Networked Predictive Control of Uncertain Constrained Nonlinear Systems: Recursive Feasibility and Input-to-State Stability Analysis

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***Abstract*—In this paper, the robust state feedback stabilization of uncertain discrete-time constrained nonlinear systems in which the loop is closed through a packet-based communication network is addressed. In order to cope with model uncertainty, time-varying transmission delays, and packet dropouts (typically affecting the performances of networked control systems), a robust control scheme combining model predictive control with a network delay compensation strategy is proposed in the context of non-acknowledged UDP-like networks. The contribution of the paper is twofold. First, the issue of guaranteeing the recursive feasibility of the optimization problem associated to the receding horizon control law has been addressed, such that the invariance of the feasible region under the networked closed-loop dynamics can be guaranteed. Secondly, by exploiting a novel characterization of regional Input-to-State Stability in terms of time-varying Lyapunov functions, the networked closed-loop system has been proven to be Input-to-State Stable with respect to bounded perturbations.**

***Index Terms*—Networked Control Systems, Nonlinear Control, Model Predictive Control.**

I. Introduction

In the past few years, applications in which sensor data and actuator commands are sent through a shared communication network have attracted increasing attention in control engineering, since network technologies provide a convenient way to remotely control large distributed plants [2], [15], [49]. Major advantages of these systems, usually referenced to as Networked Control Systems (NCS's), include low cost, reduced weight and power requirements, and simple installation and maintenance. Conversely, NCS's are affected by the dynamics introduced by both the physical link and the communication protocol, that, in general, need to be taken in account in the design of the control architectures.

As many applications converge in sharing computing and communication resources, issues of scheduling, network delays, and data losses will need to be dealt with systematically. In particular, the random nature of transmission delays in shared networks makes it difficult to analyze stability and performances of the closed-loop systems. Remarkably, random delays are inherently related with the problem of data losses in NCS's. Indeed the stringent bounds imposed on time-delays by closed-loop stability requirements lead to the necessity to discard those packets arriving later than a maximum tolerable delay threshold. In addition, when the design of feedback control systems concerns wireless sensor networks, the implicit assumption of data availability no longer holds, as data packets are randomly dropped and delayed.

While classical control theories provide many analytical results to design the various components of the control system, they critically rely on the assumption that the underlying communication technology is ideal. In the networked communication setting, with possibly shared resources, neglecting network-induced perturbations such as delays and packet losses can eventually compromise the stability of the closed-loop system, if no proper provisions are adopted.

Various control strategies have been presented in the literature to design effective NCS's for linear time-invariant systems [11], [24], [38], [42] in presence of lossy or delayed communications. In particular, many recent results are focused on characterizing the stability properties of the closed-loop NCS's in a stochastic framework when static state-feedback control laws or LQG policies are adopted in presence of random transmission delays and packet dropouts [7], [10], [16], [39].

Besides the development of inherently stable controllers for these systems, another important aspect in the deployment of an effective NCS is the choice of the communication protocol to be used. In this regard, the packet structure of most transmission networks has important implications from the control point of view [45]. For example, when shared resources are used, it is not possible to increase arbitrarily the data transfer rate, due to the subsequent increase of network congestion, delays and packet dropouts. An effective way to overcome this limitation consists in using protocols which allow to transmit fewer but more informative packets [1], [11]. Thus, large data packets can be used to collect multiple sensors data and send predictions on future control inputs, without significantly increasing the network load [36], [40]. Predictive NCS schemes have been effectively used to compensate for network delays occurring on the measurement channel [28], or in presence of etherogenous measurements collected by both point-to-point wired instruments and distributed networked sensors (see [26] and [25], which also report a detailed stability analysis for the overall distributed system based on Lyapunov methods). Recently, also the delays occurring in the controller-to-actuator link have been considered by several authors (see the recent contribution [14] and the references therein). Finally, in the case of distributed control configurations with networked sensors and actuators, it is necessary to take into account

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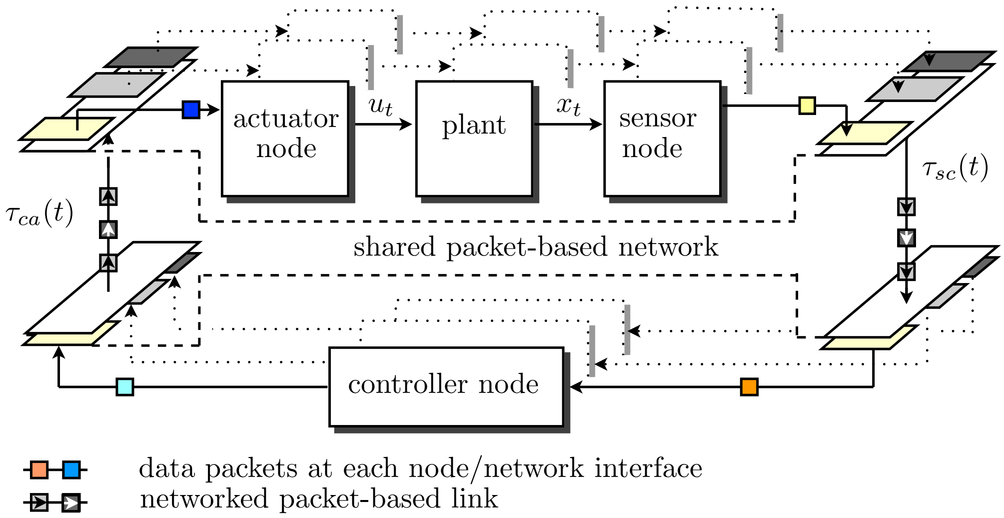

Fig. 1: Scheme of a NCS with multiple loops closed through a shared packet-based network with delayed data transmission.

the simultaneous presence of transmission delays and packet dropouts in both the up-link and down-link channels, [24].

The basic layout of an NCS with multiple loops sharing a packet-based communication network is depicted in Fig. 1, where, in order to distinguish the time delays in the sensor-to-controller and controller-to-actuator links, the network has been partitioned in two segments affected by delays $\tau_{sc}(t)$ and $\tau_{ca}(t)$, respectively.

When strict bounds on data delays and losses can be assumed and large data packets are allowed, Model Predictive Control (MPC) strategies have been proposed to cope with the design of a stabilizing NCS [7], [41], due to their intrinsic features of generating a future input sequence that can be transmitted within a single data packet.

While the aforementioned existing control design techniques rely on linear process models, if the system to be controlled is subject to constraints and nonlinearities, the formulation of an effective networked control strategy becomes really a hard task [37]. In this framework, the present paper provides theoretical results that motivate, under suitable assumptions, the combined use of nonlinear MPC with a Network Delay Compensation (NDC) strategy [4], in order to cope with the simultaneous presence of constraints, model uncertainties, time-varying transmission delays, and data-packet losses. The proposed methods, compared to the existing model-based delay compensation approaches for discrete-time systems (see [36], and the references therein), allows to cope with non-acknowledged UDP-like networks, by introducing the concept of reduced-horizon optimization in the MPC formulation. Moreover, compared to recent contributions on non-acknowledged predictive NCS (see e.g. [43] and [12]), it also allows to enforce hard constraints on state and input variables despite bounded transition uncertainty, by exploiting ideas from constraint-tightening nonlinear MPC.

In the current literature, for the specific class of MPC schemes which impose a fixed terminal constraint set, X_f, as a stabilizing condition, the robustness of the overall closed-loop system, in absence of transmission delays, has been shown to depend on the invariance properties of X_f, (see [19], [22] and [35]). In this regard, by resorting to invariant set theoretic arguments [5], [19], this paper aims to show that the devised NCS can robustly stabilize a nonlinear constrained system even in presence of data transmission delays and model uncertainty. In particular, the issue of recursive feasibility in constrained networked nonlinear MPC, first addressed in [34], in this paper is shown to be key point to prove the Input-to-State Stability (ISS) of the scheme w.r.t. additive perturbations. Indeed, by exploiting a novel regional characterization of ISS in terms of time-varying Lyapunov functions (the regional ISS for the time-invariant case has been introduced in [27], while semi-global results for time-varying discrete-time systems are given in [18], [20]), the closed-loop system is shown to be ISS with respect to the aforementioned class of disturbances, also in presence of unreliable networked communications.

The paper is organized as follows: in Section II, some useful definitions and stability notions are introduced, together with a novel preliminary result concerning the regional characterization of ISS in terms of time-varying Lyapunov functions. Then, by posing some assumptions on the communication network and on the system to be controlled, a control scheme for non-acknowledged UDP-like networks, based on the combined use of a delay compensation strategy and model predictive control (MPC–NDC), is presented in Section III. The recursive feasibility of the scheme and the stability properties of the closed-loop system are then analyzed in Section IV. Finally, a simulation example is presented in Section V to show the effectiveness of the proposed networked control methodology.

II. Notations, Definitions, and Preliminary Results

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer, and the non-negative integer sets of numbers, respectively. The Euclidean norm is denoted as $|\cdot|$. For any discrete-time sequence $\boldsymbol{\upsilon} : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$, define $\|\boldsymbol{\upsilon}\| \triangleq \sup_{k\geq 0}\{|\upsilon_k|\}$ and $\|\boldsymbol{\upsilon}_{[\tau]}\| \triangleq \max_{0\leq k\leq \tau}\{|\upsilon_k|\}$, where υ_k denotes the value that the sequence $\boldsymbol{\upsilon}$ takes on in correspondence with the index k. The set of discrete-time sequences

$\boldsymbol{\upsilon}$ taking values in some subset $\Upsilon \subset \mathbb{R}^m$ is denoted by $\mathcal{M}_\Upsilon$. Given a sequence $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$ and two non-negative integers $k, t \in \mathbb{Z}_{\geq 0}$, with $t \geq k$, we will denote as $\boldsymbol{\upsilon}_{k,t}$ the vector formed by the subsequence of elements indexed from k to t (i.e., $\boldsymbol{\upsilon}_{k,t} \triangleq \mathrm{col}\,(\upsilon_k, \upsilon_{k+1}, \ldots, \upsilon_{t-1}, \upsilon_t)$). Given a compact set $A \subset \mathbb{R}^n$, let ∂A denote the boundary of A. Given a vector $x \in \mathbb{R}^n$, $\mathrm{d}(x, A) \triangleq \inf\{|\xi - x|, \xi \in A\}$ is the point-to-set distance from $x \in \mathbb{R}^n$ to A. Given two compact sets $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^n$, $\mathrm{dist}(A, B) \triangleq \inf\{\mathrm{d}(\zeta, A), \zeta \in B\}$ is the minimal set-to-set distance. The difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, is denoted as $A \backslash B \triangleq \{x : x \in A, x \notin B\}$. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, the Pontryagin difference set C is defined as $C = A \backsim B \triangleq \{x \in \mathbb{R}^n : x + \xi \in A, \forall \xi \in B\}$. Given a vector $\eta \in \mathbb{R}^n$ and a scalar $\rho \in \mathbb{R}_{>0}$, the closed ball in $\mathbb{R}^n$ centered in η of radius ρ is denoted as $\mathcal{B}^n(\eta, \rho) \triangleq \{\xi \in \mathbb{R}^n : |\xi - \eta| \leq \rho\}$. The shorthand $\mathcal{B}^n(\rho)$ is used when $\eta = 0$. The symbol id represents the identity function from $\mathbb{R}$ to $\mathbb{R}$, while $\gamma_1 \circ \gamma_2$ is the composition of two functions γ_1 and γ_2 from $\mathbb{R}$ to $\mathbb{R}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, $\alpha(0) = 0$, and it is strictly increasing. It belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and is unbounded.

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \to 0} \beta(s, t) = \lim_{t \to \infty} \beta(s, t) = 0$.

Let us consider the time-varying discrete-time dynamic system

$$x_{t+1} = g(t, x_t, \upsilon_t),\ t \in \mathbb{Z}_{\geq 0},\ x_0 = \overline{x}\,, \tag{1}$$

where $g(t, 0, 0) = 0$, $\forall t \geq \overline{T}$ (with $\overline{T} \in \mathbb{Z}_{\geq 0}$) and where $x_t \in \mathbb{R}^n$ and $\upsilon_t \in \Upsilon \subset \mathbb{R}^r$, with Υ compact, denote the state and the bounded input of the system, respectively. The discrete-time state trajectory of the system (1), with initial state $x_0 = \overline{x}$ and input sequence $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$, is denoted by $x(t, \overline{x}, \boldsymbol{\upsilon}_{0,t-1}),\ t \in \mathbb{Z}_{\geq 0}$.

Definition 2.1 (RPI set): A set $\Xi \subset \mathbb{R}^n$ is a Robust Positively Invariant (RPI) set for system (1) if, for all $t \in \mathbb{Z}_{\geq 0}$, it holds that $g(t, x_0, \upsilon) \in \Xi$, $\forall x_0 \in \Xi$ and $\forall \upsilon \in \Upsilon$. □

In the following, the Regional Input-to-State Stability property, recently introduced in [27], is recalled. It is worth noting that regional results are needed in the framework of nonlinear MPC due to the impossibility to obtain, in general, global bounds on the finite horizon costs used as Lyapunov function in the stability analysis. Nonetheless, in the framework of NCS's, due to the variability of transmission delays, a time invariant formulation is not suited, therefore it is necessary to extend the regional ISS analysis in order to cope with time-varying Lyapunov functions (see [6] and [29]).

A. A regional ISS result for time-varying systems

The following definition of regional ISS is provided for time-varying discrete-time nonlinear systems of the form (1) .

Definition 2.2 (Regional ISS): Given a compact set $\Xi \subset \mathbb{R}^n$, if Ξ is RPI for (1) and if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$|x(t, \overline{x}, \boldsymbol{\upsilon}_{0,t-1})| \leq \max\left\{\beta(|\overline{x}|, t), \gamma(\|\boldsymbol{\upsilon}_{[t-1]}\|)\right\}, \tag{2}$$

$\forall t \in \mathbb{Z}_{\geq 0}, \forall \overline{x} \in \Xi$, then the system (1), with $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$, is said to be ISS for initial conditions in Ξ. □

In the literature, there exist some recent results concerning the characterization of the ISS property in terms of time-varying Lyapunov functions for perturbed (uncertain) discrete-time system [18], [20]; on the other hand, those results guarantee the ISS property in a semi-global sense with smooth ISS-Lyapunov functions and cannot be trivially used in MPC. Indeed, for systems controlled by MPC schemes, the stability analysis has to be carried out by using non-smooth ISS-Lyapunov functions [27]. Therefore, a novel regional ISS result for a family of time-varying Lyapunov functions is derived to assess the stability properties of MPC-based NCS's.

To this end, let us first consider the following definition.

Definition 2.3 (Time-varying ISS-Lyapunov Function): Given a pair of compact sets $\Xi \subset \mathbb{R}^n$ and $\Omega \subseteq \Xi$, with Ξ RPI for system (1) and $\{0\} \subset \Omega$, a function $V(\cdot,\cdot) : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a (regional) time-varying ISS-Lyapunov function in Ξ, if there exist $\mathcal{K}_\infty$-functions α_1, α_2, α_3, and $\mathcal{K}$-functions σ_1 and σ_2, such that

1) the following inequalities hold $\forall \upsilon \in \Upsilon$, with Υ compact, and $\forall t \in \mathbb{Z}_{\geq 0}$:

$$V(t, x) \geq \alpha_1(|x|), \qquad \forall x \in \Xi, \tag{3}$$

$$V(t, x) \leq \alpha_2(|x|) + \sigma_1(|\upsilon|), \qquad \forall x \in \Omega, \tag{4}$$

$$V(t{+}1, g(t, x, \upsilon)) - V(t, x) \leq -\alpha_3(|x|) + \sigma_2(|\upsilon|),\ \forall x \in \Xi\,; \tag{5}$$

2) there exist some suitable $\mathcal{K}_\infty$-functions ϵ and ρ (with ρ such that $(id - \rho)$ is a $\mathcal{K}_\infty$-function, too) and a scalar $c \in \mathbb{R}_{>0}$ such that the set

$$\Theta \triangleq \{x : V(t, x) \leq b(\overline{\upsilon}), \forall t \in \mathbb{Z}_{\geq 0}\}, \tag{6}$$

verifies the inclusion

$$\Theta \subseteq \Omega \backsim \mathcal{B}^n(c), \tag{7}$$

with $\{0\} \subset \Theta$, $b(s) \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4(s)$, $\alpha_4 \triangleq \underline{\alpha_3} \circ \overline{\alpha_2}^{-1}$, $\underline{\alpha_3}(s) \triangleq \min(\alpha_3(s/2), \epsilon(s/2))$, $\overline{\alpha}(s) \triangleq \alpha_2(s) + \sigma_1(s)$, $\sigma_4 = \epsilon(s) + \sigma_2(s)$, and $\overline{\upsilon} \triangleq \max_{\upsilon \in \Upsilon}\{|\upsilon|\}$.

□

The following remark provides some further insight into the meaning of Condition 2) in Definition 2.3 above.

Remark 2.1: Due the fact that, in Definition 2.3, the set Ξ has been assumed to be RPI, condition (7) is always verified for a suitably small compact set Υ (and hence $\overline{\upsilon}$). Setting $\overline{\xi} \triangleq \inf_{\xi \in \mathbb{R}^n \backslash \Xi}\{|\xi|\}$, and noting that $\overline{\xi}$ is strictly positive, a sufficient condition for (7) to hold is that

$$\overline{\upsilon} \leq b^{-1}\left(\alpha_1(\overline{\xi} - c_\upsilon)\right), \tag{8}$$

for some $c_\upsilon \in \mathbb{R}_{>0}$, with $c_\upsilon < \overline{\xi}$. Indeed from (8) it follows that $b(\overline{\upsilon}) \leq \alpha_1(\overline{\xi} - c_\upsilon)$. Then, $\forall \xi : |\xi| > \overline{\xi} - c_\upsilon$, it holds that $V(t, \xi) \geq \alpha_1(|\xi|) > b(\overline{\upsilon})$, which implies $\Theta \subseteq \mathcal{B}^n(\overline{\xi} - c_\upsilon) \subseteq \Xi \backsim \mathcal{B}^n(c_\upsilon)$. Due to the inherent conservativeness of the comparison function approach, in practice it turns out that the uncertainty bound given by (8) is in general smaller than that for which the invariance of Ξ can be guaranteed. On the other hand, it is anyway worth emphasizing the convergence towards the origin in presence of small uncertainty, while the robust constraint satisfaction (related to the concept of set invariance rather then to comparison inequalities) can be enforced for larger uncertainties. □

Notably, the ISS-Lyapunov inequalities (3), (4), and (5) differ from those posed in the original regional ISS formulation [27], since an input-dependent upper bound is admitted in (4) (thus allowing for a more general characterization). Moreover, with regard to the regional ISS result presented in [9], the ISS-Lyapunov function $V(t,x)$ is allowed to belong to a family of time-varying functions. Remarkably, the possibility to incorporate an input-dependent upper bound in (4) and to admit a time-varying characterization will be instrumental to characterize the ISS property for NCS's (see Section IV).

Now, consider the following assumption.

Assumption 1: For every $t \in \mathbb{R}_{>0}$, the state trajectories $x(t, \overline{x}_0, \boldsymbol{\upsilon}_{0,t-1})$ of the system (1) are continuous in $\overline{x}_0 = 0$ and $\boldsymbol{\upsilon}_{0,t-1} = 0$ with respect to the initial condition $\overline{x}_0$ and the disturbance sequence $\boldsymbol{\upsilon}_{0,t-1}$. □

Then, the characterization of the regional ISS property in terms of Lyapunov functions is given by the following result.

Theorem 2.1 (Lyapunov characterization of regional ISS): Suppose that Assumption 1 holds. If system (1) admits a (time-varying) ISS-Lyapunov function in Ξ, then it is regional ISS in Ξ and $\lim_{t\to\infty} \mathrm{d}(x(t,\overline{x},\boldsymbol{\upsilon}_{0,t-1}),\Theta)=0\,,\ \forall \overline{x}\in\Xi.$ □

The proof of Theorem 2.1 is reported in Appendix A.

III. Problem Formulation

Consider the nonlinear discrete-time dynamic system

$$x_{t+1} = f(x_t, u_t, \upsilon_t),\ \ t\in\mathbb{Z}_{\geq 0},\ \ x_0 = \overline{x}\,, \tag{9}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ the control vector, and $\upsilon_t \in \Upsilon$ is an uncertain exogenous input vector, with $\Upsilon \subset \mathbb{R}^r$ compact and $\{0\} \subset \Upsilon$. Assume that state and control variables are subject to the constraints

$$x_t \in X,\ \ t\in\mathbb{Z}_{\geq 0}\,, \tag{10}$$

$$u_t \in U,\ \ t\in\mathbb{Z}_{\geq 0}\,, \tag{11}$$

where X and U are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point. Given system (9), let $\hat{f}(x_t,u_t)$, with $\hat{f}(0,0)=0$, denote the *nominal* model used for control design purposes. Moreover, let $\hat{x}_{t+j|t},\ j\in\mathbb{Z}_{>0}$ denote the state "prediction" generated by the nominal model on the basis of the state informations at time t under the action of the control sequence $\mathbf{u}_{t,t+j-1} = \mathrm{col}[u_t,\ldots,u_{t+j-1}]$, that is,

$$\hat{x}_{t+j|t}{=}\hat{f}(\hat{x}_{t+j-1|t},u_{t+j-1}),\hat{x}_{t|t}{=}x_t,\ \ t{\in}\mathbb{Z}_{\geq 0}, j{\in}\mathbb{Z}_{>0}. \tag{12}$$

Assumption 2 (Lipschitz): The nominal map $\hat{f}(x,u)$ is Lipschitz with respect to x in X, uniformly in $u\in U$, with Lipschitz constant[1] $L_{f_x}\in\mathbb{R}_{>0},\ L_{f_x}\neq 1$. □

Introducing the *additive transition uncertainty* vector $d_t \triangleq f(x_t,u_t,\upsilon_t) - \hat{f}(x_t,u_t)$, the true state dynamics is given by

$$x_{t+1} = \hat{f}(x_t,u_t) + d_t\,,\ \ t\in\mathbb{Z}_{\geq 0},\ \ x_0 = \overline{x}\,. \tag{13}$$

Assumption 3 (Uncertainty): The transition uncertainty vector d_t belongs to the compact ball $D \triangleq \mathcal{B}^n(\overline{d})$, where

$$\overline{d} \triangleq \max_{(x,u,\upsilon)\in X\times U\times\Upsilon} |f(x,u,\upsilon) - \hat{f}(x,u)|\,.$$

□

Under Assumptions 2 and 3, the control objective consists in guaranteeing the ISS property for the closed-loop system with respect to a given class of uncertainties, while enforcing the fulfillment of constraints in presence of packet dropouts, bounded transmission delays and bounded disturbances.

Having introduced the nominal transition map $\hat{f}(x,u)$, the following important definition can now be introduced.

Definition 3.1 ($\mathcal{C}_i(X,\Xi)$): Given a set $\Xi\subseteq X$, the *i-step Controllability Set* to Ξ, $\mathcal{C}_i(X,\Xi)$, is the subset of states in X which can be steered to Ξ by a control sequence of length i, $\mathbf{u}_{0,i-1}$, under the nominal map $\hat{f}(x,u)$, subject to constraints (10) and (11), i.e.,

$$\mathcal{C}_i(X,\Xi) \triangleq \left\{ \begin{array}{l} x_0 \in X : \exists \mathbf{u}_{0,i-1}\in U^i \ \text{ such that} \\ \hat{x}(t,x_0,\mathbf{u}_{0,i-1}) \in X,\ \forall t\in\{1,\ldots,i-1\}, \\ \text{and } \ \hat{x}(i,x_0,\mathbf{u}_{0,i-1})\in\Xi. \end{array} \right\}$$

□

In the sequel, the shorthand $\mathcal{C}_1(\Xi)$ will be used in place of $\mathcal{C}_1(\mathbb{R}^n,\Xi)$ to denote the one-step controllability set to Ξ, [5]. The notion of controllability set will be used to prove the robust stability of the proposed NCS.

A. Communication Protocol

As regards the network dynamics and communication protocol, it is assumed that a set of data (packet) can be sent, at a given time instant, through the network by a node, while both the sensor-to-controller and the controller-to-actuator links are supposed to be affected by delays and dropouts due to the unreliable nature of networked communications. In order to cope with network delays, the data packets sent by the sensor node are Time-Stamped (TS) [40], that is, they contain the information on when the transmitted state measurement had been collected. Analogously, the controller node is required to attach to each data packet the time stamp of the state measurement which the computed control action relies on. The advantage of using a time-stamping policy in NCS's is well documented [3], [49]; however it requires, in general, that all the nodes of the network have access to a common system's clock, or that a proper clock synchronization service is provided by the network protocol. In our setup, we will assume that perfect clock synchronization is maintained between sensors, actuators and controller. This task can be achieved in different ways (see [48], [50], [44] and the references therein), however we will abstract from the particular method used to maintain synchronization, since we are mainly focused on the control design issues rather than on the transmission protocol and the network scheduling policy. The next section will describe how the TS mechanism can be used to compensate for transmission delays.

[1]The very special case $L_{f_x}=1$ can be trivially addressed by a few suitable modifications to the proofs of the results of the paper.

B. Network delay compensation

As mentioned in the Introduction, $\tau_{ca}(t)$ and $\tau_{sc}(t)$ denote the delays occurring in the controller-to-actuator and in the sensor-to-controller links, respectively. Moreover, $\tau_a(t)$ represents the "age" (in discrete time instants) of the control sequence used by the actuator to compute the current input and $\tau_c(t)$ the age of the state measurement which had been used by the controller at time t to compute the control actions to be sent to the actuator. Finally, $\tau_{rt}(t) \triangleq \tau_a(t)+\tau_c(t-\tau_a(t))$ is the so called *round trip time*, i.e., the age of the state measurement used to compute the input applied at time t.

The NDC strategy adopted in the present work, which relies on the one devised in [36] (originally developed for unconstrained systems nominally stabilized by a generic nonlinear controller), is based on exploiting the time stamps of the data packets in order to retain only the most recent information at the destination nodes: when a novel packet is received, if it carries a more recent time-stamp than the one already in the buffer, then it takes the place of the older one. The TS-based packet arrival management implies $\tau_a(t) \leq \tau_{ca}(t)$ and $\tau_c(t) \leq \tau_{sc}(t)$. Moreover, the NDC strategy comprises a Future Input Buffering (FIB) mechanism (also known as "playback buffer", see [21] for details), requiring that the controller node send a packeted sequence of N_c (with $N_c \in \mathbb{Z}_{>0}$) control actions to the actuator node; such a sequence must be long enough to accommodate the worst case delay or the maximum number of successive packet losses. Indeed the actuator, at the arrival of each packeted sequence, first stores the data in its internal buffer and afterwards, at each time instant t, applies a time-consistent control action to the plant, by setting $u_t = u_t^b$, where u_t^b is the $\tau_a(t)$-th element of the buffered sequence $\mathbf{u}^b_{t-\tau_a(t),t-\tau_a(t)+N_c-1}$, which, in turn, is given by

$$\begin{aligned}&\mathbf{u}^b_{t-\tau_a(t),t-\tau_a(t)+N_c-1}\\&\quad = \mathrm{col}[u^b_{t-\tau_a(t)},\ldots,u^b_t,\ldots,u^b_{t-\tau_a(t)+N_c-1}]\\&\quad = \mathbf{u}^c_{t-\tau_a(t),t-\tau_a(t)+N_c-1|t-\tau_{rt}(t)},\end{aligned}$$

where the sequence $\mathbf{u}^c_{t-\tau_a(t),t-\tau_a(t)+N_c-1|t-\tau_{rt}(t)}$ had been computed at time $t-\tau_a(t)$ by the controller on the basis of the state measurement collected at time $t-\tau_{rt}(t) = t-\tau_a(t) - \tau_c(t-\tau_a(t))$.

Due to the capability of performing synchronization, buffering operations and management of time stamped packets, the actuation device will be addressed to as "smart"actuator. For a deeper insight on the input buffering mechanism, the reader is referred to [1] and [21].

In most situations, it is natural to assume that the age of the data-packets available at the controller and actuator nodes subsume an upper bound [36], as specified by the following

Assumption 4 (Network reliability): The quantities $\tau_c(t)$ and $\tau_a(t)$ verify $\tau_c(t) \leq \overline{\tau}_c$ and $\tau_a(t) \leq \overline{\tau}_a$, $\forall t \in \mathbb{Z}_{>0}$, with $\overline{\tau}_c \in \mathbb{Z}_{\geq 0}$ and $\overline{\tau}_a \in \mathbb{Z}_{\geq 0}$ finite. □

Notably, we don't impose bounds on $\tau_{sc}(t)$ and $\tau_{ca}(t)$, allowing the presence of packet losses (infinite delay). In this way, an actuator buffer with finite length can be used.

Assumption 5 (Buffer length): The actuator buffer length, which is equal to the length of the input sequence sent by the controller to actuator, verifies $N_c \geq \overline{\tau}_a + \overline{\tau}_c + 1 = \overline{\tau}_{rt} + 1$. □

In this work we will focus mainly on the more difficult and challenging case of networks with non-acknowledged communication protocols, also known as *UDP–like* [16], in which the controller is not informed by the actuator of successful packet delivery. At the opposite, in the *TCP-like* case, the destination node is assumed to send an acknowledgment packet (ACK) of successful packet receipt to the source node. Although many control-theoretic works postulate that, after a successful packet receipt, the source node receives a deterministic notification within a single time-interval (see [36]), this assumption is typically not valid in practice. Therefore, the analysis of a UDP-like scenario can lead to more realistic results and is therefore pursued in this paper. A pictorial representation of the overall NCS layout is depicted in Figure 2.

C. State reconstruction in UDP–like networks

At time t, the computation of the control sequence to be sent to the actuator must rely on a state measurement $x_{t-\tau_c(t)}$ obtained at time $t-\tau_c(t)$. In order to recover the standard MPC formulation, the current (possibly unavailable) state x_t has to be reconstructed by means of the nominal model (12) and of the input sequence $\mathbf{u}_{t-\tau_c(t),t-1}$ applied by the smart actuator to the plant, $\mathbf{u}_{t-\tau_c(t),t-1} \triangleq \mathrm{col}[u_{t-\tau_c(t)},\ \ldots,\ u_{t-1}]$ from time $t-\tau_c(t)$ to $t-1$. The sequence $\mathbf{u}_{t-\tau_c(t),t-1}$ must be internally reconstructed by the controller by exploiting the control actions computed at the previous time instants. In this regard, the problem of delayed arrival of packeted input sequences to the actuator represents a major source of uncertainty. Indeed, due to the delays that affect the controller-to-actuator link, we must take into account that the control sequences forwarded to the actuator may not be applied entirely to the plant. This problem, commonly known in NCS literature as "prediction consistency", has been recently approached by many researchers which have proposed different solution (see [8], [43] for sampled-data NCS's and [13] for discrete-time systems). To solve this problem, we propose to modify the usual MPC algorithm by introducing the Reduced Horizon Optimal Control Problem (RHOCP), described in detail in the following section.

D. Reduced horizon optimization

The class of algorithms which the considered controller belongs to is that of MPC, in which a finite-horizon optimal control problem, based on the current state measurement, is solved at each time step to obtain a control action to be applied to the plant, thus implicitly yielding a closed-loop control scheme. With reference to the aforementioned class of controllers, in which the length of the horizon is usually kept fixed and equals the number of decision variables of the optimization problem, the proposed method relies on the solution, at each time instant t, of a RHOCP, that is, the number of decision variables is (in general) reduced by reusing some elements of previous optimizations. This concept has been introduced in [34] in the framework of discrete-time

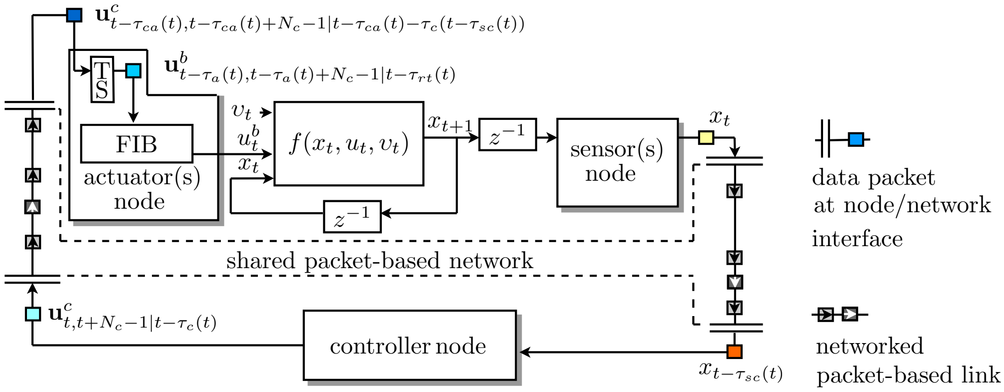

Fig. 2: Scheme of the NDC strategy. In evidence the Time-Stamping packet arrival management (TS) and the Future Input Buffering (FIB) mechanism at the actuator node.

systems and in [8] in the context of sampled-data control of continuous-time system. In particular, at time t, some of the elements of the control sequence computed at time $t-1$ are retained, while the optimization is performed only over the remaining elements by initializing the RHOCP with $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$, that, in turn, can be obtained from $\hat{x}_{t|t-\tau_c(t)}$ by prediction. The benefits due to the use of a state predictor in NCS's are deeply discussed in [36], [46], [47] and [40], [41].

With the aim to recast the formulation into a deterministic framework, such that the sequence used by the state-estimator to obtain $\hat{x}_t$ and by the predictor to obtain $\hat{x}_{t+\overline{\tau}_{rt}}$ would coincide with the true input sequence applied by the smart actuator, the optimization has to be performed over a shortened sequence $\mathbf{u}_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}$, consisting of $N_c-\overline{\tau}_{rt}$ control actions. To this end, the RHOCP has to be initialized with the predicted state $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$, obtained with the nominal model by propagating the trajectories from $x_{t-\tau_c(t)}$ with the sequence

$$\begin{aligned}&\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}\\&\quad=\mathrm{col}[\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}2|t-2-\tau_c(t-2)},u^c_{t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}],\end{aligned} \tag{14}$$

where $\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-2|t-2-\tau_c(t-2)}$ is a subsequence of $\mathbf{u}^*_{t-1-\tau_c(t-1),t-1+\overline{\tau}_{rt}-1|t-2-\tau_c(t-2)}$, retrieved from the previous step, while the control action $u^c_{t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}$ is the first element of the optimal subsequence $\mathbf{u}^\circ_{t+\overline{\tau}_{rt}-1,t+N_c-2|t-1-\tau_c(t-1)}$ obtained by solving the RHOCP at time $t-1$ (i.e., $u^c_{t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)} = u^\circ_{t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}$). Since the reduced-horizon optimization preserves the sequence $\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}$ from successive modification, it is guaranteed that the truly applied input sequence from $t-\tau_c(t)$ to $t+\tau_{rt}-1$ will coincide with the one used for reconstruction/prediction at time t, i.e. $\mathbf{u}_{t-\tau_c(t),t-1} = \mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}$.

Furthermore, we will show that the perturbed closed-loop trajectories can be enforced in the nominal constraints by providing the RHOCP with a *constraint tightening* technique [22], in which delay-dependent restrictions are introduced to guarantee the recursive feasibility of the scheme.

First, let us introduce the following sets, obtained by restricting the nominal constraint set X.

Definition 3.2 ($X_i(\overline{d})$): Under Assumptions 2 and 3, the tightened sets $X_i(\overline{d})$, are defined as

$$X_i(\overline{d}) \triangleq X \backsim \mathcal{B}^n\left(\frac{L^i_{f_x}-1}{L_{f_x}-1}\overline{d}\right),\ \forall i\in\mathbb{Z}_{>0}. \tag{15}$$

□

Now, we state the following basic RHOCP.

Problem 3.1 (RHOCP): Given a positive integer $N_c \in \mathbb{Z}_{\geq 0}$, at any time $t\in\mathbb{Z}_{\geq 0}$, let $\hat{x}_{t|t-\tau_c(t)}$ be the estimate of the current state, x_t, obtained from the last available state measurement $x_{t-\tau_c(t)}$ by the controls $\mathbf{u}_{t-\tau_c(t),t-1}$ already applied to the plant. Moreover, let $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$ be the state computed from $\hat{x}_{t|t-\tau_c(t)}$ by extending the prediction by using the input sequence computed at time $t-1$, $\mathbf{u}^c_{t,t+\overline{\tau}_{rt}-1}$. Then, given a stage-cost function h, the constraint sets $X_i(\overline{d}) \subseteq X, i \in \{\tau_c(t)+\overline{\tau}_{rt}+1,\ldots,\tau_c(t)+N_c\}$, a terminal cost function h_f and a terminal set X_f, the RHOCP consists in solving, with respect to a $(N_c-\overline{\tau}_{rt})$-steps input sequence, $\mathbf{u}_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)} \triangleq \mathrm{col}[u_{t+\overline{\tau}_{rt}|t-\tau_c(t)},\ldots,u_{t+N_c-1|t-\tau_c(t)}]$, the following minimization problem

$$\begin{aligned}&J^\circ_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)},\mathbf{u}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)},N_c-\overline{\tau}_{rt}) \triangleq\\&\min_{\mathbf{u}_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}}\Big[\sum_{l=t+\overline{\tau}_{rt}}^{t+N_c-1} h(\hat{x}_{l|t-\tau_c(t)},u_{l|t-\tau_c(t)})\\&\qquad+\ h_f(\hat{x}_{t+N_c|t-\tau_c(t)})\Big]\end{aligned}$$

subject to the

i) nominal dynamics (12);

ii) input constraints $u_{t-\tau_c(t)+i|t-\tau_c(t)} \in U$, with $i \in \{\tau_c(t)+\overline{\tau}_{rt},\ldots,\tau_c(t)+N_c-1\}$;

iii) restricted state constraints $\hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)} \in X_i(\overline{d})$, with $i\in\{\tau_c(t)+\overline{\tau}_{rt}+1,\ldots,\tau_c(t)+N_c\}$;

iv) terminal state constraint $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f$.

□

In the overall control algorithm, the sequence of control actions forwarded by the controller to the actuator are constructed by appending the solution of the RHOCP to the control sequence computed at time $t-1$, that is

$$\mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} \triangleq \mathrm{col}[\mathbf{u}^c_{t,t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}] \,.$$

The following definitions will be used in the rest of the paper.

Definition 3.3 ($X_{MPC}(\tau)$): Given a non-negative integer $\tau \in \mathbb{Z}_{\geq 0}$, the *feasible set with τ-delay restriction* is denoted with $X_{MPC}(\tau)$ and is defined as:

$$X_{MPC}(\tau) \triangleq \left\{ \overline{x}_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, \overline{\mathbf{u}}_{0,N_c-1} \in U^{N_c} : \\ \hat{x}(i, \overline{x}_0, \overline{\mathbf{u}}_{0,i-1}) \in X_{\tau+i}(\overline{d}), \forall i \in \{1,\ldots,N_c\} \\ \text{and } \hat{x}(N_c, \overline{x}_0, \overline{\mathbf{u}}_{0,N_c-1}) \in X_f \end{array} \right\} \tag{16}$$

The set $X_{MPC}(0)$ is denoted as X_{MPC} for short. □

Definition 3.4 (Feasible sequence at time t): Given a delayed state measurement $x_{t-\tau_c(t)}$, available at time t to the controller, let us consider the prediction $\hat{x}_{t|t-\tau_c(t)}$ of the actual state x_t obtained by the nominal model and by the actual control sequence applied from time $t-\tau_c(t)$ to $t-1$, $\mathbf{u}_{t-\tau_c(t),t-1}$, which is known to the controller. Moreover, consider a sequence of N_c control actions $\overline{\mathbf{u}}^c_{t,t+N_c-1}$ and its two subsequences $\overline{\mathbf{u}}^c_{t,t+\overline{\tau}_{rt}-1}$ and $\overline{\mathbf{u}}^c_{t+\overline{\tau}_{rt},t+N_c-1}$ such that $\overline{\mathbf{u}}^c_{t,t+N_c-1} = \mathrm{col}[\overline{\mathbf{u}}^c_{t,t+\overline{\tau}_{rt}-1}, \overline{\mathbf{u}}^c_{t+\overline{\tau}_{rt},t+N_c-1}]$.

The input sequence $\overline{\mathbf{u}}^c = \overline{\mathbf{u}}^c_{t,t+N_c-1}$ is said *feasible at time t* if the subsequence $\overline{\mathbf{u}}^c_{t,t+\overline{\tau}_{rt}-1}$ yields to $\hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)} \in X_i(\overline{d})$, $\forall i \in \{\tau_c(t)+1,\ldots,\tau_c(t)+\overline{\tau}_{rt}\}$ and if the second subsequence satisfies all the constraints of the RHOCP initialized with $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)} = \hat{x}(\overline{\tau}_{rt}, x_{t-\tau_c(t)}, \mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1})$, where $\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1} \triangleq \mathrm{col}[\mathbf{u}_{t-\tau_c(t),t-1}, \overline{\mathbf{u}}^c_{t,t+\overline{\tau}_{rt}-1}]$. □

Remark 3.1: Note that, what we call *feasible sequence in t* is not just an input sequence which satisfies the constraints of the RHOCP (specified in the horizon $[t+\overline{\tau}_{rt}+1,\ldots,t+N_c]$), but it is required to keep the nominal trajectories inside the restricted constraints for an horizon of N_c steps from $t+1$ to $t+N_c$, that is larger than the one considered by the optimization problem. □

Now, by accurately choosing the stage cost h, the constraints $X_i(\overline{d})$, the terminal cost function h_f, and by imposing a terminal constraint X_f at the end of the control horizon, it is possible to show that the recursive feasibility of the scheme can be guaranteed for $t \in \mathbb{Z}_{>0}$, also in presence of norm-bounded additive transition uncertainties and network delays. Moreover, the devised control scheme will be proven to be Input-to-State stabilizing if the following assumptions are verified.

Assumption 6: The transition cost function $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is such that $\underline{h}(|x|) \leq h(x,u)$, $\forall x \in X,\ \forall u \in U$, where $\underline{h}$ is a $\mathcal{K}_\infty$-function. Moreover, h is Lipschitz w.r.t. x, uniformly in u, with L. constant $L_h \in \mathbb{R}_{>0}$. □

Assumption 7 (κ_f, h_f, X_f): There exist an auxiliary control law $\kappa_f(x) : X \to U$, a function $h_f(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a positive constant $L_{h_f} \in \mathbb{R}_{>0}$, a level set of h_f, $X_f \subset X$, and a positive constant $\delta \in \mathbb{R}_{>0}$ such that the following properties hold:

i) $X_f \subset X$, X_f closed, $\{0\} \in X_f$;
ii) $\kappa_f(x) \in U$, $\forall x \in X_f \oplus \mathcal{B}^n(\delta)$;
iii) $\hat{f}(x, \kappa_f(x)) \in X_f,\ \forall x \in X_f \oplus \mathcal{B}^n(\delta)$;
iv) $h_f(x)$ Lipschitz in X_f, with L. constant $L_{h_f} \in \mathbb{R}_{>0}$;
v) $h_f(\hat{f}(x,\kappa_f(x))) - h_f(x) \leq -h(x,\kappa_f(x))$, $\forall x \in (X_f \oplus \mathcal{B}^n(\delta))\backslash\{0\}$.

□

As far as the choice of the terminal set X_f is concerned, a procedure for obtaining a set X_f satisfying Assumption 7 has been proposed in [22]. First, notice that, given a locally stabilizing auxiliary state-feedback controller $\kappa_f(x)$, a control Lyapunov function $h_f(x)$ for $\hat{f}(x,\kappa_f(x))$ and a sub-level set Ω_f, RPI under $\hat{f}(x,\kappa_f(x))$ (i.e., $\Omega_f \triangleq \{x \in \mathbb{R}^n : h_f(x) \leq \overline{h}_f,\, \overline{h}_f \in \mathbb{R}_{>0}\}$ such that $\hat{f}(x,\kappa_f(x)) \in \Omega_f \backsim \mathcal{B}(\delta), \forall x \in \Omega_f$ for some $\delta \in \mathbb{R}_{>0}$), it is always possible to find a positive definite function $h(x,u)$ such that Point v) of Assumption 7 holds. Then, it has been suggested to choose $X_f = \Omega_f \backsim \mathcal{B}(\delta)$, imposing a bound on the maximal admissible uncertainties depending on δ.

Along with this procedure for the choice of X_f, in [22] the maximal admissible uncertainty is strictly related to the contractivity of Ω_f under the particular auxiliary controller $\kappa_f(x)$ (see Theorem 1 in the referenced paper). As a consequence, the requirements on $\kappa_f(x)$ (Points ii), iii) and v) of Assumption 7) limit the class of functions upon which the contractivity of the terminal set can be evaluated.

With the aim to decouple the estimation of the maximal admissible uncertainty of our scheme from the choice of $\kappa_f(x)$, the following lemma is introduced.

Lemma 3.1 (Technical): The control law $\kappa^*_f(x) : \mathcal{C}_1(X_f) \to U$ and the function $h^*_f(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as

$$\kappa^*_f(x) \triangleq \begin{cases} \kappa_f(x), & x \in X_f \oplus \mathcal{B}^n(\delta) \\ \arg\min\limits_{u \in U}\{h_f(\hat{f}(x,u))\}, & \\ & x \in \mathcal{C}_1(X_f)\backslash(X_f \oplus \mathcal{B}^n(\delta)) \end{cases} \tag{17}$$

and

$$h^*_f(x) \triangleq \begin{cases} h_f(x), & x \in X_f \oplus \mathcal{B}^n(\delta) \\ \overline{h}_f + \lambda\, \mathrm{d}(x, X_f), & x \in \mathcal{C}_1(X_f)\backslash(X_f \oplus \mathcal{B}^n(\delta)) \end{cases}$$

with

$$\lambda > \{\max_{x \in \mathcal{C}_1(X_f),\, u \in U} [h(x,u)]\}/\delta, \tag{18}$$

verify the inequality

$$h_f(\hat{f}(x,\kappa^*_f(x))) + h(x,\kappa^*_f(x)) < h^*_f(x). \tag{19}$$

□

Proof: Consider the following facts: $i)$ the control law $\kappa^*_f(x)$ steers the state from $\mathcal{C}_1(X_f)$ to X_f by a single admissible control action (i.e., $\hat{f}(x,\kappa^*_f(x)) \in X_f, \kappa^*_f(x) \in U$, $\forall x \in \mathcal{C}_1(X_f),$); $ii)$ it holds that for all $x \in \mathcal{C}_1(X_f)\backslash(X_f \oplus \mathcal{B}^n(\delta))$ the following inequality holds: $\mathrm{d}(x,X_f) > \delta$, which yields to $h^*_f(x) > \overline{h}_f + \lambda\delta$. If we choose λ according to (18), then

$$\begin{aligned} h^*_f(x) &> \overline{h}_f + \max_{x \in \mathcal{C}_1(X_f),\, u \in U} h(x,u) \\ &> h_f(\hat{f}(x,\kappa^*_f(x))) + h(x,\kappa^*_f(x)),\ \forall x \in \mathcal{C}_1(X_f), \end{aligned}$$

which finally implies (19). ■

By exploiting Lemma 3.1, we will show that the robustness of the scheme depends only on the invariant properties of X_f through the computation of $\mathcal{C}_1(X_f)$.

Now, the following Lemma ensures that the original state constraints can be satisfied by imposing to the nominal trajectories in the RHOCP the restricted constraints introduced in Definition 3.2.

Lemma 3.2 (State Constraints Tightening): Under Assumptions 2 and 3, if the state constraints $X_i(\overline{d})$, are computed as in (15) then, each feasible control sequence $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$, applied in open-loop to the perturbed system, guarantees that the true (networked/perturbed) state trajectory satisfies $x_{t+j} \in X$, $\forall j \in \{1, \ldots, N_c\}$. □

Proof: Given the state measurement $x_{t-\tau_c(t)}$, available at time t at the controller node, let us consider the combined sequence of control actions formed by: $i)$ the subsequence used for estimating $\hat{x}_{t|t-\tau_c(t)}$ (i.e., the true control sequence applied by the NDC to the plant from $t-\tau_c(t)$ to $t-1$) and by $ii)$ a feasible control sequence $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$, that is

$$\mathbf{u}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)} \triangleq \mathrm{col}[\mathbf{u}_{t-\tau_c(t),t-1}, \overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}] \,. \quad (20)$$

Then, the prediction error $\hat{e}_{t-\tau_c(t)+i|t-\tau_c(t)} \triangleq x_{t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)}$, with $i \in \{1, \ldots, N_c + \tau_c(t)\}$ and $x_{t-\tau_c(t)+i}$ obtained by applying $\mathbf{u}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)}$ in open-loop to the uncertain system (9) is upper-bounded by

$$|\hat{e}_{t-\tau_c(t)+i|t-\tau_c(t)}| \leq \frac{L^i_{f_x} - 1}{L_{f_x} - 1}\overline{d}, \ \ \forall i \in \{1, \ldots, N_c + \tau_c(t)\}$$

where $\overline{d}$ is defined as in Assumption 3. Being $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$ feasible, it holds that $\hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)} \in X_i(\overline{d}), \forall i \in \{\tau_c(t)+1, \ldots, N_c + \tau_c(t)\}$, then it follows immediately that $x_{t-\tau_c(t)+i} = \hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)} + \hat{e}_{t-\tau_c(t)+i|t-\tau_c(t)} \in X$. ■

Due to the fact that the control sequence computation is based on a finite-horizon optimization which relies on predictions performed with a nominal model, the proposed control scheme can be viewed as a non-standard MPC combined with a NDC strategy. To gain further insight on the proposed control scheme, we refer the reader to Figure 3.

E. Formalization and implementation of the MPC–NDC scheme for UDP—like networks

The overall control scheme for NCS based on non-acknowledged UDP-like networks will now be described in detail by the Procedure 3.1 below, giving the sequence of operations that have to be performed by the NCS components. [2]

In qualitative terms, the sensor node, the controller, and the smart actuator are in charge of processing information and forming suitably structured data packets, by using some internal storage buffers and computational resources. In this regard, we will neglect the issue of quantization raised by the numerical implementation of the procedure.

[2]The low-level UDP–like communication protocol, in charge for packet routing and synchronization, is considered as a service provided by the network ”transparently” to the components of the NCS.

In the sequel, we will denote as $\mathbf{P}_{sc}$ and $\mathbf{P}_{ca}$ the data packets sent by the sensor to the controller and by the controller to the actuator respectively. For the sake of clarity, all the packets will be referred to as data structures of the form $\mathbf{P} = \{\mathbf{P}.data, \mathbf{P}.time\}$ containing a *data field* and a *time stamp field*. Moreover, denoting as $\mathbf{M}_a$ the overall storage memory of the smart actuator, we assume that $\mathbf{M}_a$ is structured in buffers: $i)$ $\mathbf{M}_a.\mathbf{u} \in \mathbb{R}^m \times N_c$, which is used to store a sequence of N_c future control actions and $ii)$ $\mathbf{M}_a.T \in \mathbb{Z}_{\geq 0}$, which contains the time stamp of the information stored in $\mathbf{M}_a.\mathbf{u}$.

The storage memory of controller node $\mathbf{M}_c$, in turn, is structured in buffers: $i)$ $\mathbf{M}_c.\mathbf{u} \in \mathbb{R}^m \times (\overline{\tau}_c + \overline{\tau}_{rt})$, which is used to store the inputs applied to the plant from time $t - \overline{\tau}_c$ to $t-1$ and the future control action used for prediction until $t + \overline{\tau}_{rt} - 1$; $ii)$ $\mathbf{M}_c.x \in \mathbb{R}^n$, which stores the last available state measurement and $iii)$ $\mathbf{M}_c.T \in \mathbb{Z}_{\geq 0}$, which contains the time stamp relative to $\mathbf{M}_c.x$.

Finally, let us denote as ”←” a data assignment operation. Given a buffer (array) $\mathbf{B}$ containing N elements, let us denote as $\mathbf{B}(i)$ the i-th element of the array, with $i \in \{1, \ldots, N\}$. Given a buffer $\mathbf{B}$ containing M sequences of N elements each, let us denote as $\mathbf{B}(i,j)$ the j-th element of the i-th sequence, with $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$. Then, the following procedure can be outlined.

Procedure 3.1 (MPC–NDC scheme for UDP–like networks): Assume that, starting from time instant $t = 0$, the initial condition x_0 is known.

Initialization

1 Given x_0, $\mathbf{M}_c.x \leftarrow x_0$;
2 $\mathbf{M}_a.\mathbf{u} = \mathbf{M}_c.\mathbf{u} \leftarrow \bar{\mathbf{u}}_{0,N_c-1}$, with $\bar{\mathbf{u}}_{0,N_c-1}$ feasible for x_0;
3 $\mathbf{M}_a.T = \mathbf{M}_c.T \leftarrow 0$.

Sensor node

1 for $t \in \mathbb{Z}_{\geq 0}$
2 form the packet $\left\{ \begin{array}{l} \mathbf{P}_{sc}.x \leftarrow x_t \\ \mathbf{P}_{sc}.T \leftarrow t \end{array} \right.$;
3 send $\mathbf{P}_{sc}$.

Controller node

1 for $t \in \mathbb{Z}_{\geq 0}$
2 if a packet $\mathbf{P}_{sc}$ arrived
3 if $\mathbf{P}_{sc}.T > \mathbf{M}_c.T$
4 $\mathbf{M}_c.x \leftarrow \mathbf{P}_{sc}.x$; $\quad (= x_{t-\tau_c(t)})$
5 $\mathbf{M}_c.T \leftarrow \mathbf{P}_{sc}.T$; $\quad (= t - \tau_c(t))$
6 considering that $\mathbf{M}_c.x = x_{t-\tau_c(t)}$, compute the prediction $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$ by using (12) and the input sequence $\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1}$, which can be retrieved from $\mathbf{M}_c.\mathbf{u}$ (see Line 9);
7 solve the RHOCP initialized with $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$, obtaining $\mathbf{u}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}$;
8 form $\mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} = \mathrm{col}[\mathbf{u}^c_{t,t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}]$;
9 store $\mathbf{M}_c.\mathbf{u} \leftarrow \mathrm{col}[\mathbf{M}_c.\mathbf{u}(2), .., \mathbf{M}_c.\mathbf{u}(\overline{\tau}_{rt}), u^c_{t+\overline{\tau}_{rt}|t-\tau_c(t)}]$
10 form the packet $\left\{ \begin{array}{l} \mathbf{P}_{ca}.\mathbf{u} \leftarrow \mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} \\ \mathbf{P}_{ca}.T \leftarrow t \end{array} \right.$;
11 send $\mathbf{P}_{ca}$.

Actuator node

1 for $t \in \mathbb{Z}_{\geq 0}$

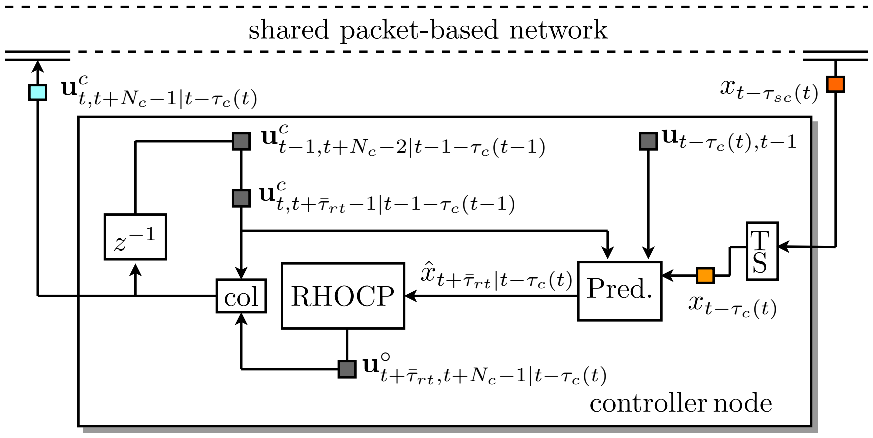

Fig. 3: Scheme of the mechanism used to compute the control sequence, based on prediction (Pred.) and reduced horizon optimization (RHOCP). We enhance the input sequences used to perform the prediction, $\mathbf{u}_{t-\tau_c(t),t-1}$ and $\mathbf{u}^c_{t,t+\bar{\tau}_{rt}-1|t-1-\tau_c(t-1)}$, and the control sequence computed by the reduced horizon optimization, $\mathbf{u}^\circ_{t+\bar{\tau}_{rt},t+N_c-1|t-\tau_c(t)}$. It is important to notice that the sequence $\mathbf{u}_{t-\tau_c(t),t-1}$ is known to the controller even in absence of aknowledgements thanks to the formalism of the reduced horizon optimization, which guarantees the consistence of the prediction.

if a packet $\mathbf{P}_{ca}$ arrived
  if $\mathbf{P}_{ca}.T > \mathbf{M}_a.T$
    $\mathbf{M}_a.\mathbf{u} \leftarrow \mathbf{P}_{ca}.\mathbf{u}$; $(= \mathbf{u}^c_{t-\tau_a(t),t-\tau_a(t)+N_c-1|t-\tau_{rt}(t)})$;
    $\mathbf{M}_a.T \leftarrow \mathbf{P}_{ca}.T$; $\quad (= t-\tau_a(t)\ )$.
apply the control action $u_t = \mathbf{M}_a.\mathbf{u}(t-\mathbf{M}_a.T+1)$. $(= u^c_{t|t-\tau_{rt}(t)})$

□

Notably, the proposed algorithm does not rely on acknowledgments, thus overcoming the limitation of previous networked model-based and predictive control approaches (see [33] and [36]) which are based upon the assumption of deterministic acknowledgment reception.

In the next section, the robust stability properties of the proposed control scheme will be analyzed in presence of transmission delays and model uncertainty.

IV. Recursive Feasibility and Regional Input-to-State Stability

The following important result states the *recursive feasibility* of the combined MPC–NDC scheme.

Theorem 4.1 (Invariance of the feasible set): Assume that at time instant t the control sequence computed by the controller, $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$, is feasible. Then, in view of Assumptions 2-7, if the norm bound on the uncertainty verifies

$$\overline{d} \leq \min_{k\in\{0,\overline{\tau}_c\}} \left\{ \min\left[\frac{L_{f_x}-1}{L_{f_x}^{N_c+k} - L_{f_x}^{N_c-1}} \mathrm{dist}\left(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f\right), \frac{L_{f_x}-1}{L_{f_x}^{N_c+k}-1} \mathrm{dist}\left(\mathbb{R}^n \backslash X_{k+N_c}(\overline{d}), X_f\right) \right] \right\}, \tag{21}$$

then, the recursive feasibility of the scheme in ensured for every time instant $t+i, \forall i \in \mathbb{Z}_{>0}$, while the closed-loop trajectories are confined into X. Hence, the feasible set X_{MPC} is RPI under the closed-loop networked dynamics w.r.t. bounded uncertainties. □

Proof: the proof consists in showing that if, at time t, the input sequence computed by the controller $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$ is feasible in the sense of Definition 3.4, then for the perturbed system evolving under the action of the MPC–NDC scheme there exists a feasible control sequence at time instant $t+1$. Finally, the recursive feasibility will follow by induction. First, notice that Points ii) and iii) of Assumption 7 together imply that $\mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f) \geq \delta > 0$. Now, the proof will be carried out in three steps.

i) $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_f$: Let us consider the sequence $\mathbf{u}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)}$ defined in (20). It is straightforward to prove that the norm difference between the predictions $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$ and $\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)}$ (initialized by $x_{t-\tau_c(t)}$ and $x_{t+1-\tau_c(t+1)}$), respectively obtained by applying to the nominal model the sequence $\mathbf{u}^*_{t-\tau_c(t),t-\tau_c(t)+j-1|t-\tau_c(t)}$ and its subsequence $\mathbf{u}^*_{t+1-\tau_c(t+1),t-\tau_c(t)+j-1|t-\tau_c(t)}$, can be upper-bounded as

$$|\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}| \leq L_{f_x}^{j-i} \sum_{l=0}^{i-1} L_{f_x}^{l} \overline{d}, \tag{22}$$

where we set $i = \tau_c(t) - \tau_c(t+1) + 1$ and $j \in \{i, \ldots, N_c + \tau_c(t)\}$. Considering now the case $j = N_c + \tau_c(t)$, then (22) yields to $|\hat{x}_{t+N_c|t-\tau_c(t)+i} - \hat{x}_{t+N_c|t-\tau_c(t)}| = |\hat{x}_{t+N_c|t+1-\tau_c(t+1)} - \hat{x}_{t+N_c|t-\tau_c(t)}| \leq (L_{f_x}^{N_c+\tau_c(t)} - L_{f_x}^{N_c+\tau_c(t)-i})/(L_{f_x}-1)\overline{d}$. If the following inequality holds $\forall k \in \{1, \ldots, \overline{\tau}_c\}$

$$\overline{d} \leq \frac{L_{f_x}-1}{L_{f_x}^{N_c+k} - L_{f_x}^{N_c-1}} \mathrm{dist}\left(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f\right),$$

then, $\hat{x}_{t+N_c|t+1-\tau_c(t+1)} \in \mathcal{C}_1(X_f)$, irrespective of the values of $\tau_c(t)$ and $\tau_c(t+1)$. Hence, there exists a control action $\overline{u}_{t+N_c|t+1-\tau_c(t+1)} \in U$ which can steer the state vector from $\hat{x}_{t+N_c|t+1-\tau_c(t+1)}$ to $\hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_f$.

Note that, a possible choice can be $\overline{u}_{t+N_c|t+1-\tau_c(t+1)} = \kappa_f^*(\hat{x}_{t+N_c|t+1-\tau_c(t+1)})$, with κ_f^* defined as in (17).

ii) $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_j(\overline{d}) \Rightarrow \hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in X_{j-i}(\overline{d})$, with $i = \tau_c(t) - \tau_c(t+1) + 1$ and $\forall j \in \{\tau_c(t)+1, \ldots, N_c + \tau_c(t)\}$. Consider the predictions $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$ and $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i}$(initialized by $x_{t-\tau_c(t)}$ and $x_{t-\tau_c(t)+i}$, respectively), obtained by the sequence $\mathbf{u}^*_{t-\tau_c(t),t-\tau_c(t)+j-1|t-\tau_c(t)}$ and by its subsequence $\mathbf{u}^*_{t-\tau_c(t)+i,t-\tau_c(t)+j-1|t-\tau_c(t)}$, respectively. Assuming that $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X \backsim \mathcal{B}^n((L^j_{f_x} - 1)/(L_{f_x} - 1)\overline{d})$, let us introduce $\eta \in \mathcal{B}^n((L^{j-i}_{f_x} - 1)/(L_{f_x} - 1)\overline{d})$. Let $\xi \triangleq \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \eta$; then, in view of Assumption 2 and thanks to (22), it follows that

$$\begin{aligned} |\xi| &\leq |\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_{rt}(t)+j|t-\tau_c(t)}| + |\eta| \\ &\leq (L^j_{f_x} - 1)/(L_{f_x} - 1)\overline{d}, \end{aligned} \tag{23}$$

hence, $\xi \in \mathcal{B}^n((L^j_{f_x} - 1)/(L_{f_x} - 1)\overline{d})$. Since $\hat{x}_{t-\tau_c(t)+j|t} \in X_j(\overline{d})$, it follows that $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \xi = \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} + \eta \in X$, $\forall \eta \in \mathcal{B}^n((L^{j-i}_{f_x} - 1)/(L_{f_x} - 1)\overline{d})$, yielding to $\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in X_{j-\tau_c(t)+\tau_c(t+1)-1}(\overline{d})$.

iii) $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_{N_c+\tau_c(t+1)}(\overline{d})$; Thanks to Point i), there exists a feasible control sequence at time $t+1$ which yields to $\hat{x}_{t+1+N_c|t+1-\tau_c(t+1)} \in X_f$. If $\overline{d}$ satisfies

$$\overline{d} \leq \min_{j \in \{N_c, \ldots, N_c + \overline{\tau}_c\}} \left\{ \frac{L_{f_x} - 1}{L^j_{f_x} - 1} \text{dist}(\mathbb{R}^n \backslash X_j(\overline{d}), X_f) \right\},$$

it follows that $\hat{x}_{t+1+N_c|t+1-\tau_c(t+1)} \in X_{N_c+\tau_c(t+1)}$, irrespective of the value of $\tau_c(t+1)$.

Then, under the assumptions posed in the statement of Theorem 4.1, given $x_0 \in X_{MPC}$, and being $\tau_c(0) = 0$ (i.e., at the first time instant, the actuator buffer is initialized with a feasible sequence) in view of Points i)–iii) it holds that, at any time $t \in \mathbb{Z}_{>0}$, a feasible control sequence does exist and can be chosen as $\overline{\mathbf{u}}^c_{t+1,t+N_c+1|t+1-\tau_c(t+1)} = \text{col}[\overline{\mathbf{u}}^c_{t+1,t+N_c-1|t-\tau_c(t)}, \overline{u}_{t+N_c|t+1-\tau_c(t+1)}]$. Therefore the recursive feasibility of the scheme is ensured. ∎

Remark 4.1 (Invariance of X_{MPC}): Given a delayed state measurement $x_{t-\tau_c(t)}$, if there exists a feasible sequence $\bar{\mathbf{u}}_{t,t+N_c-1}$ at time t, we have that $\hat{x}_{t|t-\tau_c(t)} = \hat{x}(t, \bar{x}_{t|t-\tau_c(t)}, \bar{\mathbf{u}}_{t-\tau_c(t),t-1})$ verifies $\hat{x}_{t|t-\tau_c(t)} \in X_{MPC}(\tau_c(t))$, since $\bar{\mathbf{u}}_{t,t+N_c-1}$ satisfies all the constraints specified in (16) with $i = \tau_c(t)$. Thus, proving that the scheme is recursively feasible (that is, given a feasible sequence at time t, there exists a feasible sequence at time $t+1$), would prove that $\hat{x}_{t+1|t+1-\tau_c(t+1)}$, will belong to $X_{MPC}(\tau_c(t+1))$, whatever be the value of $\tau_c(t+1)$ in the set $\{0, \ldots, \overline{\tau}_c\}$. Without loss of generality, assume that $\tau_c(t+1) = 0$, then it holds that $x_{t+1} = \hat{x}_{t+1|t+1} \in X_{MPC}$. Assuming that the initial condition $\bar{x}_0$, at time $t = 0$, is known to the controller (i.e.,$\tau_c(0) = 0$) and that the sequence stored in the actuator buffer is feasible, by induction it follows that

$$x_t \in X_{MPC}, \ \forall t \in \mathbb{Z}_{\geq 0}. \tag{24}$$

We can conclude that X_{MPC} is RPI for the NCS driven by the MPC-NDC scheme. □

Now, the following main stability result can be proved.

Theorem 4.2 (Regional Input-to-State Stability): Under Assumptions 2-7, if the bound on uncertainties verifies (21), then, system (13), controlled by the proposed MPC–NDC strategy, is regional ISS in X_{MPC} with respect to additive perturbations $d_t \in \mathcal{B}^n(\overline{d})$. □

Proof: Recalling the assumption that, at time $t = 0$, the FIB contains a feasible control sequence and that the RHOCP preserves the past computed control actions up to the $\overline{\tau}_{rt}$-th one, then, in a worst case situation, the system will be driven in open-loop for $\overline{\tau}_{rt}$ time instants (see Procedure 3.1). As far as the ISS property is concerned, this observation implies that the bound on the trajectories after $\overline{\tau}_{rt}$ should depend on $x_{\overline{\tau}_{rt}}$ and the regional ISS inequality (2) has to be modified as follows:

$$\begin{aligned} &|x(t + \overline{\tau}_{rt}, \overline{x}_{\overline{\tau}_{rt}}, \boldsymbol{\upsilon}_{\overline{\tau}_{rt}, t+\overline{\tau}_{rt}-1})| \\ &\quad \leq \max\left\{\beta(|\overline{x}_{\overline{\tau}_{rt}}|, t), \gamma(\|\boldsymbol{\upsilon}_{[t+\overline{\tau}_{rt}-1]}\|)\right\}, \end{aligned} \tag{25}$$

$\forall t \in \mathbb{Z}_{\geq 0}, \forall \overline{x}_{\overline{\tau}_{rt}} \in \Xi$, where $\overline{x}_{\overline{\tau}_{rt}}$ is the state at time $\overline{\tau}_{rt}$ after the system has been driven for $\overline{\tau}_{rt}$ steps by the open-loop policy stored in the buffer at time $t = 0$. In view of previous consideration, the proof consists in showing that there exist a ISS-Lyapunov function $V(t + \overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}})$ for the closed-loop system. To this end, let us define the following positive-definite function $V^\circ : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

$$\begin{aligned} &V^\circ(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}) \\ &\triangleq J^\circ_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}, N_c - \overline{\tau}_{rt}). \end{aligned}$$

Notice that V° corresponds to the optimal cost subsequent to the reduced horizon optimization. Now, consider the following candidate time-varying ISS-Lyapunov function $V : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$:

$$\begin{aligned} &V(t + \overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}}) \\ &\quad \triangleq J_{FH}(x_{t+\overline{\tau}_{rt}}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}, N_c - \overline{\tau}_{rt}) \\ &\quad = \sum_{l=t+\overline{\tau}_{rt}}^{t+N_c-1} h(\hat{x}_{l|t+\overline{\tau}_{rt}}, u^\circ_{l|t-\tau_c(t)}) + h_f(\hat{x}_{t+N_c|t+\overline{\tau}_{rt}}) \end{aligned} \tag{26}$$

where $\hat{x}_{t+\overline{\tau}_{rt}+j|t+\overline{\tau}_{rt}}$, $j \in \{1, \ldots, N_c - \overline{\tau}_{rt}\}$ are obtained using the nominal model initialized with $\hat{x}_{t+\overline{\tau}_{rt}|t+\overline{\tau}_{rt}} = x_{t+\overline{\tau}_{rt}}$ and the sequence $\mathbf{u}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}$ (which is optimal for $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$ and not for $x_{t+\overline{\tau}_{rt}}$). Notice that, since $\mathbf{u}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}$ is not computed in correspondence of $x_{t+\overline{\tau}_{rt}}$, but exploiting a past state information $x_{t-\tau_c(t)}$, V becomes a time-varying function of the state. We will show in the following that $V(t + \overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}})$ verifies the ISS inequalities with time-invariant bounds.

Now, let us point out that, in view of (22), $x_{t+\overline{\tau}_{rt}} \in \Omega \triangleq X_f \backsim \mathcal{B}^n((L^{\overline{\tau}_c+\overline{\tau}_{rt}}_{f_x} - 1)/(L_{f_x} - 1)\overline{d})$ implies $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)} \in X_f$ irrespective of the specific value of $\tau_c(t)$. Then, by Assumption 7, the control sequence $\tilde{\mathbf{u}}_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)} \triangleq \text{col}[\ \kappa_f(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}),\ \kappa_f(\hat{x}_{t+\overline{\tau}_{rt}+1|t-\tau_c(t)}),\ \ldots,\ \kappa_f(\hat{x}_{t+N_c-1|t-\tau_c(t)})]$ is feasible for the RHOCP, hence the set X_{MPC} is not empty.

Our objective consists in finding a suitable comparison function to upper bound the candidate time-varying ISS-Lyapunov function $V(t+\overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}})$. By adding and subtracting $V^\circ(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)})$ to the right-hand side of (26), we obtain

$$\begin{aligned}&V(t+\overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}})\\ &\leq \sum_{l=t+\overline{\tau}_{rt}}^{t+N_c-1}[h(\hat{x}_{l|t+\overline{\tau}_{rt}}, u^\circ_{l|t-\tau_c(t)}) - h(\hat{x}_{l|t-\tau_c(t)}, u^\circ_{l|t-\tau_c(t)})]\\ &\quad + h_f(\hat{x}_{t+N_c|t+\overline{\tau}_{rt}}) - h_f(\hat{x}_{t+N_c|t-\tau_c(t)})\\ &\quad + J^\circ_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt}, t+N_c-1|t-\tau_c(t)}, N_c-\overline{\tau}_{rt})\,.\end{aligned} \tag{27}$$

In view of Assumptions 2 and 6 and thanks to (22), the following inequalities hold:

$$\begin{aligned}&\sum_{l=t+\overline{\tau}_{rt}}^{t+N_c-1} |\, h(\hat{x}_{l|t-\tau_c(t)}, u^\circ_{l|t-\tau_c(t)}) - h(\hat{x}_{l|t+\overline{\tau}_{rt}}, u^\circ_{l|t-\tau_c(t)})\,|\\ &\qquad \leq L_h \frac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}\frac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1}||\boldsymbol{d}_{[t+\overline{\tau}_{rt}-1]}||\,.\end{aligned} \tag{28}$$

Moreover

$$\begin{aligned}&|h_f(\hat{x}_{t+N_c|t+\overline{\tau}_{rt}}) - h_f(\hat{x}_{t+N_c|t-\tau_c(t)})|\\ &\quad \leq L_{h_f}\frac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}L_{f_x}^{N_c-\overline{\tau}_{rt}-1}||\boldsymbol{d}_{[t+\overline{\tau}_{rt}-1]}||\,,\end{aligned} \tag{29}$$

and

$$\begin{aligned}&J_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt}, t+N_c-1|t-\tau_c(t)}, N_c-\overline{\tau}_{rt})\\ &\leq J_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, \tilde{\mathbf{u}}_{t+\overline{\tau}_{rt}, t+N_c-1|t-\tau_c(t)}, N_c-\overline{\tau}_{rt})\\ &= \sum_{l=t+\overline{\tau}_{rt}}^{t+N_c-1} h\left(\tilde{x}_{l|t-\tau_c(t)}, \tilde{u}_{l|t-\tau_c(t)}\right) + h_f(\tilde{x}_{t+N_c|t-\tau_c(t)})\,,\end{aligned} \tag{30}$$

where, given $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)} \in X_f$, $\forall j \in \{1,\ldots,N_c-\overline{\tau}_{rt}\}$ we set

$$\begin{aligned}&\tilde{x}_{t+\overline{\tau}_{rt}+j|t-\tau_c(t)}\\ &\quad = \hat{f}(\tilde{x}_{t+\overline{\tau}_{rt}+j-1|t-\tau_c(t)}, \kappa_f(\tilde{x}_{t+\overline{\tau}_{rt}+j-1|t-\tau_c(t)})) \in X_f\,.\end{aligned}$$

Considering that

$$\begin{aligned}&\sum_{l=t+\overline{\tau}_{rt}}^{t+N_c-1} h\left(\tilde{x}_{l|t-\tau_c(t)}, \tilde{u}_{l|t-\tau_c(t)}\right) + h_f(\tilde{x}_{t+N_c|t-\tau_c(t)})\\ &\qquad\qquad \leq h_f(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}),\end{aligned}$$

then, the following bound can be established

$$\begin{aligned}&J_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt}, t+N_c-1|t-\tau_c(t)}, N_c-\overline{\tau}_{rt})\\ &\quad \leq L_{h_f}\frac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}||\boldsymbol{d}_{[t+\overline{\tau}_{rt}-1]}|| + h_f(x_{t+\overline{\tau}_{rt}}).\end{aligned} \tag{31}$$

Finally, in view of (28), (29), and (31) we have

$$\begin{aligned}&V(t+\overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}})\\ &\leq \frac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}(L_h\frac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_{h_f}L_{f_x}^{N_c-\overline{\tau}_{rt}-1} + L_{h_f})\\ &\quad \times||\boldsymbol{d}_{[t+\overline{\tau}_{rt}-1]}|| \;+\; h_f(x_{t+\overline{\tau}_{rt}})\\ &\leq \alpha_1(|x_{t+\overline{\tau}_{rt}}|) + \sigma_1(||\boldsymbol{d}_{[t+\overline{\tau}_{rt}-1]}||),\end{aligned} \tag{32}$$

$\forall x_{t+\overline{\tau}_{rt}} \in X_f, \forall \mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(\overline{d})}$, where

$$\begin{aligned}&\alpha_1(s) \triangleq L_{h_f}|s|\\ &\sigma_1(s) \triangleq \frac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}(L_h\frac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_{h_f}L_{f_x}^{N_c-\overline{\tau}_{rt}-1} + L_{h_f})s.\end{aligned}$$

From Assumption 6, we have

$$V(t+\overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}}) \geq \underline{h}(x_{t+\overline{\tau}_{rt}}),\ \ \forall x_{t+\tau_{rt}} \in X_{MPC}\,. \tag{33}$$

Then, owing to (32) and (33), the ISS inequalities (3) and (4) hold with $\Xi = X_{MPC}$ and $\Omega = X_f \backsim \mathcal{B}^n((L_{f_x}^{\overline{\tau}_{rt}}-1)/(L_{f_x}-1)\overline{d})$, respectively. Moreover, in view of Point i) in the proof of Theorem 4.1, given the (feasible) control sequence computed at time t, $\mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} = \mathrm{col}[\mathbf{u}^c_{t,t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}, \mathrm{u}^\circ_{t+\overline{\tau}_{rt}, t+N_c-1}]$, the sequence

$$\begin{aligned}&\bar{\mathbf{u}}^c_{t+1,t+N_c|t+1-\tau_c(t+1)}\\ &\quad = \mathrm{col}[\mathbf{u}^c_{t+1,t+N_c-1|t-\tau_c(t)}, \kappa^*_f(\hat{x}_{t+N_c|t+1-\tau_c(t+1)})]\end{aligned}$$

with κ^*_f defined as in (17), is a feasible sequence at time $t+1$. The subsequence $\mathbf{u}^c_{t+\overline{\tau}_{rt}+1,t+N_c|t-\tau_c(t)}$ along the reduced horizon gives rise to a cost which verifies the inequality

$$\begin{aligned}&J_{FH}(\hat{x}_{t+\overline{\tau}_{rt}+1|t+1-\tau_c(t+1)}, \mathbf{u}^c_{t+\overline{\tau}_{rt}+1,t+N_c|t-\tau_c(t)}, N_c-\overline{\tau}_{rt})\\ &\leq J^\circ_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt}, t+N_c-1|t-\tau_c(t)}, N_c-\overline{\tau}_{rt})\\ &\quad - h(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, u^\circ_{t|t-\tau_c(t)})\\ &\quad + \sum_{l=t+\overline{\tau}_{rt}+1}^{t+N_c-1}[h(\hat{x}_{l|t+1-\tau_c(t+1)}, u^\circ_{l|t-\tau_c(t)})\\ &\qquad\qquad\qquad\qquad - h(\hat{x}_{l|t-\tau_c(t)}, u^\circ_{l|t-\tau_c(t)})]\\ &\quad + h(\hat{x}_{t+N_c|t+1-\tau_c(t+1)}, \kappa^*_f(\hat{x}_{t+N_c|t+1-\tau_c(t+1)}))\\ &\quad + h_f(\hat{x}_{t+N_c+1|t+1-\tau_c(t+1)}) - h_f(\hat{x}_{t+N_c|t-\tau_c(t)})\,.\end{aligned} \tag{34}$$

Now, by (27), (28), and (29), we obtain

$$\begin{aligned}&V(t+\overline{\tau}_{rt}+1, x_{t+\overline{\tau}_{rt}+1}) \leq\\ &\quad J_{FH}(\hat{x}_{t+\overline{\tau}_{rt}+1|t+1-\tau_c(t+1)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt}+1,t+N_c|t-\tau_c(t)}, N_c-\overline{\tau}_{rt})\\ &\quad + \frac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}[L_h\frac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_{h_f}L_{f_x}^{N_c-\overline{\tau}_{rt}-1}]||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}||,\end{aligned} \tag{35}$$

and

$$\begin{aligned}&J_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_{rt}, t+N_c-1|t-\tau_c(t)}, N_c-\overline{\tau}_{rt})\\ &\leq V(t+\overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}})\\ &+ \frac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}[L_h\frac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_{h_f}L_{f_x}^{N_c-\overline{\tau}_{rt}-1}]||\boldsymbol{d}_{[t+\overline{\tau}_{rt}-1]}||.\end{aligned} \tag{36}$$

In view of Point v) of Assumption 7 and thanks to Lemma 3.1, considering that $|\hat{x}_{t+N_c|t+1-\tau_c(t+1)} - \hat{x}_{t+N_c|t-\tau_c(t)}| \leq L_{f_x}^{N_c-1}(L_{f_x}^{\overline{\tau}_c}-1)/(L_{f_x}-1)||\boldsymbol{d}_{[t-\tau_c(t)]}||$, we have

$$\begin{aligned}&h(\hat{x}_{t+N_c|t+1-\tau_c(t+1)}, k^*_f(\hat{x}_{t+N_c|t+1-\tau_c(t+1)}))\\ &+ h_f(\hat{x}_{t+N_c+1|t+1-\tau_c(t+1)}) - h_f(\hat{x}_{t+N_c|t-\tau_c(t)})\\ &\qquad \leq h^*_f(\hat{x}_{t+N_c|t+1-\tau_c(t+1)}) - h^*_f(\hat{x}_{t+N_c|t-\tau_c(t)})\\ &\qquad\quad + h^*_f(\hat{x}_{t+N_c|t-\tau_c(t)}) - h_f(\hat{x}_{t+N_c|t-\tau_c(t)})\\ &\qquad \leq L^*_{h_f}L_{f_x}^{N_c-1}\frac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1}||\boldsymbol{d}_{[t]}||\,.\end{aligned} \tag{37}$$

where we have used the fact that $h^*_f(\hat{x}_{t+N_c|t-\tau_c(t)}) = h_f(\hat{x}_{t+N_c|t-\tau_c(t)})$ for $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f$ and where $L^*_{h_f} \triangleq \max\{L_{h_f}, \lambda\}$, with λ defined in (18). Then, considering that $||\boldsymbol{d}_{[t]}|| \leq ||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}||$, the following inequalities follow from

(34) by using (35), (36), and (37):

$$\begin{aligned}
&V(t+\overline{\tau}_{rt}+1, x_{t+\overline{\tau}_{rt}+1}) - V(t+\overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}}) \\
&\le -h(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, u^{\circ}_{t|t-\tau_c(t)}) \\
&+ \sum_{l=t+\overline{\tau}_{rt}+1}^{t+N_c-1} [\, h(\hat{x}_{l|t+1-\tau_c(t+1)}, u^{\circ}_{l|t-\tau_c(t)}) \\
&\qquad\qquad\qquad -h(\hat{x}_{l|t-\tau_c(t)}, u^{\circ}_{l|t-\tau_c(t)})\,] \\
&+[\, L^{*}_{h_f} L_{f_x}^{N_c-1} \tfrac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} + 2\tfrac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1} \\
&\quad \times (L_h \tfrac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_{rt}-1})]||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}||.
\end{aligned} \tag{38}$$

Moreover, by considering that

$$\begin{aligned}
&\sum_{l=t+\overline{\tau}_{rt}+1}^{t+N_c-1} h(\hat{x}_{l|t+1-\tau_c(t+1)}, u^{\circ}_{l|t-\tau_c(t)}) - h(\hat{x}_{l|t-\tau_c(t)}, u^{\circ}_{l|t-\tau_c(t)}) \\
&\le L_h \sum_{l=\overline{\tau}_{rt}+1}^{N_c-1} L_{f_x}^{l-1}(L_{f_x}^{\overline{\tau}_c}-1)/(L_{f_x}-1)||\boldsymbol{d}_{[t]}|| \\
&\le L_h (L_{f_x}^{\overline{\tau}_c}-1)/(L_{f_x}-1) \sum_{l=\overline{\tau}_{rt}}^{N_c-2} L_{f_x}^{l}||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}|| \\
&\le L_h \tfrac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} L_{f_x}^{\overline{\tau}_{rt}} \tfrac{L_{f_x}^{N_c-\overline{\tau}_{rt}-1}-1}{L_{f_x}-1}||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}||,
\end{aligned}$$

inequality (38) yields

$$\begin{aligned}
&V(t+\overline{\tau}_{rt}+1, x_{t+\overline{\tau}_{rt}+1}) - V(t+\overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}}) \\
&\le -h(x_{t+\overline{\tau}_{rt}}, u^{\circ}_{t|t-\tau_c(t)}) \\
&+L_h \tfrac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} L_{f_x}^{\overline{\tau}_{rt}} \tfrac{L_{f_x}^{N_c-\overline{\tau}_{rt}-1}-1}{L_{f_x}-1}||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}|| \\
&+[L^{*}_{h_f} L_{f_x}^{N_c-1} \tfrac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} \\
&+2\tfrac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}(L_h \tfrac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_{rt}-1})]||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}||\,.
\end{aligned}$$

Finally, by using Point iv) of Assumption 7, the third ISS inequality can be obtained:

$$\begin{aligned}
&V(t+\overline{\tau}_{rt}+1, x_{t+\overline{\tau}_{rt}+1}) - V(t+\overline{\tau}_{rt}, x_{t+\overline{\tau}_{rt}}) \\
&\le -\underline{h}(|x_{t+\overline{\tau}_{rt}}|) \\
&\quad +[L_h \tfrac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} L_{f_x}^{\overline{\tau}_{rt}} \tfrac{L_{f_x}^{N_c-\overline{\tau}_{rt}-1}-1}{L_{f_x}-1} + L^{*}_{h_f} L_{f_x}^{N_c-1} \tfrac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} \\
&\quad +2\tfrac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}(L_h \tfrac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_{rt}-1})]||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}|| \\
&\le -\alpha_2(|x_{t+\overline{\tau}_{rt}}|) + \sigma_2(||\boldsymbol{d}_{[t+\overline{\tau}_{rt}]}||),
\end{aligned} \tag{39}$$

$\forall x_{t+\overline{\tau}_{rt}} \in X_{MPC}, \forall \boldsymbol{d} \in \mathcal{M}_{\mathcal{B}^n(\overline{d})}$, where

$$\begin{aligned}
\alpha_2(s) &\triangleq \underline{h}(s) \\
\sigma_2(s) &\triangleq [L_h \tfrac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} L_{f_x}^{\overline{\tau}_{rt}} \tfrac{L_{f_x}^{N_c-\overline{\tau}_{rt}-1}-1}{L_{f_x}-1} + L^{*}_{h_f} L_{f_x}^{N_c-1} \tfrac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} \\
&\quad +2\tfrac{L_{f_x}^{\overline{\tau}_c+\overline{\tau}_{rt}}-1}{L_{f_x}-1}(L_h \tfrac{L_{f_x}^{N_c-\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_{rt}-1})].
\end{aligned}$$

Finally, in view of (32), (33), and (39), it is possible to conclude that the closed-loop system is regionally ISS in X_{MPC} with respect to $d \in \mathcal{B}^n(\overline{d})$. ■

Before reporting some simulation results, the following final remark is in place.

Remark 4.2: It is worth noting that the above important stability result involves some conservative assumptions and arguments. A possible source of conservativeness is condition (21) on the uncertainty. In practice, despite the fact that a (possibly small) value of $\bar{d}$ can always be computed, the numerical computation of $\bar{d}$ can be difficult if the various sets involved, like X and X_f, do not take on specific geometric structures (for example convex polyedra, see [31]); indeed, the numerical computation may lead to small robustness margins, especially due to the use of the Lipschitz Assumption 2 that is needed because of the generality of the functional structure of the nominal map $\hat{f}$. □

V. Simulation results

Consider the undamped single-link flexible-joint pendulum depicted in Fig. 4.

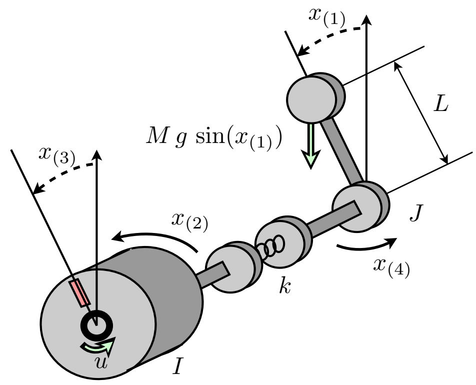

Fig. 4: The single-link flexible-joint pendulum.

The closed-loop behavior of the forward-Euler discretized version of this nonlinear system is simulated first in nominal conditions and then under the simultaneous presence of model uncertainty and unreliable communications between sensors, controller, and actuators:

$$\begin{cases}
x_{(1)_{t+1}} = x_{(1)_t} + T_s x_{(2)_t} \\
x_{(2)_{t+1}} = x_{(2)_t} - \dfrac{T_s}{I}\left[MgL\sin(x_{(1)_t}) + k\left(x_{(1)_t} - x_{(3)_t}\right)\right] \\
x_{(3)_{t+1}} = x_{(3)_t} + T_s x_{(4)_t} \\
x_{(4)_{t+1}} = x_{(4)_t} + \dfrac{T_s}{J}\left[k\left(x_{(1)_t} - x_{(3)_t}\right) + u\right]
\end{cases} \tag{40}$$

where $x_0 = \overline{x}$, $t \in \mathbb{Z}_{\ge 0}$, $x_{(i)_t}, i \in \{1, \dots, 4\}$ denotes the i-th component of the vector x_t, $T_s = 0.05\,s$ is the sampling interval, $I = 0.25\,kg \cdot m^2$ the inertia of the arm, $J = 2\,kg \cdot m^2$ the rotor inertia, $g = 9.8\,ms^2$ the gravitational acceleration, $M = 1\,kg$ the mass of the link, $L = 0.5\,m$ the distance between the rotational axis and the center of gravity of the pendulum-arm, $k = 20\,N \cdot m/rad$ the stiffness coefficient of the link. The Lipschitz constant of the transition function is $L_{f_x} = 1.1267$. The control objective consists in stabilizing the system towards the (open-loop unstable) 0-state equilibrium, while keeping the trajectories within some prescribed bounds.

The following auxiliary linear controller is used $\kappa_f(x) = [-55.92 \;\; -7.46 \;\; 124.01 \;\; 19.22] \cdot x$, with $X_f = \{x \in \mathbb{R}^4 : x^T \cdot P_f \cdot x \le 1\}$, $h_f(x) = 10^5(x^T \cdot P_f \cdot x)$ and

$$P_f = 10^3 \begin{bmatrix} 1.3789 & -0.0629 & -1.7904 & -0.1508 \\ -0.0629 & 0.0186 & 0.1404 & 0.0074 \\ -1.7904 & 0.1404 & 3.1580 & 0.2216 \\ -0.1508 & 0.0074 & 0.2216 & 0.0292 \end{bmatrix}$$

The predictive controller has been set up with control sequence length $N_c = 12$, and quadratic stage cost $h(x) = x^T \cdot Q \cdot x + Ru^2$, where $Q = \text{diag}(10, 0.1, 0.1, 0.1)$ and $R = 10^{-3}$. To compute the ellipsoidal terminal set and the quadratic terminal cost, the procedure described in Section 5 of [30] has been employed. The aforementioned method can also provide a conservative measure of the contractivity of the terminal set under the nonlinear closed-loop map which, together with inequality (21), yields to the following conservative uncertainty bound: $\overline{d} \leq 4.5098 \cdot 10^{-10}$. An extensive simulation campaign has shown (expectedly) that the developed control strategy can handle disturbances which are several degree of magnitude larger that this value. Therefore, besides computing the robust uncertainty bound provided by the theoretical results, which allows to check the correct choice of terminal set and penalty function (guaranteeing the stability of the system in the networked framework for small disturbances), also simulations tests, in different operating conditions, are needed to evaluate, in a non-conservative way, the robustness of the strategy for the particular application.

In the uncertain/unreliable networked scenario, a UDP–like protocol has been simulated, with delay bounds $\overline{\tau}_c = \overline{\tau}_a = 5$, while the nominal model is subject to the parametric uncertainty $M_{nom} = 1.05M$. The timing diagrams of the simulated networked packet-based communication links are given in Figure 5. Notice that, due to the use of a TS strategy, the networks delays τ_{ca} and τ_{sc} have been decoupled from the age of information used in the nodes τ_a and τ_c, retaining only the packets which carry on the most recent information.

Finally, Fig. 6 compares the trajectories of the state variables obtained by scheme developed for UDP networks (solid) with the ones obtained by the TCP-oriented algorithm presented in [33] (dashed). The prescribed bounds on the state trajectories and on the control variable are shown by dotted lines. Notably, the constraints are fulfilled and the recursive feasibility of the scheme is guaranteed even in absence of acknowledgments (in the UDP scenario). At the opposite, if a network delay compensation strategy is not used, then system (40), controlled by a nominal MPC, becomes unstable even for small delays $\overline{\tau}_c = \overline{\tau}_a = 2$, as shown in Fig. 7.

CONCLUDING REMARKS

In this paper, a networked control scheme, based on the combined use of MPC with a network delay compensation strategy in the context of non-acknowledged UDP-like networks, has been designed with the aim to stabilize towards an equilibrium a constrained nonlinear discrete-time system, affected by unknown perturbations and subject to delayed packet-based communications in both sensor-to-controller and controller-to-actuator links. The characterization of the robust stability properties of the devised scheme represents a significant contribution in the context of nonlinear networked control systems, since it establishes the possibility to enforce the robust satisfaction of constraints under unreliable networked communications in the feedback and command channels, also in presence of model uncertainty. Moreover, the problem of guaranteeing the recursive feasibility of the constrained optimization problem associated to the predictive control has been addressed. Finally, by exploiting a novel characterization of the regional Input-to-State Stability in terms of time-varying Lyapunov functions, the networked closed-loop system has been proven to be regionally ISS with respect to bounded perturbations.

Future research efforts will be devoted to extend the proposed methodology to more general MPC cost functions and to distributed systems (see [9]). Moreover, several important issues deserve further research, like, for example, the removal of the assumption about the synchronization of all components in the NCS, the possibility of addressing the case where not all state variables are available for measurements and the conservativeness of the robustness stability margin. Regarding this latter aspect, in the case of linear $\hat{f}$ and X and X_f convex polyhedra with a finite number of vertices some explicit solutions can be found [31]. Finally, future research will also

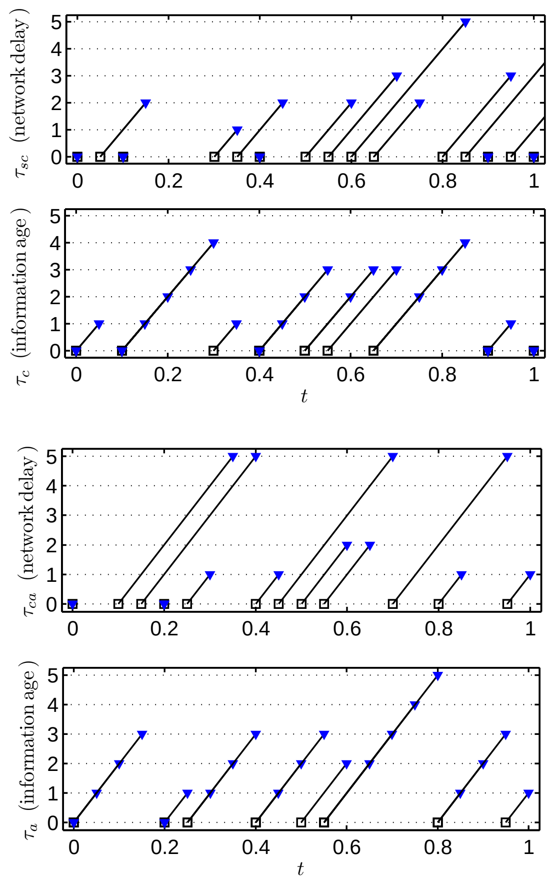

Fig. 5: Timing diagrams of feedback/control communication links and information age at the control and sensors nodes during the simulation. Each slanted segment in τ_{ca} and τ_{sc} diagrams represents a successfully delivered data packet form the sending time (square) to the arrival time (triangle). The length of each segment represents the age of the packet at the receipt instant. In τ_c and τ_a diagrams the triangles represent the age of the information retained in each node thanks to the TS strategy while the slanted segments allow to graphically evaluate the sending time.

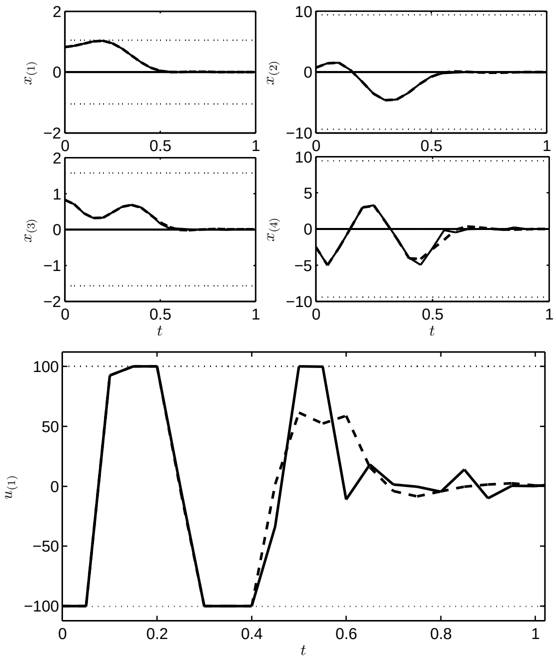

Fig. 6: State and Input trajectories of system (40) controlled by the proposed strategy for UDP-like networks (solid) compared with the trajectories obtained with the method for TCP-like protocols presented in [33](dashed), relying on deterministic acknowledgments. The proposed algorithm allows to preserve stability in absence of acknowledgments.

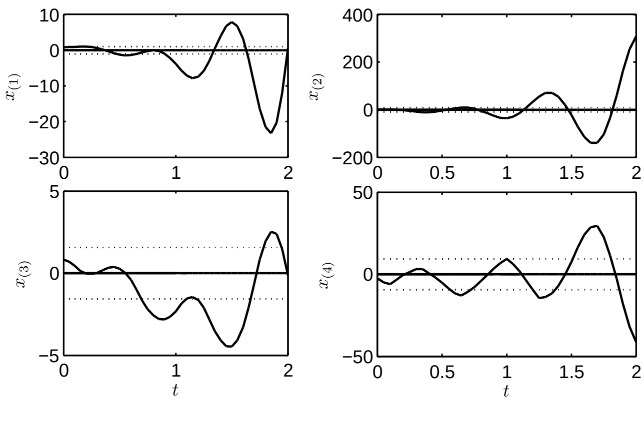

Fig. 7: Trajectories of the state variables for system (40) controlled by a nominal NMPC, without constraint tightening and delay compensation ($\overline{\tau}_c = \overline{\tau}_a = 2$). Feasibility gets lost and instability occurs.

address the extension of the stability analysis to the case where errors affect the optimization results at each time instant (some preliminary results in the non-networked case have been presented in [32]).

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A. Appendix

In order to prove Theorem 2.1, let us introduce the following definitions.

Definition 1.1 (UAG in Ξ): Given a compact set $\Xi \in \mathbb{R}^n$ including the origin as interior point, the system (1), with $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$, satisfies the Uniform Asymptotic Gain (UAG) property in Ξ, if Ξ is a RPI set for system (1) and if there exists a $\mathcal{K}$-funtion γ such that for any arbitrary $\epsilon \in \mathbb{R}_{>0}$ and $\forall\, \overline{x}_0 \in \Xi$, $\exists T^{\epsilon}_{\overline{x}_0} \in \mathbb{Z}_{\geq 0}$ finite such that $|x(t, \overline{x}_0, \boldsymbol{\upsilon}_{0,t-1})| \leq \gamma(||\boldsymbol{\upsilon}_{[t-1]}||) + \epsilon$, for all $t \geq T^{\epsilon}_{\overline{x}_0}$. □

Definition 1.2 (LS): System (1) satisfies the Local Stability (LS) property if for any arbitrary $\epsilon \in \mathbb{R}_{>0}$, $\exists \delta \in \mathbb{R}_{>0}$ such that $|x(t, \overline{x}_0, \boldsymbol{\upsilon}_{0,t-1})| \leq \epsilon$, $\forall t \in \mathbb{Z}_{\geq 0}$, for all $|\overline{x}_0| \leq \delta$ and all $\boldsymbol{\upsilon} \in \mathcal{M}_{\mathcal{B}^r(\delta)}$. □

It can be proven that, if a system satisfies both the UAG in Ξ and the LS properties, and if the trajectories are bounded, it is ISS in Ξ (see [27]). In particular, the trajectories are bounded if the set Ξ is RPI under g for all the possible realizations of uncertainties. Hence, the following result can be stated.

Lemma 1.1 ([27]): Suppose that the origin is a stable equilibrium for (1). System (1) is ISS in Ξ if and only if the properties UAG in Ξ and LS hold, and Ξ is RPI. □

We point out that, if Assumption 1 also holds, then the LS property is redundant. Indeed, under Assumption 1, if the system (1) is UAG in Ξ, then it verifies the LS property. Let us now prove Theorem 2.1. To this end, let $\bar{x} \in \Xi$. The proof will be carried out in three steps

1) First, we are going to show that the set Θ defined in (6) is RPI for the system. From the definition of $\overline{\alpha}_2(s)$ it follows that $\alpha_2(|x|) + \sigma_1(|\upsilon|) \leq \overline{\alpha}_2(|x| + |\upsilon|)$. Therefore $V(t, x) \leq \overline{\alpha}_2(|x| + |\upsilon|)$ and hence $|x| + |\upsilon| \geq \overline{\alpha}_2^{-1}(V(t, x))$. Moreover, thanks to Point 2) of Definition 2.3, there exists a $\mathcal{K}_\infty$-function ϵ such that

$$\alpha_3(|x|) + \epsilon(|\upsilon|) \geq \underline{\alpha}_3(|x| + |\upsilon|) \geq \alpha_4(V(t, x)).$$

Considering the transition from (t, x) to $(t+1, g(t, x, \upsilon))$, we have

$$\begin{aligned} &V(t+1, g(t, x, \upsilon)) - V(t, x) \\ &\quad \leq -\alpha_4(V(t, x)) + \sigma_4(|\upsilon|),\ \forall x \in \Omega, \forall \upsilon \in \Upsilon, \forall t \in \mathbb{R}_{\geq 0}. \end{aligned} \tag{A-1}$$

Let us assume now that $x \in \Theta$. Then $V(t, x) \leq b(\overline{\upsilon})$; this implies $\rho \circ \alpha_4(V(t, x)) \leq \sigma_4(\overline{\upsilon})$. Without loss of generality, assume that $(id - \alpha_4)$ is a $\mathcal{K}_\infty$-function, otherwise pick a bigger α_2 so that $\underline{\alpha}_3 < \overline{\alpha}_2$. Then, after some algebra, we have

$$\begin{aligned} &V(t+1, g(t, x, \upsilon)) \\ &\quad \leq -(id - \rho) \circ \alpha_4\left(b(\overline{\upsilon})\right) + b(\overline{\upsilon}) - \rho \circ \alpha_4\left(b(\overline{\upsilon})\right) + \sigma_4(\overline{\upsilon}). \end{aligned}$$

From the definition of b, it follows that $\rho \circ \alpha_4\left(b(\overline{\upsilon})\right) = \sigma_4(\overline{\upsilon})$ and, owing to the fact that $(id - \rho)$ is a $\mathcal{K}_\infty$-function, we obtain

$$V(t+1, g(t, x, \upsilon)) \leq (id - \rho) \circ \alpha_4\left(b(\overline{\upsilon})\right) + b(\overline{\upsilon}) \leq b(\overline{\upsilon}).$$

By induction it is possible to show that, $V(t, x(t, \bar{x}_0, \boldsymbol{\upsilon}_{0,t-1})) \leq b(\overline{\upsilon})$, $\forall \bar{x}_0 \in \Theta$, $\forall t \in \mathbb{Z}_{\geq 0}$, that is $x_t \in \Theta, \forall t \in \mathbb{Z}_{\geq 0}$. Hence Θ is RPI for system (1).

2) Next, we are going to show that the state, starting from $\Xi \backslash \Theta$, tends asymptotically to Θ. Firstly, if $x \in \Omega \backslash \Theta$, then $\rho \circ \alpha_4\left(V(t, x)\right) \geq \sigma_4(\overline{\upsilon})$. From $\alpha_3(|x|) + \epsilon(|\upsilon|) \geq \alpha_4\left(V(t, x)\right)$, we obtain $\rho\left(\alpha_3(|x|) + \epsilon(|\upsilon|)\right) > \sigma_4(\overline{\upsilon})$. Being $(id - \rho)$ a $\mathcal{K}_\infty$-

function, it holds that $id(s) > \rho(s),\ \forall s \in \mathbb{R}_{>0}$, then

$$\begin{aligned}\alpha_3(|x|)+\epsilon(\overline{v}) &> \alpha_3(|x|)+\epsilon(|v|) > \rho(\alpha_3(|x|)+\epsilon(|v|))\\ &> \sigma_4(\overline{v}) = \epsilon(\overline{v})+\sigma_2(v),\ \ \forall x \in \Omega\backslash\Theta, \forall v \in \Upsilon,\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}V(t+1,g(t,x,v))-V(t,x) &\leq -\alpha_3(|x|)+\sigma_2(\overline{v})+\sigma_3(\overline{v})\\ &< 0,\ \ \forall x \in \Omega\backslash\Theta, \forall v \in \Upsilon.\end{aligned} \tag{A-2}$$

Moreover, in view of (6), $\exists \overline{c} \in \mathbb{R}_{>0}$ such that for all $x^{'} \in \Xi\backslash\Theta$ there exists $x^{''} \in \Omega\backslash D$ such that $\alpha_3(|x^{''}|) \leq \alpha_3(|x^{'}|) - \overline{c}$. Then, from (A-2) it follows that $-\alpha_3(|x^{'}|)+\overline{c} \leq -\alpha_3(|x^{''}|) < -\sigma_2(\overline{v})-\sigma_3(\overline{v}),\ \forall x^{'} \in \Xi\backslash\Omega,\ \forall x^{''} \in \Omega\backslash\Theta$. Then,

$$\begin{aligned}V(t+1,g(t,x,v))-V(t,x) &\leq -\alpha_3(|x|)+\sigma_2(\overline{v})+\sigma_3(\overline{v})\\ &< -\overline{c},\quad \forall x \in \Xi\backslash\Omega, \forall v \in \Upsilon.\end{aligned}$$

Hence, for any $\overline{x}_0 \in \Xi$, there exists $T^{\Omega}_{\overline{x}_0} \in \mathbb{Z}_{\geq 0}$ such that $x_{T^{\Omega}_{\overline{x}_0}} = x(T^{\Omega}_{\overline{x}_0}, \overline{x}_0, \boldsymbol{v}) \in \Omega$, that is, starting from Ξ, the region Ω will be reached in finite time. Now, we will prove that starting from Ω, the state trajectories will tend asymptotically to the set Θ. Since Θ is RPI, it holds that $\lim_{j\to\infty} \mathrm{d}\left(x(T^{\Omega}_{\overline{x}_0}+j, \overline{x}_{T^{\Omega}_{\overline{x}_0}}, \boldsymbol{v}), \Theta\right) = 0$. Otherwise, posing $t = T^{\Omega}_{\overline{x}_0}$, if $x_t \notin \Theta$, then we have that $\rho \circ \alpha_4(V(t,x)) > \sigma_4(\overline{v})$; moreover, from (A-2) it follows that

$$\begin{aligned}&V(t+1,g(t,x,v))-V(t,x)\\ &\quad \leq -\alpha_4(V(t,x))+\sigma_4(\overline{v})\\ &\quad \leq -(id-\rho)\circ\alpha_4\circ\alpha_1(|x|), \forall x \in \Omega\backslash\Theta, \forall v \in \Upsilon\,.\end{aligned}$$

Then, we can conclude that $\forall \epsilon^{'} \in \mathbb{R}_{>0}$, $\exists T^{\Theta}_{\overline{x}_0} \geq T^{\Omega}_{\overline{x}_0}$ such that $V(T^{\Theta}_{\overline{x}_0}+j, x_{T^{\Theta}_{\overline{x}_0}+j}) \leq \epsilon^{'} + b(\overline{v}),\ \forall j \in \mathbb{Z}_{\geq 0}$. Therefore, starting from Ξ, the state will arrive arbitrarily close to Θ in finite time and the state trajectories will tend to Θ asymptotically. Hence $\lim_{t\to\infty} \mathrm{d}(x(t,\overline{x}_0,\boldsymbol{v}_{0,t-1}),\Theta) = 0,\ \forall \overline{x}_0 \in \Xi, \forall \boldsymbol{v} \in \mathcal{M}_\Upsilon$.

3) The present part of the proof is intended to show that system (1) is regionally ISS in the sub-level set $\mathcal{N}_{[V,\overline{e}]}$, where $\overline{e} \triangleq \max\{e \in \mathbb{R}_{>0} : \mathcal{N}_{[V,e]} \in \Omega\}$, having denoted with $\mathcal{N}_{[V,e]} \triangleq \{x \in \mathbb{R}^n : V(t,x) \leq e, \forall v \in \Upsilon, \forall t \in \mathbb{Z}_{\geq 0}\}$ a sub-level set of V for a specified $e \in \mathbb{R}_{\geq 0}$. Note that $\overline{e} > b(\overline{v})$ and $\Theta \subset \mathcal{N}_{[V,\overline{e}]}$. Since the region Θ is reached asymptotically from Ξ, the state will arrive in $\mathcal{N}_{[V,\overline{e}]}$ in finite time, that is, given $\overline{x}_0 \in \Xi$ there exists $T^{\mathcal{N}_{[V,\overline{e}]}}_{\overline{x}_0}$ such that $V\,(T^{\mathcal{N}_{[V,\overline{e}]}}_{\overline{x}_0}+j\,,\, x_{T^{\mathcal{N}_{[V,\overline{e}]}}_{\overline{x}_0}+j}) \leq \overline{e},\ \forall j \in \mathbb{Z}_{\geq 0}\,.$ Hence, the region $\mathcal{N}_{[V,\overline{e}]}$ is RPI. Now, proceeding as in the Proof of Lemma 3.5 in [17], for any $\overline{x}_0 \in \mathcal{N}_{[V,\overline{e}]}$, there exist a $\mathcal{KL}$-function $\hat{\beta}$ and a $\mathcal{K}$-function $\hat{\gamma}$ such that $V(t,x_t) \leq \max \hat{\beta}\,(V(0,\overline{x}_0),t)\,,\, \hat{\gamma}(\|\boldsymbol{v}_{[t-1]}\|),\ \forall t \in \mathbb{Z}_{\geq 0}, \forall \boldsymbol{v} \in \mathcal{M}_\Upsilon$, with $x_t \in \mathcal{N}_{[V,\overline{e}]}$ and where $\hat{\gamma}$ can be chosen as $\hat{\gamma} = \alpha_4^{-1}\circ\rho^{-1}\circ\sigma_4$. Hence, considering that $\hat{\beta}(r+s,t) \leq \hat{\beta}(2r,t)+\hat{\beta}(2s,t), \forall (s,t) \in \mathbb{R}^2_{\geq 0}$ (see [23]), it follows that

$$\alpha_1(|x_t|) \leq \max\{\hat{\beta}(2\alpha_2(|\overline{x}_0|),t)+\hat{\beta}\,(2\sigma_1(|v_0|),t), \hat{\gamma}(\|\boldsymbol{v}_{[t-1]}\|)\},$$

$\forall t \in \mathbb{Z}_{\geq 0}, \forall \overline{x}_0 \in \mathcal{N}_{[V,\overline{e}]}, \forall v \in \mathcal{M}_\Upsilon$. Now, let us define the $\mathcal{KL}$-functions $\tilde{\beta}(s,t) \triangleq \alpha_1^{-1}\circ\hat{\beta}(2s,t)$, $\beta(s,t) \triangleq \tilde{\beta}(\alpha_2(s),t)$, and the $\mathcal{K}$-functions $\tilde{\gamma}(s) \triangleq \alpha_1^{-1}\circ\hat{\gamma}(s)$ and $\gamma(s) \triangleq \tilde{\beta}(\sigma_1(s),0)+\tilde{\gamma}(s)$, we have that

$$\begin{aligned}|x_t| &\leq \max \tilde{\beta}\,(\alpha_2(|\overline{x}_0|),t)+\tilde{\beta}\,(\sigma_1(|v_0|),t)\,,\, \tilde{\gamma}(\|\boldsymbol{v}_{[t-1]}\|)\\ &\leq \tilde{\beta}\,(\alpha_2(|\overline{x}_0|),t)+\tilde{\beta}\,(\sigma_1(|v_0|),t)+\tilde{\gamma}(\|\boldsymbol{v}_{[t-1]}\|)\\ &\leq \beta\,(|\overline{x}_0|,t)+\gamma(\|\boldsymbol{v}_{[t-1]}\|),\end{aligned} \tag{A-3}$$

$\forall t \in \mathbb{Z}_{\geq 0}, \forall \overline{x}_0 \in \mathcal{N}_{[V,\overline{e}]}, \forall v \in \mathcal{M}_\Upsilon$. Hence, by (A-3), the system (1) is ISS in $\mathcal{N}_{[V,\overline{e}]}$ with ISS-asymptotic gain γ. Considering that starting from Ξ the set $\mathcal{N}_{[V,\overline{e}]}$ is reached in finite time, the ISS in $\mathcal{N}_{[V,\overline{e}]}$ implies the UAG in Ξ.

Now, thanks to Lemma 1.1, Assumption 1, the UAG in Ξ implies the LS, as well, in Ξ, and hence the regional ISS property in Ξ, thus proving Theorem 2.1.

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