OBSERVER THEORY AND CONTROL SYSTEM DESIGN

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ABSTRACT

This thesis deals with the control by feedback of linear time invariant dynamic systems of finite order in a deterministic environment. The complete state is assumed not to be available for measurement and two main solutions to the problem are examined. A minor solution applies when the unmeasured states correspond to actuator dynamics, which are ignored in the design of a Lyapunov type controller. Conditions are found for which the closed loop system containing the actuator dynamics is stable. The alternative approach of constructing estimates of the unmeasured states is reviewed and extended. An improved method of designing reduced observers is presented and the existence of degenerate observers having arbitrary feasible poles is investigated. The effect on quadratic performance indices of using state estimates in otherwise optimal control laws is examined and two useful theorems are found.

A design method is proposed that generates stable compensators for stable plant. A third approach to the problem of incomplete state measurement seeks optimal feedback gains for the measured plant outputs, and a new algorithm for finding the optimum is

discussed. A new form of reduced model, the parallel path model, is considered.

Finally it is shown that a combination of model reduction, observer theory and constrained gain optimization is a very effective approach to the design of linear control systems.

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CHAPTER 1

INTRODUCTION TO OBSERVER THEORY AND CONTROL SYSTEM DESIGN

1.1 Introduction

Over the past thirty years a classical theory of control system design has developed founded on single input transfer functions. Its main quantitative tools are root locus techniques, Nyquist diagrams, Bode plots [H3] and finally Wiener's spectral factorization [W1] approach to the design posed as an optimization problem. This body of theory very adequately deals with the design of single input systems, but all techniques except the last are graphical. Multi-input systems tend to be too complex for pencil and paper graphical methods to be useful, and for any reasonably sized plant a computer must be used to obtain quantitative results. Unfortunately, digital computers are ill-suited to graphical techniques and a more appropriate approach has been found in the state space description of dynamic systems [Z1].

For systems of finite order the relevant tools for control design within the framework of state space have been found to be modal control and optimal control with quadratic costs, and a fairly complete theory has been developed in both fields [R1, W4, W3]. An obstacle to the application of these theories is that they lead to control laws which require measurements of the complete plant state vector. A solution to this difficulty is to use estimates of the unmeasured states. These may be optimal estimates provided by a Kalman filter [K3] for a Gaussian white noise environment or estimates provided by a Luenberger observer [L1, L2] for the deterministic case.

This thesis follows the spirit implied by the use of root locus, Bode and Nyquist plots in that consideration of noise in the dynamic system to be controlled is suppressed during the calculation of a trial feedback law. Throughout the thesis we consider time invariant linear plant of finite order and examine methods of finding trial control laws that perform adequately in a deterministic environment, though in practice a trial law may be rejected because of its behaviour in a particular stochastic environment. The methods we have investigated and improved upon have the common feature of avoiding graphical techniques and of being readily implemented on a digital computer.

1.2 Outline of Thesis

Chapter 1 serves as a general introduction and explains the motivation for this research. The unconstrained quadratic cost optimal control problem is introduced in Chapter 2 and because time invariant feedback laws are sought, we deal only with the infinite time problem. Because of the indirect relation between the specified costs on state and control and the characteristics of the corresponding optimal closed loop system, we examine two approaches connecting cost functions and closed loop poles. The first uses an exponential time weighting on the costs, the second is the implicit modelling method of specifying a cost function such that closed loop poles approximate desired values. In Chapter 2 we also discuss control laws based on Lyapunov functions of stable plant and show that this type of law provides a partial solution to the problem of unmeasured states.

The theory of Luenberger observers is reviewed in Chapter 3 and an improved design technique for reduced observers is presented. An extension of the theory to degenerate observers is developed in Chapter 4. Chapter 5 gives a somewhat academic solution to the problem of designing stable compensators using observer theory.

The use of state estimates from an observer in an otherwise optimal control law is examined in Chapter 6 with respect to the effect it has on the relevant quadratic cost function.

In Chapter 7 we discuss the common approach to designing controllers with incomplete state feedback of hill-climbing on the parameters of a fixed feedback structure for a suitably defined cost function. We show that the optimal solution has a well-defined structure, which suggests a method of obtaining the solution. Conditions are found under which the method quickly converges to the optimum. The idea of a parallel path simplified dynamic model is discussed and it is shown how the methods of model reduction, observer theory and hill-climbing on a fixed structure may be combined in an effective technique for the design of linear time-invariant multi-input plant.

1.3 Contributions of the Thesis

The main contributions of the thesis are the following.

The Riccati solution in Chapter 2 of the unconstrained optimal control of linear systems with time exponentially weighted quadratic costs on state and control, is original. Also the technique whereby dynamic controllers produced by the implicit modelling method are reduced to proportional feedback controllers is new. The stability theorems concerning Lyapunov controls which enter the plant through error or actuator dynamics are believed to be original.

An improved method of designing reduced order observers is presented in Chapter 3, and the theory of degenerate observers in Chapter 4 is new, as is the theory of stable compensators derived from observers in Chapter 5. The theorems of Chapter 6 describing the deterioration of system performance when state estimates are used in otherwise optimal control laws are original.

In Chapter 7 the extension of Kleinman's algorithm [K8] for solving the steady state Riccati equation to the problem of finding optimal gains for incomplete state feedback is new. To the author's knowledge, reduced models, in the form of parallel path models, have not appeared in the literature before despite their advantages in some situations.

CHAPTER 2

SELECTION OF CONTROL LAWS ASSUMING ALL PLANT STATES ARE MEASURED

2.1 Introduction

In this chapter we briefly discuss three methods of choosing time invariant feedback control laws for the regulation of multi-input, linear, time-invariant plant when all the plant states are measured. The three methods are pole-shifting designs, optimal control with quadratic costs and a control based on Lyapunov functions of stable plant. These objectives lead to important state space control design techniques because the corresponding feedback laws have analytic or easily programmed computer solutions. Furthermore, multi-input plant are handled as easily as single input, which is an advantage over classical design techniques based on Bode, Nyquist and root locus diagrams. However, frequency domain concepts remain a valuable tool for the evaluation of designs obtained from state space methods, particularly in the consideration of the effect of additional dynamics in a plant transmission path.

The plant dynamics are assumed to be adequately described by an n-state, r-input system

$$\dot{x} = Ax + Bu , \tag{2.1.1}$$

and the control laws are of the type

$$u = Kx . \tag{2.1.2}$$

The following definitions are important in linear system theory.

Definition 2.1.1

The dynamics (2.1.1), or equivalently the pair (A, B), is controllable [K2] if the (n, n.r)-matrix defined to be

$$(B, AB \dots A^{n-1} B) \tag{2.1.3}$$

has rank n.

Definition 2.1.2

The pair (A, B) is defined to be stabilizeable [W3] if the subspace of state space not spanned by the columns of the array (2.1.3) is stable. This subspace ~~is invariant under A and~~ is called the uncontrollable subspace.

Definition 2.1.3

A set of poles will be called feasible if complex poles occur as conjugate pairs.

The following concepts are fundamental in relating the state vector x to the output m-vector,

$$y = Hx. \tag{2.1.4}$$

Definition 2.1.4

The dynamics (2.1.1), (2.1.4) or equivalently the pair (A, H), is defined to be observable if the pair (A', H') is controllable.

Definition 2.1.5

The pair (A, H) is defined to be detectable [W3] if the pair (A', H') is *stabilizeable*.

2.2 Pole Shifting Control Laws

The plant dynamics (2.1.1) are regulated to the origin by the control law (2.1.2). Rosenbrock [R4] in an early paper suggested a method of altering r poles of the plant when r independent inputs are available. The method may be applied when only r plant outputs are measured, but then has the disadvantage that the remaining (n-r) poles move in an uncontrolled manner as the r selected poles are moved to desirable locations.

Kalman [K6] has obtained a more general result from his theory of canonical forms of single input systems, namely that if the pair (A, B) is controllable, a feedback control (2.1.2) exists that gives arbitrary specified feasible poles to the closed loop system. Anderson and Luenberger [A1] extend the result to multi-variable systems with a proof by construction. Simon [S3] discusses the problem thoroughly and gives a useful and computationally efficient technique for updating the control law as poles are moved singly or as complex pairs. Wonham [W4] and Rosenbrock [R1] give a general and complete analysis of the problem. Mayne and Murdock [M3] give an alternative computational procedure for the progressive design of feedback gains to achieve arbitrary feasible poles.

The location of the specified poles is best determined in an application by trial and error based on an output transient response criterion. An advantage of this approach to feedback design is that the generated eigenvalue and eigenvector data allows an eigensystem first and second order perturbation analysis [V3, N4, D3] to be made with very little extra computational effort.

Apart from Rosenbrock's approximate theory [R4] and Simon's [S3] method of eliminating feedback from (r-1) states if r inputs are available, the theory requires feedback from all states of the plant. Usually not all plant states are available the control laws are implemented by feeding back estimates of plant states. This introduces a dynamic element such as a Kalman filter [K3] or an observer in the feedback path; the theory of observers is developed in Chapter 3.

2.3 The Unconstrained Quadratic Cost and Linear Dynamics Optimisation Problem

Often feedback control is designed to optimise a performance criterion of the closed loop plant. The optimisation problem may be posed within a deterministic or stochastic framework, but we will restrict consideration to deterministic problems. Popular performance indices have been based on the plant output error in following a step input in desired output, and have included integral error squared, integral of the absolute value of the error and weightings of the cost integrand with powers of time,

The minimisation of integral error squared is mathematically the most tractable and is the index most often considered in analytical design techniques. It has a particularly convenient formulation and solution in the state space domain.

Consider the case where the dynamic equations for the n-state, r-input and m-output linear plant may be put in the form

$$\dot{x} = Ax + Bu , \qquad (2.3.1)$$

by a suitable choice of co-ordinate system for control and state space. The usual problem formulation is to require that the plant

(2.3.1) be regulated to the origin, and a performance index J that is useful is ,

$$J \quad = \quad x'(t_f)Q_f\, x(t_f) + \int_{t_o}^{t} (x'Qx + u'Ru)dt \qquad (2.3.2)$$

In general the matrices A, B, Q and R are time-varying, but we will be interested in the design of time-invariant feedback controls for time-invariant plant, so that these matrices will be assumed to be constant. In addition t_f is assumed to be infinite and t_o zero. The performance index for closed loop stable systems becomes

$$J \quad = \int_o^\infty (x'dx + u'Ru)\, dt\ . \qquad (2.3.3)$$

Kalman [K1] has shown that the control that minimises J exists and is generated by a constant stable feedback law

$$u \quad = \quad - R^{-1} B'Px\ , \qquad (2.3.4)$$

and

$$J \quad = \quad x'(o)\, Px(o)\ , \qquad (2.3.5)$$

where P is the steady state solution to the Riccati equation

$$\dot{P} \quad = \quad Q + A'P + PA - PBR^{-1}B'P\ , \qquad (2.3.6)$$

if the pair (A, B) is controllable and the pair (A, H) is observable where H is any solution to

$$H'H = Q , \tag{2.3.7}$$

and

$$Q > 0 ,$$

$$R > 0 .$$

Wonham [W3] has relaxed the existence conditions to the requirement that the pair (A, B) be stabilizeable ~~and the pair (A, H) detectable~~.

Evaluation of Steady State Solution of the Time Invariant Riccati Equation

Direct evaluation of the steady state solution of (2.3.5) by integration is often inefficient in computing time. This has been noted by Kalman [11] who suggested using the canonical pair of linear equations associated with the Riccati equation (2.3.6)

$$\lambda = Px , \tag{2.3.8}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{pmatrix}\begin{pmatrix} x \\ \lambda \end{pmatrix} . \tag{2.3.9}$$

The transition matrix φ of (2.3.9) is calculated for a convenient period T and partitioned into blocks of dimension (n, n)

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} .$$

The steady state solution P of (2.3.6) is given by the limit as k tends to infinity of the recursive relation

$$P_o \;=\; \text{any P.S.D symmetric matrix},$$

$$P_{k+1} \;=\; (\varphi_{21} + \varphi_{22}P_k)(\varphi_{11} + \varphi_{12}P_k)^{-1} \; . \qquad (2.3.10)$$

This algorithm replaces the integration of the (n, n)-dimensioned matrix equation (2.3.6) over an 'infinite' time by the integration of a (2n, 2n)-dimensioned matrix equation derived from (2.3.9) over any convenient period T. In most cases the inversion of (2.3.10) is less time consuming than integrating (2.3.6) over a period T and the overall algorithm should be more efficient than direct integration. Potter [P3], MacFarlane [M4] and Vaughan [V1] have analysed the behaviour of the canonical system (2.3.9) in terms of its eigenvalues and eigenvectors. Assuming a well-defined problem, the poles of (2.3.9) are symmetric with respect to the imaginary axis in the complex plane, the stable poles being those of the closed loop system (2.3.1) with optimal feedback (2.3.4).

Because the algorithm (2.3.10) or integration of (2.3.6) corresponds to backward time trajectories of (2.3.9), the stable modes dominate and the asymptotic relation (2.3.8) is merely a statement that (x, λ) belong to the stable invariant subspace of (2.3.9) parameterised by x. This suggests a way of obtaining P with computational effort on integration replaced by effort on eigenvector determination. Blackburn and Bidwell [B7] report that numerical results using this method are very promising. Blackburn [B5] has analysed a Newton-Raphson iterative solution for P, but the method suffers the disadvantage of dealing with large Jacobian matrices.

An appealing method of successive approximations in policy space [B8] has been proposed by Kleinman [K8]. The algorithm is guaranteed to produce monotonic convergence and has quadratic convergence near the optimum. A minor disadvantage of the method is that a stable control is required as an initial approximation for the first iteration. If the plant's rightmost poles are on the imaginary axis the algorithm can be started by replacing the plant equations (2.3.1) by

$$\dot{x} = (A - \alpha I)x + Bu , \qquad (2.3.11)$$

with

$$0 < \alpha \ll 1 .$$

Zero feedback will then be a stable control law for this auxiliary system. At or near the optimum of the auxiliary system, its closed loop poles may be expected to be to the left of $-\alpha$ in the complex plane so that the current control for the auxiliary system would be stable for the plant. Also if α is small the optimal control for the auxiliary system will equal the optimal control of the plant to first order.

Finally we note that the first solution to the minimisation of performance index (2.3.3) was proposed by Wiener [W1] for the single input case and involved spectral factorisation of a scalar Fourier transform. Youla [Y1] and Davis [D1] give methods for the spectral factorisation of matrix Fourier transforms so that Wiener's approach may be used for the multi-input case also. However, the formulation of the problem as one of spectral factorisation is not conveniently solved by computer and in fact the steady state solution of the steady state Riccati equation has been proposed as a practical method of matrix spectral factorisation [A1] , [H1] .

2.4 Choice of Costs for the Optimisation Problem

2.4.1 Introduction

The choice of Q and R in the performance index

$$J = \int_0^\infty (x'Qx + u'Ru)dt , \qquad (2.4.1)$$

is itself a problem. Given a plant with m outputs y of interest,

$$\dot{x} = Ax + Bu , \qquad (2.4.2)$$

$$y = Hx , \qquad (2.4.3)$$

the desired closed loop control is rarely specified to minimise a given J, but rather in terms of some less quantifiable characteristics of the transient response, peak overshoots and constraints on the magnitude of control input. The usual procedure in the initial stages of a control design assuming all states are measured, is to choose a Q and R more or less arbitrarily and find the optimal control and corresponding closed loop plant trajectories from a given initial condition or step disturbance on desired output. The cost on control is adjusted in the light of these trajectories to more heavily or lightly penalise controls that were above or well under the specified constraints. Similarly, adjustment of Q may give more desirable characteristics of the closed loop transients, but it is conceptually more difficult to see what adjustment should be made to achieve a certain effect. Some progress in this problem of the choice of Q has been made by examining the behaviour of the closed loop poles as elements in Q are varied. This approach of using root locus techniques to design optimal systems was first proposed by Chang [C3], who considered single input systems with a single costed output, and later discussed by Kalman [K1].

Tyler and Tuteur [T1] present a useful analysis of what happens to optimal closed loop multi-input system poles as single elements of Q are varied, but emphasis is placed on diagonal Q. The results of Sections 2.4.2 and 2.4.3 are original contributions that give additional guidance for allocating suitable values to Q.

2.4.2 Exponential weighting of costs

A more indirect method for achieving a desired characteristic of the closed loop optimal system poles has been independently found by the author, though the result has been found elsewhere [A4]. The result will be presented in the form of a Theorem.

Theorem 2.4.1

The control which minimises the performance index

$$J = \int_0^\infty e^{2\lambda t}(x'Qx + u'Ru)\, dt , \tag{2.4.10}$$

$$\lambda > 0 , \tag{2.4.11}$$

$$Q = H'H , \tag{2.4.12}$$

$$\dot{x} = Ax + Bu , \tag{2.4.13}$$

exists and is generated by a time invariant feedback law

$$u = Kx , \tag{2.4.14}$$

if the pair (A, B) is controllable and the pair (A, H) is observable. The optimal J is finite and the closed loop poles of (2.4.13), (2.4.14) are to the left of $-\lambda$ in the complex plane. The above conditions may be relaxed to a requirement that the pairs (A ,B) and (A , H) be respectively stabilizeable and detectable.

Proof

Make the transformation

$$z = e^{\lambda t}x , \tag{2.4.15}$$

$$v = e^{\lambda t}u , \tag{2.4.16}$$

so that

$$\dot{z} = \lambda e^{\lambda t}x + e^{\lambda t}\dot{x}$$

$$= \lambda e^{\lambda t}x + e^{\lambda t}(Ax + Bu)$$

$$= (A + \lambda I)z + Bv \ . \qquad (2.4.17)$$

The performance index J becomes

$$J = \int_0^\infty (z'Qz + v'Rv)dt \ . \qquad (2.4.18)$$

The minimisation of J in (2.4.18) with respect to v and subject to the dynamics (2.4.17) is achieved with a constant feedback law

$$v = Kz \ , \qquad (2.4.19)$$

if $((A + \lambda I),B)$ and $((A + \lambda I),H)$ are stabilizeable and detectable pairs [W3] .

A sufficient condition for the above requirement is that (A, B) and (A, H) be controllable and observable pairs, for consider the arrays

$$(B,\ AB,\ A^2B,\ \ldots\ A^{n-1}B)\ , \qquad (2.4.20)$$

$$(B, (A + \lambda I)B,\ (A + \lambda I)^2B,\ \ldots\ (A + \lambda I)^{n-1}B). \qquad (2.4.21)$$

The pair $((A + \lambda I), B)$ is stabilizeable by definition if it is controllable and therefore if the array (2.4.21) has full rank. However, the array (2.4.20) may be constructed from the array (2.4.21) by taking appropriate linear combinations of columns, so that the arrays have the same rank. The pair (A, B) is controllable by assumption so that (2.4.20) and hence (2.4.21) has full rank.

The optimal feedback K in (2.4.19) is given by

$$
\begin{aligned}
P &> 0 , \\
K &= -R^{-1}B'P , \\
0 &= Q + (A + \lambda I)'P + P(A + \lambda I) - PBR^{-1}B'P \qquad (2.4.22)
\end{aligned}
$$

Then J is finite and

$$
\begin{aligned}
J &= z'(o)\, Pz(o) \\
&= x'(o)\, Px(o) . \qquad (2.4.23)
\end{aligned}
$$

(2.4.23) follows from the transformation (2.4.15).

At any time t, the optimal control is

$$
\begin{aligned}
u &= e^{-\lambda t} v \\
&= e^{-\lambda t} Kz
\end{aligned}
$$

$$= \quad e^{-\lambda t} K \, e^{\lambda t} x$$

$$= \quad Kx \, , \tag{2.4.24}$$

so that the optimal control for the performance index (2.4.10) is generated by a constant feedback law.

Finally, the minimisation of J given by (2.4.18) subject to (2.4.17) with the control law (2.4.19) means that $(A + \lambda I + BK)$ is stable [K1] . Adding λI to a matrix corresponds to moving the origin an amount $(-\lambda)$ in the complex plane plot of matrix eigenvalues, so that the optimal closed loop dynamics (2.4.13), (2.4.24), that is ,

$$\dot{x} \quad = \quad (A + BK)x \, , \tag{2.4.25}$$

has poles to the left of $-\lambda$ in the complex plane.

This result is expected because otherwise the integral (2.4.10) would not exist.

Comment

The steady state Riccati equation (2.4.22) shows that the solution of the minimisation of J (2.4.10) is identical to that of

$\underline{J}$,

$$\underline{J} \triangleq \int_0^\infty (x'\underset{\sim}{Q}x + u'Ru)\,dt \ , \tag{2.4.26}$$

where

$$\underset{\sim}{Q} \triangleq (Q + 2\lambda P) \ . \tag{2.4.27}$$

The control law

$$u = Kx$$

is therefore optimal for an ordinary quadratic cost function which, however, is not known a priori.

One interpretation of the procedure is that the equivalent quadratic cost $\underset{\sim}{Q}$ in (2.4.26), (2.4.27) is automatically selected to achieve the desired effect of placing the closed loop poles.

2.4.2 Approximate Modal Control via Optimisation - Implicit Modelling Method

Because a great deal of experience is available of the effect of pole-zero closed loop configurations on system output transient behaviour an 'implicit modelling' method has been suggested by Bass and Gura [B1] , where the theory of optimal control and modal control merge. A cost function is selected for the single-input-single-output plant

$$\dot{x} = Ax + Bu , \tag{2.4.28}$$

$$y = Hx , \tag{2.4.29}$$

with performance index

$$J = \int_0^\infty (xQx + uRu + \|Cx + Eu\|^2)dt , \tag{2.4.30}$$

so that the closed loop poles approximate desired prespecified positions.

Schultz and Melsa [S2] discuss the method in detail, again for single-input-single-output plant. We give an outline of the argument.

First consider the case of a plant where the transfer function from u to y contains no zeros and the set of stable desired closed loop poles is specified by the roots of a characteristic equation

$$s^n + \sum_0^{n-1} a_i s^i = 0 . \tag{2.4.31}$$

Because there are no zeros in the transfer function from u to y, the n^{th} derivative of y is the first that contains a term in u. Consider the performance index which therefore has the form (2.4.30),

$$J = \int_0^\infty ((y^{(n)} + \sum_0^{n-1} a_i y^{(i)})^2 + ru^2)dt \qquad (2.4.32)$$

Assuming that the plant is controllable a feedback control law K_M exists [W4] that realises the closed loop poles specified by (2.4.31). The performance index (2.4.32) for this law becomes

$$J = \int_0^\infty ru^2 \, dt$$

$$\triangleq J_M(r) ,$$

and as r tends to zero $J_M(r)$ tends to zero. For any value of r the optimal performance index $J^o(r)$ satisfies

$$J_M(r) > J^o(r) . \qquad (2.4.33)$$

Denoting the optimal law by $K^o(r)$, it follows that along the two different trajectories corresponding to the optimal and non-optimal laws beginning from the same initial condition,

$$\int_0^\infty r(K^o(r)x)^2 dt < \int_0^\infty r(K_M x)^2 dt . \qquad (2.4.34)$$

So that from (2.4.33) and (2.4.34) as r tends to zero, $J^o(r)$ tends to zero and therefore y and its derivatives tend to satisfy (2.4.31). It is also seen from (2.4.34) that the control cost of the optimal law is lower than the modal control cost at all times, but in the limit as r tends to zero, $K^o(r)$ tends to K_M and the control costs must approach the same value. The approach to modal control through optimization represents finding a compromise between the cost of deviation from achieving the desired closed loop poles and the cost on control.

When the transfer function between u and y contains zeros a lower derivative of y than the n^{th} is the first to contain a term in u. In this case derivatives of u appear in the performance index (2.4.32). This type of performance index has been considered by Moore and Anderson [M6] who show that it may be treated in the usual way by considering the highest derivative of u appearing in the integrand as a new control variable, and augmenting the system state with u and its lower derivatives. If the original system is controllable from u, the augmented system is controllable from the highest derivative of u. This introduces extra system poles and asymptotically as the cost on control is reduced, the poles of the closed loop tend to coincide with the zeros of the costed output [B1] , [K1] , [T1] , and the remaining poles tend to a Butterworth configuration.

The same considerations apply when the reference characteristic equation (2.4.31) has degree less than n. We may avoid introducing derivatives of u by choosing a performance index

$$J = \int_0^\infty ((y^{(p)} + \sum_0^{p-1} a_i y^{(i)})^2 + ru^2)dt \,, \qquad (2.4.35)$$

appropriate to a reference characteristic equation

$$s^{(p)} + \sum_0^{p-1} a_i s^i = 0 \,, \qquad (2.4.36)$$

$$0 < p < n \,.$$

The number p of specified desired closed loop poles may be restricted so that no derivatives of u appear in (2.4.35).

Alternatively, derivatives of u may be avoided by considering a different plant output, but this is a somewhat artificial device if the purpose of the theory is to use experience of transient responses for simple pole-zero configurations to obtain desired responses for particular plant outputs.

Till now, the discussion has been restricted to single-input-single-costed-output systems. The same ideas are applicable for multi-input plant and the potential advantages of the method then become apparent. Simple modal control of multi-input plant [S3] operates the inputs in fixed ratios so that

$$u = u_o v ,$$

where v is a scalar and the ratio vector u_o is such that the pair (A, Bu_o) is controllable. This leads to a gain matrix

$$u = Kx ,$$

$$K = u_o k' ,$$

where k is such that $(A + Bu_o k')$ has specified poles. In this case K has unit rank and would lead to a more constrained control action than if K had full rank. This follows because k measures the component of the plant state along one direction, and therefore the feedback control utilises information about one component in state space alone. If K had full rank r, information about the state in an r-dimensioned subspace would be used. It would appear that the full potential of a multi-input control is not realised if K does not have full rank. Full rank in K may be induced in the modal control design technique by an r stage process. At the i^{th} stage a control ratio vector $u_o(i)$ is assumed independent of $\{u_o(j) : 0 < j < i\}$, and a set of poles specified.

The plant matrix is taken to be

$$A(i) = A + \sum_{1}^{i-1} Bu_o(j)k'(j) ,$$

and a feedback row vector $k'(i)$ found so that $(A(i) + Bu_o(i)k'(i))$ has the specified poles. At the last stage the specified poles are those desired for the final closed loop system matrix $(A + BK)$, where K is likely to have full rank and is given by

$$K = \sum_{1}^{r} u_o(i)\, k'(i) .$$

This is a very indirect method and if it is desired that the degrees of freedom of a multi-variable system for achieving specified closed loop poles is to be used in obtaining some desired distribution of control effort among the inputs, it is not clear how it should be done this way.

The approach of optimal control is very well suited to solving this type of problem. A performance index may include costing on the plant inputs whether or not the plant inputs remain controls, or become states in an augmented system.

Consider the reference characteristic equation and associated performance index

$$s^p + \sum_{o}^{p-1} a_i s^i = 0 , \qquad (2.4.37)$$

$$J = \int_o^\infty (\| y^{(p)} + \sum_{o}^{p-1} a_i y^{(i)} \|^2 + u'Ru) \, dt . \qquad (2.4.38)$$

Suppose that p is such that no derivatives of u occur in (2.4.38), then the distribution of control effort in forcing the (multi-variable) output to approximately satisfy (2.4.37) may clearly be varied by varying the relative control costs in R. As the cost on an input is raised in (2.4.38), the corresponding gains in the optimal feedback law will decrease and the outputs will tend to deviate more from (2.4.37), but this tendency may be reduced and eventually reversed as the cost on the remaining inputs is decreased.

Interpretation of the situation is more complex when p in (2.4.37) is such that derivatives of u occur in (2.4.38). In that case the highest derivatives of u appearing in (2.4.38) must be regarded as system control variables, and the remaining derivatives regarded as being added components in an augmented state vector. Suppose that the augmented system has state vector $\underline{x}$ of dimension (n+q) and control vector $\underline{u}$. All elements of x will appear in $\underline{x}$, some or all of the elements of the plant input u appear in $\underline{x}$ and the remainder in $\underline{u}$.

The performance index (2.4.38) will have the form ,

$$J = \int_0^\infty \left(\|\underline{x}\|^2_{Q_1} + \|\underline{u}\|^2_{R_1} + \|C\underline{x} + D\underline{u}\|^2 \right) dt \; . \qquad (2.4.39)$$

Changes in costs on plant input are implemented by changes in the appropriate elements of Q_1 and R_1. We might expect by analogy with the single input case that as R_1 tends to zero, some of the system closed loop poles tend to those defined by (2.4.37). Tyler and Tuteur [T1] have analysed the movement of closed loop poles of optimal multi-variable systems as one element of the cost matrix varies, however we are also interested in a more detailed knowledge of the system behaviour with respect to the distribution of plant input effort. We recall that the system state vector has been augmented to dimension (n+q) and this scheme for approximately achieving specified values for some of the closed loop poles is not comparable with modal control, which does not alter the system dimension. We therefore move on to the next Section where it is shown that in simple cases specified poles and distribution of plant input effort may be approximately achieved and the system dimension is not changed.

2.4.3 An Extension to the Theory of Implicit Modelling

An original theory is presented for simple cases by which a constant feedback gain matrix is obtained for the given plant, such that specified closed loop pole locations and distribution of plant input effort are approximately achieved.

Suppose that the costing procedure of the previous section leads to a closed loop system of increased dimension (n+q) that has among its poles a set approximating the desired poles.

Suppose for simplicity that q=r and that the first derivative of each plant input occurs in (2.4.39). The optimal control is

$$\dot{u} = K\underline{x}$$

$$\triangleq (K_1 \; K_2) \binom{x}{u} \; . \tag{2.4.40}$$

We may select any set S of q elements of real and complex pair modes and obtain the corresponding set of reciprocal row eigenvectors for the closed loop augmented system. Suppose that these q row vectors are the rows of a matrix V, and V is partitioned

$$V = (V_1, V_2),$$

where V, V_1 and V_2 have dimensions (q, n+q), (q, n) and (q, q).

An initial condition $\binom{x}{u}$ of the augmented system has zero component of the q modes S, and lies in the invariant subspace corresponding to the complement of S if

$$(V_1\ V_2)\binom{x}{u} = 0 .$$

Then

$$u = -V_2^{-1}V_1 x , \qquad (2.4.41)$$

provided that the inverse exists.

However, if an initial condition of the augmented closed loop system lies in an invariant subspace, the state trajectory will remain in that subspace. Therefore (2.4.41) will hold along the complete trajectory if it holds for an initial condition.

We may regard (2.4.41) as a relation by which the n-vector x parameterises vectors $\binom{x}{u}$ of dimension (n+q) lying in an invariant subspace. Consider an initial condition x of the plant and consider a feedback law ,

$$\begin{aligned} u &= -V_2^{-1}V_1 x \\ &\triangleq Kx . \end{aligned}$$

The modes of the closed loop system matrix (A + BK) are then the modes of the closed loop augmented (n+q)-state system omitting the selected q modes S.

The situation is very similar in the general case. Suppose that the augmented system has partitioned state and control

$$\underline{x} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix},$$

$$\underline{u} = \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix},$$

where $\underline{x}_1 = x$, $\underline{x}_2$ contains some elements of u, say u_2, and derivatives of u_2. The elements of $\underline{u}_1$ are the remaining elements u_1 of u, and $\underline{u}_2$ is the group of the highest derivatives greater than zero appearing in (2.4.39). Let $\underline{x}_1$, $\underline{x}_2$, $\underline{u}_1$, $\underline{u}_2$ have dimensions n, q, r_1 and r_2.

Again we may select q modes S with the objective of discarding them from the augmented system dynamics by choice of an appropriate initial condition subspace. Suppose that the reciprocal row eigenvector matrix V is again partitioned $V = (V_1, V_2)$, so that initial conditions having zero S mode components satisfy

$$(V_1, V_2)\begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} = 0 .$$

Then by definition of $\underline{x}_1$,

$$\begin{aligned}\underline{x}_2 &= -V_2^{-1} V_1 \underline{x}_1 \\ &= -V_2^{-1} V_1 x ,\end{aligned} \tag{2.4.42}$$

if the inverse exists.

The optimal control

$$\underline{u} = \underline{K}\,\underline{x} ,$$

may be rewritten for $\underline{x}$ in the invariant subspace (2.4.42) as

$$\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix} = \begin{pmatrix} \underline{K}_{11} & \underline{K}_{12} \\ \underline{K}_{21} & \underline{K}_{22} \end{pmatrix} \begin{pmatrix} x \\ \underline{x}_2 \end{pmatrix} ,$$

so that

$$\underline{u}_1 = (\underline{K}_{11} - \underline{K}_{12} V_2{}^{-1} V_1) x . \tag{2.4.43}$$

Also, by definition, the group u_2 of plant inputs is contained in $\underline{x}_2$ so that for $\underline{x}$ lying in the subspace defined by (2.4.42)

$$u_2 = - H V_2{}^{-1} V_1 x , \tag{2.4.44}$$

where H is a selection matrix having each row zero except for a unit clement. Then (2.4.43) and (2.4.44) may be rewritten as

$$u = \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix}$$

$$= \begin{pmatrix} (\underline{K}_{11} - \underline{K}_{12} V_2^{-1} V_1) \\ (-HV_2^{-1} V_1) \end{pmatrix} x ,$$

that is

$$u \triangleq Kx . \tag{2.4.45}$$

For any initial conditions x on the plant we may choose a plant input u given by feedback law (2.4.45), and the corresponding (x, u) trajectories generated by the n-state plant dynamics will be identical to (x, u) trajectories of the (n+q) dimensioned augmented system. Furthermore, the poles of (A + BK) will be the poles of the augmented system apart from the rejected poles S.

It is clear that one of the difficulties of the approach to modal control via an optimization problem, namely an increase in the dimension of the dynamics, may be overcome by selecting a set of modes and rejecting them. The remaining difficulty concerns the distribution of control effort in the feedback law (2.4.45). The difficulty arises in the contraction of the dimension of the dynamics and the introduction of feedback terms involving functions of eigenvectors of closed loop systems, (2.4.43) and (2.4.44). In simple cases it is possible to predict the effect on these functions of making cost parameter changes in the elements of R_1 and Q_1 of (2.4.39) concerning costing on plant inputs.

Consider the particular simple situation where no derivatives of plant input higher than the first occur ,

$$J = \int_0^\infty (u'_2 R_2 u_2 + u'_1 R_1 u_1 + \dot{u}'_2 R_3 \dot{u}_2$$

$$+ (y^{(p)} + \sum_0^{p-1} a_i y^{(i)})^2) \, dt \qquad (2.4.46)$$

$$\triangleq \int_0^\infty (u'_2 R_2 u_2 + u'_1 R_1 u_1 + \dot{u}'_2 R_3 \dot{u}_2$$

$$+ \text{quad.fn.}(x, u_1, u_2, \dot{u}_2)) \, dt \, . \qquad (2.4.47)$$

Here the augmented system state variables are x and u_2, and control variables u_1 and $\dot{u}_2$. We assume that the plant output y is a scalar, though this is not basic to the argument.

Suppose that R_3 is small in (2.4.46) and that R_2 is large, so that large values of u_2 cause a rapid accumulation of cost in J. There will be a tendency for the optimal $\dot{u}_2$ to quickly reduce large values of u_2. If the cost on u_1 is small enough, u_1 may assume values such that the last term in (2.4.46) is small, and consequently the closed loop optimal system will have modes closely approximating those specified by

$$s^p + \sum_0^{p-1} a_i s^i = 0 \, .$$

Fast modes will be associated with the fast reduction of u_2 from large values, the speed depending on the relative costs in the integrand. Finally we might expect that in this case, the invariant subspace complement to these fast modes will have small u_2 values because otherwise u_2 would decrease quickly, indicating the presence of fast modes. Also the higher the cost on u_2, the smaller should be the magnitude of u_2 present in the slower modes.

Briefly summarising, we may expect that if augmented system controls are lightly costed, a group of closed loop system poles will approximate a set of desired poles and there will be q 'fast' poles. The dimension of the system may be reduced to n by suppressing the 'fast' modes so that a control law (2.4.45) is obtained for the plant inputs. The magnitude of the gains on individual plant inputs are adjustable by varying the cost weighting on them in the performance index (2.4.46).

This behaviour is clearly illustrated in Table 2.4.1 of the following example.

It is emphasised that a more careful analysis is required if higher derivatives of plant input occur in the performance index, and also if the performance index is in terms of a vector plant output, but the main ideas should carry over. Should a restriction to low order derivatives of plant input emerge for the performance index, it is considered that this is not of great importance. This is because the number of derivatives required depends on the number of specified poles; but transient responses are usually thought of in terms of a dominant complex pair, which would lead to the specification of two poles and to first order derivatives at most. We suggest that in most situations only two or three poles would be specified explicitly. Finally the control (2.4.45) is not optimal for the performance index (2.4.46). Optimization of J with respect to K for a given initial condition x is a non-quadratic problem and the optimal K will depend on the initial condition assumed. The design procedure should be regarded merely as a systematic method of shaping transient responses and apportioning control effort between plant inputs.

Example

Consider the plant shown in Figure 2.4.1,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{aligned}$$

The plant output to be controlled is x_1.

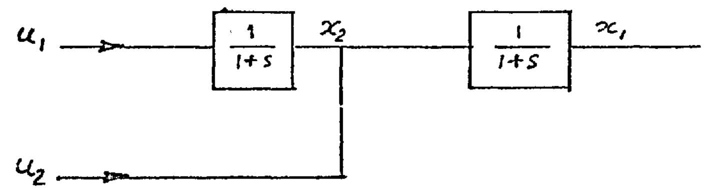


Figure 2.4.1 Plant dynamics of example.

Suppose that it is desired to achieve closed loop poles $(-2\mp 3i)$. This may be done by modal control using u_1 alone to give a unique control

$$u_1 = -10x_1 - 2x_2 ,$$

which will be called the modal control. The characteristic equation corresponding to the desired poles is

$$s^2 + 4s + 13 = 0 .$$

To apply the theory of Section 2.4 the performance index (2.4.46) is specified as

$$\begin{aligned} J &= \int_0^\infty ((\ddot{x}_1 + 4\dot{x}_1 + 13x_1)^2 + u'Ru)\, dt \\ &= \int_0^\infty ((10x_1 + 4x_2 + u_1 + 3u_2 + \dot{u}_2)^2 + u'Ru)\, dt . \end{aligned}$$

The corresponding augmented system is

$$\underline{x} = (x_1, x_2, u_2)' ,$$

$$\dot{\underline{x}} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \underline{u},$$

$$\underline{u} = (u_1, \dot{u}_2)' ,$$

with performance index

$$J = \int_0^\infty ((10\underline{x}_1 + 4x_2 + 3\underline{x}_3 + \underline{u}_1 + \underline{u}_2)^2 + r_1\underline{u}_1^{\,2} + r_2\underline{x}_3^{\,2})\, dt .$$

Scalars r_1 and r_2 control the costing on the plant inputs u_1 and u_2. Table 2.4.1 summarises the results of minimising J for a variety of costs on control. The achieved system poles are close enough to the specified values to verify the theory. It is interesting to note that the distribution of control effort, is strongly influenced by the costs on plant input in the way indicated by the theory.

Cost on control	Feedback $u = BG$	Eigenvalues of $(A + BG)$
$u_1^2 + u_2^2$	$\begin{pmatrix} -1.10 & -1.30 \\ -3.57 & 12.7 \end{pmatrix}$	$-2.13 \mp 3.02i$
$u_1^2 + 3.16u_2^2$	$\begin{pmatrix} -2.83 & .546 \\ -3.27 & 3.46 \end{pmatrix}$	$-2.36 \mp 3.00i$
$.1u_1^2 + u_2^2$	$\begin{pmatrix} -6.12 & -.706 \\ -2.46 & .763 \end{pmatrix}$	$-2.58 \mp 3.17i$
modal control $u_2 = 0$	$\begin{pmatrix} -10. & -2. \\ 0 & 0 \end{pmatrix}$	$-2.00 \mp 3.00i$

Table 2.4.1 Approximate modal control via an optimal control problem.

2.5 Lyapunov Control for Stable Plant

2.5.1 Introduction

We have already discussed modal and optimal controls for linear time-invariant plant. Lyapunov control is a third type of control law that is amenable to computer design techniques in the context of state space, because an exact non-iterative calculation gives a feedback law which is guaranteed to be stable. Like the previous laws, Lyapunov control requires *as defined later* feedback from all the system states.

Consider a stable linear plant with n-vector state x and r-vector input u ,

$$\dot{x} = Ax + Bu . \qquad (2.5.1)$$

Because A is stable, a Lyapunov function V(x) [H2] may be defined for any symmetric $Q > 0$,

$$A'P + PA = -Q , \qquad (2.5.2)$$

$$V(x) = x'Px . \qquad (2.5.3)$$

and

$$\dot{V}(x) = -x'Qx \qquad (2.5.4)$$

$$< 0 , \forall x \neq 0 .$$

If the control v is chosen as

$$u = - RB'Px , \tag{2.5.5}$$

for $R > 0$, $V(x)$ remains a Lyapunov function and

$$\begin{aligned} \dot{V}(x) &= - x'(Q + PBRB'P)x \\ &< 0 , \forall x \neq 0 . \end{aligned} \tag{2.5.6}$$

Definition 2.5.1

Feedback laws (2.5.5) for stable plant and obtained from Lyapunov functions, will be called Lyapunov controls.

Equation (2.5.2) is linear in the $\frac{n(n+1)}{2}$ distinct elements of P and may be solved directly. The number of independent unknowns has been shown by Barnett and Storey [B4] to be $\frac{n(n-1)}{2}$ and a reduced set of equations can be found to determine them. Jameson [J3] proposes an algebraic solution for P that is based on the annihilation properties of the matrix characteristic equation; the computational effort mainly involves obtaining n powers of n^{th} order matrices. Several approximate algorithms have been proposed, Smith [S1] uses a bilinear transformation to convert (2.5.2) to a discrete time analogy which is solved by evaluation of a finite number of terms of an infinite matrix power series. Davison [D4] has suggested that the transition matrix F of A corresponding to a small time interval Δt be used, so that

$$P = \int_0^\infty e^{A't} Q e^{At} dt$$

$$\approx \sum_0^\infty F'^i Q F^i \; \Delta t$$

$$\approx \sum_0^N F'^i Q F^i \; \Delta t \; ,$$

where N is sufficiently large. The power series is very efficiently evaluated by the recursive relations ,

$$F_1 = F \, ,$$

$$P_1 = Q \Delta t \, ,$$

$$F_{k+1} = F_k^{\,2} \, ,$$

$$P_{k+1} = P_k + F_k' P_k F_k \, ,$$

$$P = \lim_{k \to \infty} P_k \, .$$

In practice no more than 10 or 13 iterations are required, which corresponds to

$$N = 2^9 \text{ or } 2^{12} \, .$$

The author feels that for small systems no method shows distinct advantages except that convergence failure of a power series gives an automatic indication that the system is unstable. For $n > 10$ the iterative methods are probably preferable with respect to computing time and storage.

Unless otherwise stated R in (2.5.5) will be assumed diagonal. The closed loop dynamics (2.5.1) and (2.5.5) may be written as ;

$$\dot{x} = Ax + Bu \tag{2.5.7}$$

$$y = B'Px \tag{2.5.8}$$

$$u = -Ry \tag{2.5.9}$$

The variable y may be regarded as a system output independent of gain variations in R.

The development of equations (2.5.2) to (2.5.6) shows that the control law (2.5.9) is stable for all positive gains in the feedback loops, that is for all $R \geqslant 0$.

This type of control law has been proposed by Kalman and Bertram [K5] and more recently by Barnett and Storey [B4] , and it is well known that because the feedback law (2.5.8), (2.5.9) is stable for all diagonal $R \geqslant 0$, R may be a function of x and represent non-linear positive gains in the feedback paths from the selected measured output y of (2.5.8) to the plant input.

In theory the closed loop plant remains stable as the gains R tend to infinity; the characteristics of the closed loop plant under these conditions are examined in the next Section. Section 2.5.3 presents an investigation of the effect on system stability of the presence of actuator dynamics that are absent when the Lyapunov control law is obtained.

2.5.2 Characteristics of High Gain Lyapunov Control

Lyapunov controls remain stable as the gains R tend to infinity, and it is interesting to analyse the corresponding behaviour of the closed loop poles.

It is convenient to transform the plant state space so that the dynamics

$$\dot{x} = Ax + Bu$$

have the form ,

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,u$$

$$\triangleq \begin{pmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} u \, . \qquad (2.5.10)$$

The partitioning of (2.5.10) is defined by the required form of $\underline{B}$. Again for convenience we will drop the bar under the variables in (2.5.10) and refer to $\underline{x}$, $\underline{A}$, $\underline{B}$ as x, A, B, so that the plant dynamics become

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} u \ . \qquad (2.5.11)$$

P and Q of (2.5.2) are partitioned consistently with (2.5.11) so that a Lyapunov control for (2.5.11) becomes

$$\begin{aligned} u &= -RB'Px \\ &= -R(P_{11}x_1 + P_{12}x_2) \ . \qquad (2.5.12) \end{aligned}$$

The corresponding closed loop dynamics become

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}-RP_{11} & A_{12}-RP_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ . \qquad (2.5.13)$$

We use the following notation for conciseness.

<u>Notation</u> The set of eigenvalues of matrix X will be denoted by $\Lambda\{X\}$.

We note that $P > 0$, so that $P_{11} > 0$, $P_{11}^{\frac{1}{2}} > 0$ and therefore $P_{11}^{-\frac{1}{2}}$ exists. Also

$$\Lambda\{RP_{11}\} = \Lambda\{P_{11}^{\frac{1}{2}}RP_{11}^{\frac{1}{2}}\} \ ,$$

so that $\Lambda\{-R\ P_{11}\}$ are real and tend to $-\infty$ as $R > 0$ tends to ∞. Therefore by Gershgorin's theorem [G1] $\Lambda\{A_{11}-R_1\}$ tends to $-\infty$.

For $\|R\| \gg \|A_{11}\|$, $\|A_{21}\|$, $\|A_{22}\|$ the dynamics (2.5.13) satisfy the conditions for a weakly coupled system [M2], so that as R tends to infinity the closed loop eigenvalues tend to $\Lambda\{(A_{11}-R\ P_{11})\}$ and $\Lambda\{A_{22} - A_{21}(A_{11}-R\ P_{11})^{-1}(A_{12}-R\ P_{12})\}$, that is to $-\infty$ and $\Lambda\{A_{22} - A_{21}P_{11}^{-1}P_{12}\}$. The poles of $(A_{22} - A_{21}P_{11}^{-1}P_{12})$ are independent of R for large R, but must be stable because the Lyapunov control is stable for all R. The poles may or may not be slow, they are certainly finite and we cannot expect high gain Lyapunov control to give arbitrarily fast system dynamics.

2.5.3 Stability of Lyapunov Controls for a Class of Errors in Plant Dynamics

We have seen that Lyapunov control of the plant dynamics,

$$\dot{x} = Ax + Bu \tag{2.5.14}$$

$$y = B'Px \tag{2.5.15}$$

$$u = -Ry, \tag{2.5.16}$$

is stable for all linear and non-linear $R \geqslant 0$. We continue to consider diagonal R and examine the effect of introducing dynamics in the feedback path so that

$$u(s) = -RE(s)\ y(s), \tag{2.5.17}$$

where $E(s)$ is a diagonal transfer function matrix having a finite number of poles.

The transfer function matrix $E(s)$ may occur as actuator dynamics in applying the desired plant input $u = -Ry$, or as a representation of an assumed form of error dynamics relating a process model (2.5.1) to the true process dynamics.

Definition 2.5.2

A stable transfer function having a finite number of poles and lying in the fourth quadrant of the complex plane will be called a fourth quadrant transfer function. A matrix of such transfer functions as elements will also be called fourth quadrant.

The main result of this Section is the following theorem.

Theorem 2.5.1

The dynamics (2.5.14), (2.5.15) and (2.5.17) are stable for all diagonal $R \geqslant 0$, and all diagonal fourth quadrant $E(s)$.

The proof of Theorem 2.5.1 depends on several lemmas.

Definition 2.5.3

The return difference transfer function matrix $T(s)$ of a system

(2.5.14) and

$$u = -Kx, \tag{2.5.18}$$

is defined

$$T(s) = I + K(sI - A)^{-1}B. \tag{2.5.19}$$

Lemma 2.5.1

For any A, B and K the set of closed loop poles, $\Lambda\{A - BK\}$ of the dynamics (2.5.14) and (2.5.18) is contained in the union of the set of zeros of $|T(s)|$ and $\Lambda\{A\}$.

Proof

The set of closed loop poles, $\lambda(A - BK)$ equals the set of zeros of $|sI - (A - BK)|$. However, we may write,

$$\begin{aligned} |sI - (A - BK)| &= |(sI - A)(I + (sI - A)^{-1}BK)| \\ &= |sI - A| \cdot |I + (sI - A)^{-1}BK| \\ &= |sI - A| \cdot |I + K(sI - A)^{-1}B| \\ &= |sI - A| \cdot |T(s)|. \end{aligned} \tag{2.5.20}$$

This is an identity in s, the LHS is an n^{th} order polynomial. Consequently the RHS is a polynomial and the denominator of $|T(s)|$

arising from the term $(sI - A)^{-1}$ in $T(s)$ must be cancelled by some or all of the factors of $\left| sI - A \right|$. Whether a particular open loop pole is involved in the cancelling or not depends on whether its associated eigenvector is present in none of the columns of B, or is measured by none of the rows of K. This is clearly seen in those cases where A has distinct poles and the eigenvectors system of A is chosen as the system state space co-ordinate basis. However, the details of the cancellation are not relevant to the proof, and (2.5.20) shows that zeros of $\left| sI - (A - BK) \right|$ must be contained in the union of $\Lambda\{A\}$ and the set of zeros of $T(s)$.

Lemma 2.5.2

If $E(s)$ and A are stable and there exists $R_1 \geqslant 0$ such that the system (2.5.14), (2.5.15), (2.5.17) is unstable with $R = R_1$, then there exists $R_2 \geqslant 0$ such that when $R = R_2$ the same dynamics has at least one pole on the positive imaginary axis.

Proof

Under the assumption of finite dimensioned state, it is always possible to find a state space realisation of the dynamics (2.5.17),

$$\dot{x}_1 = A_1 x_1 + B_1 y$$

$$u = RH_1 x_1 + RD_1 y \, .$$

The closed loop dynamics (2.5.14), (2.5.15), (2.5.17) therefore may be written

$$\begin{pmatrix} \dot{x} \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} A-BRD_1B'P & BRH_1 \\ B_1B'P & A_1 \end{pmatrix} \begin{pmatrix} x \\ x_1 \end{pmatrix} . \tag{2.5.21}$$

The poles of the closed loop system are the eigenvalues of the system matrix of (2.5.21) and these lie in the LHP for zero R by assumption. By hypothesis there exists a real R_1 such that at least one pole lies in the RHP when $R = R_1$. Consider the behaviour of the poles when $R = \alpha R_1$ as α varies continuously between 0 and 1.

The eigenvalues of the system matrix of (2.5.21) vary continuously with the matrix elements, and therefore with the elements of R, and therefore with α. The root locus of the system poles as α varies starts in the LHP for $\alpha = 0$, and at least one branch finishes in the RHP when $\alpha = 1$ and $R = R_1$. At some intermediate value of α, $\alpha = \alpha_2$, a branch of the root locus must cross the imaginary axis. As all the matrix variables in the system matrix are real, its eigenvalues are real or occur as complex conjugate pairs, so that there is at least one crossing of the positive imaginary axis for $\alpha = \alpha_2$,

$$0 \leqslant \alpha_2 \leqslant 1 ,$$

and then

$$R_2 = \alpha_2 R_1 \geqslant 0 ,$$

since $R_1 \geqslant 0$ by hypothesis.

The lemma is therefore proved.

Corollary 2.5.1

If there exists no $R \geqslant 0$ such that the return difference $T(s)$ of the dynamics (2.5.14), (2.5.15) and (2.5.17) has at least one zero on the positive imaginary axis, then there exists no $R \geqslant 0$ such that the dynamics are unstable.

The next lemma concerns loop gain matrices for the Lyapunov control of dynamics (2.5.14), (2.5.15) and (2.5.16) which are rewritten for convenience in terms of the variable v,

$$\dot{x} = Ax + Bu$$

$$v = -RB'Px$$

$$u = v .$$

Instead of considering the loop gain from u to v with the loop broken at the input u, it is convenient to break the loop in such a way as to preserve symmetry. The equivalent closed loop dynamics may always be formed for R symmetric and $R \geqslant 0$,

$$\dot{x} = Ax + BR^{\frac{1}{2}} v_1 \qquad (2.5.22)$$

$$v_1 = -R^{\frac{1}{2}}B'Px , \qquad (2.5.23)$$

and the corresponding loop gain $L_1(s)$ is,

$$L_1(s) = R^{\frac{1}{2}}B'P(sI - A)^{-1}BR^{\frac{1}{2}} , \qquad (2.5.24)$$

where negative feedback is assumed.

Definition 2.5.3

A matrix x is positive real [A9] if for all complex vectors w ,

$$\mathrm{Re}(w^{*}Xw) > 0 \ , \ \forall \ w \neq 0.$$

Lemma 2.5.3

For all symmetric $R \geqslant 0$ the loop gain $L_1(s)$ of (2.5.24) is positive real.

Proof

For any X ,

$$\mathrm{Re}(w^{*}Xw) = \tfrac{1}{2}w^{*}Xw + \tfrac{1}{2}(w^{*} \, x \, w)^{*}$$

$$= \tfrac{1}{2}w^{*}(X^{*} + X)w \ . \qquad (2.5.25)$$

Then (2.5.24) and using (2.5.2) gives ,

$$L_1{}^{*} + L_1 = R^{\frac{1}{2}}B'((sI - A)^{*-1}P + P(xI - A)^{-1})BR^{\frac{1}{2}}$$

$$= R^{\frac{1}{2}}B'(sI - A)^{*-1}(P(sI - A) + (sI - A)^{*}P)(sI - A)^{-1}BR^{\frac{1}{2}}$$

$$= R^{\frac{1}{2}}B(sI - A)^{*-1}(- PA - A'P)(sI - A)^{-1}BR^{\frac{1}{2}}$$

$$= R^{\frac{1}{2}}B'(sI - A)^{*-1}Q(sI - A)^{-1}BR^{\frac{1}{2}} \ , \qquad (2.5.26)$$

which is PD since Q is PD. The lemma is proved because of (2.5.25).

Corollary 2.5.2

The eigenvalues of $L_1(s)$ lie in the RHP for all P and for all PSD symmetric R. For consider a unit magnitude eigenvector w_i, corresponding to an eigenvalue λ_i of $L_1(s)$; then

$$w_i^* w_i = 1 ,$$

$$L(s)w_i = \lambda_i w_i ,$$

and

$$w_i^* L w_i = \lambda_i .$$

By Lemma 2.5.3, L_1 is positive real and therefore ,

$$\mathrm{Re}(\lambda_i) > 0 ,$$

and λ_i lies in the RHP.

Lemma 2.5.4

For all diagonal gain matrices $R \geqslant 0$ and for all diagonal transfer function matrices E(s) with diagonal elements that are identical fourth quadrant transfer functions, the dynamics (2.5.14), (2.5.15), (2.5.17) are stable.

Proof

The dynamics are rewritten for convenience in terms of v,

$$\dot{x} = Ax + Bu , \tag{2.5.27}$$

$$v = - RB'Px , \tag{2.5.28}$$

and

$$u(s) = E(s)\ v(s) . \tag{2.5.29}$$

Suppose that each diagonal element of E equals e(s), then

$$E(s) = e(s)\ I \tag{2.5.30}$$

It is convenient to choose for a return difference stability analysis that point in the multi-variable feedback loop at which the return difference $T_2(s)$ involves $L_1(s)$ of (2.5.24) ,

$$\begin{aligned} T_2(s) &= I + E(s)\ R^{\frac{1}{2}}B'P(sI - A)^{-1}BR^{\frac{1}{2}} \\ &= I + E(s)\ L_1(s) \\ &= I + e(s)\ L_1(s) . \end{aligned}$$

The determinant of $T_2(s)$ is now considered. $T_2(s)$ is a m x m matrix.

$$\left| T_2(s) \right| = \prod_{i=1}^{m} \lambda_i(T_2(s))$$

$$= \prod_{i=1}^{m} (1 + \lambda_i(e(s)\ L_1(s)))$$

$$= \prod_{i=1}^{m} (1 + e(s)\ \lambda_i(L_1(s))) \ . \tag{2.5.31}$$

No factor in (2.5.31) may equal zero for any value of s on the positive imaginary axis, because $\mathrm{Re}\lambda_i(L_1(s)) > 0$ for all s and all i from Corollary 2.5.2, and e(s) will represent a rotation in the complex plane of at most 90°. This is true for all diagonal $R \geqslant 0$ and therefore there exists no diagonal gain matrix $R \geqslant 0$ for which $\left| T_2(s) \right|$ is zero. Since the open loop system is stable the lemma is proved by Lemma 2.5.1.

Lemma 2.5.5

The dynamics (2.5.27), (2.5.28) and (2.5.29) are stable for all diagonal $R \geqslant 0$ and for all diagonal fourth quadrant transfer function matrices E(s), such that the diagonal elements of E(s) not equal to unity are identical.

Proof

Suppose without loss of generality, that the first $r_1 < r$ diagonal elements of E(s) are unity. Denote the first r_1 elements of u by u_1

and the remaining $(r - r_1)$ elements by u_2, and partition the matrices $E(s)$, R and B according to

$$E(s) = \begin{pmatrix} I & 0 \\ 0 & E_2(s) \end{pmatrix} , \tag{2.5.32}$$

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} , \tag{2.5.33}$$

and

$$B = (B_1 \quad B_2) . \tag{2.5.34}$$

Consider the system with the first r_1 loops closed. The dynamics are then,

$$\begin{aligned} \dot{x} &= Ax + B_1u_1 + B_2u_2 \\ &= (A - B_1R_1B_1{}'P)x + B_2u_2 \\ &\triangleq \overline{A}\, x + \overline{B}\, \overline{u} , \end{aligned} \tag{2.5.35}$$

where $\overline{A}$, $\overline{B}$ and $\overline{u}$ are defined by equation (2.5.35). The matrix $\overline{A}$ is stable since unforced trajectories of (2.5.35) have Lyapunov function

$$V(x) = x'Px ,$$

and

$$\begin{aligned} \dot{V}(x) &= - x'(Q + PB_1R_1B_1{}'P)x \\ &< 0 , \quad \forall\, x \neq 0. \end{aligned}$$

Consider the system (2.5.35) and close the remaining $(r - r_1)$ loops which contain the dynamic elements of E(s). The dynamics are then ,

$$\dot{x} = \bar{A} x + \bar{B} \bar{u} ,$$

$$\bar{v} = - R_2 B_2{}' P x$$

$$= - \bar{R} \bar{B} P x ,$$

and

$$\bar{u}(s) = E_2(s) \bar{v} .$$

These dynamics satisfy the conditions of Lemma 2.5.4, and therefore the closed loop system is stable. The lemma is therefore proved.

Corollary 2.5.3

The dynamics of lemma 2.5.5 are stable if in addition any or all of the diagonal elements of E(s) are zero.

This follows because the zero diagonal elements of E(s) may be put equal to unity and the corresponding elements of R put equal to zero without changing the dynamics. The conditions for Lemma 2.5.5 hold and the system is stable.

We may now prove the theorem which is restated.

Theorem 2.5.1

The dynamic system (2.5.14), (2.5.15), (2.5.17) is stable for all diagonal $R \geqslant 0$ and all diagonal fourth quadrant $E(s)$.

Proof

Call the dynamic system of the theorem S_E. The non-dynamic diagonal elements of $E(s)$ must be zero or real positive numbers which may be put equal to unity by scaling the appropriate diagonal elements of R, and R will remain PSD. If all the dynamic elements of $E(s)$ are equal, the theorem is immediately true by Lemma 2.5.5. If not, a fourth quadrant single-input-single-output transfer function $g(s)$ is constructed which has phase angle θ more negative at any frequency than the phase angle of every diagonal element of $E(s)$, evaluated at the same frequency. It is always possible to do this.

Consider a diagonal transfer matrix $G(s)$ of the same dimension as $E(s)$, and put those diagonal elements of $G(s)$ corresponding to zero diagonal elements of $E(s)$ equal to zero. The remaining diagonal elements of $G(s)$ are given the values unity or $g(s)$, and the set of these elements will be called the subset G_{DO} of diagonal elements of $G(s)$.

Consider the hypothetical system S_G with dynamics (2.5.14), (2.5.15) and (2.5.17) except that $G(s)$ replaces $E(s)$. By Lemma 2.5.5 S_G is stable for all diagonal $R \geqslant 0$ and for any allocation of values unity and $g(s)$ to the subset G_{DO}.

Consider a particular feedback loop of S_E that contains one of the dynamic elements of $E(s)$, call it FB_1, and re-index the system inputs of S_E and S_G so that this element may be referred to as $E_{11}(s)$ and the corresponding element of $G(s)$ as $G_{11}(s)$.

The loop FB_1 in S_G is now broken and the scalar return difference $T_1(s)$ examined as a function of frequency. This is equivalent to looking at the Nyquist plot of the single-input-single-output system defined by the break in the feedback loop FB_1.

Suppose that the loop transfer function for this single-input-single-output system is ,

$$g_1(s) \quad \text{when} \quad G_{11}(s) = 1 \ ,$$

and $\bar{g}_1(s)$ when $G_{11}(s) = g(s)$.

Then

$$\bar{g}_1(s) = g(s)\, g_1(s) .$$

Because the system S_G has been proved stable for all diagonal gain matrices $R \geqslant 0$, both when $G_{11}(s) = 1$ and when $G_{11}(s) = g(s)$, S_G will be stable for any fixed $R_{ii} \geqslant 0$, $\forall\, i \neq 1$, for all values of $R_{11} \geqslant 0$.

Therefore the Nyquist diagrams for $g_1(s)$ and $\bar{g}_1(s)$ do not cut the negative real axis, as is illustrated in Figure 2.5.1

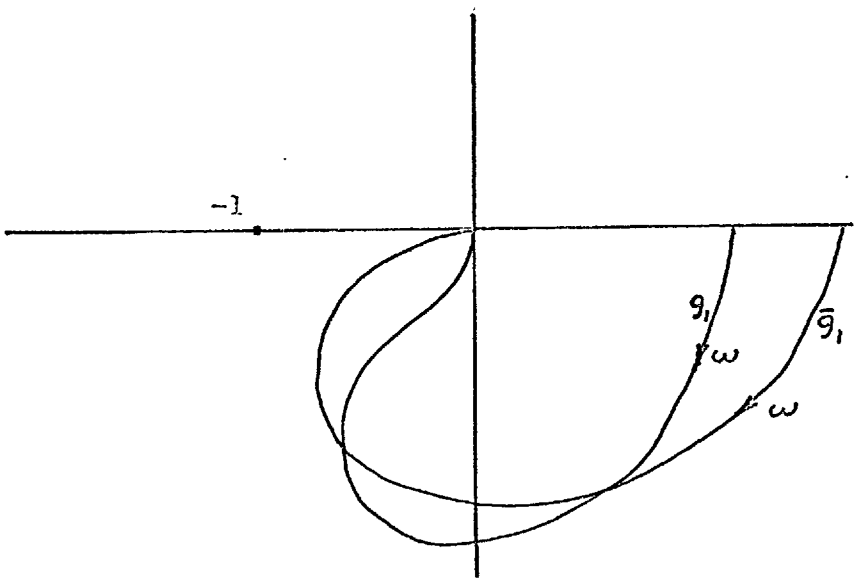


Figure 2.5.1 Transfer functions when a feedback loop is broken. ($\bar{g}_1 = g \cdot g_1$) Both g and $\bar{g}$ are stable for all positive gains.

This follows from single-input-single-output stability theory. Alternatively by Corollary 2.5.1, the relevant return differences $T_1(s)$ and $\bar{T}_1(s)$, are non-zero for all finite positive gains R_{11} at all frequencies;

$$T_1(s) = 1 + R_{11}g_1(s),$$

and

$$\bar{T}_1(s) = 1 + R_{11}\bar{g}_1(s)\,;$$

Therefore neither $g_1(s)$ nor $\bar{g}_1(s)$ cross the negative real axis.

Replace $G_{11}(s)$ by $E_{11}(s)$ and call the corresponding loop transfer function $g_1^{\,o}(s)$,

$$g_1^{\,o}(s) = E_{11}(s)\, g_1(s).$$

Then $g_1^{\,o}(s)$ does not cross the negative real axis on a Nyquist diagram, because by the constructed properties of $g(s)$, at any frequency $g_1^{\,o}(s)$ is obtained from $g_1(s)$ by a smaller clockwise rotation than is $\bar{g}_1(s)$. Magnitude changes induced by $E_{11}(s)$ are irrelevant for this analysis.

Therefore S_G with FB_1 closed, that is with all loops closed and $G_{11}(s) = E_{11}(s)$, is stable for all values of $R_{11} \geqslant 0$, and therefore for all diagonal $R \geqslant 0$.

Define a new subset G_{D1} of the diagonal elements of G(s) as the subset G_{DO} with the element $G_{11}(s)$ omitted. Then the result holds for any allocation of the values unity and g(s) to the elements of G_{D1}.

We complete the proof by induction.

Hypothesis

Suppose that (k-1) diagonal elements of G(s) belonging to the subset G_{DO} have been put equal to the elements in the corresponding positions on the diagonal of E(s); the remaining elements of G_{DO} are said to define the subset $G_{D(k-1)}$. Suppose further that it is true that S_G is stable for all diagonal $R \geqslant 0$ and for all allocations of the values unity and g(s) to the diagonal elements of G(s) belonging to $G_{D(k-1)}$.

It is seen that the hypothesis has been proved for k = 1. It remains to show that if the hypothesis is true for (k-1) it is also true for k.

Select a feedback loop FB_k corresponding to a diagonal element of G belonging to the set $G_{D(k-1)}$, and re-index the inputs of S_G and S_E

so that this element may be referred to as $G_{kk}(s)$. Define G_{DK} to be the subset $G_{D(k-1)}$ of diagonal elements of $G(s)$ with the element $G_{kk}(s)$ omitted.

Then by hypothesis S_G is stable when $G_{kk}(s) = 1$ and when $G_{kk}(s) = g(s)$, for all diagonal $R \geqslant 0$ and for all allocations of the values unity and $g(s)$ to the elements of G_{Dk}.

The loop FB_k may now be broken and a single-input-single-output transfer function analysis undertaken in precisely the same way as was done for G_{11}. In this way the system S_G is seen to be stable for all diagonal $R \geqslant 0$ when,

$$G_{kk}(s) = E_{kk}(s),$$

and for all allocations of the values unity and $g(s)$ to the elements of G_{Dk}.

Therefore if the hypothesis is true for (k-1) it is true for k. However, the hypothesis has been proved for k = 1 and therefore the hypothesis holds for k = 1, 2, 3 ..... until the set G_{Dk} is empty. At this point the diagonal elements of $G(s)$ belonging to the set G_{DO} have all been put equal to the elements in the corresponding positions of $E(s)$.

Then

$$G(s) = E(s)$$

and the dynamics of S_G are identical to those of S_E, by construction. Because S_G is stable for all diagonal $R \geqslant 0$ then so is S_E, and therefore the theorem is proved.

A Result with Non-Linear Gains

Consider a system with dynamics

$$\dot{x} = Ax + Bu \; , \tag{2.5.36}$$

$$u = N(v) \; , \tag{2.5.37}$$

$$y = RB'Px \; , \tag{2.5.38}$$

and

$$\dot{v} = - Dv + Dy \; , \tag{2.5.39}$$

where

(1) $N(v)$ is a nonlinearity lying in the third and fourth quadrants and has bounded derivatives.

(2) $D = \text{Diag}\ (d_1, \ldots d\ )$ and is PD .

(3) R is a diagonal PD matrix.

(4) A is stable.

(5) P satisfies (2.5.2).

Theorem 2.5.2

The above system is stable.

The proof follows by application of the multi-variable Popov criterion quoted below.

Popov Criterion [J1]

Given a strictly stable square transfer function matrix G(s) and a diagonal, non-interacting matrix non-linearity N, whose diagonal elements $n_i(a_i)$ have bounded derivatives and satisfy

$$0 < n_i(a_i) < k_i a_i \ , \qquad (2.5.40)$$

define K, Q, F(jw) and P(w) ,

$$K = \text{Diag}\ (k_1, \ldots k_n) \ , \qquad (2.5.41)$$

$$Q_c = \text{Diag}\ (q_1, \ldots q_n) \ , \qquad (2.5.42)$$

$$F(jw) = (I + jwQ)G(jw) + K^{-1}, \qquad (2.5.43)$$

and

$$P_c(w) = F(jw) + F^*(jw). \qquad (2.5.44)$$

If there exists a Q such that

$$P_c(w) > 0 \quad \forall\ w \ , \qquad (2.5.45)$$

the closed loop system with the nonlinearity N in the feedback paths, and with negative feedback, is stable.

Proof of Theorem

Define a new diagonal non-linear matrix N_R that incorporates R with N. Then N_R also satisfies (2.5.40) because R is constant and PD.

The transfer function matrix G(s) of the Popov criterion from (2.5.36), (2.5.38) and (2.5.39) ,

$$G(s) = (sI + D)^{-1}DB'P(sI - A)^{-1}B$$

$$= (sD^{-1} + I)^{-1}B'P(sI - A)^{-1}B . \qquad (2.5.46)$$

Consider as a trial Q_c in (2.5.42)

$$Q_c = D^{-1}.$$

Then (2.5.43) and (2.5.46) give F(jw) ,

$$F(jw) = (I + jwD^{-1})(jwD^{-1} + I)^{-1}. \\ B'P(jwI - A)^{-1}B + K^{-1}$$

$$= B'P(jwI - A)^{-1}B + K^{-1}. \qquad (2.5.47)$$

Then (2.5.44) and (2.5.47) give $P_c(w)$,

$$P_c(w) = F(jw) + F^{*}(jw)$$

$$= B'(jwI - A)^{*-1}(- PA - A'P)(jwI - A)^{-1} B \\ + 2K^{-1}$$

$$= \quad B'(jwI - A)^{*-1}Q(jwI - A)^{-1}B + 2K^{-1}.$$

Because A has all its eigenvalues in the LHP, Q is PD and the columns of B are independent by assumption,

$$P_c(w) > 0 , \quad \forall\, w ,$$

even as K^{-1} tends to zero.

The Popov criterion is satisfied and Theorem 2.5.2 is proved.

Comment

The set of linear gains is included in the set of nonlinear gains so that Theorem 2.5.2 proves a special case of Theorem 2.5.1, because the dynamics (2.5.39) inserted in the feedback paths are fourth quadrant. However, the dynamics (2.5.39) form only a small subset of the set of dynamics lying in the fourth quadrant and Theorem 2.5.1 provides a more general result than Theorem 2.5.2 applied to linear systems.

The results of Theorems 1 and 2 are interesting because they are directly applicable to the situation of control being applied to a plant through actuators which have their own dynamics.

If desired plant inputs are determined from a Lyapunov control feedback law, even though actuator dynamics have not been included in the plant dynamics but are known to satisfy the conditions of theorem 2.5.1 or 2.5.2, the closed loop system is stable.

2.5.4 Stability of a Class of Riccati Controls for a Class of Errors in Plant Dynamics

The results of Section 2.5.3 are interesting and are probably most useful in assessing the effects of actuator dynamics on stability. It is interesting to see the implications of Theorems 2.5.1 and 2.5.2 on optimal controls that minimise performance indices of the type ,

$$J = \int_0^\infty (x'Qx + u'Ru)\,dt$$

for

$$\dot{x} = Ax + Bu , \tag{2.5.48}$$

$$R > 0 \text{ diagonal.}$$

Suppose that the optimal control [K1] is

$$\begin{aligned} u &= -R^{-1}B'Px \\ &\triangleq Kx , \end{aligned} \tag{2.5.49}$$

where P satisfies the associated steady state Riccati equation,

$$(A + BK)'P + P(A + BK) = -(Q + K'RK).$$

$J = x'Px$ is a Lyapunov function for the matrix $(A + BK)$, and also [A3] for the dynamics with non-linear gain $N(x)$,

$$\dot{x} = Ax + Bv ,$$

$$v = -N(x)Kx ,$$

where $N(x)$ is diagonal and each diagonal element $N_{ii}(x)$ is continuous and satisfies

$$N_{ii}(x) > \tfrac{1}{2}, \quad \forall x .$$

We may consider other actuator irregularities besides memoryless non-linearities by first recognising the relationship between the optimal control (2.5.49) and Lyapunov control.

Consider the system

$$\dot{x} = (A - \tfrac{1}{2}BK)x + Bu . \tag{2.5.50}$$

We have seen that it is stable with Lyapunov function $x'Px$ and therefore it is a candidate for Lyapunov control of the form

$$u = -\underline{R}\,B'Px \tag{2.5.51}$$

$$\underline{R} \geqslant 0 .$$

We shall consider diagonal $\underline{R}$. We note that the optimal control (2.5.49) with dynamics (2.5.48) may be interpreted as a Lyapunov control (2.5.51) with $\underline{R} = \frac{1}{2}R^{-1}$ for the dynamics (2.5.50), and may then immediately prove the following theorem.

Theorem 2.5.3

The closed loop dynamic system,

$$\dot{x} = Ax + Bu , \tag{2.5.52}$$

$$y = Kx , \tag{2.5.53}$$

$$u(s) = E(s)\, y(s) , \tag{2.5.54}$$

where K is the optimal feedback law (2.5.49) is stable if E(s) is a diagonal stable transfer function matrix whose diagonal elements lie in the fourth quadrant of the complex plane to the right of $Re(s) = \frac{1}{2}$, and have a finite number of poles.

Proof

The dynamics (2.5.52), (2.5.53), (2.5.54) are equivalent to

$$\dot{x} = (A - \tfrac{1}{2}BK)x + Bu , \tag{2.5.55}$$

$$y = Kx \,, \tag{2.5.56}$$

$$u(s) = E_1(s)\, y(s) \tag{2.5.57}$$

where $E_1(s)$ is a diagonal fourth quadrant transfer function matrix, and the control (2.5.56) is a Lyapunov control. The conditions for Theorem 2.5.1 hold and the system is therefore stable.

2.6 Conclusion

Three types of control law have been discussed with the common property that feedback is required from all plant states. These laws are convenient to work with in state space computer design techniques because they are solutions to well-defined problems. There is a difficulty in translating general specifications on system performance into mathematical objectives. Modal control requires some trial and error in fixing desired closed loop poles to attain indirectly related performance criteria. Similarly when a law is based on optimal control, the design process tends to involve selection of the performance index parameters on a trial and error basis, but it is usually clear what change to make in the costing to obtain a desirable change in transient response. This is not necessarily so in the design of modal control laws. We have discussed the implicit modelling method of assigning costs to a Riccati problem so that the optimal closed loop system has poles approximating specified values. This method can lead to a controller with dynamic elements, and a philosophy has been given by which a dynamic element in the controller can be avoided; an example has been given that supports the ideas. A less specific method of obtaining a desired characteristic of a closed loop optimal system weights the cost on state and control exponentially. The resulting problem leads to the standard Riccati equation, and the optimal closed loop poles are faster or more stable than the exponential weighting.

We have ascertained the properties of Lyapunov control in some detail because of its tolerance with respect to stability of the presence of actuator dynamics not considered in the design process. The actuators form part of the plant dynamics, so that in the case of Lyapunov control and if the conditions of Theorems 2.5.1 and 2.5.3 hold, the need to measure the complete plant state may be slightly relaxed. We have discovered in Section 2.5 that Lyapunov control imposes constraints on closed loop system poles, which may make Riccati or modal control more appropriate in a given application. We must then measure the full plant state or use estimates of unmeasured states. The following chapters develop a theory of the observers that may be used to provide the required state estimates.

CHAPTER 3

OBSERVERS FOR DETERMINISTIC SYSTEMS

3.1 Introduction

As discussed in the previous chapter, an obstacle occurs in implementing controls obtained by modern control theory because not all plant states are measured.

In the stochastic case of a linear plant with Gaussian noise on the input and output, the Kalman filter [K3] provides optimal state estimates for the plant. A control can be synthesized from these estimates without affecting system closed loop poles. However, as the noise tends to zero the Kalman filter tends to differentiate the plant output to obtain state estimates. Luenberger [L1], [L2] has proposed a filter theory which does not rely on optimization theory and filter dynamics are more directly specified by the designer. An advantage of the Luenberger observer theory is that observers of reduced dynamics may be constructed, sometimes of greatly reduced dynamics.

Sections 3.1 and 3.2 contain introductory material, Section 3.3 presents a convenient theory and design technique for (n-m)-state observers that has not appeared in the literature before. The remaining sections briefly consider some of the frequency domain characteristics of Luenberger filters or observers.

The Observer Equations

Consider the linear plant

$$\dot{x} = Ax + Bu \,, \tag{3.1.1}$$

$$y = Hx \,, \tag{3.1.2}$$

with n states, r inputs and m outputs.

Consider also p-vectors $\hat{z}$, z and Δz such that

$$z = Lx \,, \tag{3.1.3}$$

$$\Delta z = \hat{z} - z \,, \tag{3.1.4}$$

$$\dot{\hat{z}} = D\hat{z} + THx + Cu \,. \tag{3.1.5}$$

Thus

$$\begin{aligned} \Delta\dot{z} &= \dot{\hat{z}} - L\dot{x} \\ &= D\hat{z} + THx + Cu - L(Ax + Bu) \\ &= D\Delta z + (DL + TH - LA)x + (C - LB)u \,. \end{aligned} \tag{3.1.6}$$

The dynamics (3.1.6) become

$$\Delta\dot{z} = D\Delta z \tag{3.1.7}$$

if and only if

$$DL - LA + TH = 0 \,, \tag{3.1.8}$$

and

$$C - LB = 0 \,. \tag{3.1.9}$$

Consider a trajectory of (3.1.1) with initial condition x_o and a trajectory of (3.1.5) with initial condition $\hat{z}_o$. These trajectories define a trajectory in Δz defined by (3.1.4) which has dynamics (3.1.6). If (3.1.8) and (3.1.9) are satisfied the trajectory of Δz evolves according to (3.1.7) and Δz will decay exponentially with time if D is stable. In this case, by definitions (3.1.3) and (3.1.4), the trajectory in $\hat{z}$ is such that $\hat{z}$ tends to Lx independently of the plant input u. We may then regard $\hat{z}$ as an estimate of Lx and Δz an estimation error.

Definition 3.1.1 The p-state system (3.1.5) with state variable $\hat{z}$ is called an observer or Luenberger observer [L1], [L2].

Definition 3.1.2 The equations (3.1.3) to (3.1.9) are basic to observer theory and will be called "the set of observer equations"; and a matrix set L, D, T, C satisfying them will be called a "solution to the observer equations".

Families of solutions

A solution to the observer equations generates a family of solutions. Consider a new variable $\hat{w}$ obtained from a non-singular transformation of $\hat{z}$

$$\hat{w} = W\hat{z} , \tag{3.1.10}$$

$$\hat{z} \quad = \quad W^{-1}\hat{w} \; . \tag{3.1.11}$$

Then $\hat{w}$ is the state of an observer because (3.1.3) to (3.1.9) with (3.1.10) give,

$$\begin{aligned} w \quad &\triangleq \quad WLx \\ &\triangleq \quad \bar{L}x \; , \end{aligned} \tag{3.1.12}$$

$$\Delta w \quad \triangleq \quad \hat{w} - w \; , \tag{3.1.13}$$

and

$$\begin{aligned} \dot{\hat{w}} \quad &= \quad WDW^{-1}\hat{w} + WTHx + WCu \\ &\triangleq \quad \bar{D}\hat{w} + \bar{T}Hx + \bar{C}u \; , \end{aligned} \tag{3.1.14}$$

where (3.1.14) defines $\bar{D}$, $\bar{T}$ and $\bar{C}$.

Then it is easily shown that

$$\Delta\dot{w} \quad = \quad \bar{D}\Delta w \; , \tag{3.1.15}$$

$$D\bar{L} - \bar{L}A + \bar{T}H \quad = \quad 0 \; , \tag{3.1.16}$$

and

$$\bar{C} - \bar{L}B \quad = \quad 0 \; . \tag{3.1.17}$$

Equation (3.1.14) shows that $\bar{D}$ has the same poles as D. The observers may be regarded as equivalent because (3.1.10), (3.1.11) show that no information is lost by running one of the systems (3.1.5) or (3.1.14) instead of the other.

An unsuitable solution to the observer equations

It is noted that D and T may be specified and if the eigenvalues of D are different from those of A, (3.1.8) may be uniquely solved for L [J3] . Solution of (3.1.9) for C is then trivial, and the observer dynamics (3.1.5) are then completely specified. This approach is not useful because L is not pre-specified and the variable z

$$z = Lx$$

which is estimated by $\hat{z}$ may be of no interest.

The alternative approach is to specify L rather than D in the design procedure. This leads naturally to two classes of observer, n-state and (n-m)-state, and the corresponding solutions of the observer equations are given later in Sections 3.2 and 3.3.

Chapter 4 gives a more detailed analysis of the difficult problem of finding observers of dimension less than (n-m) that provide estimates of desired functions of plant states.

A useful stability property of observers with respect to feedback systems

Consider the n-state, r-input m-output plant (3.1.1) and (3.1.2) ,

$$\dot{x} = Ax + Bu \qquad (3.1.18)$$

$$y = Hx , \qquad (3.1.19)$$

and the p-state observer for which (3.1.3) to (3.1.9) hold so that ,

$$z = Lx \, , \tag{3.1.20}$$

$$\Delta z = \hat{z} - z \, , \tag{3.1.21}$$

$$\dot{\hat{z}} = D\hat{z} + THx + Cu \, , \tag{3.1.22}$$

and

$$\dot{\Delta z} = D\Delta z \, . \tag{3.1.23}$$

Consider a desired feedback law

$$u = Kx \tag{3.1.24}$$

such that it may alternatively be expressed

$$u = K_1 y + K_2 \hat{z} \, . \tag{3.1.25}$$

An estimate $\hat{u}$ of the desired feedback control may be defined.

$$\hat{u} = K_1 y + K_2 z \, , \tag{3.1.26}$$

and (3.1.26), (3.1.18) and (3.1.22) with $u = \hat{u}$ form the dynamic equations of the interacting system of observer and plant. Equations (3.1.21) and (3.1.23) allow the dynamics to be written in terms of plant state x and estimation error Δz ,

$$\frac{d}{dt}\begin{bmatrix} x \\ \Delta z \end{bmatrix} = \begin{bmatrix} A+BK & BK_2 \\ 0 & D \end{bmatrix} \begin{bmatrix} x \\ \Delta z \end{bmatrix} \, . \tag{3.1.27}$$

It is seen that the poles of the dynamics (3.1.27) are those of the ideal closed loop system, specified by the desired law (3.1.24) as

$$\dot{x} = (A + BK)\,x\,, \tag{3.1.28}$$

together with the observer poles determined by D.

This well known [L1], [L2] property of observers provides most of the motivation for the study of observer theory. The form (3.1.27) of the closed loop plant and observer composite system shows explicitly that if the estimation error is zero, it remains zero and plant trajectories are identical to those of the ideal system (3.1.28). It is emphasized that this result applies for the general p-state observer, as long as the ideal control law (3.1.24) may be expressed as (3.1.25) in terms of the plant output and estimated variable. Sections 3.2 and 3.3 prove that if the plant is observable, n-state and (n-m)-state stable observers exist for which any ideal control law may be expressed as (3.1.25). Consequently any observable and controllable plant may be stabilized. A more detailed examination of the properties of observers in feedback paths is given in Chapters 5 and 6.

Observers used in feedback paths may be regarded as compensators; the stability of compensators is considered in Chapter 5. Chapter 6 examines the "degradation in system performance" when the approximated ideal control is optimal with respect to a quadratic performance index.

3.2 The n-State Observer

Consider the problem of constructing an n-state observer for the plant with dynamics (3.1.1) and (3.1.2). We are interested in finding the conditions for which stable observers exist, that is, for which the matrix D is stable.

The following existence theorem was first presented in [L1], [L2]. For convenience we introduce the definition.

Definition 3.2.1 A set of poles is feasible if complex poles occur in conjugate pairs.

Theorem 3.2.1

A necessary and sufficient condition for an n-state observer to exist for the plant (3.1.1) and (3.1.2) and to provide an estimate $\hat{x}$ of the plant state and to have prespecified feasible poles is that the pair (A, H) be observable.

Proof

Consider L in (3.1.3) to be defined

$$L = I , \tag{3.2.1}$$

then necessary conditions (3.1.9) and (3.1.8) determine C and D ,

$$C = B , \tag{3.2.2}$$

$$D = A - TH . \tag{3.2.3}$$

The dynamics of an observer are defined in terms of the plant data by (3.1.5), (3.2.1), (3.2.2) and (3.2.3) where T remains to be chosen.

The form of the observer dynamics follows as

$$\frac{d}{dt}\hat{x} = (A - TH)\,\hat{x} + Bu + Ty . \tag{3.2.4}$$

The poles of the observer dynamics (3.2.4) are the eigenvalues of D which are also the eigenvalues of D'. A necessary and sufficient condition for a T' to exist to give an arbitrary prespecified set of feasible eigenvalues to D' is that the pair (A', H') be controllable [A1] . By the definition of controllable and observable pairs [K4], (A', H') is controllable if and only if (A, H) is observable.

<u>Corollary 3.2.1</u>

A necessary and sufficient condition for a stable n-state observer to exist for the plant (3.1.1), (3.1.2) is that the pair

(A, H) be detectable. This follows by applying the above theorem to the observable invariant subspace [K4] of the pair (A, H) ; the unobservable invariant subspace is stable by the definition of detectable pairs.

Comment

The theorem and corollary concern special case observers with L in (3.1.3) being the unit matrix, but as noted in Section 3.1 a whole family of observers having similar dynamics may be generated. From (3.1.12) it is seen that observers may be generated having $\bar{L}$ in (3.1.3) equal to any non-singular matrix. Often the observer is used to provide an estimate $\hat{v}$ of v

$$v = Kx ,$$

and this may be done by defining

$$\hat{v} = K\hat{x} .$$

In this case the observer may be regarded as a filter with inputs the plant input u and plant output y, and output the desired estimate $\hat{v}$. Consider two observers in the same family one generated from the other by a non-singular transformation between their states

$$\hat{w} = Wz.$$

The above equation indicates that $\hat{w}$ and $\hat{z}$ are merely the state variables of different state realizations of the matrix transfer function of the filter.

Before discussing further properties of n-state observers we move on to the examination of the conditions for the existence of (n-m)-state observers.

3.3 The Reduced Observer

Notation For the purposes of this section the plant dynamics (3.1.1) and (3.1.2) will be written ,

$$\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,u\,, \tag{3.3.1}$$

$$y = \bar{H}\,\bar{x}\,. \tag{3.3.2}$$

We recall that the plant has n states, r inputs and m outputs. In the interests of simplicity it is desirable to have observers of order lower than n, but in addition Chapter 6 shows that the (n-m)-state observer may have an intrinsic advantage over n-state observers with respect to trajectory performance indices of integrated quadratic cost. Luenberger [L1], [L2] has shown that in the case of an observable system, it is always possible to design an (n-m)-state observer having arbitrary feasible poles.

We present an alternative derivation of this result which leads to a new and simpler (n-m)-state observer design method [C2] than that suggested by Luenberger.

The matrix $\bar{H}$ of (3.3.2) is assumed to be of maximum rank, that is the outputs are independent. It is then always possible [K4] to make a state transformation with (3.3.4) defining a non-singular W ,

$$\begin{aligned} x &= W\,\bar{x} \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned} \tag{3.3.3}$$

$$
\begin{aligned}
y &= \bar{H}\,\bar{x} \\
&= \bar{H}\,W^{-1}x \\
&\triangleq Hx \\
&\triangleq x_1 \,.
\end{aligned} \tag{3.3.4}
$$

Then

$$
\begin{aligned}
\dot{x} &= W\,\bar{A}\,W^{-1}x + W\,\bar{B}\,\mathbf{u} \\
&\triangleq Ax + Bu
\end{aligned} \tag{3.3.5}
$$

$$
\triangleq \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \,, \tag{3.3.6}
$$

where A and B are defined by (3.3.5). The matrix partitioning is defined by (3.3.4), (3.3.3).

We note that x_1 is an m-vector and x_2 is an (n-m)-vector. The above transformation is very convenient for the following theory.

<u>Lemma 3.3.1</u>

The pair (A_{22}, A_{12}) in the plant equations (3.3.6) is observable if the pair $(\bar{A}, \bar{H})$ of (3.3.1) and (3.3.2) is observable.

<u>Proof</u>

Observability is preserved under non-singular state transformations so that (A, H) of (3.3.5) , and (3.3.4) is an observable pair.

Assume that (A_{22}, A_{12}) is not an observable pair, then Kalman [K4] has shown that there exists a transformation matrix S such that,

$$S^{-1} A_{22} S = \underline{A} \tag{3.3.7}$$

$$\triangleq \begin{pmatrix} \underline{A}_{11} & 0 \\ \underline{A}_{21} & \underline{A}_{22} \end{pmatrix}, \tag{3.3.8}$$

and

$$A_{12} S = \underline{H} \tag{3.3.9}$$

$$= (\underline{H}_1, 0) . \tag{3.3.10}$$

The dimensions of the matrices are :

A_{22} (n-m, n-m) , A_{12} (m, n-m) , S (n-m, n-m) ,

$\underline{A}$ (n-m, n-m) , $\underline{A}_{11}$ (n_1, n_1) , $\underline{A}_{21}(n_2, n_1)$,

$\underline{A}_{22}(n_2, n_2)$, $\underline{H}$ (m, n-m) and $\underline{H}_1$ (m, n_1) ,

where

$$n_1 + n_2 = n-m . \tag{3.3.11}$$

The transformation $\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ is applied to the plant equations (3.3.4), (3.3.5) so that using (3.3.7) to (3.3.10) ,

$$\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}^{-1} A \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} = \begin{pmatrix} A_{11} & \underline{H} \\ S^{-1}A_{21} & \underline{A} \end{pmatrix} \tag{3.3.12}$$

$$= \begin{pmatrix} x & 0 \\ x & \underline{A}_{22} \end{pmatrix} . \tag{3.3.13}$$

The x's in (3.3.13) denote matrices whose precise values are irrelevant to the argument, and the block partitioning of (3.3.13) is different from that of (3.3.12).

Define a state variable w ,

$$w = \begin{pmatrix} I & 0 \\ 0 & S^{-1} \end{pmatrix} x \quad , \qquad (3.3.14)$$

Then with partitioning as in (3.3.13) the plant dynamics for zero input from (3.3.5), (3.3.13) and (3.3.14) become ,

$$\dot{w} = \begin{pmatrix} x & 0 \\ x & \bar{A}_{22} \end{pmatrix} w \ , \qquad (3.3.15)$$

and

$$y = (x \quad 0)\, w \ . \qquad (3.3.16)$$

Equations (3.3.15), (3.3.16) show that the plant is unobservable which is contrary to hypothesis, and therefore (A_{22}, A_{12}) must be an observable pair.

<u>Theorem 3.3.1</u>

If and only if the pair (A, H) is observable, an (n-m)-state observer exists for the plant (3.3.4), (3.3.5), provides an estimate $\hat{x}_2$ of the unmeasured plant state component, and may be given prespecified feasible poles.

Proof

The theorem is proved by construction. By hypothesis the instantaneous values of the variables y and u are available. Information is provided about x_2 through its interaction in the dynamics of x_1 which is measured directly from (3.3.4)

$$x_1 = y .$$

Define the m-vector z related to x_2 through equation (3.3.6),

$$z \triangleq \dot{x}_1 - A_{11} x_1 - B_1 u \qquad (3.3.17)$$

$$= A_{12} x_2 . \qquad (3.3.18)$$

We proceed as if z were available as a measurement and consider the dynamics of x_2,

$$\dot{x}_2 = A_{22} x_2 + A_{21} x_1 + B_2 u$$

where the inputs x_1 and u are known. The form of an (n-m)-state Luenberger observer for this (n-m)-state system with assumed output z,

$$z = A_{12} x_2 ,$$

is given by Section 3.1,

$$\dot{\hat{x}}_2 = A_{22} \hat{x}_2 + A_{21} x_1 + B_2 u + T(z - A_{12} \hat{x}_2)$$

$$\ldots\ldots (3.3.19)$$

and from (3.3.17)

$$\dot{\hat{x}}_2 = (A_{22} - TA_{12})\hat{x}_2 + (A_{21} - TA_{11})x_1$$

$$+ (B_2 - TB_1)u + T\dot{x}_1 \ . \qquad (3.3.20)$$

Consideration of the unmeasured variable $\dot{x}_1$ is avoided by defining a new variable w and its estimate $\hat{w}$,

$$w = x_2 - Tx_1 \ , \qquad (3.3.21)$$

$$\hat{w} = \hat{x}_2 - Tx_1 \ . \qquad (3.3.22)$$

The dynamics of $\hat{w}$ are obtained from (3.3.20) and (3.3.22) as ,

$$\begin{aligned}\dot{\hat{w}} &= \dot{\hat{x}}_2 - T\dot{x}_1 \\ &= (A_{22} - TA_{12})\hat{x}_2 + (A_{21} - TA_{11})x_1 + (B_2 - TB_1)u \\ &= (A_{22} - TA_{12})\hat{w} + (A_{22}T + A_{21} - TA_{11} - TA_{12}T)x_1 \\ &\quad + (B_2 - TB_1)u \ . \qquad (3.3.23)\end{aligned}$$

Once a matrix T and an initial condition on $\hat{w}$ have been selected, equation (3.3.23) may be integrated in conjunction with plant trajectories because the variables x_1 and u are then available. An estimate $\hat{x}_2$ is obtained from $\hat{w}$ through (3.3.22) , and (3.3.21) shows that $\hat{x}_2 = x_2$ when $\hat{w} = w$;

$$\hat{x}_2 = \hat{w} + T x_1 . \qquad (3.3.24)$$

Equations (3.3.23) and (3.3.24) together define an (n-m)-state dynamic system that provides an estimate $\hat{x}_2$ of x_2. We call this system an observer of reduced dynamics, an (n-m)-state observer or a reduced observer.

The behaviour of $\hat{w}$ is examined by defining an error vector Δw ,

$$\Delta w = \hat{w} - w . \qquad (3.3.25)$$

Equations (3.3.25), (3.3.23), (3.3.21) and (3.3.6) give

$$\begin{aligned}
\Delta\dot{w} &= \dot{\hat{w}} - \dot{w} \\
&= (A_{22} - TA_{12})(\Delta w + w) + (A_{22}T + A_{21} - TA_{11} - \\
&\quad - TA_{12}T)x_1 + (B_2 - TB_1)u - \dot{x}_2 + T\dot{x}_1 \\
&= (A_{22} - TA_{12})\Delta w + (A_{22} - TA_{12} - A_{22} + TA_{12})x_2 \\
&\quad + (-(A_{22} - TA_{12})T + A_{22}T + A_{21} - TA_{11} - TA_{12}T \\
&\quad - A_{21} + TA_{11})x_1 + (B_2 - TB_1 - B_2 + TB_1)u \\
&= (A_{22} - TA_{12})\Delta w . \qquad (3.3.26)
\end{aligned}$$

If an estimation error Δx_2 is defined

$$\Delta x_2 = \hat{x}_2 - x_2 , \tag{3.3.27}$$

Then (3.3.24), (3.3.21) and (3.3.25) give

$$\begin{aligned} \Delta x_2 &= \hat{w} + T x_1 - x_2 \\ &= \hat{w} - w \\ &= \Delta w , \end{aligned} \tag{3.3.28}$$

so that Δw may be associated with the plant state estimation error Δx_2.

The error dynamics (3.3.26) and the observer dynamics (3.3.23) have the same system matrix $(A_{22} - TA_{12})$. The proof of the theorem requires that T exists that gives this matrix arbitrary feasible eigenvalues.

By Lemma 3.3.1 (A, A_{12}) is an observable pair, in which case T may always be so chosen [A1] .

Remarks

A value for T to stabilize the matrix $(A_{22} - TA_{12})$ may alternatively be determined by the formulation and solution of an auxiliary optimal control problem. T will then be the transpose of a steady state Riccati gain matrix.

Luenberger [L1], [L2] restricts the observer matrix $(A_{22} - TA_{12})$ to have no eigenvalues in common with those of the plant dynamics matrix $\bar{A}$ in (3.3.1). This restriction is unnecessary in the above reduced observer design method. A simple example illustrates this.

<u>Example</u>

The dynamics of the system are given in the form (3.3.6) , and consideration of input effects will be omitted ;

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

with eigenvalues -1, -2 .

Then

$$(A_{22} - TA_{12}) = -3 - T \ .$$

The (scalar) matrix T may be chosen T = -2 or T = -1, which give the eigenvalue of $(A_{22} - TA_{12})$ a value of -1 or -2 respectively. The dynamics of an observer obtained by substituting for A and T = -2 in

(3.3.23) become

$$\dot{\hat{w}} = (A_{22} - TA_{12})\hat{w} + (A_{22}T + A_{12} - TA_{11} - TA_{12}T)x_1$$

$$+ (B_2 - TB_1)u$$

$$= - \hat{w} + 0.\ x_1 + (B_2 - TB_1)u\ .$$

In this case (3.3.24) is

$$\hat{x}_2 = \hat{w} - 2\,x_1\ .$$

It is seen that the plant output x_1 does not enter the dynamics of the observer but does affect the estimate $\hat{x}_2$. Equation (3.3.26) still holds, and for the example the errors Δw and Δx_2 decrease exponentially as designed irrespective of plant input,

$$\Delta x_2(t) = e^{-t}\,\Delta x_2(0)\ .$$

The example emphasizes the following point for observer and plant with common eigenvalues :

with respect to information about plant modes corresponding to the common eigenvalues, the observer is liable to degenerate to an open loop model excited by the plant input alone. Then estimation errors associated with these modes will decay naturally at a rate determined by the mode pole. It is not thought worthwhile to pursue this point in detail.

3.4 The Frequency Domain Characteristics of Observers

Introduction

As indicated in Section 3.1, the usual reason for desiring the estimate of some unmeasured function of the plant state is that the function is required for feedback purposes. A link between state space control synthesis techniques and the frequency domain was derived by Kalman [K1] as the "Kalman equation for optimal systems" for the case of single input optimal control laws. Similarly insight into the effects of observers is obtained by analysis from the frequency domain point of view. The results of this section are perhaps not surprising, but the explicit analytic derivations are thought to be new.

Definitions

The given plant has input r-vector u, output m-vector y and an n-vector state. In the analysis of the (n-m)-state observer effects, it is convenient to choose a co-ordinate basis for the plant state such that (3.4.4) holds and the partitioning (3.4.2), (3.4.3) is introduced. The plant dynamics are ,

$$\dot{x} = Ax + Bu \tag{3.4.1}$$

$$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \, , \tag{3.4.2}$$

$$y = Hx \tag{3.4.3}$$

$$= x_1 \, , \tag{3.4.4}$$

and

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{3.4.5}$$

Suppose that some function v of plant is desired

$$v = Kx \, , \tag{3.4.6}$$

which in the case of the reduced observer analysis will be regarded as

$$v = K_1 x_1 + K_2 x_2 \, . \tag{3.4.7}$$

Suppose also that the rows of K are not spanned by the rows of H, that is v is independent of plant output y, and that an observer is used to construct an estimate $\hat{v}$ of v ,

$$\hat{v} = K\hat{x} \, . \tag{3.4.8}$$

The frequency domain viewpoint is introduced by defining the transfer functions L(s), $L(\hat{s})$ such that

$$v(s) = L(s)\, u(s) \, , \tag{3.4.9}$$

and

$$\hat{v}(s) = \hat{L}(s)\, u(s) \, . \tag{3.4.10}$$

Theorem 3.4.1 When the estimate $\hat{x}$ in (3.4.8) is provided by a n-state Luenberger observer,

$$\hat{L}(s) = L(s) \, .$$

Theorem 3.4.2 When the estimate $\hat{x}$ in (3.4.8) is provided by a (n-m)-state reduced observer,

$$\hat{L}(s) = L(s).$$

These theorems are not surprising and a heuristic proof for both is given together with algebraic proofs. The algebraic proofs of this Section may be useful with their explicit presentation of cancellation processes, because at these points an indication is given of the magnitude effects of parameter variations from nominal values.

Heuristic Support for Theorems 3.4.1 and 3.4.2

For the purpose of this section the open loop plant is assumed to be stable. For both n- and (n-m)-state stable observers the estimate $\hat{x}$ of x tends to x as time increases irrespectively of the plant input by (3.1.7). After all transients have decayed when a sinusoidal input u is applied to the open loop plant,

$$\hat{x} = x,$$

and from (3.4.8) and (3.4.9)

$$\hat{v} = v.$$

This is true for all sinusoidal inputs and it follows that Fourier transforms of $\hat{v}$ and v are then equal. This with equation (3.4.10) indicates that the relation ,

$$\hat{L}(s) = L(s) ,$$

holds for both types of observer.

Proof of Theorem 3.4.1

The Laplace transforms of (3.4.3) and (3.4.6) are required ,

$$y(s) = H(sI - A)^{-1} B u(s) , \qquad (3.4.11)$$

and

$$v(s) = K(sI - A)^{-1} B u(s) . \qquad (3.4.12)$$

The dynamics of an n-state Luenberger observer have been shown in Section 3.2 to be ,

$$\dot{\hat{x}} = D\hat{x} + Bu + Ty , \qquad (3.4.13)$$

and

$$D = A - TH , \qquad (3.4.14)$$

where T is a suitable gain matrix.

The Laplace transform of (3.4.13) is

$$\hat{x}(s) = (sI - D)^{-1}(Bu(s) + Ty(s)) . \qquad (3.4.15)$$

Use of (3.4.8), (3.4.15) and (3.4.14) leads to the following manipulations ,

$$\hat{v}(s) = K\hat{x}(s)$$

$$= K(sI - D)^{-1}(B + TH(sI - A)^{-1}B)u(s)$$

$$= K(sI - D)^{-1}(I + TH(sI - A)^{-1})Bu(s)$$

$$= K(sI - D)^{-1}(I - (D - A)(sI - A)^{-1})Bu(s)$$

$$= K(sI - D)^{-1}(I - (sI - sI + D - A)(sI - A)^{-1})Bu(s)$$

$$= K(sI - D)^{-1}(I - I + (sI - D)(sI - A)^{-1})Bu(s)$$

so that

$$\hat{v}(s) = K(sI - A)^{-1} Bu(s) . \tag{3.4.16}$$

Equations (3.4.16), (3.4.12), (3.4.9) and (3.4.10) show that

$$\hat{L}(s) = L(s) ,$$

which completes the proof of Theorem 3.4.1. Note that the transfer function (3.4.15) is essentially a steady state representation and the analysis is meaningful for stable D, corresponding to stable observers.

Proof of Theorem 3.4.2

Section 3.3 has shown that the form of the dynamics of an (n-m)-state observer for the plant equations (3.4.2) and (3.4.4) is ,

$$\dot{\hat{z}} = D\hat{z} + Ex_1 + Cu \;, \tag{3.4.17}$$

$$\hat{x}_2 = \hat{z} + T_1 x_1 \;, \tag{3.4.18}$$

$$D = A_{22} - T_1 A_{12} \;, \tag{3.4.19}$$

$$E = A_{21} + DT_1 - T_1 A_{11} \;, \tag{3.4.20}$$

and

$$C = B_2 - T_1 B_1 \;, \tag{3.4.21}$$

where T_1 is any matrix such that D has stable eigenvalues.

Define and partition Z ,

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \tag{3.4.22}$$

$$= \begin{pmatrix} sI_1 - A_{11} & -A_{12} \\ -A_{21} & sI_2 - A_{22} \end{pmatrix}^{-1} , \tag{3.4.23}$$

where I_1 and I_2 are unit matrices of appropriate order.

Then

$$\begin{pmatrix} sI_1 - A_{11} & -A_{12} \\ -A_{21} & sI_2 - A_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} \quad (3.4.24)$$

The Laplace transforms of x_1 and x_2 for zero initial conditions are ,

$$x_1(s) = (Z_{11}B_1 + Z_{12}B_2)u(s) \ , \quad (3.4.25)$$

and

$$x_2(s) = (Z_{21}B_1 + Z_{22}B_2)u(s) \ . \quad (3.4.26)$$

The transform of (3.4.18) together with (3.4.17), (3.4.20) and (3.4.21) gives ,

$$\begin{aligned} \hat{x}_2(s) &= \hat{z}(s) + T_1x_1(s) \\ &= (sI_2 - D)^{-1}(Cu(s) + Ex_1(s)\ ) + T_1x_1(s) \\ &= (sI_2 - D)^{-1}((B_2 - T_1B_1)u(s) + \\ &\qquad (-T_1A_{11} + A_{21} + DT_1 + (sI_2 - D)T_1)x_1(s)) \end{aligned}$$

$$\ldots\ldots \quad (3.4.27)$$

Substitute for $x_1(s)$ from (3.4.25) ,

$$\hat{x}_2(s) = (sI_2 - D)^{-1}(B_2 - T_1B_1 +$$

$$(-T_1A_{11} + A_{21} + sI_2T_1)(Z_{11}B_1 + Z_{12}B_2))u(s)$$

$$= (sI_2 - D)^{-1}(B_2 - T_1B_1 - T_1(sI_1 - A_{11})(Z_{11}B_1 + Z_{12}B_2)$$

$$+ A_{21}(Z_{11}B_1 + Z_{12}B_2))u(s) \; . \qquad (3.4.28)$$

Terms $(sI_1 - A_{11})Z_{11}$, $(sI_1 - A_{11})Z_{12}$, $A_{21}Z_{11}$ and $A_{21}Z_{12}$ in (3.4.28) are replaced by using the relation (3.4.24), so that

$$\hat{x}_2(s) = (sI_2 - D)^{-1}(B_2 - T_1B_1 +$$

$$T_1((I_1 + A_{12}Z_{21})(B_1 + A_{12}Z_{22}B_2) +$$

$$(sI_2 - A_{22})Z_{21}B_1 + ((sI_2 - A_{22})Z_{22} - I_2)B_2)u(s)$$

$$= (sI_2 - D)^{-1}(sI_2 - A_{22} + T_1A_{12})(Z_{21}B_1 + Z_{22}B_2)u(s)$$

$$\ldots\ldots \quad (3.4.29)$$

Then (3.4.19), (3.4.29) and (3.4.26) give

$$\hat{x}_2(s) = (Z_{21}B_1 + Z_{22}B_2)u(s) \qquad (3.4.30)$$

$$= \quad x_2(s) \, , \tag{3.4.31}$$

so that from (3.4.7) and (3.4.8)

$$\hat{v}(s) = K_1 x_1(s) + K_2 \hat{x}_2(s)$$

$$= K_1 x_1(s) + K_2 x_2(s)$$

$$= v(s) \, . \tag{3.4.32}$$

This concludes the proof of Theorem 3.4.2.

Frequency effects of observers used in feedback paths

The return difference of a closed loop system has proved to be a useful guide in the classical design of single-input feedback control systems, because of its central position in the sensitivity analysis of these systems. A sensitivity theory of multi-input systems exists, both for optimal and non-optimal controls, and again the return difference plays a central role. It is therefore interesting to analyse the characteristics of return differences when feedback control is synthesised via an observer.

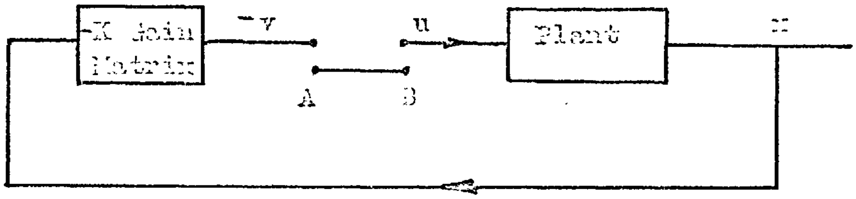


Figure 3.4.1 **Open loop configuration of feedback system with all states measured.**

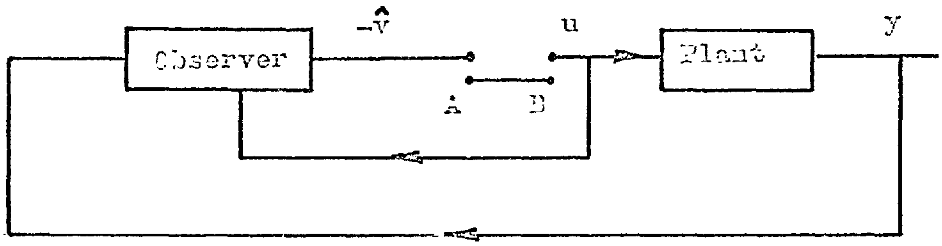


Figure 3.4.2 **Open loop operation of plant and observer.**

Figure 3.4.1 illustrates a plant with all states measured and the feedback path corresponding to the closed loop control (the negative feedback (3.4.33) is introduced to retain the classical form of return difference) ,

$$u = - v \text{ ,} \tag{3.4.33}$$

and

$$v = Kx \text{ .} \tag{3.4.34}$$

A natural break in the closed loop path is at the plant input, and the corresponding return difference T(s) is then defined by (3.4.35) ,

$$u(s) - (- v(s)) = T(s)\, u(s) \text{ .} \tag{3.4.35}$$

Substitution from (3.4.34) and (3.4.9) gives

$$T(s) = I + L(s) \text{ .} \tag{3.4.36}$$

Now consider the plant where not all state variables are measured, but the plant is observable so that either an n or (n-m)-state observer may be constructed to implement the control law ,

$$u = - \hat{v} \text{ .} \tag{3.4.37}$$

If the return difference $\hat{T}(s)$ is defined by the break in the feedback path shown in Figure 3.4.2 then $\hat{T}(s)$ satisfies

$$u(s) - (- \hat{v}(s)) = \hat{T}(s)\, u(s) , \tag{3.4.38}$$

and by Theorems 3.4.1 and 3.4.2 ,

$$\hat{T}(s) = T(s) . \tag{3.4.39}$$

This return difference or the associated loop gain indicates sensitivities of closed loop plant performance with respect to changes in dynamics in the connection joining (-v) to u. This is a useful result and shows that on any frequency test such as a multi-variable Nyquist criterion [R2] for linear dynamics in the link AB, or the multi-variable Papov criterion [J1] for a non-linear gain in the link AB, the control law (3.4.37) obtained from a state estimate is equivalent to the ideal law (3.4.33), (3.4.34) with full state measurement. This conclusion requires that there are no errors in the plant equations (3.4.1), (3.4.3) and that the plant input u which is output by the link AB of Figure 3.4.2 closing the loop, is measured precisely to drive the observer dynamics (3.4.13) or (3.4.17).

In many situations the measurement of the plant input variable u is not used to drive the observer, but rather u is assumed to be the desired value $(-\hat{v})$. The observer may then be called a compensator and has the configuration shown in Figure 3.4.3. The n state compensator has dynamics given by (3.4.13) and (3.4.34) as,

$$\dot{\hat{x}} = (D - BK)\hat{x} + Ty \; , \tag{3.4.40}$$

$$\hat{v}_c = K\hat{x} \; . \tag{3.4.41}$$

The (n-m) state compensator has dynamics given by (3.4.17) and (3.4.34) as ,

$$\dot{\hat{z}} = (D - CK_2)\hat{z} + (E - K_1 - K_2T_1)x_1 \; , \tag{3.4.42}$$

and

$$\hat{v}_c = (K_1 + K_2T_1)x_1 + K_2\hat{z} \; . \tag{3.4.43}$$

The subscript c in v_c denotes the compensator configuration.

Closed loop operation of the plant is initiated by closing the link AB of Figure 3.4.3.

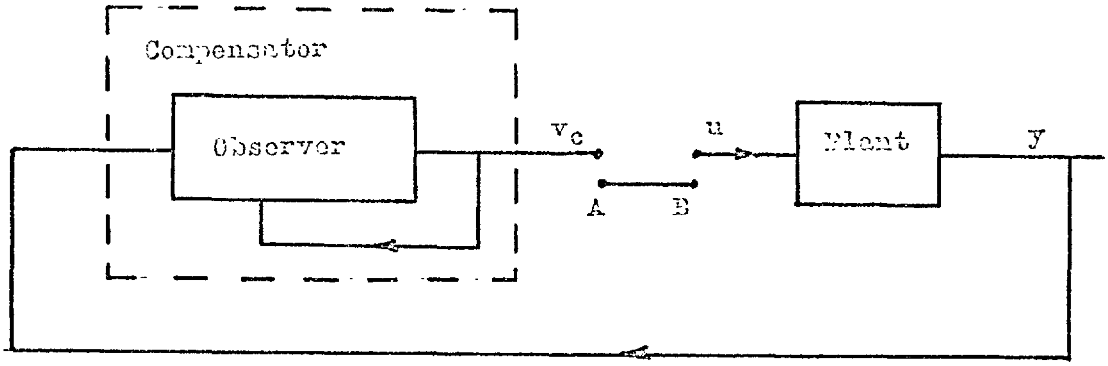


Figure 3.4.3 Open loop operation of plant and compensator derived from observer.

In any real application the plant dynamics will differ from those assumed and this can to some extent be allowed for by considering error dynamics in the link AB of Figures 3.4.1 and 3.4.3. The loop gain from u to $\hat{v}_c$ is $L_c(s)$ defined in terms of the transfer functions of the compensators ,

$$\hat{v}_c(s) = K(sI - (D - BK))^{-1} T\, y(s)$$

$$\triangleq F(s)\, y(s) , \qquad (3.4.44)$$

and using the subscript r for the reduced state observer-compensator,

$$\hat{v}_c(s) = \left[K_1 + K_2T_1 + K_2(sI - (D - CK_2))^{-1}(E - K_1 - K_2T_1)\right]y(s)$$

$$\triangleq F_r(s)\, y(s) \tag{3.4.45}$$

since by definition the plant output is denoted by x_1 and y.

If P(s) is the plant transfer function

$$y(s) = H(sI - A)^{-1}\, Bu(s)$$

$$= P(s)\, u(s)\ , \tag{3.4.46}$$

then (5.4.41) and (5.4.43) become

$$\hat{v}_c(s) = F(s)\, P(s)\, u(s)$$

$$\triangleq L_c(s)\, u(s)\ , \tag{3.4.47}$$

and

$$\hat{v}_c(s) = F_r(s)\, P(s)\, u(s)$$

$$\triangleq L_{cr}(s)\, u(s)\ . \tag{3.4.48}$$

The ideal loop gain L(s) is given as

$$v(s) = K(sI - A)^{-1}\, B\, \ddot{u}(s)$$

$$= L(s)\, u(s)\ . \tag{3.4.49}$$

It is clear that the loop gains $L_c(s)$ and $L_{cr}(s)$ do not equal the ideal loop gain $L(s)$. An explicit relation between $L_c(s)$ or $L_{cr}(s)$ and $L(s)$ is obtainable.

Define the transfer functions $M_1(s)$, $M_2(s)$ from the observer equations for n or (n-m)-state observers as the case may be, such that

$$\hat{v}(s) = M_1(s)\, y(s) + M_2(s)\, u(s) \tag{3.4.50}$$

$$= (M_1(s)\, P(s) + M_2(s))\, u(s)$$

$$= L(s)\, u(s)\ . \tag{3.4.51}$$

This observer is used in the feedback configuration of Figure 3.4.3 with the ideal control ,

$$u = -v\ ,$$

$$v = Kx\ .$$

Again using subscript c for compensator configuration, (3.4.50) gives

$$\hat{v}_c(s) = M_1(s)\, y(s) - M_2(s)\, \hat{v}_c(s)\ ,$$

so that with (3.4.46)

$$\hat{v}_c(s) = (I + M_2(s))^{-1} M_1(s)\, y(s)$$

$$= (I + M_2(s))^{-1} M_1(s)\, P(s)\, u(s)$$

$$\triangleq L_c(s)\, u(s) \; . \tag{3.4.52}$$

Then substituting $M_1(s)\, P(s)$ from (3.4.51) into (3.4.52) ,

$$L_c(s) = (I + M_2(s))^{-1}(L(s) - M_2(s)) \; . \tag{3.4.53}$$

An expression connecting return differences of the compensator observer configuration is derived ,

$$T_c(s) = I + L_c(s)$$

$$= (I + M_2(s))^{-1} (I + M_2(s) + L(s) - M_2(s))$$

$$= (I + M_2(s))^{-1} T(s) \; , \tag{3.4.54}$$

where T(s) is the ideal return difference.

Equations (3.4.53) and (3.4.54) emphasize the difference in the situations when all plant states are measured and when an observer is used in its most useful role as a compensating element in a feedback path. The inequality of L_c or L_{cr} and L arises from the minor loop feedback from the plant input u to the observer dynamics (Figure 3.4.3), which is required so that the stringent equality $\hat{L}(s) = L(s)$ of Theorems 3.4.1 and 3.4.2 is satisfied. Because of this, nonlinearities or linear dynamics in the link AB of Figure 3.4.3 have different closed loop effects from the same nonlinearities or dynamics in the link AB of Figure 3.4.1.

Sections 3.2 and 3.3 gave a straightforward method of obtaining an arbitrary set of feasible poles for the observer dynamics, and in Section 3.1 it was proven that observers in feedback paths had the desirable property that the observer-plant composite system had the poles of the observer and the ideal plant. From consideration of that property alone it may seem that not much is lost by not having measurements of all plant states. This section shows the fundamental disadvantages associated with the use of observers with respect to plant parameter changes of a particular type in comparison with the situation when all states are measured.

3.5 Sensitivity of Observers to Errors

In any application of an observer in estimating unmeasured states of a plant, what in fact is attempted is the estimation of the states of a plant model which may be expected to be an approximate description of plant dynamic behaviour. We will consider n and (n-m)-state observers used in an open loop configuration for small perturbations of plant model parameters, but will not undertake a very full analysis. First order perturbation effects are calculated in a straightforward manner for particular examples and will not be dealt with here. A qualitative idea of perturbation effects is desired.

n-state observer

The observer equation of the plant model equations (3.1.1), (3.1.2) is

$$\dot{\hat{x}} = (A - TH)\hat{x} + Ty + Bu \, . \tag{3.5.1}$$

It is now assumed that the plant equations have the same order as the model but are ,

$$\dot{x} = \underline{A}x + \underline{B}u \, , \tag{3.5.2}$$

$$y = \underline{H}x. \tag{3.5.3}$$

The estimate error Δx

$$\Delta x = \hat{x} - x$$

has dynamics

$$\dot{\Delta x} = \hat{x} - x$$

$$= (A - TH)\Delta x + (A - \underline{A} - T(H - \underline{H}))x + (B - \underline{B})u \qquad \ldots\ldots \quad (3.5.4)$$

$$\triangleq (A - TH)\Delta x + (\Delta A - T\Delta H)x + \Delta Bu \,. \qquad (3.5.5)$$

Equation (3.5.5) defines ΔA, ΔH and ΔB. Figure 3.5.1 gives the block diagram of (3.5.5), the most important qualitative feature is that the perturbation of the plant output matrix acts through the gain T.

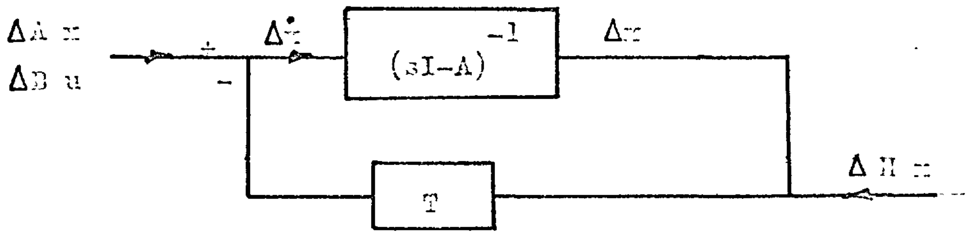


Figure 3.5.1 n-state Observer Estimation Error Dynamics for Perturbed Plant.

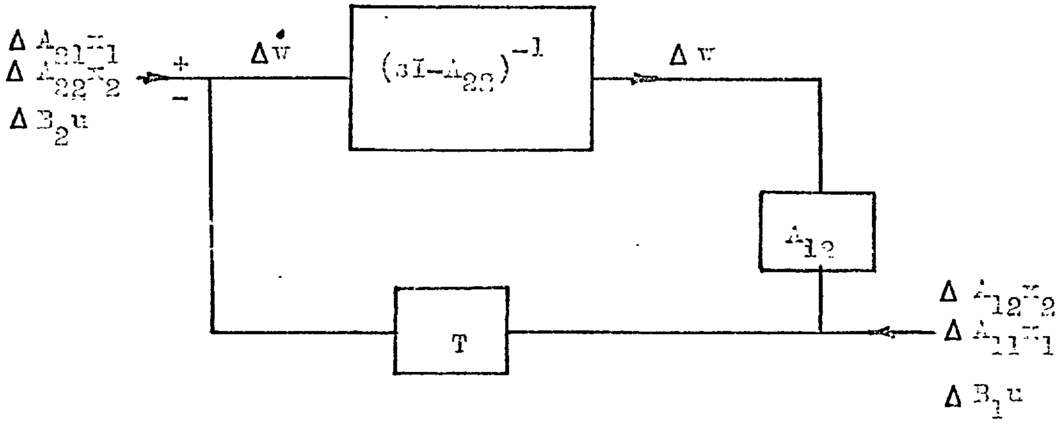


Figure 3.5.2 (n-m)-state Observer Estimation Error Dynamics for Perturbed Plant.

(n-m)-state observers

The observer corresponding to the nominal plant dynamics is

$$\dot{\hat{w}} = (A_{22} - TA_{12})\hat{w} + (A_{22}T + A_{21} - TA_{11} - TA_{12}T)x_1$$
$$+ (B_2 - TB_1)u \text{ .} \tag{3.5.6}$$

Suppose that the plant dynamics have a perturbed value

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \underline{B}_1 \\ \underline{B}_2 \end{pmatrix} u \text{ .} \tag{3.5.7}$$

Then

$$w = x_2 - Tx_1$$

$$\Delta w = \hat{w} - w$$

$$\begin{aligned}
\Delta\dot{w} &= \frac{d}{dt}\hat{w} - \dot{x}_2 + T\dot{x}_1 \\
&= (A_{22} - TA_{12})\Delta w + (A_{22} - \underline{A}_{22} - T(A_{12} - \underline{A}_{12}))x_2 \\
&\quad + (A_{21} - \underline{A}_{21} - T(A_{11} - \underline{A}_{11}))x_1 + (B_2 - \underline{B}_2 - T(B_1 - \underline{B}_1))u \\
&\triangleq (A_{22} - TA_{12})\Delta w + (\Delta A_{22} - T\Delta A_{12})x_2 \\
&\quad + (\Delta A_{21} - T\Delta A_{11})x_1 + (\Delta B_2 - T\Delta B_1)u\ . \qquad (3.5.8)
\end{aligned}$$

The block diagram of (3.5.8) is shown in Figure 3.5.2. The essential difference between the n and (n-m)-state observers is that the latter is affected by a perturbation containing a control term which acts through the observer gain T. It may be expected that if control magnitudes are large, arising from an attempt to obtain fast plant responses, this term will adversely affect the estimate produced by the (n-m)-state observer. The quadratic term in T in (3.5.6) is seen to be internal to the observer, so that it is not subject to plant variations and should create no fundamental problem. We conclude that under normal conditions and in the deterministic case, there is no priori reason why one type of observer has better sensitivity properties than the other for time invariant plant parameter variations.

CHAPTER 4

DEGENERATE OBSERVERS

4.1 Introduction

We have seen in Chapter 3 that n-and (n-m)-state observers may always be found for observable plant and that the observers may be given arbitrary feasible poles. In this chapter we examine the possibility of obtaining observers of dimension less than (n-m) to provide estimates of a desired vector function of state. In this case the observer will depend on the desired estimate, unlike the n-or (n-m)-state observers.

Definition 4.1.1 Observers having dimension less than (n-m) and providing desired estimates of functions of plant states will be called degenerate observers.

Previous interesting work by Luenberger [L2] considers the case of a multi-output plant and scalar state function for which an estimate is required; it is shown there that a suitable degenerate observer always exists having arbitrary feasible poles and dimension $(p_o - 1)$ where p_o is the observability index of the plant.

We extend these results by developing a theory of degenerate observers and consider some of the problems involved in the design of multi-variable degenerate observers of least order with arbitrary located poles. The investigation is partially successful since sufficiency conditions for the existence of a degenerate observer with arbitrarily locatable poles are found. A design method based on the sufficiency conditions is proposed, but it will not necessarily lead to observers of least order.

The major part of the design of a degenerate observer providing an estimate of a 2-vector function of the states of a 5-state plant, is presented to illustrate the use and advantages of the theory.

4.2 Dynamic Equations of Plant and Degenerate Observer

Consider an n-state plant with r measured inputs u and m measured outputs y, and such that its state space realisation has the form ,

$$\dot{x} = Ax + Bu \tag{4.2.1}$$

$$y = Hx \tag{4.2.2}$$

$$= x_1 \, . \tag{4.2.3}$$

Suppose that an estimate $\hat{v}$ is required for the state n_v-vector function v, where

$$n_v < n-m \, ,$$

$$v = Kx$$

$$\triangleq K_1x_1 + K_2x_2 \, , \tag{4.2.4}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

It may be assumed that the variables y and v are independent, because otherwise there exists a variable v_o of smaller dimension than v such that v is a function of y and v_o. An estimate $\hat{v}$ could then be found from an estimate $\hat{v}_o$ of v_o. It is always possible to find an (n-m)-state observer that provides an estimate $\hat{v}$; the problem is to examine conditions for the existence of suitable observers of lower dimension.

Form of Degenerate Observer

A general form of degenerate observer of dimension n_w , $n_v \leqslant n_w \leqslant (n-m)$ that has all measured plant variables as inputs is

$$\frac{d}{dt}\hat{w} = D\hat{w} + Ex_1 + Cu \ , \qquad (4.2.5)$$

$$\hat{v} = \underline{H}\hat{w} + \underline{G}x_1 \ , \qquad (4.2.6)$$

where $\hat{w}$ is an estimate of the variable

$$\begin{aligned} w &= Wx \\ &= W_1x_1 + W_2x_2 \ , \end{aligned} \qquad (4.2.7)$$

and there is a constraint on W such that

$$v = \underline{H}w + \underline{G}x_1 \ , \qquad (4.2.8)$$

that is

$$K = \underline{H}W + \underline{G}H \ . \qquad (4.2.9)$$

We define the estimate errors

$$\Delta v = \hat{v} - v \qquad (4.2.10)$$

$$\Delta w = \hat{w} - w \qquad (4.2.11)$$

so that

$$\Delta v = \underline{H}\Delta W \; . \tag{4.2.12}$$

The error dynamics in Δw are

$$\begin{aligned}\dot{\Delta w} &= \frac{d}{dt}\hat{w} - \dot{w} \\ &= D\hat{W} + Ex_1 + Cu - W(Ax + Bu) \\ &= D\Delta\hat{W} + (DW_1 + E - W_1A_{11} - W_2A_{21})x_1 \\ &\quad + (DW_2 - W_1A_{12} - W_2A_{22})x_2 \\ &\quad + (C - W_1B_1 - W_2B_2)u \; .\end{aligned} \tag{4.2.13}$$

This reduces to

$$\dot{\Delta w} = D\Delta w \tag{4.2.14}$$

if and only if

$$C = W_1B_1 + W_2B_2 \; , \tag{4.2.15}$$

$$E = W_1A_{11} + W_2A_{21} - DW_1 \; , \tag{4.2.16}$$

and

$$DW_2 = W_1A_{12} + W_2A_{22} \; . \tag{4.2.17}$$

Equations (4.2.15), (4.2.16) are satisfied by using them to define C and E. Then the existence of $\underline{H}$, W, $\underline{G}$ and D such that (4.2.9) and

(4.2.17) are satisfied determines the existence of a suitable observer, provided that the error dynamics (4.2.14) are satisfactory. In order that there be a choice in the dynamics of D, there must be degrees of freedom in the D that satisfy (4.2.17). Degrees of freedom may be introduced by allowing W_1 to vary, say to W_1^* so that (4.2.7) defines a new variable w^*,

$$w^* = W_1^* x_1 + W_2 x_2 , \qquad (4.2.18)$$

and $\hat{v}$ is recovered from $\hat{w}^*$ according to

$$\hat{v} = \underline{H}W^* + (\underline{G} + \underline{H}W_1 - \underline{H}W_1^*)x_1 \qquad (4.2.19)$$

$$\triangleq \underline{H}W^* + \underline{G}^* x_1 .$$

Equation (4.2.17) may be manipulated to

$$(DW_2 - W_1^* A_{12}) = W_2 A_{22} , \qquad (4.2.20)$$

so that D and W_1 may be regarded as free variables. (4.2.20) has an interesting interpretation described in Section 4.3.

We summarise the results obtained so far.

First Form for Sufficiency Conditions for the Existence of Degenerate Observer

A degenerate observer exists for the dynamics

$$\begin{aligned} \dot{x} &= Ax + Bu , \\ y &= Hx \\ &= x_1 , \end{aligned} \tag{4.2.21}$$

and provides an estimate $\hat{v}$ of v

$$\begin{aligned} v &= Kx \\ &= K_1 x_1 + K_2 x_2 , \end{aligned} \tag{4.2.22}$$

if there exists D, $\underline{H}$, W, $\underline{G}$ satisfying

$$(DW_2 - W_1 A_{12}) = W_2 A_{22} , \tag{4.2.23}$$

$$K = \underline{H}W + \underline{G}H. \tag{4.2.24}$$

The observer dynamics are

$$\begin{aligned} \frac{d}{dt}\hat{w} &= D\hat{w} + (W_1 A_{11} + W_2 A_{21} - DW_1)x_1 \\ &\quad + (W_1 B_1 + W_2 B_2)u , \end{aligned} \tag{4.2.25}$$

$$\hat{v} = \underline{H}\hat{w} + \underline{G}x_1 , \tag{4.2.26}$$

and the estimation errors evolve according to

$$\Delta\dot{w} = D\Delta w , \tag{4.2.27}$$

$$\Delta v = \underline{H}\Delta w . \tag{4.2.28}$$

Furthermore the observer poles may be prespecified if the degrees of freedom in choosing D to satisfy (4.2.23) allow arbitrary feasible poles of D to be achieved.

We note that no solution may exist for a trial dimension n_w of w and in this case the dimension must be increased. We are guaranteed that a suitable observer exists for $n_w = (n-m)$ by the theory of Chapter 3.

4.3 Reformulation of Existence Conditions

Condition (4.2.23) may be interpreted as a requirement that the time rate of change $\dot{w}$ of the n_w-vector w depends only on w, u and x_1, for

$$
\begin{aligned}
w &= W_1x_1 + W_2x_2 \,, \\
\dot{w} &= W_1(A_{11}x_1 + A_{12}x_2) + (W_1B_1 + W_2B_2)u \\
&\quad + W_2(A_{21}x_1 + A_{22}x_2) + Dw - Dw \\
&= (W_1A_{11} + W_2A_{21} - DW_1)x_1 + Dw \\
&\quad + (W_1A_{12} + W_2A_{22} - DW_2)x_2 \\
&\quad + (W_1B_1 + W_2B_2)u \,.
\end{aligned}
$$

Degrees of freedom in D in solutions (D, W_1) of (4.2.23) may be introduced in different ways.

A special situation holds when the desired estimate is a scalar; this is discussed in Section 4.5 where it is apparent that there are difficulties in extending the approach to the multivariable case.

An approach that may be used in the multivariable case follows from the Lemma 4.3.1.

Lemma 4.3.1

The existence of linear combinations of the output of the unforced plant such that their time rate of change depends only on x_1 and w, is a sufficient condition that a solution of the observer equations has degrees of freedom in D.

Proof

Consider the (n_c, m) matrix L_1 whose rows form a basis of maximum rank for all such linear combinations. That is, any such linear combination of plant output may be expressed as $\alpha' L_1 x_1$ where α' is a row vector. Then by hypothesis there exist L_2, L_3 such that

$$\frac{d}{dt}(L_1 x_1) = L_3 x_1 + L_2 w , \qquad (4.3.1)$$

$$= (L_1 + L_2 W_1) x_1 + L_2 W_2 x_2 . \qquad (4.3.2)$$

But the plant dynamics (4.2.1) give for zero input ,

$$L_1 \dot{x}_1 = L_1 A_{11} x_1 + L_1 A_{12} x_2 , \qquad (4.3.3)$$

so that equating coefficients of x_2 in (4.3.2) and (4.3.3) ,

$$(L_1 A_{12} - L_2 W_2) = 0 . \qquad (4.3.4)$$

If T is any (n_w, n_2)-matrix, (4.3.4) may be premultiplied by T and added to (4.2.16) to give ,

$$(D + TL_2)W_2 - (W_1 + TL_1)A_{12} = W_2A_{22}$$

so that if $(D, \underline{G}, W_1, W_2)$ is any solution of (4.2.16) then so is $(D^*, \underline{G}^*, W_1^*, W_2)$ given by

$$D^* = D + TL_2 , \tag{4.3.5}$$

$$W_1^* = W_1 + TL_1 , \tag{4.3.6}$$

$$\underline{G}^* = \underline{G} + \underline{H}W_1 - H_1W_1^* . \tag{4.3.7}$$

This concludes the proof of the lemma.

Alternative Approach to Degenerate Observer Equations

Lemma 4.3.1 leads to a straightforward approach to the theory of degenerate observers that closely parallels the theory of (n-m)-state observer design. This approach is presented as a theorem.

Theorem 4.3.1

If there exists a plant n_w-variable w such that the plant n_v-variable v to be estimated is expressable in terms of w and the plant output y, and the dynamics of w depend only on y, w and the plant input u, a degenerate observer exists. The existence of linear combinations of y with dynamics depending only on y, w and u is a sufficient condition for the existence of degrees of freedom in the dynamics of the observer. If the pair (D, L_2) to be later defined is observable, the observer dynamics may be given arbitrary feasible poles.

Proof

Consider the plant

$$\dot{x} = Ax + Bu \tag{4.3.8}$$

$$y = Hx \tag{4.3.9}$$

and suppose that it is desired to estimate v, and that

$$v = Kx , \tag{4.3.10}$$

$$w = Wx , \tag{4.3.11}$$

$$K = \underline{G}H + \underline{H}W \;, \qquad (4.3.12)$$

$$\dot{w} = Dw + Ex_1 + Cu \;. \qquad (4.3.13)$$

By hypothesis there exist linear combinations of plant output y such that

$$\begin{aligned} L_1\dot{y} &= L_1H\dot{x} \\ &= L_3y + L_2w + L_1B_1u \end{aligned} \qquad (4.3.14)$$

The device used in designing reduced observers of treating Ly as if it is directly measured suggests a form of n_w-state observer for the n_w-vector w ,

$$\begin{aligned} \frac{d}{dt}\hat{w} &= D\hat{w} + Ex_1 + Cu \\ &\quad + T(L_1\dot{y} - L_3y - L_2\hat{w} - L_1B_1u) \;. \end{aligned} \qquad (4.3.15)$$

Rearranging ,

$$\begin{aligned} \frac{d}{dt}(\hat{w} + TL_1y) &= (D - TL_2)\hat{w} + (E - TL_3)y + (C - TL_1B_1)u \\ &= (D - TL_2)(\hat{w} + TL_1y) \\ &\quad + (E - TL_3 - DTL_1 + TL_2TL_1)y \\ &\quad + (C - TL_1B_1)u \;. \end{aligned} \qquad (4.3.16)$$

Define a new estimation variable $\hat{w}^*$,

$$\hat{w}^* = \hat{w} + TL_1 y \, , \tag{4.3.17}$$

then the equations of the observer become,

$$\frac{d}{dt}\hat{w}^* = (D - TL_2)\hat{w}^* + (E - TL_3 - DTL_1 + TL_2TL_1)y$$

$$+ (C - TL_1B_1)u \, , \tag{4.3.18}$$

$$\hat{w} = \hat{w}^* - TL_1 y \, , \tag{4.3.19}$$

and therefore from (4.3.10) and (4.3.12)

$$\hat{v} = (\underline{G} - \underline{H}TL_1)y + \underline{H}\hat{w}^* . \tag{4.3.20}$$

It follows from (4.3.13) and (4.3.15) that an estimation error Δw satisfies

$$\Delta w \triangleq \hat{w} - w$$

$$= \hat{w}^* - w^*$$

$$\Delta\dot{w} = (D - TL_2)\Delta w \, . \tag{4.3.21}$$

The observer dynamics (4.3.18) may be given arbitrary feasible poles when the pair (D, L_2) defined in (4.3.13) and (4.3.14) is observable.

Comment

Note that when L_1 is the unit matrix of order m, the situation is identical with that of the design of an ordinary reduced observer of state dimension (n-m).

4.4 Application of the Theory of Degenerate Observers

Consider the problem of designing an observer of least order to provide an estimate of the n_v-vector v ,

$$v = Kx . \tag{4.4.1}$$

We assume that the matrix $\binom{K}{H}$ has full rank so that the observer has order of at least n_v. A systematic procedure follows for obtaining a w which leads to an observer of low order but not necessarily stable or having locatable poles.

Define the sets of vectors $\{k_i\}$ and $\{h_i\}$ and the sets of scalars $\{v_i\}$ and $\{x_i\}$

$$v = (v_1, v_2 \dots v_{n_v})' , \tag{4.4.2}$$

$$x = (x_1, x_2 \dots x_n)' , \tag{4.4.3}$$

$$K' = (k_1, k_2 \dots k_{n_v}) , \tag{4.4.4}$$

and

$$H' = (h_1, h_2 \dots h_m) . \tag{4.4.5}$$

A suitable w with dynamics that satisfy (4.3.13) and related to v by the form (4.3.12) is obtained by constructing the $(m+n_v, n)$ array (4.4.6) with vector elements ,

$$\begin{bmatrix}
h_1 & A'h_1 & \cdot & \cdot & & & & & & A'^{n-1}h_1 \\
h_2 & A'h_2 & \cdot & \cdot & & & & & & A'^{n-1}h_2 \\
\cdot & \cdot & \cdot & \cdot & & & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & & & & & \cdot \\
h_m & A'h_m & \cdot & \cdot & & & & & & 0 \\
k_1 & A'k_1 & \cdot & \cdot & \cdot & \cdot & \cdot & & 0 & 0 \\
k_2 & A'k_2 & \cdot & \cdot & A'^{p_2-1}k_2 & 0 & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & & & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & & & & & \cdot \\
k_{n_v} & A'k_{n_v} & \cdot & \cdot & & & \cdot & \cdot & \cdot & 0
\end{bmatrix} . \qquad (4.4.6)$$

The generation of vector elements A'^jk_i in the row containing k_i ceases at the first power p_i of A' at which the vector $A'^{p_i}k_i$ is linearly dependent on the vector elements of the columns to the left of column (p_i+1), together with the vector elements of the column

(p_i+1) above the row containing k_i. The remainder of the row is completed with null n-vectors.

By construction there exist sets of scalars $\{a_{ij}(j)\}$ and $\{b_{ij}(j)\}$ such that ,

$$A'^{p_i} k_i + \sum_{q=1}^{n_v} \sum_{j=0}^{p_i(q)} a_{jq}(i)(A')^j k_q$$

$$+ \sum_{q=1}^{m} \sum_{j=0}^{p_i} b_{jq}(i)\, A'^j h_q \quad = \quad 0 \, , \tag{4.4.7}$$

for $i = 1, 2 \ldots n_v$, where

$$p_i(q) \quad = \quad p_i \qquad \text{if} \quad q > i \, , \tag{4.4.8a}$$

$$\quad = \quad p_i - 1 \qquad \text{if} \quad q \leqslant i \, . \tag{4.4.8b}$$

Define the notation

$$x_i^{(j)} \quad = \quad \frac{d^j}{dt^j}(x_i) \, . \tag{4.4.9}$$

The set of equations (4.4.7) when transposed and multiplied on the left by the state x may be interpreted for zero plant input as ,

$$v_i^{(p_i)} + \sum_{q=1}^{n_v} \sum_{j=0}^{p_i(q)} a_{jq}(i)\, v_q^{(j)}$$

$$+ \sum_{q=1}^{m} \sum_{j=0}^{p_i} b_{jq}(i)\, y_q^{(j)} = 0 . \qquad (4.4.10)$$

Because $v_i^{(p_i)}$ depends on no derivatives of order higher than p_i, this set of differential equations has a state space realisation derived in the appendix Section 4.8 of this chapter ,

$$\dot{w} = Dw + Ey , \qquad (4.4.11)$$

and

$$v = \underline{H}w + \underline{G}y . \qquad (4.4.12)$$

The vector w is related to the original state vector as indicated by (4.8.22), (4.8.23) of the appendix ,

$$w = Wx . \qquad (4.4.13)$$

When the plant has an input u ,

$$\begin{aligned} \dot{w} &= W\dot{x} \\ &= W(Ax + Bu) \\ &= Dw + Ey + WBu . \end{aligned} \qquad (4.4.14)$$

Equations (4.4.13), (4.4.14) have the form required by Theorem 4.3.1 to guarantee the existence of an observer but its dynamics may not be suitable. This leads to the main difficulty in the design of degenerate observers and it has not been completely solved. It is clear that in this case the dimension of w must be increased, which should give more degrees of freedom in the observer dynamics. When n_w reaches (n-m) the theory of (n-m)-state observers applies and there is always a satisfactory solution.

It is not obvious what is the most appropriate method for choosing a new w of increased dimension. The algorithms proposed contain a degree of arbitrariness, and it is clear that there is a need for further investigation directed towards obtaining observers of least order.

Algorithm A

(1) Increase the dimension of v by adjoining an arbitrary row to K in such a way that v and y remain independent.

(2) Calculate a vector w by the method using array (4.4.6).

(3) Check the observability of (D, L_2) and return to step (1) if necessary.

Algorithm B

In step (1) of algorithm A choose the row adjoined to K as a linear combination of the rows of HA, $v_{(n_v+1)} = \ell' HA$ say where ℓ is an arbitrary vector. This ensures that $\ell' y$ will be a function of w and y, and there is a possibility that this will lead to observable (D, L_2).

Algorithm C

The vector v remains unchanged. The dimension of w is increased by suppressing linear combinations of $A'^{j}H$ from the expanding basis used for the test of linear independence in the construction of the array (4.4.6). Again these linear combinations may be arbitrary and it is not clear which to choose.

Algorithm D

This proposal is similar to algorithm C except that linear combinations of $A'^{j}K$ are suppressed rather than those of $A'^{j}H$, in the test for finding $\{p_i\}$ in the array (4.4.6).

Comments on the Algorithms

None of the algorithms is completely satisfactory because of the arbitrary decisions that must be made. As the dimension of w increases and w is maintained independent of y, the dimension of w at which (D, L_2) is observable must be less than or equal to (n-m).

When n_w equals (n-m) the matrix $\binom{H}{W}$ is invertible and the plant state could be taken as $\binom{y}{w}$ and a (n-m)-state observer may always be found which has arbitrary pole locations. In general any combination of the four algorithms may be used during a design attempt and the arbitrariness in the method is compounded. The special cases scalar v and $v = x_2$, $n_v = (n-m)$ do not offer guidance. It is shown in Section 4.5 that essentially algorithm D is appropriate to the scalar case. The algorithms are not needed when $n_v = (n-m)$ because immediately (D, L_2) is observable.

4.5 A Comparison with Other Reduced Observers

The connection between (n-m)-state reduced observers and this type of degenerate observer is apparent from the similarity between the development of Theorem 4.3.1 in deriving degenerate observer dynamics and the derivation of the (n-m)-state observer of Section 3.3. When the dimension of v or w becomes (n-m), the two observer design methods are equivalent.

Luenberger [L2] provides a systematic technique for designing a degenerate observer that provides an estimate of a scalar function of the plant states. The plant equations are transformed to a canonical structure represented by blocks of dynamics, each block outputs one plant output, and the blocks are interconnected by paths from their outputs only. At an intermediate stage of the design algorithm, ordinary reduced observers are found for each block; this is possible because of the derived structure. The block states represent a transformation of the plant states in the original basis, and the desired scalar function of plant states is expressed as a function of the block states. The reduced observer for each block provides a scalar estimate for that block's contribution to the desired scalar function, and these sub-observers may be regarded as multi-input single-output transfer function operators. If it is arranged that these transfer functions all have the same poles, because their outputs are to be summed to provide the desired scalar estimate, the same effect may be achieved by summing the numerators and creating a single multi-input single-output transfer function of the same order.

Luenberger's scheme is valuable because it gives insight into the design process. Bass and Gura achieve the same final design but without gaining much insight. It is interesting to achieve the same results again using the theory of Sections 4.3 and 4.4.

The observability index p_o of the plant (4.2.1), (4.2.2) is defined as the least integer such that the matrix H_o ,

$$H_o' = (H', A'H' \ldots A'^{p_o - 1} H') , \tag{4.5.1}$$

has n independent columns, and because the plant is assumed to be observable

$$p_o \leq n . \tag{4.5.2}$$

Define the square matrix $\overline{H}_o'$ whose columns are the first n independent columns taken from H_o', then $\overline{H}_o$ is invertible.

The argument closely follows that of Section 4.4.

Suppose that a scalar estimate

$$v = k'x \tag{4.5.3}$$

is required, and define

$$K_o' = (k, A'k \dots A'^{p_o-2}k) . \tag{4.5.4}$$

Because $\bar{H}_o$ is invertible we may write for arbitrary vector ℓ, (4.5.5) which parallels (4.4.7) ,

$$\begin{aligned}(k'A^{p_o-1} + \ell'K_o) &= (k'A^{p_o-1} + \ell'K_o)\bar{H}_o^{-1}\bar{H}_o \\ &\triangleq (\ell_3' + \ell'K_o\bar{H}_o^{-1})\bar{H}_o .\end{aligned} \tag{4.5.5}$$

Post-multiply both sides of (4.5.5) by x and apply (4.5.3), (4.2.2) to obtain

$$v^{(p_o-1)} + \sum_1^{p_o-1} \ell_i\, v^{(i-1)} = (\ell_3' + \ell'K_o\bar{H}^{-1})\begin{pmatrix} y_{j_1}^{(i_1)} \\ y_{j_m}^{(i_m)} \end{pmatrix} \tag{4.5.6}$$

This is a $(p_o-1)^{th}$ order differential equation with inputs linear combinations of the elements of y and their first (p_o-1) derivatives.

The vector ℓ is arbitrary and may be chosen to give arbitrary poles to (4.5.6) consistent with real ℓ.

When ℓ is specified (4.5.6) may be realised by a (p_o-1) state system (4.5.7), (4.5.8) as discussed in Appendix (4.8) of this chapter ,

$$\dot{w} \quad = \quad Dw + \overline{C}y \ , \tag{4.5.7}$$

$$v \quad = \quad \overline{H}w + \overline{G}y \tag{4.5.8}$$

where the poles of D are those of (4.5.6).

The system (4.5.7), (4.5.8) has the form (4.3.16) where degrees of freedom in D have been introduced through ℓ directly in such a way that pole allocation is simply done before the relation

$$w \quad = \quad Wx$$

is found.

4.6 Example

This example demonstrates a straightforward application of the ideas in designing degenerate observers with arbitrarily specified poles. The conditions of Theorem 4.3.1. are satisfied immediately without augmenting the estimation vector w.

A 5-state, 2-input, 2-output plant is considered and numerical values in the dynamics have been taken to be small integers for convenient calculation. The example is not worked through completely, it is considered sufficient to find solutions D and L_2 and show that (D, L_2) is an observable pair.

The plant dynamics are given arbitrary values denoted by (.) for those variables that do not affect the dynamics of the observer in this example.

The example is interesting because a normal (n-m)-state observer has dimension 3, the Luenberger observer estimating a scalar function would require 2 states, the multivariable degenerate observer giving a 2-vector estimate has 2 states. It is apparent that there is an advantage in using the 2-state multivariable observer in this example. All the above observers have arbitrary feasible poles.

Consider the plant ,

$$\dot{x} = \begin{bmatrix} \cdot & \cdot & 0 & 0 & 1 \\ \cdot & \cdot & 2 & 1 & 1 \\ \cdot & \cdot & 1 & 0 & 2 \\ \cdot & \cdot & 0 & 1 & 1 \\ \cdot & \cdot & 1 & 2 & 1 \end{bmatrix} x + \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} u , \qquad (4.6.1)$$

with required estimate

$$v = \begin{bmatrix} \cdot & \cdot & 1 & 0 & 1 \\ \cdot & \cdot & 0 & 1 & 2 \end{bmatrix} x , \qquad (4.6.2)$$

and plant outputs

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} x . \qquad (4.6.3)$$

The design requires the construction of the array (3.4.6) which is a rearrangement of the elements of Table (4.6.4), symbollically in column 1, and numerically in columns 3 to 7 .

H	y_1	1	0	0	0	0
	y_2	0	1	0	0	0
K	v_1	.	.	1	0	1
	v_2	.	.	0	1	2
HA	y_1	.	.	0	0	1
	y_2	.	.	2	1	1
KA	v_1	.	.	2	2	3
	v_2	.	.	2	5	3

Table (4.6.4)

Column 2 of Table (4.6.4) indicates the variables obtained when column 1 is post-multiplied by x.

By inspection of Table (4.6.4) it is seen that $\dot{v}_1$ and $\dot{v}_2$ are expressable as functions of y, v and $\dot{y}$, so that by Theorem 4.3.1 a degenerate observer exists. The design may begin by putting the estimation vector w equal to v. It remains to investigate the observer dynamics.

A linear combination L_1x_1 is sought such that L_1x_1 when the plant is unforced is a linear combination of v and x_1. Table (4.6.5) with notation similar to that of Table (4.6.4) is constructed as an aid in obtaining the required linear dependencies.

y_1	1	0	0	0	0
y_2	0	1	0	0	0
$v_1 - y_1$	.	.	1	0	0
$v_2 - 2y_1$	.	.	0	1	0
y_1	.	.	0	0	1

<u>Table (4.6.5) : Convenient linear row combinations of Table (4.6.4)</u>

Denote a function of y by $f(y)$; $f(y)$ need not represent the same function at each appearance.

From Table (4.6.5) and Table (4.6.4) y_2 may be obtained ,

$$\dot{y}_2 = 2(v_1 - \dot{y}_1) + (v_2 - 2\dot{y}_1) + \dot{y}_1 + f(y) , \qquad (4.6.6)$$

and re-arranging ,

$$(3\dot{y}_1 + \dot{y}_2) = 2v_1 + v_2 + f(y) . \qquad (4.6.7)$$

The L_1 and L_2 of (4.3.14) are from (4.6.7) .

$$L_1 = (3 \quad 1) , \qquad (4.6.8)$$

and

$$L_2 = (2 \quad 1) . \qquad (4.6.9)$$

Table (4.6.4) and (4.6.5) are used again to obtain ,

$$\dot{v}_1 = 2(v_1 - \dot{y}_1) + 2(v_2 - 2\dot{y}_1) + 3\dot{y}_1 + f(y)$$

$$= 2v_1 + 2v_2 - 3\dot{y}_1 + f(y) , \qquad (4.6.10)$$

$$\dot{v}_2 = 2(v_1 - \dot{y}_1) + 5(v_2 - 2\dot{y}_1) + 3\dot{y}_1 + f(y)$$

$$= 2v_1 + 4v_2 - 9\dot{y}_1 + f(y) , \qquad (4.6.11)$$

so that

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \end{pmatrix} \dot{y}_1 + f(y) . \qquad (4.6.12)$$

The state space representation of (4.6.12) is ,

$$\dot{w} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} w + f(y) , \qquad (4.6.13)$$

and

$$v = w - \begin{pmatrix} 3 \\ 9 \end{pmatrix} y_1 . \qquad (4.6.14)$$

The D of (4.3.13) is from (4.6.13) ,

$$D = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} . \qquad (4.6.15)$$

The pair (D, L_2) may now be tested for observability. The observability array is constructed ,

$$\begin{pmatrix} L_2 \\ L_2 D \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 9 \end{pmatrix} ,$$

and found to have non-zero determinant. Therefore (D, L_2) is an observable pair. The form of the degenerate observer is (4.3.18), (4.3.20); numerical values have been found for D, L_1 and L_2 and because (D, L_2) is observable the observer dynamics (4.3.18) can be given arbitrary pole locations. E and C are found by comparing equation (4.6.13) with (4.3.13) once numerical values are given to the undefined parameters of the plant dynamics (4.6.1), (4.6.2). The observer equations (4.3.18), (4.3.20) are then found by direct substitution.

4.7 Conclusion

The design theory of degenerate observers that are required to provide a scalar estimate is complete. The theory for the multivariable case is less satisfactory, and although a systematic design method has been suggested, it by no means always achieves the observer of least order. This investigation has revealed some of the properties of multivariable degenerate observers, but further research is required to discover better methods of introducing and using degrees of freedom in the observer dynamics matrix D.

4.8 Appendix

This appendix discusses the state space realisation of transfer functions that occur in the design of degenerate observers, and specifically the differential equation (4.4.7).

Scalar Transfer Functions

Consider the differential equation (4.8.1) where u and v are scalars and u is taken as the independent variable ,

$$v^{(p)} + \sum_{o}^{p-1} a_i v^{(i)} = \sum_{i=0}^{p} b_i u^{(i)} . \tag{4.8.1}$$

A convenient state space realisation of (4.8.1) is shown in Figure 4.8.1, and with a p-state w, input u, and output v, has dynamics

$$\dot{w} = Dw + Bu , \tag{4.8.2}$$

$$v = w_p + b_p u , \tag{4.8.3}$$

$$D = \begin{pmatrix} 0 & & & -a_o \\ 1 & \ddots & & \vdots \\ & & 1 & -a_{p-1} \end{pmatrix} , \text{ companion form ,} \tag{4.8.4}$$

and

$$B = (b_o, b_1, \ldots b_{p-1}) . \tag{4.8.5}$$

It is easily verifiable from Fig. 4.8.1 that this state space realisation leads to differential equation (4.8.1).

Denote the transfer function from x_{p-r} to x_p with the system open circuited at the input of the r^{th} integrator from the left as L_r .

Then,

$$L_n = 1 ,$$

$$L_{p-1} = \frac{\frac{1}{s} L_r}{(1 + \frac{1}{s} L_r a_{p-r-1})}$$

$$= \frac{1}{(\frac{s}{L_r} + a_{p-r-1})} ,$$

and therefore by induction

$$L_o = \frac{1}{(s^p + \sum a_i s^i)} . \qquad (4.8.6)$$

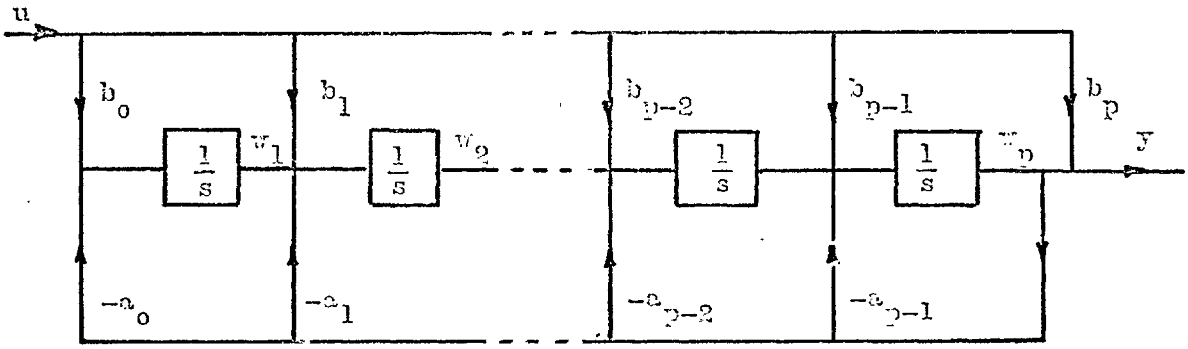


Figure 4.8.1 State space realization

$$\text{of} \quad y(s) = \frac{\sum_o^p b_i s^i}{s^p + \sum_o^{p-1} a_i s^i} \quad u(s) .$$

Referring to Fig. 4.8.1, L_o is the transfer function from the input of the leftmost integrator to the output y, and block diagram manipulation rules indicate that an input u(s) along the r^{th} input path from the left is equivalent to an input s u(s) along the $(r-1)^{th}$ input path. Therefore an input $(\sum_o^p b_i s^i)u(s)$ feeding into the leftmost integrator is equivalent to the input u(s) entering the system through the gains $(b_o, \ldots, b_n)$. A deeper analysis of this type of manipulation is given by Zadeh and Desoer [Z1] .

Multiple Input Transfer Functions

Consider an independent input vector u with components $(u_1, \ldots, u_r)$ so that ,

$$v^{(p)} + \sum_{i=o}^{p-1} a_i v^{(i)} = \sum_{i=1}^{r} \sum_{j=0}^{p} b_{ji} u_i^{(j)} , \tag{4.8.7}$$

then the state space realisation is (4.8.2), (4.8.4) with (4.8.8), (4.8.9),

$$v = w_p + \sum_{i=1}^{r} b_{pi} u_i , \tag{4.8.8}$$

and

$$B = \begin{bmatrix} b_{11} & b_{1r} \\ \cdot & \cdot \\ \cdot & \cdot \\ b_{p-1,1} & b_{p-1,r} \end{bmatrix} . \tag{4.8.9}$$

This follows from the linearity of state trajectories with respect to system inputs.

Realisation of the Set (4.4.10) of Differential Equations

Consider the set of differential equations (4.4.10) rewritten as (4.8.10) ,

$$v_i^{(p_i)} + \sum_{q=1}^{n_v} \sum_{j=0}^{p_i(q)} a_{jq}(i)\, v_q^{(j)}$$

$$+ \sum_{q=1}^{m} \sum_{j=0}^{p_i} b_{jq}(q)\, y_q^{(j)} = 0 , \qquad (4.8.10)$$

for $i = 1, 2, \ldots n_v$,

and $p_i(q)$ defined by (4.4.5).

Consider now the differential equation corresponding to a particular value of i. This may be regarded as a $(p_i)^{th}$ order system with inputs $(y_q : q = 1, 2, .. m)$,

$$(v_q : \; q = 1, 2, \ldots r \; ; \; q \neq i)$$

and derivatives of these variables of order not greater than p_i. It follows that there exists a (p_i)-dimensioned state space realisation of this differential equation of the form (4.8.2), (4.8.4), (4.8.8) and (4.8.9) with inputs $(y_q : q = 1, 2, .. m)$ and $(v_q : q = 1, 2, .. r; q \neq i)$, and output v_i. Similar blocks of dynamics exist for each i : $i = 1, 2, .. r$, and because the interaction between the blocks contains no dynamics the interconnected set of blocks may be regarded as a

$(\sum_i p_i)$-state realisation of (4.8.10) with independent inputs $(y_q : q = 1, 2, .. m)$ and outputs $(v_i : i = 1, 2, .. r)$.

This is the desired state representation of (4.8.10) and may be written,

$$\dot{w} = Dw + Ey , \tag{4.8.11}$$

and

$$v = \underline{H}w + \underline{G}y . \tag{4.8.12}$$

<u>Transformation relating state W to original state basis</u>

The vectors v and y of (4.8.10) are defined in terms of the plant state x ,

$$y = Hx , \tag{4.8.13}$$

$$v = Kx , \tag{4.8.14}$$

with unforced plant dynamics

$$\dot{x} = Ax . \tag{4.8.15}$$

It is required to find W such that w of (4.8.12) is given by

$$w = Wx . \tag{4.8.16}$$

The dynamics (4.8.11), (4.8.12) are observable because each of the component dynamic blocks is observable. This follows because each block has the form (4.8.2), (4.8.4) and (4.8.8) where measured variables v and y replace u in (4.8.8).

Define

$$\underline{H}D^{i-1} w = F_i x \, . \tag{4.8.17}$$

Then

$$\underline{H}D^{i-1} w = F_i x \, ,$$

so that from (4.8.11) and (4.8.13)

$$\underline{H}D^{i} w = (F_i A - EH)x \, ,$$

and therefore

$$F_{i+1} = (F_i A - EH) \, . \tag{4.8.18}$$

Starting with

$$F_1 = (K - \underline{G}H) \, , \tag{4.8.19}$$

the set of equations (4.8.20) may be constructed using (4.8.18) ,

$$\begin{bmatrix} \underline{H} \\ \underline{H}D \\ \cdot \\ \cdot \\ \underline{H}D^{\rho-1} \end{bmatrix} w = \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ F_p \end{bmatrix} x \, . \tag{4.8.20}$$

Since the system (4.8.11), (4.8.12) is observable p independent rows of the left hand side matrix may be selected and the resulting equations written as

$$H_o\, w \;=\; F_o\, x\ , \tag{4.8.21}$$

may be solved for w ,

$$w \;=\; H_o^{-1}\, F_o\, x\ , \tag{4.8.22}$$

and therefore

$$W \;=\; H_o^{-1}\, F_o . \tag{4.8.23}$$

Equation (4.8.23) is the required relation.

CHAPTER 5

STABLE COMPENSATORS OBTAINED FROM OBSERVERS

5.1 Introduction

We have seen in Chapter 3 that it is possible to design observers that provide an estimate of unmeasured plant states if the plant is observable. Furthermore, if an unrealizeable desired control contains feedback from unmeasured plant states and is implemented by a realizeable control which substitutes plant state estimates for the unmeasured plant states in the control law, the closed loop realizeable system has the same poles as the closed loop unrealizeable system together with the observer poles. This property holds if estimates are substituted for some or all of the measured plant states. The realizeable control is a good approximation to the unrealizeable control with respect to stability and transient response of the plant, for when the initial condition state estimation errors are zero the realizeable and unrealizeable controls generate identical plant trajectories. Observers used in this way may be regarded as compensators and have the configuration of Figure 3.3. As shown below the compensator and its component observer have different dynamics. The observer and the closed loop system of compensator and plant are designed to be stable, but the stability of the corresponding compensator may only be determined by stability analysis of particular cases. Classical single-input-single-output design techniques [H3] avoid the problem by working with stable compensation blocks at all times. Unfortunately the classical techniques are not easily extended to the multivariable case, though Rosenbrock [R2] has made some progress in multivariable compensator design using the inverse Nyquist array and graphical techniques. The general problem of the design of stable compensators is by no means solved. The particular solution of this Chapter is straightforward but restricted in application to plants stabilizeable from their measured outputs and by the form of control law that is required.

This work is original, though the idea that the stable dynamic system and Lyapunov function

$$\dot{x} = Ax ,$$

$$V(x) = x' Px ,$$

$$P > 0 ,$$

generate a family of stable systems

$$\dot{x} = (A - P^{-1}C)x ,$$

$$C \geqslant 0 ,$$

has been proposed by Barnett and Storey [B4], but not in the context of the design of stable compensators. Whether or not it is acceptable to have an unstable compensator must depend on the particular application. It is certainly desirable to have state space techniques for generating compensators that are guaranteed to be stable.

Plant, Observer and Compensator Equations

The plant has a state n-vector x, output m-vector y, and input r-vector u with dynamics

$$\dot{x} = Ax + Bu , \tag{5.1.1}$$

and

$$y = Hx . \tag{5.1.2}$$

We suppose that an unrealizeable, stable control law

$$u = Kx , \tag{5.1.3}$$

is desired but is implemented by substituting a state estimate $\hat{x}$ for x

$$u = K\hat{x} . \tag{5.1.4}$$

The dynamics of an n-state observer for the n-state plant have been derived as (3.2.5) rewritten

$$\frac{d}{dt}\hat{x} = (A - TH)\hat{x} + Bu + Ty , \tag{5.1.5}$$

and the estimation error has dynamics given by (3.1.7) and (3.2.3) as

$$\frac{d}{dt}\Delta x = (A - TH)\Delta x . \tag{5.1.6}$$

The dynamics of the closed loop composite system of plant and observer have been obtained (3.1.27) as

$$\frac{d}{dt}\begin{pmatrix} x \\ \Delta x \end{pmatrix} = \begin{pmatrix} A+BK & BK \\ 0 & A-TH \end{pmatrix}\begin{pmatrix} x \\ \Delta x \end{pmatrix} , \tag{5.1.7}$$

and the corresponding compensator dynamics are obtained from (5.1.4) and (5.1.5) as

$$\frac{d}{dt}\hat{x} = (A - TH + BK)\hat{x} + Ty \tag{5.1.8}$$

and

$$u = K\hat{x} . \tag{5.1.9}$$

The compensator has state variable $\hat{x}$, input y and output u .

By the choice of K and T the matrices (A - TH) and (A + BK) are stable, the stability of (A - TH + BK) has not entered the design of the observer gain T or the control gain K, and the compensator dynamics could possibly be unstable. A similar difficulty occurs when (n-m)-state compensators are derived from (n-m)-state observers.

The following Sections present a method for the design of a class of stable compensators.

5.2 A Stable n-State Compensator

The compensator is based on observer theory but does not approximate a given control law (5.1.3). The stable observer, compensator and closed loop plant dynamics are obtained simultaneously.

We examine the form of control law

$$u = L_2\hat{x} + L_1 y \ , \tag{5.2.1}$$

so that

$$u = L_2 \Delta x + (L_2 + L_1 H)x \ , \tag{5.2.2}$$

where L_1 and L_2 are free parameters. The observer dynamics (5.1.5) become,

$$\frac{d}{dt}\hat{x} = (A + BL_2 - TH)\hat{x} + (T + BL_1)y \ , \tag{5.2.3}$$

and the associated compensator equations are

$$\frac{d}{dt}\hat{x} = (A + BL_2 - TH)\hat{x} + (T + BL_1)y \ , \tag{5.2.4}$$

and

$$u = L_2\hat{x} + L_1 y \ . \tag{5.2.5}$$

The closed loop system is given by (5.1.1), (5.1.6) and (5.2.2) as

$$\frac{d}{dt}\binom{x}{\Delta x} = \begin{pmatrix} A+B(L_2+L_1H) & BL_2 \\ 0 & A-TH \end{pmatrix} \binom{x}{\Delta x} . \tag{5.2.6}$$

For our purposes, a suitable choice of L_1, L_2 and T give stable closed loop plant dynamics and a stable compensator. That is, the matrices $(A + B(L_2 + L_1H))$, $(A - TH)$ and $(A + BL_2 - TH)$ are required to be stable.

It is shown by construction that a sufficient condition that satisfactory L_1, L_2 and T can be found is that a stable control law ,

$$u = Jy \tag{5.2.7}$$

$$= JHx , \tag{5.2.8}$$

is known, so that $(A + BJH)$ is stable. A Lyapunov function, $V(x)$ for the plant with control law (5.2.8) may always be found as ,

$$V(x) = x'Px , \tag{5.2.9}$$

where P satisfies

$$(A + BJH)'P + P(A + BJH) = - Q , \tag{5.2.10}$$

and Q is an arbitrary symmetric positive definite matrix.

The following values are assigned to L_1, L_2 and T for arbitrary symmetric positive semidefinite R_1 and R_2 ,

$$L_1 = J , \tag{5.2.11}$$

$$L_2 = - R_1 B'P , \tag{5.2.12}$$

and

$$T = P^{-1}H' R_2 - BJ . \tag{5.2.13}$$

Because $Q > 0$ in (5.2.10) then $P > 0$ and the inverse of P exists. The control laws (5.2.11), (5.2.12), and (5.2.13) are shown to be stable by analysing the error, plant and the corresponding compensator dynamics

$$\frac{d}{dt} \hat{x} = (A + BJH - BR_1B'P - P^{-1}H'R_2)\hat{x} + P^{-1}H'R_2y , \tag{5.2.14}$$

$$u = Jy - R_1B'P\hat{x} . \tag{5.2.15}$$

Stability of the error dynamics

The rate of change of a trial Lyapunov function $V(\Delta x)$,

$$V(\Delta x) = \Delta x'P\Delta x ,$$

is considered and using (5.1.6), (5.2.13) and (5.2.10) we have ,

$$\begin{aligned}
\dot{V}(\Delta x) &= \Delta\dot{x}'P\,\Delta x + \Delta x'P\,\Delta\dot{x} \\
&= \Delta x'((A - TH)'P + P(A - TH))\,\Delta x \\
&= \Delta x'((A + BJH)'P + P(A + BJH) - 2H'R_2H)\,\Delta x \\
&= \Delta x'(-Q - 2H'R_2H)\,\Delta x \\
&\leqslant -\Delta x'Q\,\Delta x \qquad (5.2.16) \\
&< 0 \,. \qquad (5.2.17)
\end{aligned}$$

The positive definite function $V(\Delta x)$ decreases for all Δx and is therefore a Lyapunov function. The matrix $(A - TH)$ is stable and (5.2.16) shows that in a sense [B4] , the error is likely to have a faster response than $(A + JH)$.

Stability of Closed Loop Plant

Equation (5.2.6) shows that the closed loop dynamics are stable if $(A + B(L_2 + L_1H)$ is stable, since $(A - TH)$ is stable.

Stability is analysed by considering an autonomous system ,

$$\dot{x} = (A + B(L_2 + L_1H))x \ , \qquad (5.2.18)$$

and testing a trial Lyapunov function $V(x)$,

$$V(x) = x'Px \ . \qquad (5.2.19)$$

Then (5.2.19), (5.2.18), (5.2.11), (5.2.12) and (5.2.10) give ,

$$\begin{aligned} \dot{V}(x) &= \dot{x}'Px + x'P\dot{x} \\ &= x'(A + B(- R_1B'P + JH))'Px \\ &\quad + x'P(A + B(- R_1B'P + JH)x \\ &= x'((A + BJH)'P + P(A + BJH))x \\ &\quad - 2x'PBR_1B'Px \\ &= x'(- Q - 2PBR_1B'P)x & (5.2.20) \\ &\leqslant - x'Qx & (5.2.21) \\ &< 0 \ . & (5.2.22) \end{aligned}$$

Equation (5.2.22) shows that V(x) is a Lyapunov function for the dynamics (5.2.18), which again are likely to be faster than those corresponding to the control law (5.2.7).

Stability of the Compensator

The positive definite matrix P also defines a suitable trial Lyapunov function to test the stability of the compensator dynamics (5.2.4), the autonomous form of which is ,

$$\dot{\hat{x}} = (A + BL_2 - TH)\hat{x} \; . \tag{5.2.23}$$

Then

$$V(\hat{x}) = \hat{x}'P\hat{x} \; , \tag{5.2.24}$$

and use of (5.2.23), (5.2.12), (5.2.13) and (5.2.10) gives ,

$$\begin{aligned} \dot{V}(\hat{x}) &= \dot{\hat{x}}'P\hat{x} + \hat{x}'P\dot{\hat{x}} \\ &= \hat{x}'((A + BL_2 - TH)'P + P(A + BL_2 - TH))\hat{x} \\ &= \hat{x}'((A + BJH)'P + P(A + BJH))\hat{x} \\ &\quad - 2\hat{x}'(PBR_1B'P + H'R_2H)\hat{x} \\ &= \hat{x}'(- Q - 2PBR_1B'P - 2H'R_2H)\hat{x} \end{aligned} \tag{5.2.25}$$

$$\leqslant \; - x'Qx \tag{5.2.26}$$

$$< \; 0 \, . \tag{5.2.27}$$

The trial function $V(\hat{x})$ is therefore a Lyapunov function for the dynamics (5.2.23) and the compensator is stable.

<u>Example</u>

A stable compensator is designed for the simple stable plant

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \, ,$$

and

$$y = (1 \quad 0) \, x \, .$$

These equations correspond to (5.1.1) and (5.1.2). Because the plant is stable we may take

$$J = 0 \, ,$$

in (5.2.7). The matrix Q in (5.2.10) is arbitrarily chosen ,

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$> 0 \, ,$$

and (5.2.10) may be solved for P ,

$$P = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 0.8 & -0.4 \\ -0.4 & 1.2 \end{pmatrix}.$$

Substitution into (5.2.12) and (5.2.13) for P and P^{-1} gives

$$\begin{aligned} L_2 &= -R_1B'P \\ &= -R_1(0.5 \quad 1) , \\ T &= P^{-1}H'R_2 \\ &= \begin{pmatrix} 0.8 \\ -0.4 \end{pmatrix} R_2 . \end{aligned}$$

The designed control law (5.2.1) involving L_1 and L_2 defines K in (5.1.4) so that in the example ,

$$\begin{aligned} u &= L_2x \\ &\triangleq Kx . \end{aligned}$$

The matrices (A + BK), (A - TH) and (A + BK - TH) governing the closed loop system and compensator dynamics (5.1.6) and (5.1.8) become

$$(A + BK) = \begin{pmatrix} 0 & 1 \\ -1-0.5R_1 & -1-R_1 \end{pmatrix},$$

$$(A - TH) = \begin{pmatrix} -0.8R_2 & 1 \\ 0.4R_2-1 & -1 \end{pmatrix},$$

and

$$(A + BK - TH) = \begin{pmatrix} -0.8R_2 & 1 \\ -1-0.5R_1+.4R_2 & -1-R_1 \end{pmatrix}.$$

The above matrices have characteristic equations respectively

$$s^2 + (1 + R_1)s + (1 + 0.5R_1) = 0 ,$$

$$s^2 + (1 + 0.8R_2)s + (1 + 0.4R_2) = 0 ,$$

and

$$s^2 + (1 + R_1 + 0.8R_2)s + (1 + 0.5R_1 + 0.4R_2 + 0.8R_1R_2) = 0 ,$$

which have roots in the left half plane for all positive R_1 and R_2.

5.3 An (n-m)-State Stable Compensator

It is possible to design an (n-m)-state stable compensator for an initially stable plant, however no degrees of freedom exist for the design parameters that are analogues to T in (5.1.5). Consider the dynamic system (5.1.1) and (5.1.2), where the state co-ordinate basis is chosen so that (5.1.2) becomes

$$\begin{aligned} y &= Hx \\ &= x_1 , \end{aligned} \tag{5.3.1}$$

where the m-vector x_1 and (n-m)-vector x_2 are defined by

$$x = \binom{x_1}{x_2} . \tag{5.3.2}$$

It has been previously shown in Chapter 3 that an (n-m)-state observer that provides an estimate $\hat{x}_2$ of the unmeasured state component x_2 , has the form

$$\begin{aligned} \dot{\hat{z}} = {} & (A_{22} - TA_{12})\hat{z} + (A_{22}T + A_{21} - TA_{11} - TA_{12}T)x_1 \\ & + (B_2 - TB_1)u , \end{aligned} \tag{5.3.3}$$

and

$$\hat{x}_2 = (\hat{z} + Tx_1) . \tag{5.3.4}$$

Subscripted matrices are submatrices defined by the partitioning (5.3.2)

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} , \tag{5.3.5}$$

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} , \tag{5.3.6}$$

and

$$K = (K_1 \quad K_2) . \tag{5.3.7}$$

The feedback law (5.2.3) may be implemented as ,

$$u = K_1x_1 + K_2\hat{x}_2 \tag{5.3.8}$$

$$= (K_1 + K_2T)x_1 + K_2\hat{z} . \tag{5.3.9}$$

Equations (5.3.3) and (5.3.9) combine to give the dynamic compensator used for feedback ,

$$\dot{\hat{z}} = (A_{22} - TA_{12} + (B_2 - TB_1)K_2)\hat{z} + (A_{22}T + A_{21} -$$

$$TA_{11} - TA_{12}T + (B_2 - TB_1)(K_1 + K_2T))x_1 . \tag{5.3.10}$$

Define the vector z and the error Δz ,

$$z = x_2 - Tx_1 , \tag{5.3.11}$$

and

$$\Delta z = \hat{z} - z . \tag{5.3.12}$$

The complete closed loop dynamics of the system and compensator may then be expressed as

$$\begin{pmatrix} \dot{x} \\ \Delta\dot{z} \end{pmatrix} = \begin{pmatrix} A+BK & BK \\ 0 & A_{22}-TA_{12} \end{pmatrix} \begin{pmatrix} x \\ \Delta z \end{pmatrix} , \tag{5.3.13}$$

It is clear that the closed loop system dynamics (5.3.13) and associated compensator dynamics (5.3.10) are stable if the following matrices are stable ,

$$(A + BK) , \tag{5.3.14}$$

$$(A_{22} - TA_{12}) , \tag{5.3.15}$$

and

$$(A_{22} - TA_{12} + (B_2 - TB_1)K_2). \tag{5.3.16}$$

A Stable Design of (5.3.14), (5.3.15) and (5.3.16)

Consider the set of values for K, T and the associated auxiliary matrices P, Q, R, P_{22} and P_{21} defined as :

$$K \quad = \quad - RB'P \; , \tag{5.3.17}$$

$$T \quad = \quad - P_{22}^{-1} P_{21} \; , \tag{5.3.18}$$

$$A'P + PA \quad = \quad - Q \; , \tag{5.3.19}$$

$$P \quad = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} , \tag{5.3.20}$$

$$Q \quad = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \quad \text{symmetric PD} \; , \tag{5.3.21}$$

and

$$R \quad \geqslant \quad 0 \; . \tag{5.3.22}$$

Then the matrices (5.3.14), (5.3.15) and (5.3.16) are stable .

Proof

A is assumed to be stable, and then (5.3.19) and (5.3.21) give

$$P \quad > \quad 0 \quad \text{symmetric} \; . \tag{5.3.23}$$

Consider the matrix $(A + BK)$.

Then

$$(A + BK)'P + P(A + BK) = - Q - 2PB'RBP \tag{5.3.24}$$

$$\leqslant - Q$$

$$< 0 , \tag{5.3.25}$$

so that $(A + BK)$ is a stability matrix.

Consider the expansion of the rightmost diagonal block of the matrix equation (5.3.19) , and apply (5.3.18), then ,

$$- Q_{22} = (P_{21}A_{12} + P_{22}A_{22})' + (P_{21}A_{12} + P_{22}A_{22})$$

$$= (P_{22}^{-1}P_{21}A_{12} + A_{22})'P_{22} + P_{22}(P_{22}^{-1}P_{21}A_{12} + A_{22})$$

$$= (A_{22} - TA_{12})'P_{22} + P_{22}(A_{22} - TA_{12}) . \tag{5.3.26}$$

Equations (5.3.26), (5.3.21) and (5.3.23) show that $(A_{22} - TA_{12})$ is stable.

Finally, consider the expansion of the rightmost diagonal block of the matrix inequality (5.3.25) and use (5.3.18) ,

$$
\begin{aligned}
0 \quad > \quad & (P_{21}(A_{12} + B_1K_2) + P_{22}(A_{22} + B_2K_2))' + \\
& (P_{21}(A_{12} + B_1K_2) + P_{22}(A_{22} + B_2K_2)) \\
= \quad & (P_{22}(A_{22} - TA_{12} + (B_2 - TB_1)K_2)' + \\
& (P_{22}(A_{22} - TA_{12} + (B_2 - TB_1)K_2) \, . \qquad (5.3.27)
\end{aligned}
$$

Equations (5.3.27) and (5.3.23) show that $(A_{22} - TA_{12} + (B_2 - TB_1)K_2)$ is stable, which concludes the proof.

Comment

For this section the matrix A has been assumed to be stable. This assumption may be relaxed by using the following device if a stabilizing control

$$u = Jx_1$$

is known; replace A by

$$\begin{pmatrix} A_{11} + B_1J & A_{12} \\ A_{21} + B_2J & A_{22} \end{pmatrix}$$

throughout the section.

Example

The simple example of Section 4 is used. Substitution for P and P^{-1} from the previous example (5.3.17) and (5.3.18) give ,

$$K = -R(.5 \quad 1) ,$$

$$T = -.5 ,$$

and then

$$(A+BK) = \begin{pmatrix} 0 & 1 \\ -1-0.5R & -1-R \end{pmatrix}.$$

This matrix has been shown to be stable for all positive R in the previous example .

Also,

$$(A_{22}-TA_{12}) = -.5 ,$$

so that this is stable.

Finally ,

$$(A_{22} + B_2K_2 - T(A_{12} + B_1K_1)) = -.5 - R$$

and the compensator is therefore stable.

Conclusion

The method presented enables a stable compensator to be designed for an initially stable plant, or a plant that can be stabilized by feeding back the available outputs.

The feedback control obtained is constrained to have the form of Lyapunov control so that the stability properties and limitations on transient response discussed in Sections 2.5.2 and 2.5.3 will apply. It would be desirable to have similar results for Riccati gains, but the algebraic convenience of Lyapunov control laws in achieving stability of $(A + B(L_2 + L_1H))$, $(A - TH)$ and $(A + BL_2 - TH)$ by choice of L_1, L_2 and T via a common Lyapunov function does not apply.

The (n-m)-state stable compensator solution of Section 5.3 allows little variation in design parameters, and in n-state compensators the effect of varying R_1, R_2 in (5.2.14), (5.2.15) is reduced by the minor loop feedback. These considerations indicate that the compensators may be of theoretical interest only.

CHAPTER 6

OBSERVERS AND QUADRATIC COSTS FOR DETERMINISTIC SYSTEMS

6.1 Introduction

We have seen that if the plant is observable we may design an observer such that the error in the estimate will decay irrespective of input according to the dynamics ,

$$\Delta\dot{x} = D\,\Delta x\ , \tag{6.1.1}$$

in the case of an n-state observer and

$$\Delta\dot{x}_2 = D_2\,\Delta x_2\ , \tag{6.1.2}$$

in the case of an (n-m)-state observer.

Quadratic performance criteria play an important role in modern control theory, as they offer a quantitative measure of controller performance which is amenable to mathematical analysis and calculation. Meir and Anderson [M5] have studied the effects of optimal (Kalman) estimators and non-optimal n-state estimators in the feedback path of an n-state dimension, linear plant for a Gaussian white noise environment. The performance of filters of arbitrary order in the feedback path, again in the same idealised noisy environment, may be evaluated numerically using sets of computer programs as described in [B7]. Little progress has been made in the theory of the performance of filters of arbitrary dimension in either the stochastic or deterministic environment.

Newman [N2] and Bongiorno and Youla [B6] have made recent contributions to the theory of the degradation of system performance when an observer is used in the feedback path to approximate an optimal control law. Both references prove the decoupling property of (6.2.2) and (6.2.3); however, both place more emphasis on the state of the observer than on the desired plant estimate. The author believes that the analysis presented here gives a better understanding of the underlying mechanism, and in addition it treats a slightly more general performance index where Q_1 in (6.1.4) is non-zero.

The main contributions of this Section are Theorems 6.2.1 and 6.2.3 which give conditions for the degree to which system performance degradation arising from synthesizing a control from an observer estimate of plant state, may be reduced by increasing the observer speed of response. It is shown that often, in a sense to be defined, the performance degradation cannot be decreased beyond an easily calculated lower bound estimate greater than zero. The estimate, however, is not the greatest lower bound.

Mathematical Formulation

The plant dynamics are again expressed in a co-ordinate basis such that the m plant outputs y form the first m components of the n-vector state ,

$$\dot{x} = Ax + Bu$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \ , \qquad (6.1.3)$$

$$y = x_1 \ .$$

A performance index J is defined

$$J = \int_0^\infty (x'Qx + 2x'Q_1u + u'Ru)dt \ , \qquad (6.1.4)$$

and an ideal control law ,

$$u = Kx \qquad (6.1.5)$$

$$= K_1x_1 + K_2x_2 \ , \qquad (6.1.6)$$

minimises J which is then defined by

$$J^{o} \quad \triangleq \quad x'(o)P\,x(o) \; . \tag{6.1.7}$$

Matrices Q, Q_1, R are assumed to be such that the optimum exists and is stable so that P is finite. It will be assumed that P is positive definite (P.D.).

Control law (6.1.5) is unrealizeable because the state x_2 is unmeasured and the realizeable laws (6.1.8), (6.1.9) are used as approximations ,

$$u \quad = \quad K\hat{x} \tag{6.1.8}$$

and for reduced observers

$$u \quad = \quad K_1 x_1 + K_2 \hat{x}_2 \; . \tag{6.1.9}$$

We wish to see the effect on the system performance index J of using control laws (6.1.8), (6.1.9) which may be written

$$u \quad = \quad Kx + K\Delta x \; , \tag{6.1.10}$$

and

$$u \quad = \quad Kx + K_2 \Delta x_2 \; . \tag{6.1.11}$$

Effect of n-state Observer

The dynamics of the n-state observer and plant may be represented by (6.1.1) and (6.1.3) which are coupled through (6.1.10) to give ,

$$\frac{d}{dt}\begin{pmatrix} x \\ \Delta x \end{pmatrix} = \begin{pmatrix} A + BK & BK \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ \Delta x \end{pmatrix} . \tag{6.1.12}$$

$$\triangleq \bar{A}\begin{pmatrix} x \\ \Delta x \end{pmatrix} .$$

Under the control law (6.1.10) the cost integrand of (6.1.4) becomes ,

$$\begin{pmatrix}\text{cost} \\ \text{integrand}\end{pmatrix} = (x' \quad \Delta x') \begin{pmatrix} Q+K'RK+Q_1K+K'Q_1' & Q_1K+K'RK \\ K'Q_1'+K'RK & K'RK \end{pmatrix} \begin{pmatrix} x \\ \Delta x \end{pmatrix} \tag{6.1.13}$$

$$\triangleq (x' \quad \Delta x') \bar{Q}\begin{pmatrix} x \\ \Delta x \end{pmatrix} . \tag{6.1.14}$$

Then

$$J = \int_0^\infty (x' \quad \Delta x') \bar{Q}\begin{pmatrix} x \\ \Delta x \end{pmatrix} dt \tag{6.1.15}$$

$$\triangleq (x' \quad \Delta x') \begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix} \begin{pmatrix} x \\ \Delta x \end{pmatrix} \tag{6.1.16}$$

$$= (x' \ \ \Delta x') \, \overline{P} \binom{x}{\Delta x} \ . \tag{6.1.17}$$

The integral (6.1.15) exists because $\overline{A}$ is stable if both (A + BK) and D are, which is assumed. The optimal J^o has been assumed to be greater than zero for all non-zero initial conditions on x, and therefore so is J of (6.1.15). Because the cost integrand is positive and the system stable we must have [H2] ,

$$\overline{Q} \geqslant 0 \ , \tag{6.1.18}$$

$$\overline{P} \geqslant 0 \ , \tag{6.1.19}$$

and

$$\overline{A}'\overline{P} + \overline{P}\,\overline{A} = -\overline{Q} \ . \tag{6.1.20}$$

The solution $\overline{P}$ of (6.1.20) is considerably simplified by the following observations.

F or initial conditions such that

$$\Delta x = 0$$

the dynamics (6.1.12) show that Δx remains zero. The control law (6.1.10) then equals the ideal control law (6.1.5), the presence of the observer has no effect and the performance index J for the trajectories corresponding to this class of initial condition must equal the optimal J^o.

Substitution of zero Δx in (6.1.16) gives for all x,

$$J^o = x'Px$$

$$= x'\overline{P}_{11}x \; , \qquad (6.1.21)$$

and therefore

$$\overline{P}_{11} = P \; . \qquad (6.1.22)$$

The submatrix $\overline{P}_{21}$ is now evaluated. The expansion of (6.1.16) is ,

$$J = x'\overline{P}_{11}x + 2x'\overline{P}_{12}\,\Delta x + \Delta x'\overline{P}_{22}\,\Delta x \; . \qquad (6.1.23)$$

Because of (6.1.19) $\overline{P}_{22}$ must satisfy

$$\overline{P}_{22} \geqslant 0 \; , \qquad (6.1.24)$$

and then

$$\overline{P}_{22} + I > 0 \; . \qquad (6.1.25)$$

Selection of Δx such that

$$\Delta x = -(\overline{P}_{22} + I)^{-1}\overline{P}_{12}'\,x \; , \qquad (6.1.26)$$

and substitution into (6.1.22) and use of (6.1.21) gives ,

$$J = J^o + x'\overline{P}_{12}(\overline{P}_{22} + I)^{-1}(-2(I + \overline{P}_{22})$$

$$+ \overline{P}_{22})(\overline{P}_{22} + I)^{-1}\overline{P}_{12}'\,x$$

$$= J^o + x'\overline{P}_{12}(\overline{P}_{22} + I)^{-1}(-2I - \overline{P}_{22})(\overline{P}_{22} + I)^{-1}\overline{P}_{12}' x .$$

$$\ldots. \quad (6.1.27)$$

$$\triangleq J^o + \Delta J .$$

From (6.1.25) for all x

$$\Delta J \leqslant 0 , \qquad (6.1.28)$$

and for non-zero $\overline{P}_{12}$ there exist non-zero x such that

$$\Delta J < 0 .$$

This is impossible because J^o is optimal for such an x and therefore

$$\overline{P}_{12} = 0 . \qquad (6.1.29)$$

With $\overline{P}_{11}$ and $\overline{P}_{12}$ given by (6.1.22) and (6.1.29), $\overline{P}_{22}$ is the only unknown since $\overline{P}$ is symmetric. Equation (6.1.30) is the bottom diagonal block of the matrix equation (6.1.20) and has solution $\overline{P}_{22}$,

$$D'\overline{P}_{22} + \overline{P}_{22} D = - K_2'RK_2 . \qquad (6.1.30)$$

Expression (6.1.23) with (6.1.29) and (6.1.21) gives

$$J \quad = \quad x'Px + \Delta x'\overline{P}_{22}\,\Delta x \tag{6.1.31}$$

$$\triangleq \quad J^{o} + \Delta J \,. \tag{6.1.32}$$

Equations (6.1.31) and (6.1.30) are an important simplification in determining the effect of the observer on the performance index (6.1.4). This simplification depends on the ideal control law being optimal.

Effect of (n-m)-State Observer

Equations (6.1.12) to (6.1.17) become ,

$$\frac{d}{dt}\begin{pmatrix} x \\ \Delta x_2 \end{pmatrix} = \begin{pmatrix} A + BK & BK_2 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} x \\ \Delta x_2 \end{pmatrix} \tag{6.1.33}$$

$$\triangleq \quad \underline{A}\begin{pmatrix} x \\ \Delta x_2 \end{pmatrix}, \tag{6.1.34}$$

$$\text{Cost integrand} = (x' \;\Delta x_2')\begin{pmatrix} Q+K'RK+Q_1K+K'Q_1' & Q_1K_2 + K'RK_2 \\ K_2'Q_1'+K_2'RK & K_2'RK_2 \end{pmatrix}\begin{pmatrix} x \\ \Delta x_2 \end{pmatrix} \tag{6.1.35}$$

$$\triangleq \quad (x' \;\Delta x_2')\,\underline{Q}\begin{pmatrix} x \\ \Delta x_2 \end{pmatrix}, \tag{6.1.36}$$

and

$$J = \int_0^\infty (x' \quad \Delta x_2') \; \underline{Q} \begin{pmatrix} x \\ \Delta x_2 \end{pmatrix} dt \tag{6.1.37}$$

$$\triangleq (x' \quad \Delta x_2') \begin{pmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}_{21} & \underline{P}_{22} \end{pmatrix} \begin{pmatrix} x \\ \Delta x_2 \end{pmatrix} \tag{6.1.38}$$

$$\triangleq (x' \quad \Delta x_2') \; \underline{P} \begin{pmatrix} x \\ \Delta x_2 \end{pmatrix} . \tag{6.1.39}$$

Using equations (6.1.33) to (6.1.39) and the fact that the control law (6.1.11) is optimal for zero Δx_2, the procedure used in the case of an n-state observer may be followed to prove that

$$\underline{P}_{11} = P , \tag{6.1.40}$$

$$\underline{P}_{12} = 0 , \tag{6.1.41}$$

and $\underline{P}_{22}$ is given by

$$D_2' \underline{P}_{22} + \underline{P}_{22} D_2 = - K_2' Q K_2 . \tag{6.1.42}$$

However, a completely algebraic derivation which could also have been used for the n-state case is presented for completeness.

From (6.1.39), (6.1.37) and (6.1.34) $\underline{P}$ is given by the solution of ,

$$\underline{A}'\underline{P} + \underline{P}\,\underline{A} = -\underline{Q}\ , \tag{6.1.43}$$

or in expanded form

$$\begin{pmatrix} (A+BK)' & 0 \\ K_2'B' & D_2' \end{pmatrix} \begin{pmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}_{21} & \underline{P}_{22} \end{pmatrix} \ +$$

$$\begin{pmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}_{21} & \underline{P}_{22} \end{pmatrix} \begin{pmatrix} (A+BK) & BK_2 \\ 0 & D_2 \end{pmatrix} = -\begin{pmatrix} Q+K'RK+Q_1K+K'Q_1' & Q_1K_2 + K'RK_2 \\ K_2'Q_1'+K_2'RK & K_2'RK_2 \end{pmatrix}.$$

$$\ldots\ldots \quad (6.1.44)$$

The upper diagonal block of (6.1.44) is ,

$$(A + BK)'\underline{P}_{11} + P_{11}(A + BK) = -(Q + K'RK + Q_1K + K'Q_1')\ , \tag{6.1.45}$$

and (6.1.45) with the definition of J in (6.1.4) give

$$\underline{P}_{11} = P\ . \tag{6.1.46}$$

The upper off-diagonal block of (6.1.44) is

$$(A + BK)'\underline{P}_{12} + \underline{P}_{11}BK_2 + \underline{P}_{12}D_2 = - K'RK_2 - Q_1K_2 \ , \qquad \text{which}$$

becomes

$$(A + BK)'\underline{P}_{12} + \underline{P}_{12}D_2 = 0 \ , \tag{6.1.47}$$

from (6.1.46) and because the optimal K satisfies

$$K = - R^{-1}(Q_1' + B'P) \ . \tag{6.1.48}$$

The matrices (A + BK) and D_2 are stable by assumption and therefore no eigenvalue of D_2 can be the negative of an eigenvalue of (A + BK). Equation (6.1.47) then has a unique solution [J3], [G1, Chapter 8] and because the left hand side is zero, the solution $\underline{P}_{12}$ is zero, which proves (6.1.41). As in the case of an n-state observer, this simplifying result depends on the ideal control being optimal. The lower diagonal block of (6.1.44) gives (6.1.42).

6.2 Inherent Performance Degradation Induced when State Estimates Are Used in Optimal Control Laws

The performance cost increment above the optimum has been defined to be ΔJ for non-zero initial estimate error, so that

$$J = J^{o} + \Delta J \,, \tag{6.2.1}$$

$$\Delta J = \Delta x' \overline{P}_{22} \, \Delta x \,, \tag{6.2.2}$$

$$\Delta J = \Delta x_2' \underline{P}_{22} \, \Delta x_2 \,, \tag{6.2.3}$$

where (6.2.2) and (6.2.3) correspond to n and (n-m)-state observers respectively.

We are naturally interested in having small ΔJ for a given initial estimation error, and therefore it would be desirable to make $\overline{P}_{22}$, $\underline{P}_{22}$ small.

The right hand sides of (6.1.30) and (6.1.42) are fixed with respect to the observer design problem so that $\overline{P}_{22}$ and $\underline{P}_{22}$ are determined by the observer error dynamic matrices D and D_2 alone.

An important question is : can ΔJ be made arbitrarily small for all estimation errors of unit norm? The following theorem gives the answer.

6.2.1 Theorem

Consider an n-state plant with an m-vector output x_1 such that

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \ , \end{aligned}$$

and performance index J ,

$$J = \int_0^\infty (x'Qx + 2x'Q_1u + u'Ru)dt \ .$$

Consider the (ideal) optimal control law ,

$$u = Kx \ ,$$

for which J is J^o and the (non-ideal) approximating control law generated by an (n-m)-state observer,

$$u = K_1x_1 + K_2\hat{x}_2 \ ,$$

for which

$$J = J^o + \Delta J \ ,$$

where

$$\Delta J = \Delta x_2'\underline{P}_{22}\,\Delta x_2 \ ,$$

and Δx_2 is the initial estimation error.

Use of the non-ideal control law introduces an irreducible performance degradation in the sense that $\mathrm{tr}(\underline{P}_{22})$ has a finite lower bound independent of the observer if the rows of A_{12} do not span the rows of K_2.

Proof

Temporarily denote $K_2'RK_2$ by S, then (5.3.42) gives

$$\mathrm{tr}(\underline{P}_{22}) = \mathrm{tr}(\int_0^{\infty} \exp(D_2't)S \exp(D_2t)\, dt$$

$$= \mathrm{tr}(S^{\frac{1}{2}} \int_0^{\infty} \exp(D_2t) \exp(D_2't)\, dt\, S^{\frac{1}{2}}) \qquad (6.2.4)$$

The integrand of (6.2.4) is now examined by considering an auxiliary (n-m)-state dynamic system and performance index J_1 ,

$$\dot{w} = D_2'\, w$$

$$= (A_{22}' - A_{12}'\, T_1')\, w \;, \qquad (6.2.5)$$

$$J_1 = \int_0^{\infty} w'\, w\, dt \qquad (6.2.6)$$

$$\triangleq w'(o)\, L\, w(o) \;,$$

so that

$$L = \int_0^\infty \exp(D_2 t)\ \exp(D_2' t)\ dt \ . \qquad (6.2.7)$$

(6.2.5) comes from (3.3.23).

We examine the effect of different matrices T_1 on the cost L by introducing a control v in the dynamics (6.2.5),

$$v = T_1' w \ ,$$

$$\dot{w} = A_{22}' w - A_{12}' v \ .$$

Suppose that the columns of K_4' are a minimal ortho-normal basis for the columns of A_{12}' . Then no dynamic information is lost by introducing an equivalent input $\underline{v}$ and writing

$$w = A_{22}' - K_4' \underline{v}$$

where K_4' has maximum rank.

It is always possible to make a nonsingular transformation such that

$$w = W \underline{w} , \qquad (6.2.8)$$

$$W'W = I ,$$

$$\dot{\underline{w}} = W^{-1} A_{22}' W \underline{w} + \binom{I}{0} \underline{v}$$

$$\triangleq C \underline{w} + \binom{I}{0} \underline{v}$$

$$\triangleq \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \underline{w}_1 \\ \underline{w}_2 \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} \underline{v} , \qquad (6.2.9)$$

$$J_1 = \int_0^\infty \underline{w}' \, W'W \, \underline{w} \, dt$$

$$= \int_0^\infty \underline{w}' \, \underline{w} \, dt . \qquad (6.2.10)$$

Note that all matrices are finite.

We now prove that there exists a lower bound on J_1 for all non-zero initial $\underline{w}_2$. Because the input $\underline{v}$ is not costed, impulse functions are allowable and this will be interpreted that $\underline{w}_1$ may be instantaneously adjusted to any value, and may be therefore considered to be a control variable. The dynamics (6.2.9) may then be written

$$\dot{\underline{w}}_2 = C_{22}\underline{w}_2 + C_{21}\underline{w}_1 \tag{6.2.11}$$

and the performance index J_1 could be minimised with respect to $\underline{w}_1$. Because the matrices in (6.2.11) are finite the minimum J_1° exists, is finite, and greater than zero for any non-zero initial condition $\underline{w}_2$.

Explicit Lower Bound on J_1°

An explicit lower bound on J_1° may be obtained in terms of $\|\underline{w}_2\|$ by condensing the dynamics (6.2.11) to a scalar equation associated with the norms of $\underline{w}_2$ and $\underline{w}_1$.

Trajectories of (6.2.11) that minimise J_1 are generated by an optimal control of the form

$$\underline{w}_1 = \underline{K}\,\underline{w}_2 \,. \tag{6.2.12}$$

Consider an optimal trajectory TR1 for which $J_1 = J_1^{\circ}$, and associate the scalar variables w_1^{*}, w_2^{*} with $\|\underline{w}_1\|$ and $\|\underline{w}_2\|$ along

TR1 so that ,

$$w_2^* = \| \underline{w}_2 \| \quad , \tag{6.2.13}$$

$$\dot{w}_2^* = \frac{d}{dt} \| \underline{w}_2 \| \; , \tag{6.2.14}$$

$$w_1^* = \| \underline{w}_1 \| \tag{6.2.15}$$

$$\triangleq K^*(t)\, w_2^* \; , \tag{6.2.16}$$

and

$$\dot{w}_2^* = -\, C_{22}^* w_2^* - C_{21}^* w_1^* + v^* \; , \tag{6.2.17}$$

where

$$C_{22}^* \triangleq \| C_{22} \| \quad , \tag{6.2.18}$$

$$C_{21}^* \triangleq \| C_{21} \| \quad , \tag{6.2.19}$$

and $K^*(t)$ is defined by (6.2.15), (6.2.16) and therefore

$$K^*(t) \geqslant 0 \; .$$

Introduce a performance index J^* for the dynamics (6.2.17) ,

$$J^* \triangleq \int_0^\infty (w_2^{*2} + w_1^{*2})\, dt \; . \tag{6.2.20}$$

Along the trajectory TR1

$$J_1^{\circ} = \int_0^{\infty} \underline{w}' \, W'W \, \underline{w} \, dt$$

$$\geqslant \int_0^{\infty} \|\underline{w}\|^2 \, dt$$

$$= \int_0^{\infty} (\|\underline{w}_1\|^2 + \|\underline{w}_2\|^2) \, dt$$

$$= \int_0^{\infty} (w_2^{*2} + w_1^{*2}) \, dt$$

$$= \int_0^{\infty} (1 + K(t)^2) w_1^{*2} \, dt$$

$$\triangleq J_1^{*} . \qquad (6.2.21)$$

The optimal trajectory TR1 is stable and by construction

$$w_1^{*} , C_{22}^{*}, C_{21}^{*} > 0 ,$$

and

$$w_2^{*} , K^{*}(t), v^{*} \geqslant 0 .$$

Because v^* is positive, it has a destabilizing effect for positive w_2^* on the dynamics (6.2.17) rewritten as ,

$$w_2^* = - C_{22}^* w_2 - C_{21}^* x_1^* + v^*$$

$$= - (C_{22}^* + C_{21}^* K^*(t)) w_2^* + v^* , \qquad (6.2.22)$$

and therefore $J_2^* \triangleq J^*$ evaluated along a trajectory of (6.2.22) with zero v^* but having the same initial condition w_2^* satisfies

$$J_2^* < J_1^* .$$

However, we may find an explicit solution to the problem of minimising J^* with respect to w_1^* for zero v^* and dynamics (6.2.17). The optimum for initial condition w_2^* is ,

$$J^{*o} = \frac{\sqrt{2C_{22}^{*\,2} + C_{21}^{*\,2}}}{C_{21}^*} w_2^{*2} \qquad (6.2.23)$$

$$\triangleq p^* w_2^{*2} .$$

Any optimal trajectory of (6.2.11) with initial condition $\underline{w}_2$ must satisfy ,

$$
\begin{aligned}
J_1^{o} &= J_1^{*} \\
&> J_2^{*} \\
&> J_2^{*o} \\
&= p^{*} w_2^{*2} \\
&= p^{*} \| \underline{w}_2 \|^2 \,. \qquad (6.2.24)
\end{aligned}
$$

We recall that

$$
\begin{aligned}
tr(P_{22}) &= tr(S^{\frac{1}{2}} L S^{\frac{1}{2}}) \,, \\
S &= K_2{}' R K_2 \,, \\
\underline{w} &= W' w \,, \\
W'W &= I \,,
\end{aligned}
$$

and

$$
\begin{aligned}
J_1 &= w' L w \\
&\geqslant p^{*} \underline{w}_2{}' \underline{w}_2 \,.
\end{aligned}
$$

The r non-zero eigenvalues of S must have eigenvectors made up of linear combinations of the rows of K_2.

Suppose that these eigenvalues are denoted by the set $\left\{ s_i : i = 1 \,..\, r \right\}$ with associated eigenvectors $\left\{ e(i) : i = 1 \,..\, r \right\}$ and that $\left\{ e(i) \right\}$ have components $\left\{ e_2(i) : i = 1 \,..\, r \right\}$ normal to the subspace spanned by the rows of A_{12}.

Then

$$\mathrm{tr}(P_{22}) = \sum_{i=1}^{r} s_i \, e'(i) \, L \, e(i) \tag{6.2.25}$$

$$\geq p^{*} \sum s_i \, e_2'(i) \, e_2(i) \, .$$

The derivation of this inequality is independent of the observer gain T_1 so that there exists no T_1, even as $\|T_1\| \to \infty$, for which the inequality does not hold.

The matrices A, K_2, R completely determine (6.2.25) and are assumed to be fixed, so that if the rows of K_2 are not spanned by the rows of A_{12},

$$\mathrm{tr}(P_{22}) \geq p^{*} \sum_{i=1}^{r} s_i \, \|e_2(i)\|^2$$

$$\triangleq C^{*}$$

$$> 0 \, .$$

6.2.2 Corollary.

Consider an n-state plant, with an m-output y such that

$$\dot{x} = Ax + Bu$$

$$y = Hx$$

and performance index

$$J = \int_0^\infty (x'Qx + 2x'Q_1u + u'Ru)\, dt$$

with optimal J^o under the (ideal) optimal control law

$$u = Kx .$$

An n-state observer provides an estimate $\hat{x}$ of x and the ideal control law is approximated by

$$u = K\hat{x} ,$$

which for initial estimate error Δx results in

$$J = J^o + \Delta J$$

$$= J^o + \Delta x'\overline{P}\,\Delta x .$$

Then an irreducible performance degradation is introduced in the sense that there exists a C^* such that for all n-state observers

$$\begin{aligned} \mathrm{tr}(\overline{P}) &\geq C^{*} \\ &> 0 . \end{aligned}$$

Proof

The matrix $\overline{P}$ is defined by

$$D'\overline{P} + \overline{P}D = -K'RK ,$$

where

$$D = A - TH$$

and the matrix T is the only design variable.

It may be shown that by replacing the matrices D_1, T_1, A_{22}, A_{12}, $\underline{P}_{22}$ by the matrices D, T, A, H, $\overline{P}$ in the proof of Theorem 6.2.1 that if the rows of K are not spanned by the rows of H then there exists a C^{*} such that

$$\begin{aligned} \mathrm{tr}(\overline{P}) &\geq C^{*} \\ &> 0 . \end{aligned}$$

We note that if the rows of K are spanned by the rows of H, an observer is unnecessary.

6.2.3 Theorem

Consider the plant, performance index and optimal ideal control law of Theorem 6.2.1.

Suppose that the rows of A_{12} span the rows of K_2. Define a matrix K_3 with rows orthogonal to those of K_2 and such that the rows of K_2 and K_3 form a minimal basis for the rows of A_{12}.

Consider the system

$$\dot{w} = A_{22}'w - A_{12}'v,$$

then by construction the same dynamics may be effected by a new forcing term $\underline{v}$,

$$\dot{w} = A_{22}'w - (K_2' \quad K_3')\underline{v}.$$

Transform the above system so that

$$\underline{w} = Ww ,$$

$$WK_2' = \binom{I}{0} ,$$

and

$$\dot{\underline{w}} = W\,A_{22}'W^{-1}\underline{w} - W(K_2'K_3')\,\underline{v}$$

$$\triangleq \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} \underline{w}_1 \\ \underline{w}_2 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & K_4 \end{pmatrix}\begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix}.$$

If the pair (C_{22}, K_4) is controllable or stabilizable, the performance degradation introduced by using the control law

$$u = K_1 x_1 + K_2 \hat{x}_2 ,$$

may be arbitrarily reduced by stable observers in the sense that

$$\Delta J = \Delta x_2' \underline{P}_{22} \Delta x_2 ,$$

and for any $C^* > 0$ there exist observers for which

$$0 < \operatorname{tr}(\underline{P}_{22}) < C^* .$$

<u>Proof</u>

As in Theorem 6.2.1 denote $K_2'RK_2$ temporarily by S.

We recall (6.2.4) and (6.2.7) ,

$$\operatorname{tr}(\underline{P}_{22}) = \operatorname{tr}(S^{\frac{1}{2}} \int_0^\infty \exp(D_2 t) \exp(D_2' t) dt \, S^{\frac{1}{2}})$$

$$\triangleq \operatorname{tr}(S^{\frac{1}{2}} L \, S^{\frac{1}{2}}) \qquad (6.2.26)$$

and associate a dynamic system

$$\dot{w} = D_2' w$$

$$= (A_{22}' - A_{12}' T_1') w \qquad (6.2.27)$$

$$= \quad A_{22}'w + A_{12}'v \tag{6.2.28}$$

$$v \quad = \quad T_1'w \ . \tag{6.2.29}$$

By hypothesis we may write

$$\dot{w} \quad = \quad A_{22}'w + (K_2' \quad K_3')\,\underline{v} \ ,$$

and introduce the state transformation

$$\underline{w} \quad = \quad Ww \ ,$$

so that

$$\dot{\underline{w}} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} \underline{w}_1 \\ \underline{w}_2 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & K_4 \end{pmatrix}\begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix} . \tag{6.2.30}$$

Define

$$J_1 \quad = \quad \int_0^\infty w'w \ dt \tag{6.2.31}$$

$$= \quad \int_0^\infty \underline{w}' \ W^{-1}{}'W^{-1}\underline{w} \ dt \ .$$

The implication of allowing the observer gain matrix T_1 in (6.2.27) to tend to infinity is that the magnitude of $\underline{v}$ in (6.2.29) or $\underline{v}_1$ in (6.2.30) may tend to infinity. In this case feedback laws exist

$$\underline{v}_1 = K_5 \underline{w}$$

that generate impulse like inputs that may force $\underline{w}_1$ to zero value in an arbitrarily short time, and maintain it there.

The contribution to J_1 of a finite non-zero initial condition on $\underline{w}_1$ can therefore be made arbitrarily small by suitable gains on $\underline{v}_1$.

In this case the dynamics of $\underline{w}_2$ tend to

$$\dot{\underline{w}}_2 = C_{22}\underline{w}_2 + K_4\underline{v}_2 \,, \qquad (6.2.32)$$

which can be made stable with some feedback

$$\underline{v}_2 = K_6\underline{w}_2 \,,$$

because (C_{22}, K_4) has been assumed to be a controllable or stabilizable pair.

The feedback laws

$$\underline{v}_1 = K_5\underline{w} \,,$$

$$\underline{v}_2 = K_6\underline{w}_2 \,,$$

may be rewritten

$$\underline{v} = \underline{T}_1'\underline{w} .$$

But for any such $\underline{T}_1$ the same dynamic effect may be achieved by a T_1' in (6.2.29), so that there exist observer gains T_1 for which the observer dynamics

$$D_2 = (A_{22} - T_1A_{12})$$

are stable, and the contribution to J_1 of a non-zero initial $\underline{w}_1$ can be made arbitrarily small.

Now consider an initial condition w of (6.2.29) such that w lies entirely within the subspace spanned by the columns of K_2' ,

$$w = K_2'b ,$$

$$\underline{w} = W w$$

$$= WK_2'b ,$$

and because by hypothesis

$$WK_2' = \binom{I}{0} ,$$

then

$$\underline{w}_1 = b ,$$

and

$$\underline{w}_2 = 0 .$$

Therefore for this type of initial condition, J_1 can be made arbitrarily small.

Using the same notation as Theorem 6.2.1 the eigenvectors corresponding to non-zero eigenvalues of $S^{\frac{1}{2}} = (K_2'RK_2)^{\frac{1}{2}}$ must be of the form

$$e(i) = K_2'b_i$$

and therefore from (6.2.25)

$$\mathrm{tr}(\underline{P}_{22}) = \sum_{i=1}^{r} s_i \, e(i)' \, L \, e(i) \, . \tag{6.2.33}$$

All terms in (6.2.33) can be made arbitrarily small by a suitable choice of gains on $\underline{v}$ in (6.2.29) and hence by a suitable choice of T_1 in (6.2.28).

That is, for any $C^{*} > 0$ there exists observers for which

$$0 < \mathrm{tr}(\underline{P}_{22}) < C^{*} \, .$$

6.3 Comparison with Other Work and Comments on Results

Reference [B6] claims a result which conflicts with Corollary 6.2.2, and the error in the analytic support for the claim is demonstrated in this Section.

Support for Claim

For an n-state observer we have non-singular L with dimension (n, n) connecting observer state z and plant state estimate $\hat{x}$

$$\hat{x} = L^{-1} z = x + \Delta x,$$

$$z = z_o + \Delta z,$$

$$z_o = Lx,$$

$$\Delta\dot{z} = D\,\Delta z,$$

$$\therefore\ \Delta\dot{x} = L^{-1} DL\ \Delta x$$

$$\triangleq \bar{D}\,\Delta x$$

$$= (A - TH)\,\Delta x, \tag{6.3.1}$$

and

$$\Delta J = \int_0^\infty \Delta x' K' R K\, \Delta x\, dt$$

$$\triangleq \int_0^\infty \Delta x'\, S\, \Delta x\, dt. \tag{6.3.2}$$

Suppose that $\underline{s}$ is the maximum eigenvalue of S. Then the argument of [B6] runs,

$$0 \leqslant J \leqslant \underline{s} \int_0^\infty \|\exp(\overline{D}t)\ \Delta x(t_o)\|^2\, dt \,.$$

Using the properties

$$\|\exp(\overline{D}t)\,\Delta x(t_o)\| \leqslant \|\exp(\overline{D}t)\| \quad \|\Delta x(t_o)\| \;,$$

and

$$\|\exp(\overline{D}t)\| \leqslant C \exp(\overline{d}t) \;, \tag{6.3.3}$$

where C is an appropriate constant and $\overline{d} < 0$ is greater than the real part of the eigenvalue of D with largest real part ($\overline{D}$ is similar to D), one gets from (5.3.83) ,

$$0 \leqslant J \leqslant \underline{s} \|\Delta x(t_o)\|^2 \left(\frac{C^2}{-2\overline{d}}\right) . \tag{6.3.4}$$

The only term in the right hand side of (6.3.4) which depends on D is $\overline{d}$. Clearly $\Delta J \longrightarrow 0$ as $\overline{d} \longrightarrow -\infty$.

<u>Error in the Analysis</u>

But this is a false conclusion because in (6.3.4) C depends on $\overline{D}$ from (6.3.3); $\overline{D}$ varies through the gain T in (6.3.1) in order that it may be similar to D, so that $\bar{D}$ is a function of D. Therefore both C and $\bar{d}$ in (6.3.4) are functions of D. This invalidates the proof given.

Comments on Theorems 6.2.1 and 6.2.3

It is interesting to note that if the desired control cannot be synthesized from the plant output and its first derivative, the conditions for Theorem 6.2.1 hold and there is an inherent performance degradation in the deterministic optimal control problem.

The Theorems consider the situation where observer gains may approach infinity. There are several reasons why the performance limits indicated by the Theorems cannot be attained in practice. Firstly, perfect arithmetic will not be achieved internally in the observer and numerical errors arising there will cause deviations from the theoretical trajectories.

Differences between the plant dynamics and the equations assumed for the plant are unavoidable with respect to the order of the equations as well as individual parameter values, and these discrepancies sooner or later would be a limiting factor that has not entered our analysis. Finally the deterministic environment is itself an idealization. The plant output will be noisy and in so far as the observer tends to differentiate as its dynamics are made faster, the effect of the output disturbances will become more significant. It the output noise is Gaussian, white and with zero mean the Kalman filter will give optimal expected costs [J4] .

Though very high observer gains can never be implemented in practice, the author feels that the results of this chapter are of theoretical interest and may give an overall idea of what can and cannot be achieved with closed loop control when not all plant states are measured. It is emphasized that the concept of performance degradation used throughout the analysis reflects the system performance from all initial conditions of estimation errors of the plant state.

CHAPTER 7

SIMPLE CONTROLLERS FOR HIGH ORDER PLANT

7.1 Introduction

One of the main problems of designing control systems is that the available dynamic models of the plant merely approximate reality. For plant of any complexity it is necessary to use a computer at some stage in the design process, and this has motivated the development of design tools like modal control, Riccati control, Kalman filters and Luenberger observers that incorporate easily programmable, non-iterative algorithms. However, these state space design methods when applied to large plant models lead to controllers that are known by experience to be unnecessarily complex. There are two approaches to overcoming the difficulty, (a) to retain the same plant model and simplify the controller by constraining its structure, or (b) to simplify the model of the plant dynamics and use the algorithmic state space design methods. A computer search for a satisfactory control based on (a) is usually organised as the search for the minimum of an optimisation problem. In this case, a practically complete description of the plant dynamics is retained and there is every prospect that the plant and controller will have the same performance as the plant model and controller. For complex plant the iterative procedure of hill-climbing to optimal feedback parameters is likely to require large amounts of computing time compared with procedure (b), unless some structure can be found for the problem. The disadvantage of (b) is that information about plant dynamics is discarded so that it is possible for closed loop full and simplified model dynamics to differ unless precautions are taken. These precautions are difficult to define a priori and are catered for in the design process by manipulating upper limits on feedback gains or cost on control to maintain agreement between the closed loop full and reduced model dynamics. This problem of agreement is very similar to that of the plant and plant model dynamics agreement, and understanding one problem is helpful for the other. Since methods (a) and (b) lead to more or less automatic limitations on input magnitudes, it is probable that the closed loop plant performance will agree with the closed loop full model performance.

In Section 7.2 we present a design method of type (a) and find a structure for the hill-climbing problem, by which conditions are obtained that convergence to the optimal feedback parameters will be fast.

Section 7.3 discusses a method for obtaining the invariant subspace of the dominant modes of sampled data plant models.

Section 7.4 introduces the concept of a parallel path model and uses the method of Section 7.3 to obtain parallel path models of reduced dimension.

The theories of hill-climbing on a fixed feedback structure, model reduction and observer theory are combined in Section 7.5 to give an overall philosophy for the design of simple controllers for complex plant, where not all state variables are measured.

7.2 Gain Optimization from Available Plant Outputs

7.2.1 A Convenient Performance Index

Consider a closed loop plant with dynamics

$$\dot{x} = Ax + Bu \tag{7.2.1}$$

$$y = Hx \tag{7.2.2}$$

$$u = Ky , \tag{7.2.3}$$

and suppose that K is to be found to minimise the cost of an infinite time trajectory with initial condition x_o

$$J(x_o,K) = \int_0^{\infty} (x'Qx + u'Ru)dt , \tag{7.2.4}$$

for $Q \geqslant 0$, $R > 0$.

Rekasius [R7] has considered a min-max problem over all initial x_o. Unless H is invertible the optimal K is a function of x_o. A more general performance index on K is specified in terms of a set of initial conditions $\{x(i) : i = 1, \ldots N\}$ and a cost J defined,

$$J \triangleq \sum_i J(x(i),K)$$

$$= tr(X_o'PX_o) , \tag{7.2.5}$$

where

$$X_o \triangleq (x(1), x(2) \ldots x(N)), \tag{7.2.6}$$

$$(A + BG)'P + P(A + BG) = - Q - G'RG , \tag{7.2.7}$$

and

$$G \triangleq KH . \tag{7.2.8}$$

Necessary conditions at the optimum

We shall find the necessary conditions at the optimum for K to minimise J (7.2.5), and establish a hill-climbing algorithm from them. The problem has an interesting structure which may give insight into the requirements of simplified dynamic models used for feedback design.

The first order perturbation of (7.2.7) is with the definition

$$A_G \triangleq A + BKH , \tag{7.2.9}$$

$$A_G' \Delta P + \Delta P A_G = \Delta G'(RG + B'P) + (RG + B'P)' \Delta G \tag{7.2.10}$$

$$\triangleq \Delta Q .$$

We obtain $\frac{\partial J}{\partial K}$ by evaluating the first order change ΔJ in J (7.2.5) with respect to a change ΔG in G. From (2.7.10)

$$\Delta P = \int_0^{\infty} \exp (A_G't) \Delta Q \exp (A_G t) dt,$$

so that to first order

$$\Delta J = \operatorname{tr}(X'\Delta PX)$$

$$= 2\operatorname{tr}\left(\int_0^\infty X_o'\exp(A_G't)(RG + B'P)'\Delta G \exp(A_G t)X_o\,dt\right)$$

$$= 2\operatorname{tr}\left(\int_0^\infty \exp(A_G t)X_oX_o'\exp(A_G't)\,dt\,(RG + B'P)'\Delta G\right)$$

$$= 2\operatorname{tr}(T(RG + B'P)'\Delta G). \tag{7.2.11}$$

Where $T \triangleq \int_0^\infty \exp(A_G t)X_oX_o'\exp(A_G't)\,dt.$ (7.2.12)

However, $\Delta G = \Delta KH$ so that

$$\Delta J = 2\operatorname{tr}(T(RG + B'P)'\Delta KH)$$

$$= 2\operatorname{tr}(\Delta K'(RG + B'P)TH'). \tag{7.2.13}$$

For $\Delta J = 0$ for all $\Delta K \neq 0$ it is necessary that

$$\frac{\partial J}{\partial K} = (RG + B'P)TH' = 0\,. \tag{7.2.14}$$

The matrix T defined by (7.2.12) is invertible if the dynamic system

$$\dot{x} = A_G x + X_o v\,, \tag{7.2.15}$$

is controllable [K10] from input v, where X_o is the set of initial conditions (7.2.6) over which J is minimised.

When H and X_o are invertible T must be invertible and (7.2.14) gives the usual condition for the Riccati controller

$$G = -R^{-1}B'P.$$

For the case when H has maximum rank $m < n$ (7.2.14) becomes

$$RKHTH' + B'PTH' = 0$$

or

$$K = -R^{-1}B'PTH'(HTH')^{-1}, \qquad (7.2.16)$$

at the optimum where we assume that inverses exist.

Interpretation of necessary conditions

The expression (7.2.16) is usefully interpreted in the following way. Consider an estimate x^* of x to be obtained as a function of the measured output y so that

$$x^* = Z\,y. \qquad (7.2.17)$$

Consider the problem of finding Z to minimise J^* defined in terms of the closed loop trajectories with initial conditions X_o,

$$J^{*} = \sum_{X_o} \int_0^{\infty} \|x^{*} - x\|^2_{M'M} \, dt \tag{7.2.18}$$

$$= \operatorname{tr} \int_0^{\infty} X_o' \exp(A_G' t)(H'Z' - I)M'M$$

$$\exp(A_G t)\, X_o \, dt \; ,$$

so that from (7.2.12)

$$J^{*} = \operatorname{tr} (H'Z' - I)M'M(ZH - I)T$$

$$= \operatorname{tr} \; M(ZH - I)T(ZH - I)'M'.$$

Then if we define $Z^{*} = MZ$ and minimise J^{*} wrt. Z^{*}, we have

$$\frac{\partial J^{*}}{\partial Z^{*1}} = \frac{\partial}{\partial Z^{*1}} \operatorname{tr} ((Z^{*}H - M)T(Z^{*}H - M)')$$

$$= HT(Z^{*}H - M)'$$

which is zero when

$$Z^{*1} = (HTH')^{-1}HTM'$$

or $$Z^{*} = MTH'(HTH')^{-1}.$$

Therefore since $Z^{*} = MZ$, the Z that minimises J^{*} is given by

$$Z = TH'(HTH')^{-1} . \tag{7.2.19}$$

When M is invertible (7.2.19) is a unique solution, otherwise (7.2.19) is one of an infinite number of solutions.

Comparing (7.2.17), (7.2.19) and (7.2.16) it is seen that at the optimum, the incomplete state feedback control has the form

$$\begin{aligned} u &= - R^{-1}B'PTH'(HTH')^{-1}y \\ &= - R^{-1}B'P\,x^{*} . \end{aligned} \tag{7.2.20}$$

x^{*} is an 'optimal' estimate and in particular, putting $M = - R^{-1}B'P$ in (7.2.18) we see that along the constrained feedback trajectories

$$J^{*} = \sum_{X_0} \int_0^{\infty} \left\| u + R^{-1}B'Px \right\|^2 dt$$

is minimised. Since the Riccati control with full state measurement is $u = - R^{-1}B'Px$, it is clear that there is a close connection between the full and incomplete state measurement optimization problems.

This is an encouraging result and suggests the approach followed in the succeeding Section for obtaining the solution of the constrained optimization problem, however we first examine the term $(HTH')^{-1}$.

Notation

Paralleling the definition of the matrix of plant initial conditions

$$X_o \triangleq (x(1), x(2), \dots x(N))_{t=o} ,$$

we introduce the matrices X(t), Y(t) corresponding to plant closed loop trajectories corresponding to X_o,

$$X(t) \triangleq \exp (A_G t) X_o \tag{7.2.21}$$

$$Y_X(t) \triangleq HX(t) \tag{7.2.22}$$

We now show that singularity of (HTH') is merely a consequence of linear dependence among the plant output trajectories $Y_X(t)$. From (7.2.12)

$$\begin{aligned} HTH' &= H\int_o^\infty \exp (A_G t)X_o X_o' \exp (A_G' t)\, dt\, H' \\ &= \int_o^\infty H\, X(t)\, X'(t)\, H' dt \\ &= \int_o^\infty Y_X(t)\, Y_X'(t)\, dt . \end{aligned} \tag{7.2.23}$$

Therefore (HTH') is singular if there exists a constant β such that Y'(t) β is zero for all $t \geqslant 0$. That is for each closed loop trajectory starting from the initial conditions X_o, $y'(t)\beta = 0$ and the plant outputs are dependent. There is then indeterminacy in choosing Z of (7.2.17) and in the context of noise-free systems one or more plant outputs could be neglected with no loss of information.

We will assume that plant outputs are independent so that $(HTH')^{-1}$ always exists.

7.2.2 Algorithm for Constrained Optimization

The similarity between conditions at the optimum of the constrained and unconstrained problems has been noted. One successful method of solving the steady state Riccati equation is due to Kleinman [K8]. It is presented in the literature for the continuous time case but we rederive it for discrete time, because the author feels that its convergence properties when small perturbations in the control are present are more easily obtained. This will be helpful in discussing the convergence of a similar algorithm for constrained gains.

The discrete time Kleinman algorithm

Consider a sampled data system

$$x_{k+1} = Ax_k + Bu_k \,, \tag{7.2.24}$$

$$y_k = Hx_k \,, \tag{7.2.25}$$

with performance index

$$J = \sum_{o}^{\infty} (x'Qx + u'Ru)_k \,. \tag{7.2.26}$$

The minimisation of J with respect to u leads to the discrete time Riccati equation, which at steady state is ,

$$P = Q + G'RG + (A + BG)'P(A + BG) \tag{7.2.27}$$

$$= Q + A'PA - A'PB(R + B'PB)^{-1}B'PA \ , \tag{7.2.28}$$

when

$$G = -(R + B'PB)^{-1}B'PA \ . \tag{7.2.29}$$

The Kleinman algorithm is based on (7.2.27) and (7.2.29) rather than (7.2.28). Suppose that a stable control law G(k) is available, then we may define

$$J = x_o'P(k)x_o \ , \tag{7.2.30}$$

$$P(k) = Q + G'(k)\,RG(k) + A_G(k)'P(k)\,A_G(k) \ , \tag{7.2.31}$$

$$A_G(k) = A + BG(k). \tag{7.2.32}$$

Now consider a new control law

$$G(k+1) = -(R + B'P(k)B)^{-1}B'P(k)A \ . \tag{7.2.33}$$

We show that the corresponding P(k+1) satisfies $P(k+1) \leq P(k)$ and that $P(k+1) \neq P(k)$ if $G(k+1) \neq G(k)$. The approach taken is convenient for later developments.

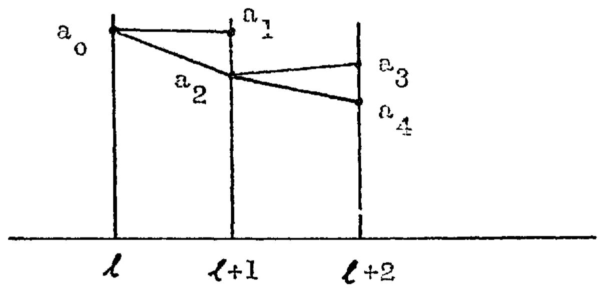


Figure 7.2.1 Trajectories for different applications of control laws G(k) and G(k+1)

Consider a trajectory with initial condition x_o and law G(k+1) for $0 \leqslant j \leqslant \ell$, and suppose that the state $x_\ell = a_o$.

At time (index) ℓ we consider two alternatives; to apply G(k) to make $x_{\ell+1} = a_1$ (see Figure 7.2.1) or to apply G(k+1) to make $x_{\ell+1} = a_2$. Law G(k) is to apply for $\ell < j < \infty$. The first choice gives a cost to ∞ from x_ℓ of $J = a_o{}'P(k)\, a_o = x_\ell{}'P(k)\, x_\ell$. The second choice gives a cost to ∞ from x_ℓ,

$$J = (x'Qx + u'Ru + (Ax + Bu)'P(k)(Ax + Bu))_\ell$$

$$u_\ell = G(k+1)\, x_\ell.$$

But G(k+1) has been chosen to minimise the above expression for all x so that the second choice gives a lower cost to ∞ from x_ℓ than the first. We conclude that at any time index ℓ, and any value for x_ℓ, extending the application of G(k+1) by a sampling interval reduces the cost to infinity. By induction, when $G(k+1)$ is applied for $0 \leqslant j \leqslant \infty$, the total cost is less than for $G(k)$. This is true for all initial conditions and therefore $P(k+1) \leqslant P(k)$. Finally there is always a cost improvement for some initial conditions when $G(k+1) \neq G(k)$.

It also follows from the quadratic nature of the cost that the gain update $G(k+1)$ may be modified in the Kleinman algorithm by using a gain $(\alpha G(k+1) + (1 - \alpha)G(k))$, $0 < \alpha \leqslant 1$ and a cost decrease will always occur.

The Kleinman algorithm uses (7.2.32) to (7.2.34) recursively to obtain the non-increasing sequence $\{P(0), P(1), \ldots\}$ and companion gains sequence $\{G(0), G(1), \ldots\}$. At the optimum $\frac{\partial J}{\partial G} = 0$ so that if $G(k) = G_{(\infty)} + \beta \Delta G : 0 < \beta \ll 1$, then $P(k) = (P_{(\infty)} + (\text{terms in } \beta^2))$ and $G(k+1) = (G_{(\infty)} + (\text{terms in } \beta^2))$. The sequences therefore converge quadratically near the optimum.

We note that if at any stage of the Kleinman algorithm the law $G(k+1) = (-(R + B'P(k)B)^{-1}B'P(k)A + \beta\Delta G)$ is used containing a small error $\beta \Delta G$, the cost decrease in choosing the law $G(k+1)$ rather than $G(k)$ at time ℓ is unaltered to first order in β. Consequently the cost decrease when $G(k+1)$ replaces $G(k)$ over the whole trajectory has zero first order variation with respect to control perturbations.

The continuous time algorithm follows by allowing the sampling time to tend to zero and obtaining expressions for the discrete time dynamics in terms of continuous time dynamics parameters. The Kleinman algorithm is readily derived for the dynamics (7.2.1) and cost (7.2.4) as ,

$$P(k) = \left[\int_0^{\infty} \exp\,(A_G{}'t)(Q + G'RG)\exp\,(A_G t)\; dt \right]_k$$

$$G(k+1) = -R^{-1}B'P(k) .$$

It follows also that in the continuous case, the cost decrease will be unaffected to first order by perturbations to the updated control law G(k+1).

Algorithm 7.2.1 - Constrained minimization

The continuous time Kleinman algorithm suggests a parallel algorithm for solution of the necessary conditions (7.2.16). A stable gain K_o is required to start the recursive process. The k^{th} stage of the algorithm is :

Step 1 : For a stable gain K_k calculate

$$P(k) = \left[\int_0^{\infty} \exp\,(A_G{}'t)(Q + G'RG)\; \exp\,(A_G t)\; dt \right]_k$$

$$T(k) = \left[\int_0^{\infty} \exp\,(A_G t)X_o X_o{}'\; \exp\,(A_G{}'t)\; dt \right]_k .$$

Step 2 : Calculate

$$\hat{K}(k+1) = \left[- R^{-1}B'PTH'(HTH')^{-1}\right]_k$$

Step 3 : Calculate for $0 < \alpha < 1$

$$\hat{K}(k+1) = K(k) + \alpha(\hat{K}(k+1) - K(k)) ,$$

and choose α such that

$$tr\ (P(k+1)\ X_oX_o') < tr\ (P(k)\ X_oX_o') ,$$

and return to Step 1.

It is very easy to derive the discrete time analogue of the above algorithm for the dynamics (7.2.24), (7.2.25) with performance index (7.2.26). If P and T become the discrete time counterparts to the integral expressions of Step 1, then Step 2 is given by

$$\hat{K}(k+1) = \left[- (R + B'PB)^{-1}B'PATH'(HTH')^{-1}\right]_K ,$$

and step 3 is unchanged. However, the analysis will be done for the continuous time case, and characteristics of the algorithm will be very similar for the continuous and discrete time forms.

Generation of Descent Directions

The algorithm may converge to a true minimum, a local minimum or a saddle point since the problem is non-quadratic. However, descent directions are generated, which may be shown by the following first order analysis.

From (7.2.13) and Steps 2 and 3 of the algorithm,

$$
\begin{aligned}
\tfrac{1}{2}\Delta J &= \mathrm{tr}\,(\Delta K'(RG + B'P)TH') \\
&= \alpha \mathrm{tr}\,(HT(H'K'R + PB)(-R^{-1}B'PTH'(HTH')^{-1} - K)) \\
&= -\alpha \mathrm{tr}\,(HT(RKH + B'P)'R^{-1}(B'PTH' + RK(HTH')) \\
&\qquad (HTH')^{-1}) \\
&= -\alpha \mathrm{tr}\,((B'PTH' + RKHTH')'R^{-1} \\
&\qquad (B'PTH' + RKHTH')(HTH')^{-1}
\end{aligned}
\tag{7.2.34}
$$

Since $R^{-1} > 0$, $(HTH')^{-1}$ is multiplied by a positive semi-definite symmetric matrix, also $(HTH')^{-1} > 0$ if the inverse exists as assumed. The eigenvalues of the matrix product (7.2.23) are therefore real and $\geqslant 0$.

Not all eigenvalues are zero and therefore the trace in (7.2.25) is non zero, unless (B'PTH' + RKHTH') is zero. Therefore it is guaranteed that $\Delta J < 0$ for changes in gain K indicated by the algorithm, if the step size parameter α is small enough for the first order analysis to be valid.

<u>Conditions for fast convergence of algorithm 7.2.1</u>

We will say that the convergence rate of the algorithm is fast if it is comparable to that of the Kleinman algorithm. The $m < n$ plant output y is available for feedback, and ~~[K1] has proven that feedback is required from all states for optimal trajectories~~, we shall assume that at all times the dominant dynamics of the plant have dimension m. This is equivalent to an assumption that a good reduced order model of the closed loop plant exists.

Consider a closed loop matrix $A_G(k)$ at the k^{th} stage of algorithm and suppose that it has two distinct groups of fast and slow modes. Suppose that the initial condition set X_o contains substantial components of all slow modes and not excessive components of the fast modes. We suppose further that the fast modes have small associated costs.

By our assumption the trajectories lie mainly in the slow mode subspace. The effect of the term $\left[TH'(HTH')^{-1}\right]_k$ of the algorithm, as explained in the interpretation of the necessary conditions, is to form an optimal estimate $x^* = Zy$, so that x^* will very nearly lie in the slow mode subspace.

Suppose that the slow and fast modes have components γ_1 and γ_2 and that we choose a modal co-ordinate basis in state space. So that

$$y = \underline{H}_1 \gamma_1 + \underline{H}_2 \gamma_2 , \tag{7.2.35}$$

$$\frac{d}{dt}\begin{bmatrix}\gamma_1 \\ \gamma_2\end{bmatrix} = \begin{bmatrix}\underline{A}_{11} & 0 \\ 0 & \underline{A}_{22}\end{bmatrix} \begin{bmatrix}\gamma_1 \\ \gamma_2\end{bmatrix} + \begin{bmatrix}\underline{B}_1 \\ \underline{B}_2\end{bmatrix} \Delta u , \tag{7.2.36}$$

$$J = (\gamma_1, \gamma_2) \begin{bmatrix}\underline{P}_{11}(k) & \underline{P}_{12}(k) \\ \underline{P}_{21}(k) & \underline{P}_{22}(k)\end{bmatrix} \begin{bmatrix}\gamma_1 \\ \gamma_2\end{bmatrix} ,$$

$$\ldots\ldots \quad (7.2.37)$$

where we write Δu because γ_1, γ_2 are modes of the closed loop system.

$$\dot{x} = (A + BK(k)H)x + B\Delta u . \tag{7.2.38}$$

An accepted approximation [D2] in (7.2.36) is that

$$\gamma_2 = -\underline{A}_{22}^{-1}\underline{B}_2 \Delta u$$

$$\triangleq S \Delta u , \tag{7.2.39}$$

where possibly S is small, either because $\|\underline{B}_2\| \ll 1$, or $\|\underline{A}_{22}\| \gg 1$.

Under our assumptions $\underline{P}_{12}(k)$ and $\underline{P}_{22}(k)$ are small, $\underline{H}_2$ and S are possible small.

We will compare a Kleinman and constrained gain update on gains and subscript variables with K and C respectively.

The current constrained control at the k^{th} iteration of algorithm 7.2.1 is

$$u_C(k) = \left[- R^{-1}B'\underline{P}\,\underline{T}\,\underline{H}'(\underline{H}\,\underline{T}\,\underline{H}')^{-1}\right]_{k-1} H\,y$$

$$\approx \left[- R^{-1}B' \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} \gamma_1 + \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} \underline{H}_1^{-1}\underline{H}_2\gamma_2\right]_{k-1} .$$

We define

$$\Delta P \triangleq P(k) - P(k-1) ,$$

$$\Delta u_K \triangleq u_K(k+1) - u_C(k)$$

$$\approx - R^{-1}B' \left[\begin{bmatrix} \Delta P_{11} \\ \Delta P_{21} \end{bmatrix} \gamma_1 + \begin{bmatrix} P_{12}(k+1)-P_{12}(k)\underline{H}_1^{-1}\underline{H}_2 \\ P_{22}(k+1)-P_{22}(k)\underline{H}_1^{-1}\underline{H}_2 \end{bmatrix} \gamma_2\right]$$

$$\triangleq \Delta K_{K1}\gamma_1 + \Delta K_{K2}\gamma_2 , \qquad (7.2.40)$$

$$\Delta u_C \triangleq u_C(k+1) - u_C(k)$$

$$\approx -R^{-1}B' \left[\begin{bmatrix} \Delta P_{11} \\ \Delta P_{21} \end{bmatrix} \gamma_1 + \begin{bmatrix} \Delta P_{11} \\ \Delta P_{21} \end{bmatrix} \underline{H}_1^{-1} \underline{H}_2 \gamma_2 \right]$$

$$\triangleq \Delta K_{C1}(\gamma_1 + \underline{H}_1^{-1}\underline{H}_2 \gamma_2) , \qquad (7.2.41)$$

where it is seen that

$$\Delta K_{C1} = \Delta K_{K1} .$$

The above equations are essentially open loop relations. We use the fast mode steady-state approximation (7.2.39) to obtain closed loop relations,

$$\Delta u_C \approx \Delta K_{C1}(\gamma_1 + \underline{H}_1^{-1}\underline{H}_2 \gamma_2)$$

$$\approx \Delta K_{C1}(\gamma_1 + \underline{H}_1^{-1}\underline{H}_2 S\Delta u_C) ,$$

so that

$$\Delta u_C \approx (I - \Delta K_{C1} \underline{H}_1^{-1} \underline{H}_2 S)^{-1} \Delta K_{C1}\gamma_1 . \qquad (7.2.42)$$

Also

$$\Delta u_K \approx \Delta K_1\gamma_1 + \Delta K_2\gamma_2$$

$$\approx \Delta K_1\gamma_1 + \Delta K_2 S \Delta u_K ,$$

so that

$$\Delta u_K \approx (I - \Delta K_2 S)^{-1} \Delta K_1\gamma_1 . \qquad (7.2.43)$$

The approximation signs approach equality as the optimal estimate of plant state tends to measure the slow modes only, and also as (7.2.39) tends to be a better description of the fast modes. (7.2.42) and (7.2.43) show that if ΔK_2, S, H_2 are small, Δu_C tends to Δu_K along the closed loop trajectories when fast modes have taken up their steady state approximations (7.2.39). The initial fast mode transient has been omitted from the analysis, but if fast modes are not heavily costed and do not have excessively large initial conditions, changes in the total cost from the initial conditions X_o will not be significantly affected by changes in fast mode transients.

Therefore the cost decrease arising from the constrained gain algorithm update tends to equal that of the Kleinman update. We note that $\Delta K_1, \Delta K_2$ depend on the cost on control, so that as R decreases there will be greater discrepancy between Δu_K and Δu_C. The cost decrease has been shown to be independent of perturbations in Δu to first order and this tends to diminish the effect of differences between Δu_K and Δu_C.

We conclude that when there are m plant measurements y which are independent with respect to initial conditions X_o defining a performance index J (7.2.5), the cost decreases resulting from the gain updates of algorithm 7.2.1 tend to be the same as for the Kleinman algorithm.

Our analysis has used a modal co-ordinate basis for the closed loop system at the k^{th} stage. At different stages, these will be different modal decompositions. The role of the T matrix in (7.2.12) may be interpreted as an updated implicit specification of a reduced order model of the closed loop system as the feedback gains are changed. Mitra [M7, M8] has stressed the importance of a "W-matrix" in model reduction theory, and this is very similar to our T matrix. As the conditions for fast convergence are invalidated, possibly by specifying a low cost on control resulting in high gains, the algorithm still provides descent directions. The rate of convergence may be associated with the existence of dominant dynamics at the optimum or with how well the plant output measures dominant dynamics at the optimum. Consequently convergence rate indicates the closeness between the constrained and unconstrained optima from the given initial conditions. If the open loop system has $m_1 < m$ say, slow modes forming a distinct group that causes most of the open loop performance cost, we would expect that the value of the cost function would at first quickly decrease with the iteration number and then level out to a slow descent. In other words, we would expect that the cost function would quickly converge to a near optimal value, even if the gains do not. The reason for this is that the plant outputs are capable of providing a good measure of the dominant slow modes, and therefore the updated control at each iteration will closely approximate the Kleinman update, as far as the slow modes are concerned. As the dominant modes are assumed to contribute most of the cost, the cost decrease per iteration will tend to keep in line with the cost decrease of the Kleinman iteration, until the feedback from the m_1 slow modes is near optimal. Then the effect of fast mode dynamics will become significant in seeking cost decreases and the question of fast convergence to the optimal gain arises. If there is slow convergence, we have argued that there is either no dominant group of modes at the optimum or that there is a dominant group of greater than or less than m modes. In either case it may be profitable to use more or fewer than m plant outputs for the control scheme.

There is little practical justification of seeking the optimal constrained gains to great accuracy for two reasons. The first is that the description of the plant dynamics (7.2.1), (7.2.2) is an approximation, the second is that in practice gain values can only be set approximately to desired values. Near the optimum the cost is fairly insensitive to gain changes and this is a desirable feature of the controller. Convergence of the algorithm is a convenient unambiguous theoretical stopping condition for the computation in those cases where it is easily achieved.

7..3 Implementation of the algorithm

Weighting function description of plant dynamics

The algorithm may be used directly when the plant dynamics information is given in weighting function form. Given a feedback law at the k^{th} iteration

$$u = K(k)\,y,$$

the state space approach specifies the $(k+1)^{th}$ law as

$$K(k+1) = K(k) + \alpha(\hat{K}(k+1) - K(k)) \;, \qquad (7.2.44)$$

where

$$\hat{K}(k+1) = \left[- \; R^{-1}B'PTH'(HTH')^{-1}\right]_k . \qquad (7.2.45)$$

The matrices P(k) and T(k) are calculated, substituted into (7.2.45) and K(k+1) is obtained from (7.2.44). This approach is helpful in the interpretation of the conditions at the optimum and is easily programmed. As an alternative, the matrices (B'PTH') and (HTH') in (7.2.38) may be found by calculations based on the trajectories starting from the initial conditions X_o, if the state cost $x'Qx$ arises from a costing $y'Ly$ on the measured plant output. Then $Q = H'LH$, say.

We define $Y_B(t)$ to be the weighting function matrix of the closed loop plant so that

$$Y_B(t) \triangleq \exp((A + BKH)t)B$$

$$= \exp(A_G t)B \,. \qquad (7.2.46)$$

Using (7.2.39), (7.2.40) and the definitions (7.2.7), (7.2.12) for P and T, and introducing $Y_X(t)$ defined by (7.2.22) we have ,

$$B'PTH' = B'\left(\int_0^\infty \exp(A_G't)(Q + G'RG)\exp(A_G t)dt\right)\left(\int_0^\infty \exp(A_G s)X_oX_o' \exp(A_G's)ds\right)H'$$

$$= \int_0^\infty\int_0^\infty Y_B'(t)L_G Y_X(t+s)Y_X'(s)dt\, ds \,, \qquad (7.2.47)$$

where

$$H'L_G H \triangleq Q + G'RG$$

$$= H'(L + K'RK)H. \qquad (7.2.48)$$

A suitable expression (7.2.23) has been found previously for the term HTH' which is rewritten as

$$HTH' = \int_0^\infty Y_X(t)\, Y_X'(t)\, dt \,. \qquad (7.2.49)$$

Equations (7.2.47), (7.2.49) show that the expression (7.2.46) may be evaluated by output trajectory information alone, so that the algorithm could be applied to infinite dimensional systems. It also follows that information for the gain updating algorithm (7.2.44), (7.2.45) can be obtained using an analogue computer simulation of the system.

Description of simple digital computer program of Algorithm 7.2.1

A program based on this algorithm has been written in which the matrix P is calculated by the purely algebraic method of [J3], and no check is made on the positive definiteness of P to ensure a stable feedback gain. The Kleinman algorithm ensures that stable feedback laws are generated for step size parameter $\alpha = 1$, but Algorithm 7.2.1 is an approximation and precautions should be taken that the gain remains in a stable region. We use the simplest precaution in the programmed implementation of the algorithm of adjusting α so that the norm of indicated gain changes is less than a prespecified constant. Otherwise α is initially taken as $\alpha = 1$ and halved until a decrease in performance index is achieved. A new search direction is then obtained.

Convergence is assumed when the norm of the indicated change in gain is less than a prespecified value.

This crude approach is justified by the desire to prove the potential of the method, rather than to produce a finished and efficient computer hill-climbing program.

Several numerical examples have been worked using the program and we briefly present the results.

Simple example

Consider the 3-state, single-input, 2-output plant to be called 'plant A',

$$x = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u ,$$

$$y = Hx$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x .$$

with two performance indices of the form ,

$$J_1 = \int_0^\infty (x_1{}^2 + u^2)\, dt ,$$

and

$$J_2 = \int_0^\infty (x_1{}^2 + .1u^2)\, dt ,$$

from initial condition

$$X_o = (0, 1, 0)' .$$

Using the established notation, it is desired to

$$\mathrm{Min}_K \; (\mathrm{tr}\; (X_o{}'PX_o)) \; ,$$

$$u \;=\; Ky \; .$$

For comparison of closed loop performance and convergence rate we consider another problem concerning a plant B, with dynamics

$$x \;=\; \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \; ,$$

$$y \;=\; Hx$$

$$\;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \; ,$$

and the same performance index. We note that Algorithm 7.2.1 here becomes the Kleinman algorithm [K8].

The results of applying Algorithm 7.2.1 to both problems are shown in Table 7.2.1.

It is seen that the number of iterations required for convergence to the optimal gain for plant A is excellent in comparison with the number required for plant B. The optimal gains are similar which is what is expected on an intuitive basis. The performance index was defined in terms of one initial condition but the resultant closed loop control gives costs that are close to the optimal of the "ideal system" plant B for all initial conditions.

Table 7.2.1 Performance of simple program of Algorithm 7.2.1 for example 7.2.1

Quantity	Plant A	Plant B	Plant A	Plant B
Cost	$J_1 = \int (x_1^2+u^2)dt$		$J_2 = \int (x_2^2+.1u^2)dt$	
X_o	(0, 1, 0)'	(0, 1)'	(0, 1, 0)'	(0, 1)'
Initial G	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Initial J	.25	.25	.25	.25
Optimal G	$\begin{bmatrix} .1918 \\ .1949 \end{bmatrix}'$	$\begin{bmatrix} .2163 \\ .1974 \end{bmatrix}'$	$\begin{bmatrix} 1.107 \\ .917 \end{bmatrix}'$	$\begin{bmatrix} 1.378 \\ .938 \end{bmatrix}'$
Optimal J	.2013	.1974	.1029	.0938
No. of search directions	4	4	6	5
No. of P evaluations for solution to 10^{-3}	4	4	6	5

The closed loop costs from all states for plants A and B are

$$J_{1A} = x_A' \begin{bmatrix} .480 & .221 & .020 \\ .221 & .201 & .020 \\ .020 & .020 & .002 \end{bmatrix} x_A ,$$

$$J_{1B} = x_B' \begin{bmatrix} .476 & .217 \\ .217 & .197 \end{bmatrix} x_B ,$$

$$J_{2A} = x_A' \begin{bmatrix} .421 & .150 & .013 \\ .150 & .103 & .010 \\ .013 & .010 & .001 \end{bmatrix} x_A ,$$

and

$$J_{2B} = x_B' \begin{bmatrix} .405 & .138 \\ .138 & .094 \end{bmatrix} x_B .$$

It is seen that the difference between J_A, J_B increases as the cost on control decreases, and the state present in plant A but neglected in plant B then has an increasing influence on plant trajectories. The agreement between J_A and J_B is still reasonable enough to indicate that the algorithm would converge well for a smaller cost on control. The closed loop eigenvalues of plant A for the smaller cost on control are $(-\,9.03,\ -\,1.49 \mp 1.07i)$.

A more complex example

A more complex example is taken to illustrate fast convergence of Algorithm 7.2.1. The system is a linear 8-state, single input, 4-output boiler model that has been used by Ellis and White [E1] to demonstrate modal control, by incomplete state feedback. The dynamics are

$$x = Ax + Bu ,$$

$$y = Hx .$$

The reader is referred to reference [E1] for a physical description of the state variables. Briefly the state variables in order are perturbations from a working point of the following :- drum pressure, range pressure, air pressure, air flow, integral action on air flow, fuel flow, integral action on fuel flow and fan actuator output.

The outputs to be controlled are range pressure (x_2) and steam flow f_t,

$$f_t = .5261 (x_1 - x_2) .$$

Ellis and White show that y is a good representation of the four slow modes that are roughly separated from the rest to form a distinct group, the set of eigenvalues of A being approximately (-.0019, -.09, -.1, -.166, -.38 $\mp$ 3i, -.499, -1.17). We therefore expect Algorithm 7.2.1 to show an acceptable convergence rate for suitable performance indices.

$$
A = \begin{bmatrix}
-.0121516 & .021516 & 0. & 0. & -.001138 & .662 & 0. & 0. \\
.132 & -.1469 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -.4241 & 0. & 0. & 0. & 0. & .5561 \\
0. & 0. & -.516 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 2.7073 & 0. & -.4995 & 0. & 0. & 0. \\
0. & 0. & .5166 & 0. & 0. & -1.834 & .1207 & 0. \\
0. & 0. & .516 & 0. & 0. & -1.332 & 0. & 0. \\
0. & 0. & -.2346 & .0909 & 0. & 0. & 0. & -.4546
\end{bmatrix},
$$

$$
B = (0. \quad 0. \quad 0. \quad -1. \quad 0. \quad 0. \quad 0. \quad -.4546)',
$$

$$
H = \begin{bmatrix}
1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 1. & 0.
\end{bmatrix}
$$

We have tried several performance indices costing range pressure, steam flow and control. Typically,

$$J(1) = \int_0^\infty (x_2^2 + .3(x_1 - x_2)^2 + .1u^2)dt \ ,$$

$$X_o(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

and

$$J(2) = \int_0^\infty (x_2^2 + (x_1 - x_2)^2 + .1u^2)dt \ ,$$

$$X_o(2) = H' \ ,$$

and finally

$$J(3) = \int_0^\infty (x_2^2 + (x_1 - x_2)^2 + .01u^2)dt$$

$$X_o(3) = H' \ .$$

The corresponding optimal gains are given in Table 7.2.2 with Ellis and White's modal control. Table 7.2.3 shows the typical convergence rate of the algorithm. There was no problem starting with zero (stable) control. During the course of the calculations it was apparent that the performance index was insensitive to gain changes. We note that J(1) is defined in terms of two initial conditions and J(2), J(3) in terms of four. In all cases the P matrix at the optimum showed that the control was good for all initial conditions. For the purposes of a rough comparison Figure 7.2.1 shows the response of the boiler to a step demand on steam flow with constant feedforward adjusted

Table 7.2.2 Constrained optimal and modal gains

P.I.	Cost	Gain
J(1)	6.674	-(2.338, .7792, .5600, .2289)
J(2)	13.130	-(3.097, .4341, .6384, .5340)
J(3)	11.684	-(12.11, 1.572, 1.873, 1.720)
modal	-	-g(.8083, .1200, .4041, .4108)

Table 7.2.3 Typical convergence to constrained optimum for example

Quantity	Iteration No. 1	2	3	4	5
	modal control to start				
J(2)	13.378	13.140	13.1305	13.1301	-
$\Vert G_{k+1}-G_k \Vert$	-	.91	.166	.003	(exit)
	G(2) optimal to start				
J(3)	12.366	11.733	11.686	11.6837	11.6832
$\Vert G_{k+1}-G_k \Vert$	-	14.0	3.9	.82	.027 (exit)

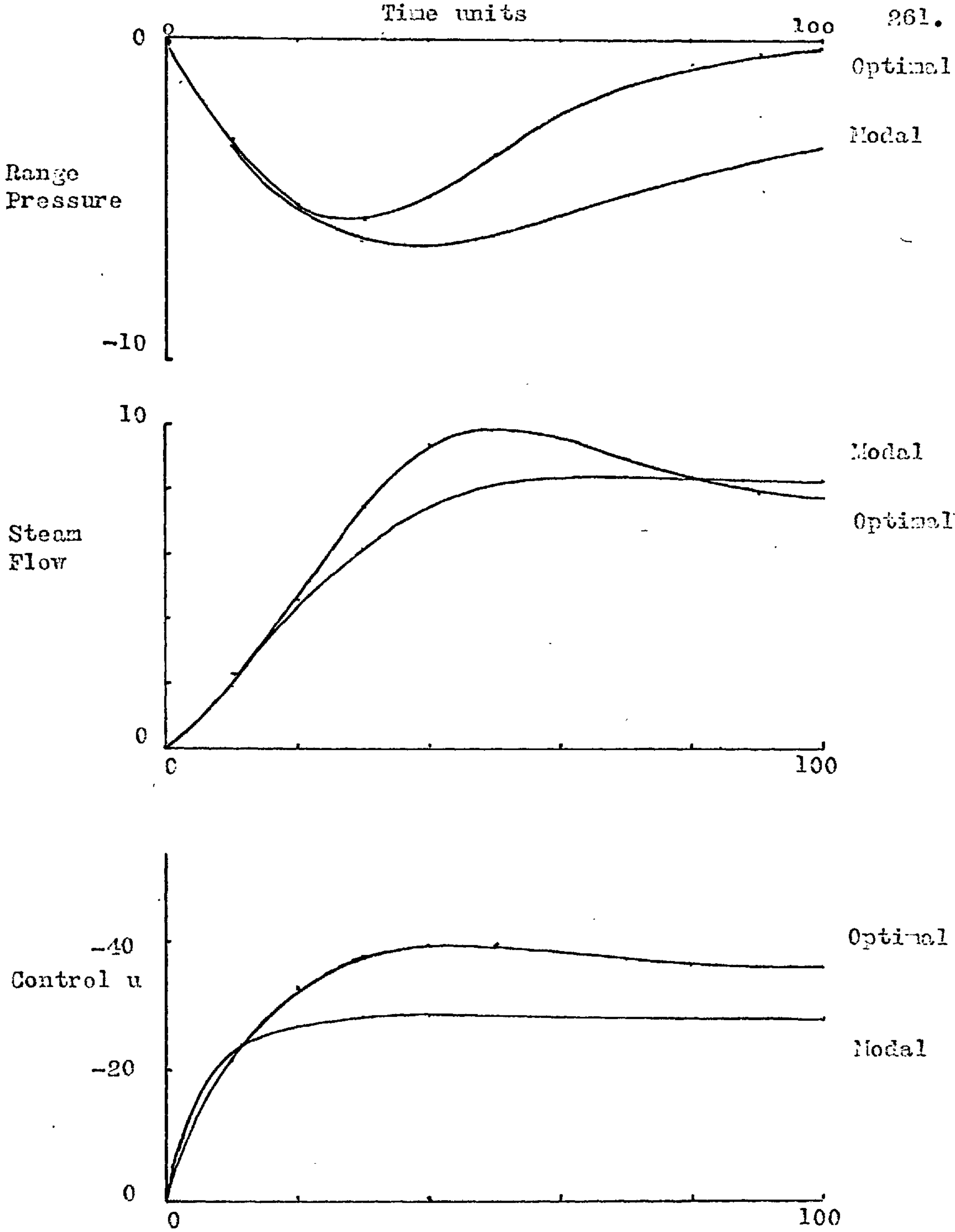


Figure 7.2.1 Comparison of modal (g=20) and constrained optimal control (J(1)).

to give zero offset of range pressure, for modal and an optimal constrained feedback. The modal trajectories are insensitive to gain increases, and the optimal trajectories corresponding to J(1), J(2), J(3) are very similar despite the increasing control effort expended.

7.2.5 Conclusion

We have proposed an optimization problem and algorithm for designing feedback controllers using incomplete state feedback, and suggested when the algorithm may be expected to show near quadratic convergence. Fast convergence is achieved in the trial examples. The optimization problem is valid whether or not conditions are suited to fast convergence, and descent directions are always generated.

With further research on improving the algorithm, possibly by finding the minimum along search directions, it should prove to be a valuable tool for designing feedback controllers for plant that have dominant dynamics, but for which it is difficult to produce reduced order models. An attraction of the algorithm is that the computational effort per iteration does not depend on the number of plant inputs or outputs, nor on the number of trajectory initial conditions included in the overall performance index.

We have indicated that should the algorithm not converge quickly near the optimum, the plant output is not a good measure of the dominant dynamic characteristics of the plant. Methods of dealing with the problem of inadequate plant outputs are discussed in Section 7.5.

7.3 Generation of Invariant Subspace of Dominant Modes

7.3.1 Introduction

One method of model reduction which has wide acceptance is Davison's modal reduction technique [D2] which restricts the dynamics of the plant to its dominant invariant subspace. We extend the idea of using dominant invariant subspaces to the parallel path form of reduced model introduced in the next section, but in anticipation discuss methods of generating the required invariant subspaces.

The problem is formulated for a sampled data n-state system

$$x_{k+1} = Ax_k + Bu_k , \tag{7.3.1}$$

and the invariant subspace, S_D say, corresponding to n_1 dominant modes of A is sought. It is also required to find the component of B lying in S_D. One solution to this problem is to find individual eigenvectors corresponding to the dominant modes of A. It is always useful to do an eigenvalue analysis of a system unless the approximate distribution of eigenvalues is known beforehand. A thorough account of the computational schemes for finding eigensystems is given by Wilkinson [W5]. The most common methods involve complex arithmetic but these may not be satisfactory when multiple eigenvalues occur in such a way that the Jordon form of A is non-diagonal.

An iterative method has been developed that avoids complex arithmetic and identifies S_D directly without finding individual eigenvectors. It has subsequently been found to be a general application of a method described by Wilkinson [W5 (p. 599 onwards)] for finding the subspace spanned by a group of close modes. We give a brief description of the idea and note an interesting connection with one of the methods for finding the steady state solution to the Riccati equation. The component of B lying in S_D may be found in several ways.

7.3.2 Power method of identifying the dominant invariant subspace

The principle in finding the dominant invariant subspace of dimension q of A (7.3.1) is to construct an (n x q) matrix V_o and consider the matrix V_N,

$$V_N = A^N V_o , \quad N \gg 1 . \tag{7.3.2}$$

Assuming that the components of the columns of V in S_D are independent, the columns of V_N tend to be composed entirely of the dominant modes of A, and therefore tend to span S_D. An obstacle to a simple application of this principle is that the columns of V_N tend to be composed entirely of the most dominant mode of A, so that V_N tends to have rank less than q.

It is very convenient to introduce the concept of a subspace trajectory under given dynamics.

Definition 7.3.1

Consider as a set of P initial conditions the independent columns of a (n, p) matrix, X say. Then X and the dynamics (7.3.1) define p independent trajectories which at any sampling time, t_k say, define a subspace, S_k say. Then the sequence $\{S_1, S_2, \ldots\}$ will be called a subspace trajectory. Also denote this set of p trajectories by the sequence $\{X_1, X_2, \ldots\}$ where the i-th column of X_j represents the state of the i-th trajectory at time t_j.

This concept provides the means whereby the obstacle of loss of rank of V_N as N increases is overcome. The subspace S_D spanned by V_N is desired, and the principle of the power method states that S_D is the steady state subspace of a subspace trajectory with the subspace spanned by V_o as initial condition. However, a subspace may be defined by an infinite number of different bases and it may be advantageous to describe a subspace trajectory by different vector trajectory basis sets over different time intervals. The choice of different vector bases is akin to scaling and may be done for the purpose of preserving rank.

A structure of defining subspaces suitable for the ultimate purpose of model reduction is also convenient for the scaling necessary for a well-conditioned representation of subspace trajectories.

Suppose that a p dimensional subspace S in E^n is spanned by the p independent, real vectors $(w_1, \dots w_p)$. Put

$$W = (w_1, w_2, \dots w_p) , \tag{7.3.3}$$

and any n-vector x in S may be written for some p-vector α

$$x = W\alpha . \tag{7.3.4}$$

x and W may be partitioned such that x_1 is a p-vector, W_1 is a (p, p) matrix and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ,$$

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} .$$

If W_1 is non-singular (7.3.4) gives

$$\alpha = W_1^{-1} x_1$$

and

$$x = \begin{bmatrix} I \\ W_2 \, W_1^{-1} \end{bmatrix} x_1 \tag{7.3.5}$$

$$\triangleq \begin{bmatrix} I \\ M \end{bmatrix} x_1 . \tag{7.3.6}$$

Vectors in S are then parameterised by their components x_1 and S itself is defined by M. We will write S as S(M) to emphasize this. Clearly the representation (7.3.6) of S is of full rank.

Consider the subspace trajectory $\{S_1, S_2 \ldots\}$ under the unforced dynamics (7.3.1), and suppose that S_k is parameterised by M_k in the form (7.3.6). Then if M_{k+1} corresponds to S_{k+1}, there is a non-linear recursive relation between M_{k+1} and M_k. This is found by partitioning A of (7.3.1) to correspond to (7.3.6). By definition S_{k+1} is spanned by the columns of $A\begin{bmatrix} I \\ M_k \end{bmatrix}$, but

$$A\begin{bmatrix} I \\ M_k \end{bmatrix} = \begin{bmatrix} A_{11}+ A_{12}M_k \\ A_{21}+ A_{22}M_k \end{bmatrix} , \tag{7.3.7}$$

so that from (7.3.5)

$$M_{k+1} = (A_{21} + A_{22}M_k)(A_{11} + A_{12}M_k)^{-1} . \tag{7.3.8}$$

(7.3.8) is the desired non-linear recursive relation and is well-defined if the inverse exists. This is merely a requirement that x_1 can parameterise the subspace S_{k+1}. If the matrix $(A_{11} + A_{12}M_k)$ tends to become singular, a more suitable choice of state components may be made to parameterise the subspace S_{k+1}. This is done by choosing any p independent rows of the matrix on the RHS of (7.3.7). The subspace trajectory evolution may be then continued in terms of the relation (7.3.8) where the partition blocks A_{ij} are redefined to correspond to the appropriate re-indexing (or re-ordering) on the state components.

The above procedure provides a well-conditioned description of the subspace trajectory at all times as it evolves, and therefore removes the main obstacle to the power method of obtaining dominant subspaces.

Convergence of the method

The rate of convergence of the sequence of (n-p, p) matrices $\{M_1, M_2 \ldots M_k \ldots\}$ to a limit depends on the ratio of the magnitude of the least dominating mode to that of the greatest dominated mode, and the transient decays as a geometric series of this ratio. Clearly if there is no p dimensioned dominant subspace, the subspace trajectory and hence the sequence $\{M_k\}$ cannot converge. The most common condition for this to occur is expected to be when a (p+1) dominant subspace exists, with a complex pair being the least dominant modes. In this situation the dimension of the desired invariant subspace must either be increased or reduced. If it is assumed that all dominated modes apart from the marginal complex pair have decayed, the invariant subspace of increased dimension is spanned by the columns of $\begin{bmatrix} I \\ M_k \end{bmatrix}$ and $\begin{bmatrix} I \\ M_{k+1} \end{bmatrix}$. The form (7.3.6) is easily regained for the augmented subspace.

7.3.3 A connection with a Riccati equation solution

The above method of obtaining an invariant subspace has an interesting special case, the steady state solution of the Riccati equation. As has been discussed in Chapter 2, the Riccati equation may be represented by a pair of n-vectors (x, λ) with stable and unstable poles symmetrically placed about the imaginary axis. The Riccati solution $P(t)$ merely represents a continuous time description of a subspace trajectory in (x, λ)-space parameterised by x in the relation (2.3.8). The canonical pair dynamics are transformed to a sampled data version and the recursion (2.3.10) is identical with (7.3.8) with P_k and Q_{ij} replacing M_k and A_{ij}.

7.3.4 Reduced dynamics corresponding to dominant invariant subspace

If we denote the dominant subspace of A (7.3.1) by S_D with a corresponding M_D, then for any x in S_D,

$$\begin{aligned} x_{1,k+1} &= A_{11}\, x_{1,k} + A_{12}\, x_{2,k} \\ &= (A_{11} + A_{12}M_D)\, x_{1,k} \\ &\triangleq A_R\, x_{1,k}\,. \end{aligned} \qquad (7.3.9)$$

A_R in (7.3.9) is the system matrix of states in the dominant subspace parameterised by x_1.

The most straightforward way of getting the dominant mode component of B in (7.3.1) is to find the reciprocal dominant invariant subspace spanned by the columns of $\begin{pmatrix} I \\ \underline{M}' \end{pmatrix}$ say. Suppose that the dominant eigenvectors and reciprocal eigenvectors are the columns of W and $\underline{W}'$, scaled so that

$$\underline{W}\,W \;=\; I. \tag{7.3.10}$$

Suppose that

$$\begin{aligned}
\begin{bmatrix} I \\ M_D \end{bmatrix} \;&\triangleq\; W\,D\,, \\
(I,\ \underline{M}_D) \;&\triangleq\; \underline{D}\,\underline{W}\,, \\
B \;&\triangleq\; W\,E + (\text{components in non-dominant modes}) \\
&\triangleq\; B_D \;+ (\text{components in non-dominant modes})
\end{aligned}$$

Then

$$\begin{aligned}
(I,\ \underline{M}_D)\begin{bmatrix} I \\ M_D \end{bmatrix} \;&=\; \underline{D}\,\underline{W}\,W\,D \\
&=\; \underline{D}\,D\,,
\end{aligned}$$

and

$$\begin{aligned}
B_D \;&=\; W\,\underline{W}\,B \\
&=\; \begin{bmatrix} I \\ M_D \end{bmatrix} D^{-1}\underline{D}^{-1}(I,\ \underline{M}_D)\,B \\
&=\; \begin{bmatrix} I \\ M_D \end{bmatrix} \left((I,\ \underline{M}_D)\begin{bmatrix} I \\ M_D \end{bmatrix}\right)^{-1}(I,\ \underline{M}_D)\,B.
\end{aligned} \tag{7.3.11}$$

This is considered to be the most satisfactory derivation of B_D, an alternative procedure seems from experience to be more liable to computational error. The theory of this is simple for the special case we consider. S_D is the dominant subspace and suppose S_C is its complement invariant subspace. Consider a column b of the matrix B (7.3.1) and suppose that (A, b) is controllable. Define

$$b \triangleq b_1 + b_2 \qquad (7.3.12)$$

where b_1 is in S_D and b_2 in S_C. Because (A, b) is assumed controllable any x may be expressed as

$$x = \sum_0^{n-1} \alpha_i A^i b , \qquad (7.3.13)$$

where α_i are the components of an n-vector α. Suppose that x_D in S_D is defined so that

$$x_D = \binom{I}{M_D} \beta , \qquad (7.3.14)$$

and (7.3.13) is solved for α for $x = x_D$.

Then

$$x_D = \sum_0^{n-1} \alpha_i A^i b_1 + \sum \alpha_i A^i b_2 ,$$

but x_D and b_1 belong to the same invariant subspace and b_2 is in its invariant complement. Therefore

$$x_D = \sum_0^{n-1} \alpha_i A^i b_1 . \qquad (7.3.15)$$

(7.3.15) may be used as the basis of a numerical procedure for finding b_1, but (7.3.13) must be solved for an n-vector α.

Computational experience

The algorithm has worked successfully on a variety of small systems using a subspace initial condition arbitrarily chosen to correspond to $M_1 = 0$. The largest numerical example that has been tried was a sampled data version of the 8-state boiler model used in the example of the previous section.

Conclusion

A simple method of identifying the dominant invariant state subspace of specified dimension of a sampled data system has been developed. The corresponding reduced system dynamics and control matrices are also found. The method is iterative, a matrix of the order of the sought subspace must be inverted at each iteration any Jordon form configuration may be handled. Very little computer storage is required so that the method is considered to be simple enough and general enough to be useful in practice.

7.4 Parallel Path Models

We propose a method of model reduction that is especially suited to discrete time systems. It allows for fast mode effects by incorporating a weighting function truncated to a small number of terms. The proposed reduced model form is shown in Figure 7.4.1 where one path transmits slow mode input-output effects, the other path

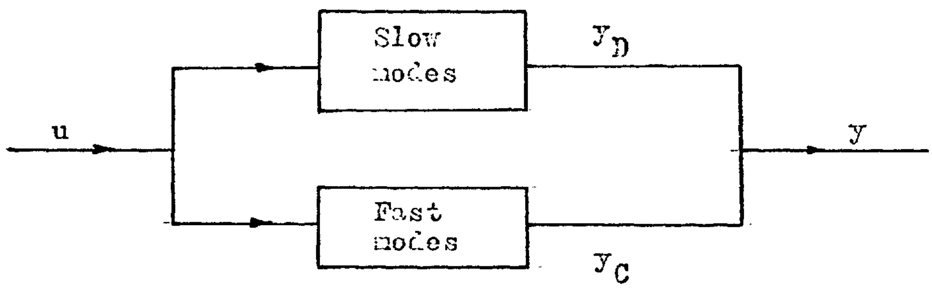


Figure 7.4.1 Form of parallel path model

corresponds to the fast mode transmission through the plant. It is proposed that the slow mode dynamics be modelled by a low order state space system, and fast mode effects be modelled by a weighting function. If the fast modes decay quickly enough, their effect can be well represented by a discrete time weighting function truncated to very few terms. This model reduction scheme provides an alternative to the Davison method [D2] because the dynamic effect of fast modes is represented to some extent. Also the effects of many fast modes can be accumulated into a small number of terms in the weighting function. There is a model complexity tradeoff determined by the number of fast modes, their rate of decay and the number of plant inputs, in the comparison of a Davison reduced model and one of the proposed form. Mathematically the model form is

$$w_{k+1} = A_R w_k + B_R u_k \,, \tag{7.4.1}$$

$$y_{1k} = H_1 w_k \,, \tag{7.4.2}$$

$$y_{2k} = \sum_{i=0}^{r} G_i \, u_{k-i} \,, \tag{7.4.3}$$

$$y_k = y_{1k} + y_{2k} \,, \tag{7.4.4}$$

where w_k represents the slow mode state of the reduced order model and subscript R denotes reduced order quantities. The above form may be put into the more conventional notation,

$$\underline{w}_{k+1} = \underline{A}_R \, \underline{w}_k + \underline{B}_R \, u_k \,, \tag{7.4.5}$$

$$y_k = \underline{H} \, \underline{w}_k + \underline{D} \, u_k \,, \tag{7.4.6}$$

by augmenting the dimension of the reduced model to include the controls $(u_{k-1}, \dots u_{k-r})$ in the model state. Values for the barred matrices in (7.4.5), (7.4.6) are then simply found. Preliminary computational results are encouraging, a detailed investigation of this form has not been undertaken. Possibly the true value of the idea is that it offers an alternative rationale for the forms of compensation proposed in the next section.

7.5 Philosophy of Controller Design

7.5.1 Introduction

Throughout the thesis we have dealt with time invariant, finite dimension, state space descriptions of plant, and have examined several separate blocks of control theory, which are now shown to combine into a method for the design of time invariant feedback controllers. The important question of controller specifications and practical constraints on the controller cannot be discussed outside the framework of specific design examples. We shall assume that the design problem has been condensed to the problem of optimizing a suitable quadratic performance index, possibly to be updated or redefined when the solution has been found, in order to produce desirable changes in properties of the solution. Methods of choosing the performance index have been discussed in Chapter 2. Our restriction to finite order plant is not rigid because the theory of gain optimization of Section 7.2, though developed within the context of finite order systems, is applicable to multi-variable systems of infinite order.

It would seem that idealised linear plant models occur as one of three types.

Type I is derived as precisely as possible from the laws of physics, and is usually far too complex to simulate and unnecessarily complex for a control study. Type II is a description of plant dynamics that is considered to be adequate for the purposes of a control study, and the difference in behaviour between it and the type I model is assumed to be insignificant. Type III is crude and there are significant differences in behaviour between it and the type II model, but feedback structures are often obtained by such models. Sometimes the feedback gains are obtained as well, and then the performance of the designed controller must be checked using the type II dynamics. This classification of a model may depend on the duty required of it, which is usually related to the speed of response desired of the closed loop plant, which again is related to the cost on control.

7.5.2 Complete state measure on models of types II and III

If the complete state is measured on a type II model the required feedback is obtained from the solution to a Riccati equation. If there is incomplete state measure, but such that the unmeasured states correspond to dynamic actuators on the inputs, a Lyapunov control is easily calculated and may be suitable if the corresponding constraints on closed loop transients are acceptable.

It remains to consider what can be done when an accurate type II model with incomplete state measure, and crude type III model with complete state measure are available. Firstly a control law can be obtained by the methods of Chapter 2 for the crude model and checked against the more accurate one a posteriori.

A more precise but compact model for discrete time plant may be to add a parallel path to the crude model to account for some fast mode effects, and then to design a control law, again by the methods of Chapter 2. This leads to a dynamic compensator, and if the parallel path model is an adequate representation of plant dynamics the compensator will be satisfactory.

An alternative which the author prefers is the use of Algorithm 7.2.1 to obtain optimal incomplete state feedback gains for the type II model. The existence of the crude model having complete state measure indicates that convergence to the optimum will be fast. As the performance index is varied the optimal feedback law will be updated with very little computational effort. This algorithm on current experience is highly effective, and is also more versatile than appears at first sight. We demonstrate this by the following theory.

Design of multivariable proportional plus integral controllers

This design procedure is a modification of Johnson's [J2] for complete state measure. Given a system with constant equivalent disturbance w, and cost J,

$$\dot{x} = Ax + B(u + w), \tag{7.5.1}$$

$$y = Hx \ , \tag{7.5.2}$$

$$J = \int_0^\infty (x'Qx + \dot{u}'R\dot{u})dt \ , \tag{7.5.3}$$

$$\underline{u} = u + w \ ,$$

then at the optimum

$$\dot{u} \triangleq K_1 x + K_2 \underline{u} \tag{7.5.4}$$

$$= K_1 x + K_2 (B'B)^{-1} B'(\dot{x} - Ax)$$

$$\triangleq K_1 x + \underline{K}_2 (\dot{x} - Ax). \tag{7.5.5}$$

Since a crude model of full state measure is assumed to exist, the plant output y and $\dot{y}$ must contain most of the information in (7.5.5). Then there exist L_1, $\underline{L}_2$, A_y analogues to K_1, $\underline{K}_2$, A such that

$$\dot{u} \approx L_1 y + \underline{L}_2 (\dot{y} - A_y y) \tag{7.5.6}$$

so that

$$\frac{d}{dt}(u - \underline{L}_2 y) \approx (L_1 - \underline{L}_2 A_y)y$$

$$\triangleq L_3 y .$$

Therefore along optimal trajectories there exist $\underline{L}_2$, L_3 such that

$$u(t) \approx \underline{L}_2 y(t) + \int_0^t L_3 y \, dt . \qquad (7.5.7)$$

The Kleinman algorithm [K8] has fast convergence to the optimal K_1, K_2 of (7.5.4), and there exist $\underline{K}_2$, K_3 analogues to $\underline{L}_2$, L_3 in (7.5.7) algebraically related to K_1, K_2. Therefore the algorithm 7.2.1 will have fast convergence (near quadratic near the optimum) to the optimal values of $\underline{L}_2$, L_3 for the problem,

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ Hx \end{bmatrix} ,$$

$$y = Hx ,$$

$$u(t) = \underline{L}_2 y(t) + L_3 \int_0^t y \, dt \qquad (7.5.8)$$

$$= \underline{L}_2 y(t) + L_3 z(t) ,$$

$$J = \sum_{X_o} \int_0^\infty (x'Qx + \dot{u}'R\dot{u}) \, dt,$$

X_o = a set of specified initial conditions,

then

$$\text{Min}_{\underline{L}_2,\, L_3}\left[\; J \;\right].$$

The constrained optimal control law (7.5.8) is of proportional plus integral form.

Incomplete state feedback of different outputs to different inputs

In the presentation of the theory of gain optimization of incomplete state feedback using Algorithm 7.2.1 and a suitable cost function we have dealt with the form

$$u \quad = \quad Ky \; .$$

The update algorithm for K is based on the new estimate K of the optimal K which always indicates a descent direction,

$$\hat{K} \quad = \quad - R^{-1}B'PTH'(HTH')^{-1}.$$

We may impose different constraints on the control form by defining selections of outputs y_1, y_2 and selections of inputs u_1, u_2 ,

$$y_1 \quad \triangleq \quad H_1 x \; ,$$

$$y_2 \quad \triangleq \quad H_2 x \; ,$$

$$u_1 \quad \triangleq \quad K_1 y_1 \; ,$$

and

$$u_2 \quad \triangleq \quad K_2 y_2 \; ,$$

where there may be some overlap between y_1 and y_2. It is easy to show that in exact correspondence with the relation for K, we have

$$\hat{K}_1 = -R_1^{-1}B_1'PTH_1'(H_1TH_1')^{-1}$$

$$\hat{K}_2 = -R_2^{-1}B_2'PTH_2'(H_2TH_2')^{-1}$$

using obvious notation for R_1, R_2, B_1, B_2.

It is clear that the results can be generalised to sets $\{u_1, u_2 \dots u_q\}$ and $\{y_1, y_2 \dots y_q\}$.

It is anticipated that this type of control constraint may occur when systems are coupled and some uncomplicated control interaction between them is desirable. This idea has not yet been tested computationally.

Incomplete state feedback of unstable plant

The proposed method of obtaining incomplete state feedback controllers requires a feasible, stable initial control law to start Algorithm 7.2.1. If a stable feasible gain $u = Ky$ is not known, the following device may overcome the obstacle provided that a stable, feasible law exists.

For the plant dynamics

$$\dot{x} = Ax + Bu ,$$

$$y = Hx ,$$

$$u = Ky ,$$

introduce an extra plant input v so that

$$\dot{x} = Ax + Bu + B_v v ,$$

$$v = K_v x ,$$

and a cost is defined

$$J_v = \sum_{X_o} \int_0^{\infty} (x'Qx + u'Ru + v'R_v v)\, dt .$$

v is chosen such that (A, B_v) is a controllable pair so that a stable law K_v exists. The algorithm may then be started and organised according to the theory of the previous sub-section to optimise K and K_v. As R_v tends to infinity, at the optimum K_v should be small and K a stable law, if one exists. In addition K should be nearly optimal for the cost

$$J = \sum_{X_o} \int_0^{\infty} (x'Qx + u'Ru)\, dt ,$$

and the optimum found with little computational effort.

7.5.3 Incomplete state measure on Type III models

When the crude model has incomplete state measure, the control problem is more difficult and there may be essential degradation of performance as discussed in Chapter 6 even if complex observers are used. Apart from determining optimal constrained gains which remains a valid approach, there seem to be two solutions to the controller design problem. Both lead to dynamic compensators.

Introduction of input derivatives

The first approach follows the spirit of Pearson [P1] and Ferguson and Rekasius [F1] where input derivatives are introduced, the number depending upon the observability index of the plant. Both papers work with the type II accurate model and propose a precise method that may lead to unnecessarily complex controllers. Our contribution is to apply their ideas to the design of simple controllers. We work with the accurate model but assume that the crude model exists. Consider the plant

$$\dot{x} = Ax + Bu \,,$$

$$y = Hx \,,$$

$$J = \sum_{X_0} \int_0^{\infty} (x'Qx + u'Ru + \dot{u}'S\dot{u})\, dt \qquad (7.5.9)$$

$$X_0 = \text{a set of initial conditions.}$$

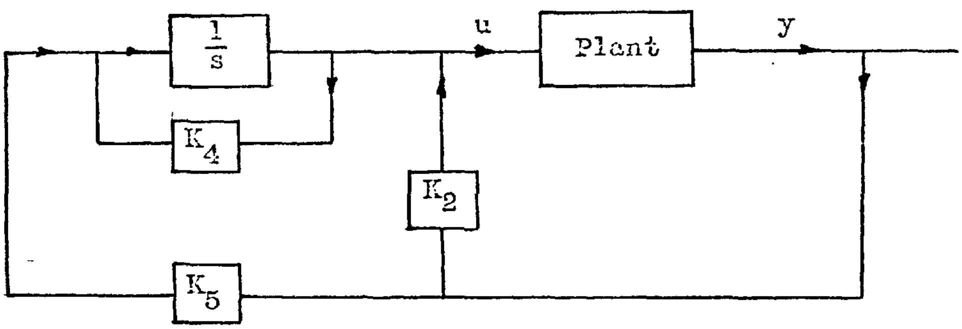
Consider a particular case where the vectors y, $z = (\dot{y} - HBu)$ would give adequate information about the state of the crude model. Then by the theory we have developed, Algorithm 7.2.1 will have fast convergence to constrained gains K_1, K_2, K_3 that optimise J where,

$$\dot{u} \quad = \quad K_1 y + K_2 z + K_3 u \tag{7.5.10}$$

so that

$$\begin{aligned} \frac{d}{dt}(u - K_2 y) &= K_1 y + (K_3 - K_2 HB)u \\ &= (K_3 - K_2 HB)(u - K_2 y) + (K_1 + (K_3 - K_2 HB)K_2)y \\ &\triangleq K_4(u - K_2 y) + K_5 y \,. \end{aligned} \tag{7.5.11}$$

The control law (7.5.10) may be implemented by (7.5.11) as shown in Figure 7.5.1.



<u>Figure 7.5.1</u> <u>Compensator realization of feedback from y and y.</u>

We note that y and u are vectors, and that in some cases first derivatives of some y components may be omitted, in other cases higher derivatives of y are required to describe the dominant dynamics. Then, higher derivatives of u are introduced as the control variable in J (7.5.9), the principle of eliminating derivatives of y from feedback by block diagram manipulations is unchanged. We note that Figure 7.5.1 has the same form as a degenerate observer, discussed in Chapter 4, and that the optimal K_1, K_2 may possibly tend to a low order multi-variable degenerate observer.

Introduction of observer based on crude model

We have available crude and accurate state space models of the plant dynamics and the crude model has incomplete state measure. It is reasonable that a low dimensioned observer obtained from the crude model and used with the accurate one should together with the output y provide a good characterization of the dominant dynamics of the plant. A control can be formulated as a feedback law from the plant and observer outputs, and the constrained gain optimization approach used to get values for the gains. An example illustrates the usefulness and pitfalls of the method.

Example

The simple example of Section 7.2 is considered with the change that only one output is measured.

The plant dynamics are

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u \ ,$$

where x_1 alone is measured as shown in Figure 7.5.2, and a crude model is taken to be

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ,$$

and the corresponding reduced observer with state x_4, from (3.3.23) with T = 9 is

$$\dot{x}_4 = -10x_4 - 81x_1 + u \ ,$$

$$\hat{x}_2 = x_4 + 9x_1 \ .$$

$\hat{x}_2$ is an estimate of the crude model state x_2. Incorporating the crude observer into the full dynamics we have

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -10 & 0 \\ -81 & 0 & 0 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \\ 1 \end{bmatrix} u \ ,$$

with output

$$y = \begin{bmatrix} x_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & 0 & 0 & 1 \end{bmatrix} x \, .$$

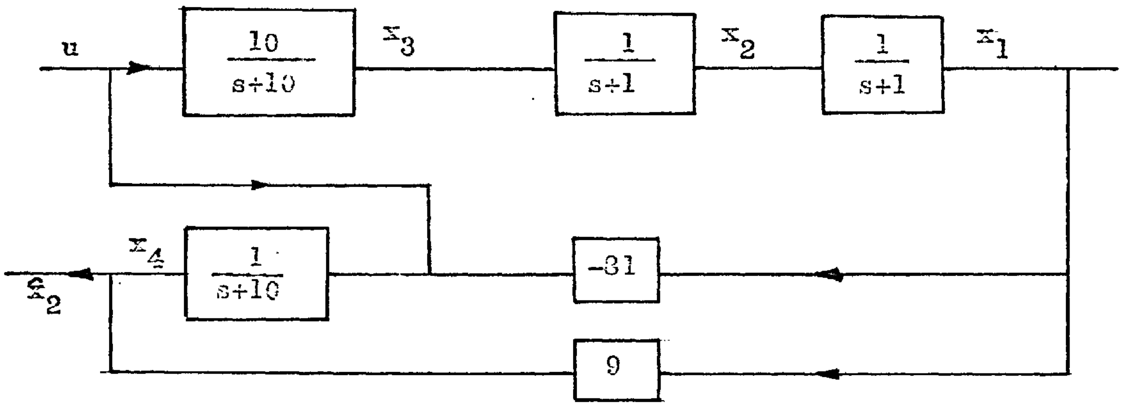


Figure 7.5.2 Plant and simple observer

A performance index is defined

$$J = \int_0^{\infty} (x_1^{\,2} + .1u^2)\, dt \, .$$

J was optimized with respect to feedback from x_1 and x_2 for two initial conditions X_o. The two solutions are compared below with respect to optimal gains K and P matrices ($J = x'Px$) ,

$$X_o'(1) = (1 \quad 0 \quad 0 \quad 0),$$

$$K(1) = (.6507 \quad .1147),$$

$$P(1) = \begin{bmatrix} .424 & .15 & .013 & .0001 \\ .15 & .15 & .015 & -.0015 \\ .013 & .015 & .0015 & -.0002 \\ .0001 & -.0015 & -.0002 & .0001 \end{bmatrix},$$

and

$$X_o'(2) = (0 \quad 1 \quad 0 \quad 0),$$

$$K(2) = (.907 \quad 1.267),$$

$$P(2) = \begin{bmatrix} 1.01 & .106 & .005 & .07 \\ .106 & .107 & .01 & -.005 \\ .005 & .01 & .001 & -.001 \\ .07 & -.005 & -.001 & .008 \end{bmatrix}$$

Comments on example

The optimal costs for the constrained optimization problems, $P_{11}(1) = .424$ and $P_{22}(2) = .107$ compare favourably with the unconstrained optimal costs of the crude model, $P_{11} = .405$ and $P_{22} = .094$, and for which the optimal control is $(-1.378x_1 - .938x_2)$. We note that $P_{11}(2)$ and $P_{22}(1)$ are not close to the optimal values, in fact $P_{11}(2)$ is about twice its open loop value. This effect is the result of the mismatch of the plant initial state x_2 and the initial observer estimate $\hat{x}_2$. The initial condition $x' = (1 \quad 0 \quad 0 \quad 0)$ implies that $x_2 = 9$ when $x_2 = 0$, and costs due to estimation error accumulate as the wrong estimate for x_2 is fedback strongly.

This may or may not be important depending on the disturbances that are likely to act on the system, but it does emphasize the importance of looking at all initial condition costs, which is easily done by reference to the P matrix. There is a tradeoff between speed of observer dynamics and the error effects and the next level of complexity in controller design using observers, is the solution of the most suitable observer. We leave this as a topic for further research.

7.6 Suggestions for Further Research

Some potential research topics have been touched upon in the previous section. Further computational experience in designing incomplete state feedback using Algorithm 7.2.1 is required to confirm its early promise as a powerful design tool. The relevance of allocating feedback from different outputs to different plant inputs should be explored with respect to the control of interacting subsystems. The last example clearly illustrates that research is required into the specification satisfactory design objectives when observers are intended to be used in constructing a dynamic compensator. These objectives should define what is a suitable, or the most suitable, observer.

The thesis has been solely concerned with deterministic control problems and it is felt that the proposed method of solution of the adequate incomplete state feedback laws is a contribution to design techniques. A parallel theory clearly exists for the state estimation problem, and a fruitful topic of investigation should be to find out how far the ideas we have developed are applicable to controller design in a stochastic environment. In the same way that full state feedback gives an often unnecessarily complex controller in the deterministic case, it is felt that the use of a Kalman filter and Riccati controller is often an unnecessarily complex design procedure in the stochastic environment.

REFERENCES

A1 B.D.O. ANDERSON and D.G. LUENBERGER
"Design of multivariable feedback systems", Proc. IEEE, Vol. 114, No. 3, 1967.

A2 B.D.O. ANDERSON
"An algebraic solution to the spectral factorization problem", IEEE, AC-12, 1967, pp. 410-414.

A3 B.D.O. ANDERSON and J.B. MOORE
"Tolerance of nonlinearities in time-varying optimal systems", Electronics Letters, Vol. 3, No. 6, June 1967.

A4 B.D.O. ANDERSON
Private Communication, 1968.

B1 R.W. BASS and I. GURA
"High order system design via state-space considerations", JACC, 1965, pp. 311-318

B2 BELLMAN
"Upper and lower bounds for the solution of the matrix Riccati equation", J.Math'l.Anal. and Applic., 17, 1967, pp. 373-379.

B3 R.W. BROCKETT and M.D. MESAROVIC
"The reproducibility of multivariable systems", JACC, 1965, pp. 481-486.

B4 S. BARNETT and C. STOREY
"Some results on the sensitivity and synthesis of asymptotically stable linear and non-linear systems", Automatica, Vol. 4, 1968, pp. 187-194.

B5 T.R. BLACKBURN
"Solution of the algebraic matrix Riccati equation via Newton-Raphson iteration", JACC, 1968, pp. 940-945.

B6 J.J. BONGIORNO and D.C. YOULA
"On observers in multi-variable control systems", Int.J.Control, Vol. 8, No. 3, 221-243.

B7 T.R. BLACKBURN and J.C. BIDWELL
"Some numerical aspects of control engineering computations", JACC, 1968, pp. 203-207.

REFERENCES (continued)

B8 R. BELLMAN
"Dynamic Programming", Princeton University Press, 1957.

C1 C.T. CHEN, C.A. DESOER, A. NIEDERLINSKI
"Simplified conditions for controllability and observability of linear time invariant systems", IEEE Trans, Vol. AC-11, 1966, pp. 613-614.

C2 S.D.G. CUMMING
"Design of observers of reduced dynamics", Electronics Letters, Vol. 5, No. 10, pp. 213-214.

C3 S.S.L. CHANG
"Synthesis of Optimum Control Systems", McGraw-Hill, 1961.

D1 M.C. DAVIS
"Factorising the spectral matrix", IEEE, AC-8, 1963, pp. 296-305.

D2 E.J. DAVISON
"A method for simplifying linear dynamic systems", IEEE, AC-11, pp. 93-101, Jan. 1966.

D3 C.A. DESOER
"Perturbations of eigenvalues and eigenvectors of a network", Proc. of the Fifth Annual Allerton Conference on Circuit and System Theory, 1967, pp. 8-11.

D4 E.J. DAVISON and F.T. MAN
"The numerical solution of $A'Q + QA = -C$", IEEE, AC-13, No. 4, August 1968, p. 448.

E1 J.K. ELLIS and G.W.T. WHITE
"An introduction to modal analysis and control", Control (U.K.), Vol. 9, April, May, June 1965.

F1 J.D. FERGUSON and Z.V. REKASIUS
"Optimal linear control systems with incomplete state measurements", Proc. of the Sixth Annual Allerton Conference on Circuit and System Theory, 1968, pp. 671-679.

G1 F.R. GANTMACHER
"The Theory of Matrices", Vol. 1, Chelsea Publishing Co. (1960).

REFERENCES (continued)

H1 B.L. HO and R.E. KALMAN
"Spectral factorisation using the Riccati equation", 4th Annual Allerton Conference on Circuit and System Theory, October 1966, pp. 388-399.

H2 W. HAHN
"Theory and Application of Lyapunov's Direct Method", Prentice-Hall (1963).

H3 HOROWITZ
"Synthesis of Feedback Systems", Academic Press, 1963.

J1 E.I. JURY and B.W. LEE
"The absolute stability of systems with many nonlinearities", Automation and Remote Control, Vol. 26, No. 6, 1965, pp. 943-961.

J2 C.D. JOHNSON
"Optimal control of the linear regulator with constant disturbances", IEEE, August 1968, pp. 416-421.

J3 A. JAMESON
"Solution of the equation AX + XB = C by inversion of an M x M of N x N matrix", SIAM J.Appl.Math., Vol. 16, No. 5, September 1968, pp. 1020-1023.

J4 P.D. JOSEPH and J.T. TOU
"On linear control theory", AIEE Trans., Applications and Industry, Vol. 80, pp. 193-196.

K1 R.E. KALMAN
"When is a linear control system optimal?", Trans. ASME Journal of Basic Engineering, D, pp. 51-60, March 1964.

K2 R.E. KALMAN
"On the general theory of control systems", Proc. of First IFAC Congress, Moscow, 1960.

K3 R.E. KALMAN and R.S. BUCY
"New results in linear filtering and prediction theory", Trans. ASME (J.Basic Eng.), March 1961, pp. 95-108.

K4 R.E. KALMAN
"On structural properties of linear constant multivariable systems", paper 6A, Third Congress of IFAC, London 1966.

REFERENCES (continued)

K5 R.E. KALMAN and J.E. BERTRAM
"Control system analysis and design via the 'second method' of Lyapunov", Trans. ASME, Vol. 82, 1960.

K6 R.E. KALMAN
"Mathematical description of linear dynamic systems", J.SIAM, A, 1963, pp. 152-192.

K7 R.E. KALMAN
"Contributions to the theory of optimal control", Bol.Soc.Mat. Mexicana, 1960.

K8 D.L. KLEINMAN
"On an iterative technique for Riccati equation computations", IEEE, Vol. AC-13, Feb. 1968, pp. 114-115.

K9 D.L. KLEINMAN and M. ATHANS
"The design of suboptimal linear time-varying systems", IEEE, AC-13, No. 2, 1968, pp. 150-167.

K10 R.E. KALMAN, Y.C. HO and K.S. NARENDRA
"Controllability of linear dynamical systems", Contr. Diff. Eqns., 1963, 1, (2), pp. 189-213.

K11 R.E. KALMAN, T.S. ENGLAR
"A users manual for the automatic synthesis program", NASA Report No. CR-475, June 1966.

L1 D.G. LUENBERGER
"Observing the state of a linear system", IEEE, Trans., 1964, MIL-8, pp. 74-80.

L2 D.G. LUENBERGER
"Observers for multivariable systems", IEEE Trans., 1966, AC-11, pp. 190-197.

M1 C.W. MERRIAM III
"Optimisation Theory and the Design of Feedback Control Systems", McGraw-Hill, 1964.

M2 R.D. MILNE
"The analysis of weakly coupled dynamical systems", Int.J. of Control, Vol.2, No. 2, 1965, pp.

REFERENCES (continued)

M3 D.Q. MAYNE and P. MURDOCH
"Model control of linear time invariant systems", Int. J. Control, 1969, 8.

M4 A.J.G. MACFARLANE
"An eigenvector solution to the optimal linear regulator problem", Journal of Electronics and Control, Vol. 14, No. 6, 1963.

M5 L. MEIER and B.D.O. ANDERSON
"Performance of suboptimal linear control systems", Proc. IEE, Vol. 114, No. 1, January 1967, pp. 124-128.

M6 J.B. MOORE and B.D.O. ANDERSON
"Optimal linear control systems with input derivative constraints", Proc. IEE, Vol. 114, No. 12, December 1967, pp. 1987-1990.

M7 D. MITRA
"On the Reduction of Complexity of Linear Dynamical Models", UKAEA Report, AEEW-R520, 1967.

M8 D. MITRA
"W matrix and the geometry of model equivalence and reduction", Proc. IEE, Vol. 116, June 1969, p. 66.

N1 NEWTON, GOULD and KAISER
"Analytical Design of Feedback Controllers", Wiley (1957).

N2 M.M. NEWMAN
"Optimal and suboptimal control using an observer when some of the state variables are not measurable", Int.J.Control, 1969, Vol. 9, No. 3, 281-290.

N3 H. NICHOLSON
"Optimal linear control with input derivatives", Proc. IEE, Vol. 115, No. 6, June 1968, pp. 877-878.

N4 H. NICHOLSON
"Eigenvalue and state-transition sensitivity of linear systems", Proc. IEE, Vol. 114, No. 12, Dec. 1967.

REFERENCES (continued)

P1 J.B. PEARSON
"Compensator design for dynamic optimisation", Department of Electrical Engineering Report, Rice University Texas (approx. 1967), unpublished.

P2 M.J.D. POWELL
"An efficient method for finding the minimum of a function of several variables without calculating derivatives.

P3 J. POTTER
"Matrix quadratic solutions", SIAM J. Appl. Math., May 1966, Vol. 14, No. 3.

R1 H.H. ROSENBROCK
"The assignment of closed loop poles", UMIST, Control System Centre Report No. 20, October 1967.

R2 H.H. ROSENBROCK
"Design of multivariable control systems using the inverse Nyquist array", UMIST Control Systems Centre Report No. 48, 1969.

R3 H.H. ROSENBROCK
"On the design of linear multivariable control systems", IFAC Paper 1A, London, 1966.

R4 H.H. ROSENBROCK
"Distinctive problems of process control", Chemical Engineering Progress, Vol. 58, September 1962.

R5 J.J. RISSANEN
"Performance deterioration of optimum systems", IEEE, AC July 1966, pp. 530-532.

R6 J. RISSANEN and R. DURBECK
"On performance bounds for control systems", ASME, Journal of Basic Engineering, June 1967, pp. 311-314.

R7 Z.V. REKASIUS
"Optimal linear regulators with incomplete state feedback", IEEE Trans., June 1967, pp. 296-299.

REFERENCES (continued)

S1 R.A. SMITH
"Matrix equation XA + BX = C", SIAM J.Appl.Math., Vol. 16, No. 1, 1968, pp. 198-201.

S2 D.G. SCHULTZ and J.L. MELSA
"State Functions and Linear Control Systems", McGraw-Hill, 1967.

S3 J.D. SIMON
"Theory and application of modal control", Case Institute, Ph.D. Thesis, November 1967.

T1 J.S. TYLER and F.B. TUTEUR
"The use of a quadratic performance index to design multivariable control systems", IEEE Trans., AC-11, No. 1, January 1966, pp. 84-92.

V1 D.R. VAUGHAN
"A negative exponential solution for the linear optimal regulator problem", JACC, 1968, pp. 717-725.

V2 W.G. VOGT
"Transient responses from Lyapunov stability equations", JACC, 1965.

V3 Van NESS
"Sensitivity of large multi-loop control systems", IEEE, Trans., July 1965, pp. 308-315.

W1 N. WIENER
"The Extrapolation, Interpolation and Smoothing of Stationary Time Series", John Wiley and Sons (1958).

W2 W.S. WIDNALL
"Applications of optimal control theory to computer controller design", Research Monograph 48, MIT Press (1968).

W3 W.M. WONHAM
"On matrix quadratic equations and matrix Riccati equations", Center for Dynamic Systems, Brown University, Technical Report 67-5, 1967.

W4 W.M. WONHAM
"On pole assignment in multi-input controllable linear systems", Center for Dynamical Systems, Brown University, Report 67-2, Feb. 1967.

REFERENCES (continued)

W5 J.H. WILKINSON
"The Algebraic Eigenvalue Problem", Oxford University Press (1965).

Y1 D.C. YOULA
"On the factorisation of rational matrices", IEEE Trans., IT-7, 1961, pp. 172-189.

Z1 L.A. ZADEH and C.A. DESOER
"Linear System Theory", McGraw-Hill, (1968).