# DEVELOPMENT OF VORTICITY PATTERNS

# IN STRATIFIED FLUIDS


by

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## ABSTRACT

A three-part study is made of the development of vorticity patterns in the flow of inviscid, stratified fluids.

The first part ~~comprises~~ consists of a study of aspects of the hydrodynamic stability of swirling flows. Features of the instability mechanisms of two dimensional flow of a homogeneous fluid, for which circulation can take on the role of a pseudo-stratifier, and of a non-gravitating fluid possessing a radial density stratification are highlighted by seeking analytical solutions to the normal mode stability problem for particular basic flow configurations. The effect of a radial gravity field is then incorporated into the analysis and an indication given of the limited physical significance of this case to small scale cloud patterns in the earth's equatorial atmosphere. A study is also made of the stability of a baroclinic equatorial atmosphere to perturbations that are axi-symmetric with respect to the earth's axis of rotation.

Part II is devoted to a study of the possible effects of stirring processes in rotating fluids. A survey of

existing theories on this topic is presented, and a distinction drawn between mechanical stirring of a homogeneous fluid by an external agency and thermal stirring of a stratified fluid due to a hydro-thermodynamic instability. A heuristic model is developed to determine the form of the Reynolds stress induced by mechanical stirring, whilst Ertel's 'potential vorticity' theorem is employed to elucidate the quintessential properties of thermal stirring.

In the last part of the work numerical integrations are performed to trace the time development of the initial distribution of sets of point vortices. The investigation is undertaken to obtain an increased understanding of the advective development of strong swirling motion in the two dimensional flow of a homogeneous fluid.

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# CHAPTER 1

## Hydrodynamic stability of two dimensional swirling flows of an inviscid, homogeneous fluid

### I. Introduction and Outline

This chapter is devoted to a study of some of the distinctive aspects of the hydrodynamic stability of two dimensional swirling flows of an inviscid, homogeneous fluid confined between two concentric cylinders.

In Section II the general stability criteria for such flows are re-examined and juxtaposed, for comparison purposes, with the stability criteria of plane shear flow bounded by two parallel plates. These general criteria were developed most notably by Rayleigh [(1880), (1916)], Fjortoft [1950], and Howard and Gupta [1962].

The criteria indicate that there exists an instability mechanism associated with an inflexion point in the angular velocity and velocity profiles respectively of the two types of flows, and that for the swirling flow there is also an instability mechanism related to the radial variation of angular momentum. The former mechanism may be regarded as the extension to continuous velocity profiles of the well known instability of a vortex sheet, whilst Rayleigh likened the latter mechanism to a density stratification in the presence of a radially acting gravity field. We also note in Section II that there is a dissident feature in the stability criteria of the two problems for two dimensional normal mode perturbations in the plane of the mean motion.

To highlight this feature we seek in Section III an analytical solution to the stability problem for a particular swirling flow distribution subject to two dimensional normal mode perturbations in the plane of the mean motion. The solution obtained is compared with the results of a relevant experimental study conducted by Hide and Titman [1967].

In Section V a 'narrow gap' approximation is employed to enable us to derive analytical results for the instability of two particular swirling flow distributions to three dimensional perturbations.

## II. The Perturbation Differential Equations and the Stability Criteria

Conducting our analysis in cylindrical polar coordinates $(r,\theta,z)$ for the swirling flow problem, and in cartesian coordinates $(x,y,z)$ for the plane flow problem, we assume basic states of the following form,

$$(u,v,w) = (0, V(r), 0) \quad \text{for the swirling flow}$$

and $$(u,v,w) = (U(z), 0, 0) \quad \text{for the plane flow.}$$

$(u,v,w)$ representing the velocities in the $(r,\theta,z)$ and $(x,y,z)$ directions for the respective flows.

The appropriate equations of motion are linearised with respect to small perturbations of the above basic flows, and the perturbation quantities are assumed to take the normal mode form,

$$\begin{bmatrix} u' \\ v' \\ w' \\ p' \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} (r) \quad e^{i(\sigma t + m\theta + kz)} \quad \text{for the swirling flow.}$$

and

$$\begin{bmatrix} u' \\ v' \\ w' \\ p' \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} (y) \quad e^{i(\sigma t + \ell x)} \quad \text{for the plane flow.}$$

(Three dimensional perturbations are not considered for the plane flow since a result due to Squire may be invoked to prove that two dimensional disturbances of the above form are the most unstable.)

It proves possible to eliminate all dependent variables except one from both systems of linearised equations and the resulting differential equations governing the stability of the basic flows to small perturbations are given by,

$$D[rf(D\phi)] - [\frac{1}{r} + m(\gamma^{-1})D(f\mathcal{G}) - 2k^2(\gamma^{-2})(fV\mathcal{G})]\phi = 0 \qquad (1)$$

for the swirling flow

where $\mathcal{G} = (\frac{dV}{dr} + \frac{V}{r})$ , $D = \frac{d}{dr}$ , $f = \frac{1}{m^2+k^2r^2}$ , $\gamma = (\sigma+m\frac{V}{r})$

and $\phi = ru$ .

and the 'Rayleigh Stability Equation',

$$\beta[D_2^2-\ell^2]w - \ell[D_2^2U]w = 0 \qquad (2)$$

for plane flow

where $D_2 = \frac{d}{dz}$ , $\beta = (\sigma+U\ell)$

The appropriate boundary conditions for these differential equations are

$$\phi = 0 \quad \text{at} \quad r = R_1 \;,\; R_2 \quad \text{for equation (1)}$$

$$\text{and} \quad w = 0 \quad \text{at} \quad z = z_1,\; z_2 \quad \text{for equation (2)}$$

$R_1$ and $R_2$ being the radii of the cylinders bounding the flow, and $z_1$ and $z_2$ denote the location of the bounding plane in the plane flow problem.

If we set $k$ (the axial wave number) to **zero** in equation (1) the differential equation reduces to

$$\gamma[D_1 D-(\tfrac{m}{r})^2]\phi - (\tfrac{m}{r})[D\mathcal{S}]\phi = 0 \qquad (3)$$

$$\text{where} \quad D_1 = (\frac{d}{dr} + \frac{1}{r})$$

This equation resembles the Rayleigh stability equation, curvature being the only differentiating factor. This resemblance suggests that the previously mentioned stratifying effect of the circulation is associated with the toroidal component of the normal mode. With $k = 0$ we may anticipate that the stability features of a swirling flow may well be analogous to those of a plane flow.

This conjecture is supported by consideration of the general stability criteria for the two types of flow.

Necessary conditions for instability of swirling flow [see e.g. Howard and Gupta (1961)] are,

(i) $2k^2(V\mathcal{G}) - \frac{1}{4}m^2 r[D(\frac{V}{r})']^2 < 0$ somewhere in $R_1 < r < R_2$

(ii) $mD[f\mathcal{G}] - 4k^2(\gamma\mathcal{G}f)\frac{1}{|\gamma|^2}(\sigma_r + m\frac{V}{r})$

must change sign within $R_1 < r < R_2$

(iii) With $k = 0$, $\min^m(\frac{V}{r}) < -(\frac{\sigma r}{m}) < \max^m(\frac{V}{r})$

With $m = 0$, $\sigma_r = 0$
$V\mathcal{G} < 0$ somewhere in $R_1 < r < R_2$

And if $\sigma_i \neq 0$, there is an upper bound on the growth rate

(iv) $\sigma_i^2 \leq$ maximum $(\frac{r^2}{m^2+k^2r^2})[\frac{1}{4}m^2[D(\frac{V}{r})]^2 - 2k^2(\frac{V}{r})\mathcal{G}]$

For plane flow the available criteria governing the nature of the disturbed flow are,

(a) Squire's Theorem. The most unstable disturbances are those with zero wave-number in the cross flow direction.

and the necessary conditions for instability,

(b) $(D_2^2U)$ must change sign within $z_1<z<z_2$

(c) $D_2^2U\ (U-U_s) < 0$ somewhere in $(z_1<z<z_2)$ , where $U_s$ is the flow velocity at a point where $(D_2^2U)$ is zero.

If $U$ is a monotonic function and $(D_2^2U)$ is zero only once in $(z_1<z<z_2)$ then condition (c) may be rewritten as

$$(D_2^2U)\ (U-U_s) \leq 0 \text{ everywhere within the flow field} \quad (4)$$

Thus we may anticipate an unstable flow only if there exists a relative maximum of vorticity within the flow field.

We now make a brief comparison of these two sets of stability criteria.

It is not possible to construct an analogue of Squire's Theorem for swirling flows, but if $(V\Psi)$ is always positive in $(R_1<r<R_2)$ then condition (i) suggests that three dimensional disturbances may be more stable than those with $k$ set to zero, whilst (iv) shows that decreasing $k$ increases the upper bound on the growth rate.

With $k = 0$, condition (ii) states that the occurrence of an extremal of vorticity is a necessary condition for instability. We note that the stratifying rôle of the circulation plays no part in this criterion, which is analogous to condition (b) for plane flow.

If we now seek the analogue of condition (c) for swirling flows we see that there is an essential difference in the analogue relation. The required condition was originally derived by Fjortoft [1950], using a variational principle, and has thereafter lain fallow. It may be derived from (1) as follows:-

Multiply (1) by $\phi^*$, the complex conjugate of $\phi$, and integrate from $R_1$ to $R_2$.

$$\int_{R_1}^{R_2} rf\,|D\phi|^2\,dr + \int_{R_1}^{R_2}\left[\frac{1}{r} + m\gamma^{-1}D(f\mathcal{G}) - 2k^2\gamma^{-2}(fV\mathcal{G})\right]|\phi|^2\,dr = 0$$

Considering cylindrical disturbances (i.e. $k = 0$), the real and imaginary parts of this equation yield,

$$\sigma_i \int_{R_1}^{R_2} \left[|\gamma|^{-2}\,D(\mathcal{G})\right]|\phi|^2\,dr = 0 \qquad (5)$$

and

$$\frac{1}{m}\int_{R_1}^{R_2}\left[|\gamma|^{-2}(\sigma_r + m\frac{V}{r})(D\mathscr{P})\right]|\phi|^2 dr = -\int_{R_1}^{R_2}\left[\frac{1}{m^2}\, r\,|D\phi|^2 + \frac{1}{r}|\phi|^2\right]dr \tag{6}$$

Therefore if $\sigma_i \neq 0$ , then

$$\frac{1}{m}\left[\sigma_r + m(\frac{V}{r})_I\right]\int_{R_1}^{R_2}|\gamma|^{-2}(D\mathscr{P})\,|\phi|^2\, dr = 0 \tag{7}$$

where I is a point within the flow field where $(D\mathscr{P}) = 0$ .

Subtracting (6) from (7) we obtain

$$\int_{R_1}^{R_2}\left[|\gamma|^{-2}(D\mathscr{P})\left(\frac{V}{r} - (\frac{V}{r})_I\right)|\phi|^2\right]dr = -\int\left[\frac{1}{m^2}\, r\,|D\phi|^2 + \frac{1}{r}|\phi|^2\right] dr$$

$$< 0 \tag{8}$$

The analogue of condition (c) follows immediately. In this case, if $(\frac{V}{r})$ is a monotonic function of $r$, and $D(\mathscr{P})$ vanishes only once in $(R_1 < r < R_2)$ , the necessary condition for instability is that,

$$\left[\frac{V}{r} - \left(\frac{V}{r}\right)_I\right] (D\mathcal{G}) \leq 0 \qquad \text{everywhere in} \quad (R_1 < r < R_2) \qquad (9)$$

A novel feature of this condition compared to that expressed by equation (4) is that it may be satisfied for a relative maximum or minimum of vorticity. This point is illustrated by the velocity/vorticity profiles of Figures 1.

These figures also demonstrate that a velocity profile resembling a slightly diffused vortex sheet can exist in curved flow for both a relative maximum and minimum of vorticity. The discrepancy in the conclusions drawn from equations (4) and (9) is merely indicative of this fact.

The general stability criteria quoted above do not yield any precise information of the instability, if any, of the velocity profiles shown in Figs. (1), when they are subjected to a small disturbance. It is conceivable that the character of the instability of these flows could be markedly different for three dimensional perturbations since the toroidal component of the disturbance would activate the stratifying effect of the circulation and the velocity profile of Fig. 1(b) has a larger 'stable stratification'.

One significant result regarding the behaviour of profiles of these types to cylindrical perturbations emerges from an examination of the effect of the addition, or subtraction of a purely solid body rotation ($\Omega$) to the flow. The stability of differential equation (3) retains the same form when derived for a system of axes rotating about the z axis with a uniform angular velocity ($\Omega$), with $(\frac{V}{r})$ now denoting the angular velocity relative to the rotating frame of reference. Hence the addition of a purely solid body rotation to the flow leaves the stability properties unchanged. The problem considered is the next Section emphasizes this point and also serves to illustrate the effect of curvature on the instability of a flow to cylindrical perturbations.

## III. <u>The Stability of a Particular angular velocity distribution to cylindrical disturbances</u>

Simple velocity profiles have to be chosen if we are seeking analytical solutions of equation (3). We will consider the flow inside a cylinder of unit radius, with the flow consisting of three distinct regions, inner and outer regions of solid body rotation separated by an intermediate shear layer giving an overall, piecewise continuous velocity profile.

Therefore our vorticity field takes the form,

$$\mathscr{S}_{I} = 1 \qquad \text{in} \quad 0 \leq r < a$$

$$\mathscr{S}_{III} = P = \text{(positive constant)} \quad \text{in} \quad b < r \leq 1 \qquad (10(a))$$

and hence

$$\mathscr{S}_{II} = (b^2 - a^2)^{-1}(Pb^2 - a^2) = Q \quad \text{in} \quad a < r < b$$

The corresponding velocity distribution is

$$V_I = \tfrac{1}{2}r$$

$$V_{III} = \tfrac{1}{2}Pr \qquad (10(b))$$

and

$$V_{II} = \tfrac{1}{2}(b^2 - a^2)^{-1}[(Pb^2 - a^2)r + \tfrac{1}{r}a^2b^2(1-P)]$$

These velocity and vorticity profiles are illustrated in Fig. (2).

Note that if (i) $1 > P$ then $Q < P$

(ii) $P > 1$ then $Q > P$

Thus we have a relative minimum and a relative maximum of vorticity within the flow field in cases (i) and (ii) respectively, provided $Q$ is a positive constant. From the general stability criteria we know that the flow is stable to axi-symmetric perturbations if $Pb^2 > a^2$ .

For cylindrical disturbances (i.e. $k = 0$) in a region of constant vorticity the stability differential equation reduces to,

$$D^2\phi + \frac{1}{r} D\phi - \left(\frac{m}{r}\right)^2 \phi = 0 \qquad (11)$$

This equation must be satisfied in each of the three regions.

The required boundary conditions stipulate that $u$ is finite at $r = 0$ , and zero at $r = 1$ , and that the interfacial pressures and radial displacements must be continuous.

$$\left.\begin{array}{llll} \text{i.e.} & \phi = 0 & \text{at } r = 0, 1 & \text{(a)} \\ & \phi \text{ continuous} & \text{at } r = a, b & \text{(b)} \\ \text{and} & & & \\ \left[r\gamma(D\phi) - (m\mathcal{S}\phi)\right] & \text{continuous} & \text{at } r = a \;\; b & \text{(c)} \end{array}\right] (12)$$

The expression for the perturbation pressure $(p)$ in

(12c), was derived from the equations of motion linearised with respect to a small perturbation.

For $m \neq 0$, the solution of (11) is given by

$$\phi = Ar^{m} + Br^{-m}$$

Therefore the perturbation functions $\phi$ in the three regions may be written, using (12(a)), as,

$$\phi_{I} = A_{1} r^{m}$$

$$\phi_{II} = A_{2} r^{m} + B_{2} r^{-m}$$

$$\phi_{III} = A_{2}(r^{m} - r^{-m})$$

Conditions (12(b)) and (12(c)) applied at $r = a$ give,

$$A_{2} a^{2m}(1-Q) + B_{2}[-(2\sigma+m) + (1-Q)] = 0$$

and at $r = b$ they give,

$$A_{2}[c^{-1}(2\sigma+mP) + b^{m}(Q-P)] + B_{2}[c^{-1}(2\sigma+mP) + b^{-m}(Q-P)] = 0$$

where $c = (b^{m}-b^{-m})$

The consistency condition for these two linear equations yields a quadratic in $\sigma$, whose solution may be written after considerable manipulation in the form,

$$4\sigma = (b^2-a^2)^{-1}(1-P)[-\{(a^{2m}-1)b^2 + m(b^2-a^2)(1+P)(1-P)^{-1} -(b^{2m}-1)a^2\} \pm (\{(a^{2m}-1)b^2 + m(b^2-a^2) + (b^{2m}-1)a^2\}^2 -4a^{2m}(abc)^2)^{1/2}] \quad (13)$$

The flow will be unstable if $\sigma$ is complex,

$$\text{i.e. if} \quad [4a^{2m}(abc)^2 - \{(a^{2m}-1)b^2 + m(b^2-a^2) + (b^{2m}-1)a^2\}^2]>0 \quad (14)$$

This condition is independent of whether $1 \gtrless P$ ; that is, independent of whether the extremal of vorticity within the flow is a relative maximum or minimum. Thus the piecewise continuous, velocity profile considered here exhibits the instability feature deduced from expression (9) for continuous velocity profiles. Furthermore we deduce from (13) and (14) that the instability (if it exists) of the two flows with $P = (1\pm\alpha)$ are identical except for a numerical difference in the values of the cyclic frequency of the unstable perturbations.

A consideration of the results one would expect if region I was assumed stationary, together with the comments that were made regarding the effect of an additional, purely solid body rotation of the whole system, would lend support to the above result.

The cyclic frequency $(-\sigma_r)$ and the growth rate $(\sigma_i)$ of an unstable perturbation are given by the expressions,

$$-4\sigma_r = \pm\ \alpha\ (b^2-a^2)^{-1}[\ (a^{2m}-1)b^2+m(b^2-a^2)(2\pm\alpha)(1\mp\alpha)^{-1}-(b^{2m}-1)a^2] \tag{15}$$

and

$$4\sigma_i = \pm\ \alpha\ (b^2-a^2)^{-1}[4a^{2m}(abc)^2 - \{(a^{2m}-1)b^2+m(b^2-a^2) +(b^{2m}-1)a^2\}^2\ ]^{1/2} \tag{16}$$

We can immediately deduce from (16) the following stability features dependent upon the azimuthal wave no. (m),

(i) $m = 1$ neutral for all $0 \leq a \leq 1$ , $a \leq b$

(ii) $m \geq 1$ neutral for $a = b$ , and $a, b \neq 1$

(iii) $m > 1$ stable for $a = 0$ , $b \neq 0$

(iv) $m > 1$ stable for $b = 1$ , $a \neq 1$

The second deduction may seem incongruous in view of the expected instability of a vortex sheet. However the flow model under investigation is such that, although in the limit of $(b-a)$ tending to zero the velocity configuration tends to vortex sheet (if $P \neq 1$), yet the stipulation that $a = b$ forces the further condition $(P = 1)$ on the model. Thus when $a = b$ the whole flow field rotates with the same angular velocity. These deductions also indicate the singular behaviour of the eccentric mode associated with the azimuthal wave number $(m = 1)$.

Neutral curves must satisfy the equation

$$4a^{2m}(abc)^2 - \left[(a^{2m}-1)b^2 + m(b^2-a^2) + (n^{2m}-1)a^2\right]^2 = 0 \qquad (17)$$

Neutral curves were computed from (17) in terms of the parameters 'a' and 'b' for fixed 'm'. The calculation was carried out for a range of values $(a,b)$, with $m$ taking integral values from 2 to 10. A selection of the results are plotted on the $(b,a)$ plane in Figure (3).

The flow configuration is unstable within the domain circumscribed by the line $(a = b)$, and the neutral curve for $m = 2$. As $(b-a)$ is decreased from its value on the $(m = 2)$ curve successively higher values of the azimuthal wave number $(m)$ become unstable. This is consistent with the expectation that the vortex sheet created as $(b-a)$ tends to zero would be unstable for all wave numbers (excluding $m = 1$).

Equation (16) may now be used to compute the values of the growth rate at unstable stations in the $(b,a)$ plane. In Figures (4) values of $[4\sigma_i(1-P)^{-1}]$ are plotted against 'm' for various values of $(b-a)$ and fixed 'a'. These diagrams indicate that the wave number and magnitude of the growth rate of the most unstable mode increase as the velocity profile tends to a vortex sheet.

Another interesting feature of the neutral curve diagram is the stabilising effect of large curvature.

## IV. Comparison of Theory with Experimental Observations

Instability of a 'quasi-cylindrical' form was observed by Hide and Titman [1967] in their experimental study of shear layers in a rotating fluid. Shear layers with a basic, swirl velocity profile resembling that considered in Section III were set up in their experiments, and shown to be stable or unstable depending on the values of certain flow parameters. Hide and Titman were able to construct a 'marginal' stability relation between these parameters.

Another salient feature of their results was that perturbed flows with a form attributable to wave-numbers $m = 2$ to $8$ were observed when the basic flow had a relative minimum of vorticity, whilst only an eccentric perturbed mode appeared when the extremum was a maximum. Their results also indicate that for the range of flow parameters used in their experiments the azimuthal wave-number observed was almost independent of the width of the shear layer, but directly related to the difference in the angular velocity of rotation within the layer.

Hide and Titman's experiments were carried out with the shear layers situated at radii $0.2 < (a,b) < 0.55$, where 'a' and 'b' are taken to be defined as in Section III. The results of Section III indicate that the basic flow is unstable in most of the region in the $(b,a)$ plane delineated by the above values, with the perturbation wave numbers ~~almost independent of~~ highly dependent upon the width of the shear layer $(b-a)$, and totally independent of the difference in the angular velocity of rotation within the shear layer.

A theoretical analysis attempting to explain the stability of the observed flow for a certain range of flow parameters within the above range of (b,a) has recently been proposed by Busse [1968]. His analysis is based on the assumption that the so-called Proudman-Taylor constraint of the rotation ensures that the zero-order perturbation flow has no axial (z) dependence in regions removed from solid boundaries perpendicular to the z direction. By imposing an Ekman suction relation on the interior flow at these boundaries he is able to partially incorporate the effect of viscous dissipation, and thus account for the marginal stability relation deduced from experiment.

However his analysis, which is an extension of the problem considered in Section III is also unable to account for the stability properties of the flow that are dependent upon the nature of the vorticity extremum within the flow. It has previously been mentioned that three dimensional perturbations might account for this differential behaviour. An examination of the results of Section III in conjunction with the first necessary condition for instability quoted in Section II lends credence to this conjecture.

## V. Further Consideration of Three Dimensional Perturbations

A brief attempt is made in this section to examine the stability of some two dimensional, swirling flows to three dimensional perturbations. A 'narrow gap' approximation is employed to simplify the stability differential equation (1), but only limited success is obtained from seeking analytical solutions of the resulting equation.

To justify the adoption of a 'narrow gap' approximation we consider the swirling flow to be confined between two concentric cylinders radii, $R_1$ and $R_2$ , such that,

$$(R_2-R_1) \ll \tfrac{1}{2}(R_1+R_2) \qquad (18)$$

This stipulation enables us to use cartesian coordinates $(x,y,z)$ , and by working in a frame of reference rotating with uniform angular velocity $(\Omega)$ we are able to retain a curvature effect in a modified form.

Our cartesian coordinates $(x,y,z)$ are defined in relation to the original cylindrical polars $(r,\Theta z^r)$ as follows,

$$x = r - R \qquad \text{where} \quad R = \tfrac{1}{2}(R_1+R_2)$$
$$y = R\theta \qquad\qquad (19)$$
$$z = z^{*}$$

The governing equations in the new rotating frame of reference are

$$\underline{v}_t + (\underline{v}.\nabla)\underline{v} + 2\,(\underline{k}\wedge\underline{v}) = -\frac{1}{\rho}(\nabla p) + (R\Omega^2)\,\underline{i} \quad \text{(a)}$$
$$\nabla.\underline{v} = 0 \qquad\qquad (20)$$

where $(\underline{i},\underline{j},\underline{k})$ are unit vectors in the $(x,y,z)$ directions. Equation 20(a) is a vector form of the momentum equations in the $(x,y,z)$ directions, and equation 20(b) is the mass continuity relation for an incompressible fluid.

For a basic flow state $(0,V(x),0)$ the analogue of the stability differential equation (1), obtained in the usual manner from equations (20), is

$$L^2(\nabla^2 u) - (V_{xx})L(u_y) + 2\Omega(2\Omega+V_x)\,u_{zz} = 0 \qquad (21)$$

$$\text{where} \quad L = \frac{\partial}{\partial t} + V\frac{\partial}{\partial y}$$

$$\text{and} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Considering normal mode perturbations of the form,

$$u = \chi(x)\, e^{i(\sigma t+ny+kz)} \tag{22}$$

equation (21) may be rewritten as follows

$$\chi_{xx} - \chi[(n^2+k^2) + n\gamma^{-1}(V_{xx}) - k^2(\gamma^{-2})2\Omega(2\Omega+V_x)] = 0 \tag{23}$$

The instability mechanisms attributable to the distribution of the vorticity gradient and the circulation, which were salient features associated with equation (1), are also intrinsic in equation (23).

The stability of two types of flow are obtained below from an examination of equation (23), whereas the original equation (1) would not have been so amenable to analytical treatment.

Type 1: Three Dimensional Perturbations of a Uniform Shear Flow. $(V_{xx} = 0)$

Introducing a new independent variable defined by

$$I = G(\rho+V) \qquad \text{where} \quad G^2 = (V_x)^{-2}(n^2+k^2)$$
$$\text{and} \quad \rho = \chi^{-1}\sigma$$

then equation (23) becomes

$$\chi_{II} - \chi[1 - \frac{\mathcal{S}}{I^2}] = 0 \tag{24}$$

$$\text{where} \quad \mathcal{S} = \frac{2\Omega(2\Omega+V_x)}{(V_x)^2} \quad (\frac{k}{n})^2$$

This equation, whose solution may be written in terms of Modified Bessel functions, has to be solved subject to inviscid boundary conditions imposed at $x = \pm h$ .

$$\text{i.e. } \chi = 0 \quad \text{at} \quad I = -\alpha[1 - \frac{\rho}{hV_x}] \tag{25}$$

$$\text{and at} \quad I = \alpha[1 + \frac{\rho}{hV_x}]$$

$$\text{where} \quad \alpha = h(n^2+k^2)^{1/2}$$

In deriving these boundary conditions we have assumed $V = 0$ (relative to the rotating system) at $x = 0$ .

The solu‥ion of an equation with the form of (24), subject to boundary conditions (25) has been extensively analysed by Kuo [1963]. His results indicate that the flow is ·nstable for $\mathcal{S} < \bar{\mathcal{S}}(\alpha)$ , and $\bar{\mathcal{S}}(\alpha)$ is shown to have the following behaviour: $\bar{\mathcal{S}} = -.75$ for $\alpha = 0$ , decreasing to $(-2)$ as $\alpha$ increases to $(1.2)$, and thereafter $\bar{\zeta}$ remains constant for further increase of $\alpha$ . Kuo also indicates that the most preferred mode corresponds to the largest value of $(-\zeta)$ .

Thus we may conclude that inviscid considerations of cylindrical Couette flow show that three dimensional perturbations may be unstable, but the largest growth rates are always associated with the two dimensional axi-symmetric perturbations.

Type 2: Three Dimensional Perturbations of a Vortex Sheet.

It would be interesting to obtain an understanding of the interplay of the destabilizing effect of a vorticity extremum of a basic, swirling flow and the ~~de~~stabilizing effect of the circulation. Both these mechanisms are involved when the flow is subjected to three dimensional perturbations.

Solutions of equation (23) can be obtained for some 'three regioned', piecewise continuous velocity profiles, with the middle region forming a vorticity extremum. However, the consistency conditions obtained for the perturbed flow are heavily weighted with Bessel functions, and extensive computations would be required to ascertain the stability features of the solution.

We resort to solving the simplest possible problem involving both a vorticity extremum and circulation effects, namely we consider the problem of a vortex sheet.

Consider the following basic velocity profile,

Region I : $V = V_o$ for $0<x<d_1$.

Region II : $V = 0$ for $-d_2<x<0$ .

In both regions equation (23) reduces to

$$\chi_{xx} - \chi[(n^2+k^2) - (\gamma^{-1}\Omega 2\ k)^2] = 0 \tag{26}$$

with $\gamma = \sigma$ in Region I

and $\gamma = \tilde{\sigma} = (\sigma+nV_o)$ in Region II

The usual inviscid boundary conditions requiring the vanishing of normal velocity at a rigid wall and continuity of radial displacement and pressure at a fluid interface may be written as follows for this problem

$$\chi = 0 \quad \text{at} \quad x = -d_2,\ d_1$$

$$\text{and} \quad \left(\frac{\chi}{\gamma}\right) \quad \text{and} \quad [\gamma\chi_x - (nV_x)\chi] \quad \text{continuous at} \quad x = 0 \tag{27}$$

The consistency condition obtained by solving the appropriate form of equation (26) in the two regions, subject to the boundary conditions (27), has the form

$$\sigma^2 f_2 \tanh (f_1 d_1) + \tilde{\sigma}^2 f_1 \tanh (f_2 d_2) = 0 \qquad (28)$$

$$\text{where} \quad f_1^{\,2} = \left[(n^2+k^2) - (\tilde{\sigma}^{-1}) 2\Omega k\right]^2$$

$$\text{and} \quad f_2^{\,2} = \left[(n^2+k^2) - (\sigma)^{-1} 2\Omega k\right]^2$$

Rather than treat equation (28) directly we confine our attention to an examination of the stability of the vortex sheet in the limit of $d_1$ and $d_2$ tending to infinity. This stipulation gives a simplified consistency equation,

$$\sigma^2 f^2 + \tilde{\sigma}^2 f_1 = 0 \qquad (29)$$

Squaring equation (29) a bi-quadratic for $\rho \; (= {}^{\sigma}/n)$ is obtained which may be re-arranged in the following form,

$$q^2 + q(2V_o^{\,2} - g) + \tfrac{1}{2}V_o^{\,2}(V_o^{\,2} - g) = 0 \qquad (30)$$

$$\text{where} \quad g = \frac{4\Omega^2}{(n^2+k^2)} \left(\frac{k}{n}\right)^2$$

$$\text{and} \quad q = \rho(\rho + V_o)$$

Solving (30) for $q$ , and then $\rho$ we obtain the following expression,

$$2\rho = -V_o \pm \sqrt{\left[ (2g-3V_o^2) \pm (2V_o^4-2gV_o^2+g^2)^{1/2} \right]} \quad (31)$$

Examination of (31) reveals that the vortex sheet is always unstable (i.e. $\rho$ is complex) for $n \neq 0$ , irrespective of the magnitude of the basic rotation ($\Omega$) of the system. Thus the destabilizing effect of the vortex sheet's vorticity can always overcome the possible stabilizing effect of the circulation in the particular problem considered here.

# CHAPTER 2

# Hydrodynamic Stability of Two Dimensional Swirling Flow with a Transverse Density Stratification

## I. Introduction and Summary

It is known from the work of Miles [1961] and Howard and Gupta [1963] that inertial instability of a parallel flow due to an extremum of vorticity can be suppressed for a range of velocity profiles by suitable stratifications, viz. density in the presence of a transverse gravity field in the case of plane shear flow, and the circulation of a swirl velocity in the case of axi-symmetric perturbations of an axial flow between concentric cylinders.

Another example of inertial instability, that of two dimensional swirling flow to cylindrical perturbations when there is an extremum of vorticity within the flow field, was the subject of investigation in the previous chapter. The introduction of a radially acting gravity field and a density stratification of the fluid to such a flow would certainly lead to results similar to those for plane shear flow in a gravity field, provided only cylindrical perturbations were considered.

In the next section the general stability criteria for two dimensional, inviscid, swirling flows between two concentric cylinders are extended to include the effect of a radial density stratification. This stratification is introduced by considering the fluid to be heterogeneous but non-gravitating. It is shown that a radial increase of density coupled with a suitable circulation can itself produce a stabilizing effect, even for cylindrical perturbations. This effect is again able to suppress inertial instability of the swirl velocity for a range of velocity profiles and density stratification. Conversely a radial decrease of density obviously introduces another mechanism for centrifugal instability. The general stability criteria obtained are derived subject to a Boussinesq type assumption.

In Section III it is shown that at least for certain flow configurations and select perturbation modes the necessary conditions for instability embodied in these general criteria are also sufficient conditions for instability.

The Boussinesq type assumption is not invoked in

Section IV, and it proves possible to obtain stability criteria for the flows when the density is proportional to a power of the radial distance.

## II. Derivation of the Stability Differential Equation and General Stability Criteria

The pertinent equations for an incompressible heterogeneous, inviscid, non-gravitating fluid in cylindrical polar coordinates $(r,\theta,z)$ are the three momentum equations, a mass continuity relation denoting the incompressibility of the fluid, and a density advection equation stating that the density is merely advected with the fluid. The latter relation takes the mathematical form,

$$\rho_t + u\rho_r + \frac{v}{r}\rho_\theta + w\rho_z = 0 \qquad (1)$$

where $(u,v,w)$ are the velocities in the $(r,\theta,z)$ directions respectively and $\rho$ is the density.

We have as our basic state of motion an azimuthal flow $\underline{v} = (0,V(r),0)$ and a radial variation of density $\rho = \bar{\rho}(r)$ . Linearising the governing equations with respect to small perturbations $(\underline{v}', \rho', p')$ from the basic state and allowing the perturbation variables to take the normal mode form

$$u' = u(r)\, e^{i(\sigma t + m\theta + kz)} \text{ etc.}$$

we can eliminate all dependent variables except 'u' from the resulting set of equations. Then the stability e equation takes the form,

$$D[rf(D\phi)] + [rf\beta(D\phi)] - \phi[\frac{1}{r} + m\gamma^{-1}\frac{1}{\bar{\rho}} D(\bar{\rho} f \varphi) - \gamma^{-2}\{2k^2(V\varphi f) + \beta(\frac{V}{r})^2\}] = 0 \quad (2)$$

where $\beta = \frac{1}{\bar{\rho}}(\bar{\rho}_r)$, and the other symbols retain the meaning attached to them in Chapter 1.

The boundary conditions are again

$$\phi = 0 \quad \text{at} \quad r = 0 \text{, and/or at the bounding cylinder(s)} \quad (3)$$

Most of the work of this chapter will be subject to an assumption akin to the Boussinesq assumption, which is frequently employed in gravitational convective problems. We will assume that the variation in $\rho$ is small compared to $\rho$ itself so that the effect of the variation of density on the inertia may be neglected except for the centrifugal effect it induces in the radial momentum equation. Subject to this assumption equation (3) simplifies to,

$$D[rf(D\phi)] - \phi[\frac{1}{r} + m\gamma^{-1}D(f\mathcal{G}) - \gamma^{-2}\{2k^2(V\mathcal{G}f) + \beta(\frac{V}{r})^2\}] = 0 \quad (4)$$

We will refer to (4) as the stability equation for a quasi-Boussinesq fluid. This differential equation is the same as equation (1.1) except for the additional term involving $\beta(\frac{V}{r})^2$. An important feature of this term, compared with the other term with $(\gamma^{-2})$ as a factor is that it is independent of the wavenumbers 'm' and 'k'.

The similarity of these equations implies that we need only extend the method employed by Howard and Gupta [1963] to derive the general stability criteria. Utilizing their technique the following expressions are easily obtained,

$$\sigma_i \int_{R_1}^{R_2} [r^3 f|\psi'|^2 + r(1 - D(rf))|\psi|^2 + \{2k^2(\gamma\mathcal{G}f) + \beta(\frac{V}{r})^2 - \frac{1}{4}rm^2(D(\frac{V}{r}))^2\}|\gamma|^{-2}|\psi|^2]dr = 0 \quad (5)$$

$$\sigma_i \int_{R_1}^{R_2} [m\, D(f\varphi) - 2\{2k^2(\gamma\varphi f) + \beta(\frac{V}{r})^2\}(\sigma_r + m\frac{V}{r})\ |\gamma|^{-2}]\ |\gamma|^{-2}|\psi|^2\, dr = 0 \qquad (6)$$

$$\int_{R_1}^{R_2} |(\gamma)|^2[r^3 f\,|D\psi|^2 + r(1-D(rf))|\psi|^2]dr + \int_{R_1}^{R_2} m\gamma\ 2(\frac{V}{r})D(f)\ |\psi|^2 dr = \int_{R_1}^{R_2} [2k^2(\gamma\varphi f) + \beta(\frac{V}{r})^2]\,|\psi|^2\, dr \qquad (7)$$

From (5) we deduce that,

a necessary condition for instability is that,

$$[2k^2(V\varphi f) + \beta(\frac{V}{r})^2 - \frac{1}{4}\, frm^2(D(\frac{V}{r}))^2] < 0 \text{ somewhere in } (R_1 < r < R_2) \qquad (8)$$

and there is an upper bound on the growth rate given by,

$$\sigma_1^2\ Min^m[1-D(rf)] \leq Max^m[\frac{1}{4}\, m^2 f(D(\frac{V}{r}))^2 - \frac{1}{r}\beta(\frac{V}{r})^2 - \frac{2}{r}k^2(\gamma\varphi f) \qquad (9)$$

Again from (6) we may deduce that,

a necessary condition for instability is that

$$[mD(f\mathcal{G}) - 2\{2k^2(\gamma\mathcal{G}f) + \beta(\tfrac{V}{r})^2\}(\sigma_r+m\tfrac{V}{r})\ |\gamma|^{-2}]$$

must disappear somewhere in $(R_1<R<R_2)$ (10)

and further if $D(f\mathcal{G}) \neq 0$ in $(R_1<r<R_2)$ then

$$\sigma_i^2 \leq |\gamma|^2 \leq 2\mathrm{Max}^m\Big[[mD(\sigma\mathcal{G})]^{-1}\{2k^2(V\mathcal{G}f)+\beta(\tfrac{V}{r})^2\}(\sigma_r+m\tfrac{V}{r})\Big] \tag{11}$$

An examination of (7) yields the following relation governing the frequency of a perturbation with either m or k set to zero.

$$\ell \leq (-\sigma_r) \leq n \tag{12}$$

where $n = \mathrm{Max}^m(m^V/r)$

and $\ell = \mathrm{Min}^m(m^V/r)$

Finally if $\beta$ and $(V\mathcal{G})$ are greater than, or equal to, zero in $(R_1<r<R_2)$ then equation (17) is of the form for which the 'Semi-Circle Theorem' of Howard [1961] can be shown to be valid if $m = 0$. For this situation we can assert that $\sigma = (\sigma_r+i\sigma_i)$ is such that,

$$[\sigma_r + \tfrac{1}{2}(\ell+n)]^2 + \sigma_i^2 \leq \tfrac{1}{4}(\ell-n)^2 \tag{13}$$

i.e. $(-\sigma)$ the complex frequency must lie inside a semi-circle in the upper half plane with the circle's diameter given by $(n-\ell)$.

For cylindrical perturbations equation (8) gives a Richardson type criterion for flow stability.

$$R_i = \frac{1}{r}\beta\left(\frac{p}{Dp}\right) \geq {}^1/4 \quad \text{everywhere ensures stability} \qquad (14)$$

with $p = \left(\frac{V}{r}\right)$ .
Swirl velocity is thus seen to have a dual role even for cylindrical perturbations in this instance. It is the possible seat of inertial instability for an extremum of vorticity within the flow field, whilst coupled with the density stratification it can provide a possible stabilizing effect on such an instability for a range of velocity profiles. We note also that in this case the addition of a purely solid body rotation in the presence of a radial density stratification could certainly alter the stability features of some flows.

Two interesting special cases of (8) are,

(i) For solid body rotation the condition is,

$$\left[4k^2(rf) + \beta\right] < 0 \quad \text{somewhere in} \quad (R_1 < r < R_2) \qquad (15)$$

(ii) For axi-symmetric perturbations the condition may be written,

$$D[\bar{\rho}(rV)^2] < 0 \quad \text{somewhere in} \quad (R_1 < r < R_2) \qquad (16)$$

Condition (15) suggests that for this case cylindrical perturbations may be the most unstable if $\beta$ is negative, whilst (16) expresses the previously known modifying effect of density variation on centrifugal instability.

## III. Instability of a quasi-Boussinesq fluid for certain flow configurations

To show that the necessary condition for instability expressed by (8) for a quasi-Boussinesq fluid is also a sufficient condition for at least some velocity and density configurations we consider two special cases.

Case 1: Cylindrical perturbations of a basic state given by

$$\left(\frac{V}{r}\right) = \Omega \quad (= \text{constant})$$

$$\beta = \frac{1}{\rho}(\bar{\rho}_r) = cr \quad \text{and} \quad c \quad \text{is a constant.}$$

For a basic state of this form equation (4) becomes

$$D^2\phi + \frac{1}{r}D\phi + m^2\phi[c\Omega^2\gamma^{-2} - \frac{1}{r^2}] = 0 \qquad (17)$$

This equation has to be solved subject to the boundary conditions given in (3). The two boundary conditions can not be satisfied simultaneously if $(c\Omega^2\gamma^{-2})$ is negative, i.e. if the solution of (17) is written in terms of modified Bessel Functions. Hence the general solution of (17) is given by

$$\phi = A\ J_m(pr) + B\ Y_m(pr) \qquad (18)$$

where $p^2 = [m^2\Omega^2\ c^2(\gamma)^{-2}]$, and $J_m$ and $Y_m$ are Bessel Functions of the first and second kind of order $m$.

Substituting the boundary conditions for the flow inside a cylinder radius 'a' into equation (18) we have that

$$J_m(pa) = 0$$

and therefore $$m^2\gamma^{-2}c^2\Omega^2 = (\alpha_m)_n^2 \qquad (19)$$

where $(\alpha_m)_n$ is the nth root of $J_m(pr)$

With $(c\Omega^2\gamma^{-2})$ positive it follows that,

(a) For $c$ positive, $\gamma^2$ is positive and real and hence the flow is stable.

(b) For $c$ negative, $\gamma^2$ is negative and real.

$$\left.\begin{array}{l} \text{Hence } (-\sigma_r) = m\Omega \\ \text{and } \sigma_i \neq 0 \quad \text{i.e. the flow is unstable} \end{array}\right\} \quad (20)$$

Thus for a radial decrease of density the flow is unstable to cylindrical perturbations. Furthermore we can deduce from (19) and (20) that for specified values of $\Omega$, $a$ and $c$ (negative) that

$$\text{Max}^m \, (\sigma_i)^2 = \left\{ |c| \, \Omega^2 a \right\} \max^m \left[ \frac{m}{(\alpha_m)_n} \right]^2$$

Thus the largest growth rates are associated with the largest azimuthal wavenumbers.

A similar result may be derived for the flow between concentric cylinders.

<u>Case 2</u>: Axi-symmetric perturbations of the flow between two concentric cylinders when the basic state has the following special form.

$$\frac{V}{r} = \Omega \quad (= \text{constant})$$

$$\beta = \frac{1}{\rho} (\bar{\rho})_r = \frac{c}{r} \quad \text{with } c \text{ taken to be a constant.}$$

In this case keeping $u = (\phi/r)$ as the dependent variable the form of the differential equation (4) becomes

$$D^2u + \frac{1}{r} Du + u[k^2\{(\frac{\Omega}{\gamma})^2(4+c) - 1\} - \frac{1}{r^2}] = 0 \qquad (21)$$

In view of the nature of the boundary conditions (3) we write the general solution of (21) in the form,

$$u = AJ_1(pr) + BY_1(pr)$$

and take $p^2 = k^2\left[(\frac{\Omega}{\gamma})^2(4+c)-1\right]$ to be positive.
It follows that the flow is stable or unstable depending on whether

$$(4+c) \gtrless 0$$

If the flow is unstable then $\sigma_r = 0$ and we can deduce also that the largest growth rates occur as the axial wavenumber $(k)$ tends to infinity.

Both these examples show that the condition given in (15) is a necessary and sufficient condition for instability of these two particular flows.

## IV. Some results for the Instability of a non-Boussinesq Fluid

The quasi-Boussinesq restrction was originally invoked to reduce the complexity of the stability differential equation given by (2). In this section we derive some stability criteria for cylindrical and axi-symmetric perturbations of a non-Boussinesq fluid when the density stratification factor $(\beta)$ is inversely proportional to radial distance.

Case 1. Two dimensional cylindrical perturbations of the flow between concentric cylinders for a basic swirl velocity $V(r)$ and a density stratification given by

$$\bar{\rho} = \rho_o \, r^{(n-1)}$$

so that $\beta = \frac{1}{r}(n-1)$

If we apply the following transformation of variables to the form of (2) appropriate to this problem,

$$\phi = \left[r^{-(1/2)(n-1)}\right]\psi$$

$$r = e^x$$

Then the stability differential equation reduces to,

$$\psi_{xx} - \psi[F] = 0$$

where $$F = m^2 + m\gamma^{-1}\left[r(D\mathscr{Y}) + \mathscr{Y}(n-1)\right] - m^2\gamma^{-2}\left[(n-1)\left(\frac{V}{r}\right)^2\right] + \frac{1}{4}(n-1)^2$$

whilst the boundary conditions become

$$\psi = 0 \quad \text{at} \quad x = \log_e R_1$$

$$\text{and at} \quad x = \log_e R_2$$

Applying the integral method due to Howard, which was also used in Section II of this chapter, we obtain the following necessary condition for instability,

$$\left[\beta\left(\frac{V}{r}\right)^2 - \frac{1}{4}r\left(D\left(\frac{V}{r}\right)\right)^2\right] < 0 \quad \text{somewhere in the field of flow}$$

This relation is of the same form as that given by (8) for cylindrical perturbations of a quasi-Boussinesq fluid.

Case 2. Axi-symmetric perturbations of the same basic state. Proceeding in the same manner as for Case 1 but keeping $u = (\phi/r)$ as the dependent variable, we arrive at the following necessary condition for instability,

$$[2\frac{V}{r^a}(\mathcal{G}) + \beta(\frac{V}{r})^2] < 0 \quad \text{somewhere in the field of flow} \qquad (22)$$

Again this relation has the same form as that given by (8) for axi-symmetric perturbations of a quasi-Boussinesq fluid.

We can further note that for this case when the fluid is in solid body rotation the stability differential eqaution can be solved immediately. The operative form of equation (2) is

$$D^2u + \frac{n}{r}Du + u[k^2\{(\frac{\Omega}{\gamma})^2(n+3)-1\} - \frac{1}{r^2}(2-n)] = 0 \qquad (23)$$

With a change of dependent variable to $\psi$, where

$$u = r^{-(1/2)(n-1)}\,\psi$$

equation (23) becomes a Bessel equation, and as for equation (21) it is possible to infer that the flow is unstable if

$$(n+3) < 0$$

Thus condition (22) is also a sufficient condition for instability of this particular basic state.

# CHAPTER 3

## Centrifugal Effects in Stratified, gravitating swirling flows.

### 1. Introduction

In the previous chapter the stability of a non-gravitating, swirling fluid possessing a radial density stratification was examined for small perturbations of the basic flow. Particular attention was paid to the centrifugal effects associated with the density stratification.

To investigate the role of the centrifugal terms in the inviscid limit of a real fluid we must relax the non-gravitating condition. However radial stratification in the presence of a non-radial gravity field requires a concomitant shear in the swirl velocity, and this complicates the stability analysis of the linearised perturbation equations.

In this chapter we study the following two problems, the effect of the introduction of a radial gravity field and secondly the stability of axi-symmetric perturbations of a swirling fluid with a radial and axial density stratification in the presence of an axial gravity field.

The effect of a radial gravity field is easily incorporated within the framework of the analysis of the previous section. An indication is given of the limited physical significance of this particular case to certain small scale motions of the earth's equatorial atmosphere.

In the second problem the central issue is the relative importance of the gravity field and the centrifugal terms as possible instigators of convective and centrifuging motions respectively. The theory is developed only for axi-symmetric perturbations, whereas the insight gained from the results of the previous chapter suggest that it would be judicious to consider axially symmetric motions.

## II. The effect of a radial gravity field on Centrifugal Instability

Only a slight modification of the analysis of the previous chapter is necessitated by the introduction of a radially inward acting (say) gravity field $g = g(r)$ . In this case the radial equation of motion is given by

$$u_t + uu_r + \frac{v}{r} u_\theta + wu_z - \frac{v^2}{r} = -\frac{1}{\rho} p_r - g(r) \qquad (1)$$

The resulting differential equation determining the stability of the basic state to small normal mode perturbations becomes [c.f. II(2)],

$$D[rfD\phi] + (rf\beta)D\phi - \phi[\frac{1}{r} + m(\gamma\ell)^{-1} D(\bar{\rho}f\mathscr{S}) - \gamma^{-2}\{2k^2(V\mathscr{S}f) + \beta(\frac{V}{r})^2 - \frac{g}{r})\}] = 0 \qquad (2)$$

This equation is again subject to the same boundary conditions and obviously has stability criteria allied to those derived in the previous chapter. In particular the necessary condition for instability corresponding to II (8) is,

$$[2k^2(V\mathscr{S}f) + \beta((\frac{V}{r})^2 - \frac{g}{r}) - \frac{1}{4} rfm^2(D(\frac{V}{r}))^2] < 0 \qquad (3)$$

somewhere in the field of flow

Thus for small perturbations the role of the density stratification is seen to be dependent upon the relative magnitude of the angular velocity of the swirling flow and the strength of the gravity field. For a velocity profile with both $D(\frac{V}{r})$ and $(V\mathcal{S})$ positive, and a potentially unstable density distribution, i.e. the second term in (3) taken to be negative, condition (3) again suggests that three dimensional disturbances $(k\neq 0)$ are more stable than purely cylindrical disturbances $(k=0)$ .

To set the problem in a geophysical context we consider the motion relative to a system of axes rotating with uniform angular velocity $(\Omega)$ , and we again employ the narrow gap approximation first introduced in the last section of Chapter I.

We assume a basic state with the velocity in the cartesian coordinates $(x,y,z)$ defined as before, given by

$$\underline{v} = (0,\ V(x),\ 0)$$

and a density distribution $\rho = \bar{\rho}(x)$ .
We further assume $V_x$ and $\beta = (\frac{1}{\bar{\rho}}\, \bar{\rho}_x)$ to be constant for mathematical convenience and to preclude instability associated with vorticity extremae.

The stability equation derived from the linearised system of equations is,

$$\mathcal{L}^2[\nabla_h^2 u + u_{zz}] + 2\Omega(2\Omega+V_x)u_{zz} - (g_1\beta)\nabla_h^2(u)$$

$$+ \beta[\mathcal{L}^2(u_x) - (2\Omega+V_x)\mathcal{L}(u_y)] = 0 \qquad (4)$$

where $u$ is the radial perturbation velocity

$$\nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\mathcal{L} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial y}$$

and $g_2 = (g - R\Omega^2 - 2\Omega V)$

In subsequent work we will assume $g_1 \approx g$ = constant. Considering normal mode perturbations of the form

$$u = U(x)\ e^{i(\sigma t+ny+kz)}$$

equation (3) becomes,

$$\psi_{xx} - [\tfrac{1}{4}\beta^2 + (n^2+k^2) + \tfrac{1}{\gamma}(2\Omega+V_x)\beta n - \frac{1}{\gamma^2}\{2\Omega k^2(2\Omega+V_x)$$

$$-g\ \beta(n^2+k^2)\}]\ \psi = 0 \qquad (5)$$

where $\psi = (e^{1/2\beta x})\, u$

and $\gamma = (\sigma + nV)$

Vanishing of the normal velocity at rigid boundaries placed at $x = \pm d$ gives the boundary conditions

$$\psi = 0 \quad \text{at} \quad x = \pm\, d \tag{6}$$

The Boussinesq form of (5) is obtained by omitting the term involving $(\gamma)^{-1}$ .

For a non-rotating system $(\Omega = 0)$ the **problem** posed by equations (5) and (6) is the stability of a constant shear flow of a stratified layer of gravitating fluid bounded above and below by rigid walls. A detailed treatment of the Boussinesq form of this problem for a compressible atmosphere was undertaken by Kuo [1963].

Briefly, Kuo's analysis shows that the stability properties of such a flow may be described in terms of a modified Richardson Number $(\mathfrak{J})$ and a vectorial dimensionless wave number $(\alpha)$ , which for the problem posed by the Boussinesq, non-rotating form of equation (8) may be written as,

$$\bar{J} = -[1 + (\tfrac{k}{n})^2]\, g\beta(V_x)^{-2}$$

and (7)

$$\alpha = \tfrac{1}{2}\, d\, (n^2+k^2)^{1/2}$$

As mentioned in Chapter I, Kuo showed that the flow is unstable for $\bar{J}$ less than $\bar{J}_o(\alpha)$, with $\bar{J}_o$ taking the value $-0.75$ for $\alpha = 0$, decreasing gradually to $(-2)$ as $\alpha$ increases to $(1.2)$, and thereafter remaining constant as $\alpha$ is further increased. Thus two dimensional perturbations in the direction of the flow $(k = 0)$, corresponding to the cylindrical perturbations of our previous analysis, become unstable only when $J = -[g\beta(V_x)^{-2}]$ is below a finite negative limit, namely the value of $\bar{J}_o(\alpha)$ quoted above.

Kuo concludes that when $J$ is only slightly negative, roll type convective motions with $n = 0$ will be the most preferred mode of motion, and he asserts that this result offers a possible explanation for the occurrence of certain cloud formations in the earth's atmosphere.

His results are also valid for the rotating, Boussinesq form of equation (5) provided a further modification is made to the Richardson Number given in (7). For a rotating system the operative form of the Richardson Number is

$$J^{*} = \left[2\Omega(2\Omega+V_x)\left(\frac{k}{n}\right)^2 - g\beta\left[1+\left(\frac{k}{n}\right)^2\right]\right] (V_x)^{-2} \qquad (8)$$

The following deductions arise as a direct consequence of this new Richardson Number representation,

(a) For $(g\beta) < 0$ the flow may be unstable to perturbations with $k \neq 0$ for large, negative values of $V_x$ .

(b) Assuming $\left[2\Omega(2\Omega+V_x) - (g\beta)\right] > 0$ , and proceeding to increase the value of $(g\beta)$ from zero, the system is first stable, and then unstable only to two dimensional perturbations in the direction of flow $(k = 0)$ .

This second result is in keeping with the inference that was drawn from the condition expressed in (3).

A Physical Implication of the form of the new Richardson Number: Deduction (b) offers a subtle variation to the physical interpretation of the theory proposed by Kuo. He suggested that the formation of roll type convective cloud motions, with their axes aligned along the direction of the mean wind in the atmosphere, may be attributed to the growth of unstable perturbation modes engendered by the effect of a shear flow on an unstable stratification. This proposal is certainly not tenable for a rotating system unless,

$$[2\Omega(2\Omega + V_x) - g\beta] < 0 \qquad (9)$$

The Boussinesq form of (5) is a reasonable approximation to the stability equation of a purely zonal, vertically sheared flow of the earth's equatorial atmosphere, provided our attention is confined to perturbed motions with horizontal length scales not greater than 10 kilometers, and $\beta$ is taken to represent $[-\frac{1}{\Theta}\Theta_x]$, where $\Theta$ is the potential temperature. Our cartesian coordinate system $(x,y,z)$ points along the vertical and to the East and North respectively. The restriction on the length scales enables us to neglect the component of the Coriolis force proportional to the sine of the latitude.

For $|V_x| \gg \Omega$, condition (9) shows that the sign of $V_x$ is of crucial importance in determining the mode by which the atmosphere will respond to a nascent, unstable stratification. With $V_x$ large and negative the parallel mode $(n = 0)$ would dominate, whilst the theory predicts that for $V_x$ negative with $|V_x| << \Omega$, and for $V_x$ positive the cross wind mode $(k = 0)$ would appear.

Cross wind modes do occur in the atmosphere, and when they assume the form of billow clouds their existence is probably attributable to a Kelvin-Helmoltz type instability. The results of the above theory also suggest an instability mechanism that could give rise to a cross wind mode in subtropical and equatorial latitudes, and its appearance would not necessarily assume the form of billow clouds.

Some details of the cloud formations in these regions are given by Riehl and Malkus [1964] and Plank [1966]. Cloud rolls aligned parallel to the wind are frequently found embedded in the Trade Easterlies, where $V_x$ is negative. Plank notes that the cross wind modes were observed to occur most frequently at low wind velocities (less than 5 metres/sec) in conditions not necessarily associated with K-H type instability. Care is required in the interpretation of these observations since the destabilizing mechanism in the tropical atmosphere could also be associated with saturation of the air coupled with a suitable 'equivalent' potential temperature distribution. Furthermore it should be noted that cloud rolls aligned at a small angle to the wind have also been attributed to an instability of the surface Ekman layer (e.g. Faller [1965]).

To obtain definitive evidence of the stabilizing effect of the rotation without recourse to Kuo's analysis, one need only consider the solution of (5) when there is no basic motion relative to the rotating system.

Equation (5) then takes the form,

$$\psi_{xx} + m^2 \psi = 0 \qquad (10)$$

where $m^2 = -\frac{1}{4}\beta^2 - \ell^2 - 2\Omega\sigma^{-1}\beta n + \sigma^{-2}[4\Omega^2 k^2 - g\beta\ell^2]$

and $\ell^2 = (n^2 + k^2)$

Applying the boundary conditions $\psi = 0$ at $x = 0, d$ to the solution of (10), and seeking the consistency condition of the resulting two homogeneous equations, we obtain a quadratic in $\sigma$. The solution of this quadratic is,

$$[\tfrac{1}{4}\beta^2 + \ell^2 + q^2](\tfrac{\sigma}{n}) = -\Omega\beta \pm\sqrt{\Omega^2\beta^2 + [\tfrac{1}{4}\beta^2 + \ell^2 + q^2][4\Omega^2(\tfrac{k}{n})^2 - g\beta(\tfrac{\ell}{n})^2]} \qquad (11)$$

where $q = \frac{p\pi}{d}$, $p$ taking integer values.

For negative values of $\beta$ the solution describes neutral, internal gravity waves, whose phase and group velocities may easily be determined. However for the conventional gravitationally unstable situation ($\beta$ positive) we note that the waves are not necessarily amplified. The condition for amplification being that

$$g\beta[1 + (\tfrac{k}{n})^2] > [\tfrac{1}{4}\beta^2 + \ell^2 + q^2]^{-1} (\Omega\beta)^2 + 4\Omega^2(\tfrac{k}{n})^2 \quad (12)$$

In deriving this result the Boussinesq assumption was not invoked and the stabilizing effect of the non-Boussinesq terms is represented by the first term on the right hand side, whilst the second term denotes the similar effect of the rotation. Both these stabilizing effects are of interest because theoretical treatments of small scale convection in the atmosphere invariably disregard or approximate the terms in the hydro-thermodynamic equations that contribute to these effects.

The stabilizing effect of rotation referred to above is, in a geophysical context, due to the Coriolis component of rotation that is proportional to the cosine of latitude. The stabilizing effect of the component due to the sine of latitude for thermally driven convective motion of larger horizontal length scales is already well known.

We also note that the result concerning the possible importance of the cross wind mode is again evident in expression (12).

Finally we mention that for a basic state comprising of an axial (South $\longrightarrow$ North) velocity field $\underline{v} = [0,0,W(x)]$ and the same density distribution $\beta$ , the stability equation takes on the form,

$$\psi_{xx} - [\tfrac{1}{4}\beta^2 + \ell^2 + \gamma^{-1}(2\Omega n+W_x)\beta - \gamma^{-2}\{ 2\Omega (2\Omega k^2+W_x nk)-g\beta\ell^2 \}] = 0 \qquad (13)$$

The same boundary conditions must be satisfied and under the Boussinesq assumption once more, the operative form of the Richardson Number for this flow is,

$$\bar{J}^{*} = \left[2 \ [2\Omega(\tfrac{k}{n})^2+W_x(\tfrac{k}{n})] - g\beta[1+(\tfrac{k}{n})^2] \right] (W_x)^{-2} \qquad (14)$$

Again we may infer that for $(g\beta)$ small and positive the $k = 0$ modes would be the likeliest candidates for instability, and for this flow such modes would have their axes aligned along the direction of the mean flow.

## III. Stability of a stratified, rotating fluid to axisymmetric perturbations in the presence of an axial gravity field

In this section we again consider the motion relative to a system of axes rotating with uniform angular velocity $\Omega$ , and employ the narrow gap approximation. Here we replace the cylindrical polar coordinates $(r,\theta,z)$ by a cartesian coordinate system $(x,y,z)$ with $x$ pointing in the $\theta$ increasing direction, $y$ pointing radially inward. Gravity will be taken to act in the negative $z$ direction.

The governing equations for an incompressible, heterogenous fluid in this rotating frame of reference are,

$$\frac{D(\underline{v})}{Dt} + 2\Omega(\underline{k}\wedge\underline{v}) = -\frac{1}{\rho}\nabla p - g\underline{k} - (R\Omega^2)\underline{j}$$

$$u_x + v_y + w_z = 0 \qquad (1)$$

$$\frac{D}{Dt}(\rho) = 0$$

where $\quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

and $(\underline{i},\underline{j},\underline{k})$ are unit vectors in the $(x,y,z)$ directions respectively.

and $R$ is the mean radial distance within the flow field.

A basic state $\underline{v} = [U(y,z),0,0]$ and $\rho = \rho_o(y,z)$ must satisfy the relation,

$$2\Omega U_z = g(S_y) - (R\Omega^2 + 2\Omega U)S_z \qquad (2)$$

where $S_y$ and $S_z$ represent $\left[\frac{1}{\rho_o}(\rho_o)_y\right]$ and $\left[\frac{1}{\rho_o}(\rho_o)_z\right]$ respectively.

A relation of this form, prescribing a shear flow that is induced by the stratification, has been termed a 'thermal wind' equation in meteorological literature. The existence of a relation of this form is a poignant reminder of the care required in attaching a physical interpretation to the results of the previous chapter.

Hereafter we will assume that,

$$\left(\frac{2U}{\Omega R}\right) \ll 1$$

so that the term involving $(2\Omega U)$ may be neglected in (2). It follows that the relative contributions of the vertical and horizontal stratification to the shear of the thermal wind is represented by the value of the dimensionless number F given by,

$$F = \left(\frac{R\Omega^2}{g}\right)\left(\frac{S_z}{S_y}\right) \quad . \qquad (3)$$

F is the product of the rotational Froude Number and the ratio of the vertical and horizontal density stratifications.

Stability Analysis:

Linearise the system of equations (1) with respect to small, axi-symmetric perturbations $(\frac{\partial}{\partial x} = 0)$ of the basic state given by equation (2). Applying a quasi-Boussinesq assumption, that is, heterogeneity is reflected only in the buoyancy and centrifuging terms, we have the following set of equations for the perturbation quantities,

$$u_t = (2\Omega - U_y)v - (U_z)w \qquad (a)$$

$$v_t + (2\Omega)u = -\frac{1}{\bar{\rho}} P_y - \frac{\rho}{\bar{\rho}}(R\Omega^2) \qquad (b)$$

$$w_t = -\frac{1}{\bar{\rho}} P_z - \frac{\rho}{\bar{\rho}}(g) \qquad (c) \quad (4)$$

$$v_y + w_z = 0 \qquad (d)$$

$$\rho_t + v(\rho_o)_y + w(\rho_o)_z = 0 \qquad (e)$$

where $(u,v,w,p,\rho)$ are the perturbation velocities, pressure and density respectively, and $\rho_o \simeq$ (constant) $= \bar{\rho}$ .

The form of (4d) enables us to introduce a stream function $(\psi)$ such that

$$v = \psi_z \ , \quad w = -\psi_y \tag{5}$$

Substituting for $v$ and $w$ in terms of $\psi$ in equations (4), and then eliminating all perturbation quantities except $\psi$ from the equations yields the following equation for $\psi$ ,

$$[2\Omega(2\Omega - U_y) - (R\Omega^2)S_y + \frac{\partial^2}{\partial t^2}]\psi_{zz} + 2(gS_y)\psi_{yz} - [gS_z - \frac{\partial^2}{\partial t^2}]\psi_{yy} = 0 \tag{6}$$

For solid walls at $y = \pm a$ and $z = 0,d$ , the inviscid boundary conditions are simply,

$$\psi = 0 \quad \text{at all boundaries} \tag{7}$$

Subject to the approximations already imposed equation (6) holds for an arbitrary velocity field $U = U(y,z)$ that satisfies the Thermal Wind relation. However to obtain an explicit solution of (6) subject to the boundary conditions (7) we impose the following restrictions on the basic state,

$U_y = 0$ , and $(S_y)$ and $(S_z)$ assume constant values.

Under these restrictions the coefficients in the partial differential (6) are constants and we may assume a normal mode form for ,

$$\psi = \chi\ (y)\ e^{i(\sigma t+nz)}$$

Substituting this expression for $\psi$ in (6) and solving the resulting ordinary differential equation with the boundary conditions (7) leads to the following expression for $\sigma$ ,

$$\sigma^2 + (R\Omega^2)S_y - (2\Omega)^2 = -(gS_z+\sigma^2)^{-1}\ (gS_y)^2 \qquad (8)$$

For negative values of $\sigma^2$ we conclude that the flow is unstable. We examine the form of equation (8) for various density configurations and limiting cases.

For hydrostatic perturbed motion $(S_z)$ is necessarily negative and $|gS_z| >> |\sigma^2|$ , so that (8) may be rewritten as,

$$\sigma^2 = 4\Omega^2[1 + (4\Omega^2 S_z)^{-1} g(S_y)^2 - \tfrac{1}{4}(RS_y)] \qquad (9)$$

This relation indicates that rotation has a stabilizing effect, pure gravitational buoyancy effects are always destabilizing, whilst the role of centrifugal 'buoyancy' depends upon the sign of $(S_y)$. The ratio of centrifugal and gravitational effects is given by the number $F$. When $S_y < 0$ then $F < 1$ (i.e. $U_z$ negative) is a necessary condition for instability.

To compare the work with the theory of the previous chapter we examine the form of (8) with $(S_z) = 0$,

$$\sigma^4 - \sigma^2\Omega^2(4-RS_y) - g(S_y)^2 = 0$$

Therefore

$$\sigma^2 = \Omega^2(4-RS_y)\left[1 \pm \left(1 + 4\Omega^{-4}(4-RS_y)^{-2}(gS_y)^2\right)^{1/2}\right] \quad (10)$$

For $S_y \neq 0$ $(S_z = 0)$ the fluid is unstable, as opposed to the non-gravitating result that $(RS_y) < 4$ for instability. Thus when there is an axial gravity field the results of the previous chapter are ~~stultified~~ invalid.

When $(S_y) > 0$, asymmetric perturbations would also provide an instability mechanism. A superficially attractive argument for the situation $(S_y) > 0$ is that a very large stable vertical stratification would so inhibit vertical motion that two dimensional asymmetric motion would result. The stability analysis of equation (1) for three dimensional perturbations does not yield a differential equation for a single perturbation quantity, however mere inspection of the equations for the vertical momentum and density advection indicate that the perturbed motion must be three dimensional.

A feature of equation (8) that highlights a basic difference between the centrifugal and gravitational destabilising mechanisms is that the former is only associated with the horizontal density stratification. Hence when $S_y = 0$, equation (8) indicates that the flow is stable or unstable depending merely on whether the density decreases or increases in the $z$ direction.

Axi-symmetric instability of a thermally stratified fluid is a meteorologically significant problem. Heterogeneity is replaced by compressibility and inhomogeneity in the atmospheric situation, and this increases the complexity of the relevant hydro-thermodynamic equations. Partly to circumvent these additional difficulties, only the hydrostatic problem is usually discussed in the meteorological literature. This enables a quasi-Lagrangian coordinate system to be employed with pressure replacing the vertical height coordinate. The centrifugal term is accounted for implicitly in such a system but it is no longer a simple matter to formulate the boundary conditions at the earth's surface. An equation resembling (6) is obtained from which a necessary condition for instability may be derived. A somewhat similar problem is considered in the next chapter.

# CHAPTER 4

## Axi-Symmetric Motion in the Earth's Equatorial Atmosphere

### I. Introduction

In the preceeding chapter it was shown that the terms in the equations of motion involving the component of the Earth's rotation proportional to the cosine of latitude could play an important role in a fluid dynamical motion of the atmosphere which is of meteorological interest. This result served to motivate the work of the present chapter, wherein an examination is undertaken of the possible importance of these terms in another geophysical fluid motion.

The terms referred to above have invariably been omitted in theoretical studies of geophysical fluid motions, and their neglect has been termed the 'Traditional Approximation'. This approximation has usually been based on an order of magnitude argument, but Phillips [1966] proposed a tentative 'rationale' for the validity of the approximation in almost all meteorologically significant fluid motions.

An attempt is made in this chapter to determine the role of these terms in a specific problem which arises in the theoretical studies of the general circulation of the earth's atmosphere. These studies deal with the convective instability of the atmosphere whose characteristic mean state comprises of an horizontal temperature gradient and a predominantly stable vertical stratification of potential temperature. This mean stratification ~~being~~ is largely attributable to the convective response of the atmosphere to solar radiative heating which has a distinct latitudinal variation.

The first theoretical contributions to the study of the general circulation considered motions that were axi-symmetric with respect to the earth's axis of rotation. The circulation comprised of air rising at low latitudes, the region of greatest solar heat input, and descending in polar regions. These theories were unable to account for the large scale features of the observed circulation, and this failure led to the study of the stability of the mean state to asymmetric perturbations. It was shown that small, asymmetric, geostrophic perturbations of the basic, zonal, thermal flow would amplify. The results were directly applicable to the cyclone scale motions of the westerly wind regime of the mid-latitudes.

Large scale motion of the atmosphere in low latitudes on the other hand has always been taken to be axi-symmetric, although its detailed structure remains largely undetermined due to the inadequacy of the data available for these regions. Theoretical work [Kuo (1956)] based on the available data emphasizes the weak nature of the mean meridional circulation. This circulation ~~being~~ is regarded as a forced motion driven by smaller scale eddy motions and non-adiabatic heating. Kuo also derives a criterion for the existence of a stronger, free axi-symmetric convection deriving its energy directly from the mean thermal flow, and he points out that this criterion is generally not satisfied in the atmosphere.

In this chapter we examine the form taken by this criterion when there exists an horizontal temperature gradient across the equator. This temperature distribution is of particular interest in view of the tentative hypothesis that could be drawn from the following result:- A non-rotating, inviscid fluid contained between two horizontal planes, possessing a stable vertical stratification, an horizontal temperature gradient, and a basic state of zero motion is unstable to small, normal mode perturbations that have a non-zero wave-number in the direction of the thermal gradient.

As a prelude to the consideration of the problem posed above we derive in the next section the full thermal wind relation for zonal flow on a rotating sphere due to latitudinal and vertical thermal stratification.

## II. The basic, zonal, Thermal Flow

An essential pre-requisite of a stability analysis of some given steady fluid motion to small perturbations is that the prescribed basic state is an allowable solution of the fundamental equations governing the behaviour of the particular fluid under consideration. Here we consider the steady, basic state of a zonal 'thermal wind' arising from vertical and latitudinal thermal stratification of an inviscid fluid on a rotating globe.

Referring the motion to a system of axes rotating with constant angular velocity $\Omega$ , then the basic state must satisfy the following equations,

$$(2\Omega\sin\phi)U + r\Omega^2\sin\phi\,\cos\phi + \frac{1}{r}\,U^2\tan\phi = -\,\frac{1}{\rho r}\,p_\phi \qquad (1)$$

and

$$-(2\Omega\cos\phi)U - \frac{1}{r}U^2 = -\,\frac{1}{\rho}\,p_r - g - r\Omega^2\cos^2\phi \qquad (2)$$

These equations ~~being~~ are the reduced form of the equations of motion when the spherical coordinates $(r,\phi,\chi)$ , with associated velocities $(w,v,u)$ , represent directions along the radius, latitude and azimuth, respectively. For such a system the basic state represented in (1) and (2) is given by $\underline{v} = (0,0,U)$ and $\rho = \rho(r,\Theta)$ .

Eliminating the pressure $(p)$ from equations (1) and (2) we obtain,

$$2\Omega[\sin\phi\,U_r + \cos\phi\,\frac{1}{r}\,U_\phi] + 2\frac{U}{r}[\tan\phi\,U_r + \frac{1}{r}\,U_\phi]$$

$$= -\,(\frac{1}{\rho}\,\rho_r)\,[F] + (\frac{1}{\rho r}\,\rho_\phi)[g_1] \qquad (3)$$

where $F = r\Omega^2\sin\phi\,\cos\phi + 2\Omega U\,\sin\phi + \frac{1}{r}U^2\tan\phi$

and $g_1 = g - r\Omega^2\cos^2\phi - 2\Omega U\,\cos\phi - \frac{1}{r}\,U^2$

In deriving (3) the only approximation made was that $g$

was taken to be a constant.

Assuming that the equation of state of the atmosphere is adequately represented by the ideal gas equation,

$$p = \rho RT$$ where $R$ is the universal gas const. and $T$ the absolute temperature (4)

Then using equations (1), (2) and (4) we obtain the following expressions for the density **gradients**

$$\left(\frac{1}{\rho}\,\rho r\right) = -\left(\frac{1}{T}\,T_r\right) - \frac{1}{RT}\,[g_1]$$

and

$$\left(\frac{1}{\rho r}\,\rho_\phi\right) = -\left(\frac{1}{Tr}\,T_\phi\right) - \frac{1}{RT}[F] \qquad (5)$$

Alternatively they may be written in terms of the potential temperature defined by $\Theta = T\,(\frac{P}{p})^{(\gamma)^{-1}(\gamma-1)}$ where $\underline{P}$ is a reference pressure.

Then

$$\left(\frac{1}{\rho}\,\rho_r\right) = -\left(\frac{1}{\Theta}\Theta_r\right) - \frac{1}{\gamma RT}\,[g_1]$$

$$\left(\frac{1}{\rho r}\,\rho_\phi\right) = -\left(\frac{1}{\Theta r}\,\Theta_\phi\right) - \frac{1}{\gamma RT}\,[F] \qquad (6)$$

Substituting the relations expressed by (5) and (6) in equation (3) we arrive at the following thermal wind relations,

$$2\Omega[\sin\phi\ U_r + \cos\phi\ \frac{1}{r}\ U_\phi] + 2\ \frac{1}{r}U[\tan\phi\ U_r + \frac{1}{r}\ U_\phi]$$

$$= (\frac{1}{r}T_r)[F] - (\frac{1}{Tr}T_\phi)[g_1] \qquad (7)$$

$$= (\frac{1}{\Theta}\ \Theta_r)[F] - (\frac{1}{\Theta r}\ \Theta_\phi)\ [g_1]$$

A steady zonal flow satisfying (7) is therefore a consistent basic state for a stability analysis of an inviscid fluid whose equation of state satisfies the ideal gas equation.

The 'traditional approximation' referred to in the introduction was not employed in the above derivation and in consequence the term $[\,2\Omega\cos\phi\ \frac{1}{r}U_\phi]$ appears in equation (7). Typical orders of magnitude of $(U_r)$ and $(\frac{1}{r}\ U_\phi)$ for the atmosphere suggest that, apart from an extremely narrow zonal band centred at the equator,

$$[\sin\phi\ U_r\ ]\ /\ [\ \cos\phi\ \frac{1}{r}\ U_\phi] \gg 1 \qquad (8)$$

The restriction on the validity of condition (8) is a salutary warning of the care required in the study of equatorial motion. The equator itself ~~being~~ is a dynamically unique region of the atmosphere because there the vertical component of the earth's rotation vanishes.

At the equator itself ($\phi = 0$) equation (7) may be written with sufficient accuracy as,

$$-\left[\frac{1}{a} U_{\phi}\right]\Big|_{\phi=0} = \frac{g}{2\Omega}\left(\frac{1}{Ta} T_{\phi}\right)\Big|_{\phi=0} \qquad (9)$$

where $a$ = earth's equatorial radius.

The left hand side of (9) represents the vertical component of the relative vorticity of the zonal flow a at the equator.

If $\left(\frac{1}{Ta} T_{\phi}\right)\Big|_{\phi=0} \sim O(10^{-8})$, then $\left(\frac{1}{a} U_{\phi}\right)\Big|_{\phi=0} \sim O(10^{-3})$.

The value assigned here to the horizontal thermal gradient is comparable with typical mid-latitude values of the same quantity. We note that it results in an exceedingly large value for the vertical component of the relative vorticity, and implies that there would be a narrow band of large horizontal shear of the zonal velocity straddling the equator.

This aspect of the zonal, thermal flow will play a central role in the theory developed in the succeeding section.

## III. Derivation of the Stability Criterion

We confine our attention to the stability of zonal flow in equatorial latitudes of the earth's atmosphere and we employ a $\beta$ plane approximation to the equations of motion. Veronis [1963] gives a systematic derivation of the equations for a $\beta$ plane centred at the equator and his model is used in the present work.

The adoption of a $\beta$ plane system of coordinates requires further justification, because Phillips [1966] indicated that in a mathematical study of an atmosphere motion, for which the terms involving the component of rotation proportional to the cosine of latitude are considered important, it would be preferable to work in Spherical Polar Coordinates. The basis for Phillips' caution is that, when these terms are retained in a $\beta$ plane type approximation of the original equations, the resulting system has the inherent deficiency of lacking a relation corresponding to the 'angular momentum conservation' principle.

With regard to the present work we may note that, consistent with the explicit approximations made by Veronis, his $\beta$ plane equations do conserve angular momentum. Furthermore, it is anticipated that a similar result to that obtained in this section would be obtained by working the problem through in spherical polar coordinates.

In the $\beta$ plane equations used below we have replaced the expression $(2\Omega\phi)$ by $(2\frac{\Omega}{a}y)$, an approximation which is justifiable within the framework of Veronis' analysis.

The $\beta$ plane equations of motion, the continuity equation, and the equation expressing the adiabatic nature of the flow are,

$$\underline{v}_t + (\underline{v}.\nabla)\underline{v} + 2\beta y(\underline{k}\wedge\underline{v}) + 2\Omega(\underline{j}\wedge\underline{v}) = -\frac{1}{\rho}\nabla p - (g-a\Omega^2)\underline{k} - (y\Omega^2)j \quad \text{(a)}$$

$$\rho_t + (v.\nabla)\underline{v} + \rho(\nabla.\underline{v}) = 0 \quad \text{(b)} \quad (10)$$

$$(\ln\theta)_t + (v.\nabla)(\ln\theta) = 0 \quad \text{(c)}$$

where $(\underline{i},\underline{j},\underline{k})$ are unit vectors in the $(x,y,z)$ directions pointing east, north and vertically respectively.

$\underline{v} = (u,v,w)$ are the associated velocities
$\beta = \frac{1}{a}(\Omega)$
and $\theta$ is the potential temperature.

A basic steady zonal flow given by $v = (U,0,0)$ with $\rho_o = \rho_o(y,z)$ must satisfy the following equations

$$(2\beta y)U + y\Omega^2 = -\frac{1}{\rho_o}(\rho_o)_y \tag{12}$$

$$0 = -\frac{1}{\rho_o}(\rho_o)_z - [g_1] \tag{13}$$

where $g_1 = [g - a\Omega^2 - 2\Omega U]$

From (12) and (13) we obtain the thermal wind relation,

$$(2\beta y)U_z + (2\Omega)U_y = \frac{1}{\rho_o}(\rho_o)_y\,[g_1] - \frac{1}{\rho_o}(\rho_o)_z[y\Omega^2 + (2\beta y)U]$$
$$= -\frac{1}{\theta_o}(\theta_o)_y[g_1] + \frac{1}{\theta_o}(\theta_o)_z[y\Omega^2 + (2\beta y)U] \tag{14}$$

These expressions are the forms of equations (3) and (7) for the $\beta$ plane approximation.

Linearising the equations of motion, the continuity equation and the adiabatic equation with respect to small, axi-symmetric perturbations they become, after some minor manipulation,

$$u_t + vU_y + wU_z - (2\beta y)v + 2\Omega w = 0 \qquad (15)$$

$$v_t \qquad + (2\beta y)u = -\left(\frac{p}{\rho_o+\rho}\right)_y - p\frac{1}{(\rho_o+\rho)^2}(\rho_o)_y$$

$$- \left(\frac{\rho}{(\rho_o+\rho)}\right)[F_1] \qquad (16)$$

$$w_t \qquad - (2\Omega)u = -\left(\frac{p}{(\rho_o+\rho)}\right)_z - p\frac{1}{(\rho_o+\rho)^2}(\rho_o)_z$$

$$- \frac{\rho \cdot 1}{(\rho_o+\rho)}[g_1] \qquad (17)$$

$$s_t + v(\rho_o)_y + w(\rho_o)_z + (\rho_o+\rho)[v_y+w_z] = 0 \qquad (18)$$

$$s_t + v(S_y) + w(S_z) = 0 \qquad (19)$$

where $(u,v,w,p,\rho,\Theta)$ are the perturbation variables,

and $S = p_n(\frac{1}{\Theta_o}\Theta)$ , $S_y = \left(\frac{1}{\Theta_o}\right)(\Theta_o)_y$ , $S_z = \frac{1}{\Theta_o}(\Theta_o)_z$ .

$$F_1 = [y\Omega^2 + (2\beta y)U]$$

We now make several explicit assumptions regarding the nature of the basic and perturbed motions:-

(a) $\rho \sim (\Delta\rho_o)_H \ll \rho_o$ , where $(\Delta\rho_o)_H$ is the horizontal variation of density of the basic state.

(b) $(\Delta\theta_o)_H \ll (\Delta\theta_o)_V \ll \theta_o$

(c) $\frac{1}{\theta_o}(\theta_o)_z \ll \frac{1}{\rho_o}(\rho_o)_z$

(d) Time scale of the motion is taken to be large enough to exclude sound and gravity waves.

These stipulations enable (16), (17) and (18) to be approximated to,

$$v_t + (2\beta y)u = -\left(\frac{p}{\rho_o}\right)_y + F(s) + \left(\frac{p}{\rho_o}\right) S_y \qquad (16b)$$

$$-(2\Omega)u = -\left(\frac{p}{\rho_o}\right)_z + g(s) \qquad (17b)$$

$$v_y + w_z = \alpha w \qquad (18b)$$

where $\alpha = -\frac{1}{\rho_o}(\rho_o)_z$

In writing these equations we have used the relations

$$\frac{1}{\rho_o}(\rho_o)_y = -\frac{1}{\gamma}\frac{\rho_o}{p_o}[y\Omega^2 + (2\beta y)U] - S_y \qquad (20)$$

and $$\frac{1}{\rho_o}(\rho_o)_z = -\frac{1}{\gamma}\frac{\rho_o}{p_o}[g - a\Omega^2 - (2\Omega)U] - S_z \qquad (21)$$

which are derived using the ideal gas equation.

A term $(\frac{p}{\rho_o})$ $(s_z)$ has been dropped from the right hand side of (17b), its neglect being justified by consideration of equation (11) and assumption (c). To this degree of approximation we have also replaced $(g_1)$ by $(g)$ and $F_1$ by $F$ $(=y\Omega^2)$, whilst assumption (a) has allowed us to write

$$(\frac{1}{\gamma}\frac{p}{\rho_o} - \frac{p}{\rho_o}) \approx s$$

A further simplification results from the introduction of a new vertical coordinate $\tilde{p} = \tilde{p}(z)$, chosen such that $\tilde{p}$ approximates closely to $p_o$, the pressure distribution of the basic state.

Then $z = z(\tilde{p})$

and $\frac{\partial}{\partial z} = - \tilde{\rho} g \frac{\partial}{\partial \tilde{p}}$ where $\tilde{\rho} = \tilde{\rho}(z)$ is related to $\tilde{p}$ by the hydrostatic relation.

With this new vertical coordinate equation (18b) may be rewritten in the form,

$$v_y + \tau_{\tilde{p}} = 0 \qquad (18c)$$

where $\tau = -\tilde{\rho} g w$

This enables us to introduce a stream function $\psi$ such that,

$$v = \psi_{\tilde{p}} \ , \quad \tilde{\tau} = - \psi_y \tag{22}$$

Then on assuming that $S_y$ is not a function of $z$ , a single differential equation for $\psi$ can be obtained from the system of equations (15-19). Allowing the perturbation quantities to have a time dependence of the form $\exp\{i\sigma t\}$ , the resulting equation for $\psi$ has the form,

$$A\psi_{\tilde{p}\tilde{p}} + 2H\psi_{y\tilde{p}} + B\psi_{yy} + C(y,\tilde{p},\psi_y,\psi_{\tilde{p}}) = 0 \tag{23}$$

where $A = \left[2\beta y(2\beta y - U_y) + F(S_y) - \sigma^2\right]$

$$2H = \left[2\beta y.(U_{\tilde{p}} - 2\Omega\frac{1}{\tilde{\rho}g}) - 2\Omega(\frac{1}{\tilde{\rho}g})(2\beta y - U_y) - S_{\tilde{p}}(F) + \frac{1}{\tilde{\rho}g}S_y g\right]$$

$$B = -\frac{1}{\tilde{\rho}g}\left[gS_{\tilde{p}} + 2\Omega(U_{\tilde{p}} - 2\Omega\frac{1}{\tilde{\rho}g})\right.$$

For the flow bounded by vertical latitudinal walls at

$y = \pm a$ , and horizontal walls at $z = 0$ and $h$ , the requirement that the normal component of velocity vanishes at these walls leads to the boundary conditions

$$\begin{aligned} \psi &= 0 \quad \text{at} \quad y = \pm a \\ \psi &= 0 \quad \text{at} \quad \tilde{p}(0) \text{ and } \tilde{p}(h) \end{aligned} \tag{24}$$

Alternatively the upper boundary condition could be replaced by the condition of vanishing velocity at $\tilde{p} = 0$ $(z = \infty)$.

Since a linear, elliptic, homogeneous, partial differential equation with the form of (23) can not have a maximum within the bounding surfaces its solution would be identically zero if it satisfied the boundary conditions specified in (24). Hence a necessary condition for the existence of a perturbed motion is that equation (23) is non-elliptic $(H^2 \geq AB)$ at least somewhere within the flow region.

i.e. $\frac{1}{4}[-2\beta y(2\Omega + U_z) - 2\Omega(2\beta y - U_y) + g(S_y) + F(S_z)]^2$

$$\geq [2\beta y(2\beta y - U_y) + F(S_y) - \sigma^2][2\Omega(2\Omega + U_z) + gS_z]$$

somewhere inside $y = \pm a$ ; $z = 0, h$ .

Using the thermal wind relation (14) the above expression may be rewritten as follows,

$$\sigma^2 \geq 2\beta y(2\beta y-U_y) + (S_y)F - [g(S_y)-2\Omega(2\beta y-U_y)]^2 [g(S_z)+2\Omega(2\Omega+U_z)]^{-1} \qquad (25)$$

Thus a necessary condition for the instability of the zonal flow to axi-symmetric perturbations is that,

$$[2\beta\,(\mathcal{S}_A) + F(S_y)] \leq [\,g(S_y) - 2\Omega(\mathcal{S}_A)]^2\,[g(S_z) + 2\Omega(\eta_A)]^{-1} \qquad (26)$$

somewhere inside $y = \pm a$ ; $z = 0, h$ .

where $\mathcal{S}_A$ and $\eta_A$ are the components of the absolute vorticity of the basic zonal flow in the $z$ and $y$ directions respectively.

This expression is an extension of the necessary condition for the existence of 'free' axi-symmetric convective motion [see e.g. Kuo (1956)] that renders it valid for equatorial flows.

A similar criterion may be derived for oceanic flows. Neglecting molecular diffusive processes again, it is possible to formulate an advective conservation equation for 'apparent' temperature (i.e. the variation of density due to both thermal and saline effects is taken into account). Then subject only to the Boussinesq and hydrostatic assumptions we obtain the analogue of equation (23),

$$A\psi_{zz} + 2H\psi_{yz} + B\psi_{yy} + G(y,z,\psi_y,\psi_z) = 0 \qquad (27)$$

$$\text{with } A = (2\beta y)\eta_A + (S_y)F - \sigma^2$$

$$H = (2\Omega)\varphi_A - (S_y)g$$

$$B = (2\Omega)\eta_A + (S_z)g$$

$$G = [(2\beta)\eta_A - S_z\Omega^2]\psi_z$$

and nos $(S_y)$ and $(S_z)$ represent $\left[-\frac{1}{\rho_o}(\rho_o)_y\right]$ and $\left[-\frac{1}{\rho_o}(\rho_o)_z\right]$ respectively. Condition (26) follows directly.

## IV. Comments on the Criterion

The criterion expressed in equation (26) is merely a necessary condition for instability and this limits its possible significance. For any consistent basic state the value of the terms on both the left and right hand side of (26) vanish at the equator. This marginal satisfaction of the necessary condition for instability may be contrasted with the extra-equatorial form of condition (26) which precludes the possibility of free axi-symmetric convection in the atmosphere, except in the vicinity of fronts where the horizontal temperature gradients may be substantial. Condition (26) can not be a sufficient condition for instability for general flow situations since it embraces the situation when the atmosphere is homogenous.

It was noted in the discussion of the basic state that the magnitude of $\left[ U_y \big|_{y=0} \right]$ would be substantial for only moderate values of $\left[ S_y \big|_{y=0} \right]$, and this implies that condition (26) might then be strongly satisfied in the immediate neighbourhood of the equator. If this implied instability then the resulting circulations would act to vitiate the strong satisfaction of the instability condition by contriving to reduce the value of $(S_y)$ near the equator.

Hence a stipulation that the mean zonal state of the equatorial troposphere should violate, or marginally satisfy, condition (26) would imply that the latitudinal thermal gradient in this region would be small throughout the year, irrespective of the equatorially asymmetric heat input during the inter-equinox periods.

The singular behaviour at the equator of a zonal, thermal flow due to an equatorially, asymmetric thermal stratification may also be important in other situations. Equatorial, zonal streams of fluid moving faster than the flow further to the north and south are known features of the major oceans of the Earth and of the circulations of the atmospheres of the Sun and the planet Jupiter.

In the earth's oceans the equatorial streams take the form of eastward, sub-surface currents flowing along, and centred on, the equator, in contrast to the surface flow which follows the atmospheric wind pattern. In the Pacific Ocean the undercurrent is known to be approximately 300 Kilometres wide and is confined to the thermocline region.

It is unlikely that the undercurrent is a manifestation of axi-symmetric motion arising out of the requirement that the mean zonal thermal flow should everywhere violate the condition expressed by the analogue of (26) for the oceans. However, it is intriguing to consider the possible modifying effect such a requirement would have on the undercurrent. The requirement would certainly imply that the isopycnics be almost horizontal at the equator, whilst the zonal component of the flow would satisfy the following equation in the vicinity of the equator,

$$(2\beta y)U_z + (2\Omega)U_y = - g(S_y) \qquad (28)$$

A mean density stratification with a maximum (minimum) at the equator would represent an easterly (westerly) flow, whilst the reverse would be true for the second term.

It is pertinent to note that recorded features of the undercurrent indicate that the isopycnics are horizontal at the equator and that the traditional thermal flow approximation can not always account for the observed flow structure of the current.

# CHAPTER 5

## Review of Theories on Mixing Processes in Rotating Fluids

### I. Introduction and Outline

Scorer (1965,1966) outlined a novel approach to the study of turbulent mixing processes in rotating fluids. His work initiated a resurgence of interest in this particularly polemic field of study, and the recent experimental work of Gough and Lynden-Bell (1968) will serve to further stimulate theoretical studies.

Scorer suggested that when a body of fluid possessing a vertical component of vorticity is stirred the momentum would be redistributed so as to reduce the absolute vorticity in the interior of the fluid and concentrate it at the boundaries. In particular he states that if a portion of fluid in solid body rotation is stirred it would tend to become more like a potential vortex with energy being transferred from the eddies to the mean motion.

It is known from Statistical Mechanics that zero momentum transport occurs for solid rotation in gas flows.

However no violation of the second law of thermodynamics occurs if solid rotation of the mean flow produces momentum fluxes in a turbulent fluid for there must anyway be a steady input of mechanical energy to sustain the turbulence.

Implicit in Scorer's studies are two different forms of stirring: small scale mechanical stirring induced by an external agency, and thermal stirring of virtually any scale arising directly from a thermo-hydrodynamic instability of the fluid. Vertical injection of gas bubbles into a liquid rotating in a horizontal plane is an example of the former type of stirring mechanism and the concept of anvil cloud stirring of the earth's atmosphere is an example of the latter. These two types of stirring which we shall refer to as forced and free gurgitation respectively will be treated separately in the present work.

Following some general comments on the nature of Reynolds stresses the remaining sections of this chapter are devoted to a resumé of the early developments in the study of mixing processes in rotating fluids and a brief review of Scorer's theory. A survey of this field of study was also prepared recently by Bretherton and Turner (1968).

## II. Reynolds Stresses

Before discussing the early transport theories of turbulence it is necessary to introduce the concept of Reynolds Stresses.

For the moment we work in Cartesian coordinates. Consider that the velocity $\underline{v}$ of a homogeneous, turbulent fluid be separated into a mean flow $\underline{V}$ and a fluctuating part $\underline{v}'$ by employing an averaging process defined by,

$$\underline{V} = \frac{1}{2T} \int_{-T}^{+T} \underline{v}\, dt$$

where $T$ is a time large compared to the time scale of the turbulent fluctuations and small compared to the time scale of the development of the mean flow.

We substitute the expression $(\underline{V}+\underline{v}')$ for $\underline{v}$ in the inviscid momentum equations. Then on time averaging the resulting equations in the manner indicated above and employing the incompressibility condition we find that the mean flow $(\underline{V})$ satisfies the following equation,

$$\underline{V}_t + (\underline{V}.\nabla)\underline{V} = -\nabla\frac{p}{\rho} + \phi$$

where in tensor notation $\phi$ is given by

$$\phi_i = \frac{1}{\rho} \frac{\partial}{\partial x_j} (\tau_{ij})$$

and $\tau_{ij} = -\rho\overline{u_i u_j}$ is a symmetric second order tensor that is termed the Reynolds stress.

We note that the mean flow obeys an equation of exactly the same form as the Cauchy equation of motion,

$$\underline{v}_t + (\underline{v}.\nabla)\underline{v} = -\nabla\frac{p}{\rho} + \frac{\partial}{\partial x_j} (T_{ij})$$

where $T_{ij}$ is an undefined second order tensor. Thus the influence of the turbulence on the mean flow can be interpreted as a stress $(\tau_{ij})$ set up by the fluctuating motion and acting on the mean flow.

A measure of understanding of the possible limitation upon the form of the Reynolds Stress may be gleaned from a consideration of the assumptions made in deriving the constitutive equations of a 'Newtonian type fluid'. A postulation that the fluid conserves its angular momentum is equivalent to restricting $T_{ij}$ to be symmetrical. For a Newtonian fluid the further assumptions are made that the stress is homogeneous, isotropic, a linear function of the deformation tensor $(e_{ij})$, and that the stress vanishes when $e_{ij} \equiv 0$ (e.g. in solid body rotation). If the effect of the turbulent fluctuations on the mean velocity field is to differ in any respect from the effect of viscous diffusion of a Newtonian fluid then at least one of the assumptions mentioned above can not be applicable to the turbulent motion.

Our present study is concerned with mixing processes in a rotating fluid and it may be shown that in cylindrical polar coordinates $(r,\theta,z)$, the equation of motion for the mean swirl velocity $(V)$ of a body of fluid in cylindrical rotation about the $z$ axis is given by,

$$\frac{\partial V}{\partial t} = \frac{1}{\rho r^2}\frac{\partial}{\partial r}\left(r^2 \tau_{r\theta}\right)$$

with the Reynolds stress $\tau_{r\theta} = -\rho\overline{u'v'}$,

where $u'$ and $v'$ are the velocity components of the fluctuating motion along the radius and azimuth.

We also note that the Reynolds stress and the vorticity $(\varphi')$ of the fluctuating motion are connected by the following relation,

$$\frac{\partial}{\partial r}\left(r^2 \tau_{r\theta}\right) = \rho r^2\, \overline{\varphi' u'}$$

where $\varphi' = \left\{\frac{1}{r}\frac{\partial}{\partial r}(rv') - \frac{1}{r}\frac{\partial u'}{\partial \theta}\right\}$

## III. Historical Resumé

The early work on mixing processes in turbulent fluid motions culminated in the Transport theories of Prandtl and Taylor. Prandtl proposed a 'Momentum Transfer Hypothesis' in 1925, and the 'Vorticity Transfer Hypothesis' was formulated as an alternative theory by Taylor in 1932 after having been originally introduced by him in 1915.

These theories were purely phenomenological in character and were based on an analogy with the Kinetic Theory of Gases. The fluid in turbulent motion was conceived as consisting of a swarm of turbulent elements moving across the direction of mean motion through a medium possessing the mean properties of the flow. These elements were assumed to detach themselves from their mother layer, travel a finite distance through the medium along an eddy trajectory before rejoining the mean flow. At the moment of detachment the element was assumed to possess all the physical properties of the mother layer except for an additional velocity component perpendicular to the mean flow. On rejoining the mean flow it was assumed that they lost this velocity component and contributed with their properties to the layer in which they were stopped. Since they carried a component of momentum in the direction of the mean flow they could give rise to a Reynolds Stress in shear flow.

Prandtl assumed that it was the component of momentum in the direction of the mean flow that was conserved along the eddy trajectory, while Taylor assumed the conservation of vorticity. The former theory implies that the turbulent elements move without being acted upon by the transitory pressure gradients associated with the turbulent motion. They must however be acted upon by the gradient of the mean pressure, which in the case of a cylindrically rotating fluid acts radially. Hence for this situation Prandtl assumed that the physical entity conserved along the eddy trajectory was the angular momentum.

Both Prandtl and Taylor specified a relation between the local Reynolds stress and the local mean velocity field by the use of an 'eddy viscosity'. For cylindrically rotating flows we may formalise their Transport Theories as follows:

(a) Prandtl's Momentum Transfer hypothesis.

Consider a turbulent element at radius $r$, having the mean tangential velocity $V$ and the fluctuating velocity components $(u',v',w')$.

Then the Reynolds Stress is given by,

$$\tau_{r\theta} = -\rho\,\overline{u'v'}$$

$$= = -\rho\beta\,\overline{(u')^2}^{\,1/2}\;\overline{(v')^2}^{\,1/2}$$

$$= -\left\{\rho\overline{(v')^2}^{\,1/2}\,r\right\}\left\{\beta\,\overline{(u')^2}^{\,1/2}/r\right\}$$

where $\beta = \dfrac{\overline{u'v'}}{\overline{(u')^2}^{\,1/2}\;\overline{(v')^2}^{\,1/2}}$ is the correlation coefficient.

and $\left\{\rho\,\overline{(v')^2}^{\,1/2}\,r\right\}$ is the fluctuating component of the angular momentum.

Assuming that the fluid element leaving the layer at $r$, conserves its angular momentum over a small 'mixing length' $\ell_1$ corresponding to the eddy trajectory. Then taking the gradient of the angular momentum be constant over the mixing length we have,

$$\left\{\rho\overline{(v')^2}^{\,1/2}\,r\right\} = \ell_1\,\frac{\partial}{\partial r}\,(\rho r V)$$

Hence $$\tau_{r\theta} = -\rho\left\{\ell_1\,\overline{(u')^2}^{\,1/2}\,\beta\right\}\left(\frac{\partial V}{\partial r} + \frac{V}{r}\right)$$

$$= \rho\,\nu_1\left(\frac{\partial V}{\partial r} + \frac{V}{r}\right) \qquad (1)$$

where $\nu_1$ is the eddy viscosity.

(b) Taylor's Vorticity Transfer hypothesis.

In this case we assume that the gradient of vorticity is constant over a small mixing length $\ell_2$ corresponding to the eddy trajectory. Then the variation $\Omega'$ of the mean vorticity at a fixed radial position may be written,

$$|\Omega'| = \left| \ell_2 \frac{\partial}{\partial r} \left( \frac{\partial V}{\partial r} + \frac{V}{r} \right) \right|$$

Hence the Reynolds Stress may be deduced from the relation,

$$\frac{1}{\rho r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) = \overline{\Omega' u'}$$

$$= (\overline{\ell_2 u'}) \frac{\partial}{\partial r} \left( \frac{\partial V}{\partial r} + \frac{V}{r} \right)$$

$$= \nu_2 \frac{\partial}{\partial r} \left( \frac{\partial V}{\partial r} + \frac{V}{r} \right) \qquad (2)$$

where $\nu_2$ is the eddy viscosity.

A steady state cylindrically rotating mean flow maintained by turbulent fluctuations must satisfy the relation,

$$r^2 \tau_{r\theta} = \text{constant.} \qquad (3)$$

It is clear from (1), (2) and (3), that there is a contrast in the predictions of the two theories regarding the form of the mean steady state. Some cogent remarks of Taylor (1932) help to clarify the reason for this dichotomy. Taylor showed that Prandtl's representation of the Reynolds Stress is invalid for a cylindrically rotating mean flow if the turbulent fluctuations are also two dimensional and in the plane of the mean motion. At the same time Taylor relaxed his unilateral support of the vorticity transfer hypothesis when he suggested that Prandtl's theory might be expected to apply in the case of the flow between concentric rotating cylinders if the turbulence consisted entirely of ring shaped vortices that were symmetrical about the common axis. He further notes that such motions do occur for certain specified values of the rotation of the inner and outer cylinders.

Evidence of this toroidal motion was obtained in the results of the stability analysis undertaken in Chapter 1. In the same context we recall that inertial instability associated with an extremum of vorticity in a cylindrically rotating flow would tend to favour purely two dimensional velocity fluctuations in the plane of the mean motion. Thus the correct form of the Reynolds Stress is probably highly dependent upon the nature of the mechanism that induces the turbulent fluctuations.

It was left to Wasiutynski (1946) in his monumental work on astrophysical fluid dynamics to indicate the manner in which the Momentum Transport theory should be applied to rotating fluid motions. He obtained the following expression for the Reynolds stress $(\tau_{r\theta})$ by a pseudo-molecular argument,

$$\frac{1}{\rho}\left\{\tau_{r\theta}\right\} = \overline{(u')^2\tau}\left(\frac{\partial V}{\partial r} + \frac{V}{r}\right) - \overline{2(v')^2\tau}\left(\frac{V}{r}\right)$$

where $\tau$ is a typical time scale for the velocity fluctuations.

For purely radial fluctuations $\left(\overline{(v')^2} = 0\right)$, this expression yields

$$\frac{1}{\rho}\ \tau_{r\theta} = (u')^2\tau\left(\frac{\partial V}{\partial r} + \frac{V}{r}\right)$$

This expression is in accordance with Prandtl's formula given in equation (1).

For isotropic fluctuations, $\overline{(u')^2\tau} = \overline{(v')^2\tau} = \mu$, we have,

$$\tau_{r\theta} = \frac{\mu}{\rho}\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right)$$

This is the form taken by the ordinary Newtonian viscous stress tensor.

It is clear that if turbulent fluctuations in the azimuthal direction occur in a rotating fluid then the effect of these fluctuations, as well as those in the radial direction, must be taken into account if a Momentum Transport Theory is being used to derive an expressions for the stress $\tau_{r\theta}$ . Thus Wasiutysnki's work underlines the possible importance of anisotropic mixing in determining the form of the stress.

Wasiutynski's clarification of the mathematical nature of the ~~Elssiveness of~~ difference between the Transport Theories was utilized by fluid dynamicists studying large mixing processes in astro- and geo-physical fluid motions. Theories based on the anisotropy of solar turbulence were developed to account for the differential rotation of the sun, whilst Rossby (e.g. 1947) and Raethjen (e.g. 1951) offered explanations for certain features of the motion of the earth's atmosphere based on modifications of Taylor and Prandtl's theories.

Both Rossby and Raethjen were concerned with adiabatic eddy motions and they argued that the atmosphere's gravitationally stable stratification of potential temperature constrained such motions to take place in isentropic surfaces. This line of reasoning led to the concept of 'lateral mixing' in isentropic surfaces. Rossby adopted a Vorticity Transfer approach postulating that thermally driven exchange in the form of lateral mixing would lead to a redistribution and equalisation of the vertical component of absolute vorticity. Raethjen on the other hand advanced the view that when the isentropic surfaces of the atmosphere are inclined in a meridional direction (say) the lateral mixing is not isotropic but intensified in the meridional direction. The degree of this intensification determining the state of vanishing angular momentum flux, and obviously this must lie between the two extremes, rigid rotation and motion with constant angular momentum.

## IV. Scorer's Hypothesis

In his theory on the stirring of a rotating fluid by turbulent processes Scorer (1965) did not adopt a conventional mixing length approach. First he noted that 'in the molecular motion of a gas ....., there is no neutral state dividing a stable state in which molecules oscillate about a mean motion and an unstable one in which the displacement of a molecule sets off a new pattern of motion.' On the other hand, he points out that there are many fluid motions which do possess a neutral state between a stable regime in which an internal perturbation would experience a restoring force and an unstable regime in which a perturbation would be amplified and the fluid would move away from the initial state.

Bearing in mind this distinction, Scorer advanced the view that, if a fluid in a stable state is forcibly stirred by an external source of energy then the fluid system would tend to move towards a new state in which the restoring force on displaced fluid elements would be less, and if the stirring continued the ultimate state would be the neutral one. These arguments were particularised to the case of a cylindrically rotating fluid, for which the hypothesis implies that forced stirring would redistribute the vorticity so that it would be reduced to zero in the interior, and concentrated on the boundaries if the stirred region is annular, and on the boundary and at the centre if the stirring takes place inside a cylindrical region.

Scorer assumed that the stirring of the rotating fluid takes the form of either radial or vertical impulses so that no net external tangential force is applied to the fluid. Radial impulses generate toroidal eddies and Scorer noted that the mixing of angular momentum induced by these eddies would produce the velocity distribution of a potential vortex. Vertical impulses generate eddies which are toroidal with respect to the local axial direction. These eddies derive their kinetic energy from the stirring agency. Scorer reasoned that if the eddies spread out (or shrink) horizontally in the plane of the mean motion before the displaced elements are integrated with their new environment, then some of this energy is used to do work against the centrifugal restoring forces of the stream. The energy consumed by this process is permanently lost to the eddies and must be fed into the mean motion or dissipated during the smaller scale mixing accompanying the integration process. The ratio of the increase in energy of the mean motion to the amount of energy dissipated would then be a measure of the efficiency of an imposed form of stirring as a vortex producing mechanism.

Scorer suggested that anvil type cloud convection in the earth's atmosphere would be particularly efficient in this respect. Anvil type eddies were modelled as thin vertical cylinders rising through a rotating fluid and spreading out horizontally into a flat disc shape at some level through which they could not penetrate. If the whole system had a uniform positive vorticity initially then an eddy's vertical vorticity is reduced on spreading and the eddy will be moving faster than the circumambient fluid on the side of the disc nearest the axis of rotation and slower on the side remote from the axis. Hence when the fluid disc is mixed with the neighbouring environment forward momentum will be transferred radially inwards.

In view of the distinction already remarked upon regarding the significance of purely toroidal eddies as opposed to fully three dimensional ones it would be desirable to rigorously justify Scorer's contention that the fore and aft portions of the fluid disc do not contribute ~~nothing~~ on the average to the redistribution of the mean angular momentum. It is worth noting that since the stirring mechanism envisaged produces essentially three dimensional eddies no special consideration need be given to the fact that two dimensional eddies in the plane of the mean motion are not subjected to a restoring force if the mean motion is one of solid body rotation.

Scorer was wary of trying to formulate a mathematical representation of the mixing process pointing out that the quantity 'transferred' would depend upon the local mean flow properties of the fluid, the strength of the restoring force field, the energy put into the stirring motion, and the efficiency of the eddy form in producing a well mixed state.

The meteorological implications of the hypothesis were also discussed (Scorer [1965], [1966]). It was suggested that the spawning of the incipient hurricane, and certain features of the jet stream and frontogenesis could be accounted for by the vorticity concentration produced along the boundary of a region of the atmosphere subjected to the stirring effects of cloud-scale convection.

The recent work of Gough and Lynden-Bell (1968) was undertaken with the specific purpose of determining the effect of imposed stirring on a rotating fluid. In view of Scorer's hypothesis it is interesting to record that they observed a vorticity expulsion from the stirred region.

# CHAPTER 6

## Simple Models of the Gurgitation of Rotating Fluids

A naive theoretical model is developed in the first section of this chapter to determine the form taken by the Reynolds Stresses set up by forced gurgitation of a cylindrically rotating fluid. An interpretation is given of the form of the expression derived for the $(r,\theta)$ component of the Stress tensor and a study of certain facets of this expression is undertaken in the succeeding section. In the third section we seek to elaborate upon the distinction between forced and free gurgitation and some general comments are made regarding the effect of free gurgitation on the mean velocity field.

### I. Reynolds Stresses for a simple model of Forced Gurgitation

An approach based on the Kinetic Theory of Gases is used to calculate the Reynolds Stresses on the mean flow of a cylindrically rotating, homogeneous and incompressible fluid which is forcibly stirred. The stirring is assumed to induce fluid elements to detach themselves repeatedly from the mean flow with randomly distributed velocities, travel with those velocities for a short eddy convection time $(\tau)$, and then integrate themselves into their new environment. This approach is a consistent extension of Prandtl's Momentum Transfer hypothesis to rotating flows, and is based on the same assumption that the motion of the turbulent elements is unaffected by the associated transitory pressure gradients The present study has at most a heuristic value:- if certain observed effects of turbulent mixing can be interpreted in terms of this naive mathematical model, then one has a better chance of understanding the actual physical process.

We denote the velocity components of a fluid element at the time of detachment from the mean flow by $(u', V(r)+v', w')$ where $V(r)$ is the mean swirl velocity and $(u', v', w')$ are the turbulent velocity components in cylindrical polar coordinates $(r, \theta, z)$. We assert that $u'$ and $v'$ are uncorrelated so that the odd moments $\overline{u'}$, $\overline{u'v'}$, $\overline{u'^3}$, $\overline{u'^2v'}$ etc. vanish at the time $(t-\tau)$ of detachment. The Reynolds Stress $(\tau_{r\theta})$ is given by the correlation of the radial and azimuthal turbulent fluctuations at the later time (t) .

$$\text{i.e.} \qquad \tau_{r\theta} = -\rho\,\overline{u'v'}\Big|_t$$

Consider a turbulent element that is detached from the mean flow at $N(r-\Delta r, 0, 0)$ and travels to $P(r,\phi,z)$ in the time $\tau$ . Then the velocity of the element at N and P is given by,

$$\text{Velocity at } N\ (\underline{v}_N) = [u',\ (V_N+v'),\ w'] \qquad (1)$$

$$\text{Velocity at } P\ (\underline{v}_P) = [\,u'\cos\phi + (V_N+v')\sin\phi\ ,\ -u'\sin\phi+(V_N+v')\cos\phi\ ,\ w'] \qquad (2)$$

and we have the following expressions for $(\Delta r)$ and $\sin\phi$.

$$(\Delta r) = r(1-\cos\phi) + u'\tau \qquad (3)$$

$$r\sin\phi = (V_N+v')\tau \qquad (4)$$

For mathematical convenience it is assumed that over a radial distance comparable to a typical length scale of the mixing process the mean swirl velocity has a linear radial variation, so that we may write,

$$V_N = V_P - (\Delta r)\frac{\partial V_P}{\partial r} \qquad (5)$$

where $V_P$ is the mean swirl velocity at $P$ .

Thus the expression obtained for the Reynolds stress will be strictly valid only for a fluid in solid body rotation. This ~~trenchant~~ physical limitation is not too restrictive in the present work since our main interest lies in examining the possibility of a body of fluid moving away from a state of uniform rotation due to the interaction of the fluctuating motion and the mean velocity field.

Algebraic manipulation of relations (3) and (4) with $V_N$ represented by (5) yields the following equation for $\cos\phi$ .

$$[1+\varepsilon^2]\cos^2\phi + 2\varepsilon[\delta+\gamma-\varepsilon(1+\beta)]\cos\phi$$
$$+ [(\delta+\gamma)^2-(1-\varepsilon^2)-2\varepsilon(\delta+\gamma-\beta\varepsilon)-2\varepsilon\beta(\delta+\gamma-\tfrac{1}{2}\beta\varepsilon)] = 0 \qquad (6)$$

where $\beta = \frac{u'\tau}{r}$ , $\gamma = \frac{v'\tau}{r}$ , $\delta = (\frac{V_P}{r})\tau$ , $\varepsilon = (\frac{\partial V_P}{\partial r})\tau$

The solution of equation (6) for $\cos\phi$ may be written as follows,

$$[1+\varepsilon^2]\cos\phi = -\varepsilon[(\delta+\gamma)-\varepsilon(1+\beta)]$$

$$\pm\sqrt{\left\{1-(\delta+\gamma)^2+2\varepsilon(\delta+\gamma-\beta\varepsilon)+2\varepsilon\beta(\delta+\gamma-\tfrac{1}{2}\beta\varepsilon)\right\}} \quad (7)$$

We now make two further assumptions regarding the form of the eddy mixing process. First the mixing is assumed to be strong so that the eddy convection time $(\tau)$ is much less than the period of rotation of the mean flow,

$$\text{i.e.} \quad \{\varepsilon \sim \delta\} \ll 1 \quad (8)$$

Secondly the typical horizontal length scale of the turbulent fluctuations is taken to be considerably smaller that the mean radial distance of the eddy during its lifetime. Hence we have,

$$\beta \ll 1 \quad \text{and} \quad \gamma \ll 1 \quad (9)$$

These last two inequalities imply that our theory will not be applicable to the core region of a cylindrically rotating fluid.

Inequalities (8) and (9) enable us to rewrite (7) in the following unambiguous form,

$$\cos \phi = 1+\varepsilon^2 - \tfrac{1}{2}(\delta+\gamma)^2 + \varepsilon\beta(\delta+\gamma) + O(\tau^4) \qquad (10)$$

This relation can be used to obtain expressions for $(\Delta r)$ and $\sin \phi$ in terms of $\beta$ , $\gamma$ , $\delta$ and $\varepsilon$ . On substituting the resulting expressions in (2) we obtain the following equation for $\underline{v}_P = (u_P, v_P, w_P)$ .

$$u_P = u' + \{u'\tau^2\} [\varepsilon_1^2 - \tfrac{1}{2}\delta_1^2 - 2\varepsilon_1\delta_1 + (u'\tau) \tfrac{1}{r} \varepsilon_1\delta_1]$$

$$+ \{v'\tau\} [2\delta_1 + \varepsilon_1^3\tau^2 - \tfrac{3}{2}\varepsilon_1\delta_1\tau^2] + \{(v')^2\tau\} [\tfrac{1}{r} - \tfrac{3}{2} \tfrac{1}{r} \varepsilon_1\delta_1\tau^2 -$$

$$\tfrac{1}{2} \tfrac{1}{r^2} (v'\tau) \varepsilon_1] - \{(V_P+v')\tau\} [-\varepsilon_1^3\tau^2 + \tfrac{1}{2}\varepsilon_1\delta_1^2 + (v'\tau^2) \tfrac{1}{r^2} \varepsilon_1\delta_1 +$$

$$\tfrac{1}{2} (v'\tau)^2 \tfrac{1}{r^2} \varepsilon_1] + \{u'v'\tau\} [- \tfrac{1}{r}\delta_1 - \tfrac{2}{r}\varepsilon_1] +$$

$$\{(u')^2 v'\tau^3\} \tfrac{1}{r^2}\varepsilon_1 - \tfrac{1}{2} \{u'(v')^2\tau^2\} \tfrac{1}{r^2} + r\delta_1^2\tau + O(\tau^4) \qquad (11)$$

$$v_P = \{u'\tau\} [-\delta_1 - \varepsilon_1^3\tau^2 + \tfrac{1}{2}\varepsilon_1\delta_1^2\tau^2 + \varepsilon_1^2\delta_1\tau^2 + (u'\tau)\tfrac{1}{r}\varepsilon_1]$$

$$- \{v'\tau^2\} [\varepsilon_1\delta_1 + \tfrac{1}{2r}(v')\varepsilon_1]$$

$$+ [V_p + v' - (u'\tau)\varepsilon_1][1 + \varepsilon_1^2\tau^2 - \tfrac{1}{2}\delta_1^2\tau^2 - (v'\tau^2)\tfrac{1}{r}\delta_1$$

$$- \frac{1}{2r^2}(v'\tau)^2 + (u'\tau^3)\tfrac{1}{r}\varepsilon_1\delta_1 + (u'v'\tau^3)\frac{1}{r^2}\varepsilon_1]$$

$$- (u'v'\tau)\tfrac{1}{r} + \{u'(v')^2\tau^3\}\frac{1}{2r^2}\varepsilon_1 - \{(u')^2v'\tau^3\}\frac{1}{r^2}\varepsilon_1^2$$

$$+ r\,\varepsilon_1^3\tau^2 - \tfrac{r}{2}\varepsilon_1\delta_1^2\tau^2 + O(\tau^4) \qquad (12)$$

$$w_P = w' \qquad (13)$$

with $\varepsilon_1 = \frac{1}{\tau}\varepsilon$ , $\delta_1 = \frac{1}{\tau}\delta$ .

To determine the Reynolds stress $(\tau_{r\theta})$ at the point P we must calculate the correlation $\{\overline{u_P v_P}\}$. Recalling that the odd moments of $u'$ and $v'$ vanish at time $(t-\tau)$ , and stipulating that there is no net radial mass flux (i.e. $\overline{u_P} = 0$ ) , we obtain the following expression for the stress from (11) and (12).

$$\overline{u_P v_P} = -\frac{1}{\rho}\tau_{r\theta}$$

$$= -\left\{\overline{(u')^2\tau}\right\}(\varepsilon_1+\delta_1) + \left\{\overline{(v')^2\tau}\right\}(2\delta_1)$$

$$+\left\{\overline{(u')^2(v')^2\tau^3}\right\}\frac{1}{r^2}\left(\frac{13}{2}\varepsilon_1+\frac{3}{2}\delta_1\right) - \left\{\overline{(v')^4\tau^3}\right\}\frac{1}{r^2}\left(\frac{3}{2}\varepsilon_1+\frac{5}{2}\delta_1\right)$$

$$+\left\{\overline{(u')^2\tau^3}\right\}\left[-\frac{5}{2}\delta_1^3 + \frac{11}{2}\varepsilon_1\delta_1^2 + 2\varepsilon_1^2\delta_1\right]$$

$$+\left\{\overline{(v')^2\tau^3}\right\}\left[-\frac{3}{2}\delta_1^3 - \frac{11}{2}\varepsilon_1\delta_1^2 + 2\varepsilon_1^2\delta_1\right]$$

$$+ O(\tau^4) \qquad (14)$$

If the stirring is assumed to produce an isotropic mixing process (i.e. $\overline{(u')^2} = \overline{(v')^2}$) then the Reynolds stress ($\tau_{r\theta}$) may be written thus,

$$\tau_{r\theta} = +\rho\left[\mu\ell_{r\theta} - \nu\ 4\left(\frac{V}{r}(\mathcal{G})\right)\ \ell_{r\theta} - \eta\frac{1}{r^2}(2\mathcal{G} - 3\ell_{r\theta})\right] + O(\tau^4) \qquad (15)$$

where $\mu$, $\nu$ and $\eta$ represent the eddy coefficients $\left\{\overline{(u')^2\tau}\right\}$, $\left\{\overline{(u')^2\tau^3}\right\}$ and $\left\{\overline{(u')^4\tau^3}\right\}$ respectively, and

$\mathcal{G} = \left(\frac{\partial V}{\partial r} + \frac{V}{r}\right)$ is the axial component of vorticity for the flow whilst $\ell_{r\theta} = (\frac{\partial V}{\partial r} - \frac{V}{r})$ is the $(r,\theta)$ component of the deformation tensor for the flow.

It was noted in Chapter 5 section II that assumptions invoking the symmetry, isotropy, homogeneity and linearity of the stress tensor formed the basis of the derivation of the constitutive equation of a Newtonian fluid. Viewed in comparison with these assumptions the form of the Reynolds stress derived above has several interesting features. Wasiutynski's result regarding the effect of anisotropy on the form of the Reynolds stress is again evident in the terms of order $\tau$ in the expression (14) derived above. Moreover the terms of order $\tau^3$ indicate the manner in which inhomogeneous and non-linear effects enter the expression (15) for the Reynolds Stress.

Confining our attention to the case of isotropic mixing, it is possible to offer the following interpretations of the two types of $\tau^3$ terms in equation (15). The term involving $\nu$ can be regarded as contributing to a reduction of the eddy viscosity $\mu$ to $\mu^*$ where,

$$\mu^* = [\mu - 2\nu \{2\frac{V}{r} \mathcal{S}\}]$$

Inequalities (8) and (9) ensure that $\mu^*$ is positive within the range of validity of the present theory. If we assume that centrifugal effects cause a fluid element that is displaced radially in a swirling flow to oscillate about its original radial location, then to a first approximation the frequency of this oscillation is give by $\left\{2\frac{V}{r}\varphi\right\}^{1/2}$ . Hence we are led to attribute the reduction in the effective eddy viscosity to a tendency of the turbulent elements to oscillate about their original radial positions due to centrifugal effects. This interpretation suggests that centrifugal effects contribute to the Reynolds stress even in an isotropic mixing process. We may also compare this interpretation with the well-known damping effect of a gravitationally stable density stratification on the turbulence in a mean horizontal, shear flow. In the latter situation the relevant frequency would be the Brunt-Väisälä frequency.

The inhomogeneity of the term involving $\eta$ in expression (15) for the Reynolds stress is certainly a novel feature. An illuminating interpretation is to regard this term as a couple stress, whereby adjacent rings of fluid exert a torque upon one another. It is clear that, for a fluid possessing such a stress form, solid body rotation is a permissible steady state only if $\eta$ is not a function of the radial distance, for then adjacent rings of fluid exert equal and opposite torques upon one another.

The nature of the problem examined here is such that the Reynolds stress is inherently symmetrical, and this serves to distinguish the above pseudo-couple stresses from those that occur in conjunction with an anti-symmetric stress tensor in the study of certain special fluids.

## II. Physical Implications of the Reynolds Stress

In this section we examine some of the physical implications of the particular form of the Reynolds stress derived in the last section.

We begin by calculating the time development of the mean swirl velocity of a cylindrically rotating flow, confined within an annular region, when the flow is subjected to a fully anisotropic mixing process. The flow is considered to be in solid body rotation initially, and the eddy viscosity is assumed constant. In view of the frequent allusions in theoretical studies to the possible importance of anisotropy in the mixing motions of rotating fluids it seems worthwhile performing such a calculation.

The azimuthal equation of motion of a cylindrically, rotating turbulent fluid is given by,

$$\frac{\partial V}{\partial t} = \frac{1}{\rho r^2} \cdot \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) \qquad (16)$$

and for a fully anisotropic mixing process we have,

$$\tau_{r\theta} = \rho K \left(\frac{\partial V}{\partial r} + \frac{V}{r}\right)$$

where $K = \overline{(u')^2 \tau}$ is assumed constant in the present problem.

To trace the time development of the mean swirl velocity (V) from its initial state of solid body rotation we require the solution of (16) subject to boundary conditions imposed at the inner and outer radii, a and b respectively, of the annular stirred region. Regarding the fluid as inviscid in the unstirred areas then the boundary conditions must stipulate that no angular momentum can be transferred out of the stirred regions,

$$\text{i.e. } \frac{1}{r}\frac{\partial}{\partial r}(rV) = 0 \quad \text{at} \quad r = a \quad \text{and} \quad r = b \tag{17}$$

We rewrite equation (16) as follows,

$$\frac{\partial}{\partial t}(rV) = K\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(rV)\right) \tag{18}$$

This equation is analogous to the Heat Conduction Equation for the temperature, when the temperature is merely a function of radial distance and time. The angular momentum $(rV)$ of the present problem corresponds to the temperature and $K$ takes on the role of a thermometric conductivity.

The solution of (18) subject to the boundary conditions (17) is [see Carslaw and Jaeger § 14.8, example IV],

$$(rV) = \frac{2}{(b^2-a^2)}\int_a^b r'\, f(r')\, dr'$$

$$+ \frac{\pi}{4}\sum_{n=1}^{\infty}\left(\frac{1}{F(\alpha_n)}\right).\exp\left\{-K(\alpha_n)^2 t\right\}.[J_1(b\alpha_n)]^2 C_o(r,\alpha_n)\int_a^b 2\pi r' f(r')\, C_o(r',\alpha_n)\, dr' \tag{19}$$

where the $(\alpha_n)$ are the positive roots of the equation,

$$J_1(a\alpha)\,Y_1(b\alpha) - Y_1(a\alpha)J_1(b\alpha) = 0$$

whilst $F(\alpha_n) = [J_1(a\alpha_n)]^2 - [J_1(b\alpha_n)]^2$

and $C_o(r, \alpha_n) = \alpha_n[J_o(r\alpha_n)Y_1(a\alpha_n) - Y_o(r\alpha_n)J_1(a\alpha_n)]$

The conventional notation for Bessel Functions is being used above. $f(r)$ is the distribution of the angular momentum at the time $t = 0$, and is given by,

$$f(r) = \Omega r^2$$

where $\Omega$ = the uniform angular velocity of the initial state.

It can be shown that

$$\int_a^b r^3 Y_o(r\alpha_n)dr$$

$$= 2\left(\frac{b^2}{\beta_n}\right)^2 [Y_o(\beta_n) + (\tfrac{1}{2}\beta_n - \frac{2}{\beta_n})Y_1(\beta_n)] - 2\left(\frac{ab}{\beta_n}\right)^2 [Y_o(k\beta_n) + (\tfrac{1}{2}k\beta_n - \frac{2}{k\beta_n})Y_1(k\beta_n)]$$

where $b\alpha_n = \beta_n$ and $k = \frac{a}{b}$

A similar result can be derived for the required companion integral involving $J_o(r\alpha_n)$ .
Using these results we may write (19) in the following form,

$$(rV) = \frac{\Omega}{2}(b^2+a^2)$$

$$+\pi^2\left(\frac{\Omega}{2}\right)\sum_{n=1}^{\infty} \exp\left\{-\beta_n^2\tau\right\}.J_1^2(\beta_n).[J_1^2(k\beta_n)-J_1^2(\beta_n)]^{-1}.$$

$$[J_o(\beta_n{}^r/b)\ Y_1(k\beta_n) - Y_o(\beta_n{}^r/b)J_1(k\beta_n)].C'(k,\beta_n) \quad (20)$$

where $C'(k,\beta_n) = 2Y_1(k\beta_n)[b^2J_o(\beta_n) - a^2\{J_o(k\beta_n) +$

$(\frac{1}{2}k\beta_n - \frac{2}{k\beta_n})J_1(k\beta_n)\}] - 2J_1(k\beta_n)[b^2Y_o(\beta_n) -$

$a^2\{Y_o(k\beta_n) + (\frac{1}{2}k\beta_n - \frac{2}{k\beta_n})Y_1(k\beta_n)\}]$

and $\tau = \frac{\kappa t}{b^2}$ .

The radial distribution of angular momentum was calculated for a fixed value of $k$ $(= \frac{1}{10})$ and various values of $\tau$. The calculation proved tractable because of the rapid convergence of the series in expression (20) for moderate values of $\tau$. The results are given in Fig.(5), where values of the non-dimensional quantity $\left(\frac{rV}{\Omega b^2}\right)$ have been plotted against $(^r/b)$. As might have been anticipated from consideration of the initial state and the boundary conditions, the initial redistribution of angular momentum towards the final state proceeds most rapidly in the outer region of the stirred annulus. It is evident from Fig.(5) that a substantial proportion of the redistribution has been completed in the time $\tau = 0.1$.

From the results obtained it is possible to estimate the pressure difference $(\Delta P)_\tau$ across the annulus at the time $\tau$. Calculating this pressure difference using the relation,

$$(\Delta P)_\tau = \int_a^b \rho \frac{1}{r^3} (rV)_\tau^2 \, dr$$

we obtain, for $k = \frac{1}{10}$,

$$(\Delta P)_{\tau=0} \approx \frac{1}{2}(\rho\Omega^2 b^2)$$

$$(\Delta P)_{\tau=.03} \approx 7.4(\rho\Omega^2 b^2)$$

$$(\Delta P)_{\tau\to\infty} \approx \frac{1}{8k^2}(\rho\Omega^2 b^2)$$

For values of

$$\rho = 10^{-3}\ \text{grm.cm}^{-3}\ ,\quad \Omega = 3.5 \times 10^{-5}\text{sec.}^{-1}\ \text{and}\ b = 200\text{kms.}$$

the pressure differences at $\tau$ equals 0 , 0.03 and infinity are approximately 0.2 , 3.5 , and 5.0 millibars respectively.

These calculations are of particular interest because of Scorer's suggestion that the incipient hurricane could be created by an angular momentum redistribution similar to the redistribution which has been time traced in the above calculation. It is felt that, despite the major assumption that the eddy coefficient is constant, the present mathematical model may be taken to represent the gross features of the pressure drop and time development of Scorer's 'Hurricane Spawning' mechanism. For cumulonimbus stirring of the atmosphere we would have the following typical values,

$$u' \sim 3 \text{ metres/sec} \quad \text{and} \quad \tau \sim 1.8 \times 10^3 \text{ secs}$$

so that $K \sim 10^4 \text{ (metres)}^2\text{/sec.}$

Thus using the results noted above we deduce that a meteorologically significant pressure drop of 3 mb, corresponding to $\tau = .03$, would occur in approximately two days. These considerations, although lending support to Scorer's conclusions, do not of themselves substantiate the anisotropic nature of the Reynolds stress.

The value of the present heuristic theory for the Reynolds stress due to forced gurgitation must be judged on its ability (or inability) to explain observed effects. We will therefore attempt to interpret the experimental results of Gough and Lynden-Bell (1968) using this theory.

In essence their experiments involved the gurgitation of a cylindrical region of water which was initially rotating with an uniform angular velocity about the axis of the cylindrical region. The apparatus comprised of a beaker, containing the water to be stirred, floating at the free surface of a larger, co-axial, water contaiing vessel, which was rotating with the same angular velocity. Gurgitation was brought about by the release of gas bubbles from 'Alka-Seltzer' tablets immersed in the water of the inner beaker. An increase in the angular velocity of this beaker was observed shortly after the commencement of gurgitation, and the experimenters concluded that vorticity, and with it angular momentum, was being expelled from the stirred liquid.

Typical order of magnitude estimates of the large eddies produced by the bubbles were,

$$\tau \sim 1 \text{ sec} , \qquad u' \sim 1 \text{ cm/sec.}$$

whilst the inner beaker had a radius of 10.2 cms, and an initial angular velocity of $(\frac{2\pi}{15.5})$ sec$^{-1}$. Hence characteristic values of the dimensionless parameters of the theory presented in section I are,

$$(\varepsilon \sim \delta) \approx 0.4 \qquad (21)$$

and the inequalities given in (9) for $\beta$ and $\gamma$ indicate that the theory can not account for a core region within a distance of approximately one centimetre from the axis of rotation of the liquid.

An interesting interpretation can be given to the restrictions imposed on the validity of the theory by the inequalities expressed in (8) and (9) for a 'bubble stirring' source of forced gurgitation. Assuming that the eddy lifetime is not less than the lifetime of the generating bubble, then

$$\tau^2 > \frac{h}{g}$$

where $h$ is depth of water through which the bubble has to rise.

Hence $\delta$ must be such that,

$$\delta^2 > \Omega^2 \frac{h}{g}$$

$$> \left(\frac{\Omega^2 r_o}{g}\right)\left(\frac{h}{r_o}\right)$$

where $r_o$ is the initial radial distance of the bubble from the axis of rotation.

Now $\left\{ \left(\frac{\Omega^2 r_o}{g}\right)\left(\frac{h}{r_o}\right)\right\}^{1/2}$ may be regarded as approximately the ratio of the time the bubble takes to reach the surface of the water to the time it would take to approach the axis of rotation. Hence the restriction that $\delta \ll 1$ implies that the theory is not valid if the bubbles travel an appreciable radial distance during their ascent through the fluid.

In this sense $\delta$ is also one measure of the anisotropy of this particular stirring mechanism. With $\delta \gg 1$ the anisotropy would certainly tend to create a motion akin to a potential vortex in the stirred fluid. This tendency is cogently illustrated in the experiments of Turner and Lilly (1963).

In Gough and Lynden-Bell's experiments we have the following characteristic value,

$$\left\{ \frac{\Omega^2 h}{g} \right\} \sim (10^{-4}) \qquad (22)$$

so that the anisotropy is not expected to dominate the mixing process. More significantly, at the commencement of stirring the anisotropy would mean that the inner beaker would be subjected to a deceelerating torque, and this deceleration would be several orders of magnitude smaller than the experimentally observed acceleration. Hence we will assume that the mixing is isotropic, and calculate the initial torque (T) exerted on the beaker using equation (15).

$$T = (b\pi)\ \rho \left\{ \overline{(u')^4\ \tau^3} \right\} \Omega . d$$

$$\approx 18 \text{ cm/dyne}$$

where d = depth of stirred fluid.

This torque arises from the radial stratification of the eddy coefficient $\eta$ . Since the moment of inertia (I) of the beaker and its contents was 7000 grm/cm$^2$, the initial acceleration is given by,

$$\frac{T}{I} = 2.6 \times 10^{-3} \text{ sec.}^{-2} \qquad (23)$$

This result compares favourably with the value $1.4 \times 10^{-3}$ sec.$^{-2}$, which was the acceleration observed experimentally when the mixing due to the gurgitation first became effective.

This successful prediction of the initial acceleration suggests that further attention be given in the study of forced gurgitation of rotating fluids to the possible existence of inhomogeneous terms in the Reynolds Stress.

We also note that the model examined in the first section of this chapter gives the following expression for the aximuthal eddy flux of a scalar quantity $\phi = \phi(r,t)$ ,

$$\overline{\phi\, v_P} = \frac{\partial\phi}{\partial r} \left[ \overline{(u'\tau)^2} \left(\frac{\partial V}{\partial r} + \frac{V}{r}\right) - \overline{(v'\tau)^2} \left(\frac{V}{r}\right) \right] . \qquad (24)$$

Thus it is seen that rotation can produce an azimuthal flux of the scalar quantity $\phi$ . This effect was noted by Welander (1966).

## III. A note on Free Gurgitation

A distressing feature of previous studies of mixing processes in rotating fluids is the failure to adequately distinguish between forced and free gurgitation. Some of the early theoretical treatments of the general circulation of the earth and the sun may be criticised in this ~~report~~ respect, and again only a nebulous distinction is drawn between the two mechanisms in the work of Scorer, and Gough and Lynden-Bell. In this brief note we argue that to regard the self-induced thermal stirring of a rotating fluid as a forced gurgitation deprives the motion of its quintessential properties.

Whereas forced gurgitation was regarded as being set up and maintained by the steady input of mechanical energy to the fluid by an external agency, there may be a transient free gurgitation of a thermally active fluid which is not associated with the input of external energy. In the latter case the motion is driven directly by the release of potential energy as the fluid moves away from an initial thermally unstable flow configuration, and the motion would persist until the fluid attained a new equilibrium state. Sustained free gurgitation could be achieved by the input of thermal energy by a conductive or radiative process.

An appreciation of the interplay of rotation and thermal stratification in a transient fluid motion may be obtained by consideration of the 'potential vorticity' of fluid elements. Ertel (1942) showed that if a scalar quantity $s = s(x,y,z,t)$ can be assigned to each element of fluid such that

$$\frac{d}{dt}(s) = Q \tag{25}$$

where $Q = Q(x,y,z,t)$ is the rate of generation of $s$ per unit volume and time, then

$$\frac{d}{dt}\left\{ \frac{\underline{\omega}\cdot\nabla s}{\rho} \right\} = \frac{1}{\rho}\left[ (\underline{\omega}.\ Q) + \nu(\nabla_s\cdot\nabla^2\underline{\omega}) \right] \tag{26}$$

where $\underline{\omega}$ and $\nu$ are the vorticity and the coefficient of viscosity respectively of the fluid, and the standard notation for symbols in cartesian coordinates has been used in both (25) and (26).

Equation (26) shows that if the fluid is inviscid $(\nu = 0)$, the potential vorticity, $\left[ \frac{\underline{\omega}\cdot\nabla s}{\rho} \right]$, is a conservative quantity for each fluid element if $s$ is also a conservative quantity. For adiabatic and pseudo-adiabatic motion of an ideal gas, $s$ may be taken to represent the potential and the equivalent potential temperature, $\Theta$ and $\Theta_E$ respectively, and $Q$ may then be set to zero. Again for an heterogeneous, incompressible and inviscid fluid, for which the density remains constant along the fluid trajectories, equation (26) becomes,

$$\frac{d}{dt}\left[ \underline{\omega} . \nabla \rho \right] = 0 \tag{27}$$

We restrict our attention for the moment to flows for which equation (27) is valid. Now let us consider the following simple flow configuration: A fluid is at rest relative to a frame of reference rotating with uniform angular velocity $\Omega$ about the vertical axis through $r = 0$, and is confined within an annular region $(a < r < b)$ bounded above and below by horizontal walls at $z = 0$ and $z = (b-a)$. The fluid is assumed to possess a gravitationally unstable vertical density stratification $(\beta)$ at the initial instant.

Potential energy is released as the fluid moves away from this unstable equilibrium configuration and this energy must reappear as kinetic energy of motion. It follows from equation (27) that if the fluid could move to a state of purely horizontal motion with an inverted stable density stratification $(-\beta)$ then the flow vorticity would also become reversed to $(-\Omega)$ . Considerations of the conservation of energy and angular momentum of the flow indicate that this is not in general a permissible flow state. Nevertheless this example is indicative of the important effect rotation has on thermal convection in a fluid.

Again let us consider the readjustment of the same fluid from an unstable baroclinic configuration. Let the fluid possess a radially decreasing density distribution, a stable vertical density stratification and be in geostrophic equilibrium at the initial instant. The 'baroclinic eddies' which are formed derive their energy from the potential energy associated with the horizontal density gradient. We assume that they in turn convert their kinetic energy into that of the mean zonal motion and that a new purely azimuthal geostrophic motion is attained in which the vertical stability has been increased everywhere. The it follows from (27) that the value of the vertical vorticity must be decreased. If this motion was treated as a forced gurgitation of a homogeneous fluid we would have had to invoke the concept of an anisotropic eddy viscosity.

As a free gurgitation problem Scorer's 'Hurricane Spawning Mechanism' must result from the time development of a portion of the atmosphere according to the equation,

$$\frac{d}{dt}\left[\frac{\underline{\omega}\cdot\ \theta_E}{\rho}\right] = \mathcal{M} \qquad (28)$$

where $\mathcal{M}$ is the change of the potential vorticity due to diffusive and radiative processes.

If the air is initially at rest relative to the earth then for free gurgitation to take place $(\frac{\partial \theta_E}{\partial z})$ must be negative and the air must be saturated somewhere in the fluid.

One of the most controversial aspects of Scorer's hypothesis is the explicit suggestion that the small scale cumulus and anvil type cloud convection can directly parturate the larger scale hurricane motion. The latent energy associated with the water vapour content of a meso-scale region of the tropical atmosphere is certainly sufficient to spawn an incipient hurricane. However the magnitude of $\mathcal{M}$ in equation (28) is difficult to assess in cloud convection situations, and hence there is no means of directly determining the efficiency of such a gurgitating motion. Nevertheless the possibility of parturition of the form suggested by Scorer is not ruled out.

To conclude our discussion of gurgitating motions it would be instructive to seek experimental and/or theoretical evidence to corroborate 'the principle' advanced by Scorer that continued gurgitation of a fluid would result in the fluid moving towards a neutral state between a stable and an unstable flow regime. Below we examine the results of various studies with this objective in mind.

For the flow between two horizontal, non-rotating, parallel plates, with the lower plate maintained at a higher temperature than the upper plate, it has been noted experimentally that at highly super-critical Rayleigh Numbers the mean temperature of the interior fluid assumes almost a constant value, and the temperature gradients are confined to thin boundary layers at the two plates. In the analogous case of the supercritical swirling flow between two concentric cylinders, with th inner one rotating and the outer one stationary, it has been observed that the gradient of angular momentum is confined to boundary layers on the cylinders.

We may interpret these observations in terms of 'the principle' as follows:- If the vertical (radial) gradient of temperature (angular momentum) is negative then the flow is unstable to exchange processes which conserve temperature (angular momentum) and the turbulent motion can gain energy from the interchange. Large eddies, which are the least subject to loss of energy due to dissipative effects, could then traverse the fluid region. The interior fluid attains a state of 'non-diffusive' neutral equilibrium and the unstable gradients are confined to the neighbourhood of the boundaries where eddies sufficiently small to dissipate the energy exist.

If in the cylinder experiments the inner cylinder is kept at higher temperature than the outer so that the temperature distribution has a stabilizing role (c.f. Chapter 2) then experimental results indicate that the interior fluid is characterized by constancy of both angular momentum and temperature. Thus the motion acts to reduce the stabilizing effect of the density gradient in the interior fluid.

Similarly solid body rotation of the two parallel plates about a vertical axis in the gravitational convection problem also imposes an additional constraint on the motion. The effect of the rotation, acting via the Coriolis forces, on thermal circulations in a vertical plane is to induce horizontal circulations. These circulations have the same energy source as the convective circulations and they too must be maintained against frictional dissipation. In a sophisticated mathematical treatment of this convection problem Veronis (1959) showed that the finite amplitude motions generate a non-linear vorticity which tends to counteract the vorticity generated by the imposed constraint of rotation.

Further evidence of the tendency of the fluid to react to an imposed constraint by generating internal motions to counteract the external restraint may be inferred from the results of certain dishpan experiments undertaken by Fultz et al. (1959). In the Hadley (axi-symmetric) flow regime they state that an analysis of the free surface velocity field shows that the fluid attempts to attain a constant angular momentum profile on the portion of the surface removed from the bounding cylinder. This is in accord with the setting up of a neutral state in the interior fluid characterised by the potential vorticity $[\underline{\omega}.\nabla\rho]$ approaching zero in that region.

# CHAPTER 7

## Time Development of Vorticity Patterns

### I. Introduction and Theory

The occurrence of vigorous, swirling flows in geophysical fluid motions has always aroused the aesthetic and practical interest of fluid dynamicists. Innumerable mechanisms have been suggested for the generation of such motions. In this study we confine our attention to what is perhaps the conceptually simplest mechanism. Two dimensional flow of a homogeneous, incompressible, inviscid fluid is considered, and we examine the possibility of the advective development of a strong, swirling motion.

Motion of this type is governed by the vorticity advection equation,

$$\varphi_t + u\,\varphi_x + v\,\varphi_y = 0 \qquad (1)$$

where $$\varphi = \psi_{xx} + \psi_{yy} \qquad (2)$$

and $\psi$ is a stream function such that,

$$u = -\psi_y \qquad \text{and} \qquad v = \psi_x \tag{3}$$

To ascertain the ability of certain initial distributions to produce vorticity concentrations, i.e. strong swirling motions, during the evolution of the flow, we require the solution of equation (1) for $\varphi$ as a function of space $(x,y)$ and time $(t)$ for a given initial distribution.

Strong advective development is essentially a non-linear process and we will therefore have to resort to a numerical technique to solve equation (1). A modified form of this equation is employed in numerical weather forecasting so that techniques for obtaining its solution are highly developed. These methods are based on finite difference techniques and are not well suited to follow the development of strong shear layers since the numerical procedure introduces a pseudo-diffusion effect. Moreover our prime objective is to obtain a better physical understanding of advective vorticity concentration and not necessarily to obtain the best possible forecast of the time development of the motion.

Bearing these comments in mind we will represent the motion by discrete vortex elements, i.e. the continuous vorticity distribution is approximated by a finite set of parallel rectilinear vortex filaments of infinitesimal cross-section and finite strength. This representation may be formalised mathematically as follows:-

Assuming the fluid to be unbounded the solution of (2) may be written,

$$\psi(x,y,t) = \iint G(x,y;\ x',y')\,\zeta(x',y',t)\,dx'\,dy' \qquad (4)$$

where G is the Green's function

$$G = \frac{1}{2\pi} \log r \qquad (5)$$

with $r^2 = (x-x')^2 + (y-y')^2$

Replacing the continuous vorticity distribution ($\zeta$) by a finite set (n) of point vortices, then the discrete analogues of (4) and (5) are,

$$\psi_i = \sum_{\substack{j=1 \\ i \neq j}}^{n} \mu_j \log r_{ij} \qquad (6)$$

$$\text{with} \quad r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 \tag{7}$$

The point vortices follow the motion materially and their velocities are given by,

$$\frac{dx_i}{dt} = -\frac{\partial \psi_i}{\partial y_i} = -\sum_{i \neq j} \mu_j \frac{y_i - y_j}{r_{ij}^2} \tag{8}$$

and

$$\frac{dy_i}{dt} = \frac{\partial \psi_i}{\partial x_i} = \sum_{i \neq j} \mu_j \frac{x_i - x_j}{r_{ij}^2} \tag{9}$$

Equation (1) has thus been replaced by a set of $2n$ ordinary differential equations, and these equations are quite amenable to numerical integration.

Introducing an energy function,

$$H = \sum_{i > j}\sum \mu_i \mu_j \log r_{ij} \,, \tag{10}$$

then we may rewrite (8) and (9) in the Hamiltonian form

$$\mu_i \frac{dx_i}{dt} = -\frac{\partial H}{\partial y_i} \tag{11}$$

and $$\mu_i \frac{dy_i}{dt} = \frac{\partial H}{\partial x_i} \qquad (12)$$

Kirchoff showed that for the system of equations (11) and (12) the four quantities $H, \sum \mu_i x_i$, $\sum \mu_i y_i$, and $\sum \mu_i(x_i^2 + y_i^2)$ are conserved. These quantities correspond to the energy, linear momentum and angular momentum respectively of a system of point vortices, and some important consequences may be deduced from their invariance. In discussing these consequences we will assume that $\mu_i$ $(i = 1,n)$ is positive.

The energy function $H$ involves the distances separating the point vortices. Since $H$ = constant, it follows that the approach of one pair of vortices must be accompanied by a recession of other pairs. Again the conservation of $\sum \mu_i(x_i^2 + y_i^2)$ implies that if a concentration of point vortices develops in some region during the flow evolution then there must also be an accompanying divergence of other vortices away from this region. We may note that the compensating motion could be confined to the removal of a few weak vortices a large distance. For symmetric motion, which will be our chief concern, the invariance of both $\sum \mu_i x_i$ and

$\sum \mu_i y_i$ follows from the symmetry alone.

The commonest example of the advective development of strong swirling motion is the rolling up of an uniform vortex sheet into regularly spaced vortices. Mathematical treatment of such a motion has been mainly restricted to numerical studies of discrete arrays of point vortices with a uniform one dimensional distribution of these being assumed to represent approximately a continuous vortex sheet.

Investigations undertaken by Birkhoff and Fisher (1959) and Van der Voreen (1965) showed that an initially one dimensional array of point vortices perturbed sinusoidally do not roll up smoothly. Van de Voreen also showed that the round off error in the numerical computation offsets the better representation obtained by increasing the number of point vortices taken to approximate to a continuous vortex sheet. The round off errors generate microscopic disturbances which amplify rapidly and swamp the macroscopic rolling up process. These disturbances may be a manifestation of a physical instability of the point vortex system that has been catalysed by round-off errors.

In view of the apparently smooth rolling up of vortex sheets in natural phenomena, the results of the studies mentioned above indicate that point vortex representation is not a satisfactory method of simulating the motion of a perturbed vortex sheet. Nevertheless these studies provide us with an useful insight to the study of the advective development of swirling motion, and in the next section we again pursue this line of investigation.

## II. Numerical Study of the Concentration of Point Vortices

Numerical integrations of equations (11) and (12) were performed to trace the time development of the initial distribution of point vortices. The integrations were carried out by means of a second order Runge-Kutta method with the time step chosen so small as to give no appreciable change in the motion when it was halved. A check was also kept on the magnitude of the conservative energy function $H$ , and its percentage change was found to be less than $10^{-2}$ in all the experiments.

Kirchoff's fourth conservation property, $\sum \mu_i (x_i^2+y_i^2)$ = constant, suggests that an infinite vortex sheet of uniform strength may not be highly efficient in engendering a concentration of point vortices because of the necessary divergence of other point vortices away from the region of concentration.

Here we present the results obtained for two simple types of initial distributions of point vortices, both of which it was anticipated would effectively produce point vortex concentrations.

Type A. :

A set of numerical experiments was performed to trace the evolution of a vortex sheet of non-uniform strength, the vortex sheet being represented by point vortices of varying strength placed at equal distances along the x axis (say) at time t = 0.

Among the forms chosen for the variation in strength ($\omega$) of the sheet was the following,

$$\begin{aligned} \omega &= \omega_0 \left(1 + \cos \frac{\pi}{\ell} x\right) \qquad \text{for} \quad |x| \leq \ell \\ \omega &= 0 \qquad\qquad\qquad\qquad \text{for} \quad |x| \geq \ell \end{aligned} \qquad (13)$$

The strength ($\mu$) of an individual point vortex at $x = 3$ (say) was <u>either</u> (a) read off directly from equations (13) <u>or</u> (b) taken to be the integrated value $\int_{2.5}^{3.5} \omega \, dx$ .

Figures (6) and (7) record the time development of a half strip of the vortex sheet with the above form of vorticity distribution and the strength distributions (a) and (b) respectively. Similar results were obtained for other analogous distributions, and the more striking qualitative features of the results are discussed below.

There is an initial period of concentration (see Figures 6a, 6b, 7a, 7b) in which the strong inner vortices perform fairly tight inward spirals, whilst the weaker outer vortices traverse a slightly outward spiral. However the approach of the two central vortices, one in each half strip, during this process induces these vortices to accelerate in their motion around the centre $[(x,y) = (0,0)]$. The point vortex representation of the vortex sheet deteriorates rapidly at this stage. On further development of the motion the central vortices arrive at a location dominated by the influence of the other 'half strip' of the vortex sheet, and at this stage the innermost vortices begin to diverge away from the centre, although the rolling up process continues. (see Figures 6c, 6d, 7c, 7d). The transient concentration process that has been isolated by these simple experiments merits further investigation.

Type B. :

The previous set of experiments were repeated with two additional vortices placed symmetrically at $(x,y) = (\ell,-m)$ and $(-\ell,m)$ , where $\ell$ and $m$ assume prescribed positive values. A typical initial distribution is shown in Figure (8a). Only half of the symmetrical distribution is illustrated. During the development of the motion it was hoped that the new vortices would continuously aid point vortex concentration, first by directing the inner vortices towards the centre and later by dragging the weak outer vortices further away.

The results of the numerical computations indicate that the motion of the vortices was irregular and thus the concept of vortex sheet representation has to be abandoned in this case. Nevertheless the motions are of intrinsic interest and the results obtained for the development of the distribution illustrated in Figure (8a) are shown in the succeeding diagrams. A qualitative feature that was common to all Type B experiments was the splitting of the point vortices into three groups, an inner group comprising of the strong vortices centred around the origin and two outer groups of weak vortices.

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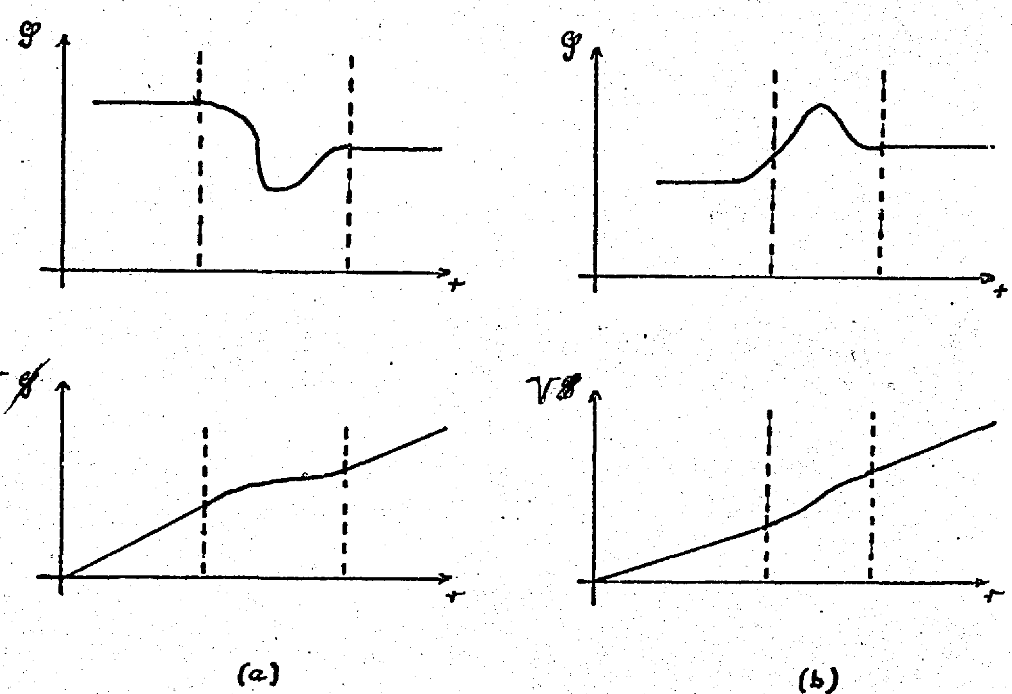
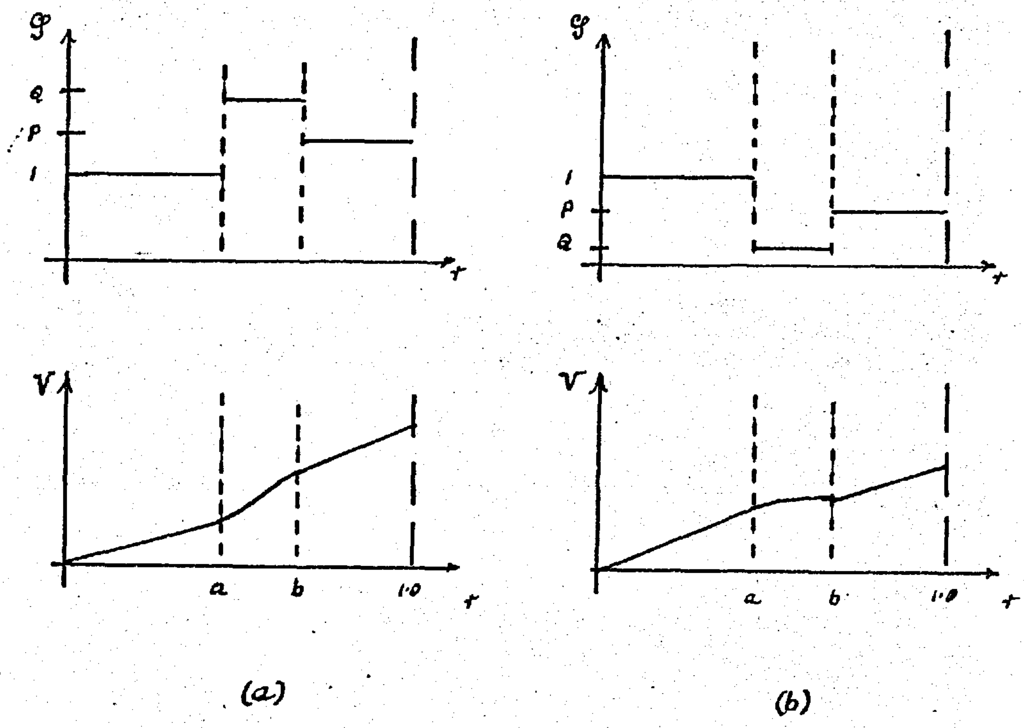



Sketches of the vorticity ($\zeta$) and velocity ($V$) profiles for curved flows possessing vorticity extremae.

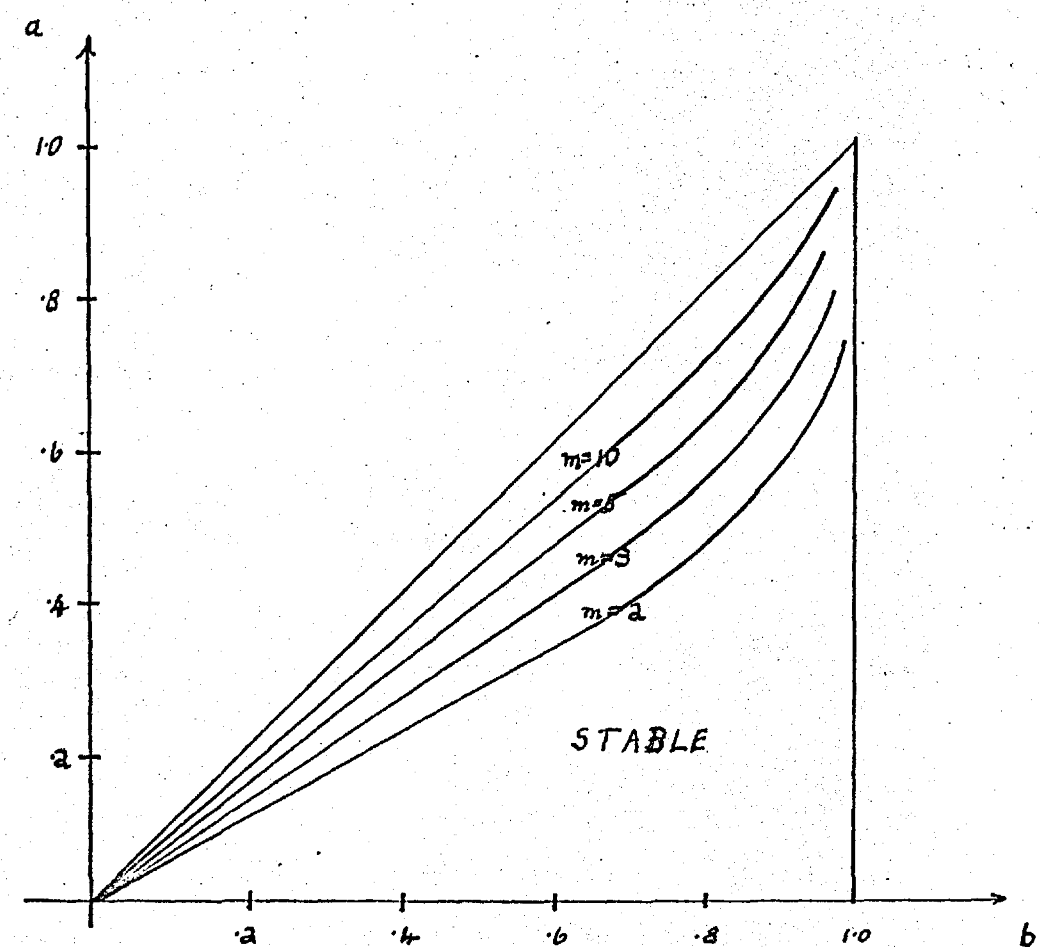
FIGURE 1.




Possible vorticity and velocity distributions represented by equations (10).

FIGURE (2)

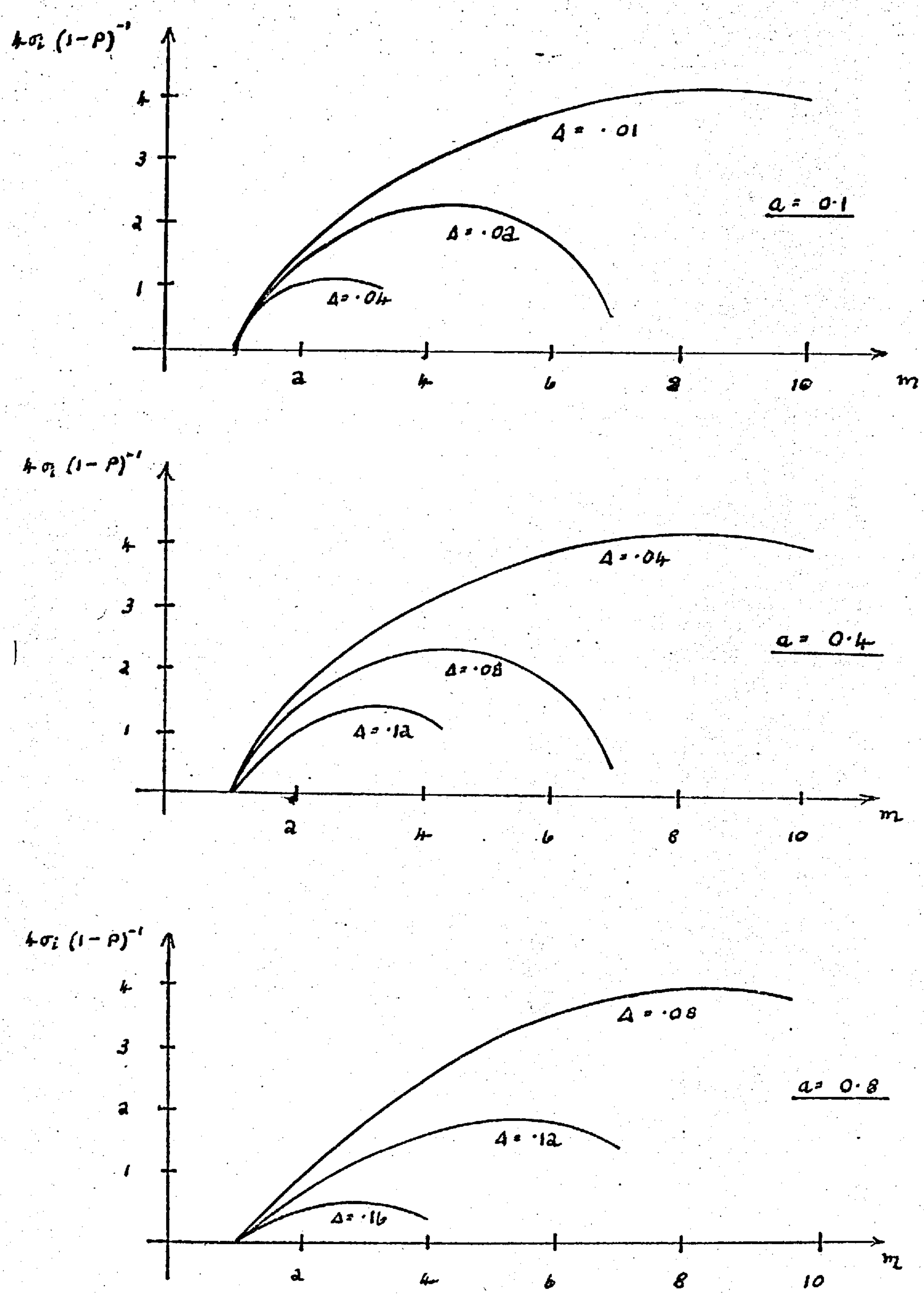
Neutral Curve Diagram.




Plot of neutral curves in the (b, a) plane for a selection of azimuthal wave numbers (m).

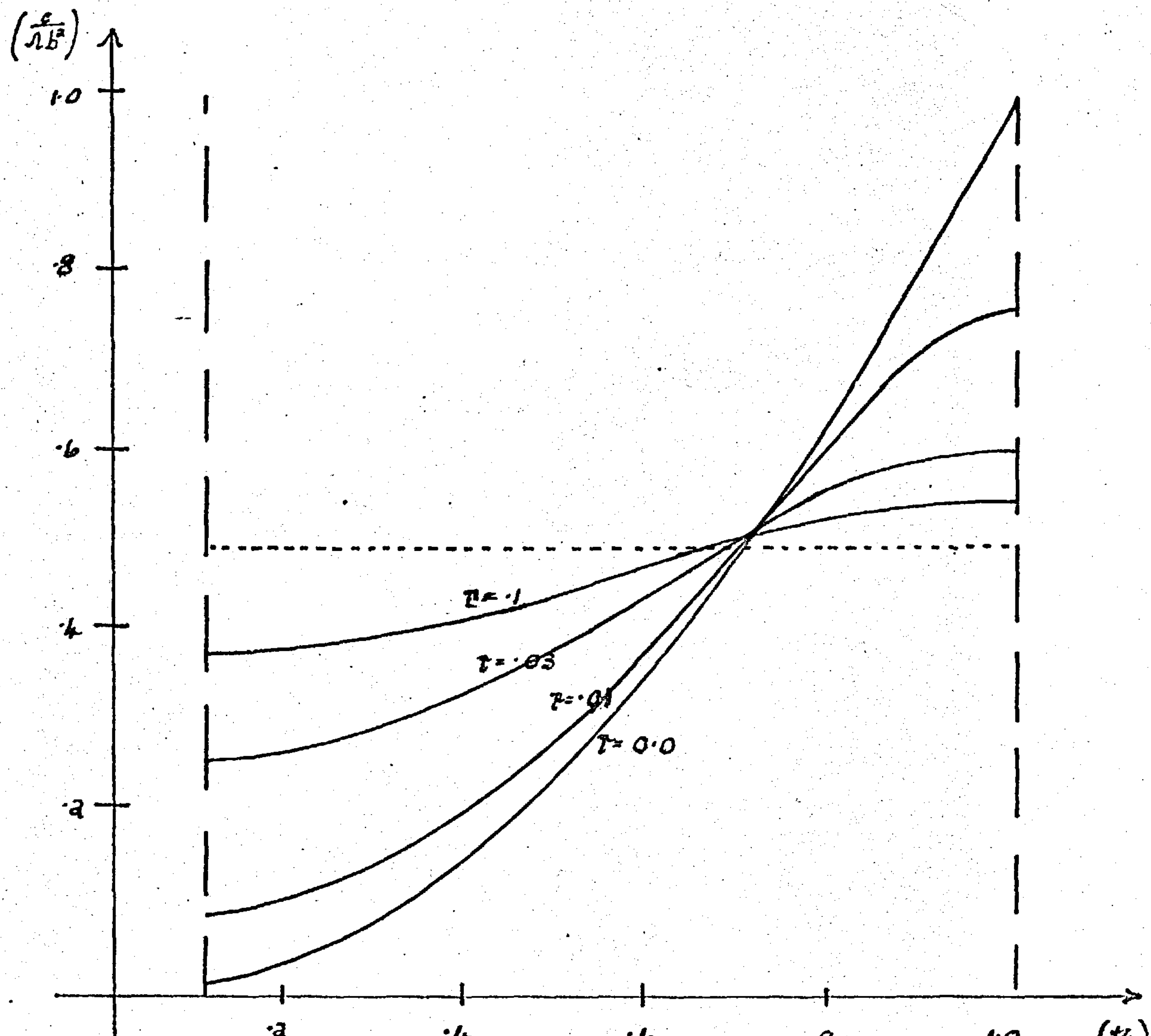
FIGURE (3).

# Growth Rate Diagrams.




Plot of dimensionless growth rate $\{4\sigma_i (1-P)^{-1}\}$ against azimuthal wave number $(m)$ for given value of '$a$' and various values of $\Delta = (b-a)$.

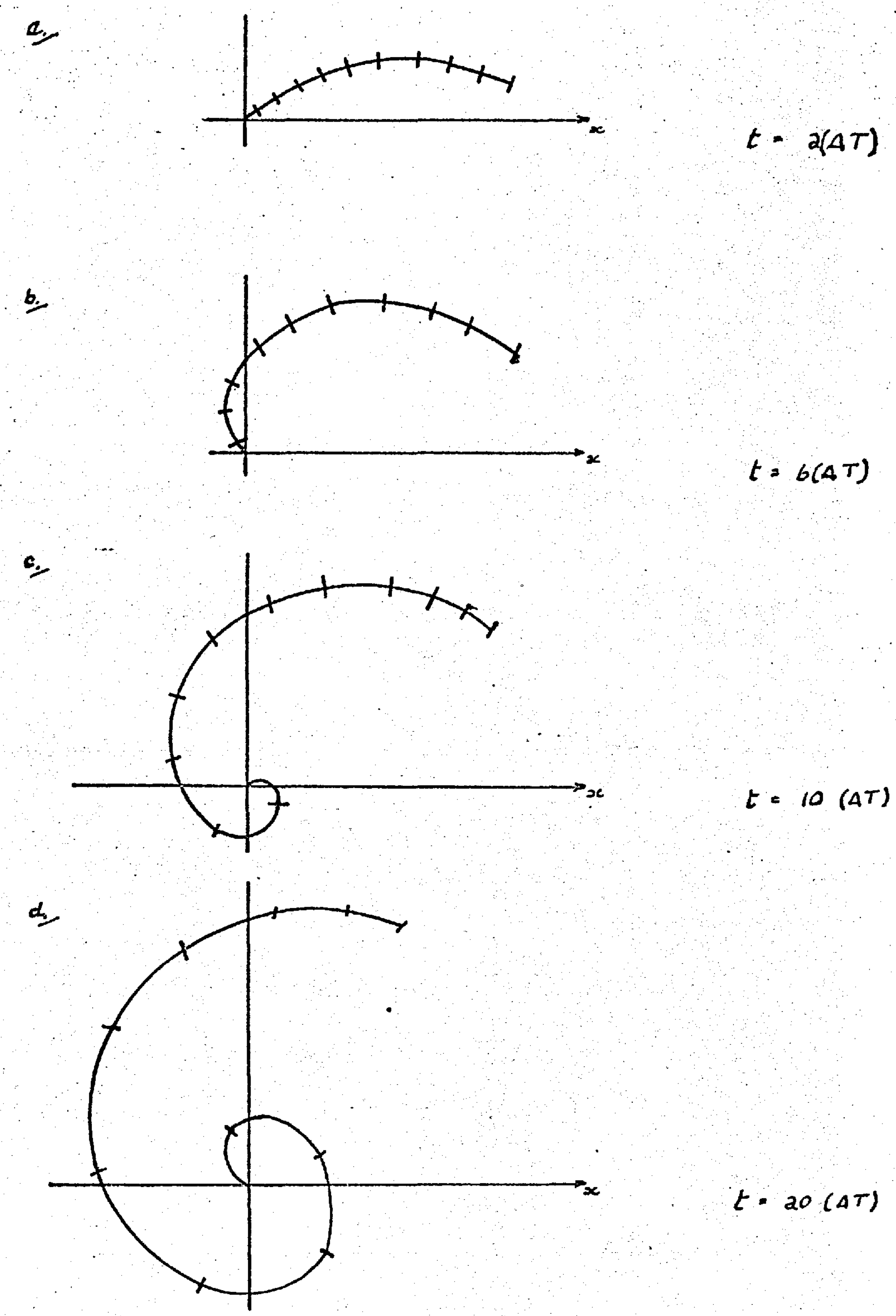
Angular Momentum Redistribution.




Plot of angular momentum against radial distance in the annular region $0{\cdot}1 \le (r/b) \le 1{\cdot}0$ at various times $(\tau)$.

FIGURE 5.

SUCCESSIVE POSITIONS OF VORTEX SHEET (HALF STRIP).




Cosine variation in strength (s) and a type 'a' representation.

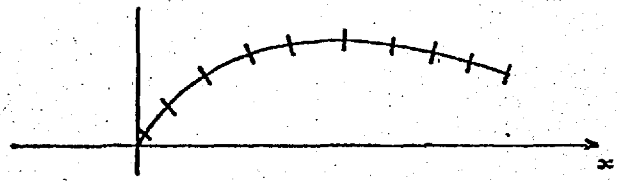
$$s = 2\left(1 + \cos \pi x/4\right)$$

Time step (ΔT) of computation = 0·025.

FIGURE (6).

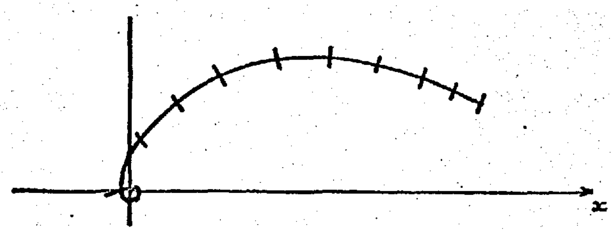
SUCCESSIVE POSITIONS OF VORTEX SHEET (HALF STRIP).

a.



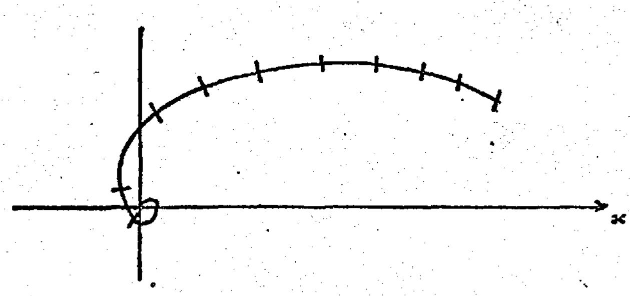

$t = 32(\Delta T)$

b.



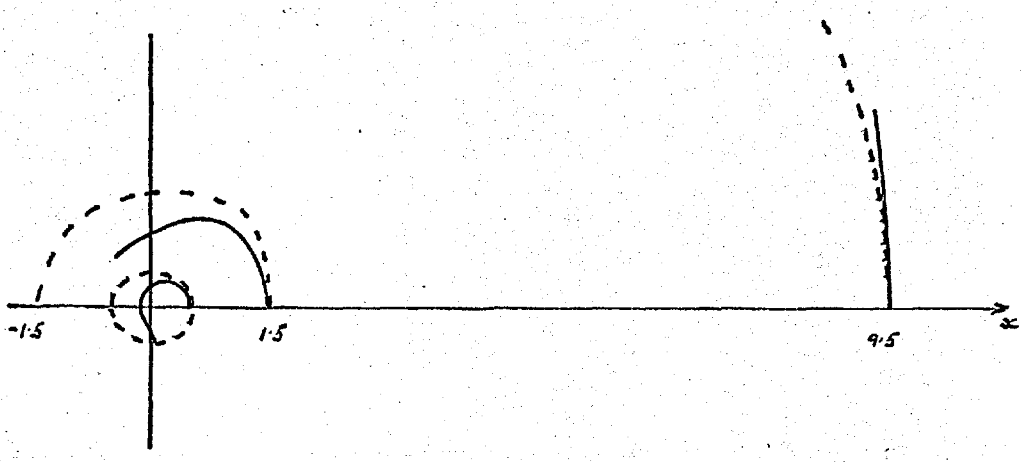

$t = 48\ (\Delta T)$

c.




$t = 40\ (\Delta T)$

d. TRAJECTORIES OF INNER AND OUTER VORTICES.



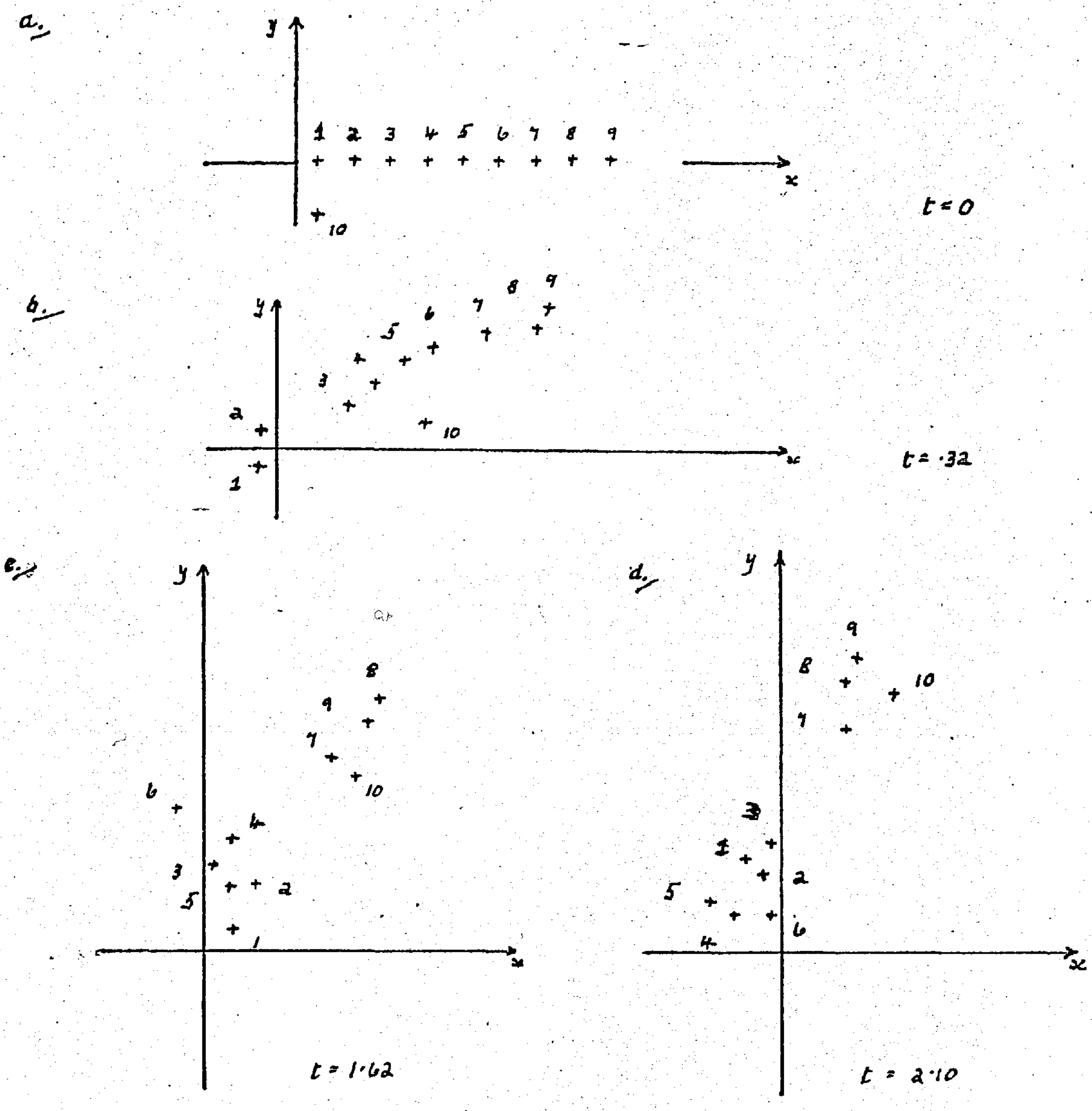

Cosine variation in strength (s) and type 'b' representation.

$$s = 2\left(1 + \cos \pi x/10\right)$$

Time step $(\Delta T)$ of computation = ·012

FIGURE (7).

FLOW EVOLUTION OF A POINT VORTEX DISTRIBUTION FOR A TYPE 'B' EXPERIMENT.




Strength of point vortices.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 4·7 | 4·7 | 4·65 | 4·65 | 4·2 | 3·8 | 3·2 | 2·8 | 2·4 | 2·0. |

FIGURE (8).