SOME FORCED WAVE PROBLEMS IN FLUID MECHANICS

by

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ABSTRACT

Three topics relating to the theory of aerodynamic sound are considered. The first is a mathematical study of the similarity between the behaviour of surface gravity waves on a shallow layer of water and sound waves in a compressible gas. It is concluded that various qualitative features of the two wave fields are the same, but that a complete quantitative similarity is not possible. In the second study, the current formulation of propeller noise theory is considered in the light of the more general aerodynamic sound theory. This reveals that some important sound sources have been overlooked in propeller theory, and their effect is discussed. Some extensions are also made to existing results concerning the frequency spectrum of a sound source in circular motion. The third chapter shows how the use of generalised functions leads to a greater understanding of the fundamental theory of aerodynamic sound. The classical results of this theory are more simply derived using the new methods. Also new results are obtained concerning the sound fields generated by rapidly moving sources, and these provide a better description of the

fields than those results employed hitherto. In particular, the sound fields generated by rapidly convected turbulence and rigid surfaces moving at high speeds are discussed.

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CONTENTS.

CHAPTER 1. INTRODUCTION

This thesis is entitled "Some forced wave problems in fluid mechanics" and studies three problems related to the theory of aerodynamic sound. That theory is concerned with sound fields that are essentially a by-product of a more basic aerodynamical flow. For example, the original application of the theory was to the noise generated by turbulent jets. The view of such situtations that is adopted in aerodynamic sound theory is that the basic flow generates or forces the sound waves, and the problem is to relate feature of the observed sound field to known parameters of the basic flow. It is in this sense that the problems discussed below are forced wave problems.

The three problems that make up this thesis are not entirely the author's own work. Some significant contributions have been made by his supervisor Dr Ffowcs Williams, and although it is difficult to trace the origin of the various ideas, the author estimates that two thirds of the material results from his own research. Each problem is studied in a separate chapter, which is written in the form of a paper which the author, in conjunction with Dr. Ffowcs Williams, has or hopes to publish upon each topic. Each chapter contains its own introduction and conclusions, so that only a few brief comments about the problems will be made at this stage. All the references have been compiled into a single list which appears at the end of the thesis.

The first paper, which forms Chapter 2, is entitled "Shallow water wave generation by unsteady flow," and is already published in the Journal of Fluid Mechanics Vol. 31 pp.779-88 (March 1968). The aim of this paper is to investigate quantitatively the remarkable 'shallow water analogy' between the behaviour of surface gravity waves on a shallow layer of water, and sound waves in a compressible gas. This qualitative similarity raises the possibility that some phenomena in the generation and propagation of sound can be simulated in the laboratory on a shallow water table. This has two experimental advantages. Firstly, unlike sound waves the surface waves on a layer of water are easily visualised; and secondly, the wavespeed of the water waves is small so that it is easy to simulate the sound fields generated by supersonic flows. However, an obvious, and mathematically crucial, difference between the two wave fields is that the shallow water waves propagate in two dimensions, whereas sound waves are three dimensional. Consequently chapter 2 is devoted to a study of the modifications that occur to familiar results in aerodynamic sound theory when the waves are constrained to propagate in two dimensions.

Chapter 3 is entitled "Some theory relating to the noise of rotating machinery". This is a modified version of a report with the same title that was presented to the Noise Committee of the Aeronautical Research Council in January 1968. Most of the modifications are of a minor nature, but section 4 has been rewritten as a result of some further thoughts upon its contents. This chapter is concerned with the sound field generated by aircraft propellers or aeroengine compressors, and falls naturally into two parts. In the first part, the current formulation of propeller noise generation is discussed in the light of aerodynamic sound theory. This discussion reveals that in modern rotating machinery, there are noise sources present that have not so far been considered in any analysis of the sound field, and it is shown that these sources may be responsible for a significant component of the sound field. In the second part, the sound field of a rotating multipole source of arbitrary strength is considered, and a formula is derived that relates the frequency spectrum of the observed sound to that of the multipole source strength. This extends the present theory which only applies to a rotating source of constant strength.

The final, and most significant, chapter is entitled "Sound generation by turbulence and surfaces in arbitrary motion". This is to be published as a Philosophical Transaction of the Royal Society. This chapter attempts to demonstrate the increased understanding of aerodynamic sound that can be obtained by the extensive use of generalised functions. The use of these functions has two advantages. Firstly, it allows the effect of an arbitrarily moving surface to be studied more simply than is possible using classical methods. Secondly, alternative expressions for the sound field can be derived which in some circumstances yield more accurate estimates of its magnitude than do the expressions used hitherto. Consequently, this chapter begins by deriving the solution of the equation governing aerodynamic sound generation, in the presence of a moving surface. This solution has not previously been given. The chapter continues by illustrating how alternative forms of the solution can be used to give a satisfactory account of the sound field generated by a turbulent eddy moving in an unbounded space, and also to reveal the effects of source acceleration. The chapter is concluded by a discussion of the sound generating effects of a rapidly moving surface, and again these results appear to be new. As a subsidiary problem, Kirchhoff's solution of the three dimensional scalar wave equation is extended to a moving surface.

CHAPTER 2. SHALLOW WATER WAVE GENERATION BY UNSTEADY FLOW

Summary

Small amplitude waves on a shallow layer of water are studied from the point of view used in aerodynamic sound theory. It is shown that many aspects of the generation and propagation of water waves are similar to those of sound waves in air. Certain differences are also discussed. It is concluded that shallow water simulation can be employed in the study of some aspects of aerodynamically generated sound.

1. Introduction

The theory of aerodynamic sound initiated by Lighthill is built upon the equations of mass and momentum conservation. These yield a three-dimensional wave equation which describes the generation and propagation of sound waves. It is shown in this paper that the generation and propagation of waves on a shallow layer of water is governed by a two-dimensional wave equation similar to Lighthill's and that, in many respects, these waves behave like sound waves. This similarity enables sound waves to be simulated and visualized in the laboratory. The ability to see the waves makes it possible to study their development and interactions in detail, a study which would be extremely difficult with aerial sound waves. Furthermore, since the propagation speed of water waves is small, high Mach number situations can be examined easily.

Gravity waves on a finite depth of water are dispersive and cannot in general provide a very good model for sound waves, which are non-dispersive. However, at a particular mean depth (0.5 cm), surface tension effects render the water layer practically non-dispersive, thus minimizing this difficulty. Indeed waves on a shallow water layer are an excellent simulation of two-dimensional aerial waves, and they also share many outstanding features in common with three-dimensional waves. The similarity is brought out in detail in the following analysis which considers turbulence as a source of shallow water waves. The philosophy underlying the derivation of the equations is the same as that used in the sound theory (Lighthill 1952). The waves are regarded as a by-product of a more complicated flow, and the problem is to estimate the waves generated by it. The flow is assumed to be known, and acts as a source of waves which radiate into the undisturbed water. From this point of view, the resultant forced wave equation, although rather artificially manufactured, is a correct description of the field.

The general solution of the shallow water equation demonstrates that water and sound waves are alike in that both are generated by the same distribution of sources, and that at large distances from the sources both are waves of constant profile radiating out at the constant wave speed. Of course, to be energy conserving in two dimensions, the amplitude of the water waves falls off only as the square root of the radiation distance, but this difference is minor. A more important difference is that the water wave amplitude depends upon a time integral of the source strength, a result not found in three dimensions. Consequently, this amplitude has a dimensional dependence different from that of sound waves.

The shallow water equations are derived and written as an inhomogeneous wave equation in §2. In §3, this equation is solved in the sense of Curle (1955). That is, the radiation field is given explicitly in terms of a known distribution of surface quadrupoles and a line distribution of dipoles, whose strength involves the field quantity. The quadrupole terms are regarded as specified in a moving reference frame in §4. The convective effects are described and the lack of a distinct singularity at the Mach wave condition is accounted for as being an essential difference between the two- and three-dimensional theories. The paper is concluded with a brief summary of the similarities that exist between aerial sound waves and shallow water waves and discusses the possibility for effective similation of aerodynamically generated sound on a shallow water table.

2. Equations of Motion

Before the equations of motion can be derived, we must demonstrate that a particular depth exists at which surface tension effects render the water layer a practically non-dispersive medium. To do this, we examine the formula for the wave speed c of small amplitude waves of wave-number k. This is given by Milne-Thomson (1960, p.409) in the form

$$c^2 = \left(\frac{g}{k} + \frac{Sk}{\rho_w}\right)\tanh kh_o, \tag{2.1}$$

ρ_w is the density of the water, S the surface tension coefficient, and h_o the mean depth. For small kh_o, this formula can be expanded in a rapidly converging power series,

$$c^2 = gh_o + \left(\frac{S}{\rho_w h_o} - \frac{gh_o}{3}\right)(kh_o)^2 + O(kh_o)^4. \tag{2.2}$$

In the absence of surface tension, the variation of c^2 from its zero wave-number value gh_o is of order $(kh_o)^2$, but with surface tension, h_o can be chosen so that

$$\frac{S}{\rho_w h_o} - \frac{gh_o}{3}$$

is zero, leaving an error of order $(kh_o)^4$. For this critical depth, the wave speed is constant for all but the shortest waves, and a wave equation with constant c correctly describes the motion. For water at room temperature this critical depth is 0.48 cm, and all references to shallow water refer to this depth.

The appropriate formsof the equations of motion are derived by integrating the usual equations vertically through the water layer, whose variable depth is denoted by h . The water is assumed to be inviscid. The subscripts α , β, γ range over the values 1,2 (the two horizontal directions), and repeated subscripts imply a tensor summation over these values. Small letters denote quantities at a point, whereas capital letters denote the average value through the depth, e.g. the average velocity U_α is

$$U_\alpha = \frac{1}{h}\int_o^h u_\alpha dx_3 \ . \qquad (2.3)$$

Consider first the integrated continuity equation

$$\int_o^h \left(\frac{\partial u_\alpha}{\partial x_\alpha} + \frac{\partial u_3}{\partial x_3}\right) dx_3 = 0 \ ,$$

or

$$\frac{\partial}{\partial x_\alpha}\int_o^h u_\alpha dx_3 - \frac{\partial h}{\partial x_\alpha}\left[u_\alpha\right]_{x_3=h} + \left[u_3\right]_{x_3=h} = 0 \ , \qquad (2.4)$$

as $\left[u_3\right]_{x_3=0}$ is zero. $\left[u_3\right]_{x_3=h}$ is the particle velocity at the surface, so that it is equal to $[Dh/Dt]_{x_3=h}$, or

$$[u_3]_{x_3=h} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_\alpha}[u_\alpha]_{x_3=h} \ . \qquad (2.5)$$

These equations combine to give the mean form of the continuity equation,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_\alpha}(hU_\alpha) = 0 \ . \qquad (2.6)$$

We similarly integrate the α-component of the momentum equation

$$\int_0^h \left(\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + u_3 \frac{\partial u_\alpha}{\partial x_3}\right)dx_3 = -\int_0^h \frac{1}{\rho_w}\frac{\partial p}{\partial x_\alpha}\, dx_3,$$

which leads to the result,

$$\frac{\partial}{\partial t}(hU_\alpha) + \frac{\partial}{\partial x_\beta}(hU_\alpha U_\beta) = -\int_0^h \frac{1}{\rho_w}\frac{\partial p}{\partial x_\alpha}\, dx_3. \qquad (2.7)$$

$hU_\alpha U_\beta$ has been written for

$$\int_0^h u_\alpha u_\beta dx_3 \ ,$$

although notationally this is not strictly consistent. The vertical momentum equation is used to eliminate the pressure from (2.7). As

$$\frac{Du_3}{Dt} = -g - \frac{1}{\rho_W} \frac{\partial p}{\partial x_3} ,$$

then
$$\frac{1}{\rho_W}(p-p_s) = g(h-x_3)+\int_{x_3}^{h} \frac{Du_3}{Dt}\, dx'_3 , \qquad (2.8)$$

where p_s is the pressure in the water just below the surface, which, owing to the surface tension, differs from the atmospheric pressure p_a . This difference is given by the Laplace formula, and for surfaces which deviate only slightly from a plane (i.e. the amplitude of the displacement is small compared with the horizontal length scale), can be written (Landau & Lifshitz 1959, p.233)

$$p_a - p_s = S \frac{\partial^2 h}{\partial x_\alpha^2} . \qquad (2.9)$$

Substitution of (2.8) and (2.9) into (2.7) gives

$$\left.\begin{array}{l} \frac{\partial}{\partial t}(hU_\alpha)+ \frac{\partial}{\partial x_\beta}(hU_\alpha U_\beta) = - gh \frac{\partial h}{\partial x_\alpha} + \frac{Sh}{\rho_W} \frac{\partial^3 h}{\partial x_\alpha \partial x_\gamma \partial x_\gamma} - \frac{\partial B}{\partial x_\alpha} , \\ \text{in which} \quad B = \int_0^h dx_3 \int_0^h \frac{Du_3}{Dt}\, dx'_3 . \end{array}\right\} (2.10)$$

Now
$$h\frac{\partial^3 h}{\partial x_\alpha \partial x_\gamma \partial x_\gamma} = \frac{\partial}{\partial x_\alpha}\left(h\frac{\partial^2 h}{\partial x_\gamma^2}+ \frac{1}{2}\left(\frac{\partial h}{\partial x_\gamma}\right)^2\right) - \frac{\partial}{\partial x_\gamma}\left(\frac{\partial h}{\partial x_\alpha} \frac{\partial h}{\partial x_\gamma}\right) , \qquad (2.11)$$

which gives the momentum equation in the form

$$\frac{\partial}{\partial t}(hU_\alpha) = -\frac{\partial}{\partial x_\beta}(hU_\alpha U_\beta + P_{\alpha\beta}) ,$$

where $$P_{\alpha\beta} = \delta_{\alpha\beta}\left[\frac{gh^2}{2} + B - \frac{S}{\rho_w}\left\{h\frac{\partial^2 h}{\partial x_\alpha^2} + \frac{1}{2}\left(\frac{\partial h}{\partial x_\alpha}\right)^2\right\}\right] + \frac{S}{\rho_w}\frac{\partial h}{\partial x_\alpha}\frac{\partial h}{\partial x_\gamma}\delta_{\gamma\beta} \qquad (2.12)$$

Equations (2.6) and (2.12) are the required shallow water equations. The term hU_α can be eliminated by cross-differentiation, and after some rearrangement, this leads to the equation,

$$\left.\begin{aligned} \frac{\partial^2 h}{\partial t^2} - gh_o\frac{\partial^2 h}{\partial x_\alpha^2} &= \frac{\partial^2 T_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} , \\ \text{where} \quad T_{\alpha\beta} &= hU_\alpha U_\beta + P_{\alpha\beta} - gh_o h\delta_{\alpha\beta} . \end{aligned}\right\} \qquad (2.13)$$

This equation is the two-dimensional inhomogeneous wave equation, and governs the generation and propagation of shallow water waves. It can be shown that at points in the wave field (for which the wave speed is $(gh_o)^{\frac{1}{2}}$)$T_{\alpha\beta}$ is zero to second order in the wave amplitude provided that the non-dispersive depth is chosen. Then the linear terms in B and $(S/\rho_w)h(\partial^2h/\partial x_\alpha^2)$ exactly cancel. Also, in a region of convected turbulence, an order of magnitude analysis shows that the dominant term of $T_{\alpha\beta}$ is $hU_\alpha U_\beta$. The form of the shallow water equations, and these results about $T_{\alpha\beta}$, show a remarkable similarity to Lighthill's theory of sound, a similarity which might be exploited by modelling certain aerodynamic problems on a shallow layer of water. Accordingly, in the next section we seek a solution of (2.13) in exactly the same sense that the Lighthill-Curle (1955) equations represent a solution to the aerodynamic problem.

3. General theory

In the theory of aerodynamic sound developed by Lighthill (1952, 1954, 1962, 1963), the equations of motion of a compressible gas are written in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 , \qquad (3.1)$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_i}(\rho u_i u_j + p_{ij}) = 0 . \qquad (3.2)$$

ρ is the density, u_i the velocity component in the x_i direction, p_{ij} the compressive stress tensor, and the subscripts i,j range over the values 1, 2 and 3. Lighthill combined these equations to yield the three-dimensional inhomogeneous wave equation,

$$\left.\begin{aligned} \frac{\partial^2\rho}{\partial t^2} - a_0^2\,\frac{\partial^2\rho}{\partial x_i^2} &= \frac{\partial^2 T_{ij}}{\partial x_i\,\partial x_j} \;, \\ \text{where}\quad T_{ij} &= \rho u_i u_j + p_{ij} - a_0^2\rho\delta_{ij} \;, \end{aligned}\right\} \tag{3.3}$$

and a_0 is the speed of sound in the gas at rest. This equation governs the generation and propagation of sound waves; it shows how the sound is equivalent to that generated by a volume distribution of quadrupoles of strength density T_{ij}. The influence of solid boundaries upon the sound was investigated by Curle (1955), who used the standard Kirchhoff solution of equation (3.3) to show how surface stresses are acoustically equivalent to a surface distribution of acoustic dipoles. Although Curle's result was only derived for finite surfaces, its more general validity is easily established.

The situation for shallow water is very similar. We have already seen how the equations of motion can be written in the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_\alpha}(hU_\alpha) = 0 \;, \tag{3.4}$$

$$\frac{\partial}{\partial t}(hU_\alpha) + \frac{\partial}{\partial x_\alpha}(hU_\alpha U_\beta + P_{\alpha\beta}) = 0 \;, \tag{3.5}$$

which lead to the wave equation

$$\frac{\partial^2 h}{\partial t^2} - c^2 \frac{\partial^2 h}{\partial x_\alpha^2} = \frac{\partial^2 T_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} , \tag{3.6}$$

where $c^2 = gh_0$. To solve this equation, we observe that Lighthill's (1952) equation (3.3) reduces to it if ρ and T_{ij} are independent of the co-ordinate x_3 . Curle's general solution of equation (3.3) expresses ρ in terms of volume and surface integrals of T_{ij} and p_{ij} . Consequently, for ρ to be independent of x_3, T_{ij}, p_{ij} , and the geometrical situation, must all be independent of x_3 . We conclude that solutions of (3.6), in the presence of a contour Γ , are identical to the solutions of (3.3) in the presence of a cylinder S erected on Γ, and in which T_{ij} and p_{ij} do not depend on x_3 . In this situation, if V denotes the volume exterior to S , then Curle's solution becomes

$$4\pi c^2 (h(x_1,x_2,t)-h_0) = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \int_V \left[T_{\alpha\beta}\right] \frac{dV(\underline{y})}{R} + \frac{\partial}{\partial x_\alpha} \int_S \left[P_\alpha\right] \frac{dS(\underline{y})}{R} . \tag{3.7}$$

R is the distance of the field point $\underline{x}$ from the source point $\underline{y}$ and is given by

$$R^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2 . \tag{3.8}$$

The square brackets imply the integrand is to be evaluated at the retarded time $t' = t - (R/c)$. P_α is written for $\ell_\beta P_{\alpha\beta}$, ℓ_β being the direction cosines of the normal to Γ (and S) .

As the only dependence of the integrands upon y_3 is through the retarded time, the y_3 integration is effectively a time integration. Accordingly, y_3 is replaced by t' as the independent variable, and (3.7) becomes

$$2\pi c^2(h(\underline{x},t)-h_0) = \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\int dA \int_{-\infty}^{t-r/c} \frac{T_{\alpha\beta}(\underline{y},t')\,c\,dt'}{[c^2(t-t')^2-r^2]^{\frac{1}{2}}} + \frac{\partial}{\partial x_\alpha}\int d\Gamma \int_{-\infty}^{t-r/c} \frac{P_\alpha(\underline{y},t')\,c\,dt'}{[c^2(t-t')^2-r^2]^{\frac{1}{2}}} \quad (3.9)$$

Here r is the two-dimensional radiation distance, given by

$$r^2 = (x_1-y_1)^2 + (x_2-y_2)^2 , \quad (3.10)$$

and dA , $d\Gamma$ are two-dimensional area and line elements respectively. The contour integral is taken around Γ , and the area integral over the area external to Γ. This form of the solution has the advantage that no reference is made to the three-dimensional model used to generate it. This result can also be obtained from

Volterra's solution of equation (3.6) (see Jones 1964, p.42), but the given derivation is more straightforward.

Equation (3.9) is the fundamental result of the theory. It is an expression for the depth in terms of the known quantities $T_{\alpha\beta}$ and P_α, for any arbitrary fluid motion about a static solid surface. From the form of this equation, it follows that the waves are the same as those produced by quadrupole sources of strength density $T_{\alpha\beta}$, distributed over the region external to Γ plus dipole sources of strength P_α distributed around Γ. This equation also shows that the waves induced at any field point not only depend on the strength of the sources at a time r/c earlier, but on all previous times as well. On the other hand, the presence of the square root factor weights each contribution differently, and, since it is singular at $t' = t-r/c$, the main contribution comes from that region. In the far field, near these values of t', the square root can be approximated by $\{2r(ct-ct'-r)\}^{-\frac{1}{2}}$, and the time integral consequently yields an expression of the form $r^{-\frac{1}{2}}F(t-r/c)$. This implies that in the far field the waves from each source are waves of constant profile travelling at speed c, and whose amplitude falls off like $r^{-\frac{1}{2}}$. This type of behaviour is also found in linear theory for the conical wave field about a supersonic projectile (Whitham 1950). It is clearly so for sources of an oscillatory nature, where the method of stationary phase furnishes a precise form for F. We conclude that in the far field, water waves behave very similarly to sound waves, being produced by a similar distribution of dipoles and quadrupoles, and propagating in the same manner.

The result that the depth depends upon a time integral of $T_{\alpha\beta}$ or P_{α} merits further comment. It suggests that in any analysis featuring order of magnitude estimations, a typical time will be included. This does not usually happen in the three-dimensional theory. The typical time often varies with the parameters of the situation, and will result in a parametric dependence different from that obtained in three dimensions. Thus, some results of aerodynamic sound theory cannot be taken over directly to shallow water theory, but must be reconsidered in the light of (3.9). To illustrate this, the two-dimensional waves generated by a region of turbulence are examined in the next section. The corresponding three-dimensional theory is well known, being the basis of jet noise theory, so comparisons are easily made.

4. Two-dimensional waves generated by convected turbulence.

Before the two-dimensional theory of waves produced by convected turbulence is developed, it is worthwhile briefly discussing the three-dimensional theory. This has been developed by Lighthill (1952, 1954, 1962, 1963) and Ffowcs Williams (1963). The latter considered the problem of a jet aircraft flying at a Mach number N, emitting a turbulent exhaust whose eddies move at Mach number M. He found that the mean square density fluctuation observed in the far field varies as

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho^2} \ \frac{\ell^2}{R_0^2} M^7(M+N)|1+N\cos\phi|^{-1}|1-M\cos\theta|^{-5}. \quad (4.1)$$

R_0 is the mean distance from the observer to the turbulence, ℓ is the typical turbulence length scale, and θ , ϕ are angles specifying the direction of the convective motion. Along the lines $(1-M\cos\theta) = 0$, where the above result is not valid, a separate analysis gave the variation as

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho^2} \quad \frac{\ell^2}{R_0^2} M^2(M+N)|\ 1+N\ \cos\phi|^{-1}. \qquad (4.2)$$

By assuming a particular form for the unknown correlation function which arose in his integral, Ffowcs Williams was able to evaluate it exactly, and deduced that for all M, the density fluctuation varies as

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho^2} \quad \frac{\ell^2}{R_0^2} M^7(M+N)\ |1+N\ \cos\phi|^{-1}((1-M\ \cos\theta)^2+b^2M^2)^{-5/2}, \qquad (4.3)$$

b being a small numerical constant. However, the general validity of (4.3) is not easily established, requiring detailed appeal to the theory of generalized functions. Nevertheless, a two-dimensional result similar to (4.3) can be obtained from Ffowcs Williams's equations, without recourse to such methods.

In deriving this two-dimensional result, the following convention is adopted. Vectors denoted by capital letters are three-dimensional vectors, whereas small letters denote vectors in the two-dimensional plane $X_3 = 0$. $\underline{k}$ denotes the unit vector normal to this plane. In his three-dimensional theory, Ffowcs Williams considers a region of turbulence which convects through space at a velocity $-a_0\underline{N}$, and which is composed of eddies travelling at a velocity $+a_0\underline{M}$. His equation (1.29) shows that the leading term of the mean square density fluctuation observed in the far field is given by the expression

$$\overline{(\rho-\rho_0)^2}(\underline{X},t) \sim \frac{1}{16\pi^2 a_0^8}\iint \frac{(X_i-Y_i)(X_j-Y_j)(X_k-Y_k)(X_\ell-Y_\ell)}{\left|\,|\underline{X}-\underline{Y}| + \underline{N}.(\underline{X}-\underline{Y})\right| \; \left|\,|\underline{X}-\underline{Y}|-\underline{M}.(\underline{X}-\underline{Y})\right|^5}$$

$$\times \frac{\partial^4}{\partial\tau^4}P_{ijk\ell}\,(\underline{H},\underline{\Lambda},\tau)dV(\underline{H})dV(\underline{\Lambda}) \,. \quad (4.4)$$

$P_{ikj\ell}$ is the covariance of the stress tensor T_{ij}, $\underline{Y}$ is defined by the equation

$$\underline{Y} = \underline{H} - a_0\underline{M}t + \underline{N}|\underline{X}-\underline{Y}| \;, \qquad (4.5)$$

and the two volume integrals are to be taken over the turbulent region. The retarded time τ is defined in terms of the other variables, and will be discussed at a later stage.

To obtain the two-dimensional result, this expression is applied to the situation in which there is no variation of the strength and geometry of the sources with X_3 , and in which the vectors $\underline{N}$ and $\underline{M}$ are two-dimensional vectors $\underline{n}$ and $\underline{m}$. This use of the leading term of the three-dimensional expression to furnish the far field expression in two dimensions is valid as long as the predicted result is not zero. Because of the symmetry about the plane $X_3 = 0$, each volume integral is calculated over half the space, and the answer doubled. By introducing the simplifying notation $\underline{R} = (\underline{X}-\underline{Y})$, $\underline{r} = (\underline{x}-\underline{y})$, $R = |\underline{R}|$ and $r = |\underline{r}|$ (consequently $\underline{R} = \underline{r} + \underline{k}H_3$ and $R^2 = r^2 + H_3^2$) expression (4.4) becomes in shallow water terminology

$$\overline{(h-h_0)^2}(\underline{x},t) \sim \frac{1}{4\pi^2 c^8}\iint \frac{r_\alpha r_\beta r_\gamma r_\delta}{|R+\underline{n}.\underline{r}|\;|R-\underline{m}.\underline{r}|^5}\,\frac{\partial^4}{\partial\tau^4}P_{\alpha\beta\gamma\delta}(\underline{n},\underline{\lambda},\tau)$$

$$\times\, dA(\underline{n})dA(\underline{\lambda})dH_3 d\Lambda_3 \;, \quad (4.6)$$

the range of integration for H_3 and Λ_3 is $(0,\infty)$.

The expression for the retarded time τ , given by Ffowcs Williams in his equation (1.29), was derived on the assumption that the eddy size was small compared with the radiation distance. Clearly, this approximation is not valid in the two-dimensional situation, where the eddies are infinite cylinders. In this case, the exact expression for τ (given in his equation (1.12)) can be approximated by

$$\tau = \frac{\underline{\lambda}.\underline{r}+\frac{1}{2}(H_3^2-\Lambda_3^2)}{c\ |R-\underline{m}.\underline{r}|} . \qquad (4.7)$$

This expression is valid for all values of H_3 and Λ_3. The integrand only depends on Λ_3 through τ, and consequently the Λ_3 integral is effectively an integration over τ. Thus Λ_3 is replaced by τ as the independent variable, and expression (4.6) becomes,

$$\overline{(h-h_0)^2}(\underline{x},t) \sim \frac{1}{4\pi^2c^7}\iint \frac{r_\alpha r_\beta r_\gamma r_\delta}{|R+\underline{n}.\underline{r}|\ |R-\underline{m}.\underline{r}|^4}\frac{\partial^4}{\partial\tau^4}P_{\alpha\beta\gamma\delta}(\underline{n},\underline{\lambda},\tau)$$

$$\times \frac{dA(\underline{n})dA(\underline{\lambda})dH_3d\tau}{\{H_3^2-2[c\tau|R-\underline{m}.\underline{r}|\ -\underline{\lambda}.\underline{r}]\}^{\frac{1}{2}}} . \qquad (4.8)$$

The τ integral goes from $-\infty$ up to the zero of the square root.

A turbulent eddy is typically of spatial dimension ℓ and life-time ℓ/bcm, where $m = |\underline{m}|$ and b is a small numerical factor. As the covariance $P_{\alpha\beta\gamma\delta}(\underline{n}\ \underline{\mu},\tau)$ is negligible unless $\underline{\lambda}$ and τ are within these ranges, the introduction of the scaled variables

$$\underline{\mu} = \frac{\lambda}{\ell} \quad \text{and} \quad T = \frac{bcm}{\ell}\tau \qquad (4.9)$$

reveals the dependence of the integral upon these scales. Furthermore, because this scaling places the significant values of $P_{\alpha\beta\gamma\delta}(\underline{\eta},\underline{\lambda},\tau)$ within constant and equal ranges of $\underline{\mu}$ and T, the transformed function $P'_{\alpha\beta\gamma\delta}(\underline{\eta},\underline{\mu},T)$ does not retain any major distinction between the new axes. Consequently, it is possible to rotate these axes without complicating the task of estimating the magnitude of $P'_{\alpha\beta\gamma\delta}(\underline{\eta},\underline{\mu},T)$, and this freedom to rotate axes allows the integral to be further simplified. The zero of the square root defines a plane in the $(\underline{\mu},T)$ space, and the integration is to be carried out over the volume on one side of it. The rotated axes are chosen to be the normal to this plane, and any two axes parallel to it. The equation of the plane, in normal form, is

$$f^{-1}|R-\underline{m}.\underline{r}|\,T-f^{-1}bm\underline{\mu}.\underline{r} = f^{-1}\frac{bmH_3^2}{2\ell}, \qquad (4.10)$$

where f is the normalizing factor $\{|R-\underline{m}.\underline{r}|^2+b^2m^2r^2\}^{\frac{1}{2}}$. If the new axes are denoted by ζ_1,ζ_2,ζ_3; ζ_3 being the normal axis, then ζ_3 is given by

$$\zeta_3 = f^{-1}|R-\underline{m}.\underline{r}|\,T-f^{-1}bm\underline{\mu}.\underline{r} . \qquad (4.11)$$

The derivatives are related by

$$\frac{\partial}{\partial T} = a_1 \frac{\partial}{\partial \zeta_1} + a_2 \frac{\partial}{\partial \zeta_2} + f^{-1} \left| R - \underline{m}.\underline{r} \right| \frac{\partial}{\partial \zeta_3} \tag{4.12}$$

a_1, a_2 being factors determined by the particular choice of ζ_1 and ζ_2. Knowledge of these is not necessary, since they vanish after integration over ζ_1 and ζ_2. Equation (4.8) reduces to the simpler form,

$$\overline{(h-h_0)^2}(\underline{x},t) \sim \frac{b^3 m^3}{4\pi^2 c^4 \ell} \iint \frac{r_\alpha r_\beta r_\gamma r_\delta}{|R+\underline{n}.\underline{r}|\ f^4} \frac{\partial^4 P'_{\alpha\beta\gamma\delta}}{\partial \zeta_3^4} (\underline{n},\underline{\zeta}) \frac{dA(\underline{n})dH_3 dV(\underline{\zeta})}{\left[H_3^2 - \frac{2\ell f}{bm}\zeta_3\right]^{\frac{1}{2}}} \tag{4.13}$$

For values of H_3 such that $H_3^2 >> 2\ell f/bm$, the square root in the denominator can be approximated by H_3. These values of H_3 contribute nothing to the integral, since after this approximation, the ζ_3 integral goes to zero. We conclude that only values of H_3 such that $H_3^2 \sim 2\ell f/bm$ contribute to the integral. For large values of r, this contribution range only increases as $(\ell r)^{\frac{1}{2}}$, and as $R^2 = r^2 + H_3^2$, it follows that R can be approximated by r in the far field. If H_3 is scaled by the factor $2\ell f^{\frac{1}{2}}/bm$ to standardize its contribution range, then (4.13) becomes

$$\overline{(h-h_0)^2}(\underline{x},t) \sim \frac{b^3m^3}{4\pi^2c^4\ell} \iint \frac{r_\alpha r_\beta r_\gamma r_\delta}{|r+\underline{n}.\underline{r}|((r-\underline{m}.\underline{r})^2+b^2m^2r^2)^2} \frac{\partial^4 P'_{\alpha\beta\gamma\delta}}{\partial\zeta_3^4}(\underline{n},\underline{\zeta})$$

$$\times \frac{dA(\underline{n})dH_3'dV(\underline{\zeta})}{(H_3'^2-\zeta_3)^{\frac{1}{2}}} . \qquad (4.14)$$

The parametric variation of $\overline{(h-h_0)^2}$ quickly follows from this equation. As $P'_{\alpha\beta\gamma\delta}(\underline{n},\underline{\zeta})$ is a mean square of the tensor $T_{\alpha\beta}$, it varies as $h^2c^4m^4$. The $\underline{n}$ area integral yields a typical source area, which varies as $\ell^2([m+n]/m)$. If $\underline{m}.\underline{r}$ and $\underline{n}.\underline{r}$ are written as $mr\cos\theta$ and $nr\cos\phi$ respectively, then the two-dimensional result equivalent to (4.3) is

$$\overline{(h-h_0)^2} \sim \overline{h^2}\,\frac{\ell}{r_0}\,m^6(m+n)|1+n\cos\phi|^{-1}((1-m\cos\theta)^2+b^2m^2)^{-2} . \qquad (4.15)$$

This result differs from the three-dimensional result (4.3) in two respects. First, the inverse square law of sound intensity is replaced by a first power law, as is to be expected upon considerations of energy. Secondly, the 'Lighthill eight power law' is here replaced by the seventh power law, coupled with a corresponding change in the directional factor. This result has also been found by Obermeier (1967) in his study of two-dimensional aerodynamic sound. This is entirely due to the infinite length of the eddies, which makes retarded time differences crucial in determining the effective volume of each eddy. This contrasts with the sound theory, where such time differences are usually unimportant in this respect.

5. Conclusion

The main conclusion to be drawn from this analysis is that qualitatively, water waves behave in a very similar manner to sound waves. Both radiate out from their sources at a constant speed, preserving their profile in the far field. Furthermore, they are generated by the same equivalent system of dipoles and quadrupoles. As a consequence of this, for both types of waves, a region of turbulence generates a directional field, whose intensity varies as a high power of the eddy convective speed. The analysis shows that quantitative results are slightly different for the shallow water waves, the intensity increasing with the seventh power of flow velocity and the fourth power of the Doppler factor $(1-m\cos\theta)^{-1}$. Though a peak is found at the Mach wave condition, no singularity is evident, even in the first approximation which is then valid at all speeds. The similarity of this result with recent developments in the theory of aerodynamic sound generation leads to the possibility that turbulence generated shallow water waves can form a satisfactory and easily visualized simulation of aerodynamic noise problems of a rather intractible kind. Indeed some experiments have already been attempted and the qualitative similarity with the aerodynamic problem is very evident. That there should also be a means of making the similarity quantitative is the main outcome of this work, though it should be emphasized that certain properties of the two-dimensional wave field distort any complete analogy with the three-dimensional problem.

CHAPTER 3. Theory relating to the Noise of Rotating Machinery

Summary

This paper discusses and extends the theory of sound generation by multibladed single stage fans operating in a free field. The results would also be applicable to shrouded fans provided that the shroud dimensions are small compared with the acoustic wavelength. The main point advanced is that it is inappropriate to regard the sound generation question as a boundary value problem governed by the homogeneous wave equation. Inhomogeneities of the equation caused by a finite velocity field in the vicinity of the fan induced a quadrupole distribution whose effect is studied. It is concluded that this effect is negligible only for low speed few bladed fans in the first harmonics. For multibladed high speed fans the quadrupole effects are important; both through the potential and turbulent velocity fields. In the absence of turbulence the inhomogeneous potential field may generate more sound than does the rotation of the steady blade loads, whereas the presence of turbulence provides a mechanism by which

the potential field around the fan is scattered as sound. The theory of this mechanism is developed, and it becomes evident that it is vastly more important in generating the blade passage frequency sound heard near the axis of single stage fans than any mechanism so far suggested. The paper then goes on to develop the theory of fan noise radiation at frequencies not necessarily related to the blade passage frequency, and is concluded by a formula relating the power spectral density of the fluctuating forces on a rotating blade to the spectral description of the radiation field.

1. Introduction

The earliest theoretical study of the noise generated by rotating machinery was the work of Gutin (1936), who analysed the sound produced by a two-bladed aeroplane propeller. He discovered that the forces exerted by the propeller on the surrounding air generate the sound; they are equivalent to acoustic dipoles. Consequently, Gutin studied the following model. The air is subjected to forces distributed over the disc swept out by the propeller; each point on the disc experiences a constant thrust and torque when a blade passes that point, and no forces at other times. This system of forces is Fourier analysed, and the Fourier components of the sound field obtained. This model essentially yields an analysis of the sound produced by a force, or acoustic dipole, of constant absolute strength rotating in a circle. The sound is composed of a series of discrete tones, whose frequencies are multiples of the blade passage frequency. Both in quality and quantity the sound predicted by this theory is in agreement with the experimental evidence of Gutin's time.

The Gutin model has survived to the present time with very little change. Garrick and Watkins (1954) have extended Gutin's analysis to account for the forward motion of the propeller, and Lowson (1965), using more modern analytical techniques, has re-obtained Gutin's results, and also extended them to helicopter main rotor noise. On the other hand, the nature of the sound produced by modern fans and axial compressors, with their large number of blades and higher rotational speeds, bears very little resemblance to Gutin's model. The noise generated by modern rotating machinery is composed of broad band noise distributed over a very wide spectrum, plus a series of superimposed discrete tones at multiples of the blade passage frequency, and sometimes of the lower disc rotation frequency. The intensity of the broad band noise tends to follow a sixth power law of some typical velocity, and this clearly indicates that it is generated by random fluctuations in blade forces. These in turn are caused by such factors as the irregular shedding of vortices at the blade trailing edge (Lilley, 1961; Bragg and Bridge, 1964), and the interaction of the blade with patches of turbulence in the oncoming stream. Gutin's analysis is clearly inapplicable to such situations, and to date no extension of his theory has been made that derives the general features of the noise from the random forces on a rotating blade. The discrete tones are usually taken to be generated by the mean force operating as in Gutin's model and by periodic inhomogeneities in the flow field, but the theoretically predicted level of these tones for single stage fans usually falls short of experimental observation.

Both the work of Lilley (1961) and Sharland (1964) has gone a long way to close the gap between theory and experiment, and Hulse et al (1966) have made a major contribution in recognising that imperfect propagation of the sound can account for substantial amplification of the blade passage frequency sound. None the less, there exists no rigorous theoretical treatment which is compatible with experimental evidence, and an attempt to close this gap is described below. In § 2 it is argued that acoustic quadrupoles not considered in the Gutin model may be important as sound sources, and their effect upon discrete tone generation is discussed in § 3 and § 4. Finally, § 5 is devoted to developing the formulae necessary to analyse the spectrum of a rotating multipole of arbitrary strength.

2. Sound sources in many bladed rotors

The work of Gutin leads to an emphasis being placed on the distribution of pressure over the blade surface as being the prime source of propeller noise. The fluid is always treated as a perfect acoustical medium, propagating the pressure fluctuations generated by the blade as sound waves. The mathematical formulation of this outlook is to assume that the pressure satisfies the homogeneous wave equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \frac{\partial^2 p}{\partial x_i^2} = 0 \, , \qquad (2.1)$$

and is to be solved for in a region exterior to the blade, in terms of the given surface pressure distribution. As a mathematical boundary value problem, this is associated with the name of Kirchhoff, but as a description of propeller noise generation, it is natural to call this the Gutin formulation. Of course, Gutin did not set up the problem in this way, nor has it yet been tackled from this viewpoint, possibly because no correct solution of the Kirchhoff problem with moving boundaries has been published.

In reality, however, a fluid only behaves as a perfect acoustical medium if the fluid velocity is everywhere small compared with the speed of sound. Since surfaces moving at speeds comparable with the speed of sound will induce fluid velocities of the same magnitude, it is not valid to treat the fluid as a perfect acoustical medium. A more exact specification must be sought. Such a specification is supplied by Lighthill's work on aerodynamic noise (Lighthill 1952, 1954). This shows that a correct equation describing the behaviour of the fluid is the forced wave equation for the density,

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} , \tag{2.2}$$

where $T_{ij} = \rho u_i u_j + p_{ij} - c^2 \rho \delta_{ij}$.

This equation means that the real fluid behaves like an ideal acoustical medium, which contains quadrupole sound sources of strength density T_{ij} . Thus Gutin's specification of the fluid is incomplete, but it preceded Lighthill's formulation by 20 years. These extra quadrupoles, which represent the sound generated by the fluid flow around the blade, are the only major difference between the two formulations. For although the two governing equations also differ in that one determines the pressure and the other the density, for an ideal fluid these are simply related by $\Delta p = c^2 \Delta \rho$ and so in Gutin's formulation the pressure could be replaced by the density.

From the mathematical point of view, the solution of Lighthill's equation consists of a particular integral plus a complementary function. The particular integral represents the sound generated by the distribution of quadrupole sources, and the complementary function represents the effects of the boundaries, and is identical with the solution obtained from Gutin's formulation. His work implies that the effect of the boundaries is equivalent to a surface distribution of acoustic dipoles whose strength equals the force P_i exerted on the fluid by the surface element. This is precisely the result obtained by Curle (1955) from a formal solution of Lighthill's equation, and although his result is only valid for stationary surfaces, unpublished work of the authors shows it to remain true when the surfaces are in motion; the boundaries are now equivalent to a moving distribution of dipoles. To summarise, theoretical considerations show that the sound produced by a propeller or fan can be regarded as being generated by quadrupoles of strength density T_{ij} distributed throughout the volume exterior to the blade, plus dipoles of strength density P_i distributed over the blade surface; theoretical work to date has ignored the quadrupole sources, and as will be shown, has consequently missed what seems to be the most efficient generator of discrete tone fan noise in many bladed machines.

It should be emphasised at this point that the statement that quadrupoles are less efficient than dipoles as sound sources cannot be invoked to dismiss the quadrupole sound; such an argument is only valid if a typical Mach number is small, but how small it must be in a given situation can only be determined after analysis. The quadrupole sound may depend upon other large factors which swamp the effect of Mach number. It cannot be assumed a priori that the sound produced by the quadrupoles is negligible compared with the dipole sound.

The importance of the quadrupoles as a source of broad band noise is difficult to assess. This is because broad band noise is generated by the random fluctuations of the sound sources, and no published analysis predicts the general characteristics of broad band noise from a model of fluctuating sources in a rotational motion. Consequently our discussion of this point must be restricted to a few general remarks. A dimensional analysis of stationary sources shows that fluctuating forces generate a sound intensity that varies as the sixth power of a typical velocity, whereas fluctuations in the quadrupole strength produces an eighth power variation. The work in § 5 suggests that such dimensional variations are also valid for rotating sources, and so the experimentally observed sixth power dependence indicates that the fluctuating forces are the dominant source of broad band noise. A possible explanation for this is that the flow induced around a blade by its motion is mainly laminar, the only significantly turbulent region being the blade wake. Thus, there is only a small distribution of quadrupoles with a randomly fluctuating strength, and the noise it generates is negligible compared with that produced by the force fluctuations. The presence of further rotors or stator rows increases the unsteadiness of the flow, but it is thought that such interactions between fixed and moving blade rows produce essentially periodic variations in the flow, and not a high level of genuine turbulence. Such periodic variations only contribute to the discrete tones. Thus, it appears likely that the quadrupoles are not a significant source of broad band noise.

3. Discrete tone generation by the potential flow field

The situation for the discrete tones is different. If the fluid temperature remains nearly constant throughout the motion, the quadrupole strength density T_{ij} can be approximated by the Reynolds stress $\rho u_i u_j$. The flow field set up by the blade motion has particle speeds near the blade surface of the same order as the blade speed U, and consequently the quadrupole source strength in this region is of order ρU^2. The quadrupoles are of this magnitude in a region surrounding the blade and rotating with it. Just as the mean force rotating in a circle produces pure tones, so does the mean Reynolds stress. Because the mean surface pressure is also of magnitude ρU^2, the essential difference between the two terms is contained in their dipole-quadrupole character. To obtain an idea of their relative importance, a simplified situation will be studied.

In general, the sound field is generated by dipoles and quadrupoles distributed over the surface and throughout the volume respectively. The central simplification in this model is to assume that these distributed sources can be replaced by a single rotating point source whose strength is equal to the integrated strength of the distributed sources. Such an assumption is difficult to justify but without it any analysis is very difficult, and requires detailed knowledge of the flow around the blade. It is further assumed that the two types of source rotate in the same circle. Only the axial component of the force (the thrust) and the longitudinal quadrupole aligned along the axis will be considered. Expressions for the sound fields of rotating point dipoles and quadrupoles of constant strength have been given by Lowson (1965), and can be analysed into a Fourier series, whence the following expressions are obtained. For the far field sound pressure of a dipole, the absolute magnitude of the nth harmonic (based on <u>rotational</u> frequency) is given by

$$D_n = \frac{n\Omega xT}{2\pi cr^2} J_n\left(\frac{nMy}{r}\right) . \qquad (3.1)$$

Here Ω is the rotational angular velocity, and T is the dipole strength equal to the total thrust exerted by a single blade. M is the rotational Mach number $\frac{\Omega R}{c}$, R being the radius of the circle. x and y are components of the distance r from the centre of rotation to the observer; x is the axial component, and y is the component in a direction perpendicular to the axis lying in the plane defined by the axis and the observer. For a quadrupole of strength Q, which equals the integrated mean Reynolds stress, the equivalent harmonic content is

$$Q_n = \frac{n^2\Omega^2 x^2 Q}{2\pi c^2 r^3} J_n\left(\frac{nMy}{r}\right) . \tag{3.2}$$

If the blade span is s and its chord l, then a representative area is sl, and volume sl^2. The total thrust T can be expressed as a thrust coefficient C_T multiplied by $\rho U^2 sl$, and similarly Q is expressed as $C_Q \rho U^2 sl^2$. The coefficients C_T and C_Q are both of order one. For a rotor with B equally spaced blades, the sound pattern of each blade is out of phase with those of other blades, and all harmonics which are not multiples of B cancel. For the remaining harmonics $n = mB$, and the ratio of quadrupole and dipole components of the sound field is

$$\frac{mB\Omega l}{c} \frac{x}{r} \frac{C_Q}{C_T} = mBM \frac{l}{R} \frac{x}{r} \frac{C_Q}{C_T} . \tag{3.3}$$

Although this ratio contains the Mach number, it also contains mB , and for a system with a large number of blades, this ratio may be large, especially for the higher harmonics. As an illustration, let us compute the magnitude of this ratio for the propeller quoted by Gutin, and for a modern engine compressor. The figures are only tentative, as chord size never seems to be recorded, and the effective radius R is not known. It is assumed that C_Q/C_T if of order unity. For Gutin's two bladed propeller values of 0.2 for l/R and 0.5 for M seem reasonable; for an observer at an angle of 30° from the axis listening to the fundamental note, this ratio is 0.17. Thus for Gutin's propeller, it appears that the quadrupoles are negligible as sources of discrete tones below about the 10th harmonic. On the other hand, for a compressor with 40 blades, figures of 0.2 for l/R and 0.8 for M are possible, and then the ratio is about 5.5 for the fundamental and increases in direct proportion to the harmonic number. This suggests that the mean Reynolds stress may be an important source of discrete tones for many bladed compressors. However, it is to be emphasised that the relative importance of the two sources depends upon the observer's position, and factors such as C_Q/C_T whose magnitudes are largely unknown. Also, only a single component of each source has been analysed. Thus, the numbers derived above should only be taken to suggest that in many bladed machinery both the surface forces and the Reynolds stresses can act through the same mechanism of sound generation to comparable effect.

Although the foregoing analysis indicates that the mean Reynolds stress may be an important source of discrete tones, it suffers from the same deficiency as the mean force, namely the predicted harmonic content depends on the Bessel function $J_{mB}(\frac{mBMy}{r})$. These functions are extremely small when the order mB is large, especially near the axis of rotation; at 30° to the axis of a 40 bladed rotor it is of magnitude 10^{-13}. To illustrate the significance of this, let us calculate the acoustical efficiency η of the rotor, i.e., the fraction of the mechanical power supplied to the blades transmitted as sound power. The mechanical power W is the product of the drag force, the velocity, and the number of blades; $W = C_D \rho U^2 slUB$, where C_D is the drag coefficient and is of the order 10^{-1}. For the dipole tones, the total sound power $W\eta$ generated by all the blades is of order

$$W\eta = 4\pi c^2 \frac{p^2}{\rho c} = \frac{m^2 B^4 \Omega^2 \rho U^4 s^2 l^2}{\pi c^3} J^2_{mB} , \qquad (3.4)$$

and hence the acoustical efficiency is

$$\eta = \frac{m^2 B^3 M^3 sl}{\pi C_D R^2} J^2_{mB} . \qquad (3.5)$$

For the fundamental tone of a 40 blade system, ℓ is of the order 10^{-22} which is an impossibly low value. Thus for multibladed systems it appears that another generating mechanism is at work. The Bessel functions are an unavoidable consequence of a source rotating in a circle; they are the remains of a large cancellation that is imperfect as a result of retarded time differences. To avoid these functions, a source must be sought that fluctuates at the blade passage frequency, but which does not rotate in a circle.

4. Turbulence as a source of discrete tones

The realisation that the quadrupoles are important sound sources gives some scope for finding an alternative mechanism of discrete tone generation. The Reynolds stress quadrupoles are of strength density $\rho u_i u_j$ and distributed throughout the volume exterior to the blades. The flow through a rotor disc is principally a potential flow, but with a certain level of turbulence superimposed upon it. This turbulence may originate from upstream obstacles in the flow, or from the rotor blades themselves. To bring out this feature, the velocity u_i is split into a potential flow velocity U_i and a turbulence velocity v_i ; the quadrupole strength then becomes

$$\rho u_i u_j = \rho U_i U_j + \rho(U_i v_j + U_j v_i) + \rho v_i v_j \quad . \tag{4.1}$$

The first of these terms is the source effect of the potential field, and has already been discussed in the previous section. Similarly, the third term represents a purely turbulence source, and has also been mentioned, although its effect not properly assessed. However the second term represents an interaction between the potential and turbulent fields, and as such has not yet been mentioned as a possible source of sound. In fact, the potential velocity may be further divided into a convection velocity through the rotor, and a purely oscillatory velocity induced by the repeated passage of rotor blades. It is with the interaction between the oscillatory and turbulence velocity fields that this section is concerned.

The properties of an interaction source term such as $\rho U_i v_j$ are a combination of those of its two components, and are deduced as follows. The potential velocity U_i can be assumed to fluctuate essentially sinusoidally at the blade passage (radian) frequency ΩB . It is of the order of the blade speed U for a distance of the order of the blade chord l either side of the rotor disc. Beyond this region, the potential velocity, and hence the interaction source, is negligible. On the other hand, the turbulence velocity v_i is random in space and time, and must be treated stochastically. The turbulence can be regarded as being composed of a number of eddies; the velocity being well correlated throughout each eddy. In general, these well correlated regions convect downstream with a velocity U_c. This convection velocity is not necessarily the downstream fluid velocity, but is the velocity of a frame of reference in which the turbulence changes most slowly. The time taken for an eddy of dimension L to pass a fixed point is clearly LU_c^{-1}, and in this time, the number of blades N that rotate past that point is

$$N = \frac{L}{U_c} \frac{\Omega B}{2\pi} = \frac{B}{2\pi} \frac{L}{R} \frac{M}{M_c} . \qquad (4.2)$$

N represents the number of blades that chop through the eddy as it convects past the rotor. The magnitude of N depends upon various parameters, but it is instructive to proceed on the likely assumption that it is much greater than unity. The lifetime of an eddy is usually greater than the time it takes to convect past a given point LU_c^{-1}; the eddy travels more than its own diameter before losing its identity. Thus at a fixed point, the turbulence velocity changes in a time scale set by the convection time LU_c^{-1}, rather than the eddy lifetime, and so a typical (radian) frequency of this velocity is $2\pi U_c L^{-1}$. The condition that N is large then implies that the blade passage frequency is much greater than a typical turbulence frequency, and it follows that a product of the potential and turbulence velocities fluctuates at the higher blade passage frequency. Thus we see that the distributed interaction quadrupoles $\rho U_i v_j$ are of a significant magnitude in a region either side of the rotor, they can be grouped into coherent eddies, and fluctuate mainly at the blade passage frequency.

For a quadrupole source, the frequency spectrum observed in the far field is the spectrum of the quadrupole source strength multiplied by ω^4 . In our situation, the quadrupole strength is $\rho U_i v_j$, and as we have seen this fluctuates mainly at the blade passage frequency. In fact it can be shown that the spectrum of such a product, one term of which varies sinusoidally at frequency ΩB , is just the spectrum of the other component shifted by ΩB . Thus the width of the source spectrum is the width of the spectrum of its turbulence component, but its mean frequency is shifted to ΩB . As the width of the turbulence spectrum is of the order of its typical frequency $2\pi U_c L^{-1}$, the bandwidth $\omega^{-1}\Delta\omega$ of the source spectrum is

$$\frac{\Delta\omega}{\omega} = \frac{2\pi U_c}{L\Omega B} = \frac{1}{N} . \qquad (4.3)$$

Thus for large N , the source spectrum is very narrow, and centred on the blade passage frequency. The spectrum observed in the far field is this spectrum modified by ω^4 , which can be regarded as constant throughout the narrow band. Thus the observed spectrum is also very narrow, and this mechanism is essentially a source of discrete tones. On the other hand if N is not large, the source spectrum is broader, and is further distorted by the factor ω^4 ; the sound is then relatively broad band, the maximum bandwidth being of order unity. Thus the nature of the radiated sound depends upon the magnitude of N ; for large N the sound is blade passage tones, but for small N it is broad band noise. Clearly N increases with the number of blades, but it may require that M_c is small compared with M for it to be very large. This would occur if the rotor were not operating at its design condition.

To assess the intensity of the sound generated by this mechanism, we must estimate the magnitude of the basic integral describing the sound field of a quadrupole distribution. One possible simplification is to neglect the retarded time differences across the eddy, but this is a bad assumption for the following reason. Although the time taken for the sound to travel across the eddy is a small fraction of the eddy lifetime, it is a large fraction of the periodic time of the potential velocity. Thus although points on opposite sides of an eddy have velocities that are well correlated instantaneously, the sound they produce is significantly out of phase when it reaches the observer. That is, the eddies are no longer acoustically compact. In this situation, the effective volumetric scale is established by the eddy area and the distance travelled by a sound wave in a characteristic period (Ffowcs Williams 1963), and the best expression for the far field sound pressure of a non-compact quadrupole distribution is

$$p(t) = \frac{1}{4\pi}\int \left[\frac{\partial^2(\rho U_i v_j)}{\partial y_i \partial y_j}\right] \frac{dScd\tau}{r} . \qquad (4.4)$$

Here the square brackets indicate that the integral is to be evaluated over the surface of the sphere S, $r = c(t-\tau)$.

The magnitude of the sound pressure produced by one eddy can be estimated from this integral. It is assumed that the eddy dimension L is the same as the blade chord length l. In the region either side of the rotor, the potential velocity is of order U and the turbulent velocity αU. A space derivative is equivalent to division by the chord length l. The eddy area is l^2, and the relevant characteristic time is $2\pi(\Omega B)^{-1}$, so that the sound pressure produced by a single eddy is of magnitude

$$p = \frac{\rho U^2 \alpha}{1^2 4\pi r} \cdot \frac{1^2 c 2\pi}{\Omega B} = \frac{\rho U \alpha c R}{2 r B} . \qquad (4.5)$$

The eddy is a coherent source and so the sound it generates is the square of this expression. The total volume of the region under discussion is $2\pi Rsl$, which thus contains $2\pi Rsl^{-2}$ turbulent eddies. The total mean square sound pressure generated by all the eddies is

$$\overline{p^2} = \frac{\rho^2 U^2 \alpha^2 c^2 R^2}{4 r^2 B^2} \cdot \frac{2\pi R s}{1^2} = \frac{\pi \rho^2 U^2 \alpha^2 c^2 R^3 s}{2 B^2 r^2 1^2} . \qquad (4.6)$$

An acoustical efficiency for this mechanism can be calculated as before, and it reduces to

$$\eta = \frac{2\pi^2 \alpha^2}{C_D M B^3} \left(\frac{R}{l}\right)^3 . \qquad (4.7)$$

This equation is restricted to frequencies such that the wavelength of sound at the blade passage frequency is not significantly larger than the blade chord, so that it would be misleading to apply this result when both B and M are small. However, for high speed many bladed fans this formula is relevant. If the previously quoted values are used, and a one per cent turbulence level is assumed, this efficiency turns out to be of the order of 10^{-5}. This figure appears very reasonable, but rather on the high side, but again the analysis has been very crude, and could easily be in error by a factor of ten.

One conclusion from the above analysis is inevitable. In situations where N is large, this mechanism is far more likely to be the cause of discrete tones near the axis of multibladed machinery than any Gutin type of mechanism. Equation (4.6) also shows that the sound intensity generated by this mechanism, whether it be discrete tone or broad band, varies as the square of the rotational Mach number, instead of the usual sixth power variation valid at low rotational speeds. It is evident from the above discussion that this mechanism operates when the blades chop through the wakes of upstream obstacles. Such wake-chopping situations are known to be powerful sources of discrete tones, but it is seen that it is not necessary to seek unsteadiness in the blade forces as the sole cause of high amplitude discrete tones.

5. Radiation from rotating multipoles at frequencies not necessarily related to rotation frequencies

The generation and propagation of sound is governed by the inhomogeneous wave equation:

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \frac{\partial^2 \rho}{\partial x_i^2} = Q(\underline{x}, t) \; . \tag{5.1}$$

Here $Q(\underline{x},t)$ is the source strength density at $(\underline{x},t)$ which may or may not be concentrated in the form of a delta function singularity. In the general aerodynamic noise problem with boundaries, the sources may include simple, dipole and quadrupole elements, so that:

$$Q(\underline{x},t) = S(\underline{x},t) + \frac{\partial D_i(\underline{x},t)}{\partial x_i} + \frac{\partial^2 T_{ij}(\underline{x},t)}{\partial x_i \, \partial x_j} \; . \tag{5.2}$$

S, D_i and T_{ij} are respectively the strength density of the three source types. The solution of (5.1) in the distant radiation field can be written in either real time or spectral form:

$$\rho(\underline{x},t) = \frac{1}{4\pi c^2 r} \int_V Q(\underline{y}, \, t - \frac{r}{c}) d\underline{y} \; , \tag{5.3}$$

$$\rho(\underline{x},\omega) = \frac{e^{-2\pi i \omega r/c}}{4\pi c^2 r} Q(- \frac{\omega}{c} \hat{\underline{r}}, \; \omega) \; . \tag{5.4}$$

r is the distance separating source and observer and $\hat{\underline{r}}$ is the unit vector in the radiation direction. $\rho(\underline{x},\omega)$ is the generalised Fourier transform of $\rho(\underline{x},t)$:

$$\rho(\underline{x},\omega) = \int_{-\infty}^{\infty} \rho(\underline{x},t)\, e^{-2\pi i\omega t}\, dt\,, \tag{5.5}$$

and $Q(\underline{k},\omega)$ is the wave number, frequency, spectral component of the source field, being the four-dimensional Fourier transform of $Q(\underline{x},t)$;

$$Q(\underline{k},\omega) = \iint Q(\underline{x},t)\, e^{-2\pi i\underline{k}\cdot\underline{x}}\, e^{-2\pi i\omega t} d\underline{x}\, dt\,. \tag{5.6}$$

In terms of the spectral components of the multipole strength,

$$Q(\underline{k},\omega) = S(\underline{k},\omega) + 2\pi i k_i D_i(\underline{k},\omega) - (2\pi)^2 k_i k_j T_{ij}(\underline{k},\omega)\,, \tag{5.7}$$

and the particular component generating the acoustic wave at frequency ω propagating in the direction $\hat{\underline{r}}$ is:

$$Q(-\tfrac{\omega}{c}\hat{\underline{r}},\omega) = S(-\tfrac{\omega}{c}\hat{\underline{r}},\omega) - 2\pi i\,\tfrac{\omega}{c}\,\hat{r}_i D_i(-\tfrac{\omega}{c}\hat{\underline{r}},\omega) - (2\pi)^2\tfrac{\omega^2}{c^2}\,\hat{r}_i\hat{r}_j T_{ij} \times (-\tfrac{\omega}{c}\hat{\underline{r}},\omega)\,. \tag{5.8}$$

$\hat{r}_i$ is the component of the unit vector $\hat{\underline{r}}$ in the direction i.

The object of this section is to develop the theory of fan noise at frequencies which are not necessarily multiples of the blade passage frequency, and for this we can, without any loss of generality, regard the sources as concentrated at a point rotating in the $x_3 = 0$ plane at radius R with frequency Ω. Then $T_{ij}(\underline{x},t)$, $D_i(\underline{x},t)$, and $S(\underline{x},t)$ can be written symbolically as:

$$\left.\begin{matrix} T_{ij} \\ D_i \\ S \end{matrix}\right\} (\underline{x},t) = q(\phi,t)\ \delta(\underline{x} - \underline{R}) \ . \tag{5.9}$$

$q(\phi,t)$ is the time dependent strength of the source which occupied angular position ϕ from the x axis at time $t = 0$, and $\underline{R}$ is the position vector of that source at time t. From equation (5.6), the four dimensional Fourier transform of this source density is:

$$\left.\begin{matrix} T_{ij} \\ D_i \\ S \end{matrix}\right\} (\underline{k},\omega) = \int_{-\infty}^{\infty} q(\phi,t)\ e^{-2\pi i\underline{k}\cdot\underline{R}}\ e^{-2\pi i\omega t}\ dt \ . \tag{5.10}$$

Now $q(\phi,t)$ may be expressed in terms of its generalised Fourier transform (or spectral components) as,

$$q(\phi,t) = \int_{-\infty}^{\infty} q(\phi,\alpha)e^{2\pi i\alpha t}\ d\alpha \ , \tag{5.11}$$

and the radiating component of the source spectrum becomes:

$$\left.\begin{matrix} T_{ij} \\ D_i \\ S \end{matrix}\right\} \left(-\frac{\omega}{c}\,\hat{\underline{r}},\omega\right) = \iint_{-\infty}^{\infty} q(\phi,\alpha)\, e^{-2\pi i t(\omega-\alpha)}$$

$$e^{2\pi i \frac{\omega}{c} R \sin\theta \cos(2\pi\Omega t+\phi)} \, dt\, d\alpha \, . \qquad (5.12)$$

Here $\underline{k}.\underline{R}$ has been set equal to $-\frac{\omega}{c}\,\hat{\underline{r}}.\underline{R}$ which is $-\frac{\omega}{c} R \sin\theta \cos(2\pi\Omega t+\phi)$, θ being the angle at which the wave is propagating measured relative to the fan axis, which is $\theta = 0$.

The time (t) integration can now be performed, (Jones 1966, p.137),

$$\left.\begin{matrix} T_{ij} \\ D_i \\ S \end{matrix}\right\} \left(-\frac{\omega}{c}\,\hat{\underline{r}},\omega\right) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q(\phi,\alpha) e^{-in\frac{\pi}{2}}\, e^{+i\phi\left(\frac{\omega-\alpha}{\Omega}\right)} J_n$$

$$\times \left(-\frac{2\pi\omega R \sin\theta}{c}\right) \delta\left(\frac{\omega-\alpha}{\Omega} - n\right) \frac{d\alpha}{\Omega} \, ,$$

$$= \sum_{n=-\infty}^{\infty} q(\phi,\omega-n\Omega) e^{-in\left(\frac{\pi}{2}-\phi\right)} J_n$$

$$\times \left(-\frac{2\pi\omega R \sin\theta}{c}\right) \, , \qquad (5.13)$$

where J_n is the Bessel function of the first kind.

The Gutin theory is simply derived as a special case of this result. That deals with the field established by point sources of constant absolute strength. If we take by way of example the field induced in the plane $x_2 = 0$ by the torque dipole, then only D_1 need be retained. If D_1 is expressed as

$$D_1(\underline{x},t) = D \cos(2\pi\Omega t + \phi - \frac{\pi}{2})\, \delta(\underline{x}-\underline{R}) , \qquad (5.14)$$

then for this source

$$q(\phi,\alpha) = \frac{D}{2}\left\{ e^{i(\phi - \frac{\pi}{2})}\, \delta(\alpha-\Omega) + e^{-i(\phi - \frac{\pi}{2})}\, \delta(\alpha+\Omega) \right\}, \qquad (5.15)$$

and

$$D_1(-\frac{\omega}{c}\,\hat{\underline{r}},\omega) = \frac{D}{2}\sum_{n=-\infty}^{\infty} e^{-i(n+1)(\frac{\pi}{2} - \phi)}\, \delta(\omega-(n+1)\Omega) J_n \times (-\frac{2\pi\omega R \sin\theta}{c})$$

$$+ \frac{D}{2}\sum_{n=-\infty}^{\infty} e^{-i(n-1)(\frac{\pi}{2} - \phi)}\, \delta(\omega-(n-1)\Omega) J_n \times (-\frac{2\pi\omega R \sin\theta}{c}) , \qquad (5.16)$$

$$= -\frac{Dc}{2\pi\omega R \sin\theta}\sum_{n=-\infty}^{\infty} n e^{-in(\frac{\pi}{2} - \phi)}$$

$$\times\ \delta(\omega-n\Omega) J_n(-\frac{2\pi\omega R \sin\theta}{c}) . \qquad (5.17)$$

Equation (5.17) is obtained by re-ordering the summations in (5.16), and employing a well known Bessel function recurrence relation. This source field generates a radiation field given by equations (5.4) and (5.8), equal to:

$$\frac{e^{-2\pi i\omega r/c}}{4\pi c^2 r}\left(-\,2\pi i\,\frac{\omega}{c}\sin\theta\right)D_1\left(-\,\frac{\omega}{c}\,\hat{\underline{r}},\omega\right)\;, \tag{5.18}$$

so that the result for the Gutin theory becomes:

$$\rho(\underline{x},\omega) = \frac{e^{-2\pi i\omega r/c}}{4\pi c^2 r}\,\frac{iD}{R}\sum_{n=-\infty}^{\infty} n\, e^{-i\,n(\frac{\pi}{2}-\phi)}$$

$$\times\;\delta(\omega-n\Omega)J_n\left(-\,\frac{2\pi\omega R\sin\theta}{c}\right)\;. \tag{5.19}$$

This is the well-known result for the spectrum function, showing the spectrum of each dipole to be discrete at harmonics of the disc rotation frequency. If there are B identical sources spaced in an exactly regular array around a circle at angular intervals $2\pi/B$, then the total effect of all sources will involve the sum

$$\left.\begin{aligned}\sum_{\mathcal{S}=1}^{B} e^{i\mathcal{S}n2\pi/B} &= B\;; &\quad n &= mB\\ &= 0\;; &\quad n &\neq mB\;,\end{aligned}\right\} \tag{5.20}$$

where m is an integer. The magnitude of the spectral level generated by B sources then becomes:

$$\left| \rho(\underline{x},\omega) \right| = \left| \frac{D}{4\pi c^2 r} \frac{B^2}{R} \sum_{m=-\infty}^{\infty} m\delta(\omega - mB\Omega) J_{mB}\left(\frac{2\pi\omega R \sin\theta}{c}\right) \right| , \quad (5.21)$$

which is Gutin's discrete spectrum at harmonics of the blade passage frequency. Of course imperfect duplication of the source or blade spacing would nullify the above summation, and leave a finite level at harmonics of the disc rotation frequency.

We return now to the general case and give three formulae for the spectral components of the distant radiation field in terms of the spectral level of the source strength. The first is for a point monopole with spectrum $s(\phi,\alpha)$, the second for a point dipole with spectrum $d_i(\phi,\alpha)$ and the third for a point quadrupole with spectrum $t_{ij}(\phi,\alpha)$. By spectrum we mean of course the generalised Fourier transform of the time dependent source strength. In each case the point source is rotating in a circular path of radius R at frequency Ω and occupies angular position ϕ at time $t = 0$.

(1) Monopole:

$$\rho(\underline{x},\omega) = \frac{e^{-2\pi i\omega r/c}}{4\pi c^2 r} \sum_{n=-\infty}^{\infty} s(\phi,\omega - n\Omega) e^{-in(\frac{\pi}{2} - \phi)} J_n\left(-\frac{2\pi\omega R \sin\theta}{c}\right) ; \quad (5.22)$$

(2) Dipole:

$$\rho(\underline{x},\omega) = -\frac{e^{-2\pi i\omega r/c}}{4\pi c^2 r}\frac{2\pi i\omega}{c}\hat{r}_i \sum_{n=-\infty}^{\infty} d_i(\phi,\omega-n\Omega)e^{-in(\frac{\pi}{2}-\phi)}$$

$$\times J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right) ; \qquad (5.23)$$

(3) Quadrupole:

$$\rho(\underline{x},\omega) = -\frac{e^{-2\pi i\omega r/c}}{4\pi c^2 r}(2\pi)^2\frac{\omega^2}{c^2}\hat{r}_i\hat{r}_j \sum_{n=-\infty}^{\infty} t_{ij}(\phi,\omega-n\Omega)$$

$$\times e^{-in(\frac{\pi}{2}-\phi)} J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right) . \qquad (5.24)$$

All these forms involve the spectral level which is a generalised function that cannot be given a unique meaning. To make these expressions practically useful, one must form power spectral density relations to predict statistics of the radiation field in terms of statistics of the source strength. This is done by multiplying both sides of these equations by their conjugates, averaging, and normalising by a factor common to both sides whose magnitude need not be considered and can be taken as unity. We will deal particularly with the dipole equation to develop a formula relating the power spectral density of a time dependent dipole strength, such as might be induced by vortex shedding on a fan blade, to the power spectral density of the density fluctuation in the distant radiation field, $P(\underline{x},\omega)$.

$$P(\underline{x},\omega) = \frac{\omega^2}{4c^6r^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overline{d_r(\phi,\omega-n\Omega)d_r(\phi,-\omega+m\Omega)}\, e^{-i(n-m)(\frac{\pi}{2}-\phi)}$$

$$\times\, J_n\left(-\frac{2\pi\omega R \sin\theta}{c}\right) J_m\left(-\frac{2\pi\omega R \sin\theta}{c}\right) \cdot \quad (5.25)$$

Here d_r has been written for the dipole component in the particular radiation direction $\hat{r}_i d_i$. If we now assume that all Fourier elements are uncorrelated unless they are conjugates, and this is so in a stationary field, we can reduce the double sum to a single summation over n ;

$$\left.\begin{aligned} \overline{d_r(\phi,\alpha)d_r(\phi,\beta)} &= D_r(\phi,\alpha) \quad ; \quad \alpha = -\beta \\ &= 0 \qquad\qquad ; \quad \alpha \neq -\beta \end{aligned}\right\} \qquad (5.26)$$

where $D_r(\phi,\alpha)$ is the power spectral density of the dipole strength in the radiation direction. Then:

$$P(\underline{x},\omega) = \frac{\omega^2}{4c^6r^2} \sum_{n=-\infty}^{\infty} D_r(\phi,\omega-n\Omega) J_n^2\left(-\frac{2\pi\omega R \sin\theta}{c}\right) . \qquad (5.27)$$

This formula can be simplified in two special cases when the dipole spectral density is not a function of n and the summation can be carried out exactly. Firstly, if the source is not rotating, $\Omega = 0$ and the Bessel function sum is unity,

$$P(\underline{x},\omega)\Big|_{\Omega=0} = \frac{\omega^2}{4c^6r^2} D_r(\phi,\omega) \quad . \qquad (5.28)$$

This is the well known radiation field of a stationary point dipole. Secondly, if the dipole spectrum is flat; its variation is that of white noise. Again the Bessel functions sum is unity to give a radiation field power spectral density proportional to ω^2 :

$$P(\underline{x},\omega) = \frac{\omega^2}{4c^6r^2} D_r(\phi) \quad . \qquad (5.29)$$

This result is evidently unaffected by blade rotation.

In general no great simplification of the above result (5.27) is possible. It can be seen that a source at frequency α beats with harmonics of the disc rotation frequency $n\Omega$ to radiate at frequency $(\alpha+n\Omega)$. A dipole with a discrete spectrum radiates a discrete spectrum at these beat frequencies. A dipole with a narrow band spectrum radiates in narrow bands about these beat frequencies, so that no definite tone is heard. On the other hand a spectrum that is flat over a wide frequency range tends to the white noise case discussed above, where there is no hint of a tone. The most general conclusion therefore seems to be that the width of the spectral peaks which are radiated at beat frequencies are directly proportional to the bandwidth of the dipole frequency spectrum, and the strength of the narrow band content is inversely proportional to the spectral bandwidth.

CHAPTER 4. Sound Generation by Turbulence and Surfaces in Arbitrary Motion

Summary

The Lighthill-Curle theory of aerodynamic sound is extended to include arbitrary convective motion. The Kirchhoff description of a homogeneous wave field in terms of surface boundary conditions is also generalized to surfaces in arbitrary motion. The extension is at variance with the two previously published accounts of this problem which are erroneous. When both the bounding surfaces and the turbulence and compact relative to the radiated length scales, the turbulence is acoustically equivalent to a volume distribution of moving quadrupoles and the surfaces to dipole and monopole distributions. At low convective speed, their field increases as inverse powers of the Doppler factor $|1-M_r|$. Convective acceleration generally gives rise to new source terms at this condition. At the Mach wave condition when the Doppler factor is singular, both the turbulence and surfaces are non-compact and are acoustically equivalent to monopole distributions. Convective acceleration

then tends to limit the radiation. At this condition the surface sources are quite unrelated to the low speed sources, being second order in the field variable contrasting with the linear low speed terms. At high supersonic convective speeds, the field is dominated by an intensive beaming along the directions of Mach wave emission that lie normal to the surface. The magnitude of the field then varies inversely as the Gaussian surface curvature. If the surface has only single curvature the field is proportional to $r^{-1/2}$ and if it is locally plane at this condition, the field no longer decays with distance travelled. There are indications that the surface induced intensity increases as the square of surface speed at high supersonic speeds.

1. Introduction.

Lighthill (1952, 1954) has shown how the problem of aerodynamic sound can be posed as an acoustic analogy in which the turbulence provides a quadrupole distribution in an ideal atmosphere at rest. He described the general properties of the induced field and developed the dominant effect of steady low speed solenoidal source convection. The field increases as $|1-M_r|^{-3}$, M_r being the Mach number at which the source approaches the field point. Ffowcs Williams (1963) extended this theory to account for high speed steady solenoidal convection and showed that relatively intense but finite fields were radiated in the Mach wave direction where the Doppler factor, $|1-M_r|^{-1}$, is singular. Curle (1955) gave the general effect of static surfaces, showing them to be equivalent to surface dipole distributions. Lowson (1965) developed the theory of point multipoles in arbitrary convective motion inferring without proof that moving aerodynamic surfaces could be modelled as moving point dipoles. In this paper general expressions are developed for the equivalent sources of arbitrarily moving aerodynamic bounding surfaces with adjacent turbulent flow. It transpires that only at low subsonic speeds can surface effects be treated unambiguously by Lowson's model. This paper then goes on to discuss general features of the radiated field by source distributions in arbitrary motion and to discuss general features of the sound generated by turbulent flow around high speed aerodynamic surfaces.

Curle's (1955) extension of Lighthill's theory to account for surface effects made use of the Kirchhoff boundary solution to the homogeneous wave equation. In principle a theory accounting for surfaces in arbitrary motion might start with extensions of the Kirchhoff problem to moving surfaces published by Morgans (1930) and Kromov (1963). However these theories are mutually incompatible and both contain errors, Kromov's a fundamental error at an early stage in his analysis, and Morgans' a minor inconsistency at a later stage. A correct extension of the Kirchhoff problem is therefore not available as a starting point but is one of the results of this paper (equation 5.3).

The following analysis makes extensive appeal to generalized function theory. Generalized forms of the field variables are established to hold over a continuous infinite space. In that part of the space occupied by fluid they are equal to the real field variables, but in regions within the surfaces they have a well defined simple form. Discontinuities across the surfaces account for concentrated surface source distributions. In section 2, equations governing the generalized density fluctuation are arranged in an inhomogeneous wave equation of the Lighthill type, with quadrupole, dipole and monopole inhomogeneities, the latter two being concentrated on the bounding surfaces, if any.

Section 3 describes four alternative descriptions of the radiation field of arbitrarily moving multipole distributions. One of the forms (equation 3.21) is a generalization to a distribution of the result given by Lowson (1965) for a point source. However this particular form suffers from extreme interpretational difficulties in regions where $|1-M_r|^{-1} \to \infty$. It is a particular object of this paper to deal effectively with this Mach wave regime and for this the alternative forms, equations (3.22), (3.23) and (3.24), are far more appropriate; they contain no singularity at the Mach wave condition. These results are developed further in section 4 to represent surface multipole distributions in arbitrary motion, equation (4.6) being the general result. In this equation one singular point remains. The denominator of this equation vanishes if the surface approaches the field point along its normal at exactly sonic speeds. This condition is a special condition treated separately in section 7 where it is shown that an intense beaming of sound can then arise. Section 5 is devoted to a statement of the formal general solutions of the Kirchhoff and aerodynamic sound problems that emerge from an application of the foregoing analysis.

The formal solution is discussed in section 6 insofar as it affects the field generated by the distributed volume sources. The previous results of the Lighthill theory emerge as special cases as does the extension to cover high Mach number convection. Unsteadiness of the convective field is shown to induce additional source terms but also a moderating influence on the Mach wave field. Equation (6.12) gives a continuously valid general expression for the parametric dependence of the density field, and is identical to that obtained for special models of the flow by Ribner (1962) and Ffowcs Williams (1963). Equations (6.14) and (6.15) give the analogous expressions when the field point is close to the source distribution and when the source is undergoing convective acceleration.

Section 7 is devoted to a discussion of the general effects of moving surfaces. At low speed when the sources are all compact they are equivalent to quadrupoles distributed on the internal volume together with a dipole of strength equal to the applied force plus the displaced inertia, a result shown explicitly in (7.4). At high speeds when the sources are no longer compact the situation is very different, the appropriate expression then being equation (7.16), though for ease of demonstration, this is a particular form for steady solenoidal convective motion. The significant feature of this result is that the important source terms at high speed are completely divorced from those dominant at low speed. The surface effect is entirely controlled by the viscous term and by a Lighthill's turbulence stress tensor. The one remaining singularity of that equation is then discussed to show how the surface curvature plays a crucial role in the high speed problem and how an intense beam of sound can be radiated from high speed surfaces.

Finally the paper is concluded with some qualitative implications of the theory to the question of sound generation by high speed aerodynamic machines.

2. Derivation of the Governing Equations.

The theory of aerodynamic sound is built upon the equations of mass and momentum conservation of a compressible fluid. These equations are valid in the region exterior to any closed internal surfaces that may be present, and can be combined to give an inhomogeneous wave equation governing the generation and propagation of sound waves in that region. (Lighthill 1952, Curle 1955). Such a situation is essentially inhomogeneous in space, in that these equations are valid in the volume outside the surfaces, but are meaningless elsewhere. However, spatial homogeneity can be restored if this situation is abandoned in favour of the following one. An unbounded fluid is envisaged, but one which is partitioned into regions by mathematical surfaces that exactly correspond to the real surfaces. The motion of the new fluid on and outside the mathematical surfaces is defined to be completely identical with the real motion, whereas the interior flow can be specified arbitrarily. Thus the original situation is embedded in a more general one, and any problem in the real fluid is matched by one in the unbounded fluid, their respective solutions being identical in the exterior region. The interior motion is usually assumed to be very simple, and consequently does not match the exterior flow at the boundaries. Mass and momentum sources have to be introduced to maintain these discontinuities, and these ultimately act as sound generators. The equations governing the unbounded fluid are then conservation equations with sources, and are valid everywhere in space, thus restoring homogeneity to the problem. The mathematical description of the unbounded fluid is aided by the use of generalised functions (Jones 1966), which enable the discontinuities to be handled quite simply. The new mass and momentum equations for a fluid with discontinuities are derived by an extension of the usual techniques.

Consider a fixed volume of fluid V enclosed by a surface Σ. Suppose V is divided into regions 1 and 2 by a surface of discontinuity S encroaching on region 2 with velocity $\underline{v}$. S may consist of several closed surfaces. Let $\underline{\ell}$ be the outward normal from V, and let $\underline{n}$ be normal to S going from region 1 to region 2. The superscripts 1 and 2 refer to the two regions, and an overbar implies that the variable is to be regarded as a generalised function valid throughout V, e.g. $\overline{\rho}$ is equal to $\rho^{(1)}$ in $V^{(1)}$ and $\rho^{(2)}$ in $V^{(2)}$. If ρ represents the fluid density, then the rate of change of mass within V is,

$$\frac{\partial}{\partial t}\int_V \overline{\rho}\, dV = \frac{\partial}{\partial t}\int_{V^{(1)}} \rho^{(1)}\, dV + \frac{\partial}{\partial t}\int_{V^{(2)}} \rho^{(2)}\, dV . \tag{2.1}$$

The two regions have a moving boundary S, so that for each region

$$\frac{\partial}{\partial t} \int_{V^{(1)}} \rho^{(1)} dV = -\int_{\Sigma^{(1)}} (\rho u_i)^{(1)} \ell_i d\Sigma - \int_S [\rho(u_i - v_i)]^{(1)} n_i dS \, , \tag{2.2}$$

where u_i is the component of the fluid velocity in the direction x_i $(i = 1, 2, 3)$, and a repeated suffix implies a summation over these values. Hence the rate of change of the total mass within V is

$$\frac{\partial}{\partial t} \int_V \overline{\rho} dV = -\int_{\Sigma} (\overline{\rho u_i}) \, \ell_i \, d\Sigma + \int_S [\rho(u_i - v_i)]_{(1)}^{(2)} n_i dS \, , \tag{2.3}$$

the symbol $[\quad]_{(1)}^{(2)}$ meaning the difference of the contents between regions 2 and 1. By applying the divergence theorem, equation (2.3) can be written

$$\int_V \left(\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\overline{\rho u_i})\right) dV = \int_S [\rho(u_i - v_i)]_{(1)}^{(2)} \, n_i \, dS \, . \tag{2.4}$$

If an equation of the form $f = 0$ defines the surface S, and is such that $f < 0$ in region 1 and $f > 0$ in region 2, then a surface integral over S can be replaced by a volume integral over V with the integrand multiplied by the generalised function $\delta(f) \left\{ \left(\frac{\partial f}{\partial x_j}\right)^2 \right\}^{1/2}$ (see § 3). Here $\delta(f)$ is the one dimensional delta function, which is zero everywhere except where $f = 0$. Now, $n_i \left\{ \left(\frac{\partial f}{\partial x_j}\right)^2 \right\}^{1/2}$ is equal to $\frac{\partial f}{\partial x_i}$, so that (2.4) leads to the generalised mass equation

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{\rho u_i}) = [\rho(u_i - v_i)]_{(1)}^{(2)} \delta(f) \frac{\partial f}{\partial x_i} . \qquad (2.5)$$

This equation implies that, as far as mass conservation is concerned, to maintain the unbounded fluid in its defined state, a shell distribution of mass sources is required, whose strength is the difference between the mass flux requirements of each region. In the same manner, the generalised momentum equation can be deduced,

$$\frac{\partial}{\partial t}(\overline{\rho u_i}) + \frac{\partial}{\partial x_j}(\overline{\rho u_i u_j} + \overline{p_{ij}}) = [p_{ij} + \rho u_i(u_j - v_j)]_{(1)}^{(2)} \delta(f) \frac{\partial f}{\partial x_j} . \qquad (2.6)$$

Here p_{ij} is the compressive stress tensor. Equations (2.5) and (2.6) are the general forms of the equations governing the unbounded fluid, and are valid throughout space. If there are no discontinuities, the mass and momentum sources vanish, leaving the usual conservation equations. It is emphasised that the only restriction placed upon the surface S is one of smoothness, it can move in an arbitrary fashion, and change its shape or orientation.

To investigate an arbitrary sound field in the presence of a moving surface S, the superscript (1) is taken to refer to the region of the unbounded fluid that corresponds to the volume inside S . In this region, the fluid is assumed to be at rest with density ρ_o and pressure p_o . These values of density and pressure are those that would be found in the real fluid were it at rest. As the stress tensor p_{ij} has the same mean value $p_o\delta_{ij}$ in both regions, this constant vanishes from equation (2.6), and the symbol p_{ij} can be reinterpreted as the difference of the stress tensor from its mean value. The interior condition is then $p_{ij} = 0$. It is also assumed that in all practical situations, the surface S is impermeable, so that in the exterior region $u_n = v_n$. As $\underline{n}$ is defined to go from region 1 to region 2, it represents the outward normal from S . After replacing the interior variables by their assigned values, and dropping the superscript (2), the mass and momentum equations become

$$\left.\begin{aligned} \frac{\partial\overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{\rho u_i}) &= \rho_o v_i\ \delta(f)\frac{\partial f}{\partial x_i}, \\ \frac{\partial\,\overline{\rho u_i}}{\partial t} + \frac{\partial}{\partial x_j}(\overline{\rho u_i u_j} + \overline{p_{ij}}) &= p_{ij}\ \delta(f)\frac{\partial f}{\partial x_j}. \end{aligned}\right\} \quad (2.7)$$

To obtain the wave equation governing the generation and propagation of sound, $\overline{\rho u_i}$ is eliminated from equations (2.7), to give:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x_i^2}\right)(\overline{\rho-\rho_o}) = \frac{\partial^2\overline{T_{ij}}}{\partial x_i\partial x_j} - \frac{\partial}{\partial x_i}\left(p_{ij}\ \delta(f)\frac{\partial f}{\partial x_j}\right) + \frac{\partial}{\partial t}\left(\rho_o\ v_i\ \delta(f)\frac{\partial f}{\partial x_i}\right). \quad (2.8)$$

The dependent variable has been changed to the generalised density perturbation $\overline{\rho-\rho_0}$, as this is a measure of the sound amplitude. The generalised function $\overline{T_{ij}}$ is equal to Lighthill's stress tensor $T_{ij} = \rho u_i u_j + p_{ij} - c^2(\rho-\rho_0)\delta_{ij}$ outside any surfaces, and is zero within them.

Equation (2.8) shows that in general sound can be regarded as generated by three source distributions. The first of these is a distribution of acoustic quadrupoles of strength density T_{ij} distributed throughout the region exterior to the surfaces (Lighthill 1952). This is supplemented by surface distributions of acoustic dipoles of strength density $p_{ij}n_j$ (Curle 1955), and if the surfaces are moving, by further surface distributions of sources essentially monopole in character representing a volume displacement effect. It is to be emphasised that although these may not be the physical origin of the sound, they do completely specify the field. This equation remains true if shock discontinuities are present; these can be treated in the same way, but as mass and momentum fluxes are continuous across a shock, there are no extra sources to be included in (2.8). However, $\overline{T_{ij}}$, now contains discontinuities other than at physical boundary surfaces.

Equation (2.8) can be obtained by a second method. A generalised function is set up that equals the required function in the relevant region, and is zero elsewhere. Such a function is formed with the aid of Heavyside's unit function $H(f)$ defined to be unity where $f>0$ and zero where $f<0$. Thus if f is a function that is positive in the region of interest and negative elsewhere, the required generalised form of ψ is $\psi H(f)$. We use this form to solve the homogeneous scalar wave equation,

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x_i^2} = 0 \qquad (2.9)$$

in the presence of a moving surface. This problem is an extension of that solved by Kirchhoff, and will be referred to by that name. It can easily be shown by direct differentiation that $\psi H(f)$ satisfies the equation,

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x_j^2}\right)\psi H(f) = -c^2 \frac{\partial}{\partial x_i}\left(\psi \delta(f)\frac{\partial f}{\partial x_i}\right) - \frac{\partial}{\partial t}\left(\psi v_i \delta(f)\frac{\partial f}{\partial x_i}\right)$$

$$-\left(c^2 \frac{\partial \psi}{\partial x_i} + v_i \frac{\partial \psi}{\partial t}\right) \delta(f) \frac{\partial f}{\partial x_i} . \qquad (2.10)$$

Thus the Kirchhoff problem is converted into that of solving the generalised wave equation with shell sources. Knowledge of ψ, $\frac{\partial \psi}{\partial n}$ and $\frac{\partial \psi}{\partial t}$ on the surface completely defines the source distribution, and these can be assumed to be given as the boundary conditions of the original problem. The density fluctuation $\rho-\rho_0$ of a compressible fluid satisfies Lighthill's inhomogeneous wave equation, and so this latter process would lead to an equation similar to (2.10) with an extra source $H(f)\,\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$ on the right hand side. This equation is an alternative governing equation for the generation of aerodynamic sound. The formal equivalence of the two equations can be established if it is noted that $\overline{\rho-\rho_o}$ and $(\rho-\rho_o)H(f)$ represent the same generalised functions, and the ordinary mass and momentum equations are employed together with the condition that the surface is impermeable.

Of the two methods of deriving equation (2.8) the first is the most efficient and physically illuminating. However, the latter method shows the relationship between aerodynamic sound, and the scalar wave field of a moving surface. Clearly the only mathematical difference between these problems is the extra volume distribution of sources in aerodynamic sound generation. These sources are negligible if the fluid velocity is everywhere small, since T_{ij} is of second order in velocity. Consequently the two wave fields are essentially identical for a slowly moving surface. However, if the surface moves at high speed, its motion induces large fluid velocities, and these extra sources can no longer be neglected; the two wave fields are then different.

3. Field of a multipole distribution in arbitrary motion.

The basic equation to be solved is a generalised wave equation of the form

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x_i^2} = \frac{\partial^n Q_{ij..}}{\partial x_i \partial x_{j..}} , \qquad (3.1)$$

ϕ and $Q_{ij..}$ being generalised functions. This equation governs the field of a distribution of nth order multipoles of strength density $Q_{ij..}$. The solution is well known (Jones 1964 p.38)

$$\phi(\underline{x},t) = \frac{1}{4\pi c^2}\int_{-\infty}^{+\infty} \frac{\partial^n Q_{ij..}(\underline{y},\tau)}{\partial y_i \partial y_{j..}} \frac{\delta(\tau - t + \frac{r}{c})}{r} d\underline{y}\, d\tau , \qquad (3.2)$$

where $r = |\underline{x}-\underline{y}|$ is the distance from the source point $\underline{y}$ to the field point $\underline{x}$. A basic property of such convolution integrals is that derivatives can be interchanged, so that (3.2) can immediately be rewritten as

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int_{-\infty}^{+\infty} Q_{ij..}(\underline{y},\tau)\delta(\tau - t + \frac{r}{c}) \frac{d\underline{y}d\tau}{r} . \qquad (3.3)$$

The usual way of presenting this result is to perform the integration over τ that yields the familiar retarded time result

$$4\pi c^2\phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int Q_{ij..}(\underline{y},\ t - \frac{r}{c}) \frac{d\underline{y}}{r} , \qquad (3.4)$$

the integral being taken over all space.

So far it has been assumed that the source strength $Q_{ij..}$ is known as a function of the fixed coordinate system $\underline{y}$, corresponding to a view that the waves are generated by a system of stationary sources. However, it is often desirable to regard the sources as in motion, as this greatly simplifies the task of specifying the source strength. For example, the strength of the shell sources corresponding to a moving surface is clearly more easily specified in a co-ordinate system moving with the surface rather than in any fixed reference frame. To allow for this change of emphasis, Lagrangian coordinates $\underline{\eta}$ are introduced which move with the sources; the sources are then at rest in the $\underline{\eta}$ space. If the source convection velocity is written as $c\underline{M}$, the Lagrangian coordinates are related to the fixed system by the equation

$$\underline{y} = \underline{\eta} + \int^{\tau} c\underline{M}(\underline{\eta}, \tau')d\tau' . \qquad (3.5)$$

By writing the source strength as $q_{ij..}$ when referred to the $\underline{\eta}$ frame, the field integral (3.3) becomes

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int_{-\infty}^{+\infty} q_{ij..}(\underline{\eta},\tau)\delta(\tau - t + \frac{r}{c}) J \frac{d\underline{\eta}\, d\tau}{r} , \quad (3.6)$$

where r is now a function of τ ;

$$r = \left| \underline{x} - \underline{\eta} - \int^{\tau} c\underline{M}(\underline{\eta}, \tau')d\tau' \right| . \quad (3.7)$$

The Jacobian of the transformation J can be related to the convection velocity $c\underline{M}$ by the equation, (Aris 1962 p.84),

$$J = \exp \left\{ \int^{\tau} \mathrm{div}\ c\underline{M}(\underline{\eta}, \tau')\, d\tau' \right\} , \quad (3.8)$$

the divergence being taken with respect to the $\underline{y}$ frame.

The time variable τ is now replaced by a new length variable λ , the two being related by

$$\left. \frac{\partial \lambda}{\partial \tau} \right|_{\underline{\eta}} = c\alpha , \quad (3.9)$$

The reasons for this are twofold. Firstly, this unifies the dimensions of the field integral (3.6); and secondly, the arbitrary scaling factor α enables important space and time scales to be equated, considerably simplifying the interpretation of later results. This change yields the equation

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int_{-\infty}^{+\infty} q_{ij..}(\underline{\eta},\lambda)\delta(\tau(\underline{\eta},\lambda) - t + \frac{r}{c}) J \frac{d\underline{\eta}\, d\lambda}{rc\alpha} . \quad (3.10)$$

Although equation (3.3) is one expression for the radiation field, alternative expressions can be given if some results of generalised function theory are employed. These concern volume integrals containing delta functions, and are firstly (Jones 1966 p.262)

$$\int_{-\infty}^{+\infty} F(\underline{z})\ \delta(g(\underline{z}))d\underline{z} = \int_{\Sigma_k} \left\{ \frac{F}{\left|\frac{\partial g}{\partial z_k}\right|} \right\} (\underline{z}^*)d\underline{z}_k \ , \qquad (3.11)$$

where $\underline{z}^*$ is a point on the hypersurface Σ defined by $g(\underline{z}^*) = 0$; Σ_k is the projection of Σ on to the coordinate hyperplane with normal in the k direction, an element of which is denoted by $d\underline{z}_k$. Secondly (Jones 1966 p.263)

$$\int_{-\infty}^{+\infty} F(\underline{z})\delta(g(\underline{z}))d\underline{z} = \int_{\Sigma} \left\{ \frac{F}{|\text{grad } g|} \right\} (\underline{z}^*)\ d\Sigma \ , \qquad (3.12)$$

where $|\text{grad } g|^2 = (\frac{\partial g}{\partial z_k})^2$, and thirdly

$$\int_{-\infty}^{+\infty} F(\underline{z})\delta(g(\underline{z}))d\underline{z} = \sum_{k=1}^{m} \int_{\Sigma_k} \left\{ \frac{F\left|\frac{\partial g}{\partial z_k}\right|}{|\text{grad } g|^2} \right\} (z^*)d\underline{z}_k \ . \qquad (3.13)$$

In (3.11) and (3.13) repeated suffices are not implicitly summed. Expression (3.12) implies that a volume integral containing a delta function is the same as a surface integral modified by the factor $|\text{grad } g|^{-1}$, a result that would be obtained by working formally and utilising the well known property of the delta function

$$\int_{-\infty}^{+\infty} F(g)\,\delta(g)\,dg = F(o) . \qquad (3.14)$$

The other two results are alternative expressions for calculating this surface integral, by either projecting Σ on to one coordinate hyperplane, or by resolving it on to all the hyperplanes. The projection of Σ fails if it lies perpendicular to the chosen hyperplane, and corresponds to $\frac{\partial g}{\partial z_k}$ vanishing in equation (3.11). The other two expressions never fail for a smooth surface, because $|\text{grad } g|$ cannot then be zero. Although these three expressions yield identical values when integrated analytically, some may be more useful than others for estimating the magnitude of the radiated field (see § 5).

To utilise these expressions in the evaluation of the radiation integral (3.10), the relevant components of $|\text{grad } g|$ must be calculated. g in equation (3.10) represents $\tau - t + \frac{r}{c}$ so that

$$\left.\frac{\partial g}{\partial \lambda}\right|_{\underline{\eta}} = \left.\frac{\partial g}{\partial \tau}\right|_{\underline{\eta}} \left.\frac{\partial \tau}{\partial \lambda}\right|_{\underline{\eta}} = \frac{1}{c\alpha}\left(1 + \frac{1}{c}\left.\frac{\partial r}{\partial \tau}\right|_{\underline{\eta}}\right) . \qquad (3.15)$$

However, from equation (3.7) it follows that $\left.\frac{\partial r}{\partial \tau}\right|_{\eta}$ is $-cM_r$, M_r being the component of $\underline{M}$ in the direction of the radiation vector $\underline{r} = (\underline{x}-\underline{y})$. Thus

$$\frac{\partial g}{\partial \lambda} = \frac{1}{c\alpha}\left\{1-M_r\right\} . \tag{3.16}$$

Similarly,

$$\left.\frac{\partial g}{\partial \eta_i}\right|_{\lambda} = \frac{\partial g}{\partial \eta_i} + \left.\frac{\partial g}{\partial \tau}\right|_{\lambda} \left.\frac{\partial \tau}{\partial \eta_i}\right._{\lambda} = \frac{1}{c}\frac{\partial r}{\partial \eta_i} + (1-M_r)\left.\frac{\partial \tau}{\partial \eta_i}\right|_{\lambda} . \tag{3.17}$$

Again from equation (3.7) it follows that

$$\left.\frac{\partial r}{\partial \eta_i}\right|_{\tau} = -\frac{r_i}{r} - \frac{r_j}{r}\int^{\tau} c\,\frac{\partial M_j}{\partial \eta_i}\,d\tau' , \tag{3.18}$$

so that

$$\left.\begin{aligned} \left.\frac{\partial g}{\partial \eta_i}\right|_{\lambda} &= -\frac{1}{c}\left\{\frac{r_i}{r} + a_i\right\} , \\ \text{where}\quad a_i &= \frac{r_j}{r}\int^{\tau} c\,\frac{\partial M_j}{\partial \eta_i}\,d\tau' - c(1-M_r)\left.\frac{\partial \tau}{\partial \eta_i}\right|_{\lambda} . \end{aligned}\right\} \tag{3.19}$$

Hence

$$\left.\begin{aligned} |\,\text{grad } g\,| &= \frac{1}{c\alpha}\left\{(1-M_r)^2 + \alpha^2(1 + 2a_r + a_i^2)\right\}^{\frac{1}{2}} \\ &= \frac{1}{c\alpha}\left\{(1-M_r)^2 + \beta^2\right\}^{\frac{1}{2}} . \end{aligned}\right\} \tag{3.20}$$

Alternative expressions for the radiation integral can now be given. By choosing the k direction to coincide with the λ (time) axis, result (3.11) yields the expression,

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int \frac{q_{ij..}(\underline{\eta},\lambda^*)}{|1-M_r|} \frac{J}{r} d\underline{\eta} , \qquad (3.21)$$

λ^* being the value of λ corresponding to the retarded time $\tau^* = t - \frac{r}{c}$. The effect of source convection is clearly revealed in this result; convection effectively increases the source strength by a factor $|1-M_r|^{-1}$, the Jacobian J accounting for any divergence of the source during the motion. If other axes are chosen to coincide with the k direction, further expressions result which avoid the difficulties of the condition $|1-M_r| = 0$. For example, if the spatial component of the normal to Σ is chosen to coincide with the k direction, the radiation integral can be written as

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int \frac{q_{ij..}(\underline{\eta}^*,\lambda)}{\{1+2a_r+a_i^2\}^{1/2}} \frac{J}{r\alpha} d\underline{\eta}_k d\lambda , \qquad (3.22)$$

a form appropriate to the estimation of Mach wave generation by supersonic flows (Ffowcs Williams 1963). If result (3.12) is employed, this leads to the formula

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int \frac{q_{ij..}(\underline{\eta}^*,\lambda^*)}{\{(1-M_r)^2+\beta^2\}^{1/2}} \frac{J}{r} d\Sigma , \qquad (3.23)$$

whereas (3.13) gives the alternative result

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int \frac{|1-M_r|}{\{(1-M_r)^2+\beta^2\}} q_{ij..}(\underline{\eta},\lambda^*) \frac{J}{r} d\underline{\eta}$$

$$+ \sum_{k=1}^{3} \frac{\partial^n}{\partial x_i \partial x_{j..}} \int \frac{\alpha \left| \frac{r_k}{r} + a_k \right|}{\{(1-M_r)^2+\beta^2\}} q_{ij}(\eta_k^*,\lambda) \frac{J}{r} d\underline{\eta}_k d\lambda \ . \qquad (3.24)$$

These three results are equivalent expressions for the solution of equation (3.1), and represent the field of a distribution of nth order multipoles. Although they are valid for any generalised distribution of sources, when the sources are concentrated on a shell, $Q_{ij..}$ contains a further delta function which renders them of little practical use without further reduction.

4. Field of a surface multipole moving with arbitrary speed.

An inspection of the equations (2.8) and (2.10) reveals that the source strength corresponding to a shell distribution of multipoles is better written as $Q_{ij..}(\underline{y},\tau)|\text{grad}_{\underline{y}} f|\delta(f(\underline{y},\tau))$. In these equations $\frac{\partial f}{\partial y_i}$ can always be replaced by $|\text{grad}_{\underline{y}} f| n_i$. The subscript $\underline{y}$ is a reminder that the gradient is taken in the $\underline{y}$ frame, and does not include the τ component of f. The convection velocity $c\underline{M}$ of the surface sources is clearly the surface velocity $\underline{v}$, and it follows that f, regarded as a function of $\underline{\eta}$ is independent of τ. The early analysis of § 3 follows through unaltered, and the basic equation corresponding to (3.10) for a shell source is

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_j ..} \int q_{ij..}(\underline{\eta},\lambda)|\text{grad}_{\underline{y}} f|\delta(f)\delta(g) \frac{J}{rc\alpha} d\underline{\eta} d\lambda, \tag{4.1}$$

where g has again been written for $\tau - t + \frac{r}{c}$.

One way of reducing expression (4.1) is by repeated use of the general results (3.11) to (3.13). By employing the first formula with the k direction coinciding with the λ axis, the intermediate result is obtained,

$$4\pi c^2 \phi(\underline{x},t) = \frac{\partial^n}{\partial x_i \partial x_j ..} \int \frac{q_{ij..}(\underline{\eta},\lambda^*)}{r|1-M_r|} J|\text{grad}_y f|\delta(f)\, d\underline{\eta}. \tag{4.2}$$

The remaining delta function is unaffected by this operation as f is independent of λ. This form of the integral allows a further use of the general results, and if (3.12) is now employed, a result analogous to (3.21) is obtained:

$$4\pi c^2 \phi(\underline{x}, t) = \frac{\partial^n}{\partial x_i \partial x_j ..} \int \frac{q_{ij..}(\underline{\eta}, \lambda)}{r|1-M_r|} J \frac{|\mathrm{grad}_{\underline{y}} f|}{|\mathrm{grad}_{\underline{\eta}} f|} dS(\underline{\eta}) . \quad (4.3)$$

The factor $J|\mathrm{grad}_{\underline{y}} f||\mathrm{grad}_{\underline{\eta}} f|^{-1}$ appearing in this integral is no more than the ratio of the area elements of the surface S in the $\underline{y}$ and $\underline{\eta}$ spaces (just as J is the ratio of volume elements), and is denoted by A. This ratio can be related to the two dimensional, or surface, divergence of the convection velocity $c\underline{M}$, in the same way as J is related to its three dimensional divergence. Differential expressions for this surface divergence are given in works on differential geometry or tensor calculus (Weatherburn 1927, 1950). If the surface is unextended in the motion, A is unity. Thus again convection of a surface source effectively increases its strength by a factor $|1-M_r|^{-1}$, A accounting for any expansion of the surface.

However, a more general expression corresponding to (3.23) can be obtained. Just as a volume integral containing one delta function $\delta(g)$ is equivalent to an integral over the subspace $g = 0$, so an integral with two delta functions

$\delta(g)$ and $\delta(f)$ is equivalent to an integral over the subspace $g = 0$, $f = 0$ (Gelfand and Shilov 1964, p.239). If this subspace is denoted by σ, then it can be shown that the basic integral (4.1) can be written

$$4\pi c^2 \phi(\underline{x}, t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int_\sigma q_{ij..}(\underline{\eta}, \lambda) \, |\mathrm{grad}_{\underline{y}} f| \, \frac{J}{r\alpha c} \, \frac{d\sigma}{D} , \quad (4.4)$$

where $$D^2 = (\nabla g)^2 (\nabla f)^2 - (\nabla f . \nabla g)^2 . \quad (4.5)$$

The symbol ∇ is the usual gradient operator in the four dimensional $\underline{\eta}, \lambda$ space. If the four dimensional normals to the hypersurfaces $g = 0$ and $f = 0$ are inclined at an angle Θ, then D can be rewritten as $|\mathrm{grad}_{\underline{\eta}} f| \, |\mathrm{grad}_{\underline{\eta}} g| \sin \Theta$. Alternatively, as f does not depend upon λ, D takes the form $|\mathrm{grad}_{\underline{\eta}} f| \left\{ \left(\frac{\partial g}{\partial \lambda}\right)^2 + \left(\frac{\partial g}{\partial \eta_i}\right)^2 \sin^2\theta \right\}^{1/2}$, θ being the angle between the $\underline{\eta}$ components of ∇f and ∇g. Thus the alternative general expression for the field of a shell multipole is

$$4\pi c^2 \phi(\underline{x}, t) = \frac{\partial^n}{\partial x_i \partial x_{j..}} \int \frac{q_{ij..}(\underline{\eta} \, \lambda)}{\left\{(1-M_r)^2 + \beta^2 \sin^2\theta\right\}^{1/2}} \, \frac{A}{r} \, d\sigma . \quad (4.6)$$

Other expressions can be obtained if this integral over the two dimensional subspace σ is projected or resolved into integrals over other subspaces, as for the hypersurface Σ . Equation (4.3) could be obtained in this way. However, other such expressions are extremely complicated, and are only useful in simplified circumstances, and are not given here.

The basic expression (4.6) for shell sources differs in one very important respect from the corresponding result (3.26) for volume sources. The integral becomes singular at the condition $\sin \Theta = 0$. This occurs whenever the surface S moves towards the field point at the wave speed c with its normal parallel to the radiation direction. This condition corresponds exactly to that of specifying the boundary conditions on a characteristic of the original governing differential equation, and a failure in the solution is to be expected. Although such singularities seriously complicate the analysis of shell sources, undue emphasis should not be placed upon this difficulty, as there are many practical situations in which this condition does not occur, and as will be shown in § 7, the singularities are integrable in the aerodynamic sound problem.

5. General solutions of the governing equations.

The integral expressions for the radiated fields of arbitrary volume or surface multipoles developed in the previous sections can be used to write down the formal solutions of our two basic wave equations. Clearly these solutions can be given in many different forms, depending upon the exact formula chosen to evaluate each radiation integral. However, only those of general importance are recorded below. The first form of solution for equation (2.8) is obtained by utilising expressions (3.21) and (4.3) to reduce the various radiations integrals.

$$4\pi c^2(\rho(\underline{x},t)-\rho_o) = \frac{\partial^2}{\partial x_i \partial x_j}\int \left[\frac{T_{ij}\ J}{r\,|1-M_r|}\right] d\underline{\eta} - \frac{\partial}{\partial x_i}\int \left[\frac{p_{ij}n_j\ A}{r\,|1-M_r|}\right] dS(\underline{\eta})$$

$$+\frac{\partial}{\partial t}\int \left[\frac{\rho_o\ v_n}{r\,|1-M_r|}\right] dS(\underline{\eta})\ , \qquad (5.1)$$

where the square brackets imply that the contents are to be evaluated at the retarded time given implicitly by $\tau = t - \frac{r}{c}$. The volume integral is taken over the region exterior to the surfaces. Alternatively, if equations (3.23) and (4.6) are employed in the reduction of the radiation integrals, the following expression results;

$$4\pi c^2(\rho(\underline{x},t)-\rho_o) = \frac{\partial^2}{\partial x_i \partial x_j}\int \frac{\overline{T_{ij}}\ J}{\{(1-M_r)^2+\beta^2\}^{1/2}}\ \frac{d\Sigma}{r}$$

$$- \frac{\partial}{\partial x_i}\int \frac{p_{ij}n_j A}{\{(1-M_r)^2+\beta^2\sin^2\theta\}^{1/2}}\ \frac{d\sigma}{r}$$

$$+ \frac{\partial}{\partial t}\int \frac{\rho_o v_n A}{\{(1-M_r)^2+\beta^2\sin^2\theta\}^{1/2}}\ \frac{d\sigma}{r}\ . \qquad (5.2)$$

$\overline{T_{ij}}$ is zero at points corresponding to the inside of S.

These two expressions are important because they correspond to two different approaches to the generation of aerodynamic sound. In the first view, the spatial and temporal natures of the sources are treated as essentially different, regarding the sound as generated by a spatial distribution of time varying sources. Equation (5.1) is the solution incorporating this outlook. Any estimation of the radiated sound based on this equation clearly runs into serious difficulties whenever the factor $1-M_r$ vanishes, unless the equation is very carefully interpreted. Such an interpretation leads to the emphasis being transferred from the spatial to the temporal distribution of the sources (Ffowcs Williams 1963). These difficulties, however, stem from the separation of space and time, and disappear if such a distinction is abandoned in favour of a unified four dimensional approach that regards the sources as distributed throughout the whole of space-time. Equation (5.2) reveals that the emphasis is now always to be placed upon the distribution of sources over a hypersurface of variable orientation. The previously troublesome vanishing of $1-M_r$ now simply implies that the hypersurface lies parallel to the time axis, immediately showing the importance of the temporal distribution of the sources, and allowing an unambiguous estimate to be made of the radiated sound field. Although the unified approach may be unfamiliar, this initial disadvantage is outweighed by its ability to shed light on situations of considerable complexity when posed within the more familiar framework. This is illustrated in the following section. However, a word of caution is necessary concerning the unified solution (5.2). The reason for developing this solution is to have a form that allows easy estimation of the sound field around the Mach wave condition $(1-M_r) = 0$. Before such an estimation can be carried out analytically, the field point derivatives must always be taken under the integral sign, where they become equivalent to differential operators acting upon the integrand. If there are any surfaces present, these operators act upon the discontinuities in $\overline{T_{ij}}$ to produce delta function singularities, and ultimately further integrals over σ. Thus if the sound field of a rapidly moving surface is investigated with the aid of (5.2), the whole equation must be considered and not just the last two integrals. This phenomenon does not occur with equation (5.1), since the extra σ integrals vanish when Σ coincides with the $\underline{\eta}$ space. On the other hand, equation (5.1) is useless in regions where $1-M_r$ vanishes. This importance of the first integral in (5.2) on mathematical grounds is in addition to its physical importance as a result of the high fluid velocities induced by the surface motion.

The essential character of the Kirchhoff problem is the derivation of an analytical expression for the wave field in terms of the given boundary conditions. It is automatically assumed that the resulting integrals can be evaluated exactly, and the more sophisticated expressions for the radiated field serve no useful purpose. Thus the basic solution of the scalar wave equation with boundary conditions prescribed on a moving surface is;

$$4\pi c^2 \Psi(\underline{x},t) = -c^2 \frac{\partial}{\partial x_i}\iint\left[\frac{\Psi n_i A}{r\left|1-v_{\frac{r}{c}}\right|}\right] dS(\underline{\eta}) - \frac{\partial}{\partial t}\iint\left[\frac{\Psi v_n A}{r\left|1-v_{\frac{r}{c}}\right|}\right] dS(\underline{\eta})$$

$$- \iint\left[\left(c^2 \frac{\partial\Psi}{\partial n} + v_n \frac{\partial\Psi}{\partial\tau}\right) \frac{A}{r\left|1-v_{\frac{r}{c}}\right|}\right] dS(\underline{\eta}) , \qquad (5.3)$$

where *v* is the surface velocity. Two previously published attempts at this problem are in error (Morgans 1930, Kromov 1963).

6. Sound field of an aerodynamic source in arbitrary motion.

The unified four dimensional approach can be used to advantage in investigating the far sound field generated by a single aerodynamic source in arbitrary motion. Not only do previously obtained results emerge quite naturally from this approach, but also new results concerning the effect of source acceleration are apparent. An aerodynamic source models the noise generating properties of a turbulent eddy, which is equivalent to a quadrupole source of strength density T_{ij}. Consequently, the source is assumed to be coherent within spatial and temporal scales corresponding to the correlation length and lifetime of a turbulent eddy. It is further assumed that there are no surfaces present, and that the convection velocity is uniform throughout the source, so that $\underline{M}$ is independent of $\underline{\eta}$. It follows that J is unity and the vector $\underline{a}$ of equation (3.19) is zero. Equation (5.2) then shows that the density fluctuations in the sound field are given by the integral,

$$4\pi c^2(\rho-\rho_o) = \frac{\partial^2}{\partial x_i \partial x_j} \int \frac{T_{ij}}{\{(1-M_r)^2+\alpha^2\}^{1/2}} \frac{d\Sigma}{r} \; . \qquad (6.1)$$

To estimate the magnitude of these fluctuations, it is clear that not only is knowledge of the integrand required, but also a measure of the 'area' $d\Sigma$. This is the area in which Σ intersects the source region, and varies with the orientation and shape of Σ. These in turn vary with the convection velocity. For example, if the source is at rest, $d\Sigma$ is determined solely by the spatial length scale ℓ, but in the Mach wave condition $(1-M_r) = 0$, Σ lies parallel to the time axis and $d\Sigma$ now depends upon a typical temporal scale $\Delta\tau$. However, by choosing α to equate the temporal (λ) scale $\alpha c\Delta\tau$ and the spatial scale ℓ, the area $d\Sigma$ is always of magnitude ℓ^3 regardless of the orientation or shape of Σ. A universal estimation of the radiation integral can then be accomplished for all convection velocities. Since these spatial and temporal scales are only defined approximately, α is also only an approximate order of magnitude, and must be regarded as constant throughout the integral. Before the detailed estimation can be completed, however, the derivatives with respect to $\underline{x}$ must be taken within the integral. The integrand's explicit dependence upon $\underline{x}$ through the factor r^{-1} is not important, as this only produces near field terms. Instead, the far field contribution arises from the dependence of the integral upon the variable position of the hypersurface Σ. To differentiate such an integral, it is best replaced by a volume integral containing a delta function, and the following lemma employed;

$$\int_{-\infty}^{+\infty} F(\underline{z})\delta'(g)d\underline{z} = -\int_{-\infty}^{+\infty} \frac{\partial}{\partial z_i}\left(\frac{\partial g}{\partial z_i} \quad \frac{F}{|\text{grad } g|^2}\right) \delta(g)\ d\underline{z}\ . \tag{6.2}$$

If $\underline{N}$ is written for the unit vector normal to the constant g hypersurface,

$$N_i = |\text{grad } g|^{-1} \frac{\partial g}{\partial z_i}\ , \tag{6.3}$$

and the integral in (6.2) can be written

$$-\int_{-\infty}^{+\infty} \left\{ \frac{N_i}{|\text{grad } g|} \frac{\partial F}{\partial z_i} + F \frac{\partial}{\partial z_i} \left(\frac{N_i}{|\text{grad } g|}\right) \right\} \delta(g)d\underline{z}\ . \tag{6.4}$$

In the far field of a source whose convection velocity is independent of $\underline{\eta}$, the latter term in (6.4) reduces to its time component, so that after a double application of this result the radiation integral can be written

$$4\pi c^2(\rho-\rho_o) = \int \left[\frac{\alpha}{\{(1-M_r)^2+\alpha^2\}^{\frac{1}{2}}} \frac{\partial}{\partial N} + \frac{\partial}{c\partial\tau}\left(\frac{1-M_r}{(1-M_r)^2+\alpha^2}\right)\right]^2 T_{rr} \frac{d\Sigma}{r\{(1-M_r)^2+\alpha^2\}^{1/2}}\ . \tag{6.5}$$

Here T_{rr} is written for $T_{ij}\hat{r}_i\hat{r}_j$, where $\underline{\hat{r}}$ is the unit radiation vector.

Our previous remarks indicate that α is to be chosen to equate the relevant temporal (λ) and spatial scales of of the intersection area $d\Sigma$. The correct choice of temporal scale is crucial in estimating the Mach wave radiation since when $(1-M_r)= 0$, $d\Sigma$ depends directly upon the time scale $\Delta\tau$. Away from this condition errors in the choice of $\Delta\tau$ are less critical. If Σ is plane, then it can intersect the source region over its whole lifetime, which thus defines the appropriate time scale. A typical lifetime of a turbulent eddy is ℓ/bcM , where b is a small numerical constant, and so the value of α based on this timespan is bM . The condition that Σ be plane is equivalent to the convection velocity being constant; if the source accelerates Σ is curved, and can no longer intersect the source region over the total lifetime but only for a much smaller time. Physically, this implies that the time spent by the eddy in the region from which Mach waves can be observed may be the total eddy lifetime, but alternatively may be a shorter time if the source rapidly accelerates through this region. This shorter time scale is determined as follows. The spatial realisation of Σ is the contracting sphere $r = c(t-\tau)$, which passes through the source at a relative radial velocity of $c(1-M_r)$. In an interval of time $\Delta\tau$, this sphere moves a distance

$$\int_0^{\Delta\tau} c(1-M_r)d\tau \qquad (6.6)$$

through the source, and will have passed completely through the source when this distance equals the radial length scale ℓ. This time scale is only important near the condition $(1-M_r) = 0$, so that $\Delta\tau$ must be determined by the equation

$$\left| c \frac{\partial}{\partial \tau} (1-M_r) \right| \frac{\Delta\tau^2}{2} = \ell . \qquad (6.7)$$

Since $\Delta\tau$ is to be set equal to $\ell/c\alpha$ for space and time scales to be equal, α is given by;

$$\alpha^2 = \frac{\ell^2}{c^2 \Delta\tau^2} = \frac{\ell}{2c} \left| \frac{\partial M_r}{\partial \tau} \right| . \qquad (6.8)$$

If M_r is written as $\underline{M}.\hat{\underline{r}}$, it follows from equation (3.7) that

$$\frac{\partial M_r}{\partial \tau} = \frac{\partial \underline{M}}{\partial \tau} \cdot \hat{\underline{r}} + \underline{M} \cdot \frac{\partial \hat{\underline{r}}}{\partial \tau}$$

$$= \frac{\partial \underline{M}}{\partial \tau} \cdot \hat{\underline{r}} - \frac{c(M^2-1)}{r} , \qquad (6.9)$$

and hence the value of α based on this timespan is

$$\alpha = \left\{ \frac{\ell}{2c} \left| \frac{\partial \underline{M}}{\partial \tau} \cdot \hat{\underline{r}} - \frac{c(M^2-1)}{r} \right| \right\}^{\frac{1}{2}} . \qquad (6.10)$$

Thus this acceleration mechanism produces two time scales; the first is due to the basic acceleration of the source limiting the emission time, whereas the second arises from the changing geometry of the situation.

Some of the possible field regimes associated with an aerodynamic source can now be discussed. If the source is in steady rectilinear motion, in the far field the only relevant value of α is that based on eddy lifetime. For this model, the second part of the operator in (6.5) vanishes, and the density fluctuations are given by the simple integral

$$4\pi c^2(\rho-\rho_o) \simeq \int \frac{\alpha^2}{\left\{(1-M_r)^2+\alpha^2\right\}^{3/2}} \frac{\partial^2 T_{rr}}{\partial N^2} \frac{d\Sigma}{r} . \qquad (6.11)$$

The magnitude of this integral is easily established. The measure of $d\Sigma$ and the normal derivative $\frac{\partial}{\partial N}$ are typically of magnitude ℓ^3 and ℓ^{-1} respectively. The turbulence stress tensor T_{ij} can be approximated by its first term $\rho u_i u_j$, which is of order $\rho c^2 M^2$, and the overall magnitude of the density fluctuations is

$$4\pi c^2(\rho-\rho_o) \simeq c^2\rho \frac{\ell}{r} \frac{b^2 M^4}{\left\{(1-M_r)^2+b^2M^2\right\}^{3/2}} . \qquad (6.12)$$

The mean square density fluctuation, which is proportional to the sound intensity, thus varies as

$$\overline{(\rho-\rho_o)^2} \sim \overline{\rho^2} \frac{\ell^2}{r^2} \frac{b^4 M^8}{\left\{(1-M_r)^2+b^2M^2\right\}^3} . \qquad (6.13)$$

This result embodies in a single formula the features of the sound field of a uniformly convected eddy that are well understood in connection with jet noise theory. At low speeds, the intensity varies as the eighth power of the convection speed, coupled with a directional factor $|1-M_r|^{-6}$. At high speeds near the Mach wave condition $(1-M_r) = 0$, this power law is replaced by a second power variation at a much higher overall level, since b is small. Away from the Mach wave condition, the intensity still varies with the square of the convection speed, but the overall level is much reduced. These variations differ slightly from their counterparts in the theory of jet noise because equation (6.13) refers to a single turbulent eddy. The results for a distribution of eddies can be obtained upon multiplication by the number of eddies heard concurrently, and this has the effect of reducing the power 3 in the denominator of (6.13) to a value of 5/2.

Nearer the source in the Mach wave regime, the relevant value of α is not that based upon eddy lifetime, but that defined by the changing geometry $\left\{\frac{\ell}{2r}(M^2-1)\right\}^{1/2}$. Again estimates of the Mach wave field can be based on equation (6.5) but now both factors in the operator are equally important, and vary dimensionally like ℓ^{-1}. This leads to a mean square density fluctuation

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho^2}\ \frac{\ell}{r}\ \frac{M^4}{M^2-1}. \tag{6.14}$$

This result differs by a factor M from that given by Ffowcs Williams (1965) because that estimate was based on a downstream rather than a radial length scale, but the two results are entirely equivalent. Thus the Mach wave field nearer the source falls off more slowly than it does in the very distant field. This variation of the (linear) density as $r^{-1/2}$ is also found in the conical wave field of a slender supersonic projectile, where the time scale is again set by the geometry. Again it should be remarked that (6.14) is the result for a single eddy, a distribution of eddies has a different dimensional variation.

It is clear that estimation of the Mach wave emission for an aerodynamic source in arbitrary motion is simple using the techniques outlined here. A choice is to be made between the possible emission time scales, and the relevant one used in the estimation of equation (6.5). If the eddy lifetime provides the appropriate time scale, the Mach wave field is given in equation (6.13). Alternatively, if acceleration effects are more important, the density fluctuations in the Mach wave field vary as

$$\overline{(\rho-\rho_o)^2} \sim \overline{\rho^2} \; \frac{\ell}{r^2} \; \frac{M^4}{\left| \frac{\partial \underline{M}}{\partial \tau} \cdot \frac{\underline{r}}{c} - \frac{M^2-1}{r} \right|} \; . \qquad (6.15)$$

It is significant that one general effect of source acceleration is to reduce the Mach wave strength by limiting the time spent by the source in the emission region. Together with this, there are Mach wave terms with a direct dependence on acceleration, as is clear from the time derivative term in equation (6.5). This substantiates the impression gained from Lowson's (1965) analysis that acceleration effects may well be important at high speeds, although his expression is unsuited to the field when $1-M_r$ tends to zero.

7. Sound field of a moving surface.

The analysis of the previous section is essentially concerned with the sound field generated by an aerodynamic source such as a turbulent eddy. However, many sound fields are generated by surface motion at high speeds, and it is to such fields that this section is devoted. Most practical examples of high speed sound generating surfaces are rigid bodies, and for such bodies some preliminary simplifications of the general results can be made. A rigid body has a clearly defined convection velocity, and the simplest choice for the moving reference frame $\underline{\eta}$ is a Cartesian frame fixed in the body. This $\underline{\eta}$ frame may be translated and rotated from the fixed $\underline{y}$ frame, but it always remains Cartesian and always moves with a solenoidal convection velocity. Hence J and A are unity. Also,

$$\left(\frac{\partial g}{\partial \eta_i}\right)^2 = \left(\frac{\partial g}{\partial y_i}\right)^2 = \frac{1}{c^2}\,, \tag{7.1}$$

so that β reduced to the scale factor α. Further, the angle θ defined in § 4 to be the angle between the spatial component of the normal to Σ and the normal to the surface, is the same whether measured in the $\underline{\eta}$ or $\underline{y}$ frames. In addition to these simplifications in the coordinate system, one other change is necessary. At present, the source term that represents the volume displaced by the surface is written in the monopole form. This form would be appropriate were the surface expanding, but is clearly inappropriate here. Instead, the surface is now equivalent to higher order volume sources, and this is brought out by a rearrangement of the original term. The convection velocity $\underline{v}$ is clearly well defined for points within as well as on the surface, and as this field is solenoidal, the following relationship holds;

$$\frac{\partial}{\partial t}\left(v_i \frac{\partial H}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}\left\{ \mathring{v}_i(1-H)\right\} + \frac{\partial^2}{\partial x_i \partial x_j}\left\{ v_i v_j (1-H)\right\} . \quad (7.2)$$

Here $\underline{\mathring{v}}$ is the Lagrangian time derivative of the velocity and so represents the acceleration of the source. The monopole surface source is thus equivalent to dipoles and quadrupoles distributed throughout the volume within the surface.

The two results (5.1) and (5.2) give alternative expressions for the sound field. (5.2) is more general in that it involves no subtle limiting process in the region $(1-M_r) \sim 0$, and can be treated as an expression in which 'time retardation' effects are always negligible. On the other hand in general the subspace σ is somewhat abstract, so that it is not easy to analyse the whole of equation (5.2) in the manner developed in § 6. Further objections to such an analysis will appear presently. However, in a particular, and very broad class of problems, equation (5.2) is practically indistinguishable from (5.1). That is when the source distributions are compact. A distribution of dimension ℓ is compact if the time taken by the sound in crossing the distribution $(1-M_r)^{-1} \frac{\ell}{c}$ is very much less than the source time scale $\Delta\tau$. This criterion is equivalent to the condition that the acoustic wavelengths by much greater than ℓ, or alternatively that the value of α based on $\Delta\tau$ is a negligible fraction of $1-M_r$. Hence equation (5.2) is essentially the same as its formal limit when this ratio is zero; this limit is equation (5.1) which thus furnishes an adequate description of the sound field;

$$4\pi c^2(\rho-\rho_o) = \frac{\partial^2}{\partial x_i \partial x_j}\int \left[\frac{T_{ij}}{r|1-M_r|}\right] d\underline{\eta} - \frac{\partial}{\partial x_i}\int\left[\frac{p_{ij}n_j}{r|1-M_r|}\right] dS(\underline{\eta})$$

$$- \frac{\partial}{\partial x_i}\int_{V_o}\left[\frac{\rho_o \dot{v}_i}{r|1-M_r|}\right] d\underline{\eta} + \frac{\partial^2}{\partial x_i \partial x_j}\int_{V_o}\left[\frac{\rho_o v_i v_j}{r|1-M_r|}\right] d\underline{\eta} . \tag{7.3}$$

The last two integrals are to be evaluated over the volume inside the surface, V_o. The spatial extent of the source distributions determine whether they can properly be replaced by a point source acting at a mean position; for those sources that may be so reduced, equation (7.3) becomes,

$$4\pi c^2(\rho-\rho_o) = \frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{Q_{ij}+\rho_o v_i v_j V_o}{r|1-M_r|}\right] - \frac{\partial}{\partial x_i}\left[\frac{P_i+\rho_o \dot{v}_i V_o}{r|1-M_r|}\right] . \tag{7.4}$$

Q_{ij} represents the integrated strength of the external quadrupoles, and P_i is the total force exerted by the surface on the fluid. V_o is the volume enclosed by the surface. These results have been obtained and discussed by Lowson (1965), except that he takes no account of the volume displacement effect. However, as (7.4) shows, and as is already well known, this effect is easily accounted for by supplementing the main dipole and quadrupole by a further $\rho_o \dot{v}_i V_o$ and $\rho_o v_i v_j V_o$ respectively. If the quadrupole and dipole distributions have similar spatial and temporal scales, the quadrupole distribution can be neglected in regions where equation (7.4) is a good approximation to the field, since its effect is smaller than that of the dipole distribution by the factor $(1-M_r)^{-1}\,\dfrac{\ell}{c\Delta\tau}$.

For such distributions the density is given by

$$4\pi c^2(\rho-\rho_o) = -\frac{\partial}{\partial x_i}\left[\frac{P_i+\rho_o \dot{v}_i V_o}{r|1-M_r|}\right] . \tag{7.5}$$

Of course, situations can be envisaged in which the scales appropriate to the two distributions differ; the quadrupoles may then be important sound sources and the whole of equation (7.4) must be retained.

The above results apply to compact source distributions. They apply to the sound fields generated at low rotational speeds by aircraft propellers, aeroengine fans and helicopter rotors. This is because the relevant time scales in these situations vary inversely with the convection speed, and the sources are compact at low speeds. The usual dimensional analysis of (7.5) shows that the sound intensity then varies as the sixth power of the convection velocity, and this agrees with practical observation. The above results may also apply to surfaces travelling at high supersonic speeds in a region where $M_r >> 1$. However as speeds increase typical time scales decrease, and it is not known whether practical source distributions remain compact at these high speeds. If they do, a dimensional analysis of (7.5) shows that the sound intensity now only increases as the square of the convection speed.

The above results fail if the source distributions are non-compact, i.e. the acoustic wavelengths are comparable with the source dimension. This type of distribution occurs in the investigation of the field of a supersonically moving surface in a region where $1-M_r$ is small. It may also occur for more moderate convection speeds if a mechanism is present which produces very short time scales. For non-compact sources, equation (7.3) is no longer a proper description of the sound field. It is still a valid formal expression, but retarded time differences can no longer be neglected. Instead, a better description of the field in these regions would be obtained by taking the limit of equation (5.2) as $1-M_r$ tended to zero. This equation is obviously more involved than (7.3), and a general analysis of it is very difficult. This is because the general equation (5.2) must first be cast into a form suitable for estimation requirements, and then the limit $1-M_r$ tending to zero taken, if all effects of acceleration are to be retained. Consequently, for simplicity it is to be assumed here that there are no significant effects of acceleration, so that $1-M_r$ can be set to zero at the outset. Also for the sake of simplicity, the time variable τ will be used here, rather than its scaled equivalent λ.

In the limit $(1-M_r) = 0$, the hypersurface Σ lies parallel to the time axis. The spatial realisation of is the contracting sphere $r = c(t-\tau)$, which is denoted by Ω. Thus $d\Sigma$ becomes $c d\Omega d\tau$. This sphere cuts the real surface S in a curve Γ, and it follows from the definition of σ that $d\sigma$ becomes $c d\Gamma d\tau$. Hence the limit of (5.2) as $1-M_r$ tends to zero is

$$4\pi c^2(\rho-\rho_o) = \frac{\partial^2}{\partial x_i \partial x_j} \int \left\{\overline{T_{ij}} + \rho_o v_i v_j (1-H(f))\right\} \frac{c d\Omega \, d\tau}{r}.$$

$$- \frac{\partial}{\partial x_i} \int \frac{p_{ij} n_j}{\sin \theta} \frac{c d\Gamma \, d\tau}{r} . \qquad (7.6)$$

The first integral in (7.6) is to be taken over the surface of the Ω sphere, whereas the second is to be integrated around the contour Γ in which Ω cuts S. Here θ is the angle between the outward normal to S and the radiation direction. This latter direction is the inward normal to Ω so that the angle θ used here is the supplement of that defined in § 4.

For estimation purposes, the field point derivatives must again be taken under the integral sign. The only dependence of the quadrupole integral upon $\underline{x}$ in the far field is through the position of the sphere Ω which has $\underline{x}$ as its centre. The differentiation of such an integral is accomplished by replacing the surface integral by a volume integral containing a delta function, and utilising the lemma (6.2). In the notation of this section, this leads to the result

$$\frac{\partial}{\partial x_i} \int F \, d\Omega = \int \frac{\partial}{\partial r} (F \, \hat{r}_i) d\Omega \quad , \qquad (7.7)$$

where, to avoid notational confusion, $\frac{\partial}{\partial r}$ is the derivative in the radiation direction, and not in the direction of increasing r. For the quadrupole integral, this radiation derivative acts upon the discontinuous functions $\overline{T_{ij}}$ and $\rho_o v_i v_j (1-H)$, and so produces δ function singularities:

$$\left.\begin{aligned} \frac{\partial}{\partial r}(\overline{T_{ij}}) &= H(f)\frac{\partial T_{ij}}{\partial r} + T_{ij}\,\delta(f)\frac{\partial f}{\partial r}, \\ \frac{\partial}{\partial r}\left\{\rho_o v_i v_j (1-H(f))\right\} &= (1-H(f))\frac{\partial \rho_o v_i v_j}{\partial r} - \rho_o v_i v_j \delta(f)\frac{\partial f}{\partial r}. \end{aligned}\right\} \quad (7.8)$$

From the geometry of the situation, it follows that

$$\left.\begin{aligned} \frac{\partial f}{\partial r} &= |\text{grad } f| \cos\theta, \\ \text{and} \qquad d\Omega &= \frac{df\, d\Gamma}{|\text{grad } f| \sin\theta}, \end{aligned}\right\} \quad (7.9)$$

and it is clear that after a single derivative has been taken under the integral sign, the density field can be expressed as,

$$4\pi c^2(\rho-\rho_o) = \frac{\partial}{\partial x_i}\int\left\{H\frac{\partial T_{ir}}{\partial r} + (1-H)\frac{\partial(\rho_o v_i v_r)}{\partial r}\right\} c\,\frac{d\Omega\, d\tau}{r}$$

$$+ \frac{\partial}{\partial x_i}\int\left\{(T_{ir}-\rho_o v_i v_r)\cos\theta - p_{in}\right\}\frac{c\,d\Gamma\, d\tau}{r\sin\theta}. \quad (7.10)$$

Here p_{in} has been written for $p_{ij}n_j$. This process can be repeated to yield a further contour integral, and equation (7.10) becomes

$$4\pi c^2(\rho-\rho_o) = \int \left\{ H \frac{\partial^2 T_{rr}}{\partial r^2} + (1-H) \frac{\partial^2(\rho_o v_r v_r)}{\partial r^2} \right\} \frac{c d\Omega \, d\tau}{r}$$

$$+ \frac{\partial}{\partial x_i} \int \left\{ (T_{ir}-\rho_o v_i v_r)\cos\theta - p_{in} \right\} \frac{c d\Gamma \, d\tau}{r \sin\theta}$$

$$+ \int \cos\theta \frac{\partial}{\partial r} (T_{rr}-\rho_o v_r v_r) \frac{c d\Gamma \, d\tau}{r \sin\theta} . \qquad (7.11)$$

Equation (7.11) requires further reduction, as it still contains field point derivatives acting upon the contour integral. Again the integral only depends upon $\underline{x}$ through the position of Γ ; if the field point changes the sphere now cuts S in a different contour. As S is an arbitrary surface, it must be assumed that Γ also changes its shape and curvature, and the form of the lemma that enables these derivatives to be evaluated is slightly different from (7.7). If $\underline{m}$ is a unit vector on the surface S normal to Γ such that $\underline{m}.\underline{r} \geq 0$, $\underline{n}$, $\underline{m}$ and $\underline{\Gamma}$ are mutually orthogonal, and,

$$\frac{\partial}{\partial x_i} \int F \, d\Gamma = \int \operatorname{div}(\underline{m} \, F \, \hat{r}_i) \frac{d\Gamma}{\sin\theta} . \qquad (7.12)$$

The proof of this lemma is similar to that of (7.7), again accomplished by replacing the contour integral by a volume integral containing two delta functions. By utilising the condition $v_r = c$ throughout the region of interest, (7.11) immediately becomes,

$$4\pi c^2(\rho-\rho_o) = \int H \frac{\partial^2 T_{rr}}{\partial r^2} \frac{cd\Omega\, d\tau}{r}$$

$$+\int \left(\text{div}\left\{ \frac{m}{\sin\theta}\left[(T_{rr}-c^2\rho_o)\cos\theta - p_{rn}\right]\right\} + \cos\theta\, \frac{\partial T_{rr}}{\partial r}\right)\frac{cd\Gamma d\tau}{r\sin\theta}. \qquad (7.13)$$

The complicated form of the contour integral in (7.13) can be simplified if it is assumed that viscosity acts in all situations of interest. Then there is no slip between the fluid and the surface, and the radiation component of the fluid velocity u_r is equal to the equivalent component of the surface velocity v_r. From the definition of T_{ij} it follows that where $v_r = c$,

$$(T_{rr}-c^2\rho_o)\cos\theta - p_{rn} = p_{rr}\cos\theta - p_{rn}. \qquad (7.14)$$

The right hand side of (7.14) can be further simplified if the compressive stress tensor p_{ij} is written as the sum of a pressure term $p\delta_{ij}$ and a viscous stress tensor E_{ij}.

We define a new unit vector $\underline{q}$ to be perpendicular to both the contour Γ and the radiation direction $\hat{\underline{r}}$ such that $\underline{q}.\underline{m} \geq 0$. Thus $\underline{r}$, $\underline{q}$ and $\underline{\Gamma}$ are mutually orthogonal. Then it can be shown that

$$(\sin\theta)^{-1}(p_{rr}\cos\theta - p_{rn}) = E_{ij}r_i q_j = E_{rq}\,, \qquad (7.15)$$

and (7.13) becomes,

$$4\pi c^2(\rho-\rho_o) = \int H\frac{\partial^2 T_{rr}}{\partial r^2}\frac{c\,d\Omega\,d\tau}{r} + \int\left\{\text{div}\,E_{rq}\underline{m} + \cos\theta\frac{\partial T_{rr}}{\partial r}\right\}\frac{c\,d\Gamma\,d\tau}{r\sin\theta}\,. \qquad (7.16)$$

This result, as might be expected, is not simple. It shows that for non-compact sources, the density field is related to various surface and contour integrals in the real space, and once again demonstrates the importance of the relevant time scale in determining the magnitude of the radiated field. An important conclusion can be drawn from (7.16). The earlier results derived for compact sources emphasise the importance of such parameters as the surface pressure and displaced volume in determining the magnitude of the radiated field. Also the presence of the Doppler factor $(1-M_r)^{-1}$ in these results suggests that as $1-M_r$ approaches zero these parameters become even more important. Equation (7.16) shows that this is not true. In fact these parameters that are so important at low speeds play no part in determining the magnitude of the Mach wave field of a high speed surface. Instead, the details of the fluid flow around the surface, through their effect on T_{ij}, are critical in defining the Mach wave strength. This conclusion is clearly of great importance in the design of quiet high speed machinery. In contrast to these qualitative deductions, it becomes apparent from (7.16) that it is very difficult to infer any quantitative results using analytical methods though any particular model could be studied numerically. In fact, it is not possible in general to deduce a simple dimensional variation of the field strength with the convection Mach number M. The reason for this is that the condition $M_r = 1$ implies that the observer is at an angle $\cos^{-1}\frac{1}{M}$ to the direction of motion of the source. Thus the variation of the field strength with M is essentially the directional variation of the integrals in (7.16). Unless the latter can be exposed, a proper description of the variation of the sound field with convection speed is not possible.

It has been assumed above that the sphere Ω and the surface S intersect at a non-zero angle θ, and it remains to discuss the behaviour of (7.16) when this angle vanishes. In fact the contour integral in (7.16) still remains finite, because as $\sin\theta$ vanishes so does Γ. The reason for this is that provided Ω and S are not identically curved, they only have a common normal (which is the condition for $\sin\theta$ to vanish) if they touch at a single point P. Thus when $\sin\theta$ vanishes, the contour of intersection shrinks to P. The smooth surface S can always be approximated by an ellipsoid near P, and the curve Γ is then a small ellipse. For this contour the integral may be evaluated analytically, and the main contribution to the sound field from this integral is

$$\int \frac{\partial T_{rr}}{\partial r} \frac{2\pi c \, d\tau}{r\left\{(K_1 + \frac{1}{r})(K_2 + \frac{1}{r})\right\}^{1/2}} . \qquad (7.17)$$

K_1 and K_2 are the principal curvatures of S at P, and are measured positive if the centre of curvature is on the side of Γ opposite to $\underline{x}$. The nature of this result is important for it shows that the curvature of the surface is critical in determining the character of the field along the normal at P. If the surface has double curvature, in the far field the factor r^{-1} is small compared with the two curvatures, and consequently the field strength falls off as r^{-1} with an overall magnitude proportional to $(K_1K_2)^{-1/2}$. Thus for a surface that is only slightly curved at P, this contribution to the field can be large, and may be the most important source of far field sound. This term is even more important if the surface has only a single curvature, for (7.17) shows that the field strength now only falls off as $r^{-1/2}$, and is ultimately more important than any source whose field varies as r^{-1}. Finally if the surface at P is locally plane, the field along the normal at P does not vary at all with distance, and an intense beam is produced. Equation (7.17) is not valid if either term in the denominator is zero, indicating that S and Ω have identical curvatures so that $\underline{x}$ lies on the focal point of S, but it is likely that there is then a genuine singularity in the field. This type of dependence of the field upon curvature is also found in high frequency diffraction theory. This is to be expected, since one effect of surface motion is that all radiated wavelengths become smaller by the factor $(1-M_r)$, and in our limit of $1-M_r$ tending to zero, the surface simply acts as an ultra high frequency radiator. Nonetheless, it is interesting to see these results appearing without the use of harmonic analysis.

8. Conclusion.

The paper gives in section 5 formal expressions for the density field radiated by turbulence in the presence of arbitrarily moving surfaces. The most general results are expressed as hypersurface integrals, which, though rather abstract, are generally computable if the source field and surface boundary conditions are known in space and time. The latter sections are devoted to particular features of these general results that might have practical significance. The now well known effects of uniformly convected turbulence emerge quite naturally without any assumption regarding detailed flow statistics. Convective acceleration induces additional sources, at low speed this being the principal effect of acceleration. At the Mach wave condition however, convective acceleration also tends to limit the Mach wave strength by a restriction on the time for which the source can remain in the Mach wave phase.

The deductions regarding the sound of high speed surfaces are quite new and could not easily be deduced by analogy with previously published accounts. The most significant effects are twofold. Firstly, it is shown that features governing the surface generated sound at low speed, i.e. the applied force and displaced inertia, play no part at all in the high speed problem. Secondly, at very high speeds, surface curvature is crucial in determining the strength of an intense beam that radiates in the direction for which the surface normal coincides with the Mach wave direction. A further point that is implied by a dimensional analysis of equations (7.5) and (7.16), is that the intensity of the field at high supersonic speed increases only as the square of surface velocity. This implies that the acoustic output of a high speed (supersonic) machine increases less rapidly than the mechanical power and that the acoustic efficiency falls off inversely as speed increases. However, this cannot be said with extreme conviction, there being some doubt that the component of the stress tensor T_{rr}, in (7.16), can reasonably be set proportional to $\rho_0 U^2$. These points emerge from the analysis of a surface in rectilinear motion, chosen here as the simplest example that illustrates the general effects of the results presented in section 5. We do not expect that these qualitative deductions will be seriously modified by a more complete evaluation of those results for the effects are clearly recognizable, but in a more abstract way, in that section. We would expect therefore that these points have some bearing on the noise radiated by the supersonically moving blades of an aero-engine fan, and we expect that blade curvature will then significantly influence what may well be the dominant part of the radiation field, the Mach wave radiated along the surface normal coinciding with the Mach angle.

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