John Rhys Fox.

CORRIGENDA Ph.D. 1969.

Page 27, line 8. In the sentence commencing "A little thought.....", delete that part following the semi-colon.

Page 44, line 10 should read :

"If $K^{\mu\nu}$ has the form (2.27a), $k_{\mu}K^{\mu\nu}$ is the sum of" (non-zero and independent............)

Page 49 2nd.last line. For eq. 109 read eq. (2.35).

Additional material,

Page 7 line 10. After the sentence concluding "..... equations (8') and (9')".

add:

"On the right hand side of the first we have set the derivatives of the Lagrange multipliers equal to zero".

Pahe 15 line 17 In the right-hand side of eq. (46) insert the operator $\Box$ between the two fields X.

Page 25 lines 2-10. Delete this paragraph.

MASSLESS PARTICLES IN S-MATRIX THEORY

by

JOHN RHYS FOX

A thesis presented for the degree of
Doctor of Philosophy
of the University of London
and the Diploma of Imperial College.

Department of Physics,
Imperial College of Science & Technology,
LONDON.

September 1969.

ABSTRACT

We present a discussion of the S-matrix theory of massless particles.

Starting from the transformation properties, under the Lorentz-group, of massless particle states, we progress to the derivation of massless particle wave functions. The possible choices for the representations of the Lorentz-group to which these belong are seen to be restricted.

For the photon, the polarisation vector is an unsuitable wave function, and in order to retain Lorentz-covariance of the S-matrix, the gauge condition must be imposed on the M-function. We circumvent this difficulty by using an antisymmetric tensor wave function. The M-function analysis is then much simpler than usual, and we discuss some examples to illustrate the method. A similar treatment may be applied to the case of the graviton.

We then turn to the massless particle crossing matrix, rederiving it in the covariant helicity formalism. We show that for consistency, the elements of the crossing matrix and the scattering angle cosines must be analytic at coincident channel-thresholds, such as occur in processes involving massless particles.

Finally we derive sum-rules, both superconvergent and finite-energy, for nucleon-Compton scattering. Integrals over total photo-production cross-sections are related to the parameters of the cross-channel Regge-poles, or to the electromagnetic parameters of the nucleon.

As additional material, we present a paper "A New Formulation of Quantum Electrodynamics" in which we introduce antisymmetric tensor potentials. The Weinberg programme for electrodynamics is here carried out with covariant, local, causal fields. No unphysical particles are present, nor is there

any indefinite metric. However, the difficulties of an axiomatic quantum field theory of electrodynamics are not completely resolved.

PREFACE

The work embodied in this thesis was carried out in the Department of Theoretical Physics, Imperial College of Science and Technology, University of London, between October 1966 and August 1969, under the supervision of Professor P.T. Matthews.

The material presented is original except where otherwise indicated in the text by clear implication or specific reference, and has not previously been submitted for any other degree in any university. It is based largely on four original papers, listed in the references, one of which is included at the end of the thesis as additional material.

I would like to thank Professor Matthews and many other members of or visitors to this department, in particular Dr. H.F. Jones and Professor G. Feldman, for numerous helpful discussions and suggestions, and for their continued interest.

I am happy to acknowledge the support of a Commonwealth Scholarship during the last three years.

CONTENTS

		Page.
INTRODUCTION		6.
I.	MASSLESS PARTICLE STATES AND WAVE FUNCTIONS.	9.
	Poincare Group Generators and Representations.	9.
	Transformation Properties of Massless Particle States.	13.
	Auxiliary Operators and Wave Functions.	17.
	Spin-One Wave Functions.	25.
	Spin-One Projection Operators.	30.
II.	PHOTON M-FUNCTIONS.	34.
	Four-Vector M-Functions and the Gauge Condition.	36.
	Antisymmetric Tensor M-Functions.	43.
	Examples of Photonic Processes.	45.
	Perturbation Theory and Pole Diagrams.	55.
	Charge Conservation.	62.
	Photon Propagators.	64.
III.	THE MASSLESS PARTICLE CROSSING MATRIX.	66.
	The Crossing Matrix for Helicity Amplitudes.	66.
	The Covariant Helicity Formalism.	70.
	Massless Particles.	73.
	The Crossing Matrix.	75.
	Determination of $\Delta(x,i,j)$, $(m_i = 0)$.	78.
	Appendix.	88.
IV.	SUM RULES IN NUCLEON COMPTON SCATTERING.	89.
	Amplitudes, the Crossing Matrix, and the Optical Theorem.	89.
	The Partial Wave Expansion and Regge Poles.	92.
	Superconvergence Relations.	94.
	Finite-Energy Sum Rules.	96.
	Using the Sum Rules.	99.
REFERENCES.		102.
ADDITIONAL MATERIAL.		
	"A New Formulation of Quantum Electrodynamics".	

INTRODUCTION

Over the last ten years, there has been considerable development of a pure S-matrix theory of strong-interactions from basic postulates of Lorentz-invariance, analyticity, etc. Although first introduced to handle the strong-interactions, the methods should have general application to all strengths of interaction.

With each of the non-strong forces, gravitational, weak and electromagnetic, is associated a massless particle, respectively the graviton, neutrino and photon.

In constructing an S-matrix theory incorporating massless particles, there are two basic problems. The first, which we do not discuss in this thesis, may be described as "the infrared problem". Each pole in the scattering amplitude sits at the beginning of an infinite number of cuts due to an indeterminate number of exchanged massless particles. The second is the gauge problem, appearing only in gravitational and electromagnetic processes, as usually formulated. The difficulty is largely artificial, as we shall show, and arises from the use of unsuitable photon and graviton wave-functions. The gauge-condition on the M-functions is necessary to ensure Lorentz invariance of the S-matrix.

We introduce antisymmetric tensor wave-functions for photons and gravitons, so that the corresponding M-functions are free of constraints of the gauge-type. The analysis is much simpler than usual. In fact conservation laws do impose some conditions on the M-functions, but these are easily dealt with, as we shall see.

Because the graviton gauge problem is exactly analogous to that for the photon, we make little mention of the former throughout our discussion. We do not mention the neutrino specifically because it is conventionally described by a wave-function which causes no trouble, nor is there any conservation law, analogous to that for charge, which imposes constraints on neutrino M-functions. In this respect, the masslessness of the neutrino is of no particular significance, and the construction of M-functions proceeds as in the case of massive particles.

The matters outlined above are treated in Chapter 2, which is preceded by an introductory chapter on the transformation laws for massless particle states and massless particle wave functions.

In the third chapter, we study another aspect of massless particle S-matrix theory, the helicity amplitude crossing matrix. This has previously been found by several workers, but we present a novel derivation in terms of the elegant covariant helicity formalism. The analysis raises the question as to how certain kinematic factors should be continued past channel-threshold branch points. This question being resolved as far as the preceding analysis is concerned, it still remains to be answered in the case of the elements of the crossing matrix itself. We show that for consistency between the behaviour of helicity amplitudes in two channels near the boundary of the physical region and the elements of the crossing matrix, the cosines of the scattering angles and the crossing angles must be analytic at the coincident thresholds that occur in a massless particle scattering process.

In the final chapter we make some slight use of the crossing matrix analysis in a discussion of forward nucleon-Compton scattering. Superconvergent and finite-energy sum rules are derived and, through the optical theorem, are expressed in terms of total photoproduction cross-sections. The assumptions are analyticity and t-channel Regge-pole dominance of the amplitudes at high energy. The crossing matrix is used to relate the s-channel amplitudes, in terms of which the sum rules are written, with the t-channel amplitudes, whose high-energy behaviour, on the Regge assumption, is most easily seen.

The references to all chapters conclude the thesis. Though possibly not exhaustive, particularly in respect of the work of Chapter 4, we believe them to be comprehensive and to include the more important sources.

As additional material, we include the preprint "A New Formulation of Quantum Electrodynamics", in which the Weinberg programme for electrodynamics is carried out with covariant, local, causal fields. No unphysical particles appear, nor is there any indefinite metric. However, the difficulties of an axiomatic theory of electrodynamics are not entirely resolved.

CHAPTER I

MASSLESS PARTICLE STATES AND WAVE FUNCTIONS.

We begin by deriving the Lorentz-transformation properties of massless particle states. As these states are labelled by eigenvalues of the Poincaré group invariants, we preface our treatment with a short discussion of this group and its generators.

After deriving the transformation properties of the states, we introduce the wave-functions and auxiliary operators (fields), showing that these operators for a massless particle of helicity j transform covariantly only if they belong to the $(0,j)$ representation of the Lorentz group. The wave-functions in such cases are given.

We study in particular the antisymmetric tensor representation $(1,0) \oplus (0,1)$ for massless spin-1 particles of helicity ± 1, writing down some of the properties of the wave functions $\varepsilon^{\lambda}_{\mu\nu}(p)$.

Poincaré Group Generators and Representations.

The Poincaré group has the ten generators $J_{\mu\nu}$, P_μ, with $J_{\mu\nu} = -J_{\nu\mu}$, which satisfy the commutation relations

$$
\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= -i\left(g_{\mu\rho} J_{\nu\sigma} + g_{\nu\sigma} J_{\mu\rho} - g_{\mu\sigma} J_{\nu\rho} - g_{\nu\rho} J_{\mu\sigma} \right) \\
[J_{\mu\nu}, P_\rho] &= i\left(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu \right) \qquad (1.1) \\
[P_\mu, P_\nu] &= 0
\end{aligned}
$$

The $J_{\mu\nu}$ are generators of rotations in 4-space. The set $J_i \equiv \frac{1}{2}\varepsilon_{ijk} J_{jk}$ have the commutation relations of angular momentum operators, and the $K_i \equiv J_{i0}$ are generators of boosts (rotations in the 0i plane). The P_μ have the commutation relations of translation generators and are identified with the components of the four-momentum.

The irreducible unitary representations of the group[1,2] are labelled by eigenvalues of the invariants, of which there are two;

$$P^2 \equiv P^\mu P_\mu \quad , \quad W^2 \equiv W^\mu W_\mu$$

where W_μ is the Pauli-Lubanski four-pseudovector

$$W_\mu \equiv -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$$

One can verify that P^2 and W^2 commute with all the generators.

We may write W_μ in the alternative form

$$\begin{aligned} W_0 &= \underline{J}\cdot\underline{P} \\ \underline{W} &= P_0\underline{J} + \underline{P}\times\underline{K} \end{aligned} \qquad (1.2)$$

with the $\underline{J}$ and $\underline{K}$ defined above.

The mass-zero representations are characterised by $P^2=0$. Then there are two kinds of representations. Those with $W^2 = \alpha \neq 0$, and those with $W^2 = 0$. In the latter case the representation is not completely specified by P^2 and W^2. However, $P^\mu W_\mu = 0$ (always), and $P^2 = W^2 = 0$ in our representation, so that W_μ must be proportional to P_μ

$$W_\mu = \Lambda P_\mu \qquad (1.3)$$

where W_μ , Λ and P_μ are representations of operators. Λ is in fact the helicity operator as may be seen by applying eq. (1.3) to an eigenstate of momentum

$$W_\mu \,|p,\cdots\rangle = \Lambda\, p_\mu \,|p,\cdots\rangle \tag{1.4}$$

and in particular, $W_0 = \Lambda p_0$ (1.5)

but $W_0 = \underline{J}\,.\,\underline{P}$

so that $\Lambda = \dfrac{\underline{J}\,.\,\underline{p}}{p_0}$, (1.6)

the helicity operator.

The eigenvalues of Λ ,P^2 and W^2 now serve to characterise the representation.

Let the four-momentum of the massless particle be $(p_0, 0, 0, p_0)$ in a suitable frame. Then $W_3 = W_0$, and W_1 and W_2 commute.

Defining

$$\Lambda_\pm \equiv \frac{W_1 \pm i W_2}{p_0} \equiv \Pi_1 \pm i \Pi_2 \quad , \tag{1.7}$$

$$\begin{aligned} W^2 \equiv W^\mu W_\mu &= W_0^2 - W_3^2 - W_1^2 - W_2^2 \\ &= -p_0^2\, \Lambda_+ \Lambda_- \end{aligned} \tag{1.8}$$

Also from the commutation relation

$$[W_\mu, W_\nu] = -i\,\varepsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \tag{1.9}$$

we may deduce that

$$\begin{aligned}[]
[\Lambda, \Lambda_\pm] &= \pm\Lambda_\pm \\
[\Lambda_+, \Lambda_-] &= 0
\end{aligned} \tag{1.10}$$

Therefore Λ, Π_1 and Π_2 generate the group of rotations and translations in a two-dimensional Euclidean space, (but this is of no physical significance).

Now let us denote the eigenstates of W^2 and Λ for momentum p by $|p, \alpha, \lambda\rangle$ or simply $|\alpha, \lambda\rangle$, since W_μ and P_μ commute.

$$\begin{aligned}
W^2 |\alpha, \lambda\rangle &= \alpha\,|\alpha, \lambda\rangle \\
\Lambda\,|\alpha, \lambda\rangle &= \lambda\,|\alpha, \lambda\rangle
\end{aligned} \tag{1.11}$$

$\Lambda_\pm$ are clearly ladder operators, so that

$$\Lambda\Lambda_\pm |\alpha, \lambda\rangle = (\lambda \pm 1)\Lambda_\pm |\alpha, \lambda\rangle \tag{1.12}$$

The spectrum of λ is then of the form $\lambda_0 + \ell$, ℓ any positive or negative integer and $1 > \lambda_0 \geqslant 0$.

Now $\quad \langle\alpha, \lambda|\,\Lambda\,|\alpha, \lambda'\rangle = \lambda\,\delta_{\lambda,\lambda'}$

and

$$\langle\alpha, \lambda|\,\Lambda_\pm\,|\alpha, \lambda'\rangle = \begin{cases} a^+_\lambda\,\delta_{\lambda,\lambda'+1} \\ a^-_{\lambda+1}\,\delta_{\lambda,\lambda'-1} \end{cases} \tag{1.13}$$

Therefore $\quad \alpha = \langle\alpha,\lambda|\, W^2 \,|\alpha,\lambda\rangle = -p_0^2 \langle\alpha,\lambda|\,\Lambda_+\Lambda_-\,|\alpha,\lambda\rangle$

$$= -p_0^2 \langle\alpha,\lambda|\,\Lambda_+\,|\alpha,\lambda-1\rangle\langle\alpha,\lambda-1|\,\Lambda_-\,|\alpha,\lambda\rangle$$

$$= -p_0^2\, a_\lambda^+ a_\lambda^- \qquad (1.14)$$

In a unitary representation, W_μ is hermitian, so that $\Lambda_+^\dagger = \Lambda_-$ and $a_\lambda^+ = a_\lambda^{-*}$. Therefore $\alpha = -p_0^2|a_\lambda|^2 \leq 0$. If $a_\lambda^+ = a_\lambda^- = 0$ for all λ , $\alpha = 0$ and $\Lambda_\pm = 0$ in the representation. Consequently $W_1 = W_2 = 0$ and $W_\mu = \Lambda P_\mu$. Since in this case Λ commutes with $\Lambda_\pm$, it is an invariant, and the representation is therefore one-dimensional in the spin. The eigenvalue should be integer or half-integer if the representation functions are to be single or double-valued. A particle with spin s may exist in the two helicity states corresponding to the eigenvalues $\pm s$ of Λ , but under proper Lorentz transformations, the representations corresponding to the two states are separately irreducible.

Transformation Properties of Massless Particle States.

Now let us see how the massless particle states with $\alpha=0$, denoted by $|k,\lambda\rangle$, transform under the Lorentz Group[3,4].

There is a restricted group of transformations $R^\mu{}_\nu$, called the "little-group" which leaves the four-momentum k invariant.

$$R^\mu{}_\nu k^\nu = k^\mu \qquad (1.15)$$

This induces a unitary transformation of the states

$$U[\mathcal{R}]\,|k,\lambda\rangle = \sum_{\lambda'} D_{\lambda'\lambda}[\mathcal{R}]\,|k,\lambda'\rangle \tag{1.16}$$

where the functions $D_{\lambda'\lambda}[\mathcal{R}]$ must satisfy the group-property

$$\sum_{\lambda''} D_{\lambda\lambda''}[\mathcal{R}_1]\,D_{\lambda''\lambda'}[\mathcal{R}_2] = D_{\lambda\lambda'}[\mathcal{R}_1\mathcal{R}_2] \tag{1.17}$$

We can see from eqs. (1.15) and (1.16) that the little group is generated by Λ, Π_1 and Π_2, for these commute with the momentum operator. Alternatively, we may deduce the little-group from eq. (1.15).

An infinitesimal $\mathcal{R}^{\mu}{}_{\nu}$ has the form

$$\mathcal{R}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{1.18}$$

with $\omega^{\mu\nu} = -\omega^{\nu\mu}$ since $\mathcal{R}$ is a Lorentz transformation. If k is (k,0,0,k), we deduce that the only non-zero components of $\omega^{\mu\nu}$ are

$$\omega^{12} = -\omega^{21} = \theta$$

$$\omega^{10} = -\omega^{01} = \omega^{13} = -\omega^{31} = \chi_1$$

$$\omega^{20} = -\omega^{02} = \omega^{23} = -\omega^{32} = \chi_2 \tag{1.19}$$

Recalling that the unitary operator corresponding to the general infinitesimal Lorentz transformation is of the form

$$U[1+\omega] = 1 + \frac{i}{2}\,\omega^{\mu\nu} J_{\mu\nu} \qquad (1.20)$$

we see that

$$\begin{aligned} U[\mathcal{R}] &= 1 + i\theta J_{12} + i\chi_1(J_{10}+J_{13}) + i\chi_2(J_{20}+J_{23}) \\ &= 1 + i\theta\Lambda + i\chi_1\Pi_1 + i\chi_2\Pi_2 \qquad (1.21) \end{aligned}$$

so that Λ, Π_1 and Π_2 generate the little group.

But we are interested in one-dimensional representations in which

$$\Pi_1 = \Pi_2 = 0 \qquad (1.22)$$

so that a general $\mathcal{R}$ in the little group transforms $|k,\lambda\rangle$ into

$$\begin{aligned} U[\mathcal{R}]\,|k,\lambda\rangle &= \exp\left\{ i\,\Theta[\mathcal{R}(\theta,\chi_1,\chi_2)]\,\Lambda \right\} |k,\lambda\rangle \\ &= \exp\left\{ i\lambda\Theta \right\} |k,\lambda\rangle \qquad (1.23) \end{aligned}$$

where the angle Θ is some function of θ, χ_1 and χ_2 such that

$$\Theta \sim \theta \qquad (1.24)$$

for infinitesimal $\mathcal{R}$, by eq. (1.21).

Furthermore, by eq. (1.17)

$$\Theta[\mathcal{R}_1] + \Theta[\mathcal{R}_2] = \Theta[\mathcal{R}_1\mathcal{R}_2] \tag{1.25}$$

We may now define the state of a particle of general momentum p, helicity λ, in terms of the state with momentum $k = (k,0,0,k)$. We define

$$|p,\lambda\rangle \equiv U[\mathcal{L}(p)]\,|k,\lambda\rangle \tag{1.26}$$

where $U[\mathcal{L}(p)]$ is the unitary operator corresponding to the Lorentz transformation $\mathcal{L}(p)$ which takes k_μ into p_μ.

We remove the ambiguity in defining $\mathcal{L}(p)$ by requiring it to be of the form

$$\mathcal{L}^\mu{}_\nu(p) = \mathcal{R}^\mu{}_\lambda(\hat{p})\,B^\lambda{}_\nu(|\underline{p}|) \tag{1.27}$$

where $B(|\underline{p}|)$ is a boost along the z-axis taking k to $(|\underline{p}|,0,0,|\underline{p}|)$ and $\mathcal{R}(\hat{p})$ is the rotation taking $(|\underline{p}|,0,0,|\underline{p}|)$ to p. The non-zero components of $B(|\underline{p}|)$ are

$$B^1{}_1 = B^2{}_2 = 1$$

$$B^3{}_3 = B^0{}_0 = \cosh\phi(|\underline{p}|)$$

$$B^3{}_0 = B^0{}_3 = \sinh\phi(|\underline{p}|),$$

$$\phi(|\underline{p}|) = \ln(|\underline{p}|/k) \tag{1.28}$$

The states $|p,\lambda\rangle$ have the normalisation

$$2\pi\,\delta(p^2)\langle p',\lambda'|p,\lambda\rangle = (2\pi)^4\,\delta^4(p-p')\,\delta_{\lambda,\lambda'} \tag{1.29}$$

A general Lorentz transformation $L^\mu{}_\nu$ now transforms $|p,\lambda\rangle$ into

$$U[L]\,|p,\lambda\rangle = U[L]\,U[\mathcal{L}(p)]\,|k,\lambda\rangle$$

$$= U[\mathcal{L}(Lp)]\,U[\mathcal{L}^{-1}(Lp)L\mathcal{L}(p)]\,|k,\lambda\rangle \tag{1.30}$$

But $\mathcal{L}^{-1}(Lp)L\mathcal{L}(p)$ leaves k invariant, so that it belongs to the little group and by eq. (1.23) we may rewrite this

$$U[L]\,|p,\lambda\rangle = U[\mathcal{L}(Lp)]\exp\left\{i\lambda\Theta[\mathcal{L}^{-1}(Lp)L\mathcal{L}(p)]\right\}|k,\lambda\rangle$$

$$= \exp\{i\lambda\Theta\}\,|Lp,\lambda\rangle \tag{1.31}$$

Auxiliary Operators and Wave Functions.

We now introduce the creation operator $a^+(p,\lambda)$ which, when acting on the vacuum state, creates the state $|p,\lambda\rangle$. The operators $a(p,\lambda)$ and $a^+(p,\lambda)$ satisfy the Bose or Fermi commutation relations

$$2\pi\,\delta(p^2)\left[a(p,\lambda),\,a^+(p',\lambda')\right]_\pm = (2\pi)^4\,\delta^4(p-p')\,\delta_{\lambda,\lambda'} \tag{1.32}$$

which is consistent with the normalisation of the states $|p,\lambda\rangle$.

The transformation law for the $a^{+}(p,\lambda)$ may be deduced from eq. (1.31)

$$U[L]\, a^{+}(p,\lambda)\, U^{-1}[L] = \exp\left\{ i\lambda\Theta[\mathcal{L}^{-1}(Lp)\, L\, \mathcal{L}(p)]\right\} a^{+}(Lp,\lambda) \tag{1.32a}$$

hence

$$U[L]\, a(p,\lambda)\, U^{-1}[L] = \exp\left\{ i\lambda\Theta[\mathcal{L}^{-1}(p)\, L^{-1}\mathcal{L}(Lp)]\right\} a(Lp,\lambda) \tag{1.32b}$$

The transformation phase factor $\exp\{i\lambda\Theta[R]\}$ depends both on the parameters of the transformation L and the momentum of the state. We may define an auxiliary operator[4,5]

$$A_{\alpha}(p)$$

with α a Lorentz-group representation label such that

$$U[L]\, A_{\alpha}(p)\, U^{-1}[L] = S_{\alpha}{}^{\beta}(L)\, A_{\beta}(Lp)\,. \tag{1.33}$$

The transformation matrix depends only on the parameters of the transformation. The relation between $A_{\alpha}(p)$ and $a(p,\lambda)$ is

$$A_{\alpha}(p) = u_{\alpha}(p,\lambda)\, a(p,\lambda) \tag{1.34}$$

which defines $u_{\alpha}(p,\lambda)$, the "wave function" (the representation must contain the spin λ).

Now $\langle\alpha|\, U[\mathcal{L}(p)]\,|\lambda\rangle\; a(p,\lambda)$ transforms under L as

$$U[L]\,\langle\alpha|\,U[\mathcal{L}(p)]|\lambda\rangle\, a(p,\lambda)\, U^{-1}[L]$$

$$= \langle\alpha|\,U[\mathcal{L}(p)]\,|\lambda\rangle\langle\lambda|\,U[\mathcal{L}^{-1}(p)]\,U[L^{-1}]\,U[\mathcal{L}(Lp)]\,|\lambda\rangle\, a(Lp,\lambda)$$

$$= \langle\alpha|\,U[L^{-1}]\,|\beta\rangle\langle\beta|\,U[\mathcal{L}(Lp)]\,|\lambda\rangle\, a(Lp,\lambda) \qquad (1.35)$$

so we see that

$$u_\alpha(p,\lambda) = \langle\alpha|\,U[\mathcal{L}(p)]\,|\lambda\rangle$$

$$= \langle\alpha|\,U[\mathcal{L}(p)]\,|\beta\rangle\langle\beta|\lambda\rangle \qquad (1.36)$$

Here $\langle\beta|\lambda\rangle$ is the constant spinor $u_\beta(k,\lambda)$ where k is our standard momentum, or, more simply, $u_\beta(\lambda)$. Further

$$u_\alpha(Lp,\lambda) = \langle\alpha|\,U[\mathcal{L}(Lp)]\,|\beta\rangle\, u_\beta(\lambda)$$

$$= \langle\alpha|\,U[L]\,U[\mathcal{L}(p)]\,U[\mathcal{L}^{-1}(p)]\,U[L^{-1}]\,\times$$

$$\times\, U[\mathcal{L}(Lp)]\,|\beta\rangle\langle\beta|\lambda\rangle$$

$$= \langle\alpha|\,U[L]\,|\beta\rangle\,\langle\beta|\,U[\mathcal{L}(p)]\,|\lambda\rangle\,\times \qquad (1.37)$$

$$\times\, \exp\{i\lambda\Theta[\mathcal{L}^{-1}(p)L^{-1}\mathcal{L}(Lp)]\}$$

since $R = \mathcal{L}^{-1}(p)L^{-1}\mathcal{L}(Lp)$ is in the little group of k.

$$\therefore\; u_\alpha(Lp,\lambda) = D_{\alpha\beta}[L]\, u_\beta(p,\lambda)\exp\{i\lambda\Theta\} \qquad (1.38)$$

In particular, if we choose $p = k$ and $L = \mathcal{L}(q)$ we see that the wave function for general momentum q is defined in terms of $u_\alpha(\lambda)$ by

$$u_\alpha(q,\lambda) = D_{\alpha\beta}[\mathcal{L}(q)]\, u_\beta(\lambda) \tag{1.39}$$

If we insert this into eq. (1.38) on each side we see that

$$D_{\alpha\beta}[\mathcal{L}(Lp)]\, u_\beta(\lambda)$$

$$= D_{\alpha\beta}[L]\, D_{\beta\gamma}[\mathcal{L}(p)]\, u_\gamma(\lambda) \exp\{i\lambda\Theta[\mathcal{R}]\} \tag{1.40}$$

i,e,

$$D_{\alpha\beta}[\mathcal{L}^{-1}(p) L^{-1} \mathcal{L}(Lp)]\, u_\beta(\lambda) = \exp\{i\lambda\Theta[\mathcal{L}^{-1}(p)L^{-1}\mathcal{L}(Lp)]\}\, u_\alpha(\lambda) \tag{1.41}$$

or

$$D_{\alpha\beta}[\mathcal{R}]\, u_\beta(\lambda) = \exp\{i\lambda\Theta[\mathcal{R}]\}\, u_\alpha(\lambda) \tag{1.42}$$

for any $\mathcal{R}$ in the little group of k.

This places a restriction on the possible representations $|\alpha\rangle$ of the Lorentz-group as we now show[4].

For an infinitesimal $\mathcal{R}$ we have

$$D_{\alpha\beta}[\mathcal{R}(\theta,\chi_1,\chi_2)] =$$

$$= \langle\alpha|[1 + i\theta J_3 + i\chi_1(K_1 - J_2) + i\chi_2(K_2 + J_1)]|\beta\rangle \tag{1.43}$$

Eq. (1.42) now requires, since $\Theta \sim \theta$ for infinitesimal $\mathcal{R}$,

$$\langle\alpha|\ J_3\ |\beta\rangle\, u_\beta(\lambda) = \lambda\, u_\alpha(\lambda) \qquad (1.43a)$$

$$\langle\alpha|(K_1 - J_2)\,|\beta\rangle\, u_\beta(\lambda) = 0 \qquad (1.43b)$$

$$\langle\alpha|(K_2 + J_1)\,|\beta\rangle\, u_\beta(\lambda) = 0. \qquad (1.43c)$$

We now recall that we may rewrite $\underline{J}$ and $\underline{K}$ in terms of other operators M and N

$$\underline{M} = \tfrac{1}{2}(\underline{J} + i\underline{K})$$
$$\underline{N} = \tfrac{1}{2}(\underline{J} - i\underline{K}) \qquad (1.44)$$

which have the decoupled commutation relations

$$[M_i, M_j] = i\,\varepsilon_{ijk}\, M_k$$
$$[N_i, N_j] = i\,\varepsilon_{ijk}\, N_k \qquad (1.45)$$
$$[\underline{M}, \underline{N}] = 0 .$$

A representation of the Lorentz-group may then be labelled by the two spin values of these sets, m and n.

A little rearrangement of eqs. (1.43b) and (1.43c) gives us

$$\langle\alpha|\ (M_1 - iM_2)\,|\beta\rangle\, u_\beta(\lambda) = 0 \qquad (1.46a)$$

$$\langle\alpha|\ (N_1 + iN_2)\,|\beta\rangle\, u_\beta(\lambda) = 0 \qquad (1.46b)$$

while (1.43a) reads

$$\langle\alpha|\,(M_3+N_3)\,|\beta\rangle\, u_\beta(\lambda) \;=\; \lambda\, u_\alpha(\lambda) \qquad (1.46c)$$

The commutation relations of the Ms and Ns, together with eqs. (1.46a,b) then show that $u_\beta(\lambda)$ is an eigenvector of M_3 and N_3, having the lowest eigenvalue of M_3 and the highest of N_3:

$$M_3\, u(\lambda) \;=\; -m\, u(\lambda)$$

$$N_3\, u(\lambda) \;=\; n\, u(\lambda) \qquad (1.47)$$

Together with eq. (1.46c) this implies

$$\lambda \;=\; n-m \qquad (1.48)$$

so that $u_\alpha(\lambda)$ only exists as defined when the representation $|\alpha\rangle \equiv |m,n\rangle$ is such that $\lambda=n-m$. We define a right-handed (left-handed) particle of spin j to have helicity $\lambda = +j\ (-j)$. Possible representations for a right handed particle are therefore

$$(0,j)\ ,\ \left(\tfrac{1}{2}, j+\tfrac{1}{2}\right), \ \ldots$$

and for a left handed particle

$$(j,0)\ ,\ \left(j+\tfrac{1}{2}, \tfrac{1}{2}\right), \ \ldots$$

Notice that the vector representation $(\tfrac{1}{2},\tfrac{1}{2})$ is not amongst the possible representations for a spin-one particle. Consequently the wave function corresponding to this representation, the polarisation vector, does not yield a covariant auxiliary operator.

We may now choose a suitable representation and within it calculate the wave-functions (or "spinors") $u_\alpha(p,\lambda)$.
If we label the states $|\alpha\rangle$ by the M_3 and N_3 eigenvalues a,b,

$$u_{ab}(\lambda) = \delta_{a,-m}\,\delta_{b,n} \quad . \tag{1.49}$$

However if we use tensorial labels, for example $|\mu\nu\rangle$, then $u_{\mu\nu}(\lambda)$ is rather arbitrary.

We saw earlier that $\mathcal{L}(p)$ could be written in the form

$$\mathcal{L}(p) = \mathcal{R}(\hat{p})\, B(|p|) \tag{1.27}$$

where $B(|p|)$ was the boost up the z-axis, and $\mathcal{R}(\hat{p})$ the rotation that took the z-axis into the direction $\hat{p}$.

Now

$$U[\mathcal{L}(p)] = U[\mathcal{R}(\hat{p})]\, U[B(|p|)] \tag{1.50}$$

$$= U[\mathcal{R}(\hat{p})]\, e^{-i\eta K_3} ,$$

$$\eta = \ln(|p|/k)$$

From eqs. (1.44) we see that

$$iK_3 = M_3 - N_3 \tag{1.51}$$

and we know that

$$(M_3 - N_3)\,u(\lambda) = (-m-n)\,u(\lambda) \tag{1.52}$$

Therefore $\langle\alpha|\, U[B(|p|)]|\beta\rangle\, u_\beta(\lambda)$

$$= e^{(m+n)\eta}\, u_\alpha(\lambda)$$
$$= \left(\frac{|p|}{k}\right)^{m+n} u_\alpha(\lambda) \qquad (1.53)$$

Also

$$\langle\alpha|\, U[R(\hat{p})]|\beta\rangle = D_{\alpha\beta}[R(\hat{p})] \qquad (1.54)$$

an ordinary rotation matrix.

Therefore

$$u_\alpha(p,\lambda) = \langle\alpha|\, U[\mathcal{L}(p)]|\beta\rangle\, u_\beta(\lambda)$$
$$= D_{\alpha\beta}[R(\hat{p})] \left(\frac{|p|}{k}\right)^{m+n} u_\beta(\lambda) \qquad (1.55)$$

For the simplest representation for spin j, (0,j) or (j,0) as the case may be,

$$u_\alpha(p,\lambda) = D_{\alpha\beta}[R(\hat{p})] \left(\frac{|p|}{k}\right)^{j} u_\beta(\lambda)$$
$$= \langle\alpha|a,b\rangle\, D_{ab,a'b'}[R] \left(\frac{|p|}{k}\right)^{j} u_{a'b'}(\lambda) \qquad (1.56)$$

In the representation (0,j) this may be shortened to

$$u_\alpha(p,\lambda) = \langle\alpha|j'\rangle\, D_{j',j}[R] \left(\frac{|p|}{k}\right)^{j} \qquad (1.57)$$

since then

$$u_{ab}(\lambda) = \delta_{a,0}\,\delta_{b,j} \qquad (1.58)$$

The rotation matrix D is a spin j matrix and is well-known so that all one needs to calculate are the $\langle \alpha | j \rangle$.

Spin-One Wave Functions.

As mentioned previously, if $|\alpha\rangle$ is labelled by tensor indices, these values are not clear and are in fact arbitrary. This state of affairs occurs when we use, for example, the anti-symmetric tensor representation to describe a massless spin one particle.

In this case, a suitable convention may be extracted from the following considerations.

Let the particle be right-handed, that is $\lambda = 1$ and $m=0, n=1$. Furthermore, the eigenvalue of N_3 is +1. N_+ increases N_3 by 1 and corresponds in a sense with the state with $N_3 = +1$.

Now

$$N_+ = \tfrac{1}{2}\left(J_{23} + i J_{31} - i J_{10} + J_{20} \right) \tag{1.59}$$

so that we may associate the state

$$\tfrac{1}{2}\left(|23\rangle + i|31\rangle - i|10\rangle + |20\rangle \right)$$

with the state $|N_3 = 1\rangle$. Here $|\mu\nu\rangle = -|\nu\mu\rangle$.
In a similar way we associate $\tfrac{1}{2}(|12\rangle - i|30\rangle)$ with $|N_3 = 0\rangle$ and $\tfrac{1}{2}(|23\rangle - i|31\rangle - i|10\rangle - |20\rangle)$ with $|N_3 = -1\rangle$.
We may then calculate $\langle \mu\nu | j \rangle$.

In the particular case that p=k, R(p) = 1, the wave function $u_{\mu\nu}(\lambda)$ is just $\langle \mu\nu | 1 \rangle$ which may be written

$$u_{\mu\nu}(\lambda) \sim \frac{k_\mu \varepsilon_\nu^\lambda(k) - k_\nu \varepsilon_\mu^\lambda(k)}{k} \tag{1.60}$$

where

$$\varepsilon_\mu^\lambda(k) = \frac{1}{\sqrt{2}}(0, i, -1, 0) \tag{1.61}$$

with ε_0 and ε_3 arbitrary but equal. We set them equal to zero for convenience, and this characterises the "radiation gauge".

ε_μ^λ(k) is the conventional standard polarisation vector for a right-handed spin-one particle.

A left handed particle has the polarisation vector

$$\varepsilon_\mu^{-\lambda}(k) = \frac{1}{\sqrt{2}}(0, -i, -1, 0)\,. \tag{1.62}$$

One may deduce from eq. (1.57) that the wave function $u_{\mu\nu}(p,\lambda)$ for general momentum p is of the form

$$u_{\mu\nu}(p,\lambda) \sim \frac{p_\mu \varepsilon_\nu^\lambda(p) - p_\nu \varepsilon_\mu^\lambda(p)}{k} \tag{1.63}$$

where $\varepsilon_\mu^\lambda(p) \equiv \varepsilon_\mu^\lambda(\hat{p}) \equiv \mathcal{R}(\hat{p})_\mu{}^\nu\, \varepsilon_\nu^\lambda(k)$

$$= \mathcal{L}(p)_\mu{}^\nu\, \varepsilon_\nu^\lambda(k) \tag{1.64}$$

since B(|p|) has no effect on ε(k). Clearly, any invariant multiple of eq. (1.63) is an equally possible choice for $u_{\mu\nu}(p,\lambda)$. In particular we define $\varepsilon_{\mu\nu}^{\lambda}(p)$ by

$$\varepsilon_{\mu\nu}^{\lambda}(p) \equiv p_{\mu}\varepsilon_{\nu}^{\lambda}(p) - p_{\nu}\varepsilon_{\mu}^{\lambda}(p) \tag{1.65}$$

We now discuss the properties of the $\varepsilon_{\mu}^{\lambda}$ and $\varepsilon_{\mu\nu}^{\lambda}$ more fully. Conventionally, $\varepsilon_{\mu}^{\lambda}(p)$ is defined to be such that

$$p^{\mu}\varepsilon_{\mu}^{\lambda}(p) = 0 \tag{1.66}$$

This condition is insufficient to determine $\varepsilon_{\mu}^{\lambda}(p)$, so we may choose a particular gauge. In our case we choose the radiation gauge, in which $\varepsilon_{0}^{\lambda}(p) \equiv 0$. The polarisation vector is then unique. A little thought shows, however, that it does not transform like a four-vector[6,7]; in particular one can perform a sequence of Lorentz transformations whose product is unity under which the polarisation vector is not invariant (there is a change of gauge). In fact, $\varepsilon_{\mu}^{\lambda}(p)$ transforms with one of the infinite-dimensional representations of the Lorentz group[7,8,9].

$$\begin{aligned}\Lambda_{\mu}{}^{\nu}\varepsilon_{\nu}^{\pm}(p) &= \left[\mathcal{L}(\Lambda p)\mathcal{L}^{-1}(\Lambda p)\Lambda\mathcal{L}(p)\varepsilon^{\pm}(k)\right]_{\mu} \\ &= \left[\mathcal{L}(\Lambda p)\mathcal{R}\,\varepsilon^{\pm}(k)\right]_{\mu}\end{aligned} \tag{1.67}$$

where $\mathcal{R}$ is in the little group of k and has the general form[7]

$$\mathcal{R}^{\mu}{}_{\nu} = \begin{bmatrix} 1+\frac{1}{2}\chi^2 , & \chi_1 , & \chi_2 , & -\frac{1}{2}\chi^2 \\ \chi_1\cos\Theta+\chi_2\sin\Theta , & \cos\Theta , & \sin\Theta , & -\chi_1\cos\Theta-\chi_2\sin\Theta \\ -\chi_1\sin\Theta+\chi_2\cos\Theta , & -\sin\Theta , & \cos\Theta , & \chi_1\sin\Theta-\chi_2\cos\Theta \\ \frac{1}{2}\chi^2 , & \chi_1 , & \chi_2 , & 1-\frac{1}{2}\chi^2 \end{bmatrix} \tag{1.68}$$

where $\quad \chi^2 \equiv \chi_1^2 + \chi_2^2$

Then

$$R_\mu{}^\nu \varepsilon_\nu^\pm(k) = \exp(\pm i\Theta)\varepsilon_\mu^\pm(k) + X_\pm k_\mu \tag{1.69}$$

where

$$X_\pm = \frac{\pm i X_1 - X_2}{\sqrt{2}\, k}$$

Eq. (1.67) then becomes:

$$\begin{aligned} \Lambda_\mu{}^\nu \varepsilon_\nu^\pm(p) &= \mathcal{L}(\Lambda p)_\mu{}^\nu \left[\exp(\pm i\Theta)\varepsilon_\nu^\pm(k) + X_\pm k_\nu\right] \\ &= \exp(\pm i\Theta)\,\varepsilon_\mu^\pm(\Lambda p) + X_\pm (\Lambda p)_\mu \end{aligned} \tag{1.70}$$

Setting $\mu = 0$, we find

$$\Lambda_0{}^\nu \varepsilon_\nu^\pm(p) = X_\pm (\Lambda p)_0 \tag{1.71a}$$

$$\therefore \quad X_\pm = \frac{\Lambda_0{}^\nu \varepsilon_\nu^\pm(\hat{p})}{|\Delta p|} \tag{1.71b}$$

$$\left(\Lambda_\mu{}^\nu - \frac{(\Lambda p)_\mu \Lambda_0{}^\nu}{|\Delta p|}\right)\varepsilon_\nu^\pm(\hat{p}) = \exp(\pm i\Theta)\,\varepsilon_\mu^\pm(\hat{\Lambda p}) \tag{1.72}$$

which incidentally shows that Θ does not depend on $|p|$.

$\varepsilon_\mu^\pm(\Lambda p)$ is still in the radiation gauge.

The auxiliary operator $\varepsilon_\mu^\pm(p)\, a(p,\pm)$ then transforms as an infinite dimensional representation of the Lorentz group, and not as a four vector.

Some of the properties of the $\varepsilon_\mu^\pm(p)$ are:

$$p^\mu \varepsilon_\mu^\pm(p) = 0 \tag{1.73}$$

$$\varepsilon_0^\pm(p) = 0 \tag{1.74}$$

$$\varepsilon_\mu^{\pm *}(p) = \varepsilon_\mu^\mp(p) \tag{1.75}$$

$$\varepsilon_\mu^{\pm *}(p)\, \varepsilon^{\pm\mu}(p) = -1 \tag{1.76}$$

$$\varepsilon_\mu^\pm(p)\, \varepsilon^{\pm\mu}(p) = 0 \tag{1.77}$$

$$\varepsilon_\mu^\pm(p)\, \varepsilon_\nu^{\pm *}(p) = -\tfrac{1}{2} g_{\mu\nu} + \frac{p_\mu \bar{p}_\nu + p_\nu \bar{p}_\mu}{4|\underline{p}|^2} \mp \frac{i\, \varepsilon_{\mu\nu\alpha\beta}\, p^\alpha \bar{p}^\beta}{4|\underline{p}|^2} \tag{1.78}$$

where $\bar{p}^\beta = g^{\beta\beta} p^\beta$ (unsummed)

Consequently,

$$p^\mu \varepsilon_{\mu\nu}^\pm(p) = 0 \tag{1.79}$$

$$\varepsilon_{\mu\nu}^{\pm *}(p) = \varepsilon_{\mu\nu}^\mp(p) \tag{1.80}$$

$$\varepsilon_{\mu\nu}{}^{\rho\sigma} \varepsilon_{\rho\sigma}^\pm(p) = \mp 2i\, \varepsilon_{\mu\nu}^\pm(p) \tag{1.81}$$

$$\Pi_{\mu\nu,\rho\sigma}^\pm(p) \equiv \varepsilon_{\mu\nu}^\pm(p)\, \varepsilon_{\rho\sigma}^{\pm *}(p) = \tfrac{1}{2}\Pi_{\mu\nu,\rho\sigma}(p) \pm \tfrac{i}{2}\Lambda_{\mu\nu,\rho\sigma}(p) \tag{1.82}$$

where[10]

$$\Pi_{\mu\nu;\rho\sigma}(p) = \left[-p_\mu p_\rho g_{\nu\sigma} - p_\nu p_\sigma g_{\mu\rho} + p_\mu p_\sigma g_{\nu\rho} + p_\nu p_\rho g_{\mu\sigma}\right] \tag{1.83}$$

and

$$\Lambda_{\mu\nu,\rho\sigma}(p) = \frac{1}{2p_0^2}\left[-p_\mu p_\rho \varepsilon_{\nu\sigma\alpha\beta} - p_\nu p_\sigma \varepsilon_{\mu\rho\alpha\beta} + p_\mu p_\sigma \varepsilon_{\nu\rho\alpha\beta} + p_\nu p_\rho \varepsilon_{\mu\sigma\alpha\beta}\right] p^\alpha \bar{p}^\beta \tag{1.84}$$

A detailed inspection of $\Lambda_{\mu\nu,\rho\sigma}(p)$ shows that is is actually covariant for $p^2=0$. Those terms in $p_0^2 \Lambda_{\mu\nu,\rho\sigma}(p)$ which are non-zero are actually proportional to p_0^2 or $\underline{p}^2$, which equals p_0^2 if $p^2=0$. In fact

$$\Lambda_{\mu\nu,\rho\sigma}(p) = p^\alpha\left[p_\mu \varepsilon_{\alpha\nu\rho\sigma} + p_\nu \varepsilon_{\mu\alpha\rho\sigma} - p_\rho \varepsilon_{\mu\nu\alpha\sigma} - p_\sigma \varepsilon_{\mu\nu\rho\alpha}\right] \tag{1.85}$$

which is covariant for $p^2 \neq 0$ as well.

An exactly similar analysis applies to the case of gravitons, massless particles of spin 2. The wave functions corresponding to the $(2,0) \oplus (0,2)$ representation are

$$\varepsilon^{\pm}_{\mu\nu,\rho\sigma}(p) \equiv \varepsilon^{\pm}_{\mu\nu}(p)\, \varepsilon^{\pm}_{\rho\sigma}(p)$$

The usual symmetric tensor representation (1,1) is excluded.

Spin-One Projection Operators.

We note that

$$\overline{\Pi}^{\pm}_{\mu\nu,\rho\sigma}(p) \equiv \frac{-1}{2p^2}\, \Pi^{\pm}_{\mu\nu,\rho\sigma}(p) \tag{1.86}$$

are orthogonal projection operators

$$\overline{\Pi}^{\pm}_{\mu\nu,\rho\sigma}(p)\, \overline{\Pi}^{\pm\,\rho\sigma,}{}_{\kappa\lambda}(p) = \overline{\Pi}^{\pm}_{\mu\nu,\kappa\lambda}(p) \tag{1.87}$$

$$\overline{\Pi}^{\pm}_{\mu\nu,\rho\sigma}(p)\, \overline{\Pi}^{\mp\,\rho\sigma,}{}_{\kappa\lambda}(p) = 0 \tag{1.88}$$

Another projection operator is

$$\overline{\overline{\Pi}}_{\mu\nu,\rho\sigma}(p) \equiv \frac{-1}{2p^2}\,\Pi_{\mu\nu,\rho\sigma}(p) \tag{1.89}$$

Other relations are, with

$$\bar{\Lambda}_{\mu\nu,\rho\sigma}(p) \equiv \frac{-1}{2p^2}\,\Lambda_{\mu\nu,\rho\sigma}(p) \tag{1.90}$$

$$\bar{\Lambda}_{\mu\nu,\rho\sigma}(p)\,\bar{\Lambda}^{\rho\sigma,}{}_{\kappa\lambda}(p) = -\overline{\overline{\Pi}}_{\mu\nu,\kappa\lambda}(p) \tag{1.91}$$

$$\bar{\Lambda}_{\mu\nu,\rho\sigma}(p)\,\overline{\overline{\Pi}}^{\rho\sigma,}{}_{\kappa\lambda}(p) = \bar{\Lambda}_{\mu\nu,\kappa\lambda}(p) \tag{1.92}$$

$$\overline{\overline{\Pi}}_{\mu\nu,\rho\sigma}(p)\,\bar{\Lambda}^{\rho\sigma,}{}_{\kappa\lambda}(p) = \bar{\Lambda}_{\mu\nu,\kappa\lambda}(p) \tag{1.93}$$

$$\bar{\Lambda}_{\mu\nu,\rho\sigma}(p)\,\overline{\overline{\Pi}}^{\pm\,\rho\sigma,}{}_{\kappa\lambda}(p) = \mp\frac{i}{2}\,\overline{\overline{\Pi}}^{\pm}_{\mu\nu,\kappa\lambda}(p) \tag{1.94}$$

etc.

It is easy to show that

$$\varepsilon_{\pm}^{\mu\nu\,*}(p)\,\overline{\overline{\Pi}}^{\pm}_{\mu\nu,\rho\sigma}(p) = \varepsilon_{\rho\sigma}^{\pm\,*}(p) \tag{1.95}$$

$$\varepsilon_{\mp}^{\mu\nu\,*}(p)\,\overline{\overline{\Pi}}^{\pm}_{\mu\nu,\rho\sigma}(p) = 0 \tag{1.96}$$

$$\varepsilon_{\pm}^{\mu\nu\,*}(p)\,\overline{\overline{\Pi}}_{\mu\nu,\rho\sigma}(p) = \varepsilon_{\rho\sigma}^{\pm\,*}(p) \tag{1.97}$$

$$\varepsilon_{\pm}^{\mu\nu\,*}(p)\,\bar{\Lambda}_{\mu\nu,\rho\sigma}(p) = \mp i\,\varepsilon_{\rho\sigma}^{\pm\,*}(p) \tag{1.98}$$

We may rewrite eqs. (1.95) and (1.96) by re-expressing $\Lambda_{\mu\nu,\rho\sigma}(p)$ through eq. (1.93)

$$\varepsilon_{\pm}^{\mu\nu *}(p)\,\tfrac{1}{2}\left[\overline{\Pi}_{\mu\nu,\rho\sigma}(p) \pm i\,\overline{\Pi}_{\mu\nu,\kappa\lambda}(p)\,\overline{\Lambda}^{\kappa\lambda,}{}_{\rho\sigma}(p)\right] = \varepsilon^{\pm *}_{\rho\sigma}(p) \qquad (1.99)$$

$$\varepsilon_{\mp}^{\mu\nu *}(p)\,\tfrac{1}{2}\left[\quad '' \qquad\qquad '' \qquad\qquad '' \quad\right] = 0 \qquad (1.100)$$

that is,

$$\varepsilon_{\pm}^{\mu\nu *}(p)\,\overline{\Pi}_{\mu\nu,\kappa\lambda}(p)\left(\tfrac{1}{2} g^{\kappa}{}_{\rho}\, g^{\lambda}{}_{\sigma} \pm \tfrac{i}{2}\,\overline{\Lambda}^{\kappa\lambda,}{}_{\rho\sigma}(p)\right) = \varepsilon^{\pm *}_{\rho\sigma}(p) \qquad (1.101)$$

$$\varepsilon_{\mp}^{\mu\nu *}(p) \quad '' \quad \left(\quad '' \qquad\qquad '' \quad\right) = 0 \qquad (1.102)$$

But

$$\varepsilon_{\pm}^{\mu\nu *}(p)\,\overline{\Pi}_{\mu\nu,\rho\sigma}(p) = \varepsilon^{\pm *}_{\rho\sigma}(p) \qquad (1.103)$$

so that

$$\varepsilon_{\pm}^{\mu\nu *}(p)\;\tfrac{1}{2}\left(g_{\mu\rho}\,g_{\nu\sigma} \pm i\,\overline{\Lambda}_{\mu\nu,\rho\sigma}(p)\right) = \varepsilon^{\pm *}_{\rho\sigma}(p) \qquad (1.104)$$

$$\varepsilon_{\mp}^{\mu\nu *}(p)\;\tfrac{1}{2}\left(\quad '' \qquad\qquad '' \quad\right) = 0 \qquad (1.105)$$

In other words, against the photon wave function,

$$\tfrac{1}{2}\, g_{\mu\rho}\, g_{\nu\sigma} \pm \tfrac{i}{2}\,\overline{\Lambda}_{\mu\nu,\rho\sigma}(p) \qquad (1.106)$$

are equivalent to $\overline{\Pi}^{\pm}_{\mu\nu,\rho\sigma}(p)$.

Furthermore, by virtue of eq. (1.81),

$$\frac{1}{2}\left(g_{\mu\rho}\, g_{\nu\sigma} \mp \frac{i}{2}\, \varepsilon_{\mu\nu\rho\sigma}\right) \qquad (1.107a)$$

and $$\frac{1}{4}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho} \mp \varepsilon_{\mu\nu\rho\sigma}\right) \qquad (1.107b)$$

are also equivalent to $\Pi^{\pm}_{\mu\nu,\rho\sigma}(p)$ against the wave functions. These operators project outgoing photon helicity ± 1 respectively.

All these results have been deduced from the forms for $\Pi(p)$ and $\Lambda(p)$ given above. However, let us write eq. (1.95) say with $\Pi^{\pm}(p)$ given by its definition $\varepsilon^{\pm}_{\mu\nu}(p)\varepsilon^{\pm *}_{\rho\sigma}(p)$.

Then

$$-\frac{1}{2p^2}\, \varepsilon^{\mu\nu *}_{\pm}(p)\left(\varepsilon^{\pm}_{\mu\nu}(p)\, \varepsilon^{\pm *}_{\rho\sigma}(p)\right) = \varepsilon^{\pm *}_{\rho\sigma}(p) \qquad (1.108)$$

But

$$-\frac{1}{2p^2}\, \varepsilon^{\mu\nu *}_{\pm}(p)\, \varepsilon^{\pm}_{\mu\nu}(p) = 1 + \frac{p\cdot\varepsilon^{\pm}(p)\; p\cdot\varepsilon^{\pm *}(p)}{p^2} \qquad (1.109)$$

where $p\cdot\varepsilon^{\pm}(p) = p\cdot\varepsilon^{\pm *}(p) = 0$ and $p^2 = 0$ for free photons. For consistency between eqs. (1.95) and (1.108) we must take the indeterminate expression $\dfrac{p\cdot\varepsilon^{\pm}(p)\; p\cdot\varepsilon^{\pm *}(p)}{p^2}$ to be zero (as it would be for $p^2 \neq 0$).

CHAPTER 2.

PHOTON M-FUNCTIONS.

Let us consider the scattering amplitude for some process which for the sake of definiteness we take to be two-particle to two-particle. The s,t, and u channels are defined by

$$s: \; A + B \rightarrow C + D$$

$$t: \; \bar{D} + B \rightarrow C + \bar{A}$$

$$u: \; A + \bar{D} \rightarrow C + \bar{B}$$

where $\bar{A}$ is the antiparticle of A, etc.

Let the particles have momenta p_i and helicities λ_i. Then, for example, the s-channel amplitude

$$T^s\{p_i, \lambda_i\} = \langle p_c, \lambda_c | \langle p_D, \lambda_D | \, T \, | p_A, \lambda_A \rangle | p_B, \lambda_B \rangle \tag{2.1}$$

where T is the scattering amplitude operator in the space of states.

The s-channel helicity amplitude[11] is defined in terms of the two-particle helicity states $| p_A, p_B, \lambda_A, \lambda_B \rangle$ and $| p_c, p_D, \lambda_c, \lambda_D \rangle$ subject to

$$\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_c + \mathbf{p}_D = 0 \tag{2.2}$$

$$p_A + p_B = p_c + p_D \tag{2.3}$$

It is written

$$T^{s}_{\lambda_c,\lambda_D;\lambda_A,\lambda_B}(s,t,u) = \langle p_C\, p_D, \lambda_C \lambda_D \,|\, T \,|\, p_A\, p_B, \lambda_A \lambda_B \rangle \tag{2.4}$$

where we have indicated explicitly that the amplitude is a function of the invariants s, t and u.

Now we may also write the amplitude as the contraction of a function[12] $M^{\alpha\beta\gamma\delta}\{p_i\}$ with the wave functions $u_\alpha(p_A,\lambda_A)$ etc., where $\alpha,\beta,\gamma,\delta$ are auxiliary group indices.

The s-channel helicity amplitude is then given by

$$T^{s}_{\lambda_c,\lambda_D;\lambda_A,\lambda_B}(s,t,u) =$$

$$= \bar{u}_\gamma(p_C,\lambda_C)\,\bar{u}_\delta(p_D,\lambda_D)\,M^{\alpha\beta\gamma\delta}\{p_i\}\,u_\alpha(p_A,\lambda_A)\,u_\beta(p_B,\lambda_B) \tag{2.5}$$

with the conditions (2.2), (2.3). $M^{\alpha\beta\gamma\delta}\{p_i\}$ is called the M-function and is assumed to be free of kinematic zeros or singularities (KZF and KSF).

In this chapter, we investigate the M-functions for processes involving massless particles, devoting most of our attention to the case of photonic processes.

This is because of the traditional use of a four-vector M-function to describe a scattering involving a photon.[13-18,24] We have already seen, however, that a four-vector wave-function is not suitable for describing the photon. It turns out that one can use such a description if gauge-invariance is imposed[7], and it is this matter that we discuss first. Similar remarks apply to the use of symmetric tensor wave functions to describe gravitons.

We then introduce antisymmetric tensor wave- and M-functions[19]. Such M-functions do not need to satisfy a gauge condition and so are far simpler to construct. From these we are able to derive conventional M-functions satisfying the gauge-condition, and these are free of kinematic zeros and singularities. In fact this method of constructing four-vector M-functions is simpler than the usual one.

We treat some photonic processes to illustrate the power of the tensor method.

We then show the connection between this analysis and the antisymmetric tensor form of perturbation theory recently proposed by the author[10], finding a condition on the M-functions which follows from charge conservation. Finally we write down the photon propagator in the antisymmetric tensor formulation.

Four-Vector M-Functions and the Gauge Condition.

Before studying the M-function in detail, let us derive the gauge condition for photonic processes when the photon is described by a four-vector wave function. The M function correspondingly carries a four-vector index, or several such if more than one photon is involved.

However, we shall consider the case of one photon with momentum k and helicity $\lambda = \pm 1$. We denote the amplitude by

$$T_\lambda (k, p)$$

where p refers to the momenta and helicities of the other particles in the process. From the transformation law of helicity states eq. (1.31), and eq. (2.1) we find that[7]

$$T_{\pm}(k,p) = \exp\{\pm i\,\Theta(k,\Lambda)\}\, T_{\pm}(\Lambda k,\Lambda p) \qquad (2.6)$$

Now

$$T_{\pm}\ (k,p) \ = \ \varepsilon^{\pm *}_{\mu}(k)\ M^{\mu}(k,p) \qquad (2.7)$$

where we have suppressed the massive particle wave-functions and M-function indices, and assumed the photon to be outgoing.

$M^{\mu}(k,p)$ is a true four-vector

$$\Lambda^{\mu}{}_{\nu}\, M^{\nu}(k,p) \ = \ M^{\mu}(\Lambda k,\Lambda p) \qquad (2.8)$$

but $\varepsilon^{\pm *}_{\mu}(k)$ is not, for

$$\Lambda^{\rho}{}_{\mu}\ \varepsilon^{\mu *}_{\pm}(k) \ = \ \exp\{\pm i\,\Theta(k,\Lambda)\}\left[\varepsilon^{\rho *}_{\pm}(\Lambda k) - \frac{(\Lambda k)^{\rho}}{|\underline{k}|}\Lambda_{\nu}{}^{0}\,\varepsilon^{\nu *}_{\pm}(\Lambda k)\right] \qquad (2.9)$$

These two equations in conjunction with eqn. (2.7) show that T transforms as

$$T_{\pm}(k,p) \ = \ \exp\{\pm i\,\Theta\}\left[\varepsilon^{\mu *}_{\pm}(\Lambda k) - (\Lambda k)^{\mu}\Lambda_{\nu}{}^{0}\,\varepsilon^{\nu *}_{\pm}(\Lambda k)\right] M_{\mu}(\Lambda k,\Lambda p)$$

$$= \ \exp\{\pm i\,\Theta\}\ T_{\pm}\ (\Lambda k,\Lambda p) \ - \qquad (2.10)$$

$$-\exp\{\pm i\,\Theta\}\,(\Lambda k)^{\mu}\Lambda_{\nu}{}^{0}\,\varepsilon^{\nu *}_{\pm}(\Lambda k)\, M_{\mu}(\Lambda k,\Lambda p)$$

which gives the correct transformation law eq. (2.6) only if

$$k^{\mu} M_{\mu}(k,p) = 0 \tag{2.11}$$

the gauge condition. ("Gauge", because this is equivalent to requiring that the amplitude be invariant under the gauge transformation $\varepsilon_{\mu}^{\pm}(k) \rightarrow \varepsilon_{\mu}^{\pm}(k) + \chi^{\pm} k_{\mu}$.) .

(In the spin-2 case (gravitons), symmetric tensor wave functions are not "permissible" (see Chap.1). However, they may be used if the symmetric tensor M-function satisfies the gauge-condition[7] $k^{\mu}k^{\nu}M_{\mu\nu} = 0$, in exact analogy with the photonic case).

The M-function may be expanded in terms of a spinor-tensor basis K_i^{μ} transforming as four-vectors, with invariant (amplitude) coefficients. It is possible to choose the K_i^{μ} so that the invariant amplitudes are, like the M-function, free of kinematic zeros and singularities.

$$M^{\mu}(s,t,u) = \sum_i K_i^{\mu} A_i(s,t,u) \tag{2.12}$$

Such K_i^{μ} are polynomials in the four-momenta and other objects, such as γ^{μ}, $g^{\mu\nu}$, $\varepsilon^{\mu\nu\rho\sigma}$, which carry four-vector indices.

Two such bases K_i^{μ} and J_i^{μ} are equivalent[14] (definition) if

$$K_i^{\mu} = \sum_j a_{ij} J_j^{\mu} \tag{2.13}$$

where the a_{ij} are kinematic functions and $\det(a_{ij})$ is KSF and KZF.

A basis K_i^μ is defined to be minimal[14] if the determinant of the transformation matrix (a_{ij}) from any other basis has no zeros (for momenta on their mass shells, but possibly complex). A minimal basis is clearly unique up to equivalence.

Now if the M-function is KSF and KZF, the invariant amplitudes associated with a minimal basis are KSF and KZF.

They are clearly KZF, for a polynomial basis is, by construction, KSF. That is, no invariant amplitude need have a zero to cancel a singularity of a K_i^μ as there are no such singularities.

Now let an invariant amplitude, say A_1, have a kinematic singularity. Then K_1^μ has a kinematic zero, since the M-function is KSF. Let the singularity in A_1 be of the form

$$A_1 = \frac{B}{f(s,t,\mu)} \tag{2.14}$$

where B is KSF and KZF and $f(s,t,\mu)$ is KSF but has a zero.

We may define another basis J_i^μ by

$$J_1^\mu = \frac{K_1^\mu}{f(s,t,\mu)}\,,$$

$$J_i^\mu = K_i^\mu \quad , \quad i = 2,3,\ldots \tag{2.15}$$

Then

$$K_i^\mu = \sum_j a_{ij} J_j^\mu \tag{2.16}$$

and

$$\det(a_{ji}) = f(s,t,\mu) \tag{2.17}$$

which has a zero, so that K_i^μ is not a minimal basis.

Therefore A_1 cannot have a kinematic singularity. The existence of such a minimal basis has been proved by Hepp[20] and Williams[21]. For processes involving only massive particles, such a basis is usually fairly easy to write down by inspection[22]. When a photon is present, however, such a basis will not in general, yield an M-function satisfying the gauge-condition.

One then proceeds as follows[15,16]. Find a minimal basis K_i^μ with corresponding invariant amplitudes A_i. Now take a projection operator

$$\mathcal{G}_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu k_\nu}{k \cdot p} \tag{2.18}$$

where k is the photon momentum and p some other momentum in the problem. then $K'_{i\mu} \equiv \mathcal{G}_{\mu\nu} K_i^\nu$ is a gauge-invariant basis,

$$k_\mu K'^\mu_i = 0 \tag{2.19}$$

but the function

$$M'^\mu = \sum_i K'^\mu_i A_i \tag{2.20}$$

is not KSF, because of the kinematic singularity in $\mathcal{G}_{\mu\nu}$, unless some of the A_i are related or have zeros so that these singularities cancel.

The constraints so placed on the A_i are a direct consequence of gauge-invariance or, equivalently, charge conservation.

We now form a new set of covariants $\tilde{K}_i^\mu$ by taking linear combinations of the K'^μ_i, with kinematic coefficients, so that the new set is KSF. A minimal basis $\tilde{K}_i^\mu$ satisfying the gauge condition will then be associated with a KSF and KZF set of invariant amplitudes by previous arguments. In practice, a minimal basis $\tilde{K}_i^\mu$ is obtained in the following manner.

1. Find a minimal basis K_i^μ not necessarily satisfying the gauge-condition.

2. Form the new basis $K'^\mu_i = \mathcal{G}^\mu{}_\nu K_i^\nu$. Then $k_\mu K'^\mu_i = 0$. If K_i^μ already satisfies the gauge condition, $K'^\mu_i = K_i^\mu$. Note that there may be fewer K'^μ_i than K_i^μ. In particular, terms proportional to p^μ or k^μ vanish.

3. Take linear combinations of the K'^μ_i in as many ways as possible, with KSF coefficients, to yield KSF covariants $\tilde{K}_i^\mu$. There will be a limited number of these. Multiply the remaining covariants K'^μ_i by $k.p$ to yield KSF covariants $\tilde{K}_i^\mu$.

We could have formed the basis $\tilde{K}_i^\mu$ simply by multiplying each $K_i'^{\mu}$ by $k \cdot p$, but examination of the transformation matrix between the bases obtained in these two ways shows that this other method does not yield a minimal basis.

We do not offer a proof that the basis obtained by the rules 1,2,3 is minimal, but in practice it may be seen to be so.

Let us now consider a process with two photons, with momenta k_μ and k'_ν . The M-function $M^{\mu\nu}$ must then satisfy

$$k_\mu M^{\mu\nu} = k'_\nu M^{\mu\nu} = 0 \tag{2.21}$$

The procedure for finding a minimal basis $\widetilde{K}_i^{\mu\nu}$ is similar to that already outlined. At stage 3, we take linear combinations of the

$$K_i'^{\mu\nu} \equiv \mathcal{G}^{\mu}{}_{\rho}\, \mathcal{G}^{\nu}{}_{\sigma}\, K_i^{\rho\sigma} \tag{2.22}$$

which now have second order singularities. $\mathcal{G}_{\mu\nu}$ is most conveniently chosen to be $g_{\mu\nu} - \frac{k'_\mu k_\nu}{k.k'}$. After taking linear combinations and multiplying by k.k' to remove second order singularities, we take further linear combinations and multiplications by k.k' to remove first order singularities. This process can become quite time-consuming.

Antisymmetric Tensor M-Functions.

An alternative and more satisfactory way of describing a photon is by means of the anti-symmetric tensor wave-function $\varepsilon^{\lambda}_{\mu\nu}(k)$ derived in the last chapter[19,23]. It transforms as

$$\Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}\varepsilon^{\pm}_{\rho\sigma}(k) = \exp\{\pm i\,\Theta(k,\Lambda)\}\,\varepsilon^{\pm}_{\mu\nu}(\Lambda k) \tag{2.23}$$

so that $$T_{\pm}(k,p) \equiv \varepsilon^{\pm *}_{\mu\nu}(k)\,M^{\mu\nu}(k,p) \tag{2.24}$$

transforms in the correct manner eq.(2.6) if $M^{\mu\nu}(k,p)$ is a tensor. No gauge condition is necessary for covariance. As $\varepsilon^{\lambda}_{\mu\nu}(k)$ is antisymmetric in μ,ν we may take $M^{\mu\nu}$ to be antisymmetric in these indices too (it cannot be symmetric).

$M^{\mu\nu}$ may now be expanded in terms of a minimal basis and invariant amplitudes A_i. As soon as such a basis is found, our work is finished, for we do not have the tedious labour of converting it to a gauge-invariant one. In fact we may easily find a minimal gauge-invariant four-vector basis $\tilde{K}^{\mu}_{i}$ by the following method[19]. Define $\tilde{K}^{\mu}_{i}$ by

$$\varepsilon^{\pm}_{\mu\nu}(k)\,K^{\mu\nu}_{i} \equiv \varepsilon^{\pm}_{\mu}(k)\,\tilde{K}^{\mu}_{i} \; . \tag{2.25}$$

Then

$$\tilde{K}^{\mu}_{i} = k_{\nu}\left(K^{\nu\mu}_{i} - K^{\mu\nu}_{i}\right) \tag{2.26}$$

Clearly $k_\mu \tilde{K}^\mu_i = 0$, so that $\tilde{K}^\mu_i$ is gauge-invariant. Further it is KSF because $K^{\mu\nu}_i$ is, and KZF as we shall see in a moment.

$K^{\mu\nu}_i$ is antisymmetric in μ and ν and has the form

$$B^\mu C^\nu - B^\nu C^\mu \tag{2.27a}$$

or

$$\varepsilon^{\mu\nu\rho\sigma} B_\rho C_\sigma \tag{2.27b}$$

where B and C are independent four-vectors. If either B or C is k, then $\varepsilon_{\mu\nu}(k) K^{\mu\nu} = 0$, so that we do not admit k as a candidate for B and C.

If $K^{\mu\nu}$ has the form (2.27a), $k_\mu K^{\mu\nu}$ and $k_\nu K^{\mu\nu}$ are non-zero and independent four-vectors, free of common kinematic zeros, since $K^{\mu\nu}$ is KZF. Therefore $\tilde{K}^\mu$ is KZF.

If $K^{\mu\nu}$ has the form (2.27b), $\tilde{K}^\mu = -2\varepsilon^{\mu\nu\rho\sigma} k_\nu B_\rho C_\sigma$ which is KZF because B and C cannot contain kinematic zeros if $K^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} B_\rho C_\sigma$ is to be KZF as required.

This method of constructing conventional gauge-invariant vector-covariants is much faster than that outlined earlier, particularly if more than one photon is involved, as we shall see in some examples. Admittedly the minimal basis takes a little longer to write down, involving as it does two indices for each photon instead of one, but the rest of the work consists merely in partially contracting these covariants with the photon momenta, as in eq. (2.26). We shall see in the examples that bases constructed in this way are equivalent to the usual ones as we must expect (both being KSF and KZF, and by virtue of the definition of equivalence).

To construct the minimal basis $K_i^{\mu\nu}$, we construct all antisymmetric forms (2.27a) and (2.27b) in independent 4-vectors B and C (not k) without any scalar kinematic factors such as k.P, $\not{\phi}$, $\frac{1}{k.P}$ etc. This set may be restricted by considerations[22] of invariance under the discrete transformations P,C,T but we shall not discuss this matter in general.

We again offer no proof that such a procedure does yield a minimal basis, but in practice it may be seen to do so.

Examples of Photonic Processes.

We now come to a consideration of some examples of photonic processes, all with an ingoing photon and some with an outgoing one as well. Let the ingoing photon have momentum k, and its wave function carry indices μ and ν . The other ingoing particle has momentum p. The outgoing boson has momentum k' and indices, if any, ρ and σ . The other outgoing particle, a baryon, or in the case of pion Compton scattering, the pion, has momentum p'.

Also

$$P = \tfrac{1}{2}(p+p') \ , \quad Q = \tfrac{1}{2}(k+k') \ , \quad \Delta = k-k' \tag{2.28}$$

Pion Photoproduction.[13-15,19,24]

The M-function carries two indices μ and ν . The covariants are in the space of Dirac matrices and are pseudotensors because the pion is pseudoscalar. This is most easily arranged by a factor γ_5 in each covariant.

Now, since

$$k^{\mu}\varepsilon^{\pm}_{\mu\nu}(k) = k^{\nu}\varepsilon^{\pm}_{\mu\nu}(k) = 0 \,, \tag{2.29}$$

$2Q+\Delta$

is equivalent to zero, and we do not use this vector in constructing covariants. It is in fact barred by our rules. We can use Q say and omit Δ.

Then, possible independent polynomial covariants are

$$\begin{aligned} K_{i\,\mu\nu} = \; & \gamma_5[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu] \,, \; \gamma_5[P_\mu Q_\nu - P_\nu Q_\mu] \,, \\ & \gamma_5[P_\mu\gamma_\nu - P_\nu\gamma_\mu] \,, \; \gamma_5[Q_\mu\gamma_\nu - Q_\nu\gamma_\mu] \,, \end{aligned} \tag{2.30}$$

and

$$M^{\mu\nu} = \sum_{i=1}^{4} K_i^{\mu\nu} A_i(s,t,u) \,. \tag{2.31}$$

We were able to find only four independent covariants according to our rules, corresponding to the four independent amplitudes of the process. However, $\not{Q}\gamma_5[P_\mu\gamma_\nu - P_\nu\gamma_\mu]$ for example appears to be a fifth independent covariant. It is barred by our rules and is in fact is found to be dependent when contracted with the nucleon spinors and the photon wave function. This is an example of an "equivalence theorem".[22]

Corresponding to these $K_i^{\mu\nu}$ we may find the four-vector covariants $\widetilde{K}_i^{\mu}$ through eq. (2.26).

They are

$$\tilde{K}_1^\mu = 4\gamma_5 \not{k} \gamma^\mu - 4k^\mu ,$$

equivalent to $4\gamma_5 \not{k}\gamma^\mu$ by eq. (1.73),

$$\tilde{K}_2^\mu = 2\gamma_5 (k.P Q^\mu - k.Q P^\mu) ,$$

$$\tilde{K}_3^\mu = 2\gamma_5 (k.P \gamma^\mu - \not{k} P^\mu) ,$$

$$\tilde{K}_4^\mu = 2\gamma_5 (k.Q \gamma^\mu - \not{k} Q^\mu) , \tag{2.32}$$

which are equivalent to the set M_A, M_B, M_C and M_D of Chew, Goldberger, Low and Nambu[13]. The relation is

$$\tilde{K}_1 = -4M_A , \quad \tilde{K}_2 = \tfrac{1}{2} M_B , \quad \tilde{K}_3 = M_D + 2M_A , \quad \tilde{K}_4 = M_C . \tag{2.33}$$

A third equivalent set is given in ref. 15.

The conventional derivation of these gauge-invariant vector covariants is displayed in refs. 13-15.

As in the usual analysis,[13-15, 25, 26], both the t-channel pion pole and the s- and u-channel nucleon poles appear in the same invariant amplitude, A_2. This double dynamical pole structure is a consequence of the fact that photons interact with a conserved quantity, charge. The independent amplitudes correspond to the various different charge and moment couplings of the photon to the $NN\pi$ current.

Rho meson photoproduction.[15,19]

We describe the produced rho meson by the polarisation vector $\varepsilon_\rho^{\lambda *}(k')$, with $k'^\rho \varepsilon_\rho^{\lambda *}(k') = 0$. Therefore $2Q-\Delta = 2k'$ is equivalent to zero against the rho wave function, so that Q and Δ are not independent when they bear an index ρ. As in pion-photoproduction Q and Δ are not independent when they bear an index μ or ν either.

We therefore omit Δ in constructing covariants.

From a count of helicity amplitudes we find that there are twelve independent ones, so there should be twelve independent covariants and invariant amplitudes. In fact there are fifteen possible polynomial covariants, three of which are dependent when contracted with the nucleon and photon wave functions.

The fifteen covariants $K_{i\,\mu\nu,\rho}$ are

$$\begin{array}{ll}
(1),(2),(3) & \sigma_{\mu\nu}\,(P_\rho\,;Q_\rho,\gamma_\rho)\,, \\
(4),(5),(6) & (P_\mu Q_\nu - P_\nu Q_\mu)(P_\rho,Q_\rho,\gamma_\rho)\,, \\
(7),(8),(9) & (\gamma_\mu P_\nu - \gamma_\nu P_\mu)(P_\rho,Q_\rho,\gamma_\rho)\,, \\
(10),(11),(12) & (\gamma_\mu Q_\nu - \gamma_\nu Q_\mu)(P_\rho,Q_\rho,\gamma_\rho)\,, \\
(13),(14),(15) & g_{\nu\rho}(P_\mu,Q_\mu,\gamma_\mu) - g_{\mu\rho}(P_\nu,Q_\nu,\gamma_\nu)\,.
\end{array} \tag{2.34}$$

The corresponding invariant amplitudes $A_1 \ldots\ldots A_{15}$ are KSF and KZF.

Against the photon wave function we find the equivalence

$$2K_5 = k.Q\,K_{13} - k.P\,K_{14}\;. \tag{2.35}$$

Between the photon and nucleon wave functions we have two equivalence theorems* [22] (and here we assume neutral rho for simplicity):

$$mK_1 - K_6 - K_8 - mK_9 - K_{10} - mK_{13} = P^2(K_3 - K_{15}) \tag{2.36}$$

---

*There is in fact a third theorem involving covariants which we have not written down because they contain, for example, a $\not{Q}$.

$$2m K_{11} = \nu \left(K_1 - mK_3 - K_9 - K_{13} - K_{14} + mK_{15} \right) + k.Q \left(K_{12} + K_{13} + K_{14} \right) + \tfrac{1}{4}\left(\mu^2 + t \right) K_2 \ , \qquad (2.37)$$

where $\nu = P.Q = \frac{s-u}{4}$, m is the nucleon mass, and μ is the rho meson mass.

We now have three superfluous covariants and we would like to remove them. Equations (2.35) and (2.37) enable us to remove K_5 and K_{11}, while eq. (2.36) allows any one of the covariants appearing on the LHS to be eliminated. Let us choose K_6.

Then

$$M^{\mu\nu,\rho} = \sum_{i=1}^{15} K_i^{\mu\nu,\rho} A_i \qquad (2.38)$$

$$= \sum_{\substack{i=1 \\ i \neq 5,6,11}}^{15} K_i^{\mu\nu,\rho} B_i \qquad (2.39)$$

where

$$B_1 = A_1 + m A_6 + \frac{\nu}{2m} A_{11}$$

$$B_2 = A_2 + \frac{\mu^2 + t}{8m} A_{11} \qquad (2.40)$$

etc.

This new set of twelve amplitudes B_i is clearly KSF and KZF. However, had we chosen to eliminate K_{13} say, rather than K_5, by eq. 109, B_5 would not have been KSF for then

$$B_5 = A_5 + \frac{2}{k.Q} A_{13} \qquad (2.41)$$

which has a kinematic singularity. If the B_i are to be KSF and KZF we can eliminate via any particular equivalence theorem only a covariant which appears in that theorem with a constant coefficient (the theorem to be in such a form that none of the coefficients has a singularity)[15].

It is simple to derive a conventional KSF and KZF set of covariants $\widetilde{K}_i^{\mu,\rho}$. Partial contraction of the $K_i^{\mu\nu,\rho}$ yields

$$
\begin{aligned}
(\widetilde{K}_1, \widetilde{K}_2, \widetilde{K}_3) &= 2\not{k}\gamma^\mu\,(P^\rho, Q^\rho, \gamma^\rho)\\
(\widetilde{K}_4, \widetilde{K}_5, \widetilde{K}_6) &= 2\,(k.P\,Q^\mu - k.Q\,P^\mu)(P^\rho, Q^\rho, \gamma^\rho)\\
(\widetilde{K}_7, \widetilde{K}_8, \widetilde{K}_9) &= 2\,(\not{k}P^\mu - k.P\,\gamma^\mu)\,(P^\rho, Q^\rho, \gamma^\rho)\\
(\widetilde{K}_{10}, \widetilde{K}_{11}, \widetilde{K}_{12}) &= 2\,(\not{k}Q^\mu - k.Q\,\gamma^\mu)\,(P^\rho, Q^\rho, \gamma^\rho)\\
(\widetilde{K}_{13}, \widetilde{K}_{14}, \widetilde{K}_{15}) &= 2(k_\nu g^{\mu\rho} - k^\rho g_\nu{}^\mu)(P^\nu, Q^\nu, \gamma^\nu)\,. \qquad (2.42)
\end{aligned}
$$

The equivalence theorems still hold when K_i is replaced by $\widetilde{K}_i$. Our set $\widetilde{K}_i$, with $\widetilde{K}_5$ eliminated, is equivalent to the set given in ref.15.

We now turn our attention to two-photon processes, where the power of this method is even more marked.

<u>Pion Compton Scattering.</u> 14,15,17,19,25,27.

We describe the outgoing photon by the wave function $\varepsilon^{\pm *}_{\rho\sigma}(k')$. The covariants carry four tensor indices $\mu\nu,\rho\sigma$. Because $k_\mu = (Q+\frac{\Delta}{2})_\mu$ and $k'_\rho = (Q-\frac{\Delta}{2})_\rho$, and by virtue of

$$k^\mu \varepsilon^{\pm}_{\mu\nu}(k) = k'^\rho \varepsilon^{\pm *}_{\rho\sigma}(k') = 0 \qquad (2.43)$$

Q and Δ are not independent when they bear a μ or ν index, nor when they bear a ρ or σ index either. We therefore omit Δ in constructing covariants.

The process is supposed invariant under the discrete transformations P,C and T.

Invariance under PT places on the covariants the restriction[22]

$$K_{i\,\mu\nu,\rho\sigma}(P,Q) = B^{-1} K^{T}_{i\,\rho\sigma,\mu\nu}(P,Q)\,B \tag{2.44}$$

and C invariance (in the t channel) the restriction

$$K_{i\,\mu\nu,\rho\sigma}(P,Q) = C\; K^{T}_{i\,\rho\sigma,\mu\nu}(-P,-Q)\;C^{-1} \tag{2.45}$$

where

$$B^{-1}\gamma_\mu B = -C\gamma_\mu C^{-1} = \gamma_\mu^T \tag{2.46}$$

and transposition of K refers to its Dirac matrix character.

Furthermore, it turns out that against the photon wave functions, covariants containing Q are dependent on those without, so we do not use Q in constructing covariants. The reason for this dependence is due basically to the fact that the photon polarisation vector has only one independent component in the three-space defined by the scattering, so that it "sees" only one independent four-momentum. In the t-channel centre of mass frame, this independent momentum is P_μ , for both photons.

For pion Compton scattering, then, the only two covariants are

$$g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}$$

and

$$g_{\mu\rho} P_\nu P_\sigma + g_{\nu\sigma} P_\mu P_\rho - g_{\mu\sigma} P_\nu P_\rho - g_{\nu\rho} P_\mu P_\sigma \,. \qquad (2.47)$$

The corresponding invariant amplitudes are KSF and KZF, and are equal in number to the independent helicity amplitudes.

We shall now find conventional covariants $\tilde{K}_i^{\mu,\rho}$ by the prescription

$$\varepsilon^{\lambda}_{\mu\nu}(k)\, \varepsilon^{\lambda' *}_{\rho\sigma}(k')\, K_i^{\mu\nu,\rho\sigma} = \varepsilon^{\lambda}_{\mu}(k)\, \varepsilon^{\lambda' *}_{\rho}(k')\, \tilde{K}_i^{\mu,\rho} \qquad (2.48)$$

They are

$$\tilde{K}_1^{\mu,\rho} = 4\,(k.k'\, g^{\mu\rho} - k'^{\mu} k^{\rho}) = -2t\, g'^{\mu\rho}$$

$$\tilde{K}_2^{\mu,\rho} = 4\,(k.k'\, P^{\mu}P^{\rho} + k.P\, k'.P\, g^{\mu\rho} - k'.P\, P^{\mu} k^{\rho} - k.P\, k'^{\mu} P^{\rho})$$

$$= -2t\, P'^{\mu} P'^{\rho} + 4\nu^2 g'^{\mu\rho} \qquad (2.49)$$

where $g'_{\mu\rho} = g_{\mu\rho} - \dfrac{k'_\mu k_\rho}{k.k'}$, $t = -2k.k'$

and $P'_\mu = g'_{\mu\nu} P^\nu$

These two covariants are equivalent to the usual ones, and are minus twice the $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2''$ respectively of reference 15.

Nucleon Compton Scattering.[14-19,28-30.]

Again, because of PT and C invariance, not both Q and Δ can appear in the covariants. We therefore omit Δ. Again, those covariants containing Q are found to be dependent on those without, and cannot form part of a minimal set. There should be six independent covariants and in fact only six can be found. In the conventional analysis there are eight, two of which are dependent through the spinor equivalence theorems mentioned earlier.

The six covariants $K_{i\ \mu\nu,\rho\sigma}$ are

$$\begin{aligned}
&(1), (2), (3), (4) \qquad g_{\mu\rho}X_{\nu\sigma} + g_{\nu\sigma}X_{\mu\rho} - g_{\mu\sigma}X_{\nu\rho} - g_{\nu\rho}X_{\mu\sigma} \\
&\quad \text{with} \quad X_{\mu\nu} = g_{\mu\nu}\,,\ P_\mu P_\nu\,,\ P_\mu\gamma_\nu + P_\nu\gamma_\mu\,,\ [\gamma_\mu,\gamma_\nu]_- \ \text{resp.} \\
&(5) \qquad \{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\}_+ \\
&(6) \qquad P_\mu P_\rho\sigma_{\nu\sigma} + P_\nu P_\sigma \sigma_{\mu\rho} - P_\mu P_\sigma \sigma_{\nu\rho} - P_\nu P_\rho\sigma_{\mu\sigma}\ .
\end{aligned} \tag{2.50}$$

The associated invariant amplitudes A_i are KSF and KZF. The corresponding $\tilde{K}_{i\ \mu,\rho}$ are found to be

$$\begin{aligned}
\tilde{K}_1 &= -4t\,g'_{\mu\rho} \qquad\qquad \tilde{K}_2 = -2t\,P'_\mu P'_\rho + 4\nu^2 g'_{\mu\rho} \\
\tilde{K}_3 &= -2t\,(P'_\mu\gamma'_\rho + P'_\rho\gamma'_\mu) + 8\nu \not{Q}\, g'_{\mu\rho} \\
\tilde{K}_4 &= 8m\not{Q}\, g'_{\mu\rho} - 8\nu\, g'_{\mu\rho} - t\,[\gamma'_\mu,\gamma'_\rho] \\
\tilde{K}_5 &= 8\nu[\gamma'_\mu,\gamma'_\rho] + 8m(\gamma'_\mu\not{Q}\gamma'_\rho - \gamma'_\rho\not{Q}\gamma'_\mu) + 4t\,g'_{\mu\rho} \\
&\qquad - 16\,(P'_\mu\not{Q}\gamma'_\rho + \gamma'_\mu\not{Q}P'_\rho) \\
\tilde{K}_6 &= 2\nu^2[\gamma'_\mu,\gamma'_\rho] + 8m\,P'_\mu P'_\rho\not{Q} \\
&\qquad - 4\nu(P'_\mu\not{Q}\gamma'_\rho + \gamma'_\mu\not{Q}P'_\rho) - 4m\nu\,(P'_\mu\gamma'_\rho + P'_\rho\gamma'_\mu)\ .
\end{aligned} \tag{2.51}$$

These are not in the most convenient form for calculation but are exhibited in this way for comparison with earlier work[14,16].

A more convenient form would be in terms of unprimed quantities but in either case a certain amount of labour is involved reducing, for example, $[\not{K}, \not{K}']$ to $4m\not{Q}-4\nu$ (between spinors). This sort of operation is not necessary in the conventional approach, but is by no means so arduous that the conventional analysis is preferable.

The relationship between the $\tilde{K}_i$ and the equivalent sets $\tilde{\mathcal{H}}_i$ of ref.16 and $\mathcal{L}_i$ of ref.14 is

$$\begin{aligned}
\tilde{K}_1 &= -4\tilde{\mathcal{H}}_1 &&= 16\mathcal{L}_1 \\
\tilde{K}_2 &= -2\tilde{\mathcal{H}}_2 - P^2\tilde{\mathcal{H}}_1 &&= 4P^2\mathcal{L}_1 + 8\mathcal{L}_5 \\
\tilde{K}_3 &= -2\tilde{\mathcal{H}}_3 &&= 8m\mathcal{L}_1 + 8\mathcal{L}_4 \\
\tilde{K}_4 &= \tilde{\mathcal{H}}_5 &&= 8\mathcal{L}_3 \\
\tilde{K}_5 &= -\frac{2}{m}\tilde{\mathcal{H}}_6 + 4\tilde{\mathcal{H}}_1 &&= -16\mathcal{L}_1 - \frac{16}{m}\mathcal{L}_2 \\
\tilde{K}_6 &= 2m\tilde{\mathcal{H}}_4 - \nu\tilde{\mathcal{H}}_1 - \frac{t}{8}\tilde{\mathcal{H}}_5 &&= 4\nu\mathcal{L}_1 + 4P^2\mathcal{L}_3 + 8m\mathcal{L}_6 \,.
\end{aligned} \tag{2.52}$$

Consequently the relationship between our invariant amplitudes A_i and those of ref.16, $\tilde{A}_i$, is

$$\begin{aligned}
A_1 &= -\frac{1}{4}\tilde{A}_1 + \frac{P^2}{8}\tilde{A}_2 - \frac{\nu}{8m}\tilde{A}_4 - \frac{m}{2}\tilde{A}_6 \\
A_2 &= -\frac{1}{2}\tilde{A}_2 \\
A_3 &= -\frac{1}{2}\tilde{A}_3 \\
A_4 &= \tilde{A}_5 + \frac{t}{16m}\tilde{A}_4 \\
A_5 &= -\frac{m}{2}\tilde{A}_6 \\
A_6 &= \frac{1}{2m}\tilde{A}_4 \,.
\end{aligned} \tag{2.53}$$

We shall be making use of this analysis when we consider sum-rules for nucleon-Compton scattering in a later chapter.

Perturbation theory and pole diagrams.

One aspect of the electromagnetic processes we have been considering has not yet been touched upon. This is charge conservation. Weinberg[7] has shown that charge conservation in any process follows from the gauge-condition on the M-function for that process with one extra soft photon, assuming that the S-matrix has the same pole structure as in perturbation theory, in the soft photon limit[31].

In so far as this is a valid S-matrix assumption, the proof is S-matrix theoretic. Remember that the gauge condition was a consequence of Lorentz-covariance.

When we express the S-matrix in terms of antisymmetric tensor photon wave- and M-functions, no subsidiary condition on the M-function is necessary to ensure Lorentz covariance. How, then, is charge conservation expressed in terms of such M-functions? The partially reduced M-functions, obtained from the antisymmetric tensor ones, satisfy the gauge condition, so we must conclude that the tensor M-functions already satisfy a charge conservation condition. To see how they might not, we must turn to perturbation theory, expressed in terms of antisymmetric tensor fields.

We have recently developed[10] a formulation of quantum electrodynamics in terms of "tensor potentials" $\chi^{\pm}_{\mu\nu}(x)$ which are related to the field strengths $F^{\pm}_{\mu\nu}(x)$ by

$$\Box X^{\pm}_{\mu\nu}(x) = F^{\pm}_{\mu\nu}(x) \tag{2.54}$$

The fields $F^{\pm}_{\mu\nu}(k)$ are proportional to the wave functions $\varepsilon^{\pm}_{\mu\nu}(k)$, while $X^{\pm}_{\mu\nu}(k)$ are proportional to $\frac{\varepsilon^{\pm}_{\mu\nu}(k)}{k^2}$, which we denote by $\xi^{\pm}_{\mu\nu}(k)$.

The conventional amplitudes in the Landau gauge are obtained if the coupling of these fields at a vertex takes the form

$$\frac{e}{2}\, \xi_{\mu\nu}(k) \left(k^{\mu} C^{\nu} - k^{\nu} C^{\mu} \right) \tag{2.55}$$

or

$$e\, \varepsilon_{\mu\nu}(k) \left(\frac{k^{\mu} C^{\nu} - k^{\nu} C^{\mu}}{2k^2} \right) \tag{2.56}$$

as the case may be. Here C^{μ} is the conventional coupling, for example γ^{μ} at a photon-fermion-fermion vertex. The wave functions may be part of a photon propagator or they may be external. In the latter case, there is a term in the coupling which is not present in the conventional theory.

The expressions (2.55) and (2.56) reduce to

$$e\, \varepsilon_{\mu}(k) \left(C^{\mu} - \frac{k.C}{k^2} k^{\mu} \right) \tag{2.57}$$

compared with the conventional $e\, \varepsilon_{\mu}(k) C^{\mu}$.

If one can overlook for a moment the fact that $k^2 = 0$, $\left(C^\mu - \frac{k.C}{k^2}k^\mu\right)$ is clearly gauge invariant. Furthermore, $k.\varepsilon(k) = 0$ for any k^2, so the second term of (2.57) is zero for any $k^2 \neq 0$, and may, if so desired, be defined to be zero at $k^2=0$ where it is formally indeterminate (c.f. eq. 1.109 et. seq). Nevertheless, such a term is not very desirable.

Let us re-express the problem. Using the external photon wave function $\varepsilon_{\mu\nu}(k)$, the form of the M-function obtained in perturbation theory is

$$M^{\mu\nu} = \sum_i \frac{(k^\mu C_i^\nu - k^\nu C_i^\mu)}{2k^2} A_i \tag{2.58}$$

where A_i are invariant amplitudes and C_i^μ conventional four-vector covariants, such as γ^μ, p^μ, etc. These covariants are not those we used in the last section.

When this $M^{\mu\nu}$ is contracted with $\varepsilon_{\mu\nu}(k)$, there are terms proportional to $\frac{k.\varepsilon(k)}{k^2}$ in addition to the usual ones.

Now in conventional perturbation theory, we would have obtained the M-function

$$M^\mu = \sum_i C_i^\mu A_i \tag{2.59}$$

<u>with exactly the same</u> A_i [10], and this must satisfy

$$k_\mu M^\mu = 0 \quad , \tag{2.60}$$

that is, $$\sum_i (k.C_i) A_i = 0 \quad . \tag{2.61}$$

Contracting the k factors in $\varepsilon_{\mu\nu}(k)$ with $M^{\mu\nu}$, we obtain the partially contracted M-function M'^{μ}

$$\varepsilon_{\mu\nu}(k)\, M^{\mu\nu} \equiv \varepsilon_{\mu}(k)\, M'^{\mu}$$

$$= \varepsilon_{\mu}(k)\Big(\sum_i C_i^{\mu} A_i - \sum_i k.C_i A_i \frac{k^{\mu}}{k^2}\Big) \tag{2.62}$$

$$= \varepsilon_{\mu}(k) \sum_i C_i^{\mu} A_i \qquad \text{by eq. (2.61)}$$

$$= \varepsilon_{\mu}(k)\, M^{\mu} \tag{2.63}$$

The coefficient of $\frac{k^{\mu}}{k^2}$ in M'^{μ} vanishes if the conventional four-vector M-function satisfies the gauge condition, that is, if charge is conserved. So we conclude that charge-conservation imposes on the M'-function M'^{μ} the condition that no terms in $\frac{k^{\mu}}{k^2}$ appear. It may therefore be possible, if this condition is satisfied, to re-express $M^{\mu\nu}$ in terms of covariants which are free of $\frac{1}{k^2}$ terms.

To find such covariants, we consider the partially reduced forms

$$C^{\mu} - k.C\, \frac{k^{\mu}}{k^2} \quad . \tag{2.64}$$

Clearly

$$k.C_1\Big(C_2^{\mu} - k.C_2 \frac{k^{\mu}}{k^2}\Big) - k.C_2\Big(C_1^{\mu} - k.C_1 \frac{k^{\mu}}{k^2}\Big) \tag{2.65}$$

is free of terms in $\frac{1}{k^2}$, being just

$$(k.C_1)\, C_2^{\mu} - (k.C_2)\, C_1^{\mu} \quad . \tag{2.66}$$

Possibly multiplication by a suitable factor will remove such a term. For example,

$$\not{k}\left(\gamma^\mu - \not{k}\frac{k^\mu}{k^2}\right) = \not{k}\gamma^\mu - k^\mu \tag{2.67}$$

But

$$\begin{aligned} &\varepsilon_\mu(k)\,\left(k.C_1\, C_2^\mu - k.C_2\, C_1^\mu\right) \\ \text{and} \qquad &\varepsilon_\mu(k)\,\left(\not{k}\gamma^\mu - k^\mu\right) \end{aligned} \tag{2.68}$$

can be expressed in the forms

$$\begin{aligned} &\tfrac{1}{2}\,\varepsilon_{\mu\nu}(k)\left(C_1^\mu C_2^\nu - C_1^\nu C_2^\mu\right) \\ \text{and} \qquad &\tfrac{1}{2}\,\varepsilon_{\mu\nu}(k)\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \end{aligned} \tag{2.69}$$

and we recognise these new covariants as the ones introduced earlier. They do not involve k.

The process described here for deriving covariants free of $\frac{1}{k^2}$ terms from perturbation theoretic covariants is very similar to that used in the conventional analysis[14,15]. There the covariants C_i^μ are multiplied by gauge-projection operators $g_{\mu\nu} - \frac{k_\mu p_\nu}{k\cdot p}$ and then linear combinations of the resulting covariants C'^μ_i are taken to remove $\frac{1}{k\cdot p}$ terms.

In our work, it is as though we had used a gauge-projection operator $g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ on the C_i^μ to get the forms (2.64) then taken linear combinations to remove $\frac{1}{k^2}$ terms.

Let us now see what we have done in terms of the unreduced covariants $\dfrac{k_\mu C_\nu - k_\nu C_\mu}{2k^2}$ that appear in eq. (2.58).

The linear combination that we took in (2.65) is now

$$K^{\mu\nu} = k.C_1 \frac{(k^\mu C_2^\nu - k^\nu C_2^\mu)}{2k^2} - k.C_2 \frac{(k^\mu C_1^\nu - k^\nu C_1^\mu)}{2k^2} \tag{2.70}$$

and this is not free of terms in $\frac{1}{k^2}$. We notice, however, that

$$K_{\mu\nu} = \overline{\Pi}_{\mu\nu,\rho\sigma}(k) \frac{(C_1^\rho C_2^\sigma - C_1^\sigma C_2^\rho)}{2} \tag{2.71}$$

with $\overline{\Pi}_{\mu\nu,\rho\sigma}(k)$ defined in eq. (1.89).

Therefore

$$\varepsilon^{\pm}_{\mu\nu}(k) K^{\mu\nu} = \varepsilon^{\pm}_{\mu\nu}(k) \frac{(C_1^\mu C_2^\nu - C_1^\nu C_2^\mu)}{2} \tag{2.72}$$

by eq. (1.97) so that the covariant

$$\frac{1}{2}(C_1^\mu C_2^\nu - C_1^\nu C_2^\mu) \tag{2.73}$$

is equivalent to $K^{\mu\nu}$, and is clearly free of $\frac{1}{k^2}$ terms.

In a similar fashion, the unreduced form of the LHS of (2.67) is

$$\not{k} \frac{(k^\mu \gamma^\nu - k^\nu \gamma^\mu)}{2k^2} \tag{2.74}$$

which may be rewritten

$$\frac{1}{2} \overline{\Pi}^{\mu\nu,}{}_{\rho\sigma}(k) (\gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho) \quad . \tag{2.75}$$

This is equivalent to $\frac{\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu}{2}$ by virtue of (1.97) and is free of $\frac{1}{k^2}$ terms.

Notice that we can simply construct helicity convariants such that when the wave-function is contracted with them, only one helicity state gives a non-zero result. For example,

$$\varepsilon_{\pm}^{\mu\nu\,*}(k)\,\overline{\Pi}^{+}_{\mu\nu,\rho\sigma}(k)\,K^{\rho\sigma} = \begin{cases}\varepsilon^{+\,*}_{\mu\nu} K^{\mu\nu} \\ 0\end{cases} \qquad (2.76)$$

We call $(\overline{\Pi}^{+}K)^{\mu\nu}$ a helicity covariant.

Now (in obvious matrix notation),

$$\overline{\Pi}^{+}K = \overline{\Pi}\,\tfrac{1}{2}(1 + i\bar{\Lambda})\,K \qquad (2.77)$$

by eqs. (1.82) and (1.93), which is equivalent to

$$\left(\tfrac{1}{2}(1 + i\bar{\Lambda})K\right)^{\mu\nu} \equiv \tfrac{1}{2}\left(g^{\mu}{}_{\rho}g^{\nu}{}_{\sigma} + i\bar{\Lambda}^{\mu\nu,}{}_{\rho\sigma}\right)K^{\rho\sigma} \qquad (2.78)$$

(see eqn. 1.106). Another equivalent form is

$$\tfrac{1}{2}\left(g^{\mu}{}_{\rho}g^{\nu}{}_{\sigma} - \tfrac{i}{2}\varepsilon^{\mu\nu}{}_{\rho\sigma}\right)K^{\rho\sigma} \qquad (2.79)$$

(see eq. 1.107a), for example

$$\tfrac{1}{2}(B^{\mu}C^{\nu} - B^{\nu}C^{\mu}) - \tfrac{i}{4}\varepsilon^{\mu\nu}{}_{\rho\sigma}(B^{\rho}C^{\sigma} - B^{\sigma}C^{\rho}) \; . \qquad (2.80)$$

The perturbation theoretic covariants

$$\varepsilon^{\mu\nu}{}_{\rho\sigma}\left(\frac{k^{\rho}B^{\sigma} - k^{\sigma}B^{\rho}}{2k^{2}}\right) \qquad (2.81a)$$

and

$$\varepsilon^{\mu\nu}{}_{\rho\sigma}\left(\frac{k^\rho C^\sigma - k^\sigma C^\rho}{2k^2}\right) \qquad (2.81b)$$

give rise naturally to the covariant

$$\varepsilon^{\mu\nu}{}_{\rho\sigma}\left(B^\rho C^\sigma - B^\sigma C^\rho\right) \qquad (2.82)$$

in the same way as the $\dfrac{k^\mu C_i^\nu - k^\nu C_i^\mu}{2k^2}$ give rise to the $\dfrac{C_i^\mu C_j^\nu - C_i^\nu C_j^\mu}{2}$ (2.73).

Charge Conservation.

For Lorentz-covariance of the S-matrix a conventional M-function must satisfy k.M = 0. Charge is then conserved. If the invariant amplitudes are to be unconstrained by the gauge-condition, the corresponding covariants should each satisfy this condition.

If the conventional M-function is gauge-invariant, then charge is conserved and the antisymmetric tensor M-function can have no $\frac{1}{k^2}$ terms. If the invariant amplitudes are to be unconstrained, the corresponding covariants must each be free of $\frac{1}{k^2}$ terms, as were those introduced at the beginning of the chapter.

The proof of the equivalence of charge-conservation and freedom of the M-function from $\frac{1}{k^2}$ terms has rested upon results for four-vector M-functions. We shall present a direct proof of this equivalence for any scattering process with a soft extra photon, but as it follows closely that given by Weinberg[7] to show that k.M = 0 implies charge conservation, we present this latter argument first.

Let the S-matrix for the basic scattering process be S_{fi}. The ingoing particles have momenta p_i, charges e_i and the outgoing particles have momenta p_j, charges e_j. The soft photon has momentum k, helicity λ, and is outgoing (say). For simplicity let the charged particles be scalar. Then the M-function for the process with emission of the soft extra photon is proportional to (and this incorporates the pole structure assumption referred to earlier)[31]

$$M^{\mu} \sim S_{fi}\left[\sum_j \frac{e_j}{k \cdot p_j} p_j^{\mu} - \sum_i \frac{e_i}{k \cdot p_i} p_i^{\mu}\right]. \tag{2.83}$$

Then $k.M=0$ implies

$$\sum_i e_i = \sum_j e_j \quad , \tag{2.84}$$

that is, the total ingoing charge equals the total outgoing charge. The basic scattering S_{fi} may be any process at all, so charge is conserved in any process, when charge is defined as the soft-photon coupling constant.

Now in an antisymmetric tensor formulation, the S-matrix $S_{fi}^{\lambda}(k)$ for the process with an extra soft photon is proportional to

$$\varepsilon_{\mu\nu}^{\lambda *}(k)\left[\sum_j \frac{e_j}{k.p_j}\left(\frac{k^{\mu} p_j^{\nu} - k^{\nu} p_j^{\mu}}{2k^2}\right) - \sum_i \frac{e_i}{k.p_i}\left(\frac{k^{\mu} p_i^{\nu} - k^{\nu} p_i^{\mu}}{2k^2}\right)\right]$$

$$\tag{2.85}$$

$$= \varepsilon_{\mu}^{\lambda *}(k)\left[\sum_j \frac{e_j}{k.p_j}\left(p_j^{\mu} - \frac{k.p_j\, k^{\mu}}{k^2}\right) - \sum_i \frac{e_i}{k.p_i}\left(p_i^{\mu} - \frac{k.p_i\, k^{\mu}}{k^2}\right)\right].$$

It is easy to see that the $\frac{k^{\mu}}{k^2}$ terms have the coefficient

$$\sum_i e_i - \sum_j e_j$$

which vanishes if and only if charge is conserved.

Although this shows that the absence of $\frac{1}{k^2}$ terms in the M-function implies charge conservation in any process, we cannot directly conclude that charge conservation implies the absence of $\frac{1}{k^2}$ terms in the M-function when the photon is hard, for the form given above was derived in the soft-photon limit. A similar remark applies to Weinberg's demonstration that charge conservation implies the gauge condition on the photonic four-vector M-function[1].

However, if the M-function is supposed to describe equally well both hard and soft photon scattering, the gauge condition or absence of $\frac{1}{k^2}$ terms must hold for all photon energies, as the M-function is expanded in terms of covariants which separately satisfy the appropriate condition at any energy.

(Similar arguments for the case of gravitons[7] show that universality of the graviton coupling constant and conservation of momentum imply that the four-index M-functions are free of $\frac{1}{(k^2)^2}$ terms, as appear in the perturbation theoretic couplings[10]. Graviton four-index covariants will therefore be of the form $K^{\mu\nu} K^{\rho\sigma}$ where $K^{\mu\nu}$ is an anti-symmetric tensor covariant such as (2.77), KSF and KZF)

Photon Propagators.

In writing down the amplitude corresponding to a diagram involving internal photon exchange, we need to know the form of the propagator. This is well known in the four-vector description, and we now give it when the photon is described by an anti-symmetric tensor wave function.

If $\varepsilon_{\mu\nu}^{\lambda}(k)$ is used, the propagator for helicity $\pm$ 1 photons is[10]

$$\frac{\varepsilon^{\pm}_{\mu\nu}(k)\,\varepsilon^{\pm *}_{\rho\sigma}(k)}{k^2+i\varepsilon} \tag{2.86}$$

$$= \frac{\Pi^{\pm}_{\mu\nu,\rho\sigma}(k)}{k^2+i\varepsilon} \tag{2.87}$$

$$= \frac{1}{2(k^2+i\varepsilon)}\left[\Pi_{\mu\nu,\rho\sigma}(k) \pm i\,\Lambda_{\mu\nu,\rho\sigma}(k)\right] \quad . \tag{2.88}$$

With couplings of the form $\frac{k^\mu C^\nu - k^\nu C^\mu}{k^2}$ at either end of the internal line, the term in $\Lambda_{\mu\nu,\rho\sigma}(k)$ vanishes. With the Pauli-type couplings $\sigma_{\mu\nu}$ at each end, the two helicity states give complex conjugate contributions to the amplitude.

Weinberg[4] has given the general expressions for massless particle propagators in terms of the spin matrices in $(j,0)\oplus(0,j)$ bases. They are not manifestly covariant however, and for particles of spin one or more are more awkward to use than those bearing tensor indices.

THE MASSLESS PARTICLE CROSSING MATRIX.

The Crossing Matrix for Helicity Amplitudes.

The helicity amplitudes for a given scattering process are defined in a specific channel. The sets of helicity amplitudes for different channels are related and between them exist orthogonal transformations, the crossing matrices. These have been derived by Trueman and Wick for processes where all particles are massive[32-35] and by Trueman and Mueller for processes with one or more massless particles[36,37].

We indicate the general idea of these derivations. The S-channel helicity amplitude is defined, up to a phase, to be

$$T^{s}_{\lambda_c,\lambda_D;\lambda_A,\lambda_B}(s,t,u) \equiv \langle p_c,\lambda_c|\langle p_D,\lambda_D|\, T\, |p_A,\lambda_A\rangle|p_B,\lambda_B\rangle \tag{3.1}$$

where $p_c + p_D = p_A + p_B$, the centre of mass condition.

The generalised s-channel helicity amplitude $T^{s}_{\lambda_c\lambda_D;\lambda_A\lambda_B}\{p_i\}$ is defined in the same way[32], but without the c.m. condition. Under a Lorentz transformation Λ , the state $|p_i, \lambda_i\rangle$ transforms, as we saw in Chapter 1, as

$$U(\Lambda)|p_i,\lambda_i\rangle = \sum_{\lambda_i'} D^{s_i}_{\lambda_i'\lambda_i}[R(p_i,\Lambda)]\,|\Lambda p_i,\lambda_i'\rangle \tag{3.2}$$

where s_i is the spin of particle i, and $R(p_i,\Lambda)$ is the Wigner rotation.

Because the T-matrix is Lorentz-invariant, we see that

$$T^{s}_{\lambda_c\lambda_D;\lambda_A\lambda_B}\{p_i\} =$$

$$= \sum_{\lambda'_i} D^{*}_{\lambda'_c\lambda_c}(R_c)\, D^{*}_{\lambda'_D\lambda_D}(R_D) \times \tag{3.3}$$

$$\times\, T^{s}_{\lambda'_c\lambda'_D;\lambda'_A\lambda'_B}\{\Lambda p_i\}\, D_{\lambda'_A\lambda_A}(R_A)\, D_{\lambda'_B\lambda_B}(R_B)\,.$$

A similar expression holds for the t-channel generalized helicity amplitude $T^{t}_{\mu_c\mu_{\bar{A}};\mu_{\bar{D}}\mu_B}(p_c,-p_A;-p_D,p_B)$,where we will still be in the s-channel physical region:

$$T^{t}_{\mu_c\mu_{\bar{A}};\mu_{\bar{D}}\mu_B}(p_c,-p_A;-p_D,p_B) =$$

$$= \sum_{\mu'_i} D^{*}_{\mu'_c\mu_c}[R(p_c,\Lambda)]\, D^{*}_{\mu'_{\bar{A}}\mu_{\bar{A}}}[R(-p_A,\Lambda)] \times$$

$$\times\, T^{t}_{\mu'_c\mu'_{\bar{A}};\mu'_{\bar{D}}\mu'_B}(\Lambda p_c,-\Lambda p_A;-\Lambda p_D,\Lambda p_B) \times \tag{3.4}$$

$$\times\, D_{\mu'_{\bar{D}}\mu_{\bar{D}}}[R(-p_D,\Lambda)]\, D_{\mu'_B\mu_B}[R(p_B,\Lambda)]\,.$$

Now the factors $D[R(-p_D,\Lambda)]$ and $D^{*}[R(-p_A,\Lambda)]$ are determined by analytic continuation from positive timelike values of $-p_D$, $-p_A$ on the respective mass shells. The values will depend on the path of continuation. One determination is[32]

$$D^{*}_{\mu'\mu}[R(-p_A,\Lambda)] = (-1)^{\mu'-\mu}\, D_{\mu'\mu}[R(p_A,\Lambda)] \quad , \tag{3.5}$$

$$D_{\mu'\mu}[R(-p_D,\Lambda)] = (-1)^{\mu-\mu'}\, D^{*}_{\mu'\mu}[R(p_D,\Lambda)] \quad . \tag{3.6}$$

Then $$T^{s}_{\lambda_c \lambda_D ; \lambda_A \lambda_B}(p_c, p_D ; p_A, p_B)$$

and $$T^{t}_{\lambda_c \lambda_A ; \lambda_D \lambda_B}(p_c, -p_A ; -p_D, p_B)$$

transform in the same way, so that if there is any connection between them, they are equal up to a factor.

We then let the momenta satisfy the t-channel c.m. condition, so that (up to a numerical factor)

$$T^{s}_{\lambda_c, \lambda_D ; \lambda_A, \lambda_B}(p_c, p_D ; p_A, p_B) = T^{t}_{\lambda_c, \lambda_{\bar{A}} = \lambda_A ; \lambda_{\bar{D}} = \lambda_D, \lambda_B}(s, t, u) \, . \tag{3.7}$$

Now let Λ in eq. (3.3) be such that Λp_i satisfy the s-channel c.m. condition. Then

$$T^{s}_{\lambda_c \lambda_D ; \lambda_A \lambda_B}\{p_i\} =$$
$$= \sum_{\lambda'_i} D^{*}_{\lambda'_c \lambda_c}(R_c) \, D^{*}_{\lambda'_D \lambda_D}(R_D) \times$$
$$\times \, T^{s}_{\lambda'_c \lambda'_D ; \lambda'_A \lambda'_B}(s, t, u) \, D_{\lambda'_A \lambda_A}(R_A) \, D_{\lambda'_B \lambda_B}(R_B) \, . \tag{3.8}$$

Inserting this into eq. (3.7) we obtain

$$T^{t}_{\lambda_c \lambda_{\bar{A}} ; \lambda_{\bar{D}} \lambda_B}(s, t, u) =$$
$$= \sum_{\lambda'_i} D^{*}_{\lambda'_c \lambda_c}[R(p_c, \Lambda)] \, D^{*}_{\lambda'_D \lambda_{\bar{D}}}[R(p_D, \Lambda)] \times$$
$$\times \, T^{s}_{\lambda'_c \lambda'_D ; \lambda'_A \lambda'_B}(s, t, u) \, D_{\lambda'_A \lambda_{\bar{A}}}[R(p_A, \Lambda)] \, D_{\lambda'_B \lambda_B}[R(p_B, \Lambda)] \, , \tag{3.9}$$

and it remains only to calculate the D factors on the right hand side. Because Λ takes the t channel cm frame into the s channel cm frame, it will in general be complex. The arguments of the D functions are the crossing angles and Leader[38] gives expressions for their sines and cosines.

In continuing this expression from say the s-physical region to the t-physical region*, the momenta p_i and the transformation Λ will all change. The final form will depend on the path of continuation, as did the relation between s and t channel generalized helicity amplitudes.

For a massive particle, there are two possible ways of continuing p from positive-timelike to negative-timelike values[32]. $|\underline{p}|$ has branch points at $p_0 = \pm m$ and these should be avoided. p_0 can then continue from a value $> m$ to a value $< -m$ by going into the complex plane, crossing the real axis inside or outside the interval $(-m,m)$, and the two types of path yield different relations between

$$D\,[R(p,\Lambda)] \quad \text{and} \quad D\,[R(-p,\Lambda)]\ .$$

p_0 should cross the real axis in the continuation so that we end up on the t-physical sheet.

For a massless particle, there is only one type of path and[36]

$$D^*_{\mu\lambda}\,[R(p,\Lambda)] = D_{-\mu\,-\lambda}\,[R(-p,\Lambda)] \tag{3.10}$$

---

* We assume that such a path of continuation, avoiding the branch points at the channel thresholds, exists. For massive particle processes, the existence has been proved by S. Bros, H. Epstein and V. Glaser, Commun.Math.Phys. 1, 240 (1965).

Using this relation in eqs. (3.3) and (3.4) we see that the s- and t-channel generalized helicity amplitudes with opposite values of the massless particle helicity transform in the same way. Furthermore, since a Lorentz transformation cannot change the helicity of a massless particle, the corresponding D-function in eq. (3.9) is proportional to $\delta_{\lambda\lambda'}$ (see also eq. 1.31)

The result is that the helicity of a massless particle is reversed in crossing from one channel to the other if and only if it is outgoing in one channel and ingoing in the other.

A rather more transparent way to derive the crossing matrix is through the covariant helicity formalism of Feldman and Matthews[39]. For massive particles this has been done by Feldman and King[35]. We present here the derivation for massless particles which has certain features of interest.[40]

We start by discussing the covariant helicity (felicity) formalism with reference to the scattering process defined as at the beginning of Chapter 2 with no restriction on the particle masses.

The Covariant Helicity Formalism.

We recall that the Pauli-Lubanski four-vector operator $W_\mu(i)$ for particle i is defined by

$$W_\mu(i) = -\tfrac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,J^{\nu\rho}(i)\,P^\sigma(i) \tag{3.11}$$

where for each i, $J^{\nu\rho}(i)$ and $P^\sigma(i)$ obey the commutation relations of a Poincare algebra, $P_\mu(i)$ is the four-momentum operator for particle i, and $J_{\nu\rho}(i)$ its rotation and boost operators, which we rewrite in the form

$$\underline{J} \equiv (J_{23}\,,\, J_{31}\,,\, J_{12})$$

$$\underline{K} \equiv (J_{10}\,,\, J_{20}\,,\, J_{30}) \quad . \tag{3.12}$$

As we shall always be concerned with eigenstates of the momentum operators $P_{\mu}(i)$, we replace the latter by their eigenvalues $p_{\mu}(i)$ in what follows.

The s-channel felicity operators for particles A and B are defined[39] by

$$F^{s}(A) \equiv \frac{2}{\Delta(s,A,B)} W_{\mu}(A)\, p^{\mu}(B) = \frac{2}{\Delta} W_{\mu}(A)\left[p^{\mu}(B) + p^{\mu}(A)\right] \tag{3.13}$$

$$F^{s}(B) \equiv \frac{2}{\Delta(s,A,B)} W_{\mu}(B)\, p^{\mu}(A) = \frac{2}{\Delta} W_{\mu}(B)\left[p^{\mu}(A) + p^{\mu}(B)\right] \quad . \tag{3.14}$$

Using eqs. (3.11) and (3.12) we may write eq. (3.13) as

$$F^{s}(A) = \frac{2}{\Delta(s,A,B)}\Big[\underline{J}(A)\,.\,\big(p^{\circ}(B)\,\underline{p}(A) - p^{\circ}(A)\,\underline{p}(B)\big) - \\ - \underline{K}(A)\,.\,\big(\underline{p}(B)\times\underline{p}(A)\big)\Big] \tag{3.15}$$

and similarly for eq. (3.14):

$$F^{s}(B) = \frac{2}{\Delta(s,A,B)}\Big[\underline{J}(B)\,.\,\big(p^{\circ}(A)\,\underline{p}(B) - p^{\circ}(B)\,\underline{p}(A)\big) - \\ - \underline{K}(B)\,.\,\big(\underline{p}(A)\times\underline{p}(B)\big)\Big] \tag{3.16}$$

where $\underline{p}(i)$ is the space part of $p_{\mu}(i)$ and

$$\Delta^{2}(s,A,B) = s^{2} + m_A^{4} + m_B^{4} - 2s\,m_A^{2} - 2s\,m_B^{2} - 2m_A^{2}m_B^{2} \quad .$$

We may rewrite eqs. (3.15) and (3.16) by introducing right-handed sets of unit three-vectors $\hat{\underset{\sim}{p}}$, $\underset{\sim}{m}$ and $\underset{\sim}{n}$ for each of particles A and B.

Here $\underset{\sim}{m}(A)$ is the unit vector in the direction of $\underset{\sim}{p}(B)\times\underset{\sim}{p}(A)$; $\underset{\sim}{n}(A)$ is the unit vector in the direction of $\underset{\sim}{p}(A)\times\underset{\sim}{m}(A)$ and similarly for $\underset{\sim}{m}(B)$ and $\underset{\sim}{n}(B)$. In terms of these vectors,

$$F^s(A) = \frac{2}{\Delta(s,A,B)}\Big[\underset{\sim}{J}(A).\hat{\underset{\sim}{p}}(A)\big(|\underset{\sim}{p}(A)|\,p_0(B) - p_0(A)\,\underset{\sim}{p}(B).\hat{\underset{\sim}{p}}(A)\big) - $$
$$- |\underset{\sim}{p}(A)\times\underset{\sim}{p}(B)|\Big(\underset{\sim}{J}(A).\underset{\sim}{n}(A)\frac{p_0(A)}{|\underset{\sim}{p}(A)|} + \underset{\sim}{K}(A).\underset{\sim}{m}(A)\Big)\Big] \tag{3.17}$$

$$F^s(B) = \;.\;.\;.\;.\quad (A \leftrightarrow B)\quad . \tag{3.18}$$

Further, the t-channel felicities of $\bar{A}$ and B are defined by

$$F^t(\bar{A}) \equiv \frac{2}{\Delta(t,A,C)}\, W_\mu(\bar{A})\, p^\mu(c) \tag{3.19}$$

$$F^t(B) \equiv \frac{2}{\Delta(t,B,D)}\, W_\mu(B)\, p^\mu(\bar{D})\quad . \tag{3.20}$$

Now, since the felicity operator is Lorentz invariant, it may be calculated in any convenient frame. In the s-channel c.m. frame, $\underset{\sim}{p}(A) = -\underset{\sim}{p}(B)$, so that

$$F^s(A) = \frac{2}{\Delta(s,A,B)}\, W_0(A)\,(p^0(A) + p^0(B))$$
$$= \frac{2\sqrt{s}}{\Delta(s,A,B)}\, \underset{\sim}{J}(A)\cdot\underset{\sim}{p}(A) \tag{3.21}$$
$$= \underset{\sim}{J}(A).\,\hat{\underset{\sim}{p}}(A)\quad ,$$

and

$$F^s(B) = \underline{J}(B) \cdot \hat{\underline{p}}(B) \; , \tag{3.22}$$

which are respectively the s-channel c.m. helicities of particles A and B.

Similarly, calculating $F^t(\bar{A})$, and $F^t(B)$ in the t-channel c.m. frame, we find that they are respectively the t-channel c.m. helicities of particles $\bar{A}$ and B.

Massless Particles.

Let us now take particle B to be massless. Its eigenstates are $|p(B), \lambda_B\rangle$ where the helicity λ_B is fixed, and is the eigenvalue of the helicity operator $\Lambda(B)$, the generator of rotations in the little group E(2) of $p(B)$. The two translation generators are Π_m and Π_n, which have zero eigenvalues in the one-dimensional representations of E(2) to which massless particles belong. In terms of the vectors $\underline{m}(B)$ and $\underline{n}(B)$ introduced earlier,

$$\begin{aligned} \Pi_m &= \underline{J}(B) \cdot \underline{m}(B) - \underline{K}(B) \cdot \underline{n}(B) \\ \Pi_n &= \underline{J}(B) \cdot \underline{n}(B) + \underline{K}(B) \cdot \underline{m}(B) \; . \end{aligned} \tag{3.23}$$

For $p(B)$ physical (real, with $p_0(B) > 0$), $\Lambda(B)$ is given by $\underline{J}(B) \cdot \hat{\underline{p}}(B)$, and its matrix elements, for all $p(B)$, are

$$\Lambda_{\lambda_B, \mu_B} \equiv \langle p(B), \lambda_B | \Lambda(B) | p(B), \mu_B \rangle = \lambda_B \, \delta_{\lambda_B, \mu_B} \; , \tag{3.24}$$

where we have omitted the momentum delta function.

Now $F^s(B)$ acts in the space of two-particle states of A and B, but if $P(A)$ is replaced by its eigenvalue $p(A)$ in $F^s(B)$, we may think of it as a single-particle operator in the space of states of B. With this in mind, we see from eq. (3.18) that $F^s(B)$ is a linear combination of $\underline{J}(B)\cdot\hat{\underline{p}}(B)$ and Π_n (since $p_0(B)=|\underline{p}(B)|$), and conclude that its eigenstates are just the helicity eigenstates of B, at least for p(B) physical.

We find

$$F^s_{\lambda_B,\mu_B}(B) \equiv \langle p(B),\lambda_B|\,F^s(B)\,|p(B),\mu_B\rangle = \frac{s-m_A^2}{\Delta(s,A,B)}\,\lambda_B\,\delta_{\lambda_B,\mu_B} \tag{3.25}$$

for p(B) physical, so if $s>m_A^2$, $F^s_{\lambda_B,\mu_B}(B)=\Lambda_{\lambda_B,\mu_B}(B)$. We may now fix p(B) and continue in p(A) so that $s-m_A^2$ becomes negative. In this case, $\Lambda_{\lambda,\mu}(B)$ is clearly unaffected, but $F^s_{\lambda,\mu}(B)$ may change sign unless $\Delta(s,A,B)$ is taken to continue as $s-m_A^2$. With this choice, $F^s_{\lambda,\mu}(B)$ and $\Lambda_{\lambda,\mu}(B)$ are identical for all s, and p(B) physical*. Now $\Lambda_{\lambda,\mu}(B)$ is by definition invariant under any complex Lorentz transformation (which may take p(B) to an unphysical value), and $F^s_{\lambda,\mu}(B)$ is invariant by construction, so that $F^s_{\lambda,\mu}(B)$ and $\Lambda_{\lambda,\mu}(B)$ are

---

*We could have made the other choice. The felicity and helicity would then have been of opposite sign for $s<m_A^2$, and the felicity amplitudes and crossing matrix would be different from, though simply related to, the helicity amplitudes and crossing matrix. Our impending calculation of the helicity crossing matrix would be altered in detail, but not in its result.

identical for all p(B) and s, and the eigenstates of the two operators are identical (in our representation).

As $F^s_{\lambda,\mu}(B)$ and $\Lambda_{\lambda,\mu}(B)$ are identical in this representation we may find an explicit form for $\Lambda(B)$ from eq. (3.18):

$$\Lambda_{\lambda,\mu}(B) = F^s_{\lambda,\mu}(B)$$

$$= \langle p(B),\lambda_B| \frac{2}{s-m_A^2} \Big[\frac{\underline{J}(B)\cdot\underline{p}(B)}{p_0(B)} \Big(p_0(A)p_0(B) - \frac{p_0(B)^2}{|\underline{p}(B)|^2}\underline{p}(A)\cdot\underline{p}(B) \Big) - \tag{3.26}$$

$$- |\underline{p}(A)\times\underline{p}(B)|\,\Pi_n \Big] | p(B),\mu_B\rangle$$

$$= \langle p(B),\lambda_B| \frac{\underline{J}(B)\cdot\underline{p}(B)}{p_0(B)} | p(B),\mu_B\rangle \quad .$$

Therefore $\frac{\underline{J}\cdot\underline{p}}{p_0}$ is the correct form for the massless particle helicity operator, which agrees with the argument of Chapter 1. If Λ is expressed in the form $\frac{\underline{J}\cdot\underline{p}}{(\underline{p}^2)^{1/2}}$, the sign of the square root is uniquely determined.

It is interesting to note that in those representations in which $\Lambda(B)$ and $F^s(B)$ are not identical, namely those with $W^2 \neq 0$ in which the massless particle has continuous spin and Π_m, Π_n are non-zero, the helicity is not Lorentz-invariant, as it differs from the invariant felicity by the non-invariant Π_n term.

The Crossing Matrix.

We now consider the behaviour of particle B under crossing. We calculate $F^s(B)$ and $F^t(B)$ in the same frame, so that the momenta p(B) appearing in the two expressions are the same. A convenient choice is an s-channel c.m. frame such that the scattering is in the yz-plane and $p_\mu(B) = (p,0,0,p)$.

From eq. (3.22) we find

$$F^s(B) = J_3(B) . \tag{3.27}$$

From eq. (3.20), expanded in the manner of eqs. (3.15) and (3.16) we find

$$\begin{aligned} F^t(B) &= \frac{2}{\Delta(t,B,D)} \left[J_3(B)\left(p_0(\bar{D}) - p_3(\bar{D})\right)p - \Pi_n(B)\, p_2(\bar{D})\, p \right] \\ &= J_3(B) , \end{aligned} \tag{3.28}$$

in the representation, with our choice of sign of $\Delta(t,B,D)$ for $t < m_D^2$. Therefore

$$F^s(B) = F^t(B) , \tag{3.29}$$

and the eigenstates are related, up to a phase by

$$| p(B), \lambda_s \rangle = | p(B), \lambda_t = \lambda_s \rangle , \tag{3.30}$$

where λ_s and λ_t are the s- and t- channel c.m. helicities of particle B (for a massless particle, the helicity is the same in any frame, and so is the same as the c.m. helicity).

From eq. (3.30) we see that the crossing relation between the s- and t- channel helicity amplitudes, in the spin space of particle B, is, apart from a possible phase

$$T^s_{\cdot\,\cdot\,;\,\cdot\,\lambda_B}(s,t,\mu) = T^t_{\cdot\,\cdot\,;\,\cdot\,\lambda_B}(s,t,\mu) , \tag{3.31}$$

since the general-frame t-channel helicity amplitude is the same as the c.m. frame helicity amplitude as far as the massless particle label is concerned.

Now let particle A, rather than B, be massless and let us calculate $F^s(A)$ and $F^t(\bar{A})$ in an s-channel c.m. frame, with the scattering in the yz-plane and $p_\mu(A) = -p_\mu(\bar{A}) = (q,0,0,q)$.

From eq. (3.21)

$$F^s(A) = J_3(A) \tag{3.32}$$

From eq. (3.19), expanded in the manner of eqs. (3.15) and (3.16)

$$\begin{aligned} F^t(\bar{A}) &= \frac{2}{\Delta(t,A,C)}\left[-J_3(\bar{A})\left(p_0(C)-p_3(C)\right)q + \Pi_n(\bar{A})\,q\,p_2(\bar{D})\right] \\ &= J_3(\bar{A}) \end{aligned} \tag{3.33}$$

in our representation.

The spin operators $\underline{J}(A)$ and $\underline{J}(\bar{A})$ of particle and antiparticle are related by

$$\underline{J}(A) = -\underline{J}^T(\bar{A}) \; . \tag{3.34}$$

Hence $J_3(A) = -J_3(\bar{A})$, as the representation is one-dimensional (and in any case, diagonal for J_3). Therefore

$$F^s(A) = -F^t(\bar{A}) \tag{3.35}$$

and the eigenstates are related up to a phase by

$$|\,p(A),\lambda_s\rangle = |\,p(\bar{A}) = -p(A),\ \lambda_t = -\lambda_s\rangle \tag{3.36}$$

Therefore the s-channel c.m. helicity of A is opposite to the t-channel c.m. helicity of A and the crossing relation, in the spin space of particle A is, up to a phase

$$T^{s}_{\cdot\,\cdot\,;\,\lambda_A\,\cdot}(s,t,u) = T^{t}_{\cdot\,-\lambda_A\,;\,\cdot\,\cdot}(s,t,u) \; . \tag{3.37}$$

Eqs. (3.31) and (3.36) show that the c.m. helicity of the massless particle changes if and only if the particle is crossed in going from one channel to the other. This statement is independent of the particle momentum or the values of s,t, and u, because the helcities are. In other words, the crossing angle is either 0 or π, and is constant in the entire (s,t,u) plane.

The extension of this analysis to processes involving more than one massless particle is trivial, as the reasoning applies to each particle independently. The crossing-matrix in the product space of the (remaining) massive particles has been given by Trueman and Wick[32].

Determination of $\Delta(x,i,j)$, ($m_i = 0$).

We would like to comment on the determination of $\Delta(x,i,j)$ below the pseudo-threshold $x = (m_i - m_j)^2$ to be used in the expressions for the centre-of-mass scattering angles and the crossing angles of the massive particles in the reaction. Leader's expressions[38] for the cosines of the crossing angles may be used for the massive particles, with the appropriate masses set equal to zero, because the masslessness of other particles makes no essential difference to the reasoning used to obtain these expressions. However, there is an ambiguity in the interpretation of $\Delta(x,i,j)$.

In earlier sections of this paper we used the analytic determination of $\Delta(x,i,j)$ when m_i or m_j was zero, but this choice was not essential to our result; it merely enabled us to arrive at this result more easily. Consequently, we must decide all over again how we are to interpret Δ in the cosine expressions.

We show that, when a massless particle is present, we are forced by consistency considerations to choose $\Delta(x,i,j),(m_i=0)$, to be analytic in the x-plane. With this determination of Δ, Leader's expressions for the crossing angles are valid for the massless particles as well.

The boundary of the physical region has six asymptotic portions, on each of which (with the possible exception of the lines s = 0, t = 0, u = 0) the crossing and c.m. scattering angles are constant and separately equal to 0 or π. We label these portions s+, s–, t+, t–, u+, u–, where s+ is that portion of the boundary of the s-physical region on which $\cos\theta_s = +1$, $s \to \infty$, etc.

Near such a portion, to within a phase,

$$T^{s}_{\lambda_C\lambda_D;\lambda_A\lambda_B}(s,t,\mu) = T^{t}_{\lambda'_C\lambda'_{\bar{A}};\lambda'_{\bar{D}}\lambda'_B}(s,t,\mu) + \Sigma \tag{3.38}$$

where the set $\{\lambda'\}$ is determined in terms of the set $\{\lambda\}$ by the crossing angles, and Σ goes to zero on the boundary, and at least as fast as $T^{t}_{\{\lambda'\}}(s,t,\mu)$ if this amplitude vanishes there. Because Σ is a linear combination of independent t-channel helicity amplitudes, Σ and $T^{t}_{\{\lambda'\}}(s,t,\mu)$ cannot in general 'conspire' to make the right-hand side of eq. (3.38) vanish faster than $T^{t}_{\{\lambda'\}}$ on the boundary.

Furthermore, near this portion of the boundary[11]

$$T^{s}_{\lambda_C\lambda_D;\lambda_A\lambda_B}(s,t,\mu) \propto (\sin\theta_s)^{|\lambda_C-\lambda_D\mp\lambda_A\pm\lambda_B|}, \tag{3.39}$$

according as $\cos\theta_s = \pm 1$, and

$$T^{t}_{\lambda_C' \lambda_{\bar{A}}' ; \lambda_{\bar{D}}' \lambda_B'}(s,t,u) \;\; \propto \;\; (\sin\theta_t)^{|\lambda_C' - \lambda_{\bar{A}}' \mp \lambda_{\bar{D}}' \pm \lambda_B'|} , \qquad (3.40)$$

according as $\cos\theta_t = \pm 1$, so that in view of our previous remarks, the exponents in the right-hand sides of eqs. (3.39) and (3.40) must be equal. This implies an intimate connection between the crossing and scattering angles.

Let us take particle A as massless, and consider the crossing from s- to t-channel. Let us order the masses of the particles: $m_D > m_B > m_C > m_A = 0$. Then, with the definitions (consistent with our previous definitions of s and t channels)

$$\cos\theta_s = \frac{s(t-u) + m_B^2(m_D^2 - m_C^2)}{\Delta(s,A,B)\,.\,\Delta(s,C,D)} ,$$

$$\cos\theta_t = \frac{t(s-u) + m_C^2(m_D^2 - m_B^2)}{\Delta(t,B,D)\;\Delta(t,A,C)} , \qquad (3.41)$$

$$\cos\theta_u = \frac{u(s-t) + m_D^2(m_C^2 - m_B^2)}{\Delta(u,B,C)\;\Delta(u,A,D)} ,$$

which are unambiguous above the relevant channel thresholds, we see that along*

---

*We refer the reader to Figs. 1 and 2.

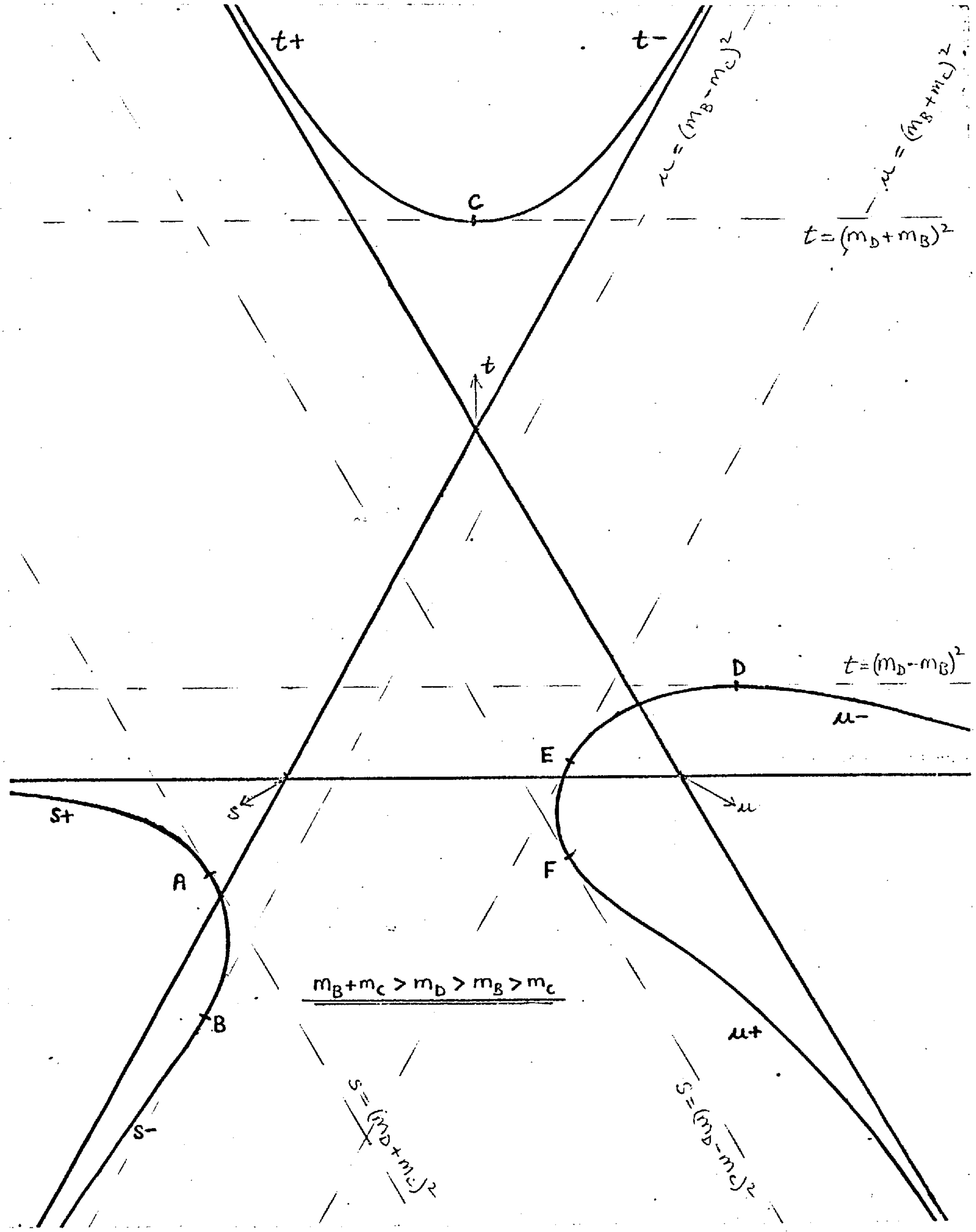

FIGURE I

The boundary of the physical region.

$m_B + m_C > m_D > m_B > m_C > m_A = 0.$

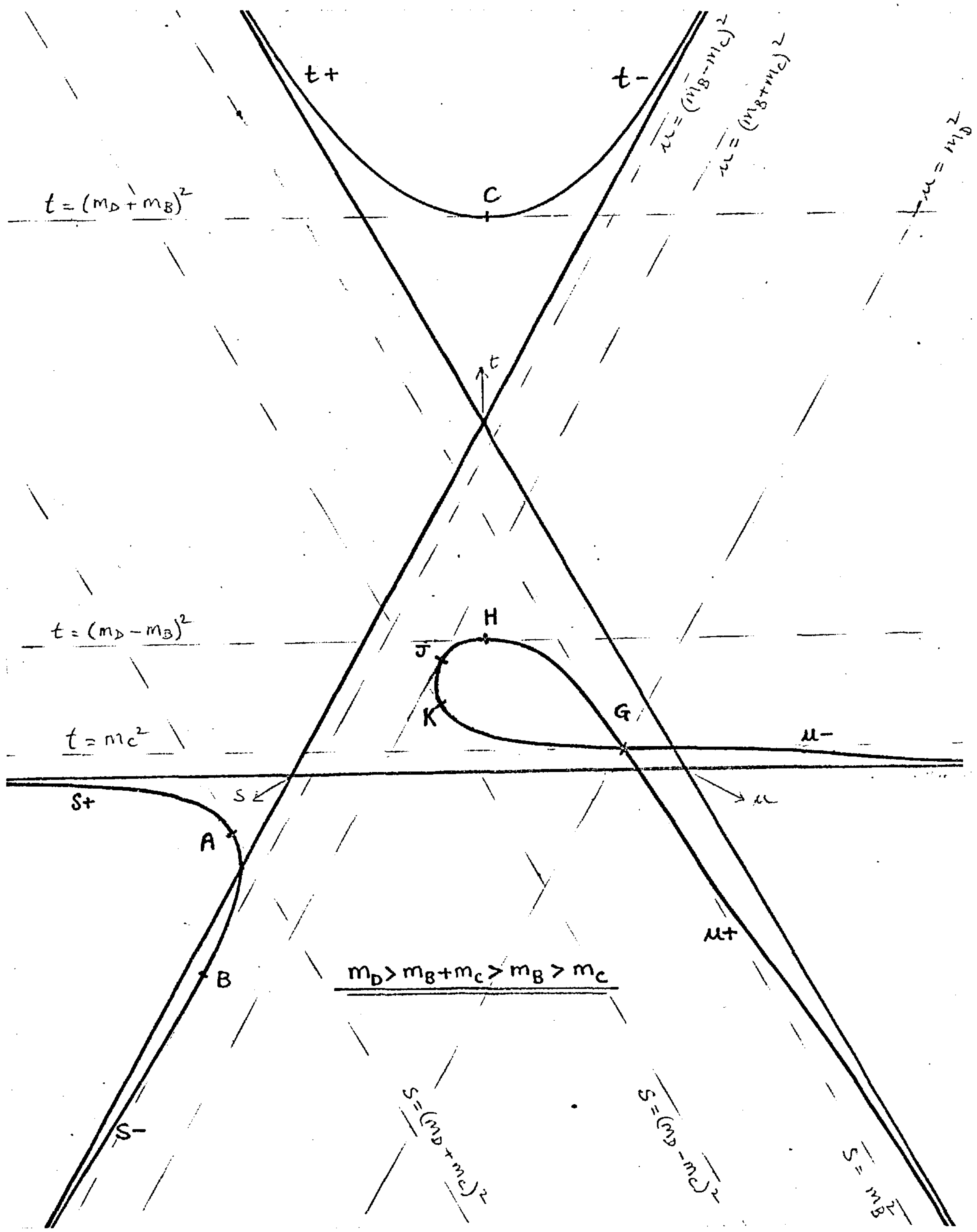

FIGURE 2.

The boundary of the physical region.

$m_D > m_B + m_C > m_B > m_C > m_A = 0.$

$$
\begin{array}{llll}
s+, & s \to +\infty, & t \to 0-, & u \to -\infty, \\
s-, & s \to +\infty, & t \to -\infty, & u \to 0+, \\
t+, & s \to 0-, & t \to +\infty, & u \to -\infty, \\
t-, & s \to -\infty, & t \to +\infty, & u \to 0-, \\
u+, & s \to 0+, & t \to -\infty, & u \to +\infty, \\
u-, & s \to -\infty, & t \to 0+, & u \to +\infty.
\end{array}
\tag{3.42}
$$

This information enables us to determine easily the sign of the cosines on any of the six portions of the boundary.

From Leader, we have the expressions for the cosines of the massive particle crossing angles:

$$
\begin{aligned}
\cos \chi_B &= \frac{(s+m_B^2)(t+m_B^2-m_D^2) - 2m_B^2(m_C^2+m_B^2-m_D^2)}{\Delta(s,A,B)\,\Delta(t,B,D)} \\
\cos \chi_C &= \frac{(s+m_C^2-m_D^2)(t+m_C^2) - 2m_C^2(m_C^2+m_B^2-m_D^2)}{\Delta(s,C,D)\,\Delta(t,A,C)} \\
\cos \chi_D &= \frac{(s+m_D^2-m_C^2)(t+m_D^2-m_B^2) + 2m_D^2(m_C^2+m_B^2-m_D^2)}{\Delta(s,C,D)\,\Delta(t,B,D)} .
\end{aligned}
\tag{3.43}
$$

Now we can choose particular determinations for the Δ and see if they are consistent. We find $\cos\theta_s$ and $\cos\theta_t$ on each of the six portions of the boundary. The equality of the exponents in eqs. (3.39) and (3.40) with the appropriate signs gives us $\{\lambda'\}$ in terms of $\{\lambda\}$ and

TABLE I

Values of the cosines of the crossing angles on the different portions of the boundary.

		s+	s−	t+	t−	u+	u−
	$\cos\theta_s$	+	−	∓	∓	∓	±
	$\cos\theta_t$	∓	∓	+	−	±	∓
	$\cos\chi_A$	−	−	−	−	−	−
from eqs. (3.39)	$\cos\chi_B$	±	±	−	+	∓	±
and (3.40), see	$\cos\chi_C$	−	+	±	±	±	∓
text	$\cos\chi_D$	±	∓	±	∓	+	+
	$\cos\chi_B$	−	−	±	∓	∓	±
from eq. (3.43)	$\cos\chi_C$	±	∓	−	−	±	∓
	$\cos\chi_D$	−	+	−	+	+	+

therefore $\cos\chi_B$, $\cos\chi_C$ and $\cos\chi_D$ (because $\cos\chi_A \equiv -1$). We can then compare these values of the cosines of the crossing angles with the values obtained from eq. (3.43).

The results are presented in table 1, which assumes the conventional determination of $\Delta(x,i,j)$ for $m_i, m_j \neq 0$, that is, the one which* gives $\Delta > 0$ for $x < (m_i - m_j)^2$. The two determinations of $\Delta(x,i,j)$,$(m_i = 0)$, are presented together; when there is more than one entry, the upper corresponds to $\Delta = |x - m_j^2|$, the lower to $\Delta = x - m_j^2$. (The entries are all +1 or -1; we give only the signs.)

It may be seen in table 1 that the lower entries, but not the upper, are consistent between lines 4-6 and lines 7-9. This means that, for consistency, $\Delta(x,i,j)$,$(m_i = 0)$, must be taken as $x - m_j^2$, the analytic form of Δ in this case. Therefore, $\Delta(s,A,B) = s - m_B^2 = 2p(A)\cdot p(B)$ always; $\Delta(t,A,C) = t - m_C^2 = -2p(A)\cdot p(C)$.

Had we used the other (analytic) determination of $\Delta(x,i,j)$, $(m_i, m_j \neq 0)$, our conclusion about the m_i=0 case would have been unaltered. This is a reflection of the fact that there are two ways of continuing, in momentum, massive particle c.m. helicity states[32,36].

---

*Here, x is taken to vary along paths which connect the upper side of the right-hand cut of the S-matrix in the x-plane with the lower side of the left-hand cut. Thus if $\Delta(x,i,j)$, $(m_i, m_j \neq 0)$ is greater than zero for x below pseudothreshold, it is analytic in the x-plane cut from $-\infty$ to $(m_i - m_j)^2$ and $(m_i + m_j)^2$ to $+\infty$ (rather than from pseudothreshold to threshold).

The conclusions are unaltered when the analysis is extended to the remaining, finite portions of the physical region boundary (on each of which it is still true that the cosines are constant and separately equal to ± 1).

With the above determinations of the functions Δ , all the crossing angle cosines are correctly given by Leader's expressions; that for $\cos\chi_A$ reducing to -1 identically.

Analogous conclusions are obtained from a consideration of s-u and t-u crossing, and when more than one massless particle is present.

It is of some interest to examine more closely the finite portions of the physical region boundary referred to above. This is because the cosines of the scattering angles and the cosines of the crossing angles do not change sign at the same points.

There are two cases to be considered (a) $m_B+m_C > m_D > m_B > m_C > m_A = 0$ and (b) $m_D > m_B+m_C > m_B > m_C > m_A = 0$ when there is a fourth physical region corresponding to the decay of $\bar{D}$ into $\bar{B}+\bar{A}+C$. These situations are depicted in Figs. 1 and 2.

The boundary of the physical region has as tangents the highest and lowest thresholds in each channel[41]. Furthermore, if in each channel both ingoing thresholds lie lower than both outgoing thresholds or vice versa, then the fourth finite region exists and has as tangents the second highest and second lowest thresholds in each channel. (See also the Appendix to this Chapter).

In general (except possibly along s=o, t=o, u=o) the scattering angle cosines are ± 1 along the boundary. Moving along the boundary the i channel scattering angle cosine changes sign at the points where the i channel thresholds are tangent to the boundary. In Fig. 1 for example, $\cos\theta_s$ changes sign at points A and F, $\cos\theta_t$ at C and D,

$\cos\theta_u$ at B and E. The numerators of the expressions for the cosines change sign at these points. The denominators also vanish but do not change sign. The same applies to the crossing angle cosines: $\cos\chi_B$ changes sign at C and D, $\cos\chi_c$ at A and F, $\cos\chi_D$ at A,F,C,D. It is simple to verify that as one moves along the boundary, the sign changes at each point occur in just the right expressions to maintain consistency.

The second case is rather different, for now the boundary has as tangents the thresholds involving the massless particle, and it crosses these instead of coming away on the same side. In Fig.2 this happens at point G. Moving along u- through G towards K or along u+ through G towards H, the cosines do not change sign. At all other points A,B,C,H,J,K the behaviour is as previously stated. Once again we obtain consistency on the finite portions AB,GH,HJ, JK, KG, but only because of the analytic determination of $\Delta(x,i,j)$, $(m_i = 0)$.

For any given process, the crossing matrix may also be calculated by contracting the M-function with particle spinors satisfying the channel conditions. The helicity amplitudes in two channels can then be related through the invariant amplitudes, but this is a tedious procedure.

The implication of the result derived above for photons is

$$\varepsilon_\mu^\lambda(p) = \varepsilon_\mu^{-\lambda *}(-p) = \varepsilon_\mu^\lambda(-p) \qquad (3.44)$$

But by eq. (1.64)

$$\varepsilon_\mu^\lambda(\hat{p}) = -\varepsilon_\mu^\lambda(-\hat{p}) \qquad \text{for } p_0 > 0 \quad , \qquad (3.45)$$

so that ε_μ^λ is a function of p/p_0 $\left(\text{c.f } \Lambda = \frac{J\cdot p}{p_0}\right)$.

APPENDIX.

The channel thresholds are tangents to the boundary of the physical region in the s-t plane, as has been remarked in ref.41. However, we have not seen a proof of this, so give a simple one here.

The boundary of the physical region is the curve $\Phi(s,t) = 0$ where one form of Φ is

$$\Phi(s,t) = \frac{1}{4s}\Delta^2(s,A,B)\Delta^2(s,C,D)\sin^2\theta_s \quad .$$

The tangent to this curve is parallel to lines of constant s, such as the s channel thresholds, when $\dfrac{\partial\Phi(s,t)}{\partial t} = 0$

But

$$\frac{\partial\Phi(s,t)}{\partial t} = -\Delta(s,A,B)\Delta(s,C,D)\cos\theta_s \quad .$$

Therefore $\Phi = \dfrac{\partial\Phi}{\partial t} = 0$ implies that

$$\Delta(s,A,B)\Delta(s,C,D) = 0$$

since not both $\cos\theta_s$ and $\sin\theta_s$ can be zero. Therefore the tangent is one of the s-channel thresholds, and all channel-thresholds are tangents (but not necessarily at points in the real s-t plane).

A similar argument holds for the other channels.

CHAPTER 4

SUM RULES IN NUCLEON COMPTON SCATTERING.

We come now to an application of some of the preceding work. It is the derivation of sum rules for nucleon-Compton scattering amplitudes on the assumption that these satisfy fixed -t dispersion relations and are dominated at high-energies by the contributions from Regge-pole exchange in the crossed (t) channel.

It is easy to deduce[42] the high -s behaviour of t-channel helicity amplitudes on the Regge-assumption. The sum-rules however involve s-channel amplitudes, and we consequently need to know the crossing-matrix between the two sets of amplitudes.

Alternatively we may express the s-channel helicity amplitudes in terms of the invariant amplitudes, and the high -s behaviour of the latter immediately becomes clear.

We consider foward nucleon-Compton scattering, relating the amplitudes, via the optical theorem, to photoproduction cross-sections. The sum-rules equate integrals over these cross-sections to Born-term parameters, or the parameters of the leading Regge-pole in the cross-channel.

Amplitudes, the Crossing Matrix, and the Optical Theorem.

At t=0 there are two independent s-channel $(\gamma N \to \gamma N)$ helicity amplitudes[43-46] $T^{s}_{1,\frac{1}{2};1,\frac{1}{2}}(s,t,u)$ and $T^{s}_{1,-\frac{1}{2};1,-\frac{1}{2}}(s,t,u)$ which we denote by T_A and T_P respectively, and two independent t-channel $(N\bar{N} \to \gamma\gamma)$ helicity amplitudes $T^{t}_{1,-1;\frac{1}{2},\frac{1}{2}}(s,t,u)$ and $T^{t}_{1,-1;\frac{1}{2},-\frac{1}{2}}(s,t,u)$ which we denote by $T_{2,0}$ and $T_{2,1}$ in the usual way.

Near t=0, the crossing matrix[32,38] gives

$$T_{2,0} = T_{-2,0} + O(t^{1/2}) = \tfrac{1}{2}\left(T_P + T_A\right) + O(t^{1/2})$$

$$T_{2,1} = -T_{-2,1} + O(t^{1/2}) = \tfrac{1}{2}\left(T_P - T_A\right) + O(t^{1/2}) \,. \tag{4.1}$$

An examination of the relation between the helicity amplitudes and the invariant amplitudes[15,16] shows that

$$\begin{aligned} T_{2,0}(s,0,u) &\sim \nu^2\left(mA_2(s,0,u) + 2A_3(s,0,u)\right) \\ T_{2,1}(s,0,u) &\sim \nu^2 m A_6(s,0,u) \,. \end{aligned} \tag{4.2}$$

In fact, $T_{2,0}$ and $T_{2,1}$ are related to the usual forward Compton amplitudes[43] $f_1(\nu)$ and $f_2(\nu)$ by

$$\begin{aligned} T_{2,0}(s,0,u) &= 8\pi m f_1(\nu) \\ T_{2,1}(s,0,u) &= -8\pi m \nu f_2(\nu) \end{aligned} \tag{4.3}$$

where $\nu = \frac{s-u}{4m}$, equal at t=0 to the photon lab. energy, and m is the nucleon mass.

The kinematic-singularity-free amplitudes $\hat{T}_{\lambda,\mu}(s,t,u)$ are given[47,48] by

$$\hat{T}_{\lambda,\mu}(s,t,u) = t^{-1}(t-4m^2)^{-\frac{1}{2}} \left(\frac{1+Z_t}{2}\right)^{-|\lambda+\mu|}\left(\frac{1-Z_t}{2}\right)^{-|\lambda-\mu|} \times \\ \times\; T_{\lambda,\mu}(s,t,u) \,, \tag{4.4}$$

where $Z_t \equiv \cos\theta_t = \dfrac{s-u}{\sqrt{t(t-4m^2)}}$.

This may also be seen from the invariant amplitude expansion[16].

Under crossing $s \leftrightarrow u$, in which the two photons are interchanged, the process remains $\gamma N \rightarrow \gamma N$ and

$$\hat{T}_{2,0}(s,t,u) = \hat{T}_{-2,0}(u,t,s)\,, \quad \hat{T}_{2,1}(s,t,u) = -\hat{T}_{-2,1}(u,t,s) \tag{4.5}$$

We form the new amplitudes

$$\begin{aligned} \hat{T}_+(s,t,u) &\equiv \hat{T}_{2,0} + \hat{T}_{-2,0} \quad , \\ \hat{T}_-(s,t,u) &\equiv \hat{T}_{2,1} + \hat{T}_{-2,1} \quad , \end{aligned} \tag{4.6}$$

which are respectively even and odd under crossing.

At t=0

$$\begin{aligned} \hat{T}_+(\nu,t=0) &= (2m\nu)^{-2}\left[T_{2,0}(\nu,0) + T_{-2,0}(\nu,0)\right] \\ \hat{T}_-(\nu,0) &= (2m\nu)^{-2}\left[T_{2,1}(\nu,0) - T_{-2,1}(\nu,0)\right] \ . \end{aligned} \tag{4.7}$$

The crossing properties of $\hat{T}_\pm(\nu,0)$ are also clear from the behaviour of A_2, A_3, A_6 under $s \leftrightarrow u$ $(\nu \leftrightarrow -\nu, \text{at } t=0)$. They are respectively even, even and odd, since K_2, K_3 and K_6 are respectively even, even and odd under the interchange of initial and final photon indices.

The definitions show that $\hat{T}_\pm(\nu,0)$ are parity-conserving amplitudes[11] and their asymptotic dependence on ν is two powers lower than that of the $T_{\lambda,\mu}$ in terms of which they are defined.

The optical theorem for the s-channel amplitudes reads

$$\mathrm{Im}\, T_{P,A}(\nu,0) = 2m\nu\, \sigma_{P,A}(\nu) \ , \tag{4.8}$$

where $\sigma_P(\nu)$ and $\sigma_A(\nu)$ are the photoproduction cross sections defined in ref. 49. By eqs. (4.1), (4.7) and (4.8) we deduce

$$\mathrm{Im}\,\hat{T}_{\pm}(\nu,0) = (2m\nu)^{-1}\left(\sigma_P(\nu) \pm \sigma_A(\nu)\right) . \tag{4.9}$$

Finally, we require the t-channel isospin amplitudes in terms of the s-channel γp and γn amplitudes. The amplitudes $T_{\gamma p} \pm T_{\gamma n}$ correspond to isospin 0 and 1 respectively in the t-channel.

The Partial-Wave Expansion and Regge Poles.

The t-channel amplitudes have the partial-wave expansion

$$T_{\lambda,\mu}(s,t,u) = \tfrac{1}{2}\sum_J{}' (2J+1)\Big[T^{J+}_{\lambda,\mu}(t)\left(d^J_{\lambda,\mu}(z_t) + (-1)^{\mu} d^J_{-\lambda,\mu}(-z_t)\right) + \\ + T^{J-}_{\lambda,\mu}(t)\left(d^J_{\lambda,\mu}(z_t) - (-1)^{\mu} d^J_{-\lambda,\mu}(-z_t)\right)\Big] \tag{4.10}$$

where $T^{J\pm}_{\lambda,\mu}(t) = \langle JM\lambda_c\lambda_D|T(t)|JM\lambda_A\lambda_B\rangle$,J even (odd), with $\lambda_c - \lambda_D = \lambda$, $\lambda_A - \lambda_B = \mu$.

The symmetry[11] of $d^J_{\lambda,\mu}(z_t)$ and $T^J_{\lambda,\mu}(t)$ may be used to show the equivalence of (4.10) with the familiar form

$$T_{\lambda,\mu}(s,t,u) = \sum_J{}' (2J+1)\, T^J_{\lambda,\mu}(t)\, d^J_{\lambda,\mu}(z_t) . \tag{4.11}$$

We now Reggeise the amplitudes $T_{\lambda,\mu}$ by performing the Sommerfeld-Watson transform on the partial-wave expansion (4.10) and obtain[50,51] the representation,

$$T_{\lambda,\mu}(s,t,u) = \sum_{poles}{}' \frac{2\alpha_i(t)+1}{\sin\pi\alpha_i(t)}\, e^{i\pi\alpha_i(t)} \beta^+_i(t)\left(d^{\alpha_i(t)}_{\lambda,\mu}(z_t) + (-1)^{\mu} d^{\alpha_i(t)}_{-\lambda,\mu}(-z_t)\right) + \\ + \sum_{poles}{}' \frac{2\alpha_j(t)+1}{\sin\pi\alpha_j(t)}\, e^{i\pi\alpha_j(t)} \beta^-_j(t)\left(d^{\alpha_j(t)}_{\lambda,\mu}(z_t) - (-1)^{\mu} d^{\alpha_j(t)}_{-\lambda,\mu}(-z_t)\right) \tag{4.12}$$

+ background integral.

At high ν, $d^{\alpha_i(t)}_{\lambda,\mu}(z_t) \sim \nu^{\alpha_i(t)}$. It follows from eq. (4.4) that the amplitudes $\hat{T}_{\pm}(\nu,t)$ have asymptotic behaviour $\nu^{\alpha_m(t)-2}$ where $\alpha_m(t)$ is the greatest of the $\alpha_i(t)$. Alternatively we could have made a partial wave expansion in the t-channel of the invariant amplitudes and deduced the ν -dependence of $\hat{T}_{\pm}$ via eq. (4.7). We assume that this behaviour persists at t=0 where, in fact, Z_t is infinite. The $\alpha_i(t)$ appearing in eq. (4.12) for any given process depend on the relevant selection rules.

By conservation of B, Y, Q, I, C and P at each vertex, the Regge trajectory must have quantum number assignments B = Y = Q = 0, I=0 or 1, C even. The amplitudes $\hat{T}_{\pm}(\nu,0)$ are parity-conserving and have contributions only from normal and abnormal trajectories respectively (a normal trajectory has the same parity as signature, while the parity and signature of an abnormal trajectory are opposite. This is a natural extension of the concept of normal and abnormal particles, respectively).

Thus we find that trajectories contributing to $\hat{T}_{+}(\nu,0)$ are neutral bosons with zero hypercharge, isospin 0 or 1, C even, and normal parity. The most important are the two vacuum trajectories* P,P', and the A_2 trajectory.

---

*There has been some discussion as to whether the P trajectory couples to two photons at t=0. The explicit factor $\alpha_P(t)-1$ in the amplitude would appear to decouple the trajectory at this point[52]. More recently, Abarbanel et. al.[53] have pointed out that a fixed pole in the angular moment plane at J=1 can restore the Pomeranchuk contribution. We therefore include it in our analysis.

Trajectories contributing to $\hat{T}_-(\nu,0)$ differ only in having abnormal parity. The π and η are forbidden in this amplitude by C conservation at the $N\bar{N}$-pole vertex, so that only the D and A_1 may be exchanged.

SUPERCONVERGENCE RELATIONS.

By the definition of ν and $\hat{T}_\pm(\nu,t)$ the amplitudes $\nu^{-2N-1}\hat{T}_+(\nu,t)$ and $\nu^{-2N}\hat{T}_-(\nu,t)$, with integral $N \geq 0$, are odd under crossing $s \leftrightarrow u$. Furthermore for $t \leq 0$, they are bounded at large ν by $|\nu|^{-1-\delta}$, $\delta > 0$. This follows from (4.4) and (4.12) and the fact that for $\hat{T}_+(\nu,t)$, $\alpha_m(t) \leq 1$ and for $\hat{T}_-(\nu,t)$, $\alpha_m(t) < 1$, $t \leq 0$ (see preceding section).

Therefore we are able to write the superconvergence relations

$$\int_{\nu_0}^{\infty} d\nu \ \nu^{-2N-1} \, \mathrm{Im}\, \hat{T}_+(\nu,t) = \tfrac{1}{2}\pi \times \text{sum of Born term residues.} \tag{4.13}$$

and correspondingly for $\nu^{-2N}\hat{T}_-(\nu,t)$.

At t=0 we may use the optical theorem, eq. (4.9) and write eq. (4.13) as

$$\begin{aligned} \int_{\nu_0}^{\infty} d\nu \ \nu^{-2N-2} \left(\sigma_{\bar{p}}(\nu) + \sigma_A(\nu)\right) &= m\pi \times \text{Born term residues,} \\ \int_{\nu_0}^{\infty} d\nu \ \nu^{-2N-1} \left(\sigma_{\bar{p}}(\nu) - \sigma_A(\nu)\right) &= m\pi \times \text{Born term residues.} \end{aligned} \tag{4.14}$$

The most familiar of these relations is for the amplitude $\hat{T}_-(\nu,0)$.

Expressed in the form (4.14) , and using eqs. (4.1), (4.3), (4.6) and (4.7) , it is[49].

$$\int_{\nu_0}^{\infty} \frac{d\nu}{\nu} \left(\sigma_P(\nu) - \sigma_A(\nu)\right) = -4\pi^2 f_2(0) = \frac{2\pi^2\alpha}{m^2} \kappa^2 , \tag{4.15}$$

where $\alpha = \frac{1}{137}$ and κ is the anomalous magnetic moment of the nucleon. This is the Drell-Hearn sum rule derived in the manner suggested by Choudhury[46].

The next most familiar relation is for $\nu^{-1}\hat{T}_+(\nu,0)$, and in the form (4.14) is

$$\int_{\nu_0}^{\infty} d\nu \, \nu^{-2} \left(\sigma_P(\nu) + \sigma_A(\nu)\right) = -2\pi^2 f_1''(0) = 2\pi^2 m \left(\alpha_E + \alpha_M\right) , \tag{4.16}$$

which is the original Gell-Mann, Goldberger and Thirring[43] subtracted dispersion relation for $f_1(\nu)$, taken at $\nu = 0$, and where α_E and α_M are the electric and magnetic polarizabilities of the nucleon.[54,55]

In general, the superconvergence relations (4.14) yield

$$\int_{\nu_0}^{\infty} d\nu \, \nu^{-2N-2} \left(\sigma_P(\nu) + \sigma_A(\nu)\right) = \frac{-4\pi^2}{(2N+2)!} f_1^{(2N+2)}(0) , \tag{4.17}$$

$$\int_{\nu_0}^{\infty} d\nu \, \nu^{-2N-1} \left(\sigma_P(\nu) - \sigma_A(\nu)\right) = \frac{-4\pi^2}{(2N)!} f_2^{(2N)}(0) ,$$

the values of the right hand sides being dependent on the detailed structure of the nucleon.

Finite-Energy Sum Rules.

The amplitudes $\nu^{2N+1}\hat{T}_+(\nu,t)$ and $\nu^{2N+2}\hat{T}_-(\nu,t)$, $N\geqslant 0$, are also odd under crossing $s \leftrightarrow u$. They are not superconvergent but have asymptotic behaviour in ν of $\nu^{\alpha_m(t)+2N-1}$ and $\nu^{\alpha_m(t)+2N}$ respectively. However we may form the superconvergent amplitudes $\nu^{2N+1}(\hat{T}_+ - \hat{T}_{+R})$ and $\nu^{2N+2}(\hat{T}_- - \hat{T}_{-R})$ where $\hat{T}_{\pm R}$ is that (analytic) part of $\hat{T}_\pm$ which is not bounded by $\nu^{-2N-2-\varepsilon}$ and $\nu^{-2N-3-\varepsilon}$ respectively. In the Regge-pole model, $\hat{T}_{\pm R}$ is the sum of the (analytic) contributions from all allowed poles with $\alpha(t) > -2N$ and -2N-1 respectively.

We therefore write[56, 57]

$$\int_0^\infty d\nu\, \nu^{2N+1}\, \mathrm{Im}\left[\hat{T}_+(\nu,t) - \hat{T}_{+R}(\nu,t)\right] = 0 \,, \tag{4.18}$$

and similarly for $\nu^{2N+2}\hat{T}_-(\nu,t)$. There are no Born terms in these amplitudes, excepting $\nu\hat{T}_+$. Since the integrand in (4.18) is $O(\nu^{-1-\delta})$, by construction, we have

$$\int_{\nu_0}^A d\nu\, \nu^{2N+1}\, \mathrm{Im}\,\hat{T}_+(\nu,t) = \int_0^A d\nu\, \nu^{2N+1}\, \mathrm{Im}\,\hat{T}_{+R}(\nu,t) + O(A^{-\delta}) \,, \tag{4.19}$$

and similarly for $\hat{T}_-(\nu,t)$.

Making the approximation $\hat{T}_R(\nu,t) = (\text{const.})\,\nu^{\alpha_m(t)-2}$ where $\alpha_m(t)$ corresponds to the leading trajectory, eq. (4.19) becomes[57,58]

$$S_{2N} \equiv \frac{1}{A}\int_{\nu_0}^A d\nu \left(\frac{\nu}{A}\right)^{2N+1} \mathrm{Im}\,\hat{T}_+(\nu,t) = \frac{\mathrm{Im}\,\hat{T}_{+R}(A,t)}{\alpha_m(t)+2N}\left(1 + O(A^{-\delta'})\right) ,$$

$$S_{2N+1} \equiv \frac{1}{A}\int_{\nu_0}^A d\nu \left(\frac{\nu}{A}\right)^{2N+2} \mathrm{Im}\,\hat{T}_-(\nu,t) = \frac{\mathrm{Im}\,\hat{T}_{-R}(A,t)}{\alpha_m(t)+2N+1}\left(1 + O(A^{-\delta''})\right). \tag{4.20}$$

At t=0, and large A, we find finally

$$2m\,A.S_{2N} \equiv \frac{1}{A}\int_{\nu_0}^{A} d\nu \left(\frac{\nu}{A}\right)^{2N} \left(\sigma_P(\nu)+\sigma_A(\nu)\right) = \frac{\sigma_P(A)+\sigma_A(A)}{\alpha_m(0)+2N},$$

$$2m\,A.S_{2N+1} \equiv \frac{1}{A}\int_{\nu_0}^{A} d\nu \left(\frac{\nu}{A}\right)^{2N+1} \left(\sigma_P(\nu)-\sigma_A(\nu)\right) = \frac{\sigma_P(A)-\sigma_A(A)}{\alpha_m(0)+2N+1}. \tag{4.21}$$

A similar expression for πN scattering was derived by Gatto[59].

A slight modification to (4.18) must be made when the plus signs are taken and N=0. This is because $\nu \hat{T}_+(\nu,0)$ has a Born term $(4\pi/m)(f_1(\nu)/\nu)$ which contributes to the right-hand sides of eqs. (4.18)-(4.21). In (4.21) this contribution appears as $4\pi^2 f_1(0)/A$. Since $\sigma_P(\nu)+\sigma_A(\nu)$ has asymptotic behaviour $\nu^{\alpha_m(0)-1}$, and is presumably dominated by the P and A_2 trajectories which have $\alpha(0)>0$, the Born term will be, in comparison, of no significance at high energies. Our result (4.21) is therefore correct for any $N \geqslant 0$.

The most interesting of the relations (4.21) are those with small N in which the higher-energy cross sections are not heavily weighted. In particular, the sum rules

$$\frac{1}{A}\int_{\nu_0}^{A} d\nu \left(\sigma_P(\nu)+\sigma_A(\nu)\right) = \frac{\sigma_P(A)+\sigma_A(A)}{\alpha_m(0)},$$

$$\frac{1}{A}\int_{\nu_0}^{A} d\nu \left(\frac{\nu}{A}\right) \left(\sigma_P(\nu)-\sigma_A(\nu)\right) = \frac{\sigma_P(A)+\sigma_A(A)}{\alpha_m(0)+1}, \tag{4.22}$$

may possibly be evaluated with some accuracy. We discuss these matters further in the next section.

In concluding this section we point out that superconvergence relations may be cast into the finite energy sum rule form. We illustrate this with an example.

The amplitude $\hat{T}_-(\nu,0)$ is superconvergent and

$$\int_{\nu_0}^{A} d\nu\, \mathrm{Im}\,\hat{T}_-(\nu,0) + \int_{A}^{\infty} d\nu\, \mathrm{Im}\,\hat{T}_-(\nu,0) = \frac{\pi^2\alpha}{m^3} K^2 . \tag{4.23}$$

In the second integral, we approximate $\hat{T}_-(\nu,0)$ by the Regge form as before. In the integration the contribution from the infinite upper limit vanishes, since $\int^{\nu} d\nu'\, \mathrm{Im}\,\hat{T}_-(\nu',0) = O(\nu^{-\delta})$, and we obtain

$$\int_{\nu_0}^{A} \frac{d\nu}{\nu}\left(\sigma_P(\nu)-\sigma_A(\nu)\right) + \frac{\sigma_P(A)-\sigma_A(A)}{1-\alpha_m(0)} = \frac{2\pi^2\alpha}{m^2} K^2 \tag{4.24}$$

We may therefore evaluate the Drell-Hearn sum-rule with data up to a finite energy only, since the term $[\sigma_P(A)-\sigma_A(A)]/[1-\alpha_m(0)]$ provides the necessary correction once $\alpha_m(0)$ is known. For the I=0 channel, the leading trajectory is D, and for the I=1 channel, A_1. The corresponding quantities in (4.24) are $(\sigma_P-\sigma_A)_{\gamma p} \pm (\sigma_P-\sigma_A)_{\gamma n}$, $\alpha_D(0)$, $\alpha_{A_1}(0)$ and $K_p^2 \pm K_n^2$. For the mixed isospin channels of pure γp or pure γn scattering, the leading trajectory is the A_1. (4.24) therefore applies to either protons or neutrons with $\alpha_m(0) = \alpha_{A_1}(0) \approx 0$. We conclude therefore that the reasonable agreement of the two sides of (4.15) obtained in ref. 49 with data up to 900 MeV is not likely to be upset by higher-energy data, since at this energy the correction term is already small.

Using the Sum Rules.

Sum rules of the form (4.21) and (4.24) provide a very physical means of either checking analyticity (from which they follow) or determining the Regge parameters $\alpha_m(0)$.

Dolen, Horn and Schmid[58] propose finding $\alpha_m(t)$ from the ratio $S_{2N}: S_{2N+2M} = (\alpha_m(t)+2N+2M):(\alpha_m(t)+2N)$ or similarly with S_{2N+1}, where the S_{2N} are to be calculated from low-energy data. At t=0, this is particularly simple by virtue of the optical theorem (see eq. (4.21)). We think, however, that more accurate results might be obtained (at t=0) using the single sum rule S_{2N}, since higher moment rules, which are necessarily introduced by the ratio method, weight preferentially the higher energy data which is less precise. The only advantage of the ratio method is that $\sigma_p(A)\pm\sigma_A(A)$ need not be accurately known insofar as it appears in the right-hand side of (4.21) , but since it must be known if the left-hand side is to be evaluated, the advantage is minimal, at any rate in the asymptotic energy region.

At lower energies, where $\sigma_p(A)\pm\sigma_A(A)$ is showing some direct-channel resonance fluctuations, it may still be possible to use the ratio method, but not a single sum-rule unless a modified value of $\sigma_p(A)\pm\sigma_A(A)$ is used on the right-hand side. This modified value may be obtained by drawing a smooth curve of the form $\nu^{\alpha-1}$ through the experimental curves. The reason for this procedure may be seen by examining (4.18). This says that $\operatorname{Im}\nu^n\hat{T}_{\pm R}(\nu,0)$ is in a sense the smoothed-out absorptive part of $\nu^n\hat{T}_{\pm}(\nu,0)$ (see ref 58 for comments on this and related points) and eq. (4.19) says that the (Regge) form $\nu^{\alpha-1}$ drawn to fit the experimental curves may be expected to be a good approximation to $\nu \, \mathrm{Im} \, \hat{T}_{\pm R}(\nu, 0)$ which is the quantity appearing on the right of (4.20) and which is to be found. The conditions that (4.20) be good do not necessarily require that A be above the resonance region, whereas eq. (4.21) as it stands is valid only in the asymptotic region of the cross sections.

As we remarked earlier, the most interesting sum rules are S_0 and S_1, in addition to the two superconvergence sum rules (4.15) and (4.16). There is some hope of evaluating them with reasonable accuracy over the resonance region, if not now, in the forseeable future.

In order to determine Regge parameters $\alpha_m(0)$ we make the isospin decompositions of S_0 and S_1:

$$\frac{1}{A}\int_{\nu_0}^{A} d\nu \, \left(\sigma_P(\nu) + \sigma_A(\nu)\right)_{\gamma p + \gamma n} = \frac{\left(\sigma_P(A) + \sigma_A(A)\right)_{\gamma p + \gamma n}}{\alpha_P(0)}, \quad I = 0,$$

$$\frac{1}{A}\int_{\nu_0}^{A} d\nu \, \left(\sigma_P(\nu) + \sigma_A(\nu)\right)_{\gamma p - \gamma n} = \frac{\left(\sigma_P(A) + \sigma_A(A)\right)_{\gamma p - \gamma n}}{\alpha_{A_2}(0)}, \quad I = 1,$$

$$\frac{1}{A}\int_{\nu_0}^{A} d\nu \left(\frac{\nu}{A}\right) \left(\sigma_P(\nu) - \sigma_A(\nu)\right)_{\gamma p + \gamma n} = \frac{\left(\sigma_P(A) - \sigma_A(A)\right)_{\gamma p + \gamma n}}{\alpha_D(0) + 1}, \quad I = 0,$$

$$\frac{1}{A}\int_{\nu_0}^{A} d\nu \left(\frac{\nu}{A}\right) \left(\sigma_P(\nu) - \sigma_A(\nu)\right)_{\gamma p - \gamma n} = \frac{\left(\sigma_P(A) - \sigma_A(A)\right)_{\gamma p - \gamma n}}{\alpha_{A_1}(0) + 1}, \quad I = 1, \tag{4.25}$$

where $(\sigma)_{\gamma p \pm \gamma n}$ means $\sigma^{\gamma p} \pm \sigma^{\gamma n}$. The appropriate trajectories were determined in a previous section.

An evaluation of these four sum rules should provide values for $\alpha_P(0)$, $\alpha_{A_2}(0)$, $\alpha_D(0)$ and $\alpha_{A_1}(0)$; S_0 should be easier to calculate than S_1 because it involves unpolarised cross sections only. Neither the I=0 or I=1 parts of a sum rule S_i will be easy to evaluate separately because of the grave lack of γn photoproduction data. At present we can hope only to evaluate the sum rules (4.22) corresponding to the mixed isospin process $\gamma p \to$ total and in which $\alpha_m(0)$ will be $\alpha_P(0)$ and $\alpha_{A_1}(0)$ respectively, since $\alpha_P(0) > \alpha_{A_2}(0)$ and $\alpha_{A_1}(0) > \alpha_D(0)$ (if these latter two trajectories are parallel).

Total photoproduction[60] data up to 5 GeV photon lab. energy yield, via eq. (4.22) a value of $\alpha_P(0)$ of approximately .85 . As the total cross section appears to be more or less constant from 2 GeV upwards, we can expect the value of $\alpha_P(0)$ obtained to approach 1 as higher and higher energy data are taken into account.

We see therefore that the easiest sum rule to use provides the least interesting information viz. the value $\alpha_P(0)$, and with less accuracy than other methods. The data are not good enough to calculate $\alpha_{A_1}(0)$. The sum rules are therefore at the moment more interesting as a check on the postulate of analyticity, using known values of the Regge parameters, than as a means of calculating these parameters, which are more readily found from phenomenological fits to hadron-hadron scattering amplitudes.

REFERENCES

1. E.P. Wigner, Ann.Math. 40, 149 (1939).

2. S. Schweber, "An Introduction to Relativistic Quantum Field Theory" Evanston, Illinois, 1961.

3. E.P. Wigner, in "Theoretical Physics" IAEA, Vienna, 1963.

4. S. Weinberg, Phys. Rev. 134, B882 (1964).

5. G. Feldman and P.T. Matthews, Ann. Phys. 40, 19 (1966).

6. D. Zwanziger, Phys. Rev. 113, B1036 (1964).

7. S. Weinberg, Phys. Rev. 135, B1049 (1964).

8. V. Bargmann and E.P. Wigner, Proc.Natl.Acad.Sci. 34, 211 (1948).

9. C.M. Bender, Phys. Rev. 168, 1809 (1968).

10. J.R. Fox, "A New Formulation of Quantum Electrodynamics" Imperial College Preprint ICTP/68/38.

11. M. Jacob and G.C. Wick, Ann. Phys. 7, 404 (1959).

12. H. Stapp, Phys. Rev. 125, 2139 (1962).

13. G. Chew, M. Goldberger, F. Low and Y. Nambu, Phys. Rev. 106, 1345 (1957).

14. W.A. Bardeen and W.K. Tung, Phys. Rev. 173, 1423 (1968).

15. H.F. Jones and M.D. Scadron, "Covariant M-Functions for Higher Spin II: Photons". Imperial College Preprint ICTP/67/26.

16. H.F. Jones and M.D. Scadron, Nucl. Phys. B10, 17 (1969).

17. K. Yamamoto, Phys. Rev. 169, 1353 (1968).

18. A.C. Hearn and E. Leader, Phys. Rev. 126, 789 (1962).

19. J.R. Fox, "Photon M-Functions" Imperial College Preprint ICTP/68/39.

20. K. Hepp, Helv. Phys. Acta 36, 355 (1963).

21. D.N. Williams, unpublished.

22. H.F. Jones and M.D. Scadron, Phys. Rev. 173, 1734 (1968).

23. S. Weinberg, Phys. Rev. 138, B988 (1965).

24. J. Ball, Phys. Rev. 124, 2014 (1961).

25. K. Bardacki and H. Pagels, Phys. Rev. 166, 1783 (1968).

26. F. Henyey, Phys. Rev. 170, 1619 (1968).

27. V. Singh, Phys. Rev. Letts., 19, 730 (1967).

28. V. Singh, Phys. Rev. 165, 1532 (1968).

29. R. Prange, Phys. Rev. 110, 240 (1958).

30. J.S. Vaishya, Nuovo Cimento 51A, 536 (1967).

31. F.E. Low, Phys. Rev. 110, 974 (1958).

32. T.L. Trueman and G.C. Wick, Ann. Phys. 26, 322 (1964).

33. I.J. Muzinich, J. Math. Phys. 5, 1481 (1964).

34. G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. 46, 239 (1968).

35. G. Feldman and M. King, Nuovo Cimento 60, 86 (1969).

36. A.H. Mueller and T.L. Trueman, Phys. Rev. 160, 1306 (1967).

37. J.P. Ader, M. Capdeville, and H. Navelet, Nuovo Cimento 56, 315 (1968).

38. E. Leader, Phys. Rev. 166, 1599 (1968).

39. G. Feldman and P.T. Matthews, Phys. Rev. 168, 1587 (1968).

40. J.R. Fox, Nucl. Phys. B11, 186 (1969).

41. T.W.B. Kibble, Phys. Rev. 117, 1159 (1960).

42. T.L. Trueman, Phys. Rev. Letts. 17, 1198 (1966).

43. M. Gell-Mann, M.L. Goldberger and W. Thirring, Phys. Rev. 95, 1612 (1954).

44. F.E. Low, Phys. Rev. 96, 1428 (1954).

45. M. Gell-Mann and M.L. Goldberger, Phys. Rev. 96, 1433 (1954).

46. S.R. Choudhury, Phys. Rev. Letts. 19, 96 (1967).

47. Y. Hara, Phys. Rev. 136, B507 (1964).

48. L.L. Wang, Phys. Rev. 142, 1187 (1966).

49. S. Drell and A.C. Hearn, Phys. Rev. Letts 16, 908 (1966).

50. F. Calogero, J.M. Charap and E.J. Squires, Ann.Phys. 25, 325 (1963)

51. M. Gell-Mann, M. Goldberger, F.E. Low, E. Marx, and F. Zachariasen. Phys. Rev. 133, B145 (1964).

52. H.D.I. Abarbanel and S. Nussinov, Phys. Rev. 158, 1462 (1967).

53. H.D.I. Abarbanel, F.E. Low, I.J. Muzinich, S. Nussinov and J.H. Schwarz, Phys. Rev. 160, 1329 (1967).

54. V. Goldansky, O. Karpukhin, A. Kutsenko and V. Pavlovskaya, Nucl. Phys. 18, 473 (1960).

55. S.R. Choudhury & D.Z. Freedman, Phys. Rev. 168, 1739 (1968).

56. K. Igi and S. Matsuda, Phys. Rev. Letts. 18, 625 (1967).

57. A. Logunov, L. Soloviev and A. Tavkhelidze, Phys.Letts. 19, 402 (1967)

58. R. Dolen, D. Horn and C. Schmid, Phys. Rev. Letts. 19, 402 (1967).

59. R. Gatto, Phys. Rev. Letts. 18, 803 (1967)

60. D.E.S.Y. Report 68/31, June 1968.

----------oOo----------

ICTP/68/38

A NEW FORMULATION OF QUANTUM ELECTRODYNAMICS

By

J. R. FOX

Physics Department
Imperial College
LONDON SW7

JULY 1969

The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research OAR through the European Office of Aerospace Research, United States Air Force.

We formulate quantized electromagnetic field theory in terms of "tensor potentials". None of the usual difficulties, such as indefinite metric or unphysical particles arises. Gauge considerations do not enter. The compatibility with axiomatic field theory is discussed.

Our fields interact locally with the electromagnetic current and the resultant quantum electrodynamics reproduces exactly the usual results.

A similar solution to the problems of spin-2 graviton theory is indicated.

I. INTRODUCTION

It has long been recognised that the usual formulation of quantized electromagnetic theory in terms of the potentials $A_\mu(x)$ suffers from theoretical difficulties (not least that of possibly not existing within the framework of axiomatic field theory[1,2]) due basically to the fact that a four-vector wave-function is unsuitable for the description of a massless spin-1 particle[3]. The work of Weinberg[3,4] exposes this fact and shows how, in the usual treatment, Lorentz-covariance of the S-matrix is ensured by the imposition of the gauge condition on the M-functions, equivalent to the conservation of charge. In spite of the elegance of this analysis, it unfortunately remains true that such a non-trivial theory is incompatible with axiomatic field theory*. Similar remarks apply to the usual theory of spin-2 gravitons.

We learn from Weinberg that we can avoid these difficulties if we describe the photon in its two helicity states by means of fields transforming with $(a,a+1) \oplus (a+1,a)$ representations of the Lorentz group. The simplest and obvious choice is $(1,0) \oplus (0,1)$ and this corresponds to fields antisymmetric in two indices, such as the electromagnetic field tensor $F_{\mu\nu}(x)$. In spin-2 graviton theory, the corresponding choice is the Riemann tensor $R_{\mu\nu\rho\sigma}(x)$ with suitable covariant subsidiary conditions.

In Section II we discuss briefly the tensor $F_{\mu\nu}(x)$, concluding as usual that it is unsuitable, as the interaction Lagrangian is non-local.

In section III we introduce the "tensor potentials", showing that the usual minimal coupling is local in terms of these. Their properties are deduced and, in section IV, the classical theory discussed.

---

*See Footnote 9, reference 2.

The tensor potentials are quantized, in the manner of Weinberg[3], in section V. The commutation relations and the momentum-space propagators are written down, and the positive definiteness of the Hamiltonian and the metric of the space of states of the theory is demonstrated.

In section VI we discuss the compatibility with axiomatic field theory, concluding that our theory may almost be cast into this form. We show that arguments similar to those in ref. (1) which might be expected to imply triviality do not actually do so because the two-point functions are not Green's functions and do not satisfy the equations of motion. An outline of a rigorous theory, with comments on this last point, is given in the Appendix.

We further discuss in section VI the derivation of our theory from a Lagrangian formalism.

In section VII we modify the Feynman rules, but find that the matrix elements of the conventional theory are reproduced exactly.

Section VIII deals with certain indeterminate terms in the matrix elements, arising from the external photon wave functions. The absence of these terms is shown to be equivalent to the gauge-invariance of the conventional theory.

Section IX deals with charge conservation.

Finally, in section X, the extension of the preceding work to higher-spin massless fields is indicated.

II. THE ELECTROMAGNETIC FIELD TENSOR

A theory formulated in terms of $F_{\mu\nu}(x)$ suffers from none of the drawbacks mentioned above, and if suitable couplings with other fields are chosen, yields exactly the usual results. The use of the corresponding tensor wave-function*[4,5]

$$\varepsilon_{\mu\nu}{}^{\lambda}(k) \equiv k_{\mu}\ \varepsilon_{\nu}{}^{\lambda}(k) - k_{\nu}\varepsilon_{\mu}{}^{\lambda}(k)$$

in S-matrix theory simplifies considerably the usual M-function analysis, as we shall see in a subsequent paper, because gauge considerations no longer enter.

The crux of the matter is the choice of coupling. The usual interaction Lagrangian is

$$\mathcal{L}_{int} = j^{\mu}(k)A_{\mu}(k) \tag{1}$$

and this is reproduced by the form, in terms of $F_{\mu\nu}(k)$,

$$\mathcal{L}_{int} = -j^{\mu}(k)\ \frac{k^{\nu}}{k^{2}}\ F_{\mu\nu}\ (k) \tag{2}$$

since

$$F_{\mu\nu}(k) = k_{\mu}A_{\nu}(k) - k_{\nu}A_{\mu}(k) \qquad \text{and} \qquad k^{\mu}j_{\mu}(k) = 0.$$

With this form of coupling we are able to describe soft-photon processes and long range forces, contrary to the usual statements[2,5], but in configuration space this interaction is non-local, being

---

*This form is a consequence of Lorentz covariance[3,4], and not of the known relation between $A_{\mu}(k)$ and $F_{\mu\nu}(k)$. Because of this relation, however, we may identify $\varepsilon_{\mu}{}^{\lambda}(k)$ with the usual polarisation vector.

$$\mathcal{L}_{\text{int}}(x) = -i\, j^{\mu}(x)\, \frac{\partial^{x}}{\Box} F_{\mu\nu}(x) \tag{3}$$

It would therefore appear that we have merely exchanged one problem for another, albeit a more tractable one, by using the electromagnetic fields.

III. THE TENSOR POTENTIALS

Although forced to abandon $F_{\mu\nu}(x)$, we would still like to formulate our theory in terms of an antisymmetric tensor field. We notice that if we define a new field $X_{\mu\nu}(x)$, the "electromagnetic tensor potential", by

$$X_{\mu\nu}(x) \equiv \frac{1}{\Box} F_{\mu\nu}(x) \tag{4}$$

the interaction Lagrangian (3) takes the local* (and minimal) form

$$\mathcal{L}_{int} = - i \; j_{\mu}(x)\partial_{\nu}X^{\mu\nu}(x) \tag{5}$$

Notice that current conservation is not necessary for the invariance of this interaction.

We now elaborate on the definition of $X_{\mu\nu}(x)$. By (4) we mean precisely

$$X_{\mu\nu}(x) = \int d^4x' \; \bar{D}(x-x') \; F_{\mu\nu}(x') \tag{6}$$

The Green's function $\bar{D}(x)$ is half the sum of the advanced and retarded functions, and we indicate the reason for this choice when we consider field commutation relations in the quantized theory.

Clearly

$$\Box X_{\mu\nu}(x) = F_{\mu\nu}(x) \tag{7}$$

so that Maxwell's equations read

$$\Box \partial_{\mu}X^{\mu\nu}(x) = 0 \tag{8}$$

---

*We wish to stress this point. The interaction is local in terms of our fundamental fields $X_{\mu\nu}(x)$.

$$\Box \; \varepsilon_{\mu\nu\rho\sigma} \, \partial^{\nu} X^{\rho\sigma}(x) = 0 \tag{9}$$

Furthermore, $X_{\mu\nu}$ may be split up[5] into its self-dual and anti-self dual parts $X^{\pm}_{\mu\nu}(x)$ which correspond to the two helicity or polarization states of the field.

$$X_{\mu\nu}(x) = X^{+}_{\mu\nu}(x) + X^{-}_{\mu\nu}(x)$$

and (7), (8) and (9) become

$$\Box \; X^{\pm}_{\mu\nu}(x) = F^{\pm}_{\mu\nu}(x) \tag{7'}$$

$$\Box \; \partial^{\mu} X^{\pm}_{\mu\nu} = 0 \qquad (8'), (9')$$

IV. THE CLASSICAL LAGRANGIAN THEORY

In terms of the classical fields $X^{\pm}_{\mu\nu}(x)$ we may write the free Lagrangian

$$
\begin{aligned}
&= -\tfrac{1}{4}\sum_{\pm}\left(\Box X^{\pm}_{\mu\nu}\,\Box\, X^{\pm\mu\nu} + \lambda^{\pm\mu}\partial^{\rho}\Box X^{\pm}_{\mu\rho}\right) \\
&= -\tfrac{1}{4}\left(F_{\mu\nu}F^{\mu\nu} + \sum_{\pm}\lambda^{\pm\mu}\,\partial^{\rho}F^{\pm}_{\mu\rho}\right)
\end{aligned}
\qquad (10)
$$

since $F^{\pm}_{\mu\nu}F^{\mp\mu\nu} = 0$. $\lambda^{\pm\mu}$ are Lagrange multipliers.

The Euler-Lagrange equations of motion derived from this Lagrangian by variation with respect to $X^{\pm}_{\mu\nu}(x)$ and $\lambda^{\pm}_{\mu}$ are[6]

$$
\Box\,\Box\, X^{\pm}_{\mu\nu}(x) = 0, \quad \Box\,\partial^{\mu}\, X^{\pm}_{\mu\nu}(x) = 0 \qquad (11)
$$

the second of these being just Maxwell's equations (8') and (9'). The Lagrangian eq. (10) reduces to the usual form when Maxwell's equations are satisfied.

There are twenty-four independent canonical fields $X^{\pm}_{\mu\nu}(x)$ $\partial_{o}X^{\pm}_{\mu\nu}(x)$ and their corresponding canonically conjugate momenta[6]

$$
\begin{aligned}
\Pi^{\pm}_{\mu\nu}(x) &= \tfrac{1}{2}\,\partial_{o}F^{\pm}_{\mu\nu}(x) \\
\Pi^{(o)\pm}_{\mu\nu}(x) &= -\tfrac{1}{2}\,F^{\pm}_{\mu\nu}(x)
\end{aligned}
\qquad (12)
$$

which satisfy the usual Poisson bracket conditions. A typical one is

$$
\left[X^{+}_{\mu\nu}(x),\ \Pi^{+}_{\rho\sigma}(x)\right] = i(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \qquad (13)
$$

The Hamiltonian density $\mathcal{H}$ is given (when equations (8') and (9') are satisfied) by

$$
\mathcal{H} = \sum_{\pm}\left[\tfrac{1}{2}(\partial_{o}F^{\pm}_{\mu\nu})\partial_{o}X^{\pm\mu\nu} - \tfrac{1}{2}(\partial_{o}{}^{2}X^{\pm}_{\mu\nu})F^{\pm\mu\nu} - \tfrac{1}{4}F^{\pm}_{\mu\nu}F^{\pm\mu\nu}\right] \qquad (14)
$$

V. THE QUANTIZED THEORY

We define the local quantized Weinberg fields $X^{\pm}_{\mu\nu}(x)$ in the usual way[4,5]

$$X^{\pm}_{\mu\nu}(x) = -\int \underline{d}^4k\ \underline{\delta}^+(k^2)\left[e^{-ik\cdot x}\ \xi^{\pm}_{\mu\nu}(k)\ a^{\pm}(k) + e^{ik\cdot x}\ \xi^{\mp *}_{\mu\nu}(k) a^{\mp\dagger}(k)\right] \tag{15}$$

where

$$\underline{d}^m k \underline{\delta}^{n+}(k^2) = (2\pi)^{n-m}\ d^m k \delta^n(k^2)\theta(k_o),$$

$a^{\pm}(k)$ are the annihilation operators for photons of momentum k and helicity ± 1, and $\xi^{\pm}_{\mu\nu}(k)$ are the corresponding wave functions which are in this case

$$\xi^{\pm}_{\mu\nu}(k) = \frac{k_\mu \varepsilon^{\pm}_\nu(k) - k_\nu \varepsilon^{\pm}_\mu(k)}{k^2} = \frac{\varepsilon^{\pm}_{\mu\nu}(k)}{k^2} \tag{16}$$

The factor of $\frac{1}{k^2}$ is introduced to ensure the correct relation between $X_{\mu\nu}(x)$ and $F_{\mu\nu}(x)$ as usually defined. However, because of this factor, $X^{\pm}_{\mu\nu}(x)$ is not perhaps well defined by eq. 15 as we have $\frac{\delta(k^2)}{k^2}$ in the integral. We therefore take the $\frac{1}{k^2}$ outside the integral as the operator $\frac{-1}{\Box}$, so that we interpret eq. (15) as

$$X^{\pm}_{\mu\nu}(x) = -\int d^4x'\bar{D}(x-x')\int \underline{d}^4k\underline{\delta}^+(k^2)\left[e^{-ik\cdot x'}\ \varepsilon^{\pm}_{\mu\nu}(k)a^{\pm}(k) + e^{ikx'}\ \varepsilon^{\mp *}_{\mu\nu}(k)a^{\mp\dagger}(k)\right] \tag{15'}$$

We remark that these fields are true tensors[5]. The particular Green's function $\bar{D}(x)$ is chosen so as to yield causal commutators between the fields $X_{\mu\nu}$. These we now discuss.

The commutators of the fields $X^{\pm}_{\mu\nu}(x)$ follow simply from the basic relations

$$\left[a^\lambda(k), a^{\lambda'\dagger}(k')\right] \underline{\delta}^+(k^2) = \underline{\delta}^4(k-k')\delta^{\lambda\lambda'} \tag{17}$$

and the properties[4] of the polarisation vectors $\varepsilon_\mu^\lambda(k)$. These satisfy

$$k\cdot\varepsilon^\lambda(k) = 0, \quad \varepsilon_\mu^\lambda(k) = \bar{\varepsilon}_\mu^{\lambda *}(k)$$
$$\varepsilon_{\mu\nu}^\pm(k)\varepsilon_{\rho\sigma}^{\pm *}(k) \equiv \Pi_{\mu\nu,\rho\sigma}^{\ \pm}(k) = \tfrac{1}{2}\Pi_{\mu\nu,\rho\sigma}(k) \pm \frac{i}{2}\,\Lambda_{\mu\nu,\rho\sigma}(k) \tag{18}$$

where

$$\Pi_{\mu\nu,\rho\sigma}(k) = \left[-\,k_\mu k_\rho g_{\nu\sigma} - k_\nu k_\sigma g_{\mu\rho} + k_\mu k_\sigma g_{\nu\rho} + k_\nu k_\rho g_{\mu\sigma}\right] \tag{19}$$

and

$$\Lambda_{\mu\nu,\rho\sigma}(k) = \frac{1}{2k_o{}^2}\Big[k_\mu k_\rho \varepsilon_{\nu\sigma\alpha\beta} + k_\nu k_\sigma \varepsilon_{\mu\rho\alpha\beta} - k_\mu k_\sigma \varepsilon_{\nu\rho\alpha\beta} - k_\nu k_\rho \varepsilon_{\mu\sigma\alpha\beta}\Big]\,k^\alpha \bar{k}^\beta \tag{20}$$

where* $\bar{k}^\beta = g^{\beta\beta}k^\beta$ (unsummed).

We notice that $\Lambda_{\mu\nu,\rho\sigma}(k)$ is covariant if $k^2=0$, as expected, and may be rewritten, after detailed analysis,

$$\Lambda_{\mu\nu,\rho\sigma}(k) = -\,k^\alpha(k_\mu\varepsilon_{\alpha\nu\rho\sigma} + k_\nu\varepsilon_{\mu\alpha\rho\sigma} - k_\rho\varepsilon_{\mu\nu\alpha\sigma} - k_\sigma\varepsilon_{\mu\nu\rho\alpha}) \tag{21}$$

This form provides a covariant extension of $\Lambda(k)$ to off-shell momentum values.

The commutation relations are, then,

$$\left[X_{\mu\nu}^\pm(x),\, X_{\bar{\rho}\sigma}^{\pm\dagger}(y)\right] = \int d^4x'd^4y'\ \bar{D}(x-x')\bar{D}(y-y')\int \underline{d}^4k\,\underline{\delta}^+(k^2)\ \times$$
$$\times\ \Pi_{\mu\nu,\rho\sigma}^\pm(k)\left[e^{-ik\cdot(x'-y')} - e^{ik\cdot(x'-y')}\right] \tag{22}$$

---

*Our metric is $g_{oo} = 1$, $g_{ii} = -1$.

$$= i\int d^4x'd^4y' \; \bar{D}(x-x')\bar{D}(y-y') \; \Pi^{\pm}_{\mu\nu,\rho\sigma}(-i\partial) \; D(x'-y') \tag{23}$$

Notice that the result is the same whether we use the definition eq. (15') for the fields, or instead manipulate formally with the forms eq. (15), finally interpreting a factor $\frac{1}{k^4}$ in the k-integration as $\int d^4x'd^4y'\bar{D}\,\bar{D}$ ....

We write down a further commutation relation, that between the fields $X_{\mu\nu}(x)$ which are self-conjugate since $X^{\pm\dagger}_{\mu\nu}(x) = X^{\mp}_{\mu\nu}(x)$.

$$\left[X_{\mu\nu}(x),\, X_{\rho\sigma}(y)\right] = i \int d^4x'd^4y' \; \bar{D}(x-x')\bar{D}(y-y') \; \Pi_{\mu\nu,\rho\sigma}(-i\partial)D(x'-y') \tag{24}$$

It is straightforward to show that all these commutators vanish for x-y spacelike, but they would not (necessarily) do so if we replaced $\bar{D}(x)$ by $D^{ret}(x)$, say.

The propagators in momentum space are

$$\frac{\xi^{\pm}_{\mu\nu}(k)\xi^{\pm *}_{\rho\sigma}(k)}{(k^2+i\varepsilon)} = \frac{\Pi^{\pm}_{\mu\nu,\rho\sigma}(k)}{k^4(k^2+i\varepsilon)}$$

or, summed over helicity states,

$$\frac{\Pi_{\mu\nu,\rho\sigma}(k)}{k^4(k^2+i\varepsilon)}$$

Finally we show that the Hamiltonian of the free fields has a positive definite expectation value, and that the states of the theory, obtained from the vacuum by applying polynomials in the smeared free fields, have a positive-definite metric.

With our fields defined as in (15) or (15') we are able to derive all the quantities appearing in the expression (14) for the

Hamiltonian density. After some calculation, its vacuum expectation value is found to be positive definite :

$$<0|\mathcal{H}(x)|0> \;=\; 12 \int_{k_o>0} \underset{\sim}{d}^3k \;\omega(\underset{\sim}{k}) \tag{25}$$

where

$$\omega(\underset{\sim}{k}) = k_o = |\underset{\sim}{k}| .$$

The smeared field $X^{\lambda}_{\mu\nu}(f)$ is defined by

$$X^{\lambda}_{\mu\nu}(f) = \int d^4x \; X^{\lambda}_{\mu\nu}(x) \; f(x) \tag{26}$$

where f(x) is some test function. The one photon state $X^{\lambda}_{\mu\nu}(f)|0>$ has the (norm)^2 (see also the Appendix)

$$<0|X^{\lambda\dagger}_{\mu\nu}(f) \; X^{\lambda}_{\mu\nu}(f)|0> \;=\; \int \underset{\sim}{d}^4k \; \frac{\delta^+(k^2)}{k^4} |\tilde{f}(k)|^2 \; \Pi^{\pm}_{\mu\nu,\mu\nu}(k) \tag{27}$$

where

$$\tilde{f}(k) = \int d^4x \; e^{ik\cdot x} f(x) \tag{28}$$

Now

$$\begin{aligned} \Pi^{\lambda}_{\mu\nu,\mu\nu}(k) &= - g_{\mu\mu}k_\nu^2 - g_{\nu\nu}k_\mu^2 + 2g_{\mu\nu}k_\mu k_\nu \\ &\geq 0 \end{aligned} \tag{29}$$

equality only holding if μ,ν and k are such that $\varepsilon^{\lambda}_{\mu\nu}(k) = 0$. The metric is then clearly positive-definite. It is straightforward to extend the analysis to the multi-photon states

$$\ldots\ldots \; X^{\lambda_i}_{\mu_i\nu_i}(f_i) \;\ldots\; |0> .$$

In fact we shall find in the next section and in the appendix that these states do not lie in the Hilbert space, this being spanned by states obtained by applying polynomials in $\Box X_{\mu\nu}$ to the vacuum. However the metric is still positive-definite as may be seen by appropriately modifying eqs. (26) and (27).

VI. AXIOMATIC FIELD THEORY

The fields $X^{\pm}_{\mu\nu}(x)$ appear at first sight to be consistent with an axiomatic formulation,[8,9,*] for they have causal commutators, transform covariantly, and the two point functions

$$W^{\pm}_{\mu\nu,\rho\sigma}(x-y) \equiv \langle 0|X^{\pm}_{\mu\nu}(x)X^{\pm\dagger}_{\rho\sigma}(y)|0\rangle \tag{30}$$

are the boundary values of the functions, analytic in the extended tube,

$$W^{\pm}_{\mu\nu,\rho\sigma}(\xi) = i\int d^4x'd^4y'\ \bar{D}(\xi-x')\bar{D}(-y')\ \Pi^{\pm}_{\mu\nu,\rho\sigma}(-i\partial)D^{+}(x'-y') \tag{31}$$

This form is obtained using the translational invariance of (30).

Recently Strocchi has shown[1,2] that in any theory of electrodynamics based on vector potentials, whether covariant or not, the condition that the two point functions be boundary values of functions analytic in the extended tube <u>and satisfy the same equations of motion as the potentials</u> leads to a trivial theory,

$$\langle 0|F_{\mu\nu}(x)F_{\rho\sigma}(y)|0\rangle = 0 \tag{32}$$

We will extend Strocchi's analysis to our two point functions $W^{\pm}_{\mu\nu,\rho\sigma}(x-y)$ and show that if they satisfy Maxwell's equations

$$\Box\ \partial^{\mu}W^{\pm}_{\mu\nu,\rho\sigma}(x-y) = 0 \tag{33}$$

we obtain a trivial theory.

If $W^{\pm}_{\mu\nu,\rho\sigma}(\xi)$ is analytic in the extended tube, and covariant, then by the Araki-Hepp theorem[10] and some extra reasoning outlined in ref. 1,

---

*See also the Appendix.

it can be expressed in the form

$$W^{\pm}_{\mu\nu,\rho\sigma}(\xi) = \tfrac{1}{2}\Pi_{\mu\nu,\rho\sigma}(-i\partial)\ D^{\pm}_{\pi}(\xi) \pm \frac{i}{2}\ \Lambda_{\mu\nu,\rho\sigma}(-i\partial)D^{\pm}_{\Lambda}(\xi) \tag{34}$$

where $\Pi(-i\partial)$ and $\Lambda(-i\partial)$ are defined in eqs. 18-20 and the functions $D^{\pm}_{\pi}(\xi)$ and $D^{\pm}_{\Lambda}(\xi)$ are invariant, $D_i(\Lambda\xi) = D_i(\xi)$. In fact this is not the most general form of $W(\xi)$, but we consider only those covariants $\Pi(-i\partial)$ and $\Lambda(-i\partial)$ arising from photon fields. $W^{\pm}_{\mu\nu,\rho\sigma}(x-y)$ is given by the boundary value of eq. (34).

Now if

$$\Box\ \partial^{\mu}W^{\pm}_{\mu\nu,\rho\sigma}(x-y) = 0 \tag{35}$$

$$\Box\ \Box\left[(\partial_{\rho}g_{\nu\sigma}-\partial_{\sigma}g_{\nu\rho})\ D^{\pm}_{\pi}(x) \pm i\ \partial^{\alpha}\varepsilon_{\alpha\nu\rho\sigma}D^{\pm}_{\Lambda}(x)\right] = 0 \tag{36}$$

by eq. (34).

Further, if

$$W^{\pm}_{\nu\sigma}(x-y) \equiv \langle 0|\partial^{\mu}X^{\pm}_{\mu\nu}(x)\partial^{\rho}X^{\pm\dagger}_{\rho\sigma}(y)|0\rangle \tag{37}$$

satisfies the same equation of motion as $\partial^{\mu}X^{\pm}_{\mu\nu}(x)$,

$$\text{i.e.}\quad \Box\ W^{\pm}_{\nu\sigma}(x-y) = 0\ , \tag{38}$$

then, since from eq. (34)

$$W^{\pm}_{\nu\sigma}(x) = \tfrac{1}{2}(\Box g_{\nu\sigma} - \partial_{\nu}\partial_{\sigma})\Box D^{\pm}_{\pi}(x) \tag{39}$$

we find that

$$(\Box g_{\nu\sigma} - \partial_{\nu}\partial_{\sigma})\Box\Box D^{\pm}_{\pi}(x) = 0\ . \tag{40}$$

This implies, by a lemma proved in ref. 1,

$$\Box\,\Box\, D^{\pm}_{\pi}(x) = \text{constant} \tag{41}$$

so that eq. (36) now reads

$$\partial^{\alpha}\varepsilon_{\alpha\nu\rho\sigma}\Box\Box D^{\pm}_{\Lambda}(x) = 0 \quad . \tag{42}$$

Now, from the definition of $X^{\pm}_{\mu\nu}(x)$,

$$\mathcal{F}^{\pm}_{\mu\nu,\rho\sigma}(x-y) \equiv \langle 0|F^{\pm}_{\mu\nu}(x)F^{\pm\dagger}_{\rho\sigma}(y)|0\rangle = \Box\Box W^{\pm}_{\mu\nu,\rho\sigma}(x-y) \tag{43}$$

and it is clear from eqs. (34), (41) and (42) that

$$\mathcal{F}^{\pm}_{\mu\nu,\rho\sigma}(x-y) = 0 \tag{44}$$

which implies a trivial theory. <u>However</u>, with the fields $X^{\pm}_{\mu\nu}(x)$ defined in eqs. (15) and (15'), the two-point functions eg. (30) do <u>not</u> satisfy Maxwell's equations nor do they satisfy the Klein-Gordon equation

$$\Box\,\Box\, W^{\pm}_{\mu\nu,\rho\sigma}(x-y) = 0 \tag{45}$$

even though the fields themselves appear to satisfy both these equations as operator identities (but see the Appendix).

This is not perhaps surprising in some respects, as the Green's functions are not the W's but

$$G^{\pm}_{\mu\nu,\rho\sigma}(x-y) = \langle 0|X^{\pm}_{\mu\nu}(x)\Box X^{\pm\dagger}_{\rho\sigma}(y)|0\rangle \tag{46}$$

and these satisfy the two equations.

We conclude that the Strocchi-style analysis given above does not apply to our theory, as eqs. (35), (36), (38), (40)-(42), and (44) are not correct.

Nevertheless, there remains the apparent contradiction that

$$\Box\,\Box <0|X^{\pm}_{\mu\nu}(x)X^{\pm\dagger}_{\rho\sigma}(y)|0> \neq 0 \,,$$
$$\partial^{\mu}\Box <0|X^{\pm}_{\mu\nu}(x)X^{\pm\dagger}_{\rho\sigma}(y)|0> \neq 0 \qquad (47)$$

whereas our fields as defined in eq. (15)satisfy $\Box\Box X^{\pm}_{\mu\nu}(x) = \Box\,\partial^{\mu} X^{\pm}_{\mu\nu}(x) = 0$. We discuss this at greater length in the Appendix. The explanation is that none of the equations (11) can hold as operator identities on states formed by applying polynomials in the fields $X_{\mu\nu}$ to the vacuum state. But since these equations are operator identities on the vacuum state, (47) shows that $X_{\mu\nu}|0>$ cannot even be a state in the Hilbert space (its scalar product with the zero vector being non-zero). For the physical states we must take those obtained by applying to the vacuum polynomials in $\Box X_{\mu\nu}$, and no contradictions then arise. Eqs. (11) hold as operator identities on such states.

All this shows that the fields $X_{\mu\nu}(x)$ are not really of much interest in an axiomatic approach. Their chief virtue is that they allow the Weinberg programme to be carried through with covariant fields satisfying causal commutation relations, and a local interaction Lagrangian.

Finally we would like to point out that our theory should in principle be derivable from a Lagrangian formalism such as that outlined in Section IV. The field commutation relations should then follow from the equal-time canonical commutation relations together with a number of assumptions about the form of the fields in momentum space. The S-matrix could then be calculated by the usual techniques. Of course this would do nothing to avoid the above criticisms.

VII. THE FEYNMAN RULES

The S-matrix is given in terms of the interaction Hamiltonian as

$$S = P\left[\exp\left(-i \int d^4x\, \mathcal{H}_{int}(x)\right)\right] \tag{48}$$

In Weinberg field theory this is an assumption.

We are then able to derive a set of Feynman rules for calculating the scattering amplitudes and these of course differ slightly from those which one uses in the conventional theory of electrodynamics. We give the necessary modifications :

(a) For an external incoming (outgoing) photon we must now use the wave function $\xi^{\lambda}_{\mu\nu}(k)$ $(\xi^{\lambda*}_{\mu\nu}(k))$.

(b) For each photonic vertex, we now put $\frac{e}{2}(k_\mu C_\nu - k_\nu C_\mu)$, where before we had simply eC_μ ($C_\mu = \gamma_\mu$, for example).

(c) For each internal photon line, we now insert the propagator $\dfrac{\Pi_{\mu\nu,\rho\sigma}(k)}{k^4(k^2+i\varepsilon)}$ (We could write down the separate contributions of the two helicity states, but they turn out to be equal with the couplings given in (b). With Pauli-type couplings $\mu\sigma_{\mu\nu}$ at the vertices, the two helicity states give complex-conjugate contributions).

Using these modified rules, it is not hard to see that we obtain exactly the same expressions as in the conventional theory with the Landau propagator $\dfrac{g_{\mu\nu} - \dfrac{k_\mu k_\nu}{k^2}}{k^2+i\varepsilon}$, with the sole exception that at an <u>external</u> photon vertex, the coupling reduces to

$$\varepsilon^{\pm}_{\mu}(k)\left(C^\mu - k\cdot C\frac{k^\mu}{k^2}\right) \tag{49}$$

which differs from the usual $\varepsilon\cdot C$ by the term $k\cdot C\,\frac{\varepsilon\cdot k}{k^2}$.

Now for $k^2=0$, $\frac{\varepsilon\cdot k}{k^2}$ is indeterminate, but for other values of k^2 it is zero. We may therefore argue that it vanishes for $k^2=0$ also, as a limit from $k^2\neq 0$. However, we cannot altogether ignore this second term as it becomes non-zero under Lorentz-transformation ($\varepsilon_\mu(k)$ is not a four-vector[4]), cancelling an equal and opposite term which appears when $\varepsilon\cdot C$ is transformed. In other words, $\varepsilon\cdot C$ is not a scalar, but $\varepsilon^\lambda_\mu(C^\mu - k\cdot C\,\frac{k^\mu}{k^2})$ is, to within a phase, and in another frame, obtained by the Lorentz transformation Λ, has the value

$$\varepsilon^{\pm}_\mu(\Lambda k)(\Lambda C)^\mu = e^{\pm\, i\theta(\Lambda,\hat{\underline{k}})}\ \varepsilon^{\pm}_\mu(k)C^\mu \tag{50}$$

so that the matrix elements of the usual theory are exactly reproduced (providing always that the $\frac{k\cdot\varepsilon}{k^2}$ terms give no contribution).

VIII FURTHER CONSIDERATIONS

Even though the $\frac{k^\mu}{k^2}$ parts of the couplings, or covariants as we shall call them from now on, need not contribute to the matrix elements, we may still try to remove them in some way.

To do so, we may form new covariants, free of such parts, by taking suitable combinations of the original ones, in fact the same combinations as are necessary, in the usual analysis, to give gauge-invariant covariants. We see this in the following way:

Let the complete set of conventional non-gauge-invariant covariants for a given process be $K_1^\mu \ldots K_n^\mu$ with corresponding invariant amplitudes $A_1, \ldots A_n$ such that the M-function is given by

$$M^\mu = \sum_i K_i^\mu A_i \tag{51}$$

Following our rules, we find an M-function

$$M^{\mu\nu} = \tfrac{1}{2} \sum_i (k^\mu K_i^\nu - k^\nu K_i^\mu) A_i \quad , \tag{52}$$

the A_i and K_i^μ being the same as before. We define M'^μ to be such that

$$\varepsilon_\mu^\lambda(k) M'^\mu = \xi_{\mu\nu}^\lambda(k) M^{\mu\nu} \tag{53}$$

Therefore

$$M'^\mu = \sum_i K_i'^\mu A_i$$

where

$$K_i'^\mu = K_i^\mu - k\cdot K_i \frac{k^\mu}{k^2} \tag{54}$$

and the A_i are still the same.

Now let C_{ij} be kinematic factors, free of singularities (but possibly possessing zeros) such that the covariants

$$\bar{K}_i = \sum_j C_{ij} K_j \tag{55}$$

satisfy $k \cdot \bar{K}_i = 0$, that is, they are conventional gauge invariant covariants. Then the same combinations $\bar{K}_i'$ of our covariants K_i' are

$$\bar{K}_i'^{\mu} = \sum_j C_{ij} K_j'^{\mu} = \sum_j C_{ij} K_j^{\mu} - \frac{k^{\mu}}{k^2} \sum_j C_{ij} k \cdot K_j$$

$$= \sum C_{ij} K_j^{\mu} \tag{56}$$

by definition of the C_{ij}, so that

$$\bar{K}_i'^{\mu} = \bar{K}_i^{\mu} \tag{57}$$

which have no term in $\frac{k^{\mu}}{k^2}$, and are the conventional gauge-invariant forms.

It may or may not be possible to construct n covariants free of $\frac{k^{\mu}}{k^2}$ terms in this way. If not, we cannot remove all such terms from the M-function. If we can construct n such covariants, then the M-function is expressible with these only and is clearly identical with the conventional one which is, of course, gauge invariant.

More succinctly, by virtue of eq. (51) and eq. (54)

$$M'^{\mu} = \sum_i K_i^{\mu} A_i - \frac{k^{\mu}}{k^2} k \cdot M \tag{58}$$

if M'^{μ} has no term in $\frac{k^{\mu}}{k^2}$, it is equal to M and $k \cdot M = 0$. Conversely, if $k \cdot M = 0$, M'^{μ} has no term in $\frac{k^{\mu}}{k^2}$, and is equal to M^{μ}. Thus the absence of $\frac{k^{\mu}}{k^2}$ terms in our M-function M'^{μ} is equivalent to gauge-invariance of the <u>conventional</u> M-function.

IX. CHARGE CONSERVATION

Our tensor potential theory is not a gauge-theory. Charge conservation is not necessary for covariance and must be imposed a posteriori. In the Weinberg theory[5], charge conservation is a necessary and sufficient condition for the theory to be covariant, and is expressible as $k\cdot M = 0$. But we saw in the last section that this implied the absence of $\frac{k^\mu}{k^2}$ terms in M'^μ and vice versa, so that in our theory charge is conserved if and only if M'^μ contains no terms in $\frac{k^\mu}{k^2}$. If charge is conserved, then $M'^\mu = M^\mu$ and the conventional M-function is therefore reproduced exactly by our rules.

The equivalence between absence of the $\frac{k^\mu}{k^2}$ terms and charge conservation is seen very clearly when we construct M'^μ, by our rules, for the emission of one soft extra photon in some scattering process[11]. The coefficient of $\frac{k^\mu}{k^2}$ in M'^μ is proportional to the difference between the total ingoing and total outgoing charges. If charge is conserved no $\frac{k^\mu}{k^2}$ term appears, and vice versa.

On the other hand, if our fields interacted with non-conserved quantities, M'^μ would have a term in $\frac{k^\mu}{k^2}$, giving an indeterminate contribution to the matrix elements (which we might, however, "define away").

X. HIGHER SPIN MASSLESS FIELDS

A similar formulation can be made of quantized spin-2 graviton theory. A suitable fundamental field is $S_{\mu\nu\rho\sigma}(x)$, related to the Riemann tensor $R_{\mu\nu\rho\sigma}$ by

$$\Box\Box\, S_{\mu\nu\rho\sigma}(x) = R_{\mu\nu\rho\sigma}(x) \tag{59}$$

The self-dual and anti-self-dual parts of $S_{\mu\nu\rho\sigma}(x)$, which may be projected out locally and covariantly, then describe gravitons of helicity ± 2. The quantization of the fields is achieved by imposing the basic commutation relation, eq. (17), on the graviton creation and annihilation operators.

$S_{\mu\nu\rho\sigma}(x)$ can interact locally with the energy-momentum tensor $T^{\mu\nu}(x)$ to reproduce the usual $fT^{\mu\nu}g_{\mu\nu}$ form.

We find that $\frac{k_\mu k_\nu}{k^4}$ terms are absent from the partially-reduced two-index M-functions if and only if the gravitational coupling constant is universal.

In the same way we may formulate field theories of massless particles of spin-3 or higher. Such fields can interact at zero momentum, contrary to a statement in ref. 4, and can therefore give rise to macroscopic fields. However $(k^2)^{-j}$ terms cannot in general be removed from the partially-reduced M-functions by any conservation or universality principle. We could remove such terms by using different wave functions, but then the fields would not interact at zero-momentum, and would give rise to a non-renormalizable theory.

XI. CONCLUSION

We have presented a solution to the difficulties inherent in Weinberg's approach to quantum electrodynamics[5]. Our tensor potentials are local covariant fields which interact locally with the electromagnetic current. They have causal commutation relations. No gauge considerations enter, and there are no unphysical particles. The Hamiltonian and the metric of the space of physical states are positive definite.

The fields satisfy their equations of motion as operator identities on the physical states, and would be compatible with axiomatic field theory but for the fact that the states generated by applying them to physical states do not lie in the Hilbert space. The physical states are in fact generated by the fields $F_{\mu\nu}(x)$. However, the PCT theorem is not lost.

Alternatively, if we had introduced the tensor potentials through an axiomatic formulation, the theory would have been trivial. This represents an extension of the analysis of refs. 1 and 2.

Subject to these reservations, the theory appears to be, in principle, derivable from a Lagrangian formalism.

The Feynman rules for calculating scattering matrix elements are similar to those in conventional quantum electrodynamics, and by virtue of charge conservation, the usual matrix elements are recovered exactly.

A similar analysis in the same spirit may be applied to higher-spin massless field theories.

* * * * * * *

ACKNOWLEDGEMENTS

The author would like to thank Professors P.T. Matthews and G. Feldman, and Drs. T.W.B. Kibble and C.J. Isham for helpful discussions. He acknowledges the support of a Commonwealth Scholarship.

APPENDIX

The Hilbert space $\mathcal{H}$ of free photon states is the direct sum[9].

$$\mathcal{H} = \oplus \sum_{n=o}^{\infty} \mathcal{H}^{(n)} \tag{A1}$$

of Hilbert spaces $\mathcal{H}^{(n)}$ which are subspaces of those states with n photons.

A vector Ψ in $\mathcal{H}$ is given by the sequence

$$\{\psi^{(o)}, \psi^{(1)}, \psi^{(2)}, \quad \ldots\ldots\}$$

of vectors $\psi^{(n)} \in \mathcal{H}^{(n)}$, with the scalar product in $\mathcal{H}$

$$(\Phi, \Psi) = \sum_{n=o}^{\infty} (\Phi^{(n)}, \Psi^{(n)}) \tag{A2}$$

where $(\Phi^{(n)}, \Psi^{(n)})$ is the scalar product in $\mathcal{H}^{(n)}$.

Only those sequences such that

$$(\Psi, \Psi) < \infty \tag{A3}$$

are in $\mathcal{H}$.

$\mathcal{H}^{(o)}$ is the (one-dimensional) space of complex numbers, $\mathcal{H}^{(1)}$ the space of square-integrable functions of momentum with anti-symmetric indices μ, ν etc.

The scalar product in $\mathcal{H}^{(n)}$ is then

$$(\Phi^{(n)}, \Psi^{(n)}) = \int \cdots \int \underline{d}^4 p_1 \cdots \underline{d}^4 p_n \underline{\delta}^+(p_1^2) \cdots \underline{\delta}^+(p_n^2) \times$$

$$\times \Phi^{(n)*}_{\mu_1 \nu_1 \ldots \mu_n \nu_n}(p_1 \cdots p_n) \Psi^{(n)}_{\mu_1 \nu_1 \ldots \mu_n \nu_n}(p_1 \cdots p_n) \tag{A4}$$

where $\Psi^{(1)}_{\mu\nu}(p)$ is the single-photon wave function.

Now it is a disease of zero-mass theories that the class of functions square-integrable on the mass hyperboloid, viz. the light cone, contains only the zero element due to singularities at infinity and at the vertex of the light cone. We may attempt to remedy this by, for example, defining our class of functions with respect to a measure somewhat smeared in mass. We shall not investigate this problem, and shall continue to use singular measures. Any meaningful result obtainable, such as

$$\int \underline{d}^4 k \underline{\delta}^+(k^2)\, k^2 = 0 \tag{A6}$$

will surely hold in some sense in the rigorous formalism.

The smeared fields $X^{\pm}_{\mu\nu}(f)$ are defined as linear functionals on a suitable space of test functions $f(x)$, and the Hilbert space of states should be obtainable by applying to the vacuum state polynomials in the smeared fields.

In particular, the one photon state is given by

$$X^{\lambda\dagger}_{\rho\sigma}(f)|0\rangle = \int \underline{d}^4 k\ \underline{\delta}^+(k^2)\xi^{\lambda *}_{\rho\sigma}(k)\tilde{f}(k)a^{\lambda\dagger}(k)|0\rangle \tag{A7}$$

where $\tilde{f}(k)$ has been defined in eq. (28).

Now let us apply the operator $\partial^\mu \Box X^\lambda_{\mu\nu}(g)$ to this state. We have

$$\partial^\mu \Box X^\lambda_{\mu\nu}(g)|k,\lambda\rangle = \int \underline{d}^4 p \underline{\delta}^+(p^2)\varepsilon^\lambda_\nu(p)\tilde{g}(-p)a^\lambda(p)a^{\lambda\dagger}(k)|0\rangle$$

$$+ \int \underline{d}^4 p \underline{\delta}^+(p^2)p^2\varepsilon^{\lambda *}_\nu(p)\tilde{g}(p)a^{-\lambda\dagger}(p)|k,\lambda\rangle \tag{A8}$$

The second integral vanishes because of the $p^2\delta\ (p^2)$ factor, while the first, by virtue of eq. (17), is

$$\int \underset{\sim}{d}^4 p \underset{\sim}{\delta}^4(p-k) p^2 \varepsilon_\nu^\lambda(p) \tilde{g}(-p)|0\rangle = k^2 \varepsilon_\nu^\lambda(k) \tilde{g}(-k)|0\rangle \tag{A9}$$

This is not zero, nor need it be, since $|k,\lambda\rangle$ is not normalizable and thus not a state in our Hilbert space. However, as soon as the state $|k,\lambda\rangle$ is put on-shell, $k^2=0$, as happens in an integration over k, the right hand side of (A9) vanishes.

Finally we find

$$\partial^\mu \Box X^\lambda_{\mu\nu}(g) X^{\lambda\dagger}_{\rho\sigma}(f)|0\rangle = \int \underset{\sim}{d}^4 k \underset{\sim}{\delta}^+(k^2) k^2 \tilde{f}(k) \xi^{\lambda *}_{\rho\sigma}(k) \tilde{g}(-k) \varepsilon^\lambda_\nu(k)|0\rangle \tag{A10}$$

which is not zero, because the $\frac{1}{k^2}$ factor in $\xi^\lambda_{\rho\sigma}(k)$ cancels the explicit k^2 in the integrand.

Therefore,

$$\langle 0|\partial^\mu \Box X^\lambda_{\mu\nu}(g) X^{\lambda\dagger}_{\rho\sigma}(f)|0\rangle \neq 0 \tag{A11}$$

But

$$\partial^\mu \Box X^\lambda_{\mu\nu}(g)|0\rangle = \int \underset{\sim}{d}^4 k \underset{\sim}{\delta}^{\mp}(k^2) k^2 \varepsilon^{\lambda *}_\nu(k) \tilde{g}(k)|0\rangle = 0 \tag{A12}$$

so that $X^{\lambda\dagger}_{\rho\sigma}(f)|0\rangle$ cannot even be in the Hilbert space, as its scalar product with the zero vector is non-zero.

In a similar way, we find that the states $\partial^\mu \Box X^{\lambda\dagger}_{\mu\nu}(f)|0\rangle$ cannot be in the Hilbert space either. However the equations of motion (11) are satisfied as operator identities on the space of states obtained from the vacuum by applying polynomials in the smeared fields $\Box X^\lambda_{\mu\nu}(f) \equiv F^\lambda_{\mu\nu}(f)$. This is simple to verify and comes about because the wave-functions for these fields have no $\frac{1}{k^2}$ factor.

Of course, these conclusions show that the fields $X_{\mu\nu}(x)$ are of little interest in this axiomatic approach. This is not to say, however, that the two-point functions involving them are not meaningful;

we have given one in eq. (31), calculated from the representation eqs. (15) and (15'). Nor do we lose axiomatic field theory results, such as PCT. This theorem will apply to the fields $\Box X_{\mu\nu}(x) \equiv F_{\mu\nu}(x)$ and their n-point functions. Consequently the n-point functions of the fields $X_{\mu\nu}(x)$, having meaning as distributions, if not as scalar products in a Hilbert space, will satisfy the PCT condition[9] :

$$<0|X_{\mu_1\nu_1}(x_1)\cdots X_{\mu_n\nu_n}(x_n)|0> = <0|X_{\mu_n\nu_n}(-x_n)\cdots X_{\mu_1\nu_1}(-x_1)|0> \tag{A13}$$

as may be seen by repeated application of eq. (6) to

$$<0|F_{\mu_1\nu_1}(x_1)\cdots F_{\mu_n\nu_n}(x_n)|0> = <0|F_{\mu_n\nu_n}(-x_n)\cdots F_{\mu_1\nu_1}(-x_1)|0> \tag{A14}$$

Finally, we remark that the analysis in section VI and in this appendix suggests that in formulating quantized electromagnetic theory axiomatically without unphysical particles, we cannot have both a local minimal Lagrangian and the equations of motion as operator identities on the (physical) states generated by operation of the fields on the vacuum state.

REFERENCES

1. F. Strocchi, Phys. Rev. 162 (1967) 1429.

2. F. Strocchi, Let. al Nuovo Cim. 1 (1969) 169.

3. S. Weinberg, Phys. Rev. 134 (1964) B882.

4. S. Weinberg, Phys. Rev. 135 (1964) B1049.

5. S. Weinberg, Phys. Rev. 138 (1965) B988.

6. A.O. Barut & G. Mullen, Annals of Phys. 20 (1962) 203.

7. A. Pais & G.E. Uhlenbeck, Phys. Rev. 79 (1950) 145.

8. A.S. Wightman, Phys. Rev. 101 (1956) 860.

9. R. Streater & A.S. Wightman, PCT, Spin and Statistics and All That, (New York, 1964).

10. H. Araki, Prog. of Theor. Phys. Suppl. 18 (1961) 83.

11. F.E. Low, Phys. Rev. 110 (1958) 974.