THE THREE-NUCLEON PROBLEM

A Thesis presented

for the Diploma of

Imperial College of Science and Technology

by

STEPHEN PAK-YUEN LEE

Department of Mathematics,
Imperial College,
LONDON, S.W.7.

March, 1969.

ABSTRACT

The Faddeev equations for three particles is given a new basis of representation according to the group SU(3). We obtain three new results. Firstly, the Faddeev equations take the form of a coupled set of one-variable integral equations which can be reduced to a finite set using Smith's criterion of simultaneous togetherness for a three-particle system. Secondly, by using the iterated Faddeev equations for particles interacting with a Yukawa potential, we can ensure that the SU(3) kernel is L^2 or 'Hilbert-Schmidt' with only a point spectrum of boundstate-poles. Thirdly, a new approach to include spin and isospin is undertaken. With the help of Omnes's symmetric angular momentum reduction, we show how the SU(3) kernels can be evaluated in practice. The case of the three nucleons in the boundstate of the triton is treated in detail.

To my wife, Lai-Mui

PREFACE

In this thesis, group theoretical methods are used to classify the states of three nucleons. The representation offered by these states is then used in the Faddeev equation to solve the triton boundstate problem. In order to present the theory in a way uninterrupted by details of calculations, I have tried to include only results in the text. A somewhat extensive Appendix is therefore provided to cover these calculations. A particular appendix is referred to in the text by its number in squared brackets.

I am much indebted to my supervisor, Dr. S. Hochberg, for suggesting this problem and for his guidance throughout the preparation of this thesis. Further acknowledgements are due for useful discussions with Professors L. Castillejo and G.E.H. Reuter and Drs. A.R. Edmonds and T.A. Osborn. The constant encouragement I get from my wife has also been invaluable to me. My thanks are also due to Mr. R. Sibbel for proof reading of part of the manuscript and to Mrs. S.A. Thomas for typing it.

A studentship from the Henry Lester Trust is most gratefully acknowledged.

CONTENTS

		Page
ABSTRACT		2
DEDICATION		3
PREFACE		4
CONTENTS		5
CHAPTER 1.	INTRODUCTION	6
CHAPTER 2.	CLASSIFICATION OF THREE-PARTICLE STATES	10
CHAPTER 3.	DESCRIPTION OF THE COORDINATES	16
CHAPTER 4.	THE SU(3) REPRESENTATION	
	I. The SU(3) States	23
	II. Some properties of the SU(3) states	32
CHAPTER 5.	FADDEEV'S EQUATIONS IN THE SU(3) REPRESENTATION	
	I. The angular momentum reduction of Omnes	38
	II The reduced Faddeev equations	44
CHAPTER 6.	THE SPIN-ISOSPIN STATES	53
CHAPTER 7.	GENERALIZATION TO INCLUDE SPIN	64
CHAPTER 8.	CONCLUSION	79
REFERENCES		81
APPENDICES INDEX		87

CHAPTER 1 INTRODUCTION

Until only recently,[1] the application of Faddeev's theory[2] for a three-particle system was dealt with by assuming that the particles interact in pairs through non-local separable potentials.[3] One main reason for making this assumption is due to the large number of variables involved. Even Omnes's method[4] of symmetrical angular momentum reduction still leaves three integrating variables in the final equations. Osborn[5] succeeded in reducing Omnes's result for the non-iterated Faddeev equations to two variables and actually solved the equations for the idealized case of three spinless bosons interacting through a simple Yukawa potential. Although reasonable results were obtained, it required a rather complicated numerical method: in particular, the integral equations have variable limits, and it seems difficult to generalize the method to nucleons interacting with spin-isospin dependent potentials. Our aim in this thesis is to obtain solutions of Faddeev's equations by solving only one-variable integral equations so that even when spin-dependent local potentials were used, the calculation could still be performed on a medium-sized computer.

Our method proceeds by solving the Faddeev equations in the SU(3) representation of three-particle states. Classification of three-particle states has been discussed elegantly by Dragt[6] and others.[7] Simonov,[8] on the other hand, expanded the three-particle wavefunction in terms of six-dimensional surface harmonics. Using the Schrödinger equation in configuration space, he showed that for the triton boundstate problem the eigenvalue, λ^2, of the squared generalized angular momentum tensor, Λ^2, in six-dimensional space, first introduced by Smith,[9] together with another quantum number μ completely classify the harmonics. It was also shown[10] that the generalized partial wave amplitudes are only significant for small values of λ. This is reminiscent of two-particle scattering problems at low energy when only small ℓ need be considered.

This thesis has been arranged as follows. Chapter 2 gives a short account of Dragt's work, and is brief enough to introduce the notations and formulae used later. The reader is well recommended to read the original paper, especially on the group aspect of the subject. Chapter 3 is a description of the angular variables used in parametrizing S_5, the manifold of a five-dimensional sphere on which we construct irreducible representations (I.R.s) of SU(3). The construction, in differential forms on S_5, in configuration space of the two Casimir operators, Λ^2 and S, whose eigenvalues characterize an I.R. of SU(3) is undertaken in Section I of Chapter 4. We also give their eigenfunctions classified in terms of the SO(3) subgroup. We are then able to show the one-to-one correspondence between I.R.s of SU(3) and the surface harmonics on

S_5. Section II of this chapter starts with a Fourier transformation to momentum representation followed by a discussion on the orthogonality, normalization, etc. of such states. We then show how to construct an alternative set of SU(3) states which have simple transformation properties under the symmetry group of three objects, S_3, for certain values of the total angular momentum. We also give here a relation between the generalized partial wave amplitudes in the two representations, configuration and momentum. Thus we are able to draw on the results of Simonov[11] to justify, at least for the boundstate problem, that only a small number of partial wave amplitudes in momentum representation are significant. Chapter 5, Section I contains some pertinent results of Faddeev's theory and a modified Omnes angular momentum analysis. In Section II we write Faddeev's equations in the SU(3) representation. The result is already a set of coupled integral equations in one variable. We simplify them by specializing to the case of three spinless identical particles and taking $\lambda \leq 4$, we obtain for the boundstate problem, just two coupled equations.

Chapters 6 and 7 are devoted to generalization to include spin and isospin. In Chapter 6, we classify the states of three nucleons in spin-isospin space. The method used is again group theoretic: The multispinor carrying I.R.s of SU(2) is analysed by means of the symmetry group S_3. As algebraic treatment of the symmetry group is very difficult, we use the diagrammatic technique of

Young. In Chapter 7 we apply the SU(3) representation to the Faddeev equation for the boundstate wavefunction of the triton.

CHAPTER 2 CLASSIFICATION OF THREE-PARTICLE STATES

In configuration space, the state of a three-particle system can be characterized by their coordinates $\underline{r}_i$. These can be reduced in the CM system to two relative vectors $\underline{r}^{(1)}$ and $\underline{r}^{(2)}$ if we take an orthogonal transformation such that

$$(\underline{r}_1, \underline{r}_2, \underline{r}_3) \longrightarrow (\underline{r}^{(1)}, \underline{r}^{(2)}, \underline{r}^{(3)})$$

with

$$\begin{aligned} \underline{r}^{(1)} &= \frac{1}{\sqrt{2}}(\underline{r}_2 - \underline{r}_1), \\ \underline{r}^{(2)} &= \frac{1}{\sqrt{6}}(2\underline{r}_3 - \underline{r}_1 - \underline{r}_2), \\ \underline{r}^{(3)} &= \frac{1}{\sqrt{3}}(\underline{r}_1 + \underline{r}_2 + \underline{r}_3) = 0 . \end{aligned} \tag{2.1}$$

We note that $\underline{r}^{(1)}$ and $\underline{r}^{(2)}$ are in the directions of the usual relative vectors commonly used in three-particle problem. They are, however, normalized so that

$$\underline{r}^{(1)^2} + \underline{r}^{(2)^2} = \underline{r}_1^{\,2} + \underline{r}_2^{\,2} + \underline{r}_3^{\,2} = \underline{r}^2 \tag{2.2}$$

where $\underline{r} = (\underline{r}^{(1)}, \underline{r}^{(2)})$ will be treated as a six-dimensional vector. We also note that $\underline{r}^2$ is invariant under SO(6) for which the Lie algebra $\mathcal{L}_o$ is parametrized by the 15 antisymmetric 6 x 6 matrices

$$R_{ij} = |i)(j| - |j)(i| , \quad i,j = 1,\ldots 6 , \tag{2.3}$$

where $|i)$ denotes a six-component column vector in a real vector space whose i^{th} component is unity, whilst others zero. $(i|$ is the corresponding row vector. The algebra $\mathcal{L}_o$ is given by the commutation rules:

$$[R_{ij}, R_{mn}] = 0, \qquad i \neq j \neq m \neq n,$$
$$[R_{ij}, R_{jk}] = R_{ik}, \qquad (2.4)$$
$$R_{ij} = -R_{ji} .$$

We will be interested in elements of $\mathcal{L}_o$ which are stable under the transformations of S_3. The advantage of working with such a subalgebra is that operators and I.R.s constructed from it will automatically have simple symmetry properties under S_3. This is particularly useful for introducing spins and statistics into the system. The subalgebra $\mathcal{L}_1$ is nothing but U(3), the elements of which are

$$J_{ij} = R_{ij} + R_{i+3,j+3}, \qquad i,j \leq 3, \; i \neq j,$$
$$K_{ij} = R_{i,j+3} - R_{i+3,j}, \qquad i,j \leq 3 . \qquad (2.5)$$

This is nine-dimensional. If we extract from $\mathcal{L}_1$ the linear Casimir operator of U(3),

$$S = \tfrac{1}{2} \sum_{i=1}^{3} K_{ii}, \qquad (2.6)$$

then the remaining eight elements form the Lie algebra $\mathcal{L}_2$ for SU(3). For a quantum mechanical system we need a realization of $\mathcal{L}_o$, $\mathcal{L}_1$ and $\mathcal{L}_2$ as Lie algebras of Hermitian operators on three-particle state vectors. Denoting by $\underline{p} = (\underline{p}^{(1)}, \underline{p}^{(2)})$ the corresponding six-dimensional momentum vector, the quantum analogy of R_{ij} is a set of operators Λ_{ij} with the following properties:

$$[\Lambda_{ij}, \underline{r}] = iR_{ij}\underline{r} ,$$

$$[\Lambda_{ij}, \underline{p}] = iR_{ij}\underline{p} , \qquad (2.7)$$

and the Λ_{ij} are given by

$$\Lambda_{ij} = r_i p_j - r_j p_i . \qquad (2.8)$$

It is easily seen that $\underline{r}$ and $\underline{p}$ are Hermitian and canonically conjugate, that is,

$$[r_i, p_j] = i\hbar\,\delta_{ij} . \qquad (2.9)$$

The commutation rules of $\mathcal{L}_o$ for Λ_{ij} are the Hermitian analogue of (2.4):

$$[\Lambda_{jk}, \Lambda_{lm}] = 0 , \qquad j \neq k \neq l \neq m ,$$

$$[\Lambda_{jk}, \Lambda_{kl}] = -i\hbar\Lambda_{jl} , \qquad (2.10)$$

$$\Lambda_{jk} = -\Lambda_{kj} .$$

In this representation, the Λ_{ij} satisfy the bilinear identity

$$\Lambda_{ij}\Lambda_{kl} + \Lambda_{il}\Lambda_{jk} + \Lambda_{ik}\Lambda_{lj} \equiv 0 , \quad i \neq j \neq k \neq l . \qquad (2.11)$$

The quadratic Casimir operator Λ^2 for O(6) which is also the square of the grand angular momentum tensor is

$$\Lambda^2 = \tfrac{1}{2} \sum_{i,j} (\Lambda_{ij})^2 . \qquad (2.12)$$

Then, it can be easily verified the relation[4]

$$\Lambda^2 = r^2(2mT - p_r^2 + 5i\hbar r^{-1} p_r) \qquad (2.13)$$

where T and p_r are the operators for the total kinetic energy of the system and the linear momentum associated with r. In configuration space, they are of course given by

$$T = -\frac{\hbar^2}{2m}(\nabla^{(1)^2} + \nabla^{(2)^2}) = -\frac{\hbar^2}{2m} \nabla_6^2 \qquad (2.14)$$

$$p_r = -i\hbar\frac{\partial}{\partial r} \qquad (2.15)$$

We have used $\nabla^{(i)2}$ for the Laplace operator associated with $\underline{r}^{(i)}$ and ∇_6^2 for the Laplace operator in six dimensions. Notice that using the relative normalized vectors $\underline{r}^{(i)}$, we can factor out $-\hbar^2/2m$, with m the mass of each particle. For three non-interacting particles traversing straight line trajectories Λ^2 will have eigenvalues, $\lambda(\lambda+4)\hbar^2$ say, which are good quantum numbers. With $\underline{p} = \hbar\underline{k}$, we can deduce from Eqn.(2.13) that for given λ and k the minimum value of r, say r_o, is given by

$$\sqrt{\lambda(\lambda+4)} = k\, r_o \,. \qquad (2.16)$$

Hence r_o has the property of an impact parameter for a three-particle system. In passing, we will use k instead of p for the rest of this work.

The elements of $\mathcal{L}_1$ in terms of the Λ_{ij} are the quantum analogue of (2.5):

$$\begin{aligned} J_{ij} &= \Lambda_{ij} + \Lambda_{i+3,j+3}, \quad i,j \leq 3, \quad i \neq j\,, \\ K_{ij} &= \Lambda_{i,j+3} - \Lambda_{i+3,j}\,, \quad i,j \leq 3\,. \end{aligned} \qquad (2.17)$$

Using the bilinear identity (2.11), we can express Λ^2 entirely in terms of elements belonging to $\mathcal{L}_1$ for

$$\Lambda^2 = \tfrac{1}{2}\sum_{i,j}(\Lambda_{ij})^2 = \tfrac{1}{2}\sum_{i,j}(J_{ij}^2 + K_{ij}^2) + \sum_{i,j}^{3}(\Lambda_{i,j+3}\Lambda_{i+3,j} - \Lambda_{i,j}\Lambda_{i+3,j+3})$$

$$= \tfrac{1}{2}\sum_{i,j}(J_{ij}^2 + K_{ij}^2) - S^2. \qquad (2.18)$$

Therefore Λ^2 must be the quadratic Casimir operator for $\mathcal{L}_1$; together with S, their eigenvalues specify an I.R. of SU(3). Finally we give the Lie algebra of $\mathcal{L}_1$ as

$$[J_i, J_k] = i\,\varepsilon_{jkl} J_l\,,$$

$$[J_i, K_{kl}] = i\,\varepsilon_{jkm} K_{ml} + i\,\varepsilon_{jlm} K_{km}\,, \qquad (2.19)$$

$$[K_{ij}, K_{mn}] = i(\delta_{im} J_{jn} + \delta_{in} J_{jm} + \delta_{jm} J_{in} + \delta_{jn} J_{im})\,,$$

where $J_i = \frac{1}{2}\varepsilon_{ijk} J_{jk}$, and are therefore seen to be the generators for SO(3). We have chosen to decompose SU(3) in terms of this subgroup because, then, the vectors of a given I.R. will be characterized by the eigenvalues of J, J_z and possibly another cubic Casimir operator $\hat{\Omega}$ to remove any further degeneracy.(6)

CHAPTER 3 DESCRIPTION OF THE COORDINATES

Since we will be interested to construct I.R. of SU(3) on S_5 carrying representations of SO(3), it is natural to use the Euler angles[12] α, β and γ as three of the angular variables. As is by now a well-adopted procedure, we can take the three vertices of the vectors $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$ as forming a triangle with body-fixed axes $(\underline{u},\underline{v},\underline{w})$: $\underline{u},\underline{v}$ in the plane of the triangle, $\underline{w} = \underline{u} \wedge \underline{v}$. Omnes parametrized the shape of this triangle in momentum space by $|\underline{k}_i|$. In our case, in order to treat the three particles on equal footing as much as possible, we use the Dalitz-Fabri[13] coordinates r, ρ and ϕ. Consider an equilateral triangle of unit altitude (see Fig.1), with O as centroid; if we denote the distances of an interior point from the sides of the triangle by $\underline{r}_i^2/\underline{r}^2$, we see that Eqn.(2.2) is automatically satisfied. The magnitudes of the vectors $\underline{r}_i$ are then given by

$$\underline{r}_i^2 = \frac{1}{3} r^2 (1 + \rho\, \xi_i), \qquad (3.1)$$

and so

$$\underline{r}_{jk}^2 = r^2 (1 - \rho\, \xi_i) , \qquad (3.2)$$

where $\underline{r}_{jk} = (\underline{r}_k - \underline{r}_j)$,

$$\begin{aligned} \xi_1 &= \cos(\phi - \tfrac{2}{3}\pi) , \\ \xi_2 &= \cos(\phi + \tfrac{2}{3}\pi) , \qquad (3.3) \\ \xi_3 &= \cos \phi . \end{aligned}$$

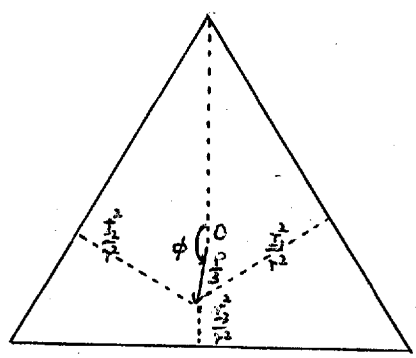


Fig.1. The Dalitz-Fabri coordinates

Now in the CM system, $\underline{r}^{(3)} = 0$. This means if the vectors $\underline{r}_i$ are to be real, the sides of the triangle (not the equilateral triangle) have to satisfy the usual triangular inequality:

$$|r_{ki} - r_{ij}| \leq r_{jk} \leq r_{ki} + r_{ij} \tag{3.4}$$

Using Eqns. (3.2) and (3.3), this condition is satisfied if $\rho^2 \leq 1$ and we therefore choose the fifth angular variable to be ψ such that

$$\rho = \cos 2\psi . \tag{3.5}$$

In keeping with our attempt to treat the three particles symmetrically, we choose the body-fixed axes $(\underline{u}, \underline{v}, \underline{w})$ as follows. Imagine unit mass at each vertex of the triangle. We take $\underline{u}$ and $\underline{v}$ to coincide with the two principal axes of inertia. In other words, we require

$$\sum_i (\underline{r}_i . \underline{u})(\underline{r}_i . \underline{v}) = 0 . \tag{3.6}$$

However, this does not define $\underline{u}$ and $\underline{v}$ uniquely for the condition does not specify the directions of $\underline{u}$ and $\underline{v}$ in space. In Zickendraht,[14] while maintaining both alternatives, the range of ϕ was taken to be $0 \leq \phi \leq 4\pi$ and a one-to-two correspondence between $\underline{r}$ and the set $(r, \psi \phi \alpha\beta\gamma)$ was obtained. We can obtain a one-to-one correspondence with the prescriptions[1]:

$$\begin{aligned} \underline{r}^{(1)} &= r(\cos\psi \sin\frac{\phi}{2}\,\underline{u} - \sin\psi \cos\frac{1}{2}\phi\,\underline{v}) , \\ \underline{r}^{(2)} &= r(\cos\psi \cos\frac{\phi}{2}\,\underline{u} + \sin\psi \sin\frac{1}{2}\phi\,\underline{v}) . \end{aligned} \tag{3.7}$$

We remark that for given r, ψ and ϕ, the last equation defines the angle ε between $\underline{r}^{(2)}$ and $\underline{u}$ unambiguously. Finally, the body-fixed axes $(\underline{u},\underline{v},\underline{w})$ are related to the space-fixed axes $(\underline{i},\underline{j},\underline{k})$ by

$$\begin{bmatrix} \underline{u} \\ \underline{v} \\ \underline{w} \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & -\sin\alpha\cos\beta\cos\gamma-\cos\alpha\sin\gamma & \sin\beta\cos\gamma \\ \cos\alpha\cos\beta\sin\gamma+\sin\alpha\cos\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\cos\alpha\cos\beta & \sin\alpha\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} . \qquad (3.8)$$

In summary, we have obtained a one-to-one correspondence between $\underline{r}$ and the set $(r,\psi\phi\alpha\beta\gamma)$ which we will denote collectively as $\underline{C}$. Given $\underline{C}$, we find $\underline{r}$ in terms of $\underline{u}$ and $\underline{v}$ by Eqn. (3.7). The actual directions in space are then given by Eqn. (3.8). The ranges of the variables in $\underline{C}$ are

$$\underline{C}, \quad \begin{array}{l} 0 \leq r \leq \infty , \\ 0 \leq \psi \leq \pi/4 , \\ 0 \leq \phi \leq 2\pi , \\ 0 \leq \alpha \leq 2\pi , \\ 0 \leq \beta \leq \pi , \\ 0 \leq \gamma \leq 2\pi , \end{array} \qquad (3.9)$$

So far, we have used $\underline{r} = (\underline{r}^{(1)},\underline{r}^{(2)})$ with $\underline{r}^{(2)}$ along $\underline{r}_3$. It will be seen that both for performing Omnes's angular momentum reduction and for the study of the symmetry properties of the functions carrying the I.R.s, we will require representations expressed in terms of $\underline{C}$ of the other two six-dimensional vectors $\underline{r}_1 = (\underline{r}_1^{(1)}, \underline{r}_1^{(2)})$ and $\underline{r}_2 = (\underline{r}_2^{(1)}, \underline{r}_2^{(2)})$, with $\underline{r}_1^{(2)}$ along $\underline{r}_1$ and $r_2^{(2)}$ along $\underline{r}_2$. Note that instead of introducing yet another symbol for the six-dimensional vector, we prefer to use the same as for the three-dimensional vector of the individual particle. As can be easily verified, $(\underline{r}^{(1)}, \underline{r}^{(2)})$ transform as the two-dimensional representation of S_3,[15] that is,

$$(12)\begin{bmatrix} \underline{r}^{(1)} \\ \underline{r}^{(2)} \end{bmatrix} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \underline{r}^{(1)} \\ \underline{r}^{(2)} \end{bmatrix} \text{, etc.} \tag{3.10}$$

Thus we find

$$\begin{bmatrix} \underline{r}_1^{(1)} \\ \underline{r}_1^{(2)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \underline{r}^{(1)} \\ \underline{r}^{(2)} \end{bmatrix},$$

and

$$\begin{bmatrix} \underline{r}_2^{(1)} \\ r_2^{(2)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \underline{r}^{(1)} \\ \underline{r}^{(2)} \end{bmatrix} \tag{3.11}$$

These together with Eqn. (3.7) allow the angles ξ_1, ξ_2 between $\underline{r}_1^{(2)}$ and

$\underline{u}$, $\underline{r}_2^{(2)}$ and $\underline{u}$ respectively to be determined.

As can be obtained by Jacobian calculations, we give here the volume element $d\underline{r} = \prod_{i=1}^{6} dr_i$ in some of the coordinates used later:

$$d\underline{r} = 3\frac{\sqrt{3}}{8}\, d\underline{\omega}\, dR$$

$$= \frac{1}{8} r^5 dr \cos 2\psi\, d(\cos 2\psi) d\phi\, dR \qquad (3.12)$$

$$= r^5 dr\, d\Omega\,(\hat{\underline{r}})$$

where

$$d\underline{\omega} = d\underline{r}_1^{\,2} d\underline{r}_2^{\,2} d\underline{r}_3^{\,2} ,$$

$$dR = d\alpha \sin\beta\, d\beta\, d\gamma , \qquad (3.13)$$

$$d\Omega(\hat{\underline{r}}) = \frac{1}{8} \cos 2\psi\, d(\cos 2\psi) d\phi\, dR .$$

We now consider the transformation properties of C under S_3. First take the internal coordinates r, ψ and ϕ. It is obvious from definitions that r and ψ are invariants while ϕ transforms as

$$
\begin{aligned}
e\phi &= \phi \ , \\
(12)\phi &= -\phi \ , \\
(23)\phi &= -\phi + \frac{4\pi}{3} \ , \\
(31)\phi &= -\phi - \frac{4\pi}{3} \ , \\
(123)\phi &= \phi + \frac{4\pi}{3} \ , \\
(132)\phi &= \phi + \frac{2\pi}{3} \ ,
\end{aligned}
\tag{3.14}
$$

where e denotes the identity element, (ij) a transposition and (ijk) a cycle. With regard to the changes in the external variables α, β and γ, we note, by definition of the body-fixed axes in Eqn.(3.7), that the transformation in ϕ and the Euler angles are coupled. (3.14) has been chosen so that under exchanges of any pair, the changes in α, β and γ are the same:[2]

$$
\begin{aligned}
\alpha &\longrightarrow \alpha \ , \\
\beta &\longrightarrow \beta - \pi \ , \\
\gamma &\longrightarrow \pi - \gamma \ .
\end{aligned}
\tag{3.15}
$$

In momentum space, there is a complete analogy with the configuration space coordinates. We use $\underline{k}$ in place of $\underline{r}$ with same meaning attached to the suffices. However, to simplify the notation, we introduce the new variables $(\underline{\eta}_i, \underline{\xi}_i)$, instead of $(\underline{k}_i^{(1)}, \underline{k}_i^{(2)})$, as the relative momentum vectors. Where no confusion can arise, we use the same symbols for the angular variables. In Chapter 4, section II, when we consider the Fourier transformation to momentum space, both coordinates will appear, then we use $\hat{\underline{r}}$, $\hat{\underline{k}}$ to denote both the six-dimensional unit vectors and their associated angular variables.

CHAPTER 4. THE SU(3) REPRESENTATION

1. The SU(3) States

We have seen in Chapter 2 that three-particle states can be classified by SU(3). Now each I.R. of SU(3) is characterized by the two Cartan indices[16] (λ_1, λ_2) and we saw that both Λ^2 and S commute with all elements of $\mathcal{L}_2$. Hence by Schur's lemma,[15] their eigenvalues, say $\lambda(\lambda+4)\hbar^2$ and $2\mu\hbar$ respectively, denote an I.R. and must be related to the Cartan indices. Indeed, it can be shown that[6]

$$\lambda = \lambda_1 + \lambda_2 \, ,$$
$$\mu = \frac{\lambda_1 - \lambda_2}{2} \, . \qquad (4.1)$$

From now on we will use (λ, μ) to denote an I.R. To obtain representations of these I.R.s as functions on S_5, we require the differential operator analogies of Λ^2 and S in our coordinates $\underline{C}$. These differential operators in other angular variables have been used before by Beg and Ruegg[17] to construct harmonic functions of SU(3) on S_5. Nelson[18] used a set of coordinates similar to ours, but he analysed the group SU(3) in terms of the usual SU(2) subgroup of unitary symmetry type in particle-physics;[19] and therefore his results are not useful to us. In principle, Λ^2 and S can be constructed from Eqns.(2.18) and (2.6). This is very easy for S but extremely tedious for Λ^2. An alternative method to obtain Λ^2 is through the Laplace-Betrami operator on the manifold S_5.[20] However, Λ^2 constructed in this way does not show up the operators of the SO(3) subgroup explicitly and hence is unsuitable for interpretation.

Using the definitions for K_{ij} and Λ_{ij}, the operator S can be expressed in terms of $\underline{r}^{(1)}$ and $\underline{r}^{(2)}$. The result is

$$S = -i\hbar\left(\underline{r}^{(1)} \cdot \frac{\partial}{\partial \underline{r}^{(2)}} - \underline{r}^{(2)} \cdot \frac{\partial}{\partial \underline{r}^{(1)}}\right) . \tag{4.2}$$

Introducing the complex vector $\underline{z}$ and its complex conjugate $\underline{z}^*$

$$\begin{aligned} \underline{z} &= \underline{r}^{(2)} + i\underline{r}^{(1)} = re^{i\frac{\phi}{2}}(\cos\psi\, \underline{u} - i\sin\psi\, \underline{v}) , \\ \underline{z}^* &= \underline{r}^{(2)} - i\underline{r}^{(1)} = re^{-i\frac{\phi}{2}}(\cos\psi\, \underline{u} + i\sin\psi\, \underline{v}) , \end{aligned} \tag{4.3}$$

we see that the simple exponential dependence on ϕ allows S to be constructed without recourse to a complete coordinate transformation.[3] Thus we find

$$S = 2i\hbar\frac{\partial}{\partial\phi} . \tag{4.4}$$

For Λ^2, we use Eqn.(2.13) which, in configuration space, is

$$\Lambda^2 = -\hbar^2 r^2 \left(\nabla_6^{\,2} - \frac{1}{r^5} \frac{\partial}{\partial r} \left(r^5 \frac{\partial}{\partial r} \right) \right) . \qquad (4.5)$$

The advantage of this approach is that when we express $\nabla_6^{\,2}$ in terms of $\underline{C}$, we can build in the angular momentum operators of SO(3). In these coordinates, $\nabla_6^{\,2}$ also separates into a part containing the Euler angles and another for the other variables. In Gallina et al.,[21] $\nabla_6^{\,2}$ for S-wave was considered. Zickendraht[14] whose method we follow shows the separation in the general $L \neq 0$ case. The coordinates used are similar to ours but the choice of the body-fixed axes is different, as discussed in the last chapter. We carry the transformation from $\underline{r}$ to $\underline{C}$ in steps. First the original frame S_o with axes $(\underline{i}, \underline{j}, \underline{k})$ is rotated by Euler angles α_1, β_1 and γ_1 to S_1 such that $\underline{z}_1$ is along $\underline{r}^{(2)}$, $\underline{x}_1$ in the plane of the triangle. This is the same rotation as Omnes. To bring the $(\underline{x}_1, \underline{y}_1)$ plane into the plane of the triangle, we rotate S_1 about $\underline{x}_1$ by $\pi/2$ so that $\underline{y}_2$ of the new frame S_2 is now along $\underline{r}^{(2)}$; the Euler angles of S_2 are α_2, β_2, and γ_2. Then we introduce the coordinates r, ψ and ϕ. Finally we rotate S_2 into S whose axes are $(\underline{u}, \underline{v}, \underline{w})$ by rotating about $\underline{z}_2$ the angle $(\frac{\pi}{2} - \epsilon)$. The Euler angles of S are, by definition, α, β and γ. Note that we have used the same numerical lable i for the frame S_i and the quantities associated with i.

For example, $\underline{z}_i$ is the z-axis of the frame S_i. The result of this transformation is that for Λ^2 we have

$$\Lambda^2 = -\left(\frac{\partial^2}{\partial\psi^2} + \frac{4\cos 4\psi}{\sin 4\psi}\frac{\partial}{\partial\psi} + \frac{4}{\cos^2 2\psi}\frac{\partial^2}{\partial\phi^2}\right) + \Big[\frac{2\underline{L}^2}{\sin^2 2\psi} +$$

$$\left(\frac{1}{\cos^2 2\psi} - \frac{2}{\sin^2 2\psi}\right) L_{\underline{w}}^2 + 4i\hbar\frac{\sin 2\psi}{\cos^2 2\psi} L_{\underline{w}}\frac{\partial}{\partial\phi} +$$

$$\frac{\cos 2\psi}{\sin^2 2\psi}\left(L_+^2 - L_-^2\right)\Big] \tag{4.6}$$

where $\underline{L} = (L_{\underline{u}}, L_{\underline{v}}, L_{\underline{w}})$ is the angular momentum operator with respect to the S frame and

$$L_{\underline{w}} = -i\hbar\frac{\partial}{\partial\gamma},$$

$$L_{\underline{\pm}} = L_{\underline{u}} \pm iL_{\underline{v}} = i\hbar e^{\mp i\gamma}\left(\frac{1}{\sin\beta}\frac{\partial}{\partial\alpha} - i\frac{\partial}{\partial\beta} \mp \cot\beta\frac{\partial}{\partial\gamma}\right). \tag{4.7}$$

This can be identified with the Laplace-Betrami operator on S_5 with same $\underline{C}$.[7] Since it is also the angular part of the six-dimensional Laplace operator, its eigenfunctions are surface harmonics $S^{\nu}_{\lambda}(\hat{\underline{r}})$ on S_5 of degree λ,[22] that is,

$$\Lambda^2 S_\lambda^\nu(\hat{\underline{r}}) = \lambda(\lambda+4)\hbar^2 S_\lambda^\nu(\hat{\underline{r}}) \qquad (4.8)$$

where ν represents the set of labels characterizing the independent surface harmonics of degree λ. The total number of such surface harmonics is

$$h(\lambda) = \frac{(\lambda+3)!(\lambda+2)}{12\lambda!} . \qquad (4.9)$$

It is easily shown that the $S_\lambda^\nu(\hat{\underline{r}})$ are also eigenfunctions of S. By definition, the surface harmonics are related to the harmonic polynomials $P_\lambda^\nu(\underline{r})$ of degree λ by

$$S_\lambda^\nu(\hat{\underline{r}}) = \frac{1}{r^\lambda} P_\lambda^\nu(\underline{r}) \qquad (4.10)$$

with

$$\nabla_6^2 P_\lambda^\nu(\underline{r}) = 0 . \qquad (4.11)$$

In terms of the complex vectors $\underline{z}$ and $\underline{z}^*$, $P_\lambda^\nu(\underline{r})$ can be written as[23]

$$P_\lambda^\nu(\underline{r}) = P_\lambda^\nu(\underline{z},\underline{z}^*) = \sum_{\substack{\sum a_i = p \\ \sum b_i = q \\ p+q=\lambda}} C_{a_1a_2a_3}^{b_1b_2b_3} (z_1^*)^{a_1} (z_2^*)^{a_2} (z_3^*)^{a_3} (z_1)^{b_1} (z_2)^{b_2} (z_3)^{b_3} , \qquad (4.12)$$

and we note that for given λ, the range of $(p-q)/2$ is

$$-\frac{\lambda}{2}, \quad -\frac{\lambda}{2}+1, \; \ldots\ldots, \; \frac{\lambda}{2}-1, \; \frac{\lambda}{2} \; .$$

The coefficients are of course determined by Eqn. (4.11). Using Eqn. (4.3) we find that the $S_\lambda^\nu(\hat{\underline{r}})$ are eigenfunctions of S with eigenvalues $\mu = (p-q)/2$. Indeed (p,q) correspond to the Cartan indices (λ_1, λ_2).

Writing the surface harmonics as $S_\lambda^{\mu,\nu}(\hat{\underline{r}})$ with ν now denoting the remaining labels, we see that on S_5 the (λ, μ) I.R. of SU(3) is carried by the surface harmonics $S_\lambda^{\mu,\nu}(\hat{\underline{r}})$. It also follows that I.R.s of SU(3) form a complete orthogonal set on S_5. From the classification of vectors belonging to a given (λ, μ) I.R., ν consists of L, M and the eigenvalues, w say, of the cubic operator $\hat{\Omega}$. However, for L = O,1 and some L ≠ O, $S_\lambda^{\mu LM}(\hat{\underline{r}})$ is multipliicity free in which case we need not consider w.[24] In any case, it is best not to require the $S_\lambda^\nu(\hat{\underline{r}})$ to be the eigenfunctions of $\hat{\Omega}$ as w is in general irrational and their eigenfunctions are difficult to be expressed in closed forms. We are satisfied if they are all the independent solutions of Eqn.(4.8). The SU(3) representation of a three-particle state in configuration space is then given by

$$\Psi_o(k^2, r, \hat{\underline{r}}) = u_\lambda(k,r)\, S_\lambda^{\mu, LM}(\hat{\underline{r}}) \; . \qquad (4.13)$$

The radial part satisfies the equation,

$$\left[\frac{1}{r^5}\frac{\partial}{\partial r}\left(r^5\frac{\partial}{\partial r}\right) - \frac{\lambda(\lambda+4)}{r^2} + k^2\right]u_\lambda(k,r) = 0 \qquad (4.14)$$

the solutions of which are $(kr)^{-2}J_{\lambda+2}(kr)$ and $(kr)^{-2}N_{\lambda+2}(kr)$ with $J_{\lambda+2}(kr)$ and $N_{\lambda+2}(kr)$ the Bessel functions of the first and second kind respectively. We require all the surface harmonics $S_\lambda^{\mu,LM}(\hat{\underline{r}})$ satisfying the eigen-equations ;

$$S\,S_\lambda^{\mu,LM}(\hat{\underline{r}}) = 2\mu\hbar\,S_\lambda^{\mu,LM}(\hat{\underline{r}}) , \qquad (4.15)$$

$$\underline{L}^2 S_\lambda^{\mu,LM}(\hat{\underline{r}}) = L(L+1)\hbar^2\,S_\lambda^{\mu,LM}(\hat{\underline{r}}) , \qquad (4.16)$$

$$L_z S_\lambda^{\mu,LM}(\hat{\underline{r}}) = M\hbar\,S_\lambda^{\mu,LM}(\hat{\underline{r}}) . \qquad (4.17)$$

By the Peter-Weyl theorem,[26] $S_\lambda^{\mu,LM}(\hat{\underline{r}})$ can be expanded in terms of matrix elements of the SO(3) subgroup, that is,

$$S_\lambda^{\mu,LM}(\hat{\underline{r}}) = \sum_K G_{\lambda\mu}^{LMK}(\psi,\phi)\,D_{MK}^L(R) \qquad (4.18)$$

where $D_{MK}^L(R) = e^{iM\alpha}d_{MK}^L(\beta)\,e^{iK\gamma}$ is the rotational matrix in the notation of Edmonds.[27] It is obvious that $S_\lambda^{\mu,LM}(\hat{\underline{r}})$ in this form satisfies Eqns.(4.16) and (4.17). If we write

$$G^{LM,K}_{\lambda\mu}(\psi,\phi) = N_{\Omega}\, g^{LM,K}_{\lambda\mu}(\psi)\, e^{-i\mu\phi} \quad , \tag{4.19}$$

Then Eqn.(4.15) is also satisfied and hence we have

$$S^{\mu,LM}_{\lambda}(\hat{\underline{r}}) = N_{\Omega}\; e^{-i\mu\phi} \sum_{K} g^{LM,K}_{\lambda\mu}(\psi)\, D^{L}_{MK}(R) \tag{4.20}$$

where N_{Ω} is a normalization constant such that

$$\int S^{*\mu,LM\kappa}_{\lambda}(\hat{\underline{r}})\, S^{\mu',L'M'\kappa'}_{\lambda'}(\hat{\underline{r}})\, d\Omega(\hat{\underline{r}}) = \delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta_{LL'}\delta_{MM'}\delta_{\kappa\kappa'} \tag{4.21}$$

with κ the parameter to remove any further degeneracy.

Using Eqn.(4.20) for $S^{\mu,LM}_{\lambda}(\hat{\underline{r}})$ in Eqn.(4.8) gives a set of coupled equations for the $g^{LM,K}_{\lambda\mu}(\psi)$:

$$\Big[\frac{\partial^2}{\partial\psi^2} + \frac{4\cos 4\psi}{\sin 4\psi}\frac{\partial}{\partial\psi} - \frac{4\mu^2}{\cos^2 2\psi} - \frac{2}{\sin^2 2\psi}(L(L+1)-K^2) - 4\mu K\frac{\sin 2\psi}{\cos^2 2\psi} + \lambda(\lambda+4)\Big]\, g^{LM,K}_{\lambda\mu}(\psi)$$

$$= \frac{\cos 2\psi}{\sin^2 2\psi}\left(C^{L}_{K+1}\, g^{LM,K+2}_{\lambda\mu}(\psi) + C^{L}_{K-1}\, g^{LM,K-2}_{\lambda\mu}(\psi)\right) \tag{4.22}$$

with

$$C^{L}_{K} = \left[(L+K+1)(L+K)(L-K)(L-K+1)\right]^{1/2} . \tag{4.23}$$

The solutions to this are difficult to obtain for general L.[28] In Zickendraht a method was devised to construct all the $G^{LM,K}_{\lambda\mu}(\psi)$ which satisfy Eqn.(4.22) for $L \leq 2$ and some for $L \geq 3$. Nevertheless the differential system provides the following useful information:

i) It does not depend on M at all and we will therefore leave it out in future.

ii) For given λ, μ and L, the $g^{L,K}_{\lambda\mu}(\psi)$ with even and odd K are coupled separately.

iii) In general for $L \geq 2$, there will be more than one solution for given λ, μ, L and K. Having found the independent solutions, we can ortho-normalize them and since a general SU(3) state is classified by the set $(k^2, \lambda\mu\, LMw)$ and there is a one-to-one correspondence (apart from the k^2) between this and a $S^{\mu,\nu}_{\lambda}(\hat{\underline{r}})$ surface harmonic, the constructed ortho-normal solutions of Eqn.(4.22) labelled by say $g^{L,K}_{\lambda\mu\kappa}(\psi)$, form surface harmonics $S^{\mu,LMK\kappa}_{\lambda}(\hat{\underline{r}})$ which span the subspace of the three-particle states on S_5 with given $(k^2, \lambda\mu LM)$ and hence must be related to the state labelled by w by an orthogonal transformation. A three-particle state can therefore be classified by the set $(k^2, \lambda\mu LM\kappa)$. Such states need not be eigenstates of the cubic operator $\hat{\Omega}$. However, the important thing is that they form a complete set for a three-particle system.

iv) Restricting to the case when κ is redundant and using Eqns.(4.21) and (4.23) it is easily shown that

$$g^{L,K}_{\lambda\mu} = g^{L,-K}_{\lambda-\mu} \quad . \tag{4.24}$$

v) By Eqn.(4.1), replacing μ by $-\mu$ means exchanging λ_1 and λ_2 which in turn implies going to the adjoint representation of a SU(3) I.R. It follows that, for given k^2, λ, L and M, the multiplicity of the states[24] with μ and $-\mu$ is the same.

We conclude this section by giving the solution for $g^{L,K}_{\lambda\mu}$ for L = O as this only is important for the boundstate problem. Eqn.(4.22) reduces to just one equation which can be solved in terms of the Jacobi polynomials giving[5]

$$S_{\lambda\mu}(\hat{\underline{r}}) = N_{\Omega} \; e^{-i\mu\phi}(\cos 2\psi)^{|\mu|} \; P^{|\mu|,0}_{\frac{1}{2}(\frac{\lambda}{2} - |\mu|)}(1-2\cos^2 2\psi) \quad . \tag{4.25}$$

II. Some Properties of the SU(3) States

In the last section, we have undertaken to construct I.R.s of SU(3) in configuration space. It is obvious from the symmetry of $\underline{p}$ and $\underline{r}$ in Eqns.(2.7), (2.8) and (2.9) that apart from the radial part, I.R.s in momentum space take exactly the same form: we only have to replace the angular variables by their momentum space counterparts. We now give an alternative method to obtain the momentum space representations by a Fourier transformation.

For notational convenience, we use Dirac's ket or bra to represent a state. Again we use ν to denote any remaining labels; a general normalized SU(3) state in configuration space therefore is

$$\langle \underline{r} | k^2, \lambda \nu \rangle = \frac{1}{\sqrt{2}} \frac{J_{\lambda+2}(kr)}{r^2} S_\lambda^\nu(\hat{\underline{r}}) . \qquad (4.26)$$

The normalization has been chosen so that orthogonality of the states takes the form

$$\langle k^2, \lambda \nu | k'^2, \lambda' \nu' \rangle = \delta(k^2 - k'^2)\, \delta_{\lambda\lambda'} \delta_{\nu\nu'} \qquad (4.27)$$

which also determines the completeness of the states to be

$$\sum_{\lambda\nu} \int |k^2, \lambda \nu\rangle \, 2k dk \, \langle k^2, \lambda \nu| = 1 . \qquad (4.28)$$

The Fourier transformation is

$$\langle \underline{k}' | k^2, \lambda \nu \rangle = \int \langle \underline{k}' | \underline{r} \rangle d\underline{r} \langle \underline{r} | k^2, \lambda \nu \rangle , \qquad (4.29)$$

where $\langle \underline{r} | \underline{k}' \rangle = (2\pi)^{-3} e^{i\underline{k}' \cdot \underline{r}}$ is the properly normalized plane wave state with the six-dimensional vectors $\underline{k}$ and $\underline{r}$. Like the partial wave decomposition of the plane wave state in two-particle problem, it can also be expanded in terms of the suitably standardized Gegenbauer polynomials, [5] $C_\lambda^2(\hat{\underline{k}} \cdot \hat{\underline{r}})$, for a six-dimensional space. These polynomials are simple generalizations of the well-known Legendre polynomials. The expansion is

$$\langle \underline{r}|\underline{k}'\rangle = \frac{1}{2\pi^3} \sum_{\lambda'} (\lambda' + 2) \frac{i^{\lambda'}}{(k'r)^2} J_{\lambda'+2}(k'r) C^2_{\lambda'}(\hat{\underline{k}}'.\hat{\underline{r}}) \quad . \qquad (4.30)$$

Then, apart from possibly an irrelevant factor of i, we have in the momentum representation the normalized state

$$\langle \underline{k}' |k^2, \lambda \nu\rangle = \sqrt{2}\, \frac{\delta(k'^2-k^2)}{k'^2} S^{\nu}_{\lambda}(\hat{\underline{k}}') \; . \qquad (4.31)$$

In deriving this, we have used the following results:(29), [5]

$$\int_0^{\infty} J_{\lambda+2}(k'r)\, J_{\lambda+2}(kr) r dr = 2\delta(k'^2-k^2) \; , \qquad (4.32)$$

$$\int C^2_{\lambda}(\hat{\underline{k}}.\hat{\underline{r}})\, S^{\nu}_{\lambda'}(\hat{\underline{r}})\, d\Omega(\hat{\underline{r}}) = \delta_{\lambda\lambda'} \frac{2\pi^3}{\lambda+2} S^{\nu}_{\lambda}(\hat{\underline{k}}) \; . \qquad (4.33)$$

In discussing symmetry properties, it is more suitable to introduce a new set of surface harmonics denoted by $S^{LM}_{\lambda\mu_i \iota k}$ where $\iota = 1,2$ and μ_i is either positive or zero and has the following meaning: For λ even,

$$\mu_1 = 3n + 1 \; ,$$
$$\mu_2 = 3n + 2 \; , \quad n = 0,1,2,\ldots \; , \qquad (4.34i)$$
$$\mu_3 = 3n;$$

and for λ odd,

$$\mu_1 = 3n + \frac{5}{2},$$
$$\mu_2 = 3n + \frac{1}{2}, \qquad n = 0,1,2 \ldots, \qquad (4.34ii)$$
$$\mu_3 = 3n + \frac{3}{2}.$$

The $S^{LM}_{\lambda\mu_i\iota\kappa}$ written shortly as $S_{\lambda\nu}$ are defined as:

for L even,

$$S^{LM}_{\lambda\mu_i 1\kappa} = \frac{i}{\sqrt{2}}(S_\lambda^{\mu_i, LM\kappa} - S_\lambda^{-\mu_i, LM\kappa}),$$
$$S^{LM}_{\lambda\mu_i 2\kappa} = \frac{1}{\sqrt{2}}(S_\lambda^{\mu_i, LM\kappa} + S_\lambda^{-\mu_i, LM\kappa}) \times \begin{cases} 1 & i \neq 1 \\ -1 & i = 1 \end{cases}; \qquad (4.35i)$$

and for L odd,

$$S^{LM}_{\lambda\mu_i 1\kappa} = \frac{1}{\sqrt{2}}(S_\lambda^{\mu_i, LM\kappa} + S_\lambda^{-\mu_i, LM\kappa}) \times \begin{cases} 1 & i \neq 1 \\ -1 & i = 1 \end{cases},$$
$$S^{LM}_{\lambda\mu_i 2\kappa} = -\frac{i}{\sqrt{2}}(S_\lambda^{\mu_i, LM\kappa} - S_\lambda^{-\mu_i, LM\kappa}). \qquad (4.35ii)$$

Note that these surface harmonics can be constructed because of remark v) after Eqn.(4.22). They are orthonormal in $(\lambda\mu_i\iota\kappa LM)$ and by construction form a complete set on S_5. By appending the radial part to them, we obtain an alternative complete set of SU(3) states. The orthogonality and completeness relations for these new states are the same as Eqns.(4.27) and (4.28) respectively with the proviso that ν now represents the set $(\mu_i \iota \kappa LM)$. From now on we will use these new states. For states with L values such that κ is redundant, we can use the symmetry properties of $\underline{C}$ and Eqn.(4.24) to show that for $i = 3$ the states $|k^2, \lambda \mu_i 1LM>$ and $|k^2, \lambda \mu_i 2LM>$ are asymmetric and symmetric respectively whereas for other i values, the pair transform as the two-dimensional representation of S_3.[6] The restriction in the L values follows from the multiplicity of states with given λ, μ_i, L and M, which means that Eqn. (4.24) is not precise enough.

To end the discussion on the symmetry properties of $S_{\lambda\nu}$, we consider spatial inversion P. Under this operation, only γ changes to $\gamma+\pi$. Hence we have

$$P\,|k^2, \lambda\nu> \;=\; (-1)^K\;|k^2, \lambda\nu> . \qquad (4.36)$$

But the spatial parity of a SU(3) state is also given by $(-1)^\lambda$ and depending on λ only.(6) Therefore, for given λ, the summation over K in $S_{\lambda\nu}$ is over either even or odd values. In particular, for L = O, K = O, only positive parity λ even states are possible.

By completeness of the states $|k^2, \lambda \mu_i\ LM\kappa>$, we can express a general three-particle interacting state $|\bar{\Psi}>$ in momentum space as

$$<\underline{k}\ |\ \bar{\Psi}> = \sum_{\lambda\nu} \int <\underline{k}|k'^2, \lambda\nu>2k'dk'<k'^2, \lambda\nu\ |\bar{\Psi}>$$

$$= \sum_{\lambda\nu} \frac{\chi_{\lambda\nu}(k)}{k^2}\ S_{\lambda\nu}(\hat{\underline{k}}) \qquad (4.37)$$

where $\chi_{\lambda\nu}(k) = \sqrt{2}<k^2, \lambda\nu|\bar{\Psi}>$ is the generalized partial wave amplitude. Analogously, in configuration space, we have $u_{\lambda\nu}(r)$ defined by

$$<\underline{r}\ |\ \bar{\Psi}> = \sum_{\lambda\nu} \frac{u_{\lambda\nu}(r)}{r^2}\ S_{\lambda\nu}(\hat{\underline{r}}) \ . \qquad (4.38)$$

It follows from Eqns.(4.29) and (4.30) (with S_{λ}^{ν} replaced by $S_{\lambda\nu}$) that the two amplitudes are related by

$$\chi_{\lambda\nu}(k) = \int J_{\lambda+2}(kr)\ u_{\lambda\nu}(r)\ r\ dr \qquad (4.39)$$

from which, with the help of Eqn.(4.32), we have

$$\int |\chi_{\lambda\nu}(k)|^2\ k\ dk = \int |u_{\lambda\nu}(r)|^2\ r\ dr\ . \qquad (4.40)$$

That is, their contributions to the normalization integral are the same, as expected.

CHAPTER 5 FADDEEV'S EQUATIONS IN SU(3) REPRESENTATION

I. The angular momentum reduction of Omnes

It has long been recognized that due to the disconnectedness of the kinematics of a three-particle system, the Lippman-Schwinger (L-S) equation has a δ-function in its kernel which persists upon iterations[30] and therefore prevents any iterated kernels to form a completely continuous integral operator in any function space.[31] Faddeev rewrote the L-S equation for the three-particle interacting state $|\Psi>$ in operator form as

$$\begin{bmatrix} |\Psi^{(1)}> \\ |\Psi^{(2)}> \\ |\Psi^{(3)}> \end{bmatrix} = \begin{bmatrix} |\Psi_o^{(1)}> \\ |\Psi_o^{(2)}> \\ |\Psi_o^{(3)}> \end{bmatrix} - \begin{bmatrix} \cdot & G_o(z)T_1(z) & G_o(z)T_1(z) \\ G_o(z)T_2(z) & \cdot & G_o(z)T_2(z) \\ G_o(z)T_3(z) & G_o(z)T_3(z) & \cdot \end{bmatrix} \begin{bmatrix} |\Psi^{(1)}> \\ |\Psi^{(2)}> \\ |\Psi^{(3)}> \end{bmatrix} , \quad (5.1)$$

with

$$|\Psi> = |\Psi_o> + \sum_{i=1}^{3} |\Psi^{(i)}> \quad (5.2)$$

where $|\Psi_o^{(i)}>$ and $|\Psi_o>$ are known asymptotic states.[2] $G_o(z)$ is the free three-particle Green's function, z a complex parameter and $T_i(z)$ is the transition operator of the i-two-particle subsystem in three-particle Hilbert space. In momentum representation, the kernel of the operator $T_i(z)$ is

$$\langle \underline{k} | T_i(z) | \underline{k}' \rangle = \delta(\underline{\xi}_i - \underline{\xi}_i{}') \langle \underline{\eta}_i | t_i(z - \underline{\xi}_i'^2) | \underline{\eta}_i{}' \rangle \qquad (5.3)$$

with

$$\langle \underline{\eta}_i | t_i(z - \underline{\xi}_i'^2) | \underline{\eta}_i{}' \rangle = t_i(\underline{\eta}_i, \underline{\eta}_i{}'; z - \underline{\xi}_i{}'^2) \qquad (5.4)$$

as the two-body transition amplitude of the i-subsystem. Because of the δ-function in $\langle \underline{k} | T_i(z) | \underline{k}' \rangle$, the matrix-operator in Eqn.(5.1) is still not completely continuous though its kernel can be bounded in space of square-integrable functions (L^2).[32] However, for particles interacting with Yukawa potential, it can be shown that the squared matrix-operator is completely continuous in L^2 for all z including the positive real axis.[33] Therefore, in contradistinction to all previous practical applications of Faddeev's equations, we use the iterated equation[34]

$$|\bar{\Psi}^{(i)}\rangle = |\tilde{\bar{\Psi}}_o^{(i)}\rangle + K^{(i,j)}(z) \; |\bar{\Psi}^{(j)}\rangle , \qquad (5.5)$$

where $|\tilde{\bar{\Psi}}_o^{(i)}\rangle = |\bar{\Psi}_o^{(i)}\rangle - \sum_{j \neq i} G_o(z) T_i(z) \; |\bar{\Psi}_o^{(j)}\rangle$ (5.6)

and $K^{(i,j)}(z)$ is the matrix of operators

$$K(z) = \begin{bmatrix} G_o T_1 G_o T_2 + G_o T_1 G_o T_3 & G_o T_1 G_o T_3 & G_o T_1 G_o T_2 \\ G_o T_2 G_o T_3 & G_o T_2 G_o T_1 + G_o T_2 G_o T_3 & G_o T_2 G_o T_1 \\ G_o T_3 G_o T_2 & G_o T_3 G_o T_1 & G_o T_3 G_o T_1 + G_o T_3 G_o T_2 \end{bmatrix} \tag{5.7}$$

Complete continuity of the operator $K^{(i,j)}(z)$ in L^2 is implied if the Schmidt norm for its kernel $\langle \underline{k} | K^{(i,j)}(z) | \underline{k}' \rangle$ exists, that is,

$$\| \langle \underline{k} | K^{(i,j)}(z) | \underline{k}' \rangle \|_s = \int | \langle \underline{k} | K^{(i,j)}(z) | \underline{k}' \rangle |^2 \, d\underline{k} \, d\underline{k}' < \infty \, . \tag{5.8}$$

The advantage of Eqn.(5.5) over the non-iterated Eqn.(5.1) is that the Fredholm theory now strictly applies; in particular, the resolvent of K when it exists is given by

$$(1-K(z))^{-1} = \frac{\Delta}{\delta} \tag{5.9}$$

where δ, Δ are the modified Fredholm determinant and first Fredholm minor.

It will be seen in the next sub-section that the introduction of SU(3) representation for the Faddeev equation (5.5) is best done through an intermediate step when the operator K(z) is first expressed in a representation diagonalised in $\underline{\omega} = (\underline{k}_1{}^2, \underline{k}_2{}^2, \underline{k}_3{}^2)$, the total angular momentum L, and its components M and K about the space- and body-fixed axes respectively. Such three-particle state is denoted the ket $|\underline{\omega}, LMK\rangle$. The same system, in a representation diagonalised in the particles' momenta can also be represented by the ket $|\underline{k}_i\rangle$ where $\underline{k}_i = (\underline{\xi}_i, \underline{\eta}_i)$ is the six-dimensional vector. For a given configuration of the three particles in momentum space, we can choose to express the state in any one of the six-dimensional vectors, $\underline{k}_i$; hence the states $|\underline{k}_i\rangle$ for $i = 1,2,$ and 3 are actually equivalent. The transformation between the states $|\underline{k}_i\rangle$ and $|\underline{\omega}, LMK\rangle$ is given by

$$\langle \underline{k}_i' | \underline{\omega}, LMK\rangle = A\delta(\underline{\omega}' - \underline{\omega}) D^L_{MK}(R') . \qquad (5.10)$$

The constant A is determined by the orthogonality condition which is chosen to be

$$\langle \underline{\omega}, LMK | \underline{\omega}', L'M'K'\rangle = \delta(\underline{\omega} - \underline{\omega}')\delta_{LL'}\delta_{MM'}\delta_{KK'} . \qquad (5.11)$$

Then A is given by

$$A^2 = \frac{2L+1}{3\sqrt{3}\pi^2} , \qquad (5.12)$$

and the completeness relation is

$$\sum_{LMK} \int |\underline{\omega}, LMK\rangle d\underline{\omega} \langle \underline{\omega}, LMK| = 1 . \qquad (5.13)$$

The operator $T_i(z)$ in this representation is

$<\underline{\omega}, LMK|T_i(z)\;|\underline{\omega}', L'M'K'>$

$$= \left(\frac{3\sqrt{3}A}{8}\right)^2 \int \delta(\underline{\omega}-\underline{\omega}^*)\delta(\underline{\omega}''-\underline{\omega}')\delta(\underline{\xi}_i^*-\underline{\xi}_i'')<\underline{\eta}_i^*|t_i(z-\underline{\xi}_i''^2)\;|\underline{\eta}_i''> D^{*L}_{MK}(R^*) \times$$

$$D^{L'}_{M'K'}(R'')d\underline{\omega}^*\; d\underline{\omega}''\; dR^*\; dR'' \qquad (5.14)$$

with
$$\underline{\xi}_i^2 = \frac{3}{2}\omega_i \;,$$

$$\underline{\eta}_i^2 = \frac{1}{2}(2\omega_j + 2\omega_k - \omega_i)\;.$$

In order to use the $\delta(\underline{\xi}_i^*-\underline{\xi}_i'')$ in evaluating the matrix element in Eqn.(5.14), we choose the coordinate frame, say $\tilde{S}_i$ (for i = 3, $\tilde{S}_3$ is same as S_1 in Chapter 4) such that the body-fixed z-axis is along $\underline{\xi}_i''$, the y-axis normal to the triangle. The component of L along the body fixed z-axis is therefore that along $\underline{\xi}_i''$. We denote it by K_i so that the new ket is $|\underline{\omega}, LMK_i>$ depending on i. The matrix element $<\underline{\omega}, LMK_i|T_i(z)|\underline{\omega}', L'M'K_i'>$ is then found to be[7]

$$<\underline{\omega}, LMK_i|T_i(z)|\underline{\omega}', L'M'K_i'> = 4\sqrt{2}\pi^2 \frac{\delta(\omega_i-\omega_i')}{\omega_i'^{1/2}}\; \delta_{LL'}\delta_{MM'}\delta_{K_iK_i'} \times$$

$$\sum_{l'} t_{i,l'}(\eta_i^2, \eta_i'^2;\; z-\xi_i'^2)\; Y_{l'K_i'}(\delta_i,0)\; Y_{l'K_i'}(\delta_i',0) \;, \qquad (5.15)$$

where $t_{i,l'}(\eta_i^2, \eta_i'^2;\; z-\xi_i'^2)$ is the l'-partial-wave off-shell transition amplitude of the i-subsystem, $Y_{lm}(\Theta,\phi)$ the spherical harmonics; δ_i, δ_i' are the angles between $\underline{\eta}_i$ and $\underline{\xi}_i$, $\underline{\eta}_i'$ and $\underline{\xi}_i'$ respectively, so that

$$\cos\delta_i = (\omega_k - \omega_i)\left[\omega_i(2\omega_i + 2\omega_k - \omega_i)\right]^{-1/2}, \qquad (5.16)$$

and $\cos\delta_i'$ defined similarly with $\underline{\omega}'$ replacing $\underline{\omega}$.

To remove the dependence on i of the state $|\underline{\omega}, LMK_i\rangle$, we carry the rotation which takes $\tilde{S}_i$ into S. In terms of Euler angles, this is

$$R(\alpha,\beta,\gamma) = R(\tfrac{\pi}{2}, -\tfrac{\pi}{2}, -\epsilon_i) \qquad (5.17)$$

recalling that ε_i is the angle between $\underline{\xi}_i$ and the body fixed x-axis. The transformation property of $|\underline{\omega} LMK_i\rangle$ under SO(3) then gives[12]

$$|\underline{\omega}, LMK\rangle = \sum_{K_i''} D^L_{KK_i''}(\tfrac{\pi}{2}, -\tfrac{\pi}{2}, -\varepsilon_i)\,|\underline{\omega}, LMK_i''\rangle \qquad (5.18)$$

Hence, we have finally

$$\langle\underline{\omega}, LMK|T_i(z)|\underline{\omega}', L'M'K'\rangle$$

$$= 4\sqrt{2}\pi^2 \frac{\delta(\omega_i - \omega_i')}{\omega_i'^{1/2}}\,\delta_{LL'}\delta_{MM'} \sum_{l'K_i''} t_{i,l'}(\eta_i^2, \eta_i'^2; z - \xi_i'^2) \times$$

$$D^{*L}_{KK_i''}(\tfrac{\pi}{2}, -\tfrac{\pi}{2}, -\varepsilon_i)\, D^L_{K'K_i''}(\tfrac{\pi}{2}, -\tfrac{\pi}{2}, -\varepsilon_i) Y_{l'K_i''}(\delta_i, 0) Y_{l'K_i''}(\delta_i', 0)\,.$$

(5.19)

II. The reduced Faddeev equations

To avoid encumbering the formulae, we consider the homogeneous equation of Eqn.(5.5). In the SU(3) representation, it reads

$$\langle k^2,\lambda\nu\,|\,\bar{\Psi}^{(i)}\rangle = \sum_{j=1}^{3}\ \sum_{\lambda'\nu'}\int \langle k^2,\lambda\nu\,|\,K^{(i,j)}(z)\,|\,k'^2,\lambda'\nu'\rangle\, 2k'\ dk'\ \langle k'^2,\lambda'\nu'\,|\,\bar{\Psi}^{(j)}\rangle$$

$$i,j = 1,2,3 \ . \qquad (5.20)$$

We recall that ν represents the set $(\mu_i \imath LM\kappa)$. Eqn.(5.20) is already a set of coupled one-variable equations in k'. The operator $K^{(i,j)}$ is usually known in the representation when $\underline{k} = (\underline{\eta}, \underline{\xi})$ is diagonalised. If we try to calculate the kernel direct from

$$\langle k^2\lambda\nu\,|\,K^{(i,j)}(z)\,|\,k'^2,\lambda'\nu'\rangle$$

$$= \int \langle k^2\lambda\nu\,|\,\underline{k}^*\rangle d\underline{k}^* \langle \underline{k}^*|\,K^{(i,j)}(z)\,|\,\underline{k}''\rangle d\underline{k}'' \langle \underline{k}''\,|\,k'^2,\lambda'\nu'\rangle\ , \qquad (5.21)$$

we find that this involves a ten-fold, non-trivial integration. To complicate matters further, the iterated kernel $\langle \underline{k}^*|\,K^{(i,j)}(z)\,|\,\underline{k}''\rangle$ itself contains a six-fold integration. Pustovalov et al.[35] have derived a complete set of surface harmonics like our $S^{LM}_{\lambda\mu_i\imath\kappa}(\hat{\underline{k}})$, but in terms of $\underline{z}$ and $\underline{z}^*$ of Eqn.(4.3) (or equivalently in terms of $\underline{k}$). While expansions of wavefunctions satisfying Schrödinger's equations in configuration space in these harmonic functions

~~have been amply~~ justified in practical calculations of three-[11] and four-[36] particle boundstate wavefunctions and binding energies, the introduction of such surface harmonics in Eqn.(5.21) involves the abovementioned integrations. Now our $S^{LM}_{\lambda\mu_i\iota\kappa}(\hat{\underline{k}})$ have the dependence on α, β and γ separated out already in $D^L_{MK}(R)$, it is natural to obtain $K^{(i,j)}(z)$ in a representation such that the SO(3) element is again separated out. Such representation is afforded by Omnes's angular momentum analysis. In place of introducing complete sets of $|\underline{k}>$ in Eqn.(5.21), we use those of $|\underline{\omega}, LMK>$ to obtain

$$<k^2,\lambda\nu|K^{(i,j)}(z)|k'^2,\lambda'\nu'>$$

$$= \sum_{L^*M^*K^*} \sum_{L''M''K''} \int <k^2,\lambda\nu|\underline{\omega}^*,L^*M^*K^*>d\underline{\omega}^*<\underline{\omega}^*,L^*M^*K^*|K^{(i,j)}(z)|\underline{\omega}'',L''M''K''>d\underline{\omega}''$$

$$<\underline{\omega}'',L''M''K''|k'^2,\lambda'\nu'> . \qquad (5.22)$$

Using Eqns.(4.35), (4.31) and (4.18) to find $<\underline{k}'|k^2,\lambda\nu>$ and Eqn.(5.10) for $<\underline{k}'|\underline{\omega},LMK>$, we have for the transformation coefficient

$$<k^2,\lambda\nu|\underline{\omega}^*,L^*M^*K^*> = \frac{3\sqrt{6}\,\pi^2 A}{2L+1}\,\delta_{LL^*}\delta_{MM^*}\delta_{KK^*}\sum_K \frac{\delta(k^2-k^{*2})}{k^{*2}}\,G^{*L,K}_{\lambda\mu_i\iota\kappa}(\psi^*,\phi^*) ,$$

$$(5.23)$$

where the $G^{L,K}_{\lambda\mu_i\iota\kappa}$ are defined in terms of the $G^{L,K}_{\lambda\mu_i\kappa}$ and $G^{L,K}_{\lambda-\mu_i\kappa}$ exactly like Eqn.(4.35) for the $S^{LM}_{\lambda\mu_i\iota\kappa}$ and in arriving at Eqn.(5.23), we have used

(3.12) and the orthogonality of the rotational matrices.

$$\int D^{*L}_{MK}(R')\, D^{L*}_{M*K*}(R')\, dR' = \frac{8\pi^2}{2L+1}\, \delta_{LL*}\delta_{MM*}\delta_{KK*} \quad (5.24)$$

By the rotational invariance of $K^{(i,j)}(z)$ under SO(3), we have

$$\langle \underline{\omega}^*, L^*M^*K^* | K^{(i,j)}(z) | \underline{\omega}'', L''M''K'' \rangle$$

$$= \delta_{L^*L''}\delta_{M^*M''} \langle \underline{\omega}^*, L^*M^*K^* | K^{(i,j)}(z) | \underline{\omega}'', L''M''K'' \rangle . \quad (5.25)$$

Thus by Eqn.(5.22) and after integrating over the δ-functions, we have for the SU(3) representation of $K^{(i,j)}(z)$

$$\langle k^2, \lambda\nu | K^{(i,j)}(z) | k'^2 \lambda'\nu' \rangle$$

$$= \frac{\pi^4 A^2}{2(2L+1)^2} \sum_{KK'} \int [\langle k^2, \psi^*\phi^*, LMK | K^{(i,j)}(z) | k'^2, \psi''\phi'', LMK' \rangle$$

$$G^{*L,K}_{\lambda\mu_i\nu\kappa}(\psi^*\phi^*)\, G^{L,K'}_{\lambda'\mu_i'\nu'\kappa'}(\psi''\phi'') k^2 k'^2] d\Delta \quad (5.26)$$

where $d\Delta = \cos 2\psi^* \, d(\cos 2\psi^*) \cos 2\psi'' \, d(\cos 2\psi'') \, d\phi^* d\phi''$ (5.27)

and we have expressed the Omnes kernel in Dalitz-coordinates to emphasize that the integrations are over the angular variables. It should be noted that, with the range of ψ given in (3.9), the kernel $\langle k^2, \psi^*\phi^*, LMK | K^{(i,j)}(z) | k'^2, \psi''\phi'', LMK' \rangle$ is always defined. By introducing complete sets of SU(3) states in Eqn.(5.8), it can easily be shown that

$$\| \langle \underline{k} | K^{(i,i)}(z) | \underline{k}' \rangle \|_s = 4 \sum_{\substack{\lambda\nu \\ \lambda'\nu'}} \| \langle k^2, \lambda\nu | K^{(i,i)}(z) | k'^2, \lambda'\nu' \rangle \|_s < \infty \, .$$

Hence

$$\| \langle k^2, \lambda\nu | K^{(i,i)}(z) | k'^2 \lambda'\nu' \rangle \|_s < \infty \quad , \tag{5.28}$$

that is, the SU(3) kernel $\langle k^2, \lambda\nu | K^{(i,i)}(z) | k'^2, \lambda'\nu' \rangle$ as an integral operator in Eqn.(5.20) is completely continuous in L^2 whence all the powerful methods of function theory can be applied to it.

For the rest of this chapter, we specialize to the case of three identical bosons interacting in pairs with a simple Yukawa potential in s-state – that is, we consider only the s-state contribution to the two-particle transition amplitudes. Thus the summation over l' in Eqn.(5.19) is reduced to just the term with $l' = 0$. We are looking at the boundstate of the system with $L = 0$ and therefore κ will be redundant. Since the state $|\bar{\Psi}\rangle$ and its components $|\bar{\Psi}^{(i)}\rangle$ must be totally symmetric with respect to all transformations of S_3, in the summation over ν' in Eqn.(5.20) we need only to include $\iota' = 2$ and the set $\{\mu_3'\}$ which also implies that the summation over λ' is over $\lambda' = 0, 4, 6, 8,$ and then even integers. With the help of the orthogonality relation of the Jacobi polynomials,[5] Eqns.(4.35i) and (4.25) we have for the properly normalized surface harmonics

$$S_{\lambda\mu_3 2}(\hat{\underline{k}}) = G_{\lambda\mu_3 2}(\psi,\phi)$$

$$= \sqrt{\frac{\lambda+2}{\pi^3}}\cos\mu_3\phi(\cos 2\psi)^{\mu_3} P^{\mu_3,0}_{\frac{1}{2}(\frac{\lambda}{2}-\mu_3)}(1-2\cos^2 2\psi) \times \begin{cases} 1 \text{ for } \mu_3 \neq 0 , \\ \frac{1}{\sqrt{2}} \text{ for } \mu_3 = 0 . \end{cases}$$

(5.29)

The SU(3) kernel of Eqn.(5.26) then simplifies to

$$\langle k^2,\lambda\mu_3|K^{(i,j)}(z)|k'^2,\lambda'\mu'_3\rangle = \frac{[(\lambda+2)(\lambda'+2)]^{1/2}}{6\sqrt{3\pi}} k^2 k'^2 \times$$

$$\times \int [\langle k^2,\psi^*\phi^*|K^{(i,j)}(z)|k'^2,\psi''\phi''\rangle \cos\mu_3\phi^* \cos\mu'_3\phi''(\cos 2\psi^*)^{\mu_3}(\cos 2\psi'')^{\mu'_3} \times$$

$$\times P^{\mu_3,0}_{\frac{1}{2}(\frac{\lambda}{2}-\mu_3)}(1-2\cos^2 2\psi^*) \times$$

$$\times P^{\mu'_3,0}_{\frac{1}{2}(\frac{\lambda'}{2}-\mu'_3)}(1-2\cos^2 2\psi'')]d\Delta \times \begin{cases} 1 \text{ for } \mu_3,\mu'_3 \neq 0 , \\ \frac{1}{\sqrt{2}} \text{ for one of } \mu_3,\mu'_3 \neq 0, \\ \frac{1}{2} \text{ for } \mu_3,\mu'_3 = 0 . \end{cases} \quad (5.30)$$

where we have left out the label $l = 2$.

To proceed further, let us just confirm a labelling convention which we have hitherto adopted implicitly: If (r,s,t) is a set of particle labels in cyclic order, we use r to denote both the odd particle associated with the vector $\underline{\xi}_r$, and the two-particle sub-system formed by particles s and t. Thus from Eqn.(5.7), a typical term of $K^{(i,j)}(z)$ in Omnes's representation is denoted by $<\underline{\omega}|G_o T_r G_o T_l|\underline{\omega}'>$ and in SU(3) representation by $<k^2,\lambda\mu_3|G_o T_r G_o T_l|k'^2,\lambda'\mu_3'>$ with r and l unequal. Now for identical particles the functional dependence of the transition amplitude of the r sub-system on $(\omega_r\,\omega_s\,\omega_t)$ should be independent of r. That is, if we denote this function by t, we should have

$$t_r(\underline{\omega},\underline{\omega}') = t(\omega_r\,\omega_s\,\omega_t,\ \omega_r'\,\omega_s'\,\omega_t') \ . \tag{5.31}$$

Hence, using Eqn.(5.19) with L = l' = O, the matrix element $<\underline{\omega}|G_o T_1 G_o T_2|\underline{\omega}'>$ is given by

$$<\underline{\omega}|G_o T_1 G_o T_2|\underline{\omega}'> = \frac{32\pi^4}{\omega_1+\omega_2+\omega_3-z}\cdot\frac{1}{(\omega_1\,\omega_2')^{1/2}}\times$$

$$\int_{(\omega_1^{1/2}-\omega_2'^{1/2})^2}^{(\omega_1^{1/2}+\omega_2'^{1/2})^2}\frac{t(\omega_1\omega_2\omega_3,\omega_1\omega_2'\omega_3'';z-\frac{3}{2}\omega_1)\,t(\omega_2'\omega_3''\omega_1,\omega_2'\omega_3'\omega_1';z-\frac{3}{2}\omega_2')\,d\omega_3''}{(\omega_1+\omega_2'-z)+\omega_3''} \tag{5.32}$$

and then it can be verified that

$$<\underline{\omega}|G_o T_r G_o T_l|\underline{\omega}'> = \begin{pmatrix}1&2&3\\ r&s&t\end{pmatrix}\begin{pmatrix}2&3&1\\ l&m&n\end{pmatrix}'<\underline{\omega}|G_o T_1 G_o T_2|\underline{\omega}'> \tag{5.33}$$

where $\binom{1\ 2\ 3}{r\ s\ t}$ and $\binom{2\ 3\ 1}{l\ m\ n}'$ are permutation operators on the labels in $\underline{\omega}$ and $\underline{\omega}'$ respectively. Eqn. (5.33) takes a particularly useful form when expressed in terms of the Dalitz-coordinates. Noting that the permutation operators belong either to the cycle (123) or (132) and by the transformation properties of ϕ, we can replace Eqn.(5.33) by

$$\langle k^2, \psi\phi | G_o T_r G_o T_l | k'^2, \psi'\phi' \rangle = \langle k^2, \psi\phi + \Theta_r | G_o T_1 G_o T_2 | k'^2, \psi'\phi' + \Theta'_l \rangle \tag{5.34}$$

with $\Theta_2 = \frac{4\pi}{3}$, $\Theta_3 = \frac{2\pi}{3}$,

$\Theta'_1 = \frac{2\pi}{3}$, $\Theta'_3 = \frac{4\pi}{3}$.

For the fact that μ_3 are multiples of three and that $\langle k^2, \psi\phi | G_o T_1 G_o T_2 | k'^2, \psi'\phi' \rangle$ is periodic in its ϕ and ϕ' dependence, we can deduce from Eqn.(5.30) the relation

$$\langle k^2, \lambda\mu_3 | G_o T_r G_o T_l | k'^2, \lambda'\mu'_3 \rangle = \langle k^2, \lambda\mu_3 | G_o T_1 G_o T_2 | k'^2, \lambda'\mu'_3 \rangle . \tag{5.35}$$

This remarkable property in the SU(3) kernel $\langle k^2, \lambda\mu_3 | K^{(i,j)}(z) | k'^2, \lambda'\mu'_3 \rangle$ allows the matrix-Faddeev equation of Eqn.(5.20) for the totally symmetric boundstate $|\Psi_s\rangle$ to simplify to just a coupled set in λ' and μ'_3; by adding up the equations for $|\Psi_s^{(i)}\rangle$, we have

$$\langle k^2, \lambda\mu_3 | \bar{\Psi}_s \rangle = 4 \sum_{\lambda'\mu_3'} \int \langle k^2, \lambda\mu_3 | G_o T_1 G_o T_2 | k'^2, \lambda'\mu_3' \rangle dk'^2 \langle k'^2, \lambda'\mu_3' | \bar{\Psi}_s \rangle . \qquad (5.36)$$

By virtually repeating the same argument leading up to Eqn.(5.28), we can show that the integral operator in Eqn.(5.36) is completely continuous in L^2. This is to be contrasted with the case when the non-iterated equation of (5.1) is used. In that case, the kernel would contain the δ-function, $\delta(\omega_i - \omega_i')$, which apart from complicating the evaluation of the kernel itself would also produce the same misgivings as the δ-function in the original L-S equation.

We now bring in the only approximation in the theory in stating that only small λ' need be considered in Eqn.(5.20). This question was first discussed by Smith.[9] For the triton problem,[10] it has been shown that for all pair potentials which, for small r_{ij}, can be expanded as

$$V(r_{ij}) = \frac{a_{-1}}{r_{ij}} + a_o + a_1 r_{ij} + a_2 (r_{ij})^2 + \dots , \qquad (5.37)$$

the partial wave amplitudes $u_{\lambda\mu_i}(r)$ in configuration space satisfy the following estimates:

$$\begin{aligned} u_{2,1}(r)_{max} &\leq 9\% \; u_{o,o}(r) , \\ u_{4,0}(r)_{max} &\leq 6\% \; u_{o,o}(r) , \\ u_{6,3}(r)_{max}, u_{4,2}(r)_{max} &\leq 1\% \; u_{o,o}(r) , \\ u_{\lambda,\mu_i}(r)_{max} &\simeq \frac{1}{\lambda^{7/2}} \; u_{o,o}(r), \quad \lambda+2 \gg 1 . \end{aligned} \qquad (5.38)$$

Thus by Eqn.(4.40) and the remark following it, we can to a good approximation consider only those $\chi_{\lambda\mu_3}$ with $\lambda \leq 4$. For our totally symmetric boundstate $|\tilde{\Psi}_s\rangle$, Eqn.(5.36) becomes just two coupled equations for $\chi_{o,o}(k)$ and $\chi_{4,o}(k)$ which can then be solved and used in Eqn.(4.37) to construct the wavefunction in momentum space.

CHAPTER 6 THE SPIN-ISOSPIN STATES

We wish to construct, in spin-isospin space, all the possible states of three nucleons corresponding to given total spin, isospin (S, I) and their z-components (S_z, I_z). Moreover, we require the states to have definite symmetry properties with respect to S_3. Since the spin and isospin states can be treated analogously, we confine ourselves first to the spin states of the system. We follow the same approach as in previous chapters for the spatial classification and endeavour to construct the states by group methods. This means, in the first place, deciding the group with respect to which the system is invariant and then to find its irreducible representations. In this connection, we have used many results on the symmetry group S_3; a detail discussion of these can be found in Chapters 7, 10and 11 of Hamermesh.[15]

Let us represent the spin state of a nucleon ($S=\frac{1}{2}$) as a two-component spinor

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6.1}$$

with x_1, x_2 representing the spinor with $S_z = \frac{1}{2}, -\frac{1}{2}$ respectively. The spinor is normalized such that

$$\sum_{i=1}^{2} |x_i|^2 = 1 . \tag{6.2}$$

If the basis vectors x_i are subjected to a unitary transformation so that

$$x_i' = u_{ij} x_j , \tag{6.3}$$

we obtain another basis for the same spinor space. This unitary transformation can be made unimodular by taking out a phase factor and therefore we may regard the spinor space as providing an I.R. of the group, carried by the 2x2 unimodular unitary matrices, which is SU(2); and a nucleon state with spin $\frac{1}{2}$ is invariant with respect to it.

For a three-nucleon system, the spin space is spanned by the components of the 3-rank tensor (or a multispinor of rank 3)

$$F_{i_1 i_2 i_3} = x_{i_1}^{(1)} x_{i_2}^{(2)} x_{i_3}^{(3)} \tag{6.4}$$

where, for example, $x_{i_2}^{(2)}$ is the spinor representation of the second nucleon. This tensor is, of course, to be defined with respect to SU(2), that is

$$F'_{i_1 i_2 i_3} = u_{i_1 j_1} u_{i_2 j_2} u_{i_3 j_3} F_{j_1 j_2 j_3} , \tag{6.5i}$$

or symbolically,

$$F'_{(i)} = u_{(i)(j)} F_{(j)} \qquad (6.5ii)$$

This 2^3 dimension tensor space obviously provides a representation of SU(2). However, what we are interested in are the I.R.s of SU(2) carried in this tensor space. This means resolving the tensor $F_{i_1 i_2 i_3}$ into component-tensors which transform irreducibly under the group. Such a resolution is achieved through the commutating property of the transformation ($\underline{u} \times \underline{u} \times \underline{u}$) with the index-permutation of S_3, which is defined as follows: Let **p** be the permutation, $p = \begin{pmatrix} 1 & 2 & 3 \\ 1' & 2' & 3' \end{pmatrix}$, which when operating on the tensor $F_{i_1 i_2 i_3}$ produces another tensor pF such that

$$(pF)_{i_1 i_2 i_3} = F_{i_{1'} i_{2'} i_{3'}} \qquad (6.6i)$$

or symbolically

$$pF = F_{p(i)} \ . \qquad (6.6ii)$$

Now consider the effect of p on a transformed tensor F' :

$$(pF')_{(i)} = F'_{p(i)}$$

$$= u_{p(i)p(j)} F_{p(j)}$$

$$= u_{p(i)p(j)} (pF)_{(j)} \ .$$

The product $u_{p(i)p(j)}$ is bisymmetric and therefore when the same permutation is applied to (i) and (j), the product

$$u_{p(i)p(j)} = u_{(i)(j)}$$

Thus, we have

$$(pF')_{(i)} = u_{(i)(j)}(pF)_{(j)} \; . \qquad (6.7)$$

Hence, tensors of a particular symmetry transform among themselves under the transformation defined by Eqn.(6.5). The problem of resolving a tensor into irreducible tensors with respect to SU(2) is reduced to resolving it into tensors of definite symmetry with respect to S_3.

The I.R.s of S_3 in the regular representation can be found by forming the outer-product of three one-dimensional representations of each object. By the Young tableau method, this gives

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 3\\\hline\end{array} = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \oplus \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \qquad (6.8)$$

with $\begin{array}{|c|c|c|}\hline \;&\;&\;\\\hline\end{array}$, $\begin{array}{|c|}\hline \;\\\hline \;\\\hline \;\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline \;&\;\\\hline \;\\\cline{1-1}\end{array}$ the Young patterns (Y.P.) denoting the totally symmetric, totally asymmetric and the two-dimensional (mixed symmetric) representations respectively. A function of three objects is said to have a definite symmetry property if it is a basis function of an I.R. of S_3 in the regular representation.

The Pierce resolution of the identity element

$$e = S + A + Y + Y' \tag{6.9i}$$

provides the four idempotent operators:

$$S = \frac{1}{6} \sum_R R \,,$$

$$A = \frac{1}{6} \sum_R \delta_R R \,,$$

$$Y = \frac{1}{3} \left[e - (13) \right] \left[e + (12) \right] \,, \tag{6.9ii}$$

$$Y' = \frac{1}{3} \left[e - (12) \right] \left[e + (13) \right] \,,$$

where R is an element of S_3 and δ_R is the parity of the permutation. When these operate on any function of three objects they produce basis functions for the symmetry, asymmetric and the two-dimensional representations. We must note, however, that the basis functions generated by Y and Y' need not belong to the same two-dimensional representation. Using Eqn.(6.9) to resolve the tensor $F_{i_1 i_2 i_3}$, we have

$$F_{i_1 i_2 i_3} = F_{\begin{array}{|c|c|c|}\hline i_1 & i_2 & i_3 \\ \hline \end{array}} + F_{\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline i_3 \\ \hline \end{array}} + F_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 \\ \cline{1-1} \end{array}} + F_{\begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline i_2 \\ \cline{1-1} \end{array}} \tag{6.10}$$

where

$$F_{\begin{array}{|c|c|c|}\hline i_1 & i_2 & i_3 \\ \hline\end{array}} = \frac{1}{6} \sum_R R\, F_{i_1 i_2 i_3} \, ,$$

$$F_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}} = YF_{i_1 i_2 i_3} = F_{i_1 i_2 i_3} + F_{i_2 i_1 i_3} - F_{i_3 i_2 i_1} - F_{i_2 i_3 i_1} \, , \qquad (6.11)$$

$$F_{\begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline i_2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}} = Y' F_{i_1 i_2 i_3} = F_{i_1 i_2 i_3} - F_{i_2 i_1 i_3} + F_{i_3 i_2 i_1} - F_{i_3 i_1 i_2} \, ,$$

and

$$F_{\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline i_3 \\ \hline\end{array}} = AF_{i_1 i_2 i_3} = 0$$

because A anti-symmetries the indices i_1, i_2 and i_3. This cannot be done since there are only two values for each index and any three must have two equal indices. It can easily be checked that the first tensor in Eqn.(6.11) has four independent components while the other two each have two independent components. Thus for $F_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}}$, we have

$$F_{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}} = -F_{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}} = \frac{1}{3}(2F_{112} - F_{211} - F_{121}) ,$$

$$F_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}} = -F_{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}} = \frac{1}{3}(F_{122} + F_{212} - 2F_{221}) , \qquad (6.12)$$

and for $F_{\begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline i_2 \\ \cline{1-1}\end{array}}$:

$$F_{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}} = -F_{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}} = \frac{1}{3}(2F_{121} - F_{211} - F_{112}) ,$$

$$F_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}} = -F_{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}} = \frac{1}{3}(F_{122} + F_{221} - 2F_{212}) . \qquad (6.13)$$

We recall that the spinor of Eqn.(6.1) provides an I.R. for SU(2) and the $D^{(\frac{1}{2})}$ representation of SO(3). Now we wish to know which representations $D^{(J)}$ of SO(3) are contained in the irreducible tensors of Eqn.(6.11). The answer is very simple for SU(2): Since the Young pattern $\square$ for one spinor has $J = \frac{1}{2}$, the outer-product of two spinors $\square \otimes \square = \begin{array}{|c|}\hline ~ \\ \hline ~ \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline ~ & ~ \\ \hline\end{array}$ have, by vector addition, $J = 0, 1$. But the 2-rank tensor $\begin{array}{|c|}\hline ~ \\ \hline ~ \\ \hline\end{array}$ is asymmetric and therefore has $J = 0$ while $\begin{array}{|c|c|}\hline ~ & ~ \\ \hline\end{array}$ has $J = 1$. Next, from the product $\begin{array}{|c|}\hline ~ \\ \hline ~ \\ \hline\end{array} \otimes \square = \begin{array}{|c|c|}\hline ~ & ~ \\ \hline ~ \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline ~ \\ \hline ~ \\ \hline ~ \\ \hline\end{array}$ and the fact that for SU(2) the tensor with $\begin{array}{|c|}\hline ~ \\ \hline ~ \\ \hline ~ \\ \hline\end{array}$ is absent, we find that $\begin{array}{|c|c|}\hline ~ & ~ \\ \hline ~ \\ \cline{1-1}\end{array}$ has $J = \frac{1}{2}$. Finally, the product $\begin{array}{|c|c|}\hline ~ & ~ \\ \hline\end{array} \otimes \square = \begin{array}{|c|c|}\hline ~ & ~ \\ \hline ~ \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline ~ & ~ & ~ \\ \hline\end{array}$

implies that the right hand side has $J = \frac{1}{2}$ and $J = 3/2$; and since $\begin{array}{|c|c|}\hline & \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}$ has $J = \frac{1}{2}$, it follows that $\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}$ has $J = 3/2$. Indeed, it is easily seen that the two independent components of $F_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline \multicolumn{1}{|c|}{i_3} \\ \cline{1-1}\end{array}}$, say, have $S_z = \frac{1}{2}$ and $S_z = -\frac{1}{2}$ and thus form an equivalent spinor space on which the elements of SO(3) are represented by the Pauli matrices.

So far, we have used the elements of S_3 as index-permutations on a tensor to give irreducible tensors and tensor components whose SO(3) content is known. For example, the states with $S = S_z = \frac{1}{2}$ are carried by the $(i_1, i_2, i_3) = (1,1,2)$ components of $F_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline \multicolumn{1}{|c|}{i_3} \\ \cline{1-1}\end{array}}$ and $F_{\begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline \multicolumn{1}{|c|}{i_2} \\ \cline{1-1}\end{array}}$. These two states are orthogonal because they are constructed by operating on the tensor $F_{i_1 i_2 i_3}$ with Y and Y' which are themselves orthogonal. Since permutations of particle labels of a state cannot change its S and S_z, these two states must form the basis functions for the two-dimensional representation. In fact, by a change of base, the properly normalized basis functions

$$\Theta_1 = \frac{1}{\sqrt{2}}\left(F_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline \multicolumn{1}{|c|}{i_3} \\ \cline{1-1}\end{array}} + 2F_{\begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline \multicolumn{1}{|c|}{i_2} \\ \cline{1-1}\end{array}}\right) , \qquad (6.14)$$

$$\Theta_2 = \sqrt{\frac{3}{2}}\, F_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline \multicolumn{1}{|c|}{i_3} \\ \cline{1-1}\end{array}}$$

provide the two-dimensional representation on which the elements of S_3 are represented by the Yamanouchi matrices and we can identify the basis functions Θ_1 and Θ_2 with the Yamanouchi-symbols (Y-symbol) [121] , and [211] respectively.

Analogous considerations can be used to obtain isospin states with $I = \frac{1}{2}$, $I_z = -\frac{1}{2}$ and which transform as the Yamanouchi basis functions in the two-dimensional representation. They are

$$\pi_1 = \frac{1}{\sqrt{2}}\left(G_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 \\ \cline{1-1}\end{array}} + 2G_{\begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline i_2 \\ \cline{1-1}\end{array}}\right) \tag{6.15}$$

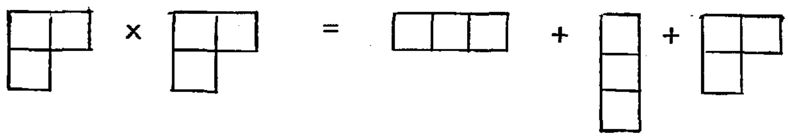
$$\pi_2 = \sqrt{\frac{3}{2}}\, G_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 \\ \cline{1-1}\end{array}}$$

with $(i_1, i_2, i_3) = (1,2,2)$ and $G_{i_1 i_2 i_3} = y^{(1)}_{i_1}\, y^{(2)}_{i_2}\, y^{(3)}_{i_3}$ is the analogue of $F_{i_1 i_2 i_3}$ with $\underline{y}$ the isospinor.

We are now in a position to construct spin and isospin functions for three nucleons with definite symmetry properties. For the triton boundstate problem, we may restrict ourselves to those states with $S = \frac{1}{2}$, $S_z = \frac{1}{2}$ and $I = \frac{1}{2}$, $I_z = -\frac{1}{2}$. The product spin-isospin space is spanned by the four basis functions

$$\Theta_i \pi_j \quad , \qquad i,j = 1 \text{ or } 2 \ , \tag{6.16}$$

and the I.R.s of S_3 contained in this is given by the inner product of the constituent two-dimensional representations :



(6.17)

The four functions, $\Theta_i \pi_j$, must therefore form the basis for one symmetric, one asymmetric and one two-dimensional representations. In fact, by a basis transformation $\underline{B}$, we can obtain a new set of four basis functions

$$\underline{\zeta} = \begin{bmatrix} \zeta_s \\ \zeta_a \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \underline{B} \begin{bmatrix} \Theta_1\pi_1 \\ \Theta_1\pi_2 \\ \Theta_2\pi_1 \\ \Theta_2\pi_2 \end{bmatrix} \tag{6.18}$$

which transform under S_3 as

$$(12) \begin{bmatrix} \zeta_s \\ \zeta_a \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \zeta_s \\ \zeta_a \\ \zeta_1 \\ \zeta_2 \end{bmatrix} , \quad (13) \begin{bmatrix} \zeta_s \\ \zeta_a \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ & & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \zeta_s \\ \zeta_a \\ \zeta_1 \\ \zeta_2 \end{bmatrix} , \text{ etc...} \tag{6.19}$$

Eqn.(6.19) and the transformation properties of the Θ_i and π_i actually determine $\underline{B}$. Thus we find

$$\underline{B} = c \begin{bmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & 1 & -1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & -1 \end{bmatrix} \qquad (6.20)$$

where c is an arbitrary constant which is fixed by normalization of the states.

In summary, we list below the spin-isospin functions of those states with $S = S_z = \frac{1}{2}$, $I = -I_z = \frac{1}{2}$ and their corresponding Y-symbols to indicate their transformation properties under S_3:

$$\begin{aligned}
\zeta_s &= \frac{1}{\sqrt{2}}(\Theta_1\pi_1 + \Theta_2\pi_2) , && [1\ 1\ 1] \ ; \\
\zeta_a &= \frac{1}{\sqrt{2}}(\Theta_1\pi_2 - \Theta_2\pi_1) , && [3\ 2\ 1] \ ; \\
\zeta_1 &= \frac{1}{\sqrt{2}}(\Theta_1\pi_2 + \Theta_2\pi_1) , && [1\ 2\ 1] \ ; \\
\zeta_2 &= \frac{1}{\sqrt{2}}(\Theta_1\pi_1 - \Theta_2\pi_2) , && [2\ 1\ 1] \ .
\end{aligned} \qquad (6.21)$$

CHAPTER 7 GENERALIZATION TO INCLUDE SPIN

In this chapter we wish to obtain, in the SU(3) representation, the Faddeev equation for the boundstate wavefunction of three nucleons interacting in pairs with more realistic spin-dependent potentials. To this end, we must construct in the product-space of spin-isospin (hereafter referred to as spin space) and momentum space the form of the complete ket vector $|\Psi\rangle$ of the system satisfying Pauli's Principle. Also, we require a generalization of the Faddeev equation to include spin.

The physical system we have in mind is the triton.[37] According to the charge independence of nuclear forces, the triton has three possible $^2S_{\frac{1}{2}}$ ($L = 0$, $S = J = \frac{1}{2}$) states: The dominant state that is fully symmetric in the space coordinates of all three nucleons, a state that is asymmetric in the interchange of space coordinates of any pair of nucleons, and a state of mixed symmetry. The other states present in the boundstate wavefunction are the three $^2P_{\frac{1}{2}}$ states, the $^4P_{\frac{1}{2}}$ state and the three $^4D_{\frac{1}{2}}$ states. There is reason to believe that the P-states are not present to any appreciable extent and that the D-states have a total probability of only a few percent. We will, therefore, consider only the $^2S_{\frac{1}{2}}$ states in the subsequent discussion.

The triton has isospin, $I = \frac{1}{2}$, and isobaric z-component, $I = -\frac{1}{2}$. The z-component of the spin, S_z, is arbitrary, so we can take $S_z = \frac{1}{2}$. Hence the spin-isospin states (hereafter referred to as spin states) of our triton are the four given in Eqn.(6.21).

We introduce a new notation $\zeta(\nu)$, $\nu = 1,2,3,4$, for the spin-states. These are defined as follows:

$$\begin{aligned} \zeta(1) &= \zeta_s , \\ \zeta(2) &= \zeta_a , \\ \zeta(3) &= \zeta_1 , \\ \zeta(4) &= \zeta_2 , \end{aligned} \tag{7.1}$$

Then, the most general state of the system is

$$|\Psi> \;=\; \sum_{\nu=1}^{4} |\Psi(\nu)>\zeta(\nu) \tag{7.2}$$

where the kets $|\Psi(\nu)>$ are, as yet, arbitrary and may be regarded simply as expansion coefficients of a vector in the four-dimensional spin space of the $\zeta(\nu)$s. Let us denote by $|\Psi_s>$ the completely symmetric, $|\Psi_a>$ the asymmetric spatial kets and by $|\Psi_1>$, $|\Psi_2>$ the spatial kets of mixed symmetry (they transform under S_3 like ζ_1 and ζ_2). The Pauli Principle, which requires the complete ket $|\Psi>$ to be fully asymmetric in exchanges of all the coordinates (spin, isospin and space) of any pair of nucleons, specifies the symmetry properties of the kets $|\Psi(\nu)>$ as follows:

$$|\bar{\Psi}(1)\rangle = |\bar{\Psi}_a\rangle, \qquad |\bar{\Psi}(2)\rangle = |\bar{\Psi}_s\rangle,$$
$$|\bar{\Psi}(3)\rangle = |\bar{\Psi}_2\rangle, \qquad |\bar{\Psi}(4)\rangle = -|\bar{\Psi}_1\rangle. \tag{7.3}$$

We wish to inquire how the kets $|\bar{\Psi}_s\rangle$, $|\bar{\Psi}_a\rangle$, $|\bar{\Psi}_1\rangle$ and $|\bar{\Psi}_2\rangle$ are represented in the SU(3) representation. Since L = O, only the states $|k^2_, \lambda\mu_i\rangle$ are required to form a basis of representation for the spatial coordinates of our system. Furthermore, with the symmetry properties of these states already known, it is easy to deduce that the necessary condition for the kets $|\bar{\Psi}_s\rangle$, $|\bar{\Psi}_a\rangle$ etc. to have the required symmetry properties in the SU(3) representation is for each ket to be represented in terms of SU(3) states of the same symmetry type only. Thus we have

$$|\bar{\Psi}_a\rangle = \sum_{\lambda\mu_3}\int |k^2_,\lambda\mu_3 1\rangle dk^2\langle k^2,\lambda\mu_3 1|\bar{\Psi}_a\rangle,$$

$$|\bar{\Psi}_s\rangle = \sum_{\lambda\mu_3}\int |k^2_,\lambda\mu_3 2\rangle dk^2\langle k^2,\lambda\mu_3 2|\bar{\Psi}_s\rangle,$$

and (7.4)

$$|\bar{\Psi}_1\rangle = \sum_{\lambda\mu_i}\int |k^2_,\lambda\mu_i 1\rangle dk^2\langle k^2,\lambda\mu_i 1|\bar{\Psi}_1\rangle,$$

$$i \neq 3.$$

$$|\bar{\Psi}_2\rangle = \sum_{\lambda\mu_i}\int |k^2_,\lambda\mu_i 2\rangle dk^2\langle k^2,\lambda\mu_i 2|\bar{\Psi}_2\rangle,$$

The condition is moreover sufficient for $|\bar{\Psi}_a\rangle$ and $|\bar{\Psi}_s\rangle$. For $|\bar{\Psi}_1\rangle$ and $|\bar{\Psi}_2\rangle$, however, sufficiency is only guaranteed if further

$$\langle k^2, \lambda\mu_i 1 | \bar{\Psi}_1 \rangle = \langle k^2, \lambda\mu_i 2 | \bar{\Psi}_2 \rangle , \qquad i \neq 3 . \tag{7.5}$$

In anticipation of using Faddeev's equations to obtain the state $|\bar{\Psi}\rangle$, let us decompose each of the "coefficients" or the "partial-wave amplitudes" of the SU(3) states in Eqn.(7.4) into three components:

$$\begin{aligned}
\langle k^2, \lambda\mu_3 1 | \bar{\Psi}_a \rangle &= \sum_{i=1}^{3} \langle k^2, \lambda\mu_3 1 | \bar{\Psi}_a^{(i)} \rangle , \\
\langle k^2, \lambda\mu_3 2 | \bar{\Psi}_s \rangle &= \sum_{i=1}^{3} \langle k^2, \lambda\mu_3 2 | \bar{\Psi}_s^{(i)} \rangle , \\
\langle k^2, \lambda\mu_i 1 | \bar{\Psi}_1 \rangle &= \sum_{i=1}^{3} \langle k^2, \lambda\mu_i 1 | \bar{\Psi}_1^{(i)} \rangle , \\
& \qquad\qquad\qquad\qquad i \neq 3 \\
\langle k^2, \lambda\mu_i 2 | \bar{\Psi}_2 \rangle &= \sum_{i=1}^{3} \langle k^2, \lambda\mu_i 2 | \bar{\Psi}_2^{(i)} \rangle ,
\end{aligned} \tag{7.6}$$

and define the ket $|\bar{\Psi}^{(i)}(\nu)\rangle$ such that

$$|\bar{\Psi}(\nu)\rangle = \sum_{i=1}^{3} |\bar{\Psi}^{(i)}(\nu)\rangle , \tag{7.7}$$

where, for example, $|\bar{\Psi}^{(i)}(4)\rangle$ is given by

$$|\bar{\Psi}^{(i)}(4)\rangle = -\sum_{\lambda\mu_i} \int |k^2, \lambda\mu_i 1\rangle dk^2 \langle k^2, \lambda\mu_i 1 | \bar{\Psi}_1^{(i)} \rangle , \; i \neq 3. \tag{7.8}$$

We can then construct the component-kets

$$|\bar{\Psi}^{(i)}\rangle = \sum_{\nu=1}^{4} |\bar{\Psi}^{(i)}(\nu)\rangle \zeta(\nu), \quad i = 1,2,3, \tag{7.9}$$

so that the complete ket of the system is

$$|\bar{\Psi}\rangle = \sum_{i=1}^{3} |\bar{\Psi}^{(i)}\rangle . \tag{7.10}$$

The spin-generalised Faddeev's equation for $|\bar{\Psi}\rangle$ is

$$|\bar{\Psi}^{(i)}\rangle = -G_o(z)\, T_i(\,|\bar{\Psi}^{(j)}\rangle + |\bar{\Psi}^{(k)}\rangle), \quad i \neq j \neq k, \tag{7.11}$$

and T_i now has the form

$$T_i = \sum_{\alpha=1}^{4} P_{i,\alpha}(jk)\, T_{i,\alpha} \tag{7.12}$$

with $P_{i,\alpha}(jk)$ the projection operator for the two-nucleon spin-isospin state denoted by α and $T_{i,\alpha}$ the transition operator of that state. The projection operators are of course given by

$$\begin{aligned} P_{i,1} &= P_\sigma^+ P_\tau^+ , \\ P_{i,2} &= P_\sigma^+ P_\tau^- , \\ P_{i,3} &= P_\sigma^- P_\tau^+ , \\ P_{i,4} &= P_\sigma^- P_\tau^- , \end{aligned} \tag{7.13}$$

with

$$P^{\pm}_{\sigma,\tau}(jk) = \tfrac{1}{2}[1 \pm (jk)_{\sigma,\tau}]$$

and $(jk)_{\sigma,\tau}$ the operators for the permutation of the spin (σ) or isospin (τ) variables of particles j and k. Thus $T_{i,2}$ and $T_{i,3}$ are respectively the triplet and singlet transition operators of the i-subsystem.

We can now verify that if, for a certain z, Eqn.(7.11) has a solution for $|\bar{\Psi}^{(i)}>$ then $|\bar{\Psi}>$ given by Eqn.(7.10) does in fact satisfy the Schrödinger equation with z the total energy of the system, that is

$$\left[H_o + (V_1 + V_2 + V_3)\right] |\bar{\Psi}> = z |\bar{\Psi}> \qquad (7.14)$$

where H_o is the free Hamiltonian and V_i is the potential between particles j and k and has the same form as T_i in Eqn.(7.12). Multiplying the Eqn.(7.11) by $1 + G_o(z)V_i$ and using the Lippman Schwinger equation for the transition operator of the i-subsystem

$$T_i = V_i - V_i G_o(z) T_i \qquad (7.15)$$

we find

$$\begin{aligned}[1 + G_o(z)V_i] |\bar{\Psi}^{(i)}> &= -G_o(z)T_i (|\bar{\Psi}^{(j)}> + |\bar{\Psi}^{(k)}>) \\ &\quad -G_o(z)(V_i - T_i)(|\bar{\Psi}^{(j)}> + |\bar{\Psi}^{(k)}>) \\ &= -G_o(z)V_i(|\bar{\Psi}^{(j)}> + |\bar{\Psi}^{(k)}>) ,\end{aligned}$$

or

$$|\bar{\Psi}^{(i)}> = -G_o(z)V_i |\bar{\Psi}> . \qquad (7.16)$$

On adding up similar equations for the other component-kets, we have

$$|\Psi\rangle = -G_o(z)(V_1 + V_2 + V_3)|\Psi\rangle \qquad (7.17)$$

which after multiplying on the right by $H_o + z1$ gives the required result.

The Faddeev equation as it is in Eqn.(7.11) is an operator equation in spin and abstract Hilbert spaces. To extract the spin states, we use Eqn.(7.9) for $|\Psi^{(i)}\rangle$ and the orthogonality of the $\zeta(\nu)$s to obtain

$$|\Psi^{(i)}(\nu)\rangle = -G_o(z)\left[\sum_{\alpha=1}^{4} \zeta(\nu)P_{i,\alpha}\zeta(\nu') \; T_{i,\alpha}\right](|\Psi^{(j)}(\nu')\rangle + |\Psi^{(k)}(\nu')\rangle) \; . \qquad (7.18)$$

We may regard the spin-states $\zeta(\nu)$ as forming a basis for the four-dimensional spin space. Then, in order to evaluate the matrix element $\zeta(\nu)P_{i,\alpha}\zeta(\nu')$ we require the matrix representations of the projection operators in this spin space. As these are expressed in terms of the transposition operators $(jk)_\sigma$ and $(jk)_\tau$, it is the matrix representations of these we want to find. Using the transformation properties of the $\zeta(\nu)$s under S_3, we find

$$(23)_\sigma = \begin{bmatrix} \cdot & \cdot & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ & & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot & \cdot \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot & \cdot \end{bmatrix}, \qquad (23)_\tau = \begin{bmatrix} \cdot & \cdot & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \cdot & \cdot & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & \cdot & \cdot \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot & \cdot \end{bmatrix},$$

$$(31)_\sigma = \begin{bmatrix} \cdot & \cdot & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \cdot & \cdot & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot & \cdot \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot & \cdot \end{bmatrix}, \quad (31)_\tau = \begin{bmatrix} \cdot & \cdot & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \cdot & \cdot & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \cdot & \cdot \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & \cdot & \cdot \end{bmatrix}, \quad (7.19)$$

$$(12)_\sigma = \begin{bmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{bmatrix}, \quad (12)_\tau = \begin{bmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{bmatrix}.$$

The matrix representations of the projection operators can now be constructed and the matrix elements evaluated. The result is that Eqn.(7.18) becomes a set of 12 coupled equations for $|\bar{\Psi}^{(i)}(\nu)\rangle$, $i = 1,2,3$ and $\nu = 1,2,3,4$:

$$\begin{bmatrix} |\bar{\Psi}^{(1)}\rangle \\ |\bar{\Psi}^{(2)}\rangle \\ |\bar{\Psi}^{(3)}\rangle \end{bmatrix} = -G_o(z) \begin{bmatrix} \cdot & I_1 & I_1 \\ I_2 & \cdot & I_2 \\ I_3 & I_3 & \cdot \end{bmatrix} \begin{bmatrix} |\bar{\Psi}^{(1)}\rangle \\ |\bar{\Psi}^{(2)}\rangle \\ |\bar{\Psi}^{(3)}\rangle \end{bmatrix} \quad (7.20)$$

where

$$|\bar{\Psi}^{(i)}\rangle = \begin{bmatrix} |\bar{\Psi}^{(i)}(1)\rangle \\ |\bar{\Psi}^{(i)}(2)\rangle \\ |\bar{\Psi}^{(i)}(3)\rangle \\ |\bar{\Psi}^{(i)}(4)\rangle \end{bmatrix}, \quad i = 1,2,3, \quad (7.21)$$

and

$$I_1 = \begin{bmatrix} \frac{1}{2}(T_{1,1}+T_{1,4}) & \cdot & \frac{\sqrt{3}}{4}(T_{1,1}-T_{1,4}) & \frac{1}{4}(T_{1,1}-T_{1,4}) \\ \cdot & \frac{1}{2}(T_{1,2}+T_{1,3}) & \frac{1}{4}(T_{1,2}-T_{1,3}) & -\frac{\sqrt{3}}{4}(T_{1,2}-T_{1,3}) \\ \frac{\sqrt{3}}{4}(T_{1,1}-T_{1,4}) & \frac{1}{4}(T_{1,2}-T_{1,3}) & \frac{1}{8}(3T_{1,1}+T_{1,2}+T_{1,3}+3T_{1,4}) & \frac{\sqrt{3}}{8}(T_{1,1}-T_{1,2}-T_{1,3}+T_{1,4}) \\ \frac{1}{4}(T_{1,1}-T_{1,4}) & -\frac{\sqrt{3}}{4}(T_{1,2}-T_{1,3}) & \frac{\sqrt{3}}{8}(T_{1,1}-T_{1,2}-T_{1,3}+T_{1,4}) & \frac{1}{8}(T_{1,1}+3T_{1,2}+3T_{1,3}+T_{1,4}) \end{bmatrix}$$

$$I_2 = \begin{bmatrix} \frac{1}{2}(T_{2,1}+T_{2,4}) & \cdot & -\frac{\sqrt{3}}{4}(T_{2,1}-T_{2,4}) & \frac{1}{4}(T_{2,1}-T_{2,4}) \\ \cdot & \frac{1}{2}(T_{2,2}+T_{2,3}) & \frac{1}{4}(T_{2,2}-T_{2,3}) & \frac{\sqrt{3}}{4}(T_{2,2}-T_{2,3}) \\ -\frac{\sqrt{3}}{4}(T_{2,1}-T_{2,4}) & \frac{1}{4}(T_{2,2}-T_{2,3}) & \frac{1}{8}(3T_{2,1}+T_{2,2}+T_{2,3}+3T_{2,4}) & -\frac{\sqrt{3}}{8}(T_{2,1}-T_{2,2}-T_{2,3}+T_{2,4}) \\ \frac{1}{4}(T_{2,1}-T_{2,4}) & \frac{\sqrt{3}}{4}(T_{2,2}-T_{2,3}) & -\frac{\sqrt{3}}{8}(T_{2,1}-T_{2,2}-T_{2,3}+T_{2,4}) & \frac{1}{8}(T_{2,1}+3T_{2,2}+3T_{2,3}+T_{2,4}) \end{bmatrix}$$

$$I_3 = \begin{bmatrix} \frac{1}{2}(T_{3,1}+T_{3,4}) & \cdot & \cdot & -\frac{1}{2}(T_{3,1}-T_{3,4}) \\ \cdot & \frac{1}{2}(T_{3,2}+T_{3,3}) & \frac{1}{2}(T_{3,2}-T_{3,3}) & \cdot \\ \cdot & -\frac{1}{2}(T_{3,2}-T_{3,3}) & \frac{1}{2}(T_{3,2}+T_{3,3}) & \cdot \\ -\frac{1}{2}(T_{3,1}+T_{3,4}) & \cdot & \cdot & \frac{1}{2}(T_{3,1}+T_{3,4}) \end{bmatrix} . \quad (7.22)$$

As in the spinless case, we assume that the subsystems interact only in s-states. Now by virtue of the projection operators $P_{i,1}$ and $P_{i,4}$ the transition operators, $T_{i,1}$ and $T_{i,4}$ are for two-nucleon states which are symmetric in spin-space. Therefore in order to satisfy Pauli's Principle, the two-nucleon interacting states projected by these operators cannot have $I = 0$. Hence $T_{i,1}$ and $T_{i,4}$ are null operators. It follows that the states

$|\bar{\Psi}^{(i)}(1)>$ for i = 1,2,3 and $|\bar{\Psi}^{(3)}(4)>$ vanish identically. By a rearrangement of rows and columns, the remaining equations can be written in the form

$$\begin{bmatrix} |\bar{\underline{\Psi}}(2)> \\ |\bar{\underline{\Psi}}(3)> \\ |\bar{\underline{\Psi}}(4)> \end{bmatrix} = \begin{bmatrix} J_{22} & J_{23} & J_{24} \\ J_{32} & J_{33} & J_{34} \\ J_{42} & J_{43} & J_{44} \end{bmatrix} \begin{bmatrix} |\bar{\underline{\Psi}}(2)> \\ |\bar{\underline{\Psi}}(3)> \\ |\bar{\underline{\Psi}}(4)> \end{bmatrix} . \qquad (7.23)$$

where $|\bar{\underline{\Psi}}(\nu)>$, with a bar to distinguish it from $|\bar{\Psi}(\nu)>$, are the volumn vectors

$$|\bar{\underline{\Psi}}(2)> = \begin{bmatrix} |\bar{\Psi}^{(1)}(2)> \\ |\bar{\Psi}^{(2)}(2)> \\ |\bar{\Psi}^{(3)}(2)> \end{bmatrix}, \qquad |\bar{\underline{\Psi}}(3)> = \begin{bmatrix} |\bar{\Psi}^{(1)}(3)> \\ |\bar{\Psi}^{(2)}(3)> \\ |\bar{\Psi}^{(3)}(3)> \end{bmatrix}, \qquad |\bar{\underline{\Psi}}(4)> = \begin{bmatrix} |\bar{\Psi}^{(1)}(4)> \\ |\bar{\Psi}^{(2)}(4)> \end{bmatrix},$$

(7.24)

and the $J_{\nu\nu'}$, given explicitly in Appendix 8, are matrices of operators.

In order to obtain a completely continuous operator, we iterate Eqn.(7.23) once and for convenience of interpretation, we revert through relation (7.3) to the kets $|\bar{\Psi}_s>$, $|\bar{\Psi}_1>$ and $|\bar{\Psi}_2>$. The final spin-generalised Faddeev equation is

$$\begin{bmatrix} |\bar{\Psi}_s> \\ |\bar{\Psi}_1> \\ |\bar{\Psi}_2> \end{bmatrix} = \begin{bmatrix} K_{ss} & K_{s1} & K_{s2} \\ K_{1s} & K_{11} & K_{12} \\ K_{2s} & K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} |\bar{\Psi}_s> \\ |\bar{\Psi}_1> \\ |\bar{\Psi}_2> \end{bmatrix} , \qquad (7.25)$$

with

$$\begin{aligned}
K_{ss} &= J_{22}^2 + J_{23}J_{32} + J_{24}J_{42} , \\
K_{s1} &= -J_{22}J_{24} - J_{23}J_{34} - J_{24}J_{44} , \\
K_{s2} &= J_{22}J_{23} + J_{23}J_{33} + J_{24}J_{43} , \\
K_{1s} &= -J_{42}J_{22} - J_{43}J_{32} - J_{44}J_{42} , \\
K_{11} &= J_{42}J_{24} + J_{43}J_{34} + J_{44}^2 , \\
K_{12} &= -J_{42}J_{23} - J_{43}J_{33} - J_{44}J_{43} , \\
K_{2s} &= J_{32}J_{22} + J_{33}J_{32} + J_{34}J_{42} , \\
K_{21} &= -J_{32}J_{24} - J_{33}J_{34} - J_{34}J_{44} , \\
K_{22} &= J_{32}J_{23} + J_{33}^2 + J_{34}J_{43} .
\end{aligned} \qquad (7.26)$$

Eqn.(7.25) can again be solved in the SU(3) representation. Taking $\lambda \leq 4$, we note from Eqn.(7.4) that $|\bar{\Psi}_s^{(i)}>$ is expressed in terms of the two states $|k^2,002>$ and $|k^2,402>$ while $|\bar{\Psi}_1^{(i)}>$ and $|\bar{\Psi}_2^{(i)}>$ involve only $|k^2,211>$ and $|k^2,212>$ respectively. We propose two methods to obtain a numerical solution.

The Direct Method. The complete equation, Eqn.(7.25), is solved as a homogeneous equation. In the SU(3) representation, there are 11 unknown partial-wave amplitudes: $\langle k^2, 002 \mid \bar{\Psi}_s^{(i)} \rangle$, $\langle k^2, 402 \mid \bar{\Psi}_s^{(i)} \rangle$ and $\langle k^2, 212 \mid \bar{\Psi}_2^{(i)} \rangle$ for $i = 1,2,3$; $\langle k^2, 211 \mid \bar{\Psi}_1^{(i)} \rangle$ for $i = 1,2$. Once solved, they can be used in Eqns.(7.6), (7.4), (7.3), (7.9) and (7.10) to reconstruct the ket $\mid \bar{\Psi} \rangle$. The wavefunction in momentum space, $\langle \underline{\eta}\ \underline{\xi} \mid \bar{\Psi} \rangle$, then follows immediately on using the momentum space representation of the SU(3) states. The binding energy of the system is, of course, the value of z when a solution exists.

If a 15-point integration formula is used for the integration over k^2, a matrix of order 165 x 165 has to be inverted to yield the binding energy and the 11 unknown functions. This is not prohibitive. However, because of relation (7.5), at least one of the five partial-wave amplitudes associated with the mixed symmetry states is not independent. Therefore, the matrix is likely to be ill-conditioned which has to be remedied.

The Iterative Method. In order to avoid the difficulty of over determinancy encountered in the Direct Method, we can solve for $\mid \bar{\Psi}_s \rangle$, $\mid \bar{\Psi}_1 \rangle$ and $\mid \bar{\Psi}_2 \rangle$ separately in an iterative procedure. The equations we want to solve are

$$\langle k^2, \lambda\mu_3 2 \mid \underline{\bar{\Psi}}_s \rangle = \sum_{\lambda'\mu'_3} \int \langle k^2, \lambda\mu_3 2 | K_{ss}(z) | k'^2, \lambda'\mu'_3\ 2\rangle dk'^2 \langle k'^2, \lambda'\mu'_3\ 2 \mid \underline{\bar{\Psi}}_s \rangle$$

$$+ \int \langle k^2, \lambda\mu_3 2 | K_{s1}(z) | k'^2, 211\rangle dk'^2 \langle k'^2, 211 \mid \underline{\bar{\Psi}}_1 \rangle \tag{7.27}$$

$$+ \int \langle k^2, \lambda\mu_3 2 | K_{s2}(z) | k'^2, 212\rangle dk'^2 \langle k'^2, 212 \mid \underline{\bar{\Psi}}_2 \rangle ,$$

$$\langle k^2, 211 \mid \underline{\bar{\Psi}}_1 \rangle = \sum_{\lambda'\mu'_3} \int \langle k^2, 211 | K_{1s}(z) | k'^2, \lambda'\mu'_3\ 2\rangle dk'^2 \langle k'^2, \lambda'\mu'_3\ 2 \mid \underline{\bar{\Psi}}_s \rangle$$

$$+ \int \langle k^2, 211 | K_{11}(z) | k'^2, 211\rangle dk'^2 \langle k'^2, 211 \mid \underline{\bar{\Psi}}_1 \rangle \tag{7.28}$$

$$+ \int \langle k^2, 211 | K_{12}(z) | k'^2, 212\rangle dk'^2 \langle k'^2, 212 \mid \underline{\bar{\Psi}}_2 \rangle ,$$

$$\langle k^2, 212 \mid \underline{\bar{\Psi}}_2 \rangle = \sum_{\lambda'\mu'_3} \int \langle k^2, 212 | K_{2s}(z) | k'^2, \lambda'\mu'_3\ 2\rangle dk'^2 \langle k'^2, \lambda'\mu'_3\ 2 \mid \underline{\bar{\Psi}}_s \rangle$$

$$+ \int \langle k^2, 212 | K_{21}(z) | k'^2, 211\rangle dk'^2 \langle k'^2, 211 \mid \underline{\bar{\Psi}}_1 \rangle \tag{7.29}$$

$$+ \int \langle k^2, 212 | K_{22}(z) | k'^2, 212\rangle dk'^2 \langle k'^2, 212 \mid \underline{\bar{\Psi}}_2 \rangle ,$$

with (λ, μ_3) taking only two sets of values, (O,O) and (4,O). The iteration scheme consists in this case of the following steps: Since the triton exists predominantly in the totally symmetric state, the homogeneous equation for $\langle k^2, \lambda\mu_3 2 \mid \underline{\bar{\Psi}}_s \rangle$ should be solvable to give the binding energy, and the wave-function in the zeroth approximation. Knowing $\langle k^2, \lambda\mu_3 2 \mid \underline{\bar{\Psi}}_s \rangle$, one then solves the inhomogeneous solutions for $\langle k^2, 211 \mid \underline{\bar{\Psi}}_1 \rangle$ and $\langle k^2, 212 \mid \underline{\bar{\Psi}}_2 \rangle$, keeping only the contributions from $\langle k^2, \lambda\mu_3 2 \mid \bar{\Psi}_s \rangle$. In the next iteration, we substitute $\langle k^2, 211 \mid \bar{\Psi}_1 \rangle$ and $\langle k^2, 212 \mid \bar{\Psi}_2 \rangle$ back in Eqn.(7.27), which is then solved as an inhomogeneous equation to find the correction to $\langle k^2, \lambda\mu_3 2 \mid \bar{\Psi}_s \rangle$ in the next approximation etc.

To conclude this chapter, we justify our method of spin-generalization by showing that if we allow the spin space to "shrink away", the homogeneous equation for $\langle k^2, \lambda\mu_3 2 \mid \bar{\Psi}_s \rangle$ in Eqn.(7.27) reduces to Eqn.(5.36) of the spinless case. With the help of the $J_{\sigma\sigma'}$ matrices, the matrix K_{ss} is easily found to be

$$K_{ss}(z) = \begin{bmatrix} M_{12} + M_{13} & M_{13} & M_{12} \\ M_{23} & M_{21} + M_{23} & M_{21} \\ M_{32} & M_{31} & M_{31} + M_{32} \end{bmatrix} \tag{7.30}$$

with

$$M_{rl} = \frac{1}{8} \{ G_o(z)T_{r,2}G_o(z)T_{l,2} + 3G_o(z)T_{r,2}G_o(z)T_{l,3} + 3G_o(z)T_{r,3}G_o(z)T_{l,2} + G_o(z)T_{r,3}G_o(z)T_{l,3} \} . \tag{7.31}$$

Each term of M_{rl} is of the same form as $G_oT_rG_oT_l$ in $K(z)$ of Eqn.(5.7) - only now the transition operators may be different. Furthermore, the kernel $\langle k^2, \lambda\mu_3 2 | M_{rl} | k'^2, \lambda'\mu_3' 2 \rangle$ is again independent of r and 1. Thus we have

$$\langle k^2, \lambda\mu_3 2 \mid \bar{\Psi}_s\rangle = \tfrac{1}{2} \sum_{\lambda'\mu_3'} \int \langle k^2, \lambda\mu_3 2 | G_o T_{1,2} G_o T_{2,2} + 3G_o T_{1,2} G_o T_{2,3} +$$

$$+ 3G_o T_{1,3} G_o T_{2,2} + G_o T_{1,3} G_o T_{2,3} | k'^2, \lambda'\mu_3' 2\rangle dk'^2 \langle k'^2, \lambda'\mu_3' 2 \mid \bar{\Psi}_s\rangle . \tag{7.32}$$

In the limit the spin space "shrinks" to zero, there is only one two-particle transition operator $T_i = T_{i,2} = T_{i,3}$ and so we find that Eqn.(7.32) reduces exactly to Eqn.(5.36). There is one remaining pleasant surprise. The zeroth approximation in the iterative method turns out to be exact. It arises because in this case, as can be easily verified, the totally symmetric state $|\bar{\Psi}_s\rangle$ is uncoupled to the mixed symmetry states in Eqn.(7.25).

CHAPTER 8 CONCLUSION

We have shown how the SU(3) representation of the three-particle states can form a basis for the full power of Faddeev's Theory to be applied in practice. In this representation, Faddeev's equations can be approximated to any desired accuracy by a finite set of coupled integral equations in one variable only. Furthermore, for particles interacting with Yukawa potential, by taking the iterated equations [Eqn.(5.6)] we have a SU(3) kernel which can be shown to form a completely continuous integral operator in L^2 and hence possesses only a point spectrum of boundstate poles. To pass from the SU(3) representation to either the momentum or configuration representation, we only have to use the functions carrying the I.R.s in the appropriate space as transformation coefficients.

It must be mentioned that insofar as we are just trying to reduce the number of variables from six to one in a three-particle problem, we could apply the method of Simonov[10,11] to expand the wavefunction in terms of six-dimensional spherical harmonics. Through the connection between the I.R.s of SU(3) and the surface harmonics on S_5, we found that both methods are equivalent. However, we believe our approach is more general and more suitable for Faddeev's equations because it suggests so naturally the form of the surface harmonics [Eqn.(4.18)], which is important for the evaluation of the SU(3) kernel from the normally known kernel in momentum representation.

For three-particle systems existing predominantly in the L = 1,0 states, and these include boundstates and low energy nucleon-deuteron scattering, the symmetric properties of our SU(3) states make it relatively simple to introduce spins and the Pauli Principle into the theory. We demonstrate this by deriving the Faddeev equation in the SU(3) representation for the boundstate wavefunction of the triton.

REFERENCES

(1) J.W. Humberston, R.L. Hall and T.A. Osborn, Phys.Lett. 27B, No.4, 195 (1968).

(2) L.D. Faddeev, Soviet Physics, JETP 12, 1014 (1961);
L.D. Faddeev, Mathematical Problems of the Quantum Theory of Scattering for a Three-particle System. English translation by J.B. Sykes, Harwell, 1963.

(3) C. Lovelace, in Strong Interactions and High Energy Physics, edited by R.G. Moorhouse (Oliver and Boyd, London, 1964);
C. Lovelace, Phys.Rev. 135, No.5B, B1225 (1964);
A.C. Phillips, Phys.Rev. 142, 984 (1966);
N.M. Petrov, S.A. Storozhenko and V.F. Kharchenko, Soviet JNP 6, No.3, 340 (1968).

(4) R.L. Omnes, Phys.Rev. 134, B1358 (1964).

(5) T. Osborn and H. Pierre Noyes, Phys.Rev.Lett. 17, 215 (1966);
T. Osborn, Faddeev's Equations for Local Potential, SLAC report No.79 (1967).

(6) A.J. Dragt, J.Maths.Phys. 6, No.4, 533 (1965).

(7) Jean-Marc Levy-Leblond and Francois Lucrat, J.Math.Phys. 6, No.10 1564 (1965);
Jean-Marc Levy-Leblond and Monique Levy-Nahas, J.Math.Phys. 6, No.10, 1571 (1965).

(8) Yu.A. Simonov, Soviet JNP 3, 461 (1966).

(9) F.T. Smith, Phys.Rev. 120, No.3, 1058 (1960).

(10) A.M. Badalyan and Yu.A. Simonov, Soviet JNP 3, No.6, 755 (1966).

(11) Yu.A. Simonov and A.M. Badalyan, Soviet JNP 5, No.1, 60 (1967).

(12) A.R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1957).

(13) R.H. Dalitz, Repts.Progr.Phys. 2, 163 (1957);
E. Fabri, Nuovo Cimento 11, 479 (1954).

(14) W. Zickendraht, Annals of Phys. 35, 18 (1965).

(15) M. Hamermesh, Group Theory (Addison-Wesley, 1962).

(16) C. Fronsdal, in Elementary Particle Physics and Field Theory, 1962, Brandeis University Summer Institute Lectures in Theoretical Physics (W.A. Benjamin Company Inc., 1963), Vol.1.

(17) Mirza A.B. Beg and Henri Ruegg, J.Math.Phys. 6, No.5, 677 (1965).

(18) T.J. Nelson, J.Math.Phys. 8, No.5, 857 (1967).

(19) M. Gourdin, Unitary Symmetries (North-Holland Publishing Co., Amsterdam, 1967).

(20) T.Y. Thomas, Concepts from Tensor-Analysis and Differential Geometry (Academic Press Inc., New York, 1961).

(21) V. Gallina, P. Nata, L. Bianchi and G. Viano, Nuovo Cimento 24, 835 (1962).

(22) A. Erdelyi, Higher Transcendental Functions Vol.2 (Bateman Manuscript Project, McGraw-Hill, 1953).

(23) N.Ya. Vilenkii, G.I. Kuznetsov and Ya.A. Smorodinskii, Soviet JNP 2, No.5, 645 (1966).

(24) Jeanne-Yvonne Pasque, J.Math.Phys. 8, No.9, 1878 (1967).

(25) G. Racah, Proceedings of the Istanbul International Summer School, 1962 (Gordon and Breach, New York, 1965).

(26) L. Pontrjagin, Topological Groups (Princeton University Press, Princeton New Jersey, 1939), Sec.29.

(27) There are about as many conventions for the rotational matrices as workers in the field! We have found it more suitable to adopt Edmonds'[12] conventions throughout this work. Hence a rotation of $R(\alpha,\beta,\gamma)$ is used in the passive sense, that is, on the coordinate frame.

(28) R.C.Witten and F.T. Smith, J.Math.Phys. 9, No.7, 1103 (1968).

(29) A. Gray and G.B. Mathew, A Treatise on Bessel Functions (MacMillan, 1895), p.80.

(30) See for example: Faddeev's first paper of Ref.2, and S. Weinberg, Phys.Rev. 133, No.1B, B232 (1964).

(31) Spectral theory of completely continuous operators can be found in any standard textbooks on Functional Analysis, see for example:

F. Riesz and B. Sz-Nagy, Functional Analysis (Frederick Ungar, New York, 1955);

B.Z. Vulikh, Functional Analysis (Pergamon Press, 1963).

On L^2 or (Hilbert-Schmidt) kernel:

F. Smithies, Integral Equations (Cambridge University Press, New York, 1958).

(32) A very useful discussion from the physics point of view on the difference between bounded and completely continuous operators can be found in the Appendix of Weinberg's paper [Ref.30] .

(33) M. Rubin, R. Sugar and G. Tiktopoulos, Phys.Rev. 159, 1348 (1967). Faddeev, using more general potentials, obtained the same result [Ref.2] with the fifth power of the operator in Eqn.(5.1) in a certain Banach space.

(34) On the possibility of having spurious non-Schrödinger homogeneous solutions, see for example,

J.V. Noble, Phys.Rev. 148, 1553 (1966);

J.V. Noble, Phys.Rev. 161, 1495 (1967);

R.G. Newton, Phys.Rev. 153, 1502 (1967).

(35) V.V. Pustovalov and Yu.A. Simonov, Soviet Physics JETP 24, No.1, 230 (1967).

(36) A.M. Badalyan, E.S. Gal'pern, V.N. Lyakhovitskii, V.V. Pustovalov, Yu.A. Simonov and E.L. Sinkov, Soviet JNP 6, No.3, 345 (1968).

(37) L.I. Schiff, Phys.Rev. 133, B802 (1964).

APPENDICES

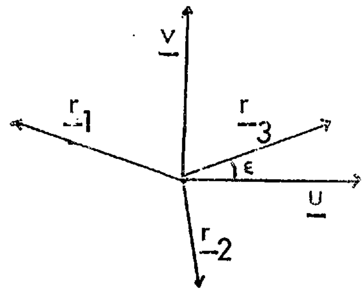
APPENDICES INDEX

		Page
1	Choice of the body-fixed axes ($\underline{u},\underline{v},\underline{w}$)	88
2	Transformation of the Euler angles α, β and γ under S_3	90
3	Construction of S	93
4	Construction of Λ^2	97
5	On orthogonal polynomials and spherical harmonics in six dimensions	105
6	On the spherical harmonics $S^{LM}_{\lambda\mu_i\iota\kappa}$	113
7	On evaluation of the matrix element $\langle \underline{\omega}, LMK \mid T_i(z) \mid \underline{\omega}', L'M'K' \rangle$	118
8	The matrices $J_{\upsilon\upsilon'}$	124

Appendix 1 Choice of the body-fixed axes $(\underline{u},\underline{v},\underline{w})$

We wish to choose the body-fixed axes $(\underline{u},\underline{v},\underline{w})$ such that

$$\sum_{i=1}^{3} (\underline{r}_i.\underline{u})(\underline{r}_i.\underline{v}) = 0. \qquad \text{(A1.1)}$$



When expressed in terms of the relative vectors, $\underline{r}^{(1)}$ and $\underline{r}^{(2)}$, (A1.1) is

$$\sum_{i=1}^{2} (\underline{r}^{(i)}.\underline{u})(\underline{r}^{(i)}.\underline{v}) . \qquad \text{(A1.2)}$$

Hence to choose $\underline{u}$ and $\underline{v}$ satisfying (A1.1) is equivalent to choosing the components of the relative vectors along $\underline{u}$ and $\underline{v}$ satisfying (A1.2). By means of Eqns.(3.1) and (3.2) we can easily obtain the other three conditions satisfied by these components:

$$(\underline{r}^{(1)}.\underline{u})^2 + (\underline{r}^{(1)}.\underline{v})^2 = \underline{r}^{(1)2} = \tfrac{1}{2} r_{ij}^2 = \frac{1}{2} r^2 (1 - \cos 2\psi \cos\phi)$$

$$= (r \cos\psi \sin \tfrac{1}{2}\phi)^2 + (r \sin\psi \cos \tfrac{1}{2}\phi)^2 , \qquad \text{(A1.3)}$$

$$(\underline{r}^{(2)}.\underline{u})^2 + (\underline{r}^{(2)}.\underline{v})^2 = \underline{r}^{(2)2} = \frac{3}{2} r_3^2 = \tfrac{1}{2}(1 + \cos 2\psi \cos\phi)$$

$$= (r \cos\psi \cos \frac{\phi}{2})^2 + (r \sin\psi \sin \tfrac{1}{2}\phi)^2 , \qquad \text{(A1.4)}$$

$$(\underline{r}^{(1)}.\underline{u})(\underline{r}^{(2)}.\underline{u}) + (\underline{r}^{(1)}.\underline{v})(\underline{r}^{(2)}.\underline{v}) = \frac{\sqrt{3}}{4}(r_2^2 + r_{31}^2 - r_1^2 - r_{23}^2) = \frac{1}{2} r^2 \cos 2\psi \sin\phi$$

$$= (r \cos\psi \sin\frac{\phi}{2})(r \cos\psi \cos\frac{\phi}{2}) - (r \sin\psi \cos\tfrac{1}{2}\phi)(r \sin\psi \sin\tfrac{1}{2}\phi). \qquad (A1.5)$$

If we solve the Eqns.(A1.2), (A1.3), (A1.4) and (A1.5) for the components we are bound to obtain more than one set of solutions. This is because the condition (A1.2) only demands $\underline{u}$ and $\underline{v}$ to be along the principal axes of inertia; it does not specify the direction in space. In any case, it is not easy to solve them in this way. This is why we have expressed the conditions in such a form so that a solution by inspection is possible. It is clear that a necessary condition for (A1.2) to be satisfied is that one of the components must be of the opposite sign. Thus, if we choose

$$(\underline{r}^{(1)}.\underline{v}) = - r \sin\psi \cos\tfrac{1}{2}\phi \qquad (A1.6)$$

so that

$$(\underline{r}^{(2)}.\underline{v}) = r \sin\psi \sin\tfrac{1}{2}\phi \qquad (A1.7)$$

and therefore by (A1.3) and (A1.4)

$$(\underline{r}^{(1)}.\underline{u}) = r \cos\psi \sin\tfrac{1}{2}\phi , \qquad (A1.8)$$

$$(\underline{r}^{(2)}.\underline{u}) = r \cos\psi \cos\tfrac{1}{2}\phi . \qquad (A1.9)$$

This is the prescription used in Eqn.(3.7) to define the body-fixed axes.

Appendix 2 Transformation of the Euler angles, α, β and γ under S_3.

We first consider the transformation of these Euler angles under the exchange of particles 1 and 2, (12). Let us denote the transformed body-fixed axes by $(\underline{u}',\underline{v}',\underline{w}')$ and the transformed Euler angles by α', β' and γ'. By standard matrix transformation theory, if the coordinates, with respect to basis vectors $\underline{e}_1$ and $\underline{e}_2$, of a fixed vector $\underline{x}$ in two dimensions is transformed by

$$\underline{x}' = \tilde{A}^{-1}\underline{x} , \tag{A2.1}$$

then the base $\underline{e} = (\underline{e}_1,\underline{e}_2)$ is transformed by

$$\underline{e}' = A\underline{e} . \tag{A2.2}$$

The vector $\underline{r}^{(2)}$ is unchanged in space under the transposition (12). Its coordinates with respect to base $(\underline{u},\underline{v})$ are, however, transformed to

$$\begin{aligned} (\underline{r}^{(2)}.\underline{u})' &= (12)\,(\underline{r}^{(2)}.\underline{u}) = (12)r\cos\psi\cos\tfrac{1}{2}\phi = +(\underline{r}^{(2)}.\underline{u}) , \\ (\underline{r}^{(2)}.\underline{v})' &= (12)\,(\underline{r}^{(2)}.\underline{v}) = (12)r\sin\psi\sin\tfrac{1}{2}\phi = -(\underline{r}^{(2)}.\underline{v}) , \end{aligned} \tag{A2.3}$$

where we have used Eqn.(3.14) for the transformation properties of ϕ. Hence by virtue of (A2.1) and (A2.2), we have

$$\begin{bmatrix} \underline{u}' \\ \underline{v}' \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} . \tag{A2.4}$$

For the other transpositions, (23) and (31), we use $\underline{r}_1^{(2)}$ and $\underline{r}_2^{(2)}$ respectively. It can then be verified that (23) and (31) induce the same transformation on $\underline{u}$ and $\underline{v}$ and therefore the changes in the Euler angles are the same. We note that (A2.4) is effected by the rotation $R(\pi,\pi,O)$ on the $(\underline{u},\underline{v},\underline{w})$ frame. Thus

$$R(\alpha' \beta' \gamma') = R(\pi,\pi,O)R(\alpha\ \beta\ \gamma) \qquad \text{(A2.5)}$$

which is equivalent to

$$R_c(-\gamma' -\beta' -\alpha') = R_c(O-\pi-\pi)R_c(-\gamma-\beta-\alpha) \qquad \text{(A2.6)}$$

where R_c denotes a rotation on the original coordinate frame S_o; that is to say, it is a rotation in the passive sense. Since the rotation matrices $D^L_{MM'}$ offer a representation of the three-dimensional rotation, the result of two successive rotations is represented by

$$D^L_{MM'}(-\gamma' -\beta' -\alpha') = D^L_{MM''}(O-\pi-\pi)D^L_{M''M'}(-\gamma-\beta-\alpha)\ , \qquad \text{(A2.7)}$$

or

$$D^{*L}_{M'M}(\alpha' \beta' \gamma') = D^{*L}_{M''M}(\pi\ \pi\ O)D^{*L}_{M''M}(\alpha\ \beta\ \gamma)\ , \qquad \text{(A2.8)}$$

since

$$D^L_{MM'}(-\gamma-\beta-\alpha) = D^{*L}_{M'M}(\alpha\ \beta\ \gamma)\ . \qquad \text{(A2.9)}$$

And on using

$$d^L_{M''M}(\pi) = (-1)^{L+M}\delta_{M'',-M} \quad , \tag{A2.10}$$

$$d^L_{M'-M}(\beta) = (-1)^{L-M} d_{M'M}(\beta-\pi) \; , \tag{A2.11}$$

we have

$$D^{*L}_{M'M}(\alpha'\beta'\gamma') = e^{-iM\alpha'} d_{M'M}(\beta-\pi) e^{iM(\gamma-\pi)} \; . \tag{A2.12}$$

Therefore

$$\begin{aligned} \alpha' &= \alpha \quad , \\ \beta' &= \beta-\pi \; , \\ \gamma' &= \pi-\gamma \; . \end{aligned} \tag{A2.13}$$

and

$$\begin{aligned} D^L_{MM'}(\alpha'\beta'\gamma') &= e^{iM\alpha} d^L_{MM'}(\beta-\pi) e^{iM'(\pi-\gamma)} \\ &= (-1)^L D^L_{M-M'}(\alpha\;\beta\;\gamma) \; . \end{aligned} \tag{A2.14}$$

Appendix 3 Construction of S

We present two methods to obtain the differential operator for S on the manifold S_5.

Method 1

By definition, we have

$$S = \frac{1}{2}\sum_{i=1}^{3} K_{ii} = \frac{1}{2}\sum(\Lambda_{i,i+3} - \Lambda_{i+3,i}) = \sum \Lambda_{i,i+3}$$

$$= -i\hbar \sum (r_i \frac{\partial}{\partial r_{i+3}} - r_{i+3}\frac{\partial}{\partial r_i}) \tag{A3.1}$$

$$= -i\hbar (\underline{r}^{(1)} \cdot \frac{\partial}{\partial \underline{r}^{(2)}} - \underline{r}^{(2)} \cdot \frac{\partial}{\partial \underline{r}^{(2)}}).$$

If we introduce the complex variables

$$\underline{z} = \underline{r}^{(2)} + i\underline{r}^{(1)} = re^{i\frac{\phi}{2}}(\cos\psi\underline{u} - i\sin\psi\underline{v}) ,$$

$$\underline{z}^* = \underline{r}^{(2)} - i\underline{r}^{(1)} = re^{-i\frac{\phi}{2}}(\cos\psi\underline{u} + i\sin\psi\underline{v}) , \tag{A3.2}$$

and therefore

$$\underline{r}^{(1)} = \frac{\underline{z} - \underline{z}^*}{2i} ,$$

$$\underline{r}^{(2)} = \frac{\underline{z} + \underline{z}^*}{2} , \tag{A3.3}$$

$$\frac{\partial}{\partial \underline{r}^{(1)}} = \frac{\partial \underline{z}}{\partial \underline{r}^{(1)}} \cdot \frac{\partial}{\partial \underline{z}} + \frac{\partial \underline{z}^*}{\partial \underline{r}^{(1)}} \cdot \frac{\partial}{\partial \underline{z}^*} = i(\frac{\partial}{\partial \underline{z}} - \frac{\partial}{\partial \underline{z}^*}) ,$$

$$\frac{\partial}{\partial \underline{r}^{(2)}} = \frac{\partial \underline{z}}{\partial \underline{r}^{(2)}} \cdot \frac{\partial}{\partial \underline{z}} + \frac{\partial \underline{z}^*}{\partial \underline{r}^{(2)}} \cdot \frac{\partial}{\partial \underline{z}^*} = (\frac{\partial}{\partial \underline{z}} + \frac{\partial}{\partial \underline{z}^*}) , \tag{A3.4}$$

then S expressed in terms of $\underline{z}$ and $\underline{z}^*$ is

$$S = -i\hbar\left[\frac{1}{2i}(\underline{z} - \underline{z}^*)\cdot\left(\frac{\partial}{\partial \underline{z}} + \frac{\partial}{\partial \underline{z}^*}\right) - \frac{1}{2}(\underline{z} + \underline{z}^*)\cdot\, i\left(\frac{\partial}{\partial \underline{z}} - \frac{\partial}{\partial \underline{z}^*}\right)\right] \qquad \text{(A3.5)}$$

$$= -\hbar\left(\underline{z}\cdot\frac{\partial}{\partial \underline{z}} - \underline{z}^*\cdot\frac{\partial}{\partial \underline{z}^*}\right) .$$

Now consider the operator $\frac{\partial}{\partial \phi}$ on a function of $\underline{z}$ and $\underline{z}^*$:

$$\frac{\partial}{\partial \phi} f(\underline{z},\underline{z}^*) = \left(\frac{\partial \underline{z}}{\partial \phi}\cdot\frac{\partial}{\partial \underline{z}} + \frac{\partial \underline{z}^*}{\partial \phi}\cdot\frac{\partial}{\partial \underline{z}^*}\right) f(\underline{z},\underline{z}^*) \qquad \text{(A3.6)}$$

But by (A3.2),

$$\frac{\partial \underline{z}}{\partial \phi} = \frac{i}{2}\underline{z} ,$$

$$\frac{\partial \underline{z}^*}{\partial \phi} = -\frac{i}{2}\underline{z}^* . \qquad \text{(A3.7)}$$

It follows that

$$\frac{\partial}{\partial \phi} = \frac{i}{2}\left(\underline{z}\cdot\frac{\partial}{\partial \underline{z}} - \underline{z}^*\cdot\frac{\partial}{\partial \underline{z}^*}\right) \qquad \text{(A3.8)}$$

which when compared with Eqn.(A3.5) gives

$$S = 2i\hbar\frac{\partial}{\partial \phi} . \qquad \text{(A3.9)}$$

Method 2

The result of the first method shows that S is independent of the Euler angles, so we might attempt to construct it by performing the coordinates transformation taking $\underset{\sim}{r}$ to $r, \psi, \phi, \alpha, \beta, \gamma$ with $\alpha = \beta = \gamma = 0$. By Eqns. (3.7) and (3.8), we see that in this case the body-fixed axes coincide with the space-fixed axes and the non-zero r_i s are

$$\begin{aligned} r_1 &= r \cos\psi \sin\frac{\phi}{2} , \\ r_2 &= -r \sin\psi \cos\frac{\phi}{2} , \\ r_4 &= r \cos\psi \sin\frac{\phi}{2} , \\ r_5 &= r \sin\psi \sin\frac{\phi}{2} . \end{aligned} \tag{A3.10}$$

These are just the components of the vector $\underset{\sim}{r}^{(1)}$ and $\underset{\sim}{r}^{(2)}$ with respect to $\underset{\sim}{u}$ and $\underset{\sim}{v}$. Because of the condition (A1.2), only three of them are independent variables which are chosen to be r_1, r_2 and r_4 so that

$$r_5 = -\frac{r_1 r_2}{r_4} . \tag{A3.11}$$

The differential operators for the two sets of coordinates are related by

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \psi} \\ \frac{\partial}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\psi \sin\frac{\phi}{2} & -\sin\psi \cos\frac{\phi}{2} & \cos\psi \cos\frac{\phi}{2} \\ -r \sin\psi \sin\frac{\phi}{2} & -r \cos\psi \cos\frac{\phi}{2} & -r \sin\psi \cos\frac{\phi}{2} \\ \frac{r}{2} \cos\psi \cos\frac{\phi}{2} & \frac{r}{2} \sin\psi \sin\frac{\phi}{2} & -\frac{r}{2} \cos\psi \sin\frac{\phi}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r_1} \\ \frac{\partial}{\partial r_2} \\ \frac{\partial}{\partial r_3} \end{bmatrix} \tag{A3.12}$$

whose inverse is

$$\begin{bmatrix} \frac{\partial}{\partial r_1} \\ \frac{\partial}{\partial r_2} \\ \frac{\partial}{\partial r_3} \end{bmatrix} = \frac{1}{2r \cos\psi \cos\frac{\phi}{2}} \begin{bmatrix} r \sin\phi & \cdot & 4\cos^2\frac{\phi}{2} \\ -r \sin 2\psi & -2\cos^2\psi & \cdot \\ r(\cos\phi + \cos 2\psi) & \sin 2\psi & -2\sin\phi \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \psi} \\ \frac{\partial}{\partial \phi} \end{bmatrix} \tag{A3.13}$$

We can now construct the differential operators for K_{ii} on the sub-manifold of S_5 (with $\alpha = \beta = \gamma = 0$). We use a prime to denote operators defined only on this sub-manifold. Thus

$$K'_{11} = \Lambda'_{14} - \Lambda'_{41} = 2\Lambda'_{14} = 2i\hbar(r_1\frac{\partial}{\partial r_4} - r_4\frac{\partial}{\partial r_1})$$

$$= -i\hbar(-2r \sin^2\psi \tan\frac{\phi}{2}\frac{\partial}{\partial r} - \sin 2\psi \tan\frac{\phi}{2} - 4\frac{\partial}{\partial\phi}) ,$$

$$K'_{22} = \Lambda'_{25} - \Lambda'_{52} = 2\Lambda'_{25} = -2 r_5 \frac{\partial}{\partial r_2} \tag{A3.14}$$

$$= -i\hbar(2r \sin^2\psi \tan\frac{\phi}{2}\frac{\partial}{\partial r} + \sin 2\psi \tan\frac{\phi}{2}) ,$$

$$K'_{33} = \Lambda'_{3,6} - \Lambda'_{6,3} = 2\Lambda'_{3,6} = 2(r_3\frac{\partial}{\partial r_6} - r_6\frac{\partial}{\partial r_3}) = 0 .$$

and therefore

$$S = \frac{1}{2}\sum_{i=1}^{3} K'_{ii} = K'_{11} + K'_{22} = 2i\hbar\frac{\partial}{\partial\phi} . \tag{A3.15}$$

Appendix 4 Construction of Λ^2

Let us first establish relation (2.13) ,

$$\Lambda^2 = r^2(2mT - p_r^2 + 5\, i\hbar\, r^{-1} p_r) , \tag{A4.1}$$

where

$$T = \frac{1}{2m} \sum_{i=1}^{6} p_i^2 \tag{A4.2}$$

is the operator for the total kinetic energy and

$$p_r = m \frac{dr}{dt} = r^{-1} \sum_i r_i p_i \tag{A4.3}$$

is the momentum operator whose commutation relation with r is

$$r p_r - p_r r = i\hbar . \tag{A4.4}$$

Using Eqn.(2.8) for Λ_{ij}, we have

$$\Lambda^2 = \tfrac{1}{2} \sum_{i,j} \Lambda_{ij}^2 = \tfrac{1}{2} \sum_{i,j} (r_i p_j - r_j p_i)^2$$

$$= \sum_{i,j} (r_i p_j r_i p_j - r_i p_j r_j p_i) . \tag{A4.5}$$

But

$$\sum_{i,j} r_i p_j r_i p_j = \sum_{i,j} r_i (r_i p_j - i\hbar\, \delta_{ij}) p_j$$

$$= r^2 2mT - i\hbar r\, p_r \tag{A4.6}$$

and

$$\sum_{i,j} r_i p_j r_j p_i = \sum_{i,j} r_i (r_j p_i - i\hbar) p_j$$

$$= -5i\hbar r p_r + r p_r r p_r$$

$$= -5i\hbar\, r p_r + r(r p_r - i\hbar) p_r$$

$$= r^2 p_r^{\,2} - 6i\hbar\, r p_r \; . \qquad \text{(A4.7)}$$

Substituting Eqns.(A4.6) and (A4.7) in Eqn.(A4.5), we obtain the required relation (A4.1). Since $T = -(\hbar^2/2m)\, \nabla_6^2$ and $p_r = i(\hbar/r) \sum_i r_i\, \partial/\partial r_i$, we have the relation

$$\Lambda^2 = -\hbar^2 r^2 \left[\nabla_6^2 - \frac{1}{r^5} \frac{\partial}{\partial r} \left(r^5 \frac{\partial}{\partial r} \right) \right] \qquad \text{(A4.8)}$$

The problem of obtaining Λ^2 in coordinates $\underset{\sim}{C}$ is therefore reduced to finding the Laplace operator ∇_6^2 in six-dimensions in the same coordinates.

We write ∇_6^2 as

$$\nabla_6^2 = \frac{1}{r^{(1)^2}} \frac{\partial}{\partial r^{(1)}} \left(r^{(1)^2} \frac{\partial}{\partial r^{(1)}} \right) + \frac{1}{r^{(2)^2}} \frac{\partial}{\partial r^{(2)}} \left(r^{(2)^2} \frac{\partial}{\partial r^{(2)}} \right) - \frac{1}{\hbar^2} \left(\frac{\underset{\sim}{l}^2_{(1)}}{r^{(1)^2}} + \frac{\underset{\sim}{l}^2_{(2)}}{r^{(2)^2}} \right) \qquad \text{(A4.9)}$$

where $\underset{\sim}{l}_{(i)}$ is the angular momentum operator associated with $\underset{\sim}{r}^{(i)}$. We have not specified the coordinate frame yet. In the original space-fixed frame S_o, it is of course given by

$$\underline{l}_{(i)}^2 = -\hbar^2 \frac{1}{\sin\Theta_i}\frac{\partial}{\partial\Theta_i}(\sin\Theta_i \frac{\partial}{\partial\Theta_i}) + \frac{1}{\sin^2\Theta_i}\frac{\partial^2}{\partial\Theta_i^2} \qquad (A4.10)$$

where Θ_i and ϕ_i are the angles of $\underline{r}^{(i)}$ in S_o. We define

$$\cos\delta = \frac{\underline{r}^{(1)} \cdot \underline{r}^{(2)}}{r^{(1)} r^{(2)}} , \qquad (A4.11)$$

$$\underline{L} = \underline{l}_{(1)} + \underline{l}_{(2)} . \qquad (A4.12)$$

Then, the last term in Eqn.(A4.9) is

$$-\frac{1}{\hbar^2}\left[\frac{\underline{L}^2}{{r^{(2)}}^2} + \left(\frac{1}{{r^{(1)}}^2} + \frac{1}{{r^{(2)}}^2}\right)\underline{l}_{(1)}^2 - 2\frac{\underline{L}\cdot\underline{l}_{(1)}}{{r^{(2)}}^2}\right] \qquad (A4.13)$$

We can now introduce the representations for the angular momentum operators $\underline{L}$ and $\underline{l}_{(1)}$ directly in S_1. This frame has Euler angles given by

$$\alpha_1 = \phi_1$$
$$\beta_1 = \Theta_1 \qquad (A4.14)$$

$$\sin\gamma_1 \cos\delta = \sin\Theta_1 \sin(\phi_1 - \phi_2)$$

In this frame, $\underline{r}^{(1)}$ is always in the $(\underline{x}_1, \underline{z}_1)$ plane, therefore $\underline{l}_{(1)} = (l_{\underline{x}_1}, l_{\underline{y}_1}, l_{\underline{z}_1})$ has to be defined in the limit when the azimuth angle, ε say, tends to zero; viz.

$$l_{\underline{x}_1} = \underset{\varepsilon \to 0}{\mathrm{Lt}}\; i\hbar\left(-\sin\varepsilon \frac{\partial}{\partial\delta} + \cot\delta\cos\varepsilon \frac{\partial}{\partial\phi}\right),$$

$$l_{\underline{y}_1} = \underset{\varepsilon \to 0}{\mathrm{Lt}}\; i\hbar\left(-\cos\varepsilon \frac{\partial}{\partial\delta} + \cot\delta\sin\varepsilon \frac{\partial}{\partial\phi}\right), \qquad \text{(A4.15)}$$

$$l_{\underline{z}_1} = -i\hbar \frac{\partial}{\partial\varepsilon} = L_{\underline{z}_1} .$$

These operators obviously satisfy the same commutational rules:

$$[\, l_{\underline{x}_1}, l_{\underline{y}_1}] = i\hbar\, l_{\underline{z}_1} , \text{ etc.} \qquad \text{(A4.16)}$$

Using (A4.15) we find

$$\underline{l}^2_{(1)} = \frac{1}{\sin^2\delta} L^2_{\underline{z}_1} ,$$

and Eqn.(A4.9) becomes

$$\nabla_6^2 = \frac{1}{r^{(1)^2}} \frac{\partial}{\partial r^{(1)}}\left(r^{(1)^2} \frac{\partial}{\partial r^{(1)}}\right) + \frac{1}{r^{(2)^2}} \frac{\partial}{\partial r^{(2)}}\left(r^{(2)^2} \frac{\partial}{\partial r^{(2)}}\right) +$$

$$+ \left(\frac{1}{r^{(1)^2}} + \frac{1}{r^{(2)^2}}\right)\left(\frac{\partial^2}{\partial \delta^2} + \cot\delta \frac{\partial}{\partial \delta}\right) \tag{A4.17}$$

$$- \frac{1}{\hbar^2}\left\{\frac{\underline{L}^2}{r^{(2)^2}} + \left[\left(\frac{1}{r^{(1)^2}} + \frac{1}{r^{(2)^2}}\right)\frac{1}{\sin^2\delta} - \frac{2}{r^{(2)^2}}\right] L^2_{\underline{z}_1}\right.$$

$$\left. + \frac{2i\hbar}{r^{(2)^2}} L_{\underline{y}_1} \frac{\partial}{\partial \delta} + \frac{2\cot\delta}{r^{(2)^2}} L_{\underline{x}_1} L_{\underline{z}_1}\right\} .$$

We go from S_1 to S_2 by taking

$$\begin{aligned} \underline{x}_2 &= \underline{x}_1 , \\ \underline{y}_2 &= \underline{z}_1 , \\ \underline{z}_2 &= -\underline{y}_1 , \end{aligned} \tag{A4.18}$$

and as a result, the last term in bracket in Eqn.(A4.17) goes over to

$$- \frac{1}{\hbar^2}\left\{\frac{\underline{L}^2}{r^{(2)^2}} + \left[\left(\frac{1}{r^{(1)^2}} + \frac{1}{r^{(2)^2}}\right)\frac{1}{\sin^2\delta} - \frac{2}{r^{(2)^2}}\right] L^2_{\underline{y}_2} - \right.$$

$$\left. - \frac{2i\hbar}{r^{(2)^2}} L_{\underline{z}_2} \frac{\partial}{\partial \delta} + \frac{2\cot\delta}{r^{(2)^2}} L_{\underline{x}_2} L_{\underline{y}_2}\right\} . \tag{A4.19}$$

The coordinates r, ψ and ϕ are now introduced in place of $r^{(1)}$, $r^{(2)}$ and δ. By Eqns.(A1.3), (A1.4) and (A4.11) we have

$$r^2 = r^{(1)^2} + r^{(2)^2},$$

$$\cos 2\psi = \frac{\left[(r^{(2)^2} - r^{(1)^2})^2 + 4\, r^{(1)^2} r^{(2)^2} \cos^2\delta\right]^{\frac{1}{2}}}{r^{(1)^2} + r^{(2)^2}}, \qquad \text{(A4.20)}$$

$$\tan\phi = \frac{2r^{(1)} r^{(2)} \cos\delta}{r^{(2)^2} - r^{(1)^2}},$$

and the differential operators are related by

$$\frac{\partial}{\partial r^{(1)}} = \frac{r^{(1)}}{r}\frac{\partial}{\partial r} + \frac{1}{2r^{(1)}}\sin 2\psi\cos\phi\frac{\partial}{\partial\psi} + \frac{1}{r^{(1)}}\frac{\sin\phi}{\cos 2\psi}\frac{\partial}{\partial\phi},$$

$$\frac{\partial}{\partial r^{(2)}} = \frac{r^{(2)}}{r}\frac{\partial}{\partial r} - \frac{1}{2r^{(2)}}\sin 2\psi\cos\phi\frac{\partial}{\partial\psi} - \frac{1}{r^{(2)}}\frac{\sin\phi}{\cos 2\psi}\frac{\partial}{\partial\phi}, \qquad (4.21)$$

$$\frac{\partial}{\partial\delta} = \tfrac{1}{2}\sin\phi\frac{\partial}{\partial\psi} - \tan 2\psi\cos\phi\frac{\partial}{\partial\phi}.$$

Then, after a tedious calculation, we obtain ∇_6^2 in these coordinates as

$$\nabla_6^2 = O_s - \frac{2}{\hbar^2 r^2(1+\cos 2\psi \cos\phi)} \Big[\underline{L}^2 + \frac{2\cos 2\psi \sin\phi}{\sin 2\psi} L_{\underline{x}_2} L_{\underline{y}_2}$$

$$+ \frac{2\cos 2\psi}{\sin^2 2\psi}(\cos 2\psi + \cos\phi) L^2_{\underline{y}_2} - 2i\hbar L_{\underline{z}_2}\left(\tfrac{1}{2}\sin\phi \frac{\partial}{\partial\psi} - \frac{\sin 2\psi}{\cos 2\psi}\frac{\partial}{\partial\phi}\right)\Big] \qquad \text{(A4.22)}$$

where O_s is the S-state operator

$$O_s = \frac{1}{r^5}\frac{\partial}{\partial r}\left(r^5 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\psi^2} + \frac{4\cos 4\psi}{\sin 4\psi} + \frac{4}{\cos^2 2\psi}\frac{\partial^2}{\partial\phi^2}\right). \qquad \text{(A4.23)}$$

Finally, we bring the axes $\underline{x}_2$ and $\underline{y}_2$ to coincide with the two principal axes of inertia of the triangle such that $\underline{r}^{(2)}$ subtends the angle ε with $\underline{u}$ in this, S, frame. This means that the third Euler angle γ is increased by $\frac{\pi}{2} - \varepsilon$ with the result that

$$\frac{\partial}{\partial\psi} \longrightarrow \frac{\partial}{\partial\psi} - \frac{\sin\phi}{(1+\cos 2\psi \cos\phi)}\frac{\partial}{\partial\gamma},$$

$$\frac{\partial}{\partial\phi} \longrightarrow \frac{\partial}{\partial\phi} - \frac{\sin 2\psi}{2(1+\cos 2\psi\cos\phi)}\frac{\partial}{\partial\gamma}. \qquad \text{(A4.24)}$$

The rotation operators with respect to S_2 and S are related by

$$L_{\underline{x}_2} = \sin\varepsilon\, L_{\underline{u}} - \cos\varepsilon\, L_{\underline{v}},$$

$$L_{\underline{y}_2} = \cos\varepsilon\, L_{\underline{u}} + \sin\varepsilon\, L_{\underline{v}}, \qquad \text{(A4.25)}$$

$$L_{\underline{z}_2} = L_{\underline{w}}.$$

So we have

$$\nabla_6^2 = O_s - \frac{1}{r^2\hbar^2}\Big[\frac{2\underline{L}^2}{\sin^2 2\psi} + \Big(\frac{1}{\cos^2 2\psi} - \frac{2}{\sin^2 2\psi}\Big)\underline{L}_w^2 + 4i\hbar\frac{\sin 2\psi}{\cos^2 2\psi}\underline{L}_w\frac{\partial}{\partial\phi} + \frac{\cos 2\psi}{\sin^2 2\psi}(\underline{L}_+^2 + \underline{L}_-^2)\Big] \tag{A4.26}$$

with

$$\underline{L}_\pm = \underline{L}_u \pm i\,\underline{L}_v \,. \tag{A4.27}$$

Using the fact that a rotation operator can be resolved into components like a vector, we find

$$\underline{L}_u = -i\hbar\left(-\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \sin\gamma\frac{\partial}{\partial\beta} + \cos\gamma\cot\beta\frac{\partial}{\partial\gamma}\right),$$

$$\underline{L}_v = -i\hbar\left(\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \cos\gamma\frac{\partial}{\partial\beta} - \sin\gamma\cot\beta\frac{\partial}{\partial\gamma}\right) \tag{A4.28}$$

$$\underline{L}_w = -i\hbar\frac{\partial}{\partial\gamma} \,.$$

Relations (4.6) and (4.7) thus follow immediately.

Appendix 5 On orthogonal polynomials and spherical harmonics in six-dimensions

In this appendix we collect together some results on orthogonal polynomials and spherical harmonics in six-dimensions details of which can be found in ref.(22). We also indicate their relevance in this thesis.

A family of quadratically integratable (L^2) functions is said to form an orthogonal system in the interval (a,b) with a weigh function w(x) which is non-negative there if for any two distinct members, $\phi_1(x)$ and $\phi_2(x)$, their scalar product vanishes, that is,

$$(\phi_1,\phi_2) = \int_a^b w(x)\phi_1\phi_2 dx = 0 . \qquad (A5.1)$$

Since the space of L^2 functions is separable, it follows that an orthogonal system consists either of a finite number or at most of a denumerable infinity of elements. Thus an orthogonal system can always be written as a sequence, $\phi_o,\phi_1\ldots$ or shortly as $\{\phi_n(x)\}$. Now every orthogonal system can be normalized by replacing $\phi_n(x)$ by $(\phi_n,\phi_n)^{-\frac{1}{2}}\phi_n(x)$, and we have an orthonormal system i.e. $\{\phi_n\}$ such that

$$(\phi_h,\phi_k) = \delta_{hk} . \qquad (A5.2)$$

If ϕ_k is a polynomial in x of exact degree k, denoted by $P_k(x)$ say, then $\{P_k(x)\}$ is a sequence of orthogonal polynomials. The interval and weigh function determine the system of orthogonal polynomials up to an arbitrary factor in each $P_k(x)$. The polynomials can be standardized by the adoption of additional requirements. Our standardization is such that for a given x_o, $P_k(x_o)$ shall have a prescribed value. The orthogonal polynomials of interest to us in this thesis, apart from the well-known Legendre polynomials, are the Jacobi (or hypergeometric) and the Gegenbauer (or ultra-spherical) polynomials for a six-dimensional sphere.

The Jacobi Polynomials

We use Szego's notation $P_n^{(\alpha,\beta)}(x)$ for the suitably standardized orthogonal polynomials associated with

$$a = -1, \quad b = 1, \quad w(x) = (1-x)^\alpha (a+x)^\beta .$$

We give below some properties of these polynomials which are used in the text:

standardization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \tag{A5.3}$$

with $\binom{m}{n} = \dfrac{m!}{n!(m-n)!}$

explicit expression

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{m=o}^{n} \binom{n+\alpha}{m}\binom{n+\beta}{n-m} (x-1)^{n-m}(x+1)^m \tag{A5.4}$$

differential equation

$$(1-x^2)y'' + [\beta-\alpha-(\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0 \qquad (A5.5)$$

where $y = P_n^{(\alpha,\beta)}(x)$.

orthogonalization

$$\int_{-1}^{+1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}\delta_{mn} \qquad (A5.6)$$

In Chapter 4, Section I, in order to obtain representations of those SU(3) states which L = 0, we have to solve the differential equation (4.20) when L = K = 0. The differential equation concerned is

$$\left[\frac{\partial^2}{\partial\psi^2} + \frac{4\cos 4\psi}{\sin 4\psi}\frac{\partial}{\partial\psi} - \frac{4\mu^2}{\cos^2 2\psi} + \lambda(\lambda+4)\right] g_{\lambda\mu}(\psi) = 0 . \qquad (A5.7)$$

We now give a method of solution. By a change of variable

$$\rho = \cos 2\psi ,$$

Eqn.(A5.7) becomes

$$\left\{(1-\rho^2)\frac{\partial^2}{\partial\rho^2} + (\frac{1}{\rho} - 3\rho)\frac{\partial}{\partial\rho} + \frac{1}{4}\left[\lambda(\lambda+4) - \frac{4\mu^2}{\rho^2}\right]\right\} g_{\lambda\mu}(\rho) = 0 . \qquad (A5.8)$$

We seek a solution of the form

$$g_{\lambda\mu}(\rho) = \rho^{|\mu|} h_{\lambda\mu}(\rho) . \qquad (A5.9)$$

Then, the differential equation for h(ρ) is

$$\rho^3 \frac{\partial^2 h}{\partial \rho^2} + (3\rho^2 - 2\mu\rho^2 - 1 - 2\mu)\frac{\partial h}{\partial \rho} + \rho\left[2\mu + \mu^2 - \tfrac{1}{4}\lambda(\lambda+4)\right]h = 0 , \qquad \text{(A5.10)}$$

where, for simplicity, we have left out the labels λ, μ on $h(\rho)$ and the modulus sign on μ .

If we change the variable again to

$$\sigma = 1 - 2\rho^2 ,$$

we have for h the differential equation

$$(1-\sigma^2)\frac{\partial^2 h}{\partial \sigma^2} + \left[-\mu-(\mu+2)\sigma\right]\frac{\partial h}{\partial \sigma} + \tfrac{1}{2}(\tfrac{\lambda}{2} - \mu)\left[\tfrac{1}{2}(\tfrac{\lambda}{2} - \mu) + \mu + 1\right] h = 0 . \qquad \text{(A5.1}$$

This can be compared with Eqn.(A5.5) for the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$; and so we find

$$h_{\lambda\mu}(\sigma) = P^{|\mu|,0}_{\frac{1}{2}(\frac{\lambda}{2} - |\mu|)}(\sigma) .$$

Thus

$$g_{\lambda\mu}(\psi) = (\cos 2\psi)^{|\mu|} P^{|\mu|,0}_{\frac{1}{2}(\frac{\lambda}{2} - |\mu|)}(1-2\cos^2 2\psi) . \qquad \text{(A5.12)}$$

The Gegenbauer polynomials

We use Gegenbauer's notation $C_\lambda^{\frac{1}{2}p}(x)$ for the suitably standardized polynomials associated with

$a = -1$, $b = 1$, $w(x) = (1-x^2)^{\frac{1}{2}(p-1)}$, $p > -1$ and for $w(x)$ positive and square-integratable.

These polynomials are generalizations of the Legendre's polynomials for a (p+2)-dimensional sphere. Since the manifold we are concerned with is the five-dimensional surface S_5, we consider the case of p = 4. The polynomials $C^2_\lambda(x)$ are standardized in accordance with

$$C^2_\lambda(1) = \binom{\lambda+3}{\lambda} . \qquad \text{(A5.13)}$$

The differential equation for $C^{\frac{1}{2}p}(x)$ is

$$(1-x^2)y'' - (p+1)xy' + \lambda(\lambda+p)y = 0 \qquad \text{(A5.14)}$$

It is obvious from the above equation that for p=1, we have the differential equation for the Legendre's function. In common with the usual partial wave expansion of a plane wave state, $e^{i\underline{k}.\underline{r}}$, in three dimensions in terms of the Legendre polynomials, we have the corresponding result for the plane wave state in six-dimensions in terms of the Gegenbauer polynomials:

$$e^{i\underline{k}.\underline{r}} = e^{ikr(\hat{\underline{k}}.\hat{\underline{r}})} = 4 \sum_\lambda \frac{(\lambda+2)i^\lambda}{(kr)^2} J_{\lambda+2}(kr)C^2_\lambda(\hat{\underline{k}}.\hat{\underline{r}}) \qquad \text{(A5.15)}$$

where $J_m(x)$ is the Bessel function of the first kind.

The following addition theorem for the Gegenbauer polynomials is also useful to us:

Let $S^l_\lambda(\hat{\underline{r}})$, l = 1,... h, be h = h (λ) [Eqn.(4.9)] linearly independent surface harmonics of degree λ, and let the S^l_λ be orthonormal on S_5 so that for $l, m = 1, 2, \ldots h$

$$\int S_\lambda^{*l}(\hat{\underline{r}}) S_\lambda^m(\hat{\underline{r}}) d\Omega(\hat{\underline{r}}) = \delta_{lm}, \qquad \text{(A5.16)}$$

then for any fixed unit vector $\hat{\underline{k}}$

$$C_\lambda^2(\hat{\underline{k}}.\hat{\underline{r}}) = \frac{\omega}{h} C_\lambda^2(1) \sum_{l=1}^{h} S_\lambda^{*l}(\hat{\underline{k}}) S_\lambda^l(\hat{\underline{r}}) \qquad \text{(A5.17)}$$

where ω is the total surface area.

Corollary. For every surface harmonics $S_\lambda(\hat{\underline{k}})$ of degree λ,

$$\int C_\lambda^2(\hat{\underline{k}}.\hat{\underline{r}}) S_{\lambda'}(\hat{\underline{k}}) d\Omega(\hat{\underline{k}}) = \delta_{\lambda\lambda'} \frac{\omega}{h} C_\lambda^2(1) S_\lambda(\hat{\underline{r}}) . \qquad \text{(A5.18)}$$

For S_5, it can easily be shown that in any polar coordinates irrespective of the choice of angular variables $\omega = \pi^3$. Consider the integral

$$\int_{-\infty}^{+\infty} \exp(-r_1^2 - r_2^2 \ldots r_6^2) dr_1 \ldots dr_6 = \left(\int_{-\infty}^{+\infty} e^{-r_1^2} dr_1\right)^6 = \int_V \exp(-r^2) dV =$$

$$= \int_V \exp(-r^2) r^5 d\Omega\, dr. \qquad \text{(A5.19)}$$

where, by definition, $d\Omega = dV/(r^5 dr)$ and is of the form $\prod_{i=1}^{5} g_i(\hat{\underline{r}})\, d\Theta_i$ with Θ_i denoting the angular variables. Using the equality of the second and last term, we get $\omega = \pi^3$.

It is interesting to use Eqn.(A5.15) and the addition theorem to obtain the δ-function, $\delta(\underline{k}-\underline{k}')$, in terms of the surface harmonics $S_\lambda^l(\hat{\underline{k}})$. We start with

$$\delta(\underline{k}-\underline{k}') = (2\pi)^{-6} \int e^{i(\underline{k}-\underline{k}')\cdot\underline{r}} d\underline{r}$$

$$= (2\pi)^{-6} \int e^{i\underline{k}\cdot\underline{r}} e^{-i\underline{k}'\cdot\underline{r}} d\underline{r} \, .$$

By Eqn.(A5.16), we have

$$\delta(\underline{k}-\underline{k}') = \frac{1}{4\pi^6} \sum_{\lambda\lambda'} \frac{i^\lambda(-i)^{\lambda'}(\lambda+2)(\lambda'+2)}{(kk')^2} \int rJ_{\lambda+2}(kr)J_{\lambda'+2}(k'r)dr \int C_\lambda^2(\hat{\underline{k}}\cdot\hat{\underline{r}})C_{\lambda'}^2(\hat{\underline{k}}'\cdot\hat{\underline{r}})d\Omega$$

$$= \frac{\delta(k-k')}{k^5} \sum_{\lambda,l} S_\lambda^l(\hat{\underline{k}})S_\lambda^{*l}(\hat{\underline{k}}') \, , \qquad \text{(A5.20)}$$

where we have used the relation

$$\int rJ_{\lambda+2}(kr)J_{\lambda'+2}(k'r)dr = 2\delta(k^2-k'^2) \, .$$

If $\underline{\xi}$ is a three-dimensional vector, the corresponding result is

$$\delta(\underline{\xi}-\underline{\xi}') = \frac{\delta(\xi-\xi')}{\xi'^2} \sum_{lm} Y_l^m(\hat{\underline{\xi}})Y_l^{*m}(\hat{\underline{\xi}}')$$

which suggests that

$$\delta(\underline{\xi}-\underline{\xi}') = \frac{2}{\xi} \delta(\xi^2-\xi'^2)\delta(\cos\Theta - \cos\Theta')\delta(\phi-\phi') \qquad \text{(A5.21)}$$

This relation is used in Appendix 7 to evaluate the matrix elements in Omnes's angular momentum analysis.

Appendix 6 On the spherical harmonics $S^{LM}_{\lambda\mu_j\iota K}$

In this appendix, we show how to arrive at the spherical harmonics $S^{LM}_{\lambda\mu_j\iota K}$ which have the following symmetry properties when k is absent: For $j = 3$, $S^{LM}_{\lambda\mu_j 1}$ and $S^{LM}_{\lambda\mu_j 2}$ are asymmetric and symmetric respectively whilst for other j values the pair transform as the two-dimensional representation of S_3.

Let us consider the effect of exchanging particle 1 and 2 on the spherical harmonics

$$S^{\mu LM}_{\lambda}(\hat{\underline{k}}) = N_{\Omega}\ e^{-\mu\phi} \sum_{K} g^{LK}_{\lambda\mu}(\psi) D^{L}_{MK}(R) \ . \tag{A6.1}$$

By the symmetry properties of ϕ and the Euler angles, we have

$$(12)S^{\mu LM}_{\lambda}(\hat{\underline{k}}) = (-1)^L N_{\Omega} e^{i\mu\phi} \sum_{K} g^{LK}_{\lambda\mu}(\psi) D^{L}_{M-K}(R) \ . \tag{A6.2}$$

On changing K to -K and using relation (4.25) that

$$g^{LK}_{\lambda\mu}(\psi) = G^{L-K}_{\lambda-\mu}(\psi) \ , \tag{A6.3}$$

we find

$$(12)S^{\mu LM}_{\lambda} = (-1)^L S^{-\mu LM}_{\lambda} \ . \tag{A6.4}$$

This suggests the following alternative set of spherical harmonics $S^{LM}_{\lambda|\mu|\iota}$ where $\iota = 1$ or 2 and defined by

$$S^{LM}_{\lambda|\mu|1}(\hat{\underline{k}}) = \frac{i}{\sqrt{2}}(S_{\lambda}^{|\mu|LM}(\hat{\underline{k}}) - S_{\lambda}^{-|\mu|LM}(\hat{\underline{k}}))$$

$$= \frac{iN_{\Omega}}{\sqrt{2}} \sum_{K} (e^{-i|\mu|\phi} g^{LK}_{\lambda|\mu|}(\psi) - e^{i|\mu|\phi} g^{LK}_{\lambda-|\mu|}(\psi)) D^{L}_{MK}(R) , \tag{A6.5}$$

$$S^{LM}_{\lambda|\mu|2}(\hat{\underline{k}}) = \frac{i}{\sqrt{2}}(S_{\lambda}^{|\mu|LM}(\hat{\underline{k}}) + S_{\lambda}^{-|\mu|LM}(\hat{\underline{k}}))$$

$$= \frac{N_{\Omega}}{\sqrt{2}} \sum_{K} (e^{-i|\mu|\phi} g^{LK}_{\lambda|\mu|}(\psi) + e^{i|\mu|\phi} g^{LK}_{\lambda-|\mu|}(\psi)) D^{L}_{MK}(R) . \tag{A6.6}$$

Let us first study the symmetry properties of $S^{LM}_{\lambda|\mu|\iota}$ when λ is even and hence $|\mu|$ is integral. We divide the set of $|\mu|$'s into three subsets $\{\mu_i\}$, $i = 1,2$ or 3, such that

$$\begin{aligned} \mu_1 &= 3n + 1 \\ \mu_2 &= 3n + 2 \qquad n = 0,1,2,\cdots , \\ \mu_3 &= 3n \end{aligned} \tag{A6.7}$$

Then, it follows from Eqn.(A6.4) that $S^{LM}_{\lambda\mu_i\iota}$, for all i, transform under (12) as

$$(12)\begin{bmatrix} S^{LM}_{\lambda\mu_i 1} \\ S^{LM}_{\lambda\mu_i 2} \end{bmatrix} = (-1)^L \begin{bmatrix} -1 & \\ & +1 \end{bmatrix} \begin{bmatrix} S^{LM}_{\lambda\mu_i 1} \\ S^{LM}_{\lambda\mu_i 2} \end{bmatrix} \qquad (A6.8)$$

Now consider the effect of (23) on $S^{LM}_{\lambda\mu_i \iota}$. Taking $j = 1$ and $\iota = 1$, we have

$$(23)\, S^{LM}_{\lambda\mu_1 1}(\hat{\underline{k}}) = \frac{iN_\Omega}{\sqrt{2}} \sum_K \left(e^{-i\mu_1(\phi - \frac{4\pi}{3})} g^{LK}_{\lambda\mu_1}(\psi) - e^{i\mu_1(\phi-\frac{4\pi}{3})} g^{LK}_{\lambda-\mu_1}(\psi) D^L_{M-K}\right.$$

(A6.9)

But

$$e^{i\mu_1 \frac{4\pi}{3}} = -\tfrac{1}{2} - i\frac{\sqrt{3}}{2} ,$$

$$e^{-i\mu_1 \frac{4\pi}{3}} = -\tfrac{1}{2} + i\frac{\sqrt{3}}{2} . \qquad (A6.10)$$

Therefore on using Eqn.(A6.3) again, we find

$$(23)\, S^{LM}_{\lambda\mu_1 1} = (-1)^{L+1}\left(-\tfrac{1}{2}S^{LM}_{\lambda\mu_1 1} + \frac{\sqrt{3}}{2} S^{LM}_{\lambda\mu_1 2}\right) . \qquad (A6.11)$$

Similar argument for $\iota = 2$ leads to

$$(23)\, S^{LM}_{\lambda\mu_1 2} = (-1)^L\left(-\frac{\sqrt{3}}{2} S^{LM}_{\lambda\mu_1 1} - \tfrac{1}{2}S^{LM}_{\lambda\mu_1 2}\right) . \qquad (A6.12)$$

Eqns.(A6.11) and (A6.12) are equivalent to

$$(23)\begin{bmatrix} S^{LM}_{\lambda\mu_1 1} \\ -S^{LM}_{\lambda\mu_1 2} \end{bmatrix} = (-1)^L \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} S^{LM}_{\lambda\mu_1 1} \\ -S^{LM}_{\lambda\mu_1 2} \end{bmatrix} . \qquad (A6.13)$$

The effects of (23) on $S^{LM}_{\lambda\mu_i \iota}$ for other i values can be treated by the same method. The results are: for i = 2

$$(23)\begin{bmatrix} S^{LM}_{\lambda\mu_2 1} \\ S^{LM}_{\lambda\mu_2 2} \end{bmatrix} = (-1)^L \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} S^{LM}_{\lambda\mu_2 1} \\ S^{LM}_{\lambda\mu_2 2} \end{bmatrix} , \qquad (A6.14)$$

and for i = 3

$$(23)\begin{bmatrix} S^{LM}_{\lambda\mu_3 1} \\ S^{LM}_{\lambda\mu_3 2} \end{bmatrix} = (-1)^L \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \begin{bmatrix} S^{LM}_{\lambda\mu_3 1} \\ S^{LM}_{\lambda\mu_3 2} \end{bmatrix} \qquad (A6.15)$$

Since the transpositions (12) and (23) can generate all the elements of S_3, Eqns.(A6.8), (A6.13), (A6.14) and (A6.15) completely specify the transformation properties of $S^{LM}_{\lambda\mu_i \iota}$ under S_3. From Eqn.(A6.13) we see that if $S^{LM}_{\lambda\mu_1 2}$ is redefined with a negative sign, then for L even, the spherical harmonics $S^{LM}_{\lambda\mu_i 1}$ and $S^{LM}_{\lambda\mu_i 2}$ do have the required symmetry properties. This leads to the definitions adopted in Eqn.(4.35i). For L odd, the extra factor of -1 suggests the definitions used in Eqn.(4.35ii).

For λ odd, $|\mu|$ is half-integral. We again divide the set of $|\mu|$'s into three sub-sets $\{\mu_i\}$, $i = 1,2$ or 3, such that

$$\begin{aligned} \mu_1 &= 3n + \frac{5}{2} , \\ \mu_2 &= 3n + \frac{1}{2} , \qquad n = 0,1,\ldots . \qquad \text{(A6.16)} \\ \mu_3 &= 3n + \frac{3}{2} , \end{aligned}$$

Then, it can be verified that the spherical harmonics defined in Eqns.(4.35i) and (4.35ii) also have the required symmetry properties.

Appendix 7   On the evaluation of the matrix element

$<\underline{\omega}, LMK|T_i(z)|\underline{\omega}', L'M'K'>$

The matrix element to be evaluated is

$$<\underline{\omega}, LMK_i|T_i(z)|\underline{\omega}', L'M'K_i'> = \left(\frac{3\sqrt{3A}}{8}\right)^2 \times$$

$$\int \delta(\underline{\omega}-\underline{\omega}^*)\delta(\underline{\omega}''-\underline{\omega}')\delta(\xi_i^*-\xi_i'')<\underline{\eta}_i^*|t_i(z-\xi_i''^2)|\underline{\eta}_i''>D^{*L}_{MK_i}(R^*)D^{L'}_{M'K'_i}(R'') \times$$

$$d\underline{\omega}^* d\underline{\omega}'' dR^* dR'' , \qquad (A7.1)$$

and we note that $<\underline{\eta}_i^*|t_i(z-\xi_i''^2)|\underline{\eta}_i''>$ depends only on η_i^{*2}, $\eta_i''^2$ and $\hat{\underline{\eta}}_i^* \cdot \hat{\underline{\eta}}_i''$ which is given by

$$\hat{\underline{\eta}}_i^* \cdot \hat{\underline{\eta}}_i'' = \cos\delta_i^* \cos\delta_i'' + \sin\delta_i^* \sin\delta_i'' \cos(\gamma^* - \gamma''). \qquad (A7.2)$$

Hence, in so far as dependence on the Euler angles is concerned, the two-particle transition amplitude depends only on

$$u = \gamma^* - \gamma'' \qquad (A7.3)$$

and so we write it as

$$<\underline{\eta}_i^*|t_i(z-\xi_i''^2)|\underline{\eta}_i''> = t_i(\eta_i^{*2}, \eta''^2, u; z-\xi_i''^2). \qquad (A7.4)$$

Using Eqn.(A5.21) for $\delta(\xi_i^* - \xi_i'')$ and the δ-functions $\delta(\underline{\omega} - \underline{\omega}^*)$ and $\delta(\underline{\omega}'' - \underline{\omega}')$, the integrations over $d\underline{\omega}^*$ and $d\underline{\omega}''$ can be performed trivially. The result is

$$<\underline{\omega}, LMK_i | T_i(z) | \underline{\omega}', L'M'K_i'> = (\frac{3\sqrt{6}}{8} A)^2 \frac{\delta(\xi_i^2 - \xi_i'^2)}{\xi_i'} \times$$

$$\iint t_i(\eta_i^2, \eta_i'^2, u; z - \xi_i'^2) D_{MK_i}^{*L}(R^*) D_{M'K_i'}^{L'}(R'') dR^* dR'' \delta(\alpha^* - \alpha'') \delta(\cos\beta^* - \cos\beta'') . \tag{A7.5}$$

Now, by virtue of relation (A7.3), we can choose the measure dR^* to be

$$dR^* = d\alpha^* \sin\beta^* d\beta^* du \tag{A7.6}$$

and so, after integrating over the remaining δ-functions in Eqn.(A7.5), we obtain

$$<\underline{\omega}, LMK_i | T_i(z) | \underline{\omega}', L'M'K_i'> = (\frac{3\sqrt{6}}{8} A)^2 \frac{\delta(\xi_i^2 - \xi_i'^2)}{\xi_i'} \times$$

$$\int_u \int_{R''} t_i(\eta_i^2, \eta_i'^2, u; z - \xi_i'^2) D_{MK_i}^{*L}(\alpha'', \beta'', u + \gamma'') D_{M'K_i'}^{L'}(\alpha'' \beta'' \gamma'') dR'' du . \tag{A7.7}$$

But

$$D_{MK_i}^{*L}(\alpha'', \beta'', u+\gamma'') = e^{-iM\alpha''} d_{MK_i}(\beta'') e^{-iK_i(u+\gamma'')}$$

$$= e^{-iK_i u} D_{MK_i}^{*L}(R'') , \tag{A7.8}$$

and we can integrate over dR'', using the orthogonality of the rotation matrices [Eqn.(5.24)], to obtain

$$\langle \underline{\omega}, LMK_i | T_i(z) | \underline{\omega}', L'M'K_i' \rangle = \frac{3\sqrt{3}}{4} \frac{\delta(\xi_i^2 - \xi_i'^2)}{\xi_i'} \times$$

$$\delta_{LL'} \delta_{MM'} \delta_{K_i K_i'} \int_0^{2\pi} t_i(\eta_i^2, \eta_i'^2, u; z - \xi_i'^2) e^{-iK_i'u} du \qquad (A7.9)$$

The last integral in Eqn.(A7.9) can also be evaluated. We expand $t_i(\eta_i^2, \eta_i'^2, u; z - \xi_i'^2)$ in terms of partial wave amplitudes

$$t_i(\eta_i^2, \eta_i'^2, u; z - \xi_i'^2) = 4\pi \sum_{l'm'} t_{i,l'}(\eta_i^2, \eta_i'^2; z - \xi_i'^2) Y_{l'm'}(\delta_i, \phi_i) Y^*_{l'm'}(\delta_i', \phi_i') \qquad (A7.10)$$

where (δ_i, ϕ_i) and (δ_i', ϕ_i') are the angular variables of $\underline{\eta}_i$ and $\underline{\eta}_i'$ respectively in a coordinate frame whose z-axis is along $\underline{\xi}_i$ so that $\phi - \phi' = u$. Thus

$$\int_0^{2\pi} t_i(\eta_i^2, \eta_i'^2, u; z - \xi_i'^2) e^{-iK_i'u} du$$

$$= 4\pi \sum_{l'm'} t_{i,l'}(\eta_i^2, \eta_i'^2; z - \xi_i'^2) Y_{l'm'}(\delta_i, o) Y^*_{l'm'}(\delta_i', o) \int_0^{2\pi} e^{i(m' - K_i')u} du$$

$$= 8\pi^2 \sum_{l'} t_{i,l'}(\eta_i^2, \eta_i'^2; z - \xi_i'^2) Y_{l'K_i'}(\delta_i, o) Y_{l'K_i'}(\delta_i', o) \qquad (A7.11)$$

where we have used

$$\int_0^{2\pi} e^{i(m'-K_i')u} du = 2\pi\delta_{m'K_i'} .$$

Putting $\xi_i = \sqrt{\frac{3}{2}}\, \omega_i^{\frac{1}{2}}$ and substituting Eqn.(A7.11) in Eqn.(A7.9) , we have finally

$$\langle \underline{\omega}, LMK_i | T_i(z) | \underline{\omega}', L'M'K_i' \rangle = 4\sqrt{2}\pi^2 \frac{\delta(\omega_i - \omega_i')}{\omega_i'^{\frac{1}{2}}} \delta_{LL'}\delta_{MM'}\delta_{KK_i'} \times$$

$$\sum_{l'} t_{i,l'}(\eta_i^2, \eta_i'^2; z - \xi_i'^2) Y_{l'K_i'}(\delta_i, o) Y_{l'K_i'}(\delta_i', o) . \qquad \text{(A7.12)}$$

It is perhaps worthwhile to mention the range of ω_1, ω_2 and ω_3 in the Omnes representation of three-particle states. In the centre of mass system, the condition $\sum_{i=1}^{3} \underline{k}_i = O$ has to be satisfied. We have seen, in the case of Dalitz coordinates, that this restricts $\rho^2 \leq 1$. Since $(2m\omega_k)^{\frac{1}{2}}$ represents the length of the vector $\underline{k}_k$, in order that the vectors can form a closed triangle their lengths must satisfy the triangular inequality which in terms of ω_k is

$$(\omega_i^{\frac{1}{2}} - \omega_j^{\frac{1}{2}})^2 \leq \omega_k \leq (\omega_i^{\frac{1}{2}} + \omega_j^{\frac{1}{2}})^2 . \qquad \text{(A7.13)}$$

The completeness of the states $|\underline{\omega}, LMK\rangle$ therefore refer to the set of ω_1,

ω_2 and ω_3 such that the above condition on ω_k is satisfied. Hence the completeness relation for these states should be

$$\sum_{LMK} \int_0^\infty d\omega_i \int_0^\infty d\omega_j \int_{(\omega_i^{\frac{1}{2}} - \omega_j^{\frac{1}{2}})^2}^{(\omega_i^{\frac{1}{2}} + \omega_j^{\frac{1}{2}})^2} d\omega_k \, |\underline{\omega}, LMK\rangle\langle\underline{\omega}, LMK| = 1 . \tag{A7.14}$$

The choice of which ω to be restricted in range is immaterial provided it is integrated first. In this connection it is important to find out the behaviour of δ-functions in such limits. Consider the integral

$$g(\underline{\omega}) = \int_0^\infty d\omega'_i \int_0^\infty d\omega'_j \int_{(\omega_i'^{\frac{1}{2}} - \omega_j'^{\frac{1}{2}})^2}^{(\omega_i'^{\frac{1}{2}} + \omega_j'^{\frac{1}{2}})^2} d\omega'_k \, \delta(\underline{\omega} - \underline{\omega}')f(\underline{\omega}') . \tag{A7.15}$$

To extend the range of ω'_k to cover the whole of the positive real axis, we modify the integrand by means of theta-functions so that

$$g(\underline{\omega}) = \int_0^\infty d\omega'_i \int_0^\infty d\omega'_j \int_{(\omega_i'^{\frac{1}{2}} - \omega_j'^{\frac{1}{2}})^2}^{(\omega_i'^{\frac{1}{2}} + \omega_j'^{\frac{1}{2}})^2} d\omega'_k \Big[\Theta(\omega'_k - (\omega_i'^{\frac{1}{2}} - \omega_j'^{\frac{1}{2}})^2) -$$

$$- \Theta(\omega'_k - (\omega_i'^{\frac{1}{2}} + \omega_j'^{\frac{1}{2}})^2)\Big]\delta(\underline{\omega} - \underline{\omega}')f(\underline{\omega}') \tag{A7.16}$$

which gives $g(\underline{\omega}) = \left[\Theta(\omega_k - (\omega_i^{\frac{1}{2}} - \omega_j^{\frac{1}{2}})^2) - \Theta(\omega_k - (\omega_i^{\frac{1}{2}} + \omega_j^{\frac{1}{2}})^2)\right] f(\underline{\omega})$,

that is, provided the ωs satisfy the condition (A7.13), δ-functions in these integration limits behave in the usual manner.

When we iterate the Faddeev equations in the Omnes representation, we have to consider integral of the form

$$\int_0^\infty d\omega_i' \int_0^\infty d\omega_j' \int_{(\omega_i'^{\frac{1}{2}} - \omega_j'^{\frac{1}{2}})^2}^{(\omega_i'^{\frac{1}{2}} + \omega_j'^{\frac{1}{2}})^2} d\omega_k' \; \delta(\omega_i - \omega_i')\delta(\omega_j - \omega_j') f(\underline{\omega}') \; . \qquad \text{(A7.17)}$$

It can be seen that in this case the δ-functions can be integrated first. The reason why this can be done is that the δ-functions pick up the limits for the $d\omega_k'$ integration. This is how we arrive at the limits of integrations in Eqn.(5.32).

Appendix 8 The Matrices $J_{\nu\nu'}$

$$J_{22} = G_o(z) \begin{bmatrix} & \frac{1}{2}(T_{1,2}+T_{1,3}) & \frac{1}{2}(T_{1,2}+T_{1,3}) \\ \frac{1}{2}(T_{2,2}+T_{2,3}) & \cdot & \frac{1}{2}(T_{2,2}+T_{2,3}) \\ \frac{1}{2}(T_{3,2}+T_{3,3}) & \frac{1}{2}(T_{3,2}+T_{3,3}) & \cdot \end{bmatrix}$$

$$J_{23} = J_{32} = G_o(z) \begin{bmatrix} \cdot & \frac{1}{4}(T_{1,2}-T_{1,3}) & \frac{1}{4}(T_{1,2}-T_{1,3}) \\ \frac{1}{4}(T_{2,2}-T_{2,3}) & \cdot & \frac{1}{4}(T_{2,2}-T_{2,3}) \\ -\frac{1}{2}(T_{3,2}-T_{3,3}) & -\frac{1}{2}(T_{3,2}-T_{3,3}) & \cdot \end{bmatrix}$$

$$J_{24} = G_o(z) \begin{bmatrix} \cdot & -\frac{\sqrt{3}}{4}(T_{1,2}-T_{1,3}) \\ \frac{\sqrt{3}}{4}(T_{2,2}-T_{2,3}) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$J_{33} = G_o(z) \begin{bmatrix} \cdot & \frac{1}{8}(T_{1,2}+T_{1,3}) & \frac{1}{8}(T_{1,2}+T_{1,3}) \\ \frac{1}{8}(T_{2,2}+T_{2,3}) & \cdot & \frac{1}{8}(T_{2,2}+T_{2,3}) \\ \frac{1}{2}(T_{3,2}+T_{3,3}) & \frac{1}{2}(T_{3,2}+T_{3,3}) & \cdot \end{bmatrix}$$

$$J_{34} = G_o(z) \begin{bmatrix} \cdot & -\frac{\sqrt{3}}{8}(T_{1,2}+T_{1,3}) \\ \frac{\sqrt{3}}{8}(T_{2,2}+T_{2,3}) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$J_{42} = G_o(z) \begin{bmatrix} \cdot & -\frac{\sqrt{3}}{4}(T_{1,2}-T_{1,3}) & -\frac{\sqrt{3}}{4}(T_{1,2}-T_{1,3}) \\ \frac{\sqrt{3}}{4}(T_{2,2}-T_{2,3}) & \cdot & \frac{\sqrt{3}}{4}(T_{2,2}-T_{2,3}) \end{bmatrix}$$

$$J_{43} = G_o(z) \begin{bmatrix} \cdot & -\frac{\sqrt{3}}{8}(T_{1,2}+T_{1,3}) & -\frac{\sqrt{3}}{8}(T_{1,2}+T_{1,3}) \\ \frac{\sqrt{3}}{8}(T_{2,2}+T_{2,3}) & \cdot & \frac{\sqrt{3}}{8}(T_{2,2}+T_{2,3}) \end{bmatrix}$$

$$J_{44} = G_o(z) \begin{bmatrix} \cdot & \frac{3}{8}(T_{1,2}+T_{1,3}) \\ \frac{3}{8}(T_{2,2}+T_{2,3}) & \end{bmatrix}$$