$

is smaller than

$$\frac{9}{16} \times 2^{n} w(1-x)$$

Since on the (n+1)th iteration the parameter change would be $2^{n}w(1-x)$, the "overshoot" will be prevented if the iteration procedure is stopped when $\gamma_{n-1}^{(3)}$ becomes negative.

The conditions for negative $\gamma_{n-1}^{(3)}$ when the parameter y is being adjusted are obtained in a similar fashion. Thus, from (D.65), (D.63), (D.62), (D.61) and (D.59),

$$\begin{aligned}\gamma_{n-1}^{(3)} = 2^{n-3} w(1-y)\Big\{&(a_7x^2+a_8x) \\ &+ 2(a_4+a_5x^2+a_6x)\left(y+w(1-y)-w(1-y)\times 27\times 2^{n-4}\right) \\ &+ 3(a_2x+a_3x^2)\left[y+w(1-y)+2^{n-4}(-27+16\cdot5)w(1-y)\right]\times \\ &\times\left[y+w(1-y)+2^{n-4}(-27-16\cdot5)w(1-y)\right]\Big\}\end{aligned} \tag{D.71}$$

This can be written

$$\begin{aligned}\gamma_{n-1}^{(3)} = 2^{n-3} w(1-y)\Big\{&(a_7x^2+a_8x) \\ &+2(a_4+a_5x^2+a_6x)\left[y-(2^n-1)w(1-y)-\tfrac{9}{16}2^n w(1-y)\right] \\ &+3(a_2x+a_3x^2)\left[y-(2^n-1)w(1-y)+\tfrac{5\cdot5}{16}\times 2^n w(1-y)\right]\times \\ &\times\left[y-(2^n-1)w(1-y)-1\tfrac{11}{16}\times 2^n w(1-y)\right]\Big\}\end{aligned} \tag{D.72}$$

The product of the first two brackets in the last member is greater than the first bracket in the second member. Hence $\gamma_{n-1}^{(3)}$ first becomes negative when the overshoot has already occurred

The minimum value is attained when, on the next step, the variance increases, that is, when

$$\gamma^{(1)}_{n-1} = \gamma_{n-1} - \gamma_n < 0 \tag{D.73}$$

Therefore, it is necessary to consider now the expressions for the rate of change of variance $\gamma^{(1)}_{n-1}$.

Using equations previously developed the expressions for $\gamma^{(1)}_{n-1}$ corresponding to alterations in parameters x and y are found to be, respectively,

$$\gamma^{(1)}_{n-1} = 2^{n-1}\omega(1-x)\Big\{(a_2 y^3 + a_6 y^2 + a_8 y) + 2(a_3 y^3 + a_5 y^2 + a_7 y)\times\left[x + \omega(1-x) - \tfrac{3}{4}\times 2^n \omega(1-x)\right]\Big\} \tag{D.74}$$

and

$$\gamma^{(1)}_{n-1} = 2^{n-1}\omega(1-y)\Big\{(a_7 x^2 + a_8 x) + 2(a_4 + a_5 x^2 + a_6 x)\cdot \left[y + \omega(1-y) - \omega(1-y)\times\tfrac{3}{4}\times 2^n\right]\Big\} \tag{D.75}$$

the last expression being obtained on the assumption that, approximately,

$$\frac{9}{4}\omega^2(1-y)^2\times 2^{2n-2} = \frac{27}{12}\omega^2(1-y)^2\times 2^{2n-2} \approx \frac{28}{12}\omega^2(1-y)^2\, 2^{2n-2} \tag{D.76}$$

The expressions (D.75) and (D.76) become negative when, respectively,

$$x + \omega(1-x) - \frac{3}{4}2^n\omega(1-x) < 0 \tag{D.77}$$

$$y + \omega(1-y) - \frac{3}{4}\times 2^n\omega(1-x) < 0 \tag{D.78}$$

This means that the minimum variance will be obtained when the parameters x and y are overestimated ,respectively, by

$$x^{x} = \frac{1}{4} 2^{n} \omega (1-x) \tag{D.79}$$

$$y^{y} = \frac{1}{4} 2^{n} \omega (1-y) \tag{D.80}$$

The amount of overshoot will depend on the values of the parameters at the start of a stage. The absolute minimum will be attained when n=1, or when

$$x = \frac{\omega}{2+\omega} \tag{D.81}$$

$$y = \frac{\omega}{2+\omega} \tag{D.82}$$

D.3. Output readings subject to a nonstationary correlated disturbance.

This section indicates how the approach of Section D.2 can be employed when the disturbance corrupting the output readings is characterized by a nonstationary behaviour and is represented by models of Appendix C. In particular, a second order disturbance characterized by

$$\eta_t = \eta_0 + \sum_{j=1}^{t-1} K \frac{t-j}{t} \xi_{t-j} + \xi_t \tag{D.83}$$

and

$$\eta_t' - \frac{t-1}{t}\eta_{t-1}' = (G-1)\frac{t-1}{t}\zeta_{t-1} + \zeta_t \qquad (D.84)$$

will be considered.

The proportion of the output due to the disturbance is, of course, not known and must be estimated. To this end, an approach, in a sense a dual of that used by Kopp and Orford (1963), will be employed. Kopp and Orford expressed the structural system parameters as additional state variables and included them in the enlarged state vector, as discussed in Chapter 4. The present approach, however, is to represent the initial state as additional structural parameters and estimate them together with the structural parameters of the system and the disturbance.

For the first order system, of interest in this thesis, the initial state is represented by one variable. To represent it as a structural parameter,, assume that the first output reading, y_{to}, is the sum of two contributions, $(1-\delta)y_{to}$ due to the system dynamics, and δy_{to} due to the disturbance.

Thus,

$$\frac{\eta_1}{y_4'} = \frac{\delta y_4}{(1-\delta)y_4} \qquad (D.85)$$

or,

$$\eta_1 = \frac{\delta}{1-\delta}y_4' \qquad (D.86)$$

Using (D.9), (D.83) and (D.86), the output readings are modelled by

$$y_t = g\beta\left[x_{t-1} + (1-\beta)x_{t-2} + (1-\beta)^2 x_{t-3}\right] + g\beta\frac{\delta}{1-\delta}\left[x_3 + (1-\beta)x_2 + (1-\beta)^2 x_1\right] + \sum_{j=1}^{t'-2} K\frac{t'-j}{t'}\zeta_{t'-j} + \zeta_{t'} \qquad \text{(D.87)}$$

The relation (D.87) was used in the actual estimation procedure to obtain ,recursively, the values of quasi-residuals corresponding to a given set of assumed values of the parameters g, $1-\beta$, δ, and K. For the purpose of the present discussion, however, the relation is not convenient because

a) the value of the parameter δ appears not only in the numerator but also in the denominator, and this would make the derivation of the relation for the quasi-residuals rather difficult;

b) the factor

$$\frac{\delta}{1-\delta}$$

premultiplies a constant term, independent of the time parameter; this means that this factor would , on averaging be included in the mean, and would not appear explicitly in the expressions for covariances of quasi-residuals.

The relation (D.87) will, therefore, be transformed as follows

When an assumed value

$$\delta_0 = \delta - \delta^* \tag{D.88}$$

of δ is used, the factor

$$\frac{\delta}{1-\delta} \tag{D.89}$$

becomes

$$\frac{\delta - \delta^*}{1-(\delta-\delta^*)} \tag{D.90}$$

Expanding this as a Taylor series we have

$$\frac{\delta-\delta^*}{1-(\delta-\delta^*)} = \frac{\delta}{1-\delta} - \delta^* \frac{1}{(1-\delta)^2} + \delta^{*2}\frac{1}{(1-\delta)^3} + \cdots + \ldots + (-1)^k \delta^{*k} \frac{1}{(1-\delta)^{k+1}} \tag{D.91}$$

This infinite series will converge if

$$\frac{\delta^*}{1-\delta} < 1 \tag{D.92}$$

or, if

$$\delta^* < 1-\delta \tag{D.93}$$

The sum, including the k-th derivative, is

$$S_k = \frac{\delta}{1-\delta} \times \frac{1-\left(-\frac{\delta^*}{1-\delta}\right)^{k+1}}{1+\frac{\delta^*}{1-\delta}}$$

$$= \delta \cdot \frac{1-\left(-\frac{\delta^*}{1+\delta}\right)^{k+1}}{1-\delta+\delta^*} \tag{D.94}$$

Hence the error R_k due to stopping the series at the k-th order term is

$$R_K = \frac{\delta - \delta^*}{1-(\delta-\delta^*)} - \frac{\delta}{1-\delta+\delta^*}\left[1-\left(\frac{\delta^*}{1-\delta}\right)^{K+1}\right] \tag{D.95}$$

Defining the fractional parameter change z by

$$z = \frac{\delta^*}{\delta} \tag{D.96}$$

the stability condition (D.93) is now

$$z < \frac{1}{\delta} - 1 \tag{D.97}$$

and

$$R_K = \frac{1}{\frac{1}{\delta}-1+z}\left\{1-z-\left[1-\left(\frac{z}{\frac{1}{\delta}-1}\right)^{K+1}\right]\right\} \tag{D.98}$$

which may be written,

$$R_K = -\frac{1}{\frac{1}{\delta}-1+z}\left\{z-\left(\frac{z}{\frac{1}{\delta}-1}\right)^{K+1}\right\} \tag{D.99}$$

The relation (D.99) was used to compute a grid of values of R_k for varying z and δ. The results have not been included here owing to some difficulty in showing their three-dimensional character with both sufficient clarity and detail. The investigation showed that at low values of δ (such as those used in the simulation studies) the second and higher order terms could be regarded as having a negligible effect on the accuracy of the approximation. On the other hand, with the values of δ near unity (which was the case when the theory was applied to estimation of boiler dynamics) the stability condition (D.97) becomes a limiting factor and the starting values of δ must be very near the true values if instability is to be avoided. Thus, the deviations z are small and, again, the higher order terms may be neglected. For these reasons the linear approximation

$$\frac{\delta-\delta^{x}}{1-(\delta-\delta^{x})} \approx \frac{\delta}{1-\delta} - \delta^{x}\frac{1}{(1-\delta)^{2}} \qquad \text{(D.100)}$$

introducing errors of the order of 1-2 per cent, has been adopted in the analysis given below.

In order to include the parameter in the expressions for covariances of the quasi-residuals it is necessary to make the second member of eq.(D.87) time-dependent. This may be achieved if one regards the input values x_t as being generated by a stochastic process of a form similar to that generating the disturbance η_t (This assumption is tacitly made in the method of identification of the structure of the disturbancemas discussed in Chapter 5).

The analysis of results of tests described in Chapter 6 shows that various quantities associated with the boiler operation, like steam flowrate, steam temperature or the drum pressure may be represented by a second order process of the form (D.83). Let, then, the input process x_t be represented by the model

$$x_t = x_1 + \sum_{i=1}^{t-1} K_1 \frac{t-i}{t} \zeta_{t-i} + \zeta_t . \tag{D.101}$$

Suppose that the expected value of the random process in (D.83) and that of the random process in (D.101) and denoted m_η and m_ζ respectively, are both non-zero (This assumption does not seem to be unreasonable in the light of results of Chapter 6 and method of simulation described in Chapter 5). Thus, when K in (D.83) is replaced by G, the expected values of the processes (D.83) and (D.101) are respectively given by

$$E\langle \eta_t \rangle = \eta_1 + \sum_{j=1}^{t-1} G \frac{t-j}{t} E\langle \zeta_{t-j} \rangle + E\langle \zeta_t \rangle$$

$$= \eta_1 + \frac{m_\zeta G}{t} \left[t(t-1) - \tfrac{1}{2}(t-1)t \right] + m_\zeta \tag{D.102}$$

$$E\langle x_t \rangle = x_1 + \sum_{i=1}^{t-1} K_1 \frac{t-i}{t} E\langle \zeta_{t-i} \rangle + E\langle \zeta_t \rangle$$

$$= x_1 + \frac{m_\zeta}{t} K_1 \left[t(t-1) - \tfrac{1}{2}(t-1)t \right] + m_\zeta \tag{D.103}$$

Thus, from (D.102),

$$\eta_t = \eta_1 + \frac{m_\xi}{2} G(t-1) + m_\xi \qquad \text{(D.104)}$$

and, neglecting the effect of the current random shock on the current value of x_t,

$$x_t \approx x_1 + \frac{1}{2} m_\xi K_1 (t-1) \qquad \text{(D.105)}$$

From the last relation it follows that

$$x_{t+1} \approx x_1 + \frac{1}{2} m_\xi K_1 t \qquad \text{(D.106a)}$$

and

$$x_{t-1} \approx x_1 + \frac{1}{2} m_\xi K_1 (t-1) \qquad \text{(D.106b)}$$

Since also

$$x_2 = x_1 + \frac{1}{2} m_\xi K_1 \times 1 \qquad \text{(D.107a)}$$

and

$$x_3 = x_1 + \frac{1}{2} m_\xi K_1 \times 2 \qquad \text{(D.107b)}$$

it follows that x_3, x_2 and x_1 may be expressed in terms of the time dependent quantities as

$$x_3 = x_{t+1} - \frac{1}{2} m_\xi K_1 (t-2) \qquad \text{(D.108a)}$$

$$x_2 = x_t - \frac{1}{2} m_\xi K_1 (t-2) \qquad \text{(D.108b)}$$

$$x_1 = x_{t-1} - \frac{1}{2} m_\xi K_1 (t-2) \qquad \text{(D.108c)}$$

Setting $t=t'+2$ and subsequently calling $t'=t$ in (D.87), one obtains from (D.87),(D.104) and (D.108),

$$y_{t+2} = g\beta\left[1+\frac{\delta}{1-\delta}\right]\left[x_{t+1}+(1-\beta)x_t+(1-\beta)^2x_{t-1}\right]$$
$$-g\beta\frac{\delta}{1-\delta}\;\frac{m_\zeta}{2}K_1(t-2)\left[1+(1-\beta)+(1-\beta)^2\right]$$
$$+\frac{m_\zeta}{2}G(t-1)+m_\zeta+\zeta_t \tag{D.109}$$

or,

$$y_{t+2} = g\beta\frac{1}{1-\delta}\left[x_{t+1}+(1-\beta)x_t+(1-\beta)^2x_{t-1}\right]$$
$$-g\beta\frac{\delta}{1-\delta}\;\frac{m_\zeta}{2}K_1(t-2)\left[1+(1-\beta)+(1-\beta)^2\right]$$
$$+\frac{m_\zeta}{2}G(t-1)+m_\zeta+\zeta_t \tag{D.110}$$

Suppose that the parameters $g, \beta, \delta,$ and G are estimated to be $g_0, \beta_0, \delta_0,$ and G_0 so that the relation (D.110) is

$$y_{t+2} = (g-g^*)(\beta-\beta^*)\left\{x_{t+1}+(1-\beta+\beta^*)x_t\right.$$
$$\left.+\left[(1-\beta)^2+2(1-\beta)\beta^*+\beta^{*2}\right]x_{t-1}\right\}\frac{1}{1-(\delta-\delta^*)}$$
$$-\frac{(\delta-\delta^*)}{1-(\delta-\delta^*)}(g-g^*)(\beta-\beta^*)\frac{m_\zeta}{2}K_1(t-2)\times$$
$$\times\left\{1+(1-\beta+\beta^*)+(1-\beta)^2+2(1-\beta)\beta^*+\beta^{*2}\right\}$$
$$+\frac{m_\zeta}{2}(G-G^*)(t-1)+m_\zeta+\zeta_t \tag{D.111}$$

Using the result (D.100),

$$y_{t+2} = [g\beta + g^*\beta^* - g^*\beta - g\beta^*]\left[\frac{1}{1-\delta} - \frac{\delta^*}{(1-\delta)^2}\right]^r$$

$$\times \left\{x_{t+1} + [(1-\beta)+\beta^*]x_t + [(1-\beta)^2 + 2(1-\beta)\beta^* + \beta^{*2}]x_{t-1}\right\}$$

$$- [g\beta + g^*\beta^* - g^*\beta - g\beta^*]\cdot\left[\frac{\delta}{1-\delta} - \frac{\delta^*}{(1-\delta)^2}\right]\cdot \frac{m_\zeta}{2}K_1(t-2)\,\cdot$$

$$\cdot\left\{1 + (1-\beta) + \beta^* + (1-\beta)^2 + 2(1-\beta)\beta^* + \beta^{*2}\right\}$$

$$+ \frac{m_\zeta}{2}G(t-1) - \frac{m_\xi}{2}G^*(t-1) + m_\xi + \zeta_t \qquad \text{(D.112)}$$

or,

$$y_{t+2} = g\beta\frac{1}{1-\delta}\left[x_{t+1} + (1-\beta)x_t + (1-\beta)^2 x_{t-1}\right]$$

$$- g\beta\frac{\delta}{1-\delta}\,\frac{m_\zeta}{2}K_1(t-2)\left[1 + (1-\beta) + (1-\beta)^2\right]$$

$$+ \frac{m_\zeta}{2}G(t-1) + m_\zeta$$

$$+ \varepsilon_{t+2} \qquad \text{(D.113)}$$

where the quasi-residual, including the combined effect of the disturbance and of the deviations of parameters from their true values is

$$\varepsilon_{t+2} = x_{t+1}\left\{\frac{1}{1-\delta}A - (g\beta + A)\frac{\delta^*}{(1-\delta)^2}\right\} +$$

$$+ x_t\left\{\frac{1}{1-\delta}B - (g\beta + A)(1-\beta+\beta^*)\frac{\delta^*}{(1-\delta)^2}\right\}$$

$$+ x_{t-1}\left\{\frac{1}{1-\delta}C - (g\beta + A)(1-\beta+\beta^*)^2\frac{\delta^*}{(1-\delta)^2}\right\}$$

$$+ \frac{m_\zeta K_1}{2}(t-2)\frac{\delta}{1-\delta}\left\{(g\beta + A)[\beta^* + 2(1-\beta)\beta^* + \beta^{*2}] + \right.$$

$$\left. + A[1 + (1-\beta) + (1-\beta)^2]\right\} - \frac{m_\xi}{2}G^*(t-1) + \zeta_t \qquad \text{(D.114)}$$

where A, B and C are respectively defined by (D.38), (D.39) and (D.40).

Let

$$P = \frac{1}{1-\delta} A - (g\beta + A)\frac{\delta^*}{(1-\delta)^2} \tag{D.115}$$

$$Q = \frac{1}{1-\delta} B - (g\beta + A)(1-\beta+\beta^*)^2\frac{\delta^*}{(1-\delta)^2} \tag{D.116}$$

$$R = \frac{1}{1-\delta} C - (g\beta + A)(1-\beta+\beta^*)^2\frac{\delta^*}{(1-\delta)^2} \tag{D.117}$$

$$S = -\frac{m_\xi K}{2}\frac{\delta}{1-\delta}\{(g\beta+A)[\beta^* + 2(1-\beta)\beta^* + \beta^{*2}]$$

$$+ A[1 + (1-\beta) + (1-\beta)^2]\} \tag{D.118}$$

$$T = -\frac{m_\xi G^*}{2} \tag{D.119}$$

Then the relation (D.114) is

$$\varepsilon_{t+2} = Px_{t+1} + Qx_t + Rx_{t-1} + S(t-2)$$

$$+ T(t-1) + \xi_t \tag{D.120}$$

Let the average value of the quasi-residuals be denoted by

$$\bar{\varepsilon}_{t+2} = \frac{1}{N-4}\sum_{t=2}^{N-2} \varepsilon_{t+2} \tag{D.121}$$

Then,

$$\bar{\varepsilon} = P\frac{1}{N-4}\sum_{t=2}^{N-2} x_{t+1} + Q\frac{1}{N-4}\sum_{t=2}^{N-2} x_t$$

$$+ R\frac{1}{N-4}\sum_{t=2}^{N-2} x_{t-1}$$

$$+ S\frac{N-2}{2} + T\frac{N-3}{2} \tag{D.122}$$

Assume that, approximately,

$$\frac{1}{N-4}\sum_{t=2}^{N-2} x_{t+1} = \frac{1}{N-4}\sum_{t=2}^{N-4} x_t = \frac{1}{N-4}\sum_{t=2}^{N-4} x_{t-1}$$

$$= \bar{x} \qquad \text{(D.123)}$$

and let

$$\tilde{x}_i = x_i - x \qquad \text{(D.124)}$$

$$\tilde{\varepsilon}_j = \varepsilon_j - \bar{\varepsilon} \qquad \text{(D.125)}$$

Then,

$$\tilde{\varepsilon}_{t+2} = P\tilde{x}_{t+1} + Q\tilde{x}_t + R\tilde{x}_{t-1}$$
$$+ S\left(t-2-\frac{N-3}{2}\right) + T\left(t-1-\frac{N-3}{2}\right) + \tilde{\xi}_t \qquad \text{(D.126)}$$

$$\tilde{\varepsilon}_{t+3} = P\tilde{x}_{t+2} + Q\tilde{x}_{t+1} + R\tilde{x}_t$$
$$+ S\left(t-\frac{N-3}{2}-1\right) + T\left(t-\frac{N-3}{2}\right) + \tilde{\xi}_{t+1} \qquad \text{(D.127)}$$

$$\tilde{\varepsilon}_{t+4} = P\tilde{x}_{t+3} + Q\tilde{x}_{t+2} + R\tilde{x}_{t+1}$$
$$+ S\left(t-\frac{N-3}{2}\right) + T\left(t+1-\frac{N-3}{2}\right) \qquad \text{(D.128)}$$

Let the covariances of quasi-residuals and input be

$$\gamma_\varepsilon(s) = \frac{1}{N-4}\sum_{t=2}^{N-2-s} \tilde{\varepsilon}_{t+2}\,\tilde{\varepsilon}_{t+2+s} \qquad \text{(D.129)}$$

$$\gamma_x(v) = \frac{1}{N-4}\sum_{t=2}^{N-2-v} \tilde{x}_t\,\tilde{x}_{t+v} \qquad \text{(D.130)}$$

The expressions for the covariances of the quasi-residuals will involve averages of products of the input x_t and the time parameter t, as well as the averages of products of terms containing the time parameter only. In order to obtain manageable expressions it will be assumed that all the averages of the first kind are approximated by

$$\frac{1}{N-4} \sum_{t=2}^{N-4} \tilde{x}_t \left(t - \frac{N-1}{3}\right)$$

$$= \frac{1}{N-4} \frac{m_3 K_1}{2} \sum_{t=2}^{N-4} \left(t - \frac{N+1}{2}\right)\left(t - \frac{N-1}{2}\right)$$

$$\approx \frac{m_3 K_1}{2} \times \frac{N^2+2N+15}{12} = N_2 \qquad \text{(D.131)}$$

and the averages of the second kind by

$$\frac{1}{N-4} \sum_{t=2}^{N-4} \left(t - \frac{N+1}{2}\right)\left(t - \frac{N-1}{2}\right) = \frac{N^2+2N+15}{12}$$

$$= N_1 \qquad \text{(D.132)}$$

Then,

$$\begin{aligned} \gamma_\varepsilon(0) = {} & (P^2 + Q^2 + R^2)\gamma_x(0) + 2(PQ + QR)\gamma_x(1) \\ & + PR\gamma_x(2) + S^2 N_1 + T^2 N_1 + \gamma_\xi(0) \\ & + 2(PS + QS + RS) N_2 \\ & + 2(PT + QT + RT) N_2 + 2ST N_2 \end{aligned} \qquad \text{(D.133)}$$

or,

$$\gamma_{\varepsilon}(0) = (P^2+Q^2+R^2)\,\gamma_x(0) + 2(PQ+QR)\gamma_x(1)$$
$$+PR\,\gamma_x(2) + 2(P+Q+R)(S+T)N_2$$
$$+(S+T)^2 N_1 + \gamma_{\zeta}(0) \tag{D.134}$$

Similarly,

$$\gamma_{\varepsilon}(1) = (PQ+QR)\gamma_x(0) + (P^2+Q^2+R^2+PR)\gamma_x(1)$$
$$+(PQ+QR)\,\gamma_x(2) + PR\,\gamma_x(3)$$
$$+2(P+Q+R)(S+T)N_2 + (S+T)^2 N_1 + \gamma_{\zeta}(1) \tag{D.135}$$

and

$$\gamma_{\varepsilon}(2) = PR\,\gamma_x(0) + (PQ+QR)\gamma_x(1) + (P^2+Q^2+R^2)\,\gamma_x(2)$$
$$+(PQ+QR)\gamma_x(3) + PR\,\gamma_x(4)$$
$$+2(P+Q+R)(S+T)N_2 + (S+T)^2 N_1$$
$$+\gamma_{\zeta}(2) \tag{D.136}$$

To evaluate these expressions, one needs to compute $(P^2+Q^2+R^2)$, $(P^2+Q^2+R^2+PR)$, $(PQ+QR)$, PR, $(P+Q+R)(S+T)$ and $(S+T)^2$. After some involved calculations one arrives at the following expressions (where higher order powers of β have been neglected)

$$P^2+Q^2+R^2=$$

$$= \frac{1}{(1-\delta)^2}\Big\{\beta^{*3}\left[-2g^2-2g^2\beta\right]+g^*\beta^{*3}\left[4g+8g\beta\right]+g^{*2}\beta^{*3}[-2]$$
$$+\beta^{*2}\left[-4g^2\beta+2g^2\right]+\beta^{*2}g^*\left[2g\beta-4g\right]+\beta^{*2}g^{*2}\left[2-2\beta\right]$$
$$+g^*\beta^*\left[4g\beta\right]+\beta^*g^{*2}\left[-4\beta\right]\Big\}$$

$$+\frac{\delta^{*2}}{(1-\delta)^4}\Big\{\beta^{*3}\left[2g^2-16g^2\beta\right]+g^*\beta^{*3}\left[28g\beta\right]+g^{*2}\beta^{*3}\left[2-16\beta\right]$$
$$+\beta^{*2}\left[2g^2-6g^2\beta\right]+g^*\beta^{*2}\left[-4g+4g\beta\right]+g^{*2}\beta^{*2}\left[2-6\beta\right]$$
$$+\beta^*\left[-4g^2\beta\right]+g^*\beta^*\left[8g\beta\right]+g^{*2}\beta^*\left[-4\beta\right]$$

$$-\frac{2\delta^*}{(1-\delta)^3}\Big\{\beta^{*3}\left[2g^2-18g^2\beta\right]+g^*\beta^{*3}\left[-4g+32g\beta\right]+$$
$$+g^{*2}\beta^{*3}\left[2-16\beta\right]+\beta^{*2}\left[2g^2-6g^2\beta\right]+g^*\beta^{*2}\left[-4g+11g\beta\right]$$
$$+g^{*2}\beta^{*2}\left[2-6\beta\right]+\beta^*\left[g^2\beta\right]$$
$$+g^*\beta^*\left[6g\beta\right]+g^{*2}\beta^*\left[-2\beta\right]\Big\}$$

(D.137)

$PQ + QR =$

$$= \frac{1}{(1-\delta)^2}\Big\{\beta^{*3}[4g^2-7g^2\beta] + g^*\beta^{*3}[8g\beta-8g] + g^{*2}\beta^{*3}[4-12\beta]$$
$$+\beta^{*2}[-4g^2\beta+g^2] + g^*\beta^{*2}[12g\beta-2g] + g^{*2}\beta^{*2}[1-3\beta]$$
$$+g^*\beta^*[2g\beta] + g^{*2}\beta^*[-2\beta]\Big\}$$

$$+\frac{\delta^{*2}}{(1-\delta)^4}\Big\{\beta^{*3}[3g^2-12g^2\beta] + g^*\beta^{*3}[24g\beta-6g] + g^{*2}\beta^{*3}[3-12\beta]$$
$$+\beta^{*2}[g^2-7g^2\beta] + g^*\beta^{*2}[-2g+14g\beta] + g^{*2}\beta^{*2}[1-4\beta]$$
$$+\beta^*[-2g^2\beta] + g^*\beta^*[4g\beta] + g^{*2}\beta^*[-2\beta]\Big\}$$

$$-\frac{\delta^*}{(1-\delta)^3}\Big\{\beta^{*3}[8g^2-22g^2\beta] + g^*\beta^{*3}[-16g+47g\beta] + g^{*2}\beta^{*3}[7-24\beta]$$
$$+\beta^{*2}[2g^2-15g^2\beta] + g^*\beta^{*2}[-4g+28g\beta]$$
$$+g^{*2}\beta^{*2}[1-17\beta] + \beta^*[-2g^*\beta] + g^*\beta^*[11g\beta]$$
$$+g^{*2}\beta^*[1-5\beta] \qquad \text{(D.138)}$$

$$PR =$$

$$= \frac{1}{(1-\delta)^2}\Big\{\beta^{*3}[-2g^2+3g^2\beta] + g^*\beta^{*3}[7g\beta-4g] + g^{*2}\beta^{*2}[2-4\beta]$$

$$+\beta^{*2}[g^2-4g^2\beta] + g^*\beta^{*2}[6g\beta+4g\beta-2g]$$

$$+g^{*2}\beta^{*2}[2\beta-3] + g^*\beta^*[2g\beta] + g^{*2}\beta^*[-2\beta]\Big\}$$

$$+\frac{\delta^{*2}}{(1-\delta)^4}\Big\{\beta^{*3}[2g^2-4g^2\beta] + g^*\beta^{*3}[-4g+8g\beta] + g^{*2}\beta^{*3}[2-4\beta]$$

$$+\beta^{*2}[g^2-6g^2\beta] + g^*\beta^{*2}[-2g+12g\beta]$$

$$+g^{*2}\beta^{*2}[1-6\beta] + \beta^*[-2g^2\beta]$$

$$+g^*\beta^*[4g\beta]\Big\}$$

$$-\frac{\delta^*}{(1-\delta)^3}\Big\{\beta^{*3}[-5g^2\beta+2g^2] + g^{*2}\beta^{*3}[2-4\beta] + g^*\beta^{*3}[-4g+7g\beta]$$

$$+\beta^{*2}[g^2-7g^2\beta] + g^*\beta^{*2}[-2g+15g\beta]+$$

$$+g^{*2}\beta^{*2}[2\beta-3] + \beta^*[-g^2\beta]$$

$$+g^*\beta^*[3g\beta] + g^{*2}\beta^*[-2\beta] \tag{D.139}$$

$$(P+Q+R)(S+T)$$

$$= \frac{m_\xi K_1}{2} \frac{\delta}{(1-\delta)^2} \Big\{ \beta^{*3}[13g^2 - 37g^2\beta] + g^*\beta^{*3}[-11g + 38g\beta]$$
$$+ g^{*2}\beta^{*3}[-6+7\beta] + \beta^{*2}[6g^2 - 25g^2\beta]$$
$$+ g^*\beta^{*2}[-6g + 24g\beta] + g^{*2}\beta^{*2}[3-9\beta] \Big\}$$

$$- \frac{m_\xi K_1}{2} \frac{\delta\delta^*}{(1-\delta)^3} \Big\{ \beta^{*3}[12g^2 - 39g^2\beta] + g^*\beta^{*3}[-6g + 24g\beta]$$
$$+ g^{*2}\beta^{*3}[6-18\beta] + \beta^{*2}[4g^2 - 16g^2\beta]$$
$$+ g^*\beta^{*2}[-2g + 28g\beta] + \beta^*[-5g^2\beta]$$
$$+ g^*\beta^*[13g\beta] + g^{*2}\beta^*[-5\beta] \Big\}$$

$$+ \frac{m_\xi}{2} G^* \frac{1}{(1-\delta)} \Big\{ \beta^{*3}[-g] + g^*\beta^{*3}[1] + \beta^{*2}[-3g + 2g\beta]$$
$$+ g^*\beta^{*2}[1-\beta] + \beta^*[-g^* + 3g\beta] + g^*\beta^*[5-5\beta]$$
$$+ g^*[-2\beta] \Big\}$$

$$- \frac{m_\xi}{2} G^* \frac{\delta^*}{(1-\delta)^2} \Big\{ \beta^{*3}[-g] + g^*\beta^{*3}[1] + \beta^{*2}[-3g + 3g\beta]$$
$$+ g^*\beta^{*2}[-3+3\beta] + \beta^*[-3g + 6g\beta]$$
$$+ g^*[-3\beta] + [3g\beta] \tag{D.140}$$

$$(S+T)^2 =$$

$$= \left(\frac{m_\zeta K_1}{2}\right)^2 \frac{\delta^2}{(1-\delta)^2} \Big\{ \beta^{*3}\left[12g^2 - 36g^2\beta\right] + g^*\beta^{*3}\left[-12g + 42g\beta\right]$$

$$+ g^{*2}\beta^{*3}\left[-6\beta\right] + \beta^{*2}\left[4g^2\right]$$

$$+ g^*\beta^{*2}\left[12g\beta\right] + g^{*2}\beta^{*2}\left[-12\beta\right]$$

$$+ g^*\beta^*\left[10g\beta\right] + g^{*2}\beta^*\left[-6\beta\right]\Big\}$$

$$+ \left(\frac{m_\zeta}{2}\right)^2 G^{*2}$$

$$+ 2\left(\frac{m_\zeta K_1}{2}\right)\left(\frac{m_\zeta}{2}\right)\frac{\delta}{1-\delta} G^* \Big\{ \beta^{*3}\left[-g\right] + g^*\beta^{*3}\left[1\right] + \beta^{*2}\left[3g + 3g\beta\right]$$

$$+ g^*\beta^{*2}\left[3 - 3\beta\right] + \beta^*\left[-3g + 6g\beta\right]$$

$$+ g^*\beta^*\left[3 - 6\beta\right] + g^*\left[-3\beta\right] \tag{D.141}$$

Define

$$x = \frac{g^*}{g} \tag{D.142}$$

$$y = \frac{\phi^*}{\phi} = -\frac{\beta}{1-\beta} \times \frac{\beta^*}{\beta} \tag{D.143}$$

$$z = \frac{\delta^*}{\delta} \tag{D.144}$$

$$w = \frac{G^*}{g} \tag{D.145}$$

$$P_0^{\,2} = P^2 \times \frac{1}{g^2\beta^3} \times \frac{(1-\delta)^4}{\delta^2} \tag{D.146}$$

$$Q_0^{\,2} = Q^2 \times \frac{1}{g^2\beta^3} \times \frac{(1-\delta)^4}{\delta^2} \tag{D.147}$$

$$R_0^{\,2} = R^2 \times \frac{1}{g^2\beta^3} \times \frac{(1-\delta)^4}{\delta^2} \tag{D.148}$$

$$P_0 R_0 = PR \times \frac{1}{g^2\beta^3} \times \frac{(1-\delta)^4}{\delta^2} \tag{D.149}$$

$$(S_0+T_0)^2 = (S+T)^2 \times \frac{1}{g^2\beta^3} \times \frac{(1-\delta)^4}{\delta^2} \tag{D.150}$$

$$(P_0+Q_0+R_0)(S_0+T_0)$$

$$= (P+Q+R)(S+T) \times \frac{1}{g^2\beta^2} \frac{(1-\delta)^4}{\delta^2} \tag{D.151}$$

Then the following relationships can be easily obtained.

$$P_0^2 + Q_0^2 + R_0^2$$

$$= \left(\frac{1-\delta}{\delta}\right)^2 \left\{ y^3\left[-\frac{2-6\beta}{(1-\beta)^3}\right] + y^3x\left[\frac{4+8\beta}{(1-\beta)^3}\right] + y^3x^2\left[\frac{-2}{(1-\beta)^3}\right] \right.$$

$$+ y^2\left[\frac{2-4\beta}{\beta(1-\beta)^2}\right] + y^2x\left[\frac{2\beta-4}{\beta(1-\beta)^2}\right] + y^2x^2\left[\frac{2+2\beta}{\beta(1-\beta)^2}\right]$$

$$\left. + yx\left[\frac{4}{\beta(1-\beta)}\right] + yx^2\left[\frac{-4}{\beta(1-\beta)}\right] \right\}$$

$$+ z^2 \left\{ y^3\left[\frac{2-16\beta}{(1-\beta)^3}\right] + y^3x\left[\frac{28\beta}{(1-\beta)^3}\right] + y^3x^2\left[\frac{2-16\beta}{(1-\beta)^3}\right] \right.$$

$$+ y^2\left[\frac{(2-6\beta)}{\beta(1-\beta)^2}\right] + y^2x\left[\frac{4\beta-4}{\beta(1-\beta)^2}\right] + y^2x^2\left[\frac{2-6\beta}{\beta(1-\beta)^2}\right]$$

$$\left. + y\left[-\frac{4}{\beta(1-\beta)}\right] + yx\left[\frac{8}{\beta(1-\beta)}\right] + yx^2\left[-\frac{4}{\beta(1-\beta)}\right] \right\}$$

$$- \left(\frac{1-\delta}{\delta}\right) z \left\{ y^3\left[\frac{4-36\beta}{(1-\beta)^3}\right] + y^3x\left[\frac{64\beta-8}{(1-\beta)^3}\right] + y^3x^2\left[\frac{4-32\beta}{(1-\beta)^3}\right] \right.$$

$$+ y^2\left[\frac{4-12\beta}{\beta(1-\beta)^2}\right] + y^2x\left[\frac{22\beta-8}{\beta(1-\beta)^2}\right] + y^2x^2\left[\frac{4-12\beta}{\beta(1-\beta)^2}\right]$$

$$+ y\left[\frac{2}{\beta(1-\beta)}\right] + yx\left[\frac{12}{\beta(1-\beta)}\right] + yx^2\left[-\frac{4}{\beta(1-\beta)}\right]$$

(D.152)

$P_o R_o =$

$$= \left(\frac{1-\delta}{\delta}\right)^2 \Big\{ y^3\Big[\frac{3\beta-2}{(1-\beta)^3} + y^3x\Big[\frac{7\beta-4}{(1-\beta)^3}\Big] + y^3x^2\Big[\frac{2-4\beta}{(1-\beta)^3}\Big]$$

$$+ y^2\Big[\frac{1-4\beta}{\beta(1-\beta)^2}\Big] + y^2x\Big[\frac{10\beta-2}{\beta(1-\beta)^2}\Big] + y^2x^2\Big[\frac{2\beta-3}{\beta(1-\beta)^2}\Big]$$

$$+ yx\Big[\frac{2}{\beta(1-\beta)}\Big] + yx^2\Big[-\frac{2}{\beta(1-\beta)}\Big]\Big\}$$

$$+ z^2\Big\{ y^3\Big[\frac{2-4\beta}{(1-\beta)^3}\Big] + y^3x\Big[\frac{8\beta-4}{(1-\beta)^3}\Big] + y^3x^2\Big[\frac{2-4\beta}{(1-\beta)^3}\Big]$$

$$+ y^2\Big[\frac{1-6\beta}{\beta(1-\beta)^2}\Big] + y^2x\Big[\frac{12\beta-2}{\beta(1-\beta)^2}\Big] + y^2x^2\Big[\frac{1-6\beta}{\beta(1-\beta)^2}\Big]$$

$$+ y\Big[\frac{-2}{\beta(1-\beta)}\Big] + yx\Big[\frac{4}{\beta(1-\beta)}\Big]\Big\}$$

$$-\left(\frac{1-\delta}{\delta}\right)z\Big\{ y^3\Big[\frac{2-5\beta}{(1-\beta)^3}\Big] + y^3x\Big[\frac{7\beta-4}{(1-\beta)^3}\Big] + y^3x^2\Big[\frac{2-4\beta}{(1-\beta)^3}\Big]$$

$$+ y^2\Big[\frac{1-7\beta}{\beta(1-\beta)^2}\Big] + y^2x\Big[\frac{15\beta-2}{\beta(1-\beta)^2}\Big] + y^2x^2\Big[\frac{2\beta-3}{\beta(1-\beta)^2}\Big]$$

$$+ y\Big[-\frac{1}{\beta(1-\beta)}\Big] + yx\Big[\frac{3}{\beta(1-\beta)}\Big] + yx^2\Big[-\frac{2}{\beta(1-\beta)}\Big]\Big\}$$

(D.153)

$$
\begin{aligned}
P_0 Q_0 + Q_0 R_0 &= \\
&= \left(\frac{1-\delta}{\delta}\right)^2 \left\{ y^3\left[\frac{4-7\beta}{(1-\beta)^3}\right] + y^3 x\left[\frac{8\beta-8}{(1-\beta)^3}\right] + y^3x^2\left[\frac{4-12\beta}{(1-\beta)^3}\right] \right. \\
&\qquad \left. + y^2x^2\left[\frac{1-3\beta}{\beta(1-\beta)^2}\right] + yx\left[\frac{2}{\beta(1-\beta)}\right] + yx^2\left[-\frac{2}{\beta(1-\beta)}\right] \right\} \\
&\quad + z^2 \left\{ y^3\left[\frac{3-12\beta}{(1-\beta)^3}\right] + y^3x\left[\frac{24\beta-6}{(1-\beta)^3}\right] + y^3x^2\left[\frac{3-12\beta}{(1-\beta)^3}\right] \right. \\
&\qquad + y^2\left[\frac{1-7\beta}{\beta(1-\beta)^2}\right] + y^2x\left[\frac{14\beta-2}{\beta(1-\beta)^2}\right] + y^2x^2\left[\frac{1-4\beta}{\beta(1-\beta)^2}\right] \\
&\qquad \left. + y\left[-\frac{2}{\beta(1-\beta)}\right] + yx\left[\frac{4}{\beta(1-\beta)}\right] + yx^2\left[-\frac{2}{\beta(1-\beta)}\right] \right\} \\
&\quad - \left(\frac{1-\delta}{\delta}\right) z \left\{ y^3\left[\frac{8-22\beta}{(1-\beta)^3}\right] + y^3x\left[\frac{47\beta-16}{(1-\beta)^3}\right] + y^3x^2\left[\frac{7-24\beta}{(1-\beta)^3}\right] \right. \\
&\qquad + y^2\left[\frac{2-15\beta}{\beta(1-\beta)^2}\right] + y^2x\left[\frac{28\beta-4}{\beta(1-\beta)^2}\right] + y^2x^2\left[\frac{1-17\beta}{\beta(1-\beta)^2}\right] \\
&\qquad \left. + y\left[-\frac{2}{\beta(1-\beta)}\right] + yx\left[\frac{11}{\beta(1-\beta)}\right] + yx^2\left[\frac{1-5\beta}{\beta(1-\beta)}\right] \right\}
\end{aligned}
\tag{D.154}
$$

$$
\begin{aligned}
&(P_0 + Q_0 + R_0)(S_0 + T_0) \\
&= \frac{m_\zeta K_1}{2} \frac{(1-\delta)^2}{\delta} \Big\{ y^3\Big[\frac{13-37\beta}{(1-\beta)^3}\Big] + y^3 x\Big[\frac{38\beta-11}{(1-\beta)^3}\Big] + y^3x^2\Big[\frac{7\beta-6}{(1-\beta)^3}\Big] \\
&\qquad + y^2\Big[\frac{6-25\beta}{\beta(1-\beta)^2}\Big] + y^2x\Big[\frac{24\beta-6}{\beta(1-\beta)^2}\Big] + y^2x^2\Big[\frac{3-9\beta}{\beta(1-\beta)^2}\Big]\Big\} \\
&- \frac{m_\zeta K_1}{2}(1-\delta) z \Big\{ y^3\Big[\frac{12-39\beta}{(1-\beta)^3}\Big] + y^3x\Big[\frac{24\beta-6}{(1-\beta)^3}\Big] + y^3x^2\Big[\frac{6-18\beta}{(1-\beta)^3}\Big] \\
&\qquad + y^2x\Big[\frac{28\beta-2}{\beta(1-\beta)^2}\Big] + y\Big[\frac{-5}{\beta(1-\beta)}\Big] + yx\Big[\frac{13}{\beta(1-\beta)}\Big] \\
&\qquad + yx^2\Big[\frac{-5}{\beta(1-\beta)}\Big]\Big\} \\
&+ \frac{m_\zeta}{2} \frac{(1-\delta)^3}{\delta^2} w \Big\{ y^3\Big[-\frac{1}{(1-\beta)^3}\Big] + y^3x\Big[\frac{1}{(1-\beta)^3}\Big] + y^2\Big[\frac{-3+2\beta}{\beta(1-\beta)^2}\Big] \\
&\qquad + y^2x\Big[\frac{1-\beta}{\beta(1-\beta)^2}\Big] + y\Big[\frac{3\beta-1}{\beta^2(1-\beta)}\Big] \\
&\qquad + yx\Big[\frac{5}{\beta^2}\Big] + x\Big[-\frac{2}{\beta^2}\Big]\Big\} \\
&- \frac{m_\zeta}{2} \frac{(1-\delta)^2}{\delta} zw \Big\{ y^3\Big[-\frac{1}{(1-\beta)^3}\Big] + y^3x\Big[\frac{1}{(1-\beta)^3}\Big] + y^2\Big[\frac{-3}{\beta(1-\beta)}\Big] \\
&\qquad + y^2x\Big[\frac{-3}{\beta(1-\beta)}\Big] + y\Big[\frac{6\beta-3}{\beta^2(1-\beta)}\Big] \\
&\qquad + yx\Big[\frac{3-6\beta}{\beta^2(1-\beta)}\Big] + x\Big[-\frac{3}{\beta^2}\Big] + \Big[\frac{3}{\beta^2}\Big]\Big\}
\end{aligned}
$$

(D.155)

$$
\begin{aligned}
(S_0 + T_0)^2 = \\
= \left(\frac{M\gamma K_1}{2}\right)^2 (1-\delta)^2 \Big\{ & y^3\left[\frac{12-36\beta}{(1-\beta)^3}\right] + y^3 x\left[\frac{42\beta-12}{(1-\beta)^3}\right] \\
& + y^3x^2\left[-\frac{6\beta}{(1-\beta)^3}\right] + y^2\left[-\frac{4}{\beta(1-\beta)^2}\right] \\
& + y^2 x\left[\frac{12}{(1-\beta)^2}\right] + y^2x^2\left[\frac{-12}{(1-\beta)^2}\right] \\
& + yx\left[\frac{10}{\beta(1-\beta)}\right] + yx^2\left[-\frac{6}{\beta(1-\beta)}\right] \Big\} \\
+ \left(\frac{m_\xi}{2}\right)^2 \frac{(1-\delta)^4}{\delta^2} \times \frac{1}{\beta^3} W^2 \\
+ 2\left(\frac{M\gamma K_1}{2}\right)\left(\frac{m_\xi}{2}\right) W \Big\{ & y^3\left[-\frac{1}{(1-\beta)^3}\right] + y^3 x\left[\frac{1}{(1-\beta)^3}\right] \\
& + y^2\left[\frac{-3}{\beta(1-\beta)}\right] + y^2 x\left[\frac{3}{\beta(1-\beta)}\right] \\
& + y\left[\frac{-3+6\beta}{\beta^2(1-\beta)}\right] + yx\left[\frac{3-6\beta}{\beta^2(1-\beta)}\right] \\
& + x\left[-\frac{3}{\beta^3}\right] \Big\} \qquad \text{(D.156)}
\end{aligned}
$$

and, finally,

$$P_0^2 + Q_0^2 + R_0^2 + P_0 R_0 =$$

$$= \left(\frac{1-\delta}{\delta}\right)^2 \Big\{ y^3\left[\frac{3\beta-4}{(1-\beta)^3}\right] + y^3x\left[\frac{15\beta}{(1-\beta)^3}\right] + y^3x^2\left[\frac{-4\beta}{(1-\beta)^3}\right]$$
$$+ y^2\left[\frac{3-8\beta}{\beta(1-\beta)^2}\right] + y^2x\left[\frac{12\beta-6}{\beta(1-\beta)^2}\right] + y^2x^2\left[\frac{4\beta-1}{\beta(1-\beta)^2}\right]$$
$$+ yx\left[\frac{6}{\beta(1-\beta)}\right] + yx^2\left[-\frac{6}{\beta(1-\beta)}\right]\Big\}$$

$$+ z^2\Big\{ y^3\left[\frac{4-20\beta}{(1-\beta)^3}\right] + y^3x\left[\frac{36\beta-4}{(1-\beta)^3}\right] + y^3x^2\left[\frac{4-20\beta}{(1-\beta)^3}\right]$$
$$+ y^2\left[\frac{3-12\beta}{\beta(1-\beta)^2}\right] + y^2x\left[\frac{16\beta-6}{\beta(1-\beta)^2}\right] + y^2x^2\left[\frac{3-12\beta}{\beta(1-\beta)^2}\right]$$
$$+ y\left[\frac{-6}{\beta(1-\beta)}\right] + yx\left[\frac{12}{\beta(1-\beta)}\right]\Big\}$$

$$-\left(\frac{1-\delta}{\delta}\right)z\Big\{ y^3\left[\frac{6-41\beta}{(1-\beta)^3}\right] + y^3x\left[\frac{71\beta-12}{(1-\beta)^3}\right] + y^3x^2\left[\frac{6-36\beta}{(1-\beta)^3}\right]$$
$$+ y^2\left[\frac{5-19\beta}{\beta(1-\beta)^2}\right] + y^2x\left[\frac{37\beta-10}{\beta(1-\beta)^2}\right] + y^2x^2\left[\frac{1-10\beta}{\beta(1-\beta)^2}\right]$$
$$+ y\left[\frac{1}{\beta(1-\beta)}\right] + yx\left[\frac{15}{\beta(1-\beta)}\right] + yx^2\left[\frac{-6}{\beta(1-\beta)}\right]\Big\} \qquad \text{(D.157)}$$

In the following development it will be required to normalize the covariance expressions (D.134)-(D.136) by dividing them by $\gamma_x(0)$. Since, however, the terms involving (S+T) in these expressions contain, explicitly, the number of terms N, and expressions independent of N are required, it will be necessary to calculate $\gamma_x(0)$ in terms of N. This can be achieved by using the expressions (D.105) and (D.123) for x_t and x , respectively.

From (D.123),,

$$\bar{x} = \frac{1}{N-4} \sum_{t=2}^{N-2} \left[x_1 + \frac{1}{2} m_\gamma K_1(t-2)\right]$$

$$= x_1 + \frac{1}{2} m_\gamma K_1 \frac{1}{N-4} \sum_{t=2}^{N-2} (t-2)$$

$$= x_1 + \frac{1}{2} m_\gamma K_1 \frac{N-2}{N-4} \cdot \frac{N-1}{2}$$

$$\simeq x_1 + \frac{1}{4} m_\gamma K_1 (N-1) \qquad \text{(D.158)}$$

and,

$$x_t = x_1 + \frac{1}{2} m_\gamma K_1(t-1) \qquad \text{(D.105)}$$

Therefore,

$$\tilde{x}_t = x_t - \bar{x}$$

$$= \frac{1}{4} m_\gamma K_1 \left[2t - (N+1)\right] \qquad \text{(D.159)}$$

Hence

$$\gamma_x(0) = \frac{1}{N-4}\sum_{t=2}^{N-4} \tilde{X}_t^2$$

$$= \frac{1}{N-4}\sum_{t=2}^{N-4}\left[4t^2 + (N+1)^2 - 4(N+1)t\right]\frac{1}{4} m_\zeta K_1$$

$$= \frac{1}{N-4} \cdot \frac{1}{4^2} m_\zeta^2 K_1^2\left[4\sum_{t=2}^{N-2} t^2 - 4(N+1)\sum_{t=2}^{N-2} t + (N+1)^2(N-3)\right]$$

$$= \frac{1}{N-4} \times \frac{1}{4^2} m_\zeta^2 K_1^2\Big[2 \times \frac{2(N-2)^3 + 3(N-2)^2 + (N-2)}{3}$$

$$- 2(N+1) \times (N-2)(N-1) + 4(N+1)$$

$$+ (N+1)^2(N-3)\Big]$$

$$\approx \frac{1}{16} m_\zeta^2 K_1^2 \times \frac{N^2 + 4N + 15}{3} \qquad \text{(D.160)}$$

Hence, if N is greater than, say, 100, we have from (D.132)

$$\frac{N_1}{\gamma_x(0)} = \frac{N^2 + 2N + 15}{12} \div \frac{m_\zeta^2 K_1^2 (N^2 + 4N + 15)}{48}$$

$$\approx \frac{4}{m_\zeta^2 K_1^2} \qquad \text{(D.161)}$$

Define now the normalized covariance function of the quasi-residuals by

$$\gamma_{\varepsilon}^{*}(k) = \frac{\gamma_{\varepsilon}(k) - \gamma_{\varepsilon_0}(k)}{g^2 \beta^3 \gamma_x(0)} \times \frac{(1-\delta)^4}{\delta^2} \tag{D.162}$$

and the autocorrelation function by

$$\rho_v(k) = \frac{\gamma_v(k)}{\gamma_v(0)} \tag{D.163}$$

Then,

$$\begin{aligned} \gamma_{\varepsilon}^{*}(0) = & \left(P_0^2 + Q_0^2 + R_0^2\right) + \left(P_0 Q_0 + Q_0 R_0\right) \rho_x(1) \\ & + (P_0 R_0) \rho_x(2) \\ & + 2(P_0 + Q_0 + R_0)(S_0 + T_0) \times \frac{4}{m_2^2 K_1^2} \times \frac{m_2 K_1}{2} \\ & + (S_0 + T_0)^2 \times \frac{4}{m_2^2 K_1^2} \end{aligned} \tag{D.164}$$

$$\begin{aligned} \gamma_{\varepsilon}^{*}(1) = & \left(P_0 Q_0 + Q_0 R_0\right) + \left(P_0^2 + Q_0^2 + R_0^2 + P_0 R_0\right) \rho_x(1) \\ & + \left(P_0 Q_0 + Q_0 R_0\right) \rho_x(2) + P_0 R_0 \rho_x(3) \\ & + 2(P_0 + Q_0 + R_0)(S_0 + T_0) \times \frac{4}{m_2^2 K_1^2} \times \frac{m_2 K_1}{2} \\ & + (S_0 + T_0)^2 \times \frac{4}{m_2^2 K_1^2} \end{aligned} \tag{D.165}$$

and,

$$\gamma_\varepsilon^*(2) = P_0 R_0 + (P_0 Q_0 + Q_0 R_0)\,\rho_x(1) + (P_0^2 + Q_0^2 + R_0^2)\,\rho_x(2) + (P_0 Q_0 + Q_0 R_0)\,\rho_x(3) + P_0 R_0\,\rho_x(4) + 2\,(P_0 + Q_0 + R_0) * \frac{4}{m_2^2 K^2} * \frac{m_2 K_1}{2} + (S_0 + T_0)^2 * \frac{4}{m_2^2 K_1^2} \tag{D.166}$$

Now, the autocorrelation function is a decreasing function of lag. Also, the factor involving the gain constant K is, by inspection, much smaller than either of the autocorrelations involved in the above expressions. It follows, therefore, that the covariance expressions are most sensitive to changes in parameters involved in the biggest term in these expressions. This is, of course, the first member in each expression.**

An examination of the relations (D.152)-(D.157) shows that, for a given choice of $\phi = 1-\beta$, the coefficients of the <u>odd</u> powers of any of the involved variables have not only different magnitude, but also different sign. This means that a change, in the wrong direction, of a parameter

** Except in the case of the parameter w.

does not necessarily result in the change of the covariances of lag one and two being of the same sign. In other words, it is quite possible to make a wrong change in a parameter, which would result in the decrease of the first covariance but in the increase of the second covariance. This fact has been discovered by the writer during numerous simulation studies. Only if the changes are in the right direction do the covariances of lag one and two decrease in the same sense, the covariance of lag two being smaller than the covariance of lag one. Thus, convergence to the global minimum is assured provided that the structure of the system dynamics is known, as has been assumed in this thesis. The variance expression (D.164) involves only second and third powers of the parameters and may be reduced to a simpler form so that the theory of the preceding section may be used.

For any given set of input and output readings, $\varepsilon, \delta, \beta, G, K_1, m_\gamma$, and m_ξ can be regarded as constants. Therefore, using (D.152)-(D.157) one can write the variance relation (D.164) in the form,

$$
\begin{aligned}
\gamma_{\Sigma}(0) = {} & \left(a_{01} y^3 + a_{02} y^3 x + a_{03} y^3 x^2 + a_{04} y^2 + a_{05} y^2 x \right. \\
& \left. + a_{06} y^2 x^2 + a_{07} y + a_{08} y x + a_{09} y x^2 + a_{010} x \right) \\
& + Z\left(a_{11} y^3 + a_{12} y^3 x + a_{13} y^3 x^2 + a_{14} y^2 + a_{15} y^2 x \right. \\
& \left. + a_{16} y^2 x^2 + a_{17} y + a_{18} y x + a_{19} y x^2 + a_{110} x \right) \\
& + Z^2\left(a_{21} y^3 + a_{22} y^3 x + a_{23} y^3 x^2 + a_{24} y^2 + a_{25} y^2 x \right. \\
& \left. + a_{26} y^2 x^2 + a_{27} y + a_{28} y x + a_{29} y x^2 + a_{210} x \right) \\
& + W\left(a_{31} y^3 + a_{32} y^3 x + a_{33} y^3 x^2 + a_{34} y^2 + a_{35} y^2 x \right. \\
& \left. + a_{36} y^2 x^2 + a_{37} y + a_{38} y x + a_{39} y x^2 + a_{310} x \right) \\
& + ZW\left(a_{41} + a_{42} y^3 x + a_{43} y^3 x^2 + a_{44} y^2 + a_{45} y^2 x \right. \\
& \left. + a_{46} y^2 x^2 + a_{47} y + a_{48} y x + a_{49} y x^2 + a_{410} x \right) \\
& + W^2\left(a_{51}\right)
\end{aligned}
\qquad \text{(D.167)}
$$

Therefore, when one parameter at a time is being adjusted, while the other parameters remain constant, the variance relations corresponding to such a mode of adjustment of x,y,z, and w can be obtained from (D.167) and are respectively given by

$$ {}_x\gamma_\varepsilon^*(0) = b_{10} + b_{11}x + b_{12}x^2 \tag{D.168}$$

$$ {}_y\gamma_\varepsilon^*(0) = b_{20} + b_{21}y + b_{22}y^2 + b_{23}y^3 \tag{D.169}$$

$$ {}_z\gamma_\varepsilon^*(0) = b_{30} + b_{31}z + b_{32}z^2 \tag{D.170}$$

$$ {}_w\gamma_\varepsilon^*(0) = b_{40} + b_{41}w + b_{42}w^2 \tag{D.171}$$

where $b_{42}=a_{51}$ and all other coefficients b_{ij} are functions of the coefficients a_{kl} and the other variables not being currently adjusted.

The above expressions are now in the form (D.64) or (D.65) and the theory of the preceding section can now be applied to each of the expressions (D.168) -(D.171).