THE LINEAR THEORY OF WAVE PACKETS PROPAGATING IN THE WHISTLER MODE

by

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ABSTRACT

We discuss the propagation of wave packets of the form

$$e^{i(\omega t - kz)} G(z,t)$$

in an infinite uniform plasma (in both the formal cold plasma, and hot collision-free Vlasov plasma limit), where G(z,t) is a slowly varying function of space z and time t. One can derive the equation of change of G(z,t) (in both the above stated temperature limits) for the stable or unstable case. The terms in the equation are of physical interest and clearly define the limitations of linear theory. In particular we show that by using the model of complete stirring developed by A. C. Das, changes in apparent frequency $\frac{\partial \phi}{\partial t}$ (ϕ = phase of disturbance) can occur due to sharp changes in growth rate with respect to wave number.

We then investigate the problem of Whistler mode wave propagation in a collisionless Vlasov plasma in a given non-uniform magnetic field. We choose the electric field to be of a W.K.B. form and the particle distribution to be isotropic. We

can express the perturbation in the particle distribution in terms of an integration along the zero order particle orbits (an integration over time). These orbits can be found correct to a term linear in a smallness parameter ϵ (when ϵ equals zero we arrive back at a uniform magnetic field). The charge and current density due to the perturbation are related through Maxwell's equations to the electric and magnetic field of the wave in the usual self consistent Boltzmann-Vlasov description.

We show that the contribution to the current arises from recent events in the history of a given particle because of the finite temperature of the plasma. This result leads to an expansion of slowly varying parameters which in turn gives rise to the equation governing the motion of the wave-packet. In the final chapter the monochromatic wave case is also considered and cyclotron resonance is then investigated. It is shown that typically Fresnel integrals arise. Some light is also thrown on the magnetic beach configuration discussed by Stix.

For completeness a paper on the modulation of cosmic rays, which was presented by the author at the Ninth International Conference on Cosmic Rays, is also included at the end of this thesis.

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Chapter I

INTRODUCTION

The research presented here is concerned in the main with the problem of Whistler mode wave and wave packet propagation in hot, collision free, uniform and non-uniform plasmas. The motivation for such research coming from the associated magnetospheric phenomina of whistling atmospherics and stimulated V.L.F. emissions. (The magnetosphere being that region of the earth's environment lying above the ionosphere ($\sim$ 150 km) and below the magnetopause ($\sim$ 8 earth radii) where the earth's magnetic field dominates the ambient plasma).

Whistlers or whistling atmospherics are the result of dispersion of energy derived from a lightning stroke which has travelled in a right hand polarized electromagnetic mode along the earth's magnetic field lines in the magnetosphere. The different frequency components present in the initial impulse propagate at different velocities, and the disturbance transforms into a smoothly descending (and/or rising) audio frequency signal (hence the term Whistler mode). For a review of such things as 'ducting', 'multiple hops', determination of electron densities and temperatures etc in the magnetosphere (see Helliwell (1965)).

V.L.F. emissions are also audio-frequency Whistler mode signals. Their amplitude is comparable to Whistlers and they can be divided into two main groups. The first is 'hiss' which is a continuous wide band of noise and which may persist for periods of hours down to a few seconds. The second group are the discrete emissions. They exhibit a variety of sonogram traces such as hooks, rising tones, falling tones, etc. Their frequency range is short in comparison to hiss and their duration is from .1 to several seconds. For a discussion of "triggering" of these emissions and their characteristic sonogram traces (see also Helliwell (1965).

The discussion and results presented in this introductory chapter are by and large well-known; however they are included in order that the work presented here should be as complete and self-contained as possible. Because of this the emphasis has been put on simplicity and understanding rather than rigour. Wherever possible references to the original work have been included for further reading.

We briefly introduce, and discuss points arising from the Boltzmann Vlasov description of Whistler mode wave propagation in a hot collisionless plasma. We also investigate the physics of the wave particle gyroresonance interaction with particular reference to the stability or instability of wave propagation. The work presented is really salient both for direct understanding of the problems we have attempted and also to make clear the motivation for tackling such problems.

1.1 The Boltzmann Vlasov Description of Whistler Mode Wave Propagation

We consider a hot, collisionless, infinite uniform plasma emersed in an infinite uniform magnetic field. Charge neutrality in the unperturbed state is ensured by the presence of a background ionic plasma whose motion can effectively be ignored in the description of Whistler propagation owing to the high frequency of this mode. We denote the number of electrons in the volume element $d^3\underline{r}$ centred round the point with position vector $\underline{r}$ and whose velocities lie between $\underline{v}$ and $\underline{v} + d\underline{v}$ at the time t by

$$dn = f(\underline{r},\underline{v},t)\, d^3\underline{r}\, d^3\underline{v}$$

where $d^3\underline{r} = dx.dy.dz$

$$d^3\underline{v} = dv_x.dv_y.dv_z \quad .$$

$f(\underline{r},\underline{v},t)$ being the particle distribution function, which can be considered to be the number density of particles $\frac{dn}{d^3\underline{r}\; d^3\underline{v}}$ in the six dimensional phase space whose coordinates are X, Y, Z, V_x, V_y, V_z.

We can find the equation of motion of the distribution function (equation of continuity in phase space) as follows. The distribution function changes with time because electrons constantly enter and leave a given 'volume' element $d\tau = d^3\underline{r}\; d^3\underline{v}$ in phase space. If no collisions occur then an electron with coordinates $\underline{r},\underline{v}$ at time t will have coordinates $\underline{r} + \underline{v}\,dt$, $\underline{v} + \frac{\underline{F}}{m}dt$ at the instant $t + dt$, where $\underline{F}$ is the external force acting on an electron. Thus all the electrons contained in the phase space element $d\tau$ at $\underline{r},\underline{v}$ will be found in an element $d\tau'$ at $(\underline{r} + \underline{v}\,dt,\ \underline{v} + \frac{\underline{F}}{m}\,dt)$ at the instant $t + dt$.
Hence

$$f(\underline{r},\underline{v},t)d\tau = f(\underline{r} + \underline{v}\,dt,\ \underline{v} + \frac{\underline{F}}{m}\,dt,\ t + dt)d\tau'$$

It can be shown (e.g. see Chapman and Cowling (1939) page 322) that the two volume elements $d\tau$ and $d\tau'$ will be equal for forces of the Lorentz type (i.e. $\underline{F} = \frac{q}{m}\underline{v}\wedge\underline{B}$) or forces independent of $\underline{v}$. Hence in the limit $dt \to 0$ we have:-

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{r}} + \frac{\underline{F}}{m}\cdot\frac{\partial f}{\partial \underline{v}} = 0 \qquad (1.1)$$

As we are in the collision free regime we have neglected the effect of short range coulomb collisions which make electrons jump into and out of the 'volume' element $d\tau$ during the interval dt.

Thus equation (1.1) states that f is constant following a particle trajectory in phase space which is of course Louville's theorem as applied to this dynamical system (see any standard text on Classical Mechanics).

The forces acting on electrons in the plasma are assumed electromagnetic. Thus

$$\frac{\underline{F}}{m} = \frac{-|\varepsilon|}{m}\left[\underline{E} + \frac{\underline{v}\wedge\underline{B}}{c}\right]$$

Equation (1.1) becomes

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{r}} - \frac{|\varepsilon|}{m}\left[\underline{E} + \frac{\underline{v}\wedge\underline{B}}{c}\right]\cdot\frac{\partial f}{\partial \underline{v}} = 0 \qquad (1.2)$$

which is the well known Boltzmann equation describing the behaviour of the distribution f in the collision free regime in the presence of electromagnetic forces.

The electric and magnetic fields in (1.2) obey the Maxwell equations

$$\nabla_\wedge \underline{E} = -\frac{1}{c}\frac{\partial B}{\partial t}, \qquad \nabla\cdot\underline{E} = 4\pi\rho$$
$$\nabla_\wedge \underline{B} = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\underline{j}, \quad \nabla\cdot\underline{B} = 0 \qquad (1.3)$$

We assume the set of equations (1.2) and (1.3) to be self-consistent, that is, the electromagnetic field in equations (1.3) gives rise to force $\underline{F}$ on each electron to produce the distribution f in equation (1.2) which in turn is responsible for that same electromagnetic field. Thus the plasma appears through the current and charge density in equations (1.3) as

$$\underline{j} = |\varepsilon| \int d^3\underline{v}\ \underline{V}\ (f^+ - f^-) \tag{1.4}$$

and $$\rho = |\varepsilon| \int d^3\underline{v}\ (f^+ - f^-)$$ respectively,

where the superscripts + and - refer to the ion and electron distribution functions respectively.

We now use the set of equations (1.2), (1.3) and (1.4) to determine the characteristic features of disturbances propagating in the Whistler mode. We perturb the variables f, $\underline{E}$, $\underline{B}$ as follows.

$$f = f^o(\underline{v}) + f^{(1)}(\underline{r},\underline{v},t)$$

$$\underline{B} = \underline{B}^o + \underline{b}(\underline{r},t) \tag{1.5}$$

$$\underline{E} = 0 + \underline{E}(\underline{r},t)$$

where the unperturbed electric field is zero and the uniform ambient magnetic field is in the positive z direction i.e. $\underline{B}^o = \hat{z}\, B^o$.

In the unperturbed state equation (1.2) reduces to

$$[\underline{v}_\wedge \underline{B}^o] \cdot \frac{\partial f^o(\underline{v})}{\partial \underline{v}} = 0 \qquad (1.6)$$

On transferring to cylindrical coordinates in velocity space i.e.

$$V_x = V_\perp \text{ Cos } \emptyset$$

$$V_y = V_\perp \text{ Sin } \emptyset$$

($V_\perp$ being the component of velocity perpendicular to the ambient magnetic field, and $\emptyset$ being the azimuthal angle measured about $\underline{B}^o$) equation (1.6) becomes:-

$$\frac{\partial f^o(\underline{v})}{\partial \emptyset} = 0$$

That is, the most general form possible for the ambient particle distribution is

$$f^o(\underline{v}) = f^{(o)}(v_\perp , v_z) \qquad (1.7)$$

To limit space we do not review the Landau solution (Landau (1946)) of the initial value problem, for a full discussion see Stix (1962), or Montgomery and Tidman (1964). The treatment is well known and we shall use some of the points which derive from the above references. We simply assume the perturbed quantities are in the form of a plane wave propagating along the ambient magnetic field, thus the time and position coordinates of the first order quantities in equations (1.5) appear in the form $e^{i(\omega t - kz)}$ (i.e. a function of z and t only). On putting (1.5) into (1.2) and linearizing (i.e. neglecting terms in the product of first order quantities) we arrive at the linearized Boltzmann equation

$$|\Omega|\frac{\partial f'}{\partial \phi} + i(\omega - kv_z)f' = \frac{|\varepsilon|}{m}\left[\underline{E} + \frac{\underline{v}}{c}\wedge\underline{b}\right]\cdot\frac{\partial f^o}{\partial \underline{v}} \tag{1.8}$$

where $|\Omega| = \dfrac{|\varepsilon| B^o}{mc}$

We eliminate the wave magnetic field $\underline{b}$ from equation (1.8) by using the Maxwell equation

$$\nabla\wedge\underline{E} = -\frac{1}{c}\frac{\partial \underline{b}}{\partial t}$$

That is $\underline{b} = \dfrac{c}{\omega}\underline{k}\wedge\underline{E}$ (where $\underline{k} = \hat{z}k$).

Hence equation (1.8) becomes

$$\frac{\partial f'}{\partial \phi} + \frac{i(\omega - kv_z)f'}{|\Omega|} = \frac{1}{|\Omega|}\frac{|\varepsilon|}{m}\left[\underline{E} + \underline{v}\wedge\frac{(\underline{k}\wedge\underline{E})}{\omega}\right]\cdot\frac{\partial f^o}{\partial \underline{v}} \tag{1.9}$$

On carrying out the triple vector product $\underline{v} \wedge (\underline{k} \wedge \underline{E})$ (remembering $\underline{k} = \hat{z}k$) the right hand side of equation (1.9) (which we write as S) becomes:-

$$S = \frac{1}{|\Omega|} \frac{|\mathcal{E}|}{m} \left(\frac{E_x V_x}{V_\perp} + \frac{E_y V_y}{V_\perp}\right)\left[\left(1 - \frac{kv_z}{\omega}\right)\frac{\partial f^o}{\partial v_\perp} + \frac{kv_\perp}{\omega} \frac{\partial f^o}{\partial v_z}\right] + E_z \frac{\partial f^o}{\partial V_z}$$

where we have also used the relations

$$\frac{\partial f^o}{\partial V_x} = \frac{\partial f^o}{\partial V_\perp} \frac{dV_\perp}{dV_x} , \quad \frac{dV_\perp}{dV_x} = \frac{V_x}{V_\perp} \quad \text{(and similarly for } \frac{\partial f^o}{\partial V_y}\text{)}$$

On using the cylindrical coordinates $V_x = V_\perp \text{ Cos } \phi$, $V_y = V_\perp \text{ Sin } \phi$, V_z we have $\frac{dV_\perp}{dV_x} = \text{Cos } \phi$, $\frac{dV_\perp}{dV_y} = \text{Sin } \phi$.

S now contains the factor $E_x \text{Cos } \phi + E_y \text{ Sin } \phi$ which we separate into left and right hand components. Thus:-

$$S(\phi) = \frac{1}{|\Omega|} \frac{|\mathcal{E}|}{m}\left[\left(\frac{E_x + iE_y}{2}\right)e^{-i\phi} + \left(\frac{E_x - iE_y}{2}\right)e^{+i\phi}\right]$$

$$\left[\left(1 - \frac{kv_z}{\omega}\right)\frac{\partial f^o}{\partial v_\perp} + \frac{kv_\perp}{\omega} \frac{\partial f^o}{\partial v_z}\right] + \frac{1}{|\Omega|} \frac{|\mathcal{E}|}{m} E_z \frac{\partial f^o}{\partial v_z} \tag{1.10}$$

where the dependence on the azimuthal angle ϕ has been made explicit in S. (This is important in solving equation (1.9) for f', since it contains the derivative $\frac{\partial f'}{\partial \phi}$).

Thus we rewrite equation (1.9) as

$$\frac{\partial f'}{\partial \phi} + \frac{i(\omega - kv_z)f'}{|\Omega|} = S(\phi)$$

and solve for f' by the method of Bernstein (1958). That is we notice $e^{\frac{-i(\omega - kv_z)}{|\Omega|}\int_{\phi'}^{\phi} d\phi''}$ is an integrating factor and hence the solution of (1.9) can be written

$$f' = \int_{\pm\infty}^{\phi} d\phi' \, S(\phi') \, e^{\frac{i(\omega - kv_z)(\phi' - \phi)}{|\Omega|}} \tag{1.11}$$

(To make this step trivial we differentiate equation (1.11) with respect to ϕ. First put $\frac{(\omega - kv_z)}{|\Omega|} = A$ and write equation (1.11) as

$$f' = e^{-iA\phi}\left\{\int_{\pm\infty}^{\phi} d\phi' \, e^{iA\phi'} \, S(\phi')\right\}$$

Then:-

$$\frac{\partial f'}{\partial \phi} = -iA\, e^{-iA\phi}\left\{\int_{\pm\infty}^{\phi} d\phi' \, e^{iA\phi'} \, S(\phi')\right\} + e^{-iA\phi}\left\{e^{+iA\phi} S(\phi)\right\}$$

which is precisely equation (1.9)).

On putting (1.10) into equation (1.11) and carrying out the integration we find

$$f' = \frac{|\varepsilon|}{m}\left[\frac{E_x + iE_y}{2} e^{-i\phi} + \frac{E_x - iE_y}{2} e^{+i\phi}\right]$$

$$\frac{\left[\left(1 - \frac{kv_z}{\omega}\right)\frac{\partial f^o}{\partial v_\perp} + \frac{kv_\perp}{\omega}\frac{\partial f^o}{\partial v_z}\right]}{i(\omega - kv_z - |\Omega|)} + E_z \frac{|\varepsilon|}{m} \frac{\frac{\partial f^o}{\partial v_z}}{i(\omega - kv_z)} \tag{1.12}$$

The currents associated with this perturbation can be written

$$j_z = - |\mathcal{E}| \int dv_z \int d\phi \int dV_\perp V_\perp \, [V_z f']$$

and

$$j_x \pm i j_y = - |\mathcal{E}| \int dv_z \int d\phi \int dV_\perp V_\perp \, [(V_x \pm iV_y) f'] \tag{1.13}$$

Thus the left hand, right hand, and longitudinally polarized components are uncoupled in the linear approximation. The right hand polarized component, $E_x + iE_y$, rotates in the same sense as the electrons and is known as the Whistler mode. In what follows we shall consider the propagation characteristics of this mode only.

On eliminating $\underline{b}$ from the two Maxwell equations

$$\nabla_\wedge \underline{E} = -\frac{1}{c}\frac{\partial \underline{b}}{\partial t} \quad , \quad \nabla_\wedge \underline{b} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\underline{j}$$

we have

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c}\frac{\partial^2 E}{\partial t^2} + \frac{4\pi}{c^2}\left\{ - i\omega|\mathcal{E}| \, . \int d^3\underline{V} \, [V_\perp e^{i\phi} f'] \right\}$$

(Since $\underline{E}$ and $\underline{b}$ are independent of the x and y coordinates, i.e. a function of z and t only) and hence ...

$$k^2c^2 - \omega^2 = \frac{\omega_p^2 \omega \pi}{n_o} \int_{-\infty}^{+\infty} dv_z \int V_\perp^2 dV_\perp \frac{\left[\left(1 - \frac{kv_z}{\omega}\right) \frac{\partial f^o}{\partial V_\perp} + \frac{kv_\perp}{\omega} \frac{\partial f^o}{\partial v_z} \right]}{(\omega - kv_z - |\Omega|)}$$

(1.14)

where $\omega_p^{\;2} = \frac{4\pi n_o |\mathcal{E}|^2}{m}$ and $n_o = \int d^3\underline{v}\ f^o$.

(The hot plasma dispersion relation for the Whistler mode was first derived by Vedenov, Velikhov and Sagdeev (1961)).

There is no problem in principle in carrying out the integration over $v_\perp$ in (1.14). However a great deal of thought has been given to the remaining integration over v_z which has a singularity at $v_z = \frac{\omega - |\Omega|}{k}$. The prescription for integrating past this pole was first derived by Landau. We shall use this prescription and refer the interested reader to those references already given to the Landau solution.

Equation (1.14) defines a relationship between ω and k and is known as the dispersion relation; many properties of the Whistler mode become apparent on investigating this relation. In what follows we shall assume the wave number to be real and find that the frequency ω, as defined by relation (1.14), will, in general, be complex, i.e. $\omega = \omega_o + i\gamma$. The imaginary part of the frequency γ then corresponds to waves growing or decaying exponentially in time.

One can find the growth or damping decrement by one of several procedures. For example, one can follow the method of Sudan (1963) and expand about the point of marginal stability, i.e. assuming $\gamma \ll \omega_o$ we may write equation (1.14) in the form

$$D(\omega,k) = 0 \simeq D[\omega_o,k] + i\gamma \frac{\partial D}{\partial \omega_o} [\omega_o,k]$$

and hence $$\gamma = - \frac{D_I[\omega_o,k]}{\frac{\partial D_R[\omega_o,k]}{\partial \omega_o}}$$

here we separate out the real and imaginary parts of (1.14) by using an expanded form of the Landau prescription first given by Jackson (1960).
This method allows one to arrive neatly at the relation between both ω_o and k together with γ and k, the approximations made in the process are made clear.
By inspection we may write the dispersion relation (1.14) in the form

$$k^2c^2 - \omega^2 = -\frac{\omega_p^2 \pi}{n_o} \left[\frac{\omega}{k} \int_{-\infty}^{+\infty} \frac{dv_z \, A_1(v_z)}{v_z - z} + \int_{-\infty}^{+\infty} \frac{dv_z \, A_2(v_z)}{v_z - z}\right] \qquad (1.15)$$

where

$$A_1(v_z) = \int_0^\infty dv_\perp v_\perp^2 \frac{\partial f^o}{\partial v_\perp} \quad , \quad A_2(v_z) = \int_0^\infty dv_\perp v_\perp^2 \left(v_\perp \frac{\partial f^o}{\partial v_z} - v_z \frac{\partial f^o}{\partial v_\perp}\right)$$

and

$$z = u + iv \quad \text{where} \quad u = \frac{\omega_o - |\Omega|}{k} \quad , \quad v = \gamma/_k$$

In what follows we use the approximations

$$\omega_o << |\Omega| << \omega_p \qquad (a.1)$$

$$\gamma/_{\omega_o} << 1 \qquad (a.2)$$

$$\frac{(\omega - |\Omega|)^2}{k^2} >> \langle v_z^2 \rangle \qquad (a.3)$$

where $\langle v_z^2 \rangle = \int_{-\infty}^{+\infty} \frac{dv_z v_z^2 f^o}{n_o}$

Both integrals in equation (1.15) are of the form

$$I(z) = \int_{-\infty}^{+\infty} \frac{f(x)dx}{x-z} \quad \text{where} \quad z = u + iv$$

After Jackson (1960) we write this as

$$I(z) = \sum_{n=0}^{\infty} \frac{(iv)^n}{n!} \left[P\int \frac{f^n(x)}{X - u} + i\pi f^n(u)\right] \qquad (1.16)$$

We introduce the first two terms of this expansion into (1.15) and use the following notation

$$P\int \frac{A_1 dv_z}{v_z - z} = I_1 \quad ; \quad P\int \frac{A_1' dv_z}{v_z - z} = I_1' \qquad (1.17)$$

$$P \int \frac{A_2 dv_z}{v_z - z} = I_2 \quad ; \quad P \int \frac{A_2' dv_z}{v_z - z} = I_2'$$

On using the approximations (a) it can be shown that

$$I_1 \simeq \frac{n_o}{\pi u} \quad , \quad I_1' \simeq \frac{n_o}{\pi u^2} \qquad (1.17a)$$

$$I_2 \simeq - \frac{\langle v_z^2 \rangle n_o}{\pi u^2} \quad , \quad I_2' \simeq 0$$

(See Engle (1965)

On using (1.16) and (1.17) in (1.15) we have

$$k^2c^2 - (\omega_o + i\gamma)^2 = - \frac{\omega_p^2 \pi}{n_o} \left\{ \frac{\omega_o + i\gamma}{k} [(I_1 + i\pi A_1(u)) + \frac{i\gamma}{k}(I_1' + iA_2(u))] \right.$$

$$\left. + [I_2 + i\pi A_2(u)] + \frac{i\gamma}{k}[I_2' + i\pi A_2'(u)] \right\}$$

(1.18)

we now equate the real and imaginary parts of equation (1.18). By inspection the real part reduces to

$$k^2c^2 - \omega_o^2 = - \frac{\omega_p^2 \pi}{n_o} \left[\frac{\omega_o}{k} I_1 + I_2 \right]$$

where we have neglected terms of order γ^2.

On neglecting all effects due to finite temperature i.e. putting $I_2 \simeq 0$ (see (1.17a) we have

$$k^2c^2 - \omega_o^2 = \frac{\omega_p^2 \omega_o}{|\Omega| - \omega_o} \qquad (1.19)$$

which is the cold plasma (magneto-ionic) Appleton Hartree dispersion relation for the Whistler mode (e.g. Helliwell (1965)).

We now collect imaginary terms together neglecting terms of order γ^2. By inspection we have

$$2\gamma\omega_o = \frac{\omega_p^2\pi}{n_o}\left[\frac{\gamma I_1}{k} + \frac{\omega_o}{k} A_1(u) + \frac{\gamma}{k}\omega_o I_1{}' + \pi A_2(u) + \frac{\gamma}{k} I_2{}'\right]$$

Neglecting the term on the left-hand side, (see (a.2)) we have collecting terms in γ together and using (1.17a)

$$\gamma\left[\frac{n_o}{\pi uk} + \frac{\omega_o}{k^2}\left(\frac{-n_o}{\pi u^2}\right)\right] = \pi\left[\frac{\omega_o}{k} A_1(u) + A_2(u)\right]$$

Hence

$$\gamma = \frac{\pi^2(\omega_o - |\Omega|)^2}{n_o|\Omega|}\left[\frac{\omega_o}{k}\frac{\partial f^o}{\partial v_\perp} + \left(v_\perp\frac{\partial f^o}{\partial v_z} - v_z\frac{\partial f^o}{\partial v_\perp}\right)\right] \tag{1.20}$$

where $v_z = \dfrac{\omega_o - |\Omega|}{k}$.

We now examine equation (1.20) in some detail.

1.2 The Linear Theory of the Wave Particle Gyro-resonance Interaction

The growth or damping decrement γ emerged when we separated equation (1.14) into its real and imaginary parts by use of expansion (1.16). We show that the phenomena of growth or damping is explicable in physical terms. To this end we consider the wave-particle gyro-resonance interaction between a Whistler mode wave and an electron. We shall consider the initial value problem using linearized equations. A resonance phenomenon will then be uncovered which is indicated by long term growth or decay in the oscillation amplitude of the electron. Since the total energy of the electromagnetic wave field and the plasma particles is conserved, the steady growth in oscillation amplitude of the resonant particles must result in the damping of the wave field (and vice versa).

Consider a transverse wave whose electric and magnetic fields are given by

$$\underline{E}(z,t) = (\hat{x}e_x + \hat{y}e_y)\, e^{i(\omega t - kz)} \qquad (1.21)$$

$$\underline{b}(z,t) = (\hat{x}b_x + \hat{y}b_y)\, e^{i(\omega t - kz)} \quad \text{respectively}$$

where e_x, e_y and b_x, b_y are constant amplitudes, propagation is along theambient magnetic field $\underline{B}_o = \hat{z}B_o$

(notice $e_z = b_z = 0$).

We denote the motion of an electron in the absence of the wave by the unperturbed velocity vector

$$\underline{v}^o = \hat{x}v_x^o + \hat{y}v_y^o + \hat{z}v_z^o \qquad (1.22)$$

and the velocity of the electron in the presence of the wave by

$$\underline{v} = \underline{v}^o + \underline{v}'$$

where $\underline{v}'$ is the perturbation due to the presence of the wave.

The linearized equation of motion of the electron is

$$\left[\frac{\partial}{\partial t} + \underline{v}^o \cdot \nabla\right]\underline{v} = -\frac{|\mathcal{E}|}{m}\left[\underline{E} + \frac{\underline{v}_\wedge \underline{B}^o}{c} + \frac{\underline{v}^o_\wedge \underline{b}}{c}\right] \quad (1.23)$$

The Maxwell equation $\nabla_\wedge \underline{E} = -\frac{1}{c}\frac{\partial \underline{b}}{\partial t}$ relates the amplitudes in (1.21) by

$$b_x = -\frac{kc}{\omega}e_y \ , \quad b_y = \frac{kc}{\omega}e_x \qquad (1.24).$$

The three components of equation (1.23) when written out making use of the relations (1.24) are

$$\left[\frac{\partial}{\partial t} - ikv_z^o\right]v_x + |\Omega|\, v_y = -\frac{|\mathcal{E}|}{m}\left[E_x\left(1 - \frac{kv_z^o}{\omega}\right)\right]$$

$$[\frac{\partial}{\partial t} - ikv_z^o]v_y - |\Omega| v_x = -\frac{|\varepsilon|}{m}[E_y(1 - \frac{kv_z^o}{\omega})] \quad (1.25)$$

$$[\frac{\partial}{\partial t} - ikv_z^o]v_z = -\frac{|\varepsilon|}{m}[v_x^o E_x + v_y^o E_y]\frac{k}{\omega}$$

where $|\Omega| = \frac{|\varepsilon| B^o}{mc}$.

We now separate out the right-hand polarized Whistler mode wave $E_x + iE_y = E^+$, by multiplying the second of equations (1.25) by i and adding to the first.

Thus:-

$$[\frac{\partial}{\partial t} - i(kv_z^o + |\Omega|)]v^+ = -\frac{|\varepsilon|}{m}(1 - \frac{kv_z^o}{\omega})E^+$$

We rewrite this as

$$\frac{\partial}{\partial t}[v^+ e^{-i(kv_z^o + |\Omega|)t}] = -\frac{|\varepsilon|}{m}(1 - \frac{kv_z^o}{\omega})e^{-ikz}\ e^+ .\ e^{i(\omega - |\Omega| - kv_z^o)}$$

(1.26)

where $v^+ = v_x + iv_y$, $e^+ = e_x + ie_y$.

We consider the initial value problem such that at the instant $t = 0$ $v^+ = v_+^o = v_x^o + iv_y^o$.

Hence from (1.26) we have

$$v^+ = v_+^o e^{i(|\Omega| + kv_z^o)t} - \frac{|\varepsilon|}{m}[1 - \frac{kv_z^o}{\omega}]e^+$$

$$e^{i(\omega t - kz)} \frac{1 - e^{-i(\omega - |\Omega| - kv_z^o)t}}{i(\omega - |\Omega| - kv_z^o)} \quad (1.27)$$

Since in the unperturbed state the electrons rotate in the right-hand sense about the ambient magnetic field with random phase ϕ at the gyro-frequency $|\Omega|$ we have:-

$$v_x^o = V_\perp^o \operatorname{Cos}(|\Omega| t + \phi) \; ; \quad v_y^o = V_\perp^o \operatorname{Sin}(|\Omega| t + \phi) \; .$$

The third equation of the set can thus be written

$$\frac{\partial}{\partial t}[v_z e^{-ikv_z^o t}] = -\frac{|\mathcal{E}|}{m} V_\perp^o [e_x \operatorname{Cos}(|\Omega| t + \phi) - e_y \operatorname{Sin}(|\Omega| t + \phi)]$$

$$\cdot e^{i(\omega - kv_z^o)t} \frac{k}{\omega} e^{-ikz}$$

On taking out the Whistler mode component and integrating using the initial value condition that at $t = 0$ $v_z = v_z^o$ we find

$$v_z = v_z^o e^{ikv_z^o t} - \frac{|\mathcal{E}|}{m} \frac{k}{\omega} V_\perp^o \frac{e^+}{2} \cdot e^{i(\omega t - kz)} \left[\frac{1 - e^{-i(\omega - |\Omega| - kv_z^o)t}}{i(\omega - |\Omega| - kv_z^o)}\right]$$

$$\cdot e^{-i|\Omega|t} \qquad (1.28)$$

(where we have put $\phi = 0$ for convenience).

The first point to notice is that the perpendicular and parallel velocity components of most electrons oscillate in some simple manner with constant amplitude. However the oscillation amplitudes of those electrons which have an initial parallel velocity such that

$$\omega - |\Omega| - kv_z^o = 0$$

grow linearly with time

$$\left(\text{since } \left[\frac{1 - e^{ixt}}{x}\right] \simeq \frac{ixt + \frac{(ixt)^2}{2}}{x} + \ldots \text{ for small } x\right).$$

The amplitude is of course bounded by non-linear effects which move the particle out of gyroresonance. The electrons and Whistler mode waves both rotate in the right-hand sense around the ambient field line; however the wave frequency ω lies below the electron gyrofrequency $|\Omega|$, i.e. $\omega < |\Omega|$. Thus the resonant condition $\omega - kv_z^o = |\Omega|$ means that electrons travelling in the opposite direction to the wave with velocity $v_z^o = -\frac{(|\Omega| - \omega)}{k}$ (a negative velocity) will see the wave frequency increased by the doppler effect to their own gyrofrequency. The wave vector and electron both rotate at exactly the same rate, and for these electrons the angle between the transverse velocity vector $\underline{V}_\perp$ and the transverse component of the wave will be constant. The resonant electrons will thus be embedded in a constant electric field and can interact strongly with the wave. It is the energy exchanged through this strong interaction between the resonant electrons and the wave which leads to the stability or instability of the waves.

If the wave amplitude is sufficiently small one can expect that the linear treatment will be valid for all time for non-resonant particles. However, the above solutions show that for those particles initially in cyclotron resonance, the non-linear terms must rapidly become important. At the end of this chapter, we discuss very briefly the interesting non-linear effect of particle trapping which then results, leading to "stirring" of the distribution in phase space.

It is valuable to compare the associated changes in energy and pitch angle of electrons which interact with the wave.

In the frame of the wave, the time rate of change of the magnetic field is zero, $\omega = 0$, hence there is no electric field (Equations (24)), thus in this frame the energy of the electrons cannot change.

Since the velocity of the wave is ω/k we have

$$\frac{m}{2} V_\perp^2 + \frac{m}{2} (v_z - \frac{\omega}{k})^2 = \text{constant} \qquad (1.29)$$

Suppose the wave causes small changes $\delta V_\perp$ and δV_z in the two velocity components $V_\perp$ and V_z. Then to first order, the change in energy

$$W = \frac{m}{2} V_\perp^2 + \frac{m}{2} v_z^{\ 2}$$

is $\delta W = mV_\perp \delta V_\perp + mv_2 \delta v_z$, which from (1.29) is equal to

$m\frac{\omega}{k} \delta v_z$ also first order.

i.e. $$\delta W = m \frac{\omega}{k} \delta v_z \qquad (1.30)$$

In the wave frame the pitch angle α of an electron is defined by either

$$V \cos \alpha = v_z - \frac{\omega}{k}$$

or $$V \sin \alpha = v_\perp$$

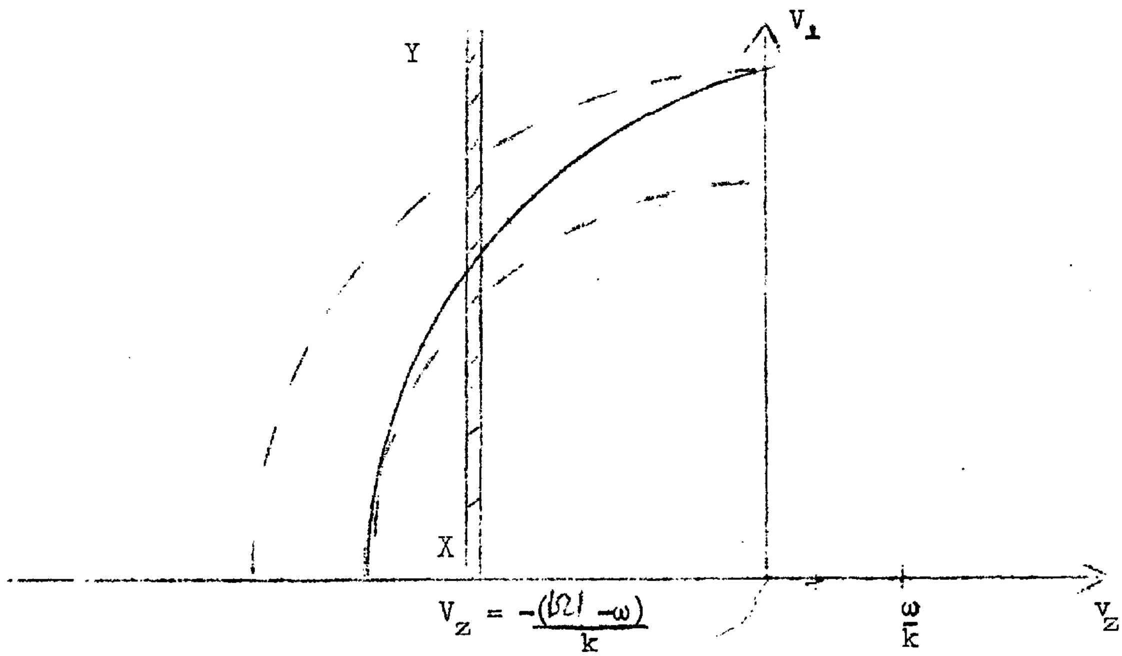
where $$V = [v_\perp^2 + (v_z - \frac{\omega}{k})^2]^{\frac{1}{2}} = \text{constant},$$

thus a small change in $v_{\parallel}$ will result in a change in pitch angle $\delta\alpha$ given by

$$-V \sin \alpha \, \delta\alpha = \delta V_z$$

i.e. $$\delta\alpha = - \frac{\delta V_z}{V_\perp} \tag{1.31}$$

After Dungey (1963), and Kennel and Petschek (1966), it is convenient to draw the following diagram in the $V_\perp$, V_z plane.



Equation (1.29) states that electrons will move on circles $v_\perp^2 + (v_z - \frac{\omega}{k})^2 = \text{constant}$, centred on $v_z = \frac{\omega}{k}$ (e.g. smooth line). However, the lines of constant energy are circles centred on the origin $v_\perp^2 + v_\parallel^2 = \text{constant}$, (e.g. dotted lines).

By inspection of the diagram, we may draw the following conclusions.

(1) For frequencies such that $\omega << |\Omega|$ the change effected by the wave is going to be primarily in pitch angle, one can see that this is going to become more pronounced the greater the inequality $\omega << |\Omega|$. For example, to go from the outer to the inner dotted line (corresponding to a small change in energy), one must move all the way along the continuous curve (a pitch angle change of $\frac{\pi}{2}$). (Of course, an almost monochromatic wave of frequency centred on ω only effects a small region such as the shaded line xy which extends over all values of $v_\perp$.

(2) We also note from the diagram that a decrease in energy of a resonant electron is going to be associated with a decrease in pitch angle. It has been shown for example by Brice (1964), that the change in pitch angle $\delta\alpha$ is related approximately to the change in energy δW by

$$\delta\alpha = + \frac{|\Omega|}{\omega} \cdot \frac{\delta W}{W}$$

for 45° pitch angle electrons. (A result easily derived from the above equations, essentially (30) and (31)).

One can also draw the same conclusions (1) and (2) from this relation.

The condition for the wave amplitude to grow is that the energy of the resonant electrons decreases, and hence the associated (relatively large) decrease in pitch angle means that a similarly large transfer of transverse to longitudinal energy has taken place for the resonant electrons. One thus expects distributions with more transverse than longitudinal energy (i.e. appropriately anisotropic pitch angle distributions), in the region of the gyroresonant velocity, to be unstable. For examples of such distributions we may take the anisotropic Maxwellian in which the transverse temperature $T_{\perp}$ is greater than the longitudinal temperature $T_{\parallel}$, or an isotropic distribution with a loss cone. Both these distributions have a suitable imbalance of longitudinal to transverse energy for instability. It was Bell and Buneman (1964) who first showed (by studying the dispersion relation) that resonant particles must have finite $V_{\perp}$ in order for wave growth to be possible.

We now return to equation (1.20) and examine the growth rate in the light of our investigation of the gyroresonance interaction.

For easiest discussion, we put equation (1.20) in the form given by Kennel and Petschek (1966) as follows.

Replace

$$(\omega - |\Omega|)^2 \quad \text{by} \quad (1 - \omega/|\Omega|)^2 |\Omega|^2 \qquad \text{(i)}$$

$$\int_0^\infty dv_\perp \; v_\perp^2 \frac{\partial f^o}{\partial v_\perp} \quad \text{by} \quad -2\int_0^\infty dv_\perp \, v_\perp \, f^o \qquad \text{(ii)}$$

$$\frac{\omega}{k} \quad \text{by} \quad \frac{1}{\frac{|\Omega|}{\omega} - 1} \cdot \frac{|\Omega| - \omega}{k} \qquad \text{(iii)}$$

and

$$\left[\int_0^\infty dv_\perp \; v_\perp^2 \left(v_\perp \frac{\partial f^o}{\partial v_z} - v_z \frac{\partial f^o}{\partial v_\perp}\right)\right]_{v_z = \frac{\omega - |\Omega|}{k}}$$

which can be written as

$$-\frac{|\Omega| - \omega}{k} \left[\int_0^\infty dv_\perp \; v_\perp \left(v_\perp \frac{\partial f^o}{\partial v_z} - v_z \frac{\partial f^o}{\partial v_\perp}\right) \frac{v_\perp}{v_z}\right]_{v_z = \frac{\omega - |\Omega|}{k}}$$

by

$$+\frac{|\Omega| - \omega}{k} \left[\int_0^\infty dv_\perp \; v_\perp \frac{\partial f^o}{\partial \alpha} \tan\alpha\right] \qquad \text{(iv)}$$

where we have used the transformation

$$V_z = V \operatorname{Sin} \alpha$$

$$V_\perp = V \operatorname{Cos} \alpha \, .$$

Making replacements (i), (ii), (iii), (iv) in equation (1.20) we have by inspection

$$\gamma = -\frac{\pi}{n_o}(1 - \omega/|\Omega|)^2 [2\pi \left(\frac{|\Omega| - \omega}{k}\right) \int_0^\infty dv_\perp v_\perp f^o]$$

$$\left[\frac{\int_0^\infty dv_\perp v_\perp \left(\frac{\partial f^o}{\partial \alpha} \tan \alpha\right)}{2\int_0^\infty dv_\perp v_\perp f^o} - \frac{1}{\frac{|\Omega|}{\omega} - 1}\right]\Bigg|_{v_\parallel = \frac{\omega - |\Omega|}{k}}$$

(1.32)

The factor in the first square bracket may be interpreted as the measure of the number of resonant electrons (it is of course a positive quantity).

The second square bracket determines the sign of γ , that is, it shows that if the distribution is isotropic with regard to pitch angle, $\frac{\partial f^o}{\partial \alpha} = 0$, at the resonant velocity then $\gamma > 0$ and only damping can result. Thus all isotropic distributions give rise to damping, this result is, of course, in agreement with the Penrose criterion as applied to the Whistler mode, Bell (1964). The first term in this bracket becomes positive when $\frac{\partial f^o}{\partial \alpha}$ becomes positive and thus has the appropriate sign for growth; if this term dominates the second, the particle distribution will be unstable.

Since $\frac{\partial f^o}{\partial \alpha} > 0$ implies that at fixed energy the number of particles increases with increasing pitch angle there will be more total energy in the distribution function at higher pitch angles. Since the wave will become unstable if this term dominates, the stability or instability of the Whistler mode wave depends only on the pitch angle anisotropy (or the unbalance of transverse longitudinal energy) in the region of gyroresonance. The magnitude of the growth or damping decrement depends on both the anisotropy and the number of resonant electrons. Both these results are in agreement with our previous discussion of the test particle gyroresonant interaction.

In view of the above discussion one may expect the following mechanism, Dungey (1963), for loss of electrons from the radiation belts. The effect of propagation of a Whistler mode wave packet on an electron is to cause its pitch angle to change, this change may be positive or negative, (depending on the phase of the wave field with respect to the perpendicular component of the particles velocity $\underline{V_\perp}$). Thus one expects the passage of a series of atmospheric Whistlers* to cause the pitch angle of a given electron to random walk (since there should be no correlation in the above mentioned phase for these atmospherics). The overall effect on the electron

. . . .

*Whistlers whose energy is derived from a lightning stroke on the earth's surface and which subsequently propagates far out into the Magnetosphere along a field line (e.g. see Helliwell (1965)).

population will thus appear as a diffusion in pitch angle (from high to low particle density). Because of the presence of a loss cone (i.e. a particle sink) in the magnetosphere the overall effect of such pitch angle diffusion will be a loss of particles from the radiation belt. Dungey (1963) and Cornwall (1964) have both obtained good quantitative agreement with the observational evidence of such a loss.

In the light of the above discussion it is not surprising to find that the effect of broad band Whistler mode noise is to produce pitch angle diffusion in the electron distribution. Kennel and Petschek (1965) have investigated quantitatively the effect of such noise (generated by the presence of a loss cone) on the radiation belts. The basic equations they used in describing the effect of Whistler mode noise were the quasilinear equations

$$\frac{\partial f}{\partial t} = \frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} \left(D \sin \alpha \frac{\partial f}{\partial \alpha} \right) \qquad (1.33)$$

where the diffusion coefficient D is given by

$$D = \frac{|\Omega|^2}{V \cos \alpha} \left\{ \frac{b_k}{B^o} \right\}^2 , \qquad (1.34)$$

$b_k^{\,2}$ being the energy per unit wave number at resonance.

They derived these equations in a rather simple though not fully rigorous manner. A fully rigorous treatment (within the limitations of quasilinear theory) has been given by Engle (1965).

These equations tell us something about the expected pitch angle distribution in the magnetosphere. Since there is a loss cone present, the ambient plasma will have an anisotropy with appropriate sign for instability and can thus generate its own Whistler mode wave amplitude noise. Kennel and Petschek pointed out that although the rate at which particles diffuse towards the loss cone depends on the magnitude of the diffusion coefficient, and hence the wave energy, equation (1.34), the shape of the steady state pitch angle distribution outside the loss cone is essentially independent of the diffusion coefficient (c.f. the temperature profile for steady state heat conduction with different materials but fixed boundary conditions). Thus, since the pitch angle anisotropy is fixed in the steady state the wave growth rate depends only on the number of resonant particles. This number must just balance injection with precipitation so that the growth rate just replaces escaping wave energy. Using this argumentthey were able to estimate an upper limit on the magnitude of stably trapped particle fluxes and also to derive the expected pitch angle profiles. Of course, although the pitch angle distribution outside the loss cone does not depend on the magnitude of the diffusion coefficient, the distribution inside depends on the diffusion time to the time of loss to the atmosphere. Therefore the measured loss cone profiles (O'Brien (1964)) estimate a diffusion coefficient and a particle lifetime. (In Kennel and Petschek's paper this comparison of pitch angle profiles is made in Section 7, essentially Figs. (5) and (6)). However, a source of contention is that in their paper they have assumed that all particles are injected into the magnetosphere with pitch angles of $\pi/2$.

We now briefly review a mechanism by which emissions in the Whistler mode can be stimulated by the passage of a wave packet (also in the Whistler mode), the mechanism has been developed by A. C. Das (to be published (1968)). This mechanism has some bearing on the work presented in the next chapter. Essentially he looked at the motion of electrons on the edge of the loss cone at or near resonance with a Whistler mode wave packet propogating along the ambient magnetic field. He then invoked Liouville's theorem to find the effect on the particle distribution and hence the ambient growth rate. A particle will resonate with a particular frequency component of the wave packet. If ψ is the angle between this frequency component of the wave magnetic field $\underline{b}$ and the component of velocity of the particle perpendicular to the ambient field $\underline{V_\perp}$, then:-

$$\psi_r = \psi_o + \int_o^t dt' \frac{d\psi}{dt'} \qquad (1.35)$$

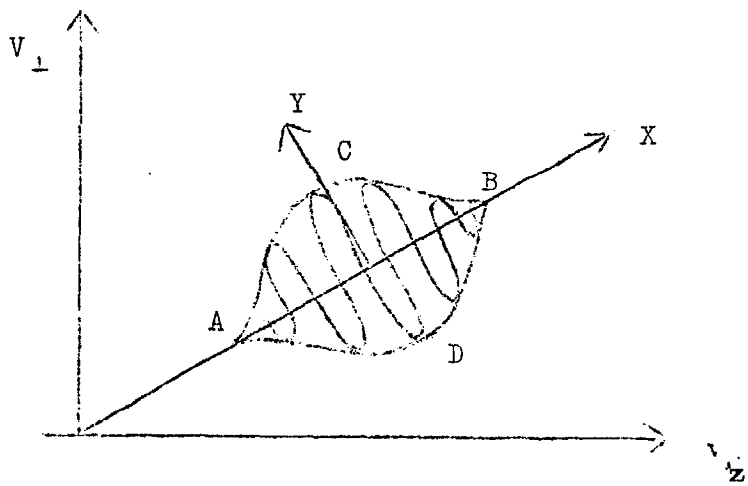
is this angle at the instant of their coincidence $t' = t$, where ψ_o is the angle ψ at $t' = 0$ and $\frac{d\psi}{dt'} = \omega - kv_{\parallel} - |\Omega|$ for an unperturbed particle trajectory.

From equation (1.35) it is easily shown that the quantitative behaviour of ψ_r is

$$\psi_r - \psi_o \sim v_{\parallel} t \qquad (1.36)$$

The effect of the wave packet on a given particle is taken to be essentially a change in pitch angle which is determined primarily by ψ_r and the amplitude of the component of the wave field in resonance with that particle (1.37).

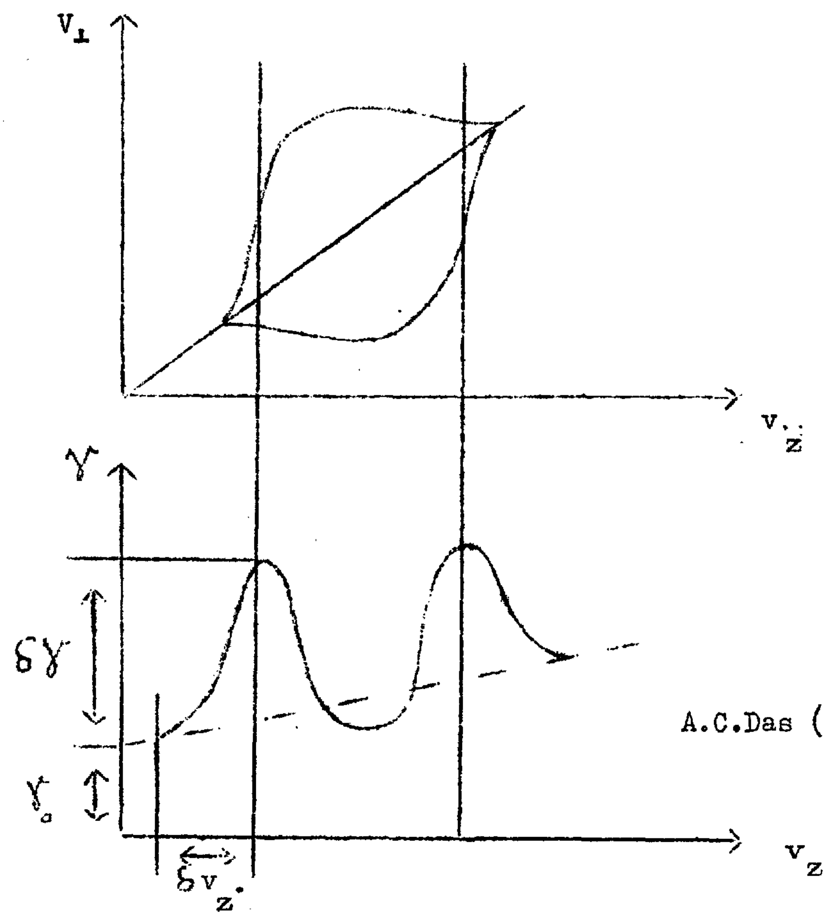
Thus from the statements (1.36) and (1.37) we can plot the figure below, which is a 'snap shot' of the disturbance of those particles in the velocity distribution on the edge of the loss cone AB some time after the wave packet has passed.



That is from (1.37) the envelope in this figure is essentially the amplitude frequency spectrum of the wave packet (taken as gaussian) while the oscillating phase is ψ_r (on replacing $V_\perp$ by vector $\underline{V_\perp}$ the diagram becomes a screw thread since ψ_o includes the phase of the electron about the ambient magnetic field).

From (1.36) we see that as time t increases the oscillations become more and more rapid and eventually those particles on the edge AB become averaged over the region represented by the two gaussians ABCD (since the fine structure must smooth out owing to the large gradients in velocity space and the subsequent operation of the Fokker Plank term). The distribution then becomes a $\mathrm{Sin}^{-1}\, y$ function (see above figure) and is joined smoothly at the two boundaries ACB and ADB. However at these boundaries one expects the rate of change of the particle distribution with respect to pitch angle, $\frac{\partial f}{\partial \alpha}$, to be particularly large. The result of this is to give an enhanced growth rate in the region where the boundary is near vertical, (i.e. AC and BD) since the growth rate depends on the integrated effect of $\frac{\partial f}{\partial \alpha}$ over the vertical $v_\perp$ coordinate, (see equation (1.32) above). This, in fact, shows up in the full computation and is reproduced qualitatively below.

(a)



A.C.Das (1968)

The orders of magnitude of γ_o, $\delta\gamma$ and δv_z labelled in this figure will be investigated and used in the next chapter, where wave packet propagation is examined carefully. However, before passing on we notice that the presence of a loss cone is essential to the mechanism and was taken as a step function in the analysis (any variation can be built up out of such step functions). The distribution was taken as slowly varying outside the loss cone and zero inside. (The edge was not really assumed sharp since then $\frac{\partial f}{\partial \alpha} = \infty$). However the rather sharp transition at the boundary is essentially responsible for the background noise (represented by the finite ambient growth shown dotted on the above figure). We also notice that non-linear effects on the particle trajectory have been neglected in this model.

In a nonuniform ambient magnetic field such an enhancement of growth rate can generate frequencies outside the wave number band located between A and B by the resonant conditions (Dowden (1962)). The mechanism could thus be responsible for the production of discrete V.L.F. emissions.

1.3 Particle Trapping and the Application of Liouville's Theorem

Firstly we investigate briefly one of the more important phenomina which arises when equations (1.25) become non-linear, that is, we shall examine the mechanism of particle trapping. In order to see how this phenomina arises we look at the last of equations (1.25) which we shall write in the non-linearized form.

$$\frac{dv_z}{dt} = -|Q| \operatorname{Sin}\psi \quad \text{where} \quad |Q| = \frac{|\varepsilon|}{m}\left|(\underline{v}_\perp^o + \underline{v}_\perp')\right|\left|\underline{b}\right|$$

and ψ is the angle between $\underline{V}_\perp$ and $\underline{b}$.

We know that for resonant electrons

$$\frac{d\psi}{dt} = \omega - kv_{\parallel\,res}^o - |\Omega| = 0.$$

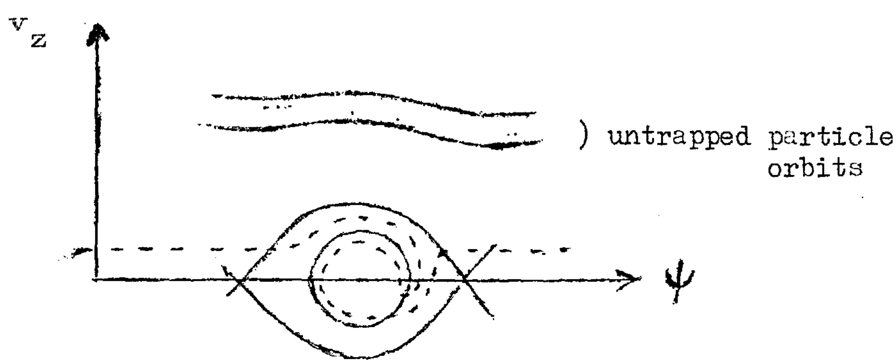
Suppose that $v_{\parallel\,res}^o$ becomes $v_{\parallel\,res}^o + v_\parallel'$ then we shall have

$$\frac{d\psi}{dt} = -kv_\parallel'$$

Now consider resonant electrons in the vicinity $\psi = \pi \pm \xi$ where ξ is small, for these electrons

$$\frac{dv_z'}{dt} = |Q| \operatorname{Sin}\xi \quad \text{(i)} \quad \text{and} \quad \frac{d\xi}{dt} = -kv_z' \quad \text{(ii)}$$

(Notice $|Q|$ depends on the perturbation wave field $\underline{b}$). These two equations are coupled and show clearly the mechanism of particle trapping. Thus equation (i) says that if $\zeta > 0$, then v_z' is increasing with respect to time, equation (ii) then says that ζ is decreasing and will go on decreasing until $\zeta < 0$; however the direction of change of ζ and v_z' are then exactly reversed. Hence ζ and v_z' oscillate about zero and form closed orbits in the phase space of ζ and v_z, these particles are referred to as trapped particles. Thus after Dungey (1963)a we may draw the diagram below (very qualitatively)



The effect on the particle distribution f can be found since Liouville's theorem states that f behaves like an incompressible fluid in phase space.

Since $f^o = f^o(V_\perp\ V_z)$, equation (1.7), the lines of constant f are horizontal in the diagram prior to the disturbance. However the closed trajectory of a given particle will stir the distribution, i.e. since f is constant following the trajectory of a particle in phase space, the dotted line being a contour of constant f, will be wound up as shown. (The Fokker-Plank term will obviously take effect and result in a smoothing of the distribution). The associated change in $V_\perp$ is very easily found from equation (1.29)

$$m[V_\perp^2 + (V_z - \omega/_k)^2] = \text{constant}.$$

Hence

$$\frac{dv_z}{dt} = \frac{k}{\omega - kv_z}\frac{d}{dt}(V_\perp^2) \quad \text{(see also Stix (1962))}$$

or $\dfrac{dv_\perp}{dt} \simeq \dfrac{|\Omega|}{kV_\perp}\dfrac{dv_z}{dt}$ where we have used the resonance condition.

An exact analytic solution for the trajectory of an electron in the field of a Whistler mode wave has been derived by Laird and Knox (1965).

Contribution of the Present Work

The contribution of the work presented in the following chapters is essentially that of deriving descriptions of the electromagnetic wave field of narrow band wave packet distrubances, with particular reference to the Whistler mode.

In Chapter (II) we investigate a wave packet solution of the formal cold plasma equations (essentially the cold plasma wave equation). Although such solutions can always by synthesised from their spectrum of plane wave solutions it is found that by demanding the disturbance satisfy the relevant wave equation, all the information about its behaviour becomes readily available. Surprisingly (in this uniform linear treatment) some new and quite interesting results and insights can be obtained, in particular when the chosen disturbance or wave packet is unstable.

In Chapter (III) we go on to investigate the same problem when the basic plasma equations are those of hot plasma theory. Stix has pointed out that formal cold plasma theory denies the very nature of hot collision free plasmas in which of course the plasma particles are not fixed (in average position) in space (as in cold plasma theory), but are almost completely unrestrained by the forces which normally operate in a fluid (e.g. collisions). That is, the particles within a given volume element at one instant can be found located over a wide region of space only seconds later. In describing rigourously wave-like phenomena in such a media (especially when it is non-uniform or time dependent) one must work in terms of the distribution function, which of course obeys Liouville's Theorem. However, one normally evades this issue (in the uniform case) by working in terms of individual Fourier components. The cold plasma theory can then be made to emerge formally by taking the limit of zero thermal speead in the particle distribution (Stix (1962)). In this chapter (i.e. Chapter (III)) we show how the cold plasma wave packet solution of Chapter (II) also satisfies the hot Boltzmann Vlasov equations. The rather crucial point that emerges in this chapter is that the "memory" of a given particle is finite and that it does not really "remember" the electromagnetic field in which it was located in the distant past. One can also see from the analysis that this dephasing of the past history of a given particle is a characteristic feature of hot plasma theory.

In Chapter (IV), we attempt a similar wave packet problem when the ambient magnetic field is non-uniform. The cold plasma theory is briefly discussed, the situation being clear. The hot plasma wave equation is then tackled in the light of Chapter (III). The behaviour of a chosenwave packet is determined on defining it to be a solution of the hot plasma wave equation. Unfortunately, the analysis is plagued by the fact that one is forced to make approximations, which must of course be self consistent.

In the final chapter (Chapter (V)), we look for monochromatic wave solutions of the hot plasma wave equation. The treatment is rather easier than that of the wave packet. A brief review of gyroresonant phenomena is then included and suggestions for further work are made.

In essence, the contribution presented here is that of showing how one may use the finite memory of a chosen particle to derive solutions of the wave equation and hence arrive at a description of wave-like phenomena in hot, collision free, non-uniform and time dependent plasmas.

Finally, for completeness, a paper on the modulation of cosmic rays, written by the author during the course of his first year of research in the field of space physics, has been included at the end of this thesis.

Chapter II

THE LINEAR THEORY OF GROWTH AND DISPERSION OF WAVE PACKETS

Introduction

In this chapter we examine the fundamentals of growth (or damping) and dispersion of wave-packets propagating in the Whistler mode along a uniform ambient magnetic field $\underline{B}^o = \hat{z}\, B^o$. That is, we examine the effect on the electromagnetic field of a given disturbance of both (a) the preferential growth (or damping) of certain frequencies and (b) the fact that the different frequency components do not travel with the same velocity in a dispersive medium.

We shall take the point of view that the dispersive effects are well described by the cold plasma approximation and we use formal cold plasma theory in the investigation of these effects. The growth (or damping) of different frequencies is, of course, by necessity a finite temperature phenomena depending on the free streaming of those particles in cyclotron resonance with that frequency component of the wave. In the next chapter it will be seen how basically the same equations evolve from the hot plasma theory.

It is well known that in the linear theory one may always use the techniques of Fourier transforms (i.e. one may use the principle of superposition of plane waves to build up any chosen disturbance). These techniques are so well known that, although we shall call freely on the insights which they give, we will give only the briefest possible review of them and refer the reader to some of the standard works.

Basically we derive the equation of change of both the phase and amplitude of the wave field of a chosen disturbance. The method of deriving these equations is novel but must, of course, have an equivalent treatment in terms of Fourier integrals. However, some new results are obtained and some interesting points raised by a close investigation of these equations. The work has been done with particular reference to V.L.F. emissions, where the instability which generates the emissions is probably due to wave-particle gyroresonance (Bell and Bunemann (1964)). As noted, this type of instability becomes operative under conditions of anisotropic particle distribution with respect to pitch angle (for example, anisotropy due to the presence of a loss cone). A typical suggestion is that the propagation of waves (e.g. atmospheric Whistlers) causes changes in the zero order distribution and this in turn gives rise to enhanced growth rates and hence emissions. The dtailed structure of the emission would seem to depend (at least in part) on the non-linear pitch angle diffusion of electrons probably in the region of the loss cone in velocity space, A. C. Das [1968].

The effect of pitch angle diffusion of particles into the loss cone has been studied and agreement achieved with some of the observational aspects of the associated precipitation, Kennel and Petschek [1966]. However, a great deal of work remains to be carried out on the details of the stimulation of emissions, the mechanism involved being presumably this same diffusion process. An investigation of the linear theory of wave-packets cannot produce a new mechanism for the stimulation of the emissions but can describe certain aspects of the causitive atmospheric and of the emission itself once it has been produced.

The concepts of wave-packets and group velocity stem directly from applying the techniques of Fourier Synthesis to the problem of finding the general solution of a given wave equation (linear partial differential equation). The solution can always be expressed in terms of a spatial transform over the spectrum of plane waves. (This spectrum is given by the Fourier transform of the initial disturbance). The plane waves are chosen in such a way as to be particular solutions of the wave equation, i.e. their frequency and wave number are related through the dispersion relation $D[\omega,k] = 0$.

Thus the general solution takes the form of a Fourier integral

$$\psi(z,t) = \int_{-\infty}^{+\infty} A(k)\, e^{i(\omega(k)t - kz)} \frac{dk}{2\pi} \qquad (I)$$

where $A(k) = a(k)\, e^{iH(k)} = \int_{-\infty}^{+\infty} \psi(z,t=0)\, e^{+ikz}\, dz$

($a(k)$, $H(k)$ being real), and $\omega = \omega(k)$ is the dispersion relation solved in terms of the frequency (assuming only one mode present).

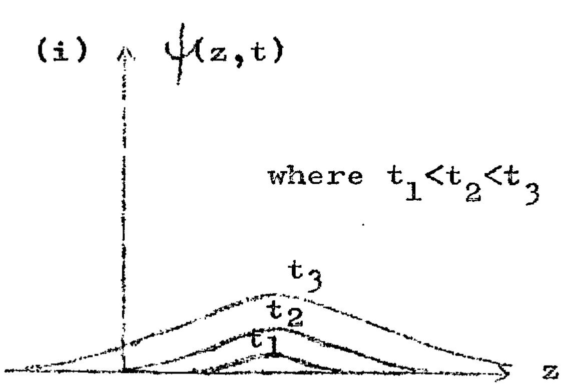
The integral (I) can seldom be carried out in practice and it is approximations arising out of attempts to evaluate it which give rise to the ideas of wave-packets and wave-trains. (The former from the method of unresolved waves and the latter from the method of stationary phase). For a full review of these approximations, see, for example, C. Eckart [1948] or J. Jackson [1963].

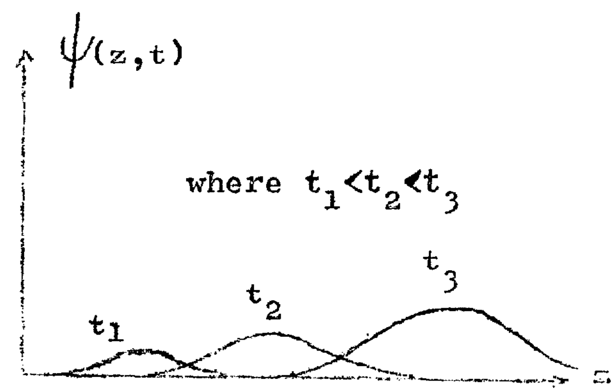
Since any spatially bounded disturbance $\psi(z,t)$ (i.e. one for which $\psi(z,t) \to 0$ as $z \to \pm\infty$) may be formed by superposing many real wave numbers in the form (I), one may ask what happens when some of these real wave numbers give rise to complex frequencies. The criterion for stability used by most authors is whether the dispersion relation $D[\omega,k] = 0$ has solutions for which real k give rise to complex frequencies $\omega(k)$ such that $\psi(z,t)$ contains terms of the form $e^{+\omega_I t}$, where $\omega_I = \text{Im}(\omega)$. If individual Fourier components of the wave-packet $\psi(z,t)$ grow without limit (with time) the wave-packet as a whole may still not become infinite at a fixed point in space (basically because $\psi(z,t)$ becomes for large t

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} A(k)\; e^{\omega_I(k)t}\; e^{i(\omega_R(k)t - kz)}$$

and as $t \to \infty$ $e^{+\omega_I t} \to \infty$. However $e^{i\omega_R(k)t}$ becomes a rapidly oscillating function of k so that $\psi(z,t)$ is an indeterminate quantity of the type $0 \times \infty$).

Twiss first pointed out that two types of instability are in fact possible, i.e. a spatial pulse may grow in two distinct ways as shown in (i) and (ii) below.

(i)


(ii)


Thus the wave-packet initially finite in z may grow without limit at every point in z (see (i)) or it may "propagate along" so the amplitude $\psi(z,t)$ eventually decreases at any fixed point in space (see (ii)). (The labelling is obviously not frame invariant, i.e. (i) becomes of the form (ii) on transforming to a frame moving in the negative z direction with uniform velocity). A physical interpretation of the difference between (i) and (ii) is that (i) has an "internal feedback" mechanism while (ii) has not. An instability of the type (i) is known as an absolute or non-convecting instability while type (ii) is called a convective instability.

A closely related problem is the interpretation of roots of the dispersion relation where ω is real and k is complex. That is, what is the sinusoidal steady state response of a system at given real frequency ω (if, in fact, this is possible for the system) when there are "normal modes" (i.e. solutions of the dispersion relation for k at same chosen ω, or vice versa) which have k complex. For example, in a passive system such as an empty wave guide the imaginary part of k must imply evanescence in, say, the positive z direction rather than amplification in the opposite (negative) z direction. In more complicated systems when there is a "pool" of energy available the situation is often not at all clear. (Notice that the terms amplifying and evanescent always refer to the case in which ω is real while k is complex). Sturrock[1958] first showed that a convective instability is the same thing as an amplifying wave, the only difference being in the form of excitation of the system being considered (i.e. pulse or sinusoidal in time repectively).

The best method of determining the physical meaning of the roots of the dispersion relation $D[\omega,k] = 0$ (see Briggs [1964]) is to consider the excitation of these waves in the infinitely long medium by a source that is of finite spatial extent (i.e. from $z = -d$ to $z = +d$) and which is zero for $t < 0$. The response outside the source is then a linear superposition of "normal modes" of the system. If the asymptotic response at fixed z is exponentially increasing with time there is an absolute instability present or if the asymptotic response for sinusoidal excitation is an oscillation at this frequency spatially increasing away from the source one has an amplifying wave (or convective instability, Sturrock [1958]). One can write the response of a one dimensional linear system in terms of its Green's function K and source function, i.e.

$$\psi(z,t) = \int dz' \int dt' \, K[t - t', z-z'] \, g(z') \, f(t')$$

where $f(t) = 0$ for $t < 0$ and for convenience the source function is taken to be of the form $g(z) \, f(t)$. One can always perform a Fourier transformation (for a spatially finite wave-packet) with respect to space for all times (because of the finite speed of propagation of the wave-packet) and a Laplace transformation with respect to time. For illustration the form taken by the transformations for the source functions $g(z)$ and $f(t)$ are

$$g(z) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \, g(k) \, e^{-ikz} \, , \quad g(k) = \int_{-\infty}^{+\infty} dz \, g(z) \, e^{ikz} \, ;$$

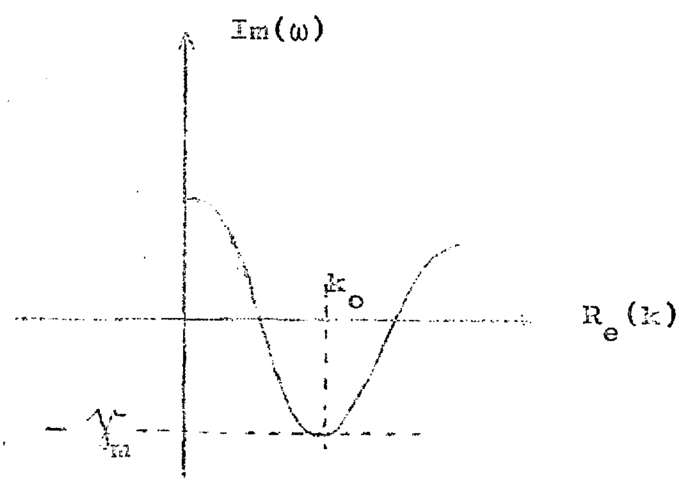
$$f(t) = \int_{-\infty-i\sigma}^{+\infty-i\sigma} \frac{d\omega}{2\pi} \, f(\omega) \, e^{i\omega t} \, , \quad f(\omega) = \int_{0}^{t} dt \, f(t) \, e^{-i\omega t}$$

where the integration over frequency ω is below all the singularities in $f(\omega)$ so that $f(t) = 0$ for $t < 0$. Thus the response of the system can then be written as

$$\psi(z,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty-i\sigma}^{+\infty-i\sigma} \frac{d\omega}{2\pi} \, G(\omega,k) \, f(\omega) \, g(k) \, e^{i(\omega t - kz)} \quad \text{(II)}$$

$G(\omega,k)$ being the transform of the Green's function. The character of the asymptotic limit of the response in time can be determined by investigating this expression and in the process the physical meaning of the roots of the dispersion relation are determined. As the method of treatment of the problem is rather outside the scope of this thesis we shall simply discuss the propagation of a spatial pulse a little further, state the criteria (for distinguishing between convective and absolute instabilities) and discuss the physical interpretation that may be put on these criteria. The interested reader is referred to the outstanding publication by Briggs [1964].

The usual definition of stability is whether real k gives rise to complex ω such that waves growing exponentially in time result. Any system which is unstable by this definition will "blow up" in amplitude even though it may appear to decrease in time at a fixed point, this is because it could convect along as it blows up. It can be shown that by letting both z and t tend to infinity in (II) at a fixed ratio (i.e. $z = z_o + Vt$) then a velocity $V = V_o$ can always be found for which the wave-packet appears to increase exponentially with time at the maximum growth rate of the unstable waves (i.e. maximum imaginary ω for real k). Thus, if the plot of Im (ω) v $R_e(k)$ is as below, then it can be shown that an observer moving with velocity $V_o = \frac{\partial}{\partial k} (\omega_R) \Big|_{k=k_o}$ will see a disturbance growing like $e^{\gamma_M t}$.



Briggs [1964] proposes V_o as the "propagation velocity" of a pulse in such an unstable medium.

The statement of the criteria is as follows:-

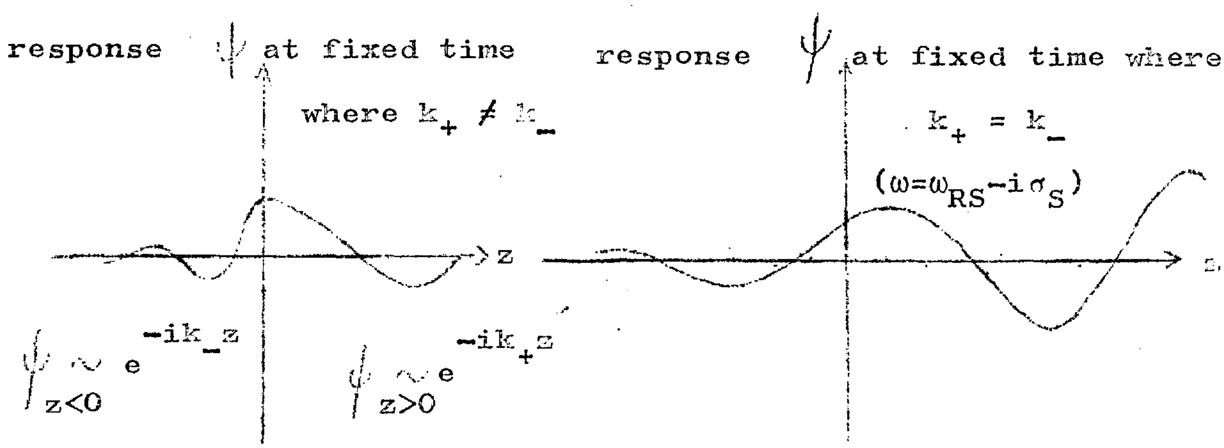
(1) To decide whether an unstable wave (i.e. one for which real k gives rise to negative imaginary ω) is unstable in the absolute or convective sense one maps the dispersion relation from the ω plane into the k plane (usually it is most convenient to map lines of constant real ω). An absolute instability is present whenever there is a double root of k for some ω in the lower half ω plane for which the two merging roots have come from different halves of the k plane (upper and lower). Only merging of roots from the upper and lower halves indicate an absolute instability, otherwise the instability is convective.

In passing we also state the criterion for the related problem of evanescent and amplifying waves, Briggs [1964].

(2) "To decide whether a given wave with complex k for real frequency ω is amplifying or evanescent, determine whether or not imaginary k has a different sign when the frequency takes on a large negative imaginary part, if it does then the wave is amplifying otherwise it is an evanescent wave."

One may interpret physically the merging of the two roots of k from the upper and lower half k plane, stated in (1) as follows:-

Consider an infinite system excited at $z = 0$, if the source has complex ω with imaginary ω larger than any growth rate of the unstable system then the waves must all decrease in amplitude (e.g. decay in amplitude implying that imaginary $k > 0$ for the response when $z > 0$) as one moves away from the source (this is demanded physically by causality). Suppose now that $k+$ and $k-$ are the wave numbers which appear for $z > 0$ and $z < 0$ respectively. For example, for $z > d$ one can close the Fourier integral in (II) in the lower half k plane; this closure allows the integral to be expressed as a sum over appropriate normal modes by the theory of residues (i.e. since the Green's function $G(\omega,k)$ has poles in the complex k plane, for some fixed complex ω on the Laplace contour, at just the "normal mode" wave numbers. These are roots of the dispersion relation $D[\omega,k] = 0$ which have k in the lower half k plane and where ω is some frequency on the Laplace contour. For the full treatment of the analytic continuation of the Fourier integral as one attempts to move the Laplace contour upwards in the ω plane in order to investigate the asymptotic response in the usual way, see Briggs [1964]). If we now imagine the growth rate of the source to decrease then for some frequency $\omega_S = \omega_{RS} - i\sigma_S$ (say) one might have k+ and k- equal. The source causes a discontinuity in the response at $z = 0$ except when $k+ = k-$, in which case the response is continuous across $z = 0$. Briggs refers to this as "spatial resonance" of the infinite system at the frequency ω_S because the presence of this response does not require a source. The situation is shown diagramatically below (after Briggs [1964]).



2.1 The Equation of Change of Wave-Packets

In this section we will not write the electric field in the form of a Fourier Integral (as in equation (I)), but choose it to be of the form

$$\underline{E}(z,t) = \mathrm{Re}\left\{ \underline{e}_o \, e^{i(\omega t-kz)} . \, G(z,t) \right\} \qquad (2.1)$$

where Re stands for the real part, and $\underline{e}_o = (\hat{x}e_x + \hat{y}ie_y)$, e_x and e_y being constant amplitudes.

In this equation G(z,t) is an unspecified envelope slowly varying in space z and time t, and is in general a complex quantity. The description slowly varying will become clear below. If we choose $e_x = e_y$ in the expression for $\underline{e}_o$ above, $\underline{E}(z,t)$ represents a plane $(\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0)$ circularly polarized (in the right handed sense) wave-packet propagating in the z direction. This choice for the electric field gives a description of most reasonable Whistler mode disturbances over a limited range of space and time. One can study the polarization properties of a more general disturbance by introducing a separate envelope for the x and y components of $\underline{E}(z,t)$.

The equation of change of the 'envelope' G(z,t) is determined as follows. On determining the dispersion relation for plane waves the electric field behaves as

$$e^{i(\omega t - kz)}$$

so that we may replace $\frac{\partial}{\partial t}$ by $i\omega$, and $\frac{\partial}{\partial z}$ by $-ik$ (2.2) in the relevant wave equation, $D[-i\frac{\partial}{\partial t}, i\frac{\partial}{\partial z}] = 0$, giving rise to the dispersion relation

$$D[\omega,k] = 0 \quad (2.3a) \quad \text{or} \quad \omega = F(k) \quad (2.3b)$$

assuming only one mode.

However from equation (2.1) we notice that

$$\frac{\partial}{\partial t} = i\omega + \tau \quad \text{and} \quad \frac{\partial}{\partial z} = -ik + \mathfrak{Z} \qquad (2.4)$$

$$\text{where} \quad \tau = e^{i\omega t}\frac{\partial}{\partial t}e^{-i\omega t}$$

$$(2.5)$$

$$\text{and} \quad \mathfrak{Z} = e^{-ikz}\frac{\partial}{\partial z}e^{ikz}$$

That is, τ and $\mathfrak{Z}$ represent operators which if used on E(z,t) determine the rate of change of G(z,t) in time and space respectively. The relations (2.4) when put in the same wave equation will give rise to the 'dispersion relation'

$$D[\omega - i\tau,\ k + i\mathfrak{Z}] = 0 \qquad (2.6a)$$

$$\text{or } \omega - i\tau = F[k + i\mathfrak{Z}] \qquad (2.6b)$$

For example, consider the cold plasma dispersion relation.

The linearized equation of motion for electrons using formal cold plasma theory is

$$m\frac{d\underline{v}'}{dt} = -|\mathcal{E}|(\underline{E} + \frac{\underline{v}\wedge\underline{B}^o}{c}) \qquad (2.7)$$

(using familiar notation) and hence

$$[\frac{d}{dt} - i|\Omega|]\underline{j} = \frac{|\mathcal{E}|^2}{m} n^o\underline{E} \qquad (2.8)$$

where the current density $\underline{j} = -|\mathcal{E}|n^o\underline{v}'$, the perturbations are functions of z and t only and $\underline{B}^o = \hat{z}B^o$.

Hence

$$\underline{j} = \frac{|\mathcal{E}|\, n^o\underline{E}}{m[\frac{d}{dt} - i|\Omega|]} \qquad (2.9)$$

The Maxwell equations

$$\nabla_\wedge\underline{E} = -\frac{1}{c}\frac{\partial\underline{b}}{\partial t} \quad \text{and}$$

$$\nabla_\wedge\underline{b} = \frac{1}{c}\frac{\partial\underline{E}}{\partial t} + \frac{4\pi}{c}\underline{j} \qquad (2.10)$$

give

$$\frac{\partial^2\underline{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial\underline{j}}{\partial t} \qquad (2.11)$$

Equation (2.9) combined with (2.11) becomes

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \frac{4\pi}{c^2}\frac{|\varepsilon|^2}{m}\frac{n^o \frac{\partial E}{\partial t}}{[\frac{d}{dt} - i|\Omega|]} \tag{2.12}$$

On using the relations (2.2) we arrive at the Appleton Hartree dispersion relation for Whistler propogation along the field line, i.e.

$$\omega^2 - k^2c^2 = \frac{\omega\,\omega_p^{\,2}}{(\omega - |\Omega|)} \tag{2.13}$$

(see also equation (19) Chapter I).

However, on using the relations (2.4) equation (2.12) becomes

$$(\omega - i\tau)^2 - (k + i\mathcal{Z})^2c^2 = \frac{\omega_p^{\,2}[\omega - i\tau]}{[\omega - i\tau] - |\Omega|} \tag{2.14}$$

We now treat $k + i\mathcal{Z}$ as a complex variable and Taylor expand F about the point k in equation (2.6b). (It is slightly simpler to use (2.6a) rather than (2.6b), the latter is, however, in the same form as (2.14)).

Hence

$$\omega - i\tau = F(k) + i\mathcal{Z}\frac{dF(k)}{dk} + \frac{(i\mathcal{Z})^2}{2!}\frac{d^2F(k)}{dk^2} + \ldots \tag{2.15}$$

or

$$\frac{\partial G(z,t)}{\partial t} = -\frac{d\omega}{dk}\frac{\partial G(z,t)}{\partial z} - \frac{i}{2!}\frac{d^2\omega}{dk^2}\frac{\partial^2 G(z,t)}{\partial z^2} \tag{2.16}$$

where we have used the identity (2.3b).
The condition that $G(z,t)$ be slowly varying is taken to mean that the expansion (2.15) is a good approximation for $F[k + i\mathcal{Z}]$.

Equation (2.16) allows for the possibility of growth or damping. That is, in equation (2.3b) $F(k)$ may be complex (though k will be assumed real throughout).
If we choose to represent the electric field in this case by

$$E(z,t) = \underset{\sim}{e}_o \, e^{i(\omega_o t - kz)} \, G_1(z,t) \qquad (2.17)$$

$$\text{where} \quad \omega = \omega_o - i\gamma \quad \text{and} \quad G_1 = e^{\gamma t} G \qquad (2.18)$$

and ω_o, γ are both real and positive, then equation (2.16) becomes

$$\frac{\partial G_1}{\partial t} = \gamma G_1 - \frac{d\omega}{dk}\frac{\partial G_1}{\partial z} - \frac{i}{2!}\frac{d^2\omega}{dk^2}\frac{\partial^2 G}{\partial z^2} \qquad (2.19)$$

Equation (2.19) becomes equation (2.16) on differentiating the product

$\frac{\partial}{\partial t}[e^{\gamma t}G]$ on the left hand side and dividing through by $e^{\gamma t}$ since γ is not a function of z.

The two 'envelopes' G and G_1 will not be labelled specifically in the work that follows since it will be obvious which envelope we are referring to from relevant equation of motion.

Obviously the method we have used here will not work for non-linear wave equations, and is equivalent to the Fourier integral treatment of the problem. This point will become clear when we solve (2.16) formally. However one can always write down the electric field in a form most suitable for the problem in hand and then use an analogous procedure to the expansion technique we have used. This may well have some advantages over starting the investigation from the Fourier integral, depending rather on the information which is being sought.

2.2 Process Determining Phase and Amplitude Changes

In this section we shall use equations (2.16) and (2.19) to acquire some useful physical insight into the processes governing wave-packet propagation. The discussion presented is likely to be helpful when one goes on to consider the same type of problem in non-uniform or time-varying plasmas; the former problem being attempted in detail in a later chapter.

Thus far we have chosen to work purely in terms of the electric field. However it is fairly obvious that we could derive a similar equation describing the behaviour of the associated magnetic field.

Thus the magnetic field $\underline{b}(z,t)$ associated with the electric field

$$\underline{E}(z,t) = \underline{e}_o \, e^{i(\omega t - kz)} \, G(z,t) \tag{2.1}$$

is

$$\underline{b}(z,t) = \underline{b}_o \, e^{i(\omega t - kz)} \, G_b(z,t) \tag{2.1a}$$

where $\underline{b}_o = (\hat{x}\, b_x + \hat{y}\, i\, b_y)$, a constant amplitude, and $G_b(z,t)$ the wave 'envelope'.

It is almost self-evident and will now be verified that the envelope $G_b(z,t)$ will obey an equation of the form (2.16) or (2.19), i.e.

$$\frac{\partial G_b}{\partial t} = \gamma G_b - \frac{d\omega}{dk}\frac{\partial G_b}{\partial z} - \frac{i}{2!}\frac{d^2\omega}{dk^2}\frac{\partial^2 G_b}{\partial z^2} \tag{2.19a}$$

Rather than going back to the beginning and reworking the treatment already given, this time eliminating $\underline{E}(z,t)$ rather than $\underline{b}(z,t)$, it is more instructive to use the relations (2.1) and (2.1a) and the Maxwell equation

$$\nabla \wedge \underline{E} = -\frac{1}{c}\frac{\partial \underline{b}}{\partial t}$$ to show that:-

$$b_x = -\frac{kce_y}{\omega}, \quad b_y = \frac{kce_x}{\omega} \quad \text{and} \tag{2.20}$$

$$G(z,t) = \hat{Q} G_b(z,t)$$

where $\hat{Q} = e^{-i(\omega t - kz)} \dfrac{(1 - \frac{i\tau}{\omega})}{(1 + \frac{i\zeta}{k})} e^{i(\omega t - kz)}$

is a differential operator.

Thus equation (2.19) may be written

$$\frac{\partial}{\partial t}[\hat{Q}G_b] - \gamma[\hat{Q}G_b] + \frac{d\omega}{dk}\frac{\partial}{\partial z}[\hat{Q}G_b] + \frac{i}{2!}\frac{d^2\omega}{dk^2}\frac{\partial^2}{\partial z^2}[\hat{Q}G_b] = 0.$$

It is a simple exercise in differentiation to show that the operators $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial z}$, $\frac{\partial^2}{\partial z^2}$ all commute with $\hat{Q}$ in this equation, thus

$$\hat{Q}\left[\frac{\partial G_b}{\partial t} - \gamma G_b + \frac{d\omega}{dk}\frac{\partial G_b}{\partial z} + \frac{i}{2!}\frac{d^2\omega}{dk^2}\frac{\partial^2 G_b}{\partial z^2}\right] = 0 \qquad (2.21)$$

from which we derive equation (2.19a) since $\hat{Q} \neq 0$.

Equation (2.16), together with $G(z,t=0)$, fully specifies $G(z,t)$ within the limits of the above approximation. The first and second terms in (2.16) show that G propagates at the group velocity, while the third term shows the effect of a spread in group velocity at each point in the disturbance. In the frame in which the group velocity is zero and provided there is no instability or damping (more precisely $\gamma = \frac{d\gamma}{dk} = \frac{d^2\gamma}{dk^2} = 0$ (2.20) equation (2.16) reduces to the Schrödinger equation for a free particle. A discussion of the free particle wave-packet is given in any standard work on Quantum Mechanics .e.g. R. Sillitto [1960], and the picture of the diffusing envelope is familiar. On writing

$$G(z,t) = A(z,t)\, e^{\phi(z,t)} \tag{2.22}$$

one may also show

$$\frac{d}{dt} \int_V dV\, A^2 = 0$$

in the same manner as conservation of probability current is established, i.e. multiply equation (2.16) by the complex conjugate of G (i.e. G^*), write down the complex conjugate equation and multiply it by G, add the two resulting equations to find

$$\frac{\partial A^2}{\partial t} + \frac{d\omega}{dk}\frac{\partial A^2}{\partial z} = -\frac{i}{2!}\frac{d^2\omega}{dk^2}\left\{G^* \frac{\partial^2 G}{\partial z^2} - G\frac{\partial^2 G^*}{\partial z^2}\right\} \tag{2.23}$$

On using the identity

$$x\,\nabla^2 y - y\,\nabla^2 x = \nabla\cdot(x\nabla y - y\nabla x)$$

on the left hand side of (2.21) and integrating over a volume V bounded by a surface s, we have

$$\frac{d}{dt}\int_V dV\, A^2 = -\frac{i}{2!}\frac{d^2\omega}{dk^2}\int_s ds \left\{G^* \frac{\partial G}{\partial z} - G\frac{\partial G^*}{\partial z}\right\} \tag{2.24}$$

where we have used Green's theorem to go from a volume to a surface integral. By letting the volume go to infinity the surface integral vanishes and the stated identity results, which now implies that the energy of the wave field is conserved.

The presence of growth terms (i.e. terms containing γ, $\frac{d\gamma}{dk}$, $\frac{d^2\gamma}{dk^2}$) give rise to some interesting effects, particularly interesting in view of the results of Das [1967]. Thus we now find the equations governing the behaviour of the amplitude A and phase ϕ of the wave packet since the effect of these growth terms is not best understood from equation (2.19) as it stands. The relation (2.19) is really two separate equations, one real and one imaginary. Thus, on separating it into its real and imaginary parts by use of equation (2.22) and rearranging the terms for more convenient discussion we find:-

$$\frac{\partial A}{\partial t} = -\frac{d\omega}{dk}\frac{\partial A}{\partial z} + \frac{d^2\omega}{dk^2}\frac{\partial \phi}{\partial z}\frac{\partial A}{\partial z} + \gamma A - \frac{d\gamma}{dk}\frac{\partial \phi}{\partial z} A + \frac{1}{2}\frac{d^2\gamma}{dk^2}\left(\frac{\partial \phi}{\partial z}\right)^2 A$$

$$- \frac{1}{2}\frac{d^2\gamma}{dk^2}\frac{\partial^2 A}{\partial z^2} + \frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 \phi}{\partial z^2} A \qquad (2.25a)$$

$$A\frac{\partial \phi}{\partial t} = -\frac{d\omega}{dk}\frac{\partial \phi}{\partial z} A + \frac{1}{2}\frac{d^2\omega}{dk^2}\left(\frac{\partial \phi}{\partial z}\right)^2 A + \frac{d\gamma}{dk}\frac{\partial A}{\partial z} - \frac{d^2\gamma}{dk^2}\frac{\partial \phi}{\partial z}\frac{dA}{dz}$$

$$- \frac{1}{2}\frac{d^2\gamma}{dk^2}\frac{\partial^2 \phi}{\partial z^2} A - \frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial z^2} \qquad (2.25b)$$

(These equations (2.25a) and (2.25b) were derived from consideration of cold plasma theory. In the next chapter we shall discuss explicitely the same problem in terms of rigorous hot plasma theory).

The two equations (2.25a) and (2.25b) are, of course, coupled. However, each term in these equations can tell us something new about the propagation of the wave-packet, the propagation characteristics of the disturbance in a given situation, depending on which terms are dominant. Thus we shall take each term and investigate its physical significance. Consider (2.25a) first.

The first, second and third terms show that there exists an effective wave number $k_{eff:}$ given by

$$k - \frac{\partial \phi}{\partial z} \qquad \text{i.e.}$$

$$k_{eff:} = \frac{\partial}{\partial z} \left\{ \text{phase of the disturbance (2.1)} \right\} \qquad (2.26)$$

and the amplitude (at each point) is propagated at the group velocity which corresponds to this wave number, i.e. the effective group velocity is

$$\left. \frac{d\omega(k)}{dk} \right|_{k = \frac{\partial}{\partial z} \left\{ \text{phase of (2.1)} \right\}}$$

Similarly the fourth, fifth and sixth terms show that the effective growth rate is also that corresponding to k_{eff} i.e.

$$\left. \gamma(k) \right|_{k = \frac{\partial}{\partial z} \left\{ \text{phase of (2.1)} \right\}}$$

The seventh term is quite interesting; it shows that the amplitude diffuses at a rate determined by the second derivative of the growth rate with respect to wave number. At a maximum of the growth rate (with respect to wave number) $\frac{d\gamma}{dk} = 0$ and $\frac{d^2\gamma}{dk^2} < 0$. Thus, if the growth rate is sharply peaked $\frac{d^2\gamma}{dk^2}$ will be large and negative and as time progresses the spectrum of the disturbance will also become progressively more sharply peaked. This sharpening of the spectrum implies a widening of the disturbance. (This result stems from the well known Fourier Transform 'uncertainty relation' $\Delta k \, \Delta x \geq 1$). Thus the continuous sharpening of the spectrum will result in a progressive broadening of the disturbance which appears in the form of positive diffusion in amplitude. Of course, if $\frac{d^2\gamma}{dk^2}$ is positive (implying damping), the opposite is true; one then has an effective broadening of the spectrum and hence a narrowing in spatial extent of the wave-packet. This case of negative diffusion results in bumps and irregularities sharpening with time, rather than smoothing out, as is the case when the "diffusion coefficient" is positive.

The eighth and final term in equation (2.25a) is of the same type as a growth or damping term. We now show that the effective growth or damping decrement, $\frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2\phi}{\partial z^2}$, has fairly clear physical origins.

Since

$$\frac{d^2\omega}{dk^2} = \frac{d}{dk}\left\{ \text{group velocity} \right\}$$

and from (2.26)

$$\frac{\partial^2\phi}{\partial z^2} = -\frac{\partial}{\partial z}\left\{ k_{eff} \right\} \text{, the product}$$

$$\frac{d^2\omega}{dk^2}\frac{\partial^2\phi}{\partial z^2} \text{ is of the form } -\frac{\partial}{\partial z}\left\{ \text{group velocity} \right\}.$$

It is natural that such a term should lead to growth or decay in amplitude. Consider some point on a wave-packet propagating in the positive z direction, where the group velocity is decreasing with respect to z, then in that region the wave will "concertina" on itself, and the amplitude must build up. The argument is reversed for points where the group velocity is increasing with respect to z. The sharper the gradient in group velocity, the more rapid the corresponding changes in amplitude. (This term has the same nature as the final equation in (2.25b), see below).

The energy W_e contained in the electromagnetic field of the wave is given by

$$W_e = \left\{ \frac{|\underline{E}(z,t)|^2 + |\underline{b}(z,t)|^2}{8\pi} \right\} \qquad (2.27),$$

the equation of motion for this quantity is going to be essentially the same as that for A^2. That is

$$W_e = \left\{ \frac{|\underline{e}_o|^2 A^2 + |\underline{b}_o|^2 A_b^2}{8\pi} \right\}$$

where we have also replaced G_b in equation (2.19a) by $A_b e^{i\phi_b}$, $|\underline{e}_o|^2$, $|\underline{b}_o|^2$ are constants. The equations of motion (2.19) and (2.19a) for A and A_b respectively are identical. (Since $|\underline{b}_o|^2 = \frac{k^2c^2}{\omega^2} |\underline{e}_o|^2$ we may often neglect the energy stored in the electric field). We may find the equation of motion for A^2 in the same manner as we derived equation (2.21). However, it is simpler to multiply equation (2.25a) by A.

Thus

$$\frac{\partial A^2}{\partial t} = 2\gamma \Big|_{k=k_{eff}} A^2 - \frac{d\omega}{dk}\Big|_{k=k_{eff}} \frac{\partial A^2}{\partial z} - \frac{d^2\gamma}{dk^2} A \frac{\partial^2 A}{\partial z^2} + \frac{d^2\omega}{dk^2} \frac{\partial^2 \phi}{\partial z^2} A^2 \qquad (2.28)$$

This equation is really an improved version of that derived by Kadomstev (1965). The discussion of the first three and the final term is not changed from our previous discussion of equation (2.25a). However we can see that the fourth term cannot be put in an appropriate form. It may be replaced by

$$-\frac{d^2\gamma}{dk^2} \left[\frac{1}{2} \frac{\partial^2 A}{\partial z^2} - \left(\frac{\partial A}{\partial z}\right)^2\right]$$

the first term in this bracket also corresponds to diffusion (in wave energy). However the second does not appear to describe a simple physical process in terms of wave energy. Since the whole term derives from diffusion in amplitude we see that for cases in which growth rate derivatives are significant it is simpler to think in terms of wave amplitude rather than wave energy.

The discussion of (2.25b) is in some ways analogous to that of (2.25a); the discussion can also be in terms of changes in frequency, $\omega + \frac{\partial \phi}{\partial t}$, of the wave packet (see below). The first, second and third terms show that the points at which the phase modulating function $\phi(z,t)$ is constant also propagates at the group velocity defined by the local wave number k_{eff} (2.26). Similarly the fourth and fifth terms show that the derivative of the growth rate is evaluated at the wave number k_{eff}, i.e.

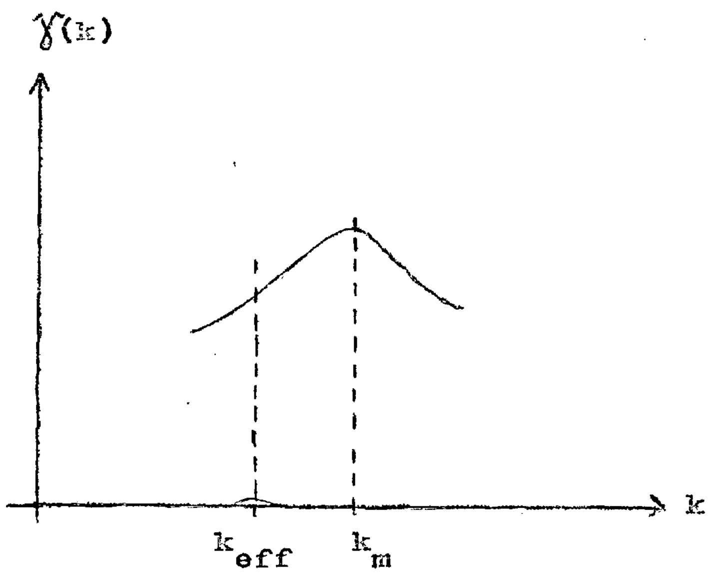
$$\left.\frac{d\hat{\gamma}}{dk}\right|_{k=k_{eff}} \frac{\partial A}{\partial z} \frac{1}{A}$$

This term has a clear physical origin. However, before investigating its roots it is worth thinking in terms of individual Fourier (or plane wave) components, in the more conventional manner, rather than total phase and amplitude. Individual wave-number components will grow at different rates, i.e. $\gamma = \gamma(k)$; the effect of this growth on the amplitude is to introduce the term $\gamma[k_{eff}]A$. It is interesting to see that the integrated effect of growth of these wave-number components on the disturbance is to produce a growth term evaluated at the local wave number k_{eff}. Thus these two descriptions of the same process are interrelated in a rather simple but not obvious way. (The appearance of $\gamma[k_{eff}]$ presumably stems from the fact that over a differential element of space δz and time δt a given particle will 'sense' that it is in a plane wave of the form

$$e^{i([\omega + \frac{\partial \phi}{\partial t}]\delta t - [k - \frac{\partial \phi}{\partial z}]\delta z)}\quad)$$

The effect of the preferential growth of different wave-number components on the phase of the wave-packet is less obvious and appears through the above term

$$\left.\frac{d\gamma}{dk}\right|_{k_{eff}} \frac{\partial A}{\partial z}\frac{1}{A}$$

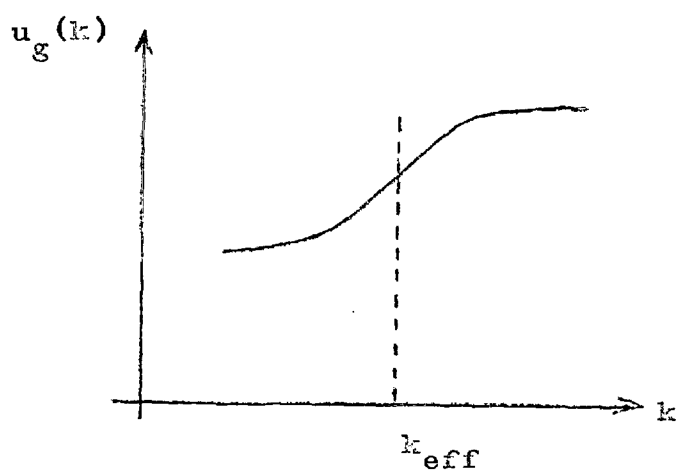


Thus if the local wave number k_{eff} corresponds to a region in which the growth rate is an increasing function of wave number ($\frac{d\gamma}{dk}$ +ve as in the figure) we expect the wave number components in the spectrum which lie slightly to the right of k_{eff}: to dominate those to the left as time progresses, leading to a progressive change in phase. This process will stop once the wave number k_{eff} coincides with that wave number corresponding to a local maximum in growth rate, i.e. k_m.

The sixth term in (2.25b) is directly analogous to the seventh term in (2.25a). Thus as the spectrum sharpens (as it will when $\frac{d^2\gamma}{dk^2}$ is large and negative, see also above) the phase modulating factor diffuses, i.e. becomes more constant over a wider range of position z. This corresponds to a more constant effective frequency, $\omega + \frac{\partial\phi}{\partial t}$ (or wave number, $k - \frac{\partial\phi}{\partial z}$) over a wider range of position. That is, the phase variation of the wave becomes more like that of a monochrometic wave as the spectrum sharpens. This process also appears in the form of positive diffusion when the growth rate is positively peaked, the effect being reversed in the case of damping.

The final term is closely related to the last term in equation (2.25a), its origin is also clear.

Thus it states that the phase of the wave packet in some given region will change rapidly if the group velocity corresponding to k is changing sharply with respect to wave number, this is best understood from the figure below. (It is reasonable to assume that on including higher order terms in (2.25b) this term would become $-\frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k=k_{eff}}\frac{\partial^2 A}{\partial z^2}\frac{1}{A}$ though this is not essential to the argument).



In the (rather exaggerated) situation depicted those wave number components to the right of k_{eff} will propagate out of the considered region faster than those components to the left. The phase or effective wave number must change rapidly as a result, rather in the same way as it must when new wave number components are 'introduced' as a result of growth.

The above discussion of these underlying processes makes equation (2.16) easier to comprehend. On the other hand these same processes would be rather difficult to forsee without the help of equation (2.16).

In the equations (2.25a), (2.25b) we notice that all the terms may be divided into two types, i.e. those closely related to the concept of group velocity, and those connected with the growth constant γ. All the terms in these equations have a reasonably clear physical origin in relation to these two ideas.

There are really three reasons which justify an investigation of the sort we have just made. The first is that it has given some simple physical insight into the mode of propagation wave packets in uniform media when the regime is linear, and in the process it has obviously made clear what the linear theory cannot do.

The second reason is that we have discussed fully all the implications inherent in the inclusion of a slowly varying envelope in some chosen disturbance when requiring that it must satisfy the uniform cold plasma wave equation. It will become clear later that such an investigation is helpful in understanding the nature of solutions (which also contain slowly varying envelopes) which we derive in a later chapter when the plasma is both hot and non-uniform.

Thirdly it is hoped that the basic equations (2.25a), (2.25b) and their discussion will be of value in the study of sonogram traces. In this respect one needs to know how the sonogram traces are related to the rate of change of phase and amplitude of the electric or magnetic field of the input signal.

A sonogram trace usually consists of the output response of a series of narrow band resonators as a function of time, the quantities plotted in two dimensional cartesian coordinates (time being the abscissa). The response of a tuned resonator to a frequency modulated or gliding tone is well established [e.g. Barber and Ursell (1948)]. Roughly the response of the resonator will become maximum when the frequency $\omega(t)$ ($= \frac{\partial}{\partial t}$ [phase of input signal]) coincides with its own resonant frequency. The resolution in time and frequency is determined by the band width of the resonators, (the narrower the band width the longer the resolution time, i.e. the wider the trace). This relation is essentially in the nature of a Fourier Transform, i.e. $\Delta\omega \; \Delta t < \mu$, where μ depends on the characteristics of the resonators used. To the author's knowledge no detailed investigation has ever been made of the response of a tuned resonator to a frequency and amplitude modulated signal. One expects the darkness of the trace to be closely related to the amplitude A, and theinteresting effects are likely to be connected with equation (2.25b) (though it is very tentatively suggested that amplitude modulation could give rise to fine structure in the observed traces by analogy with side bands). Thus one expects the signal, equation (2.1), to appear on a sonogram as a trace of small slope in a region near the frequency ω, the deviations from this frequency being due to the terms in equation (2.25b). (This is best understood by writing the phase variation of the electric field in equation (2.1) as

$$e^{i\int^{t}\omega(t')dt'} = e^{i\int^{t}[\omega + \frac{\partial\phi}{\partial t'}]dt'}\quad).$$

In the introduction we reviewed the model of complete stirring as developed by A. C. Das [1967] and we have seen how sharp peaks in the growth rate result from the interaction of a narrow band wave packet with the ambient particle distribution. Typically we may thus expect the growth rate derivatives to be large when considering this type of magnetospheric phenomina. These large derivatives not only lead to changes in frequency, $(\omega + \frac{\partial\phi}{\partial t})$, but also to a positive feedback mechanism via "diffusion" in amplitude (due to the diffusion coefficient $-\frac{1}{2}\frac{d^2\gamma}{dk^2}$). Thus if one has a situation in which energy is supplied to the wave by the particle distribution it will normally be propagated out of that region at the appropriate group velocity. However, should the "diffusion" coefficient $-\frac{1}{2}\frac{d^2\hat{\gamma}}{dk^2}$ become sufficiently large the amplitude would diffuse back and the disturbance amplitude would then grow indefinitely in that region. Under these conditions one would then presumably expect a "hiss" or non-convective type of instability.

It is not really the objective of the work presented here to make a detailed investigation of the appearance of various sonogram traces in an attempt to identify which processes are in fact dominant, nor to suggest V.L.F. generation mechanisms. However one hopefully expects the discussion of the equations (2.25a) and (2.25b) to be of value to workers attempting to find answers to such problems. (Notice in the magnetosphere one must also take into account the inhomogeniety in the ambient field; for example, the detailed appearance of hooks is only explicable in terms of both enhanced growth rates and also the inhomogeniety of the ambient field, e.g. Dowden (1962)).

In conclusion of this section it is worth briefly investigating typical orders of magnitude of quantities appearing in equation (2.16). In order to do this we use the model developed by A. C. Das (1967).

Starting from the well known cold plasma Whistler mode dispersion relation

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{\omega_p^{\,2}/\omega}{|\Omega| - \omega}$$

and assuming $\omega << |\Omega|$, one can easily show that

(i) $\frac{d\omega}{dk} \simeq 2\frac{\omega}{k} = 2V_p$

(ii) $\frac{d^2\omega}{dk^2} \simeq \frac{V_p}{k}$

Thus taking typically $k \sim 10\ km^{-1}$, $\omega/k \sim 10^5\ km\ sec^{-1}$ (e.g. see Das (1967) we have

(iii) $\frac{d\omega}{dk} \sim 10^5\ km\ sec^{-1}$, $\frac{d^2\omega}{dk^2} \sim 10^4\ km^2\ sec^{-1}$

From Figure a Chapter (I), page (40) we take typically for the labelled quantities $\delta\gamma$, $\delta v_{\parallel}$ the orders of magnitude

$\delta\gamma \sim 1\ sec^{-1}$

$\delta v_{\parallel} \sim 10\ km\ sec^{-1}$

We also take $v_{\parallel res} \sim 10^4\ km\ sec^{-1}$ (e.g. Das (1967)).

From the resonant condition $v_{\parallel} = \dfrac{\omega - |\Omega|}{k}$ we have

$$\delta k = \frac{k \, \delta v_{\parallel}}{\left.\dfrac{d\omega}{dk} - v_{\parallel}\right|_{res}}$$

and hence for the above orders of magnitude $\delta k \sim 10^{3}$ km^{-1}.

Hence roughly we may put

(iv) $\dfrac{d\gamma}{dk} \sim \dfrac{\delta\gamma}{\delta k} = 10^{3}$ km sec^{-1}

(v) $\dfrac{d^{2}\gamma}{dk^{2}} \sim \dfrac{\delta}{(\delta k)^{2}} = 10^{6}$ km^{2} sec^{-1}

Comparing (iii) with (iv) and (v) we can see that these rough calculations alone are sufficient to show that the derivatives of the growth rate can give rise to terms which can dominate the propagation characteristics of the disturbance. (Of course, if they become too big the whole expansion procedure will break down).

2.3 The Formal Solution

The formal solution of (2.16) can be found by a Fourier Transformation technique. For reasons that will become clear below we transform with respect to $k_1 = k' - k$. That is:-

$$S(k_1,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} G(z,t)\, e^{-ik_1 z}\, dz \tag{2.29}$$

$$G(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S(k_1,t)\, e^{ik_1 z}\, dk_1 \tag{2.30}$$

Thus (2.16) becomes

$$\frac{\partial S(k_1,t)}{\partial t} = \left[(-ik_1)\omega' - (ik_1)^2 \frac{i\omega''}{2}\right] S(k_1,t) \tag{2.31}$$

where $\omega' = \left.\frac{d\omega}{dk}\right|_k$, $\omega'' = \left.\frac{d\omega^2}{dk^2}\right|_k$

So:-

$$S(k_1,t) = S(k_1)\, e^{-ik_1\omega' t + \frac{i\omega''}{2} k_1^2 t} \tag{2.32}$$

where $$S(k_1) = \int_{-\infty}^{+\infty} g(z)\, e^{-ik_1 z}\, dz \tag{2.33}$$

Hence

$$G(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dz'\, g(z')\, e^{ik_1(z-z'-\omega' t)+\left(\frac{i\omega'' t}{2}\right)k_1^2}$$

(2.34)

This equation represents an integration over the spectrum of plane waves where the frequency as a function of wave number has been approximated. This will become clear if we consider a simple example.

Suppose we consider an initial envelope which is a simple gaussian centred on the origin, i.e.

$$G(z,t{=}0) = g(z) = \sqrt{\frac{\alpha}{\pi}} \cdot e^{-\alpha z^2} \tag{2.35}$$

and normalized so

$$\int_{-\infty}^{+\infty} g(z)\, dz = 1$$

Putting (2.35) in (2.34) and integrating over z' first we have

$$G(z,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk_1\, e^{-\frac{1}{2}(a-i\omega''t)k_1^2+ik_1(z-\omega't)} \tag{2.36}$$

where $a = \frac{1}{2\alpha}$.

On putting the expression in (2.1) we recognise (2.36) as the integral over the spectrum

$$\frac{1}{2\pi}\, e^{-(k_1^2/4\alpha)} \tag{2.37}$$

of plane waves where the frequency as a function of wave number has been approximated. This type of integral has been discussed rather fully by Feix (1963) and also Kadomtsev (1965). On carrying out the integration in (2.36) by completing the square we find:-

$$G(z,t) = \frac{C \exp -\left\{\frac{(z - \omega' t)^2}{2[a - i\omega'' t]}\right\}}{[a - i\omega'' t]^{\frac{1}{2}}} \tag{2.38}$$

It can, of course, be confirmed by direct substitution that (2.38) is a solution of (2.16). However the particular conditions of validity are discussed by Feix and as expected (2.38) is good in the vicinity of its maximum.

If we put $\omega = \omega_o - i\gamma_o$ and use relation (2.1) we see that

$$\underline{E} = \frac{\underline{E}_o \exp\left\{i(\omega_o t - k z) + \gamma_o t - \frac{(z - \omega_o' t)^2}{2(a - i\omega_o'' t)}\right\}}{[a - i\omega'' t]^{\frac{1}{2}}} \tag{2.39}$$

On assuming that the wave-packet has had **sufficient** time to distort so that the dominant wave number in the spectrum is such that

$\gamma_o' = 0$ and $-\gamma_o'' = \beta_o''$ a positive quantity.

Equation (2.39) becomes

$$\frac{E_o \exp\left\{\gamma_o t - \frac{(z - \omega_o' t)^2 (a - \gamma_o'' t)}{2S}\right\}}{S^{\frac{1}{2}}}$$

$$\exp i\left[\omega_o t - k z - \frac{(z - \omega_o' t)^2 \omega_o'' t}{2S} + \phi(t)\right] \tag{2.40}$$

where $S = (a - \gamma_o''t)^2 + (\omega_o''t)^2$

$$\phi(t) = \tan^{-1}\left[\frac{\omega_o''t}{(a - \gamma_o''t)}\right] \qquad (2.41)$$

condition (2.40) becomes reasonable after large time intervals we may then presumably neglect a in (2.41). We then arrive at the situation discussed by Feix (1963). That is, the amplitude is given by

$$\frac{\exp\left\{ \gamma_o t - (z - \omega_o't)^2 \left[\frac{\beta_o''}{2(\omega_o''^2 + \beta_o''^2)t}\right] \right\}}{[\beta_o''^2 + \omega_o''^2]^{\frac{1}{4}} \sqrt{t}}$$

The centre of the disturbance propagates at the group velocity while the width increases as $t^{\frac{1}{2}}$ as in a diffusion process. For group velocities $\omega_o'^2 < 2\gamma_o \dfrac{\omega_o''^2 + \beta_o''^2}{\beta_o''}$ the amplitude of the wave-packet will increase without limit at a given point, i.e. in those frames the instability is non-convective.

Chapter III

WAVE PACKETS IN A HOT UNIFORM VLASOV PLASMA

Introduction

In the previous chapter we investigated some fundamental aspects arising from wave packet propagation in uniform media. The essential point being that the medium was described in terms of cold plasma equations. That is the integrodifferential 'wave equation' of hot plasma theory was replaced by a partial differential wave equation. In this chapter we shall show how these same equations, derived in the previous chapter, can also be derived from a consideration of the full hot plasma 'wave equation'.

Suppose we consider for example the problem of V.L.F. emissions. Then if at least some V.L.F. emissions are due to wave particle gyroresonances (as is generally believed (e.g. Bell and Bunenann, 1964)) then there are many processes which can complicate the interaction and subsequent wave propagation. The most obvious complication is that the ambient magnetic field is not uniform. This fact has more than one feature, not only do the varying plasma parameters change the dispersion of the propagating disturbance (e.g. the gyrofrequency and wave number will not be independent of position) but also particles with the same $v_{\parallel}$ will resonate with a particular wave at some instant, but because these particles will have a large range of $V_{\perp}$ they will subsequently behave differently in the non-uniform ambient magnetic field. Other complications are not difficult to imagine, any or all of the ambient plasma parameters could be functions of space and time, due, for example, to the presence of other waves. (e.g. It is well known that compression of the magnetospheric boundary increases the pitch angle anisotropy of the particle distribution and makes the plasma more unstable). This type of mechanism is really a wave-wave interaction. The crucial effects on the emission probably come via the resonant particles. All these processes mentioned are likely to change the frequency of the emission and some of them may well give rise to the characteristic sonogram traces of particular emissions (e.g. Dowden (1962)).

The problem of devising a mathematical description of the propagating disturbance (i.e. finding solutions of the appropriate integro differential wave equation) in the presence of processes of this type seems a formidable task. In this section we show that the method of characteristics is a suitable technique for solving the Boltzmann-Vlasov equations in this type of work. That is, we solve the linearised collisionless Boltzmann equation in the Lagrangian system of coordinates. This method of solution involves integration over the zero order particle trajectories, in other words over their past histories. The fact that emerges from this section is that one does not have to go far back in time in evaluating the trajectory integration owing to the finite temperature of the plasma. This result seems useful since it means there is great scope for making approximations in the type of problem discussed, and since it would be impossible, in practice, to know the distant past. In order to make this point clear we consider the problem of wave-packet propagation as in the last section. We show that the equation governing its behaviour can be derived in a perfectly natural way from approximations which are seen to be typical for the type of problems we wish eventually to solve.

3.1 Wave Packet Solution of the Vlasov Equations by the Method of Characteristics

A discussion of the method of characteristics is given in most advanced plasma physics texts (for example, Stix (1962)). For the reasons we have already mentioned we shall not assume plane wave propagation in the treatment that follows.

Assuming the unperturbed distribution $f^{o}(\underline{v})$ to be independent of space and time the zero order Boltzmann equation becomes:-

$$- \frac{|\varepsilon|}{m} (\underline{v} \wedge \underline{B}_o) \cdot \nabla_{\underline{v}} f^o(\underline{v}) = 0 \qquad (3.1)$$

On transforming to cylindrical coordinates in velocity space one can see that this relation is equivalent to the statement

$$f^o(\underline{v}) = f^o(v_\perp^2, v_\parallel) \qquad (3.2)$$

where $v_\perp$ and $v_\parallel$ are the components of velocity, perpendicular and parallel to the uniform ambient magnetic field $\underline{B}_o = B_o \hat{z}$ (3.3) .

Assuming the zero order electric field to be zero and using the usual perturbation technique, i.e.

$$f(z,\underline{v},t) = f^o(v_\perp^2, v_\parallel) + f'(z,\underline{v},t)$$

$$\underline{B} = \underline{B}_o + \underline{b}(z,t)$$

$$\underline{E} = 0 + \underline{E}(z,t)$$

(The perturbations being as in the introduction, and their spatial dependence is on z only).

On linearizing the Boltzmann equation we arrive at:-

$$\frac{\partial f'}{\partial t} + \underline{v}\cdot\nabla f' - \frac{|\mathcal{E}|}{m}(\underline{v}_\wedge \underline{B}^o)\cdot\nabla_{\underline{v}} f' = \frac{|\mathcal{E}|}{m}\left(\underline{E} + \frac{\underline{v}_\wedge \underline{b}}{c}\right)\cdot\nabla_v f^o$$

The left hand side of this equation represents the time derivative of f' along zero order particle trajectories (e.g. see Stix (1962)).

Thus

$$f'(z,\underline{v},t) = \frac{|\mathcal{E}|}{m}\int_{-\infty}^{t} dt' \left[\underline{E}(z',t') + \frac{\underline{v}'_\wedge \underline{b}}{c}(z',t')\right]\cdot\frac{\partial f^o}{\partial \underline{v}'} \qquad (3.6) .$$

Here, as in the derivation of equation (2.16), we are not considering the initial value problem.

Thus in equation (3.6) we choose the electric and magnetic fields to vary as :-

$$E(z,t) = (\hat{x}\, e_x + \hat{y}\, e_y)\, e^{i(\omega t - kz)}\, G(z,t)$$

$$(3.7)$$

$$b(z,t) = (\hat{x}\, b_x + \hat{y}\, b_y)\, e^{i(\omega t - kz)}\, G_1(z,t)$$

where e_x, e_y, b_x, b_y are constant amplitudes as in the previous chapter.
(We assume E_z to be zero as it has already been shown in the introduction that the perpendicular and parallel motions are decoupled in the linear approximation).
We use the same notation as in the previous chapter:-
G and G_b are slowly varying envelopes related to the electric and magnetic fields respectively, where from the Maxwell equation

$$\nabla_\wedge \underline{E} = -\frac{1}{c}\frac{\partial \underline{b}}{\partial t} \quad \text{we can show that:-}$$

$$b_x = -\frac{kc}{\omega} e_y \; ; \quad b_y = \frac{kc}{\omega} e_x \; ; \quad b_z = 0 \qquad (3.8)$$

and

$$e^{i(\omega t - kz)} G_b(z,t) = \frac{(1 + i\mathcal{Z}/k)}{(1 - i\mathcal{V}/\omega)} G(z,t) \cdot e^{i(\omega t - kz)} \qquad (3.9)$$

Hence putting the relations (3.7) in equation (3.6) we have:-

$$f'(z,\underline{v},t) = \frac{|\varepsilon|}{m} \int_{-\infty}^{t} dt' [G(z',t')[\hat{x}e_x + \hat{y}e_y] + G_b(z',t')\underline{v}'_\wedge [\hat{x}b_x + \hat{y}b_y]] \cdot \frac{\partial f^o}{\partial \underline{v}'} e^{i(\omega t' - kz')} \qquad (3.10)$$

The integration in (3.10) is along the zero order particle trajectory. The equation of motion for electrons in the ambient magnetic field $\underline{B} = B_o \hat{z}$ is:-

$$\frac{d\underline{v}'}{dt'} = -|\Omega|\, \underline{v}' \wedge \hat{z} \quad (3.11) \quad \text{where } |\Omega| = \frac{|\varepsilon|\, B^o}{mc} \quad (3.12).$$

The solution of equation (3.11) satisfying the criterion that at $t' = t$, then $\underline{v}' = \underline{v}$ and $z' = z$ (3.13) is:-

$$\begin{aligned} V_x' &= V_\perp \cos(\theta - |\Omega| T) \\ V_y' &= V_\perp \sin(\theta - |\Omega| T) \qquad (3.14) \\ V_z' &= V_z \end{aligned}$$

and hence $z' = z - v_z T$, where $T = t - t'$ time measured backwards from the point t, so:-

$$V_x = V_\perp \cos\theta, \quad V_y = V_\perp \sin\theta \qquad (3.15).$$

From the relations (3.2) and (3.14) we find easily:-

$$\begin{aligned} \frac{\partial f^o}{\partial v_x'} &= \frac{\partial f^o}{\partial V_\perp} \cos(\theta - |\Omega| T) \\ \frac{\partial f^o}{\partial V_y'} &= \frac{\partial f^o}{\partial v_\perp} \sin(\theta - |\Omega| T) \qquad (3.16) \\ \frac{\partial f^o}{\partial v_z'} &= \frac{\partial f^o}{\partial v_z} \end{aligned}$$

It is a short step using (3.8), (3.14) and (3.16) to transform (3.10) to the unprimed coordinates and on separating out the right hand circularly polerized mode we easily arrive at

$$f'(z,\underline{v},t) = -\frac{|\varepsilon|}{m} \underline{E}_{o\perp} e^{-i\theta}\int_{+\infty}^{o} dT \left\{ \frac{\partial f^o}{\partial v_\perp} G[(z-v_{\shortparallel}T),(t-T)] \right.$$

$$\left. -\frac{k}{\omega}(v_{\shortparallel}\frac{\partial f^o}{\partial v_\perp} - v_\perp \frac{\partial f^o}{\partial v_{\shortparallel}})G_b[(z-v_{\shortparallel}T),(t-T)]\right\} e^{i(kv_{\shortparallel}+|\Omega|-\omega)T}$$

(3.17)

where

$$\underline{E}_{o\perp} = \left\{\frac{\hat{x}e_x + \hat{y}ie_y}{2}\right\} e^{i(\omega t - kz)} \qquad (3.18)$$

and we have written $t' = t-T$ in the two envelopes G and G_b, also dt' has become -dT. Hence:-

$$j(z,t) = -\frac{|\varepsilon|}{m}\int_o^{2\pi} d\theta \int_o^{+\infty} dv_\perp\, v_\perp \int_{-\infty}^{+\infty} dv_{\shortparallel}\, [e^{i\theta} v_\perp f'(z,\underline{v},t)]$$

$$= +\frac{|\varepsilon|^2}{m}\underline{E}_{o\perp}\int_{-\infty}^{+\infty} dv_{\shortparallel}\int_{+\infty}^{o} dT\left\{ gG[(z-v_{\shortparallel}T),(t-T)] - \frac{k}{\omega}[v_{\shortparallel}g-h]G_b[(z-v_{\shortparallel}T),(t-T)\right.$$

$$\cdot e^{i(kv_{\shortparallel}-\omega+|\Omega|)T} \qquad (3.19)$$

where $g(v_{\shortparallel}) = \pi\int_o^{+\infty} dv_\perp\, v_\perp^2 \frac{\partial f^o}{\partial v_\perp} = -2\pi\int_o^{+\infty} f^o v_\perp\, dv_\perp$

(3.20)

and $h(v_{\shortparallel}) = \pi\frac{\partial}{\partial v_{\shortparallel}}\int_o^{\infty} dv_\perp\, v_\perp^3 f^o$

The remainder of this section will be spent in what is essentially a discussion of (3.19). Of course, if we discuss the plane wave case (i.e. put $G=G_b=1$) equation (3.19) becomes:-

$$\underline{j}(z,t) = + \frac{|\mathcal{E}|^2}{m} \underline{E}_{o\perp} \int_{-\infty}^{+\infty} dv_{\shortparallel} \int_{+\infty}^{o} dT \left\{ \frac{g[\omega - kv_{\shortparallel}] + hk}{\omega} \right\} e^{i(kv_{\shortparallel} - \omega + |\Omega|)T}$$

(3.21)

On using (3.21) in conjunction with the two Curl Maxwell equations, as in the introduction, and integrating over T, (assuming the contribution from $T = +\infty$ is zero, see below) we arrive at the familiar dispersion relation valid for growing waves. That is, on eliminating the magnetic field $\underline{b}$ from the two Curl Maxwell equations and substituting (3.21) for the current density we find

$$\text{(a)} \quad k^2c^2 - \omega^2 = - \frac{\omega_p^{\,2} i\omega}{n_o} \int_{-\infty}^{+\infty} dv_{\shortparallel} \int_{+\infty}^{o} dT \left\{ \left(1 - \frac{kv_{\shortparallel}}{\omega}\right) g + \frac{kh}{\omega} \right\} e^{i(kv_{\shortparallel} - \omega + |\Omega|)T}$$

On carrying out the time integration in this equation, assuming that the contribution from $T = +\infty$ is zero (see below), we arrive at the familiar result

$$\text{(b)} \quad k^2c^2 - \omega^2 = \frac{\omega_p^{\,2}\omega}{n_o} \int_{-\infty}^{+\infty} dv_{\shortparallel} \frac{\left(1 - \frac{kv_{\shortparallel}}{\omega}\right) g + \frac{kh}{\omega}}{(\omega - kv_{\shortparallel} - |\Omega|)}$$

where $\text{Im}(\omega) < 0$.

This equation is valid for growing waves Im $(\omega) < 0$. Compare this equation with equation (.14) of the introductory chapter.

We shall show in what follows that, written in the form (a), this equation is valid for both growing and damped waves. Of course, written in the form (b), we analytically continue the equation into the upper half ω plane using the Landau prescription, i.e.

$$\text{(c)}\quad k^2c^2-\omega^2 = \frac{\omega_p{}^2\omega}{n_o}\int_{-\infty}^{+\infty} dv_{\parallel}\, \frac{(1-\frac{kv_{\parallel}}{\omega})g + \frac{kh}{\omega}}{(\omega - kv_{\parallel} - |\Omega|)}$$

$$+ 2\pi i\left\{(1-\frac{kv_{\parallel}}{\omega})g + \frac{kh}{\omega}\right\}\frac{\omega_p{}^2\omega}{n_o}\Bigg|_{v_{\parallel} = \frac{(|\Omega| - \omega)}{k}}$$

where Im $(\omega) > 0$, and as in the introductory chapter we may write this equation in the form valid for both growing and damped waves, i.e.

$$\text{(d)}\quad k^2c^2-\omega^2 = \frac{\omega_p{}^2\omega}{n_o}\, P\int_{-\infty}^{+\infty} dv_{\parallel}\, \frac{(1-\frac{kv_{\parallel}}{\omega})g + \frac{kh}{\omega}}{(\omega - kv_{\parallel} - |\Omega|)}$$

$$+ \pi i\left\{(1-\frac{kv_{\parallel}}{\omega})g + \frac{kh}{\omega}\right\}\frac{\omega_p{}^2\omega}{n_o}\Bigg|_{v_{\parallel} = \frac{(|\Omega| - \omega)}{k}}$$

3.2 Cut Off in the Orbit Integration due to Finite Temperature

We now look a little more closely at (3.21), thus by reversing the order of integration we notice that it has the form

$$\frac{|\varepsilon|^2}{m} \underline{E}_{o\perp} \int_{+}^{o} dT \; e^{i(|\Omega| - \omega)T} \int_{-\infty}^{+\infty} dv_{\shortparallel} F(v_{\shortparallel}) e^{iv_{\shortparallel}(kT)} \qquad (3.22)$$

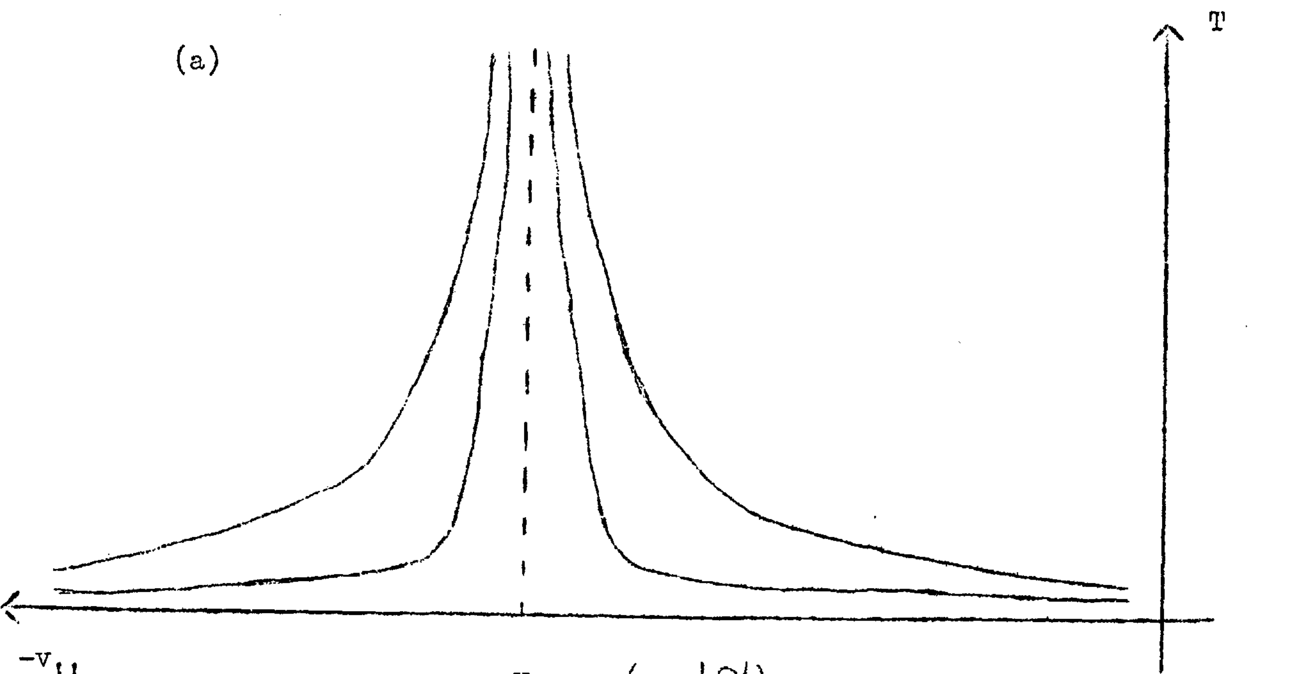
where $F(v_{\shortparallel})$ is a function of $v_{\shortparallel}$ whose width depends on the temperature of the plasma. We notice that the integration over $v_{\shortparallel}$ is of the form of a Fourier Transform, i.e.

$$\int_{-\infty}^{+\infty} dv_{\shortparallel} F(v_{\shortparallel}) e^{iv_{\shortparallel}(kT)} = \xi(kT) \qquad (3.23)$$

The wider the thermal spread in $F(v_{\shortparallel})$ the more sharply peaked its "Fourier Transform" $\xi(kT)$ will be. The integration over time which follows (3.23) will be "cut off" at some point because $\xi(kT)$ will be effectively zero. The cold plasma in which $F(v_{\shortparallel})$ is a delta function and $\xi(kT)$ finite even as $T \to \infty$ is, of course, fictitious. However, any sharp peak or gradient in the function $F(v_{\shortparallel})$ due for example to particle beams, the greater the value of T which is important in that region of $v_{\shortparallel}$. (This is analogous to the presence of more "frequency components" in the "spectrum" $\xi(kT)$ of $F(v_{\parallel})$.

We can increase our insight further by examining this result diagramatically. The diagrams (a) and (b) below are "maps in relief" of equation (3.21) in the $v_{\parallel}$,T plane.

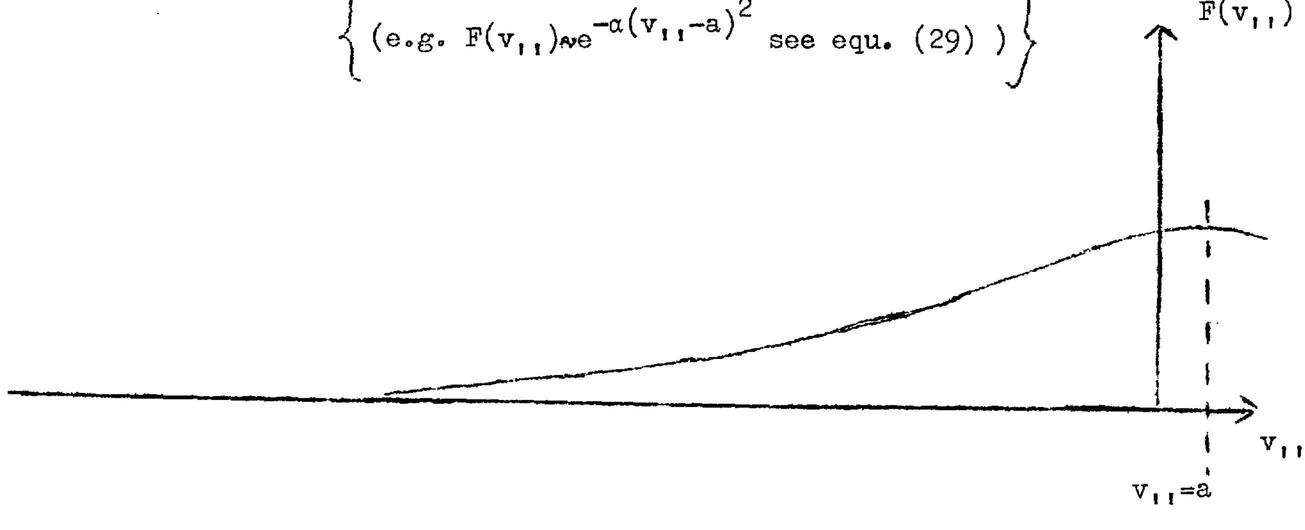
(a)



$$v_{\parallel} = \frac{-(\omega - |\Omega|)}{k} = v_{\parallel \, res.}$$

(b)

$\left\{ \text{(e.g. } F(v_{\parallel}) \sim e^{-\alpha(v_{\parallel}-a)^2} \text{ see equ. (29))} \right\}$



In diagram (a) we have plotted lines of constant phase of the integrand of equation (3.21) in the $v_{\parallel}$,T plane, these are rectangular hyperbola centered on $v_{\parallel res}$. In diagram (b) we have plotted $F(v_{\parallel})$ against $v_{\parallel}$, which is constant for all T (at least in the linear approximation - for example, c.f. quasi-linear approximation). We see that the further back in time we go, the faster the cosine of the phase oscillates. Eventually the oscillations are so rapid that the envelope $F(v_{\parallel})$ can be considered constant over one period of oscillation, the contribution to the integral (3.21) becomes zero in the same way as:-

$$\lim_{\alpha \to 0} \int F(v_{\parallel}) \cos (v_{\parallel}/\alpha)\, dv_{\parallel} \qquad (3.25)$$

($F(v_{\parallel})$ must, of course, satisfy Dirichlet's conditions, which we know on physical grounds any particle distribution will). We can again see from these diagrams why any sharp peak or kink in $F(v_{\parallel})$ results in a contribution to the current $\underline{j}(z,t)$ in equation (3.21) coming from larger values of T.

A particularly good distribution for discussion is that of the anisotropic Maxwell Boltzmann shifted in $v_{\parallel}$, i.e.

$$f^{o}(v_\perp{}^2, v_\parallel) = \frac{N\delta\sqrt{\alpha}}{\pi^{3/2}} e^{-\delta V_\perp{}^2 - \alpha(v_\parallel - a)^2} \quad (3.26)$$

where $\delta = \dfrac{m}{2KT_\perp}$, $\alpha = \dfrac{m}{2KT_\parallel}$ $\left\{\begin{array}{l} K = \text{Boltzmann's constant.} \\ T_\perp, T_\parallel \text{ are the perpendicular} \\ \text{and parallel temperatures.}\end{array}\right.$

and $N = \int_0^{2\pi} d\theta \int_0^{+\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_\parallel f^o$

This distribution is unstable when $\alpha > \delta$ (i.e. $T_\perp > T_\parallel$).

Thus

$$h(v_\parallel) = -N_1 e^{-\alpha(v_\parallel - a)^2}$$

$$g(v_\parallel) = -\frac{\alpha}{\delta} N_1 e^{-\alpha(v_\parallel - a)^2} \quad \text{where } N_1 = N\sqrt{\alpha/\pi}$$

Equation (3.21) can then be put in the form:-

$$\underline{j}(z,t) = + \frac{|\varepsilon|^2}{m} \underline{E}_{o\perp} \left\{ \frac{B_1 + k[a + \frac{1}{2\alpha} \frac{\partial}{\partial a}]}{\omega} \right\} I \quad (3.28)$$

where

$$B_1 = [\omega - ka\frac{\alpha}{\delta}] , \quad B_2 = [\alpha/\delta - 1]$$

and

$$I = \int_{+\infty}^{0} dT\, e^{i(|\Omega| - \omega)T} \int_{-\infty}^{+\infty} dv_\parallel\, e^{-\alpha(v_\parallel - a)^2 + i(kT)v_\parallel}$$

$$(3.29)$$

On putting $u = (v_\parallel - a)$, (3.29) becomes

$$I = \int_{+\infty}^{0} dT \; e^{i(|\Omega| - \omega + ka)T} \int_{-\infty}^{+\infty} du \; e^{-\alpha u^2 + i(kT)u} \qquad (3.30)$$

On completing the square in this section we have

$$I = \int_{+\infty}^{0} dT \; e^{i(|\Omega| - \omega + ka)T - \frac{(kT)^2}{4\alpha}} \int_{-\infty}^{+\infty} du \; e^{-\alpha\left[u - \frac{i(kT)}{2\alpha}\right]^2} \qquad (3.30a)$$

Written in this form we can see immediately that this double integration will converge rapidly for large values of u and T, for both growing and damped waves. For example, taking magnetospheric numbers, since $\alpha = \frac{1}{v_{T_{\parallel}}^{2}}$ where $v_{T_{\parallel}}$ = parallel thermal velocity, we can say that the integration over time T effectively cuts off when

$$\frac{(kT)^2}{4\alpha} \sim 100 \quad \text{i.e.} \quad T \sim \frac{10}{kv_{T_{\parallel}}} .$$

Typically $k \sim 10 \text{ km}^{-1}$

$v_{T_{\parallel}} \sim 10^3 \text{ km sec}^{-1}$ (Guthart (1964))

so $T \sim 10^{-2}$ sec.

In the magnetosphere we have a situation in which particles are bouncing between mirror points and drifting in longitude, if the integration over time really had to be taken back into the infinite past, $T = +\infty$, the problem would surely become intractable.

If we formally take the limit $T_{\parallel} \to \infty$ in equation (3.30) the integration over u gives rise to a delta function, i.e. $\underset{\gamma}{\xi}(kT) = \frac{1}{k} \, \delta(T)$, see equation (3.23). The contribution to the current then comes exclusively from the region $T \to +0$.

3.3 Application of 3.2 to the Wave Packet Problem

In order to limit the material in this section, we will not discuss equations (3.26) through to (3.30) further, but simply state that the integration (3.30) can be evaluated. The resulting current (equation (3.28)) is a complex quantity, and on applying Maxwell's equations we can derive an expression for the growth rate (see Appendix A).

We now show that the rapid convergence of the integration over time can be put to use in the solution of real problems. We return to equation (3.19) and derive the equation of motion of the wave packet envelope G as follows:-

We eliminate the magnetic field from the two relations (3.7) by use of the two Curl Maxwell equations.

Thus:-

$$\frac{\partial^2 \underline{E}(z,t)}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \underline{E}(z,t)}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \underline{j}}{\partial t}$$

Since we have

$\frac{\partial}{\partial t} = i\omega + \tau$ and $\frac{\partial}{\partial z} = ik + \chi$, this equation

becomes:-

$$[-k^2 G - 2ikG_z + G_{zz}] = \frac{1}{c^2}[-\omega^2 G + 2i\omega G_t + G_{tt}] + \frac{4\pi}{c^2}\frac{\partial \underline{j}}{\partial t}$$

(3.31)

where subscripts denote differentiation, e.g.

$$G_{zz}(z,t) \equiv \frac{\partial^2 G}{\partial z^2} \quad \text{etc.}$$

From equation (3.19) we have:-

$$\frac{\partial \underline{j}}{\partial t} = \frac{|\mathcal{E}|^2}{m}\underline{E}_{o\perp} i\omega[1 - i\tau/\omega]\int_{-\infty}^{+\infty} dv_{\shortparallel} \int_{+\infty}^{o} \left\{ gG - \frac{k}{\omega}[v_{\shortparallel} g - h]G_b \right\} . e^{i(kv_{\shortparallel} - \omega + |\Omega|)T} \quad (3.32)$$

and since

$$e^{i(\omega t - kz)} G_b(z,t) = \frac{[1 + i\chi/k]}{[1 - i\tau/\omega]} e^{i(\omega t - kz)} G(z,t)$$

(3.9)

equation (3.32) becomes[1]

$$\frac{\partial \underline{j}}{\partial t} = + \frac{|\mathcal{E}|^2}{m}\underline{E}_{o\perp}\int_{-\infty}^{+\infty} dv_{\shortparallel} \int_{+\infty}^{o} dT \left\{ g[i\omega + \tau] - \frac{k}{\omega}[v_{\shortparallel} g - h] i\omega [1 + i\chi/k] \right\} . G e^{i(kv_{\shortparallel} - \omega + |\Omega|)T} \quad (3.33)$$

[1] To see that we can use equation (3.9) in (3.32) imagine G_b also expanded as in (3.34).

Since the major contribution to the current comes from the region $T \to +0$ we Taylor expand

$G[(z-V_{\shortparallel}T),(t-T)]$ in equation (3.33) about the point $T = 0$ i.e.

$$G[(z-V_{\shortparallel}T),(t-T)] = G(z,t)-T[v_{\shortparallel}G_z(z,t)+G_t(z,t)] + T^2/2!\ [v_{\shortparallel}^2G_{zz}(z,t)+2v_{\shortparallel}G_{zt}(z,t)+G_{tt}(z,t)]+ \ldots \tag{3.34}$$

On putting (3.34) into (3.33) we find:-

$$\frac{\partial \underline{j}}{\partial t} = -\frac{|\mathcal{E}|^2}{m}\underline{E}_{o\perp}\left\{i\omega GA+G_tA_1+G_zA_2+G_{zz}A_3+G_{zt}A_4+G_{tt}A_5\right\} \tag{3.35}$$

where

$$A = -\int_{-\infty}^{+\infty} dv_{\shortparallel}\int_{+\infty}^{0} dT\left\{g + \frac{k}{\omega}[h-v_{\shortparallel}g]\right\} e^{i(kv_{\shortparallel}-\omega+|\Omega|)T}$$

and after some laborious rearrangement we recognise

$$\begin{aligned}
A_1 &= A + \omega\frac{\partial A}{\partial \omega}\\
A_2 &= -\omega\frac{\partial A}{\partial k}\\
A_3 &= -\frac{i\omega}{2}\frac{\partial^2 A}{\partial k^2}\\
A_4 &= i\left(\omega\frac{\partial^2 A}{\partial k\partial\omega} + \frac{\partial A}{\partial k}\right)\\
A_5 &= -\frac{i}{2}\left(2\frac{\partial A}{\partial \omega} + \omega\frac{\partial^2 A}{\partial \omega^2}\right)
\end{aligned} \tag{3.36}$$

On putting (3.35) into (3.31) and solving for $\frac{\partial G(z,t)}{\partial t}$ we have:-

$$-\left[\frac{2\omega}{c^2} + \frac{\omega_p^{\,2} iA_1}{n_o c^2}\right]G_t = i\left[-k^2 + \frac{\omega^2}{c^2} + \frac{\omega_p^{\,2}}{n_o} Ai\omega\right]G + \left[2k + \frac{\omega_p^{\,2} iA_2}{n_o c^2}\right]G_z$$

$$+ \left[1 + \frac{\omega_p^{\,2} A_3}{n_o c^2}\right]G_{zz} + \left[\frac{\omega_p^{\,2} iA_4}{n_o c^2}\right]G_{zt} + i\left[-\frac{1}{c^2} + \frac{\omega_p^{\,2} A_5}{n_o c^2}\right]G_{tt} \qquad (3.37)$$

where $\omega_p^{\,2} = \frac{4\pi n_o |\varepsilon|^2}{m}$; n_o appearing through the particle distribution f^o, i.e. $n_o = \int d^3v f^o$.

Equation (3.37) is the equation governing the behaviour of the envelope G. We can convert it into a more familiar form as follows:-

The frequency and wave number in equation (3.37) are related through the hot plasma dispersion relation. That is, on putting G = 1 in this equation ω and k are related through

$$-k^2 + \omega^2/c^2 + \frac{\omega_p^{\,2}}{c^2 n_o} i\omega A = 0, \text{ thus we put}$$

$$k^2 = \omega^2/c^2 + \frac{\omega_p^{\,2} i\omega A}{n_o c^2} = J[\omega,k] \qquad (3.38)$$

Using relations (3.36) and (3.38) equation (3.37) becomes:-

$$- J_{\omega} G_t = (2k - J_k) G_z + \frac{i}{2}(2 - J_{kk}) G_{zz} - \frac{i}{2} J_{\omega\omega} G_{tt} + i J_{k\omega} G_{zt}$$

(3.39)

where $J_{k\omega} = \dfrac{\partial^2 J}{\partial k \partial \omega}$ etc.

From equation (3.38) we also have:-

$$\frac{d\omega}{dk} = \frac{2k - J_k}{J_{\omega}} \qquad (3.40)$$

and

$$\frac{d^2\omega}{dk^2} = \frac{2 - J_{kk} - 2J_{k\omega}\omega_k - J_{\omega\omega}\omega_k^{\,2}}{J_{\omega}} \qquad (3.41)$$

where $\omega_k = \dfrac{d\omega}{dk}$.

Hence from (3.39) and (3.40)

$$G_{zt} = - \omega_k G_{zz} \qquad (3.42)$$

and

$$G_{tt} = \omega_k^{\,2} G_{zz} \qquad (3.43)$$

On putting (3.42) and (3.43) into (3.39) we have:-

$$G_t = - \left\{ \frac{2k - J_k}{J_{\omega}} \right\} G_z - \frac{i}{2} \left\{ \frac{2 - J_{kk} - 2J_{k\omega}\omega_k - J_{\omega\omega}\omega_k^{\,2}}{J_{\omega}} \right\} G_{zz}$$

which from (3.40) and (3.41) we recognise as the equation

$$\frac{\partial G}{\partial t} = - \frac{d\omega}{dk} \frac{\partial G}{\partial z} - \frac{i}{2} \frac{d^2\omega}{dk^2} \frac{\partial^2 G}{\partial z^2} \qquad \text{(equation (2.16)}$$

Discussion

In this chapter we have shown how one may use the integro-differential "wave equation" of hot plasma theory to derive the equation of motion of a chosen wave packet. In particular we have shown that only limited regions of $v_{\parallel}, T$ space are important, and the solution of many problems may well boil down to deciding which regions (diagrams of the type (a) and (b) should be particularly useful in this respect). Typical approximations will involve some sort of expansion of slowly varying parameters about the important regions in the $v_{\parallel}, T$ plane (the Taylor expansion seems an obvious choice as it has just the properties we are looking for). A typical problem has been solved and the usefulness of the technique seems established. Eventually we wish to know how the equation of motion of a wave packet is modified by the presence of the various mechanisms discussed earlier. However in this respect one should remember that in a hot collision free plasma the particles are not confined essentially within a given volume element by collisional effects, as is the case for most fluids, but travel almost freely through the plasma. Therefore any perturbation induced in a given volume element will be carried by this free streaming of the plasma particles to different localities at later times. As pointed out by Stix (Stix (1962)) one avoids this difficulty in a uniform plasma by Fourier analysing in space and time. Thus, hopefully, when the plasma is non-uniform or time-varying, the method we have used will become especially valuable. In the next chapter where we attempt the same problem in a non-uniform ambient magnetic field, the above points will be reiterated.

In the above context we discuss the basis of the method a little further before proceeding to the next chapter. The fact that we knew the integration over time in equation (3.19) or (3.32) converged rapidly for large values of T suggested that we should Taylor expand the slowly varying quantities about the point T = 0. However, having once carried out this procedure we then discovered [through the relations (3.36)] that the square brackets [involving integrations over time T] appearing in equation (3.37) were related in a simple way to the plane wave dispersion relation, [essentially equation (3.38)]. Thus the series of terms in equation (3.37) became successively smaller through the increasing order of the derivatives of the slowly varying quantity G.

In the next chapter we carry this procedure through into the case in which the ambient magnetic field is non-uniform.

In conclusion it is worth pointing out that the dephasing of the past history of particles revealed in this chapter, and the discussion of equation (2.16) in Chapter II (see also the discussion of equations (2.25a) and (2.25b) in Chapter II) clearly defines the limitations of the linear theory and in particular shows that particles will not come into phase at a later time. Thus, from these discussions (in both Chapter II and Chapter III) one cannot expect the linear theory of a given wave packet to generate emissions far removed in space and time from itself, (only a spread in size can be expected), also frequencies outside its own spectrum cannot be generated by linear processes. For the generation of new frequencies one needs non-linear effects, e.g. Das (1967), or wave-wave interactions. It is not difficult to cite cases in the literature where this point has not been fully appreciated.

Chapter IV

Wave Packets in Hot non-uniform Vlasov Plasmas

Introduction

We now investigate the problem of wave, and wave-packet propagation in a hot Vlasov plasma when the ambient magnetic field is not uniform. Before we attempt this problem it is worth discussing, without too much justification, how one might approach the same problem in the cold plasma limit. One could proceed essentially in the manner of Stix (1962) as follows

We could write down the relevant uniform cold plasma wave equation but allow the various plasma parameters which appear in it to be given functions of position; (one should really examine the approximations involved in making this step). One then has a linear partial differential equation (wave-equation) with variable coefficients. We can write this formally as

$$D\left[\frac{1}{i}\frac{\partial}{\partial t}, \frac{1}{-i}\frac{\partial}{\partial z}, \lambda_n(z)\right] = 0 \quad (4.1)$$ where $\lambda_n(z)$

are the said coefficients (the dependence is assumed to be only on the z coordinate). One might then look for a solution of (4.1) which takes the form of a propagating wave of constant frequency i.e.

$$= f(\quad k(z))e^{i(\omega t - \int^{z} k(z'')dz''} \tag{4.2}$$

where the function f, the wave number k and frequency ω are to be determined such that (4.2) put into (4.1) will give rise to the dispersion relation

$$D[\omega, k(z), \lambda_n(z)] = 0 \tag{4.3}$$

i.e. the uniform plasma dispersion relation in which local values of the plasma parameters λ_n have been inserted (again some investigation of the approximations involved, in arriving at (4.3), would have to be made). A classical example is the familiar W.K.B solution

$$\psi = \frac{\text{constant}}{\sqrt{k(z)}} e^{i(\omega t - \int^{z} k(z'')dz'')} \tag{4.4}$$

of the differential equation

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{c(z)}{b(z)}\psi = 0 \tag{4.5}$$

(i.e. second order in z).

The dispersion relation being

$$k^2(z) + \frac{c(z)}{b(z)} = 0 \qquad (4.6)$$

provided we neglect $\frac{1}{k^3}\frac{d^2k}{dz^2}$ and $\frac{1}{k^2}\frac{dk}{dz}$ with respect to unity. The original equation (4.5) may contain derivatives with respect to time, $\frac{\partial}{\partial t}$, on looking for a solution of the form (4.4) these derivatives can be replaced by $i\omega\psi$, $i\omega$ can then be absorbed into $c(z)$ (and/or $b(z)$) in equation (4.5). Cold plasma equations of the form (4.5) have been discussed previously by Stix and a brief review of his work will be given in the next chapter where we examine gyroresonance phenomena in the magnetic beach configuration.

We could now look for a solution of (4.1) which takes the form of a wave packet such as:-

$$= e^{i(\omega t - \int^{z} dz'' k(z''))} G(z,t,k(z)) \qquad (4.7)$$

where G is of the form $A\, e^{i\phi}$. The dependence of the function G on z and t in equation (4.7) is assumed weak and expresses the fact that the local amplitude, frequency and wave number are not given strictly by $f, k(z), \omega$, respectively as in equation (4.2) but are allowed to deviate in some manner throughout the disturbance. By analogy with the uniform media case one expects the wave-packet (4.7) to be the result of superposition of waves of the form (4.2) whose frequency spectrum is sharply peaked around ω .

On putting (4.7) in (4.1) one can presumably arrive at the relation:-

$$D[\omega - i\mathcal{T}, k(z) + i\mathcal{Z}, \lambda_n(z)] = 0 \qquad (4.8)$$

where $\mathcal{T} = e^{i\omega t} \frac{\partial}{\partial t} e^{-i\omega t}$,

and $\mathcal{Z} = e^{-i\int^z kdz''} \frac{\partial}{\partial z} e^{i\int^z kdz''}$ are operators,

provided one makes sufficient approximations. For example the cold plasma Whistler mode dispersion relation is

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p{}^2(z)}{\omega(\omega - \Omega(z))}$$

(where the plasma parameters now depend on z)

or in operator form

$$\frac{\left(-\frac{1}{i}\frac{\partial}{\partial z}\right)^2 c^2}{\left(\frac{1}{i}\frac{\partial}{\partial t}\right)^2} = 1 - \frac{\omega_p^{\,2}(z)}{\left(\frac{1}{i}\frac{\partial}{\partial t} - \Omega(z)\right)\left(\frac{1}{i}\frac{\partial}{\partial t}\right)} \tag{4.9}$$

for a disturbance of the form (4.7) equation (4.9) becomes

$$\frac{[(k + i\mathcal{Z})^2 + i\frac{dk}{dz}]c^2}{(\omega + i\tau)^2} = 1 - \frac{\omega_p^{\,2}(z)}{(\omega + i\tau)(\omega + i\tau - \Omega(z))}$$

in order to write this in the form (4.8) we have to neglect the term $\frac{dk}{dz}$. We shall not investigate or justify this approximation here.

Using relation (4.8) one can easily arrive at the equation

$$\frac{\partial G}{\partial t} = -\left(\frac{\partial \omega}{\partial k}\right)_z \frac{\partial G}{\partial z} \tag{4.10}$$

(see Chapter (II) section (I)) where the subscript z on $\frac{\partial}{\partial k}$ indicates that the derivative is evaluated at constant z. On using the relations

$$\frac{\partial G}{\partial z} = \left(\frac{\partial G}{\partial z}\right)_k + \left(\frac{\partial G}{\partial k}\right)_z \frac{dk}{dz} \quad \text{and} \quad \frac{\partial \omega}{\partial z} = 0 = \left(\frac{\partial \omega}{\partial z}\right)_k + \left(\frac{\partial \omega}{\partial k}\right)_z \frac{dk}{dz}$$

equation (4.10) becomes

$$\frac{\partial G}{\partial t} = -\left(\frac{\partial \omega}{\partial k}\right)_z \left(\frac{\partial G}{\partial z}\right)_k + \left(\frac{\partial \omega}{\partial z}\right)_k \left(\frac{\partial G}{\partial k}\right)_z$$

Equation (4.10) is essentially the equation derived by Kadomtsev from energy balance considerations (see equation (4.74) and its subsequent discussion at the end of this chapter). As pointed out by Stix, the inclusion of finite temperature corrections in the cold plasma wave equation necessarily involves a loss of rigour since the refractive index and dispersion relation for a hot collisionless plasma is not a local quantity. The free streaming of particles causes the effect of a perturbation at one locality to be transported to different points at later times. (Both Guthart (1964) and Das (1967)a have investigated the effect of finite temperature on the dispersion of Whistlers in the Magnetosphere and found that typically the effect is rather small). In order to attempt a description of wave and wave-packet propagation in a hot non uniform collisionless plasma one must employ the full rigour of the Boltzmann-Vlasov equations. In this chapter we attempt to derive the equation of motion of a whistler mode wave-packet in the stated regime examining carefully the approximations and assumptions which have to be applied in order to arrive at a result.

In the next chapter we discuss an attempted investigation of gyrcresonance phenomena. In particular we discuss the absorption of cyclotron waves. in the "magnetic beach" configuaration as discussed by Stix (1962) in which he attacked the problem by fitting finite temperature effects onto the cold plasma wave equation for non-uniform media.

Finally in this introduction we sketch our method of approach to the hot plasma problem before proceeding in detail. We consider the ambient particle distribution $f^{(o)}$ to be a function of the magnetic moment invariant μ and energy W . i.e. $f^{(o)} = f^{(o)}(\mu, W)$. Now consider the effect of a wave or wave-packet which has been present since sometime in the past (say $t' = -\infty$). The presence of the wave will change the magnetic moment and energy of a given particle i.e. μ and W at time $t' = -\infty$ (say) will become $W + \delta W$, $\mu + \delta\mu$ at some later time $t' = t$ (say), where $\delta\mu$ and δW are perturbations caused by the wave.

Thus by Liouville's theorem, had there been no wave present

$$f^{(o)}(\mu, W\ t' = -\infty) = f^{(o)}(\mu, W\ t' = t) .$$

Because of the wave the distribution at time t is perturbed i.e. $f^{(o)}(\mu,W)$, at $t' = -\infty$, becomes $f^{(o)}(\mu+\delta\mu, W+\delta W)$, at $t'=t$, by Liouville's theorem.

Thus writing $f^{(o)}(\mu+\delta\mu, W+\delta W) = f^{(o)}(\mu,W) + f^{(1)}$ we have

$$f^{(1)} = \frac{\partial f^{(o)}}{\partial \mu}\delta\mu + \frac{\partial f^{(o)}}{\partial W}\delta W \qquad (1)$$

On considering the associated perturbations to $V_\perp$ and V_z (i.e. $\delta V_\perp$, δV_z) we should arrive at

$$f^{(1)} = \frac{\partial f^{(o)}}{\partial V_\perp}\delta V_\perp + \frac{\partial f^{(o)}}{\partial V_z}\delta V_z \qquad (2)$$

by the same application of Liouville's theorem. Notice that the derivatives of the distribution $f^{(o)}$ are evaluated at time $t' = t$. However the perturbations $\delta\mu, \delta W$ (or $\delta V_\perp, \delta V_z$) are due to the interaction of the wave with a particular particle and are given by an integration over the past history of this interaction. We know (from Chapter (III)) that only events in the recent past contribute to the current associated with those changes owing to the finite temperature of the plasma, i.e. the charge and current density due to $f^{(1)}$ are related through Maxwell's equations to the electric and magnetic field of the wave in the usual self-consistent Boltzmann, Vlasov description (it is this final requirement of self-consistency coupled with the validity of making an expansion of slowly VARYING parameters which enables us to derive the equation of motion of the wave-packet, the procedure being similar to that in Chapter (III). Having discussed the underlying approach we proceed in detail .

SECTION (I)

Application of the Method of Characteristics to the case of Non-uniform Ambient Magnetic Fields.

If a trajectory in the ambient magnetic field parametric in t is given by $\underline{r} = \underline{r}(t)$, then the rate of change of the particle distribution $f(\underline{r},\underline{V},t)$ as we move along this trajectory is given by:-

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{d\underline{r}}{dt} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{d\underline{V}}{dt} \cdot \frac{\partial f}{\partial \underline{V}} \qquad (4.13)$$

where $\frac{d\underline{r}}{dt} = \underline{V}$.

If we put $\frac{d\underline{V}}{dt} = \frac{-|\mathcal{E}|}{mc} \underline{V} \wedge \underline{B}(r)$ into equation (4.13)

(where $\underline{B}(\underline{r})$ = ambient field, $-|\mathcal{E}|$ = charge on the particle) $\frac{Df}{Dt}$ becomes $\left(\frac{Df}{Dt}\right)_{(o)}$ the rate of change of the particle distribution as one follows a particle trajectory in the ambient field. Louiville's theorem states that this rate of change is zero for a collisionless plasma, i.e.

$$\left(\frac{Df^{(o)}}{Dt}\right)_{(o)} = 0 \qquad (4.14)$$

where $f^{(o)}$ is the particle distribution in the ambient field. To ensure condition (4.14) we choose $f^{(o)}$ to be a function of the constants of motion K_i in the ambient field, i.e. $f^{(o)} = f^{(o)}(K_i)$. Equation (4.14) becoming

$$\left(\frac{Df^{(o)}(K_i)}{Dt}\right)_{(o)} = \sum_i \frac{\partial f^{(o)}}{\partial K_i} \left(\frac{DK_i}{Dt}\right)_{(o)} = 0$$

since $\left(\frac{DK_i}{Dt}\right)_{(o)} = 0$ by definition of K_i .

We now use the familiar perturbation technique on the Vlasov equations, i e. put $f = f^{(o)} + f^{(1)}$

$$\underline{E} = \underline{E}^{(1)} \text{ (there being no ambient electric field)} \qquad (4.15)$$

$$\underline{B} = \underline{B}(r) + \underline{b}$$

$f^{(1)}$, $\underline{E}$, $\underline{b}$ being perturbations which are specified to take the form of a propagating wave or wave packet. An electron trajectory in the total field (ambient plus perturbed) is defined by

$$\frac{d\underline{V}}{dt} = \frac{-|\varepsilon|}{m} \left[\frac{\underline{V}}{c} \wedge [\underline{B}(\underline{r}) + \underline{b}] + \underline{E} \right].$$

$$\frac{d\underline{r}}{dt} = \underline{V} .$$

Thus equation (4.13) becomes:-

$$\left(\frac{D[f^{(o)}+f^{(1)}]}{Dt}\right)_{(o)} + \left(\frac{D[f^{(o)}+f^{(1)}]}{Dt}\right)_{(1)} = 0$$

where

$$\left(\frac{D}{Dt}\right)_{(1)} = \frac{-|\varepsilon|}{m} \left(\underline{E} + \frac{\underline{V}\wedge\underline{b}}{c}\right) . \frac{\partial}{\partial \underline{V}} \qquad (4.16)$$

on linearizing i.e. dropping term $\left(\frac{Df^{(1)}}{Dt}\right)_{(1)}$ which is second order in the perturbed quantities and using (4.14) we have:-

$$\left(\frac{Df^{(1)}}{Dt}\right)_{(o)} = - \left(\frac{Df^{(o)}}{Dt}\right)_{(1)} = - \sum_i \frac{\partial f^{(o)}}{\partial K_i} \left(\frac{DK_i}{Dt}\right)_{(1)} . \qquad (4.17)$$

This is the familiar linearized Boltzmann equation where the ambient field has not been assumed uniform. We can solve (4.17) by the method of characteristics i.e.

$$f^{(1)} = \sum_i \frac{\partial f^{(o)}}{\partial K_i} \int_{-\infty}^{t} dt' \left(\frac{DK_i}{Dt'}\right)_{(1)} \qquad (4.18)$$

where the integration is taken along the zero order particle trajectory and t is the instant of time t' at which the perturbation is evaluated. We can rewrite (4.18) as

$$f^{(1)} = \sum_i \frac{\partial f^{(o)}}{\partial K_i} \delta K_i \quad \text{where} \quad \delta K_i = \int_{-\infty}^{t} \left(\frac{DK_i}{Dt'} \right)_{(1)} dt' \tag{4.19}$$

We assume adiabatic theory to apply and take the constants of motion to be the energy $W = V_\perp^2 + V_z^2$ and magnetic moment $\mu = V_\perp^2/B$ (putting $\frac{m}{2} = 1$).
Equation (4.19) is thus :-

$$f^{(1)} = \frac{\partial f^{(o)}}{\partial \mu} . \delta\mu + \frac{\partial f^{(o)}}{\partial W} . \delta W \quad (4.11)$$

$$\text{where} \quad \delta\mu = \int_{-\infty}^{t} dt' . \left(\frac{D\mu}{Dt'} \right)_{(1)} = \frac{2V_\perp \delta V_\perp}{B} \quad \text{(i)} \tag{4.20}$$

$$\text{and} \quad \delta W = \int_{-\infty}^{t} dt' \left(\frac{DW}{Dt'} \right)_{(1)} = 2V_\perp \delta V_\perp + 2V_z \delta V_z \quad \text{(ii)}$$

On making the transformations:-

$$\frac{\partial f^{(o)}}{\partial W} = \frac{1}{2V_z} \frac{\partial f^{(o)}}{\partial V_z} , \quad \frac{\partial f^{(o)}}{\partial \mu} = \left(\frac{2V_\perp}{B} \right)^{-1} \frac{\left(\frac{\partial f^{(o)}}{\partial V_\perp} V_z - \frac{\partial f^{(o)}}{\partial V_z} V_\perp \right)}{V_z} .$$

$$\delta W = 2V_\perp \delta V_\perp + 2V_z \delta V_z \quad \text{and} \quad \delta\mu = \frac{2V_\perp \delta V_\perp}{B}$$

equation (4.11) becomes:-

$$f^{(1)} = \frac{\partial f^{(o)}}{\partial V_\perp} \delta V_\perp + \frac{\partial f^{(o)}}{\partial V_z} \delta V_z \tag{4.12}$$

In the next section we find the perturbationns δV_1 and δV_z in terms of an integration over the past history of a given particle. We can do this when we have specified the problem a little more, in particular when we have chosen the form of the ambient and disturbance field.

SECTION II

Derivation of the Perturbation in the Particle Distribution.

We particularise the problem by choosing the ambient magnetic field to be of the form

$$\underline{B}(\underline{r}) = B_o \left[- \frac{\epsilon}{2}(\hat{x}\, x + \hat{y}\, y)\frac{dP(z)}{dz} + (1 + \epsilon P(z))\hat{z} \right] \quad (4.22)$$

where ϵ is a smallness parameter. (If we put $\epsilon = 0$ then $\underline{B} = B_o\hat{z}$ and we return to a uniform field).

In (4.22) x, y, z form a right-handed cartesian coordinate system $\hat{x}, \hat{y}, \hat{z}$ are the respective unit vectors. $P(z)$ is some monotonically increasing or decreasing function of z, in this chapter we shall choose the former case.

We are going to look for solutions of the hot plasma wave equation in the ambient magnetic field (4.22) which are of two types. The first is that of a monochromatic wave, the second is that of a wave packet i.e. a disturbance containing a spread or spectrum of frequencies.

We shall look for monochromatic wave solutions whose electric and magnetic fields take the form:-

$$\underline{E}(z,t) = \underline{E}_0 e^{i(\omega t - \int^z k(z'')dz'')} S(z) \quad (.a)$$

$$(4.23)$$

$$\underline{b}(z,t) = \underline{b}_0 e^{i(\omega t - \int^z k(z'')dz'')} S_1(z) \quad (.b)$$

where $k(z)$ and $S(z)$ are slowly varying functions of position z, and $\underline{E}_0, \underline{b}_0$ are constant vector amplitudes.

We also look for wave packet solutions whose electric and magnetic fields take the form:-

$$\underline{E}(z,t) = \underline{E}_0 e^{i(\omega t - \int^z k(z'')dz'')} G(z,t) \quad (.i)$$

$$(4.23)$$

$$\underline{b}(z,t) = \underline{b}_0 e^{i(\omega t - \int^z k(z'')dz'')} G_1(z,t) \quad (.ii)$$

where the functions G and G_1 are now dependent on both position z and time t, we shall examine the form of the envelope G more closely in Section (IV).

In the developement of the next two sections we shall discuss primarily the solution (4.23i, ii) since we may return to the solution (4.23a, b) by making the transformations

$$G(z,t) \rightarrow S(z)$$

and

$$G_1(z,t) \rightarrow S_1(z) \; .$$

The constant vector amplitudes in the equations (4.23) are given by

$$\underline{E}_o = (\hat{x}\ell_{ox} + \hat{y}\ell_{oy}) \quad \text{and} \quad \underline{b}_o = (\hat{x}b_{ox} + \hat{y}b_{oy}) \qquad (4.24)$$

where ℓ_{ox}, ℓ_{oy} ; b_{ox}, b_{oy} are constants. $S(z)$, $G(z,t)$ and $S_1(z)$, $G_1(z,t)$ are slowly varying 'envelopes' related to the electric and magnetic fields respectively.

Near the centre of the flux tube (4.22) the wave fields (4.23) correspond approximately to propagation along the field lines.

In looking for solutions of the form (4.23) we assume that the inhomogeneity and wavelength (i.e. ϵ and k) are such that $\frac{1}{B}\frac{dB}{dz} \sim \frac{1}{k}\frac{dk}{dz} \sim \frac{1}{S}\frac{dS}{dz} \sim \frac{1}{G}\frac{dG}{dz} \ll k$.

From the Maxwell Equation

$$\nabla_\wedge \underline{E} = -\frac{1}{c}\frac{\partial \underline{b}}{\partial t} \quad \text{we have:-}$$

$$b_x = \frac{-\hat{k}}{\hat{\omega}} c\, E_y \quad \text{and} \quad b_y = \frac{+\hat{k}}{\hat{\omega}} c\, E_x \qquad (4.25)$$

where:-

$$\hat{k} = [k(z) + i\mathcal{Z}],\ \hat{\omega} = [\omega - i\mathcal{T}]$$

and $$\mathcal{Z} = e^{-i\int^{z} k(z'')dz''} \frac{\partial}{\partial z} e^{+i\int^{z} k(z'')dz''}$$

$$\mathcal{T} = e^{i\omega t} \frac{\partial}{\partial t} e^{-i\omega t}$$ are differential operators.

The amplitudes (4.24) are related by

$$b_{ox} = \frac{-k(z)}{\omega} \ell_{oy}, \quad b_{oy} = \frac{k(z)}{\omega} \ell_{ox} \qquad (4.26)$$

The equations (4.25) and (4.26) are really equivalent to the statement

$$\frac{\hat{k}}{\hat{\omega}} \cdot e^{i(\omega t - \int k(z'')dz''} \cdot G(z,t) = \frac{k(z)}{\omega} \cdot e^{i(\omega t - \int k(z'')dz'')} G_1(z,t) \qquad (4.27)$$

Equation (4.27) is a simplifying transformation enabling us to work only in the wave electric field, this makes the working considerably easier to follow. Notice that on putting $G(z,t) = S(z)$ a function of z only $\hat{\omega}$ reduces to ω (or on putting $G(z,t) =$ constant $\hat{k}$ and $\hat{\omega}$ reduce to $k(z)$ and ω respectively).

We can now find the change in $V_\perp$ and V_z caused by the wave field as follows:-

$$\left(\frac{DV_\perp}{Dt}\right)_{(1)} = \left(\frac{D(V_x{}^2 + V_y{}^2)^{1/2}}{Dt}\right)_{(1)} = \frac{V_x\left(\frac{DV_x}{Dt}\right)_{(1)} + V_y\left(\frac{DV_y}{Dt}\right)_{(1)}}{V_\perp} \tag{4.28}$$

From (4.16) we have

$$\left(\frac{DV_x}{Dt}\right)_{(1)} = \frac{-|\varepsilon|}{m}\,(1 - V_z\frac{\hat{k}}{\hat{\omega}})E_x \qquad \text{(i)}$$

$$\left(\frac{DV_y}{Dt}\right)_{(1)} = \frac{-|\varepsilon|}{m}\,(1 - V_z\frac{\hat{k}}{\hat{\omega}})\,E_y \qquad \text{(ii)} \tag{4.29}$$

$$\left(\frac{DV_z}{Dt}\right)_{(1)} = \frac{-|\varepsilon|}{m}\,(V_x b_x - V_y b_y) = \frac{-|\varepsilon|}{m}\frac{\hat{k}}{\hat{\omega}}\,(V_x E_y + V_y E_x) \qquad \text{(iii)}$$

Where we have used the relation (4.25) to put the final result in terms of the wave electric field. Putting the two equations (4.29i), (4.29ii) into equation (4.28) we have:-

$$\left(\frac{DV_\perp}{Dt}\right)_{(1)} = -\frac{|\varepsilon|}{m}\,(1-V_z\frac{\hat{k}}{\hat{\omega}})(\frac{V_x E_x + V_y E_y}{V_\perp}) \tag{4.30}$$

Hence we can solve equation (4.20i) for $\delta V_\perp$ arriving at

$$\delta V_\perp = \frac{-|\varepsilon| V_\perp}{m}\int_{-\infty}^{t} dt'\,(1 - \frac{\hat{k}'}{\hat{\omega}'}.V_z')\;\frac{(V_x'E_x' + V_y'E_y')}{V_\perp'^2} \tag{4.31}$$

where all functions of the dummy variable t' have been primed.

In a similar manner from equation (4.30) equation (4.29iii) and equation (4.20ii) we have:-

$$\delta W = -\frac{|\mathcal{E}|}{m}\int_{-\infty}^{t} dt' \left\{ 2V_{\perp}'\left(\frac{DV_{\perp}}{Dt'}\right)_{(1)} + 2V_{z}'\left(\frac{DV_{z}'}{Dt'}\right)_{(1)}\right\} =$$

$$= 2\int_{-\infty}^{t} dt' \left\{ V_x'E_x' + V_y'E_y' \right\} \qquad (4.32)$$

Solving (4.32) and (4.31) for δV_z we arrive at:-

$$\delta V_z = \frac{\delta W - 2V_{\perp}\delta V_{\perp}}{2V_z} =$$

$$= -\frac{|\mathcal{E}|}{m}\frac{1}{V_z}\int_{-\infty}^{t} dt' \left\{ 1 - \frac{\left(1 - k'/\hat{\omega}' V_z'\right)}{V_{\perp}'^2} \right\} (V_x'E_x' + V_y'E_y') \qquad (4.33)$$

The integration in equations (4.31) and (4.33) are along the particle trajectory in the ambient field (4.22).

The equation of motion of an electron in this field is

$$\frac{d\underline{V}'}{dt'} = -\frac{|\mathcal{E}|}{mc}\,\underline{V}' \wedge \underline{B}(r')$$

where $B(\underline{r}') = B_o[-\frac{\epsilon}{2}(\hat{x}x' + \hat{y}y')\frac{dP(z')}{dz'} + \hat{z}(1+P(z'))]$

i.e.

$$\frac{dV_x'}{dt'} = -\Omega_o[V_y'(1+P(z')) + \frac{\epsilon}{2}y'V_z'\frac{dP(z')}{dz'}]$$

$$\frac{dV_y'}{dt'} = \Omega_o[V_x'(1+P(z')) + \frac{\epsilon}{2}x'V'_z\frac{dP(z')}{dz'}]. \quad (4.34)$$

$$\frac{dV_z'}{dt'} = -\frac{\epsilon}{2}\Omega_o[V_y'x' - V_x'y']\frac{dP(z')}{dz'} \quad (\text{where } \Omega_o = \frac{|\varepsilon| B_o}{mc})$$

The solution of these equations to terms linear in ϵ are:-

$$V_x' = V_\perp\left\{1+\epsilon[P(z-V_z T)-P(z)]\right\}^{1/2}\text{Cos}\left\{\theta - \int^T \Omega[z(T'')]dT''\right\} \quad (.i)$$

$$V_y' = V_\perp\left\{1+\epsilon[P(z-V_z T)-P(z)]\right\}^{1/2}\text{Sin}\left\{\theta - \int^T \Omega[z(T'')]dT''\right\} \quad (.ii)$$

(4.35)

$$V_z' = V_z - \frac{\epsilon}{2}V_\perp^2/V_z \left\{P(z-V_z T)-P(z)\right\} \quad (.iii)$$

Hence:-

$$z' = z - V_z T + \frac{\epsilon}{2}V_\perp^2/V_z \int^T dT''\left\{P(z-V_z T'') - P(z)\right\} \quad (.iv)$$

where $T = t - t'$ time measured positive in the backward direction from the point t.

The solution for x', y' are not required in what follows.

The solutions (4.35) will satisfy the equations (4.34) provided terms of order ϵ^2 are neglected and they have been chosen such that at time $t' = t$

$$V_x' = V_\perp \cos\theta = V_x$$

$$V_y' = V_\perp \sin\theta = V_y \qquad (4.36)$$

$$V_z' = V_z$$

$$z' = z \;.$$

Since $\underline{E}(z,t) = \underline{E}_o e^{i(\omega t - \int^z k dz'')} G(z,t)$ (4.23)

where $\underline{E}_o = (\hat{x}\ell_{ox} + \hat{y}\ell_{oy})$

we may write:-

$$E(z',t') = \underline{E}_o e^{i\omega t} \cdot e^{-i(\omega T + \int^{z'} k dz'')} \cdot G(z', t-T)$$

where we have replaced t' by $t-T$.

Thus the term $(E_x'V_x' + E_y'V_y')$ which occurs in both (4.31) and (4.33) becomes:-

$$V_\perp \left\{1 + \epsilon[P(z - V_z T) - P(z)]\right\}_T^{1/2} \left\{\ell_{ox} \cos(\theta - \int^T dT''\Omega) + \ell_{oy} \sin(\theta - \int dT''\Omega)\right\} \cdot e^{-i(\omega T + \int^{z'} dz'' k)} G(z', t-T) \qquad (4.37)$$

We take out the left hand polarized (Whistler mode) as in the previous chapter and arrive at

$$(E_x'V_x' + E_y'V_y') = \frac{E_{o\perp}}{2} \cdot e^{-i\theta} \cdot e^{i\omega t} \left\{ R'^{1/2} . e^{i\phi'} . G(z', t-T) \right\} \quad \text{(i)}$$

where $$R' = 1 + \epsilon[P(z-V_zT) - P(z)] \quad \text{(ii)} \quad (4.38)$$

$$\phi' = \int^T dT'' \Omega - \omega T - \int^{z'} dz'' \, k(z'') \quad \text{(iii)}$$

(and $\underline{E}_{o\perp} = (\hat{x}\ell_x + \hat{y}i\ell_y)$ is a constant amplitude (where i will give rise to the Whistler mode polarization)

Thus equations (4.31) and (4.33) become:-

$$\delta V_\perp = -e^{-i\theta} \frac{|\epsilon|}{m} E_{o\perp} . e^{i\omega t} \int_{+\infty}^{o} dT \left(1 - V_z' \frac{\hat{k}'}{\hat{\omega}'}\right) \frac{e^{i\phi'}}{R'^{1/2}} G' \quad (4.39i)$$

$$\delta V_z = -e^{-i\theta} . \frac{|\epsilon|}{m} \underline{E}_{o\perp} . e^{i\omega t} \int_{+\infty}^{o} dT . \left\{ R'^{1/2} - \frac{(1 - \hat{k}'V_z')/\hat{\omega}'}{R'^{1/2}} \right\} e^{i\phi'} . G'$$

(4.39ii)

where we have used $V_\perp'^2 = V_\perp^2 R'$.

Hence

$$f^{(1)} = -e^{-i\theta} . \frac{|\epsilon|}{m} \underline{E}_{o\perp} e^{i\omega t} \int_{+\infty}^{o} dT \left\{ f_\perp^{(o)} [1 - V_z'\hat{k}'/\hat{\omega}']^{1}/R'^{1/2} + \right.$$

$$\left. + f_\parallel^{(o)} [R'^{1/2} - (1 - V_z'\hat{k}'/\hat{\omega}')^{1}/R'^{1/2}] e^{i\phi'} G' \right. \quad (4.40)$$

where $f_\perp^{(o)} = \frac{\partial f^{(o)}}{\partial V_\perp}$ and $f_\parallel^{(o)} = \frac{\partial f^{(o)}}{\partial v_z}$

Using relation (4.27) we may replace $\frac{\hat{k}'}{\hat{\omega}'} e^{i\phi'} G'$ by $\frac{k(z')}{\omega} e^{i\phi'} G_1'$, in equation (4.40) giving:-

$$f^{(1)} = - e^{-i\theta} \cdot \frac{|\varepsilon|}{m} E_{o\perp} e^{i\omega t} \int_{+\infty}^{0} dT \left\{ f_1^{(o)} [G' - V_z' \frac{k'}{\omega} G_1']^1 / {R'}^{1/2} + \right.$$

$$\left. + f_{\parallel}^{(o)} [R'^{1/2} G' - \frac{(G' - k'/\omega V_z' G_1')}{R'^{1/2}}] e^{i\phi'} \right. \qquad (4.41)$$

Thus having chosen the form of the disturbance field, equations (4.23), we have derived the associated perturbation in the particle distribution to terms linear in the smallness parameter ϵ by using the method of characteristics. The restriction to terms linear in ϵ derives from the imperfect solution for the particle orbits in the chosen ambient field. There are of course still terms in equation (4.41) which are of order ϵ^2 (i.e. product terms). These terms will be neglected consistently as we proceed.

The current is given essentially by an integration of a first moment of equation (4.41) over velocity space (see equ. (4.48) below). We expand the slowly varying quantities in this equation about the point $T = 0$, since we know that only small values of T contribute to the current. This really follows from our previous discussion, however we shall amplify this point a little further in the next section.

Section (III)

Small time Expansion of Slowly Varying Quantities

We now make an expansion of equation (4.41) which will be valid for small values of the time variable T. That is we Taylor expand the slowly varying quantities about the point $T = 0$. However before carrying out this procedure we notice that particles with $v_z > 0$ in the given ambient field will not have mirrored. However particles with $V_z < 0$ will have mirrored at the point defined by $T = T_B$ where:-

$$v_z' = 0 = V_z - \frac{\epsilon}{2} \frac{{V_\perp}^2}{V_z} \left\{ P(z - V_z T_B) - P(z) \right\} \quad \text{defines } T_B.$$

As a first step in expansion procedure we expand V_z', and z' in equations (4.35) about the point $T = 0$. Thus:-

$$V_z' = V_z + \frac{\epsilon {V_1}^2}{2} \frac{dP(z)}{dz} . T \qquad \text{from (4.35iii)}$$

$$z' = z - V_z T - \frac{\epsilon}{2} {V_1}^2 \frac{dP(z)}{dz} . T^2/2 \quad \text{from (4.35iv)}$$

where we have neglected second order derivatives of the function P.

Similarly

$$R^{1/2} \simeq [1 - \frac{\epsilon}{2} \frac{dP(z)}{dz} V_z T]$$

$$R^{-1/2} \simeq [1 + \frac{\epsilon}{2} \frac{dP(z)}{dz} V_z T]$$ where we have also used the Binomial expansion.

We expand $k(z')$ about the point $T = 0$ neglecting terms of order $\frac{dk(z)}{dz} \cdot \frac{\epsilon}{2} \frac{dP(z)}{dz}$ and $\frac{d^2k}{dz^2}$ i.e.

$$k(z') = k(z) - V_z T \frac{dk(z)}{dz} .$$

We do not expand $G(z', t-T)$ or $G_1(z', t-T)$ at present.

We expand the various terms in ϕ' $\left\{ \phi' = \int^T \Omega \, dT'' - \omega + \int^{z'} k dz'' \right)$ eq (4.38iii)$\Big\}$ as follows:-

$$\int^T dT'' . \Omega[z(T'')] = \int^T dT'' \Omega_0 \left\{ 1 + \epsilon P(z - V_z T) \right\} =$$

$$= \Omega_0 \left\{ [1 + \epsilon P(z)] T - \epsilon \frac{dP(z)}{dz} V_z T^2/2 \right\}$$

where we have expanded $P(z - V_z T)$ to first derivatives of $P(z)$ only.

Similarly:-

$$\int^{z'} dz'' k(z'') = \int^{z - (V_z T - \frac{\epsilon}{2} \frac{dP(z)}{dz} V_1^2 T^2/2)} dz'' k(z'') =$$

$$= \int^{z} dz'' . k(z'') - (V_z T - \frac{\epsilon}{2} \frac{dP(z)}{dz} V_1^{2} T^2/2) k(z) + \frac{(V_z T)^2}{2} \frac{dk(z)}{dz}$$

where terms of order $\frac{dk(z)}{dz}\ \epsilon\frac{dP(z)}{dz}$, $\frac{d^2k(z)}{dz^2}$ have been neglected.

Thus:-

$$\phi' \simeq \int^{z} dz''.k(z'') + (\Omega_o[1+\epsilon P(z)]-\omega+k(z)V_z)T- (\Omega_o\epsilon\frac{dP(z)}{dz} + \frac{dk(z)}{dz}V_z{}^2 + \frac{\epsilon}{2}\frac{dP(z)}{dz}V_\perp{}^2k(z))T^2/2 .$$

The first term in this expansion is not a primed quantity (i.e. it is independent of t') and can thus come outside the integration over T. Equation (4.41) can now be written in the form:-

$$f^{(1)}(z,\underline{V},t) = -|\mathcal{E}|\frac{e^{-i\theta}}{m}E_{o\perp}\ e^{i(\omega t - \int^{z} dz''k(z''))}$$

$$\int_{+\infty}^{o} dT.\Big\{ [f_\perp G' + (V_\perp f_{\parallel} - V_z f_\perp)\frac{k(z)}{\omega}G_1{}']$$

$$- T[\frac{\epsilon}{2}\frac{dP(z)}{dz}(2V_zG' - \frac{k(z)}{\omega}(V_\perp^2+V_z^2)G_1') + \frac{V_z^2}{\omega}\frac{dk(z)}{dz}G_1'][\frac{V_\perp f_{\parallel} - V_z f_\perp}{V_z}]$$

$$e^{i(\Omega_o[1+\epsilon P(z)]-\omega+k(z)V_z)T - i(\Omega_o\epsilon\frac{dP(z)}{dz}V_z + \frac{dk(z)}{dz}V_z{}^2 + \frac{\epsilon}{2}\frac{dP(z)}{dz}V_\perp{}^2k(z))T^2/2} \quad (4.42)$$

Hence the scheme of approximation adopted thus far is as follows:-

Equation (4.41) is correct to terms linear in the smallness parameter ϵ , while to arrive at equation (4.42) we **have consistently** neglected terms of order

$$\epsilon^2,\ \frac{\epsilon}{2}\frac{d^2P(z)}{dz^2}\ ,\ \frac{d^2k(z)}{dz^2}\ \text{ and }\ \frac{dk(z)}{dz}\frac{\epsilon}{2}\frac{dP(z)}{dz} \qquad (4.42a)$$

We label these approximations (4.42a) since they are associated with equation (4.42).

In the previous chapter we established that the integration over the time variable T cuts off owing to the finite temperature of the plasma. That is on inserting equation (4.42) into the expression for the current density (essentially a first moment in velocity space) each term in the resulting equation has the characteristic form

$$\int_{+\infty}^{0} dT\ S(T) \int_{-\infty}^{+\infty} dV_z\ \ F(V_z\ )e^{ikTV_z} \qquad \text{(see equation (3.22) chapter (III))}$$

where $F(V_z)$ is a function of V_z, whose width is determined essentially by the temperature of the plasma, hence (as in chapter (III)) the integration over time, T, 'cuts off' justifying the expansion we have already made.

Section (IV)

Wave Packet Propagation with an Isotropic Particle Distribution

In this section we investigate the problem of wave packet propagation in the hot non-uniform plasma. In the next chapter we investigate the monochromatic wave solution and discuss the problem of gyroresonant phenomena in the non-uniform ambient magnetic field with particular reference to the magnetic beach configuration discussed by Stix (1962). The treatment of the monochromatic wave solution is slightly the simpler of the two problems however the discussion of the wave particle gyroresonance is not complete and is thus included in the final chapter. For convenience this section and chapter (V) can be read independently. Thus we attempt to find what relationships ω, k(z) and G(z,t) must obey in order that (4.23i) should be a solution of the hot plasma wave equation, paying careful attention to the approximations made, the procedure is rather similar to that in the previous chapter.

The case in which the particle distribution is isotropic is simpler to deal with. All terms involving the factor $f_{\perp} V_z - f_{\parallel} \cdot V_{\perp}$ vanish and equation (4.42) reduces to:-

$$f(V',z,t) = -\frac{|\mathcal{E}|}{m}\, e^{-i\theta}\, \frac{\underline{E}(z,t)}{G(z,t)} \int_{+\infty}^{0} dT \left\{ f_{\perp}^{(o)} G[z',t-T] \right\}$$

$$e^{i(\Omega_o[1+\epsilon P(z)]-\omega+k(z)V_z)T - i(\Omega_o \epsilon \frac{dP(z)}{dz}V_z + \frac{dk(z)}{dz}V_z^{\,2} +$$

$$+ \frac{\epsilon}{2}\frac{dP(z)}{dz}V_{\perp}^{\,2}\, k(z))\frac{T^2}{2} \qquad (4.43)$$

where $\underline{E}(z,t) = \underline{E}_{o\perp}\, e^{i(\omega t - \int^{z} k(z'')dz'')}.G(z,t)$ (4.44)

and $\underline{E}_{o\perp} = (\hat{x}\mathcal{E}_{ox} + \hat{y}i\mathcal{E}_{oy})$.

We can simplify the working by first introducing some notation. We put:-

$(\Omega_o[1+\epsilon P(z)]-\omega+k(z)V_z) = \psi$ and notice that

$$\frac{d\psi}{dz} = \Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz}V_z \quad \text{and} \quad \frac{d\psi}{dk} = V_z \qquad (4.45)$$

We also put

$$\frac{\epsilon}{2}\frac{dP(z)}{dz}V_{\perp}^{\,2}k(z) = \beta .$$

On eliminating $\underline{b}$ from the two Maxwell Equations

$\nabla_{\wedge}\underline{E} = -\frac{1}{c}\frac{\partial b}{\partial t}$; $\nabla_{\wedge}\underline{b} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\underline{\dot{J}}$ and inserting

$\underline{E}(z,t)$ (equation (4.44)) into the resulting equation we arrive at:-

$$[-k^2(z) - 2iG_z(z,t)/_G - i\frac{dk(z)}{dz} + G_{zz}(z,t)/_G]\underline{E} =$$

$$= \frac{1}{c^2}[-\omega^2+2i\omega G_t(z,t)/_G + G_{tt}(z,t)/_G]\underline{E} + 4\pi/_{c^2}\frac{\partial\underline{\dot{J}}}{\partial t} \quad (4.46)$$

Where subscripts on the envelope $G(z,t)$ denote differentiation

$$\text{e.g.}\quad G_t(z,t) = \frac{\partial G(z,t)}{\partial t}\ ;\ G_{zz}(z,t) = \frac{\partial^2 G(z,t)}{\partial z^2}\ \text{etc.} \quad (4.47)$$

The current in equation (4.46) is given by:-

$$\underline{\dot{J}}(z,t) = -|\varepsilon|\int_0^{2\pi} d\Theta \int_{-\infty}^{+\infty} dV_z \int_0^{+\infty} dV_\perp . V_\perp\ [e^{i\Theta}V_\perp f^{(o)}] \quad (4.48)$$

in the usual self-consistent Boltzmann-Vlasov description.

Hence

$$\frac{\partial\underline{\dot{J}}(z,t)}{\partial t} = \frac{E(z,t)}{G(z,t)}[i\omega + \tau]|\varepsilon|^2/_m\ \pi$$

$$\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \left\{\int_0^{+\infty} dV_\perp . V_\perp{}^2\ f_\perp^{(o)} . G(z',t-T)e^{i\psi T - i(\frac{d\psi}{dz}V_z + \beta)T^2/2}\right\} \quad (4.49)$$

The presence and operation of τ has been discussed and clarified in the previous chapter.

We find the equation of motion of $G(z,t)$ by proceeding as follows. We first Taylor expand $G(z',t-T)$ in equation (4.49) about the point $T = 0$ i.e.

$$G(z',t-T) \simeq G([z-V_z T + \frac{\epsilon}{2}\frac{dP(z)}{dz} V_\perp^2 T^2/2], t-T) \tag{4.50}$$

$$\simeq G(z,t) - T[V_z G_z(z,t) + G_t(z,t)] + T^2/2!\ [V_z G_{zz}(z,t) + 2V_z G_{zt}(z,t) + G_{tt}(z,t)] \equiv G(z,t) + R \tag{4.51}$$

where we have introduced R as a short hand notation for that part of the expansion which depends on derivatives of $G(z,t)$. In this expansion we have neglected the product terms in the derivatives of $G(z,t)$ and $\epsilon\frac{dP(z)}{dz}$; for example the term $G_z(z,t)\frac{\epsilon}{2}\frac{dP(z)}{dz}V_\perp^2 T^2/2$. This really means we have replaced the right hand side of equation (4.50) by $G((z-V_z T),t-T)$.

Next we expand the exponential $e^{-i(\frac{d\psi}{dz}V_z)T^2/2}$ to terms linear in $\epsilon\frac{dP(z)}{dz}$ and $\frac{dk(z)}{dz}$ i.e. we replace it by $[1-i(\frac{d\psi}{dz}V_z)T^2/2]$.

Equation (4.46) now becomes:-

$$[-k^2(z) - 2i\frac{G_z(z,t)}{G(z,t)} - i\frac{dk(z)}{dz} + G_{zz}(z,t)/G(z,t)] =$$

$$= \frac{1}{c^2}[-\omega^2 + 2i\omega G_t(z,t)/G(z,t) + G_{tt}(z,t)/G(z,t)] +$$

$$+ \frac{\omega_p{}^2\pi}{n_o c^2 G}[i\omega + \tau] \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \left\{ \cdot \int_0^{+\infty} dV_\perp V_\perp^2 f_\perp^0 (G(z,t)+R) \right.$$

$$\left\{ e^{i\psi T - i\beta T^2/2} \right\} [1 - i \frac{d\psi}{dz} V_z T^2/2] \quad . \qquad (4.52)$$

We shall find it unnecessary to introduce any new approximations, but in order to be consistent it is helpful to review briefly and label carefully the approximations we have made thus far. Basically we have made and justified an expansion in the slowly varying quantities $k(z), B(z)$. In this expansion we have only gone to first derivatives i.e. we have assumed

$$\frac{d^2 k(z)}{dz^2} \sim \frac{d^2 P(z)}{dz^2} \sim 0 \qquad (4.53)$$

We have also neglected terms in the products of $\frac{dk(z)}{dz}$ and $\frac{1}{B}\frac{dB(z)}{dz}$ i.e. we assume:-

$$\frac{dk(z)}{dz} \epsilon \frac{dP(z)}{dz} \sim \left(\frac{dk(z)}{dz}\right)^2 \sim \left(\epsilon \frac{dP(z)}{dz}\right)^2 \sim 0 \qquad (4.54)$$

The approximations (4.53) and (4.54) were used in the derivation of the expression (4.42) for the perturbation in the particle distribution. (We have of course also neglected terms of order ϵ^2 in solving the orbit equations (4.34)). These approximations (i.e. 53 and 54) are identical with (42a) but have been relabelled in this section simply for convenience.

We next assumed an isotropic particle distribution and could thus put terms involving $f_\perp V_z - f_\parallel V_\perp$ identically equal to zero. This gave rise to equation (4.43). The other slowly varying quantity in equation (4.43) is $G(z,t)$, to arrive at the expansion $G(z,t)+R$ equation (4.51) we neglected terms like

$\frac{\partial G}{\partial z} \in \frac{dP}{dz}$, $\frac{\partial^2 G}{\partial z^2} \in \frac{dP}{dz}$ (4.55) but not second derivatives of G i.e. we assumed $\frac{\partial^2 G}{\partial z^2} >> \frac{\partial G}{\partial z} \in \frac{dP}{dz}$ (4.56) which will be valid provided the wave packet is not too dispersed. We shall also neglect terms like $\frac{\partial G}{\partial z} \cdot \frac{dk}{dz}$, $\frac{\partial^2 G}{\partial z^2} \in \frac{dP}{dz}$ (4.57) in the same manner.

Having reviewed the approximations we have made thus far (i.e. (53) to (57)) we can continue by applying them to simplify equation (4.52).

First we replace

$$(G(z,t)+R)e^{i\psi T - i\beta T^2/2}\,[1 - i\,\frac{d\psi}{dz}\,V_z\,T^2/2] \quad \text{by}$$

$$(G(z,t)+R)[e^{i\psi T - i\beta T^2/2} - e^{i\psi T}\,i\,\frac{d\psi}{dz}\,V_z\,T^2/2]$$

where we have used approximation (4.54). This term can be simplified further by use of approximations (4.56), (4.57).

$$[G(z,t)(e^{i\psi T - i\beta T^2/2} - e^{i\psi T}\,i\,\frac{d\psi}{dz}\,V_z\,T^2/2) + Re^{i\psi T}]$$

i.e. the term of order $R\,\frac{d\psi}{dz} \sim 0$ by (4.56) and (4.57) and $Re^{i\psi T - i\beta T^2/2} \simeq Re^{i\psi T}$ by (4.56).

Thus the final term in equation (4.52) becomes:-

$$\frac{\omega_p{}^2\pi}{n_o c^2 G}[i\omega + \tau]\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \int_{o}^{+\infty} dV_\perp V_\perp{}^2 f_\perp{}^{(o)}$$

$$\left\{G(e^{i\psi T - i\beta T^2/2} - e^{i\psi T}\, i\, \frac{d\psi}{dz} V_z\, T^2/2) + Re^{i\psi T}\right\}.$$

Since the operator τ operates on $G(z,t)$ and its derivatives (i.e. G and R in the above expression) the term

$\tau G\, e^{i\psi T - i\beta T^2/2}$ becomes $G_t\, e^{i\psi T}$ where we have used approximation (4.56). Similarly $\tau\, e^{i\psi T}\, i\, \frac{d\psi}{dz} V_z\, T^2/2\, G$ becomes $G_t e^{i\psi T}\, i\, \frac{d\psi}{dz} V_z\, T^2/2 \sim 0$ by (4.56) and (4.57)

Thus equation (4.52) becomes:-

$$[-k^2(z) - 2iG_{z}/G - i\frac{dk}{dz} + G_{zz}/G] = \frac{1}{c^2}[-\omega^2 + 2i\omega G_{t}/G + G_{tt}/G]$$

$$+ \frac{i\omega_p{}^2}{c^2 n_o}\int_{+\infty}^{o} dV_z \int_{+\infty}^{o} dT \int_{o}^{+\infty} dV_\perp\, V_\perp{}^2 \left\{ f_\perp{}^{(o)} e^{i\psi T - i\beta T^2/2} - \right.$$

$$\left. - e^{i\psi T}\, i\, \frac{d\psi}{dz} V_z\, T^2/2 \right\}$$

$$+ \frac{\omega_p{}^2\pi}{c^2 G n_o}[i\omega + \tau]\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \int_{o}^{+\infty} dV_\perp\, V_\perp{}^2 \left\{ f_\perp{}^{(o)} Re^{i\psi T} \right\}. \qquad (4.58)$$

This equation describes the motion of the disturbance " envelope" $G(z,t)$, it only remains to reduce it to a more intelligible form. We shall do this in stages by proceeding as follows.

We choose $k(z)$ to satisfy the relation

$$k^2(z) = \omega^2/c^2 - \frac{i\omega\omega_p{}^2\pi}{c^2 n_o} \int_{-\infty}^{+\infty} dV_z \cdot \int_{+\infty}^{0} dT \int_{0}^{+\infty} dV_\perp \, V_\perp{}^2 \Big\{ f_\perp{}^{(o)} \, e^{i\psi T - i\beta T^2/2} \Big\} \tag{4.59}$$

Notice that it is only when the term β is neglected that equation (4.59) becomes the dispersion relation for an infinite uniform plasma in which local values of the plasma parameters have been inserted i.e.

$$k^2(z) = \omega^2/c^2 - \frac{i\omega\omega_p{}^2}{n_o c^2} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \, g(V_z) \, e^{i(\Omega_o[1+\epsilon P(z)] - \omega + k(z)V_z)T} \tag{4.60}$$

where $g(V_z) = \pi \int_0^{+\infty} dV_\perp \, V_\perp{}^2 \, f_\perp{}^{(o)}$.

The term $e^{-i\beta T^2/2}$ arises in the above equations because we have retained all terms linear in the smallness parameter ϵ , though β is of order $\frac{1}{B}\frac{dB}{dz}$. Equation (4.58) now becomes:-

$$[-2iG_z - iG\frac{dk}{dz} + G_{zz}] = \frac{1}{c^2}[2i\omega G_t + G_{tt}]$$

$$+ i\frac{\omega\omega_p{}^2\pi G}{n_o c^2}\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \int_{o}^{+\infty} dV_\perp V_\perp{}^2 \left\{-f_\perp{}^{(o)} e^{i\psi T} i\, \frac{d\psi}{dz} V_z\ T^2/2\right\}$$

$$+\frac{\omega_p{}^2\pi}{n_o c^2}[i\omega + \tau]\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \int_{o}^{+\infty} dV_\perp V_\perp{}^2 \left\{ f_\perp{}^{(o)}{}_R\ e^{i\psi T}\right\} \quad (4.61)$$

We recognise the second term on the right hand side of this equation as being $-iG\frac{d}{dz}\frac{d}{dk}k^2$, by differentiation of equation (4.59).

We are thus finally left with

$$[-2iG_z + G_{zz}] = \frac{1}{c^2}[2i\omega G_t + G_{tt}] + \frac{\omega_p{}^2}{n_o c^2}[i\omega + \tau]\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \left\{g(V_z\ )Re^{i\psi T}\right\} \quad (4.62)$$

It is here that the treatment becomes very similar to that of the previous chapter. Thus on solving for $\frac{\partial G}{\partial t}(z,t)$ we can put equation (4.62) into the form:-

$$-[\frac{2\omega}{c^2} + \frac{\omega_p{}^2 iA_1}{n_o c^2}]G_t = i[-k^2 + \frac{\omega^2}{c^2} + \frac{\omega_p{}^2}{n_o} Ai\omega]G + [2k + \frac{\omega_p{}^2 iA_2}{n_o c^2}]G_z$$

$$+ [1 + \frac{\omega_p{}^2 A_3}{n_o c^2}]G_{zz} + [\frac{\omega_p{}^2 iA_4}{n_o c^2}]G_{zt} + i[-{}^1/_{c^2} + \frac{\omega_p{}^2 A_5}{n_o c^2}]G_{tt} \quad (4.63)$$

where if we choose to define:-

$$A = -\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT\ g(V_z\ ).e^{i\psi T}$$

Then:-

$$A_1 = A + \omega\frac{\partial A}{\partial\omega}$$

$$A_2 = -\omega\frac{\partial A}{\partial\omega}$$

$$A_3 = -\frac{i\omega}{2}\frac{\partial^2 A}{\partial k^2} \qquad (4.64)$$

$$A_4 = i(\omega\frac{\partial^2 A}{\partial k\partial\omega} + \frac{\partial A}{\partial k})$$

$$A_5 = -\frac{i}{2}(2\frac{\partial A}{\partial\omega}+\omega\frac{\partial^2 A}{\partial\omega^2}) \ .$$

We now put:-

$$k^2(z) - \omega^2/c^2 + \frac{i\omega_p{}^2\pi}{c^2 n_o}\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \int_0^{\infty} dV_\perp V_\perp{}^2 \left\{ f_\perp{}^o e^{i\psi T - i\beta T^2/2} \right\}$$

$$= J[\omega,\Omega(z),\ k(z),(\frac{\epsilon}{2}\ \frac{dP(z)}{dz}k(z))] \qquad (4.65)$$

(see also equation (4.59)) where $\Omega(z) = \Omega_o[1+\epsilon P(z)]$ and where $(\frac{\epsilon}{2}\ \frac{dP(z)}{dz}\ k(z))$ appears through the presence of β (although it can be neglected in most of what follows). From the equations (4.64) and equation (4.65) the equation of motion (4.63) becomes:-

$$-J_\omega\, G_t = -(2k-J_k)G_z + \frac{i}{2}(2-J_{kk})G_{zz} - \frac{i}{2} J_{\omega\omega}\cdot G_{tt} + iJ_{k\omega}G_{zt} \tag{4.66}$$

where the subscripts on J denote differentiation e.g. $J_k = \frac{\partial J}{\partial k}$, $J_{k\omega} = \frac{\partial^2 J}{\partial k \partial \omega}$, etc. and we notice that the factor $e^{-i\beta T^2/2}$ which appears in the definition of J is replaced by unity in equation (4.66) because each term contains a derivative of $G(z,t)$ (i.e. we use approximation (4.55)).

We simplify equation (4.66) further as follows:-

From equation (4.65) we have:-

$$\left(\frac{\partial\omega}{\partial k}\right)_z = \frac{2k-J_k}{J_\omega} \quad \text{and} \quad \left(\frac{\partial^2\omega}{\partial k^2}\right)_z = \frac{2-J_{kk}-2J_{k\omega}\,\omega_k - J_{\omega\omega}\cdot\omega_k^2}{J_\omega} \tag{4.67}$$

where $\omega_k = \left(\frac{\partial\omega}{\partial k}\right)_z$

Thus to lowest order equation (4.66) is:-

$\frac{\partial G}{\partial t} = -\left(\frac{\partial\omega}{\partial k}\right)_z \frac{\partial G}{\partial z}$ and hence:-

$$\frac{\partial^2 G}{\partial z \partial t} = -\frac{\partial}{\partial z}\left(\frac{\partial\omega}{\partial k}\right)_z \frac{\partial G}{\partial z} - \left(\frac{\partial\omega}{\partial k}\right)_z \frac{\partial^2 G}{\partial z^2} \tag{4.68}$$

From equation (4.65) (with $e^{-i\beta T^2/2} = 1$) one has $\omega = \omega[k(z), \Omega(z)]$ (of course $\frac{d\omega}{dz} = 0$) .

Hence:-

$$\frac{\partial}{\partial z}\left(\frac{\partial\omega}{\partial k}\right)_z = \frac{\partial^2\omega}{\partial k^2}\frac{dk}{dz} + \frac{\partial^2\omega}{\partial k\,\partial\Omega(z)}\,\Omega_o\epsilon\,\frac{dP(z)}{dz},$$

since $\Omega(z) = \Omega_o[1+\epsilon P(z)]$.

Thus the term $\frac{\partial}{\partial z}\left(\frac{\partial\omega}{\partial k}\right)_z \frac{\partial G}{\partial z}$ is neglected in equation (4.68) by approximation (4.56) and (4.57).

Thus:-

$$\frac{\partial^2 G}{\partial z \partial t} = -\left(\frac{\partial\omega}{\partial k}\right)_z \frac{\partial^2 G}{\partial z^2} \qquad \text{and similarly} \qquad \frac{\partial^2 G}{\partial t^2} = \left(\frac{\partial\omega}{\partial k}\right)_z^2 \frac{\partial^2 G}{\partial z^2} \tag{4.70}$$

On using relations (4.70) in (4.66) and identities (4.67) we find

$$\frac{\partial G}{\partial t} = -\left(\frac{\partial\omega}{\partial k}\right)_z \frac{\partial G}{\partial z} - \frac{i}{2!}\left(\frac{\partial^2\omega}{\partial k^2}\right)_z \frac{\partial^2 G}{\partial z^2} \tag{4.71}$$

Discussion

Since the factor $e^{-i\beta T^2/2}$ in equation (4.71) is replaced by unity (approx. (4.55)) the frequency ω and wave number $k(z)$ in this equation can be considered related through the dispersion relation for an infinite uniform plasma in which local values of the plasma parameters have been inserted (i.e. equation (4.60)).

We have derived the equation of motion of the wave packet envelope in a non-uniform ambient magnetic field using the approximations listed (i.e. (4.53) to (4.57)). We see that the final equation is similar to equation (2.16) chapter (II). The effort involved in trying to eliminate one or more of the listed approximations does not seem warrented, however it is of some interest to have pinpointed them clearly and to have seen how the equation of motion emerges from consideration of the hot plasma equations in the non-uniform ambient field.

We can relate equation (4.71) to the equations derived by Kadomtsev and others for wave-packet propagation in non-uniform media. To do this we examine the form of the 'envelope' $G(z,t)$ more closely. Since $G(z,t)$ contains both a position and time dependence, then for a chosen disturbance, the central frequency and wave number are not defined precisely but may be chosen within a narrow range. Thus the disturbance $\underline{E}(z,t)$ may be written as either

$$\underline{E}_{o\perp} e^{i(\omega_1 t - \int k_1(z'')dz'')} G_1(z,t) \quad \text{or} \quad \underline{E}_{o\perp} e^{i(\omega_2 t - \int k_2(z'')dz'')} G_2(z,t) \tag{4.72}$$

provided ω_1, k_1 are not much different from ω_2, k_2). This implies that

$$\underline{E}(z,t) = \underline{E}_{o\perp}\, e^{i(\omega t - \int^z k(z'')dz'')} G(z,t,k(z),\omega)$$

where the dependence of G on $k(z)$ and ω is such that the identity (4.72) is satisfied. (This dependence was not made explicit in the equations of chapter (II) and (III) though it was obviously true (see below). The derivative of $G(z,t,k(z),\omega)$ in the second term of equation (4.71) when written out in full becomes:-

$$\frac{\partial G}{\partial z} = \left(\frac{\partial G}{\partial z}\right)_k + \left(\frac{\partial G}{\partial k}\right)_z \frac{dk}{dz} \qquad (4.73)$$

In the limit of cold uniform plasma we know that G can be represented by an integration over a sharply peaked spectrum of plane waves i.e.

$$e^{i(\omega t-kz)}G(z,t) = \int dk' A(k') e^{i(\omega(k')t-k'z)}$$

(see chapter (II) section (III))

since $A(k')$ is sharply peaked at $k' = k$

$$G(z,t) \simeq A(k) \int dk' \; e^{i[\frac{d\omega}{dk'}t - z](k'-k)}$$

hence $\frac{\partial G}{\partial k}$ $(\sim \frac{\partial A}{\partial k})$ will tend to be a very large derivative

Thus we expect

$$\left(\frac{\partial G}{\partial k}\right)_z \frac{dk}{dz} \sim \left(\frac{\partial G}{\partial z}\right)_k \quad \text{in equation (4.73)}$$

We can now proceed as in the introduction to derive the equation

$$\frac{\partial G}{\partial t} = -\left(\frac{\partial \omega}{\partial k}\right)_z \left(\frac{\partial G}{\partial z}\right)_k + \left(\frac{\partial \omega}{\partial z}\right)_k \left(\frac{\partial G}{\partial k}\right)_z \qquad (4.74)$$

On including all three terms of equation (4.71) we should arrive at

$$\frac{\partial G}{\partial t} = -\left(\frac{\partial \omega}{\partial k}\right)_z \left(\frac{\partial G}{\partial z}\right)_k + \left(\frac{\partial \omega}{\partial z}\right)_k \left(\frac{\partial G}{\partial k}\right)_z - \frac{i}{2!}\left(\frac{\partial^2 \omega}{\partial k^2}\right)_z \left[\left(\frac{\partial^2 G}{\partial z^2}\right)_k + 2\left(\frac{\partial^2 G}{\partial k \partial z}\right)_{z,k}\right] \qquad (4.75)$$

where the terms in $\frac{d^2k(z)}{dz^2}$ and $(\frac{dk}{dz})^2$ must be neglected in order to be consistent in our approximations.

It is at this point that the treatment of equation (2.16) in chapter (II) becomes particularly helpful. However we need not pursue the discussion presented in chapter (II) again in this section. One may simply state that on neglecting the terms of order $\frac{1}{B}\frac{dB}{dz}$ throughout the treatment (i.e. putting $e^{-i\beta T^2/2} = 1$ in equation (4.59)) then essentially the whole discussion of equation (2.16) chapter (II) is relevant to equation (4.71) provided one notes the relation (4.73) and keeps the discussion fully consistent with the approximations (4.53) through to (4.57) (and approximation $\frac{1}{B}\frac{dB}{dz} \sim 0$).

As an example it is interesting to derive equation (1.43) of Kadomtsev (1965) from the first three terms of equation (4.75) as follows:-

We first assume the frequency has a small imaginary part γ

$$\text{i.e. } \omega[k(z),\Omega(z)] = \omega_0[k(z),\Omega(z)] - i\gamma[k(z),\Omega(z)]$$

$$\text{(where } \frac{d\omega}{dz} = 0\text{)}$$

We define $G' = e^{\gamma t}G$ then the first three terms of equation (4.75) become:-

$$\frac{\partial G'}{\partial t} = \gamma G' - \left(\frac{\partial \omega_o}{\partial k}\right)_z \left(\frac{\partial G'}{\partial z}\right)_k + \left(\frac{\partial \omega_o}{\partial z}\right)_k \left(\frac{\partial G'}{\partial k}\right)_z$$

(where terms in $\frac{d\gamma}{dk}$ have been neglected, see chapter (II))

On multiplying this by the complex conjugate of G' i.e. G'^{*} and writing down the complex conjugate equation multiplied by G' and adding the two resulting equations one finds

$$\frac{\partial A^2}{\partial t} = 2\gamma A^2 - \left(\frac{\partial \omega}{\partial k}\right)_z \left(\frac{\partial A^2}{\partial z}\right)_k + \left(\frac{\partial \omega}{\partial z}\right)_k \left(\frac{\partial A^2}{\partial k}\right)_z$$

see (1.43) of Kadomtsev (1965) (c.f. equation (2.28) chapter II)) where $G' G'^{*} = A^2$

Similarly by putting $G = Ae^{i\phi}$ one can discuss the changes in phase and amplitude in equation (4.75) by simple comparison with equations (2.25a) and (2.25b) chapter (II). To the authors knowledge the discussion presented here (and in section (II) chapter (II)) give a more generalized and easily understood treatment of wave packet propagation than previously (e.g. it contains (1.43) as a particular case) although unfortunately the detailed derivation of (4.71) was rather laborious.

Chapter V

The Propagation of Monochromatic Waves in Non uniform Plasmas

Section I Solution of the Wave Equation

In this chapter we attempt to find monochromatic wave solutions of the hot plasma 'wave equation' in the non-uniform ambient magnetic field (The problem is now fairly straightforward in the light of our treatment of wave packets in chapter (IV))

$$\underline{B}^{o}(\underline{r}) = B^{o}[-\tfrac{\epsilon}{2}(\hat{x}x+\hat{y}y)\frac{dP(z)}{dz} + (1+\epsilon P(z))\hat{z}]$$

(chapter (IV) equation (4.22))

That is we look for solutions which are of the form:-

$$\underline{E}(z,t) = (\hat{x}\ell_x+\hat{y}i\ell_y)e^{i(\omega t-\int^{z}k(z'')dz'')}S(z) \qquad (5.1)$$

where $S(z)$ is now a slowly varying function of z only, and ℓ_x, ℓ_y are constant amplitudes.

Solution (5.1) represents a monochromatic wave of frequency ω. Thus it remains to determine ω, $k(z)$ and $S(z)$, with the minimum (selfconsistent) approximation possible, such that equation (5.1) represents a solution of the 'wave equation'.

In the previous chapter we found that for an isotropic particle distribution and a disturbance of the form:-

$$\underline{E}(z,t) = (\hat{x}\ell_x + \hat{y}i\ell_y)e^{i(\omega t - \int^z k(z'')dz'')}G(z,t)$$

the perturbation in the particle distribution f' reduced to:-

$$f'(z,\underline{V},t) = -e^{-i\theta}\frac{|\mathcal{E}|}{m}\frac{\underline{E}(z,t)}{G(z,t)}\int_{\infty}^{0} dT\left\{\frac{\partial f^o}{\partial V_\perp}G[z',(t-T)]\right\}$$

$$.\,e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T}$$

$$.\,e^{-i\left\{\Omega_o\,.\,\epsilon\frac{dP(z)}{dz}V_z + \frac{dk(z)}{dz}V_z{}^2 + \frac{\epsilon}{2}\frac{dP(z)}{dz}V_\perp{}^2 k(z)\right\}T^2/2}$$

see equation (4.43) Chapter (IV).

For the monochromatic wave (equation (5.1)) we must of course replace $G(z,t)$ and $G[z',(t-T)]$ appearing in this equation by $S(z)$ and $S[z']$ respectively, i.e.

$$f'(z,\underline{V},t) = -e^{-i\theta}\frac{|\mathcal{E}|}{m}\frac{\underline{E}(z,t)}{S(z)}\int_{+\infty}^{0}\left\{\frac{\partial f^o}{\partial V_\perp}S[z']\right\}$$

$$e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T}$$

$$e^{-i\left\{\Omega_o\epsilon\frac{dP(z)}{dz}V_z + \frac{dk(z)}{dz}V_z{}^2 + \frac{\epsilon}{2}\frac{dP(z)}{dz}V_\perp{}^2 k(z)\right\}T^2/2} \qquad (5.2)$$

On eliminating the magnetic field $\underline{b}(z,t)$ associated with the electric field $\underline{E}(z,t)$ (equation (5.1)) using the two Curl Maxwell equations in the usual way we find

$$\frac{\partial^2 \underline{E}(z,t)}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \underline{E}(z,t)}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \dot{\underline{j}}}{\partial t} \qquad (5.3)$$

which on inserting equation (5.1) becomes:-

$$[-k^2(z) - 2ik(z)\frac{dS(z)}{dz}/S(z) - i\frac{dk(z)}{dz} + \frac{d^2S(z)}{dz^2}/S(z)]\underline{E}(z,t) =$$
$$= \frac{-\omega}{c^2}\underline{E}(z,t) + \frac{4\pi}{c^2}\frac{\partial \dot{\underline{j}}(z,t)}{\partial t} \qquad (5.4)$$

where:-

$$\dot{\underline{j}}(z,t) = -|\varepsilon|\int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} dV_z \int_0^{+\infty} dV_\perp V_\perp [e^{i\theta} V_\perp f'(z,\underline{V},t)] \qquad (5.5)$$

Hence using equation (5.2) in (5.5) and substituting the resulting current density in (5.4) we have:-

$$[-k^2(z)-2ik(z)\frac{dS(z)}{dz}/S(z) - i\frac{dk(z)}{dz} + \frac{d^2S(z)}{dz^2}/S(z)]\underline{E}(z,t) =$$
$$= -\frac{\omega^2}{c^2}\underline{E}(z,t) + \frac{\omega_p^2}{n_o}\frac{i\omega\pi}{c^2 S(z)}$$

$$\int_{-\infty}^{+\infty} dV_z \cdot \int_{+\infty}^{0} dT \int_0^{+\infty} dV_\perp V_\perp^2 \frac{\partial f^o}{\partial V_\perp} S[z'].e^{i(kV_z - \omega + \Omega_o[1+\epsilon P(z)])T}$$

$$e^{-i(\Omega_o \epsilon \frac{dP(z)}{dz}V_z + \frac{dk(z)}{dz}V_z^2 + \frac{\epsilon}{2}\frac{dP(z)}{dz}V_\perp^2 k(z))T^2/2} \qquad (5.6)$$

where $\omega_p^2 = \dfrac{4\pi |\varepsilon|^2 n_o}{m}$.

Before developing this equation with the minimum approximation possible, paying careful attention to the question of self consistency, it is instructive to proceed in the following manner.

Neglect the terms

$$\frac{2\,\frac{dS(z)}{dz}}{k(z)S(z)} \;,\quad \frac{\frac{dk(z)}{dz}}{k^2(z)} \;,\quad \frac{\frac{d^2S(z)}{dz^2}}{k^2(z)S(z)} \quad \text{(a)} \quad \text{with respect to}$$

unity, on the left hand side of equation (5.6).

On the right hand side we neglect terms of order

$\frac{1}{B}\frac{dB}{dz}$ (b) .

The equation then reduces to:-

$$k^2c^2-\omega^2 = \frac{-\omega_p^2 i\omega\pi}{n_o S(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \left\{ \int_0^{+\infty} dV_\perp V_\perp^2 \frac{\partial f^o}{\partial V_\perp} S[z'] \right\}$$

$$e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T}$$

$$e^{-i(\Omega_o \epsilon \frac{dP(z)}{dz} + V_z \frac{dk(z)}{dz})V_z \cdot T^2/2}$$

(5.7)

We now expand the exponential

$$e^{-i\left(\Omega_o \in \frac{dP(z)}{dz} + V_z \frac{dk(z)}{dz}\right)\frac{V_z T^2}{2}}$$ to terms linear in $\in$ and $\frac{dk(z)}{dz}$ i.e. we replace it by:-

$$\left[1 - i\left(\Omega_o \in \frac{dP(z)}{dz} + \frac{dk(z)}{dz} V_z\right)\frac{V_z T^2}{2}\right] .$$

Thus:-

$$k^2c^2-\omega^2 = \frac{-\omega_p^{\,2} i\omega}{n_o S(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \;\; g(V_z)$$

$$S(z')\left[1-i\left(\Omega_o \in \frac{dP(z)}{dz} + \frac{dk(z)}{dz}V_z\right)\frac{V_z T^2}{2}\right]$$

$$e^{i(k(z)V_z - \omega + \Omega_o[1+\in P(z)])T} \qquad (5.8)$$

where $$g(V_z) = \pi \int_o^{+\infty} dV_\perp V_\perp^{\,2} \frac{\partial f^o}{\partial V_\perp} .$$

It now only remains to expand $S(z')$ about the point $T = 0$ in this equation and simplify, bearing in mind the approximation (b). Thus:-

$$S(z') = S[z-V_z T + \frac{\in}{2}\frac{dP(z)}{dz} V_\perp^{\,2}T^2/2] \simeq$$

$$S(z) - V_z T \frac{dS(z)}{dz} + \frac{(V_z T)^2}{2} \frac{d^2S(z)}{dz^2}$$

where we have again neglected terms of order $\frac{1}{B}\frac{d\underline{B}(z)}{dz}$. see (b)

We shall also neglect the second derivative of $S(z)$ i.e.

$$\frac{d^2S(z)}{dz^2} \simeq 0 . \qquad (c)$$

Thus:-

$$[1 - i(\Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz} V_z) \frac{V_z . T^2}{2}] S(z')$$

$$\simeq S(z) - iV_z \, T(\Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz} V_z) \frac{TS(z)}{2} - V_z . T \frac{dS(z)}{dz}$$

where we have neglected the product terms

$$\frac{dS(z)}{dz} . \frac{dk(z)}{dz} , \frac{dS(z)}{dz} \Omega_o \epsilon \frac{dP(z)}{dz} \qquad (d)$$

Thus equation (5.8) becomes

$$k^2c^2 - \omega^2 = \frac{-\omega_p^2}{n_o} \frac{i\omega}{S(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \;\; g(V_z)$$

$$\left\{ S(z) - iV_z \, T(\Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz}) T \frac{S(z)}{2} - V_z \, T \frac{dS(z)}{dz} \right\} e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T} \qquad (5.9)$$

For clarity we introduce some simple notation. Put

$$\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT g(V_z) \, e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T} = \int_{-\infty}^{+\infty} dV_z . \int_{+\infty}^{o} dT \; L$$

(5.10)

and then notice the following identities:-

(5.11.i) $$\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \; g(V_z)\, iV_z T \, e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T} = \frac{\partial}{\partial k(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \; L \, .$$

(5.11.ii) $$\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \; g(V_z) \left\{ -V_z T\left(\Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz}\right)T \right\} e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T} = \frac{\partial^2}{\partial z \partial k(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \; L$$

(5.11.iii) $$\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \; g(V_z) \left\{ V_z T \right\}^2 e^{i(k(z) - \omega + \Omega_o[1+\epsilon P(z)])T} = -\frac{\partial^2}{\partial k^2(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \; L$$

On using (5.10) in (5.9) we have:-

$$k^2c^2 - \omega^2 = \frac{-\omega_p^2 i\omega}{n_o S(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \; L . \left\{ S(z) - iV_z T\left(\Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz}\right)T \cdot \frac{S(z)}{2} - V_z T \frac{dS(z)}{dz} \right\} \qquad (5.12)$$

We can choose k(z) so that

$$n^2(z) = \frac{k^2(z)c^2}{\omega^2} = 1 - \frac{\omega_p^{\,2} i\omega}{n_o}\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT.L \quad (5.13)$$

Where we recognise n(z) as the refractive index of a uniform plasma in which local values (at the position z) of the plasma parameters have been inserted, (see for example equation (a) page 96, section (I), Chapter (III), remembering of course that the ambient particle distribution f^o is now isotropic).

On using the identities (5.11i) and (5.11ii) we can see that the remaining terms in equation (5.12) reduce to:-

$$i\,\frac{S(z)}{2}\,\frac{\partial^2}{\partial z\partial k(z)}\left[\frac{\omega_p^{\,2} i\omega}{n_o}\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT.L\right] + i\,\frac{dS(z)}{dz}\,\frac{\partial}{\partial k(z)}\left[\frac{\omega_p^{\,2} i\omega}{n_o}\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dTL\right] = 0 .$$

Which by equation (5.13) can be written in the form

$$\frac{S(z)}{2}\,\frac{\partial^2}{\partial z\partial k(z)}\,[n^2(z)] + \frac{dS(z)}{dz}\,\frac{\partial}{\partial k(z)}\,[n^2(z)] = 0 . \quad (5.14)$$

The solution of this equation is

$$S(z) = \frac{\text{constant}}{\left\{\frac{\partial}{\partial k(z)}\left[n^2(z)\right]\right\}^{1/2}} \qquad (5.15)$$

Thus, to the degree of approximation we have adopted (that is approximations (a), (b), (c) and (d)), the solution becomes:-

$$\underline{E}(z,t) = (\hat{x}\ell_x + \hat{y}i\ell_y)\ \frac{e^{i(\omega t - \int^z k(z'')dz'')}}{\left\{\frac{d[n^2(z)]}{dz}\right\}^{1/2}} \qquad (5.16)$$

where $k(z)$ satisfies the uniform hot plasma dispersion relation in which local values (at the position z) of the plasma parameters have been inserted i.e. equation (5.13).

We shall return to discuss this solution (and its relation to standard W.KB) at a later stage in this chapter.

We now return to equation (5.6) and reduce it without use of the approximations (a), (b) and (c), (we shall however retain (d)).

We put $\frac{\epsilon}{2}\frac{dP(z)}{dz} V_\perp{}^2 k(z) = \beta$ (5.17) and write equation (5.6) in full, thus:-

$$-k^2(z)-2ik(z)\frac{dS(z)}{dz}/S(z) - i\,\frac{dk(z)}{dz} + \frac{d^2S(z)}{dz^2}/S(z) = -\omega^2/c^2 +$$

$$\frac{\omega_p{}^2 i\omega}{n_o c^2 S(z)}\int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \left\{\pi \int_0^{+\infty} dV_\perp V_\perp{}^2 \frac{\partial f^o}{\partial V_\perp} . S[z'] . e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T} . e^{-i(\Omega_o \epsilon \frac{dP(z)}{dz} V_z + \frac{dk(z)}{dz} V_z{}^2 + \beta)T^2/2}\right\} \quad (5.18)$$

We proceed in a similar manner to the previous treatment. That is we expand the exponential $e^{-i(\Omega_o \epsilon \frac{dP(z)}{dz} V_z + \frac{dk(z)}{dz} V_z{}^2)T^2/2}$ to terms linear in ϵ and $\frac{dk(z)}{dz}$, and we expand $S[z']$ about the point $T=0$, as before, thus

$$e^{-i(\Omega_o \epsilon \frac{dP(z)}{dz} V_z + \frac{dk(z)}{dz} V_z{}^2)T^2} . S[z']$$

$$\simeq S(z) - iV_z T(\Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz})T \frac{S(z)}{2} - V_z T \frac{dS(z)}{dz} + + \frac{(V_z T)^2}{2} \frac{d^2S(z)}{dz^2} \quad (5.19)$$

In this expansion we have neglected the product terms

$$\frac{dS(z)}{dz} . \Omega_o \epsilon \frac{dP(z)}{dz} , \frac{dS(z)}{dz} . \frac{dk(z)}{dz} , \text{ and higher order terms} \quad (5.20)$$

(essentially approximation (d)).

Thus equation (5.18) becomes:-

$$\left[-k^2(z)-2ik(z)\,\frac{dS(z)}{dz}\Big/S(z) - i\,\frac{dk(z)}{dz} + \frac{d^2S(z)}{dz^2}\right] = -\omega^2/c^2 +$$

$$\frac{\omega_p^2 i\omega}{n_o c^2 S(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \left\{ \pi \int_{o}^{+\infty} dV_\perp V_\perp^2 \frac{\partial f^o}{\partial V_\perp} \right.$$

$$\left. e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T - i\beta T^2/2} \right\}$$

$$\left\{ S(z) - iV_z\, T\left(\Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz}\right)T\, \frac{S(z)}{2} - V_z\, T\, \frac{dS(z)}{dz} \right.$$

$$\left. + \frac{(V_z\, T)^2}{2}\, \frac{d^2S(z)}{dz^2} \right\} \qquad (5.21)$$

We now choose $k(z)$ to satisfy the relation

$$k^2(z)c^2 - \omega^2 = \frac{\omega_p^2 i\omega}{n_o S(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{o} dT \left\{ \pi \int_{o}^{+\infty} dV_\perp V_\perp^2 \frac{df^o}{dV_\perp} \right.$$

$$\left. e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T - i\beta T^2/2} \right\} \qquad (5.22)$$

We use the identities (5.11) i, ii and iii, together with the approximations (5.20) to simplify the terms remaining in equation (5.21).

Thus using equation (5.22) and approximations (5.20) in equation (5.21) we have:-

$$\left[-2ik(z)\,\frac{dS(z)}{dz}\Big/S(z) - i\,\frac{dk(z)}{dz} + \frac{d^2S(z)}{dz^2}\Big/S(z)\right] =$$

$$\frac{\omega_p^2 i\omega}{n_o c^2 S(z)} \int_{-\infty}^{+\infty} dV_z \int_{+\infty}^{0} dT \left\{ \pi \int_0^{+\infty} dV_\perp V_\perp^2 \frac{\partial f^o}{\partial V_\perp} e^{i(k(z)V_z - \omega + \Omega_o[1+\epsilon P(z)])T} \right\}$$

$$\cdot \left\{ -iV_z T\left(\Omega_o \epsilon \frac{dP(z)}{dz} + \frac{dk(z)}{dz}\right)T \frac{S(z)}{2} - V_z T \frac{dS(z)}{dz} + \frac{(V_z T)^2}{2} \frac{d^2S(z)}{dz^2} \right\} \qquad (5.23)$$

Notice that the factor $e^{-i\beta T^2/2}$ has been replaced by unity by use of approximations (5.20).

Hence we may now use the three identities (5.11) to rewrite equation (5.23) as:-

$$-\frac{2ik(z)}{S(z)}\cdot\frac{dS(z)}{dz} - i\,\frac{dk(z)}{dz} + \frac{d^2S(z)}{dz^2}\cdot\frac{1}{S(z)} =$$

$$\frac{\omega_p^2 i\omega}{n_o c^2 S(z)} \left\{ i\,\frac{S(z)}{2}\left[\frac{\partial^2}{\partial z\,\partial k(z)} \int dV_z \int dT.L\right] + \frac{i\,dS(z)}{dz}\left[\frac{\partial}{\partial k(z)} \int dV_z \int dT.L\right] - i\,\frac{d^2S(z)}{dz^2}\left[\frac{\partial^2}{\partial k^2(z)} \int dV_z \int dTL\right] \right\} \qquad (5.24)$$

We use the relation (5.13), (i.e. equation (5.22) in which $e^{-i\beta T^2/2}$ is replaced by unity) to rewrite equation (5.24) as:-

$$-2ik(z)\,\frac{dS(z)}{dz}\Big/S(z) - i\,\frac{dk(z)}{dz} + \frac{d^2S(z)}{dz^2}\Big/S(z) =$$

$$\frac{1}{S(z)}\left\{ i\,\frac{S(z)}{2}\left[\frac{\partial^2 n^2(z)}{\partial z\,\partial k(z)}\right] + i\,\frac{dS(z)}{dz}\left[\frac{\partial n^2(z)}{\partial k(z)}\right] - \frac{1}{2}\,\frac{d^2S(z)}{dz^2}\left[\frac{\partial^2 n^2(z)}{\partial k^2}\right]\right\} \qquad (5.25)$$

This relation is independent of $S(z)$, however $S(z)$ must be sufficiently slowly varying that approximation (5.20) is valid. In retrospect we can see that we derived the particular value

$S(z) = \dfrac{\omega}{C\sqrt{2k(z)}}$ essentially because we used the approximation (a) (see equation (5.30) below) and which we can now see is not a fully selfconsistent approximation. Thus we have derived the solution

$$\underline{E}(z,t) = (x\ell_x + yi\ell_y)\,e^{i(\omega t - \int^z k(z'')dz'')} \qquad (5.26)$$

Where the wave number $k(z)$ is related to the frequency through the "dispersion relation" (5.22). It is only when one chooses to neglect terms of order

$\frac{1}{B}\frac{dB}{dz}$ (i.e. β) that equation (5.22) becomes the local uniform plasma dispersion relation. Thus we see that typically Fresnal integrals are going to arise in a discussion of wave propagation in a non-uniform ambient magnetic field and a discussion of resonant particle effects by means of equations (5.20) and (5.26) should prove fruitful. At this stage we review as briefly as possible some work done by Stix (1962) on the propagation of waves in inhomogeneous plasma since it bears some relation to what we have so far achieved. In particular we discuss his treatment of Whistler mode wave propagation in the magnetic beach configuration. Stix considered the problem of wave propagation in non-uniform plasmas in the region where the local cold plasma refractive index goes either to zero or infinity. It is the latter case which we are concerned with here. Of particular interest to us he considered the cyclotron damping mechanism in the region of the local (cold plasma) Whistler mode refractive index infinity.

Stix searched for cold lossless plasma dispersion relations which could be written in the form:-
$bk^2 + c = 0$ (5.27) for the homogeneous case, where b and c are constants depending on the plasma density (e.g. plasma frequency) or magnetic field strength

(e.g. cyclotron frequency) etc. He then investigated the inhomogeneous case (assuming a one dimensional z dependence only). He assumed that in the inhomogeneous case, if the variation in plasma parameters is sufficiently slow, one may replace equation (5.27) by:-

$$b(z)\, k^2(z) + c(z) = 0 \qquad (5.28)$$

where the constants b, c and wave number k are now slowly varying functions of z .

He assumed that the dispersion relation (5.28) corresponds to the differential equation

$$\frac{\partial^2 \underline{E}(z,t)}{\partial z^2} + k^2(z)\underline{E}(z,t) = 0 \qquad (5.29)$$

where $k^2(z) = -\frac{c(z)}{b(z)}$.

(One may assume the time dependence is given by $e^{i\omega t}$ i.e. investigate solutions which are in the form of monochromatic waves). Provided $c(z)$ and $b(z)$ are sufficiently slowly varying the solution of (5.29) is given by the well known W.K.B approximation i.e.

$$E(z,t) \simeq \frac{\text{constant}}{\sqrt{k(z)}}\, e^{i(\omega t \pm \int^{z} k(z'')dz'')} \qquad (5.30)$$

which is valid provided

$$\frac{1}{k^3}\frac{d^2k}{dz^2} \quad \text{and} \quad \frac{1}{k^2}\frac{dk}{dz} \tag{5.31}$$

are neglected with respect to unity on the left hand side (essentially approximation (a) used in deriving solution (5.16)).

It is interesting to contrast the solution (5.30) derived via approximation (5.31) with our solution (5.16) of the hot plasma dispersion relation which we derived essentially by use of approximation (a). (The actual quantities neglected in the two approximations (a) and (5.31) are the same when

$S(z) = \frac{\text{constant}}{\sqrt{k(z)}}$) .

The two cases where $b(z) \longrightarrow 0$ and $C(z) \rightarrow 0$ where investigated, we shall consider only the former case corresponding to $k(z) \rightarrow \infty$ (i.e. the local refractive index tending to become infinite).

In the vicinity $b(z) \longrightarrow 0$ equation (5.29) was replaced by

$$\frac{\partial^2 E}{\partial z^2} + \frac{\mu_1 E}{z' - z_0 + i\mu_2} \tag{5.32}$$

where μ_1 is a positive constant and μ_2 is a small real constant, μ_2 is introduced as a trick to ensure that E will be single-valued and finite, but has the physical significance of growth or damping. The sign of μ_2 is chosen to correspond to damping. (The introduction of the $(z-z_o)$ linear dependence of $b(z)$ in the region z_o is well established in quantum mechanics, the innovation is the introduction of $i\mu_2$). The solutions of equation (5.32) can be found and its asymptotic form (i.e. for large $z-z_o$) is the same solution (5.30). Thus one may join the solution of (5.32) valid in the region z_o to the solution of (5.29) valid in the region far from z_o, (i.e. solution (5.30)). (There may of course exist an intermediate region in which neither the solution of (5.32) nor the solution of (5.29) is very good). However by joining these two solutions Stix could investigate the complete result thus obtained. This complete solution helps to determine whether or not reflection or absorption has taken place, (one has to impose some physics on the problem to determine this e.g. boundedness of the solution beyond the turning point z_o). He then attempted to fit the treatment discussed above to the problem of Whistler mode wave propagation in an ambient magnetic field which grows progressively weaker. Physically we can argue as follows. One knows from the cold plasma Appleton Hartree dispersion relation that the local refractive index

$$n^2 = \frac{k^2c^2}{\omega^2} = 1 + \frac{\omega_p^2\omega}{|\Omega| - \omega}$$

will become infinite in the region $\omega \approx |\Omega(z)|$. That is the phase velocity $\omega/_k$ will decrease to zero. Of course one also knows that the wave will be in cyclotron resonance with electrons for which $V_{z\ res:} = -(\frac{|\Omega| - \omega}{k})$, and as ω approaches $|\Omega|$ the wave will resonate with electrons deep within the body of the distribution; rather than a tenuous stream in its high energy tail as was previously considered to be the case (i.e. $V_{Phase} >> V_{Thermal}$). One thus expects the wave to be rapidly damped out (probably before it ever reaches the region for which $\omega \simeq |\Omega|$).

One may use the solutions of equation (5.29) and (5.32) to get at least a rough analytic treatment of the problem of Whistler mode waves propagating into a region for which $\omega \simeq \Omega$. Essentially we determine the constants μ_1 and μ_2 which appear in the solution of (5.32) as follows:-

Using the hot plasma dispersion relation

$$k^2c^2-\omega^2 = \frac{\omega_p^2\omega\pi}{n_o}\int_{-\infty}^{+\infty} dV_z \int_{-\infty}^{+\infty} \frac{dV_\perp V_\perp{}^2\left\{(1-\frac{kV_z}{\omega})\frac{\partial f^0}{\partial V_\perp} + \frac{kV_\perp}{\omega}\frac{\partial f^0}{\partial V_z}\right\}}{(\omega - kV_z - |\Omega|)}$$

valid for $\mathrm{Im}(\omega) < 0$
(see equation (a) page 96, section (I) chapter (III)).

One chooses the ambient particle distribution f^{o} to be the isotropic Maxwell Boltzmann i.e.

$$f^{o}(\underline{v}) = \frac{n_o}{\pi^{3/2} V_T^{2}} \, e^{-\frac{(V_z^{2} + V_\perp^{2})}{V_T^{2}}}$$

where V_T is the characteristic thermal velocity.

Then on integrating over $V_\perp$ one has

$$\int_0^{+\infty} dV_\perp V_\perp^{2} \frac{\partial f^{o}}{\partial V_\perp} = - \frac{n_o}{\pi^{3/2} V_T} \, e^{-V_z^{2}/V_T^{2}}$$

On putting $z' = V_z / V_T$ and $\alpha(z) = \dfrac{\omega - |\Omega(z)|}{kV_T}$

the above dispersion relation may be written in the form

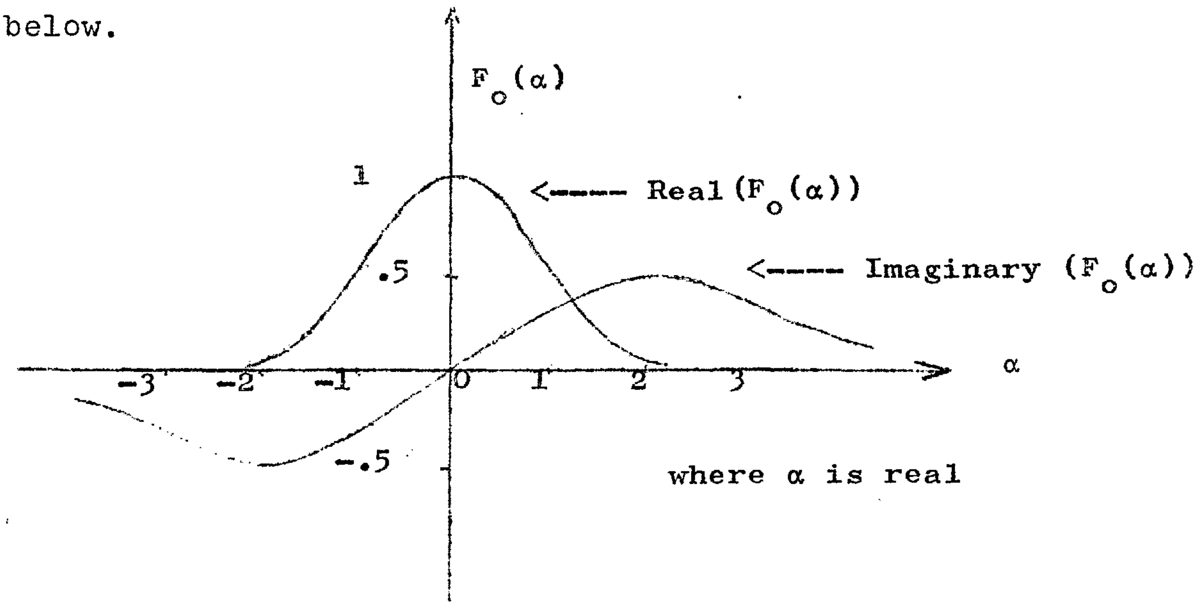
$$\frac{k^2c^2}{\omega^2} = 1 + \frac{i\omega_p^{2}}{kV_T\omega} F_o(\alpha) \qquad (5.33)$$

where $$F_o(\alpha) = - \frac{i}{\sqrt{\pi}} \int \frac{dz' e^{-z'^2}}{z' - \alpha} \qquad (5.34)$$

$F_o(\alpha)$ is the well known plasma dispersion function valid for growing waves i.e. $\mathrm{Im}(\omega) < 0$.
Equation (5.34) must of course be analytically continued into the upper half frequency plane using the Landau prescription in the usual way. Thus:-

$$F_o(\alpha) = -\frac{i}{\sqrt{\pi}} P \int_{-\infty}^{+\infty} \frac{dz' e^{-z'^2}}{z'-\alpha} + \sqrt{\pi} e^{-\alpha^2} \qquad (5.35)$$

(where P stands for principle value) is valid over the whole complex frequency plane. The plasma dispersion function has been tabulated by Fried and Conte and a plot is given by Stix (1962), which is reproduced below.



We may proceed in the same manner as Stix with only a slight change in his notation. Since we chose the ambient magnetic field to be

$$\underline{B}^o(\underline{r}) = B[-\frac{\epsilon}{2}(\hat{x}x+\hat{y}y)\frac{dP(z)}{dz} + (1+\epsilon P(z))\hat{z}] \quad \text{(see eqn (4.22) Chapter (IV))}$$

we have $|\Omega(z)| = \left| \frac{|\mathcal{E}|\underline{B}^o(\underline{r})}{mc} \right| = \left| \frac{|\mathcal{E}|B_o}{mc}(1+\epsilon P(z)) \right|$ to terms linear in ϵ .

We choose $P(z)$ to be **a** linear function of z i.e. replace $P(z)$ by $\epsilon(z-z_o)$ and also choose the gyro-frequency $|\Omega|$ equal to the wave frequency ω at the point z_o ; thus:-

$$\alpha(z) = \frac{\omega}{kV_T} \epsilon (z-z_o) .$$

We replace k^2 on the left hand side of (5.33) by $-\frac{\partial^2}{\partial z^2}$ i.e.

$$\frac{\partial^2 E}{\partial z^2} = - \frac{\omega^2}{c^2} [1 + \frac{i\omega_p^2}{kV_T \omega} F_o(\alpha(z))]E \qquad (5.36)$$

and rewrite equation (5.32) as

$$\frac{\partial^2 E}{\partial z^2} = [\frac{\mu_1(z-z_o)}{(z'-z_o)^2+\mu_2^2} - \frac{i\mu_1\mu_2}{(z-z_o)^2+\mu_2}]E \qquad (5.37)$$

The above diagram may now be interpreted as essentially a plot of the real and imaginary part of the right hand side of equation (5.36) against z in the vicinity $z = z_o$. We see that it has qualitatively the same dependence on $z - z_o$ as the right hand side of equation (5.37). To arrive at an expression for μ_1 and μ_2 Stix equated, (i) the imaginary parts at $z = z_o$ and, (ii) the real parts at large $(z - z_o)$.

Thus (i) requires

$$- \mu_1/\mu_2 = \frac{\omega_p^2 \omega}{c^2 k V_T} F_o(0) \qquad (.i)$$

and (ii) requires

$$\frac{\mu_1}{(z-z_o)} \simeq \frac{\omega^2}{c^2}\left[\frac{\omega_p^{\,2}}{\omega^2 \epsilon (z-z_o)}\right] \qquad (.ii)$$

where in deriving this result only the first term of the imaginary part of the asymptotic expansion (valid for $\alpha \gg 1$) of $F_o(\alpha)$ is used

i.e. $F_o(\alpha) = \sqrt{\pi}\, e^{-\alpha^2} + \frac{i}{\alpha}\left(1 + \frac{1}{2\alpha^2} + \ldots\right)$

hence $\mathrm{Im}[F_o(\alpha)] \simeq \frac{1}{\alpha}$, $(\alpha \gg 1)$ (see Stix (1962)).

Thus from (i) and (ii)

$$\mu_1 = \frac{\omega_p^{\,2}}{c^2} \quad \text{and} \quad \mu_2 = \frac{ka}{F_o(0)\omega\epsilon}.$$

One cannot do better than to quote Stix directly on this procedure.

"This computation of the absorption of cyclotron waves in a magnetic beach illustrates the lack of rigour mentioned in the introduction to this chapter. We have tried to write an appropriate hot-plasma wave equation in differential form. In doing so, we have had to fit the function $F_o(\alpha)$ onto the algebraic form $\frac{\mu_1}{(z'-z_o+i\mu_2)}$. However, the refractive index for a hot plasma is not a local quantity. The function $F_o(\alpha)$ which we use here is in fact a transendental function of the fourier wave number k, and k should have been replaced in some fashion by $\frac{d}{dz}$ wherever it occurs. A rigorous mathematical solution of the problem is clearly a very difficult task. Nevertheless the physical argument that cyclotron damping is a very strong process and will dominate the kinematics of the plasma in the $\omega \simeq |\Omega|$ region suggests that essentially complete absorption will occur for longitudinally propogated cyclotron waves".

Thus in this treatment of the problem one is really attempting to force cold plasma theory beyond its limits of validity to describe what are essentially hot plasma phenomena. The method we have developed in this and preceding chapters for dealing with the hot plasma "wave equation" shows that one may approach the problem afresh by means of the rigorous hot plasma theory. We have found solutions of the hot plasma wave equation; these solutions were not chosen especially for their validity in the region $\omega \simeq |\Omega(z)|$. However one may obviously look for solutions appropriate for this (and possibly an intermediate) region. On matching the asymptotic forms of these solutions one would then have a complete solution of the wave equation valid over all values of z. The question of reflection and absorption could then be investigated in a manner directly analogous to the method adopted by Stix.

Even now the problem of gyroresonance phenomena in the chosen non-uniform ambient magnetic field can be investigated by means of equations (5.26) and (5.22), this investigation would presumably require some modification in the region $\omega \simeq |\Omega(z)|$, (though possibly the more difficult part of the task as stated by Stix has been carried out).

Section (II) Conclusions and Suggestions for further Work.

The above discussion really suggests a great deal of further work that could be attempted in non-uniform ambient magnetic fields. The problem of wave and wave packet propagation in time varying media (as say in wave-wave interactions) would also appear amenable to the sort of procedure we have adopted here. The basic conclusion that one may draw is that by using the method of characteristics to solve the Boltzmann Vlasov set of equations (the constants of the particle motion are useful in achieving this step) and by establishing that only recent events in the history of a given particle is important one may find solutions of the hot plasma integro-differential wave equation in non-uniform and time dependent plasmas (this step is achieved by means of the Taylor expansion). The free streaming of the plasma particles is of course rigourously accounted for in this treatment.

APPENDIX A

On carrying out the integration in equation (3.30a) in the order indicated (where, of course ω_I can be greater than, equal to or less than zero) we find

$$I = k \cdot e^{-\alpha\left\{\frac{ka - \omega + |\Omega|}{k}\right\}^2} - 2i\frac{\pi^{\frac{1}{2}}}{k} \cdot e^{-\alpha\left\{\frac{ka - \omega + |\Omega|}{k}\right\}^2}$$

$$\times \int_0^{\sqrt{\alpha}\left\{\frac{ka - \omega + |\Omega|}{k}\right\}} dx . e^{x^2} \qquad \text{(A.1)}$$

(Stix (1962)).

This equation being valid in both the upper and lower half of the complex frequency plane.

On carrying out the integration over time first (see equation (3.29) one finds

$$I = -i \int_{-\infty}^{+\infty} dv_{\parallel} \frac{e^{-\alpha(v_{\parallel} - a)^2}}{(kv_{\parallel} - \omega + |\Omega|)} \qquad \text{(A.2)}$$

This equation being valid only if $\omega_I < 0$ (i.e. growing waves) but may be analytically continued into the lower half frequency plane using the usual Landau prescription (provided the numerator of the intergrand of equation (A.2) is an entire function of $v_{\parallel}$ in the complex $v_{\parallel}$ plane, see Montgomery and Tidman (1964), page 56). The growth or damping decrement can then be calculated (e.g. see Chapter I).

However using equation (A.1) it is obvious that one may similarly calculate the imaginary part of ω for both growing and damped waves without using the familiar analytic continuation and hence without having to stipulate that $f^{(o)}$ must be an entire function of $v_{\parallel}$ (a rather non-physical requirement).

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simple contribution to the theory of the anisotropy cosmic rays

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Abstract. Simplicity is obtained by assuming that the first invariant is preserved, that the magnetic field is spiral in the ecliptic and the solar wind radial and that $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$. A cosmic ray particle mirroring near the Sun has gradient and curvature drifts normal to the ecliptic. The displacement of the guiding centre between inward and outward crossings of the Earth's orbit is of the order of two gyroradii and is in the opposite direction to the electric field for positive particles. The day to day variations of this drift have not yet been fully investigated; an assumption is needed about the gradient of the incoming cosmic rays normal to the ecliptic. Computations of the drift are presented for a typical field.

ıs been shown by Ahluwalia and Dessler (1962) that by ıming Parker's idealized model of the spiral interplanetary netic field one expects to find an anisotropy in the cosmic intensity. Ahluwalia and Dessler describe this by the con- ion of the whole of the cosmic ray distribution with the ion of the field lines. The same result can be obtained in ıs of the electric field which is perpendicular to the eclip- The particle motion normal to the ecliptic is oscillatory so the particle energy oscillates with the gyroperiod. ıg Liouville's theorem it is then seen that the anisotropy ılts unless the incoming distribution has a particular form rn 1964) requiring a gradient. This is further discussed r.

ıssible anisotropy caused by gradient and curvature drifts ıe same electric field has been suggested by Dungey* and ınsidered here. It is found that the displacement resulting ı these drifts is of the order of a gyroradius (for certain icles) and could thus give rise to an anisotropy comparable Ahluwalia and Dessler's.

first assumption is that the magnetic moment invariant $E_\perp/B$ of the cosmic ray particle is preserved. The second the solar wind is radial and of constant speed, the stream s in the rotating frame of the Sun thus forming spirals in ecliptic plane. After Parker (1963) and assuming axial metry

$$B_r = \frac{A}{r^2}, \quad B_\phi = \frac{wA}{ur}, \quad B_\theta = 0$$

$$|\mathbf{B}| = \frac{A}{r}\left(\frac{1}{r^2} + \frac{w^2}{u^2}\right)^{1/2} \tag{1}$$

e w is the angular velocity of the Sun, u is the solar wind d. The third assumption is that $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$, E having a ntial. We assume that the three familiar electric field, ient and curvature drifts are dominant. The electric field $cE \times B/B^2$ lies in the plane of the ecliptic, does not change nergy, and is not considered here. Thus

$$v_d = \frac{v_\parallel^{\,2}}{\rho\Omega} + \frac{v_\perp^{\,2}\nabla_\perp B}{2\Omega B} \tag{2}$$

e v_d is the drift velocity normal to the ecliptic plane due rvature and gradients in the magnetic field,

$$\rho = \frac{(u^2/w^2 + r^2)^{3/2}}{2u^2/w^2 + r^2}$$

adius of curvature of a line of force calculated from equation (1). The gyrofrequency is given by

$$\Omega = \frac{eB}{mc}. \tag{3}$$

The velocity of the guiding centre along the line of force is

$$v_\parallel = v\cos\alpha = v\left(1 - \frac{B}{B_M}\right)^{1/2} \tag{4}$$

Similarly

$$v_\perp = v\sin\alpha = v\left(\frac{B}{B_M}\right)^{1/2} \tag{5}$$

$$\frac{\nabla_\perp B}{B} = \frac{1}{\rho} \tag{6}$$

where $\nabla_\perp B$ is the gradient of the magnetic field perpendicular to a line of force.

Because $B_\theta = 0$ the lines of force lie on cones of constant θ. The rate of change of heliocentric latitude θ therefore involves only v_d and $d\theta/dt = v_d/r$. The change in θ in time $ds/v_\parallel$ is $d\theta = (v_d/r)\,(ds/v_\parallel)$, therefore the angular displacement of the guiding centre is

$$\Delta\theta = 2\int_{r_M}^{r_1} \frac{v_d\,ds}{r v_\parallel} \tag{7}$$

for the total inward and return journey. Substituting (2) in (7) and (3), (4), (5) and (6) in the resulting equation we obtain

$$\Delta\theta = \frac{2vmc}{e}\int_{r_M}^{r_1} \frac{1 - B/2\,B_M}{\rho r B(1 - B/B_M)^{1/2}}\,ds \text{ (non-relativistic).}$$

Hence using the latter of equations (1) we get

$$r_1\Delta\theta = \frac{2\,vmc}{e}\,\frac{r_1^{\,2}}{A}\quad I$$

with

$$I = \int_{r_M}^{r_1} \frac{1 - B/B_M}{\rho(1 - B/B_M)^{1/2}\,(1/r^2 + w^2/u^2)^{1/2}}\,\frac{ds}{r_1}.$$

Taking $r_1 = 1$ astronomical unit, A/r^2 is B_r at the Earth's orbit, I is dimensionless and was computed for mirror points ranging from 0.1 to 0.9 A.U. in steps of 0.1 A.U.

u/w was taken as 1 A.U. and then the angle between the interplanetary field and the Sun-Earth line is 45° at the Earth, hence $B_r = B/\sqrt{2}$. Thus it can be seen that $2\sqrt{2}$ I represents the displacement of the guiding centre in units of 90° pitch angle gyroradii ($v_\perp$ replaced by v). These displacements are shown plotted against the corresponding mirror points in the

* is believed that K. G. McCracken reported on this effect the meeting of the American Physical Society at Houston xas 1963.

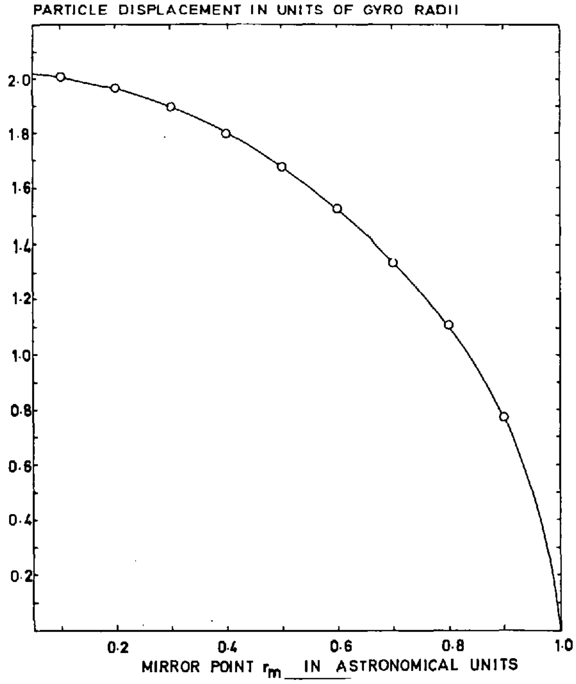

figure. It is seen that the effect is as large as Ahluwalia and Dessler's for particles of small pitch angles. The change in energy can be got from this displacement and alternatively $\int v_d E dt$ leads to the same result.

For the case of relativistic particles, provided the electric field is sufficiently small, one can multiply I by the corresponding radius of gyration, i.e. $m_0vc/(1 - v^2/c^2)^{1/2}$ e. Alternatively one can follow the factor $(1 - v^2/c^2)^{-1/2}$ through from equation (2) with m replaced by $m_0/(1 - v^2/c^2)^{1/2}$ which again is a valid substitution provided the electric field is not too large. For a relativistic particle we have initial energy $E = (m - m_0)c^2$. For a given r_M the change in energy is proportional to the gyroradius, i.e. proportional to $v/(1-v^2/c^2)^{1/2}$. The fractional change in energy

$$\frac{\Delta E}{E} \propto \frac{v[1 - (1 - v^2/c^2)^{1/2}]}{(1 - v^2/c^2)^{1/2}}.$$

Thus as v increases so does the fractional change in energy, for fixed mirror point.

It is seen that the energy change for our anisotropy is o same order for small pitch angles as Ahluwalia and De for 90° pitch angles. This suggests approximately equal tributions from each assuming an isotropic distribution hence an 1800 hr phase, though the details are still to b culated.

The anisotropy does depend on the incoming distributio total energy of a particle, including potential energy, is stant of the motion and Stern (1964) pointed out that if t distribution function f were a function of energy only th would be no anisotropy though there would be a gradien Another simple possibility is that the incoming distribu has no gradient across the ecliptic plane and there mus be an anisotropy compounded of Ahluwalia and Dessler' sotropy and that described here which is probably comp in magnitude.

The Ahluwalia and Dessler anisotropy has a maximum the particle is moving normal to the magnetic field and from the Sun, typically at 1500 hr local time, while ours its maximum when the particle is moving parallel to th towards the Sun, typically at 2100 hr. The direction of interplanetary magnetic field does vary in practice and curvature and gradient of the field involved in (2) also Consequently the local time of maximum for each contr and the relative importance of the contributions all var Since this is not in accord with experimental results it possible that some other effect is dominant for exampl non-conservative fields were present. Alternatively va tions in I with wind speed may be such as to make the insensitive to the wind speed. This possibility and a m accurate estimate of the magnitude of the anisotropy s here are being investigated. It seems certain that if a in the particle distribution does not exist Ahluwalia an Dessler's anisotropy occurs together with the anisotro cussed here.

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