WAVE MOTION AND VIBRATION INDUCED BY TURBULENT FLOW

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CONTENTS

ABSTRACT

In the first section of this thesis, the effect of a plane flexible composite boundary on the sound generated by nearby turbulence is examined. It is shown that the boundary reflects and diffracts the sound from the turbulence, and also that the inhomogeneity of the surface scatters the energy of eddy motion into propagating sound. Reflexion and diffraction processes do not appreciably increase the sound power from the turbulence, but the scattering mechanism is a powerful means of augmenting the radiation.

A general formulation of the problem of sound generation by a region of turbulence in which the fluid consists of a mixture of two phases is given in the next chapter. The theory is applied to the case of a small volume concentration of air bubbles in water, and also to that of a distribution of small dust particles in a gas. The effect of even a very small concentration of bubbles is to increase the turbulence radiation enormously, while the presence of dust particles in a gas also increases the radiation appreciably, though less drastically.

The vibration induced in a thin elastic plate by a random pressure field is then considered. General equations are derived and applied to the case of plate excitation by a turbulent boundary layer. The dependence of the vibration statistics on parameters characterising the plate and the turbulence is obtained, giving results which are analogous to the well-known results of Lighthill for the sound power generated by turbulence.

Finally, an estimate is made of the corrections caused by compressibility to the pressure and velocity fluctuations in homogeneous turbulence. These corrections are large if the volume of turbulence is large, but are negligible for low Mach number flows on any terrestrial scale.

CHAPTER 1

GENERAL INTRODUCTION

This thesis is concerned with four problems of wave generation as a by-product of turbulent flow. In all cases the turbulent flow is regarded as given, and the problem is to estimate the side-effects of the flow under the assumption that these cause a negligible back reaction on the turbulence. Very little is known, of course, about the detailed structure of most common types of turbulent flow, so that the problems must be posed in such a way as to require only a minimal specification of the properties of the turbulence. Dimensional arguments often enter the calculations, and preclude any highly accurate numerical predictions, but nonetheless these procedures have led in the past to valuable information about the gross effects to be expected in many situations of practical importance.

The pioneering work in problems of this kind was done by Lighthill in 1952-1954, in a set of three well-known papers on the sound waves generated by turbulence. Lighthill formulated an exact acoustic analogy for this strongly non-linear problem, and applied it to the estimation of the noise fields of jets and boundary layers, and to the scattering of sound resulting from the interaction of sound and shock waves with turbulence. From the simplest possible ideas about turbulence he was able to draw powerful conclusions about the intensity and directivity pattern of the radiated noise fields. Recent years have seen great advances in the development of Lighthill's basic theory, and have shown it to be much superior to all other theories - above all in the range of possible applications.

The work of the first two sections of this thesis describes the extension of Lighthill's theory to two new situations.

Chapter 2 deals with a problem which provides a link between aerodynamic noise theory and the classical theory of diffraction of acoustic waves by a solid boundary. For a long time after Lighthill's first papers, the effects of solid boundaries upon turbulence-generated sound were inadequately understood. The work described here shows that it is possible to obtain a complete solution to the aerodynamic sound problem when a plane, flexible, but non-homogeneous boundary is present in the turbulent flow. Specifically, the boundary considered consists of two half-planes which have differing elastic and inertia properties. The interaction between a plane sound wave and such a composite surface provides a typical example of diffraction theory, though the solution of the diffraction problem sheds little light on how the composite surface will affect the radiation from nearby turbulence. A new method of solving Lighthill's inhomogeneous wave equation, subject to the discontinuous boundary conditions required by the presence of the composite boundary, is given here. Previously obtained results of diffraction theory are recovered, and they show how the sound emitted by the turbulence is reflected and diffracted by the surface. These effects do not substantially increase the radiation from the turbulence. However, the solution also shows that the inhomogeneity in the surface properties acts as a scatterer, converting the strong hydrodynamic near-field of the turbulence into radiating acoustic energy.

This scattering mechanism is a powerful means of converting the energy of eddy motion into radiating sound, and from the detailed results given below it appears that such surface inhomogeneities - even though apparently small - may often make the dominant contributions to the noise fields found in practice.

In Chapter 3, the problem of sound generation by turbulent motion occurring in a two-phase fluid mixture is considered. A modification of Lighthill's methods allows us to see clearly how the interaction of the two phases produces sound waves in the fluid beyond the turbulent region. The important case of a small volume concentration of air bubbles in water is considered in detail. Such a mixture is well-known to have very startling acoustic properties, arising from the fact that the inertia of the mixture lies almost entirely in the water phase, while the compressibility lies almost entirely in the gas phase. These unusual properties naturally play an important part in the problem of sound generation by turbulence. It is shown here that the presence of only 1% of gas by volume is sufficient to increase the acoustic power output of the turbulence by about 50 dB, while at the maximum concentration, of order 10% perhaps, which can reasonably be covered by the theory, the power output is increased by about 70 dB. The two-phase mixture formed by a distribution of small dust particles in a gas is also examined. Increases in the acoustic power are again found. They are much smaller than those caused by the presence of bubbles in water, but are appreciable if the mass concentration of dust is high.

In the next section, an examination is made of the vibration induced in a thin elastic plate by an unsteady pressure field acting over part of the plate. Particular attention is given to the case of a random pressure field, and the results obtained are applied to the case when the plate is excited by pressure fluctuations typical of a turbulent boundary layer. Functional forms for the dependence of the vibration statistics on the flow and plate parameters are given, and these may be regarded as equivalent to the results of Lighthill for the aerodynamic noise problem. Convection of the pressure field is examined in a simple example. The results are interpreted as implying that, for excitation by turbulence, the normal mode analysis often used for finite plates may not be very useful in practice. For no modes seem, a priori, to be preferred, and the number of modes required to describe a turbulent pressure field may be so large as to render the method useless.

Finally, the pressure fluctuations in stationary isotropic turbulence are examined using Lighthill's wave equation. When the fluid is slightly compressible, the pressure fluctuations in a large volume of turbulence diverge, in mean square, in proportion to the linear scale of the region. In the strict homogeneous limit of infinite scale, the pressure fluctuations are bounded only by small diffusive effects, and cannot therefore be applied to real flows unless the volume of turbulence involved is enormous. Results for small and large volumes of turbulence are derived, and contrasted. All turbulent flows on a terrestrial scale seem to involve only a "small" volume, and for these flows compressibility effects are found to be negligible when the Mach number is low.

Not all of the work described in this thesis is that of the author alone. About one third of the work of Chapters 3 and 4 was done by Dr. J.E. Ffowcs Williams, and those chapters will be submitted for publication as joint papers. Chapters 2 and 5 were written entirely by the author, though of course these have also benefited greatly from many discussions with Dr. Ffowcs Williams. His help is again gratefully acknowledged.

CHAPTER 2

RADIATION FROM TURBULENCE NEAR A COMPOSITE FLEXIBLE BOUNDARY

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1. Introduction

One of the major problems in the theory of aerodynamic noise concerns the influence of boundaries upon the noise generated by a turbulent fluid flow. In only two cases so far has a satisfactory solution been achieved. In the first case, the boundary surface concerned has typical dimension small compared with a typical wavelength of the sound generated, and then an unambiguous dipole field is created by the presence of the surface, which is assumed rigid. In the second case, the boundary is supposed to be formed by an infinite plane homogeneous surface, which may respond with small-amplitude vibration to the turbulent flow over the surface. We shall here consider an extension of the results known for this second case, and so a brief history of the previous developments will first be given.

Lighthill (1952) was the first to consider the problem of noise generation by a turbulent flow in the absence of boundaries. He showed that the turbulent flow could be regarded as equivalent, acoustically, to a volume distribution of quadrupoles, the quadrupole strength being supposed known in terms of properties of the turbulence. Flows at low Mach number yield a quadrupole strength varying as the square of a velocity characteristic of the turbulence, while the radiated pressure varies as the quadrupole strength, and as the square of a typical turbulence frequency. Thus the radiated pressure varies as V^4, and the acoustic intensity as V^8. In 1955, Curle obtained the complete solution of Lighthill's equations when the effects of rigid boundaries in the flow were taken into account. He showed that a rigid boundary was acoustically equivalent to a distribution of dipoles over the surface, with a radiated intensity proportional to V^6. An analysis similar to that of Curle, in which the boundary is allowed to vibrate in response to the turbulent pressure field upon it, would show that the vibration of a flexible panel is acoustically equivalent to a surface distribution of monopoles (in addition to the dipoles and quadrupoles of Curle and Lighthill), with a radiated intensity proportional to V^4. These conclusions, based on dimensional analysis of terms which appear superficially to represent acoustic sources of essentially high efficiency, are now known to be quite incorrect in the case when a single infinite homogeneous surface is the only boundary present.

We can anticipate the correct result for this case by looking at the problem in the following way. The spectral components of the boundary layer type of pressure field (in a Fourier analysis in time, and in the plane of the boundary layer) have, for the most part, subsonic phase velocities, if the Mach number of the flow is small. These subsonic components constitute a strong near-field, but they are exponentially attenuated with distance from the flow, and cannot propagate as sound. Only the relatively few supersonic spectral components can propagate to the far-field, and the flow is basically inefficient in generating sound. Now if the presence of a rigid, or flexible boundary in the flow were to substantially increase the sound radiation, some form of interaction between the flow and surface must occur in order to scatter the subsonic near-field into propagating sound. Scattering, or "wavenumber conversion", must occur if the flow is to use the surface as a sounding board to increase its radiation. But it is difficult to see how such scattering could occur if the only boundary present were an infinite homogeneous plane, responding perhaps with vibration of small amplitude. For the problem is linear, and there are no edge-effects or discontinuities which alone could lead to wavenumber conversion in a linear problem. Consequently the flow cannot use such a boundary to augment its radiation, and the sound field must be essentially that due to the inefficient quadrupoles equivalent to the turbulent flow.

These conclusions were first rigorously deduced by Powell (1960) in the case of an infinite rigid surface. He showed, from Curle's equation together with a complementary null equation, that the surface pressure dipoles merely "reflected" the turbulence-generated sound, and hence that the V^8 law remained valid in this case. Ffowcs Williams (1965) subsequently extended Powell's work to cover the case of an infinite flexible boundary; the action of both monopoles and dipoles was here shown only to involve the reflexion of the quadrupole sound, but now with a phase change depending on the frequency and direction of the sound. The properties of the radiated sound are completely known, in principle, when the quadrupole strength and the reflexion coefficient for the surface are given. In a later paper, Ffowcs Williams (1966) also considered the radiation from turbulence near a flexible surface when a distribution of simple supports acts on the surface. Scattering does now occur, and the supports themselves act as radiators of genuine dipole sound. No specific dependence of the dipole sound upon the speed V was obtained, however, and the details of the scattering were somewhat by-passed in the method used by Ffowcs Williams. More will be said later in this paper about the influence of supports on the radiated sound.

The papers of Powell and Ffowcs Williams illustrate the great danger in assuming that one can regard the surface pressure and velocity as known (dimensionally, at any rate) independently of a knowledge of the Lighthill quadrupole strength. The estimates $p \sim \rho V^2$ and $v \sim V$ for pressure and velocity are quite inadequate, and Powell and Ffowcs Williams show how one should attempt to calculate p and v when given only the quadrupole strength. In this paper, the calculation will be carried through in a situation of much greater complexity than has been considered before. Ordinary acoustic theory does, of course, deal with the diffraction of acoustic waves by boundaries of the type considered here. In particular, Heins and Feshbach (1954) have solved the diffraction problem for a composite flexible boundary. This is not, however, the sort of problem which is of relevance to aerodynamic noise theory. We are really interested in the scattering of near-field pressure into sound, rather than the scattering of a propagating field into diffracted fields. The scattering of sound into sound is a problem of no importance in flow-noise theory, whereas the scattering of near-field pressure into sound may provide a powerful means of increasing the acoustic efficiency of a flow. The theory is therefore set up here in a manner appropriate to the flow-noise problem. As a special case, the results of Heins and Feshbach (1954) are recovered. The technique used here does, however, have the advantage of potentially greater generality than does the Wiener-Hopf method of Heins and Feshbach - even for the diffraction problem. It may thus be possible to extend results in diffraction theory by the present method, but that is left for further study.

The situation to be discussed in this paper concerns the case of an infinite boundary, composed of two semi-infinite half-planes. Each of the surfaces is assumed homogeneous, but the surfaces are supposed to differ in their inertia and elastic properties. The basic problem is to formulate a set of equations sufficient, in principle, to determine the radiated sound field entirely in terms of the one quantity which can be supposed known - viz. Lighthill's quadrupole strength $T_{ij}(\underline{x},t)$, given throughout the flow. Only when such equations are set up and solved can one hope to give a reliable dimensional analysis of the sound field.

2. The Governing Equations

Let the infinite surface to be considered occupy the (1,2) plane. A turbulent flow occupies some region in $x_3 \geq 0$, and we seek the radiated pressure fluctuation $p(\underline{x},t)$ in the region beyond the turbulence. The equations governing the radiation may be written (Ffowcs Williams, 1965),

$$p(\underline{x},t) = T_+ - \frac{1}{4\pi}\frac{\partial}{\partial x_3}\int [p_s]\frac{d\underline{y}}{r} - \frac{\rho}{4\pi}\frac{\partial}{\partial t}\int [v]\frac{d\underline{y}}{r}, \qquad (2.1)$$

$$0 = T_- + \frac{1}{4\pi}\frac{\partial}{\partial x_3}\int [p_s]\frac{d\underline{y}}{r} - \frac{\rho}{4\pi}\frac{\partial}{\partial t}\int [v]\frac{d\underline{y}}{r}. \qquad (2.2)$$

In these equations, viscous forces and nonlinear terms in surface response have been neglected, as usual. ρ is the mean fluid density, v the surface velocity in the (-3) direction. The $\underline{y}$-integration is over the whole of the (1.2) plane, and p_s denotes the surface pressure. T_+ is the contribution to $p(\underline{x},t)$ from the turbulence stresses in the real fluid, while T_- is the pressure which would be radiated to $(\underline{x},t)$ by the specular images of the stress quadrupoles T_{ij} in the (1,2) plane. T_+ and T_- are to be regarded as known from a knowledge of T_{ij}. As usual, square brackets [ ] imply evaluation at retarded time,

$$[f] \equiv f(\underline{y}, t - \frac{|\underline{x} - \underline{y}|}{a_o}),$$

a_o being the speed of sound in the distant field.

If now we are given the relation between p_s and v - i.e. if we know the response equation for the surface, we have in principle sufficient equations for the determination of p in terms of $T_\pm$. For example, in the case of a rigid surface, considered by Powell (1960), we have v = 0, so that $p = T_+ + T_-$. This is just a statement of Powell's reflexion principle. In general, however, it is not at all clear how the equations can be used, even in a formal sense, to determine $p(\underline{x},t)$, so that the first problem is to manipulate the equations into a more suitable form.

Suppose that the surface consists of two homogeneous half-planes, $y_1 > 0$ and $y_1 < 0$ say, which have different elastic and inertia constants. Suppose also that a distribution of applied stresses $q(\underline{y},t)$ acts on the surface in addition to the turbulent pressure field p_s, a positive value of q implying a stress in the +3 direction. The response of the surface can then be expressed, most conveniently in terms of the surface deflextion $\eta(\underline{y},t)$ in the -3 direction, by equations of the form

$$p_s - q = \begin{cases} F_1(\eta) & y_1 > 0 \, , \\ F_2(\eta) & y_1 < 0 \, . \end{cases} \tag{2.3}$$

F_1, F_2 are linear space and time differential operators with constant coefficients.

Using (2.3) we can express the surface integral of pressure as follows:

$$\int [p_s] \frac{d\underline{y}}{r} = \int [q] \frac{d\underline{y}}{r} + F_1 I_1 + F_2 I_2 + S \, , \tag{2.4}$$

where

$$I_1 = \int_{y_1 = 0+}^{\infty} \int_{y_2 = -\infty}^{+\infty} [\eta] \frac{d\underline{y}}{r} \, ,$$

$$I_2 = \int_{y_1 = -\infty}^{0-} \int_{y_2 = -\infty} [\eta] \frac{d\underline{y}}{r} \, ,$$

and the term S denotes a collection of line-integrals along $y_1 = 0$ from

$y_2 = -\infty$ to $+\infty$. In fact, if we take a fairly simple case, that in which

$$F_i \equiv A_i \frac{\partial^n}{\partial y_1^{\,n}} + B_i \frac{\partial^m}{\partial y_2^{\,m}} + C_i \frac{\partial^s}{\partial t^s} \qquad (i = 1,2)$$

then

$$S = \sum_{r=o}^{n-1} \frac{\partial^r}{\partial x_1^{\,r}} J_{n-r-1} \,, \qquad \text{and}$$

$$J_k = \int_{-\infty}^{+\infty} \left\{ \left[A_2 \frac{\partial^k \eta}{\partial y_1^{\,k}} \right]_{y_1=0-} - \left[A_1 \frac{\partial^k \eta}{\partial y_1^{\,k}} \right]_{y_1=0+} \right\} \frac{dy_2}{r} \,.$$

We can interpret these line integrals at a later stage.

If we now substitute (2.4) into (2.1) and (2.2), and use $v = \partial\eta/\partial t$, we have

$$p = Q_+ - \frac{1}{4\pi} \frac{\partial}{\partial x_3} (F_1 I_1 + F_2 I_2) - \frac{\rho}{4\pi} \frac{\partial^2}{\partial t^2} (I_1 + I_2), \qquad (2.5)$$

$$0 = Q_- + \frac{1}{4\pi} \frac{\partial}{\partial x_3} (F_1 I_1 + F_2 I_2) - \frac{\rho}{4\pi} \frac{\partial^2}{\partial t^2} (I_1 + I_2) \,. \qquad (2.6)$$

In these equations, modified pressure fields have been used, defined by

$$Q_{\pm} = T_{\pm} \mp \frac{1}{4\pi} \frac{\partial S}{\partial x_3} \mp \frac{1}{4\pi} \frac{\partial}{\partial x_3} \int [q] \frac{dy}{r} \,. \qquad (2.7)$$

Q_- is the pressure induced by the specular image of the system which generates Q_+, reflexion of the system of surface dipoles involving S, q requiring only a change of sign. It will be seen later that S is known if q is, so that the problem now is to understand the roles played by the terms involving I, assuming the Q known. In equations (2.4-6), the operators F_i are now taken to act on the field point $\underline{x}$, and not on the source point $\underline{y}$ as they did in (2.3).

The equations (2.5-6) are still insufficient to determine p, for we have as yet no relation between I_1 and I_2. (The relation is not needed in the case $F_1 = F_2$ treated by Ffowcs Williams, (1965)) We can obtain the required relation by taking Fourier transforms in (1,2) space, and in time, and by expressing the transforms of each of I_1, I_2 in terms of the transform $\tilde{\eta}$ of the surface deflexion alone. We use the following definition of generalised Fourier transforms:

$$g(\underline{x},t;\, x_3) \;=\; \int \tilde{g}\,(\underline{k},\omega;\, x_3)\exp i(\underline{k}.\underline{x} + \omega t)d\underline{k}\; d\omega \;,$$

with $\underline{k} = (k_1,k_2)$ and $\underline{x} = (x_1,x_2)$. We need also the following result, which is easily proved by direct calculation; if

$$\phi(\underline{x},t;\, x_3) \;=\; \int \Theta(y_1,y_2,\, t-r/a_o)\,\frac{d\underline{y}}{r}\;,\ \text{then}$$

$$\tilde{\phi}(\underline{k},\omega;\, x_3) \;=\; -\frac{2\pi i}{\psi}\exp(-ix_3\psi\,)\tilde{\Theta}(\underline{k},\omega)\;.$$

Here $\psi = \operatorname{sgn}\omega\sqrt{\dfrac{\omega^2}{a_o^2} - k^2}$ if $|\omega| > a_o k$,

$= -i\sqrt{k^2 - \dfrac{\omega^2}{a_o^2}}$ if $|\omega| < a_o k$.

This result, which is frequently useful, expresses the transform of the pressure ϕ radiated by a distribution of surface sources Θ in terms of the transform $\tilde{\Theta}$ of the sources. To apply this result here, we interpret I_1 as

$$\iint_{-\infty}^{+\infty} \left[\eta H(y_1)\right] \frac{dy}{r} , \qquad \text{and } I_2 \text{ as}$$

$$\iint_{-\infty}^{+\infty} \left[\eta H(-y_1)\right] \frac{dy}{r} ,$$

H denoting the Heaviside unit function. Use of the convolution theorem to obtain the transform of ηH then gives

$$\tilde{I}_1 = -\frac{2\pi i}{\psi} e^{-ix_3\psi}\left\{\tfrac{1}{2}\tilde{\eta} - \tfrac{1}{2}\mathcal{H}\tilde{\eta}\right\} ,$$

$$\tilde{I}_2 = -\frac{2\pi i}{\psi} e^{-ix_3\psi}\left\{\tfrac{1}{2}\tilde{\eta} + \tfrac{1}{2}\mathcal{H}\tilde{\eta}\right\} . \qquad (2.8)$$

$\mathcal{H}$ denotes the Hilbert transform operator,

$$\mathcal{H}\tilde{\eta} = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{\tilde{\eta}(k_1', k_2, \omega)}{(K_1' - k_1)} dk_1' ,$$

the integral being taken in the sense of a Cauchy principal value.

We now substitute (2.8) into (2.5-6). The transforms of the operators F_1, F_2 are denoted by $i\omega z_1$ and $i\omega z_2$ respectively, so that z_1, z_2 are the impedances of the two half-planes according to the usual convention. Also we write $z = \frac{1}{2}(z_1 + z_2)$, and denote the acoustic wave impedance

$\rho\omega/\psi$ by z_w. Then (2.5-6) become

$$(\tilde{p} - \tilde{Q}_+)e^{ix_3\psi} = \left\{\tfrac{1}{2}(z - z_w) - \tfrac{1}{4}(z_1 - z_2)\mathcal{H}\right\}\tilde{v} \ , \qquad (2.9)$$

$$\tilde{Q}_- e^{ix_3\psi} = \left\{\tfrac{1}{2}(z + z_w) - \tfrac{1}{4}(z_1 - z_2)\mathcal{H}\right\}\tilde{v} \ , \qquad (2.10)$$

$\tilde{v} = i\omega\tilde{\eta}$ being the transform of $v = \partial\eta/\partial t$.

(2.10) is a standard form of singular integral equation with Cauchy kernel, and with variable coefficients. It has a simple exact formal solution for arbitrary Q and z, and this solution will be discussed in detail later. Note that the left sides of (2.9-10) are independent of x_3. For

$$\left(\frac{\partial^2}{\partial t^2} - a_o^2 \nabla^2\right)Q_- = 0 \quad \text{for all } x_3 > 0 \ ,$$

while $\left(\frac{\partial^2}{\partial t^2} - a_o^2 \nabla^2\right)p, \ Q_+ = 0$ whenever the observation point $\underline{x}$ lies beyond the region containing turbulence sources. In terms of transforms this implies

$$\tilde{p}(\underline{k}, \omega; x_3) \sim \exp(-ix_3\psi) \ ,$$

with the use of suitable conditions at $x_3 = +\infty$, and hence $\tilde{p}\,e^{ix_3\psi}$ is independent of x_3. We may note also that $\tilde{Q}_- e^{ix_3\psi}$ is the pressure which would be exerted by the turbulence sources (or their images) on the surface $x_3 = 0$ if the flexible boundary were absent. It is not a far-field pressure, but in fact is dominated, for low Mach number flows, by the hydrodynamic near-field of the turbulence. We may therefore expect $\tilde{Q}_{-}e^{ix_3\psi}$ to be concentrated, for any frequency ω, on wavenumbers k such that $|\omega| < a_0 k$. More precisely, if V, L_0 are characteristic velocity and length scales for the turbulence, with $V \ll a_0$, then we expect $\tilde{Q}_{-}e^{ix_3\psi}$ to be concentrated around a frequency of order V/L_0 and around a wavenumber of order $1/L_0$.

3. Acoustic Sources on the Discontinuity

Before discussing (2.9-10) further, we must examine the way in which the turbulence-generated pressure T_+ is augmented to C_+, and in particular, the term S requires attention. Suppose the response operators F_i have the usual forms for thin homogeneous panels,

$$F_i \equiv m_i \frac{\partial^2}{\partial t^2} + B_i \nabla_1^4 , \qquad (i = 1,2) \qquad (3.1)$$

m denotes the panel mass per unit area, B is the bending stiffness, and dissipation in the surfaces is neglected. ∇_1^2 denotes the two dimensional Laplacian. Since the general effect of a distribution of externally applied stresses is well understood, it will be sufficient here to restrict these stresses to a line distribution along $y_1 = 0$;

$$q(\underline{y},t) = q(y_2,t)\delta(y_1) .$$

(The case $q(\underline{y},t) = \delta(\underline{y})q(t)$ is discussed in detail by Ffowcs Williams, 1966). The acceleration at $y_1 = 0$ is not infinite, so that the externally applied stress must balance the discontinuity in shear across $y_1 = 0$ which is induced by elastic forces;

$$\left| B \frac{\partial^3 \eta}{\partial y_1^3} \right| = - \left(\frac{\text{downward stress}}{\text{line density}} \right) = q(y_2,t) ,$$

where, in this section, $|f|$ denotes the discontinuity in f across $y_1 = 0$,

$$|f| \equiv f(y_1 = 0-) - f(y_1 = 0+) .$$

Now when the response operators have the forms considered here, the line integrals occurring in S involve the following as acoustic sources per unit length of $y_1 = 0$;

$$+ \left| B \frac{\partial^3 \eta}{\partial y_1^3} \right| + \frac{\partial}{\partial x_1} \left| B \frac{\partial^2 \eta}{\partial y_1^2} \right| + \frac{\partial^2}{\partial x_1^2} \left| B \frac{\partial \eta}{\partial y_1} \right|$$

$$+ \frac{\partial^3}{\partial x_1^3} \left| B\eta \right| + \frac{\partial^2}{\partial x_2^2} \left| 2B \frac{\partial \eta}{\partial y_1} \right| + \frac{\partial^3}{\partial x_2^2 \partial x_1} \left| 2B\eta \right| \qquad (3.2)$$

The orders of the sources are indicated by the number of derivatives which act on them with respect to the field point $\underline{x}$. In addition, the terms above are acted upon by the $\partial/\partial x_3$ operator (cf. equation 2.7), so that formally they represent a dipole in the 3-direction, a (3,1) quadrupole, a (3,1,1) octupole, etc. The dominant term is the dipole, and by the remarks above, the dipole strength is equal to the applied stress line density $q(y_2,t)$. Combining this with the other contribution to $\underline{Q}_+$ (equation 2.7), we see that the dominant term in the "edge sources" is a dipole distribution along $y_1 = 0$ of strength $2q(y_2,t)$.

This remains true even if the surfaces have equal impedances. A line distribution of stresses induces a shear discontinuity, and so a dipole radiation additional to that induced by the stresses themselves. This should be regarded as a qualification to Ffowcs Williams' (1966) result, that the dipole strength is equal to the applied stress. That result is true only for a stress which is smoothly distributed in a two-dimensional sense, so that the space derivatives of the surface deflexion remain continuous.

4. Formal solution of the integral equation

We turn now to the solution of equations (2.9-10). For the theory of singular integral equations with the Cauchy kernel, reference may be made to the books by Mushkelishvili (1953), and Gakhov (1966).

If $\tilde{v}$ satisfies (2.10), $\tilde{p}$ can most easily be obtained by subtracting (2.10) from (2.9) to give

$$(\tilde{p} - \tilde{Q}_+ - \tilde{Q}_-)e^{ix_3\Psi} = - z_w\tilde{v} . \qquad (4.1)$$

To obtain $\tilde{v}$, we write (2.10) in the canonical form

$$F(k_1) = A(k_1)\,\tilde{v}\,(k_1) + \frac{B(k_1)}{\pi i}\int_{-\infty}^{+\infty}\frac{\tilde{v}(k_1')dk_1'}{k_1' - k_1} , \qquad (4.2)$$

where $F = \tilde{Q}_- e^{ix_3\Psi}$, $A = \frac{1}{2}(z + z_w)$, $B = -\frac{1}{4}(z_1 - z_2)$, and the dependence upon k_2, ω is ignored for the moment. Subject to suitable conditions on A, B, which are satisfied in the case considered in detail below, (4.2) has an explicit formal solution, which can be written in the form

$$\tilde{v}(k_1) = A^*(k_1)F(k_1) - \frac{B^*(k_1)Z(k_1)}{\pi i}\int_{-\infty}^{+\infty}\frac{F(k_1')dk_1'}{Z(k_1')(k_1'-k_1)} . \qquad (4.3)$$

Here $A^* = A/(A^2 - B^2)$, $B^* = B/(A^2 - B^2)$, and $Z(k_1)$ is the "fundamental function" for (4.2). The function $\left[Z(k_1')(k_1'-k_1)\right]^{-1}$ plays a role for the singular integral equation which is analogous to that played by the resolvent kernel of a Fredholm equation with regular kernel.

The fundamental function is defined by

$$Z(k_1) = (A + B)X(k_1) = (A + B)\exp\Gamma(k_1) \quad ,$$

$$\Gamma(k_1) = \frac{1}{2\pi i}\left(\lim_{\mathrm{Im}\, k_1 \to 0+}\right)\int_{-\infty}^{+\infty} \ln\left(\frac{A-B}{A+B}\right)(k_1')\frac{dk_1'}{(k_1' - k_1)} \quad . \qquad (4.4)$$

Thus we have to evaluate the non-singular integral $\Gamma^+(k_1)$ when k_1 lies in the upper half of the complex k_1-plane, and then let Im $k_1 \to 0+$. The solution is completed, in principle, when Z is reduced to a usable form n not involving limits or singular integrals. Whether or not this can be achieved depends upon the complexity of the forms assumed for z_1, z_2, though this in no way invalidates the existence of the general solution (4.3) if the required regularity conditions are obeyed. This is the sense in which this formulation has, potentially, greater generality than the method employed by Heins and Feshbach (1954); in their formulation the impedances z_i were taken as constant at the outset, and a major reworking of the theory would be needed to encompass any other case. Here the impedances determine only the details of the solution, and not the general form. However, we have only found it possible to evaluate Z analytically for the constant impedance case of Heins and Feshbach, though it should be stressed again that the present formulation is particularly applicable to the flow-noise problem, while the Wiener-Hopf technique of Heins and Feshbach is only really suited to diffraction problems.

Suppose the surface impedances are independent of $\underline{k}$, though dependent upon ω. The surfaces are thus assumed to be formed by a system of infinitesimal decoupled mass-loaded elements, and the possibility of elastic wave propagation in the surfaces is thereby excluded. The problem of the diffraction of plane acoustic waves by such a surface has been solved by Heins and Feshbach (1954). The determination of the fundamental function Z is exactly analogous to the factor decomposition which is required by the Wiener-Hopf method.

Consider then

$$\Gamma^{+}(k_1) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln \frac{z_1 + z_w}{z_2 + z_w}(k_1') \frac{dk_1'}{(k_1' - k_1)}, \quad \text{Im } k_1 > 0 .$$

By differentiation on k_1 and integration by parts we find

$$\frac{\partial \Gamma^{+}(k_1)}{\partial k_1} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dk_1'}{k_1' - k_1} \frac{\lambda_1}{[\lambda_1 + (K^2 - k_1'^2)^{\frac{1}{2}}]} \frac{k_1'}{(K^2 - k_1'^2)}, \qquad (4.5)$$

minus a similar expression with λ_2 instead of λ_1. Here we have inserted the appropriate expression for z_w, have taken $\omega > 0$, and written $K^2 = \omega^2/a_o^2 - k_2^2$. It will appear later that it is sufficient to consider only the k_2 with $|k_2| < \omega/a_o$, and therefore K can be taken as real and positive. λ_1 and λ_2 denote $\rho\omega/z_1$ and $\rho\omega/z_2$ respectively, z_1, z_2 being regarded here as functions of ω only. The branch of $(K^2 - k_1'^2)^{\frac{1}{2}}$ is taken as that which reduces to K when $k_1' = 0$. There is thus a branch point on the real k_1'-axis, but this can be removed from the integration path by supposing K to have a small negative imaginary part. This is a familiar device (Copson, 1946), and corresponds to a slight dissipation in the fluid.

With this supposition, consider the integral involving λ_1 in (4.5) around the contour shown in Fig.1. The contributions from the circular arcs vanish in the limits of large and small radii. The integrals along the edges of the branch cut combine to give

$$\frac{\lambda_1}{\pi}\int_{-\infty}^{-K} \frac{dk_1'}{k_1' - k_1} \; \frac{1}{\left[\lambda_1^2 + (k_1'^2 - K^2)\right]} \; \frac{k_1'}{(k_1'^2 - K^2)^{\frac{1}{2}}} \quad ,$$

where we can now put Im $K = 0$. This integral can be evaluated by elementary means (Heins and Feshbach, 1954), to give

$$-\frac{\lambda_1}{\pi}\left\{\frac{\Psi(k_1) - \Psi(a_1)}{k_1 - a_1} + \frac{\Psi(k_1) - \Psi(-a_1)}{k_1 + a_1}\right\}$$

$= \quad G_1(k_1)$ say, where $a_1 = (K^2 - \lambda_1^2)^{\frac{1}{2}}$, and

$$\Psi(k_1) = \frac{1}{(K^2 - k_1^2)^{\frac{1}{2}}} \tan^{-1}\sqrt{\frac{K - k_2}{K + k_1}} \quad ,$$

$$= \frac{1}{2i}\frac{1}{(K^2 - k_1^2)^{\frac{1}{2}}} \ln\left(\frac{k_1^2 + (k_1^2 - K^2)^{\frac{1}{2}}}{K}\right) .$$

Combining this with the term $G_2(k_1)$, and with the residue contributions, we have finally, on letting Im $k_1 \to 0+$,

$$\frac{\partial \Gamma(k_1)}{\partial k_1} = G_2(k_1) - G_1(k_1) + \frac{k_1}{(K^2 - k_1^2)}\left[\frac{\lambda_1}{\lambda_1 + (K^2-k_1^2)^{\frac{1}{2}}} - \frac{\lambda_2}{\lambda_2 + (K^2-k_1^2)^{\frac{1}{2}}}\right] . \tag{4.6}$$

Then, since $\Gamma(-\infty) = 0$, we have

$$\Gamma(k_1) = \int_{-\infty}^{k_1} \frac{\partial \Gamma(u)}{\partial u}\, du ,$$

but the integral cannot be expressed in terms of known functions. Approximations for Γ can be obtained, as shown by Heins and Feshbach, and these can be used if a detailed study of the function Z is required. We note for further use, that if we set $z_1 \approx z_2$ approximately in the above, then $\Gamma = 0$, $X = 1$ and $Z = A + B$. This will provide us with an adequate approximation when the two surfaces have impedances differing by a small amount compared with the magnitude of either impedance. The next task is to invert the Fourier transforms over wavenumber, so as to obtain the far-field radiated pressure as a function of frequency.

5. Inversion of the Fourier transform; the far-field radiation

From equations (4.1) and (4.3) we have the following equation for the pressure transform in terms of the source fields;

$$\tilde{p}(\underline{k},\omega,x_3) - \tilde{Q}_+(\underline{k},\omega,x_3) = (1 - z_w A^+)e^{-ix_3\Psi} F(\underline{k},\omega)$$

$$+ \frac{e^{-ix_3\Psi} z_w B^* Z}{\pi i} \int \frac{F(k_1',k_2,\omega)\, dk_1'}{Z(k_1',k_2,\omega)(k_1'-k_1)} \, . \qquad (5.1)$$

Consider again a positive frequency ω, and write $k_o = \omega/a_o$, and $K = +\sqrt{k_o^2 - k_2^2}$ where we again assume $|k_2| < k_o$. Multiply (5.1) by $\exp ik_1x_1$, and integrate over k_1. Under suitable assumptions, the k_1,k_1' integrals in the final term can be inverted (even though one of them is singular), and after the integrals are inverted we also interchange the dummy variables k_1,k_1'. For the final term in (5.1) we then have to consider the integral

$$L = \int e^{ik_1x_1 - ix_3\sqrt{K^2-k_1'^2}} \frac{(z_w B^* Z)(k_1')dk_1'}{(k_1 - k_1')} \, , \qquad (5.2)$$

taken in the principal value sense. By the Plemelj formulae (see Mushkelishvili, 1953), L is given by

$$L = \tfrac{1}{2}(L^+ + L^-) \, ,$$

where $L^{\pm}$ are the values of L as k_1 approaches the real axis from above and below respectively. We calculate the integrals $L^{\pm}$ by a deformation of the integration path which is standard in the diffraction theory (Copson, 1946).

Consider for definiteness an observation point $\underline{x}$ with $x_1 \geq 0$, $x_1 = R\cos\Theta$, $x_3 = R\sin\Theta$, $0 \leq \Theta \leq \pi/2$. We deform the path of integration into one branch of the hyperbola given by

$$k_1' = -K\cos(\Theta + i\tau), \qquad (5.3)$$

$$\sqrt{K^2 - k_1'^2} = +K\sin(\Theta + i\tau), \text{ (see Fig.2).}$$

Again we suppose the branch points slightly displaced from the real axis by giving K a small negative imaginary part. The branch cut from -K to to $-\infty$ is now a radial one, as shown in Fig.2. $N^{\pm}$ denote the integrals $L^{\pm}$ taken along the hyperbolic path for Im k_1 = O+ and Im k_1 = O- respectively. The contributions to the integrals from the circular arcs joining the real axis to the hyperbola vanish in the limit of infinite radius.

When $-K\cos\Theta < k_1 < +\infty$ we have, from Cauchy's theorem,

$$L^+ + N^+ = 2\pi i \text{ (residue at } k_1' = k_1),$$

$$L^- + N^- = 0, \qquad \text{and hence}$$

$$L = \pi i \text{ (residue at } k_1' = k_1) - N. \qquad (5.4)$$

On the other hand, if $-\infty < k_1 < -K\cos\Theta$, we have

$$L^+ + N^+ = 0,$$

$$L^- + N^- = -2\pi i \text{ (residue at } k_1' = k_1),$$

because the pole is now circumvented in the negative sense. Hence

$$L = -\pi i \text{ (residue at } k_1' = k_1) - N. \qquad (5.5)$$

The residue contributions to L can be combined with the first term on the right of (5.1) to give a contribution

$$\int_{-\infty}^{+\infty} (1 - z_w A^*)F\, e^{ik_1x_1 - ix_3\sqrt{K^2-k_1^2}}\, dk_1$$

$$- \int_{-K\cos\Theta}^{\infty} z_w B^* F\, e^{ik_1x_1 - ix_3\sqrt{K^2-k_1^2}}\, dk_1 \;+\; \int_{-\infty}^{-K\cos\Theta} z_w B^* F\, e^{ik_1x_1 - ix_3\sqrt{K^2-k_1^2}}\, dk_1$$

Because of the factor $e^{-ix_3\sqrt{K^2-k_1^2}}$, the limits $\pm\infty$ can be replaced by $\pm K$, the remaining contributions vanishing exponentially as $x_3 \longrightarrow +\infty$. Now from the definitions of A^*, B^* we find

$$1 - z_w A^* - z_w B^* = \frac{z_1 - z_w}{z_1 + z_w} = R_1 \,,$$

$$1 - z_w A^* + z_w B^* = \frac{z_2 - z_w}{z_2 + z_w} = R_2 \,, \tag{5.6}$$

where R_1, R_2 are the usual reflexion coefficients for the two surfaces. Therefore, $\tilde{p}(\underline{k},\omega)$ contains the reflected fields $R_1\tilde{Q}_-$ or $R_2\tilde{Q}_-$ according as

$$-K \cos\Theta < k_1 < +K, \quad \text{or}$$

$$-K < k_1 < -K\cos\Theta \,.$$

This just states that the propagating components of the source field Q (i.e. those for which $|k_1| < K$) are reflected to the observation point from one surface or the other, with appropriate reflexion coefficients, according to the obvious geometric rule. These reflected pressures have precisely the interpretation placed on them by Ffowcs Williams (1965,1966).

Reflexion of turbulence-generated sound does not imply any fundamental increase in the efficiency of the turbulence flow. We shall therefore neglect these reflected pressures now, and shall consider only the scattered field, which is given by the integral along the hyperbolic path, and depends upon the integral

$$N = \int_{-\infty}^{+\infty} e^{-iKR\cosh\tau} \frac{(z_w B^* Z)(-K\cos(\Theta+i\tau))}{k_1 + K\cos(\Theta+i\tau)} iK\sin(\Theta+i\tau)d\tau . \quad (5.7)$$

Now, referring to (4.4), we have $Z = (A+B)X$, where $X \neq 0$ has only branch points as its singularities. We can therefore write

$$z_w B^* Z = \lambda_1(z_2-z_1)X/2\left[\lambda_1 + K\sin(\Theta + i\tau)\right], \text{ and then}$$

$$N = \frac{i\lambda_1(z_2-z_1)}{2}\int_{-\infty}^{+\infty} \frac{e^{-iKR\cosh\tau} X(-K\cos(\Theta + i\tau))K\sin(\Theta + i\tau)d\tau}{\left[k_1 + K\cos(\Theta + i\tau)\right]\left[\lambda_1 + K\sin(\Theta + i\tau)\right]} . \quad (5.$$

Two distinct cases now arise, representing quite different physical processes. Suppose firstly that $|k_1| < K$, i.e. consider a propagating element (k_1,k_2,ω) with supersonic phase speed in the (1,2) plane. Such elements can propagate to infinity as sound when no boundaries (or an infinite homogeneous boundary) are present. The process described by the integral N for such components is that of the scattering of radiating energy with one directional distribution into radiating energy with a different directivity - i.e. the diffraction of sound. From the viewpoint of aerodynamic noise this process is unimportant, for the diffraction can only produce a directional redistribution of the turbulence-generated sound – whereas here we hope to find evidence of the conversion of non-propagating near field energy into radiating sound. The case $|k_1| < K$ is only of interest in diffraction theory, where the directivity pattern is the most important feature. When we write

$$F(k_1, k_2, \omega) = \delta(k_1 + k_o \cos \Theta')\delta(k_2)\delta(\omega - a_o k_o)$$

we have the case of a plane monochromatic wave of wavenumber k_o incident upon the plane at an angle Θ'. The integral determining the (two-dimensional) diffracted field is then proportional to N with $k_2 = 0$ and $k_1 = -k_o \cos \Theta'$, i.e.

$$N = \frac{i\lambda_1(z_2 - z_1)}{2} \int_{-\infty}^{+\infty} \frac{e^{-ik_o R\cosh \tau} \times \left[-k_o\cos(\Theta + i\tau)\right] \sin(\theta + i\tau)d\tau}{\left[\cos(\Theta + i\tau) - \cos\Theta'\right]\left[\lambda_1 + k_o \sin(\Theta + i\tau)\right]} . \quad (5.9)$$

This integral is just that found by Heins and Feshbach (1954) by the Wiener-Hopf method, and when certain changes of notation are made, we can recover their results on the diffraction of plane waves. The transformation of the integral to a usable form is not trivial, and the method of stationary phase cannot be applied immediately as there is a singularity when $\Theta = \Theta'$ at $\tau = 0$. The necessary transformations, and the final form for the diffracted field are given in detail by Heins and Feshbach.

We turn next to the case $|k_1| > K$, which does not correspond to a diffraction problem for plane waves. This is a case which necessarily arises when the inhomogeneous wave equation has to be solved, and the radiated field is properly termed a scattered field in this case. We must emphasise that scattering of near-field pressure into sound only occurs from any kind of inhomogeneity when turbulence sources exist at distances less than about a sound wavelength from the discontinuity, or inhomogeneity. Thus, if all the turbulence sources in this problem are at least several characteristic wavelengths away from the line $y_1 = 0$, the radiated field consists of direct and reflected fields, and a diffracted field which can be treated in the manner described above. The impedance discontinuity cannot, in such a case, increase the power output from the flow. Practical cases, however, often involve such impedance discontinuities with turbulence sources distributed nearly homogeneously in the (1,2) plane (as in a turbulent boundary layer) and then scattering is certain to occur.

When $|k_1| > K$, the situation is more easy to handle than in the diffraction problem, for the integral N contains no pole on the range of integration. An application of the method of stationary phase gives at once, for $KR \longrightarrow \infty$,

$$N \sim \frac{i\lambda_1(z_2-z_1)}{2}\sqrt{\frac{2\pi}{KR}}\, e^{i\frac{\pi}{4} - iKR}\, \frac{X(-K\cos\Theta)K\sin\Theta}{(k_1 + K\cos\Theta)(\lambda_1 + K\sin\Theta)} \quad . \quad (5.10)$$

We have now to multiply the expression by $\exp(ik_2x_2)$ and integrate over k_2. The factor $\exp(-iR\sqrt{k_o^2 - k_2^2})$ shows that only those k_2 for which $|k_2| < k_o$ contribute to the far-field integral, and this is the justification for taking $|k_2| < k_o$, as was done earlier. A further application of the method of stationary phase then gives us the following result for the scattered pressure field $\hat{p}(\underline{x},\omega)$ at position $x_i = r\beta_i$ and frequency $\omega > C$;

$$\hat{p}(\underline{x},\omega) = -\frac{1}{\pi i}\iint \frac{F(k_1,k_2,\omega)}{Z(k_1,k_2,\omega)} N(k_1,k_2,\omega)\, e^{ik_2x_2}\, dk_1 dk_2$$

$$\sim -i\lambda_1(z_2-z_1)\left(\frac{k_o\beta_3}{r}\right) e^{-ik_or}\, \frac{X(-k_o\beta_1, -k_o\beta_2,\omega)}{(\lambda_1 + k_o\beta_3)}\, \hat{V}(k_o,\beta_i,\omega)\,,$$

$$\hat{V}(k_o,\beta_i,\omega) = \int_{|k_1| > K} \frac{F(k_1, -k_o\beta_2,\omega)}{Z(k_1, -k_o\beta_2,\omega)}\, \frac{dk_1}{(k_1 + k_o\beta_1)}\,. \qquad (5.11)$$

where now $K = \sqrt{k_o^2 - k_o^2\beta_2^2}$.

This is an exact asymptotic expansion for $\hat{p}(\underline{x},\omega)$, in terms of the known source field F, in the far-field $k_or \to \infty$. In the next section, the familiar kind of dimensional analysis will be applied to (5.11) to predict the dependence of the scattered field on the characteristic length and velocity scales L_o, V of the turbulent motion.

6. Dimensional Analysis for Boundary Layer Turbulence

In this section we perform a simple dimensional analysis of equation (5.11), to predict the dependence of the scattered acoustic power on the parameters characteristic of the plates, and of the turbulent flow over the plates. A boundary layer type of flow will be considered, whose internal dynamics may be regarded as incompressible if the ··ean flow Mach number is small. The flow may be characterised in the usual way by length and velocity scales L_o, V, representing a boundary layer thickness and free stream velocity, respectively. The typical frequency, both for the turbulent motion and for the emitted sound, then varies roughly as V/L_o, the typical turbulence wavenumber is of order $1/L_o$, and the acoustic wavenumber is of order M/L_o. $M = V/a_o$ is the free stream Mach number normalised on the far-field sound speed a_o.

We have already supposed the plates to have negligible bending stiffness. This is a good approximation for underwater applications, where it is common practice to represent plates by a purely mass-loaded impedance for the frequency range of general interest. To further reduce the complexity of (5.11) we shall suppose the impedance difference $|z_2 - z_1|$ to be small compared with either of $|z_2|$, $|z_1|$, - which amounts here only to a condition on the plate masses, independent of frequency or wavenumber. This is a case evidently relevant to many practical situations, though of course certain extreme situations are excluded (e.g., the case of one perfectly rigid surface and one perfectly limp surface, $z_1 = \infty$, $z_2 = o$). We can then replace z_i by the average impedance z everywhere, except in the factor (z_2-z_1), and in particular we may replace the function X by unity. As a check, we note that the resulting equation for $\hat{p}(\underline{x},\omega)$ could have been obtained, in this approximation, from a simple iteration process on equations (2.9) - 2.10).

Taking the direction cosines β_i all O(1) and neglecting some numerical factors, we then have, symbolically,

$$\hat{p}(\underline{x},\omega) \sim \lambda |z_2-z_1| \left(\frac{k_o}{r}\right) e^{-ik_o r} \frac{\hat{V}(\omega)}{\lambda + k_o} ,$$

$$\hat{V}(\omega) \sim \int_{k_o}^{\infty} \frac{F(k_1,-k_o,\omega)}{(k_1+k_o)\left(z + \frac{i\omega\rho}{\sqrt{k_1^2-k_o^2}}\right)} dk_1 . \tag{6.1}$$

Leaving aside the function $\hat{V}(\omega)$ for the moment, we have now to distinguish two cases. Suppose the panels to have the same thickness h, and to be made of materials of densities σ, $\sigma + \Delta\sigma$, with $\Delta\sigma \ll \sigma$. Then we have

$$z_1 = i\sigma h\omega, \qquad z_2 = i(\sigma + \Delta\sigma)h\omega, \qquad |\lambda| = \rho/\sigma h .$$

The ratio $|\lambda|/k_o$ is equal to $\rho a_o/|z|$, and therefore the case in which $|\lambda| \gg k_o$ corresponds to that in which the specific acoustic impedance ρa_o is large compared with the plate impedance $|z|$. This is the case relevant to underwater applications. There one is often concerned with steel plates of density $\sigma \sim 10\rho$, and with Mach numbers of order 10^{-2}. The value of $|\lambda|/k_o$ is then of order $10\ L_o/h$, and even the smallest relevant length scale L_o (the boundary layer displacement thickness δ^*) is at least several times the plate thickness h. $\delta^* = 3$ cm, h = 1 cm are perhaps typical values. In this underwater limit we have

$$\hat{p}(\underline{x},\omega) \qquad \left(\frac{h\omega^2 \Delta\sigma}{a_o r}\right) e^{-ik_o r} \hat{V}(\omega) \qquad (6.2)$$

On the other hand, the limit $|\lambda| \ll k_o$ implies that the fluid loading on the plates, represented by the specific acoustic impedance ρa_o, is negligible compared with the mass loading represented by the plate impedance $|z|$. This limit may apply in some aeronautical contexts, where in fact it is usual to ignore effects of fluid loading. In those applications, the Mach number is much higher than in underwater cases, so that taking $M = O(1)$ we have

$$|\lambda|/k_o \sim (\rho L_o/\sigma h) .$$

For aluminium plates in air, ρ/σ is of order 10^{-3}, so that $|\lambda|/k_o \lesssim 1$ unless $L_o/h \gtrsim 10^3$. The case $L_o/h \gtrsim 10^3$ may apply in practice, since the plates used in aircraft construction are exceedingly thin ($h \sim 1$ mm), so that fluid loading cannot be neglected and the previous limit (6.2) will apply. However, it is clear that the limit $|\lambda| \ll k_o$ can <u>only</u> occur in aeronautical problems, if at all, and then we would have

$$\hat{p}(\underline{x},\omega) \sim \left(\frac{\rho\omega\Delta\sigma}{\sigma r}\right) e^{-ik_o r} \hat{V}(\omega) . \qquad (6.3)$$

We consider now the function $\hat{V}(\omega)$. Since we are interested primarily in the radiation scattered by the impedance discontinuity, rather than in that induced by supports acting on the discontinuity, the support sources in F will be omitted. In fact the solution obtained in §4 is incomplete if boundary conditions on the discontinuity are prescribed through the action of externally applied forces. For example, if the boundary is supported along the discontinuity by a rib of infinite impedance, we have to satisfy the condition $v(y_1 = 0) = 0$, or equivalently,

$$\int \tilde{v}(k_1, k_2, \omega) dk_1 = 0 \qquad (6.4)$$

on the boundary velocity. To do this we have to add to the particular solution (4.3), a solution of the homogeneous integral equation, determining the arbitrary constant so as to satisfy the integral constraint (6.4). The solution of (4.2) in the form of complementary function plus particular integral is discussed in detail by Gakhov (1966) and Mushkelishvili (1953). The solution (4.3) may easily be generalised in this way, though to no great point.

With support terms omitted, we have

$$F(k_1, k_2, \omega) = \tilde{T}_-(k_1, k_2, \omega) e^{i x_3 \psi} .$$

$\tilde{T}_-$ is the Fourier transform of the pressure induced at $(\underline{x}, t)$ by the system of image sources <u>alone</u>, i.e. of

$$\frac{1}{4\pi}\frac{\partial^2}{\partial x_i\,\partial x_j}\int_{V_-} T_{ij}(\underline{y},\, t - \frac{|\underline{x}-\underline{y}|}{a_o})\frac{d\underline{y}}{|\underline{x}-\underline{y}|} \ .$$

Here the integration runs over the volume V_- occupied by the image sources. By two applications of the divergence theorem we can interchange $\underline{x}$ and $\underline{y}$ derivatives to write the image pressure as

$$\frac{1}{4\pi}\int_{V_-}\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}(\underline{y},\, t - \frac{|\underline{x}-\underline{y}|}{a_o})\frac{d\underline{y}}{|\underline{x}-\underline{y}|} \ . \qquad (6.5)$$

There are no difficulties here concerning the surface integrals arising from use of the divergence theorem. We are considering the image sources in isolation, so that the surface integrals may be taken over any distant control surface enclosing the observation point and the sources, and may be neglected on the usual assumptions. Now the integral (6.5) is proportional to the pressure exerted on a rigid plane boundary, $x_3 = 0$, by the same turbulent flow. This pressure field has been examined by Ffowcs Williams (1965a). If we take Fourier transforms (1,2,t) space, and follow his method of manipulating the resulting expressions, we readily find that

$$\tilde{T}_-(k_1,k_2,\omega)e^{ix_3\psi} \propto$$

$$\int_0^\infty \tilde{T}_{ij}(k_1,k_2,\omega,\bar{x}_3)\frac{\left[(\frac{\omega^2}{a_o^2}-k^2)^{\frac{1}{2}}\delta_{i3}+k_i\right]\left[(\frac{\omega^2}{a_o^2}-k^2)^{\frac{1}{2}}\delta_{j3}+k_j\right]}{(\frac{\omega^2}{a_o^2}-k^2)^{\frac{1}{2}}} \times$$

$$\times \ \exp\left\{-i\bar{x}_3(\frac{\omega^2}{a_o^2}-k^2)^{\frac{1}{2}}\right\} d\bar{x}_3 \ . \qquad (6.6)$$

Numerical factors have again been omitted. $\tilde{T}_{ij}$ is the (1,2,t) transform of T_{ij}, the Lighthill turbulence stress tensor. The vector k_i appearing in (6.6) is the 3-vector $(k_1, k_2, 0)$.

Now the function $\tilde{T}_{ij}$ is the transform of a typical incompressible turbulence function, and is concentrated on wavenumbers of order $1/L_o$, and on frequencies of order V/L_o. Since the Mach number M is small, we are therefore interested in the regime for which $\omega \ll a_o k$, and in this regime we have, from (6.6), $\tilde{T}_-(k_1, k_2, \omega) e^{ix_3\Psi} \propto$

$$\int_0^\infty \tilde{T}_{ij}(k_1, k_2, \omega, \bar{x}_3)(k\delta_{i3} + k_i)(k\delta_{j3} + k_j)\frac{e^{-k\bar{x}_3}}{k}\, d\bar{x}_3 \, . \qquad (6.7)$$

The typical value of $\tilde{T}_- e^{ix_3\Psi}$ now follows quickly from this equation. The exponential factor may be replaced by unity, for in the boundary layer the sources are concentrated on wavenumbers less than about $1/L_o$, while the sources are also concentrated in space in the region $0 \leq \bar{x}_3 \lesssim L_o$. The $\bar{x}_3$-integration is dimensionally equivalent to multiplication by L_o, while the transform $\tilde{T}_{ij}$ is proportional to $\rho V L_o^3$. This follows if we suppose, as usual, that the Lighthill stress tensor is dominated by contributions from the fluctuating Reynolds stresses, $T_{ij} \sim \rho V^2$. Then we have

$$\tilde{T}_- e^{ix_3\Psi} \sim \rho V L_o^3 \, , \qquad (6.8)$$

which is exactly what one would expect for the transform of a typical near-field pressure ρV^2.

To complete the determination of $\hat{V}(\omega)$, note that, since $k_1 \gg k_o$ for the dominant wavenumber $1/L_o$, the factors $(k_1 + k_o)^{-1}$ and dk_1 in (6.1) cancel, dimensionally. Also, that it is sufficient for the Mach number dependence to write

$$(z + \frac{i\omega\rho}{\sqrt{k_1^2 - k_o^2}}) \quad \text{as} \quad (\sigma h + \rho L_o)\omega \ ,$$

leaving aside the question of whether, generally, $\sigma h \gg \rho L_o$ or not.

When all these estimates are made, we readily find the following dimensional forms for the total scattered acoustic power output E from the discontinuity:

$$E \sim (\rho V^3 L_o^2) M^3 \left(\frac{h\,\Delta\sigma}{\sigma h + \rho L_o}\right)^2$$

$$\text{for } |\lambda| \gg k_o \qquad (6.9)$$

$$E \sim (\rho V^3 L_o^2) M \left(\frac{\rho L_o\,\Delta\sigma}{\sigma(\sigma h + \rho L_o)}\right)^2$$

$$\text{for } |\lambda| \ll k_o \ . \qquad (6.10)$$

Note that these are three-dimensional results, the turbulence being supposed confined to a limited region in the x_2 direction. If the turbulence extends to infinity in the x_2 direction, the problem is effectively two-dimensional, and M^3 and M in (6.9-6.10) should be replaced by M^2 and 1.

Thus the power output varies as V^6 if fluid loading is significant, and as V^4 if fluid loading is negligible. As noted in §1, these laws might be deduced immediately from the Curle type of solution to Lighthill's wave equation. We must, however, emphasise again that any such procedure is dangerous unless all quantities occurring in the expression for the radiated pressure have been expressed in terms of the Lighthill tensor T_{ij}. In this sense, Curle's equation is not a solution, and the whole object of this paper has been to show how one should attempt to obtain the proper solution. The danger of making predictions from an incomplete solution of the Curle type can be seen at once by subtracting (2.2) from (2.1). The resulting equation contains no monopole terms, and superficial examination would appear to give $E \sim V^6$, and would entirely preclude the V^4 law. On the other hand, from (2.1) alone, the monopole terms appear always to dominate at low enough Mach numbers, and to yield a V^4 law. The difficulties, of course, stem from the fact that in our problem the typical surface dimension is much larger than the typical sound wavelength, and for such cases the conventional interpretations of acoustic sources as monopoles, dipoles, etc., are valueless. Curle's aim was to apply his solution to the Aeolian tone problem, and for that problem there are no such difficulties.

A more interesting interpretation of (6.9 – 6.10) can be given in terms of the radiation from a single infinite homogeneous plate, driven by a point force $F_o \exp(i\omega t)$. Under the assumption that the bending wavenumber $k_B = \sqrt[4]{m\omega^2/B}$ is large compared with the acoustic wavenumber $k_o = \omega/a_o$

(i.e. when the bending stiffness B is small, as we have assumed), it is found (Cremer and Heckl, 1967, p.457) that the power radiated by the plate is given by

$$E = \frac{\rho a_o k_o^2 F_o^2}{2\pi \omega^2 m^2} \left[1 - \frac{\rho a_o}{m\omega} \tan^{-1} \frac{m\omega}{\rho a_o} \right] . \qquad (6.11)$$

The first term represents the radiation if fluid loading is negligible, the second term represents the correction due to fluid loading. Suppose that $\rho L_o \ll \sigma h$ for simplicity, and set

$$F_o \sim (\rho V^2)(L_o^2)(\frac{\Delta\sigma}{\sigma}) , \qquad \omega \sim \frac{V}{L_o} \qquad (6.12)$$

Then from (6.11) we find the result (6.9) by expanding the $\tan^{-1}$ function for the case of appreciable fluid loading, $\rho a_o \gg m\omega$. Also, for the case of negligible fluid loading, $\rho a_o \ll m\omega$, we find the result (6.10) by neglecting the $\tan^{-1}$ function. Thus in either case, we can interpret the radiation scattered by the discontinuity as being that produced by a homogeneous plate with small bending stiffness, driven at the turbulence frequency V/L_o, by a force of strength

$$F_o \sim \Big(\underbrace{\rho V^2}_{\text{typical pressure}}\Big) \times \Big(\underbrace{L_o^2}_{\text{typical area}}\Big) \times \Big(\underbrace{\Delta\sigma/\sigma}_{\text{relative impedance jump}}\Big) .$$

7. Conclusions

The object of this paper has been to solve Lighthill's inhomogeneous wave equation, given only the turbulence stress tensor T_{ij}, subject to the discontinuous boundary conditions appropriate to a composite flexible boundary. The boundary considered consists of two homogeneous semi-infinite planes $y_1 > 0$, $y_1 < 0$, with different impedances. To attain reasonable analytical simplicity, we have been forced in the end to neglect bending stiffness of the planes, and to suppose the planes to be merely mass-loaded. Some important aspects of the interaction between flows and surfaces are undoubtedly by-passed by this supposition, though there is no reason to doubt its usefulness in bringing out many essential points. A general formulation is attempted, in which details of the source terms and of the boundary conditions are not needed at the outset. The method yields a singular integral equation, with variable coefficients, for the Fourier transform of the boundary velocity. A great deal is known about such equations, and it is possible to write down an exact formal solution for arbitrary source terms and for arbitrary forms of the impedances of the two halves of the boundary. The solution is applied here to the problem of noise generation by a turbulent boundary layer formed on the plane. However, the method might also be used either to generalise previously obtained results in diffraction theory, or to estimate the radiation from a propellor, for example, rotating near the boundary. It would seem that the practical usefulness of such extensions is probably severely limited, as the formulae become formidably complicated.

The method given here may evidently be applied to other problems involving discontinuous boundary conditions, and it seems to be more straightforward than the usual formulation of such problems in terms of a Wiener-Hopf integral equation. Moreover, the Wiener-Hopf method is difficult to apply to problems in which sources are involved. However, the details of the solution lie mainly in the "Fundamental function" for the singular integral equation, and to that extent our method is not superior, for the determination of this function is just equivalent to obtaining the factor decomposition required by the Wiener-Hopf technique.

The formal solution obtained in this work for the radiated pressure involves a function F, which is essentially the Fourier transform of the pressure in the boundary layer. The supersonic spectral components of F (i.e. those for which $\omega > a_o k$) are those which can propagate to infinity as sound in the absence of any boundary. They form the genuine sound field of the turbulence. The solution shows that these propagating components are reflected, according to the obvious geometrical rule, with appropriate reflexion coefficients, from one half of the boundary or the other, depending upon their angle of incidence upon the boundary. In addition, a diffracted field, emanating from the discontinuity, is set up by the action of propagating components on the surface. None of these effects can substantially increase the radiation from the turbulence. Reflexion of propagating components can at most increase the acoustic power output by a factor of 4, while the diffraction implies only a directional redistribution of the turbulence-generated sound. The diffracted field generated by propagating components draws its energy, by definition, from those components, and not directly from the turbulence.

The solution also shows clearly, however, that the impedance discontinuity acts as a scatterer, or wavenumber converter. Non-propagating near-field pressure components are scattered into supersonic propagating components. This process can make use of the large energy content of the hydrodynamic near-field of the turbulence, and may appreciably increase the acoustic power output. Wavenumber conversion is the principal aspect of turbulence dynamics, of course, where it arises through the nonlinear convective terms in the Navier-Stokes equation. The sound generation problem, as formulated by Lighthill, is linear, but interaction between Fourier components at different wavenumbers can still occur through discontinuities in boundary conditions. Any such discontinuities clearly lead to convolution integrals in Fourier space, and so to scattering.

The terminology used here is different from that which is usual in acoustics (see, e.g. Morse and Ingard, 1968). Scattering usually refers to the process occurring when sound (i.e. a propagating wave-field) is incident upon a body with typical dimension small compared with a wavelength of the incoming sound. Diffraction theory is concerned with the interaction of a propagating wave with a body whose typical dimension is large compared with the sound wavelength. The term "diffraction" is retained here, implying in addition an energy-conserving exchange between incident and diffracted waves. On the other hand, it seems preferable to use the term "scattering" for the process whereby energy is converted from a passive near-field state to a propagating wave-field state.

A dimensional analysis for the case of small impedance discontinuity shows that the scattered acoustic power varies as V^4 if fluid loading on the boundary is small, and as V^6 if fluid loading is appreciable. The first limit is relevant to aerodynamic problems, the second to underwater applications. The conventional interpretation of these results as implying monopole and dipole radiation is worthless. A more illuminating interpretation follows by comparing our results with well-known results for the radiation from a single infinite homogeneous plate driven by an oscillatory point force. Setting the force frequency equal to the turbulence frequency, and the force strength equal to the product (typical pressure ρV^2) x (typical area L_o^2) x (relative impedance jump $\Delta\sigma/\sigma$) in those results, and making the appropriate simplifications for low and high fluid loading, we obtain the V^4 and V^6 laws, together with the other relevant factors. Thus, apart from the reflexion and diffraction effects noted earlier, the discontinuity acts in the same way as does a force

$$(\rho V^2 L_o^2 \frac{\Delta\sigma}{\sigma}) \exp(i\frac{Vt}{L_o}) \qquad (7.1)$$

on a homogeneous boundary.

To obtain a rough idea of the importance of the scattered radiation, compare equation (6.9), with $\sigma h \gg \rho L_o$ for simplicity, with the familiar Lighthill formula for the acoustic power output of the flow in the absence of scattering. The relevant formula, obtained by the usual dimensional analysis, is

$$E_Q \sim (\rho V^3 L_o^2) M^5 \left(\frac{V_Q}{V^*}\right) , \qquad (7.2)$$

where $V^* \sim L_o^3$ is the correlation volume, and V_Q is the volume occupied by the sources. For the boundary layer case, we have $V_Q \sim L_o S$ where S is the area of surface covered by the layer, since the typical turbulence length scale L_o is of the order of the boundary layer thickness. Then we see that the scattering process generates as much sound as do the turbulence quadrupoles in an area

$$S = L_o^2 \left(\frac{\Delta\sigma}{\sigma}\right)^2 M^{-2} \qquad (7.3)$$

of the layer. In a typical underwater situation we might have $\Delta\sigma/\sigma = 5 \times 10^{-2}$, $M = 5 \times 10^{-3}$, $L_o = 1$ ft., in which case the of S is equal to 100 sq.ft. Thus the scattering due to surface inhomogeneities is a powerful effect, and may well make the dominant contribution to many noise fields observed in practice.

Fig. I. Integration contour for fundamental function. (4.5)

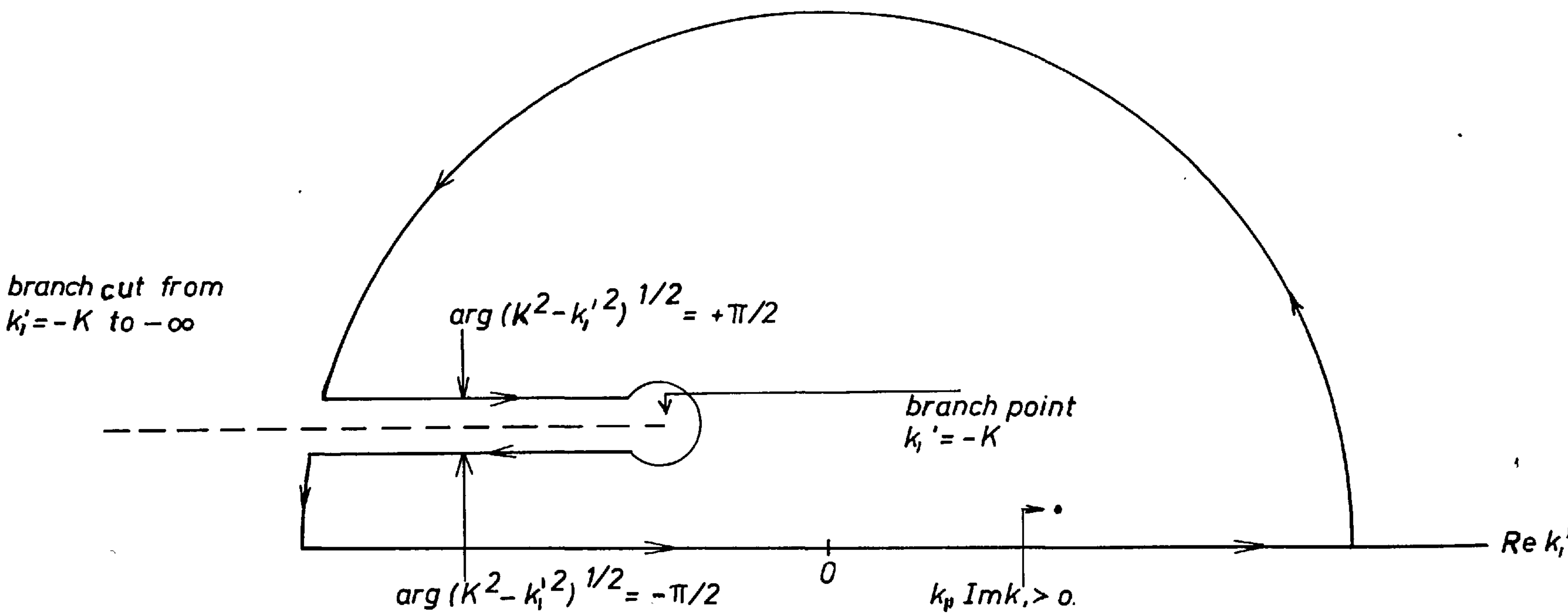
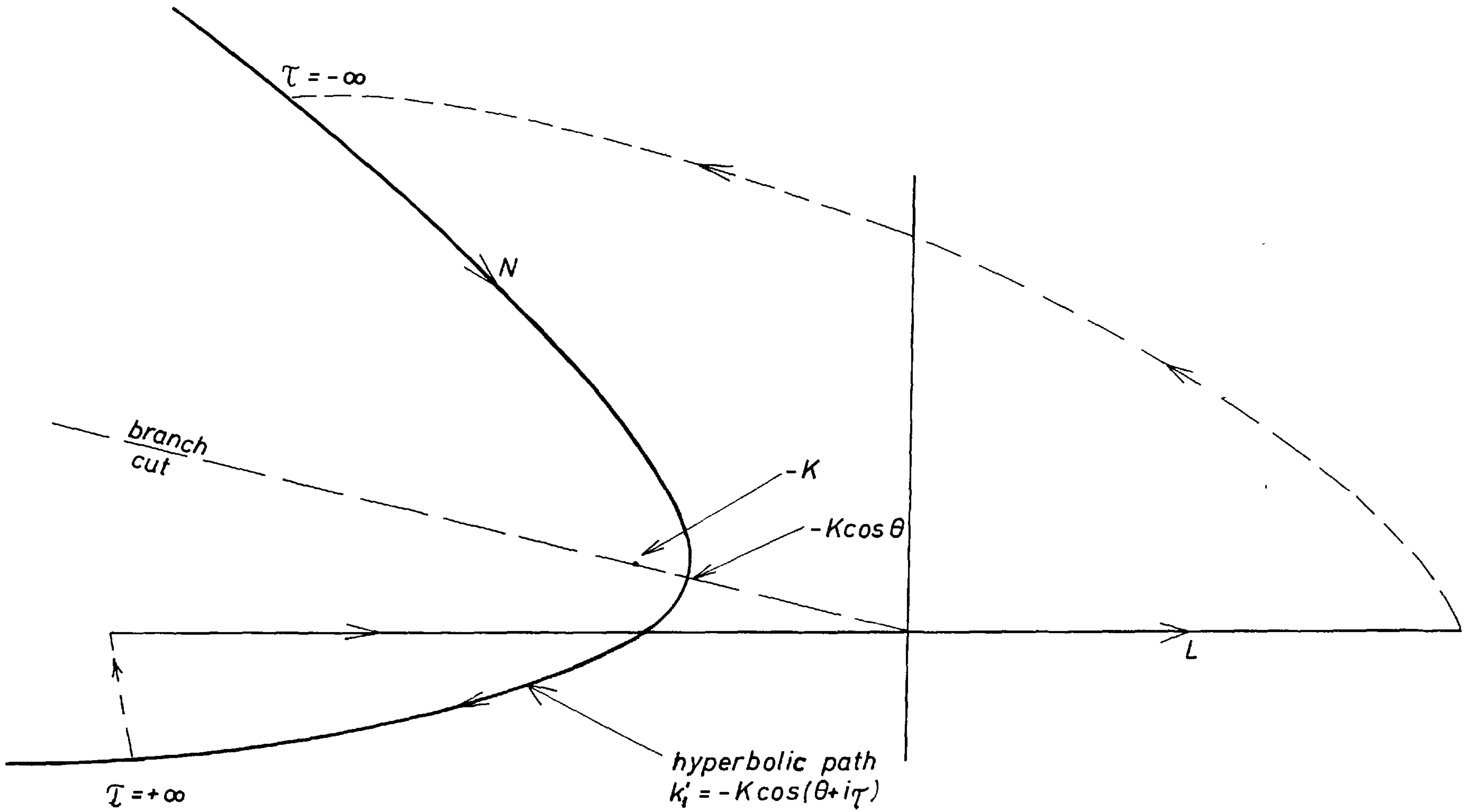


Fig. 2. Deformation of integration path for L.(5.2.)



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CHAPTER 3

SOUND GENERATION BY TURBULENT TWO-PHASE FLOW

Abstract

Sound generation by turbulent two-phase flow is considered by the methods of Lighthill's theory of aerodynamic noise. An inhomogeneous wave equation is derived, in which the effects of one phase on the other are represented by monopole, dipole and quadrupole distributions. The resulting power outputs are obtained for the case of a distribution of small air bubbles in water. The monopole radiation resulting from volumetric response of the bubbles to the turbulent pressure field overwhelms that from the quadrupoles equivalent to the turbulent flow, the increase in acoustic power output being about 70 dB for a volume concentration of 10%. The monopole radiation occurs through the forced response of the bubbles at the turbulence frequency; resonant response is shown to be impossible when the excitation is due to turbulence alone. Surface radiation arises from the edge of a cloud of bubbles. This radiation is important when the region containing bubbles is in the form of a sheet with thickness smaller than the length scale of the turbulent motion. Dipole radiation is also considered, and found to be negligible whenever monopole sources are present. In the case of a dusty gas, only dipole and quadrupole sources are present, and here it is shown that the dipole radiation is equivalent to an increase in the usual quadrupole radiation. The increase depends upon the mass concentration of dust, and is significant for mass concentrations in excess of unity.

1. Introduction

In this paper we consider the sound radiation from a finite region of turbulent or unsteady flow, in which the fluid consists of a mixture of two phases. For the most part attention is confined to the case of a small volume concentration of air bubbles in water, though the case of a gas containing small dust particles is also examined briefly. Much work has been done in the past on the radiation from a single air bubble in water (e.g. Strasberg, 1956) when various forms of excitation are responsible for the motion of the bubble. The bulk properties of a distribution of bubbles in water have also been studied, in particular the well-known drastic reduction of the sound speed caused by even a very small concentration of bubbles, and the variation of the sound speed with frequency. A review of these, and many other effects is given by Batchelor (1967). Much less has been done on the excitation of a single bubble, or a distribution of bubbles, by a turbulent pressure field. This problem is discussed here on the lines of the Lighthill (1952) theory of aerodynamic sound generation.

A Lighthill inhomogeneous wave equation is first derived, in which the action of the bubbles on the water is represented by an equivalent distribution of monopole and dipole sources, in addition to the quadrupoles acoustically equivalent to the fluctuating flow. When no boundaries are present in the flow, the acoustic power output can be found in terms of the source strengths by the usual formulae. In order to estimate the monopole source strength, the response of a single bubble in turbulent flow is then considered using familiar equations. The pressure spectrum of a turbulent flow is relatively broad, and there is the possibility that large changes in bubble volume may arise from the small spectral intensity of the pressure at the high natural resonance frequency of the bubble. This would be a difficult effect to estimate reliably, for although it is possible to give an analytical form for the pressure spectrum at high frequencies using the Kolmogorov theory of the fine-scale structure of turbulence (Batchelor, 1953), the resonance would be limited only by dissipative effects whose character is not yet properly understood. In particular, the radiation damping of a bubble at resonance when surrounded by a distribution of bubbles is difficult to analyse, since the sound speed at high frequency in the distribution is complex, and varies with frequency. A detailed consideration of these effects is, fortunately, not necessary here, for the possibility of significant resonant response under excitation by turbulence alone is ruled out (§5). The reason for this is that the length scale over which the pressure field remains coherent at the resonance frequency is found to be very small compared with the bubble radius. The phase of the pressure field then varies rapidly over the bubble surface, whereas significant volume response requires the pressure to be substantially in phase all over the surface.

For this reason, the extension given by Curle (1955) to the Lighthill theory, taking account of the effect of surfaces in the flow, is not considered. The only way in which the presence of surfaces can alter the inferences to be made about the effect of bubbles on the radiated noise, is by introducing the possibility of coherent forcing, at the resonance frequency, over length scales large compared with the bubble radius. If the behaviour of the surface is controlled entirely by the turbulent flow, this possibility is again ruled out, since the length scales of the forcing due to the surface would be of the order of those in the turbulent flow itself. If, however, the motion of the surface were controlled by some external means, we could have the possibility of coherent forcing at the resonance frequency. This is exactly what happens if, for example, the bubble is irradiated by a sound wave generated by motion of a surface (Hunter, 1967). Even then, this does not necessarily mean that resonant response is significant, in view of the high dissipation occurring in a distribution of bubbles at the resonant frequency. If such cases, in which the control of the surface behaviour by external means provides a length scale large compared with the bubble radius, are excluded, we can entirely discount the resonant response of the bubbles, and no further attention need be paid to the effect of surfaces.

Certain effects of two-phase flow are obvious, and will receive no more attention in this paper. These concern surface and volume sources in an infinite region of bubbly fluid in which the sound speed c_m is significantly lower than the sound speed c_α in pure water. According to the usual ideas of aerodynamic noise theory, the intensities of monopoles, dipole and quadrupole sources vary as c^{-1}, c^{-3} and c^{-5}, where c is the sound speed in the far-field of the sources. Therefore in this case, the power output of these sources will be increased by the factors c_α/c_m, $(c_\alpha/c_m)^3$ and $(c_\alpha/c_m)^5$ respectively, over their values for emission into pure water. However, in practice this case never arises, and one is usually concerned with situations in which the bubbly liquid occupies a region with typical dimension small compared with a sound wave length in pure water. The theory is therefore set up in a form capable of handling these cases where the fluid mixture is inhomogeneous on scales smaller than a wavelength. In this way, changes in the turbulence-generated sound are attributed to a distribution of acoustic sources, whereas the increases noted above for the infinite bubbly region are essentially connected with sound propagation over distances of many wavelengths. The physical bases for the results in the two cases are thus quite different.

In the formulation given here, monopole sources of sound arise from the forced response of the bubbles at the frequency characteristic of the turbulence. They lead to an efficiency proportional to the fifth power of Mach number, which is the variation usually ascribed to quadrupole sources. In fact it is shown that the monopole intensity is just that of the usual Lighthill quadrupoles, but augmented by the factor $(c_\alpha/c_m)^4$, which should be contrasted with the $(c_\alpha/c_m)^5$ factor referred to previously. c_α/c_m can easily exceed 10, so that the presence of bubbles in a turbulent flow will very greatly increase the acoustic power output. For the extreme case of a 10% concentration of bubbles by volume the acoustic power may be increased by about 70 dB.

Apart from effects arising from the variation of bubble volume, there is the question of whether abrupt changes in the mean concentration can produce appreciable sound. The sources corresponding to a discontinuous rise in concentration are examined in §6, where it is shown that the radiated field can be expressed in terms of a surface distribution over the interface across which the concentration changes. The radiation produced is shown to be equal to that produced by distributed sources in a volume which has one typical dimension equal to the turbulence length scale.

Dipole sources of sound arising from bubble response are also considered. As expected, they are much less efficient than the monopoles at the very low Mach numbers typical in underwater applications. The case of a dusty gas is then dealt with, in which monopole radiation cannot occur, and in which the action of the dust particles on the gas is represented entirely by a dipole distribution. Again it is shown that the presence of dust is to augment the usual quadrupole radiation. The increase in power output is less startling than that caused by bubbles, but is appreciable when the mass concentration of dust exceeds unity.

2. Lighthill equation for flow of air bubbles in water.

We consider a finite region in which unsteady or turbulent flow occurs, and in which the fluid is a mixture of water (α - phase), and a small concentration by volume of gas bubbles (β - phase). The small quantity $\beta(\underline{x},t)$ is the fraction of unit volume of the mixture which is occupied by the bubbles. ρ^{α}, ρ^{β} are the actual densities of the two phases, i.e. ρ^{α} = (mass of α -phase) / (volume occupied by α - phase). The mass of α - phase in unit volume of mixture is then $(1 - \beta)\,\rho^{\alpha}$, and the total mass per unit volume is $(1 - \beta)\,\rho^{\alpha} + \beta\rho^{\beta}$. Far from the turbulent region $\beta = 0$, and the fluid is entirely α - phase, at rest apart from small velocities induced by the passage of sound waves from the turbulence.

We choose to formulate a Lighthill equation for the density ρ^{α}. This has the advantage of displaying clearly the action of one phase on the other in terms of acoustic sources with a simple physical interpretation. In particular, monopole and dipole distributions appear, representing the effects of mass and momentum injection into the α -phase resulting from the motion in the β - phase. The same kinds of sources appear if we consider the density $(1 - \beta)\,\rho^{\alpha}$ instead of just ρ^{α} , but their interpretation is not quite so simple, and they are less easy to calculate. The alternative is to regard the fluid as a mixture, with density $\rho = (1 - \beta)\,\rho^{\alpha} + \beta\rho^{\beta}$. In this case, a conventional Lighthill equation can be derived, involving quadrupole sources only. The physical interpretation is then largely lost, and the task of relating the quadrupole strength to the flow and phase parameters is difficult, as so much is hidden, for example, in the term $p - c_{\alpha}^{2}\rho$.

We are assuming the concentration β, and the bubble radius a to be so small that meaningful values can be attached to the velocity and stress in the α - phase at all points $(\underline{x},t)$. Let u_i^α denote the velocity in the α - phase. Mass conservation for this phase is expressed by

$$\frac{\partial}{\partial t}(1 - \beta)\rho^\alpha + \frac{\partial}{\partial x_j}(1 - \beta)\,\rho^\alpha\, u_j^\alpha = 0,$$

which we write in the form

$$(2.1) \qquad \frac{\partial}{\partial t}\rho^\alpha + \frac{\partial}{\partial x_j}\rho^\alpha\, u_j^\alpha = Q.$$

Here $$Q = -\rho^\alpha\left(\frac{\partial}{\partial t} + u_j^\alpha \frac{\partial}{\partial x_j}\right)\ln(1 - \beta)$$

$$= -\rho^\alpha \frac{D_\alpha}{Dt}\ln(1 - \beta)$$

is the effective rate of mass injection density into phase α. If F_i denotes the interphase force density, the momentum equation for phase α reads

$$\frac{\partial}{\partial t}(1 - \beta)\rho^\alpha\, u_i^\alpha + \frac{\partial}{\partial x_j}\{(1 - \beta)\,\rho^\alpha\, u_i^\alpha\, u_j^\alpha + p_{ij}\} = F_i.$$

p_{ij} is the stress tensor, and is composed partly of stresses set up by the eddy motion in the α - phase, and partly of stresses set up by the response of the β - phase to the fluctuating eddy pressures. For the present there is no need to attempt to specify F_i further. We rewrite the momentum equation in the form

$$(2.2) \qquad \frac{\partial}{\partial t}\rho^\alpha\, u_i^\alpha + \frac{\partial}{\partial x_j}\{(1 - \beta)\,\rho^\alpha\, u_i^\alpha\, u_j^\alpha + p_{ij}\} = G_i,$$

where $$G_i = F_i + G_i', \quad G_i' = \frac{\partial}{\partial t}\beta\rho^\alpha\, u_i^\alpha.$$

By cross-differentiation of (2.1) and (2.2) we get the required Lighthill equation, provided we note that far from the turbulent region this equation must reduce to the homogeneous wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c_\alpha^2 \nabla^2\right)\rho^\alpha = 0,$$

where c_α is the sound speed in pure α - phase. This gives

$$(2.3) \quad \left(\frac{\partial^2}{\partial t^2} - c_\alpha^2 \nabla^2\right) \rho^\alpha = \frac{\partial Q}{\partial t} - \frac{\partial G_i}{\partial x_i} + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} ,$$

in which $T_{ij} = (1 - \beta)\, \rho^\alpha u_i^\alpha u_j^\alpha + p_{ij} - c_\alpha^2 \rho^\alpha \delta_{ij}$.

The process of sound generation by the turbulent flow is accomplished by three distinct mechanisms. Firstly, by a distribution of monopoles, of strength Q, equal to the rate of mass injection into the α - phase. Secondly, by a distribution of dipoles, of strength G_i. G_i is the effective force on the α - fluid, composed in part of the interphase force F_i, and in part of the term G_i'. The latter represents the momentum defect arising from the fact that a fraction β of the total volume is not occupied by α - phase. Finally, we have a distribution of quadrupoles of the Lighthill type, of strength T_{ij}. As usual, T_{ij} is dominated by the Reynolds stress terms, since by the definition of c_α, the fluctuations in p and $c_\alpha^2 \rho^\alpha$ cancel, approximately. Viscous contributions to p_{ij} are neglected here, just as usual. In general it is quite adequate, for the order of magnitude arguments to be used later, to approximate T_{ij} by $\rho_o^\alpha u_i^\alpha u_j^\alpha$, where the zero suffix implies an average value.

The Mach number is typical underwater applications of flow noise theory is extremely small when based on c_α (10^{-2} at most), and the usual arguments would therefore indicate that monopole sources overwhelm the dipoles, while these in turn are very much more efficient than the quadrupoles. However, in the present problem we have a great range of new parameters - for example, the radius and resonance frequency of the bubbles, the strength of the interphase force, the relaxation time for response of the bubbles to the α - motion, and the concentration β . The usual rank ordering of acoustic sources may therefore only be valid for certain restricted ranges of the above parameters. It is the object of subsequent sections to determine how the efficiency of each type of source varies with these parameters, as well as with the parameters (length and velocity scales) of the turbulent motion.

3. Volumentric Response of a Bubble to a Fluctuating Pressure Field.

In this section, we consider the volumetric response of a single bubble, immersed in infinite compressible fluid, when a fluctuating pressure field is set up in the fluid. The pressure will be regarded as uniform in space far from the bubble, though fluctuating in time. A real pressure field, with finite length scale, will behave in this way provided the bubble diameter is small compared with the length scale of pressure variation. Viscous forces and thermal diffusion effects will be neglected, with the consequence that radiation damping is the only form of dissipation which limits the response of the bubble at its resonance frequency. It will be seen in §5 that resonant response cannot occur, and therefore that the validity of this assumption is only an academic matter for our purposes.

The object is to determine the variation of bubble volume, and of the pressure scattered by the bubble, with the imposed pressure variation. The equations governing the response are well-known (see, e.g. Strasberg, 1956), so that only a brief derivation need be given here. In the undisturbed state, the bubble has internal pressure $\tilde{\omega}$ and radius a, and is surrounded by infinite fluid of density ρ_o, pressure P, and sound speed c. A pressure fluctuation p(t) is then set up uniformly in space at infinity, the bubble pressure is $p_b(t)$ and the radius R(t). T denotes the surface tension, $p_s(r,t)$ is the pressure induced by bubble response. Spherical symmetry is assumed, as it is known (Strasberg, 1956) that negligible acoustic power is contained in any mode of oscillation of the bubble other than the symmetric expansion mode.

For the pressure drop across the bubble surface we have

(3.1) $\tilde{\omega} = \frac{2T}{a} + P$, and

(3.2) $p_b = \frac{2T}{R} + P + p + p_s$ at $r = R$.

If adiabatic changes are assumed in the bubble,

(3.3) $p_b R^{3\gamma} = \tilde{\omega} a^{3\gamma}$.

There is evidence to suggest that in general changes are isothermal, so that $\gamma = 1$ effectively. This is particularly likely to be true in the circumstances when the characteristic frequency of p is small compared with the bubble resonance frequency, in which case a slow forced motion of the bubble occurs. At higher frequencies, however, changes are more likely to be adiabatic, and for this reason γ is retained.

The scattered pressure p_s is a solution of the homogeneous wave equation, vanishing at $r = \infty$. Thus

$$p_s(r,t) = \frac{F(t - r/c)}{r} \quad \text{say, so that}$$

(3.4) $$-\frac{\partial p_s}{\partial r} = \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{r}\right) p_s .$$

The gradient of p_s at $r = R$ is related to the bubble radius by the linearised equation of fluid motion,

(3.5) $$-\frac{\partial p_s}{\partial r} = \rho_o \frac{\partial q}{\partial t} = \rho_o \frac{\partial^2 R}{\partial t^2} \quad \text{at } r = R.$$

Write $R' = R - a$, and linearise (3.1) - (3.5), supposing that $|R'| << a$. Defining a resonance frequency ω_o by

$$\omega_o^2 = \frac{1}{a\rho_o}\{(3\gamma - 1)\frac{2T}{a^2} \quad \frac{3\gamma P}{a}\} ,$$

we find

$$\text{(3.7)} \qquad LR' = -\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{a}\right)\left(\frac{p}{\rho_o}\right) \text{ , in which}$$

$$L \equiv \frac{\partial^2}{\partial t^2} + \frac{a\omega_o^2}{c}\frac{\partial}{\partial t} + \omega_o^2 .$$

We find also that $R' = -\frac{1}{a\rho_o\omega_o^2}\{p + p_s(a,t)\}$,

and this gives

$$\text{(3.8)} \qquad L\, p_s(a,t) = -\frac{\partial^2 p}{\partial t^2} .$$

From (3.7) we can now find an equation for the fluctuating concentration β , in the case when we have N bubbles, each of mean radius a, in unit volume of fluid. For $\frac{\partial\beta}{\partial t} = 4\pi a^2 N \frac{\partial R}{\partial t} = \frac{3}{a}\beta_o \frac{\partial R}{\partial t}$

in linearised form, where β_o is the time-average of β . This gives

$$\text{(3.9)} \qquad L\beta = -\frac{3\beta_o}{a}\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{a}\right)\left(\frac{p}{\rho_o}\right) .$$

In this equation, ρ_o may still be taken as the density of the fluid surrounding each bubble (i.e. the α - fluid) rather than of the mixture, for a small mean concentration cannot significantly alter the density when $\rho^\beta << \rho^\alpha$. p must apparently now include not only the forcing pressure set up at infinity, but also the resultant of all the scattered pressures set up by the distribution of bubbles. Just how much p is modified by these scattered pressures is an important point, which will receive further attention.

It will be seen later that in general the bubbles may respond significantly around two very distinct frequencies, one the resonance frequency ω_0, the other a frequency characteristic of the turbulent motion. The terms involving c in (3.9) will be found to be negligible for the forced motion at the turbulence frequency whichever of c_m, c_α is used. The problem of which value of c is relevant to (3.9) only arises in the case of resonant response, which we shall see is impossible when the bubbles are excited by nearly incompressible turbulence.

4. The Sound Field from Forced Bubble Motion.

We assume for the moment that the pressure field $p(t)$ generating the bubble motion is that of a turbulent flow whose internal dynamics may be regarded as nearly incompressible. Let ℓ_o denote a correlation scale for the turbulent flow, U the mean flow velocity and u_o the r.m.s. turbulent velocity. The dominant frequency of the pressure field, measured in a fixed frame, is then of order U/ℓ_o, and this is certainly an upper limit for the typical frequency of the field $p(t)$ experienced by the bubbles. Bubbles are convected with a speed of order U, and the frequencies observed following the mean flow are generally smaller than those observed at a fixed point by a factor $\sigma = u_o/U$. The dominant frequency of $p(t)$ may therefore be taken as of order u_o/ℓ_o. Applications of flow noise theory to underwater situations commonly involve values of U of order 30 ft/sec, while $a\omega_o$ is roughly 60 ft/sec in the case of air bubbles in water at one atmosphere static pressure P. $\sigma = 5 \times 10^{-2}$ is perhaps typical, and also $a << \ell_o$, for a bubble of radius comparable with ℓ_o could not withstand the high shear across it. It follows that $u_o/\ell_o << \omega_o$, and we have a situation in which there is strong forcing but small response at the turbulence frequency, while at the much higher frequency ω_o the pressure field has relatively little spectral intensity, but the bubbles have a strong intrinsic response. The response spectrum for the bubble motion therefore has two distinct peaks, near u_o/ℓ_o and near ω_o, corresponding to forced and resonant oscillations respectively. If the resonance peak is sufficiently narrow, we may take the two effects separately, and add them in mean-square, a conclusion which can be investigated in detail if a definite analytical form for the pressure spectrum is assumed.

For the forced motion, we neglect $\partial^2/\partial t^2$ compared with ω_o^2. The terms involving c in (3.9) can both be neglected, for they are smaller than those retained by a factor of order $u_o a/c\ell_o$. This factor is extremely small even with $c = c_m$, the mixture sound speed, for c_m certainly never drops below the typical mean velocity U of order 30 ft/sec. The terms involving c represent radiation damping, and are important only in controlling the resonant response. We have then, simply

$$\beta = -\frac{3\beta_o}{(a\omega_o)^2}\left(\frac{p}{\rho_o}\right). \tag{4.1}$$

Before using this equation in the Lighthill equation (2.3), we must first justify the assumption that the pressure field forcing any particular bubble is dominated by the eddy motion pressure. Now the mean square pressure scattered by a distribution of bubbles to any point in the distribution is size-dependent, and in fact varies linearly with the typical dimension L of the turbulent bubbly region. Thus, if L is large enough the scattered pressures would appear to dominate the pressure field experienced by any bubble. However, this dependence upon size L is largely irrelevant to the problem of sound generation to distances large compared with L. The pressure reaching a bubble from bubbles further away then a wavelength λ , approximately, is a radiating sound field pressure, and its action on the bubble is exactly that of ordinary sound waves on the bubble. The bubble is essentially passive in its response, and absorbs energy, if anything. Scattering of the incoming sound field results, with a directional redistribution, and a decrease in the acoustic energy flux. The waves scattered draw their energy from the primary wave, and energies in the acoustic mode cannot be increased by the scattering. Compare Lighthill (1953), where the sound waves scattered by the interaction of a primary sound wave with turbulance draw their energy from the primary wave, and not from the turbulence. We can therefore reject the scattered pressures reaching a particular bubble, provided they originate at distances greater than λ from the bubble. That bubble can, however, scatter the near-field of any other bubble within reach (Hunter, 1967), so that modifications to p from scattered pressures originating at distances less than about a wavelength λ must be considered. Whether these modifications are significant or not is now independent of the size L of the bubbly region.

This idea has important consequences for the Lighthill (1952, 1954) theory of aerodynamic noise. A turbulent eddy radiates sound waves, with a $1/r$ variation of pressure and velocity at distances greater than a wavelength. Consequently, the mean square acoustic pressure at any point in the turbulent region increases linearly with the scale L of the region, at any rate until viscous effects limit the otherwise unbounded increase which would occur in the "compressible homogeneous turbulence" limit $L \to \infty$ (see Lighthill, 1955). When L is large, but finite, one might expect these acoustic quantities to provide a significant change in the acoustic stress tensor T_{ij}, so that the sound power output from the flow might be increased. In view of the discussion above, we see that the apparent dependence of T_{ij} upon L is irrelevant to the sound generation problem. Near-field corrections to T_{ij} may be important, as an eddy can scatter the near-field of its neighbours into sound - but these corrections really should be discussed whether or not L is very large compared with ℓ_0 or λ. The outcome of this argument appears to be that the Lighthill theory for low Mach number flows is adequate for the description of sound emission from large volumes of turbulence ($L >> \lambda$) to just the same extent that it is adequate in the case $\lambda > L \gtrsim \ell_0$.

Returning now to the question of two-phase flow, we calculate the near-field correction to p by integrating the scattered pressure of a single bubble over the distribution of bubbles occupying a sphere of radius λ about any point in the turbulent bubbly region. The wavelength is that appropriate to propagation at frequency u_o/ℓ_o and at speed c_m, the low frequency sound speed in the mixture. This will be true when $L >> \lambda$, for then the time L/c_m for propagation at speed c_m across the distance L is large compared with the time-scale ℓ_o/u_o of the source, and therefore the source radiates effectively into an <u>infinite</u> medium with speed c_m. On the other hand, if $L \lesssim \lambda$, the integration of the scattered pressures must run only over a sphere of radius L. The greatest modification of the pressure field then corresponds to the case $L >> \lambda$, and then we have $\lambda >> \ell_o >> a$ - for the minimum value of c_m we shall be concerned with is 100 ft/sec, corresponding to a concentration $\beta_o = 10^{-1}$ (see Batchelor, 1967). The integration procedure is therefore relevant on two counts. In the first place, the near-field of radius λ is large enough for a continuous distribution of bubbles to be relevant, and in the second, the near-field is so extensive that it contains many eddy volumes ℓ_o^3. This allows us to replace each eddy by a point source of strength proportional to the eddy volume ℓ_o^3.

The calculation is done at the end of this section, with the result

$$\langle p_s^2\rangle \;/\; \langle p^2\rangle \sim \frac{9\beta_o^2}{4\pi}\left(\frac{c_m}{u_o}\right)\left(\frac{u_o}{a\omega_o}\right)^4$$

The brackets < > denote average values, all quantities being assumed stationary random functions of time. With the typical values $\beta_o = 10^{-1}$, $c_m = 100$ ft/sec, $u_o/U = 5 \times 10^{-2}$, $U = 30$ ft/sec, $a\omega_o = 60$ ft/sec, which would seem to give the maximum value of $<p_s^2>$ likely to occur in any practical situation, this gives

$$<p_s^2> / <p^2> \sim 2 \times 10^{-7} .$$

Therefore it is quite adequate, for the forced motion, to assume that the pressure forcing any particular bubble is that generated by the turbulent motion alone.

We now require an estimate of the acoustic power output P_m from the region containing bubbles, whose volume is of order L^3, arising from the monopole term $\partial Q/\partial t$ in (2.3). The contribution from the forced mode only is considered here. P_m is given by

$$(4.2) \qquad P_m \sim \frac{1}{4\pi \rho_o c_\alpha} < \left(\frac{\partial Q}{\partial t}\right)^2 > \ell_o^3 L^3,$$

where $\rho_o = \rho_o^\alpha$ is the density in the very distant field. This expression has been obtained from the usual retarded-potential solution

$$(4.3) \qquad (\rho^\alpha - \rho_o)(\underline{x}, t) = \frac{1}{4\pi c_\alpha^2} \int_{\underline{y}} \frac{\partial Q}{\partial t}\left(\underline{y}, t - \frac{|\underline{x}-\underline{y}|}{c_\alpha}\right)\frac{d\underline{y}}{|\underline{x}-\underline{y}|} ,$$

on the understanding that differences in retarded-time (of order ℓ_o/c_α) corresponding to points separated by less than an eddy scale ℓ_o are negligible compared with the time-scale ℓ_o/u_o of the source field in the forced mode. This is evidently well satisfied, since the fluctuation Mach number u_o/c_α is always exceedingly small. We can express this by saying that the source field is "acoustically compact" as far as the forced mode is concerned.

Equation (4.2) is valid only if the turbulent region has typical dimension at least of order ℓ_o in all directions. It is useful to write down also the power output P_Δ obtained from (4.3) when the radiating volume has the form of a sheet of area L^2 ($L \gtrsim \ell_o$) and thickness $\Delta << \ell_o$.

$$(4.4) \qquad P_\Delta \sim \frac{1}{4\pi \rho_o c_\alpha} < (\frac{\partial Q}{\partial t})^2 > \Delta^2 \ell_o^2 L^2.$$

For the moment, however, we consider only equation (4.2).

Since β is small compared with unity, we have from (4.1),

$$(4.5) \qquad Q = - \rho^\alpha \frac{D_\alpha}{Dt} \ell n(1 - \beta) \approx \rho_o \frac{D_\alpha}{Dt} \beta = - \frac{3\beta_o}{(a\omega_o)^2} \frac{D_\alpha}{Dt} p ,$$

as we are interested at present in the sound generated by fluctuations in β , rather than that generated by variations in space of the mean concentration β_o . As already discussed, p will be taken as the fluctuation is the eddy motion pressure. The magnitude of p will be estimated as the typical fluctuation in $\frac{1}{2}\rho_o \underline{u}^2$, i.e. $p \sim \rho_o \sigma U^2$, where $\sigma = u_o/U$ is the relative turbulence level. The time differentiation D_α/Dt will be represented by the frequency multiplication u_o/ℓ_o. This is also the relevant estimate of the operation $\partial/\partial t$, although superficially one might expect $\partial/\partial t \sim U/\ell_o$. We can see this in two ways. If the derivative $\partial/\partial t$ is written as the sum of a total derivative D/Dt and a convective derivative, the total derivative is equivalent to the multiplicative operation u_o/ℓ_o, while the convective term can be shown to represent an acoustic source of essentially lower efficiency. Alternatively, transform to a frame of reference which is convected with the mean flow. In this frame the operation $\partial/\partial t$ is certainly equivalent to multiplication by u_o/ℓ_o, while other changes resulting from the transformation are negligible if the mean flow Mach number U/c_α is small. Either way, we see that only the true turbulence frequencies contribute to the acoustic power output, and that for acoustic purposes all time differentiations are equivalent to multiplication by u_o/ℓ_o. This point is emphasised by Lighthill (1954).

With these estimates, and with neglect of convective effects, except insofar as they determine the relevant frequencies, we find that

$$(4.6) \qquad P_m \sim \frac{9\beta_o^{\,2}}{4\pi} (\rho_o \, \sigma \, U^3 \, L^2) \, (\sigma M)^5 \left(\frac{c_\alpha}{a\omega_o}\right)^4 \left(\frac{L}{\ell_o}\right) ,$$

where $M = U/c_\alpha$. An efficiency can be conveniently defined by comparing P_m with the rate of working of the fluctuating pressure $\rho_o \, \sigma \, U^2$ against the mean flow U over an area L^2.

$$(4.7) \qquad \eta_m \sim \frac{9\beta_o^{\,2}}{4\pi} (\sigma M)^5 \left(\frac{c_\alpha}{a\omega_o}\right)^4 \left(\frac{L}{\ell_o}\right) .$$

The dependence of η_m on M^5 is rather surprising, being characteristic of quadrupole rather than monopole sources. It is less surprising if we remember (§2) that it was noted that the whole problem could be tackled using a quadrupole type of source only. The monopole Q is equivalent, in part, to $\partial(p - c_\alpha^2 \, \rho)/\partial t$, a quadrupole time-derivative which would occur in this alternative treatment, p and ρ now both referring to the two-phase mixture. Evidently the two forms both yield the same dependence upon M.

We have already noted that changes of volume of the bubbles are likely to take place at constant temperature when the frequency is small compared with ω_o. Thus $\gamma = 1$ effectively, and then by (3.6),

$$(a\omega_o)^2 = \left(3P + \frac{4T}{a}\right) / \rho_o .$$

Now when β_o is neither too small nor too close to unity, Batchelor's (1967) expression for the isothermal sound speed c_m at low frequencies can be written

(4.8) $$c_m^2 = (P + \frac{4T}{3a}) \,/\, \beta_o \rho_o \quad .$$

and therefore we have the following simple relation between sound speed and resonance frequency,

(4.9) $$(a\omega_o)^2 = 3\beta_o \, c_m^2$$

(4.7) can then be written in the form

(4.10) $$\eta_m \sim \frac{1}{4\pi} (\sigma M)^5 \left(\frac{c_\alpha}{c_m}\right)^4 \left(\frac{L}{\ell_o}\right) .$$

Except for the factor $(c_\alpha/c_m)^4$, this is exactly the radiation efficiency of a typical turbulence quadrupole of strength $T_{ij} \sim \rho_o \sigma U^2$. Note that the operation $\partial^2/\partial t^2$ on T_{ij} must be represented here by multiplication by u_o^2/ℓ_o^2 ; the reasons are exactly those referred to earlier. Thus the effect of bubbles in the turbulence is to increase the acoustic power output by the factor $(c_\alpha/c_m)^4$. This increase is extremely large; in fact $(c_\alpha/c_m)^4$ is of order 10^5 even when β_o is as small as 10^{-2}, while for the maximum concentration $\beta_o = 10^{-1}$ which can reasonably be encompassed by the theory, $(c_\alpha/c_m)^4$ is of order 10^7 . The acoustic power output of a flow may therefore be increased by up to 70 dB by the monopole radiation of bubbles at the turbulence frequency.

To close this section, note that the pressure p_s induced by the monopole source $\partial Q/\partial t$ at any point in the turbulent bubbly region is given by

$$p_s = \frac{1}{4\pi} \int_V \left[\frac{\partial Q}{\partial t}\right] \frac{d\underset{\sim}{y}}{r}$$

where V is the turbulent volume, and the square brackets imply evaluation at retarded-time, as in (4.3). When $u_o/c_m \ll 1$, and when the volume $V \sim L^3$ is large enough to contain many eddy volumes ℓ_o^3 , this gives

$$(4.11) \qquad \langle p_s^2 \rangle \sim \frac{1}{16\pi^2} \langle (\frac{\partial Q}{\partial t})^2 \rangle \ell_o^3 \int_V \frac{dV}{r^2}$$

$$\sim \frac{1}{4\pi} \langle (\frac{\partial Q}{\partial t})^2 \rangle \ell_o^3 L.$$

Thus, as claimed earlier, the mean square scattered pressure increases linearly with L. However, it was explained previously that if we wish to consider the sound generation problem only, the volume integration need run only over a sphere of radius λ centred on the point considered. Hence

$$\langle p_s^2 \rangle \sim \frac{1}{4\pi} \langle (\frac{\partial Q}{\partial t})^2 \rangle \ell_o^3 \lambda ,$$

and with the estimate of $\partial Q/\partial t$ made above, we quickly find the value of $\langle p_s^2 \rangle$ quoted earlier in this section.

5. Resonant Response of Bubbles

We have noted in the previous section that appreciable monopole radiation may result from the resonant response of bubbles to the small spectral density of the pressure field at the frequency ω_o. This, however, is a possibility which cannot occur when the applied pressure field p is that due to turbulent motion in a nearly incompressible fluid. The essential reason is that the turbulent pressure field cannot remain coherent in space, at the high frequency ω_o, over length scales as large as a bubble radius a. The spherically symmetric mode of oscillation of the bubble, which is the only mode which can give rise to volume change and so to monopole radiation, cannot then occur, for it can be created only when the pressure field has nearly the same phase at all points on the bubble surface.

The effective length scale for the turbulent field at frequency ω_o can be found by the following argument. The bubbles travel with a translational velocity which must be comparable with the mean velocity U. Relative to the mean flow, the bubbles have fluctuating velocities which are certainly of the order of the turbulence velocity u_o in the α-phase. The pressure fluctuations experienced by the bubbles will therefore be similar to those observed at a point following the mean flow. Now the high frequency content of a field of turbulence, relative to the mean flow, occurs mainly through the convection of an almost frozen pattern of small spatial scales (i.e. small compared with ℓ_o) by the energy-containing eddies with characteristic velocity u_o. The length scale of the pressure fluctuations at frequency ω_o is therefore of the order of the length scale which, when convected by the large eddies at speed u_o gives rise to the frequency ω_o. This gives $\ell_r \sim u_o/\omega_o$ for the "correlation scale" at frequency ω_o, following the mean motion.

With the typical values U = 30 ft/sec, $a\omega_o$ = 60 ft/sec, and $u_o/U = 5 \times 10^{-2}$ we then have

$$\frac{\ell_r}{a} \sim 2.5 \times 10^{-2} .$$

ℓ_r is thus very much smaller than a, and the possibility of coherent forcing of the bubble over its entire surface is ruled out.

It might be thought that resonant response could arise if the pressure field contained an acoustic component at frequency ω_o, generated either by the turbulent eddies themselves or by their interaction with a surface in the flow. ℓ_r would then be of the order of a wave-length λ_o at frequency ω_o and at the mixture sound speed c_m at frequency ω_o. The low-frequency value of c_m is 100 ft/sec when $\beta_o = 10^{-1}$ (Batchelor, 1967), and so $\lambda_o/a \sim 10$ in this case. Coherent forcing at the resonance frequency may then be possible, but the possibility is marginal, since the speed c_m at frequency ω_o is much less than that at zero frequency. In any case, we can exclude the resonant response to small acoustic fields from the sound generation problem by the argument used in §4. The action of sound waves on the bubble results merely in a scattering of acoustic energy, and no increase in energy output can occur. This does not quite complete the argument, for near-field scattering can occur, as we have seen. However, the scale λ_o of the near-field in this case is very small, indeed comparable with the average separation between bubbles, so that we can probably ignore this effect - which if it occurs at all, will depend critically on how many bubbles are in the near field at any instant.

Since we have now shown the resonant motion not to be significant, the problems of the relevant value of c in (3.9), and whether the neglect of viscous and thermal damping is valid, are of no interest here. Resonance, and the dissipation which limits it, are two aspects of the problem which are irrelevant when incompressible turbulence provides the excitation.

6. Radiation due to Inhomogeneities in Mean Concentration.

In the previous sections, we have considered the radiation which arises when the concentration β fluctuates abouts its mean value because of the compressibility of the bubbles. We now ignore that aspect of the problem and consider the radiation which may result from rapid spatial variation of the mean concentration. Situations commonly arise in which the bubbles form intense clouds, in which the concentration is high, surrounded by more or less clear fluid. It is obviously of interest to see whether the unsteady convection and distortion of these clouds can produce an appreciable sound field.

The concentration can be expressed as the sum of mean and fluctuating parts, $\beta = \bar{\beta} + \beta'$. The part of the monopole source strength involving β' has been dealt with in the last sections, and here we consider the monopole

$$(6.1) \qquad \frac{\partial Q}{\partial t} \approx \rho_o \frac{\partial}{\partial t} \frac{D\alpha}{Dt} \bar{\beta} .$$

We shall model the cloud-water interface as a surface of discontinuity in $\bar{\beta}$ which is convected by the bubble velocity field u_i^{β}. The interface is taken as locally plane, so that we can write

$$(6.2) \qquad \bar{\beta} = \beta_o H(y_n - y_o) ,$$

where H denotes the Heaviside unit function β_o is the constant value of the mean concentration within the cloud, y_n the coordinate normal to the interface, $y_o(t)$ the y_n - coordinate of the interface at time t. We have

$$(6.3)\qquad \frac{D_\beta}{Dt}\,\bar{\beta} = \left(\frac{\partial}{\partial t} + u_j{}^\beta \frac{\partial}{\partial y_j}\right)\bar{\beta} = 0,$$

and $\dfrac{dy_o(t)}{dt} = u_n^\beta$.

This gives $\dfrac{D_\alpha}{Dt}\,\bar{\beta} = (u_i{}^\alpha - u_i{}^\beta)\,\dfrac{\partial\bar{\beta}}{\partial y_i}$,

and then the monopole in (6.1) can be conveniently combined with that part of the momentum defect dipole $G_i{}' = \rho_o\,\partial\beta\, u_i{}^\alpha/\partial t$ which contains $\bar{\beta}$, to yield

$$(6.4)\qquad \frac{\partial Q}{\partial t} - \frac{\partial G_i{}'}{\partial y_i} = -\rho_o \frac{\partial}{\partial y_i}\frac{\partial}{\partial t}\,(u_i{}^\beta\bar{\beta}) - \rho_o \frac{\partial}{\partial t}\left\{\bar{\beta}\,\frac{\partial}{\partial y_i}(u_i^\alpha - u_i{}^\beta)\right\} .$$

The first term in (6.4) represents a dipole field arising from the random distortion of the interface by the motion of the bubbles. This dipole term will be considered further below. For the monopole term in (6.4) we estimate the divergences of $u_i{}^\alpha$, $u_i{}^\beta$ from the equation (2.1) and from the analogous equation

$$(6.5)\qquad \frac{\partial}{\partial t}\,\beta\rho^\beta + \frac{\partial}{\partial y_i}\,\beta\rho^\beta\, u_y{}^\beta = 0,$$

expressing conservation of the mass of the β - phase. Neglecting small variations in ρ^α we have

$$\frac{\partial u_i{}^\alpha}{\partial y_i} \approx \frac{D_\alpha}{Dt}\,\beta\ ,\ \frac{\partial u_i{}^\beta}{\partial y_i} \approx -\frac{1}{\beta}\frac{D_\beta}{Dt}\,\beta\ ,$$

and the latter term dominates, since $\beta \ll 1$. Then

$$\bar{\beta} \frac{\partial}{\partial y_i} (u_i^{\alpha} - u_i^{\beta}) = \bar{\beta} \cdot \frac{1}{\beta_o} \frac{D_\beta}{Dt} \beta'$$

$$= - \frac{3\bar{\beta}}{(a\omega_o)^2} \frac{D_\beta}{Dt} p ,$$

where equation (4.1) has been used to relate the fluctuating concentration β' to the pressure p. The monopole term in (6.4) then becomes

$$(6.6) \qquad - \frac{3\beta_o}{(a\omega_o)^2} \left(\frac{D_\beta}{Dt} p\right) u_n^{\beta} \quad \delta(y_n - y_o)$$

$$+ \frac{3\beta_o}{(a\omega_o)^2} \left(\frac{\partial}{\partial t} \frac{D_\beta}{Dt} p\right) H(y_n - y_o).$$

The operations D_β/Dt and D_α/Dt on the pressure p are equivalent, to the degree of accuracy possible here. Comparing the second term of (6.6) with the value of $\partial Q/\partial t$ obtained from (4.5), we see that this term involving the H-function represents the monopole sources distributed throughout the interior of the cloud. On the other hand, the sources represented by the first term of (6.6) are confined to the interface between the cloud and the clear fluid outside it. The interface is equivalent to a distribution of surface sources, of strength

$$\frac{3\beta_o}{(a\omega_o)^2} \left(\frac{D_\beta}{Dt} p\right) u_n^{\beta}$$

$$\sim \frac{3\beta_o \, \rho_o \, u_o^3 \, U}{(a\omega_o)^2 \, \ell_o} \quad \text{per unit area.}$$

The resulting efficiency of these sources is found to be equal to that produced by the monopoles distributed in a sheet whose total area is that of the interface and whose thickness is just one eddy length ℓ_o.

Thus an area S of the interface produces the same power output as do the monopole sources distributed throughout a volume ℓ_o S. The surface effects are therefore extremely large when the typical dimension of the bubbly region is comparable with ℓ_o. If the region is in the form of a thin sheet of thickness $\Delta \lesssim \ell_o$, the surface sources will dominate the radiation field. In that case, equation (4.2) represents an overestimate of the volume induced sound, and equation (4.4) should then be used.

The dipole term in (6.4) can also be expressed in terms of surface and volume distributions. Taking the dominant surface source term, the radiated density field can be shown to be given by

$$(6.8) \qquad (\rho^{\alpha} - \rho_o)\ (\underline{x},t) = \frac{\rho_o\, \beta_o}{4\pi c_{\alpha}^{\ 3}}\ \left(\frac{x_i}{r^2}\right) \int_S \left[u_i^{\ \beta}\ u_n^{\ \beta}\ \frac{\partial u_n^{\beta}}{\partial y_n}\right] dS,$$

from which the radiation efficiency follows as

$$\eta_S \sim \frac{\beta_o^2}{4\pi}\ \sigma\ M^3 .$$

The ratio of this efficiency to that of the surface monopoles is of order $\frac{1}{9}\,\sigma^{-4}\ M^{-2}\ \left(\frac{a\omega_o}{c_{\alpha}}\right)^4$,

and this is slightly greater than unity when the typical values given in §4 are again used. Therefore this form of radiation is also important when the radiating volume is in the form of a sheet with thickness less than about ℓ_o. The dipole and monopole sound fields are comparable in this case essentially because the dipole field exists independently of the response of the bubbles, and would be produced even if the bubbles were rigid and could not respond. On the other hand, the monopole surface sound field depends almost entirely on the compliance of the bubbles, and the velocities induced by bubble response are small compared with those in the turbulence which provide the convection and distortion of the interface, and hence the dipole surface sound.

7. Dipole Sources of Sound.

The term $G_i' = \partial\beta\, \rho^\alpha u_i^\alpha/\partial t$ contains contributions other than those arising from changes in mean concentration. We have, identically,

$$G_i' = \frac{\rho^\alpha\, u_i^\alpha}{1-\beta}\frac{\partial\beta}{\partial t} + \frac{\beta}{1-\beta}\frac{\partial}{\partial t}(1-\beta)\rho^\alpha\, u_i^\alpha .$$

Assume that β is small compared with unity, and use the momentum equation (2.2) to transform the last term above. Apart from the interphase force, we have

$$(7.1)\quad G_i' = \rho^\alpha\, u_i^\alpha \frac{D_\alpha}{Dt}\beta + p_{ij}\frac{\partial\beta}{\partial x_j} - \frac{\partial}{\partial x_j}\{\beta\rho^\alpha\, u_i^\alpha\, u_j^\alpha + \beta p_{ij}\} .$$

The last term in (7.1) represents a quadrupole source, whose strength certainly vanishes in the far-field where $\beta = 0$. It therefore represents a basically less efficient source than do the other terms, and may be neglected. From the estimates $D_\alpha/Dt \sim u_o/\ell_o$, $\partial/\partial x_j \sim 1/\ell_o$, $p_{ij} \sim \rho_o\, \sigma\, U^2$ we see that the remaining terms are of the same order of magnitude. We use these estimates, with equation (4.1) to relate β to p, to obtain the dipole efficiency η_d due to volumetric response of the bubbles.

$$(7.2)\qquad \eta_d \sim \frac{9\beta_o^{\,2}}{4\pi}\sigma^5 M^7 \left(\frac{c_\alpha}{a\omega_o}\right)^4 \left(\frac{L}{\ell_o}\right)$$

$$= M^2\, \eta_m,$$

where η_m is the monopole efficiency given in (4.10). The factor M^2 ensures that this kind of radiation is negligible in all cases.

Neglect of the interphase force compared with the displaced momentum is certainly valid for the case of air bubbles in water, since the bubbles have appreciable volume but negligible mass. If, however, the density of the β-phase is large compared with that of the α-phase, the interphase force may be important. This happens in the case when the α-phase is a gas, and the β-phase a distribution of rigid dust particles. The volume concentration of dust particles is supposed negligible, though the mass concentration may be appreciable. We obtain the case of a dusty gas from our general equations by letting $\beta \to 0$, $\rho^{\beta} \to \infty$, so that the mass concentration $\beta\rho^{\beta}/\rho^{\alpha}$ has a finite limit, f say. The terms $\partial Q/\partial t$ and $\partial G_i'/\partial x_i$ now vanish identically, and the influence of the dust particles on the gas is contained entirely in the interphase force F_i.

Suppose that the dust particle number density is N, and that each particle has mass m, so that $f\rho^{\alpha} = Nm$. Saffman (1962) wrote down the equations of dusty gas flow, and assumed that the force density F_i was given by a linear Stokes law,

$$F_i = KN(u_i^{\beta} - u_i^{\alpha}). \tag{7.3}$$

u_i^{β} is the velocity of the β-phase at $(\underline{x},t)$, and K is a constant proportional to the viscosity of the α-phase and to the typical particle dimension. This viscous drag force is very much larger than any forces due to virtual inertia for the kinds of system envisaged by Saffman. We do not need the specific form (7.3) here, though it is useful in that it allows us to define a relaxation time for the dust particles as $\tau = m/K$. In most practical cases τ is small compared with the characteristic time of the gas motion, and when this is so, the dust particles follow the gas motion closely. The effect of the dust particles is then to increase the effective density of the mixture from ρ^{α} to $(1 + f)\rho^{\alpha}$ without change in the other variables. In particular, the sound speed c_m in the dusty gas is given by

(7.4) $$c_m^2 = c_\alpha^2/(1 + f_o),$$

where f_o denotes an average value of f. This result is true, irrespective of the validity of (7.3), provided only that a suitable relaxation time is small compared with the time-scale of the gas motion.

Now the momentum and mass conservation equations for the dust particles may be written (Saffman, 1962),

(7.5) $$\frac{\partial}{\partial t} f\rho^\alpha u_i^\beta + \frac{\partial}{\partial x_j} f\rho^\alpha u_i^\beta u_j^\beta = - F_i,$$

(7.6) $$\frac{\partial}{\partial t} f\rho^\alpha + \frac{\partial}{\partial x_i} f\rho^\alpha u_i^\beta = 0.$$

Using these equations, the dipole and quadrupole sources in equation (2.3) may be expressed as

(7.7) $$-\frac{\partial F_i}{\partial x_i} + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} = -\frac{\partial^2}{\partial t^2} f\rho^\alpha + \frac{\partial^2 W_{ij}}{\partial x_i \partial x_j},$$

in which

$$W_{ij} = \rho^\alpha u_i^\alpha u_j^\alpha + f\rho^\alpha u_i^\beta u_j^\beta + p_{ij} - c_\alpha^2 \rho^\alpha \delta_{ij}.$$

To regard the terms in (7.7) as a monopole and a quadrupole, respectively, would be an error. For the monopole term may be rewritten as

$$-\frac{\partial^2}{\partial t^2} (1 + f)\rho^\alpha + \frac{\partial^2}{\partial t^2} \rho^\alpha$$

$$= -\frac{1}{c_m^2} \frac{\partial^2 p}{\partial t^2} + \frac{1}{c_\alpha^2} \frac{\partial^2 p}{\partial t^2} = -\frac{f_o}{c_\alpha^2} \frac{\partial^2 p}{\partial t^2},$$

since, by definition, $\frac{1}{c_m^2} = \frac{\partial}{\partial p}(1 + f)\rho^\alpha$.

(The low relaxation time limit has been assumed). Therefore the monopole strength is $O(M^2)$, rather than $O(1)$, and this monopole is equivalent to an isotropic $O(1)$ quadrupole.

We can now estimate the efficiency corresponding to the two terms in (7.7), remembering that in the low relaxation time limit we have $u_i^\alpha \sim u_i^\beta$, and that ρ^α is increased to $(1 + f)\rho^\alpha$. This applies also to the factor $\rho_o \sigma U^3 L^2$ used to normalise the efficiency. We find that

(7.8.) $$\eta_f \sim \frac{1}{4\pi}(\sigma M)^5 f_o^2 (1 + f_o)\left(\frac{L}{\ell_o}\right),$$

(7.9) $$\eta_w \sim \frac{1}{4\pi}(\sigma M)^5 (1 + f_o)\left(\frac{L}{\ell_o}\right),$$

for the efficiencies corresponding to the first and second terms on the right of (7.7), respectively. When $f_o < 1$, $\eta_w > \eta_f$, and then in virtue of (7.4),

(7.10) $$\eta \sim \frac{1}{4\pi}(\sigma M)^5 \left(\frac{c_\alpha}{c_m}\right)^2 \left(\frac{L}{\ell_o}\right).$$

The radiation efficiency is increased by the factor $(c_\alpha/c_m)^2$ by the presence of the dust, and the radiated power is increased by the factor $(c_\alpha/c_m)^4$, exactly as in the case of a suspension of air bubbles in water. However the increases are negligible, in practical terms, when $f_o < 1$. When $f_o > 1$, we have

(7.11) $$\eta \sim \frac{1}{4\pi}(\sigma M)^5 \left(\frac{c_\alpha}{c_m}\right)^6 \left(\frac{L}{\ell_o}\right),$$

so that now the efficiency is increased by the sixth power of the sound speed ratio, and the power output by the eighth power. If the typical velocity U is the same for both a clean and a dusty gas, this increase in power output is large - up to about 20 dB perhaps, for mass concentrations f_o of the order of 2 or 3 which are common in many industrial processes where dusty gases are used to increase rates of heat transfer. In some cases, however, this comparison is not relevant. For example, if the mechanical power of the flow were the same for the clean and dusty gases, as might be the case in a jet-type flow, then

$$(1 + f_o)\, U^3 = U_o^3 \;,$$

where U, U_o are the values of the same typical velocity with and without the presence of dust, respectively. The increase in power output, according to (7.11) would then only be of order $f_o^{7/3}$, instead of f_o^{4}, but should still provide an effect which is easily detectable in practice.

8. Conclusions.

The radiation properties of turbulent flow in water have been shown to be greatly modified by the presence of a small distribution of air bubbles in the turbulence. In the model used here to describe this process, the effects of the bubbles have been represented as acoustically equivalent to volume distribution of monopoles and dipoles, in addition to the quadrupoles equivalent to the fluctuating stresses in the turbulence. Monopole radiation results from the low-frequency forced volumetric response of the bubbles to the turbulent pressure field. The effect of this radiation is in all cases equivalent to an increase in the quadrupole radiation (above its value in pure water) by the factor $(c_\alpha/c_m)^4$, where c_α, c_m are the sound speeds in pure water and in the bubbly region respectively. The acoustic power output of the flow is increased by 50 dB for a 1% air/water concentration, and by 70 dB for a 10% concentration. These may be regarded as relevant figures for many practical situations.

Significant volumetric response of the bubbles at their high natural resonance frequency has been shown to be impossible when the excitation is due to nearly incompressible turbulence alone. The reason is that the length scale over which the pressure field remains coherent at the resonance frequency is found to be very small compared with the bubble radius. The exclusion of resonant response indicates that the use of linear equations to represent the bubble response is justified.

Dipole radiation arises through the displacement of fluid momentum by the gas bubbles, and through the action of the force between bubbles and fluid. The momentum displacement effect is the dominant cause of dipole radiation, but the resulting efficiency is always negligible compared with that of the monopoles.

Monopole and dipole radiation occur through the unsteady convection of the interface between the bubble/water mixture and the clear fluid outside it. In this case the radiation is generated essentially by a distribution of sources over the interface. The monopole and dipole radiation efficiencies are comparable, and are important, compared with the volume-generated sound, if the radiating volume is in the form of a sheet with thickness equal to, or smaller than an eddy scale ℓ_0. If the thickness is equal to the eddy scale, which may be of the order of one foot in practical situations, then surface and volume monopole power outputs are equal, and either overwhelms the radiation which would occur if no bubbles are present.

Finally, in the case of a suspension of dust particles in a gas, no monopole sound can be produced. Dipole radiation occurs through the action of the force exerted by the dust on the gas, and it is shown that this form of radiation is equivalent to an amplification of the quadrupole sound which occurs in a clean gas. When the mass concentration of dust exceeds unity, this increase is large - up to about 20 dB perhaps, though not nearly as large as that provided by the presence of bubbles. Moreover, the presence of a large mass concentration of dust will substantially reduce the flow speeds if the flow is governed by a source of constant power. In that case, the quadrupole sound is enhanced, in intensity, by the factor (mass concentration $f_0)^{7/3}$ over its value in a clean gas under the same mechanical power. This would still indicate that the use of dust particles in many industrial processes will make a considerable contribution to the noise level.

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CHAPTER 4

PLATE VIBRATION INDUCED BY UNSTEADY PRESSURE FIELDS

PLATE VIBRATION INDUCED BY UNSTEADY PRESSURE FIELDS

1. Introduction

In this paper, a study is made of the vibration induced in an infinite thin plate when an unsteady pressure field acts over a finite region of the plate. The use of a Green's function is evidently required, and for this purpose the well-known Green's function (H. and L. Cremer, 1948) for the time-reduced plate equation is first obtained by Fourier analysis. The method given here involves the use of a radiation condition in a certain manner, and may easily be applied to other problems (e.g., surface gravity waves on water) without the difficulties usually encountered when an appeal to fictitious damping forces is made. The complete Green's function is then obtained by a frequency integration. This function does not seem to have been found before, although it has a very simple form. A new representation of mechanical dissipation in the plate is postulated, to remove some unphysical characteristics of the Green's function. This representation gives agreement with experimentally determined dissipation laws, and is more satisfactory - particularly in real space and time - than the usual representation of dissipation through a complex elastic constant.

The excitation of the plate by a random, statistically steady, pressure field is considered in §3. Calculations have been made in the past of the resulting vibration (e.g. Ffowcs Williams and Lyon, 1963), but in the main these have assumed the pressure field to be randomly, but uniformly, distributed over the infinite area occupied by the plate. The problem is then singular, in the sense that mean-square vibration emplitudes are limited only by small dissipative effects, or by non-linear effects, and may be unrealistic in the sense that the results are not relevant to practical cases unless the pressure acts over a very large area indeed. In the problem considered here, statistical homogeneity is not required, neither is any form of damping. The steady state is achieved as a balance between the power input from the pressure acting over a finite area, and the energy loss from that area in the form of free flexural waves in the plate propagating outward to infinity.

The general results obtained are applied in §4 to the case when the pressure field exciting the plate is that of a turbulent boundary layer in incompressible fluid. The form of the important vibration statistics can be found from a knowledge of the form of the pressure spectrum function near zero wavenumber and frequency, and for this knowledge general theoretical results are available. A dependence of the power input into the plate upon V^5 at low flow speeds V, and on V^3 at high speeds is found. These dimensional laws may be regarded as analogous to the well-known V^8 and V^3 laws for the acoustic power output of turbulent flow (Lighthill, 1952; Ffowcs Williams, 1963). In both problems, the efficiency of the wave-generating process becomes constant in the high-speed limit (and this is also true if we consider gravity waves on deep water instead of elastic waves in a plate).

The dispersive nature of the waves in the plate makes it very difficult to deal adequately with the effects of convection of the pressure field. A simple example illustrating convective effects is, however, considered, and this example also provides a demonstration of the use of the complete Green's function. The interesting result is found that, when a point force with slowly varying strength is convected across the plate at speed V, energy is propagated away from the force at speed V in the direction of propagation, and also at speed V radially outward from the force. Thus, as judged from the energy propagation vector (which is all that is available), the waves generated by the force travel at most at double the convection speed. The paper ends with a discussion of this result in terms of plate excitation by turbulent eddies. The claim is made that, for such excitation, the approach to plate vibration through a normal mode representation is inappropriate, and that it is preferable to regard a finite plate as effectively infinite and devoid of modal structure. Any possibility of reverberant build-up of a modal structure is precluded by the fact that the phase of the turbulent excitation at any point cannot remain even roughly constant over the time taken by the waves generated at that point to cross the plate and return again.

2. The Green's functions for the plate motion equation

The differential equation describing the motion of a thin homogeneous elastic plate has the well-known form (e.g. Cremer and Heckl, 1967),

$$\left(m \frac{\partial^2}{\partial t^2} + B \nabla^4\right) y = p \; . \tag{2.1}$$

Here y is the plate deflexion, p the total applied force per unit area. m is the mass of the plate per unit area, and B is the bending stiffness. We write λ for the quantity $\sqrt{B/m}$, which has the dimensions of kinematic diffusivity, and k_o for the wavenumber $\sqrt{|\omega|/\lambda}$ of free flexural waves of frequency ω in the plate. The effects of mechanical damping in the plate are neglected for the moment. This will be seen to be permissible in general unless the fluctuating pressure p on the plate is either highly concentrated over a very small region, or is distributed randomly, but uniformly, over a very large area.

The Green's function for (2.1) is essentially the solution of

$$\left(\frac{\partial^2}{\partial t^2} + \lambda^2 \nabla^4\right) y = \frac{1}{m} \delta(\underline{x})\delta(t) \; , \tag{2.2}$$

and we require a solution valid for an infinite plate, with no boundaries. The solution of (2.2) is of course not unique; solution by generalised Fourier analysis gives a particular solution in the form of a singular integral. The integration path may be deformed in several ways, each giving a different interpretation of the integral, and any two results differing only by a solution of the homogeneous equation. We shall here render the solution unique by the requirement that each element of $y(\underline{x},t)$ in a Fourier time-analysis of y shall represent entirely outgoing waves as $|\underline{x}| \to \infty$. Each frequency component of $y(\underline{x},t)$ then represents the physically realistic solution of (2.1) when the applied pressure is that due to an oscillatory point force,

$$p(\underline{x},t) \propto \delta(\underline{x}) \exp i\omega t .$$

Define generalised Fourier transforms by

$$y(\underline{x},t) = \int \tilde{y}(\underline{k},\omega) \exp i(\underline{k}.\underline{x} + \omega t)\, d\underline{k}\, d\omega ,$$
$$y(\underline{x},t) = \int \tilde{y}(\underline{x},\omega) \exp i\omega t\, d\omega . \qquad (2.3)$$

The $\underline{k}$-integration is over all wavenumbers in the plane of the plate, and the frequency integration is over $(-\infty, +\infty)$. Write $r = |\underline{x}|$, $k = |\underline{k}|$, $\underline{k}.\underline{x} = kr \cos\Theta$, so that $d\underline{k} = k dk d\Theta$. Also take $\omega > 0$, and let k_o be the positive root of $k_o^4 = \omega^2/\lambda^2$. Then we have from (2.2),

$$8\pi^3 B\, \tilde{y}(\underline{x},\omega) = \int \frac{e^{ikr\cos\Theta}}{k^4 - k_o^4}\, k\, dk\, d\Theta$$

$$= I + \frac{\pi}{2k_o^2} \int_o^\infty \frac{J_o(kr)}{k + k_o}\, dk - \frac{\pi}{k_o^2} \int_o^\infty \frac{J_o(kr)}{k^2 + k_o^2}\, k\, dk , \qquad (2.4)$$

where $$I = \int_o^\infty \int_o^{2\pi} \frac{e^{ikr\cos\Theta}}{k - k_o} \frac{dk\, d\Theta}{4k_o^2} .$$

The integrals involving $J_o(kr)$ are given by Watson (1966, pp.436 and 434 respectively), as

$$\int_o^\infty \frac{J_o(kr)}{k+k_o}\, dk = \frac{\pi}{2}\left\{ \underline{H}_o(k_o r) - Y_o(k_o r) \right\} ,$$

$$\int_o^\infty \frac{J_o(kr)}{k^2+k_o^2}\, k\, dk = K_o(k_o r) ,$$

where $\underline{H}_o$ is the Struve function, K_o the modified Bessel function of the second kind. The integrand in I has a pole on the range of integration, and must be interpreted in a certain way. Firstly we write

$$\int_\Theta \int_o^\infty dk\, d\Theta = \int_\Theta \int_{-\infty}^{+\infty} dk\, d\Theta - \int_\Theta \int_{-\infty}^{o} dk\, d\Theta ,$$

and then the last integral can be evaluated as above. The first integral is equal to $\int_o^{2\pi} e^{ik_o r \cos\Theta} \int_{-\infty}^{+\infty} e^{iar\cos\Theta} \frac{da}{a} d\Theta$,

and the radiation condition referred to earlier requires that $\int_{-\infty}^{+\infty} e^{iar\cos\Theta} \frac{da}{a}$

must equal zero when $\cos\Theta > 0$. For otherwise, $y(\underline{x},t)$ would contain Fourier components of the form $\exp i(k_o r\cos\Theta + \omega t)$, representing inward propagating waves when ω , $\cos\Theta > 0$. We must therefore interpret the integral as

$$\int_{-\infty}^{+\infty} e^{i\alpha r \cos\Theta} \frac{d\alpha}{\alpha - i0} = -2\pi i\, H(-\cos\Theta) .$$

H denotes the Heaviside unit function, and the definitions and transforms of the generalised functions $(\alpha \pm i0)^m$ are given by Jones (1966).

I can then be evaluated using the result of Watson (1966, p.312),

$$\int_0^{\pi} \exp(-iz \sin\Theta) d\Theta = \pi \left\{ J_o(z) + i\, \underline{E}_o(z) \right\} ,$$

noting that $\underline{E}_o(z) = -\underline{H}_o(z)$ (Watson, p.337). This gives us finally,

$$8\pi^3 B\tilde{y}(\underline{x},\omega) = -\frac{\pi^2 i}{2k_o^2} H_o^{(2)}(k_o r) - \frac{\pi}{k_o^2} K_o(k_o r), \qquad (2.5)$$

in which $H_o^{(2)}$ denotes the Hankel function of the second kind. The corresponding result for $\omega < o$ is

$$8\pi^3 B\tilde{y}(\underline{x},\omega) = +\frac{\pi^2 i}{2k_o^2} H_o^{(1)}(k_o r) - \frac{\pi}{k_o^2} K_o(k_o r) , \qquad (2.6)$$

where again k_o is the positive root of $k_o^4 = \omega^2/\lambda^2$.

The function $\tilde{y}(\underline{x},\omega)$ is itself the Green's function for the time-reduced plate-equation,

$$(\lambda^2 \nabla^4 - \omega^2)\tilde{y}(\underline{x},\omega) = \frac{1}{m}\tilde{p}(\underline{x},\omega) .$$

This function has been found before (H. and L. Cremer, 1948), though not by this kind of method. The method given here is evidently applicable to other problems (e.g., gravity waves on deep water), and replaces the difficulties which can arise in an appeal to fictitious damping forces by an explicit appeal to a suitable radiation condition in frequency-space. We note that the radiation condition is obviously satisfied by (2.5) and (2.6) with the time factors $\exp(\pm i\omega t)$ respectively, the K_o function vanishing exponentially for $k_o r \rightarrow \infty$.

For the complete Green's function, we have now to integrate (2.5) and (2.6) over frequency ω. The quantities of physical interest are the velocity $v = \partial y/\partial t$, and $q = \nabla^2 y$; q is proportional to the bending moment. We find that

$$8\pi^3 B\, v(\underline{x},t) = 2\pi\lambda \int_o^\infty \left\{ K_o\left(\frac{r\omega^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right) + \frac{\pi}{2} Y_o\left(\frac{r\omega^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right) \right\} \sin \omega t \, d\omega$$

$$+ \pi^2\lambda \int_o^\infty J_o\left(\frac{r\omega^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right) \cos \omega t \, d\omega ,$$

and both of these integrals are tabulated by Erdelyi et al. (1954, pp.111 and 53). We obtain

$$v(\underline{x},t) = \frac{1}{4\pi m\lambda t} \sin\left(\frac{r^2}{4\lambda t}\right) , \qquad (2.7)$$

and

$$q(\underline{x},t) = - \frac{1}{4\pi B t} \cos\left(\frac{r^2}{4\lambda t}\right) . \qquad (2.8)$$

These Green's functions do not seem to have been found before. As a check, consider the result of applying the pressure

$$p(\underline{x},t) = \delta(\underline{x})e^{i\omega t}/2\pi \text{ , with } \omega > 0 \text{ .}$$

The velocity is given by

$$v(\underline{x},t) = \int_{\underline{x}'}\int_{t'=-\infty}^{t} \frac{1}{4\pi m\lambda(t-t')} \sin \frac{|\underline{x}-\underline{x}'|^2}{4\lambda(t-t')} p(\underline{x}',t')\, d\underline{x}'\, dt' \text{ ,}$$

and this can be shown to be equal to

$$e^{i\omega t}\tilde{v}(\underline{x},\omega) \text{ ,}$$

where $\tilde{v}(\underline{x},\omega) = i\omega\tilde{y}(\underline{x},\omega)$ and $\tilde{y}(\underline{x},\omega)$ is given in (2.5). Thus the use of the Green's function (2.7) leads us back to the steady-state oscillatory motion given by (2.5).

In the next section, certain unphysical aspects of these solutions will be discussed, and the modifications to them will be discussed when a new form of mechanical dissipation is postulated.

3. A new representation of mechanical damping

The Green's functions (2.7) and (2.8) are singular in the sense that $v \not\rightarrow 0$ as $r \rightarrow \infty$ for fixed $t > 0$, and $v \rightarrow \infty$ as $t \rightarrow 0+$ for any $r > 0$. This unphysical behaviour must arise through neglect of mechanical damping in the plate, and there are two ways in which this damping is usually represented. Either one adds a term $\beta\, \partial y/\partial t$ to the left hand side of (2.1), or one writes B in complex form, $B = B(1 + i\eta)$, where η is the attenuation rate. The $\beta\, \partial y/\partial t$ term is quite unrealistic, giving an amplitude decay $\exp(-\beta t/2)$ independent of frequency ω. Moreover, this term can only represent some kind of "air resistance", and it has no connection with the internal dissipation in the plate. The unphysical behaviour of the Green's functions is not removed by this kind of damping, for the behaviour arises from the presence of some Fourier components in the dispersive system with very large phase velocity (the point impulse having a white spectrum in $\underline{k}$ and in ω). A physically realistic form of dissipation must attenuate Fourier components more rapidly with increasing frequency, and it is found experimentally that in fact the amplitude of a free flexural wave of frequency ω decays like $\exp(-\omega\eta t/2)$, with η a constant. (Alternatively, the amplitude of a cylindrical wave of wavenumber k_o decays with distance like $\exp(-k_o\eta r/4)$).

Agreement with these experimental results is obtained by writing $B(1 + i\eta)$ for B (see Cremer and Heckl, 1967). The physical meaning of this substitution is not clear, however, and difficulty arises in some problems, because the poles at $\omega = \pm \lambda k^2$ are displaced from the real axis in opposite directions by the damping. We are therefore led to postulate the following modification of (2.1);

$$\left(m \frac{\partial^2}{\partial t^2} - \lambda m \eta \nabla^2 \frac{\partial}{\partial t} + B \nabla^4\right) y = p \qquad (3.1)$$

This is equivalent to letting the diffusivity λ become complex. The attenuation factor η is supposed small, so that η^2 can be neglected compared with unity. The damping force is proportional to the time rate of change of bending moment, and the dissipation may be regarded as occurring through a hysteresis loss in the bending process.

Instead of (2.5) we have now

$$8\pi^3 B \tilde{y}(\underline{x}, \omega) = \frac{\pi}{k_o^2} \left\{ K_o\left(r \sqrt{\frac{i\eta\omega}{2\lambda} - \frac{\omega}{\lambda}}\right) - K_o\left(r \sqrt{\frac{i\eta\omega}{2\lambda} + \frac{\omega}{\lambda}}\right) \right\}, \qquad (3.2)$$

for $\omega > 0$. As $r \longrightarrow \infty$ we have the wave-like structure already implied by (2.5), but with the multiplicative factor

$$\exp\left(-\frac{k_o \eta r}{4}\right),$$

while corresponding to (2.7) we have

$$v(\underline{x}, t) = \frac{1}{4\pi m \lambda t} \exp\left(-\frac{\eta}{2} \cdot \frac{r^2}{4\lambda t}\right) \sin\left(\frac{r^2}{4\lambda t} + \frac{\eta}{2}\right). \qquad (3.3)$$

This solution does not have any of the unphysical characteristics of (2.7), and the proposed damping term certainly gives agreement with the known experimental results.

In the subsequent work we shall revert to using (2.7), noting that any singular effects which may arise can easily be removed by the exponential factors given above.

4. Steady-state plate response induced by a random pressure field

We shall now consider the applied pressure field $p(\underline{x},t)$ to be a stationary random function of the time t, so that the resulting plate vibration is statistically steady in time. Homogeneity of the pressure in the space variables is not required. The object of this section is to set up general relations between the statistics of the plate vibration and those of the pressure field without further assumptions about the pressure.

The simplest way of dealing with the stationary case involves the use of the time-reduced Green's function (2.5). The solution of (2.1) may be written, with the aid of this Green's function, in the form

$$\tilde{y}(\underline{x},\omega) = \int_{\underline{x}'} G_y(\underline{x},\underline{x}',\omega)\, \tilde{p}(\underline{x}',\omega)d\underline{x}' , \qquad (4.1)$$

where $\underline{r} = \underline{x}' - \underline{x}$ and

$$G_y(\underline{x},\underline{x}',\omega) = \frac{1}{4\pi^2 m\lambda^2}\left\{ - \frac{\pi^2 i}{2k_o^2} * H_o^{(2)}(k_o r) - \frac{\pi}{k_o^2} K_o(k_o r)\right\} .$$

Equations in a corresponding notation hold for the transforms of $v = \partial y/\partial t$ and $q = \nabla^2 y$. For $k_o r >> 1$ we have the asymptotic formulae

$$G_v(\underline{x},\underline{x}',\omega) \sim \frac{\lambda}{8B}\sqrt{\frac{2}{\pi k_o r}} \ \exp\ i(\tfrac{\pi}{4} - k_o r)$$

$$G_q(\underline{x},\underline{x}',\omega) \sim \frac{i}{8B}\sqrt{\frac{2}{\pi k_o r}} \ \exp\ i(\tfrac{\pi}{4} - k_o r) \qquad (4.2)$$

for $\omega > 0$, and with the obvious change of i to $-i$ for $\omega < 0$.

Now let <....> denote an ensemble-average (or equivalently, a time-average in this case) and * denote complex conjugate. When all the variables considered are stationary random functions of t, the various Fourier coefficients are statistically orthogonal (see, e.g., Batchelor, 1953), so that

$$\langle \tilde{v}^*(\underline{x},\omega)\tilde{v}(\underline{x},\omega')\rangle = \Phi(\underline{x},\omega)\delta(\omega-\omega') . \qquad (4.3)$$

$\Phi(\underline{x},\omega)$ is the (real) transform of the one-point velocity autocorrelation,

$$\langle v(\underline{x},t)v(\underline{x},t+\tau)\rangle = \int_{-\infty}^{+\infty} e^{i\omega\tau}\,\Phi(\underline{x},\omega)d\omega , \qquad (4.4)$$

and gives the spectral density of mean-square velocity. Similarly, we define $\chi(\underline{x},\omega)$ as the spectral density of $\langle q^2\rangle$. Also we write

$$\langle \tilde{p}^*(\underline{x}',\omega)\tilde{p}(\underline{x}'',\omega')\rangle = \Psi(\underline{x}',\underline{x}'',\omega)\delta(\omega-\omega') , \qquad (4.5)$$

so that Ψ is the time transform of the two-point pressure autocorrelation,

$$\langle p(\underline{x}',t)p(\underline{x}'',t+\tau)\rangle = \int_{-\infty}^{+\infty} e^{i\omega\tau}\,\Psi(\underline{x}',\underline{x}'',\omega)d\omega . \qquad (4.6)$$

We have then

$$\tilde{v}(\underline{x},\omega) = \int_{\underline{x}'} G_v(\underline{x},\underline{x}',\omega)\,\tilde{p}(\underline{x}',\omega)d\underline{x}' ,$$

from which it follows that

$$\frac{\partial}{\partial \underline{x}'} \Phi(\underline{x},\omega) = \int_{\underline{s}} G_v^*(\underline{r},\omega) G_v(\underline{r}+\underline{s},\omega) \Psi(\underline{x}',\underline{x}'+\underline{s},\omega) d\underline{s} \ . \qquad (4.7)$$

Here the left hand side gives the contribution to $\Phi(\underline{x},\omega)$ from unit area at $\underline{x}'$. We have written $\underline{x}'' = \underline{x}' + \underline{s}$, $\underline{x}' = \underline{x} + \underline{r}$, and noted that the Green's functions depend only on the differences $\underline{x}' - \underline{x}$, $\underline{x}'' - \underline{x}$. The dependence of Ψ upon $\underline{x}'$ may be ignored for the present.

Suppose now that $r = |\underline{x}' - \underline{x}|$ satisfies

(i) $k_o r \gg 1$, i.e. $\underline{x}$ lies in the far-field of the flexural waves generated at $\underline{x}'$,

(ii) $r \gg L_p$, where L_p is a correlation scale for variations in the applied pressure.

Then we may approximate (4.7) by

$$\frac{\partial}{\partial \underline{x}'} \Phi(\underline{x},\omega) = \frac{\lambda^2}{32\pi B^2 k_o r} \cdot \int_{\underline{s}} e^{ik_o \underline{n}.\underline{s}} \Psi(\underline{x}',\underline{x}'+\underline{s},\omega) d\underline{s} \ , \qquad (4.8)$$

where $\underline{n} = -\underline{r}/r$ is a unit vector from $\underline{x}'$ to $\underline{x}$. Now define a complete pressure spectrum function by

$$\langle p(\underline{x}',t)p(\underline{x}'+\underline{s},\ t+\tau)\rangle = \int e^{i\underline{k}.\underline{s} + i\omega\tau} S_p(\underline{k},\omega;\underline{x}') d\underline{k}\, d\omega \ . \qquad (4.9)$$

Then from (4.6) we have

$$\int_{\underline{s}} e^{ik_o \underline{n}.\underline{s}} \Psi(\underline{x}',\underline{x}'+\underline{s},\omega) d\underline{s} = 4\pi^2 S_p(-\underline{n}k_o,\ \omega;\underline{x}'),$$

and when this is substituted into (4.8) we have

$$\frac{\partial}{\partial \underline{x}'} \Phi(\underline{x},\omega) = \frac{\pi}{8Bmk_o r} S_p(-\underline{n}k_o, \omega ; \underline{x}') . \qquad (4.10)$$

The contribution to the mean-square velocity at $\underline{x}$ from unit area at $\underline{x}'$ decays as $1/r$, as is required for energy balance in the steady state. Also, the effective source strength at $\underline{x}'$ for waves of frequency ω arriving at $\underline{x}$ is proportional to the pressure spectrum at $\underline{x}'$, evaluated at the free wavenumber $-\underline{n}k_o$ for flexural waves at frequency ω propagating from $\underline{x}'$ to $\underline{x}$. For the spectrum of $\langle q^2 \rangle$ we have, similarly,

$$\frac{\partial}{\partial \underline{x}'} \chi(\underline{x},\omega) = \frac{\pi}{8B^2 k_o r} S_p(-\underline{n}k_o, \omega ; \underline{x}') \qquad (4.11)$$

This gives $m\Phi(\underline{x},\omega) = B\chi(\underline{x},\omega)$, indicating, as expected, that there is equipartition of energy at each frequency between the kinetic and elastic modes of the plate. ($\frac{1}{2}m\Phi(\underline{x},\omega)$ and $\frac{1}{2}B\chi(\underline{x},\omega)$ are the spectral densities of the kinetic and elastic energies per unit area, $\frac{1}{2}m\langle v^2 \rangle$ and $\frac{1}{2}B\langle q^2 \rangle$ respectively.)

Consider next the energy balance. Let S be an area of the plate, bounded by the closed curve ℓ, with unit outward normal $\underline{\nu}$. From (2.1) we find the energy equation

$$\frac{\partial}{\partial t} \int_S E(\underline{x}',t)d\underline{x}' = \int_S p(\underline{x}',t)v(\underline{x}',t)d\underline{x}' - \oint \underline{F}(\underline{x},t).\underline{\nu}(\underline{x})d\ell(\underline{x}) \qquad (4.12)$$

where

$E = \frac{1}{2}mv^2 + \frac{1}{2}Bq^2$, the energy density,

$\underline{F} = B(v\nabla q - q\nabla v)$, the energy flux vector.

Taking a mean value, we have in the steady state,

$$\int_S \langle pv\rangle \, d\underline{x}' \;=\; \oint \langle \underline{F}\rangle . \, \underline{\nu} \, d\ell \qquad (4.13$$

Now let $\eta(\underline{x}', \omega)$, $\zeta(\underline{x}, \omega)$ denote the frequency spectra of $\langle pv(\underline{x}')\rangle$ and of $\langle F_r(\underline{x})\rangle$ respectively. Here an origin is taken at some point in the region over which the pressure acts, $\underline{x}$ denotes a point very distant from all points $\underline{x}'$ of that region, and F_r denotes the component of $\underline{F}$ normal to the curve ℓ, which is taken to be a circle of large radius $R = |\underline{x}|$. Then from the definition of $\underline{F}$, and the expressions for v, q in terms of the pressure field, we can calculate $\frac{\partial}{\partial \underline{x}'} \zeta(\underline{x}, \omega)$ in much the same way as above, subject to the two conditions on $r = |\underline{x}' - \underline{x}|$. We find

$$\frac{\partial}{\partial \underline{x}'} \zeta(\underline{x}, \omega) \;=\; \frac{\pi\lambda}{4Br} \mathcal{R} \, S_p(-\underline{n}k_o, \omega; \underline{x}') \, ,$$

and on integrating this expression round the circle ℓ centred on the origin, and letting the radius $R = |\underline{x}| \approx |\underline{x}' - \underline{x}| \longrightarrow \infty$, we have

$$\frac{\partial}{\partial \underline{x}'} \oint \zeta(\underline{x}, \omega) d\ell(\underline{x}) \;=\; \frac{\pi^2 \lambda}{2B} \, \hat{S}_p(k_o, \omega; \underline{x}') \, . \qquad (4.14)$$

Here $\hat{S}_p$ is the direction-averaged spectrum,

$$\hat{S}_p(k_o, \omega; \underline{x}') \;=\; \frac{1}{2\pi} \mathcal{R} \int_o^{2\pi} S_p(-k_o \underline{n}(\Theta), \omega; \underline{x}') d\Theta \, , \qquad (4.15)$$

Θ representing the angle between $\underline{x}' - \underline{x}$ and some fixed line as $\underline{x}$ varies round the circle ℓ. Now (4.13) is equivalent to

$$\int \eta(\underline{x}', \omega) d\underline{x}' = \oint \zeta(\underline{x}, \omega) d\ell(\underline{x}) , \quad \text{or to}$$

$$\eta(\underline{x}', \omega) = \frac{\partial}{\partial \underline{x}'} \oint \zeta(\underline{x}, \omega) d\ell(\underline{x}) ,$$

and therefore from (4.14) we have the following expression for the spectral density of the rate of work of the pressure field,

$$\eta(\underline{x}', \omega) = \frac{\pi^2 \lambda}{2B} \hat{S}_p(k_o, \omega; \underline{x}') . \qquad (4.16)$$

This is an exact expression for the rate of working of the pressure field in terms of the pressure spectrum evaluated at the free flexural wave-number, and averaged over all directions of the bending waves. A similar result for the rate of working has been found by Ffowcs Williams and Lyon (1963); in their derivation, however, the pressure field was assumed to cover the whole infinite plate, and to be spatially homogeneous. In that case, dissipation in the plate has to be invoked if the bending wave intensities are to remain finite. Here we have no need to consider mechanical damping. The rate of working of the pressure field is balanced, for a finite region in the steady state, by the outflow of energy from that region in the form of propagating elastic waves.

The formulae developed here can be used to estimate the intensities and power input at any frequency if the pressure spectrum is known at that frequency and at the free wavenumber k_o. The integration over frequency will be carried out in the next section, to obtain mean intensities induced when incompressible fluid in turbulent motion flows over the plate.

5. Plate Vibration induced by a turbulent boundary-layer

In this section, we suppose the statistically steady pressure field to be that of a turbulent boundary layer in the flow of incompressible fluid over the plate. A general theoretical result about the form of the pressure spectrum of such a flow near zero frequency and wavenumber allows us to determine the dependence of the vibration statistics upon the flow and plate parameters in the limits of high and low fluid velocity. As before, edge effects are neglected, and dissipation in the plate is also assumed negligible.

As is usual in the theory of aerodynamic noise, the boundary-layer flow will be supposed unmodified by the plate vibration, at any rate in a first approximation. The vibration is regarded as a small by-product of a given flow, which here we take to be the boundary layer over an infinite rigid plate. It can then be shown (Kraichnan, 1956; Ffowcs Williams, 1965) that the spectrum S_p of the surface pressure must approach zero like k^2 as the wavenumber $\underline{k} \longrightarrow 0$. Equivalently, the pressure field has zero correlation area in incompressible flow. Hence, for small values of $|\underline{k}|$, we have

$$S_p(\underline{k},\omega;\ \underline{x}') = A_{ij}(\omega,\underline{x}')k_i k_j + O(k^3), \qquad (5.1)$$

where the A_{ij} are determined by the distribution of velocity in the boundary layer. The frequency spectrum is,,in the main, a duplication of the wave-number spectrum, so that one might expect that

$$S_p \sim \omega^2 \qquad \text{as } \omega \longrightarrow 0\ .$$

This does not, however, occur, since Taylor's hypothesis of rigid convection is only valid for frequencies ω large compared with the mean shear. For $\underline{k}$ finite and $\omega \longrightarrow 0$, the spectrum does not generally vanish, but approaches a non-zero constant value. The zero of S_p at $\underline{k} = 0$ arises entirely from the fact that the "source term" in the Poisson equation for the pressure is a double space-derivative.

Let V, δ^* denote the mean velocity and displacement thickness, respectively, of the boundary-layer. These are taken as (constant) characteristic velocity and length scales for the flow. ρ is the constant fluid density. We assume that the pressure p has magnitude equal to the typical fluctuation in $\frac{1}{2}\rho \underline{u}^2$, i.e. $p \sim \rho \epsilon V^2$, where $\epsilon = u/V$ is the ratio of r.m.s. turbulent velocity to mean velocity. We also assume that the characteristic frequency of the pressure field is of order V/δ^*. These are the usual estimates of aerodynamic noise theory. Then we can write the following general form for the real part of the pressure spectrum;

$$\mathcal{R} S_p(\underline{k}, \omega) = \rho^2 \epsilon^2 V^3 \delta^{*3} (k_i k_j \delta^{*2}) A_{ij}\left(\underline{k}\delta^*, \frac{\omega\delta^*}{V}\right) , \qquad (5.2)$$

where A_{ij} is a dimensionless function of the dimensionless frequency $\omega\delta^*/V$ and wavenumber $\underline{k}\delta^*$. In view of (5.1) and the subsequent remarks, we have

$$A_{ij}\left(0, \frac{\omega\delta^*}{V}\right) \neq 0 , \qquad A_{ij}(\underline{k}\delta^*, 0) \neq 0.$$

Substitute (5.2) into (4.10), and use the fact that

$S_p(\underline{k},\omega) = S_p^*(-\underline{k}, -\omega)$ to get $\frac{\partial}{\partial \underline{x}'} \int_{-\infty}^{+\infty} \Phi(\underline{x},\omega)d\omega =$

$$\frac{\pi}{8} \epsilon^2 \left(\frac{\rho}{m}\right)^2 \left(\frac{\delta^*}{r}\right) R^{5/2} V^2 \int_0^\infty n_i n_j A_{ij}(-\underline{n}\sqrt{R\sigma}, \sigma)\sigma^{-\frac{1}{2}} d\sigma \quad . \qquad (5.3)$$

Here $R = V\delta^*/\lambda$ is a kind of Reynolds number based on the "plate diffusivity" λ. Two cases now arise. When $R \ll 1$, the integral in (5.3) is approximately equal to

$$\int_0^\infty n_i n_j A_{ij}(0,\sigma)\sigma^{-\frac{1}{2}} d\sigma = f(\underline{n}) \quad \text{say} \; .$$

Hence, for $R \ll 1$,

$$\frac{\partial}{\partial \underline{x}'} < V^2(\underline{x}) > = \frac{\pi}{8} f(\underline{n}) \epsilon^2 \left(\frac{\rho}{m}\right)^2 \left(\frac{\delta^*}{r}\right) V^2 R^{5/2} \qquad (5.4)$$

Secondly, if $R \gg 1$ we may put $\sigma R = q$, and then the integral in (5.3) transforms into

$$R^{-3/2} \int_0^\infty n_i n_j \; A_{ij}(-\underline{n}\; q^{\frac{1}{2}}, \tfrac{q}{R}) q^{-\frac{1}{2}} dq$$

$$\sim R^{-3/2} \int_0^\infty n_i n_j \; A_{ij}(-\underline{n}\; q^{\frac{1}{2}}, 0) q^{-\frac{1}{2}} dq$$

$$= R^{-3/2} g(\underline{n}) \quad \text{say} \; .$$

Therefore, for $R \gg 1$ we have

$$\frac{\partial}{\partial \underline{x}'} \langle v^2(\underline{x}) \rangle = \frac{\pi}{8} g(\underline{n}) \epsilon^2 \left(\frac{\rho}{m}\right)^2 \left(\frac{\delta^*}{r}\right) R V^2 \; . \qquad (5.5)$$

For the rate of working of the pressures, we write

$$\hat{S}_p(k,\omega) = \rho^2 \epsilon^2 V^3 \delta^{*3} (k\delta^*)^2 A\left(k\delta^*, \frac{\omega\delta^*}{V}\right) , \qquad (5.6)$$

in analogy with (5.2), and define α, β by

$$\alpha = \int_0^\infty \sigma \, A(0,\sigma) d\sigma \; ,$$

$$\beta = \int_0^\infty \sigma \, A(\sigma^{\frac{1}{2}}, 0) d\sigma \; . \qquad (5.7)$$

Then we find

$$\langle pv \rangle = \frac{\pi\alpha}{2} \epsilon^2 \left(\frac{\rho\delta^*}{m}\right)(\rho V^3) R^2 \qquad (R \ll 1) \qquad (5.8)$$

$$= \frac{\pi\beta}{2} \epsilon^2 \left(\frac{\rho\delta^*}{m}\right)(\rho V^3) \qquad (R \gg 1) \qquad (5.9)$$

The formulae derived above are analogous to the well-known results in aerodynamic noise theory, for the acoustic intensity of a turbulent flow in the low and high speed limits (Lighthill, 1952; Ffowcs Williams, 1963). The limits $R \ll 1$, $R \gg 1$ also have exactly the same interpretation as the corresponding limits in the acoustic problem. The turbulence and the resulting plate vibration have the same typical frequency V/δ^*, δ^* being the characteristic length over which a turbulent eddy remains coherent. The wavelength of the flexural motion corresponding to frequency V/δ^* is $\Lambda \sim \delta^* R^{-\frac{1}{2}}$, and hence $R \ll 1$ implies $\Lambda \gg \delta^*$. In the limit

$R \ll 1$, the turbulence scale δ^* is small compared with a wavelength of the flexural waves generated by the turbulence. The corresponding case in aerodynamic noise theory is referred to as the "acoustically thin" limit, and arises as a consequence of low Mach number.

An efficiency can be defined for the vibration process by the relation $\gamma = \frac{\langle pv \rangle}{\rho V^3}$; γ is a measure of the rate of energy loss from the flow into the plate as compared with the rate of advection of energy by the mean flow. The efficiency rises fairly slowly with the speed V($\gamma \sim R^2$ as $R \longrightarrow 0$), and asymptotes to a constant value as $R \longrightarrow \infty$. Two factors may prevent this ultimate state from being achieved. Firstly, the value of $\frac{\pi\beta}{2}\epsilon^2(\frac{\rho\delta^*}{m})$ determines whether the energy loss into vibration can appreciably modify the basic flow. The vibration can only be regarded as a small by-product of the flow if $\frac{\pi\beta}{2}\epsilon^2(\frac{\rho\delta^*}{m}) \ll 1$. Secondly, compressibility may become important in the case $R \gg 1$ if then the Mach number V/a_o is not small. If V/a_o is not small, compressibility will certainly affect all scales of motion, and most of all will affect the spectral components at low wavenumber and finite frequency - for these have high phase Mach number $\omega/a_o k$. Ffowcs Williams (1965) has shown how these spectral components are modified by compressibility. He shows that the k^2 decay of the pressure spectrum cannot continue to wavenumbers lower than about M/δ^*, where $M = V/a_o$. In the wavenumber range $(0, M/\delta^*)$ the pressure spectrum is approximately constant and non-zero. Without more precise details, however, it is not possible to apply these ideas in a reliable way to the problem of plate vibration, though we should note that in underwater applications the limit $R \ll 1$ applies almost always, and that the compressibility effects should be quite negligible. For the quantities m, B are related to the plate thickness h, plate density ρ_p, and the speed C_L of longitudinal waves in the plate by the formulae

$$m = \rho_p h, \qquad B = \rho_p C_L^2 h^3/12 , \qquad (5.10)$$

(Cremer and Heckl, 1967). Typical values of the various parameters for steel plates used in underwater situations are

$$\delta^* = 3 \text{ cm}, \quad h = 1 \text{ cm}, \quad \rho_p = 8 \text{ gm/cm}^3, \quad V = 10 \text{ met/sec.},$$

and then we have $R \sim 2 \times 10^{-2}$, $M \sim 7 \times 10^{-3}$. The spectral density of the pressure field in the range $(0, M/\delta^*)$ is proportional to M^2, and since also this range is extremely small, the compressibility effects on low wavenumbers are likely to be quite insignificant.

The fact that the efficiency becomes constant as $V \longrightarrow \infty$ is interesting, for this behaviour is also found in the acoustic problem (Ffowcs Williams, 1963). In that problem, source convection effects play an important role, and it is possible that similar effects may not have been adequately covered in the above work. This is a difficult problem to deal with here, since the system is dispersive, and any convection speed is "supersonic" relative to some spectral components. Eddy "Mach waves" are emitted at any frequency provided the convection speed exceeds the free flexural wave speed for that frequency. The difficulty lies in the problem of treating simultaneously these Mach waves, which depend only on supersonic convection, and the waves generated by subsonic spectral components, which depend on the time variation of the source field. Such problems do not arise in the acoustic problem, where all components of the source field are convected in the same way relative to the wave speed. Source convection will not be considered further in connection with the boundary layer, though we may note again that this effect is probably negligible in underwater applications. There the dominant components of the source field are well "subsonic", the condition $R \ll 1$ being equivalent to the statement that the convection speed V is small compared with the phase speed of flexural waves of frequency V/δ^*.

6. Waves induced by a convected pressure field

The following simple example is chosen to illustrate some effects due to convection of the pressure field, and also to demonstrate the use of the complete Green's function. Fourier time-analysis, and the use of the time-reduced Green's function may also give the result, but it seems preferable to use the more direct method given below.

Consider the motion generated by the application of a point force, of constant unit strength, convected at uniform speed (V,O) through the origin at time t = o. The pressure field is then

$$p(\underline{x},t) = \delta(x_1 - Vt)\delta(x_2) ,$$

and the velocity at $(\underline{x},t)$, in a fixed frame of reference, is given by

$$v(\underline{x},t) = \frac{1}{4\pi m\lambda} \int_{\underline{x}'} \int_{t'=-\infty}^{t} \frac{1}{t-t'} \sin \frac{|\underline{x} - \underline{x}'|^2}{4\lambda(t-t')} \quad p(\underline{x}',t')d\underline{x}'dt'$$

$$= \frac{1}{4\pi m\lambda} \int_{\tau=o}^{\infty} \frac{1}{\tau} \sin \left\{ \frac{(\eta_1 + V\tau)^2 + \eta_2^2}{4\lambda\tau} \right\} d\tau . \qquad (6.1)$$

Here ($\eta_1 = x_1 - Vt$, $\eta_2 = x_2$) are the coordinates of $\underline{x}$ relative to the current position of the force. We evaluate the integral by the method of stationary phase. The point of stationary phase occurs at

$$\tau = r/V , \quad \text{where } (\eta_1,\eta_2) \longrightarrow (r,\Theta) ,$$

and the method is valid for $(\lambda/Vr)^{\frac{1}{2}} \ll 1$. (This follows from a consideration of the second and third derivatives of the phase at $\tau = r/V$).

We find that

$$v \sim \frac{1}{4m(\pi\lambda Vr)^{\frac{1}{2}}} \sin\left[\frac{Vr}{2\lambda}(1 + \cos\Theta) + \frac{\pi}{4}\right] , \qquad (6.2)$$

and similarly that

$$q \sim \frac{-\lambda}{4B(\pi\lambda Vr)^{\frac{1}{2}}} \cos\left[\frac{Vr}{2\lambda}(1 + \cos\Theta) + \frac{\pi}{4}\right] . \qquad (6.3)$$

The effect of a time-varying strength $p_o(t)$ is merely to multiply (6.2) and (6.3) by the factor $p_o(t-r/V)$, and these formulae may then be used to give the far-field of an arbitrary convected pressure distribution in $(\underline{x},t)$.

The waves have crests defined by the lines of constant phase,

$$r + \eta_1 = \beta \text{ say } (\beta \geq 0), \quad \text{or}$$

$$\eta_2^2 = -2\beta(\eta_1 - \beta/2) \qquad (6.4)$$

For varying $\beta \in (0,\infty)$ this equation represents a family of confocal parabolas, with focus at the position of the force. The situation here is evidently very different from that which occurs when a point force is uniformly convected over deep water. The surface gravity waves thus produced have the characteristic two-family ship wave form, and are confined to a wake behind the force (Lamb, 1962, p.434).

The energy relationships have an interesting form for the bending wave case. Consider a fixed reference frame, with origin instantaneously coincident with the position of the force, and let $\hat{r}$, $\hat{x}$ be unit vectors in the radial and convection directions, respectively. The energy flux vector

$$\underline{F} = B(v\nabla q - q\nabla v)$$

is easily found to be given by

$$\underline{F} = \frac{1}{32\pi m\lambda r}(\hat{r} + \hat{x}), \qquad (6.5)$$

and the energy density

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}Bq^2 \quad \text{by}$$

$$E = \frac{1}{32\pi m\lambda V r} \qquad (6.6)$$

Thus the energy-propagation vector $\underline{c}$, giving the rate at which energy of unit density is propagated, is given by

$$\underline{c} = \underline{F}/E = V(\hat{r} + \hat{x}) \qquad (6.7)$$

This is a surprising result, and again is quite different from the behaviour found in the gravity-wave case. The rate of energy propagation at large distances depends only upon the convection speed V, and not on the parameters m, B of the plate. In the problem with a constant strength force one might expect a dependence of $\underline{c}$ upon V, and also on the dimensionless ratio Vr/λ, but evidently the latter dependence disappears for $(Vr/\lambda)^{\frac{1}{2}} \gg 1$. Also, if the strength of the force is $p_o(t)$, (6.7) is unchanged, and $\underline{c}$ does not depend on the typical frequency of the force, provided this is low enough for p_o to be regarded as slowly varying compared with the trigonometric term in (6.1).

The result (6.7) means that if one were to make a normal mode analysis of the vibration induced in a finite plate, it would be difficult to say, a priori, which set of modes would be preferred above the others independently of the details of the excitation. For any particular mode to be preferred, reverberant amplification must occur. This requires that the phase of the excitation change only slowly in the time taken by a flexural wave to cross the plate at least several times. Here, however, the waves containing most of the energy travel only at a speed comparable with the convection speed of the excitation. If one can suitably model a turbulent pressure field as a distribution of convected pressure points ("eddies"), it follows that by the time the group of waves generated by an eddy have crossed the plate once, the eddy will have been replaced by a succession of eddies with random relative phases. Reverberant amplification cannot then occur. The number of modes required to describe the vibration will then depend very much upon the details of the excitation, about which little is known in a form suitable for modal analysis. Equation (6.7) points to the relevance of the infinite plate equations for plates of finite size. The plate is effectively infinite for most of the time spent by a turbulent eddy over the plate.

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CHAPTER 5

THE SCALE EFFECT IN COMPRESSIBLE TURBULENCE

THE SCALE EFFECT IN COMPRESSIBLE TURBULENCE

1. Introduction

The density fluctuations in a fluid in turbulent motion are governed by Lighthill's (1952) well-known wave equation

$$\left(\frac{\partial^2}{\partial t^2} - a_o^2\,\nabla^2\right)\rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \quad . \qquad (1.1)$$

ρ is the density, a_o the sound speed in the fluid at rest, and $T_{ij} = \rho v_i v_j + p_{ij} - a_o^2\rho\delta_{ij}$ is Lighthill's acoustic stress tensor, v_i being the fluid velocity and p_{ij} the compressive stress tensor. This equation is usually used to predict the density fluctuations occurring far from the turbulent region, where $T_{ij} = 0$, and where the fluctuations propagate as sound waves with speed a_o. The equation is, however, formally exact, and we shall use it in this paper to estimate the fluctuations occurring in the turbulent fluid due to the passage through the fluid of turbulence-generated sound waves.

Lighthill's equation shows that we can calculate the density fluctuations by replacing the actual turbulent fluid with a perfect acoustic medium at rest, acted upon by a volume distribution of stresses T_{ij}. For this to be a meaningful way of visualising the generation of sound by turbulence, T_{ij} must be regarded as a forcing stress, essentially independent of the sound field it generates. At sufficiently low Mach numbers this requirement is generally satisfied, for we can approximate T_{ij} by $\rho_o u_i u_j$, where ρ_o is the mean density and u_i is the solenoidal velocity in the vorticity mode of the turbulence. (The resolution of low Mach number flows into vorticity, sound and entropy modes is due to Kovasznay (1953), and Chu and Kovasznay (1958)). Using this approximation, Proudman (1952) was able to calculate the acoustic power output of isotropic turbulence. The power output of unit volume of stationary isotropic turbulence is evidently independent of time, and of the position of the volume element considered. This leads to the paradox first noted by Lighthill (1955), that the density fluctuations occurring in an infinite expanse of stationary isotropic turbulence are infinite in mean square, even at very low Mach number.

The singularity in these fluctuations will become apparent later when a modified form of (1.1) is considered. In physical terms it arises from the fact that stationary isotropic turbulent flow generates sound in the same way, statistically, as a uniform volume distribution of acoustic sources whose strengths are statistically stationary in time. (This is true provided the sources are correlated over a region small compared with the region occupied by the sources.) The intensity thus produced at the centre of a spherical shell of thickness dr is proportional to $(1/r^2)4\pi r^2 dr$; hence, on integration, the intensity of sound due to a large volume of turbulence diverges in proportion to the linear scale R of the volume.

Essentially the same effect occurs when we try to calculate the intensity at a fixed distance from a large plane sheet composed of a uniform random distribution of statistically stationary sources. As the problem is two-dimensional the intensity this time diverges like ln R, where R is the radius of the sheet. This was noted by Saffman (1962).

The infinite singularity is known as the "scale effect", and is a far-field, rather than a near-field singularity. Now as Lighthill (1955) has shown, the singularity is removed once diffusive effects are taken into account. These cause the intensity produced at distance r from a source to die off like exp(-ar), and so ensure a finite intensity from an infinite volume. If unit volume of the (isotropic) turbulence has power output E, then the intensity produced at the centre of a sphere of turbulent fluid of radius R is given by

$$\frac{a_o^3}{\rho_o}\,\overline{\rho^2} = \frac{E}{\alpha}\,(1 - e^{-\alpha R}) \qquad (1.2)$$

$\overline{\rho^2}$ denotes the mean square of the density fluctuation. Letting $R \rightarrow \infty$, and using Proudman's (1952) estimate

$$E = 40\rho_o u_v^8/a_o^5 L_o \,, \qquad (1.3)$$

we have the following infinite-scale viscous limit to density fluctuations,

$$\overline{\rho^2}/\rho_o^2 = \frac{40}{\alpha L_o} M^8 \qquad (1.4)$$

here $M = u_v/a_o$, u_v is the r.m.s. velocity in the vorticity mode of the turbulence and L_o is the integral scale of the u_v-field. (1.4) is the equation given by Lighthill (1955).

It is partly the purpose of this note, however, to emphasise that (1.4) is unlikely to give a correct prediction - even in order of magnitude - of the density fluctuations under ordinary circumstances in which a turbulent motion can be regarded as approximately homogeneous and stationary. If the fluid is strictly incompressible, and occupies a limited region which is large enough for the assumption of homogeneity to be valid, then well within the turbulent region all mean square quantities are accurately given by a theory which assumes the turbulent region to be infinite in all directions. Finite compressibility, however, permits each element of fluid to act as a source of acoustic energy, and any result which is derived for an infinite medium is found to grossly overestimate the density or pressure fluctuations which occur in a finite region.

Thus, when $\alpha R \ll 1$ the appropriate form of (1.2) will be

$$\overline{\rho^2}/\rho_o^2 = 40\, M^8 \left(\frac{R}{L_o}\right), \qquad (1.5)$$

instead of (1.4). This is the inviscid small-scale limit, in which the intensity varies directly as the scale length R.

The damping factor α varies with frequency. Assuming the Stokes-Kirchhoff formula for α we have (Lighthill, 1956),

$$\alpha = \beta\omega^2/a_o^3 \qquad (1.6)$$

for a wave of frequency ω . β is the "diffusivity of sound" which, with an approximate adiabatic exponent $\gamma = \frac{3}{2}$ and Prandtl number $\frac{3}{4}$ is related to the mean kinematic viscosity ν by $\beta = 2\nu$. A "typical" frequency in both the acoustic and vorticity modes of the turbulence is u_v/L_o (see §3), so that equations (1.4) and (1.5) may now be written

$$\bar{\bar{\rho^2}}/\rho_o^2 = 20\, M^5\, Re\ , \qquad (M^3(\frac{R}{L_o})(Re)^{-1} \gg 1)\ , \qquad (1.7)$$

$$\bar{\bar{\rho^2}}/\rho_o^2 = 40\, M^8\, (\frac{R}{L_o})\ , \qquad (M^3(\frac{R}{L_o})(Re)^{-1} \ll 1)\ , \qquad (1.8)$$

Re is the turbulence Reynolds number, $Re = u_v L_o/\nu$, $M = u_v/a_o$ is the turbulence Mach number. The coefficients should be of the correct order of magnitude, but no more. The implications of these equations for practical situations will be discussed in §4. We shall see there that, even at very low values of M, the pressure fluctuations generated in the acoustic mode will be extremely large if the region of turbulence considered is large enough for (1.7) to be relevant.

2. The Damped Wave Equation

We propose now to obtain the result (1.7) directly, without using Proudman's equation (1.3), by extracting an explicit diffusive term from the acoustic stress tensor T_{ij}. The equation we shall use is

$$(\frac{\partial^2}{\partial t^2} - a_o^2\, \nabla^2 - \beta\nabla^2\, \frac{\partial}{\partial t})\rho = \frac{\partial^2 W_{ij}}{\partial x_i \partial x_j}\ , \qquad (2.1)$$

a Lighthill equation with a damping term included in the wave operator. As before, β is the diffusivity of sound, and

$$W_{ij} = T_{ij} - \beta \frac{\partial\rho}{\partial t} \delta_{ij} . \qquad (2.1)$$

The operator on the left of (2.1) is the well-known wave operator in a viscous heat-conducting fluid, (e.g. Kovasznay (1953)), while on the right we have Lighthill's stress tensor with a linearised approximation to diffusive effects removed. It is not difficult to check, from the momentum and energy equations, that the right side of (2.1) is independent of viscous and thermal diffusion effects, in a first approximation. We suggest that (2.1) represents a realistic model for calculating the intensity in homogeneous turbulence. The essential point of the equation is that β is some combination of the diffusivities, and the term $\beta\nabla^2\partial\rho/\partial t$ accounts, supposedly, for all irreversible effects.

It may now be realistic to prescribe W_{ij}, the part of T_{ij} remaining when viscous effects are explicitly removed, as a stationary random function of $\underline{x}$. Stationarity in t will also be required. The field can be supposed stationary if a random stationary driving force F_i per unit mass is added in the momentum equation. The equation (2.1) remains unaltered by this change provided div $\underline{F} = 0$. In addition to this constraint, F_i will be taken as statistically isotropic.

We can solve equation (2.1) formally by the methods of Batchelor (1953). Define Fourier-Stieltjes transforms by

$$\begin{aligned} \rho(\underline{x},t) &= \int \exp i(\underline{k}.\underline{x} + \omega t) dP(\underline{k}, \omega) , \\ W_{ij}(\underline{x},t) &= \int \exp i(\underline{k}.\underline{x} + \omega t) dR_{ij}(\underline{k}, \omega). \end{aligned} \qquad (2.2)$$

Here ρ, W_{ij} denote the fluctuations in these quantities about their mean values. The Stieltjes differentials are statistically orthogonal, so that, e.g.,

$$\overline{dP^*(\underline{k}', \omega')dP(\underline{k}, \omega)} = 0 \quad \text{if} \quad \underline{k} \neq \underline{k}' \quad \text{or} \quad \omega \neq \omega' ,$$

but

$$\overline{dP^*(\underline{k}, \omega)dP(\underline{k}, \omega)} = \Phi(\underline{k}, \omega)d\underline{k}d\omega ,$$
$$\overline{dR^*_{ij}(\underline{k}, \omega)dR_{lm}(\underline{k}, \omega)} = V_{ijlm}(\underline{k}, \omega)d\underline{k}d\omega . \qquad (2.3)$$

Thus $\Phi(\underline{k}, \omega)$ and $V_{ijlm}(\underline{k}, \omega)$ are spectral densities,

$$\bar{\rho}^2 = \int \Phi(\underline{k}, \omega)d\underline{k}d\omega ,$$
$$\overline{\left(\frac{\partial^2 W_{ij}}{\partial x_i \partial x_j}\right)^2} = \int k_i k_j k_l k_m V_{ijlm}(\underline{k}, \omega)d\underline{k}d\omega . \qquad (2.4)$$

The function $\Phi(\underline{k}, \omega)$ is entirely real, though $V_{ijlm}(\underline{k}, \omega)$ will in general be complex. However, since $V^*_{ijlm}(\underline{k}, \omega) = V_{lmij}(\underline{k}, \omega)$, the contracted form $k_i k_j k_l k_m V_{ijlm}$ is purely real.

From the differential equation (2.1) we have

$$\Phi(\underline{k}, \omega) = \frac{k_i k_j k_l k_m \; V_{ijlm}(\underline{k}, \omega)}{(a_o^2 k^2 - \omega^2)^2 + \beta^2 k^4 \omega^2} , \qquad (2.5)$$

in which $k = |\underline{k}|$. When the turbulence is isotropic we can write

$$k_i k_j k_l k_m V_{ijlm}(\underline{k}, \omega) = k^4 V(k, \omega) ,$$

with V an even function of k and of ω (Robertson, (1940)). We then have

$$\bar{p^2} = \int_{-\infty}^{+\infty} d\omega \int_{0}^{\infty} \frac{4\pi k^6 V(k,\omega)dk}{(a_o^2k^2 - \omega^2)^2 + \beta^2k^4\omega^2} . \qquad (2.6)$$

Incompressible flow results are obtained from (2.6) by putting $p = a_o^2 \rho$, and letting $\rho \longrightarrow 0$, $a_o^2 \longrightarrow \infty$ so that $a_o^2\rho$ has a finite limit. This gives

$$\bar{p^2} = \int_{-\infty}^{+\infty} d\omega \int_{0}^{\infty} 4\pi k^2 V(k,\omega)dk , \qquad (2.7)$$

which is the equation which one would obtain by applying Fourier transforms to the Poisson equation

$$\nabla^2 p = -\rho_o \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}$$

for the pressure in incompressible flow. Batchelor (1951) and Kraichnan (1956) have discussed the pressure fluctuations in incompressible isotropic turbulence using the Poisson equation. Their results are similar and will be used later.

The incompressible flow fluctuations arise essentially from the fact that the spectrum V is concentrated, for a particular frequency ω, around wave-numbers of order ω/u_v. For these wave-numbers, $a_o^2k^2 \gg \omega^2$; the term involving β in (2.6) may also be neglected, and we obtain (2.7).

There is, however, the possibility of near-resonance arising when $V(\underline{k}, \omega)$ has significant values for particular values of k, ω satisfying $\omega = a_o k$. Fluctuations generated by spectral components of the forcing field W_{ij} with $\omega = a_o k$ correspond to motions of an acoustic kind. To obtain these acoustic fluctuations we consider the integral in (2.6) over the narrow resonance peak near $k = \omega/a_o$. Take $V(k, \omega)$ as constant at the value $V(\omega/a_o, \omega)$, perform the integration over the resonance band and then let $\beta \longrightarrow 0$. This gives the following result for density fluctuations limited only by the small diffusive constant β,

$$\bar{\rho^2} = \frac{4\pi^2}{\beta a_o^5} \int_0^\infty \omega^2 V(\frac{\omega}{a_o}, \omega) d\omega \quad . \qquad (2.8)$$

This formula holds provided all frequencies ω making a significant contribution to the acoustic spectrum satisfy $\beta\omega/a_o^2 \ll 1$. We can also perform the integration of (2.6) over frequency first. Supposing this time that $\beta k/a_o \ll 1$ for all relevant wave-numbers we find

$$\bar{\rho^2} = \frac{4\pi^2}{\beta a_o^2} \int_0^\infty k^2 V(k, a_o k) dk. \qquad (2.9)$$

As expected, (2.8) and (2.9) are symmetrical in k, ω ; either is obtained from the other by putting $\omega = a_o k$, a substitution which shows that the requirements $\beta\omega/a_o^2 \ll 1$, $\beta k/a_o \ll 1$ are equivalent. Equation (2.8) will be analysed dimensionally in §4, and will be shown to be consistent with the result expressed in (1.7), obtained by Lighthill (1955) on the basis of Proudman's work.

3. Typical Frequencies and Wave-numbers

We want here to decide what are the typical frequencies and wave-numbers in the acoustic and vorticity modes of the turbulence, so as to make an estimate of the integrals in (2.8) or (2.9). The suffix s denotes a quantity representative of the irrotational acoustic mode, the suffix v a quantity representative of the solenoidal vorticity mode.

At sufficiently low Mach numbers, the turbulence can be regarded as incompressible from the point of view of the sound generation process. a_o can then play no part in determining frequencies in the vorticity mode. The only frequency which can be defined in the spectral range which contains most of the energy of the turbulence is $\omega_v = u/L_o$. The characteristic wave-number for the vorticity mode is $k_v = 1/L_o$.

Consider now Fig.1, which shows a possible contour of maximum source power spectral density. The turbulence is not convected relative to a main-stream flow, $V(k, \omega)$ is even in k and in ω, and so the contour is symmetrical about both axes. Now the only contribution to the acoustic integrals in 2.8 or 2.9 comes from points on the sonic line $\omega = a_o k$, and at sufficiently low Mach number, the source power along the sonic line OT <u>must</u> be approximately equal to that along the zero wave-number line OS. Frequencies which can contribute to the acoustic integrals lie in OS, while the only wave-numbers which can contribute are those in the range OR. Wave-numbers beyond R (i.e. those for which $k > M/L_o$) are unable to contribute to the acoustic integrals, because the frequencies to which they correspond via the sonic line are outside the frequency range which carries most of the source power. A given wave-number can only contribute if the frequency to which it is acoustically matched is less than the typical frequency $\omega_v = u_v/L_o$.

Since OT = OS as $M \longrightarrow 0$, this means that the frequency characteristic of the acoustic mode is $\omega_s = \omega_v = u_v/L_o$, while the characteristic wave-number is $k_s = M/L_o = Mk_v$. The typical wavelength is then $\lambda_s = L_o/M \gg L_o$, so that the situation we have here may be termed "acoustically thin". Turbulence scales are very much smaller than a wave-length of the sound produced by the turbulence. The acoustically thin situation is a necessary consequence of the assumption of low Mach number.

We have then the following characteristic quantities:

$$\omega_s = \omega_v = \frac{u_v}{L_o} , \quad k_v = \frac{1}{L_o} , \quad k_s = M/L_o .$$

Taking $\beta = 2\nu$ again, the approximation $\beta\omega/a_o^2 \ll 1$ of the last section will be justified if

$$2M^2(Re)^{-1} \ll 1 .$$

This is certainly well satisfied in all relevant situations, and is consistent with common experience that sound waves of moderate frequency are not significantly damped over a distance of a few wavelengths.

4. Dimensional Analysis

The function V_{ijlm} is the spectral density of the mean square fluctuation in W_{ij},

$$\overline{W_{ij}W_{lm}} = \int V_{ijlm}(\underline{k}, \omega)d\underline{k}d\omega \ . \qquad (4.1)$$

We suppose that we can approximate W_{ij} by the fluctuation in $\rho_o u_i u_j$, where u_i is the velocity in the vorticity mode. The left side of (4.1) then has the value

$$\rho_o^2 \ \overline{(u_i u_j - \overline{u_i u_j})(u_l u_m - \overline{u_l u_m})} \ .$$

We can also suppose that the one-point distribution of the u_i is Gaussian. This should give a correct order of magnitude estimate. The above mean value is then equal to

$$\rho_o^2 u_v^4 (\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})$$

when the turbulence is isotropic.

From this and (4.1) we can express a typical value $V_{ijlm}^{(o)}$ of V_{ijlm} in terms of u_v, ρ_o, L_o:

$$\rho_o^2 u_v^4 (\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) = V_{ijlm}^{(o)} L_o^{-3} \frac{u_v}{L_o} \ . \qquad (4.2)$$

The typical value of $V(k, \omega)$ is then given by

$$V^{(o)} = 2\rho_o^2 u_v^3 L_o^4 \ . \qquad (4.3)$$

If we now substitute this value in (2.8) or (2.9) and perform the integrations over ω, k using the characteristic values ω_s, k_s given in §3, we find, on writing $\beta = 2\nu$ as before, that

$$\bar{p^2}/\rho_o^2 = 40M^5 Re . \qquad (4.4)$$

The functional form of this equation is identical with that of (1.7), while the coefficients are of the same order of magnitude. Lighthill's equation (1.4), based on Proudman's estimate of the power output of isotropic turbulence, is thus confirmed by the method given here.

At low Mach numbers, the mean square acoustic pressure and velocity fluctuations are given by

$$\bar{p_s^2} = a_o^4 \bar{\rho^2} = \rho_o^2 u_v^4 (40MRe), \quad \text{and} \qquad (4.5)$$

$$\bar{p_s^2} = \rho_o^2 a_o^2 \bar{u_s^2}, \quad \text{or}$$

$$\bar{u_s^2} = u_v^2 (40M^3 Re) . \qquad (4.6)$$

To these we may add (cf. Lighthill, 1955) the values of $\bar{p_v^2}$ and u_v^2 involved in maintaining the local incompressible eddy turbulence (the vorticity mode). The pressure fluctuations in incompressible flow have been calculated by Batchelor (1951) and Kraichnan (1956), with the result

$$\bar{p_v^2} = A\rho_o^2 u_v^4 , \qquad (4.7)$$

where A = 0.34 (Batchelor), or A = 1 (Kraichnan). Addition of the two (uncorrelated) contributions gives

$$\bar{p^2} = \rho_o^2 u_v^4 (A + 40MRe) \qquad (4.8)$$

for the total pressure fluctuations, and

$$\bar{q^2} = u_v^2 (1 + 40M^3 Re)$$

for total velocity fluctuations.

We can regard these formulae as giving the first two terms in an expansion of mean square quantities in terms of Mach number.

The above results are relevant to the case of infinite scale R. Analogous results for the inviscid limit $aR \ll 1$ also follow directly from (1.8). The two sets are compared below. For consistency, Lighthill's equation (1.7) has been used, rather than (4.4). Provided $R \gg L_o$, the result (4.7) is independent of R; acoustic pressures depend upon R, since they fall off only as r^{-1} whereas the vorticity-mode pressure field falls off as r^{-3}.

Infinite Scale Viscous Limit	Finite Scale Inviscid Limit $aR \ll 1$
$\bar{p^2} = \rho_o^2 u_v^4 (A + 20MRe)$	$\bar{p^2} = \rho_o^2 u_v^4 (A + 40M^4 \frac{R}{L_o})$
$\bar{q^2} = u_v^2 (1 + 20M^3 Re)$	$\bar{q^2} = u_v^2 (1 + 40M^6 \frac{R}{L_o})$
$\bar{\rho^2} = \rho_o^2 (20M^5 Re)$	$\bar{\rho^2} = \rho_o^2 (40M^8 \frac{R}{L_o})$

The effect of compressibility with increasing Mach number will evidently show in the value of the pressure before it becomes noticeable in any other quantity. In the case of infinite scale, the acoustic pressure will greatly exceed that in the vorticity mode even at very low Mach numbers. For example, if we use the figures given by Batchelor (1953, p.124), viz. $u_v = 0.5$ met.sec^{-1}, $L_o = 100$ met. as typical in atmospheric turbulence, then we have in the case of infinite scale

$$\bar{p_s^2} \sim 10^5 \rho_o^2 u_v^4 \gg \bar{p_v^2} \quad ,$$

though $\bar{\rho^2}/\rho_o^2 \sim 10^{-6}$, so that there would still be no tendency towards cavitation.

However, although the atmosphere seems to provide the largest volume of turbulence for which any reliable figures are available, its scale length, of the order of 5 km perhaps, certainly satisfies the condition $aR \ll 1$ for the inviscid limit. The corresponding value of $\bar{p_s^2}$ is then only of order $10^{-7}\rho_o^2 u_v^4$.

There are thus two observations to be made from this example. Firstly, while the pressure fluctuation in the vorticity mode are correctly predicted, in order of magnitude, for a finite region by the infinite scale model, those in the acoustic mode are not, unless the region concerned is very large indeed. For example, the atmosphere would have to have a scale length of order 10^{16} metres for the viscous limit to apply. Secondly, the acoustic pressure fluctuations are not likely to be appreciable in the case of low

Mach number turbulence unless the turbulence occupies a region with scale vastly greater than any that can be found in a terrestrial example.

5. The Two-Dimensional Singularity

We now consider the intensity at a fixed distance z above the centre of a circular sheet of radius R. The sheet is composed of acoustic sources whose strengths Q per unit area are stationary random functions of space and time. Working this time in terms of the induced pressure fluctuation p we have

$$p = \frac{1}{4\pi}\int \frac{Q(\underline{y}, t - r/a_o)}{r} \exp(-\frac{\alpha r}{2})\, d\underline{y} \ , \qquad (5.1)$$

r being the distance from the element of area $d\underline{y}$ to the point considered. A damping factor has been included, with the same value of α as that used previously. Then

$$\bar{p^2} = \frac{1}{(4\pi)^2}\iint \frac{\overline{Q(\underline{y}, t - r/a_o)Q(\underline{y}', t-r'/a_o)}}{rr'} \times \exp\left[-\frac{\alpha}{2}(r + r')\right] d\underline{y}d\underline{y}' \ . \qquad (5.2)$$

If the sources are correlated over an area S* which is small compared with the area of the sheet, and if retarded-time differences for points separated by less than a correlation length $\sqrt{S^*}$ are small compared with the time-scale for variations in the source strength, (5.2) can be reduced to the approximate result

$$\bar{\bar{p^2}} = \frac{\bar{\bar{Q^2}}S^*}{16\pi^2} \int \frac{\exp(-ar)}{r^2}\, d\underline{y}$$

$$= \frac{\bar{\bar{Q^2}}S^*}{8\pi} \left[ei(az) - ei(ar) \right] , \qquad (5.3)$$

where $ei(x) = \int_x^\infty \frac{e^{-u}}{u}\, du$.

For the inviscid limit, we let $a \longrightarrow o$, keeping r finite, and obtain

$$\bar{\bar{p^2}} = \frac{\bar{\bar{Q^2}}S^*}{16\pi} \ln(1 + \frac{R^2}{z^2}) , \qquad (5.4)$$

since $ei(x) \sim - \ln x$ as $x \longrightarrow o$. Thus $\bar{\bar{p^2}} \longrightarrow \infty$ like $\ln R$ as $R \longrightarrow \infty$. On the other hand, for the viscous limit we let $r, R \longrightarrow \infty$ to obtain

$$\bar{\bar{p^2}} = \frac{\bar{\bar{Q^2}}S^*}{16\pi} ei(az) \qquad (5.5)$$

The divergence of the mean square pressure, as given by (5.4), is extremely slow, so that there is little possibility of finding any realistic example in which the pressure fluctuations become so large that they are limited only by viscous effects. Nonetheless, these results show that it is not possible, for example, to represent finite scale situations by models involving infinite spatial distributions of source if we want a realistic estimate of the sound produced.

The same ideas apply to the steady state wave motion produced in a thin plate, or on the surface of deep water, by a homogeneous stationary pressure field applied over the surface of the plate or the water. Steady state energy conservation requires a $1/r$ fall-off of mean square velocities, deflexions, etc., where r is distance in the surface from any source position. This $1/r$ decay then implies that the mean square velocity at any point on the surface increases with the logarithm of the typical length R over which the surface pressure acts. The response in the strict homogeneous limit $R \longrightarrow \infty$ must be limited by some kind of dissipation, or by nonlinear effects. The steady-state solution, bounded by some mechanism, is, however, not relevant to problems in which only a finite region is directly excited by externally applied pressures.

6. Conclusions

The pressure fluctuations in a compressible fluid in stationary isotropic turbulent motion depend critically on the size of the turbulent volume. Finite compressibility permits each element to act as a source of acoustic energy, and in virtue of the assumptions of stationarity and homogeneity the distribution of these sources is statistically uniform in space and time. The inverse square law of acoustics - which applies if diffusive effects are neglected - then implies that the intensity at any point diverges in proportion to the linear scale of the region. If viscous damping is included, the simple argument of §1 shows how the pressure fluctuations in an infinite medium are bounded by dissipative effects, and also define a scale length below which the effect of viscosity can reasonably be ignored. The same result is obtained if Lighthill's wave equation is modified by the inclusion of a damping factor, and then solved formally as in §§2 and 4.

The results for finite and infinite scale are very different, and the model of an infinite expanse of compressible fluid in homogeneous turbulent motion will not give results which are relevant to real flows in a volume satisfying the condition $aR \ll 1$ for the inviscid limit. In contrast, results derived for an infinite expanse of strictly incompressible fluid may be applied to the motion of a finite volume of fluid, provided only that the length R is large compared with the scale L_o of the turbulent motion.

All easily conceived turbulent motions at low Mach number which are approximately homogeneous and stationary seem to satisfy the condition for the inviscid limit - except perhaps those on astrophysical scales - unless the dissipation assumed here has been vastly underestimated. For example, one might think it more realistic to use an eddy diffusivity β_e in (2.1) instead of the molecular diffusivity β, although such an idea could not leave the right hand side of the Lighthill equation with its usual form. If this were done, and β_e were so large that the effective Reynolds number were reduced to order one, we should then find that the infinite scale acoustic pressures were somewhat <u>smaller</u> than the eddy motion pressures. However, egen though the scale R required to achieve the viscous limit would be much reduced in the case $Re \sim 1$, R would still be very much greater than any length occurring in terrestrial problems. The viscous limit seems therefore <u>never</u> to be relevant, except perhaps in astrophysical problems, even when an eddy diffusivity is used to explain the action of the small scale turbulence on the large scale sound waves it produces ($\beta_e \sim 10^7 \beta$ is assumed here as an exaggerated upper limit on β_e for atmospheric turbulence).

We conclude that the acoustic pressures occurring in turbulent motion do depend upon the scale R of the turbulent volume in the manner given by (1.8), provided M is small and R is typical of terrestrial motions. The acoustic pressures seem to be always negligible compared with those involved in maintaining the eddy turbulence.

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Fig. 1. FEATURES OF SOURCE SPECTRUM AT LOW MACH NUMBER