Portfolio Optimisation and Option Pricing in Discrete Time with Transaction Costs

A thesis presented for the degree of

Doctor of Philosophy of Imperial College London

by

Gary Sze Huat Quek

Department of Mathematics

Imperial College London

180 Queen's Gate, London SW7 2BZ

MAY 2012

Declaration

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Signed:

Copyright

Abstract

Discrete time models of portfolio optimisation and option pricing are studied under the effects of proportional transaction costs. In a multi-period portfolio selection problem, an investor maximises expected utility of terminal wealth by rebalancing the portfolio between a risk-free and risky asset at the start of each time period. A general class of probability distributions is assumed for the returns of the risky asset. The optimal strategy involves trading to reach the boundaries of a no-transaction region if the investor's holdings in the risky asset fall outside this region. Dynamic programming is applied to determine the optimal strategy, but it can be computationally intensive. In the limit of small transaction costs, a two-stage perturbation method is developed to derive approximate solutions for the exponential and power utility functions. The first stage involves ignoring the no-transaction region and transacting to the optimal point corresponding to the zero transaction costs case. Approximations of the resulting suboptimal value functions are obtained. In the second stage, these suboptimal value functions are corrected to obtain approximations of the optimal value functions and optimal boundaries at all time steps.

A discrete time option pricing model is developed based on the utility maximisation approach. This model reduces to the binomial model in the special case where the risky asset follows a binomial price process without transaction costs. Incorporating transaction costs, the utility indifference price and marginal utility indifference price of the option are observed to depend on the price of the underlying risky asset and the investor's holdings in the risky asset. The regions where these option prices do not vary with the investor's holdings in the risky asset are identified. An example illustrates how utility indifference pricing or marginal utility indifference pricing enables one to determine the bid and ask price of a European call option.

Acknowledgements

I would like to thank my supervisor Professor Colin Atkinson for his guidance and encouragement throughout the work on this thesis. It was a privilege to learn from him and I have benefited tremendously from his insightful knowledge and vast experience. His commitment and engagement in my work and personal well-being has been a constant source of motivation. I am touched by his immense generosity for which I remain very grateful.

Financial support provided by the Beit Fellowship for Scientific Research from the Beit Trust, and the Deputy Rector's Award from Imperial College London is gratefully acknowledged.

Special thanks are due to my parents and extended family for their encouragement and support. I am especially grateful to my in-laws whose practical advice and family support allowed me to focus on my research. Their tireless efforts are very much appreciated.

Lastly and most importantly, I would like to give a big and special thanks to my wife Cassandra and son Matthew. I am extremely thankful for their understanding, especially in the past few months when we had to sacrifice our family time together. Their immeasurable patience, encouragement and moral support have made this journey possible.

Table of contents

List of Figures

List of Tables

Chapter 1

Introduction

Portfolio theory and option pricing theory, which are broadly two of the most active areas of research in mathematical finance, typically involve the study of decision-making under uncertainty. For instance, the following questions arise: How much of one's wealth should be utilised for consumption or allocated between investments in stocks and bonds? How does one hedge away the risk and determine the valuation of an option? The ongoing research in these areas aims to develop more realistic models that reflect the dynamics of financial markets and accurately describe the strategies to be taken by decision-makers. The present study concentrates on analysing the impact of transaction costs in portfolio theory and option pricing theory. Pioneering research in portfolio theory and option pricing theory, which ignored the presence of transaction costs, derived exact closed-from solutions but may lead to unrealistic investment or hedging strategies. On the other hand, the inclusion of transaction costs in subsequent financial market models often resulted in equations that did not allow exact solutions. In order to solve these equations, one had to employ mathematical analysis and techniques to obtain analytical, numerical or approximate solutions.

In the first part of the thesis, we consider a multi-period portfolio selection problem where an investor chooses to allocate his wealth between a risky and risk-free asset. The investor makes a sequence of investment decisions at the start of each time period with the objective of maximising expected utility of terminal wealth. A cost that is proportional to the value of the transaction is incurred each time the investor trades in the risky asset. In order to determine the optimal investment strategy, we reduce the original problem to a sequence of more tractable sub-problems with the application of dynamic programming. Nonetheless, an exact solution is generally unavailable and it is also computationally intensive to evaluate a numerical solution. Therefore, in the limit of small transaction costs, we derive an approximate solution to the optimal investment strategy via perturbation analysis. We investigate the cases where the investor has an exponential or a power utility function.

The second part of the thesis focuses on developing and analysing a discrete time model of option pricing that incorporates proportional transaction costs. In situations where it might be impossible or unfavourable to replicate an option, we adopt an approach that is based on the maximisation of expected utility of terminal wealth. In this approach, the selling (buying) price of an option is defined as the amount of money that will make the investor indifferent, in terms of expected utilities, between trading in the market with and without a short (long) position in the option. In other words, to price the option, one has to compare the aforementioned portfolio selection problem (from the first part of the thesis) with one that incorporates the option position. An alternative approach is also investigated, where the price of an option is determined by the requirement that an infinitesimal diversion of funds into the option purchase or sale has a neutral effect on the investor's achievable utility. These approaches are generally known as utility indifference pricing and marginal utility indifference pricing respectively. We also demonstrate that, in the absence of transaction costs and with the risky asset following a binomial price process, both the utility indifference price and marginal utility indifference price reduce to the perfect replication price (from the binomial model).

In the next two sections, we review the existing literature in portfolio theory and option pricing theory with a focus on transaction costs.

1.1 Review of Portfolio Theory

The mean-variance approach in modern portfolio theory was pioneered by the seminal work of Markowitz (1952), who introduced the use of standard deviation of return as a measure of risk. This theory provided an investor with a criterion to select portfolio combinations efficiently from a given set of securities. An efficient portfolio was defined as one that maximised its mean return given a pre-specified level of risk, or equivalently, one that minimised risk given a pre-specified level of mean return. The main disadvantage of the mean-variance approach rested on it being a static one-period model. It was one in which the investor would allocate his portfolio at the start of a period and wait for the return to be realised at the end of the period without making intermediate changes to the portfolio composition. Nonetheless, it remains a popular approach that is still widely used today in asset allocation models.

1.1.1 Continuous Time Models

A more general approach to portfolio theory involved the development of dynamic or multi-period models. In these models, the investor would make a sequence of decisions with the objective of maximising expected utility. The portfolio selection and consumption problem in a continuous time setting was first studied by Merton (1969). The investor's objective was to maximise expected utility from consumption, where the price of the risky asset was assumed to be driven by a geometric Brownian motion in a frictionless market. For the case of the power or logarithmic utility function, an explicit solution was obtained using stochastic control theory. It was shown that the optimal investment strategy involved continuously rebalancing the portfolio to maintain a constant proportion of the risky asset. This constant proportion is commonly referred to as the Merton proportion.

However, continuous trading in financial markets would be prohibitively expensive due to the impact of transaction costs. Transaction costs incorporated into subsequent research were generally modelled as: a constant amount independent of the value of the transaction (i.e. constant costs); an amount proportional to the value of the transaction (i.e. proportional costs); or a fixed proportion of the entire portfolio value. Magill and Constantinides (1976) were the first to extend Merton's (1969) model to incorporate proportional transaction costs. Although their argument was heuristic, they provided the insight that "the investor trades in securities when the variation in the underlying security prices forces his portfolio proportions outside a certain region about the optimal proportions in the absence of transactions costs". Davis and Norman (1990) provided a rigorous formulation and analysis of the portfolio selection and consumption problem with proportional transaction costs by applying the theory of stochastic singular control. Their work was further generalised by Shreve and Soner (1994) with less restrictive assumptions using the theory of viscosity solutions. Taksar et al. (1988) analysed the portfolio selection problem with proportional transaction costs, which involved applying stochastic singular control to maximise the long run growth rate of the portfolio value. In the aforementioned models with transaction costs, the typical optimal strategy was not to transact when the proportion of risky asset drifted within a particular no-transaction region. When the proportion of risky asset exceeded the boundary of this region, the investor would transact instantaneously to return to the boundary. Akian et al. (1996) extended Davis and Norman's (1990) model to study the case where there was more than one risky asset. Morton and Pliska (1995) introduced a multi-asset model where the investor paid a transaction cost equal to a fixed proportion of the entire portfolio value. The investor's objective was to maximise the long run growth rate of the portfolio value and the optimal strategy was shown to be reduced to one that solved a single stopping time problem.

In general, models that incorporated transaction costs did not allow exact analytical solutions. Most of these models had to be solved by numerical methods that were often computationally intensive, especially in the case of multiple risky assets. Nonetheless, it was observed that transaction costs were small in practice relative to the value of the transactions. In the limit of small transaction costs, Atkinson and Wilmott (1995) applied the technique of perturbation analysis about the no transaction costs solution to derive an approximate solution to the Morton and Pliska (1995) model with multiple risky assets. Mokkhavesa and Atkinson (2002) derived an approximate solution to the portfolio selection and consumption problem with small transaction costs via the use of perturbation analysis for a single risky asset and an arbitrary utility function. Janecek and Shreve (2004) provided a rigorous derivation of the asymptotic expansions of the optimal value function and boundaries of the no-transaction region for an investor with the power utility function. Their work was recently extended by Gerhold et al. (2011), who obtained power series expansions of arbitrary order for the optimal value function and boundaries of the no-transaction region by using duality theory. However, all the aforementioned perturbation analyses were applied to continuous time models where the prices of the risky assets were assumed to be geometric Brownian motions.

1.1.2 Discrete Time Models

In a discrete time framework, Samuelson (1969) studied the portfolio selection and consumption problem in a multi-period model using a dynamic programming approach, which was analogous to the work of Merton (1969). He analysed the maximisation of expected utility from consumption without transaction costs, where the return of the risky asset was assumed to follow a general probability distribution. In the case of the power or logarithmic utility function, he showed that the investor's optimal strategy was to maintain a constant proportion of wealth invested in the risky asset at each time step.

Mossin (1968) analysed the dynamic portfolio selection problem (without consumption) in discrete time and considered a multi-period model without transaction costs. For example, consider a N-period model in which an investor would rebalance the portfolio between a risk-free and risky asset at the start of each time period. Suppose that W_n denotes the wealth of the portfolio and a_n denotes the dollar value invested in the risky asset at time period n. Then, the investor's wealth at time period $n + 1$ is given by

$$W_{n+1} = r_n W_n + (s_n - r_n)\, a_n, \tag{1.1}$$

where r_n is one plus the return of the risk-free asset and s_n is one plus the return of the risky asset from time period n to $n + 1$. In this model, the investor would determine his optimal investment in the risky asset a_n at the start of each time period n ($n = 0, \ldots, N - 1$) with the objective of maximising expected utility of terminal wealth

$$\mathbb{E}\left[U(W_N)\right]. \tag{1.2}$$

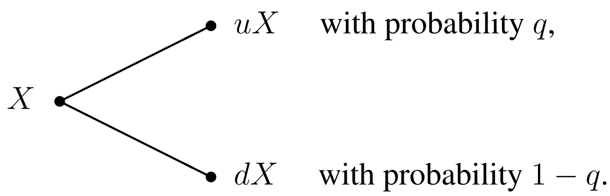
He showed that, for an investor with the power or logarithmic utility function, the optimal strategy involved making a series of single-period decisions without considering future reinvestment opportunities. This was described as a myopic strategy and represented a simplification of the problem. However, this was only a special case and it would generally not be optimal for the investor to make a decision at each time step without looking ahead. A more detailed description of this portfolio selection model is presented in Section 1.5.

Bobryk and Stettner (1999) incorporated proportional transaction costs in a discrete time model and studied the maximisation of expected utility from consumption. In the case of the power or logarithmic utility function, the optimal investment strategy was shown to be characterised by a cone shaped no-transaction region. They derived various bounds on the no-transaction region by specifying upper and lower bounds on the support of the probability measure for the returns of the risky asset. Sass (2005) embedded the Cox et al. (1979) binomial model with a general transaction costs structure that included proportional costs, constant costs and costs that were a fixed proportion of the portfolio value. He formulated the objective of maximising expected utility of terminal wealth as a Markov control problem and gave a multi-period existence result based on the solution of the dynamic programming equation. Explicit results were provided for the one-period problem in the case of a single risky asset with a binomial price process. Atkinson and Storey (2010) recently analysed the discrete time portfolio selection problem by incorporating proportional transaction costs in Mossin's (1968) model. They assumed a general class of underlying probability distributions for the returns of the risky asset and studied the problem of maximising expected utility of terminal wealth for the power utility function. Perturbation analysis was applied to obtain approximations of the optimal boundaries of the no-transaction region in the limits of small and large transaction costs. However, the approximations of the optimal boundaries were only derived up to two time steps. Atkinson and Quek (2012) investigated the case of maximising expected utility of terminal wealth for the exponential utility function. In the limit of small transaction costs, they developed a perturbation method to construct approximations of the optimal value function and optimal boundaries of the no-transaction region at all time steps of the problem. A detailed description of the multi-period portfolio selection model with proportional transaction costs will be presented in Chapters 2 and 3.

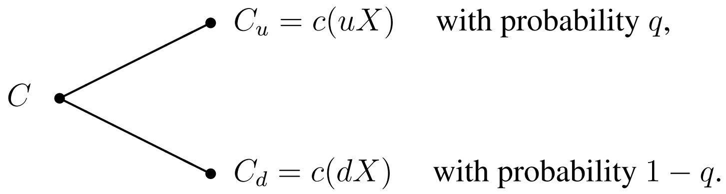
1.2 Review of Option Pricing Theory

The seminal paper by Black and Scholes (1973) led to a major breakthrough in option pricing theory. A continuous time model was developed to value a European option in a frictionless market (without transaction costs), where the price of the underlying risky asset was assumed to follow a geometric Brownian motion. This is known as the Black-Scholes option pricing model. They constructed a portfolio consisting of a long position in the underlying risky asset and a short position in the European option. Adopting a dynamic trading strategy, the portfolio was continuously rebalanced to maintain a perfectly hedged position with zero risk. In the absence of arbitrage, it was concluded that the perfectly hedged portfolio had a return that was equal to the risk-free rate of return. From this hedging and no-arbitrage argument, they derived a simple formula for computing the price of a European option, which is known as the Black-Scholes option pricing formula.

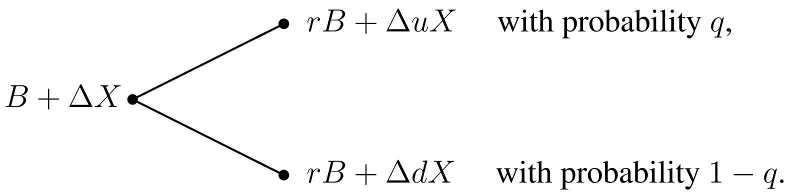
A discrete time model of option pricing by no-arbitrage methods was developed by Cox et al. (1979) where the price of the underlying risky asset was assumed to follow a multiplicative binomial tree. They constructed a portfolio consisting of the risky asset and a risk-free asset, which replicated the payoff of a European call option at its expiration date. As an illustration of this construction, consider a one-period model where the initial price of the risky asset is X. Let u denote one plus the return of the risky asset with probability q and let d denote one plus the return of the risky asset with probability $1-q$ over a single time period. Furthermore, let r denote one plus the one-period return of the risk-free asset and assume that $d < r < u$. The price of the risky asset at the end of the time period is represented by the diagram



Consider a European option expiring at the end of the time period with a general payoff function c that depends on the price of the risky asset at expiry. Let C be its initial value, C_u be its value at the end of the period if the price of the risky asset is uX and C_d be its value at the end of the period if the price of the risky asset is dX. Since the option is expiring at the end of the time period, its value will be equal to its payoff. In other words, $C_u = c(uX)$ and $C_d = c(dX)$. Therefore,



Suppose that a portfolio comprising B dollars of the risk-free asset and Δ shares of the risky asset is constructed. This will cost $B + \Delta X$ initially and at the end of the period, its value will be



The unknowns B and Δ are to be chosen so that the portfolio value replicates the option payoff at the end of the period, which require

$$rB + \Delta uX = C_u, \tag{1.3}$$

$$rB + \Delta dX = C_d. \tag{1.4}$$

Solving these equations simultaneously,

$$\Delta = \frac{C_u - C_d}{(u-d)\,X}, \quad B = \frac{uC_d - dC_u}{(u-d)\,r}. \tag{1.5}$$

The portfolio formed in this way is known as the replicating or hedging portfolio. In order to ensure that there are no risk-free arbitrage opportunities, the initial value of the option must be equal to the value of its replicating portfolio, which means that

$$C = B + \Delta X. \tag{1.6}$$

Substituting (1.5) into the above equation and simplifying, the price of the option is therefore

$$C = \frac{pC_u + (1-p)\,C_d}{r}, \tag{1.7}$$

where $p = \dfrac{r-d}{u-d}$ and $1 - p = \dfrac{u-r}{u-d}$. Note that p, which is strictly between 0 and 1, is known as the risk neutral probability.

This one-period option pricing model can be easily extended to the N-period case. Starting from the expiry date of the option and evaluating Equation (1.7) recursively backwards in time. Cox et al. (1979) showed that the price of the option is given by

$$C = \frac{1}{r^N} \sum_{i=0}^{N} \binom{N}{i} p^i \,(1-p)^{N-i}\, c(u^i d^{N-i} X), \tag{1.8}$$

where $\dbinom{N}{i} = \dfrac{N!}{i!\,(N-i)!}$ is the binomial coefficient. Furthermore, from Equation (1.5), the hedge ratio can be written as

$$\begin{aligned} \Delta &= \frac{1}{r^{N-1}\,(u-d)\,X} \\ &\times \sum_{i=0}^{N-1} \binom{N-1}{i} p^i \,(1-p)^{N-1-i} \left[c(u^{i+1} d^{N-1-i} X) - c(u^i d^{N-i} X) \right]. \end{aligned} \tag{1.9}$$

This model, known as the binomial model, presented a simple and efficient numerical procedure for valuing options. In a special limiting case, it was also shown that the discrete time binomial model converged to the continuous time Black-Scholes option pricing model.

In a continuous time framework with transaction costs, it could become ruinously expensive to continuously rebalance a portfolio in order to maintain a perfect hedge. Therefore, in the presence of transaction costs, it was necessary to consider alternative criteria in the pricing and hedging of an option. These criteria generally involved a trade-off between the transaction costs incurred in portfolio rebalancing and the bounds on the option price (i.e. the bid and ask price). For instance, reducing the hedging error through more frequent trading would incur higher transaction costs and lead to wider bid-ask spreads. Therefore, the problem became one of designing some form of an optimal pricing and hedging strategy that would be consistent with an investor's objectives. The two main approaches that addressed the issue of transaction costs were based on the concepts of replication and utility maximisation.

1.2.1 Replication Approach

Approximate Replication Strategy

Leland (1985) was the first to incorporate transaction costs in the Black-Scholes model. Rebalancing the portfolio at discrete time intervals instead of continuously, he constructed a modified hedging strategy that approximately replicated the option return inclusive of transaction costs. He also derived an explicit Black-Scholes type formula adjusted with a modified volatility parameter, which provided upper and lower bounds on the price of a European call option. Leland's pricing methodology was of practical importance due to its ease of implementation. In the limiting case where the length of the revision interval tends to zero, Kabanov and Safarian (1997) showed that contrary to the claim in Leland (1985), the hedging error in Leland's strategy was non-zero when the level of transaction costs was a constant. However, if the level of transaction costs decreased to zero as the revision interval tends to zero, the limit of the hedging error would be equal to zero. Recently, Denis (2010) extended Leland's approach of approximate hedging to study a more general class of option payoffs.

Perfect Replication Strategy

In a discrete time framework, unlike in continuous time, it was still possible to perfectly replicate an option payoff in the presence of transaction costs. Merton (1990) and Boyle and Vorst (1992) studied the replication of European call options in the binomial model when there were proportional transaction costs on trades in the risky asset. Merton considered a two-period model while Boyle and Vorst extended the analysis to multiple time periods. They constructed self financing strategies that perfectly replicated the option payoff at the expiration date. Using the usual no-arbitrage argument, the price of the option corresponded to the initial cost of constructing the replicating portfolio. The cost of replicating a long position in the option was shown to provide an upper bound for the option price, while the (negative of the) cost of replicating a short position provided a lower bound.

Palmer (2001) revisited Boyle and Vorst's (1992) model and relaxed some of the conditions imposed in their procedure to compute the lower bound. He developed an alternative algorithm to compute the cost of replicating a short position in the option without the need for those conditions.

Super-Replication Strategy

In the binomial model, Bensaid et al. (1992) investigated super-replication strategies that dominated the payoff of an option at expiration under proportional transaction costs. Among these strategies, they addressed the problem of finding the optimal (i.e. least costly) one and derived bounds on the price of the option. The upper bound was defined as the minimum initial cost of constructing a super-replicating portfolio for a long position in the option. Similarly, the lower bound was defined as the (negative of the) minimum initial cost of super-replicating a short position in the option. In some instances, they showed that the minimum cost super-replication strategy was in fact cheaper than the cost of the perfect replication strategy. However, in the case of small transaction costs or for long positions in options settled by delivery, they also showed that perfect replication was the optimal strategy. The super-replication approach was further analysed by Edirsinghe et al. (1993), who considered options with general non-convex payoffs, lot size constraints and transaction cost structures that included proportional as well as constant costs. They showed that in the presence of these trading frictions, it was no longer optimal to rebalance the portfolio in every period. Recently, Roux et al. (2008) developed an algorithm based on the super-replication approach, which they applied to the pricing and hedging of European options with arbitrary payoffs in a general discrete market model with arbitrary proportional transaction costs.

The main disadvantage of the super-replication strategy was that the cost of hedging would increase as the frequency of portfolio revisions was increased. Moreover, in the continuous time limit, Soner et al. (1995) proved the conjecture by Davis and Clark (1994) that the minimum cost super-replication strategy was the trivial buy and hold strategy. This strategy, which involved buying one share of the underlying risky asset and holding it until the expiration of the option, was of no interest to practitioners in the financial markets as it was considered too expensive.

1.2.2 Utility Maximisation Approach

Utility Indifference Pricing

Hodges and Neuberger (1989) proposed a utility maximisation approach to analyse the problem of option pricing with proportional transaction costs. They recognised that the optimal hedging strategy for an option under transaction costs should depend on an investor's risk preferences. In this approach, the selling (buying) price of an option was defined as the amount of money that would make the investor indifferent, in terms of expected utilities, between trading in the market with and without a short (long) position in the option. The resulting price of the option is known as the utility indifference price or reservation price. Davis et al. (1993) developed this idea rigorously and demonstrated that the pricing definition would reduce to the Black-Scholes price in the absence of transaction costs. They computed the option selling price for the case of the exponential utility function by solving two stochastic singular control problems with different boundary conditions. Clewlow and Hodges (1997) extended the utility maximisation approach to option pricing by incorporating a general cost function with constant and proportional costs. Constantinides and Zariphopoulou (1999) considered the case of proportional transaction costs with general risk preferences and derived in closed form an upper bound to the utility indifference selling price of a European call option. Andersen and Damgaard (1999) adopted the utility maximisation approach in a discrete time model with two risky assets, where the price vector process was assumed to evolve in a trinomial tree. They considered utility functions within the hyperbolic absolute risk aversion (HARA) class. Based on their numerical examples, they found that the utility indifference buying price of a European call option was relatively insensitive to the functional form of the HARA utility function if the initial level of absolute risk aversion was identical. Zakamouline (2006) developed a numerical procedure for computing option prices and optimal hedging strategies within the utility maximisation framework of Davis et al. (1993) with proportional as well as constant transaction costs. She carried out a comparative simulation study of various hedging strategies by comparing the mean and standard deviation of the corresponding hedging errors and concluded that the utility-based hedging strategy outperformed the others.

Marginal Utility Indifference Pricing

Davis (1997) proposed embedding the option pricing problem within a utility maximisation framework in situations where replication was either impossible or unfavourable. Using a marginal rate of substitution argument, he defined the fair price of an option as one that had a neutral effect on the investor's utility when an infinitesimal amount of initial wealth was diverted into the option at that price. This definition resulted in a unique price for the option, which is known as the marginal utility indifference price. Using this approach, Monoyios (2004) developed an efficient algorithm based on a Markov chain approximation to price European options in the presence of proportional transaction costs. A detailed description of utility indifference pricing and marginal utility indifference pricing in discrete time will be presented in Chapters 4 and 5.

The utility maximisation approach to option pricing often resulted in equations that had to be solved with computationally intensive numerical methods. In the case of small transaction costs, Whalley and Wilmott (1997) carried out a perturbation analysis of the utility indifference pricing approach of Davis et al. (1993). They reduced a three dimensional free boundary problem to an inhomogeneous diffusion equation in two independent variables. This technique increased the speed at which the optimal hedging strategy was calculated and also provided additional insights to the solution. Barles and Soner (1998) derived an option pricing formula from the model of Hodges and Neuberger (1989) by applying an asymptotic analysis of partial differential equations. Atkinson and Alexandropoulos (2006) studied the case of multiple uncorrelated risky assets and approximated the price of basket options by applying the method of perturbation analysis.

In addition to markets with transaction costs, the utility maximisation approach was also widely used to price options in incomplete markets. Examples of market incompleteness include the presence of non-traded assets or portfolio constraints. A recent reference on the theory and applications of utility indifference pricing in incomplete markets could be found in Carmona (2009).

Having presented a review of portfolio theory and option pricing theory, we now proceed to introduce the mathematical tools that will be used in the thesis. In the next two sections, we introduce the common classes of utility functions and the concept of dynamic programming.

1.3 Utility Functions

A common approach to the dynamic portfolio selection problem is one that is based on maximising expected utility. Suppose that an investor is given a number of investment assets and is to make a choice in allocating his wealth among these various assets. Having chosen an investment strategy, the investor's wealth at the end of some time horizon is determined by the random outcomes of these assets. In view of such uncertainty and risk, expected utility theory provides a way for the investor to rank his choice of investment strategy. Assume that the investor has a utility function of wealth $U(W)$ that is strictly increasing (i.e. more wealth is preferred to less wealth) and strictly concave (i.e. risk aversion). Figure 1.1 is an example of a risk averse investor's utility function. The investor will choose to allocate his wealth in a way that maximises his expected utility of terminal wealth $\mathbb{E}\left[U(W_N)\right]$. An alternative to maximising utility of wealth is to maximise utility from consumption, which arises in the study of portfolio consumption and selection problems. In the thesis, we assume that the investor has a pre-determined consumption plan and our focus is on the portfolio selection problem.

There are commonly used utility functions that fall under the classes of constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA). First, we discuss the notion of absolute risk aversion (due to Arrow (1971) and Pratt (1964)) and present the class of utility functions with constant absolute risk aversion. This is followed by a discussion of relative risk aversion and a description of the utility functions that exhibit constant relative risk aversion.

1.3.1 Absolute Risk Aversion

Consider an investor with wealth W and utility function $U(W)$. Assume that $U(W)$ is strictly increasing, strictly concave and twice continuously differentiable. Let ϵ denote a

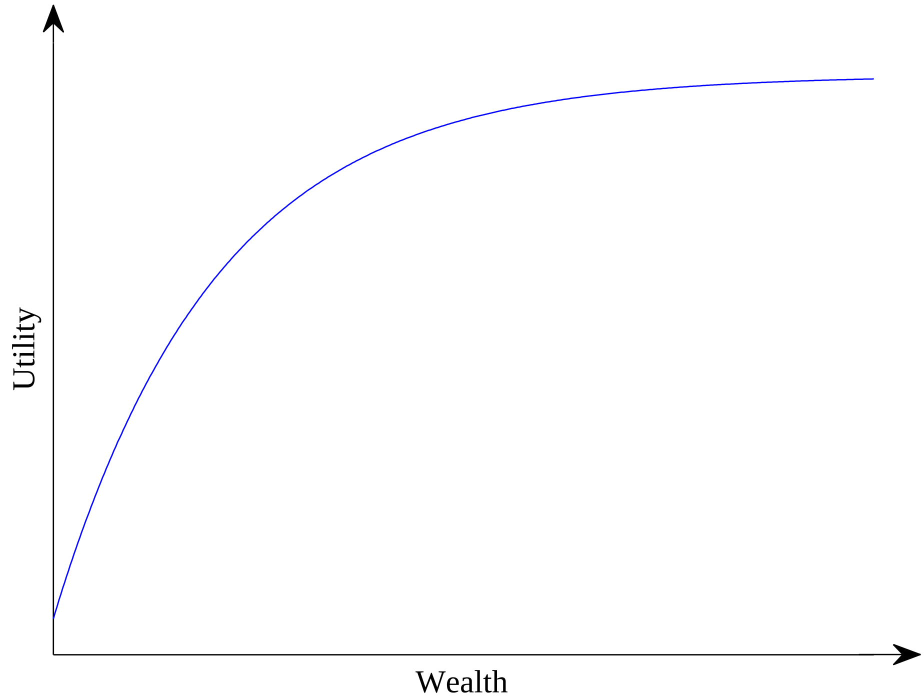


Figure 1.1: Utility Function of a Risk Averse Investor

fair gamble, where ϵ is a random outcome with $\mathbb{E}[\epsilon] = 0$ and $\mathbb{E}[\epsilon^2] = \sigma^2$. Suppose that π is the risk premium that the investor will pay to avoid a fair gamble. In other words, the investor is indifferent between receiving a (random) risk ϵ and paying a (non-random) risk premium π. Thus, we have the relationship

$$U(W - \pi) = \mathbb{E}[U(W + \epsilon)]. \tag{1.10}$$

The amount $W - \pi$ is known as the certainty equivalent of the gamble $W + \epsilon$. In order to measure the investor's aversion to risk, consider the investor's risk premium for a small risk ϵ. Assuming that ϵ is small, the Taylor expansion of Equation (1.10) about W is given by

$$U(W) - \pi U'(W) + O(\pi^2) = \mathbb{E}\left[U(W) + \epsilon U'(W) + \frac{1}{2}\epsilon^2 U''(W) + O(\epsilon^3)\right]. \tag{1.11}$$

This implies that

$$\pi \approx \frac{1}{2}\sigma^2 A(W), \tag{1.12}$$

where

$$A(W) = -\frac{U''(W)}{U'(W)} \tag{1.13}$$

is known as the Arrow-Pratt coefficient of absolute risk aversion. In other words, the risk premium is approximately proportional to the coefficient of absolute risk aversion for a small risk. We wish to obtain the class of utility functions with constant absolute risk aversion. Suppose that

$$-\frac{U''(W)}{U'(W)} = \kappa, \tag{1.14}$$

where κ is a constant. We have assumed that the investor's utility function is strictly increasing so that $U'(W) > 0$. If $\kappa < 0$, then $U''(W) > 0$ and the utility function is strictly convex (i.e. risk seeking). If $\kappa = 0$, then $U''(W) = 0$ and the utility function is linear (i.e. risk neutral). Otherwise, if $\kappa > 0$, then $U''(W) < 0$ and the utility function is strictly concave (i.e. risk averse). Thus, we assume that $\kappa > 0$ as we are interested in studying the behaviour of a risk averse investor. A class of utility functions that satisfies Equation (1.14) for $\kappa > 0$ is given by

$$U(W) = -\frac{\alpha}{\kappa}e^{-\kappa W} + \beta, \tag{1.15}$$

where $\alpha > 0$ and β are arbitrary constants. It is well known that the utility function is defined up to a positive linear transformation $U(W) \to aU(W) + b$ for a positive scalar a and a scalar b. Furthermore, the coefficient of absolute risk aversion is invariant under a positive linear transformation of the utility function. Therefore, for the ease of presentation in subsequent chapters, we assume that an investor with constant absolute risk aversion has an exponential utility function of the form

$$U(W) = -e^{-\kappa W}, \tag{1.16}$$

where $\kappa > 0$ is the (constant) coefficient of absolute risk aversion.

1.3.2 Relative Risk Aversion

We now discuss the related notion of relative risk aversion. Suppose that the risk and risk premium are viewed as a fraction of the investor's wealth. Let $\tilde{\epsilon} = \epsilon/W$ denote the proportional risk and $\tilde{\pi} = \pi/W$ denote the proportional risk premium. In this case, we have the certainty equivalence relationship

$$U((1-\tilde{\pi})W) = \mathbb{E}\left[U((1+\tilde{\epsilon})\,W)\right]. \tag{1.17}$$

For a small proportional risk, the Taylor expansion of Equation (1.17) leads one to the definition of the Arrow-Pratt coefficient of relative risk aversion

$$R(W) = -\frac{WU''(W)}{U'(W)}. \tag{1.18}$$

In order to determine the class of utility functions with constant relative risk aversion, suppose that

$$-\frac{WU''(W)}{U'(W)} = 1-\gamma, \tag{1.19}$$

where γ is a constant. Assuming that $W > 0$ and recalling that $U'(W) > 0$, the case of $\gamma > 1$ corresponds to a risk seeking investor with $U''(W) > 0$; $\gamma = 1$ corresponds to one who is risk neutral with $U''(W) = 0$; and $\gamma < 1$ corresponds to one who is risk averse with $U''(W) < 0$. Since our interest lies in studying the behaviour of a risk averse investor, we assume that $\gamma < 1$. A class of utility functions that satisfies Equation (1.19) is given by

$$U(W) = \begin{cases} \frac{\alpha}{\gamma}\left(W^{\gamma}+\beta\right) & \text{if } \gamma \neq 0, \\ \alpha \ln W + \beta & \text{if } \gamma = 0, \end{cases} \tag{1.20}$$

where $\alpha > 0$ and β are arbitrary constants. Recall that the utility function is unique up to a positive linear transformation. Therefore, for the ease of presentation, we assume that an investor with constant relative risk aversion has a power utility function of the form

$$U(W) = \frac{1}{\gamma}W^{\gamma} \tag{1.21}$$

for the case where $\gamma < 1, \gamma \neq 0$. In this case, the (constant) coefficient of relative risk aversion is equal to $(1 - \gamma)$. For the case where $\gamma = 0$, we assume a logarithmic utility function of the form

$$U(W) = \ln W. \tag{1.22}$$

It is noted that utility functions with constant absolute risk aversion or constant relative risk aversion fall within a wider hyperbolic absolute risk aversion (HARA) class of utility functions. This is a class of utility functions $U(W)$ whose coefficient of absolute risk aversion is positive and inverse to a linear function of wealth (see Merton (1971)), that is,

$$-\frac{U''(W)}{U'(W)} = \left(\frac{W}{1-\eta} + \frac{\mu}{\lambda}\right)^{-1} \tag{1.23}$$

and subject to the restrictions

$$\eta \neq 1,\ \lambda > 0,\ \frac{\lambda W}{1-\eta} + \mu > 0,\ \mu = 1 \text{ if } \eta = -\infty. \tag{1.24}$$

All members of the HARA family can be expressed in the form

$$U(W) = \frac{1-\eta}{\eta}\left(\frac{\lambda W}{1-\eta} + \mu\right)^{\eta}. \tag{1.25}$$

By a suitable choice of the parameters, one can obtain a utility function with absolute or relative risk aversion that is increasing, decreasing or constant.

1.4 Dynamic Programming

In this section, we give a slightly abbreviated description of dynamic programming as described in the book by Bertsekas (2005). The multi-period portfolio selection problem is an example of a situation where decisions are made in stages under uncertainty. The outcome of each decision, which may not be completely predictable, can be anticipated to some extent before the next decision is made. The objective is to maximise a certain value or desirable outcome. Dynamic programming is often used to deal with such a situation. A key aspect is that decisions cannot be viewed in isolation as one seeks to strike a balance between the desire for high present value and the undesirability of low future values. At each stage, the dynamic programming technique ranks decisions (via utility functions) based on the sum of the present value and the expected future value, assuming that the decision maker behaves optimally at subsequent stages. We illustrate the technique of dynamic programming with a basic optimisation model in discrete time (see Bertsekas (2005) for the full technical details).

Consider a model that comprises a discrete time dynamic system and a value function that is additive over time. The dynamic system is of the form

$$x_{n+1} = f_n(x_n, a_n, s_n), \qquad n = 0, 1, \ldots, N-1, \tag{1.26}$$

where
n indexes discrete time,
x_n is the state of the system at time period n,
a_n is the control or decision variable to be selected at time period n,
s_n is a random parameter characterised by a probability distribution that may depend on x_n and a_n but not on values of the prior random parameters $s_{n-1}, \ldots, s_0$,
f_n is a function that describes the system, and
N is the number of time periods or number of times control can be applied.

Let $g_n(x_n, a_n, s_n)$ be the value that is gained at time period n. Assuming that the value accumulates over time, the total value is

$$g_N(x_N) + \sum_{n=0}^{N-1} g_n(x_n, a_n, s_n), \tag{1.27}$$

where $g_N(x_N)$ is the terminal value at the end of the process. This value is generally a random variable due to the presence of s_n. Therefore, the problem is formulated as one of maximising the expected total value

$$\mathbb{E}\left[g_N(x_N) + \sum_{n=0}^{N-1} g_n(x_n, a_n, s_n)\right]. \tag{1.28}$$

We now define the optimal policy and the optimal value function of the problem. Consider the class Π of admissible policies that consists of a sequence of functions $\pi = \{\alpha_0, \ldots, \alpha_{N-1}\}$, where α_n maps the states x_n into controls $a_n = \alpha_n(x_n)$. Given an initial state x_0 and policy $\pi \in \Pi$, the parameters s_n and the states x_n are random variables with distributions defined through the system equation

$$x_{n+1} = f_n(x_n, \alpha_n(x_n), s_n), \qquad n = 0, 1, \ldots, N-1. \tag{1.29}$$

Thus, the expected total value of policy π starting at state x_0 is denoted by

$$J_\pi(x_0) = \mathbb{E}\left[g_N(x_N) + \sum_{n=0}^{N-1} g_n(x_n, \alpha_n(x_n), s_n)\right], \tag{1.30}$$

where the expectation $\mathbb{E}$ is taken over the joint distribution of the random variables s_n and x_n. An optimal policy π^* is defined to be one that maximises this value, that is,

$$J_{\pi^*}(x_0) = \max_{\pi \in \Pi} J_\pi(x_0). \tag{1.31}$$

The corresponding optimal value function J^* is a function that assigns to each initial state x_0 the optimal value $J^*(x_0)$ given by

$$J^*(x_0) = \max_{\pi \in \Pi} J_\pi(x_0). \tag{1.32}$$

In order to obtain the optimal policy π^* and the optimal value function J^*, we apply the technique of dynamic programming. This technique is based on Bellman's (1957) principle of optimality, which states that

> An optimal policy has the property that whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The dynamic programming algorithm, which allows one to systematically solve the problem for π^* and J^*, is stated in the following proposition.

Proposition 1.4.1. *The dynamic programming algorithm, which proceeds backward in time* *from period N-1 to period 0, is given by*

$$J_N(x_N) = g_N(x_N) \tag{1.33}$$

and

$$J_n(x_n) = \max_{a_n} \mathop{\mathbb{E}}_{s_n} \Big[g_n(x_n, a_n, s_n) + J_{n+1}\left(f_n(x_n, a_n, s_n)\right) \Big] \tag{1.34}$$

for $n = 0, \ldots, N-1$*, where the expectation is taken with respect to the probability distribution of* s_n*. For every initial state* x_0*, the optimal value* $J^*(x_0)$ *of the problem is equal to* $J_0(x_0)$ *given by the above algorithm. Furthermore, if* $a_n^* = \alpha_n^*(x_n)$ *maximises the right hand side of the algorithm (1.34) for each* x_n *and* n*, the policy* $\pi^* = \left\{\alpha_0^*, \ldots, \alpha_{N-1}^*\right\}$ *is optimal.*

Proof. The main ideas behind the proof of the proposition rest on the principle of optimality and an application of mathematical induction. For $n = 0, \ldots, N-1$, denote $\pi_n = \{\alpha_n, \ldots, \alpha_{N-1}\}$ and let $J_n^*(x_n)$ be the optimal value for the $(N-n)$-stage problem that starts at time period n with state x_n and ends at time period N, so that

$$J_n^*(x_n) = \max_{\pi_n} \mathop{\mathbb{E}}_{s_n, \ldots, s_{N-1}} \left[g_N(x_N) + \sum_{i=n}^{N-1} g_i(x_i, \alpha_i(x_i), s_i) \right]. \tag{1.35}$$

For $n = N$, define $J_N^*(x_N) = g_N(x_N)$. We use mathematical induction to show that the functions J_n^* are equal to the functions J_n generated by the dynamic programming algorithm.

By definition, we have $J_N^* = g_N = J_N$. Assume that for some n and all x_{n+1}, we have $J_{n+1}^*(x_{n+1}) = J_{n+1}(x_{n+1})$. For all x_n, since $\pi_n = \{\alpha_n, \pi_{n+1}\}$, we can express

$$J_n^*(x_n) = \max_{\{\alpha_n, \pi_{n+1}\}} \mathop{\mathbb{E}}_{s_n, \ldots, s_{N-1}} \left[g_n(x_n, \alpha_n(x_n), s_n) + g_N(x_N) + \sum_{i=n+1}^{N-1} g_i(x_i, \alpha_i(x_i), s_i) \right]. \tag{1.36}$$

The principle of optimality, which states that the tail portion (i.e. π_{n+1}) of an optimal policy is optimal for the tail problem (i.e. the $(N-n-1)$-stage problem), implies that

$$\begin{aligned} J_n^*(x_n) &= \max_{\alpha_n} \mathbb{E}_{s_n}\Bigg[g_n(x_n, \alpha_n(x_n), s_n) \\ &\quad + \max_{\pi_{n+1}} \mathbb{E}_{s_{n+1},\ldots,s_{N-1}} \left\{ g_N(x_N) + \sum_{i=n+1}^{N-1} g_i(x_i, \alpha_i(x_i), s_i) \right\} \Bigg]. \end{aligned} \tag{1.37}$$

From the definition of $J_{n+1}^*(x_{n+1})$, we have

$$J_n^*(x_n) = \max_{\alpha_n} \mathbb{E}_{s_n}\Big[g_n(x_n, \alpha_n(x_n), s_n) + J_{n+1}^*(x_{n+1}) \Big]. \tag{1.38}$$

Using the induction hypothesis, we conclude that

$$\begin{aligned} J_n^*(x_n) &= \max_{\alpha_n} \mathbb{E}_{s_n}\Big[g_n(x_n, \alpha_n(x_n), s_n) + J_{n+1}(x_{n+1}) \Big] \\ &= \max_{a_n} \mathbb{E}_{s_n}\Big[g_n(x_n, a_n, s_n) + J_{n+1}\left(f_n(x_n, a_n, s_n) \right) \Big] \\ &= J_n(x_n), \end{aligned} \tag{1.39}$$

which completes the induction. Setting $n = 0$ gives us the desired result. □

The dynamic programming algorithm thus provides one with a systematic approach to solve sequential optimisation problems under uncertainty. In special cases, this algorithm may provide exact analytical solutions for the optimal value function and optimal policy. However, in most practical cases, exact solutions are often not available and one has to rely on numerical computations of the dynamic programming algorithm. In the next section, we present an example that illustrates the application of utility functions and dynamic programming.

1.5 Portfolio Optimisation without Transaction Costs

Consider the multi-period portfolio selection model in which an investor rebalances the portfolio at successive time periods with the objective of maximising expected utility of terminal wealth. This model was studied by Mossin (1968) for a portfolio with a single risk-free and risky asset and extended in Bertsekas (2005) to include multiple risky assets. They assumed that the rebalancing of the portfolio at each time period did not incur transaction costs. We review this portfolio selection model and consider investors with the exponential and power utility functions. In each case, we start with an analysis of the one-period model before extending the results to the multi-period model via dynamic programming.

1.5.1 One-Period Model with Exponential Utility Function

Let W_0 be the initial wealth of the investor and assume that the investor's wealth is allocated among one risk-free asset and M risky assets. Let a^i be the dollar value invested in the ith risky asset. The dollar value invested in the risk-free asset is thus $W_0 - \sum_{i=1}^{M} a^i$. Suppose that s^i denotes one plus the one-period return of the ith risky asset and r denotes one plus the one-period return of the risk-free asset. Then, the wealth at the end of the period is given by

$$W_1 = rW_0 + \sum_{i=1}^{M} \left(s^i - r\right) a^i. \tag{1.40}$$

The investor's objective is to maximise the expected utility of terminal (end-of-period) wealth

$$\mathbb{E}\left[U(W_1)\right] \tag{1.41}$$

over the investments in the risky assets a^i, where the expectation $\mathbb{E}$ is taken with respect to the random variables s^i. Assume that the investor has an exponential utility function of the form

$$U(W) = -e^{-\kappa W}, \tag{1.42}$$

where $\kappa > 0$ is the coefficient of absolute risk aversion.

In this problem, the aim is to obtain the optimal investment in each risky asset so that the investor's expected utility of terminal wealth is maximised. Let $J(W_0)$ be the optimal value function defined as

$$J(W_0) = \max_{a^i} \mathbb{E}\left[U(W_1)\right]. \tag{1.43}$$

Substituting in Equations (1.40) and (1.42),

$$J(W_0) = e^{-\kappa r W_0} \max_{a^i} \mathbb{E}\left[-e^{-\kappa \sum_{i=1}^{M}\left(s^i - r\right)a^i}\right]. \tag{1.44}$$

Here, the term in W_0 is taken out of the expectation operator $\mathbb{E}$ since it is given at the initial time. The problem may now be reduced to one of maximising

$$V = \mathbb{E}\left[-e^{-\kappa \sum_{i=1}^{M}\left(s^i - r\right)a^i}\right] \tag{1.45}$$

with respect to a^i. The optimality conditions $\dfrac{\partial V}{\partial a^i} = 0$ for $i = 1, \ldots, M$ give us

$$\mathbb{E}\left[\left(s^i - r\right) e^{-\kappa \sum_{j=1}^{M}\left(s^j - r\right)a^j}\right] = 0, \tag{1.46}$$

a system of M equations in the M unknowns $a^1, \ldots, a^M$. In general, one will be required to solve this system of equations numerically. Suppose that an optimal investment strategy exists. Since the equations do not depend on the initial wealth W_0, the optimal investment in the ith risky asset is independent of W_0 and assumed to be of the form $a^i = a^{i*}$. Therefore, the optimal value function is given by

$$J(W_0) = -e^{-\kappa r W_0} \mathbb{E}\left[e^{-\kappa \sum_{i=1}^{M}\left(s^i - r\right)a^{i*}}\right]. \tag{1.47}$$

Having determined the form of the optimal strategy and optimal value function in the one-period model, we extend the analysis to the multi-period case.

1.5.2 Multi-Period Model with Exponential Utility Function

Consider a model with N periods and M risky assets, where the investor rebalances the portfolio at the start of each time period. Let W_n denote the wealth of the portfolio and a_n^i denote the dollar value invested in the ith risky asset at time period n. Let s_n^i denote one plus the return of the ith risky asset and r_n denote one plus the return of the risk-free asset from time period n to $n+1$. Therefore, the investor's wealth at time period $n+1$ is given by

$$W_{n+1} = r_n W_n + \sum_{i=1}^{M} \left(s_n^i - r_n\right) a_n^i \tag{1.48}$$

for $n = 0, \ldots, N-1$. The objective is to maximise the expected utility of terminal wealth $\mathbb{E}\left[U(W_N)\right]$, where the utility function is given in Equation (1.42). Thus, the optimal value function for the multi-period problem is defined as

$$J(W_0) = \max \mathbb{E}\left[U(W_N)\right], \tag{1.49}$$

where the maximisation is over $a_0^i, \ldots, a_{N-1}^i$ and $\mathbb{E}$ is taken with respect to the random variables $s_0^i, \ldots, s_{N-1}^i$. The dynamic programming algorithm for this problem, which proceeds backwards in time from period $N-1$ to period 0, is given by

$$J_N(W_N) = U(W_N) \tag{1.50}$$

and

$$J_n(W_n) = \max\ \mathbb{E}_n\left[J_{n+1}\left(r_n W_n + \sum_{i=1}^{M} \left(s_n^i - r_n\right) a_n^i\right)\right] \tag{1.51}$$

for $n = 0, \ldots, N-1$. Here, the maximisation is over the investments in the risky assets a_n^i and $\mathbb{E}_n$ is the conditional expectation operator taken with respect to the random variables s_n^i given the information at time period n.

At time period $N-1$, using the results from the one-period model, the optimal investment strategy for the ith risky asset is of the form

$$a_{N-1}^i = a_{N-1}^{i\,*} \tag{1.52}$$

and the optimal value function is

$$J_{N-1}(W_{N-1}) = -e^{-\kappa r_{N-1} W_{N-1}} \mathbb{E}_{N-1}\left[e^{-\kappa \sum_{i=1}^{M}\left(s_{N-1}^i - r_{N-1}\right) a_{N-1}^{i\,*}}\right]. \tag{1.53}$$

Observe that, up to a constant, $J_{N-1}(W_{N-1})$ has a functional form that is similar to $J_N(W_N) = -e^{-\kappa W_N}$. Therefore, one can apply the results from the one-period model (with the appropriate modification) to the analysis at time period $N-2$.

Thus, at time period $N-2$, the optimal investment strategy for the ith risky asset is given by

$$a^i_{N-2} = \frac{a^{i\,*}_{N-2}}{r_{N-1}} \tag{1.54}$$

and the optimal value function is

$$\begin{aligned} J_{N-2}(W_{N-2}) &= -e^{-\kappa r_{N-1} r_{N-2} W_{N-2}} \mathbb{E}_{N-1}\left[e^{-\kappa\sum_{i=1}^{M}\left(s^i_{N-1}-r_{N-1}\right)a^{i\,*}_{N-1}}\right] \\ &\quad \times \mathbb{E}_{N-2}\left[e^{-\kappa\sum_{i=1}^{M}\left(s^i_{N-2}-r_{N-2}\right)a^{i\,*}_{N-2}}\right]. \end{aligned} \tag{1.55}$$

Proceeding in a similar way, we deduce that at time period n ($n = 0, \ldots, N-2$), the optimal investment in the ith risky asset is of the form

$$a^i_n = \frac{a^{i\,*}_n}{r_{N-1}\cdots r_{n+1}}, \tag{1.56}$$

where $a^{i\,*}_n$ satisfies the system of M equations given by

$$\mathbb{E}_n\left[\left(s^i_n - r_n\right) e^{-\kappa\sum_{j=1}^{M}\left(s^j_n - r_n\right)a^{j\,*}_n}\right] = 0 \tag{1.57}$$

for $i = 1, \ldots, M$. Furthermore, the optimal value function is

$$\begin{aligned} J_n(W_n) &= -e^{-\kappa r_{N-1} r_{N-2}\cdots r_n W_n} \mathbb{E}_{N-1}\left[e^{-\kappa\sum_{i=1}^{M}\left(s^i_{N-1}-r_{N-1}\right)a^{i\,*}_{N-1}}\right] \\ &\quad \times \mathbb{E}_{N-2}\left[e^{-\kappa\sum_{i=1}^{M}\left(s^i_{N-2}-r_{N-2}\right)a^{i\,*}_{N-2}}\right]\cdots\mathbb{E}_n\left[e^{-\kappa\sum_{i=1}^{M}\left(s^i_n-r_n\right)a^{i\,*}_n}\right]. \end{aligned} \tag{1.58}$$

If we make a comparison with Equations (1.51), (1.56) and (1.58), it can be seen that the optimal investment strategy at time period n is equivalent to one that an investor will use in a single-period model to maximise $\mathbb{E}\left[U(r_{N-1}\cdots r_{n+1}W_{n+1})\right]$ over a^i_n, subject to $W_{n+1} = r_n W_n + \sum_{i=1}^{M}\left(s^i_n - r_n\right)a^i_n$. In other words, using this strategy, the investor will maximise the expected utility of wealth that arises from investing a^i_n in the risky assets at time period n and reinvesting the resulting wealth W_{n+1} entirely in the risk-free asset at the subsequent time periods. This is described as a partially myopic strategy and is one that only requires the investor to have a modest amount of foresight.

We now proceed to consider the case of an investor with the power utility function, starting with a one-period analysis.

1.5.3 One-Period Model with Power Utility Function

Consider the one-period model as described in Section 1.5.1. In this model, the investor aims to maximise $\mathbb{E}\left[U(W_1)\right]$ over the investments in the risky assets a^i, subject to $W_1 = rW_0 + \sum_{i=1}^{M}\left(s^i - r\right)a^i$.

Assume that the investor has a power utility function of the form

$$U(W) = \frac{1}{\gamma}W^{\gamma}, \tag{1.59}$$

where $\gamma < 1, \gamma \neq 0$. The optimal value function $J(W_0)$ is defined as

$$J(W_0) = \max_{a^i}\mathbb{E}\left[U\left(rW_0 + \sum_{i=1}^{M}\left(s^i - r\right)a^i\right)\right]. \tag{1.60}$$

Substituting in Equation (1.59),

$$J(W_0) = \max_{a^i}\mathbb{E}\left[\frac{1}{\gamma}\left\{rW_0 + \sum_{i=1}^{M}\left(s^i - r\right)a^i\right\}^{\gamma}\right]. \tag{1.61}$$

It is convenient to re-parametrise the problem by expressing the variable a^i as a proportion of the wealth W_0. We introduce the variable $A^i = a^i/W_0$, which represents the proportion of wealth invested in the ith risky asset. In terms of A^i, the optimal value function becomes

$$J(W_0) = W_0^{\gamma}\max_{A^i}\mathbb{E}\left[\frac{1}{\gamma}\left\{r + \sum_{i=1}^{M}\left(s^i - r\right)A^i\right\}^{\gamma}\right], \tag{1.62}$$

which is now a maximisation over the proportional investments in the risky assets A^i. Observe that with this new parametrisation, the term in W_0 is taken out of the expectation operator $\mathbb{E}$ since it is known at the initial time. The problem is now reduced to one of maximising

$$V = \mathbb{E}\left[\frac{1}{\gamma}\left\{r + \sum_{i=1}^{M}\left(s^i - r\right)A^i\right\}^{\gamma}\right] \tag{1.63}$$

with respect to A^i. The optimality conditions $\dfrac{\partial V}{\partial A^i} = 0$ for $i = 1, \ldots, M$ give us a system of M equations

$$\mathbb{E}\left[\left(s^i - r\right)\left\{r + \sum_{j=1}^{M}\left(s^j - r\right)A^j\right\}^{\gamma-1}\right] = 0 \tag{1.64}$$

with M unknowns $A^1, \ldots, A^M$. Assuming that an optimal investment strategy exists, the optimal proportion of wealth invested in the ith risky asset is of the form $A^i = A^{i*}$, which is independent of the initial wealth W_0. Therefore, the optimal value function is

$$J(W_0) = \frac{1}{\gamma}W_0^{\gamma}\mathbb{E}\left[\left\{r + \sum_{i=1}^{M}\left(s^i - r\right)A^{i*}\right\}^{\gamma}\right]. \tag{1.65}$$

Having obtained the form of the optimal strategy and optimal value function in the one-period model, we now consider the multi-period case.

1.5.4 Multi-Period Model with Power Utility Function

Recall the multi-period model as described in Section 1.5.2. The investor's objective is to maximise $\mathbb{E}\left[U(W_N)\right]$ by rebalancing the portfolio at the start of each time period, subject to $W_{n+1} = r_n W_n + \sum_{i=1}^{M}\left(s_n^i - r_n\right)a_n^i$ for $n = 0, \ldots, N-1$. Assume that the investor has a power utility function given by Equation (1.59). Similar to the one-period analysis, we re-parametrise the problem by expressing the variable a_n^i as a proportion of the wealth W_n. We introduce the variable $A_n^i = a_n^i / W_n$, which represents the proportion of wealth invested in the ith risky asset at time period n. In terms of A_n^i, we now have $W_{n+1} = W_n\left\{r_n + \sum_{i=1}^{M}\left(s_n^i - r_n\right)A_n^i\right\}$. In this case, the optimal value function for the multi-period problem is defined as

$$J(W_0) = \max \mathbb{E}\left[U(W_N)\right], \tag{1.66}$$

where the maximisation is over $A_0^i, \ldots, A_{N-1}^i$. The dynamic programming algorithm for this problem, which proceeds backwards in time, is given by

$$J_N(W_N) = U(W_N) \tag{1.67}$$

and

$$J_n(W_n) = \max\ \mathbb{E}_n \left[J_{n+1} \left(W_n \left\{ r_n + \sum_{i=1}^{M} \left(s_n^i - r_n \right) A_n^i \right\} \right) \right] \tag{1.68}$$

for $n = 0, \ldots, N-1$. Here, the maximisation is over the proportions of wealth invested in the risky assets A_n^i at time period n.

At time period $N-1$, using the results from the one-period model, the optimal investment strategy for the ith risky asset is given by

$$A_{N-1}^i = A_{N-1}^{i*} \tag{1.69}$$

and the optimal value function is

$$J_{N-1}(W_{N-1}) = \frac{1}{\gamma} W_{N-1}^{\gamma} \mathbb{E}_{N-1} \left[\left\{ r_{N-1} + \sum_{i=1}^{M} \left(s_{N-1}^i - r_{N-1} \right) A_{N-1}^{i*} \right\}^{\gamma} \right]. \tag{1.70}$$

Observe that $J_{N-1}(W_{N-1})$ has the same functional form as $J_N(W_N) = \frac{1}{\gamma} W_N^{\gamma}$ (up to a constant).

Therefore, at time period $N-2$, applying the results from the one-period model, the optimal investment strategy for the ith risky asset is given by

$$A_{N-2}^i = A_{N-2}^{i*} \tag{1.71}$$

and the optimal value function is

$$\begin{aligned} J_{N-2}(W_{N-2}) \quad &= \quad \frac{1}{\gamma} W_{N-2}^{\gamma} \mathbb{E}_{N-1} \left[\left\{ r_{N-1} + \sum_{i=1}^{M} \left(s_{N-1}^i - r_{N-1} \right) A_{N-1}^{i*} \right\}^{\gamma} \right] \\ &\times \mathbb{E}_{N-2} \left[\left\{ r_{N-2} + \sum_{i=1}^{M} \left(s_{N-2}^i - r_{N-2} \right) A_{N-2}^{i*} \right\}^{\gamma} \right]. \end{aligned} \tag{1.72}$$

In general, we deduce that at time period n ($n = 0, \ldots, N-1$), the optimal proportion of wealth invested in the ith risky asset is given by

$$A_n^i = A_n^{i\,*}, \tag{1.73}$$

where $A_n^{i\,*}$ satisfies the system of M equations

$$\mathbb{E}_n \left[\left(s_n^i - r_n \right) \left\{ r_n + \sum_{j=1}^{M} \left(s_n^j - r_n \right) A_n^{j\,*} \right\}^{\gamma - 1} \right] = 0 \tag{1.74}$$

for $i = 1, \ldots, M$. In addition, the optimal value function is

$$\begin{aligned} J_n(W_n) &= \frac{1}{\gamma} W_n^{\gamma} \mathbb{E}_{N-1} \left[\left\{ r_{N-1} + \sum_{i=1}^{M} \left(s_{N-1}^i - r_{N-1} \right) A_{N-1}^{i\,*} \right\}^{\gamma} \right] \\ &\quad \times \mathbb{E}_{N-2} \left[\left\{ r_{N-2} + \sum_{i=1}^{M} \left(s_{N-2}^i - r_{N-2} \right) A_{N-2}^{i\,*} \right\}^{\gamma} \right] \\ &\quad \times \cdots \times \mathbb{E}_n \left[\left\{ r_n + \sum_{i=1}^{M} \left(s_n^i - r_n \right) A_n^{i\,*} \right\}^{\gamma} \right]. \end{aligned} \tag{1.75}$$

It follows from the optimal strategy (Equation (1.73)) that, at each time period n, the investor behaves as if it was a one-period investment characterised by the returns r_n and s_n^i with the objective of maximising $\mathbb{E}\left[U(W_{n+1})\right]$. In other words, the investor's optimal investment strategy is obtained as a sequence of single-period decisions, with each period being treated as if it was the last one. This is known as a myopic strategy. With a myopic strategy, the investor makes his current investment decision without considering the reinvestment opportunities in the future.

In this section, we have analysed the portfolio selection problem in discrete time without transaction costs. We considered investors with the exponential utility function (i.e. constant absolute risk aversion) and the power utility function (i.e. constant relative risk aversion). In the multi-period model where the portfolio's wealth is reinvested at the start of each time period, it is observed that the investor adopts an optimal strategy similar to the one-period case. The investor with an exponential utility function follows a partially myopic strategy, which assumes that he will reinvest his subsequent wealth entirely in the risk-free asset. The investor with a power utility function adopts a myopic strategy, which ignores the fact that he will have the opportunity to reinvest his wealth in the future. Thus, in the absence of transaction costs, the solution to the portfolio selection problem is of a relatively simple form.

In the subsequent chapters, we will investigate the effects of transaction costs on the portfolio selection problem and the associated option pricing problem. We now present an overview of the thesis and the main contributions of our research, which are as follows:

1.6 Overview

In Chapter 2, we analyse the multi-period portfolio selection problem with proportional transaction costs. An investor with the exponential utility function aims to maximise his utility of terminal wealth by optimally rebalancing his portfolio among a single risk-free and risky asset at the start of each time period. The optimal strategy involves trading to reach the boundaries of a no-transaction region if the investor's risky asset holdings fall outside this region. Dynamic programming is used to compute the optimal value function and optimal boundaries of the no-transaction region, which can be computationally intensive. Previous work by Atkinson and Storey (2010) applied perturbation analysis to approximate the optimal boundaries in the limit of small transaction costs. However, they only obtained approximations for two time steps and it is not obvious whether their method can be extended to an arbitrary number of time steps.

We devise a method that allows one to systematically obtain approximations of the optimal boundaries as well as the optimal value functions at all time steps. This method consists of two stages, the first of which assumes that the investor trades to reach the Merton point at each time step when transaction costs are small. Recall that the Merton point is the optimal investment in the risky asset when there are no transaction costs. This is clearly a suboptimal strategy as the investor has ignored the presence of the no-transaction region. Nonetheless, an approximation of the suboptimal value function is derived at each time step. The second stage assumes that the investor behaves optimally by transacting to the boundaries of the no-transaction region. A sequence of corrections is then applied to the suboptimal value functions to give us the desired approximations to the optimal value functions. The approximate optimal boundaries are then derived from the condition that the first derivative of the optimal value function is continuous across the boundaries. The details of this method can be found in Chapter 2.

A feature of using the exponential utility function is that it resulted in optimal boundaries that are independent of the investor's wealth, which is often not the case in practice. A more realistic description of the investor's optimal strategy is provided by using the power utility function. In Chapter 3, we investigate the multi-period portfolio selection model with proportional transaction costs for an investor with the power utility function. We adopt a similar two-stage perturbation method to obtain approximations of the optimal value function and optimal boundaries at each time step in the rebalancing of the portfolio. However, it is more challenging to apply the perturbation analysis in this situation as the proportion of risky asset inherited at each time step depends on variations in both the return of the risky asset as well as the investor's wealth.

In Chapter 4, we present a discrete time model of option pricing that is based on the utility maximisation approach. We investigate both utility indifference pricing as well as marginal utility indifference pricing in the case without transaction costs. One advantage of this discrete time model is that the underlying risky asset is assumed to follow a general price process. Investors with the exponential and power utility functions are considered. In the special case where the risky asset follows a binomial price process, we demonstrate that both the utility indifference price and marginal utility indifference price of the option reduce to the perfect replication price from the binomial model. The binomial model is the only discrete time model where perfect replication is possible. In the case where the price of the risky asset follows a trinomial tree, we illustrate how the price of the option may be determined via an approximate replication approach. This approach values the option by constructing an approximately replicating portfolio that minimises the variance of the replication error. Finally, a comparison is made between the utility maximisation approach and the approximate replication approach.

In Chapter 5, we extend the discrete time option pricing model (based on the utility maximisation approach) by incorporating proportional transaction costs. We consider the case of an investor with the exponential utility function. With the application of dynamic programming and using some of the results from Chapter 2, we determine the utility indifference price and marginal utility indifference price of a European option. In the presence of transaction costs, the utility indifference price and the marginal utility indifference price of the option depend on the price of the underlying risky asset as well as on the investor's holdings in the risky asset. We also identify the regions where the option prices do not vary with the investor's holdings in the risky asset. Numerical results are presented for the case of a European call option, where the underlying risky asset is assumed to follow a binomial price process. We examine how utility indifference pricing and marginal utility indifference pricing allow us to obtain the bid and ask price of the option. Moreover, the utility indifference pricing approach also provides one with a natural definition of the hedging strategy.

Chapter 2

Portfolio Optimisation with Transaction Costs and Exponential Utility Function

In Chapter 1, we studied the multi-period portfolio selection problem in discrete time without transaction costs. The investor would rebalance the portfolio at successive time periods with the objective of maximising expected utility of terminal wealth. In this chapter, we incorporate proportional transaction costs in the multi-period portfolio selection problem and consider the case of a single risk-free and risky asset. A general class of underlying probability distributions is assumed for the returns of the risky asset. Assuming that the investor has an exponential utility function, we determine the optimal value function and optimal boundaries of the no-transaction region via the application of dynamic programming. In the limit of small transaction costs, a two-stage perturbation method is developed and used to derive approximations of the optimal value function and optimal boundaries at each time step of the investment process. This chapter is based on Atkinson and Quek (2012), which is forthcoming in the journal of Applied Mathematical Finance.

2.1 Market Model

Consider an investor holding a portfolio which is divided between an investment in a risk-free asset (i.e. bond) and an investment in a risky asset (i.e. stock). A financial market is considered where the prices of the assets evolve in a discrete time model with N periods.

2.1.1 Evolution of Wealth and Risky Asset

At time period n, let W_n denote the wealth of the investor and let a_n denote the dollar value of the risky asset inherited from the previous period. The investor rebalances the portfolio at time period n by buying l_n or selling m_n dollars of the risky asset. The value of the position in the risky asset, $a_n + l_n - m_n$, is assumed to grow at a random rate between time period n and $n+1$. Let s_n denote one plus the return of the risky asset between time period n and $n + 1$. Therefore, the value of the risky asset at time period $n + 1$, inherited from period n, is given by

$$a_{n+1} = s_n(a_n + l_n - m_n) \tag{2.1}$$

for $n = 0, \ldots, N - 1$.

Furthermore, let λ_n and μ_n denote the proportional costs of buying and selling the risky asset at time period n respectively. The costs of transaction reduce the wealth invested in the risk-free asset, which is equal to $W_n - (a_n + l_n - m_n) - \lambda_n l_n - \mu_n m_n$. Assume that the investment in the risk-free asset grows at a sure rate between time period n and $n + 1$. Let r_n denote one plus the return of the risk-free asset between time period n and $n + 1$. Thus, the evolution of the investor's wealth (also known as the budget equation) at time period $n + 1$ is given by

$$W_{n+1} = r_n W_n + (s_n - r_n)(a_n + l_n - m_n) - r_n\lambda_n l_n - r_n\mu_n m_n \tag{2.2}$$

for $n = 0, \ldots, N - 1$. It is convenient to write

$$W_{n+1} = r_n W_n + F_n, \tag{2.3}$$

where

$$F_n = (s_n - r_n)(a_n + l_n - m_n) - r_n\lambda_n l_n - r_n\mu_n m_n. \tag{2.4}$$

At all time steps in the rebalancing of the portfolio, observe that it will not be optimal for the investor to simultaneously buy and sell the risky asset, due to the higher costs incurred as compared to only buying or selling the asset. Therefore, it is assumed that simultaneously buying and selling of the risky asset is not allowed. The investor is then left with three possible choices, namely to buy, to sell or not to transact the risky asset. The investor's decision will affect the wealth and risky asset inherited at the next time step.

2.1.2 Risk Preference and Objective

Assume that the investor has a risk preference of the constant absolute risk aversion class. Specifically, suppose that the investor has an exponential utility function of wealth given by

$$U(W) = -e^{-\kappa W}, \tag{2.5}$$

where $\kappa > 0$ is the coefficient of absolute risk aversion.

Given an initial wealth W_0 and initial holding of risky asset a_0, the investor's objective is to maximise the expected utility of terminal wealth W_N by choosing the optimal investment strategy at each stage of the investment process. In general, the investor's optimal value function at time period n ($n = 0, \ldots, N-1$), given a wealth of W_n and a_n dollars of risky asset, is defined to be

$$J_n(W_n, a_n) = \max \mathbb{E}[U(W_N)]. \tag{2.6}$$

The maximisation is over the investments $(l_n, m_n), \ldots, (l_{N-1}, m_{N-1})$ in the risky asset, and $\mathbb{E}$ is the conditional expectation (given W_n and a_n) with respect to $s_n, \ldots, s_{N-1}$. At terminal time, the value function is given by

$$J_N(W_N, a_N) = U(W_N), \tag{2.7}$$

which constitutes the terminal condition of the problem. The investor aims to determine the optimal investment strategy $(l_0, m_0), \ldots, (l_{N-1}, m_{N-1})$ which maximises $J_0(W_0, a_0)$. In order to simplify this multi-period decision problem, it is re-written into the dynamic programming form.

2.2 Dynamic Programming

The dynamic programming algorithm for the problem, which proceeds recursively backwards in time from period $N-1$ to period 0, is given by

$$J_N(W_N, a_N) = U(W_N) \tag{2.8}$$

and

$$J_{N-k}(W_{N-k}, a_{N-k}) = \max \mathbb{E}_{N-k}[J_{N-k+1}(W_{N-k+1}, a_{N-k+1})] \tag{2.9}$$

for $k = 1, \dots, N$. Here, the maximisation is taken over the investment (l_{N-k}, m_{N-k}) in the risky asset at time period $N-k$, while $\mathbb{E}_{N-k}$ represents the conditional expectation (given W_{N-k} and a_{N-k}) with respect to s_{N-k} between time period $N-k$ and $N-k+1$.

In principle, the dynamic programming algorithm allows the investor to systematically determine the optimal strategy and value function starting from period $N-1$ and, proceeding backwards in time, to obtain the optimal solution recursively at period $N-2$ all the way to the initial time. As an illustration, consider one step before the terminal time.

Time Period $N-1$

The investor determines the optimal strategy at time period $N-1$ by considering the value function

$$J_{N-1}(W_{N-1}, a_{N-1}) = \max \mathbb{E}_{N-1}[J_N(W_N, a_N)] = \max \mathbb{E}_{N-1}\left[-e^{-\kappa W_N}\right]. \tag{2.10}$$

Using Equation (2.3),

$$\begin{aligned} J_{N-1}(W_{N-1}, a_{N-1}) &= \max \mathbb{E}_{N-1}\left[-e^{-\kappa\{r_{N-1}W_{N-1}+F_{N-1}\}}\right] \\ &= \max e^{-\kappa r_{N-1}W_{N-1}} \mathbb{E}_{N-1}\left[-e^{-\kappa F_{N-1}}\right]. \end{aligned} \tag{2.11}$$

The term in W_{N-1} is taken out of the expectation $\mathbb{E}_{N-1}$ since it is conditional upon knowledge of W_{N-1}. Furthermore, W_{N-1} does not depend on the investment decision at time period $N-1$, unlike F_{N-1} that depends on the investor's decision at period $N-1$. Therefore, express

$$J_{N-1}(W_{N-1}, a_{N-1}) = \max e^{-\kappa r_{N-1} W_{N-1}} V_{N-1}(a_{N-1}), \tag{2.12}$$

where

$$V_{N-1}(a_{N-1}) = -\mathbb{E}_{N-1}\left[e^{-\kappa F_{N-1}}\right]. \tag{2.13}$$

The problem can now be reduced to one of maximising $V_{N-1}(a_{N-1})$ with respect to the investment strategy (l_{N-1}, m_{N-1}).

The investor has a choice of three different strategies, each of which affects the definition of F_{N-1}.

1. If the investor buys $l_{N-1} > 0$ of the risky asset (i.e. $m_{N-1} = 0$),

$$F^{(B)}_{N-1} = (s_{N-1} - r_{N-1})(a_{N-1} + l_{N-1}) - r_{N-1}\lambda_{N-1} l_{N-1}. \tag{2.14}$$

2. If the investor sells $m_{N-1} > 0$ of the risky asset (i.e. $l_{N-1} = 0$),

$$F^{(S)}_{N-1} = (s_{N-1} - r_{N-1})(a_{N-1} - m_{N-1}) - r_{N-1}\mu_{N-1} m_{N-1}. \tag{2.15}$$

3. If the investor does not transact in the risky asset (i.e. $l_{N-1} = 0 = m_{N-1}$),

$$F^{(N)}_{N-1} = (s_{N-1} - r_{N-1})a_{N-1}. \tag{2.16}$$

We have used the superscripts "B", "S" and "N" to denote the investor buying, selling and not transacting in the risky asset respectively.

The problem is equivalent to finding a region where it is optimal for the investor not to transact in the risky asset. The no-transaction region is denoted by $a^-_{N-1} \leq a_{N-1} \leq a^+_{N-1}$, which represents the region between the optimal buy boundary a^-_{N-1} and the optimal sell boundary a^+_{N-1}. The buy region, where buying is optimal, is the region to the left of a^-_{N-1}. The sell region, where selling is optimal, is the region to the right of a^+_{N-1}. In order to find a^-_{N-1} and a^+_{N-1}, the value function V_{N-1} is maximised with respect to the variables l_{N-1}

and m_{N-1} respectively. Therefore, a^-_{N-1} and a^+_{N-1} are the solutions to the equations

$$\frac{\partial V_{N-1}}{\partial l_{N-1}} = \mathbb{E}_{N-1}\left[\kappa\{s_{N-1} - (1+\lambda_{N-1})r_{N-1}\}e^{-\kappa F_{N-1}}\right] = 0 \tag{2.17}$$

and

$$\frac{\partial V_{N-1}}{\partial m_{N-1}} = \mathbb{E}_{N-1}\left[\kappa\{(1-\mu_{N-1})r_{N-1} - s_{N-1}\}e^{-\kappa F_{N-1}}\right] = 0, \tag{2.18}$$

respectively. It is noted that in Equations (2.17) and (2.18), $F_{N-1} = (s_{N-1} - r_{N-1})a_{N-1}$ as $a_{N-1} = a^-_{N-1}$, $l_{N-1} = 0$ on the buy boundary and $a_{N-1} = a^+_{N-1}$, $m_{N-1} = 0$ on the sell boundary.

Having obtained the optimal buy and sell boundaries, and depending on the region which the value of the risky asset lies, the investor can optimise the strategies by choosing l_{N-1} and m_{N-1} in the following way.

1. In the buy region $a_{N-1} < a^-_{N-1}$, the investor buys $l_{N-1} = a^-_{N-1} - a_{N-1}$ of the risky asset to reach the optimal buy boundary, which results in

$$F^{(B)}_{N-1} = (s_{N-1} - r_{N-1})a^-_{N-1} - r_{N-1}\lambda_{N-1}(a^-_{N-1} - a_{N-1}). \tag{2.19}$$

2. In the sell region $a_{N-1} > a^+_{N-1}$, the investor sells $m_{N-1} = a_{N-1} - a^+_{N-1}$ of the risky asset to reach the optimal sell boundary, which leads to

$$F^{(S)}_{N-1} = (s_{N-1} - r_{N-1})a^+_{N-1} - r_{N-1}\mu_{N-1}(a_{N-1} - a^+_{N-1}). \tag{2.20}$$

It is observed that V_{N-1} is maximised by these strategies as

$$\frac{\partial^2 V_{N-1}}{\partial l^2_{N-1}} = -\mathbb{E}_{N-1}\left[\kappa^2\{s_{N-1} - (1+\lambda_{N-1})r_{N-1}\}^2 e^{-\kappa F^{(B)}_{N-1}}\right] < 0 \tag{2.21}$$

and

$$\frac{\partial^2 V_{N-1}}{\partial m^2_{N-1}} = -\mathbb{E}_{N-1}\left[\kappa^2\{(1-\mu_{N-1})r_{N-1} - s_{N-1}\}^2 e^{-\kappa F^{(S)}_{N-1}}\right] < 0 \tag{2.22}$$

in the buy and sell regions respectively.

These strategies show that F_{N-1} as given by Equations (2.16), (2.19) and (2.20) is continuous across the buy and sell boundaries at $a_{N-1} = a_{N-1}^-$ and $a_{N-1} = a_{N-1}^+$ respectively. Hence, the optimal value function V_{N-1} in the buy, sell and no-transaction regions, as denoted by

$$V_{N-1}^{(B)} = -\mathbb{E}_{N-1}\left[e^{-\kappa F_{N-1}^{(B)}}\right], \tag{2.23}$$

$$V_{N-1}^{(S)} = -\mathbb{E}_{N-1}\left[e^{-\kappa F_{N-1}^{(S)}}\right] \quad \text{and} \tag{2.24}$$

$$V_{N-1}^{(N)} = -\mathbb{E}_{N-1}\left[e^{-\kappa F_{N-1}^{(N)}}\right], \tag{2.25}$$

respectively, is also continuous across the boundaries. Furthermore, the derivative of V_{N-1} with respect to a_{N-1} in the buy, sell and no-transaction regions is given by

$$\frac{\partial V_{N-1}^{(B)}}{\partial a_{N-1}} = \mathbb{E}_{N-1}\left[\kappa r_{N-1}\lambda_{N-1}e^{-\kappa F_{N-1}^{(B)}}\right], \tag{2.26}$$

$$\frac{\partial V_{N-1}^{(S)}}{\partial a_{N-1}} = -\mathbb{E}_{N-1}\left[\kappa r_{N-1}\mu_{N-1}e^{-\kappa F_{N-1}^{(S)}}\right] \quad \text{and} \tag{2.27}$$

$$\frac{\partial V_{N-1}^{(N)}}{\partial a_{N-1}} = \mathbb{E}_{N-1}\left[\kappa(s_{N-1} - r_{N-1})e^{-\kappa F_{N-1}^{(N)}}\right], \tag{2.28}$$

respectively. A consequence of Equations (2.17) and (2.18) is that the derivative of V_{N-1} with respect to a_{N-1} is continuous across the buy and sell boundaries, that is,

$$\frac{\partial V_{N-1}^{(B)}}{\partial a_{N-1}} = \frac{\partial V_{N-1}^{(N)}}{\partial a_{N-1}} \tag{2.29}$$

at $a_{N-1} = a_{N-1}^-$, and

$$\frac{\partial V_{N-1}^{(S)}}{\partial a_{N-1}} = \frac{\partial V_{N-1}^{(N)}}{\partial a_{N-1}} \tag{2.30}$$

at $a_{N-1} = a_{N-1}^+$. It is noted that this observation provides one with another approach to solve for the optimal boundaries.

In conclusion, the optimal strategies chosen provide continuity of the value function and its derivative across the buy and sell boundaries. It is noted that the analysis at time period $N-1$ is a special case as the investor does not have the opportunity to rebalance the portfolio at period N when the investment process is terminated.

Time Period $N-k$

Applying the dynamic programming algorithm, the problem is considered after taking a step back to time period $N-2$ and assuming that the steps ahead are optimal. Having obtained the optimal strategies and value function at period $N-2$, the problem is considered by taking another step back in time to period $N-3$. Implementing the dynamic programming algorithm recursively backwards in time, the optimal strategies and value function at period $N-k$ ($k=2,\ldots,N$) are given by the following analysis.

The value function at time period $N-k$ is given by Equation (2.9) as

$$\begin{aligned} J_{N-k}(W_{N-k},a_{N-k}) &= \max \mathbb{E}_{N-k}[J_{N-k+1}(W_{N-k+1},a_{N-k+1})] \\ &= \max \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}W_{N-k+1}}V_{N-k+1}(a_{N-k+1})\right], \end{aligned} \tag{2.31}$$

where $V_{N-k+1}(a_{N-k+1})$ is optimal. Using Equation (2.3), the value function is expressed as

$$\begin{aligned} &J_{N-k}(W_{N-k},a_{N-k}) \\ &= \max \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}\{r_{N-k}W_{N-k}+F_{N-k}\}}V_{N-k+1}(a_{N-k+1})\right] \\ &= \max e^{-\kappa r_{N-1}\cdots r_{N-k}W_{N-k}}\mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}}V_{N-k+1}(a_{N-k+1})\right], \end{aligned} \tag{2.32}$$

where the term in W_{N-k} is taken out of the expectation $\mathbb{E}_{N-k}$ since it is conditional on W_{N-k}. Since W_{N-k} does not depend on the investment decision at time period $N-k$, the problem can be reduced to one of maximising the value function defined as

$$V_{N-k}(a_{N-k}) = \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}}V_{N-k+1}(a_{N-k+1})\right] \tag{2.33}$$

with respect to the variables l_{N-k} and m_{N-k}, as

$$J_{N-k}(W_{N-k},a_{N-k}) = \max e^{-\kappa r_{N-1}\cdots r_{N-k}W_{N-k}}V_{N-k}(a_{N-k}). \tag{2.34}$$

The choice of the investment strategy at time period $N-k$ affects the definition of F_{N-k} and also affects the value of the risky asset a_{N-k+1} inherited at the next period $N-k+1$.

1. If the investor buys $l_{N-k} > 0$ of the risky asset (i.e. $m_{N-k} = 0$),

$$F^{(B)}_{N-k} = (s_{N-k} - r_{N-k})(a_{N-k} + l_{N-k}) - r_{N-k}\lambda_{N-k}l_{N-k} \tag{2.35}$$

and

$$a_{N-k+1} = s_{N-k}(a_{N-k} + l_{N-k}). \tag{2.36}$$

2. If the investor sells $m_{N-k} > 0$ of the risky asset (i.e. $l_{N-k} = 0$),

$$F^{(S)}_{N-k} = (s_{N-k} - r_{N-k})(a_{N-k} - m_{N-k}) - r_{N-k}\mu_{N-k}m_{N-k} \tag{2.37}$$

and

$$a_{N-k+1} = s_{N-k}(a_{N-k} - m_{N-k}). \tag{2.38}$$

3. If the investor does not transact in the risky asset (i.e. $l_{N-k} = 0 = m_{N-k}$),

$$F^{(N)}_{N-k} = (s_{N-k} - r_{N-k})a_{N-k} \tag{2.39}$$

and

$$a_{N-k+1} = s_{N-k}a_{N-k}. \tag{2.40}$$

In order to find the optimal buy boundary a^-_{N-k} and optimal sell boundary a^+_{N-k}, the value function V_{N-k} is maximised with respect to the variables l_{N-k} and m_{N-k} respectively. The derivative of V_{N-k} with respect to l_{N-k} is given by

$$\begin{aligned}
\frac{\partial V_{N-k}}{\partial l_{N-k}} &= \mathbb{E}_{N-k}\left[\frac{\partial e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}}{\partial l_{N-k}}V_{N-k+1} + e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}\frac{\partial V_{N-k+1}}{\partial l_{N-k}}\right] \\
&= \mathbb{E}_{N-k}\Big[-\kappa r_{N-1}\cdots r_{N-k+1}\{s_{N-k} - (1+\lambda_{N-k})r_{N-k}\} \\
&\times\ e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}V_{N-k+1} + s_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}\frac{\partial V_{N-k+1}}{\partial a_{N-k+1}}\Big],
\end{aligned} \tag{2.41}$$

and the derivative of V_{N-k} with respect to m_{N-k} is given by

$$\begin{aligned}\frac{\partial V_{N-k}}{\partial m_{N-k}} &= \mathbb{E}_{N-k}\left[\frac{\partial e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}}{\partial m_{N-k}}V_{N-k+1} + e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}\frac{\partial V_{N-k+1}}{\partial m_{N-k}}\right] \\ &= \mathbb{E}_{N-k}\Bigg[\kappa r_{N-1}\cdots r_{N-k+1}\{s_{N-k}-(1-\mu_{N-k})r_{N-k}\} \\ &\times \; e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}V_{N-k+1} - s_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}\frac{\partial V_{N-k+1}}{\partial a_{N-k+1}}\Bigg].\end{aligned} \tag{2.42}$$

Therefore, a^-_{N-k} and a^+_{N-k} are the solutions to the equations

$$\begin{aligned}&\mathbb{E}_{N-k}\Bigg[-\kappa r_{N-1}\cdots r_{N-k+1}\{s_{N-k}-(1+\lambda_{N-k})r_{N-k}\} \\ &\times e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}}V_{N-k+1} + s_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}}\frac{\partial V_{N-k+1}}{\partial a_{N-k+1}}\Bigg] = 0\end{aligned} \tag{2.43}$$

and

$$\begin{aligned}&\mathbb{E}_{N-k}\Bigg[\kappa r_{N-1}\cdots r_{N-k+1}\{s_{N-k}-(1-\mu_{N-k})r_{N-k}\} \\ &\times e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}}V_{N-k+1} - s_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}}\frac{\partial V_{N-k+1}}{\partial a_{N-k+1}}\Bigg] = 0,\end{aligned} \tag{2.44}$$

respectively. In Equations (2.43) and (2.44), it is noted that $F_{N-k} = (s_{N-k} - r_{N-k})a_{N-k}$ as $a_{N-k} = a^-_{N-k}$, $l_{N-k} = 0$ on the buy boundary and $a_{N-k} = a^+_{N-k}$, $m_{N-k} = 0$ on the sell boundary. Having solved for the buy and sell boundaries, the investor's optimal strategy is as follows.

In the *buy region* $a_{N-k} < a^-_{N-k}$, the investor buys $l_{N-k} = a^-_{N-k} - a_{N-k}$ of the risky asset to reach the optimal buy boundary, which results in

$$F^{(B)}_{N-k} = (s_{N-k} - r_{N-k})a^-_{N-k} - r_{N-k}\lambda_{N-k}(a^-_{N-k} - a_{N-k}) \tag{2.45}$$

and

$$a_{N-k+1} = s_{N-k}a^-_{N-k}. \tag{2.46}$$

At time period $N-k$, having chosen the optimal strategy and allowing the wealth and risky asset to evolve to the next time step, the investor has the opportunity to optimally rebalance the portfolio at period $N-k+1$. At time period $N-k+1$, the investor has three optimal strategies for each of the three different regions delimited by the buy boundary a^-_{N-k+1} and the sell boundary a^+_{N-k+1}. Therefore, re-writing Equation (2.33) as an integral over s_{N-k} will take into account the three different regions at time period $N-k+1$ and the corresponding optimal value functions V_{N-k+1}, so that the optimal value function at period $N-k$ is expressed as

$$\begin{aligned} V^{(B)}_{N-k}(a_{N-k}) &= \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}V_{N-k+1}\right] \\ &= \int_0^{s^-_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}V^{(B)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &\quad + \int_{s^-_{N-k}}^{s^+_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}V^{(N)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &\quad + \int_{s^+_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}V^{(S)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k}. \end{aligned} \tag{2.47}$$

The function $p(s_{N-k})$ is the probability density function of the random term s_{N-k}. It is noted that V_{N-k+1} is a function of a_{N-k+1}. Furthermore, the value of the inherited risky asset a_{N-k+1} depends on s_{N-k} and a^-_{N-k} as seen in Equation (2.46), which implies that $s^-_{N-k} = \dfrac{a^-_{N-k+1}}{a^-_{N-k}}$ and $s^+_{N-k} = \dfrac{a^+_{N-k+1}}{a^-_{N-k}}$.

In the *sell region* $a_{N-k} > a^+_{N-k}$, the investor sells $m_{N-k} = a_{N-k} - a^+_{N-k}$ of the risky asset to reach the optimal sell boundary, which leads to

$$F^{(S)}_{N-k} = (s_{N-k} - r_{N-k})a^+_{N-k} - r_{N-k}\mu_{N-k}(a_{N-k} - a^+_{N-k}) \tag{2.48}$$

and

$$a_{N-k+1} = s_{N-k}a^+_{N-k}. \tag{2.49}$$

Similar to the buy region, the optimal value function at time period $N-k$ is written as

$$\begin{aligned}
V^{(S)}_{N-k}(a_{N-k}) &= \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}V_{N-k+1}\right] \\
&= \int_0^{s^-_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}V^{(B)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{s^-_{N-k}}^{s^+_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}V^{(N)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{s^+_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}V^{(S)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k}. \qquad (2.50)
\end{aligned}$$

Here, a_{N-k+1} depends on s_{N-k} and a^+_{N-k} as seen in Equation (2.49), which means that $s^-_{N-k} = \dfrac{a^-_{N-k+1}}{a^+_{N-k}}$ and $s^+_{N-k} = \dfrac{a^+_{N-k+1}}{a^+_{N-k}}$.

In the *no-transaction region* $a^-_{N-k} \le a_{N-k} \le a^+_{N-k}$, the investor does not transact in the risky asset ($l_{N-k} = 0 = m_{N-k}$) so that

$$F^{(N)}_{N-k} = (s_{N-k} - r_{N-k})a_{N-k} \qquad (2.51)$$

and

$$a_{N-k+1} = s_{N-k}a_{N-k}. \qquad (2.52)$$

Similar to the buy and sell regions, the optimal value function at time period $N-k$ is written as

$$\begin{aligned}
V^{(N)}_{N-k}(a_{N-k}) &= \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}V_{N-k+1}\right] \\
&= \int_0^{s^-_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}V^{(B)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{s^-_{N-k}}^{s^+_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}V^{(N)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{s^+_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}V^{(S)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k}. \qquad (2.53)
\end{aligned}$$

Here, a_{N-k+1} depends on s_{N-k} and a_{N-k} as seen in Equation (2.52), which means that $s^-_{N-k} = \dfrac{a^-_{N-k+1}}{a_{N-k}}$ and $s^+_{N-k} = \dfrac{a^+_{N-k+1}}{a_{N-k}}$.

The optimal strategies show that F_{N-k} as given by Equations (2.45), (2.48) and (2.51) is continuous across the buy and sell boundaries where $a_{N-k} = a^-_{N-k}$ and $a_{N-k} = a^+_{N-k}$ respectively. Hence, the optimal value function V_{N-k} as given by Equations (2.47), (2.50) and (2.53) is continuous across the buy and sell boundaries. In addition, the derivative of V_{N-k} with respect to a_{N-k} in the buy, sell and no-transaction regions is given by

$$\frac{\partial V^{(B)}_{N-k}}{\partial a_{N-k}} = \mathbb{E}_{N-k}\left[-\kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}V_{N-k+1}\right], \tag{2.54}$$

$$\frac{\partial V^{(S)}_{N-k}}{\partial a_{N-k}} = \mathbb{E}_{N-k}\left[\kappa r_{N-1}\cdots r_{N-k}\mu_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}V_{N-k+1}\right] \quad \text{and} \tag{2.55}$$

$$\begin{aligned}\frac{\partial V^{(N)}_{N-k}}{\partial a_{N-k}} &= \mathbb{E}_{N-k}\Bigg[-\kappa r_{N-1}\cdots r_{N-k+1}(s_{N-k}-r_{N-k})\\ &\times\ e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}V_{N-k+1} + s_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}\frac{\partial V_{N-k+1}}{\partial a_{N-k+1}}\Bigg],\end{aligned} \tag{2.56}$$

respectively. A consequence of Equations (2.43) and (2.44) is that the derivative of V_{N-k} with respect to a_{N-k} is continuous across the buy and sell boundaries, that is,

$$\frac{\partial V^{(B)}_{N-k}}{\partial a_{N-k}} = \frac{\partial V^{(N)}_{N-k}}{\partial a_{N-k}} \tag{2.57}$$

at $a_{N-k} = a^-_{N-k}$, and

$$\frac{\partial V^{(S)}_{N-k}}{\partial a_{N-k}} = \frac{\partial V^{(N)}_{N-k}}{\partial a_{N-k}} \tag{2.58}$$

at $a_{N-k} = a^+_{N-k}$. Therefore, the optimal strategies chosen provide continuity of the value function and its derivative across the buy and sell boundaries. Moreover, the latter continuity condition allows one to obtain the optimal buy and sell boundaries. In principle, by applying the dynamic programming algorithm, it should be possible to solve for the optimal strategies and value function recursively. However, numerical solutions are usually obtained and implementing the dynamic programming algorithm is computationally intensive.

In financial markets, proportional transaction costs are usually small. Therefore, by applying a perturbation analysis in the limit of small transaction costs in the model, analytical estimates of the optimal value function and strategies will be derived at any time period

$N-k$ ($k=1,\ldots,N$) whatever the number of steps considered. First of all, consider the special case where transaction costs are zero at all time periods.

2.3 Zero Transaction Costs

Assume that there are no transaction costs at all time periods so that $\lambda_{N-k}=0=\mu_{N-k}$. The budget equation (Equation (2.3)) becomes

$$W_{N-k+1}=r_{N-k}W_{N-k}+F_{N-k}, \tag{2.59}$$

where

$$F_{N-k}=(s_{N-k}-r_{N-k})(a_{N-k}+l_{N-k}-m_{N-k}) \tag{2.60}$$

for $k=1,\ldots,N$. The investor applies the dynamic programming algorithm to solve for the optimal strategies and value function when there are no transaction costs, starting at one step before terminal time.

Time Period $N-1$

With $\lambda_{N-1}=0=\mu_{N-1}$, F_{N-1} is now defined as

$$F^{(B)}_{N-1} = (s_{N-1}-r_{N-1})(a_{N-1}+l_{N-1}), \tag{2.61}$$
$$F^{(S)}_{N-1} = (s_{N-1}-r_{N-1})(a_{N-1}-m_{N-1}), \tag{2.62}$$
$$F^{(N)}_{N-1} = (s_{N-1}-r_{N-1})a_{N-1}. \tag{2.63}$$

Consequently, from Equation (2.17), the buy boundary a^-_{N-1} is the solution to the equation

$$\mathbb{E}_{N-1}\left[(s_{N-1}-r_{N-1})e^{-\kappa F^{(B)}_{N-1}}\right]=0, \tag{2.64}$$

where $a_{N-1}=a^-_{N-1}$ and $l_{N-1}=0$. Similarly, from Equation (2.18), the sell boundary a^+_{N-1} is the solution to the equation

$$\mathbb{E}_{N-1}\left[(r_{N-1}-s_{N-1})e^{-\kappa F^{(S)}_{N-1}}\right]=0, \tag{2.65}$$

where $a_{N-1} = a^+_{N-1}$ and $m_{N-1} = 0$.

It is observed from Equations (2.64) and (2.65) that the optimal buy boundary a^-_{N-1} and optimal sell boundary a^+_{N-1} coincide to the same optimal point, which is to be denoted by $\widetilde{a}_{N-1}$ and generally known as the Merton point. It can be seen that $\widetilde{a}_{N-1}$ satisfies the condition

$$\mathbb{E}_{N-1}\left[(s_{N-1} - r_{N-1})e^{-\kappa \widetilde{F}_{N-1}}\right] = 0, \tag{2.66}$$

where

$$\widetilde{F}_{N-1} = (s_{N-1} - r_{N-1})\widetilde{a}_{N-1}. \tag{2.67}$$

Therefore, in the absence of transaction costs, the investor's optimal strategy is to buy or sell to reach the optimal point $\widetilde{a}_{N-1}$. Moreover, if the investor is already at the point $\widetilde{a}_{N-1}$, the optimal strategy is not to transact. These optimal strategies result in

$$F^{(B)}_{N-1} = F^{(S)}_{N-1} = F^{(N)}_{N-1} = \widetilde{F}_{N-1}, \tag{2.68}$$

and from Equations (2.23) to (2.25) the optimal value function when there are no transaction costs is denoted by

$$\widetilde{V}_{N-1} = -\mathbb{E}_{N-1}\left[e^{-\kappa \widetilde{F}_{N-1}}\right]. \tag{2.69}$$

Time Period $N - k$

Applying the dynamic programming algorithm recursively backwards in time and carrying out an analysis similar to the one at period $N-1$, the optimal strategies and value functions are easily obtained for the reduced portfolio management problem with zero transaction costs. In summary, it can be shown that at each time period $N - k$ ($k = 2, \ldots, N$), the investor's strategy is to transact to reach the optimal point $\widetilde{a}_{N-k}$, which satisfies the condition

$$\mathbb{E}_{N-k}\left[(s_{N-k} - r_{N-k})e^{-\kappa r_{N-1} \cdots r_{N-k+1} \widetilde{F}_{N-k}}\right] = 0, \tag{2.70}$$

where

$$\widetilde{F}_{N-k} = (s_{N-k} - r_{N-k})\widetilde{a}_{N-k}. \tag{2.71}$$

The optimal value function at time period $N-k$ is then given by

$$\widetilde{V}_{N-k} = -\mathbb{E}_{N-1}\left[e^{-\kappa\widetilde{F}_{N-1}}\right]\mathbb{E}_{N-2}\left[e^{-\kappa r_{N-1}\widetilde{F}_{N-2}}\right]\cdots\mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}\right]. \tag{2.72}$$

It is observed that $\widetilde{V}_{N-k}$ does not vary with a_{N-k}, essentially because there is no cost incurred in buying or selling the risky asset to reach $\widetilde{a}_{N-k}$ from a_{N-k}. The optimal Merton point $\widetilde{a}_{N-k}$ and the condition that it satisfies will be used in deriving estimates of the optimal value function and strategies when transaction costs are small. It is also noted that if one makes the assumption that s_{N-k} are independent and identically distributed, and $r_{N-k} = r$ for all $k = 1, \ldots, N$, it is deduced from Equations (2.66) and (2.70) that $r^{k-1}\widetilde{a}_{N-k} = \widetilde{a}_{N-1}$. Moreover, from Equation (2.72), $\widetilde{V}_{N-k}$ is simplified to $\widetilde{V}_{N-k} = -\left\{\mathbb{E}_{N-1}\left[e^{-\kappa(s_{N-1}-r)\widetilde{a}_{N-1}}\right]\right\}^k$.

2.4 Small Transaction Costs

Assume that the costs of buying and selling are small so that $\lambda_{N-k} = O(\varepsilon)$ and $\mu_{N-k} = O(\varepsilon)$ where $\varepsilon \ll 1$ for $k = 1, \ldots, N$. Here, $O(.)$ is the usual asymptotic order symbol so that λ_{N-k} and μ_{N-k} are said to be "of the order" ε (see Appendix A for its definition). In the limit of small transaction costs, a two-stage perturbation method is developed to obtain approximations of the optimal value function and the optimal buy and sell boundaries at each time step.

From the analysis in Section 2.3, the buy and sell boundaries are observed to be equal to the Merton point at each time step when transaction costs are zero. Therefore, in the limit of small transaction costs, it is expected that the buy and sell boundaries are 'close' to the Merton point. However, in the first stage of our perturbation analysis, it shall be assumed that the investor buys or sells to reach the Merton point when transaction costs are small and the corresponding (suboptimal) value function is approximated. In the second stage, a sequence of corrections are derived to eventually provide an approximation to the (optimal) value function in the buy, sell and no-transaction regions, which then allows one to obtain an estimate of the optimal buy and sell boundaries.

2.4.1 Stage One: Transacting to the Merton Point

In order to obtain an intermediate approximation of the value function in the buy and sell regions, suppose that the buy and sell boundaries are equal to the Merton point in the limit of small transaction costs. This is an approximation because transaction costs, though small, are not equal to zero. In other words, the investor is assumed to adopt the suboptimal strategy of buying or selling to reach the Merton point at each time step.

In general, at time period $N-k$ ($k = 1, \ldots, N$), the investor's suboptimal strategy is to buy l_{N-k} or sell m_{N-k} dollars of the risky asset to reach the point $a^-_{N-k} = a^+_{N-k} = \widetilde{a}_{N-k}$. Therefore, the budget equation (2.3) is now given by

$$W_{N-k+1} = r_{N-k}W_{N-k} + \widehat{F}_{N-k}, \tag{2.73}$$

where

$$\widehat{F}_{N-k} = (s_{N-k} - r_{N-k})\widetilde{a}_{N-k} - r_{N-k}\lambda_{N-k}l_{N-k} - r_{N-k}\mu_{N-k}m_{N-k}. \tag{2.74}$$

Since $\widetilde{F}_{N-k} = (s_{N-k} - r_{N-k})\widetilde{a}_{N-k}$, it is convenient to write

$$\widehat{F}_{N-k} = \widetilde{F}_{N-k} - r_{N-k}\lambda_{N-k}l_{N-k} - r_{N-k}\mu_{N-k}m_{N-k}. \tag{2.75}$$

Furthermore, recall from Equation (2.1) that the value of the risky asset inherited in the next time step is given by

$$a_{N-k+1} = s_{N-k}\widetilde{a}_{N-k}, \tag{2.76}$$

since the investor buys or sells to reach the same point $\widetilde{a}_{N-k}$. Specifically, the investor adopts the following (suboptimal) strategy.

In the *buy region* $a_{N-k} < \widetilde{a}_{N-k}$, the investor buys $l_{N-k} = \widetilde{a}_{N-k} - a_{N-k}$ of the risky asset to reach the Merton point, which results in

$$\widehat{F}^{(B)}_{N-k} = \widetilde{F}_{N-k} - r_{N-k}\lambda_{N-k}(\widetilde{a}_{N-k} - a_{N-k}). \tag{2.77}$$

The value function at time period $N-1$, from Equation (2.23), is denoted by

$$\widehat{V}_{N-1}^{(B)} = -\mathbb{E}_{N-1}\left[e^{-\kappa \widehat{F}_{N-1}^{(B)}}\right], \tag{2.78}$$

and the value function at time period $N-k$ ($k = 2, \dots, N$), from Equation (2.47), is denoted by

$$\begin{aligned} \widehat{V}_{N-k}^{(B)}(a_{N-k}) &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1} \cdots r_{N-k+1} \widehat{F}_{N-k}^{(B)}} \widehat{V}_{N-k+1}^{(B)} p(s_{N-k}) \, \mathrm{d}s_{N-k} \\ &\quad + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1} \cdots r_{N-k+1} \widehat{F}_{N-k}^{(B)}} \widehat{V}_{N-k+1}^{(S)} p(s_{N-k}) \, \mathrm{d}s_{N-k}. \end{aligned} \tag{2.79}$$

It is noted that $\widehat{V}_{N-k+1}$ is a function of a_{N-k+1}, which depends on s_{N-k} and $\widetilde{a}_{N-k}$ via Equation (2.76). Furthermore, $\widehat{V}_{N-k+1}^{(B)}$ and $\widehat{V}_{N-k+1}^{(S)}$ are delineated by the point $a_{N-k+1} = \widetilde{a}_{N-k+1}$. Therefore, $\widetilde{s}_{N-k} = \dfrac{\widetilde{a}_{N-k+1}}{\widetilde{a}_{N-k}}$. Using Equation (2.77), Equations (2.78) and (2.79) can be expressed as

$$\widehat{V}_{N-1}^{(B)} = -e^{\kappa r_{N-1} \lambda_{N-1} (\widetilde{a}_{N-1} - a_{N-1})} \mathbb{E}_{N-1}\left[e^{-\kappa \widetilde{F}_{N-1}}\right] \tag{2.80}$$

and

$$\begin{aligned} \widehat{V}_{N-k}^{(B)} &= e^{\kappa r_{N-1} \cdots r_{N-k} \lambda_{N-k} (\widetilde{a}_{N-k} - a_{N-k})} \\ &\quad \times \left\{ \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1} \cdots r_{N-k+1} \widetilde{F}_{N-k}} \widehat{V}_{N-k+1}^{(B)} p(s_{N-k}) \, \mathrm{d}s_{N-k} \right. \\ &\quad \left. + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1} \cdots r_{N-k+1} \widetilde{F}_{N-k}} \widehat{V}_{N-k+1}^{(S)} p(s_{N-k}) \, \mathrm{d}s_{N-k} \right\}, \end{aligned} \tag{2.81}$$

respectively.

In the *sell region* $a_{N-k} > \widetilde{a}_{N-k}$, the investor sells $m_{N-k} = a_{N-k} - \widetilde{a}_{N-k}$ of the risky asset to reach the Merton point, which leads to

$$\widehat{F}_{N-k}^{(S)} = \widetilde{F}_{N-k} - r_{N-k} \mu_{N-k} (a_{N-k} - \widetilde{a}_{N-k}). \tag{2.82}$$

The value function at time period $N-1$, from Equation (2.24), is given by

$$\widehat{V}^{(S)}_{N-1} = -\mathbb{E}_{N-1}\left[e^{-\kappa \widehat{F}^{(S)}_{N-1}}\right], \tag{2.83}$$

and the value function at time period $N-k$ ($k=2,\ldots,N$), from Equation (2.50), is given by

$$\begin{aligned}\widehat{V}^{(S)}_{N-k}(a_{N-k}) &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widehat{F}^{(S)}_{N-k}}\widehat{V}^{(B)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &\quad + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widehat{F}^{(S)}_{N-k}}\widehat{V}^{(S)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k}.\end{aligned} \tag{2.84}$$

Similar to the buy region, $\widetilde{s}_{N-k} = \dfrac{\widetilde{a}_{N-k+1}}{\widetilde{a}_{N-k}}$. Using Equation (2.82), Equations (2.83) and (2.84) can be expressed as

$$\widehat{V}^{(S)}_{N-1} = -e^{\kappa r_{N-1}\mu_{N-1}(a_{N-1}-\widetilde{a}_{N-1})}\mathbb{E}_{N-1}\left[e^{-\kappa \widetilde{F}_{N-1}}\right] \tag{2.85}$$

and

$$\begin{aligned}\widehat{V}^{(S)}_{N-k} &= e^{\kappa r_{N-1}\cdots r_{N-k}\mu_{N-k}(a_{N-k}-\widetilde{a}_{N-k})} \\ &\quad \times \left\{\int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}\widehat{V}^{(B)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k}\right. \\ &\quad \left. + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}\widehat{V}^{(S)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k}\right\},\end{aligned} \tag{2.86}$$

respectively.

One observes from Equations (2.80), (2.81), (2.85) and (2.86) that the value function in the buy and sell regions at any time period $N-k$ ($k=1,\ldots,N$) is an exponential function of a_{N-k}, with a coefficient that is independent of a_{N-k}. When transaction costs are of $O(\varepsilon)$, this coefficient will be approximated by a series expansion up to $O(\varepsilon^2)$, with the analysis starting at period $N-1$ and proceeding backwards in time. The results of this approximation are hereby presented.

Time Period $N-1$

In the *buy region* $a_{N-1} < \widetilde{a}_{N-1}$, the value function is given by Equation (2.80). Recall from the zero transaction costs case that $\widetilde{V}_{N-1} = -\mathbb{E}_{N-1}\left[e^{-\kappa \widetilde{F}_{N-1}}\right]$. Therefore, the value function can be expressed as

$$\widehat{V}_{N-1}^{(B)} = e^{\kappa r_{N-1}\lambda_{N-1}(\widetilde{a}_{N-1}-a_{N-1})}\widetilde{V}_{N-1}. \tag{2.87}$$

Similarly, in the *sell region* $a_{N-1} > \widetilde{a}_{N-1}$, the value function is given by Equation (2.85), which can be expressed as

$$\widehat{V}_{N-1}^{(S)} = e^{\kappa r_{N-1}\mu_{N-1}(a_{N-1}-\widetilde{a}_{N-1})}\widetilde{V}_{N-1}. \tag{2.88}$$

The value function at time period $N-1$ is exact, which is a special case since it is one step before termination of the investment process.

Time Period $N-k$

Applying the dynamic programming algorithm by taking one step backwards in time from period $N-1$, an approximation of the value function in the buy and sell regions is obtained at period $N-2$. The analysis at period $N-3$ and subsequent time steps will proceed in the same way as the analysis at period $N-2$. In general, at time period $N-k$ ($k = 2, \dots, N$), the coefficient of the value function in the buy and sell regions is approximated up to $O(\varepsilon^2)$ as follows.

In the *buy region* $a_{N-k} < \widetilde{a}_{N-k}$, the value function is given by Equation (2.81). Expanding, simplifying and collecting terms of the same order up to $O(\varepsilon^2)$ in the coefficient of $e^{\kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}(\widetilde{a}_{N-k}-a_{N-k})}$, the value function in the buy region is approximated by

$$\widehat{V}^{(B)}_{N-k} = e^{\kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}(\widetilde{a}_{N-k}-a_{N-k})}\widetilde{V}_{N-k} \times\left\{1+\sum_{i=2}^{k}\frac{\zeta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]}+\sum_{i=2}^{k}\frac{\eta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]}\right. \left.+\sum_{\substack{i,j=2\\ i<j}}^{k}\frac{\zeta_{N-i}\zeta_{N-j}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]\mathbb{E}_{N-j}\left[e^{-\kappa r_{N-1}\cdots r_{N-j+1}\widetilde{F}_{N-j}}\right]}+O(\varepsilon^3)\right\}, \quad (2.89)$$

where $\widetilde{V}_{N-k}$ is given by Equation (2.72) from the zero transaction costs case, and the terms

$$\begin{aligned}\zeta_{N-i} &= \kappa r_{N-1}\cdots r_{N-i+1}\lambda_{N-i+1}\widetilde{a}_{N-i} \\ &\quad\times\int_0^{\widetilde{s}_{N-i}}(\widetilde{s}_{N-i}-s_{N-i})e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}p(s_{N-i})\,\mathrm{d}s_{N-i} \\ &\quad-\kappa r_{N-1}\cdots r_{N-i+1}\mu_{N-i+1}\widetilde{a}_{N-i} \\ &\quad\times\int_{\widetilde{s}_{N-i}}^{\infty}(\widetilde{s}_{N-i}-s_{N-i})e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}p(s_{N-i})\,\mathrm{d}s_{N-i}\end{aligned} \quad (2.90)$$

is of $O(\varepsilon)$, and

$$\begin{aligned}\eta_{N-i} &= \frac{1}{2}(\kappa r_{N-1}\cdots r_{N-i+1}\lambda_{N-i+1}\widetilde{a}_{N-i})^2 \\ &\quad\times\int_0^{\widetilde{s}_{N-i}}(\widetilde{s}_{N-i}-s_{N-i})^2e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}p(s_{N-i})\,\mathrm{d}s_{N-i} \\ &\quad+\frac{1}{2}(\kappa r_{N-1}\cdots r_{N-i+1}\mu_{N-i+1}\widetilde{a}_{N-i})^2 \\ &\quad\times\int_{\widetilde{s}_{N-i}}^{\infty}(\widetilde{s}_{N-i}-s_{N-i})^2e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}p(s_{N-i})\,\mathrm{d}s_{N-i}\end{aligned} \quad (2.91)$$

is of $O(\varepsilon^2)$. Recall that $\widetilde{s}_{N-i}=\dfrac{\widetilde{a}_{N-i+1}}{\widetilde{a}_{N-i}}$. An estimate of the remainder term in the above expansion is presented and shown to be bounded in Appendix B.1. It is noted that the remainder terms in subsequent expansions are similarly bounded and thus will not be presented.

In the *sell region* $a_{N-k}>\widetilde{a}_{N-k}$, the value function is given by Equation (2.86), which is essentially the same as Equation (2.81) in the buy region with λ_{N-k} replaced by $-\mu_{N-k}$.

Thus, one can immediately deduce that the value function in the sell region is approximated by

$$\begin{aligned}\widehat{V}^{(S)}_{N-k} &= e^{\kappa r_{N-1}\cdots r_{N-k}\mu_{N-k}(a_{N-k}-\widetilde{a}_{N-k})}\widetilde{V}_{N-k} \\ &\times\left\{1+\sum_{i=2}^{k}\frac{\zeta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]}+\sum_{i=2}^{k}\frac{\eta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]}\right. \\ &\left.+\sum_{\substack{i,j=2\\ i<j}}^{k}\frac{\zeta_{N-i}\zeta_{N-j}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]\mathbb{E}_{N-j}\left[e^{-\kappa r_{N-1}\cdots r_{N-j+1}\widetilde{F}_{N-j}}\right]}+O(\varepsilon^3)\right\}. \quad (2.92)\end{aligned}$$

A key feature here is the transmission of information from the buy and sell regions to the next step through the boundaries of these buy and sell regions, as seen in Equations (2.46) and (2.49), that is, $a_{N-k+1} = s_{N-k}a^{-}_{N-k}$ and $a_{N-k+1} = s_{N-k}a^{+}_{N-k}$. Even though a^{-}_{N-i} and a^{+}_{N-i} ($i = 1, \ldots, k$) are as yet unknown, replacing them by $\widetilde{a}_{N-i}$ enables one to remove any dependence on a_{N-k+j} ($j = 1, \ldots, k-1$) except for $\widetilde{a}_{N-k+j}$ throughout Equations (2.89) and (2.92). Corrections to the approximation $a^{-}_{N-i} = \widetilde{a}_{N-i} = a^{+}_{N-i}$ will be assessed later.

This strategy of transacting to the Merton point is suboptimal in the sense of maximising utility of terminal wealth, as the investor has ignored the presence of the no-transaction region. This region exists when transaction costs are non-zero, even if they are assumed to be small. However, for small transaction costs, it is expected that the no-transaction region is small. Therefore, the optimal value functions $V^{(B)}_{N-k}$ and $V^{(S)}_{N-k}$are expected to be 'close' perturbations about the suboptimal value functions $\widehat{V}^{(B)}_{N-k}$ and $\widehat{V}^{(S)}_{N-k}$ respectively. But how close are these perturbations, that is, what are the correction terms? And how should the optimal value function for the no-transaction region be approximated? These questions are addressed in the second stage of our perturbation analysis.

2.4.2 Stage Two: Estimating the Optimal Value Function and Boundaries

Recall from Section 2.3 that the buy and sell boundaries coincide to the Merton point at each time step when transaction costs are equated to zero. Therefore, one would assume that the problem in Section 2.2 with small transaction costs is a perturbation of the problem in Section 2.3 with zero transaction costs. In other words, when $\lambda_{N-k} = O(\varepsilon)$ and $\mu_{N-k} = O(\varepsilon)$ where $\varepsilon \ll 1$ for $k = 1, \ldots, N$, the optimal buy and sell boundaries are assumed to be 'close' to the Merton point so that

$$a^{-}_{N-k} = \widetilde{a}_{N-k} + \omega^{-}_{N-k} \tag{2.93}$$

and

$$a^{+}_{N-k} = \widetilde{a}_{N-k} + \omega^{+}_{N-k}, \tag{2.94}$$

where $\omega^{-}_{N-k} = O(\varepsilon)$ and $\omega^{+}_{N-k} = O(\varepsilon)$ are unknown and yet to be determined. Therefore, in the no-transaction region where $a^{-}_{N-k} \leq a_{N-k} \leq a^{+}_{N-k}$, a_{N-k} is expressed as

$$a_{N-k} = \widetilde{a}_{N-k} + \omega_{N-k}, \tag{2.95}$$

where $\omega^{-}_{N-k} \leq \omega_{N-k} \leq \omega^{+}_{N-k}$ and $\omega_{N-k} = O(\varepsilon)$. It is noted that ω^{-}_{N-k} and ω^{+}_{N-k} are assumed to be of $O(\varepsilon)$ as one would expect them to depend on the transaction costs, which are of $O(\varepsilon)$. It will be shown subsequently that this assumption is indeed self-consistent with the results that follow from the perturbation analysis.

An estimate of the optimal value function will be obtained by carrying out a perturbation analysis about the intermediate approximation derived in Section 2.4.1. The optimal value function will be approximated up to $O(\varepsilon^2)$ so as to ensure continuity of the value function and continuity of its first derivative across the boundaries. The approximation of the optimal value functions will be achieved by implementing the following procedure, starting from period $N - 1$ and proceeding backwards in time.

1. Estimate the correction term between the optimal value function and its intermediate approximation in the buy and sell regions, defined as

$$\delta^{(B)}_{N-k} = V^{(B)}_{N-k} - \widehat{V}^{(B)}_{N-k} \tag{2.96}$$

and

$$\delta^{(S)}_{N-k} = V^{(S)}_{N-k} - \widehat{V}^{(S)}_{N-k}, \tag{2.97}$$

respectively.

2. Estimate the optimal value function $V_{N-k}^{(N)}$ in the no-transaction region.
3. Repeat the procedure by taking a step backwards in time and using the previous estimates obtained in the time steps ahead.

The above procedure provides one with a systematic methodology to generate an estimate of the optimal value function *at any time step* in the rebalancing of the portfolio. It can also be verified that the optimal value function thus obtained is continuous across the buy and sell boundaries. Moreover, continuity of the optimal value function's derivative across the boundaries will enable one to obtain estimates of the optimal buy and sell boundaries. An illustration of the above method to estimate the optimal value functions and boundaries is hereby presented, starting from time period $N-1$.

Time Period $N-1$

In the *buy region* $a_{N-1} < a_{N-1}^{-}$, from Equations (2.23), (2.78) and (2.96), the correction term is given by

$$\delta_{N-1}^{(B)} = -\mathbb{E}_{N-1}\left[e^{-\kappa F_{N-1}^{(B)}} - e^{-\kappa \widehat{F}_{N-1}^{(B)}}\right]. \tag{2.98}$$

Recall that $F_{N-1}^{(B)}$ and $\widehat{F}_{N-1}^{(B)}$ are given by Equations (2.19) and (2.77) respectively, and that $a_{N-1}^{-} = \widetilde{a}_{N-1} + \omega_{N-1}^{-}$. Factorising $e^{\kappa r_{N-1}\lambda_{N-1}(\widetilde{a}_{N-1}-a_{N-1})}$ from the right-hand side of Equation (2.98), expanding its coefficient up to terms in $O(\varepsilon^2)$ and simplifying with Equation (2.66), it can be shown that

$$\delta_{N-1}^{(B)} = e^{\kappa r_{N-1}\lambda_{N-1}(\widetilde{a}_{N-1}-a_{N-1})}\left\{\xi_{N-1}^{(B)} + O(\varepsilon^3)\right\}, \tag{2.99}$$

where

$$\xi_{N-1}^{(B)} = \kappa r_{N-1}\lambda_{N-1}\omega_{N-1}^{-}\widetilde{V}_{N-1} - \frac{1}{2}\kappa^2(\omega_{N-1}^{-})^2\mathbb{E}_{N-1}\left[(s_{N-1} - r_{N-1})^2 e^{-\kappa\widetilde{F}_{N-1}}\right] \tag{2.100}$$

is of $O(\varepsilon^2)$ and independent of a_{N-1}. Therefore, the optimal value function $V_{N-1}^{(B)} = \widehat{V}_{N-1}^{(B)} + \delta_{N-1}^{(B)}$ is approximated by

$$V_{N-1}^{(B)} = e^{\kappa r_{N-1}\lambda_{N-1}(\widetilde{a}_{N-1}-a_{N-1})}\left\{\widetilde{V}_{N-1} + \xi_{N-1}^{(B)} + O(\varepsilon^3)\right\}. \tag{2.101}$$

In the *sell region* $a_{N-1} > a^+_{N-1}$, from Equations (2.24), (2.83) and (2.97), the correction term is given by

$$\delta^{(S)}_{N-1} = -\mathbb{E}_{N-1}\left[e^{-\kappa F^{(S)}_{N-1}} - e^{-\kappa \widehat{F}^{(S)}_{N-1}}\right]. \tag{2.102}$$

Recall that $F^{(S)}_{N-1}$ and $\widehat{F}^{(S)}_{N-1}$ are given by Equations (2.20) and (2.82) respectively, and that $a^+_{N-1} = \widetilde{a}_{N-1} + \omega^+_{N-1}$. One observes that Equation (2.102) is equivalent to Equation (2.98) with λ_{N-1} replaced by $-\mu_{N-1}$, and ω^-_{N-1} replaced by ω^+_{N-1}. Therefore, the correction term in the sell region is estimated by

$$\delta^{(S)}_{N-1} = e^{\kappa r_{N-1}\mu_{N-1}(a_{N-1}-\widetilde{a}_{N-1})}\left\{\xi^{(S)}_{N-1} + O(\varepsilon^3)\right\}, \tag{2.103}$$

where

$$\xi^{(S)}_{N-1} = -\kappa r_{N-1}\mu_{N-1}\omega^+_{N-1}\widetilde{V}_{N-1} - \frac{1}{2}\kappa^2(\omega^+_{N-1})^2\mathbb{E}_{N-1}\left[(s_{N-1}-r_{N-1})^2 e^{-\kappa \widetilde{F}_{N-1}}\right] \tag{2.104}$$

is of $O(\varepsilon^2)$ and independent of a_{N-1}. Therefore, the optimal value function $V^{(S)}_{N-1} = \widehat{V}^{(S)}_{N-1} + \delta^{(S)}_{N-1}$ is approximated by

$$V^{(S)}_{N-1} = e^{\kappa r_{N-1}\mu_{N-1}(a_{N-1}-\widetilde{a}_{N-1})}\left\{\widetilde{V}_{N-1} + \xi^{(S)}_{N-1} + O(\varepsilon^3)\right\}. \tag{2.105}$$

In the *no-transaction region* $a^-_{N-1} \le a_{N-1} \le a^+_{N-1}$, from Equation (2.25), the optimal value function is given by

$$V^{(N)}_{N-1} = -\mathbb{E}_{N-1}\left[e^{-\kappa F^{(N)}_{N-1}}\right], \tag{2.106}$$

where $F^{(N)}_{N-1}$ is given by Equation (2.16), and $a_{N-1} = \widetilde{a}_{N-1} + \omega_{N-1}$. Recall that ω_{N-1} is of $O(\varepsilon)$ in the no-transaction region. Expanding Equation (2.106) up to $O(\varepsilon^2)$ and simplifying with Equation (2.66), it can be shown that

$$V^{(N)}_{N-1} = \widetilde{V}_{N-1} - \frac{1}{2}\kappa^2(a_{N-1}-\widetilde{a}_{N-1})^2\mathbb{E}_{N-1}\left[(s_{N-1}-r_{N-1})^2 e^{-\kappa \widetilde{F}_{N-1}}\right] + O(\varepsilon^3). \tag{2.107}$$

Having obtained an approximation of the optimal value function in the buy, sell and no-transaction regions as given by Equations (2.101), (2.105) and (2.107) respectively, it is easily verified that it is continuous across the boundaries at $a_{N-1} = a^-_{N-1}$ and $a_{N-1} =$

a^+_{N-1}. In order to obtain an estimate of the optimal buy and sell boundaries, one applies continuity of the optimal value function's first derivative across the boundaries as seen in Equations (2.29) and (2.30), which lead to the following results.

1. At the buy boundary $a_{N-1} = a^-_{N-1}$, the condition $\dfrac{\partial V^{(B)}_{N-1}}{\partial a_{N-1}} = \dfrac{\partial V^{(N)}_{N-1}}{\partial a_{N-1}}$ implies that

$$\omega^-_{N-1} = \frac{r_{N-1}\lambda_{N-1}\widetilde{V}_{N-1}}{\kappa \mathbb{E}_{N-1}\left[(s_{N-1} - r_{N-1})^2 e^{-\kappa \widetilde{F}_{N-1}}\right]} + O(\varepsilon^2). \tag{2.108}$$

2. At the sell boundary $a_{N-1} = a^+_{N-1}$, the condition $\dfrac{\partial V^{(S)}_{N-1}}{\partial a_{N-1}} = \dfrac{\partial V^{(N)}_{N-1}}{\partial a_{N-1}}$ implies that

$$\omega^+_{N-1} = \frac{-r_{N-1}\mu_{N-1}\widetilde{V}_{N-1}}{\kappa \mathbb{E}_{N-1}\left[(s_{N-1} - r_{N-1})^2 e^{-\kappa \widetilde{F}_{N-1}}\right]} + O(\varepsilon^2). \tag{2.109}$$

It is noted that the leading order terms of ω^-_{N-1} and ω^+_{N-1} depend linearly on λ_{N-1} and μ_{N-1} respectively, which is self-consistent with the initial assumption that they are of $O(\varepsilon)$. Therefore, one can estimate the optimal buy and sell boundaries by $a^-_{N-1} = \widetilde{a}_{N-1} + \omega^-_{N-1}$ and $a^+_{N-1} = \widetilde{a}_{N-1} + \omega^+_{N-1}$ respectively.

Time Period $N - k$

Taking one step back in time and using the optimal value functions that are estimated at time period $N - 1$, an approximation of the optimal value functions at period $N - 2$ is derived via the correction terms about the suboptimal value functions. Applying the procedure recursively backwards in time, one will be able to generate an estimate of the optimal value functions at any period $N - k$ ($k = 2, \dots, N$). A general description of the perturbation analysis and the results that follow is hereby presented.

In the *buy region* $a_{N-k} < a^-_{N-k}$, an estimate of the correction term $\delta^{(B)}_{N-k} = V^{(B)}_{N-k} - \widehat{V}^{(B)}_{N-k}$

is obtained via the following analysis. Recall from Equation (2.79) that

$$\begin{aligned}\widehat{V}^{(B)}_{N-k} &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widehat{F}^{(B)}_{N-k}}\widehat{V}^{(B)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &\quad + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widehat{F}^{(B)}_{N-k}}\widehat{V}^{(S)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k}, \end{aligned} \tag{2.110}$$

where $\widehat{F}^{(B)}_{N-k}$ is given by Equation (2.77). It is noted that $\widehat{V}_{N-k+1}$ is a function of $a_{N-k+1} = s_{N-k}\widetilde{a}_{N-k}$ since by definition, the investor buys to reach the Merton point $\widetilde{a}_{N-k}$.

Consider $V^{(B)}_{N-k}$ from Equation (2.47), where $F^{(B)}_{N-k}$ is given by Equation (2.45). Here, V_{N-k+1} is a function of $a_{N-k+1} = s_{N-k}a^{-}_{N-k}$ since the investor buys to reach the boundary $a^{-}_{N-k} = \widetilde{a}_{N-k} + \omega^{-}_{N-k}$. Since the procedure is implemented backwards in time, one would have obtained an estimate of the optimal value function in the time step ahead, given by $V_{N-k+1} = \widehat{V}_{N-k+1} + \delta_{N-k+1}$. It can be verified that at each time period $N-k$, Proposition B.2.1 in Appendix B.2 is valid. Therefore, it is estimated that, up to $O(\varepsilon^2)$,

$$\begin{aligned}V^{(B)}_{N-k} &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}\left[\widehat{V}^{(B)}_{N-k+1} + \delta^{(B)}_{N-k+1}\right]p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &+ \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}\left[\widehat{V}^{(S)}_{N-k+1} + \delta^{(S)}_{N-k+1}\right]p(s_{N-k})\,\mathrm{d}s_{N-k} + O(\varepsilon^3), \end{aligned} \tag{2.111}$$

which one observes is of a similar form as Equation (2.110). Here, it is noted that $\widehat{V}_{N-k+1}$ and δ_{N-k+1} are functions of $a_{N-k+1} = s_{N-k}a^{-}_{N-k}$ where $a^{-}_{N-k} = \widetilde{a}_{N-k} + \omega^{-}_{N-k}$.

Lastly, recall that $\widehat{V}^{(B)}_{N-k+1}$ is given by Equation (2.87) or (2.89), $\delta^{(B)}_{N-1}$ by Equation (2.99), $\widehat{V}^{(S)}_{N-k+1}$ by Equation (2.88) or (2.92) and $\delta^{(S)}_{N-1}$ by Equation (2.103). Factorising $e^{\kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}(\widetilde{a}_{N-k}-a_{N-k})}$ from the right-hand side of $\delta^{(B)}_{N-k} = V^{(B)}_{N-k} - \widehat{V}^{(B)}_{N-k}$, expanding its coefficient up to terms of $O(\varepsilon^2)$ and simplifying with Equation (2.70), it can be shown after some algebra that

$$\delta^{(B)}_{N-k} = e^{\kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}(\widetilde{a}_{N-k}-a_{N-k})}\left\{\xi^{(B)}_{N-k} + O(\varepsilon^3)\right\}, \tag{2.112}$$

where

$$\begin{aligned}
\xi_{N-k}^{(B)} &= \kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}\omega_{N-k}^{-}\widetilde{V}_{N-k} + \left[\alpha_{N-k}\omega_{N-k}^{-} + \beta_{N-k}(\omega_{N-k}^{-})^2\right]\widetilde{V}_{N-k+1} \\
&\quad + \xi_{N-k+1}^{(B)}\int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \xi_{N-k+1}^{(S)}\int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \qquad (2.113)
\end{aligned}$$

is of $O(\varepsilon^2)$;

$$\begin{aligned}
\alpha_{N-k} &= -\kappa r_{N-1}\cdots r_{N-k+1}\lambda_{N-k+1} \\
&\quad \times \int_0^{\widetilde{s}_{N-k}} \{\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{a}_{N-k}(\widetilde{s}_{N-k} - s_{N-k})(s_{N-k} - r_{N-k}) + s_{N-k}\} \\
&\quad \times e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \kappa r_{N-1}\cdots r_{N-k+1}\mu_{N-k+1} \\
&\quad \times \int_{\widetilde{s}_{N-k}}^{\infty} \{\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{a}_{N-k}(\widetilde{s}_{N-k} - s_{N-k})(s_{N-k} - r_{N-k}) + s_{N-k}\} \\
&\quad \times e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \qquad (2.114)
\end{aligned}$$

is of $O(\varepsilon)$; and

$$\beta_{N-k} = \frac{1}{2}(\kappa r_{N-1}\cdots r_{N-k+1})^2 \mathbb{E}_{N-k}\left[(s_{N-k} - r_{N-k})^2 e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}\right] \qquad (2.115)$$

is of $O(\varepsilon^0)$.

In the *sell region* $a_{N-k} > a_{N-k}^{+}$, one obtains an estimate of the correction term $\delta_{N-k}^{(S)} = V_{N-k}^{(S)} - \widehat{V}_{N-k}^{(S)}$ by following a similar analysis as the buy region. Recall from Equation (2.84) that

$$\begin{aligned}
\widehat{V}_{N-k}^{(S)} &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widehat{F}_{N-k}^{(S)}} \widehat{V}_{N-k+1}^{(B)} p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widehat{F}_{N-k}^{(S)}} \widehat{V}_{N-k+1}^{(S)} p(s_{N-k})\,\mathrm{d}s_{N-k}, \qquad (2.116)
\end{aligned}$$

where $\widehat{F}_{N-k}^{(S)}$ is given by Equation (2.82). Similar to the buy region, $\widehat{V}_{N-k+1}$ is a function of

$a_{N-k+1} = s_{N-k}\widetilde{a}_{N-k}$ since by definition, the investor sells to reach the Merton point $\widetilde{a}_{N-k}$.

Recall that $V^{(S)}_{N-k}$ is from Equation (2.50), where $F^{(S)}_{N-k}$ is given by Equation (2.48). Here, V_{N-k+1} is a function of $a_{N-k+1} = s_{N-k}a^{+}_{N-k}$ since the investor sells to reach the boundary $a^{+}_{N-k} = \widetilde{a}_{N-k} + \omega^{+}_{N-k}$. Similar to the analysis in the buy region, it can be verified that Proposition B.2.2 in Appendix B.2 is valid at any time period $N-k$. Therefore, up to $O(\varepsilon^2)$,

$$V^{(S)}_{N-k} = \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}\left[\widehat{V}^{(B)}_{N-k+1} + \delta^{(B)}_{N-k+1}\right] p(s_{N-k})\,\mathrm{d}s_{N-k}$$
$$+\int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}\left[\widehat{V}^{(S)}_{N-k+1} + \delta^{(S)}_{N-k+1}\right] p(s_{N-k})\,\mathrm{d}s_{N-k} + O(\varepsilon^3). \quad (2.117)$$

Here, it is noted that $\widehat{V}_{N-k+1}$ and δ_{N-k+1} are functions of $a_{N-k+1} = s_{N-k}a^{+}_{N-k}$ where $a^{+}_{N-k} = \widetilde{a}_{N-k} + \omega^{+}_{N-k}$.

At this stage, one observes that essentially the same analysis as the buy region follows through with λ_{N-2} replaced by $-\mu_{N-2}$, and ω^{-}_{N-2} replaced by ω^{+}_{N-2}. Therefore, it can be deduced that an estimate of the correction term in the sell region is given by

$$\delta^{(S)}_{N-k} = e^{\kappa r_{N-1}\cdots r_{N-k}\mu_{N-k}(a_{N-k}-\widetilde{a}_{N-k})}\left\{\xi^{(S)}_{N-k} + O(\varepsilon^3)\right\}, \quad (2.118)$$

where

$$\begin{aligned}\xi^{(S)}_{N-k} \;=\; & -\kappa r_{N-1}\cdots r_{N-k}\mu_{N-k}\omega^{+}_{N-k}\widetilde{V}_{N-k} + \left[\alpha_{N-k}\omega^{+}_{N-k} + \beta_{N-k}(\omega^{+}_{N-k})^2\right]\widetilde{V}_{N-k+1} \\ & +\xi^{(B)}_{N-k+1}\int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \\ & +\xi^{(S)}_{N-k+1}\int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \end{aligned} \quad (2.119)$$

is of $O(\varepsilon^2)$, and α_{N-k} and β_{N-k} are given by Equations (2.114) and (2.115) respectively.

In the *no-transaction region* $a^{-}_{N-k} \leq a_{N-k} \leq a^{+}_{N-k}$, an estimate of the optimal value function is derived. Consider $V^{(N)}_{N-k}$ from Equation (2.53), where $F^{(N)}_{N-k}$ is given by Equation (2.51). In this case, V_{N-k+1} is a function of $a_{N-k+1} = s_{N-k}a_{N-k}$ as the investor does not transact in the risky asset. Since a_{N-k} lies within the no-transaction region, $a_{N-k} =$

$\widetilde{a}_{N-k} + \omega_{N-k}$ where ω_{N-k} is of $O(\varepsilon)$. Similar to the analysis in the buy and sell regions, it can be verified that Proposition B.2.3 in Appendix B.2 is valid at any time period $N-k$ so that, up to $O(\varepsilon^2)$,

$$V_{N-k}^{(N)} = \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(N)}} \left[\widehat{V}_{N-k+1}^{(B)} + \delta_{N-k+1}^{(B)}\right] p(s_{N-k})\,\mathrm{d}s_{N-k}$$
$$+ \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(N)}} \left[\widehat{V}_{N-k+1}^{(S)} + \delta_{N-k+1}^{(S)}\right] p(s_{N-k})\,\mathrm{d}s_{N-k} + O(\varepsilon^3). \quad (2.120)$$

It is noted that $\widehat{V}_{N-k+1}$ and δ_{N-k+1} are functions of $a_{N-k+1} = s_{N-k}a_{N-k}$ where $a_{N-k} = \widetilde{a}_{N-k} + \omega_{N-k}$.

It is convenient to adopt a notation that is consistent with the buy and sell regions, and to write

$$V_{N-k}^{(N)} = \widehat{V}_{N-k}^{(N)} + \delta_{N-k}^{(N)} + O(\varepsilon^3), \quad (2.121)$$

where $\widehat{V}_{N-k}^{(N)}$ denotes

$$\widehat{V}_{N-k}^{(N)} = \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(N)}} \widehat{V}_{N-k+1}^{(B)} p(s_{N-k})\,\mathrm{d}s_{N-k}$$
$$+ \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(N)}} \widehat{V}_{N-k+1}^{(S)} p(s_{N-k})\,\mathrm{d}s_{N-k}. \quad (2.122)$$

Using $\widehat{V}_{N-k+1}^{(B)}$ as given by Equation (2.80) or (2.89), $\widehat{V}_{N-k+1}^{(S)}$ as given by Equation (2.85) or (2.92), expanding up to $O(\varepsilon^2)$ and simplifying with the Equation (2.70), it is estimated that

$$\widehat{V}_{N-k}^{(N)} = \widetilde{V}_{N-k}\left\{1 + \sum_{i=2}^{k} \frac{\zeta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]} + \sum_{i=2}^{k} \frac{\eta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]}\right.$$
$$\left. + \sum_{\substack{i,j=2\\ i<j}}^{k} \frac{\zeta_{N-i}\zeta_{N-j}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]\mathbb{E}_{N-j}\left[e^{-\kappa r_{N-1}\cdots r_{N-j+1}\widetilde{F}_{N-j}}\right]}\right\}$$
$$+ \left[\alpha_{N-k}(a_{N-k} - \widetilde{a}_{N-k}) + \beta_{N-k}(a_{N-k} - \widetilde{a}_{N-k})^2\right]\widetilde{V}_{N-k+1} + O(\varepsilon^3), \quad (2.123)$$

where the terms ζ_{N-i}, η_{N-i}, α_{N-k} and β_{N-k} are given by Equations (2.90), (2.91), (2.114) and (2.115) respectively. In addition, $\delta^{(N)}_{N-k}$ denotes

$$\delta^{(N)}_{N-k} = \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}\delta^{(B)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}\delta^{(S)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k}. \tag{2.124}$$

where $\delta^{(B)}_{N-k+1}$ is given by Equation (2.99) or (2.112), and $\delta^{(S)}_{N-k+1}$ is given by Equation (2.103) or (2.118). Expanding the terms up to $O(\varepsilon^2)$ and observing that $\xi^{(B)}_{N-k+1}$ and $\xi^{(S)}_{N-k+1}$ are of $O(\varepsilon^2)$, it is estimated that

$$\delta^{(N)}_{N-k} = \xi^{(B)}_{N-k+1}\int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}p(s_{N-k})\,\mathrm{d}s_{N-k} + \xi^{(S)}_{N-k+1}\int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}p(s_{N-k})\,\mathrm{d}s_{N-k} + O(\varepsilon^3). \tag{2.125}$$

Consolidating the results of our analysis, the optimal value function in the buy, sell and no-transaction regions at time period $N-k$ ($k = 2, \ldots, N$) is approximated up to $O(\varepsilon^2)$ by

$$\begin{aligned}
V^{(B)}_{N-k} &= e^{\kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}(\widetilde{a}_{N-k}-a_{N-k})} \\
&\times\Bigg[\widetilde{V}_{N-k}\Bigg\{1+\sum_{i=2}^{k}\frac{\zeta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]}+\sum_{i=2}^{k}\frac{\eta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]} \\
&+\sum_{\substack{i,j=2\\ i<j}}^{k}\frac{\zeta_{N-i}\zeta_{N-j}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]\mathbb{E}_{N-j}\left[e^{-\kappa r_{N-1}\cdots r_{N-j+1}\widetilde{F}_{N-j}}\right]}\Bigg\} \\
&+\kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}\omega^{-}_{N-k}\widetilde{V}_{N-k}+\left[\alpha_{N-k}\omega^{-}_{N-k}+\beta_{N-k}(\omega^{-}_{N-k})^2\right]\widetilde{V}_{N-k+1} \\
&+\xi^{(B)}_{N-k+1}\int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&+\xi^{(S)}_{N-k+1}\int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}p(s_{N-k})\,\mathrm{d}s_{N-k}+O(\varepsilon^3)\Bigg],
\end{aligned} \tag{2.126}$$

$$
\begin{aligned}
V^{(S)}_{N-k} &= e^{\kappa r_{N-1}\cdots r_{N-k}\mu_{N-k}(a_{N-k}-\widetilde{a}_{N-k})} \\
&\times\Bigg[\widetilde{V}_{N-k}\Bigg\{1+\sum_{i=2}^{k}\frac{\zeta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]}+\sum_{i=2}^{k}\frac{\eta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]} \\
&+\sum_{\substack{i,j=2\\ i<j}}^{k}\frac{\zeta_{N-i}\zeta_{N-j}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]\mathbb{E}_{N-j}\left[e^{-\kappa r_{N-1}\cdots r_{N-j+1}\widetilde{F}_{N-j}}\right]}\Bigg\} \\
&-\kappa r_{N-1}\cdots r_{N-k}\mu_{N-k}\omega^{+}_{N-k}\widetilde{V}_{N-k}+\left[\alpha_{N-k}\omega^{+}_{N-k}+\beta_{N-k}(\omega^{+}_{N-k})^2\right]\widetilde{V}_{N-k+1} \\
&+\xi^{(B)}_{N-k+1}\int_{0}^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&+\xi^{(S)}_{N-k+1}\int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}p(s_{N-k})\,\mathrm{d}s_{N-k}+O(\varepsilon^3)\Bigg]
\end{aligned}
\tag{2.127}
$$

and

$$
\begin{aligned}
V^{(N)}_{N-k} &= \left[\alpha_{N-k}(a_{N-k}-\widetilde{a}_{N-k})+\beta_{N-k}(a_{N-k}-\widetilde{a}_{N-k})^2\right]\widetilde{V}_{N-k+1} \\
&+\widetilde{V}_{N-k}\Bigg\{1+\sum_{i=2}^{k}\frac{\zeta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]}+\sum_{i=2}^{k}\frac{\eta_{N-i}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]} \\
&+\sum_{\substack{i,j=2\\ i<j}}^{k}\frac{\zeta_{N-i}\zeta_{N-j}}{\mathbb{E}_{N-i}\left[e^{-\kappa r_{N-1}\cdots r_{N-i+1}\widetilde{F}_{N-i}}\right]\mathbb{E}_{N-j}\left[e^{-\kappa r_{N-1}\cdots r_{N-j+1}\widetilde{F}_{N-j}}\right]}\Bigg\} \\
&+\xi^{(B)}_{N-k+1}\int_{0}^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&+\xi^{(S)}_{N-k+1}\int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}p(s_{N-k})\,\mathrm{d}s_{N-k}+O(\varepsilon^3),
\end{aligned}
\tag{2.128}
$$

respectively. The terms ζ_{N-i}, η_{N-i}, α_{N-k} and β_{N-k} are given by Equations (2.90), (2.91), (2.114) and (2.115) respectively. The recursive terms $\xi^{(B)}_{N-k+1}$ and $\xi^{(S)}_{N-k+1}$ are given by Equations (2.113) and (2.119) respectively.

It is observed from the above expressions that the optimal value functions at time period $N-k$ are dependent on the time steps ahead up to period $N-1$, essentially via the zero transaction costs solutions $\widetilde{a}_{N-k+j}$ ($j=1,\dots,k-1$). Furthermore, they are dependent on $\xi^{(B)}_{N-k+1}$ and $\xi^{(S)}_{N-k+1}$, which implies by recursion that they depend on ω^{-}_{N-k+j} and ω^{+}_{N-k+j} ($j=1,\dots,k-1$). Therefore, in order to estimate the optimal value function at time period $N-k$, one will also require the optimal boundaries from the time steps ahead up to period $N-1$, as given by $a^-_{N-k+j} = \widetilde{a}_{N-k+j} + \omega^-_{N-k+j}$ and $a^+_{N-k+j} = \widetilde{a}_{N-k+j} + \omega^+_{N-k+j}$ $(j = 1, \ldots, k-1)$.

Using the estimates of the optimal value functions as provided by Equations (2.126) to (2.128), it is easily verified that they are continuous across the buy and sell boundaries at $a_{N-k} = a^-_{N-k}$ and $a_{N-k} = a^+_{N-k}$ respectively. In addition, the first derivative of the optimal value function is also continuous across the boundaries as seen in Equations (2.57) and (2.58). Applying this condition allows one to obtain estimates of the optimal boundaries.

1. At the buy boundary $a_{N-k} = a^-_{N-k}$, the condition $\dfrac{\partial V^{(B)}_{N-k}}{\partial a_{N-k}} = \dfrac{\partial V^{(N)}_{N-k}}{\partial a_{N-k}}$ implies that, up to $O(\varepsilon)$,

$$\omega^-_{N-k} = \frac{-\kappa r_{N-1} \cdots r_{N-k} \lambda_{N-k} \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1} \cdots r_{N-k+1} \widetilde{F}_{N-k}}\right] - \alpha_{N-k}}{2\beta_{N-k}} + O(\varepsilon^2). \tag{2.129}$$

2. At the sell boundary $a_{N-k} = a^+_{N-k}$, the condition $\dfrac{\partial V^{(S)}_{N-k}}{\partial a_{N-k}} = \dfrac{\partial V^{(N)}_{N-k}}{\partial a_{N-k}}$ implies that, up to $O(\varepsilon)$,

$$\omega^+_{N-k} = \frac{\kappa r_{N-1} \cdots r_{N-k} \mu_{N-k} \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1} \cdots r_{N-k+1} \widetilde{F}_{N-k}}\right] - \alpha_{N-k}}{2\beta_{N-k}} + O(\varepsilon^2). \tag{2.130}$$

Recall that $\widetilde{F}_{N-k}$, α_{N-k} and β_{N-k} are defined in Equations (2.71), (2.114) and (2.115) respectively. It is noted that the leading order terms of ω^-_{N-k} and ω^+_{N-k} are linear in transaction costs, which is self-consistent with the initial assumption that they are of $O(\varepsilon)$. Therefore, the optimal buy and sell boundaries are estimated by $a^-_{N-k} = \widetilde{a}_{N-k} + \omega^-_{N-k}$ and $a^+_{N-k} = \widetilde{a}_{N-k} + \omega^+_{N-k}$.

These approximations at time period $N-k$ depend on the time steps ahead only via the terms $r_{N-1}, \ldots, r_{N-k+1}, \lambda_{N-k+1}, \mu_{N-k+1}$ and $\widetilde{a}_{N-k+1}$, which are easily determined. As can be seen from the expressions above for ω^-_{N-k} and ω^+_{N-k}, and the expressions for α_{N-k} and β_{N-k}, one requires the solutions $\widetilde{a}_{N-k+1}$ and $\widetilde{a}_{N-k}$ of Equation (2.70) for the zero transaction costs case. This is the partially myopic case and each of these solutions satisfies its own version of Equation (2.70) depending on the probability distribution appropriate to that time step in the discrete process. Therefore, the results for ω^-_{N-k} and ω^+_{N-k} depend on $p(s_{N-k+1})$ to determine $\widetilde{a}_{N-k+1}$, $p(s_{N-k})$ to determine $\widetilde{a}_{N-k}$, α_{N-k} from Equation (2.114) and β_{N-k} from Equation (2.115). Having obtained estimates of the optimal buy and sell boundaries, one can therefore apply these estimates to approximate the optimal value function at any time step.

It has been shown that, in the limit of small transaction costs, perturbation analysis is successfully applied to obtain explicit approximations of the optimal value function and optimal boundaries *at any time step* in the rebalancing of the portfolio. Furthermore, these approximations are valid for a general class of underlying probability distributions for the returns of the asset prices.

2.5 Results

A simple example is considered, where the returns of the risky asset are assumed to be independent and identical Bernoulli distributions and the returns of the risk-free asset are constant at each time period, that is,

$$s_{N-k} = \begin{cases} u & \text{with probability } q, \\ d & \text{with probability } 1-q, \end{cases} \tag{2.131}$$

and

$$r_{N-k} = r. \tag{2.132}$$

The probability density function of s_{N-k} can be written as

$$p(s_{N-k}) = q\delta(s_{N-k} - u) + (1-q)\delta(s_{N-k} - d), \tag{2.133}$$

where $\delta(.)$ is the Dirac delta function. In addition, assume that the costs of buying and selling are equal and constant at each time period, i.e.

$$\lambda_{N-k} = \mu_{N-k} = \lambda. \tag{2.134}$$

Suppose that the values of the underlying parameters are chosen to be

$$N = 4,\ \kappa = 0.1,\ u = 1.5,\ d = 0.5,\ q = 0.7,\ r = 1.05$$

and λ is allowed to vary from 0 to 0.02.

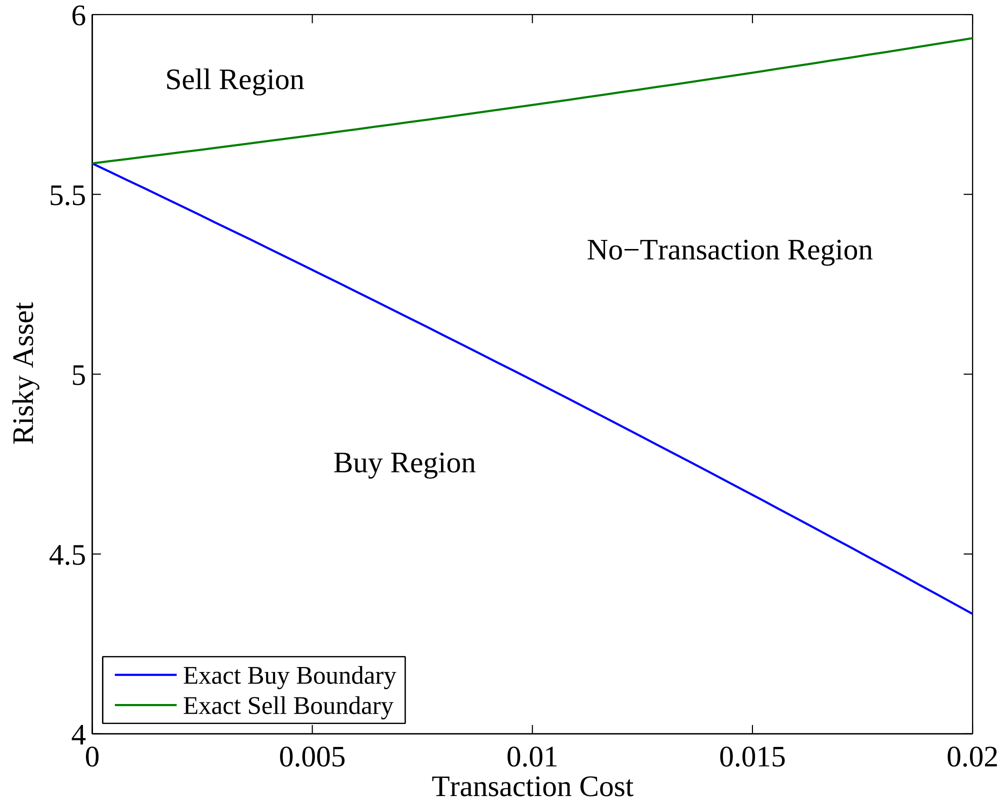


Figure 2.1: Exact Optimal Boundaries

Fig. 2.1 shows the evolution of the buy, sell and no-transaction regions with respect to transaction cost at the initial time. The horizontal axis represents the level of transaction cost, while the vertical axis represents the value of the risky asset. The lines that delineate the buy, sell and no-transaction regions are the exact optimal boundaries, which are obtained by applying the dynamic programming algorithm as described in Section 2.2. When transaction cost is zero, the optimal buy and sell boundaries converge to the Merton point $\widetilde{a}_0 = 5.59$. In this case, the investor's optimal strategy is to transact to reach this point. When transaction cost is non-zero, there exists a region where it is optimal not to transact in the risky asset. As expected, the no-transaction region increases as the level of transaction cost increases.

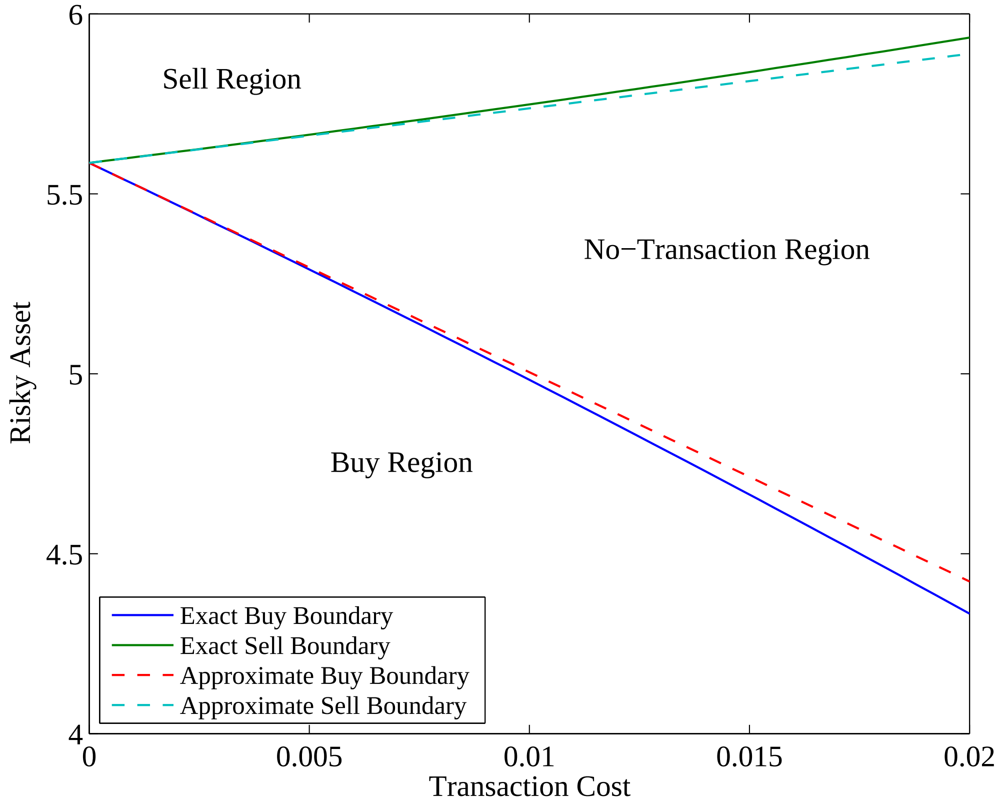


Figure 2.2: Comparison of Exact and Approximate Optimal Boundaries

Fig. 2.2 shows a comparison of the exact boundaries and the approximate boundaries as depicted by the dashed lines. The approximate boundaries are linear perturbations, given by Equations (2.129) and (2.130), about the Merton point $\widetilde{a}_0$. In the limit of small transaction costs, it can be seen in this example that the approximate buy and sell boundaries are good estimates of the exact boundaries. It is also observed that the absolute error between the exact and approximate boundaries decreases with decreasing transaction cost.

Chapter 3

Portfolio Optimisation with Transaction Costs and Power Utility Function

In the previous chapter, we studied the discrete time portfolio selection problem with proportional transaction costs. We considered the case of an investor with the exponential utility function (i.e. constant absolute risk aversion), which resulted in optimal buy and sell boundaries that were independent of wealth. However in practice, one would expect the optimal boundaries to vary with the wealth of the investor. In this chapter, we assume the more realistic case of the power utility function and carry out a perturbation analysis (in the limit of small transaction costs) to an arbitrary number of time steps. We present a method for explicitly constructing the approximations of both the optimal value function and optimal boundaries of the no-transaction region.

3.1 Market Model

Recall the market model that was described in Section 2.1. Consider a multi-period portfolio selection model with N periods. An investor holds a portfolio that is divided between one risk-free asset (i.e. bond) and one risky asset (i.e. stock), where the price of each asset evolves in discrete time. The investor is assumed to have a risk preference of the constant relative risk aversion class (i.e. power utility function). A cost that is proportional to the value of the transaction is incurred each time the investor buys or sells the risky asset. The investor's objective is to maximise the expected utility of terminal wealth by rebalancing the portfolio optimally at each step of the investment process.

At time period n, let W_n denote the wealth of the portfolio and let a_n be the dollar value of the risky asset inherited from the previous period. Therefore, the corresponding value of the risk-free asset is $W_n - a_n$. The investor rebalances the portfolio at time period n by buying l_n or selling m_n dollars of the risky asset. Suppose that s_n denotes one plus the (random) return of the risky asset from time period n to $n+1$. Thus, the value of the risky asset at time period $n+1$ inherited from period n is

$$a_{n+1} = s_n\left(a_n + l_n - m_n\right). \tag{3.1}$$

Furthermore, let λ_n and μ_n be the proportion costs of buying and selling the risky asset respectively at time period n. These costs reduce the wealth invested in the risk-free asset, resulting in a value of $W_n - (a_n + l_n - m_n) - \lambda_n l_n - \mu_n m_n$. Suppose that r_n denotes one plus the (sure) return of the risk-free asset from time period n to $n+1$. The investor's wealth at time period $n+1$ is then given by

$$W_{n+1} = r_n W_n + (s_n - r_n)\left(a_n + l_n - m_n\right) - r_n\lambda_n l_n - r_n\mu_n m_n. \tag{3.2}$$

Assume that simultaneous buying and selling of the risky asset is not allowed, since it will not be optimal due to the higher costs as compared to only buying or selling the asset. The investor is thus left with three possible choices, which is to buy, to sell or not to transact the risky asset. The investor's decision at each time step will affect the wealth and risky asset inherited at the next time step.

Suppose that the investor has a risk preference of the power utility type. Let the investor's utility of wealth be

$$U(W) = \frac{1}{\gamma}W^{\gamma}, \tag{3.3}$$

where $\gamma < 1$ and $\gamma \neq 0$. In this case, it is convenient to parametrise the problem by expressing the original variables in terms of fractions of wealth. Introduce the variables $A_n = a_n/W_n$ (i.e. fraction of wealth held in the risky asset), $L_n = l_n/W_n$ (i.e. fraction of wealth in buying the risky asset) and $M_n = m_n/W_n$ (i.e. fraction of wealth in selling the risky asset). Using these variables, Equations (3.1) and (3.2) can be rewritten as

$$A_{n+1} = \frac{s_n \left(A_n + L_n - M_n\right)}{F_n} \tag{3.4}$$

and

$$W_{n+1} = W_n F_n, \tag{3.5}$$

respectively, where

$$F_n = r_n + \left(s_n - r_n\right)\left(A_n + L_n - M_n\right) - r_n \lambda_n L_n - r_n \mu_n M_n. \tag{3.6}$$

Assume that $0 < A_n + L_n - M_n < 1$ and $0 < 1 - A_n - \left(1 + \lambda_n\right) L_n + \left(1 - \mu_n\right) M_n < 1$, which ensure that the investor's wealth $W_n > 0$.

The investor's objective is to maximise the expected utility of terminal wealth W_N given an initial wealth W_0 and initial proportion of risky asset A_0, by choosing the optimal strategy at each step of the investment process. The optimal value function at time period n ($n = 0, \ldots, N-1$) is defined to be

$$J_n(W_n, A_n) = \max \mathbb{E}\left[U(W_N)\right], \tag{3.7}$$

where $\mathbb{E}$ is the conditional expectation operator taken with respect to the random variables $s_n, \ldots, s_{N-1}$ given W_n and A_n. The maximisation is over the sequence of investments $(L_n, M_n), \ldots, (L_{N-1}, M_{N-1})$ in the risky asset. In order to obtain the investor's optimal value function $J_0(W_0, A_0)$ and corresponding optimal strategy given an initial wealth W_0 and initial proportion of risky asset A_0, we first simplify the problem by applying the principle of dynamic programming.

3.2 Dynamic Programming

The dynamic programming algorithm for the problem, which starts at period $N-1$ and proceeds recursively backwards in time, is given by

$$J_N(W_N, A_N) = U(W_N) \tag{3.8}$$

and

$$J_{N-k}(W_{N-k}, A_{N-k}) = \max \mathbb{E}_{N-k} \left[J_{N-k+1}(W_{N-k+1}, A_{N-k+1}) \right] \tag{3.9}$$

for $k = 1, \ldots, N$. At time period $N - k$, $\mathbb{E}_{N-k}$ is the conditional expectation operator with respect to the random variable s_{N-k} given W_{N-k} and A_{N-k}, while the maximisation is over the investment (L_{N-k}, M_{N-k}) in the risky asset. A detailed description of the construction of the optimal portfolio via dynamic programming can be found in Atkinson and Storey (2010). We shall only present the relevant results of this construction that will be subsequently required for our perturbation analysis in the limit of small transaction costs.

Time Period $N - 1$

This is a special case as it is one step before termination of the investment process, which means that there is no further rebalancing opportunity for the investor. Using Equation (3.5), the value function at time period $N - 1$ is written as

$$J_{N-1}(W_{N-1}, A_{N-1}) = \max \mathbb{E}_{N-1} \left[\frac{1}{\gamma} W_N^\gamma \right] = W_{N-1}^\gamma \max V_{N-1}(A_{N-1}), \tag{3.10}$$

where

$$V_{N-1}(A_{N-1}) = \mathbb{E}_{N-1} \left[\frac{1}{\gamma} F_{N-1}^\gamma \right]. \tag{3.11}$$

In Equation (3.10), W_{N-1} is taken out of $\mathbb{E}_{N-1}$, which is the expectation operator conditional on W_{N-1} and A_{N-1}. In addition, W_{N-1} does not depend on the investor's decision at time period $N - 1$. Therefore, the problem is reduced to one of maximising $V_{N-1}(A_{N-1})$ with respect to the investment strategy (L_{N-1}, M_{N-1}). The investor's choice to buy (i.e. $M_{N-1} = 0$), to sell (i.e. $L_{N-1} = 0$) or not to transact (i.e. $M_{N-1} = 0$ and $L_{N-1} = 0$) in the risky asset affects the definition of F_{N-1} as given in Equation (3.6).

The problem is one of finding the optimal no-transaction region delineated by $A_{N-1}^- \leq A_{N-1} \leq A_{N-1}^+$, where A_{N-1}^- and A_{N-1}^+ are the optimal buy and sell boundaries respectively. The region to the left of A_{N-1}^- is the optimal buy region while the region to the right of A_{N-1}^+ is the optimal sell region. The optimal boundaries A_{N-1}^- and A_{N-1}^+ are given by the first order optimality conditions

$$\frac{\partial V_{N-1}}{\partial L_{N-1}} = \mathbb{E}_{N-1}\left[\left\{s_{N-1} - (1+\lambda_{N-1})\, r_{N-1}\right\} F_{N-1}^{\gamma-1}\right] = 0 \tag{3.12}$$

and

$$\frac{\partial V_{N-1}}{\partial M_{N-1}} = \mathbb{E}_{N-1}\left[\left\{(1-\mu_{N-1})\, r_{N-1} - s_{N-1}\right\} F_{N-1}^{\gamma-1}\right] = 0, \tag{3.13}$$

respectively. In Equations (3.12) and (3.13), note that $F_{N-1} = r_{N-1} + (s_{N-1} - r_{N-1})\, A_{N-1}$ because we have $A_{N-1} = A_{N-1}^{-}$, $L_{N-1} = 0$ on the buy boundary and $A_{N-1} = A_{N-1}^{+}$, $M_{N-1} = 0$ on the sell boundary. Furthermore, it can be shown that the second order conditions $\frac{\partial^2 V_{N-1}}{\partial L_{N-1}^2} < 0$ and $\frac{\partial^2 V_{N-1}}{\partial M_{N-1}^2} < 0$ are satisfied, which ensure that these boundaries are optimal. In general, one will solve for A_{N-1}^{-} and A_{N-1}^{+} numerically.

Having determined the optimal boundaries A_{N-1}^{-} and A_{N-1}^{+}, the investor's optimal strategy and value function are as follows:

1. In the buy region $A_{N-1} < A_{N-1}^{-}$, the investor's optimal strategy is to buy $L_{N-1} = A_{N-1}^{-} - A_{N-1}$ of the risky asset to reach the optimal buy boundary A_{N-1}^{-}. The corresponding optimal value function V_{N-1} and its first derivative $\frac{\partial V_{N-1}}{\partial A_{N-1}}$ are thus given by

$$V_{N-1}^{(B)} = \mathbb{E}_{N-1}\left[\frac{1}{\gamma} F_{N-1}^{(B)\gamma}\right] \tag{3.14}$$

and

$$\frac{\partial V_{N-1}^{(B)}}{\partial A_{N-1}} = \mathbb{E}_{N-1}\left[r_{N-1}\lambda_{N-1} F_{N-1}^{(B)\gamma-1}\right], \tag{3.15}$$

where

$$F_{N-1}^{(B)} = r_{N-1} + (s_{N-1} - r_{N-1})\, A_{N-1}^{-} - r_{N-1}\lambda_{N-1}\left(A_{N-1}^{-} - A_{N-1}\right). \tag{3.16}$$

2. In the sell region $A_{N-1} > A_{N-1}^{+}$, the investor sells $M_{N-1} = A_{N-1} - A_{N-1}^{+}$ of the risky asset to reach the optimal sell boundary, so that V_{N-1} and $\frac{\partial V_{N-1}}{\partial A_{N-1}}$ are given by

$$V_{N-1}^{(S)} = \mathbb{E}_{N-1}\left[\frac{1}{\gamma} F_{N-1}^{(S)\gamma}\right] \tag{3.17}$$

and

$$\frac{\partial V_{N-1}^{(S)}}{\partial A_{N-1}} = -\mathbb{E}_{N-1}\left[r_{N-1}\mu_{N-1}F_{N-1}^{(S)\gamma-1}\right], \tag{3.18}$$

where

$$F_{N-1}^{(S)} = r_{N-1} + (s_{N-1} - r_{N-1})\, A_{N-1}^{+} - r_{N-1}\mu_{N-1}\left(A_{N-1} - A_{N-1}^{+}\right). \tag{3.19}$$

3. In the no-transaction region $A_{N-1}^{-} \leq A_{N-1} \leq A_{N-1}^{+}$, where $L_{N-1} = 0$ and $M_{N-1} = 0$ as the investor does not trade in the risky asset, V_{N-1} and $\dfrac{\partial V_{N-1}}{\partial A_{N-1}}$ are given by

$$V_{N-1}^{(N)} = \mathbb{E}_{N-1}\left[\frac{1}{\gamma}F_{N-1}^{(N)\gamma}\right] \tag{3.20}$$

and

$$\frac{\partial V_{N-1}^{(N)}}{\partial A_{N-1}} = \mathbb{E}_{N-1}\left[\left\{s_{N-1} - r_{N-1}\right\} F_{N-1}^{(N)\gamma-1}\right], \tag{3.21}$$

where

$$F_{N-1}^{(N)} = r_{N-1} + (s_{N-1} - r_{N-1})\, A_{N-1}. \tag{3.22}$$

It is noted that V_{N-1} and $\dfrac{\partial V_{N-1}}{\partial A_{N-1}}$ are continuous across the optimal buy and sell boundaries. At the optimal boundaries $A_{N-1} = A_{N-1}^{-}$ and $A_{N-1} = A_{N-1}^{+}$, continuity of the former is simply observed from Equations (3.14), (3.17) and (3.20), while the latter is a direct consequence of the first order optimality conditions (Equations (3.12) and (3.13)). We have used the superscripts "B", "S" and "N" to denote the buy, sell and no-transaction regions respectively.

Time Period $N-k$

Applying the dynamic programming algorithm recursively backwards in time allows one to construct the optimal strategy and value function at period $N-k$ ($k = 2, \ldots, N$). Using

Equation (3.5), the optimal value function given by Equation (3.9) is expressed as

$$\begin{aligned} J_{N-k}(W_{N-k}, A_{N-k}) &= \max \mathbb{E}_{N-k}\left[W_{N-k+1}^{\gamma} V_{N-k+1}(A_{N-k+1})\right] \\ &= W_{N-k}^{\gamma} \max V_{N-k}(A_{N-k}), \end{aligned} \tag{3.23}$$

where

$$V_{N-k}(A_{N-k}) = \mathbb{E}_{N-k}\left[F_{N-k}^{\gamma} V_{N-k+1}(A_{N-k+1})\right]. \tag{3.24}$$

The problem is effectively reduced to one of maximising $V_{N-k}(A_{N-k})$ with respect to L_{N-k} and M_{N-k}, assuming that $V_{N-k+1}(A_{N-k+1})$ is optimal by the principle of dynamic programming. The definitions of F_{N-k} from Equation (3.6) and A_{N-k+1} from Equation (3.4) depend on the investor's decision to buy (i.e. $M_{N-k} = 0$), to sell (i.e. $L_{N-k} = 0$) or not to transact (i.e. $L_{N-k} = 0$ and $M_{N-k} = 0$) the risky asset.

The optimal buy boundary $A_{N-k} = A_{N-k}^{-}$ and sell boundary $A_{N-k} = A_{N-k}^{+}$ satisfy the corresponding first order optimality conditions

$$\begin{aligned} \frac{\partial V_{N-k}}{\partial L_{N-k}} &= \mathbb{E}_{N-k}\Bigg[\gamma\left\{s_{N-k} - (1+\lambda_{N-k})\, r_{N-k}\right\} F_{N-k}^{\gamma-1} V_{N-k+1} \\ &\quad + s_{N-k} r_{N-k}\left(1+\lambda_{N-k} A_{N-k}\right) F_{N-k}^{\gamma-2} \frac{\partial V_{N-k+1}}{\partial A_{N-k+1}}\Bigg] = 0 \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \frac{\partial V_{N-k}}{\partial M_{N-k}} &= \mathbb{E}_{N-k}\Bigg[\gamma\left\{(1-\mu_{N-k})\, r_{N-k} - s_{N-k}\right\} F_{N-k}^{\gamma-1} V_{N-k+1} \\ &\quad - s_{N-k} r_{N-k}\left(1-\mu_{N-k} A_{N-k}\right) F_{N-k}^{\gamma-2} \frac{\partial V_{N-k+1}}{\partial A_{N-k+1}}\Bigg] = 0, \end{aligned} \tag{3.26}$$

respectively. Note that $F_{N-k} = r_{N-k} + (s_{N-k} - r_{N-k})\, A_{N-k}$ in Equations (3.25) and (3.26). Compared to the first order optimality conditions at time period $N-1$, there is an additional $\dfrac{\partial V_{N-k+1}}{\partial A_{N-k+1}}$ term due to the opportunities for the investor to rebalance the portfolio at the time steps ahead. Moreover, unlike the case where there are no transaction costs, the investor is not myopic and will take into account future rebalancing opportunities when he determines his current investment strategy. In general, one will obtain A_{N-k}^{-} and A_{N-k}^{+} by solving the dynamic programming algorithm numerically.

Having determined the optimal buy and sell boundaries, the investor's optimal strategy and value function are as follows:

In the *buy region* $A_{N-k} < A^-_{N-k}$, the investor's optimal strategy is to buy $L_{N-k} = A^-_{N-k} - A_{N-k}$ of the risky asset to reach the optimal buy boundary A^-_{N-k}. In this case, the optimal value function V_{N-k} and its first derivative $\dfrac{\partial V_{N-k}}{\partial A_{N-k}}$ are given by

$$V^{(B)}_{N-k} = \mathbb{E}_{N-k}\left[F^{(B)\gamma}_{N-k} V_{N-k+1}\right] \tag{3.27}$$

and

$$\begin{aligned}\frac{\partial V^{(B)}_{N-k}}{\partial A_{N-k}} &= \mathbb{E}_{N-k}\bigg[\gamma r_{N-k}\lambda_{N-k} F^{(B)\gamma-1}_{N-k} V_{N-k+1} \\ &\quad - s_{N-k} r_{N-k} \lambda_{N-k} A^-_{N-k} F^{(B)\gamma-2}_{N-k} \frac{\partial V_{N-k+1}}{\partial A_{N-k+1}}\bigg],\end{aligned} \tag{3.28}$$

where

$$F^{(B)}_{N-k} = r_{N-k} + (s_{N-k} - r_{N-k}) A^-_{N-k} - r_{N-k}\lambda_{N-k}\left(A^-_{N-k} - A_{N-k}\right). \tag{3.29}$$

Note that the optimal value function at the time step ahead V_{N-k+1} is a function of A_{N-k+1}. From Equation (3.4), we know that A_{N-k+1} depends on s_{N-k}, A_{N-k} and A^-_{N-k} via the equation $A_{N-k+1} = \dfrac{s_{N-k}A^-_{N-k}}{F^{(B)}_{N-k}}$. Define s^-_{N-k} and s^+_{N-k} as the values of s_{N-k} that correspond to the optimal buy boundary A^-_{N-k+1} and optimal sell boundary A^+_{N-k+1} at the time step ahead respectively. After rearranging the equations, they are explicitly given by

$$s^-_{N-k} = \frac{r_{N-k}A^-_{N-k+1}\left\{\left(1 - A^-_{N-k}\right) - \lambda_{N-k}\left(A^-_{N-k} - A_{N-k}\right)\right\}}{A^-_{N-k}\left(1 - A^-_{N-k+1}\right)} \tag{3.30}$$

and

$$s^+_{N-k} = \frac{r_{N-k}A^+_{N-k+1}\left\{\left(1 - A^-_{N-k}\right) - \lambda_{N-k}\left(A^-_{N-k} - A_{N-k}\right)\right\}}{A^-_{N-k}\left(1 - A^+_{N-k+1}\right)}. \tag{3.31}$$

Since the expectation operator $\mathbb{E}_{N-k}$ is taken with respect to the random variable s_{N-k}, Equation (3.27) can thus be written in its integral form delineated by s^-_{N-k} and s^+_{N-k}, to give

$$\begin{aligned} V_{N-k}^{(B)} &= \int_0^{s_{N-k}^-} F_{N-k}^{(B)\gamma} V_{N-k+1}^{(B)} p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &+ \int_{s_{N-k}^-}^{s_{N-k}^+} F_{N-k}^{(B)\gamma} V_{N-k+1}^{(N)} p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &+ \int_{s_{N-k}^+}^{\infty} F_{N-k}^{(B)\gamma} V_{N-k+1}^{(S)} p(s_{N-k})\,\mathrm{d}s_{N-k}, \end{aligned} \tag{3.32}$$

where $p(s_{N-k})$ is the probability density function of the random variable s_{N-k}. We will be using Equation (3.32) in the perturbation analysis that follows subsequently.

In the *sell region* $A_{N-k} > A_{N-k}^+$, the investor sells $M_{N-k} = A_{N-k} - A_{N-k}^+$ of the risky asset to reach the optimal sell boundary. Thus, V_{N-k} and $\dfrac{\partial V_{N-k}}{\partial A_{N-k}}$ are given by

$$V_{N-k}^{(S)} = \mathbb{E}_{N-k}\left[F_{N-k}^{(S)\gamma} V_{N-k+1}\right] \tag{3.33}$$

and

$$\begin{aligned} \frac{\partial V_{N-k}^{(S)}}{\partial A_{N-k}} &= \mathbb{E}_{N-k}\Bigg[-\gamma r_{N-k}\mu_{N-k} F_{N-k}^{(S)\gamma-1} V_{N-k+1} \\ &\quad + s_{N-k} r_{N-k}\mu_{N-k} A_{N-k}^+ F_{N-k}^{(S)\gamma-2} \frac{\partial V_{N-k+1}}{\partial A_{N-k+1}}\Bigg], \end{aligned} \tag{3.34}$$

where

$$F_{N-k}^{(S)} = r_{N-k} + (s_{N-k} - r_{N-k})\,A_{N-k}^+ - r_{N-k}\mu_{N-k}\left(A_{N-k} - A_{N-k}^+\right). \tag{3.35}$$

Similar to the buy region, A_{N-k+1} depends on s_{N-k}, A_{N-k} and A_{N-k}^+ via the equation $A_{N-k+1} = \dfrac{s_{N-k}A_{N-k}^+}{F_{N-k}^{(S)}}$. In this case, s_{N-k}^- (corresponding to A_{N-k+1}^-) and s_{N-k}^+ (corresponding to A_{N-k+1}^+) are explicitly given by

$$s_{N-k}^- = \frac{r_{N-k}A_{N-k+1}^-\left\{\left(1 - A_{N-k}^+\right) - \mu_{N-k}\left(A_{N-k} - A_{N-k}^+\right)\right\}}{A_{N-k}^+\left(1 - A_{N-k+1}^-\right)} \tag{3.36}$$

and

$$s_{N-k}^{+} = \frac{r_{N-k} A_{N-k+1}^{+} \left\{ \left(1 - A_{N-k}^{+}\right) - \mu_{N-k} \left(A_{N-k} - A_{N-k}^{+}\right) \right\}}{A_{N-k}^{+} \left(1 - A_{N-k+1}^{+}\right)}. \tag{3.37}$$

We can similarly express Equation (3.33) in its integral form delineated by s_{N-k}^{-} and s_{N-k}^{+} as given above.

In the *no-transaction region* $A_{N-k}^{-} \leq A_{N-k} \leq A_{N-k}^{+}$ where $L_{N-k} = 0$ and $M_{N-k} = 0$, the optimal value function V_{N-k} and its derivative $\dfrac{\partial V_{N-k}}{\partial A_{N-k}}$ are given by

$$V_{N-k}^{(N)} = \mathbb{E}_{N-k} \left[F_{N-k}^{(N)\gamma} V_{N-k+1} \right] \tag{3.38}$$

and

$$\begin{aligned} \frac{\partial V_{N-k}^{(N)}}{\partial A_{N-k}} \quad = \quad & \mathbb{E}_{N-k} \Bigg[\gamma \left(s_{N-k} - r_{N-k}\right) F_{N-k}^{(N)\gamma-1} V_{N-k+1} \\ & + s_{N-k} r_{N-k} F_{N-k}^{(N)\gamma-2} \frac{\partial V_{N-k+1}}{\partial A_{N-k+1}} \Bigg], \end{aligned} \tag{3.39}$$

where

$$F_{N-k}^{(N)} = r_{N-k} + \left(s_{N-k} - r_{N-k}\right) A_{N-k}. \tag{3.40}$$

Here, A_{N-k+1} depends on s_{N-k} and A_{N-k} via $A_{N-k+1} = \dfrac{s_{N-k} A_{N-k}}{F_{N-k}^{(N)}}$, which means that

$$s_{N-k}^{-} = \frac{r_{N-k} A_{N-k+1}^{-} \left(1 - A_{N-k}\right)}{A_{N-k} \left(1 - A_{N-k+1}^{-}\right)} \tag{3.41}$$

and

$$s_{N-k}^{+} = \frac{r_{N-k} A_{N-k+1}^{+} \left(1 - A_{N-k}\right)}{A_{N-k} \left(1 - A_{N-k+1}^{+}\right)}. \tag{3.42}$$

Similarly, we can write Equation (3.38) in its integral form delineated by s_{N-k}^{-} and s_{N-k}^{+} as given above.

It is noted that V_{N-k} and $\dfrac{\partial V_{N-k}}{\partial A_{N-k}}$ are continuous across the optimal boundaries. The former is observable from Equations (3.32), (3.33) and (3.38) while the latter is a direct consequence of the first order optimality conditions (Equations (3.25) and (3.26)). Generally, one will need to implement the dynamic programming algorithm recursively to obtain numerical solutions of the optimal boundaries A_{N-k}^{-} and A_{N-k}^{+} and the optimal value function $V_{N-k}(A_{N-k})$. However, this implementation is computationally intensive particularly when the number of time steps is large. Therefore, this has motivated us to investigate an alternative method of approximating the solutions of A_{N-k}^{-}, A_{N-k}^{+} and $V_{N-k}(A_{N-k})$. Moreover, in the case where there are no transaction costs, the solution to the portfolio selection problem is easily obtained as the optimal strategy is essentially myopic in nature. Coupled with the knowledge that transaction costs are small in practice, we therefore carry out a perturbation analysis of the small transaction costs case about the no transaction costs case. In addition to deriving more tractable approximations to the solutions, a perturbation analysis may also provide some qualitative insights to the nature of the solutions.

3.3 No Transaction Costs

Prior to the perturbation analysis in the limit of small transaction costs, we first consider the special case where there are no transaction costs incurred in buying or selling the risky asset. Setting $\lambda_{N-k} = 0 = \mu_{N-k}$ and repeating the construction of the optimal portfolio as seen in Section 3.2, we obtain the following results.

In general ($k = 1, \ldots, N$), the optimal buy boundary A_{N-k}^{-} and sell boundary A_{N-k}^{+} coincide to the same point, which is denoted by $\widetilde{A}_{N-k}$ and given by the first order condition

$$\mathbb{E}_{N-k}\left[(s_{N-k} - r_{N-k})\,\widetilde{F}_{N-k}^{\gamma-1}\right] = 0, \tag{3.43}$$

where

$$\widetilde{F}_{N-k} = r_{N-k} + (s_{N-k} - r_{N-k})\,\widetilde{A}_{N-k}. \tag{3.44}$$

This optimal point is commonly known as the Merton proportion (or point). Generally, one has to solve Equation (3.43) numerically for $\widetilde{A}_{N-k}$, which can be easily done by using standard root finding techniques. Moreover, for the case where the risky asset has a binomial price process, one will be able to obtain an explicit solution for $\widetilde{A}_{N-k}$. The investor's optimal strategy is thus to transact to the Merton proportion at each time step of the investment process. In addition, the optimal value function $\widetilde{V}_{N-k}$ is given by

$$\widetilde{V}_{N-k} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\widetilde{F}^{\gamma}_{N-1}\right]\cdots\mathbb{E}_{N-k+1}\left[\widetilde{F}^{\gamma}_{N-k+1}\right]\mathbb{E}_{N-k}\left[\widetilde{F}^{\gamma}_{N-k}\right]. \tag{3.45}$$

It is observed that the optimal value function at time period $N-k$ does not vary with the proportion A_{N-k} of risky asset inherited from the previous period, since there is no cost incurred in buying or selling the risky asset to reach the Merton proportion $\widetilde{A}_{N-k}$. The optimal strategy is a myopic one as the investor does not need to consider future rebalancing opportunities at the time steps ahead. If one further assumes that s_{N-k} are independent and identically distributed random variables and that r_{N-k} is a constant independent of k, then the Merton proportion $\widetilde{A}_{N-k}$ simplifies to a constant independent of k and the optimal value function becomes $\widetilde{V}_{N-k} = \frac{1}{\gamma}\left\{\mathbb{E}_{N-k}\left[\widetilde{F}^{\gamma}_{N-k}\right]\right\}^k$. The relatively simple solution of the Merton proportion and the optimal value function motivates one to carry out a perturbation analysis about the no transaction costs solution, in the limit of small transaction costs. Before proceeding further, it is useful to state the following results that

$$\mathbb{E}_{N-k}\left[\widetilde{F}^{\gamma}_{N-k}\right] = \mathbb{E}_{N-k}\left[r_{N-k}\widetilde{F}^{\gamma-1}_{N-k}\right] \tag{3.46}$$

and

$$\mathbb{E}_{N-k}\left[r_{N-k}\left(s_{N-k}-r_{N-k}\right)\widetilde{F}^{\gamma-2}_{N-k}\right] = -\mathbb{E}_{N-k}\left[\widetilde{A}_{N-k}\left(s_{N-k}-r_{N-k}\right)^2\widetilde{F}^{\gamma-2}_{N-k}\right]. \tag{3.47}$$

The above results are direct consequences of Equation (3.43) and will be used extensively to simplify the approximations of the value functions in the next section.

3.4 Small Transaction Costs

In practice, transaction costs are usually small compared to the value of the transactions. As observed in Section 3.3, the no transaction costs problem admits a relatively simple myopic solution. Therefore, one is motivated to analyse the small transaction costs solution as a perturbation about the no transaction costs solution. In Atkinson and Storey (2010), they obtained the leading order approximations to the optimal buy and sell boundaries for two time steps via the expansion of Equations (3.12), (3.13), (3.25) and (3.26) in the limit of small transaction costs. However, it was not obvious that a direct expansion of the first order conditions will enable one to easily obtain leading order approximations to an arbitrary number of time steps. In this section, we develop an approach to apply perturbation analysis about the no transaction costs solution in the limit of small transaction costs. The advantages of this approach over the direct expansion approach is that it allows one to systematically obtain approximations of both the optimal value function and optimal boundaries for an arbitrary number of time steps. A similar approach had been adopted in Atkinson and Quek (2012) for an investor with the exponential utility function. A feature of the exponential utility function was that it resulted in optimal boundaries that were independent of the investor's wealth, which is not usually the case in practice. A more realistic description of the investor's optimal strategy is provided by using the power utility function. Moreover, it is also more challenging to carry out the perturbation analysis in this context as the proportion of risky asset inherited at each time step depends on variations in both the return of the risky asset and the investor's wealth.

We start with approximating the optimal value function. In order to approximate the optimal value function for an arbitrary number of time steps, we adopt an approach that consists of two stages. The first stage involves making the assumption that the investor buys or sells to reach the Merton proportion at each time step when transaction costs are small. This is clearly a suboptimal strategy as the investor has ignored the presence of the no-transaction region. Consequently, an approximation of the suboptimal value function is derived at each time step. In the second stage, we assume that the investor behaves optimally by taking into account the no-transaction region. A sequence of corrections is then applied to the suboptimal value function to give us the desired approximation to the optimal value function. After approximating the optimal value function at each time step, the optimal boundaries are then estimated by imposing the condition that the first derivative of the value function is continuous across the boundaries.

Suppose that transaction costs are small such that $\lambda_{N-k} = \varepsilon\bar{\lambda}_{N-k}$ and $\mu_{N-k} = \varepsilon\bar{\mu}_{N-k}$ ($k = 1, \ldots, N$), where $\varepsilon \ll 1$, $\bar{\lambda}_{N-k} = O(1)$ and $\bar{\mu}_{N-k} = O(1)$. Here, $O(.)$ is the usual asymptotic order symbol so that $\bar{\lambda}_{N-k}$ and $\bar{\mu}_{N-k}$ are said to be "of the order" one (see Appendix A for its definition). Equivalently, λ_{N-k} and μ_{N-k} are said to be of the order ε.

We now apply a perturbation analysis in the following two stages.

3.4.1 Stage One: Transacting to the Merton Proportion

In the first stage, assume that the investor follows the suboptimal strategy of transacting to the Merton proportion. This is equivalent to assuming that both the optimal buy and sell boundaries are equal to the Merton proportion, which is suboptimal as we have effectively removed one of the investor's possible choices of not transacting in the risky asset.

In general, at time period $N-k$ ($k = 1, \ldots, N$), the investor is assumed to adopt the suboptimal strategy of buying L_{N-k} or selling M_{N-k} of the risky asset to reach the Merton proportion $\widetilde{A}_{N-k} = A^{-}_{N-k} = A^{+}_{N-k}$. Therefore, Equation (3.5) becomes

$$W_{N-k+1} = W_{N-k}\widehat{F}_{N-k}, \tag{3.48}$$

where

$$\widehat{F}_{N-k} = \widetilde{F}_{N-k} - \varepsilon\bar{\lambda}_{N-k}r_{N-k}L_{N-k} - \varepsilon\bar{\mu}_{N-k}r_{N-k}M_{N-k}. \tag{3.49}$$

Recall that $\widetilde{F}_{N-k} = r_{N-k} + (s_{N-k} - r_{N-k})\,\widetilde{A}_{N-k}$, from our analysis of the no transaction costs solution. The proportion of risky asset inherited in the next time step is now given by

$$A_{N-k+1} = \frac{s_{N-k}\widetilde{A}_{N-k}}{\widehat{F}_{N-k}}. \tag{3.50}$$

In particular, the investor's suboptimal strategy of transacting to the Merton proportion and corresponding value function in the buy and sell regions (compare with Section 3.2) are as follows:

In the *buy region* $A_{N-k} < \widetilde{A}_{N-k}$, the investor buys $L_{N-k} = \widetilde{A}_{N-k} - A_{N-k}$ of the risky asset so that

$$\widehat{F}^{(B)}_{N-k} = \widetilde{F}_{N-k} - \varepsilon\bar{\lambda}_{N-k}r_{N-k}\left(\widetilde{A}_{N-k} - A_{N-k}\right). \tag{3.51}$$

The value function now becomes

$$\widehat{V}^{(B)}_{N-1} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\widehat{F}^{(B)\gamma}_{N-1}\right] \tag{3.52}$$

at time period $N-1$, and

$$
\begin{aligned}
\widehat{V}_{N-k}^{(B)} &= \mathbb{E}_{N-k}\left[\widehat{F}_{N-k}^{(B)\gamma}\widehat{V}_{N-k+1}\right] \\
&= \int_0^{\widehat{s}_{N-k}} \widehat{F}_{N-k}^{(B)\gamma}\widehat{V}_{N-k+1}^{(B)} p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{\widehat{s}_{N-k}}^{\infty} \widehat{F}_{N-k}^{(B)\gamma}\widehat{V}_{N-k+1}^{(S)} p(s_{N-k})\,\mathrm{d}s_{N-k},
\end{aligned} \tag{3.53}
$$

at time period $N-k$ ($k=2,\ldots,N$). Here, $\widehat{s}_{N-k}$ is defined as the value of s_{N-k} that results in an inherited proportion of risky asset given by $\widetilde{A}_{N-k+1}$ (i.e. the Merton proportion). From Equation (3.50) with $s_{N-k}=\widehat{s}_{N-k}$ and $A_{N-k+1}=\widetilde{A}_{N-k+1}$, we have

$$
\widehat{s}_{N-k} = \frac{r_{N-k}\widetilde{A}_{N-k+1}\left\{\left(1-\widetilde{A}_{N-k}\right)-\varepsilon\bar{\lambda}_{N-k}\left(\widetilde{A}_{N-k}-A_{N-k}\right)\right\}}{\widetilde{A}_{N-k}\left(1-\widetilde{A}_{N-k+1}\right)}. \tag{3.54}
$$

It is convenient to define

$$
\widetilde{s}_{N-k} = \frac{r_{N-k}\widetilde{A}_{N-k+1}\left(1-\widetilde{A}_{N-k}\right)}{\widetilde{A}_{N-k}\left(1-\widetilde{A}_{N-k+1}\right)} \tag{3.55}
$$

and express

$$
\widehat{s}_{N-k} = \widetilde{s}_{N-k}\left\{1-\frac{\varepsilon\bar{\lambda}_{N-k}\left(\widetilde{A}_{N-k}-A_{N-k}\right)}{\left(1-\widetilde{A}_{N-k}\right)}\right\}. \tag{3.56}
$$

Therefore, note that $\widetilde{s}_{N-k}$ is the leading order term of $\widehat{s}_{N-k}$. It is observed that in the special case where s_{N-k} are independent and identically distributed random variables and r_{N-k} are constant in time, then $\widetilde{A}_{N-k}$ are constant in time and $\widetilde{s}_{N-k}$ reduces to r_{N-k}.

In the *sell region* $A_{N-k}>\widetilde{A}_{N-k}$, the investor sells $M_{N-k}=A_{N-k}-\widetilde{A}_{N-k}$ of the risky asset so that

$$
\widehat{F}_{N-k}^{(S)} = \widetilde{F}_{N-k} - \varepsilon\bar{\mu}_{N-k}r_{N-k}\left(A_{N-k}-\widetilde{A}_{N-k}\right). \tag{3.57}
$$

The value function now becomes

$$
\widehat{V}_{N-1}^{(S)} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\widehat{F}_{N-1}^{(S)\gamma}\right] \tag{3.58}
$$

at time period $N-1$, and

$$\begin{aligned}\widehat{V}^{(S)}_{N-k} &= \mathbb{E}_{N-k}\left[\widehat{F}^{(S)\gamma}_{N-k}\widehat{V}_{N-k+1}\right] \\ &= \int_0^{\widehat{s}_{N-k}} \widehat{F}^{(S)\gamma}_{N-k}\widehat{V}^{(B)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &\quad + \int_{\widehat{s}_{N-k}}^{\infty} \widehat{F}^{(S)\gamma}_{N-k}\widehat{V}^{(S)}_{N-k+1}p(s_{N-k})\,\mathrm{d}s_{N-k} \qquad (3.59)\end{aligned}$$

at time period $N-k$ ($k=2,\ldots,N$). In this case,

$$\widehat{s}_{N-k} = \widetilde{s}_{N-k}\left\{1 - \frac{\varepsilon\bar{\mu}_{N-k}\left(A_{N-k}-\widetilde{A}_{N-k}\right)}{\left(1-\widetilde{A}_{N-k}\right)}\right\}. \qquad (3.60)$$

It is of interest to note that the (suboptimal) value function in the buy and sell regions differ by only a change of the transaction cost variable from $\bar{\lambda}_{N-k}$ to $-\bar{\mu}_{N-k}$. Essentially, we will exploit this observation to deduce the approximation of the value function in the sell region from that in the buy region.

3.4.2 Stage One: Perturbation about the No Transaction Costs Solution

Since the parameter $\varepsilon \ll 1$ (in the limit of small transaction costs), we will derive an approximation of the suboptimal value function as a power series in terms of ε, starting from time period $N-1$ before proceeding to the general period $N-k$ case. We achieve this by perturbing the suboptimal value function about the no transaction costs solution. Recall that the no transaction costs solution is of a relatively simple form since it is characterised by a myopic investment strategy. This perturbation is carried out recursively, starting from period $N-1$ and proceeding backwards in time. In general, the perturbation at period $N-k$ depends on the perturbations from the time steps ahead. The exception is period $N-1$, since it is one step before termination of the investment process.

Time Period $N-1$

The value function in the buy region, from Equations (3.51) and (3.52), is given by

$$\widehat{V}_{N-1}^{(B)} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\left\{\widetilde{F}_{N-1} - \varepsilon\bar{\lambda}_{N-1}r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma}\right]. \qquad (3.61)$$

Expanding it as a power series in ε and using Equation (3.46), we approximate

$$\begin{aligned}\widehat{V}_{N-1}^{(B)} &= \widetilde{V}_{N-1}\bigg\{1 - \varepsilon\bar{\lambda}_{N-1}\gamma\left(\widetilde{A}_{N-1} - A_{N-1}\right) \\ &\quad + \frac{1}{2}\varepsilon^2\bar{\lambda}_{N-1}^2\alpha_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)^2\bigg\} + O(\varepsilon^3),\end{aligned} \qquad (3.62)$$

where

$$\alpha_{N-1} = \frac{\gamma\left(\gamma-1\right)r_{N-1}^2\mathbb{E}_{N-1}\left[\widetilde{F}_{N-1}^{\gamma-2}\right]}{\mathbb{E}_{N-1}\left[\widetilde{F}_{N-1}^{\gamma}\right]}. \qquad (3.63)$$

Note that the leading order term of the expansion is $\widetilde{V}_{N-1} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\widetilde{F}_{N-1}^{\gamma}\right]$, which we recall is the no transaction costs solution given by Equation (3.45). An analysis of the remainder term in the above expansion, which is shown to be bounded, can be found in Appendix C.1. The remainder terms for subsequent expansions will not be provided but are otherwise similar.

Similarly, the value function in the sell region is given by Equations (3.57) and (3.58), which differs from the value function in the buy region by a change of variable from $\bar{\lambda}_{N-1}$ to $-\bar{\mu}_{N-1}$. Therefore, it can be immediately deduced from Equation (3.62) that

$$\begin{aligned}\widehat{V}_{N-1}^{(S)} &= \widetilde{V}_{N-1}\bigg\{1 - \varepsilon\bar{\mu}_{N-1}\gamma\left(A_{N-1} - \widetilde{A}_{N-1}\right) \\ &\quad + \frac{1}{2}\varepsilon^2\bar{\mu}_{N-1}^2\alpha_{N-1}\left(A_{N-1} - \widetilde{A}_{N-1}\right)^2\bigg\} + O(\varepsilon^3).\end{aligned} \qquad (3.64)$$

Time Period $N-k$

Taking one step back to time period $N-2$, the value function in the buy region is given by Equation (3.53). Our aim is to delineate the integrals of $\widehat{V}_{N-2}^{(B)}$ by $\widetilde{s}_{N-2}$ rather than $\widehat{s}_{N-2}$, since $\widetilde{s}_{N-2}$ is the leading order term of $\widehat{s}_{N-2}$. Therefore, it is rewritten as

$$\begin{aligned}\widehat{V}^{(B)}_{N-2} &= \int_0^{\widetilde{s}_{N-2}} \widehat{F}^{(B)\gamma}_{N-2}\widehat{V}^{(B)}_{N-1}p(s_{N-2})\,\mathrm{d}s_{N-2} + \int_{\widetilde{s}_{N-2}}^{\infty} \widehat{F}^{(B)\gamma}_{N-2}\widehat{V}^{(S)}_{N-1}p(s_{N-2})\,\mathrm{d}s_{N-2} \\ &\quad + \int_{\widehat{s}_{N-2}}^{\widetilde{s}_{N-2}} \widehat{F}^{(B)\gamma}_{N-2}\left\{\widehat{V}^{(S)}_{N-1} - \widehat{V}^{(B)}_{N-1}\right\}p(s_{N-2})\,\mathrm{d}s_{N-2}, \end{aligned} \tag{3.65}$$

where $\widehat{s}_{N-2} = \widetilde{s}_{N-2}+O(\varepsilon)$ from Equation (3.56). Recall that $\widehat{V}^{(B)}_{N-1}$ and $\widehat{V}^{(S)}_{N-1}$ are functions of A_{N-1}, where $A_{N-1} = \dfrac{s_{N-2}\widetilde{A}_{N-2}}{\widehat{F}^{(B)}_{N-2}}$ from Equation (3.50). Also, when $s_{N-2} = \widehat{s}_{N-2}$, we have $A_{N-1} = \widetilde{A}_{N-1}$ by definition. We now derive an estimate of the third integral in the above equation. Applying the Mean Value Theorem for an integral,

$$\begin{aligned}&\int_{\widehat{s}_{N-2}}^{\widetilde{s}_{N-2}} \widehat{F}^{(B)\gamma}_{N-2}\left\{\widehat{V}^{(S)}_{N-1} - \widehat{V}^{(B)}_{N-1}\right\}p(s_{N-2})\,\mathrm{d}s_{N-2} \\ &= \left(\widetilde{s}_{N-2} - \widehat{s}_{N-2}\right)\widehat{F}^{(B)\gamma}_{N-2}\left\{\widehat{V}^{(S)}_{N-1} - \widehat{V}^{(B)}_{N-1}\right\}p(s_{N-2})\end{aligned} \tag{3.66}$$

evaluated at a point $s_{N-2} \in (\widehat{s}_{N-2}, \widetilde{s}_{N-2})$, that is, $s_{N-2} = \widehat{s}_{N-2} + O(\varepsilon)$. At this point, $A_{N-1} = \widetilde{A}_{N-1} + O(\varepsilon)$ as a value of s_{N-2} that is close to $\widehat{s}_{N-2}$ results in an inherited proportion of risky asset that is close to $\widetilde{A}_{N-1}$. This implies that

$$\widehat{V}^{(S)}_{N-1} - \widehat{V}^{(B)}_{N-1} = -\varepsilon\left(\bar{\mu}_{N-1} + \bar{\lambda}_{N-1}\right)\gamma\widetilde{V}_{N-1}\left(A_{N-1} - \widetilde{A}_{N-1}\right) + O(\varepsilon^2) \tag{3.67}$$

is of $O(\varepsilon^2)$. Since $\widetilde{s}_{N-2} - \widehat{s}_{N-2}$ is of $O(\varepsilon)$, we conclude that Equation (3.66) is of $O(\varepsilon^3)$.

Therefore, the value function in the buy region is approximated by

$$\widehat{V}^{(B)}_{N-2} = \int_0^{\widetilde{s}_{N-2}} \widehat{F}^{(B)\gamma}_{N-2}\widehat{V}^{(B)}_{N-1}p(s_{N-2})\,\mathrm{d}s_{N-2} + \int_{\widetilde{s}_{N-2}}^{\infty} \widehat{F}^{(B)\gamma}_{N-2}\widehat{V}^{(S)}_{N-1}p(s_{N-2})\,\mathrm{d}s_{N-2} + O(\varepsilon^3). \tag{3.68}$$

We now substitute the expressions for $\widehat{F}^{(B)}_{N-2}$, $\widehat{V}^{(B)}_{N-1}$ and $\widehat{V}^{(S)}_{N-1}$ from Equations (3.51), (3.62) and (3.64) into Equation (3.68). Expanding in powers of ε, simplifying with Equation (3.46) and collecting the common terms together, it can be shown after some algebra that $\widehat{V}^{(B)}_{N-2}$ is given by Equation (3.69) below. In order to approximate the value function in the sell region, recall that it differs from the value function in the buy region by a change of variable from $\bar{\lambda}_{N-2}$ to $-\bar{\mu}_{N-2}$. Therefore, with this change of variable, we can immediately deduce the approximation of $\widehat{V}^{(S)}_{N-2}$, which is given below by Equation (3.70).

Using Equations (3.53) and (3.59), the above analysis for the time period $N-2$ case is repeated recursively at period $N-3$, period $N-4$ and so on. In general, the suboptimal value function in the buy and sell regions at time period $N-k$ ($k=2,\ldots,N$) can be inductively shown to be approximated by

$$\begin{aligned}\widehat{V}^{(B)}_{N-k} &= \widetilde{V}_{N-k}\Bigg\{1-\varepsilon\bar{\lambda}_{N-k}\gamma\left(\widetilde{A}_{N-k}-A_{N-k}\right)+\frac{1}{2}\varepsilon^2\bar{\lambda}^2_{N-k}\alpha_{N-k}\left(\widetilde{A}_{N-k}-A_{N-k}\right)^2\\ &+\varepsilon^2\bar{\lambda}_{N-k}\left(\beta_{N-k}-\gamma\sum_{i=3}^{k}\zeta_{N-i+1}\right)\left(\widetilde{A}_{N-k}-A_{N-k}\right)+\varepsilon\sum_{i=2}^{k}\zeta_{N-i}\\ &+\varepsilon^2\left(\frac{1}{2}\sum_{i=2}^{k}\eta_{N-i}\alpha_{N-i+1}-\frac{1}{\gamma}\sum_{i=3}^{k}\zeta_{N-i}\beta_{N-i+1}+\sum_{i=4}^{k}\sum_{j=4}^{i}\zeta_{N-i}\zeta_{N-j+2}\right)\Bigg\}\\ &+O(\varepsilon^3),\end{aligned} \tag{3.69}$$

and with a change of variable from $\bar{\lambda}_{N-k}$ to $-\bar{\mu}_{N-k}$, by

$$\begin{aligned}\widehat{V}^{(S)}_{N-k} &= \widetilde{V}_{N-k}\Bigg\{1-\varepsilon\bar{\mu}_{N-k}\gamma\left(A_{N-k}-\widetilde{A}_{N-k}\right)+\frac{1}{2}\varepsilon^2\bar{\mu}^2_{N-k}\alpha_{N-k}\left(A_{N-k}-\widetilde{A}_{N-k}\right)^2\\ &+\varepsilon^2\bar{\mu}_{N-k}\left(\beta_{N-k}-\gamma\sum_{i=3}^{k}\zeta_{N-i+1}\right)\left(A_{N-k}-\widetilde{A}_{N-k}\right)+\varepsilon\sum_{i=2}^{k}\zeta_{N-i}\\ &+\varepsilon^2\left(\frac{1}{2}\sum_{i=2}^{k}\eta_{N-i}\alpha_{N-i+1}-\frac{1}{\gamma}\sum_{i=3}^{k}\zeta_{N-i}\beta_{N-i+1}+\sum_{i=4}^{k}\sum_{j=4}^{i}\zeta_{N-i}\zeta_{N-j+2}\right)\Bigg\}\\ &+O(\varepsilon^3).\end{aligned} \tag{3.70}$$

Note that in Equations (3.69) and (3.70), the summation terms are only valid in instances where the upper limit is at least as large as the lower limit. For example, when $k=2$, $\sum_{i=3}^{k}\zeta_{N-i+1}$ is not valid and assumed to be zero while $\sum_{i=2}^{k}\eta_{N-i}=\eta_{N-2}$. The corresponding definitions of α_{N-k}, β_{N-k}, ζ_{N-k} and η_{N-k} are as follows:

$$\alpha_{N-k} = \frac{\gamma(\gamma-1)r_{N-k}^2 \mathbb{E}_{N-k}\left[\widetilde{F}_{N-k}^{\gamma-2}\right]}{\mathbb{E}_{N-k}\left[\widetilde{F}_{N-k}^{\gamma}\right]}, \tag{3.71}$$

$$\begin{aligned}\beta_{N-k} \quad = \quad & \frac{\gamma r_{N-k}}{\mathbb{E}_{N-k}\left[\widetilde{F}_{N-k}^{\gamma}\right]} \\ & \times\Bigg\{\bar{\lambda}_{N-k+1}\int_0^{\widetilde{s}_{N-k}} \widetilde{F}_{N-k}^{\gamma-2}\left[s_{N-k}\widetilde{A}_{N-k} - \gamma\left(s_{N-k}\widetilde{A}_{N-k} - \widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)\right] \\ & \times p(s_{N-k})\,\mathrm{d}s_{N-k} \\ & -\bar{\mu}_{N-k+1}\int_{\widetilde{s}_{N-k}}^{\infty} \widetilde{F}_{N-k}^{\gamma-2}\left[s_{N-k}\widetilde{A}_{N-k} - \gamma\left(s_{N-k}\widetilde{A}_{N-k} - \widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)\right] \\ & \times p(s_{N-k})\,\mathrm{d}s_{N-k}\Bigg\},\end{aligned} \tag{3.72}$$

$$\begin{aligned}\zeta_{N-k} \quad = \quad & \frac{\gamma}{\mathbb{E}_{N-k}\left[\widetilde{F}_{N-k}^{\gamma}\right]} \\ & \times\Bigg\{\bar{\lambda}_{N-k+1}\int_0^{\widetilde{s}_{N-k}} \widetilde{F}_{N-k}^{\gamma-1}\left(s_{N-k}\widetilde{A}_{N-k} - \widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)p(s_{N-k})\,\mathrm{d}s_{N-k} \\ & -\bar{\mu}_{N-k+1}\int_{\widetilde{s}_{N-k}}^{\infty} \widetilde{F}_{N-k}^{\gamma-1}\left(s_{N-k}\widetilde{A}_{N-k} - \widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)p(s_{N-k})\,\mathrm{d}s_{N-k}\Bigg\},\end{aligned} \tag{3.73}$$

and

$$\begin{aligned}\eta_{N-k} \quad = \quad & \frac{1}{\mathbb{E}_{N-k}\left[\widetilde{F}_{N-k}^{\gamma}\right]} \\ & \times\Bigg\{\bar{\lambda}_{N-k+1}^2\int_0^{\widetilde{s}_{N-k}} \widetilde{F}_{N-k}^{\gamma-2}\left(s_{N-k}\widetilde{A}_{N-k} - \widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)^2 p(s_{N-k})\,\mathrm{d}s_{N-k} \\ & +\bar{\mu}_{N-k+1}^2\int_{\widetilde{s}_{N-k}}^{\infty} \widetilde{F}_{N-k}^{\gamma-2}\left(s_{N-k}\widetilde{A}_{N-k} - \widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)^2 p(s_{N-k})\,\mathrm{d}s_{N-k}\Bigg\}.\end{aligned} \tag{3.74}$$

In the first stage of the perturbation analysis, we have obtained approximations for the suboptimal value functions $\widehat{V}_{N-k}^{(B)}$ and $\widehat{V}_{N-k}^{(S)}$ in the buy and sell regions by ignoring the no-transaction region and perturbing about the no transaction costs solution. The main advantage is that we have derived the approximations, albeit suboptimal, at any time step of the investment process. However, we do not know how the no-transaction region will affect the terms in these approximations and the corrections that may be required. In the second stage of our analysis, we improve on these preliminary approximations by correcting them as we incorporate the no-transaction region.

3.4.3 Stage Two: Perturbation about the Suboptimal Value Function

In the second stage of the perturbation analysis, we reintroduce the no-transaction region in our approximation of the optimal value function. So instead of transacting to the Merton proportion at all times, the investor will buy to reach the optimal buy boundary when he falls in the buy region. Correspondingly, the investor will sell to reach the optimal sell boundary in the sell region and will choose not to trade in the no-transaction region. In the limit of small transaction costs, one would expect the optimal buy and sell boundaries to be close to the Merton proportion. Suppose that the optimal value function in the case of small transaction costs is a perturbation about the no transaction costs solution and assume that

$$A^-_{N-k} = \widetilde{A}_{N-k} + \varepsilon \omega^-_{N-k} \tag{3.75}$$

and

$$A^+_{N-k} = \widetilde{A}_{N-k} + \varepsilon \omega^+_{N-k}, \tag{3.76}$$

where ω^-_{N-k} and ω^+_{N-k} are of $O(1)$ and yet to be determined. Therefore, in the no-transaction region $A^-_{N-k} \leq A_{N-k} \leq A^+_{N-k}$, we have

$$A_{N-k} = \widetilde{A}_{N-k} + \varepsilon \omega_{N-k}, \tag{3.77}$$

where $\omega^-_{N-k} \leq \omega_{N-k} \leq \omega^+_{N-k}$. We subsequently demonstrate in our perturbation analysis that these assumptions are indeed self-consistent in this model.

At each time step, we first perturb the optimal value function about the suboptimal value function in the corresponding buy and sell regions, followed by a further perturbation about the no transaction costs solution. The aim is to correct the suboptimal value function for the terms that were left out when we assumed a strategy of transacting to the Merton proportion. Recall that for the optimal value function, $F^{(B)}_{N-k}$, $F^{(S)}_{N-k}$ and $F^{(N)}_{N-k}$ are given by Equations (3.29), (3.35) and (3.40) respectively. Since we are first perturbing the optimal value function about the suboptimal value function, using Equations (3.75), (3.76) and (3.77), they are rewritten as

$$F^{(B)}_{N-k} = \widehat{F}^{(B)}_{N-k} + \varepsilon\omega^{-}_{N-k}\left(s_{N-k} - r_{N-k}\right) - \varepsilon^2\bar{\lambda}_{N-k}\omega^{-}_{N-k}r_{N-k}, \tag{3.78}$$

$$F^{(S)}_{N-k} = \widehat{F}^{(S)}_{N-k} + \varepsilon\omega^{+}_{N-k}\left(s_{N-k} - r_{N-k}\right) + \varepsilon^2\bar{\mu}_{N-k}\omega^{+}_{N-k}r_{N-k} \tag{3.79}$$

and

$$F^{(N)}_{N-k} = \widetilde{F}_{N-k} + \varepsilon\omega_{N-k}\left(s_{N-k} - r_{N-k}\right). \tag{3.80}$$

Define the correction term in the buy and sell regions as

$$\delta^{(B)}_{N-k} = V^{(B)}_{N-k} - \widehat{V}^{(B)}_{N-k} \tag{3.81}$$

and

$$\delta^{(S)}_{N-k} = V^{(S)}_{N-k} - \widehat{V}^{(S)}_{N-k}, \tag{3.82}$$

respectively. After correcting for the optimal value function in the buy and sell regions, we proceed to obtain an approximation of the optimal value function $V^{(N)}_{N-k}$ in the no-transaction region, which will be a perturbation about the no transaction costs solution. Applying the dynamic programming principle, this sequence of corrections and approximations is achieved recursively backwards in time by using the estimates of the optimal value functions from the time steps ahead. As before, the analysis starts at time period $N-1$ before proceeding to the general period $N-k$ case.

Time Period $N-1$

In the *buy region*, recall that $\widehat{V}_{N-1}^{(B)}$ is given by Equation (3.52) and that $V_{N-1}^{(B)}$ is given by Equations (3.14) and (3.78) as

$$\widehat{V}_{N-1}^{(B)} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\widehat{F}_{N-1}^{(B)\gamma}\right] \tag{3.83}$$

and

$$V_{N-1}^{(B)} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\left\{\widehat{F}_{N-1}^{(B)} + \varepsilon\omega_{N-1}^{-}\left(s_{N-1} - r_{N-1}\right) - \varepsilon^2\bar{\lambda}_{N-1}\omega_{N-1}^{-}r_{N-1}\right\}^{\gamma}\right], \tag{3.84}$$

respectively. First, we perturb about the suboptimal solution as the correction term is given by $\delta_{N-1}^{(B)} = V_{N-1}^{(B)} - \widehat{V}_{N-1}^{(B)}$. Expanding Equation (3.84) in powers of ε up to $O(\varepsilon^2)$ and subtracting Equation (3.83), we obtain

$$\begin{aligned}\delta_{N-1}^{(B)} &= \frac{1}{\gamma}\mathbb{E}_{N-1}\Bigg[\gamma\widehat{F}_{N-1}^{(B)\gamma-1}\left\{\varepsilon\omega_{N-1}^{-}\left(s_{N-1} - r_{N-1}\right) - \varepsilon^2\bar{\lambda}_{N-1}\omega_{N-1}^{-}r_{N-1}\right\} \\ &\quad + \frac{1}{2}\gamma\left(\gamma-1\right)\widehat{F}_{N-1}^{(B)\gamma-2}\varepsilon^2\omega_{N-1}^{-2}\left(s_{N-1} - r_{N-1}\right)^2\Bigg] + O(\varepsilon^3).\end{aligned} \tag{3.85}$$

Next, we perturb about the no transaction costs solution. Substituting $\widehat{F}_{N-1}^{(B)} = \widetilde{F}_{N-1} - \varepsilon\bar{\lambda}_{N-1}r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)$ into Equation (3.85), expanding further in powers of ε and simplifying with Equations (3.43), (3.46) and (3.47), the correction term is found to be

$$\begin{aligned}\delta_{N-1}^{(B)} &= \widetilde{V}_{N-1}\varepsilon^2\Bigg\{\left[\bar{\lambda}_{N-1}\omega_{N-1}^{-}\widetilde{A}_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right) + \frac{1}{2}\omega_{N-1}^{-2}\right]\phi_{N-1} \\ &\quad - \bar{\lambda}_{N-1}\omega_{N-1}^{-}\gamma\Bigg\} + O(\varepsilon^3),\end{aligned} \tag{3.86}$$

where

$$\phi_{N-1} = \frac{\gamma\left(\gamma-1\right)\mathbb{E}_{N-1}\left[\left(s_{N-1} - r_{N-1}\right)^2\widetilde{F}_{N-1}^{\gamma-2}\right]}{\mathbb{E}_{N-1}\left[\widetilde{F}_{N-1}^{\gamma}\right]}. \tag{3.87}$$

In the *sell region*, note that $V_{N-1}^{(S)}$ is given by Equations (3.17) and (3.79) as

$$V_{N-1}^{(S)} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\left\{\widehat{F}_{N-1}^{(S)} + \varepsilon\omega_{N-1}^{+}\left(s_{N-1} - r_{N-1}\right) + \varepsilon^2\bar{\mu}_{N-1}\omega_{N-1}^{+}r_{N-1}\right\}^{\gamma}\right], \tag{3.88}$$

which is equivalent to $V_{N-1}^{(B)}$ with a change of variables from $\bar{\mu}_{N-1}$ to $-\bar{\lambda}_{N-1}$ and from ω_{N-1}^{+} to ω_{N-1}^{-}. Therefore, the correction term in the sell region is immediately deduced from Equation (3.86) to be

$$\begin{aligned}\delta_{N-1}^{(S)} \quad = \quad & \widetilde{V}_{N-1}\varepsilon^2\Bigg\{\left[\bar{\mu}_{N-1}\omega_{N-1}^{+}\widetilde{A}_{N-1}\left(A_{N-1} - \widetilde{A}_{N-1}\right) + \frac{1}{2}\omega_{N-1}^{+2}\right]\phi_{N-1} \\ & +\bar{\mu}_{N-1}\omega_{N-1}^{+}\gamma\Bigg\} + O(\varepsilon^3).\end{aligned} \tag{3.89}$$

In the *no-transaction region*, $V_{N-1}^{(N)}$ is given by Equation (3.20) and (3.80) as

$$V_{N-1}^{(N)} = \frac{1}{\gamma}\mathbb{E}_{N-1}\left[\left\{\widetilde{F}_{N-1} + \varepsilon\omega_{N-1}\left(s_{N-1} - r_{N-1}\right)\right\}^{\gamma}\right]. \tag{3.90}$$

Expanding in powers of ε up to $O(\varepsilon^2)$ and simplifying with Equation (3.43),

$$V_{N-1}^{(N)} = \widetilde{V}_{N-1}\left\{1 + \frac{1}{2}\varepsilon^2\omega_{N-1}^2\phi_{N-1}\right\} + O(\varepsilon^3). \tag{3.91}$$

Note that time period $N-1$ is a special case as it is one step before termination of the investment process. We now take one step back to time period $N-2$ and describe the analysis that is required, followed by the results for the general case.

Time Period $N-k$

Here, we describe the key ideas in deriving the correction and approximation of the optimal value function at time period $N-2$. The details of this perturbation analysis is provided in Appendix C.2. One of the key observations is that the analysis in the sell region differs from that in the buy region only by a suitable change of variables. This reduces the analysis that is required for the problem, since one can immediately deduce the result for the sell region from the buy region by a simple change of variables.

Therefore, we focus our attention to the perturbation analysis in the buy region. From Equation (3.32), $V_{N-2}^{(B)}$ is expressed in its integral form as a sum of integrals delineated by s_{N-2}^{-} and s_{N-2}^{+}. We note that $\widetilde{s}_{N-2}$ is the leading order term of s_{N-2}^{-} and s_{N-2}^{+}. Thus, our first objective is to rewrite $V_{N-2}^{(B)}$ as a sum of integrals delineated by $\widetilde{s}_{N-2}$, which is achieved by applying the mean value theorem for integrals. This makes it directly comparable with $\widehat{V}_{N-2}^{(B)}$ from Equation (3.68), which we have also expressed as a sum of integrals delineated by $\widetilde{s}_{N-2}$. In order to estimate the correction $\delta_{N-2}^{(B)} = V_{N-2}^{(B)} - \widehat{V}_{N-2}^{(B)}$, we first perturb the optimal value function $V_{N-2}^{(B)}$ about the suboptimal value function $\widehat{V}_{N-2}^{(B)}$ by taking their difference and using the approximation of the optimal value function from the time step ahead (i.e. period $N-1$ in this case). This is followed by a perturbation about the no transaction costs solution. After some long algebra and simplification, we will be able to derive the approximation of the correction $\delta_{N-2}^{(B)}$ in the buy region. The corresponding approximation of the correction term in the sell region $\delta_{N-2}^{(S)} = V_{N-2}^{(S)} - \widehat{V}_{N-2}^{(S)}$ can then be immediately deduced from the buy region by a change of variables from $\bar{\lambda}_{N-2}$ to $-\bar{\mu}_{N-2}$ and ω_{N-2}^{-} to ω_{N-2}^{+}.

After deriving the correction in the buy and sell regions, we proceed to estimate the optimal value function $V_{N-2}^{(N)}$ in the no-transaction region, which is given by Equation (3.38). Once again, by applying the mean value theorem for integrals, we can rewrite $V_{N-2}^{(N)}$ as a sum of integrals delineated by $\widetilde{s}_{N-2}$. Substituting in estimates of the optimal value function from the time step ahead, expanding in powers of ε and simplifying, we will be able to derive the approximation for the optimal value function in the no-transaction region.

Having approximated the optimal value function in the buy, sell and no-transaction regions at period $N-2$, we then proceed to the next time step. Applying the dynamic programming principle, the above analysis for the period $N-2$ case is repeated recursively backwards in time at period $N-3$ using the results from period $N-2$, at period $N-4$ using the results from period $N-3$ and so on. By induction, we are thus able to derive the results for the general period $N-k$ ($k = 2, \ldots, N$) case, which are stated below. The correction in the buy and sell regions are found to be given by

$$\begin{aligned}\delta_{N-k}^{(B)} &= \widetilde{V}_{N-k}\varepsilon^2\Bigg\{\left[\bar{\lambda}_{N-k}\omega_{N-k}^{-}\widetilde{A}_{N-k}\left(\widetilde{A}_{N-k}-A_{N-k}\right)+\frac{1}{2}\omega_{N-k}^{-2}\right]\phi_{N-k}\\ &\quad -\bar{\lambda}_{N-k}\omega_{N-k}^{-}\gamma+\omega_{N-k}^{-}\psi_{N-k}+\sum_{i=2}^{k}\theta_{N-i}\Bigg\}+O(\varepsilon^3)\end{aligned} \tag{3.92}$$

and

$$\begin{aligned}\delta_{N-k}^{(S)} &= \widetilde{V}_{N-k}\varepsilon^2\Bigg\{\left[\bar{\mu}_{N-k}\omega_{N-k}^{+}\widetilde{A}_{N-k}\left(A_{N-k}-\widetilde{A}_{N-k}\right)+\frac{1}{2}\omega_{N-k}^{+2}\right]\phi_{N-k}\\ &\quad +\bar{\mu}_{N-k}\omega_{N-k}^{+}\gamma+\omega_{N-k}^{+}\psi_{N-k}+\sum_{i=2}^{k}\theta_{N-i}\Bigg\}+O(\varepsilon^3).\end{aligned} \tag{3.93}$$

The optimal value function in the no-transaction region is given by

$$V_{N-k}^{(N)}=\widehat{V}_{N-k}^{(N)}+\delta_{N-k}^{(N)}, \tag{3.94}$$

where

$$\begin{aligned}\widehat{V}_{N-k}^{(N)} &= \widetilde{V}_{N-k}\Bigg\{1+\varepsilon^2\omega_{N-k}\psi_{N-k}+\frac{1}{2}\varepsilon^2\omega_{N-k}^2\phi_{N-k}+\varepsilon\sum_{i=2}^{k}\zeta_{N-i}\\ &\quad +\varepsilon^2\left(\frac{1}{2}\sum_{i=2}^{k}\eta_{N-i}\alpha_{N-i+1}-\frac{1}{\gamma}\sum_{i=3}^{k}\zeta_{N-i}\beta_{N-i+1}+\sum_{i=4}^{k}\sum_{j=4}^{i}\zeta_{N-i}\zeta_{N-j+2}\right)\Bigg\}\\ &\quad +O(\varepsilon^3)\end{aligned} \tag{3.95}$$

and

$$\delta_{N-k}^{(N)}=\widetilde{V}_{N-k}\varepsilon^2\sum_{i=2}^{k}\theta_{N-i}+O(\varepsilon^3). \tag{3.96}$$

Recall that α_{N-k}, β_{N-k}, ζ_{N-k} and η_{N-k} are previously defined in Equations (3.71) to (3.74). The definitions of ϕ_{N-k}, ψ_{N-k} and θ_{N-k} in the above equations are as follows:

$$\phi_{N-k}=\frac{\gamma(\gamma-1)\,\mathbb{E}_{N-k}\left[\left(s_{N-k}-r_{N-k}\right)^2\widetilde{F}_{N-k}^{\gamma-2}\right]}{\mathbb{E}_{N-k}\left[\widetilde{F}_{N-k}^{\gamma}\right]}, \tag{3.97}$$

$$\begin{aligned}
\psi_{N-k} &= \frac{\gamma}{\mathbb{E}_{N-k}\left[\widetilde{F}^{\gamma}_{N-k}\right]} \\
&\times\Big\{\bar{\lambda}_{N-k+1}\int_0^{\widetilde{s}_{N-k}} \widetilde{F}^{\gamma-2}_{N-k}\Big[s_{N-k}r_{N-k}+\gamma\left(s_{N-k}-r_{N-k}\right) \\
&\times\left(s_{N-k}\widetilde{A}_{N-k}-\widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)\Big]p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&-\bar{\mu}_{N-k+1}\int_{\widetilde{s}_{N-k}}^{\infty} \widetilde{F}^{\gamma-2}_{N-k}\Big[s_{N-k}r_{N-k}+\gamma\left(s_{N-k}-r_{N-k}\right) \\
&\times\left(s_{N-k}\widetilde{A}_{N-k}-\widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)\Big]p(s_{N-k})\,\mathrm{d}s_{N-k}\Big\},
\end{aligned} \tag{3.98}$$

and

$$\begin{aligned}
&\theta_{N-k} = \frac{1}{\mathbb{E}_{N-k}\left[\widetilde{F}^{\gamma}_{N-k}\right]} \\
&\times\Big\{\int_0^{\widetilde{s}_{N-k}}\Big[-\widetilde{F}^{\gamma-1}_{N-k}\bar{\lambda}_{N-k+1}\omega^{-}_{N-k+1}\widetilde{A}_{N-k+1}\left(s_{N-k}\widetilde{A}_{N-k}-\widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)\phi_{N-k+1} \\
&+\widetilde{F}^{\gamma}_{N-k}\left(\frac{1}{2}\omega^{-2}_{N-k+1}\phi_{N-k+1}-\bar{\lambda}_{N-k+1}\omega^{-}_{N-k+1}\gamma+\omega^{-}_{N-k+1}\psi_{N-k+1}\mathbb{I}_{\{k\neq 2\}}\right)\Big] \\
&\times p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&+\int_{\widetilde{s}_{N-k}}^{\infty}\Big[\widetilde{F}^{\gamma-1}_{N-k}\bar{\mu}_{N-k+1}\omega^{+}_{N-k+1}\widetilde{A}_{N-k+1}\left(s_{N-k}\widetilde{A}_{N-k}-\widetilde{F}_{N-k}\widetilde{A}_{N-k+1}\right)\phi_{N-k+1} \\
&+\widetilde{F}^{\gamma}_{N-k}\left(\frac{1}{2}\omega^{+2}_{N-k+1}\phi_{N-k+1}+\bar{\mu}_{N-k+1}\omega^{+}_{N-k+1}\gamma+\omega^{+}_{N-k+1}\psi_{N-k+1}\mathbb{I}_{\{k\neq 2\}}\right)\Big] \\
&\times p(s_{N-k})\,\mathrm{d}s_{N-k}\Big\},
\end{aligned} \tag{3.99}$$

where $\mathbb{I}_{\{k\neq 2\}}$ denotes an indicator function with respect to the index k.

In conclusion, we have devised a systematic method to approximate the optimal value function in the buy, sell and no-transaction regions at any time step. In this method, we initially assume that the investor adopts a suboptimal strategy of transacting to the Merton proportion in the limit of small transaction costs. The second order approximation of the suboptimal value function in the buy and sell regions is derived by perturbing about the no transaction costs solution. However, by ignoring the no-transaction region, we have missed out on some second order terms in our preliminary approximation. In order to correct our initial approximation in the buy and sell regions, we perturb the optimal value function about the suboptimal value function. This is followed by a perturbation about the no transaction costs solution. We also derive the approximation for the optimal value function in the no-transaction region by perturbing about the no transaction costs solution. This perturbation scheme is achieved by backwards recursion starting from time period $N-1$. Nonetheless, the optimal buy and sell boundaries are as yet unknown, which we determine in the next section.

3.4.4 Stage Two: Approximation of the Optimal Buy and Sell Boundaries

In Sections 3.4.1 to 3.4.3, we have derived the approximation of the optimal value function in the buy, sell and no-transaction regions at each time step. In this section, we verify that the optimal value function is continuous across the buy and sell boundaries. We further recall that the first derivative of the optimal value function should also be continuous across the optimal boundaries. An application of this condition allows us to derive estimates for the optimal buy and sell boundaries. In addition, the results that we obtain serve to demonstrate that our perturbation analysis is indeed self-consistent.

We start by considering the case at time period $N-1$. Collecting the previous results from Equations (3.62), (3.64), (3.86), (3.89) and (3.91), the optimal value function in the buy, sell and no-transaction regions at time period $N-1$ is given by $V_{N-1}^{(B)} = \widehat{V}_{N-1}^{(B)} + \delta_{N-1}^{(B)}$, $V_{N-1}^{(S)} = \widehat{V}_{N-1}^{(S)} + \delta_{N-1}^{(S)}$ and $V_{N-1}^{(N)}$ respectively. Recall that $A_{N-1} = \widetilde{A}_{N-1} + \varepsilon\omega_{N-1}$ in the no-transaction region. At the optimal buy boundary, $A_{N-1} = A_{N-1}^{-}$ and $\omega_{N-1} = \omega_{N-1}^{-}$. Thus, we can verify that

$$V_{N-1}^{(B)} = \widetilde{V}_{N-1}\left\{1 + \frac{1}{2}\varepsilon^2\omega_{N-1}^{-2}\phi_{N-1}\right\} + O(\varepsilon^3) = V_{N-1}^{(N)}. \tag{3.100}$$

Similarly, at the optimal sell boundary, where $A_{N-1} = A_{N-1}^{+}$ and $\omega_{N-1} = \omega_{N-1}^{+}$, we can also verify that $V_{N-1}^{(S)} = V_{N-1}^{(N)}$. Therefore, the optimal value function is continuous across the buy and sell boundaries up to $O(\varepsilon^2)$.

Note that we have obtained the approximation of the optimal value function up to $O(\varepsilon^2)$. This is so that we can match the first derivative of the optimal value function at the optimal buy and sell boundaries up to $O(\varepsilon)$. If we wish to obtain higher order estimates of the optimal boundaries, we will need to estimate the optimal value function beyond $O(\varepsilon^2)$. By matching the first derivative of the optimal value function at the buy and sell boundaries, we are able to derive the first order approximation of these boundaries. Therefore, in order to determine the optimal boundaries, differentiate $V^{(B)}_{N-1}$, $V^{(S)}_{N-1}$ and $V^{(N)}_{N-1}$ with respect to A_{N-1} to give

$$\begin{aligned}\frac{\partial V^{(B)}_{N-1}}{\partial A_{N-1}} &= \widetilde{V}_{N-1}\Big\{\varepsilon\bar{\lambda}_{N-1}\gamma - \varepsilon^2\bar{\lambda}^2_{N-1}\alpha_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right) \\ &\quad -\varepsilon^2\bar{\lambda}_{N-1}\omega^-_{N-1}\widetilde{A}_{N-1}\phi_{N-1}\Big\} + O(\varepsilon^3), \qquad (3.101)\end{aligned}$$

$$\begin{aligned}\frac{\partial V^{(S)}_{N-1}}{\partial A_{N-1}} &= \widetilde{V}_{N-1}\Big\{-\varepsilon\bar{\mu}_{N-1}\gamma + \varepsilon^2\bar{\mu}^2_{N-1}\alpha_{N-1}\left(A_{N-1} - \widetilde{A}_{N-1}\right) \\ &\quad +\varepsilon^2\bar{\mu}_{N-1}\omega^+_{N-1}\widetilde{A}_{N-1}\phi_{N-1}\Big\} + O(\varepsilon^3) \qquad (3.102)\end{aligned}$$

and

$$\frac{\partial V^{(N)}_{N-1}}{\partial A_{N-1}} = \widetilde{V}_{N-1}\varepsilon\omega_{N-1}\phi_{N-1} + O(\varepsilon^2). \qquad (3.103)$$

Note that the first derivative of the optimal value function in the no-transaction region is obtained by recalling that A_{N-1} and ω_{N-1} are related via $A_{N-1} = \widetilde{A}_{N-1} + \varepsilon\omega_{N-1}$.

At the buy boundary (i.e. $A_{N-1} = A^-_{N-1}$ and $\omega_{N-1} = \omega^-_{N-1}$), by equating the coefficients of ε for $\frac{\partial V^{(B)}_{N-1}}{\partial A_{N-1}}$ and $\frac{\partial V^{(N)}_{N-1}}{\partial A_{N-1}}$, we obtain

$$\omega^-_{N-1} = \frac{\bar{\lambda}_{N-1}\gamma}{\phi_{N-1}}. \qquad (3.104)$$

Similarly at the sell boundary (i.e. $A_{N-1} = A^+_{N-1}$ and $\omega_{N-1} = \omega^+_{N-1}$), by equating the coefficients of ε for $\frac{\partial V^{(S)}_{N-1}}{\partial A_{N-1}}$ and $\frac{\partial V^{(N)}_{N-1}}{\partial A_{N-1}}$, we obtain

$$\omega^+_{N-1} = \frac{-\bar{\mu}_{N-1}\gamma}{\phi_{N-1}}. \qquad (3.105)$$

Recall that ϕ_{N-1} is given by Equation (3.87). In addition, the corresponding buy and sell boundaries are estimated by $A^-_{N-1} = \widetilde{A}_{N-1} + \varepsilon\omega^-_{N-1}$ and $A^+_{N-1} = \widetilde{A}_{N-1} + \varepsilon\omega^+_{N-1}$ up to $O(\varepsilon)$.

For the general case at time period $N-k$ ($k = 2,\ldots,N$), collecting the results from Equations (3.69), (3.70), (3.92),(3.93),(3.95) and (3.96),the optimal value function in the three regions is given by $V^{(B)}_{N-k} = \widehat{V}^{(B)}_{N-k} + \delta^{(B)}_{N-k}$, $V^{(S)}_{N-k} = \widehat{V}^{(S)}_{N-k} + \delta^{(S)}_{N-k}$ and $V^{(N)}_{N-k} = \widehat{V}^{(N)}_{N-k} + \delta^{(N)}_{N-k}$. At the optimal buy boundary, $A_{N-k} = A^-_{N-k}$ and $\omega_{N-k} = \omega^-_{N-k}$. Thus, it can be verified that

$$\begin{aligned} V^{(B)}_{N-k} &= \widetilde{V}_{N-k}\Bigg\{1 + \varepsilon\sum_{i=2}^{k}\zeta_{N-i} + \varepsilon^2\sum_{i=2}^{k}\theta_{N-i} \\ &\quad +\varepsilon^2\left(\frac{1}{2}\sum_{i=2}^{k}\eta_{N-i}\alpha_{N-i+1} - \frac{1}{\gamma}\sum_{i=3}^{k}\zeta_{N-i}\beta_{N-i+1} + \sum_{i=4}^{k}\sum_{j=4}^{i}\zeta_{N-i}\zeta_{N-j+2}\right) \\ &\quad +\frac{1}{2}\varepsilon^2\omega^{-2}_{N-k}\phi_{N-k} + \varepsilon^2\omega^-_{N-k}\psi_{N-k}\Bigg\} + O(\varepsilon^3) = V^{(N)}_{N-k}. \end{aligned} \tag{3.106}$$

At the optimal sell boundary, $A_{N-k} = A^+_{N-k}$ and $\omega_{N-k} = \omega^+_{N-k}$. Similarly, it can also be verified that $V^{(S)}_{N-k} = V^{(N)}_{N-k}$ up to $O(\varepsilon^2)$.

In order to estimate the buy and sell boundaries, we differentiate $V^{(B)}_{N-k}$, $V^{(S)}_{N-k}$ and $V^{(N)}_{N-k}$ with respect to A_{N-k}, which give us

$$\begin{aligned} \frac{\partial V^{(B)}_{N-k}}{\partial A_{N-k}} &= \widetilde{V}_{N-k}\Bigg\{\varepsilon\bar{\lambda}_{N-k}\gamma - \varepsilon^2\bar{\lambda}^2_{N-k}\alpha_{N-k}\left(\widetilde{A}_{N-k} - A_{N-k}\right) \\ &\quad -\varepsilon^2\bar{\lambda}_{N-k}\left(\beta_{N-k} - \gamma\sum_{i=3}^{k}\zeta_{N-i+1}\right) - \varepsilon^2\bar{\lambda}_{N-k}\omega^-_{N-k}\widetilde{A}_{N-k}\phi_{N-k}\Bigg\} + O(\varepsilon^3), \end{aligned} \tag{3.107}$$

$$\begin{aligned} \frac{\partial V^{(S)}_{N-k}}{\partial A_{N-k}} &= \widetilde{V}_{N-k}\Bigg\{-\varepsilon\bar{\mu}_{N-k}\gamma + \varepsilon^2\bar{\mu}^2_{N-k}\alpha_{N-k}\left(A_{N-k} - \widetilde{A}_{N-k}\right) \\ &\quad +\varepsilon^2\bar{\mu}_{N-k}\left(\beta_{N-k} - \gamma\sum_{i=3}^{k}\zeta_{N-i+1}\right) + \varepsilon^2\bar{\mu}_{N-k}\omega^+_{N-k}\widetilde{A}_{N-k}\phi_{N-k}\Bigg\} + O(\varepsilon^3) \end{aligned} \tag{3.108}$$

and

$$\frac{\partial V^{(N)}_{N-k}}{\partial A_{N-k}} = \widetilde{V}_{N-k}\left\{\varepsilon\psi_{N-k} + \varepsilon\omega_{N-k}\phi_{N-k}\right\} + O(\varepsilon^2). \tag{3.109}$$

At the buy boundary (i.e. $A_{N-k} = A^-_{N-k}$ and $\omega_{N-k} = \omega^-_{N-k}$), equating the coefficients of ε for $\frac{\partial V^{(B)}_{N-k}}{\partial A_{N-k}}$ and $\frac{\partial V^{(N)}_{N-k}}{\partial A_{N-k}}$ gives us

$$\omega^-_{N-k} = \frac{\bar{\lambda}_{N-k}\gamma - \psi_{N-k}}{\phi_{N-k}}. \tag{3.110}$$

Similarly at the sell boundary (i.e. $A_{N-k} = A^+_{N-k}$ and $\omega_{N-k} = \omega^+_{N-k}$), equating the coefficients of ε for $\frac{\partial V^{(S)}_{N-k}}{\partial A_{N-k}}$ and $\frac{\partial V^{(N)}_{N-k}}{\partial A_{N-k}}$ gives us

$$\omega^+_{N-k} = \frac{-\bar{\mu}_{N-k}\gamma - \psi_{N-k}}{\phi_{N-k}}. \tag{3.111}$$

Recall that ϕ_{N-k} and ψ_{N-k} are defined in Equations (3.97) and (3.98) respectively.

The corresponding optimal buy and sell boundaries are thus given by $A^-_{N-k} = \widetilde{A}_{N-k} + \varepsilon\omega^-_{N-k}$ and $A^+_{N-k} = \widetilde{A}_{N-k} + \varepsilon\omega^+_{N-k}$ up to $O(\varepsilon)$. Therefore, in the limit of small transaction costs, we have obtained the first order approximation of the optimal boundaries at any time period $N-k$. Observe that ϕ_{N-k} is determined by the variables r_{N-k}, s_{N-k} and $\widetilde{A}_{N-k}$ at period $N-k$. The variable r_{N-k} is non-random while the random variable s_{N-k} is characterised by its probability density function $p(s_{N-k})$. Observe that in addition to the variables at period $N-k$, ψ_{N-k} also depends on the variables $\bar{\lambda}_{N-k+1}, \bar{\mu}_{N-k+1}$ and $\widetilde{A}_{N-k+1}$ at the time step ahead. Moreover, the Merton proportion $\widetilde{A}_{N-k}$ at any time period $N-k$ is determined by the specification of r_{N-k} and $p(s_{N-k})$ via Equation (3.43). Therefore, one concludes that the first order approximation of the optimal buy and sell boundaries at each time step essentially depends on the transaction costs, returns of the risk-free asset and returns of the risky asset at the current time step and one time step ahead.

In the next section, we make a few assumptions that allow us to simplify the expressions for the optimal boundaries and present a numerical example.

3.5 Results

In order to illustrate our main results with a numerical example, we make the following assumptions:

1. Assume that r_{N-k} is constant in time and say that $r_{N-k} = r$ for all k.

2. Assume that s_{N-k} are independent and identically distributed to the random variable s for all k.

3. Assume that the costs of buying and selling the risky assets are equal and constant in time and say that $\bar{\lambda}_{N-k} = \bar{\mu}_{N-k} = \bar{\lambda}$ for all k.

We focus on the results in the general case of time period $N - k$ ($k = 2, \ldots, N$) as the period $N - 1$ case is trivial. With these assumptions, we deduce from Equation (3.43) that the Merton proportion $\widetilde{A}_{N-k} = \widetilde{A}$ is a constant, where $\widetilde{A}$ satisfies the equation

$$\mathbb{E}\left[(s-r)\left\{r+(s-r)\widetilde{A}\right\}^{\gamma-1}\right] = 0. \tag{3.112}$$

Here, the expectation $\mathbb{E}$ is taken with respect to the random variable s. From Equation (3.55), we simplify the term $\widetilde{s}_{N-k} = r$. Furthermore, we simplify the expression $s_{N-k}\widetilde{A}_{N-k} - \widetilde{F}_{N-k}\widetilde{A}_{N-k+1} = (s-r)\widetilde{A}\left(1-\widetilde{A}\right)$ that appears in Equation (3.98). Therefore, we observe that $\phi_{N-k} = \phi$ and $\psi_{N-k} = \psi$ are constants, where

$$\phi = \frac{\gamma(\gamma-1)\,\mathbb{E}\left[(s-r)^2\left\{r+(s-r)\widetilde{A}\right\}^{\gamma-2}\right]}{\mathbb{E}\left[\left\{r+(s-r)\widetilde{A}\right\}^{\gamma}\right]} \tag{3.113}$$

and

$$\psi = \frac{\gamma\bar{\lambda}}{\mathbb{E}\left[\left\{r+(s-r)\widetilde{A}\right\}^{\gamma}\right]}$$
$$\times\left\{\int_0^r \left\{r+(s-r)\widetilde{A}\right\}^{\gamma-2}\left\{sr+\gamma(s-r)^2\widetilde{A}\left(1-\widetilde{A}\right)\right\}p(s)\,\mathrm{d}s\right.$$
$$\left.-\int_r^\infty \left\{r+(s-r)\widetilde{A}\right\}^{\gamma-2}\left\{sr+\gamma(s-r)^2\widetilde{A}\left(1-\widetilde{A}\right)\right\}p(s)\,\mathrm{d}s\right\}. \quad (3.114)$$

Consequently, the first order approximations of the optimal boundaries

$$A^-_{N-k} = \widetilde{A} + \varepsilon\frac{\bar{\lambda}\gamma - \psi}{\phi} \quad (3.115)$$

and

$$A^+_{N-k} = \widetilde{A} - \varepsilon\frac{\bar{\lambda}\gamma + \psi}{\phi} \quad (3.116)$$

are also constants over time. This is not a surprising observation as the Merton proportion, which corresponds to the no transaction costs case, is also a constant. Thus, in the limit of small transaction costs, one would expect the optimal buy and sell boundaries (being close to the Merton point) to be constants as well. However, note that these results only hold under the given assumptions in this example. One advantage of this portfolio selection model is that a general probability distribution is assumed for the return of the risky asset at each time step. In general, if the returns of the risky asset are not assumed to be identically distributed, then one will not expect the optimal boundaries to be constants over time.

In order to study the behaviour of the optimal boundaries numerically, we consider a simple case and assume that the parameters in our model take the values $N = 6$, $\gamma = 0.1$ and $r = 1.05$. The random variable s is assumed to have the probability density function $p(s) = 0.7 \times \delta(s - 1.5) + 0.3 \times \delta(s - 0.5)$, where $\delta(.)$ is the Dirac delta function.

In Figure 3.1, the transaction costs are allowed to vary from 0 to 0.03. This figure shows the relationship between the optimal boundaries at the initial time and transaction costs. The exact boundaries are obtained by implementing the dynamic programming algorithm numerically while the approximate boundaries are given by Equations (3.115) and

(3.116). When transaction costs are zero, the boundaries converge to the Merton proportion and the no-transaction region disappears. Therefore, in the absence of transaction costs, the investor will always trade in the risky asset to reach the Merton proportion. When transaction costs increase, the width of the no-transaction region (i.e. difference between the buy and sell boundaries) increases and the investor is less likely to trade in the risky asset. From this example, it can be observed that the approximate boundaries are good estimates of the exact boundaries in the limiting case of small transaction costs.

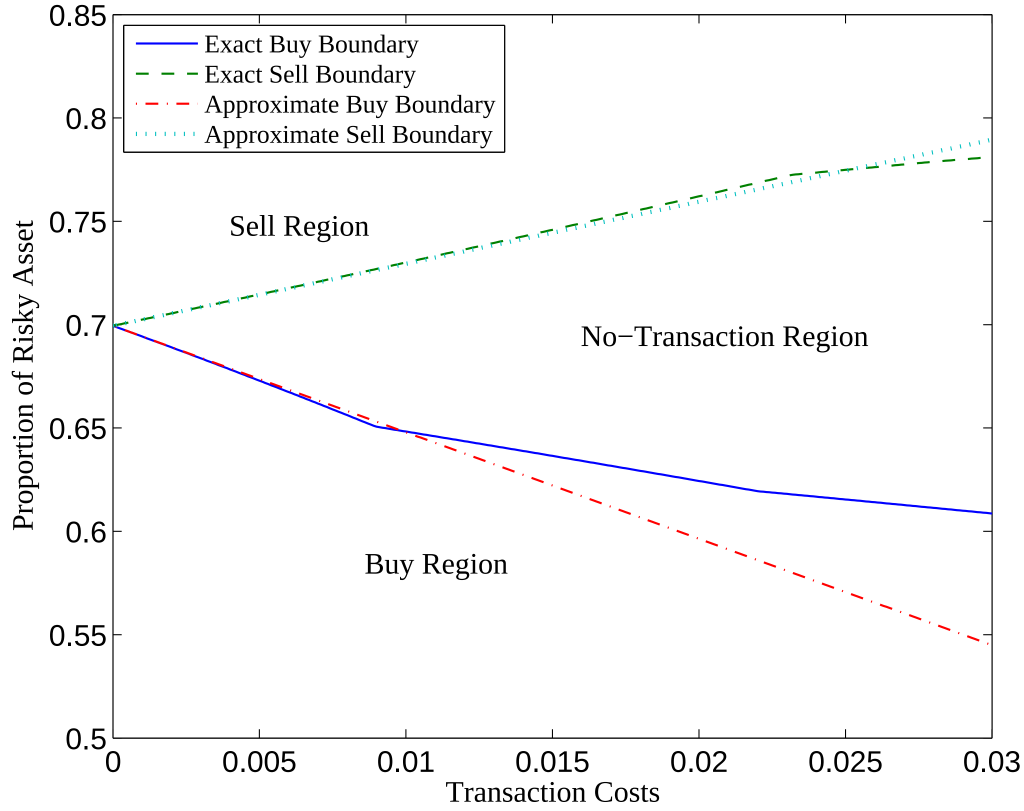


Figure 3.1: Comparison of Exact and Approximate Optimal Boundaries

In Figure 3.2, we assume that the transaction costs are fixed at 0.004 and we vary γ from 0.1 to 0.3. The level of risk aversion of the investor, which is given by $(1-\gamma)$ for the power utility function, therefore varies from 0.7 to 0.9. In this figure, we investigate how the investor's risk aversion affects the Merton proportion and the optimal boundaries at the initial time as approximated by Equations (3.115) and (3.116). It is observed that the Merton proportion and the optimal boundaries are inversely related to the level of risk aversion of the investor. As the investor becomes increasingly risk averse, he would prefer to hold less of the risky asset and more of the risk-free asset, which explains the decrease in the Merton proportion. Although the width of the no-transaction region appears to remain the same, the effects of increasing the investor's risk aversion are seen by the sell region that is widening and the buy region that is narrowing. This means that there is a greater tendency for the investor to sell rather than to buy the risky asset when his level of risk aversion is high.

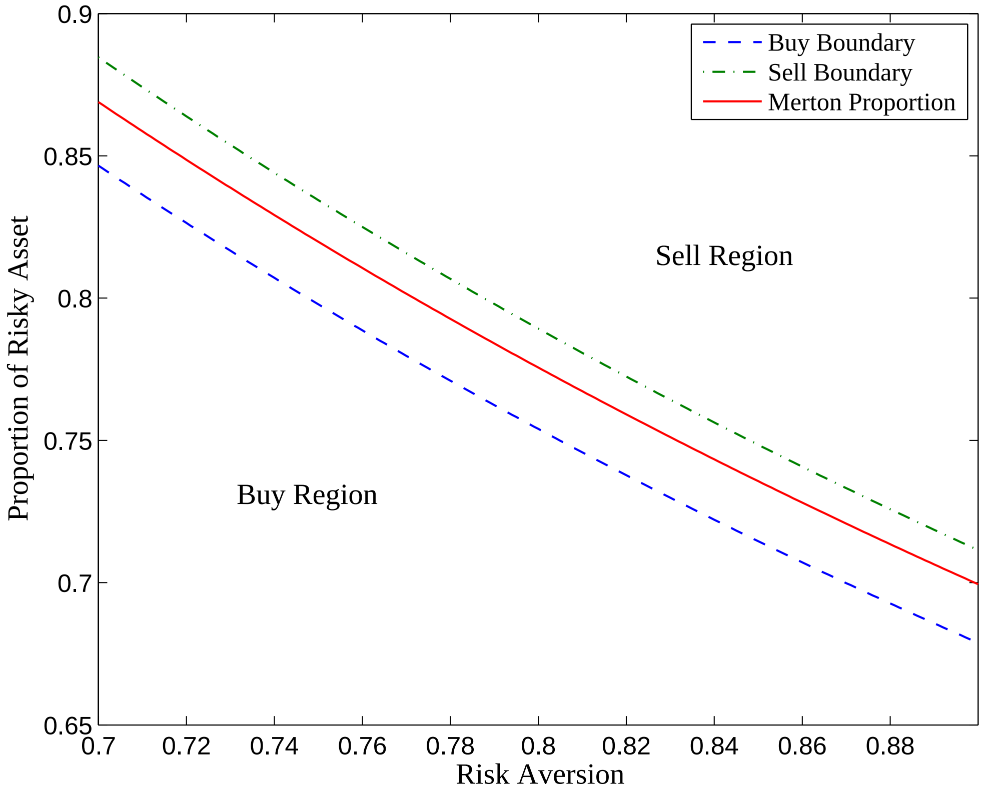


Figure 3.2: Optimal Boundaries vs Risk Aversion

In Figure 3.3, we assume that transaction costs are 0.004 and $\gamma = 0.1$. This figure shows the linear relationship between the optimal holdings in the risky asset and the wealth of the investor. Recall the parametrisation $a_{N-k} = A_{N-k}W_{N-k}$, where a_{N-k} is the dollar value invested in the risky asset, A_{N-k} is the proportion of wealth invested in the risky asset and W_{N-k} is the wealth of the investor. The Merton line $a_0 = \widetilde{A}_0 W_0$ corresponds to the optimal value to be invested in the risky asset when there are no transaction costs. The buy boundary $a_0 = A_0^- W_0$ and the sell boundary $a_0 = A_0^+ W_0$ delineate the no-transaction region, which in this case is narrow since the transaction costs are small. As the investor's wealth increases, observe that the optimal holdings in the risky asset (in dollar value terms) also increase as depicted by the increasing buy and sell boundaries, which is what one would expect in practice. Therefore, this observation represents a more realistic description of an investor's behaviour as compared to the exponential utility function, where the Merton line, buy and sell boundaries are found to be independent of the investor's wealth.

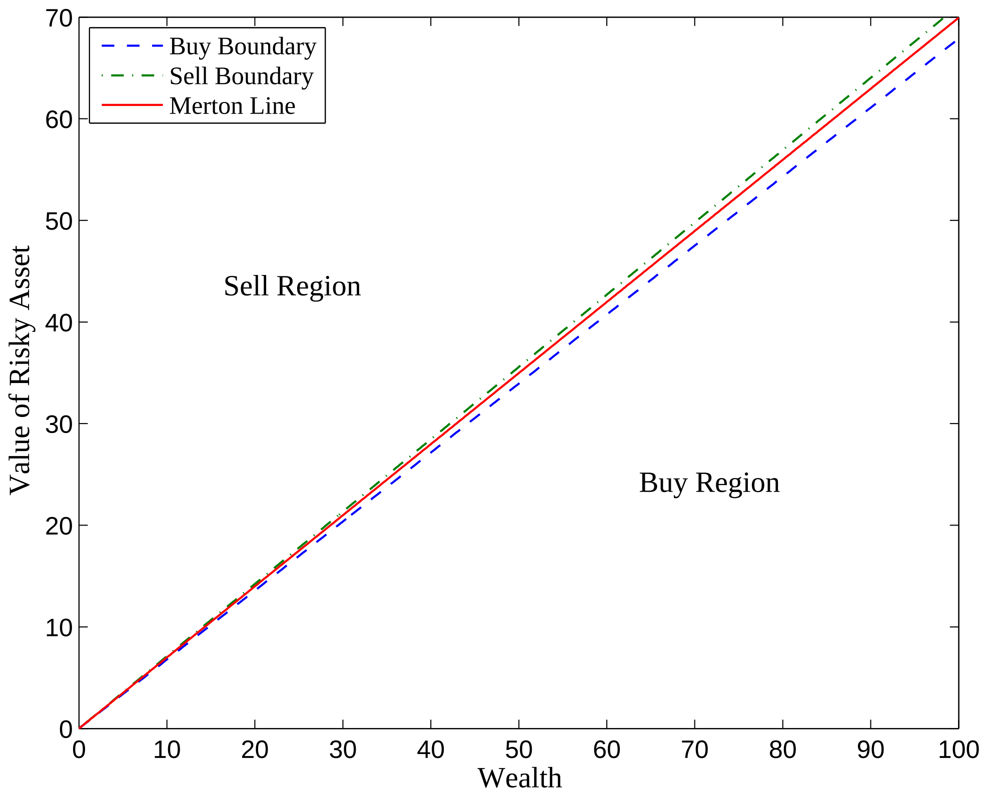


Figure 3.3: Optimal Boundaries vs Wealth

In conclusion, we have devised a perturbation method that allows one to obtain approximations of the optimal value function and optimal boundaries to an arbitrary number of time steps in the portfolio selection model. Therefore, when transaction costs are small, one does not need to compute the optimal value function and optimal boundaries via the dynamic programming algorithm, which becomes computationally intensive as the number of time steps increases. Furthermore, one has the flexibility to specify the probability distribution of the returns of the risky asset as we have kept it generic in the model.

Chapter 4

Indifference Option Pricing without Transaction Costs

In this chapter, we present two closely related approaches to option pricing that are based on the maximisation of expected utility of terminal wealth. The first of these approaches was developed by Hodges and Neuberger (1989) in a continuous time setting to price options with transaction costs. In this approach, the selling (buying) price of an option is defined as the amount of money that will make the investor indifferent, in terms of expected utilities, between trading in the market with and without a short (long) position in the option. The resulting price of the option is known as the utility indifference price. An alternative utility-based approach was proposed by Davis (1997) for a general setting in which the replication of an option was either impossible or unfavourable. In this approach, the price of an option is determined by the requirement that an infinitesimal diversion of funds into the option purchase or sale has a neutral effect on the investor's achievable utility. This definition results in a unique price for the option, which is described as the marginal utility indifference price. After defining the utility indifference price and marginal utility indifference price of an option within a discrete time framework, we specify a market model without transaction costs. The underlying risky asset is assumed to follow a general price process. We consider separately the case where the investor has an exponential utility function and a power utility function. When the price of the risky asset follows a multiplicative binomial tree, we demonstrate that both the utility indifference price and marginal utility indifference price are equivalent to the perfect replication price (from the Cox et al. (1979) binomial model). In the case where the risky asset is assumed to follow a trinomial price process, we present an approximate replication approach that may be used to price the option. A comparison is made between this approach and the utility maximisation approach.

4.1 Definitions

In this section, we provide the definitions of the utility indifference price and marginal utility indifference price of an option. Consider the discrete time portfolio selection model with N periods. Suppose that time period n ($n = 0, 1, \ldots, N-1, N$) indexes discrete time $t_0 < t_1 < \ldots < t_{N-1} < t_N$, where $t_0 = 0$ is the initial time and $t_N = T$ is the terminal time. The investor rebalances the portfolio among a risk-free asset and a risky asset at the start of each time period so as to maximise expected utility of wealth at the end of the investment horizon. Suppose that the investor's utility function U is strictly increasing and strictly concave. The investor starts with an initial wealth of $W_0 = z$ and uses the investment strategy π to form a dynamic portfolio whose value at time period n ($n = 1, \ldots, N$) is denoted by $W_n^{z,\pi}$. Given an initial wealth z, the investor's optimal value function $J(z)$ is defined to be

$$J(z) = \max_{\pi \in \Pi} \mathbb{E}\left[U(W_N^{z,\pi})\right], \tag{4.1}$$

where the maximisation is over the set Π of admissible investment strategies. The investment strategies will be described in greater details when we specify the market model in the later part of this chapter (for the case without transaction costs) and in the next chapter (for the case with transaction costs). Introduce a European option, expiring at the end of N periods (at time T) and yielding a random payoff C_N at expiry. Suppose that the investor takes a position in the option at the initial time and prices the option via utility maximisation.

4.1.1 Utility Indifference Price

In the utility indifference pricing approach first suggested by Hodges and Neuberger (1989), the selling (buying) price of an option is the amount of money that will make the investor indifferent, in terms of expected utilities, between trading in the market with and without a short (long) position in the option.

Specifically, suppose that the investor sells one European option at the initial time and proceeds to maximise expected utility of terminal wealth. In this case, the optimal value function for the portfolio with a short position in the option is defined to be

$$J^{(so)}(z) = \max_{\pi \in \Pi} \mathbb{E}\left[U(W_N^{z,\pi} - C_N)\right]. \tag{4.2}$$

The superscript "so" denotes the case of the investor selling an option at the initial time. The indifference selling price of the option with payoff C_N is defined to be the value $\nu^{(s)}$ that satisfies

$$J^{(so)}(z + \nu^{(s)}) = J(z) \tag{4.3}$$

for an initial wealth z. In other words, $\nu^{(s)}$ is the value at which the two optimal value functions J and $J^{(so)}$, defined in Equations (4.1) and (4.2) respectively, coincide.

On the other hand, suppose that the investor buys one European option at the initial time, in which case the investor's optimal value function with a long position in the option is defined as

$$J^{(bo)}(z) = \max_{\pi \in \Pi} \mathbb{E}\left[U(W_N^{z,\pi} + C_N)\right]. \tag{4.4}$$

Here, the superscript "bo" denotes the case of the investor buying an option at the initial time. Analogous to the definition of the selling price, the indifference buying price of the option with payoff C_N is defined to be the value $\nu^{(b)}$ that satisfies

$$J^{(bo)}(z - \nu^{(b)}) = J(z) \tag{4.5}$$

for an initial wealth z.

In general, note that $\nu^{(s)}$ and $\nu^{(b)}$ depend on the level of initial wealth z. However, we will see subsequently that for an investor with the exponential utility function, $\nu^{(s)}$ and $\nu^{(b)}$

are independent of z.

4.1.2 Marginal Utility Indifference Price

An alternative utility-based approach was proposed by Davis (1997), where the price of an option is determined by the requirement that an infinitesimal diversion of funds into the option purchase or sale has a neutral effect on the investor's achievable utility.

Suppose that a small amount of initial wealth z is diverted to the purchase or sale of a European option with payoff C_N. To be precise, assume that δ dollars of wealth is diverted into the option with a price of ν at the initial time. The optimal value function for the portfolio with an infinitesimal position in the option is defined to be

$$J^{(o)}(z, \nu, \delta) = \max_{\pi \in \Pi} \mathbb{E}\left[U\left(W_N^{z-\delta,\pi} + \frac{\delta}{\nu} C_N\right)\right]. \tag{4.6}$$

Here, the superscript "o" denotes the case of the investor having a position in the option at the initial time. The investor has a long (short) position in the option if $\delta > 0$ ($\delta < 0$). The price of the option is defined to be the value $\tilde{\nu}$ that satisfies the equation

$$\frac{\partial J^{(o)}}{\partial \delta}(z, \tilde{\nu}, 0) = 0. \tag{4.7}$$

From this pricing definition, one is able to derive a formula for the price of the option $\tilde{\nu}$ as follows. Expanding Equation (4.6) for a small δ,

$$J^{(o)}(z, \nu, \delta) = \max_{\pi \in \Pi} \left\{\mathbb{E}\left[U(W_N^{z-\delta,\pi})\right] + \frac{\delta}{\nu}\mathbb{E}\left[U'(W_N^{z-\delta,\pi})C_N\right] + O(\delta^2)\right\}. \tag{4.8}$$

Taking the partial derivative of $J^{(o)}$ with respect to δ, setting $\delta = 0$ and using Equation (4.7), the price of the option $\tilde{\nu}$ is given by

$$\tilde{\nu} = \frac{G(z)}{J'(z)}, \tag{4.9}$$

where $G(z)$ is defined to be

$$G(z) = \mathbb{E}\left[U'(W_N^{z,\pi^*})C_N\right]. \tag{4.10}$$

Note that π^* is the optimal investment strategy for the portfolio selection problem without a position in the option and $J(z)$ is the corresponding optimal value function as defined in Equation (4.1).

Having defined the utility indifference price and marginal utility indifference price, we proceed to develop a utility-based option pricing model in discrete time without transaction costs.

4.2 Market Model without Transaction Costs

Consider the discrete time portfolio selection model with N periods where the investor rebalances the portfolio among a risk-free asset and a risky asset at the start of each time period. Assume that trading in the risky asset does not incur any transaction costs. Let W_n be the wealth of the portfolio and a_n be the dollar value invested in the risky asset at time period n. Therefore, $W_n - a_n$ dollars are invested in the risk-free asset at time period n. Suppose that s_n denotes one plus the return of the risky asset and r_n denotes one plus the return of the risk-free asset from time period n to $n+1$. Thus, the investor's wealth at time period $n+1$ is given by

$$W_{n+1} = r_n W_n + (s_n - r_n)\, a_n \tag{4.11}$$

for $n = 0, \ldots, N-1$. Given an initial wealth W_0, the investor aims to maximise expected utility of terminal wealth by choosing the investments in the risky asset $a_0, \ldots, a_{N-1}$ optimally.

Let X_n be the price of one unit of the risky asset at time period n. The price of the risky asset at time period $n+1$ is given by

$$X_{n+1} = s_n X_n \tag{4.12}$$

for $n = 0, \ldots, N-1$. Consider a European option, expiring at the end of N periods (at time T) and yielding a payoff $C_N = c(X_N)$ that depends on the price of the risky asset X_N at expiry.

Suppose that the investor takes a position in the option at the initial time. So the question arises: What is the price at which the investor will value the option? In a complete market without transaction costs, if the price of the risky asset is assumed to follow a binomial tree, one can construct a portfolio that perfectly replicates the option payoff as demonstrated in Cox et al. (1979). This results in a unique universal price for the option that is independent of investors' risk preferences. The continuous time analogue of the binomial model is the Black-Scholes model. However, if the price of the risky asset does not follow a binomial tree, it is generally not possible to perfectly replicate the option payoff. As an example, we will consider the case where the price of the risky asset follows a trinomial tree. Therefore, there remains an element of risk that cannot be hedged away when one takes on a position in the option. So, it follows that the valuation of the option should take into account an investor's aversion to risk. In order to incorporate the investor's level of risk aversion in pricing the option, we adopt the utility maximisation approach. Moreover, in the next chapter, we will extend this approach to a market model with transaction costs.

We proceed to illustrate, using the definitions from Section 4.1, how one could determine the utility indifference price and marginal utility indifference price of a European option in the context of the exponential and power utility functions.

4.3 Exponential Utility Function

Assume that the investor has an exponential utility function of the form

$$U(W) = -e^{-\kappa W}, \tag{4.13}$$

where $\kappa > 0$. In order to obtain the utility indifference price of the option, we need to determine the optimal value functions for the portfolios with and without a position in the option.

4.3.1 Portfolio Selection without Option Position

Consider first the portfolio without a position in the option. Given an initial wealth W_0, the investor's optimal value function $J(W_0)$ is defined to be

$$J(W_0) = \max \mathbb{E}\left[U(W_N)\right]. \tag{4.14}$$

In this market model without transaction costs, the maximisation is over the investments $a_0, \ldots, a_{N-1}$ in the risky asset and the expectation is taken with respect to the random variables $s_0, \ldots, s_{N-1}$. This is a special case of the portfolio selection problem (with one risky asset) that we previously analysed in Section 1.5.2. The optimal value function $J(W_0)$ defined above is equivalent to Equation (1.49) with $M = 1$. Therefore, we quote directly from the results of the analysis found in Equations (1.56) and (1.58). At time period n ($n = 0, \ldots, N-2$), the optimal investment in the risky asset is

$$a_n = \frac{a_n^*}{r_{N-1}\cdots r_{n+1}} \tag{4.15}$$

and the optimal value function is

$$\begin{aligned} J_n(W_n) &= -e^{-\kappa r_{N-1} r_{N-2}\cdots r_n W_n}\mathbb{E}_{N-1}\left[e^{-\kappa(s_{N-1}-r_{N-1})a_{N-1}^*}\right] \\ &\quad \times \mathbb{E}_{N-2}\left[e^{-\kappa(s_{N-2}-r_{N-2})a_{N-2}^*}\right]\cdots\mathbb{E}_n\left[e^{-\kappa(s_n-r_n)a_n^*}\right], \end{aligned} \tag{4.16}$$

where a_n^* satisfies the equation

$$\mathbb{E}_n\left[(s_n - r_n)\, e^{-\kappa(s_n-r_n)a_n^*}\right] = 0. \tag{4.17}$$

We obtain $J(W_0)$ by setting $n = 0$ in Equation (4.16).

4.3.2 Portfolio Selection with Option Position

In the discussion that follows, we focus on deriving the utility indifference selling price of the option. The derivation of the buying price of the option is similar. Consider now the investor with a short position in an option with payoff $C_N = c(X_N)$. The optimal value function (adapting from Equation (4.2)) is defined to be

$$J^{(so)}(W_0, X_0) = \max \mathbb{E}\left[U(W_N - C_N)\right], \tag{4.18}$$

where the maximisation is over the investments $a_0, \ldots, a_{N-1}$ in the risky asset and the expectation is taken with respect to the random variables $s_0, \ldots, s_{N-1}$. In addition to W_0, the optimal value function also depends on X_0 as the option payoff C_N is a function of the price of the risky asset X_N.

In order to determine the optimal investment strategy and optimal value function, we apply the principle of dynamic programming. The dynamic programming algorithm for the portfolio selection problem with a short position in the option, which starts at period $N-1$ and proceeds backwards recursively in time is

$$J_N^{(so)}(W_N, X_N) = U(W_N - C_N) \tag{4.19}$$

and

$$J_{N-k}^{(so)}(W_{N-k}, X_{N-k}) = \max \mathbb{E}_{N-k}\left[J_{N-k+1}^{(so)}(W_{N-k+1}, X_{N-k+1})\right] \tag{4.20}$$

for $k = 1, \ldots, N$. The conditional expectation operator $\mathbb{E}_{N-k}$ is taken with respect to the random variable s_{N-k} given the information at time period $N-k$ and the maximisation is over the investment a_{N-k} in the risky asset.

Time Period $N-1$

From Equations (4.11) and (4.20), the optimal value function at time period $N-1$ is

$$\begin{aligned} &J_{N-1}^{(so)}(W_{N-1}, X_{N-1}) = \max \mathbb{E}_{N-1}\left[-e^{-\kappa\{W_N - C_N\}}\right] \\ &= e^{-\kappa r_{N-1} W_{N-1}} \max \mathbb{E}_{N-1}\left[-e^{-\kappa\{(s_{N-1}-r_{N-1})a_{N-1} - c(s_{N-1}X_{N-1})\}}\right]. \end{aligned} \tag{4.21}$$

The term in W_{N-1} is taken out of the conditional expectation $\mathbb{E}_{N-1}$ since it is given at time period $N-1$. In addition, the option payoff is $C_N = c(X_N) = c(s_{N-1}X_{N-1})$. The problem is now reduced to one of maximising $V_{N-1}^{(so)}(X_{N-1})$ over a_{N-1}, where

$$V_{N-1}^{(so)}(X_{N-1}) = \mathbb{E}_{N-1}\left[-e^{-\kappa\{(s_{N-1}-r_{N-1})a_{N-1}-c(s_{N-1}X_{N-1})\}}\right]. \tag{4.22}$$

Taking the partial derivative of $V_{N-1}^{(so)}$ with respect to a_{N-1} (of course with X_{N-1} held constant), the first order optimality condition $\dfrac{\partial V_{N-1}^{(so)}}{\partial a_{N-1}} = 0$ is given by

$$\mathbb{E}_{N-1}\left[(s_{N-1}-r_{N-1})\, e^{-\kappa\{(s_{N-1}-r_{N-1})a_{N-1}-c(s_{N-1}X_{N-1})\}}\right] = 0. \tag{4.23}$$

In general, one will solve this equation numerically. Observe that the optimal investment in the risky asset is independent of the wealth W_{N-1}. However, unlike the portfolio selection problem without a position in the option, the optimal investment in the risky asset depends on the price of the risky asset X_{N-1}. Suppose that a solution exists and is of the form $a_{N-1} = a_{N-1}^{(so)*}(X_{N-1})$, where $a_{N-1}^{(so)*}$ denotes the optimal investment strategy for the portfolio selection problem in which the investor has sold an option. Consequently, the optimal value function is

$$V_{N-1}^{(so)}(X_{N-1}) = -\mathbb{E}_{N-1}\left[e^{-\kappa\left\{(s_{N-1}-r_{N-1})a_{N-1}^{(so)*}(X_{N-1})-c(s_{N-1}X_{N-1})\right\}}\right]. \tag{4.24}$$

Time Period $N-2$

Similarly at time period $N-2$, the optimal value function is

$$\begin{aligned} J_{N-2}^{(so)}(W_{N-2}, X_{N-2}) &= \max \mathbb{E}_{N-2}\left[e^{-\kappa r_{N-1}W_{N-1}}V_{N-1}^{(so)}(X_{N-1})\right] \\ &= e^{-\kappa r_{N-1}r_{N-2}W_{N-2}} \max \mathbb{E}_{N-2}\left[e^{-\kappa r_{N-1}(s_{N-2}-r_{N-2})a_{N-2}}V_{N-1}^{(so)}(s_{N-2}X_{N-2})\right]. \end{aligned} \tag{4.25}$$

The problem is reduced to one of maximising $V_{N-2}^{(so)}(X_{N-2})$ over a_{N-2}, where we define

$$V_{N-2}^{(so)}(X_{N-2}) = \mathbb{E}_{N-2}\left[e^{-\kappa r_{N-1}(s_{N-2}-r_{N-2})a_{N-2}}V_{N-1}^{(so)}(s_{N-2}X_{N-2})\right]. \tag{4.26}$$

The first order optimality condition $\dfrac{\partial V_{N-2}^{(so)}}{\partial a_{N-2}} = 0$ gives us the equation

$$\mathbb{E}_{N-2}\left[(s_{N-2} - r_{N-2})\, e^{-\kappa r_{N-1}(s_{N-2}-r_{N-2})a_{N-2}} V_{N-1}^{(so)}(s_{N-2}X_{N-2})\right] = 0. \tag{4.27}$$

Assuming that the optimal investment strategy satisfying the above equation is of the form $a_{N-2} = a_{N-2}^{(so)*}(X_{N-2})$, the optimal value function is

$$V_{N-2}^{(so)}(X_{N-2}) = \mathbb{E}_{N-2}\left[e^{-\kappa r_{N-1}(s_{N-2}-r_{N-2})a_{N-2}^{(so)*}(X_{N-2})} V_{N-1}^{(so)}(s_{N-2}X_{N-2})\right]. \tag{4.28}$$

Time Period $N - k$

Proceeding in a similar way, we deduce that at time period $N - k$ ($k = 2, \ldots, N$), the optimal investment in the risky asset $a_{N-k} = a_{N-k}^{(so)*}(X_{N-k})$ satisfies the equation

$$\mathbb{E}_{N-k}\left[(s_{N-k} - r_{N-k})\, e^{-\kappa r_{N-1}\cdots r_{N-k+1}(s_{N-k}-r_{N-k})a_{N-k}} V_{N-k+1}^{(so)}(s_{N-k}X_{N-k})\right] = 0. \tag{4.29}$$

Furthermore, the optimal value function is given by

$$V_{N-k}^{(so)}(X_{N-k}) = \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}(s_{N-k}-r_{N-k})a_{N-k}^{(so)*}(X_{N-k})} V_{N-k+1}^{(so)}(s_{N-k}X_{N-k})\right] \tag{4.30}$$

or

$$J_{N-k}^{(so)}(W_{N-k}, X_{N-k}) = e^{-\kappa r_{N-1}\cdots r_{N-k}W_{N-k}} V_{N-k}^{(so)}(X_{N-k}). \tag{4.31}$$

Therefore, via solving the dynamic programming algorithm from $k = 1$ up to $k = N$, we obtain the optimal investment in the risky asset

$$a_0 = a_0^{(so)*}(X_0) \tag{4.32}$$

and the optimal value function

$$J^{(so)}(W_0, X_0) = e^{-\kappa r_{N-1} r_{N-2}\cdots r_0 W_0} V_0^{(so)}(X_0) \tag{4.33}$$

at the initial time.

4.3.3 Utility Indifference Price and Hedge

From the definition in Equation (4.3), the utility indifference selling price $\nu^{(s)}$ of the option satisfies

$$J^{(so)}\big(W_0 + \nu^{(s)}, X_0\big) = J(W_0). \tag{4.34}$$

Using the optimal value functions given by Equations (4.16) and (4.33), the utility indifference selling price $\nu^{(s)}(X_0)$ of the option is given by

$$\nu^{(s)}(X_0) = \frac{1}{\kappa r_{N-1}\cdots r_0} \ln\left\{ \frac{-V_0^{(so)}(X_0)}{\mathbb{E}_{N-1}\left[e^{-\kappa(s_{N-1}-r_{N-1})a^*_{N-1}}\right]\cdots\mathbb{E}_0\left[e^{-\kappa(s_0-r_0)a^*_0}\right]}\right\}. \tag{4.35}$$

In addition to pricing an option, the utility indifference approach also provides a natural definition of a hedging strategy. A portion of the proceeds received from selling an option at the start of the investment process would have been diverted to incremental investments in the risky asset. Therefore, by comparing the optimal investments in the risky asset for the portfolio without a position in the option $\dfrac{a_0^*}{r_{N-1}\cdots r_1}$ and with a short position in the option $a_0^{(so)*}(X_0)$, one is naturally led to define the hedge ratio $\Delta^{(s)}(X_0)$ of the option as

$$\Delta^{(s)}(X_0) = \frac{1}{X_0}\left\{a_0^{(so)*}(X_0) - \frac{a_0^*}{r_{N-1}\cdots r_1}\right\}. \tag{4.36}$$

Recall that a_0^* and $a_0^{(so)*}(X_0)$ correspond to Equations (4.17) and (4.29) respectively. Observe that, for an investor with the exponential utility function, both the utility indifference price and hedge ratio are independent of the investor's wealth.

4.3.4 Marginal Utility Indifference Price

Using the formula given by Equation (4.9), the marginal utility indifference price of an option with payoff $C_N = c(X_N)$ is

$$\tilde{\nu} = \frac{G(W_0, X_0)}{J'(W_0)}. \tag{4.37}$$

Consider the function

$$G(W_0, X_0) = \mathbb{E}\left[U'(W_N)C_N\right], \tag{4.38}$$

where the expectation is taken with respect to the random variables $s_0, \ldots, s_{N-1}$. Recall that the wealth evolves according to $W_{n+1} = r_n W_n + (s_n - r_n)\, a_n$. Moreover, note that $a_n = \dfrac{a_n^*}{r_{N-1}\cdots r_{n+1}}$ is the optimal investment strategy for the portfolio selection problem without an option, where a_n^* satisfies Equation (4.17). In addition, the price of the risky asset is given by $X_{n+1} = s_n X_n$. We evaluate $G(W_0, X_0)$ by applying the following algorithm backwards in time from period $N-1$, which is given by

$$G_N(W_N, X_N) = U'(W_N)C_N \tag{4.39}$$

and

$$G_{N-k}(W_{N-k}, X_{N-k}) = \mathbb{E}_{N-k}\left[G_{N-k+1}(W_{N-k+1}, X_{N-k+1})\right] \tag{4.40}$$

for $k = 1, \ldots, N$.

Time Period $N-1$

From Equation (4.40) at time period $N-1$, we have

$$\begin{aligned} G_{N-1}(W_{N-1}, X_{N-1}) &= \mathbb{E}_{N-1}\left[\kappa e^{-\kappa W_N} c(X_N)\right] \\ &= \mathbb{E}_{N-1}\left[\kappa e^{-\kappa\{r_{N-1}W_{N-1} + (s_{N-1} - r_{N-1})a_{N-1}\}} c(s_{N-1}X_{N-1})\right]. \end{aligned} \tag{4.41}$$

Observe that the term in W_{N-1} can be taken out of the conditional expectation $\mathbb{E}_{N-1}$ and that $a_{N-1} = a_{N-1}^*$, where a_{N-1}^* satisfies Equation (4.17). Therefore, we shall express $G_{N-1}(W_{N-1}, X_{N-1})$ as

$$G_{N-1}(W_{N-1}, X_{N-1}) = \kappa e^{-\kappa r_{N-1} W_{N-1}} H_{N-1}(X_{N-1}), \tag{4.42}$$

where

$$H_{N-1}(X_{N-1}) = \mathbb{E}_{N-1}\left[e^{-\kappa(s_{N-1} - r_{N-1})a_{N-1}^*} c(s_{N-1}X_{N-1})\right]. \tag{4.43}$$

Time Period $N - k$

Applying the algorithm (Equation (4.40)) recursively, we deduce that for the time period $N - k$ case in general,

$$G_{N-k}(W_{N-k}, X_{N-k}) = \kappa e^{-\kappa r_{N-1} \cdots r_{N-k} W_{N-k}} H_{N-k}(X_{N-k}), \tag{4.44}$$

where

$$H_{N-k}(X_{N-k}) = \mathbb{E}_{N-k}\left[e^{-\kappa(s_{N-k} - r_{N-k}) a^*_{N-k}} H_{N-k+1}(s_{N-k} X_{N-k})\right]. \tag{4.45}$$

Therefore, by applying the algorithm from $k = 1$ up to $k = N$, we obtain the function

$$G(W_0, X_0) = \kappa e^{-\kappa r_{N-1} \cdots r_0 W_0} H_0(X_0). \tag{4.46}$$

In order to obtain $J'(W_0)$, we set $n = 0$ in Equation (4.16) and differentiate it with respect to W_0. From Equation (4.37), the price of the option is thus given by

$$\tilde{\nu}(X_0) = \frac{H_0(X_0)}{r_{N-1} \cdots r_0 \mathbb{E}_{N-1}\left[e^{-\kappa(s_{N-1} - r_{N-1}) a^*_{N-1}}\right] \cdots \mathbb{E}_0\left[e^{-\kappa(s_0 - r_0) a^*_0}\right]}. \tag{4.47}$$

Similar to the utility indifference price, observe that the marginal utility indifference price is also independent of the investor's wealth. However in this case, we do not have a definition of the hedge ratio of the option.

4.3.5 Special Case of Binomial Price Process

Any good option pricing model must necessarily converge to the perfect replication model under the appropriate assumptions. In our case, the benchmark for comparison is Cox et al.'s (1979) binomial model. We make the following assumptions and show that the utility indifference pricing model reduces to the binomial model. Assume that the price of the risky asset follows a multiplicative binomial tree, that is, s_n are independent and identically distributed random variables such that

$$s_n = \begin{cases} u & \text{with probability } q, \\ d & \text{with probability } 1-q, \end{cases} \tag{4.48}$$

for $n = 0, \ldots, N-1$. Assume that the return of the risk-free asset is a constant, that is,

$$r_n = r, \tag{4.49}$$

where $d < r < u$. With these assumptions, we are able to solve for the optimal investments and optimal value functions exactly. Consequently, explicit expressions for the utility indifference price and hedge ratio of the option are obtained and compared with those from the binomial model.

Consider first the portfolio selection problem without the option position. From Equation (4.17), we obtain

$$q\,(u-r)\,e^{-\kappa(u-r)a_n^*} + (1-q)\,(d-r)\,e^{-\kappa(d-r)a_n^*} = 0 \tag{4.50}$$

for $n = 0, \ldots, N-1$. Rearranging the above equation gives us

$$a_n^* = \frac{1}{\kappa\,(u-d)} \ln\left\{\frac{q\,(1-p)}{(1-q)\,p}\right\}, \tag{4.51}$$

where

$$p = \frac{r-d}{u-d} \quad \text{and} \quad 1-p = \frac{u-r}{u-d}. \tag{4.52}$$

It follows from Equation (4.15) that the optimal investment in the risky asset at the initial time is

$$a_0 = \frac{1}{r^{N-1}\kappa\,(u-d)} \ln\left\{\frac{q\,(1-p)}{(1-q)\,p}\right\}. \tag{4.53}$$

Moreover, for all $n = 0, \ldots, N-1$,

$$\mathbb{E}_n\left[e^{-\kappa(s_n-r_n)a_n^*}\right] = qe^{-\kappa(u-r)a_n^*} + (1-q)\,e^{-\kappa(d-r)a_n^*}. \tag{4.54}$$

Substituting Equation (4.51) into the above equation and simplifying,

$$\mathbb{E}_n\left[e^{-\kappa(s_n-r_n)a_n^*}\right]=\left(\frac{1-q}{1-p}\right)^{1-p}\left(\frac{q}{p}\right)^p. \tag{4.55}$$

Therefore, it follows from Equation (4.16) that the optimal value function at the initial time is

$$J(W_0)=-e^{-\kappa r^N W_0}\left\{\left(\frac{1-q}{1-p}\right)^{1-p}\left(\frac{q}{p}\right)^p\right\}^N. \tag{4.56}$$

Next, consider the portfolio selection problem with the option position. Solving this problem requires the application of dynamic programming. The optimal investment $a_{N-1}^{(so)*}$ in the risky asset at time period $N-1$ satisfies Equation (4.23), which is given by

$$q\,(u-r)\,e^{-\kappa\left\{(u-r)a_{N-1}^{(so)*}-C_u\right\}}+(1-q)\,(d-r)\,e^{-\kappa\left\{(d-r)a_{N-1}^{(so)*}-C_d\right\}}=0, \tag{4.57}$$

where the option payoff is denoted by $C_u=c(uX_{N-1})$ and $C_d=c(dX_{N-1})$ corresponding to an "up" and "down" movement in the price of the risky asset respectively. Note that $a_{N-1}^{(so)*}$ is dependent on X_{N-1}, although we have written $a_{N-1}^{(so)*}$ instead of $a_{N-1}^{(so)*}(X_{N-1})$ for convenience. Rearranging the above equation gives us

$$a_{N-1}^{(so)*}=\frac{1}{\kappa\,(u-d)}\ln\left\{\frac{q\,(1-p)}{(1-q)\,p}\right\}+\frac{C_u-C_d}{u-d}. \tag{4.58}$$

Consequently, from Equation (4.24), the optimal value function at time period $N-1$ is

$$\begin{aligned} V_{N-1}^{(so)}(X_{N-1}) &= -\left[qe^{-\kappa\left\{(u-r)a_{N-1}^{(so)*}-C_u\right\}}+(1-q)\,e^{-\kappa\left\{(d-r)a_{N-1}^{(so)*}-C_d\right\}}\right] \\ &= -e^{\kappa\{pC_u+(1-p)C_d\}}\left(\frac{1-q}{1-p}\right)^{1-p}\left(\frac{q}{p}\right)^p. \end{aligned} \tag{4.59}$$

At time period $N-2$, substituting the above equation for $V_{N-1}^{(so)}(X_{N-1})$ into Equation (4.27), the first order optimality condition for the optimal investment $a_{N-2}=a_{N-2}^{(so)*}$ in the risky asset becomes

$$\mathbb{E}_{N-2}\left[(s_{N-2}-r_{N-2})\,e^{-\kappa\left\{r_{N-1}(s_{N-2}-r_{N-2})a_{N-2}-[pC_u+(1-p)C_d]\right\}}\right]=0. \tag{4.60}$$

Here, $C_u = c(uX_{N-1}) = c(us_{N-2}X_{N-2})$ and $C_d = c(dX_{N-1}) = c(ds_{N-2}X_{N-2})$ depend on s_{N-2} and X_{N-2}. Observe that the above equation is equivalent to Equation (4.23) with a change of variables from $r_{N-1}a_{N-2}$ to a_{N-1} and from $pC_u + (1-p)\,C_d$ to $c(s_{N-1}X_{N-1})$. Therefore, we can deduce from our period $N-1$ solution in Equation (4.58) that the period $N-2$ solution is

$$a_{N-2}^{(so)*} = \frac{1}{r\kappa\,(u-d)} \ln\left\{\frac{q\,(1-p)}{(1-q)\,p}\right\} + \frac{[pC_{uu} + (1-p)\,C_{du}] - [pC_{ud} + (1-p)\,C_{dd}]}{r\,(u-d)}, \tag{4.61}$$

where $C_{uu} = c(u^2X_{N-2})$, $C_{dd} = c(d^2X_{N-2})$ and $C_{ud} = C_{du} = c(udX_{N-2})$. In order to determine the optimal value function, substitute Equation (4.59) into Equation (4.28) to obtain

$$V_{N-2}^{(so)}(X_{N-2}) = -\mathbb{E}_{N-2}\left[e^{-\kappa\left\{r_{N-1}(s_{N-2}-r_{N-2})a_{N-2}^{(so)*} - [pC_u + (1-p)C_d]\right\}}\right]\left(\frac{1-q}{1-p}\right)^{1-p}\left(\frac{q}{p}\right)^{p}. \tag{4.62}$$

We observe again that the above equation is equivalent to Equation (4.24), up to a constant coefficient, with a change of variables from $r_{N-1}a_{N-2}^{(so)*}$ to $a_{N-1}^{(so)*}$ and from $pC_u + (1-p)\,C_d$ to $c(s_{N-1}X_{N-1})$. Therefore, we deduce from Equation (4.59) that

$$V_{N-2}^{(so)}(X_{N-2}) = -e^{\kappa\left\{p[pC_{uu}+(1-p)C_{du}]+(1-p)[pC_{ud}+(1-p)C_{dd}]\right\}}\left\{\left(\frac{1-q}{1-p}\right)^{1-p}\left(\frac{q}{p}\right)^{p}\right\}^2. \tag{4.63}$$

Proceeding recursively backwards in time, we are then able to deduce (similar to the period $N-2$ case) the optimal investment and optimal value function at any time period. We state the following results. At the initial time, the optimal investment in the risky asset is given explicitly as

$$\begin{aligned} a_0^{(so)*}(X_0) &= \frac{1}{r^{N-1}\kappa\,(u-d)} \ln\left\{\frac{q\,(1-p)}{(1-q)\,p}\right\} + \frac{1}{r^{N-1}\,(u-d)} \\ &\times \sum_{i=0}^{N-1}\binom{N-1}{i}p^i\,(1-p)^{N-1-i}\left[c(u^{i+1}d^{N-1-i}X_0) - c(u^id^{N-i}X_0)\right], \end{aligned} \tag{4.64}$$

where $\binom{N-1}{i} = \frac{(N-1)!}{i!\,(N-1-i)!}$ is the binomial coefficient. Moreover, the optimal value function is

$$V_0^{(so)}(X_0) = -\exp\left\{\kappa \sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} c(u^i d^{N-i} X_0)\right\} \left\{\left(\frac{1-q}{1-p}\right)^{1-p} \left(\frac{q}{p}\right)^{p}\right\}^N \tag{4.65}$$

or

$$J^{(so)}(W_0, X_0) = e^{-\kappa r^N W_0} V_0^{(so)}(X_0). \tag{4.66}$$

Using the definitions of the utility indifference price and hedge from Equations (4.35) and (4.36) respectively, we show that

$$\nu^{(s)}(X_0) = \frac{1}{r^N} \sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} c(u^i d^{N-i} X_0) \tag{4.67}$$

and

$$\begin{aligned} \Delta^{(s)}(X_0) &= \frac{1}{r^{N-1}(u-d) X_0} \\ &\times \sum_{i=0}^{N-1} \binom{N-1}{i} p^i (1-p)^{N-1-i} \left[c(u^{i+1} d^{N-1-i} X_0) - c(u^i d^{N-i} X_0)\right]. \end{aligned} \tag{4.68}$$

In this special case where the price of the risky asset follows a binomial tree and the risk-free asset provides a constant return, we note that the utility indifference selling price and hedge do not depend on the level of risk aversion κ of the investor. This observation is consistent with the binomial model where it is possible to perfectly replicate the payoff of the option. Since the risk associated with the option can be perfectly hedged away, the investor is not risk averse to holding a position in the option. Therefore, as one would expect, the utility indifference selling price and hedge are equivalent to the perfect replication price (Equation (1.8)) and hedge (Equation (1.9)) from the binomial model. In addition, it can be shown that the utility indifference buying price and hedge are also equal to the perfect replication price and hedge.

We now proceed to show that the marginal utility indifference price is also equivalent to the perfect replication price. Recall that the pricing formula is given by Equation (4.47). We need to determine $H_0(X_0)$ via the recursive algorithm in Equation (4.45). Starting at time period $N-1$, from Equation (4.43), we have

$$H_{N-1}(X_{N-1}) = qe^{-\kappa(u-r)a^*_{N-1}}C_u + (1-q)\,e^{-\kappa(d-r)a^*_{N-1}}C_d, \tag{4.69}$$

where $C_u = c(uX_{N-1})$ and $C_d = c(dX_{N-1})$. Recall that a^*_{N-1} corresponds to the optimal investment strategy for the portfolio without an option position. Substituting in a^*_{N-1} from Equation (4.51) and simplifying,

$$H_{N-1}(X_{N-1}) = \{pC_u + (1-p)\,C_d\}\left(\frac{1-q}{1-p}\right)^{1-p}\left(\frac{q}{p}\right)^{p}. \tag{4.70}$$

Proceeding to time period $N-2$ and comparing Equation (4.45) with Equation (4.43), we note that $H_{N-2}(X_{N-2})$ is of the same form as $H_{N-1}(X_{N-1})$ up to a constant coefficient. In general, $H_{N-k}(X_{N-k})$ is of the same form as $H_{N-k+1}(X_{N-k+1})$, which allows us to immediately deduce that

$$H_0(X_0) = \left\{\sum_{i=0}^{N}\binom{N}{i}p^i\,(1-p)^{N-i}\,c(u^i d^{N-i}X_0)\right\}\left\{\left(\frac{1-q}{1-p}\right)^{1-p}\left(\frac{q}{p}\right)^{p}\right\}^{N}. \tag{4.71}$$

Substituting the above expression for $H_0(X_0)$ and Equation (4.55) into the pricing formula given by Equation (4.47), we obtain

$$\tilde{\nu}(X_0) = \frac{1}{r^N}\sum_{i=0}^{N}\binom{N}{i}p^i\,(1-p)^{N-i}\,c(u^i d^{N-i}X_0), \tag{4.72}$$

which is as required.

4.4 Power Utility Function

Consider an investor with a power utility function of the form

$$U(W) = \frac{1}{\gamma}W^{\gamma}, \tag{4.73}$$

where $\gamma < 1, \gamma \neq 0$. In this case, it is convenient to express the investment in the risky asset as a fraction of wealth. More specifically, let $a_n = A_n W_n$, where A_n denotes the proportion of wealth invested in the risky asset at time period n. Therefore, Equation (4.11) becomes

$$W_{n+1} = \{r_n + (s_n - r_n) A_n\} W_n \tag{4.74}$$

for $n = 0, \ldots, N-1$. With this parametrisation, the investor aims to maximise expected utility of terminal wealth by choosing the proportions of wealth invested in the risky asset $A_0, \ldots, A_{N-1}$ optimally.

4.4.1 Portfolio Selection without Option Position

Consider the portfolio selection problem without a position in the option. Given an initial wealth W_0, the investor's optimal value function $J(W_0)$ is defined to be

$$J(W_0) = \max \mathbb{E}\left[U(W_N)\right], \tag{4.75}$$

where the maximisation is over the proportions of wealth $A_0, \ldots, A_{N-1}$ invested in the risky asset and the expectation is taken with respect to the random variables $s_0, \ldots, s_{N-1}$. Recall that this is a special case of the portfolio selection problem (with one risky asset) that we previously analysed in Section 1.5.4. The optimal value function $J(W_0)$ defined above is equivalent to Equation (1.66) with $M = 1$. Therefore, we quote directly from the results of the analysis found in Equations (1.73) and (1.75). At time period n ($n = 0, \ldots, N-1$), the optimal proportion of wealth invested in the risky asset is

$$A_n = A_n^* \tag{4.76}$$

and the optimal value function is

$$\begin{aligned} J_n(W_n) &= \frac{1}{\gamma} W_n^\gamma \mathbb{E}_{N-1}\left[\left\{r_{N-1} + (s_{N-1} - r_{N-1}) A_{N-1}^*\right\}^\gamma\right] \\ &\quad \times \mathbb{E}_{N-2}\left[\left\{r_{N-2} + (s_{N-2} - r_{N-2}) A_{N-2}^*\right\}^\gamma\right] \\ &\quad \times \cdots \times \mathbb{E}_n\left[\left\{r_n + (s_n - r_n) A_n^*\right\}^\gamma\right], \end{aligned} \tag{4.77}$$

where A_n^* satisfies the equation

$$\mathbb{E}_n\left[(s_n - r_n)\left\{r_n + (s_n - r_n)\, A_n^*\right\}^{\gamma-1}\right] = 0. \tag{4.78}$$

We obtain $J(W_0)$ by setting $n = 0$ in Equation (4.77).

4.4.2 Portfolio Selection with Option Position

Consider now the investor with a short position in the option with payoff $C_N = c(X_N)$. The optimal value function is defined to be

$$J^{(so)}(W_0, X_0) = \max \mathbb{E}\left[U(W_N - C_N)\right], \tag{4.79}$$

where the maximisation is over the proportions of wealth $A_0, \ldots, A_{N-1}$ invested in the risky asset and the expectation is taken with respect to the random variables $s_0, \ldots, s_{N-1}$.

The dynamic programming algorithm for the portfolio selection problem with a short position in the option, which starts at period $N-1$ and proceeds backwards recursively in time, is

$$J_N^{(so)}(W_N, X_N) = U(W_N - C_N) \tag{4.80}$$

and

$$J_{N-k}^{(so)}(W_{N-k}, X_{N-k}) = \max \mathbb{E}_{N-k}\left[J_{N-k+1}^{(so)}(W_{N-k+1}, X_{N-k+1})\right], \tag{4.81}$$

for $k = 1, \ldots, N$. The conditional expectation operator $\mathbb{E}_{N-k}$ is taken with respect to the random variable s_{N-k} given the information at time period $N-k$ and the maximisation is over the proportion of wealth A_{N-k} invested in the risky asset.

Time Period $N-1$

From Equations (4.74) and (4.81), the optimal value function at time period $N-1$ is

$$\begin{aligned} J_{N-1}^{(so)}(W_{N-1}, X_{N-1}) &= \max \mathbb{E}_{N-1}\left[\frac{1}{\gamma}\left\{W_N - C_N\right\}^{\gamma}\right] \\ &= \max \mathbb{E}_{N-1}\left[\frac{1}{\gamma}\Big\{\left[r_{N-1} + (s_{N-1} - r_{N-1})\, A_{N-1}\right] W_{N-1} - c(s_{N-1} X_{N-1})\Big\}^{\gamma}\right]. \end{aligned} \tag{4.82}$$

Note that, unlike the case of the exponential utility function, the wealth W_{N-1} does not factorise out of the conditional expectation $\mathbb{E}_{N-1}$ and so it is not possible to reduce the problem by one dimension. Taking the partial derivative of $J^{(so)}_{N-1}$ with respect to A_{N-1}, the first order optimality condition $\dfrac{\partial J^{(so)}_{N-1}}{\partial A_{N-1}} = 0$ is given by

$$\mathbb{E}_{N-1}\left[(s_{N-1} - r_{N-1})\left\{[r_{N-1} + (s_{N-1} - r_{N-1})\, A_{N-1}]\, W_{N-1} - c(s_{N-1}X_{N-1})\right\}^{\gamma-1}\right] = 0. \tag{4.83}$$

In general, one will solve this equation numerically. In this case, observe that the optimal proportion invested in the risky asset depends on both the wealth W_{N-1} and price of the risky asset X_{N-1}. Since the solution depends on two state variables instead of one, it will typically be more difficult to solve this equation numerically as compared to the case of the exponential utility function (see Equation(4.23)). Suppose that a solution to the above equation exists and is of the form $A_{N-1} = A^{(so)*}_{N-1}(W_{N-1}, X_{N-1})$, where $A^{(so)*}_{N-1}$ denotes the optimal investment strategy for the portfolio selection problem with a short position in the option. Consequently, the optimal value function is

$$\begin{aligned} J^{(so)}_{N-1}(W_{N-1}, X_{N-1}) \;=\; & \mathbb{E}_{N-1}\Bigg[\frac{1}{\gamma}\Big\{\Big[r_{N-1} + (s_{N-1} - r_{N-1})\, A^{(so)*}_{N-1}\Big]\, W_{N-1} \\ & -c(s_{N-1}X_{N-1})\Big\}^{\gamma}\Bigg]. \end{aligned} \tag{4.84}$$

For convenience, we have written $A^{(so)*}_{N-1}$ instead of $A^{(so)*}_{N-1}(W_{N-1}, X_{N-1})$.

Time Period $N-2$

At time period $N-2$, the optimal value function is

$$J^{(so)}_{N-2}(W_{N-2}, X_{N-2}) = \max \mathbb{E}_{N-2}\left[J^{(so)}_{N-1}(W_{N-1}, X_{N-1})\right]. \tag{4.85}$$

Since $W_{N-1} = \{r_{N-2} + (s_{N-2} - r_{N-2}) A_{N-2}\} W_{N-2}$, the first order optimality condition $\dfrac{\partial J^{(so)}_{N-2}}{\partial A_{N-2}} = 0$ gives us the equation

$$\mathbb{E}_{N-2}\left[(s_{N-2} - r_{N-2}) \frac{\partial J^{(so)}_{N-1}(W_{N-1}, X_{N-1})}{\partial W_{N-1}}\right] = 0. \qquad (4.86)$$

Assuming that the optimal investment strategy satisfying the above equation is of the form $A_{N-2} = A^{(so)*}_{N-2}(W_{N-2}, X_{N-2})$, the optimal value function is

$$\begin{aligned} & J^{(so)}_{N-2}(W_{N-2}, X_{N-2}) \\ &= \mathbb{E}_{N-2}\left[J^{(so)}_{N-1}\left(\left\{r_{N-2} + (s_{N-2} - r_{N-2}) A^{(so)*}_{N-2}\right\} W_{N-2}, s_{N-2}X_{N-2}\right)\right]. \end{aligned} \qquad (4.87)$$

Time Period $N - k$

Proceeding in a similar way, we deduce that at time period $N - k$ ($k = 2, \ldots, N$), the optimal proportion invested in the risky asset $A_{N-k} = A^{(so)*}_{N-k}(W_{N-k}, X_{N-k})$ satisfies the equation

$$\mathbb{E}_{N-k}\left[(s_{N-k} - r_{N-k}) \frac{\partial J^{(so)}_{N-k+1}(W_{N-k+1}, X_{N-k+1})}{\partial W_{N-k+1}}\right] = 0. \qquad (4.88)$$

Furthermore, the optimal value function is given by

$$\begin{aligned} & J^{(so)}_{N-k}(W_{N-k}, X_{N-k}) \\ &= \mathbb{E}_{N-k}\left[J^{(so)}_{N-k+1}\left(\left\{r_{N-k} + (s_{N-k} - r_{N-k}) A^{(so)*}_{N-k}\right\} W_{N-k}, s_{N-k}X_{N-k}\right)\right]. \end{aligned} \qquad (4.89)$$

Therefore, at $k = N$, we obtain via dynamic programming the optimal proportion invested in the risky asset $A_0 = A^{(so)*}_0(W_0, X_0)$ and the optimal value function $J^{(so)}(W_0, X_0) = J^{(so)}_0(W_0, X_0)$ at the initial time.

4.4.3 Utility Indifference Price and Hedge

From the definition in Equation (4.3), the utility indifference selling price $\nu^{(s)}(W_0, X_0)$ of the option satisfies the equation

$$J^{(so)}\big(W_0 + \nu^{(s)}, X_0\big) = J(W_0), \tag{4.90}$$

where the optimal value functions J and $J^{(so)}$ are given by Equations (4.77) and (4.89) respectively.

In addition to the option price, the utility indifference approach also provides a natural hedging strategy. By comparing the optimal proportionals invested in the risky asset for the portfolio selection problem without a position in the option A_0^* and with a short position in the option $A_0^{(so)*}(W_0 + \nu^{(s)}, X_0)$, one is led to a natural definition of a hedge $\Delta^{(s)}(W_0, X_0)$ for the option, which is defined to be

$$\Delta^{(s)}(W_0, X_0) = \frac{1}{X_0}\left\{\big(W_0 + \nu^{(s)}\big)\, A_0^{(so)*}(W_0 + \nu^{(s)}, X_0) - W_0 A_0^*\right\}. \tag{4.91}$$

In other words, the hedge ratio is given by the incremental holdings in the risky asset between the portfolio selection problem with and without a position in the option. Recall that A_0^* and $A_0^{(so)*}(W_0, X_0)$ correspond to Equations (4.78) and (4.88) respectively. Note that, unlike the case of the exponential utility function, the option price and hedge ratio generally depend on both the price of the risky asset as well as the wealth of the investor.

4.4.4 Marginal Utility Indifference Price

Recall from Equation (4.9) that the marginal utility indifference price of an option with payoff $C_N = c(X_N)$ is given by

$$\tilde{\nu} = \frac{G(W_0, X_0)}{J'(W_0)}, \tag{4.92}$$

where

$$G(W_0, X_0) = \mathbb{E}\left[U'(W_N) C_N\right]. \tag{4.93}$$

Recall that the wealth evolves according to $W_{n+1} = \{r_n + (s_n - r_n) A_n\} W_n$. In this case, $A_n = A_n^*$ is the optimal investment strategy for the portfolio without an option position, where A_n^* satisfies Equation (4.78). Moreover, the price of the risky asset is given by $X_{n+1} = s_n X_n$. In order to obtain $G(W_0, X_0)$, we work backwards in time from period $N-1$ using the following algorithm:

$$G_N(W_N, X_N) = U'(W_N) C_N \tag{4.94}$$

and

$$G_{N-k}(W_{N-k}, X_{N-k}) = \mathbb{E}_{N-k}\left[G_{N-k+1}(W_{N-k+1}, X_{N-k+1})\right] \tag{4.95}$$

for $k = 1, \ldots, N$.

Time Period $N-1$

From Equation (4.95) at time period $N-1$, we have

$$\begin{aligned} G_{N-1}(W_{N-1}, X_{N-1}) &= \mathbb{E}_{N-1}\left[W_N^{\gamma-1} c(X_N)\right] \\ &= \mathbb{E}_{N-1}\left[\{r_{N-1} + (s_{N-1} - r_{N-1}) A_{N-1}\}^{\gamma-1} W_{N-1}^{\gamma-1} c(s_{N-1} X_{N-1})\right]. \end{aligned} \tag{4.96}$$

Observe that the term in W_{N-1} can be taken out of the conditional expectation $\mathbb{E}_{N-1}$ and that $A_{N-1} = A_{N-1}^*$, where A_{N-1}^* satisfies Equation (4.78). Therefore, we shall express $G_{N-1}(W_{N-1}, X_{N-1})$ as

$$G_{N-1}(W_{N-1}, X_{N-1}) = W_{N-1}^{\gamma-1} H_{N-1}(X_{N-1}), \tag{4.97}$$

where

$$H_{N-1}(X_{N-1}) = \mathbb{E}_{N-1}\left[\left\{r_{N-1} + (s_{N-1} - r_{N-1}) A_{N-1}^*\right\}^{\gamma-1} c(s_{N-1} X_{N-1})\right]. \tag{4.98}$$

Time Period $N - k$

Applying the algorithm (Equation (4.95)) recursively, we deduce that for the time period $N - k$ case in general,

$$G_{N-k}(W_{N-k}, X_{N-k}) = W_{N-k}^{\gamma-1} H_{N-k}(X_{N-k}), \tag{4.99}$$

where

$$H_{N-k}(X_{N-k}) = \mathbb{E}_{N-k}\left[\left\{r_{N-k} + (s_{N-k} - r_{N-k})\, A^*_{N-k}\right\}^{\gamma-1} H_{N-k+1}(s_{N-k}X_{N-k})\right]. \tag{4.100}$$

Therefore, by applying the algorithm from $k = 1$ up to $k = N$, we obtain the function

$$G(W_0, X_0) = W_0^{\gamma-1} H_0(X_0). \tag{4.101}$$

In order to obtain $J'(W_0)$, we set $n = 0$ in Equation (4.77) and differentiate it with respect to W_0. From Equation (4.92), the price of the option is

$$\tilde{\nu}(X_0) = \frac{H_0(X_0)}{\mathbb{E}_{N-1}\left[\left\{r_{N-1} + (s_{N-1} - r_{N-1})\, A^*_{N-1}\right\}^{\gamma}\right] \cdots \mathbb{E}_0\left[\left\{r_0 + (s_0 - r_0)\, A^*_0\right\}^{\gamma}\right]}. \tag{4.102}$$

Observe that, unlike the utility indifference price, the marginal utility indifference price of the option is independent of the investor's wealth.

4.4.5 Special Case of Binomial Price Process

Similar to Section 4.3.5, assume that the risky asset follows a binomial price process and the risk-free asset has a constant return. Under these assumptions, consider the portfolio selection problem without a position in the option. From Equation (4.78), we obtain

$$q\,(u - r)\,\{r + (u - r)\, A^*_n\}^{\gamma-1} + (1 - q)\,(d - r)\,\{r + (d - r)\, A^*_n\}^{\gamma-1} = 0 \tag{4.103}$$

for $n = 0, \ldots, N-1$. Simplifying the above equation, the optimal investment strategy is

$$A_n^* = \frac{r\left(\beta - 1\right)}{\left(u-d\right)\left(1-p+p\beta\right)}, \tag{4.104}$$

where we have defined

$$p = \frac{r-d}{u-d} \quad , \quad 1-p = \frac{u-r}{u-d} \quad \text{and} \quad \beta = \left\{\frac{q(1-p)}{(1-q)p}\right\}^{\frac{1}{1-\gamma}}. \tag{4.105}$$

From Equation (4.77), we consider the general expression

$$\mathbb{E}_n\left[\left\{r_n + \left(s_n - r_n\right)A_n^*\right\}^\gamma\right] = q\left\{r + \left(u-r\right)A_n^*\right\}^\gamma + \left(1-q\right)\left\{r + \left(d-r\right)A_n^*\right\}^\gamma. \tag{4.106}$$

Using A_n^* from Equation (4.104), we simplify

$$\mathbb{E}_n\left[\left\{r_n + \left(s_n - r_n\right)A_n^*\right\}^\gamma\right] = r^\gamma\left(1-p+p\beta\right)^{1-\gamma}\left(\frac{1-q}{1-p}\right) \tag{4.107}$$

and conclude that the optimal value function is

$$J(W_0) = \frac{1}{\gamma}W_0^\gamma\left\{r^\gamma\left(1-p+p\beta\right)^{1-\gamma}\left(\frac{1-q}{1-p}\right)\right\}^N. \tag{4.108}$$

Next, we consider the portfolio selection problem with a short position in the option. At time period $N-1$, the optimal proportion invested in the risky asset $A_{N-1}^{(so)*}$ satisfies Equation (4.83), which is given by

$$\begin{aligned} &q\left(u-r\right)\left\{\left[r + \left(u-r\right)A_{N-1}^{(so)*}\right]W_{N-1} - C_u\right\}^{\gamma-1} \\ &+\left(1-q\right)\left(d-r\right)\left\{\left[r + \left(d-r\right)A_{N-1}^{(so)*}\right]W_{N-1} - C_d\right\}^{\gamma-1} = 0, \end{aligned} \tag{4.109}$$

where $C_u = c(uX_{N-1})$ and $C_d = c(dX_{N-1})$. Rearranging the above equation,

$$A_{N-1}^{(so)*} = \frac{r\left(\beta-1\right)W_{N-1} + C_u - \beta C_d}{\left(u-d\right)\left(1-p+p\beta\right)W_{N-1}} \tag{4.110}$$

Substituting the above expression for $A_{N-1}^{(so)*}$ into Equation (4.84) and simplifying, the optimal value function is

$$J_{N-1}^{(so)}(W_{N-1}, X_{N-1}) = \frac{1}{\gamma}\left\{W_{N-1} - \frac{pC_u + (1-p)\,C_d}{r}\right\}^{\gamma} r^{\gamma}\,(1-p+p\beta)^{1-\gamma}\left(\frac{1-q}{1-p}\right). \tag{4.111}$$

Comparing $J_{N-1}^{(so)}(W_{N-1}, X_{N-1})$ with $J_N^{(so)}(W_N, X_N) = \frac{1}{\gamma}\{W_N - C_N\}^{\gamma}$, we note that they are equivalent (up to a constant coefficient) with a change of variables from W_N to W_{N-1} and from C_N to $\frac{pC_u + (1-p)\,C_d}{r}$. Therefore, as we apply the dynamic programming algorithm to determine the optimal value function recursively, we are in fact solving a problem at each time step that is equivalent (up to a constant coefficient) to the one at period $N-1$ with an appropriate change of variables. Thus, we immediately deduce that the optimal value function at the initial time is

$$\begin{aligned} J_0^{(so)}(W_0, X_0) &= \frac{1}{\gamma}\left\{W_0 - \frac{1}{r^N}\sum_{i=0}^{N}\binom{N}{i} p^i\,(1-p)^{N-i}\, c(u^i d^{N-i} X_0)\right\}^{\gamma} \\ &\quad \times \left\{r^{\gamma}\,(1-p+p\beta)^{1-\gamma}\left(\frac{1-q}{1-p}\right)\right\}^{N}. \end{aligned} \tag{4.112}$$

In addition, the optimal proportion invested in the risky asset is

$$\begin{aligned} A_0^{(so)*}(W_0, X_0) &= \frac{1}{(u-d)\,(1-p+p\beta)\,W_0}\Bigg\{r\,(\beta-1)\,W_0 \\ &\quad + \frac{1}{r^{N-1}}\sum_{i=0}^{N-1}\binom{N-1}{i} p^i\,(1-p)^{N-1-i}\, c(u^{i+1} d^{N-1-i} X_0) \\ &\quad - \frac{\beta}{r^{N-1}}\sum_{i=0}^{N-1}\binom{N-1}{i} p^i\,(1-p)^{N-1-i}\, c(u^{i} d^{N-i} X_0)\Bigg\}. \end{aligned} \tag{4.113}$$

Using the definition of the utility indifference price and hedge as given in Equations (4.90) and (4.91) respectively, we show that

$$\nu^{(s)}(X_0) = \frac{1}{r^N}\sum_{i=0}^{N}\binom{N}{i} p^i\,(1-p)^{N-i}\, c(u^i d^{N-i} X_0) \tag{4.114}$$

and

$$\begin{aligned}\Delta^{(s)}(X_0) &= \frac{1}{r^{N-1}(u-d)\,X_0} \\ &\times \sum_{i=0}^{N-1}\binom{N-1}{i}p^i\,(1-p)^{N-1-i}\left[c(u^{i+1}d^{N-1-i}X_0)-c(u^id^{N-i}X_0)\right]. \quad (4.115)\end{aligned}$$

In this special case, since it is possible to perfectly replicate the option, observe that the utility indifference selling price and hedge are no longer dependent on the investor's wealth and level of risk aversion. As expected, the results that we obtain are consistent with the option price and hedge from the binomial model. Note that we will achieve the same results for the utility indifference buying price and hedge.

We now proceed to show that the marginal utility indifference price is also equivalent to the perfect replication price. Recall that the pricing formula is given by Equation (4.102). We need to determine $H_0(X_0)$ via the recursive algorithm in Equation (4.100). Starting at time period $N-1$, from Equation (4.98), we have

$$H_{N-1}(X_{N-1}) = q\left\{r+(u-r)\,A^*_{N-1}\right\}^{\gamma-1}C_u + (1-q)\left\{r+(d-r)\,A^*_{N-1}\right\}^{\gamma-1}C_d, \quad (4.116)$$

where $C_u = c(uX_{N-1})$ and $C_d = c(dX_{N-1})$. Substituting in A^*_{N-1} from Equation (4.104) and simplifying,

$$H_{N-1}(X_{N-1}) = \left\{\frac{pC_u+(1-p)\,C_d}{r}\right\}r^\gamma\,(1-p+p\beta)^{1-\gamma}\left(\frac{1-q}{1-p}\right). \quad (4.117)$$

Proceeding to time period $N-2$ and comparing Equation (4.100) with Equation (4.98), we note that $H_{N-2}(X_{N-2})$ is of the same form as $H_{N-1}(X_{N-1})$ up to a constant coefficient. In general, $H_{N-k}(X_{N-k})$ is of the same form as $H_{N-k+1}(X_{N-k+1})$, which allows us to immediately deduce that

$$\begin{aligned}H_0(X_0) &= \left\{\frac{1}{r^N}\sum_{i=0}^{N}\binom{N}{i}p^i\,(1-p)^{N-i}\,c(u^id^{N-i}X_0)\right\} \\ &\times\left\{r^\gamma\,(1-p+p\beta)^{1-\gamma}\left(\frac{1-q}{1-p}\right)\right\}^N. \quad (4.118)\end{aligned}$$

Substituting the above expression for $H_0(X_0)$ and Equation (4.107) into the pricing formula given by Equation (4.102), we obtain

$$\tilde{\nu}(X_0) = \frac{1}{r^N} \sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} c(u^i d^{N-i} X_0), \tag{4.119}$$

which is as required.

In the special case where the risky asset follows a binomial price process, we have shown that the utility indifference price and marginal utility indifference price reduce to the perfect replication price. The advantage of utility indifference pricing and marginal utility indifference pricing in our discrete time model is that they are valid for an underlying risky asset with a general price process. In the next section, we consider an example where the price of the risky asset follows a multiplicative trinomial tree and perfect replication is no longer possible. We present an approximate replication approach to pricing options in the trinomial tree. Comparisons are then made between the prices of a European call option computed from this approximate replication approach and the utility maximisation approach (for an investor with the exponential utility function).

4.5 Trinomial Price Process

Recall that s_n and r_n denote one plus the returns of the risky and risk-free assets from time period n to $n+1$ respectively. Assume that s_n ($n = 0, \ldots, N-1$) are independent and identically distributed random variables such that

$$s_n = \begin{cases} u & \text{with probability } q_u, \\ h & \text{with probability } q_h, \\ d & \text{with probability } q_d. \end{cases} \tag{4.120}$$

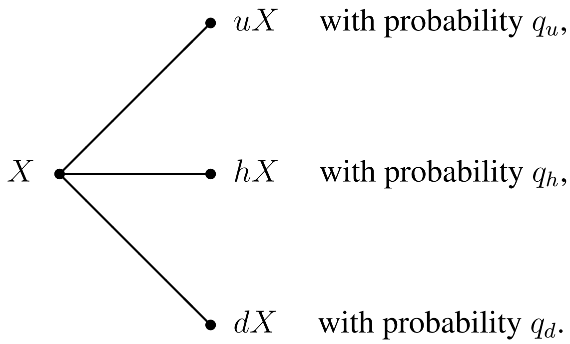
In other words, the price of the risky asset is assumed to follow a multiplicative trinomial tree. We also assume that $ud = h$ in order to obtain a re-combining tree. In addition, assume that

$$r_n = r \tag{4.121}$$

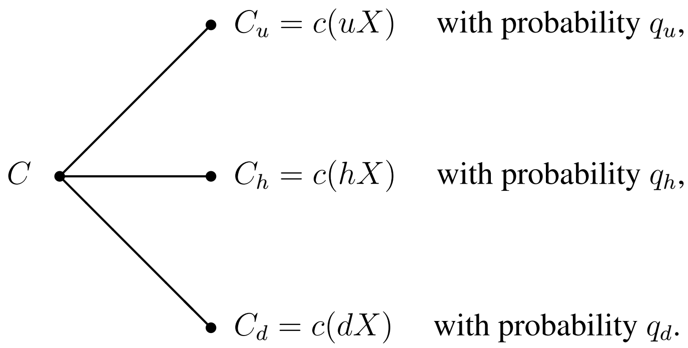
for $n = 0, \ldots, N-1$, where $d < r < u$. In other words, the risk-free asset is assumed to have a constant return. For a comparison with utility indifference pricing and marginal utility indifference pricing, we present an alternative pricing and hedging strategy that approximately replicates the option payoff.

4.5.1 Approximate Replication with Minimum Variance

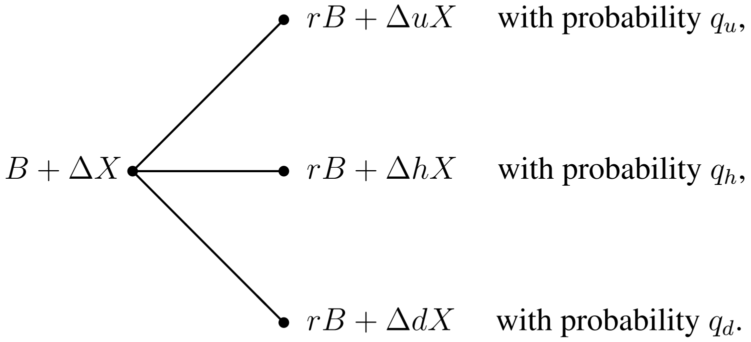
We start by considering a one-period model, where the initial price of the risky asset is X. The price of the risky asset at the end of the time period is represented by the diagram



Consider a European option with initial value C, which is to be determined. Let C_u, C_h and C_d be its end-of-period value when the corresponding price of the risky asset is uX, hX and dX respectively. Assuming that the option expires at the end of the period with a payoff function c that depends on the price of the underlying risky asset, we have



Construct a portfolio consisting of B dollars of the risk-free asset and Δ shares of the risky asset. The initial cost of setting up this portfolio is $B + \Delta X$. At the end of the time period, its value will be



Unlike the binomial model, it is not possible to form a portfolio that perfectly replicates the option payoff at expiration as it involves solving a system of three equations with only two unknowns B and Δ. Any choice of B and Δ inevitably results in a replication error ϵ, which we define as the difference between the end-of-period values of the portfolio and the option. A natural alternative strategy will be to construct an approximately replicating portfolio such that Var $[\epsilon]$ is minimised together with the constraint $\mathbb{E}[\epsilon] = 0$. Equivalently, we will choose B and Δ such that $\mathbb{E}[\epsilon^2]$ is minimised. Consider

$$\mathbb{E}\left[\epsilon^2\right] = q_u \left(rB + \Delta uX - C_u\right)^2 + q_h \left(rB + \Delta hX - C_h\right)^2 + q_d \left(rB + \Delta dX - C_d\right)^2 . \tag{4.122}$$

From the first order conditions $\dfrac{\partial \mathbb{E}[\epsilon^2]}{\partial B} = 0$ and $\dfrac{\partial \mathbb{E}[\epsilon^2]}{\partial \Delta} = 0$, we obtain the equations

$$rB + \left(q_u u + q_h h + q_d d\right) \Delta X = \left(q_u C_u + q_h C_h + q_d C_d\right) \tag{4.123}$$

and

$$\left(q_u u + q_h h + q_d d\right) rB + \left(q_u u^2 + q_h h^2 + q_d d^2\right) \Delta X = \left(q_u u C_u + q_h h C_h + q_d d C_d\right) . \tag{4.124}$$

For convenience, we define the constant $\zeta = q_u q_h \left(u - h\right)^2 + q_u q_d \left(u - d\right)^2 + q_h q_d \left(h - d\right)^2$.

Solving Equations (4.123) and (4.124) simultaneously, we find that

$$\Delta = \frac{q_u \alpha_u C_u + q_h \alpha_h C_h + q_d \alpha_d C_d}{X}, \tag{4.125}$$

where

$$\alpha_u = \left\{ q_h (u - h) + q_d (u - d) \right\} / \zeta, \tag{4.126}$$

$$\alpha_h = \left\{ -q_u (u - h) + q_d (h - d) \right\} / \zeta, \tag{4.127}$$

$$\alpha_d = \left\{ -q_u (u - d) - q_h (h - d) \right\} / \zeta. \tag{4.128}$$

In addition, we have

$$B = \frac{q_u \beta_u C_u + q_h \beta_h C_h + q_d \beta_d C_d}{r}, \tag{4.129}$$

where

$$\beta_u = \left\{ -q_h h (u - h) - q_d d (u - d) \right\} / \zeta, \tag{4.130}$$

$$\beta_h = \left\{ q_u u (u - h) - q_d d (h - d) \right\} / \zeta, \tag{4.131}$$

$$\beta_d = \left\{ q_u u (u - d) + q_h h (h - d) \right\} / \zeta. \tag{4.132}$$

It can be shown that $\mathbb{E}\left[\epsilon^2\right]$ is a minimum at these values of Δ and B. Moreover, it can also be verified that the constraint $\mathbb{E}\left[\epsilon\right] = 0$ is satisfied. This constraint implies that the expected end-of-period values of the portfolio and the option are equal. Therefore, we assume that one will expect the initial values of the portfolio and the option to be equal, that is,

$$C = B + \Delta X. \tag{4.133}$$

Substituting in Equations (4.125) and (4.129), we obtain

$$C = \frac{q_u \eta_u C_u + q_h \eta_h C_h + q_d \eta_d C_d}{r}, \tag{4.134}$$

where

$$\eta_u = \{q_h (u-h)(r-h) + q_d (u-d)(r-d)\}/\zeta, \tag{4.135}$$

$$\eta_h = \{q_u (u-h)(u-r) + q_d (h-d)(r-d)\}/\zeta, \tag{4.136}$$

$$\eta_d = \{q_u (u-d)(u-r) + q_h (h-d)(h-r)\}/\zeta. \tag{4.137}$$

Note that if we set $q_h = 0$, the trinomial tree reduces to the binomial tree and our results for B, Δ and C are consistent with the binomial model.

This one-period model can be easily extended to the multi-period case (similar to the standard computation of the binomial model). Suppose we have a N-period model and an option that expires at the end of N periods. At the expiration date, the price of the option will be equal to its payoff. Therefore, starting from period $N-1$ and working recursively backwards in time, the price of the option can be computed from repeated applications of Equation (4.134). In addition, the hedge ratio is obtained from Equation (4.125). Having presented an approximate replication approach that minimises the variance of the replication error with zero mean error, we proceed to compare this approach with the utility maximisation approach in a numerical example.

First, we recall that the utility indifference selling price and hedge are given by Equations (4.35) and (4.36) respectively. For the portfolio with a short position in the option, we are required to determine the optimal investment in the risky asset $a_0^{(so)*}(X_0)$ and the optimal value function $V_0^{(so)}(X_0)$. They are obtained by solving Equations (4.23) and (4.24) at period $N-1$, followed by Equations (4.29) and (4.30) recursively backwards in time. Note that a straightforward modification to the aforementioned equations will give us the utility indifference buying price and hedge. We also recall that the marginal utility indifference price is given by Equation (4.47), where $H_0(X_0)$ is obtained by solving Equations (4.43) and (4.45) recursively backwards in time.

4.5.2 Results

Consider a N-period option pricing model and an investor with the exponential utility function $U(W) = -e^{-\kappa W}$, $\kappa > 0$. We wish to value a European call option expiring at the end of N periods (at time T) with a strike price of K. The payoff function is given by $c(X_N) = \max(X_N - K, 0)$. Assume that the price of the risky asset follows a multiplicative trinomial tree with an initial price of X_0. Recall that the parameters of the trinomial tree are u, h, d, q_u, q_h, q_d (see Equation (4.120)) and one plus the return of the risk-free asset is a constant r (see Equation (4.121)), where $d < r < u$.

In order to specify the parameters of the trinomial tree, we assume that the trinomial price process approximates the geometric Brownian motion

$$dX(t) = \alpha X(t)dt + \sigma X(t)dZ(t) \tag{4.138}$$

with constant drift α and volatility σ, where $X(t)$ is the price of the risky asset and $Z(t)$ is a standard Brownian motion. Suppose that each time period between n and $n+1$ has an interval of $\delta t = \frac{T}{N}$. Then, we have

$$\ln\left[\frac{X(t+\delta t)}{X(t)}\right] = \left(\alpha - \frac{1}{2}\sigma^2\right)\delta t + \sigma\left\{Z(t+\delta t) - Z(t)\right\}, \tag{4.139}$$

which is a normal random variable with mean $\left(\alpha - \frac{1}{2}\sigma^2\right)\delta t$ and variance $\sigma^2\delta t$. The parameters of the trinomial tree are chosen so that

$$\mathbb{E}\left[\ln s_n\right] = q_u \ln u + q_h \ln h + q_d \ln d = \left(\alpha - \frac{1}{2}\sigma^2\right)\delta t \tag{4.140}$$

and

$$\text{Var}\left[\ln s_n\right] = q_u\left(\ln u\right)^2 + q_h\left(\ln h\right)^2 + q_d\left(\ln d\right)^2 + o(\delta t) = \sigma^2\delta t \tag{4.141}$$

for a small δt. Recall that $ud = h$ for a re-combining tree and $q_u + q_h + q_d = 1$. We use the parametrisation by Kamrad and Ritchken (1991), who proposed that:

$$u = e^{\theta\sigma\sqrt{\delta t}},\ h = 1,\ d = \frac{1}{u},$$

$$q_u = \frac{1}{2\theta^2} + \frac{1}{2\theta\sigma}\left(\alpha - \frac{1}{2}\sigma^2\right)\sqrt{\delta t},\ q_h = 1 - \frac{1}{\theta^2},\ q_d = \frac{1}{2\theta^2} - \frac{1}{2\theta\sigma}\left(\alpha - \frac{1}{2}\sigma^2\right)\sqrt{\delta t},$$

where $\theta \geq 1$ is a stretch parameter that can be chosen to determine the jump sizes and probabilities of the trinomial tree. Observe that when $\theta = 1$, $q_h = 0$ and the trinomial tree reduces to a binomial tree. In addition, let $r = e^{R\delta t}$, where R is the annualised risk-free rate. Choose the following parameter values:

$$T = 1 \text{ (year)},\ N = 6,\ X_0 = 100,\ \alpha = 0.15,\ \sigma = 0.25,\ R = 0.1,\ \theta = \sqrt{\frac{3}{2}}.$$

We compute and compare the option prices from the utility maximisation approach and the approximate replication approach for various values of the investor's risk aversion κ $(= 0.01, 0.1, 1)$ and the strike K $(= 80, 90, 100, 110, 120)$. The results are presented in Table 4.1. The minimum variance price is observed to be just slightly lower than the marginal utility indifference price. Moreover, both of these prices lie between the utility indifference buying (i.e. bid) and selling (i.e. ask) price. When the investor has low risk aversion ($\kappa = 0.01$), the utility indifference bid-ask spread is small. As the level of risk aversion increases, the utility indifference bid price decreases, the ask price increases and the bid-ask spread widens. This result is intuitive since one would expect an investor who is more risk averse to purchase the option at a lower price and sell at a higher price. However, we also observe that the marginal utility indifference price does not vary with the level of risk aversion. An explanation for this observation is to recognise that the source of risk to the investor arises from an option position that cannot be perfectly replicated. Furthermore, marginal utility indifference pricing corresponds to holding an infinitesimal position in the option, which implies that the investor faces an infinitesimal level of risk. Therefore, in this case, the marginal utility indifference price is not influenced by the investor's aversion to risk. By construction, the minimum variance price is independent of the investor's level of risk aversion.

In this chapter, we have developed utility indifference pricing and marginal utility indifference pricing in a discrete time model without transaction costs. We showed that both the utility indifference price and marginal utility indifference price reduce to the perfect replication price when the underlying risky asset is assumed to follow a binomial price process. In the case where the price of the risky asset is assumed to follow a trinomial tree, we compared the utility maximisation approach with an approximate replication approach that minimised the variance of the replication error. In the next chapter, we will investi-

Table 4.1: Option Prices in a Trinomial Tree

Utility Bid=Utility indifference buying price; Utility Ask=Utility indifference selling price; Marginal Utility=Marginal utility indifference price; Minimum Variance=Price from minimising variance of replication error.

Strike	Utility Bid	Utility Ask	Marginal Utility	Minimum Variance
$\kappa = 0.01$				
80	28.593	28.601	28.597	28.594
90	21.144	21.162	21.153	21.148
100	14.774	14.809	14.792	14.784
110	10.105	10.141	10.123	10.115
120	6.601	6.635	6.618	6.610
$\kappa = 0.1$				
80	28.557	28.635	28.597	28.594
90	21.058	21.242	21.153	21.148
100	14.604	14.959	14.792	14.784
110	9.934	10.294	10.123	10.115
120	6.442	6.778	6.618	6.610
$\kappa = 1$				
80	28.209	28.890	28.597	28.594
90	20.092	21.814	21.153	21.148
100	12.378	15.783	14.792	14.784
110	7.745	11.288	10.123	10.115
120	4.506	7.689	6.618	6.610

gate utility indifference pricing and marginal utility indifference pricing with proportional transaction costs.

Chapter 5

Indifference Option Pricing with Transaction Costs

In this chapter, we extend the utility indifference pricing and marginal utility indifference pricing approach from Chapter 4 to incorporate proportional transaction costs. We focus on an investor with the exponential utility function and use some of the results from our previous analysis of the portfolio selection problem (without an option position) in Chapter 2. A numerical example is presented for the case where the underlying risky asset follows a binomial price process.

Recall that the general definitions of the utility indifference selling price $\nu^{(s)}$, utility indifference buying price $\nu^{(b)}$ and marginal utility indifference price $\tilde{\nu}$ are given by Equations (4.3), (4.5) and (4.9) respectively. In order to obtain the utility indifference price of an option, we are required to determine the optimal value functions for the portfolios with and without a position in the option. On the other hand, we only need to consider the portfolio without an option position for the marginal utility indifference price. Therefore, let us recall the market model with proportional transaction costs as described in Section 2.1.

5.1 Market Model with Transaction Costs

Consider a multi-period portfolio selection model with N periods. Suppose that time period n ($n = 0, 1, \ldots, N-1, N$) indexes discrete time $t_0 < t_1 < \ldots < t_{N-1} < t_N$, where $t_0 = 0$

is the initial time and $t_N = T$ is the terminal time. Assume an investor who holds a portfolio that is divided between one risk-free asset and one risky asset, where the price of each asset evolves in discrete time. A cost that is proportional to the value of the transaction is incurred each time the investor buys or sells the risky asset. The investor's objective is to maximise the expected utility of terminal wealth by rebalancing the portfolio optimally at each step of the investment process.

At time period n, let W_n denote the wealth of the portfolio and let a_n be the dollar value of the risky asset inherited from the previous period. Therefore, the corresponding value of the risk-free asset is $W_n - a_n$. The investor rebalances the portfolio at time period n by buying l_n or selling m_n dollars of the risky asset. Suppose that s_n denotes one plus the (random) return of the risky asset from time period n to $n+1$. Thus, the value of the risky asset at time period $n+1$ inherited from period n is

$$a_{n+1} = s_n \left(a_n + l_n - m_n\right) \tag{5.1}$$

for $n = 0, \ldots, N-1$.

Furthermore, let λ_n and μ_n be the proportion costs of buying and selling the risky asset respectively at time period n. These costs reduce the wealth invested in the risk-free asset, resulting in a value of $W_n - (a_n + l_n - m_n) - \lambda_n l_n - \mu_n m_n$. Suppose that r_n denotes one plus the (sure) return of the risk-free asset from time period n to $n+1$. The investor's wealth at time period $n+1$ is then given by

$$W_{n+1} = r_n W_n + (s_n - r_n)\left(a_n + l_n - m_n\right) - r_n \lambda_n l_n - r_n \mu_n m_n \tag{5.2}$$

for $n = 0, \ldots, N-1$. It is convenient to write

$$W_{n+1} = r_n W_n + F_n, \tag{5.3}$$

where

$$F_n = (s_n - r_n)(a_n + l_n - m_n) - r_n \lambda_n l_n - r_n \mu_n m_n. \tag{5.4}$$

Assume that simultaneous buying and selling of the risky asset is not allowed, since it will not be optimal due to the higher costs as compared to only buying or selling the asset. The investor is thus left with three possible choices, which is to buy, to sell or not to transact the risky asset. The investor's decision at each time step will affect the wealth and risky asset inherited at the next time step.

Let X_n be the price of one unit of the risky asset at time period n. The price of the risky asset at period $n+1$ is given by

$$X_{n+1} = s_n X_n \tag{5.5}$$

for $n = 0, \ldots, N-1$. Consider a European option, expiring at the end of N periods (at time T) and yielding a payoff $C_N = c(X_N)$ that depends on the price of the risky asset X_N at expiry. Suppose that the investor takes a position in the option at the initial time and wishes to determine its value. In the presence of transaction costs, we adopt the utility maximisation approach to price the option. This is an extension of the utility indifference pricing and marginal utility indifference pricing approach from Chapter 4, with the inclusion of proportional transaction costs. In order to determine the option prices, we are required to solve the portfolio selection problem with and without a position in the option.

We focus on the case of an investor with constant absolute risk aversion (i.e. exponential utility function). Assume that the investor's utility of wealth is

$$U(W) = -e^{-\kappa W}, \tag{5.6}$$

where $\kappa > 0$ is the coefficient of absolute risk aversion.

5.2 Portfolio Selection with Option Position

Consider the portfolio selection problem where the investor sells a European option at the initial time. The option expires at the end of N periods with payoff $C_N = c(X_N)$. The optimal value function is defined as

$$J^{(so)}(W_0, a_0, X_0) = \max \mathbb{E}\left[U(W_N - C_N)\right], \tag{5.7}$$

where the maximisation is over the investments $(l_0, m_0), \ldots, (l_{N-1}, m_{N-1})$ in the risky asset and the expectation is taken with respect to the random variables $s_0, \ldots, s_{N-1}$. Recall that the superscript "so" denotes selling an option at the initial time.

The method that we adopt to solve for the optimal investment strategy and optimal value function is similar to the portfolio selection problem without the option position (see Section 2.2). Specifically, dynamic programming is again applied to reduce the original problem to a sequence of more manageable sub-problems. However, compared to the case of the portfolio without a position in the option, the optimal value function now depends additionally on the price of the risky asset due to the option payoff. This additional state variable generally makes the dynamic programming algorithm more difficult to solve. The dynamic programming algorithm, which starts at period $N-1$ and proceeds backwards in time, is given as

$$J_N^{(so)}(W_N, a_N, X_N) = U(W_N - C_N) \tag{5.8}$$

and

$$J_{N-k}^{(so)}(W_{N-k}, a_{N-k}, X_{N-k}) = \max \mathbb{E}_{N-k}\left[J_{N-k+1}^{(so)}(W_{N-k+1}, a_{N-k+1}, X_{N-k+1})\right] \tag{5.9}$$

for $k = 1, \ldots, N$. Here, the maximisation is over the investment (l_{N-k}, m_{N-k}) in the risky asset and $\mathbb{E}_{N-k}$ is the conditional expectation with respect to the random variable s_{N-k} given W_{N-k}, a_{N-k} and X_{N-k}.

Time Period $N-1$

The optimal value function at time period $N-1$ is

$$\begin{aligned} J_{N-1}^{(so)}(W_{N-1}, a_{N-1}, X_{N-1}) &= \max \mathbb{E}_{N-1}\left[U(W_N - c(X_N))\right] \\ &= \max \mathbb{E}_{N-1}\left[-e^{-\kappa\{W_N - c(X_N)\}}\right]. \end{aligned} \tag{5.10}$$

Using Equations (5.3) and (5.5), we have

$$J^{(so)}_{N-1}(W_{N-1}, a_{N-1}, X_{N-1}) = e^{-\kappa r_{N-1} W_{N-1}} \max \mathbb{E}_{N-1}\left[-e^{-\kappa\{F_{N-1} - c(s_{N-1}X_{N-1})\}}\right]. \tag{5.11}$$

The term in W_{N-1} is taken out of the conditional expectation $\mathbb{E}_{N-1}$ as it is assumed to be given at time period $N-1$. Furthermore unlike F_{N-1}, W_{N-1} does not depend on the investor's decision to buy, sell or not to transact. Therefore, we have reduced the problem to one of maximising

$$V^{(so)}_{N-1}(a_{N-1}, X_{N-1}) = \mathbb{E}_{N-1}\left[-e^{-\kappa\{F_{N-1} - c(s_{N-1}X_{N-1})\}}\right] \tag{5.12}$$

with respect to (l_{N-1}, m_{N-1}). This represents a reduction from three to two state variables as we have factored out the wealth variable W_{N-1}. Nonetheless, in comparison with the portfolio selection problem without the option position (see Equation (2.13)), we now need to take into account an additional price variable X_{N-1}. Our subsequent analysis of the optimal investment strategy and optimal value function is analogous to that found in Section 2.2, which corresponds to the case without the option position. Therefore, instead of repeating the analysis in its entirety, we will only present the essential equations and highlight the differences between the two cases.

The investor has the choice of buying, selling or not transacting in the risky asset, which affects the definition of F_{N-1}. Obtaining the optimal investment strategy is equivalent to solving for the no-transaction region denoted by $a^{(so)-}_{N-1} \leq a_{N-1} \leq a^{(so)+}_{N-1}$. The optimal buy boundary $a^{(so)-}_{N-1}$ delineates the buy and no-transaction regions while the optimal sell boundary $a^{(so)+}_{N-1}$ delineates the sell and no-transaction regions. Therefore, $a^{(so)-}_{N-1}$ is given by the condition $\dfrac{\partial V^{(so)}_{N-1}}{\partial l_{N-1}} = 0$ in the buy region and similarly, $a^{(so)+}_{N-1}$ is given by the condition $\dfrac{\partial V^{(so)}_{N-1}}{\partial m_{N-1}} = 0$ in the sell region. Specifically, $a^{(so)-}_{N-1}$ and $a^{(so)+}_{N-1}$ are the solutions to the equations

$$\mathbb{E}_{N-1}\left[\{s_{N-1} - (1+\lambda_{N-1})r_{N-1}\}\, e^{-\kappa\left\{(s_{N-1}-r_{N-1})a^{(so)-}_{N-1} - c(s_{N-1}X_{N-1})\right\}}\right] = 0 \tag{5.13}$$

and

$$\mathbb{E}_{N-1}\left[\left\{s_{N-1}-(1-\mu_{N-1})r_{N-1}\right\}e^{-\kappa\left\{(s_{N-1}-r_{N-1})a_{N-1}^{(so)+}-c(s_{N-1}X_{N-1})\right\}}\right]=0, \qquad (5.14)$$

respectively. Compared to Equations (2.17) and (2.18), the above equations contain the additional option payoff $c(s_{N-1}X_{N-1})$ that depends on the price of the risky asset X_{N-1}. Therefore, the optimal buy and sell boundaries will also depend on the price of the risky asset, which we denote as $a_{N-1}^{(so)-}(X_{N-1})$ and $a_{N-1}^{(so)+}(X_{N-1})$ to emphasise this dependence on X_{N-1}. In contrast, the optimal buy and sell boundaries for the case without the option position are independent of the price of the risky asset. Having determined the form of the optimal boundaries, we proceed to state the optimal value function and its first derivative in the three regions.

In the buy region $a_{N-1} < a_{N-1}^{(so)-}(X_{N-1})$, the investor buys $l_{N-1} = a_{N-1}^{(so)-}(X_{N-1})-a_{N-1}$ of the risky asset. So we have

$$F_{N-1}^{(B)} = (s_{N-1}-r_{N-1})\,a_{N-1}^{(so)-}(X_{N-1}) - r_{N-1}\lambda_{N-1}\left[a_{N-1}^{(so)-}(X_{N-1})-a_{N-1}\right], \qquad (5.15)$$

$$V_{N-1}^{(so,B)}(a_{N-1},X_{N-1}) = \mathbb{E}_{N-1}\left[-e^{-\kappa\left\{F_{N-1}^{(B)}-c(s_{N-1}X_{N-1})\right\}}\right], \qquad (5.16)$$

and

$$\frac{\partial V_{N-1}^{(so,B)}}{\partial a_{N-1}}(a_{N-1},X_{N-1}) = \mathbb{E}_{N-1}\left[\kappa r_{N-1}\lambda_{N-1}e^{-\kappa\left\{F_{N-1}^{(B)}-c(s_{N-1}X_{N-1})\right\}}\right]. \qquad (5.17)$$

In the sell region where $a_{N-1} > a_{N-1}^{(so)+}(X_{N-1})$, the investor sells $m_{N-1} = a_{N-1} - a_{N-1}^{(so)+}(X_{N-1})$ of the risky asset so that

$$F_{N-1}^{(S)} = (s_{N-1}-r_{N-1})\,a_{N-1}^{(so)+}(X_{N-1}) - r_{N-1}\mu_{N-1}\left[a_{N-1}-a_{N-1}^{(so)+}(X_{N-1})\right], \qquad (5.18)$$

$$V_{N-1}^{(so,S)}(a_{N-1},X_{N-1}) = \mathbb{E}_{N-1}\left[-e^{-\kappa\left\{F_{N-1}^{(S)}-c(s_{N-1}X_{N-1})\right\}}\right], \qquad (5.19)$$

and

$$\frac{\partial V_{N-1}^{(so,S)}}{\partial a_{N-1}}(a_{N-1}, X_{N-1}) = \mathbb{E}_{N-1}\left[-\kappa r_{N-1}\mu_{N-1}e^{-\kappa\left\{F_{N-1}^{(S)}-c(s_{N-1}X_{N-1})\right\}}\right]. \qquad (5.20)$$

Finally, in the no-transaction region where $a_{N-1}^{(so)-}(X_{N-1}) \leq a_{N-1} \leq a_{N-1}^{(so)+}(X_{N-1})$, we have

$$F_{N-1}^{(N)} = (s_{N-1} - r_{N-1})\, a_{N-1}, \qquad (5.21)$$

$$V_{N-1}^{(so,N)}(a_{N-1}, X_{N-1}) = \mathbb{E}_{N-1}\left[-e^{-\kappa\left\{F_{N-1}^{(N)}-c(s_{N-1}X_{N-1})\right\}}\right], \qquad (5.22)$$

and

$$\frac{\partial V_{N-1}^{(so,N)}}{\partial a_{N-1}}(a_{N-1}, X_{N-1}) = \mathbb{E}_{N-1}\left[\kappa\,(s_{N-1} - r_{N-1})\, e^{-\kappa\left\{F_{N-1}^{(N)}-c(s_{N-1}X_{N-1})\right\}}\right]. \qquad (5.23)$$

Once again, we use the superscripts "B", "S" and "N" to denote the buy, sell and no-transaction regions respectively.

Having obtained the optimal boundaries and the optimal value function at time period $N-1$, we apply the principle of dynamic programming to solve the problem at period $N-2$. In general, working recursively backwards in time, one is thus able to solve the problem at period $N-k$ ($k = 2, \ldots, N$) by using the optimal solutions from the time steps ahead.

Time Period $N-k$

In general, the optimal value function at time period $N-k$ ($k = 2, \ldots, N$) is given by

$$\begin{aligned} J_{N-k}^{(so)}(W_{N-k}, a_{N-k}, X_{N-k}) &= \max \mathbb{E}_{N-k}\left[J_{N-k+1}^{(so)}(W_{N-k+1}, a_{N-k+1}, X_{N-k+1})\right] \\ &= \max \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}W_{N-k+1}}V_{N-k+1}^{(so)}(a_{N-k+1}, X_{N-k+1})\right], \end{aligned} \qquad (5.24)$$

where $V^{(so)}_{N-k+1}(a_{N-k+1}, X_{N-k+1})$ is the optimal value function from the time step ahead. Recall that $W_{N-k+1} = r_{N-k}W_{N-k} + F_{N-k}$. Thus, we can write

$$\begin{aligned} J^{(so)}_{N-k}(W_{N-k}, a_{N-k}, X_{N-k}) &= e^{-\kappa r_{N-1}\cdots r_{N-k}W_{N-k}} \max \mathbb{E}_{N-k}\Big[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}} \\ &\quad \times V^{(so)}_{N-k+1}(a_{N-k+1}, X_{N-k+1})\Big]. \end{aligned} \tag{5.25}$$

Having taken the term in W_{N-k} out of the conditional expectation $\mathbb{E}_{N-k}$, the problem is reduced to one of maximising

$$V^{(so)}_{N-k}(a_{N-k}, X_{N-k}) = \mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}} V^{(so)}_{N-k+1}(a_{N-k+1}, X_{N-k+1})\right] \tag{5.26}$$

with respect to (l_{N-k}, m_{N-k}). Recall that a_{N-k+1}, F_{N-k} and X_{N-k+1} are given by Equations (5.1), (5.4) and (5.5) respectively. The optimal boundaries are found to satisfy equations that are similar to Equations (2.43) and (2.44), with the difference being having an additional dependence on the price of the risky asset X_{N-k}. Therefore, the optimal buy boundary $a^{(so)-}_{N-k}(X_{N-k})$ is the solution to the equation

$$\begin{aligned} &\mathbb{E}_{N-k}\Bigg[\Bigg\{-\kappa r_{N-1}\cdots r_{N-k+1}\big[s_{N-k} - (1+\lambda_{N-k})\, r_{N-k}\big] V_{N-k+1}(a_{N-k+1}, X_{N-k+1}) \\ &+ s_{N-k}\frac{\partial V_{N-k+1}}{\partial a_{N-k+1}}(a_{N-k+1}, X_{N-k+1})\Bigg\} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}}\Bigg] = 0 \end{aligned} \tag{5.27}$$

with $F_{N-k} = (s_{N-k} - r_{N-k})\, a^{(so)-}_{N-k}$, $a_{N-k+1} = s_{N-k}a^{(so)-}_{N-k}$ and $X_{N-k+1} = s_{N-k}X_{N-k}$. Meanwhile, the optimal sell boundary $a^{(so)+}_{N-k}(X_{N-k})$ is the solution to the equation

$$\begin{aligned} &\mathbb{E}_{N-k}\Bigg[\Bigg\{\kappa r_{N-1}\cdots r_{N-k+1}\big[s_{N-k} - (1-\mu_{N-k})\, r_{N-k}\big] V_{N-k+1}(a_{N-k+1}, X_{N-k+1}) \\ &- s_{N-k}\frac{\partial V_{N-k+1}}{\partial a_{N-k+1}}(a_{N-k+1}, X_{N-k+1})\Bigg\} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}}\Bigg] = 0 \end{aligned} \tag{5.28}$$

with $F_{N-k} = (s_{N-k} - r_{N-k})\, a^{(so)+}_{N-k}$, $a_{N-k+1} = s_{N-k}a^{(so)+}_{N-k}$ and $X_{N-k+1} = s_{N-k}X_{N-k}$. Having determined the form of the optimal boundaries, we proceed to state the optimal value function and its first derivative in the three regions.

In the buy region $a_{N-k} < a^{(so)-}_{N-k}(X_{N-k})$, the investor buys $l_{N-k} = a^{(so)-}_{N-k}(X_{N-k}) - a_{N-k}$ of the risky asset. So we have

$$F^{(B)}_{N-k} = (s_{N-k} - r_{N-k})\, a^{(so)-}_{N-k}(X_{N-k}) - r_{N-k}\lambda_{N-k}\left[a^{(so)-}_{N-k}(X_{N-k}) - a_{N-k}\right], \quad (5.29)$$

$$\begin{aligned} V^{(so,B)}_{N-k}(a_{N-k}, X_{N-k}) &= \mathbb{E}_{N-k}\Big[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}} \\ &\quad \times V^{(so)}_{N-k+1}\big(s_{N-k}a^{(so)-}_{N-k}(X_{N-k}), s_{N-k}X_{N-k}\big)\Big], \end{aligned} \quad (5.30)$$

and

$$\begin{aligned} \frac{\partial V^{(so,B)}_{N-k}}{\partial a_{N-k}}(a_{N-k}, X_{N-k}) &= \mathbb{E}_{N-k}\Big[-\kappa r_{N-1}\cdots r_{N-k}\lambda_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}} \\ &\quad \times V^{(so)}_{N-k+1}\big(s_{N-k}a^{(so)-}_{N-k}(X_{N-k}), s_{N-k}X_{N-k}\big)\Big]. \end{aligned} \quad (5.31)$$

In the sell region where $a_{N-k} > a^{(so)+}_{N-k}(X_{N-k})$, the investor sells $m_{N-k} = a_{N-k} - a^{(so)+}_{N-k}(X_{N-k})$ of the risky asset so that

$$F^{(S)}_{N-k} = (s_{N-k} - r_{N-k})\, a^{(so)+}_{N-k}(X_{N-k}) - r_{N-k}\mu_{N-k}\left[a_{N-k} - a^{(so)+}_{N-k}(X_{N-k})\right], \quad (5.32)$$

$$\begin{aligned} V^{(so,S)}_{N-k}(a_{N-k}, X_{N-k}) &= \mathbb{E}_{N-k}\Big[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}} \\ &\quad \times V^{(so)}_{N-k+1}\big(s_{N-k}a^{(so)+}_{N-k}(X_{N-k}), s_{N-k}X_{N-k}\big)\Big], \end{aligned} \quad (5.33)$$

and

$$\begin{aligned} \frac{\partial V^{(so,S)}_{N-k}}{\partial a_{N-k}}(a_{N-k}, X_{N-k}) &= \mathbb{E}_{N-k}\Big[\kappa r_{N-1}\cdots r_{N-k}\mu_{N-k}e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}} \\ &\quad \times V^{(so)}_{N-k+1}\big(s_{N-k}a^{(so)+}_{N-k}(X_{N-k}), s_{N-k}X_{N-k}\big)\Big]. \end{aligned} \quad (5.34)$$

Finally, in the no-transaction region where $a^{(so)-}_{N-k}(X_{N-k}) \leq a_{N-k} \leq a^{(so)+}_{N-k}(X_{N-k})$, we have

$$F^{(N)}_{N-k} = (s_{N-k} - r_{N-k})\, a_{N-k}, \tag{5.35}$$

$$\begin{aligned} V^{(so,N)}_{N-k}(a_{N-k}, X_{N-k}) &= \mathbb{E}_{N-k}\Big[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}} \\ &\quad \times V^{(so)}_{N-k+1}(s_{N-k}a_{N-k}, s_{N-k}X_{N-k})\Big], \end{aligned} \tag{5.36}$$

and

$$\frac{\partial V^{(so,N)}_{N-k}}{\partial a_{N-k}}(a_{N-k}, X_{N-k}) \tag{5.37}$$

$$= \mathbb{E}_{N-k}\Bigg[\Big\{-\kappa r_{N-1}\cdots r_{N-k+1}\,(s_{N-k} - r_{N-k})\, V^{(so)}_{N-k+1}(s_{N-k}a_{N-k}, s_{N-k}X_{N-k})$$

$$+ s_{N-k}\frac{\partial V^{(so)}_{N-k+1}}{\partial a_{N-k+1}}(s_{N-k}a_{N-k}, s_{N-k}X_{N-k})\Big\} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}\Bigg]. \tag{5.38}$$

Furthermore, the optimal value function of the original problem is given by

$$J^{(so)}_{N-k}(W_{N-k}, a_{N-k}, X_{N-k}) = e^{-\kappa r_{N-1}\cdots r_{N-k}W_{N-k}} V^{(so)}_{N-k}(a_{N-k}, X_{N-k}). \tag{5.39}$$

Applying the dynamic programming algorithm up to $k = N$, we obtain the optimal value function

$$J^{(so)}_0(W_0, a_0, X_0) = e^{-\kappa r_{N-1}\cdots r_0 W_0} V^{(so)}_0(a_0, X_0) \tag{5.40}$$

and the optimal boundaries $a^{(so)-}_0(X_0)$ and $a^{(so)+}_0(X_0)$ at the initial time. Thus, we have determined the solution to the portfolio selection problem with a short position in the option.

In the case where the investor buys an option at the initial time, the optimal value function is defined as

$$J^{(bo)}(W_0, a_0, X_0) = \max \mathbb{E}\left[U(W_N + C_N)\right], \tag{5.41}$$

where the maximisation is over the investments $(l_0, m_0), \ldots, (l_{N-1}, m_{N-1})$ in the risky asset and the expectation is taken with respect to the random variables $s_0, \ldots, s_{N-1}$. The superscript "*bo*" denotes buying an option at the initial time. A straightforward modification of the aforementioned dynamic programming algorithm will enable us to obtain the solution to the portfolio selection problem with a long position in the option. Specifically, replace C_N with $-C_N$ in Equation (5.8). Applying the dynamic programming algorithm with this modified terminal condition will give us the optimal value function

$$J_0^{(bo)}(W_0, a_0, X_0) = e^{-\kappa r_{N-1} \cdots r_0 W_0} V_0^{(bo)}(a_0, X_0) \tag{5.42}$$

and the optimal boundaries $a_0^{(bo)-}(X_0)$ and $a_0^{(bo)+}(X_0)$ at the initial time.

Moreover, if we consider the portfolio without a position in the option, the optimal value function is defined as

$$J(W_0, a_0) = \max \mathbb{E}\left[U(W_N)\right], \tag{5.43}$$

where the maximisation is over the investments $(l_0, m_0), \ldots, (l_{N-1}, m_{N-1})$ in the risky asset and the expectation is taken with respect to the random variables $s_0, \ldots, s_{N-1}$. In this case, by setting $C_N = 0$ in Equation (5.8), the optimal boundaries and optimal value function no longer depend on the price of the risky asset and (as expected) we obtain the dynamic programming algorithm from Section 2.2. Therefore, solving this algorithm gives us the optimal value function

$$J_0(W_0, a_0) = e^{-\kappa r_{N-1} \cdots r_0 W_0} V_0(a_0) \tag{5.44}$$

and the optimal boundaries a_0^- and a_0^+ at the initial time.

Having determined the optimal value functions of the portfolio selection problems with and without a position in the option, one is thus able to obtain the utility indifference price of the option.

5.3 Utility Indifference Price and Hedge

Recall from Section 4.1 that the utility indifference selling price of an option is defined to be the value $\nu^{(s)}$ that satisfies the equation

$$J^{(so)}(W_0 + \nu^{(s)}, a_0, X_0) = J(W_0, a_0) \tag{5.45}$$

for an initial W_0, a_0 and X_0. Applying the dynamic programming algorithm recursively backwards in time from period $N-1$ to period 0, $J^{(so)}(W_0 + \nu^{(s)}, a_0, X_0)$ is given by Equation (5.40) and $J(W_0, a_0)$ by Equation (5.44). From the above pricing definition, we thus have

$$e^{-\kappa r_{N-1}\cdots r_0\left\{W_0+\nu^{(s)}\right\}}V_0^{(so)}(a_0, X_0) = e^{-\kappa r_{N-1}\cdots r_0 W_0}V_0(a_0). \tag{5.46}$$

Simplifying, we obtain

$$\nu^{(s)}(a_0, X_0) = \frac{1}{\kappa r_{N-1}\cdots r_0}\ln\left[\frac{V_0^{(so)}(a_0, X_0)}{V_0(a_0)}\right]. \tag{5.47}$$

We have written the utility indifference selling price of the option as $\nu^{(s)}(a_0, X_0)$ to emphasise its dependence on a_0 and X_0. Not only does the utility indifference price of the option depend on the price of the underlying risky asset, it also depends on the investor's holdings in the risky asset. Similarly, the utility indifference buying price of the option is given by

$$\nu^{(b)}(a_0, X_0) = \frac{-1}{\kappa r_{N-1}\cdots r_0}\ln\left[\frac{V_0^{(bo)}(a_0, X_0)}{V_0(a_0)}\right]. \tag{5.48}$$

Let us investigate further the behaviour of $\nu^{(s)}(a_0, X_0)$ and $\nu^{(b)}(a_0, X_0)$ for a certain range of values of a_0. For instance, consider the utility indifference selling price $\nu^{(s)}(a_0, X_0)$ in the intersection of the buy regions of the portfolios with and without a position in the option. This intersection is given by the region $a_0 < \min\left(a_0^-, a_0^{(so)-}(X_0)\right)$ and we have

$$\nu^{(s)}(a_0, X_0) = \frac{1}{\kappa r_{N-1}\cdots r_0}\ln\left[\frac{V_0^{(so,B)}(a_0, X_0)}{V_0^{(B)}(a_0)}\right]. \tag{5.49}$$

Recall that a_0^- is the optimal buy boundary and $V_0^{(B)}(a_0)$ is the optimal value function in the buy region corresponding to the portfolio without an option position. Similarly, $a_0^{(so)-}(X_0)$ is the optimal buy boundary and $V_0^{(so,B)}(a_0, X_0)$ is the optimal value function in the buy region corresponding to the portfolio with a short position in the option.

From Equations (5.29) and (5.30) with $k = N$, we have

$$V_0^{(so,B)}(a_0, X_0) = e^{\kappa r_{N-1}\cdots r_1 r_0 \lambda_0 \left\{a_0^{(so)-}(X_0)-a_0\right\}} \times \mathbb{E}_0\left[e^{-\kappa r_{N-1}\cdots r_1 (s_0 - r_0) a_0^{(so)-}(X_0)} V_1^{(so)}\left(s_0 a_0^{(so)-}(X_0), s_0 X_0\right)\right]. \quad (5.50)$$

Similar to the above equation, we have

$$V_0^{(B)}(a_0) = e^{\kappa r_{N-1}\cdots r_1 r_0 \lambda_0 \left\{a_0^- - a_0\right\}} \mathbb{E}_0\left[e^{-\kappa r_{N-1}\cdots r_1 (s_0 - r_0) a_0^-} V_1(s_0 a_0^-)\right]. \quad (5.51)$$

Dividing Equation (5.50) by Equation (5.51), we observe that $\dfrac{V_0^{(so,B)}(a_0, X_0)}{V_0^{(B)}(a_0)}$ is independent of a_0 as the terms in a_0 cancel out. Therefore, in the region $a_0 < \min\left(a_0^-, a_0^{(so)-}(X_0)\right)$, we have shown that $\nu^{(s)}(a_0, X_0)$ does not vary with a_0. Moreover, it can also be shown that $\nu^{(s)}(a_0, X_0)$ does not vary with a_0 in the region $a_0 > \max\left(a_0^+, a_0^{(so)+}(X_0)\right)$, which corresponds to the intersection of the sell regions of the portfolios with and without a position in the option. An analogous set of results is also applicable to the utility indifference buying price of the option $\nu^{(b)}(a_0, X_0)$.

In general, we conclude that the utility indifference price does not vary with a_0 in the intersection of the buy regions (or sell regions) of the portfolios with and without a position in the option. This observation will be illustrated in the numerical example that follows.

5.4 Marginal Utility Indifference Price

In this section, we investigate the marginal utility indifference pricing approach in the presence of proportional transaction costs. Using the formula from Equation (4.9), the marginal utility indifference price of an option with payoff $C_N = c(X_N)$ is

$$\tilde{\nu} = \frac{G(W_0, a_0, X_0)}{J'(W_0, a_0)}, \tag{5.52}$$

where

$$G(W_0, a_0, X_0) = \mathbb{E}\left[U'(W_N)C_N\right]. \tag{5.53}$$

Recall that $J(W_0, a_0)$ corresponds to the optimal value function for the portfolio without a position in the option and is given by Equation (5.44). Therefore,

$$J(W_0, a_0) = e^{-\kappa r_{N-1}\cdots r_0 W_0} V_0(a_0). \tag{5.54}$$

Differentiating J with respect to W_0, we obtain

$$J'(W_0, a_0) = -\kappa r_{N-1}\cdots r_0 e^{-\kappa r_{N-1}\cdots r_0 W_0} V_0(a_0). \tag{5.55}$$

Now, consider

$$G(W_0, a_0, X_0) = \mathbb{E}\left[U'(W_N)C_N\right], \tag{5.56}$$

where the expectation is taken with respect to the random variables $s_0, \dots, s_{N-1}$. Recall that the wealth evolves like $W_{n+1} = r_n W_n + F_n$, with $F_n = (s_n - r_n)(a_n + l_n - m_n) - r_n \lambda_n l_n - r_n \mu_n m_n$ depending on the investor's decision to buy, sell or not to transact in the risky asset. Moreover, the inherited holdings in the risky asset evolves like $a_{n+1} = s_n\left(a_n + l_n - m_n\right)$ and the price of the risky asset $X_{n+1} = s_n X_n$. In this case, the investor's strategy corresponds to the optimal strategy for the portfolio selection problem without an option position from Section 2.2. In summary,

$$(l_n, m_n) = \begin{cases} (a_n^- - a_n, 0) & \text{if } a_n < a_n^-, \\ (0, a_n - a_n^+) & \text{if } a_n > a_n^+, \\ (0,0) & \text{otherwise,} \end{cases} \tag{5.57}$$

where a_n^- and a_n^+ are the optimal buy and sell boundaries that satisfy Equations (2.43) and

(2.44). Consequently,

$$F_n = \begin{cases} (s_n - r_n)\, a_n^- - r_n \lambda_n \,(a_n^- - a_n) & \text{if } a_n < a_n^-, \\ (s_n - r_n)\, a_n^+ - r_n \mu_n \,(a_n - a_n^+) & \text{if } a_n > a_n^+, \\ (s_n - r_n)\, a_n & \text{otherwise.} \end{cases} \tag{5.58}$$

We also have

$$a_{n+1} = \begin{cases} s_n a_n^- & \text{if } a_n < a_n^-, \\ s_n a_n^+ & \text{if } a_n > a_n^+, \\ s_n a_n & \text{otherwise.} \end{cases} \tag{5.59}$$

In order to evaluate $G(W_0, a_0, X_0)$, we apply the following algorithm backwards in time starting from period $N-1$:

$$G_N(W_N, a_N, X_N) = U'(W_N) C_N \tag{5.60}$$

and

$$G_{N-k}(W_{N-k}, a_{N-k}, X_{N-k}) = \mathbb{E}_{N-k}\left[G_{N-k+1}(W_{N-k+1}, a_{N-k+1}, X_{N-k+1})\right] \tag{5.61}$$

for $k = 1, \ldots, N$.

Time Period $N-1$

At time period $N-1$, we have

$$\begin{aligned} G_{N-1}(W_{N-1}, a_{N-1}, X_{N-1}) &= \mathbb{E}_{N-1}\left[\kappa e^{-\kappa W_N} c(X_N)\right] \\ &= \mathbb{E}_{N-1}\left[\kappa e^{-\kappa \{r_{N-1} W_{N-1} + F_{N-1}\}} c\big(s_{N-1} X_{N-1}\big)\right]. \end{aligned} \tag{5.62}$$

Observe that the term in W_{N-1} is taken out of the conditional expectation $\mathbb{E}_{N-1}$ so we can write

$$G_{N-1}(W_{N-1}, a_{N-1}, X_{N-1}) = \kappa e^{-\kappa r_{N-1} W_{N-1}} H_{N-1}(a_{N-1}, X_{N-1}), \tag{5.63}$$

where

$$H_{N-1}(a_{N-1}, X_{N-1}) = \mathbb{E}_{N-1}\left[e^{-\kappa F_{N-1}} c(s_{N-1} X_{N-1})\right]. \quad (5.64)$$

In terms of the buy, sell and no-transaction regions, we have

$$H_{N-1}(a_{N-1}, X_{N-1}) = \begin{cases} \mathbb{E}_{N-1}\left[e^{-\kappa F^{(B)}_{N-1}} c(s_{N-1} X_{N-1})\right] & \text{if } a_{N-1} < a^-_{N-1}, \\ \mathbb{E}_{N-1}\left[e^{-\kappa F^{(S)}_{N-1}} c(s_{N-1} X_{N-1})\right] & \text{if } a_{N-1} > a^+_{N-1}, \\ \mathbb{E}_{N-1}\left[e^{-\kappa F^{(N)}_{N-1}} c(s_{N-1} X_{N-1})\right] & \text{otherwise.} \end{cases} \quad (5.65)$$

Recall that $F^{(B)}_{N-1}$, $F^{(S)}_{N-1}$ and $F^{(N)}_{N-1}$ correspond to F_{N-1} in the buy, sell and no-transaction regions of Equation (5.58) respectively. Note that $H_{N-1}(a_{N-1}, X_{N-1})$ is continuous across the buy and sell boundaries for a given X_{N-1}.

Time Period $N - k$

Applying the algorithm (Equation (5.61)) recursively, at the general time period $N - k$ ($k = 2, \ldots, N$), we have

$$\begin{aligned} G_{N-k}(W_{N-k}, a_{N-k}, X_{N-k}) &= \mathbb{E}_{N-k}\Big[\kappa e^{-\kappa r_{N-1} \cdots r_{N-k+1} W_{N-k+1}} \\ &\quad \times H_{N-k+1}(a_{N-k+1}, X_{N-k+1})\Big] \\ &= \mathbb{E}_{N-k}\Big[\kappa e^{-\kappa r_{N-1} \cdots r_{N-k+1} \{r_{N-k} W_{N-k} + F_{N-k}\}} \\ &\quad \times H_{N-k+1}(a_{N-k+1}, X_{N-k+1})\Big]. \end{aligned} \quad (5.66)$$

Taking the term in W_{N-k} out of the conditional expectation $\mathbb{E}_{N-k}$, we write

$$G_{N-k}(W_{N-k}, a_{N-k}, X_{N-k}) = \kappa e^{-\kappa r_{N-1} \cdots r_{N-k} W_{N-k}} H_{N-k}(a_{N-k}, X_{N-k}), \quad (5.67)$$

where

$$H_{N-k}(a_{N-k}, X_{N-k}) = \mathbb{E}_{N-k}\Big[e^{-\kappa r_{N-1} \cdots r_{N-k+1} F_{N-k}} H_{N-k+1}(a_{N-k+1}, X_{N-k+1})\Big]. \quad (5.68)$$

Recall that F_{N-k} and a_{N-k+1} are given by Equations (5.58) and (5.59) respectively. In addition, $X_{N-k+1} = s_{N-k}X_{N-k}$.

Therefore, in the buy region where $a_{N-k} < a^-_{N-k}$,

$$H^{(B)}_{N-k}(a_{N-k}, X_{N-k}) = \mathbb{E}_{N-k}\Big[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(B)}_{N-k}}H_{N-k+1}(s_{N-k}a^-_{N-k}, s_{N-k}X_{N-k})\Big]. \tag{5.69}$$

In the sell region where $a_{N-k} > a^+_{N-k}$,

$$H^{(S)}_{N-k}(a_{N-k}, X_{N-k}) = \mathbb{E}_{N-k}\Big[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}}H_{N-k+1}(s_{N-k}a^+_{N-k}, s_{N-k}X_{N-k})\Big]. \tag{5.70}$$

In the no-transaction region where $a^-_{N-k} \leq a_{N-k} \leq a^+_{N-k}$,

$$H^{(N)}_{N-k}(a_{N-k}, X_{N-k}) = \mathbb{E}_{N-k}\Big[e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}}H_{N-k+1}(s_{N-k}a_{N-k}, s_{N-k}X_{N-k})\Big]. \tag{5.71}$$

Note that $H_{N-k}(a_{N-k}, X_{N-k})$ is continuous across the buy and sell boundaries. Evaluating Equation (5.68) recursively from $k = 2$ up to $k = N$, we obtain $H_0(a_0, X_0)$ and consequently

$$G(W_0, a_0, X_0) = \kappa e^{-\kappa r_{N-1}\cdots r_0 W_0}H_0(a_0, X_0). \tag{5.72}$$

Substituting Equations (5.55) and (5.72) into Equation (5.52), the marginal utility indifference price of the option is thus given by

$$\tilde{\nu}(a_0, X_0) = -\frac{H_0(a_0, X_0)}{r_{N-1}\cdots r_0 V_0(a_0)}. \tag{5.73}$$

Similar to the utility indifference price, observe that this price is independent of the investor's wealth but dependent on the investor's holdings in the risky asset. In addition, it can also be shown that $\tilde{\nu}(a_0, X_0)$ does not vary with a_0 in the buy region $a_0 < a^-_0$ or sell region $a_0 < a^+_0$ of the portfolio selection problem without a position in the option.

5.5 Results

In this section, we illustrate the use of utility indifference pricing and marginal utility indifference pricing to value a European call option with payoff $c(X_N) = \max(X_N - K, 0)$, where K is the strike price. Assume that at each time period n,

$$s_n = \begin{cases} u & \text{with probability } q, \\ d & \text{with probability } 1-q, \end{cases} \tag{5.74}$$

and

$$r_n = r, \tag{5.75}$$

where $d < r < u$. In other words, we assume that the price of the risky asset follows a binomial tree and the risk-free asset has constant returns. Suppose that the binomial price process approximates the geometric Brownian motion $dX(t) = \alpha X(t)dt + \sigma X(t)dZ(t)$ with constant drift α and volatility σ, where $X(t)$ is the price of the risky asset and $Z(t)$ is a standard Brownian motion. Let R be the annualised risk-free rate. The time interval between successive periods is $\delta t = \frac{T}{N}$, where T is the terminal time at the end of N periods. Using the parametrisation by Cox et al. (1979), we have

$$u = e^{\sigma\sqrt{\delta t}},\ d = \frac{1}{u},\ q = \frac{1}{2} + \frac{1}{2\sigma}\left(\alpha - \frac{1}{2}\sigma^2\right)\sqrt{\delta t},\ r = e^{R\delta t}.$$

In addition, assume that the costs of buying and selling the risky asset are equal and constant (i.e. $\lambda_n = \mu_n = \lambda$) at each time period n. We specify the following parameter values:

$$T = 1 \text{ (year)},\ N = 6,\ X_0 = 20,\ K = 20,\ \alpha = 0.15,\ \sigma = 0.25,\ R = 0.1,\ \lambda = 0.02.$$

Using the above values with risk aversion $\kappa = 0.05$, we compute the utility indifference selling price, the utility indifference buying price and the marginal utility indifference price of the option from Equations (5.47), (5.48) and (5.73) respectively. Recall that the option prices depend on both the initial price of the underlying risky asset X_0 as well as on the initial holdings of the risky asset a_0 in the portfolios. In order to study the dependence of the option prices on a_0, we have fixed the value of X_0. For the ease of presentation, our subsequent notations of the option prices and optimal boundaries will not include their explicit dependence on X_0.

In Figure 5.1, we plot the (European call) option prices $\nu^{(s)}(a_0)$, $\nu^{(b)}(a_0)$ and $\tilde{\nu}(a_0)$ with respect to the initial holdings of risky asset a_0. Observe that the marginal utility indifference price lies between the utility indifference buying price and selling price, since it corresponds to the case of the portfolio having an infinitesimal position in the option. In addition, the option prices are non-increasing functions of the initial holdings of the risky asset. Within certain regions (to be described in the following discussion), recall that the option prices do not vary with the initial holdings of the risky asset (see Sections 5.3 and 5.4). Otherwise, the prices are observed to decrease as the initial holdings of the risky asset increase.

Consider the marginal utility indifference price and recall that it is determined from the optimal strategy of the portfolio selection problem without a position in the option. This strategy involves not transacting when the holdings of the risky asset fall within the no-transaction region $a_0^- \leq a_0 \leq a_0^+$. Otherwise, the investor will buy to reach a_0^- in the buy region $a_0 < a_0^-$ or sell to reach a_0^+ in the sell region $a_0 > a_0^+$. It is observed from Figure 5.1 that the option price $\tilde{\nu}(a_0)$ does not vary with a_0 in the buy or sell region. However, in the no-transaction region, the option price $\tilde{\nu}(a_0)$ decreases as the value of a_0 increases. Moreover, the option has a maximum price given by $\tilde{\nu}_{max} = \tilde{\nu}(a_0^-)$ (in the buy region) and a minimum price given by $\tilde{\nu}_{min} = \tilde{\nu}(a_0^+)$ (in the sell region) over all initial holdings a_0. Therefore, if the marginal utility indifference pricing approach is adopted to value an option, the bid and ask price of the option will be defined as $\tilde{\nu}_{min}$ and $\tilde{\nu}_{max}$ respectively.

On the other hand, the utility indifference selling price is obtained from maximising the expected utility of a portfolio with and without a position in the option. Recall that the no-transaction region for the portfolio with a short position in the option is denoted by $a_0^{(so)-} \leq a_0 \leq a_0^{(so)+}$. In this case, the utility indifference selling price $\nu^{(s)}(a_0)$ does not vary with a_0 in the intersection of the buy regions of the portfolios with and without a position in the option, which is the region $a_0 < a_0^-$ as seen in Figure 5.1. In addition, $\nu^{(s)}(a_0)$ also does not vary with a_0 in the intersection of the sell regions, which is the region $a_0 > a_0^{(so)+}$. The seller of an option will be interested in the maximum utility indifference selling price over all initial holdings a_0, which is given by $\nu^{(s)}_{max} = \nu^{(s)}(a_0^-)$. At this price, an

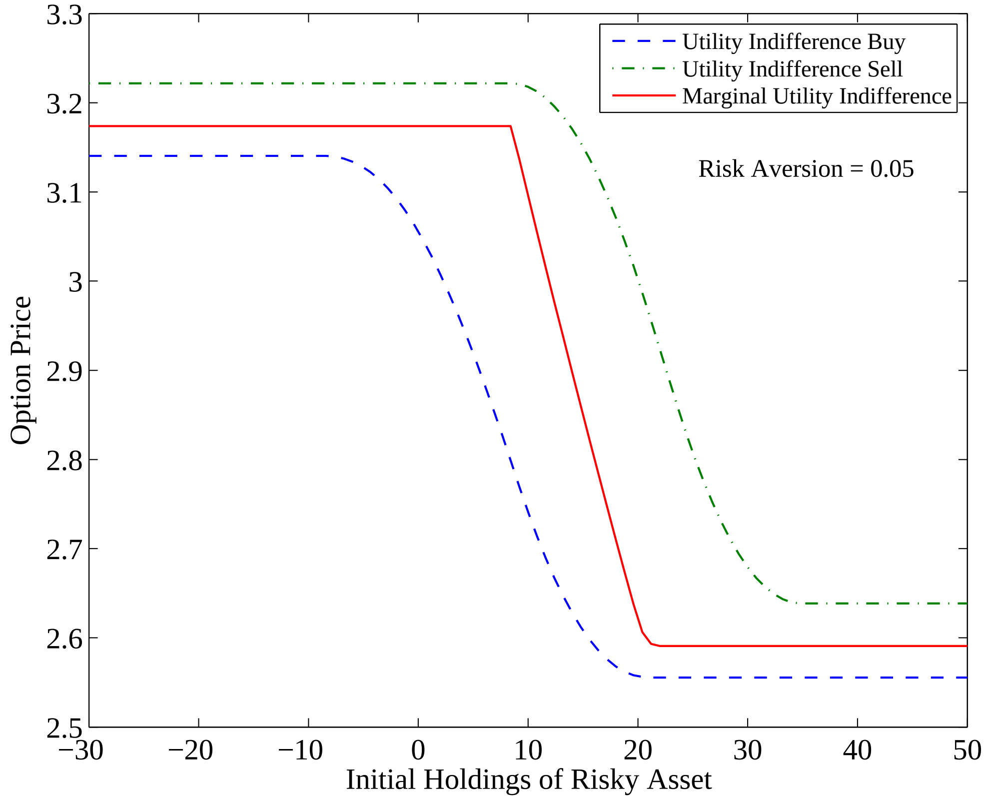


Figure 5.1: Option Price vs Initial Holdings of Risky Asset ($\kappa = 0.05$)
No-transaction region of portfolio without option $= [8.477, 21.351]$,
$\tilde{\nu}_{min} = 2.591,\ \tilde{\nu}_{max} = 3.174,\ \nu^{(b)}_{min} = 2.556,\ \nu^{(s)}_{max} = 3.222.$

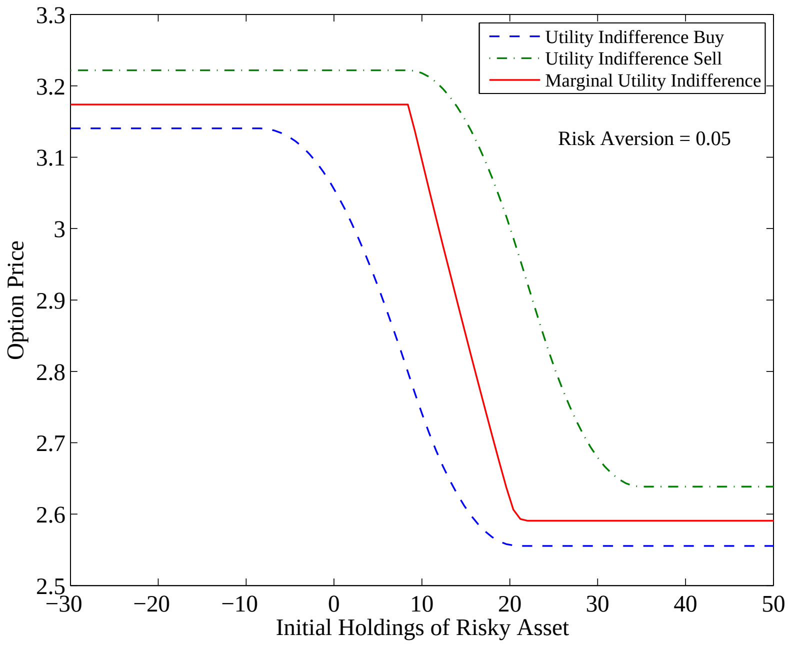
investor will be willing to sell the option regardless of his initial holdings in the risky asset. Unlike marginal utility indifference pricing, this approach also leads to a natural definition of the hedge ratio as the incremental investment in the risky asset due to the additional proceeds received from selling the option. For example, the hedge ratio corresponding to the maximum selling price $\nu^{(s)}_{max}$ is given by $\Delta^{(s)}(a_0^-) = \left(a_0^{(so)-} - a_0^-\right)/X_0$.

Similarly, for the utility indifference buying price, the no-transaction region for the portfolio with a long position in the option is given by $a_0^{(bo)-} \leq a_0 \leq a_0^{(bo)+}$. Here, as observed in Figure 5.1, the utility indifference buying price $\nu^{(b)}(a_0)$ does not vary with a_0 in the intersection of the buy regions (i.e. $a_0 < a_0^{(bo)-}$) and the intersection of the sell regions (i.e.

$a_0 > a_0^+$). From the perspective of a buyer of the option, one will be interested in achieving the minimum utility indifference buying price over all initial holdings a_0, which is given by $\nu_{min}^{(b)} = \nu^{(b)}(a_0^+)$. At this price, an investor will be willing to buy the option regardless of his initial holdings in the risky asset. Therefore, in the utility indifference pricing approach, the bid and ask price of the option are defined as $\nu_{min}^{(b)}$ and $\nu_{max}^{(s)}$ respectively.

Suppose that the coefficient of risk aversion is increased from $\kappa = 0.05$ to $\kappa = 0.1$. The results are presented in Figure 5.2. Comparing the marginal utility indifference prices, observe that the no-transaction region becomes narrower and shifts to the left. In other words, the boundaries and the width of the no-transaction region decrease in value as the investor's risk aversion increases. However, the minimum $\tilde{\nu}_{min}$ and maximum $\tilde{\nu}_{max}$ marginal utility indifference prices remain the same and is not influenced by the level of risk aversion. On the other hand, the minimum utility indifference buying price $\nu_{min}^{(b)}$ decreases while the maximum utility indifference selling price $\nu_{max}^{(s)}$ increases. In utility indifference pricing, an investor who is more risk averse would require a lower bid price and a higher ask price. Therefore, as the level of risk aversion increases, the bid-ask spread widens in the utility indifference pricing approach but it does not change in marginal utility indifference pricing.

In this chapter, we extended the utility indifference pricing and marginal utility indifference pricing approach to incorporate proportional transaction costs in a discrete time model. In the presence of transaction costs, the utility indifference price and marginal utility indifference price are shown to depend on the price of the underlying risky asset as well as on the investor's holdings in the risky asset. Moreover, we identified the regions where the option prices do not vary with the investor's holdings in the risky asset. We considered an example where the price of the risky asset follows a binomial tree and illustrated how one could determine the bid and ask price of a European call option. We also discussed the differences between utility indifference pricing and marginal utility indifference pricing.



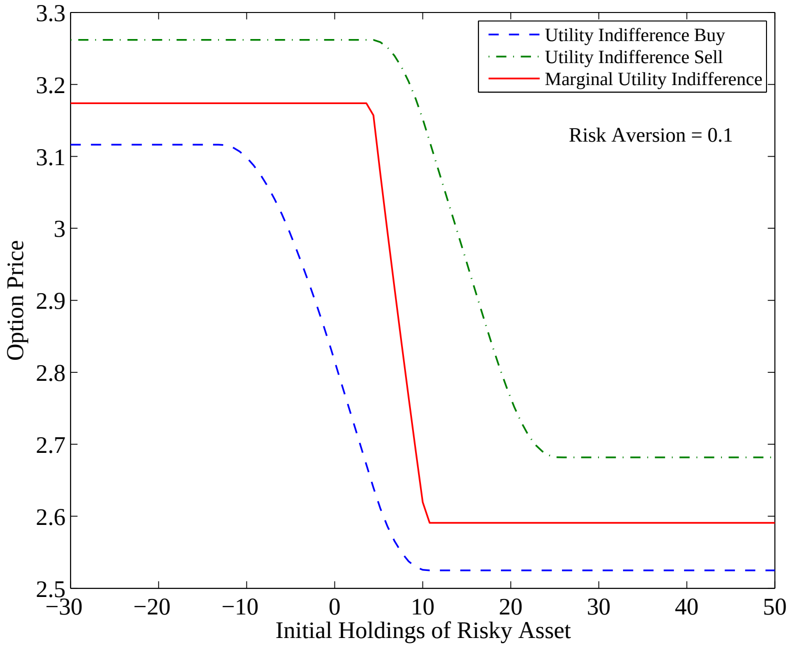


Figure 5.2: Option Price vs Initial Holdings of Risky Asset ($\kappa = 0.05$ and $\kappa = 0.1$)
$\kappa = 0.05$: No-transaction region of portfolio without option $= [8.477, 21.351]$,
$\tilde{\nu}_{min} = 2.591,\ \tilde{\nu}_{max} = 3.174,\ \nu^{(b)}_{min} = 2.556,\ \nu^{(s)}_{max} = 3.222.$
$\kappa = 0.1$: No-transaction region of portfolio without option $= [4.238, 10.675]$,
$\tilde{\nu}_{min} = 2.591,\ \tilde{\nu}_{max} = 3.174,\ \nu^{(b)}_{min} = 2.525,\ \nu^{(s)}_{max} = 3.262.$

Chapter 6

Conclusion

In this thesis, our study is focused on analysing the impact of transaction costs in portfolio theory and option pricing theory. In the first part of the thesis, we considered a multi-period portfolio selection problem where an investor allocates his wealth between a risky and risk-free asset. A general class of underlying probability distributions is assumed for the returns of the risky asset. At the start of successive time periods, the investor rebalances the portfolio with the objective of maximising expected utility of terminal wealth. A cost that is proportional to the value of the transaction is incurred each time the investor trades in the risky asset. In order to determine the investor's optimal investment strategy, dynamic programming is applied to reduce the original problem to a sequence of more manageable sub-problems. Nonetheless, an exact solution is generally not available and solving the dynamic programming algorithm numerically is computationally intensive. Therefore, in the limit of small transaction costs, perturbation analysis is applied to derive an approximate solution to the optimal investment strategy.

In Chapter 2, we analysed the case of an investor with the exponential utility function. The optimal strategy involves trading to reach the boundaries of a no-transaction region if the investor's holdings of the risky asset fall outside this region. We developed a two-stage perturbation method that enabled us to systematically obtain approximations of the optimal value functions and the optimal boundaries at all time steps. In the first stage, the investor is assumed to trade to the Merton point at each time step when transaction costs are small. The Merton point corresponds to the optimal investment in the risky asset for the portfolio selection problem without transaction costs. This is clearly a suboptimal strategy since one has ignored the presence of the no-transaction region. Nonetheless, we managed to derive an approximation of the suboptimal value function at each time step by perturbing about the zero transaction costs solution. In the second stage, the investor is assumed to behave optimally by trading to reach the boundaries of the no-transaction region. Applying a sequence of corrections to the suboptimal value functions allowed us to determine the desired approximations to the optimal value functions at all time steps. Consequently, the approximate optimal boundaries are obtained from the condition that the derivative of the optimal value function is continuous across the boundaries. It is observed that the approximate boundaries at each time step depend on the (random) returns of the risky asset at the current time step and one step ahead. In addition, they are also observed to depend on the (non-random) returns of the risk-free asset at the current time step and all the steps ahead.

In Chapter 3, we considered the case of an investor with the power utility function. A more realistic description of the optimal strategy is provided by the power utility function as compared to the exponential utility function, which resulted in optimal boundaries that are independent of the investor's wealth. We adopted a similar two-stage perturbation method to obtain approximations of the optimal value function and optimal boundaries at each time step in the rebalancing of the portfolio. In this case, it is found to be more challenging to apply the perturbation analysis, as the proportion of risky asset inherited at each time step from the previous step depends on variations in both the return of the risky asset as well as the investor's wealth. We also observed that the approximate boundaries at each time step depend on the returns of the risky and risk-free assets at the current time step and one step ahead.

In the second part of the thesis, our study is concentrated on developing and analysing a discrete time model of option pricing that incorporates proportional transaction costs. The underlying risky asset is assumed to follow a general price process. We adopted an option pricing approach that is based on the maximisation of expected utility of terminal wealth. Using the definition by Hodges and Neuberger (1989), the utility indifference selling (buying) price of an option is defined as the amount of money that will make the investor indifferent, in terms of expected utilities, between trading in the market with and without a short (long) position in the option. An alternative definition by Davis (1997) is also investigated, where the marginal utility indifference price of an option is determined by the requirement that an infinitesimal diversion of funds into the option purchase or sale has a neutral effect on the investor's achievable utility.

In Chapter 4, we developed an option pricing model in discrete time without transaction costs, which is based on the utility maximisation approach. We considered utility indifference pricing and marginal utility indifference pricing in the context of the exponential and power utility functions. An advantage of this model is that the underlying risky asset is assumed to follow a general price process. When the risky asset is assumed to follow a binomial price process, we established that both the utility indifference price and marginal utility indifference price of the option reduce to the perfect replication price from Cox et al.'s (1979) binomial model. In the case where the underlying risky asset follows a trinomial price process, perfect replication of the option is no longer possible and we illustrated how the option price may be determined via an approximate replication approach. In this approach, the option is valued by constructing an approximately replicating portfolio that minimises the variance of the replication error with a mean error of zero. Using the exponential utility function, we also obtained the utility indifference price and marginal utility indifference price of the option in this case. A comparison is then made between the utility maximisation approach and the approximate replication approach.

In Chapter 5, we extended the discrete time option pricing model from Chapter 4 by incorporating proportional transaction costs. We focused on the case of an investor with the exponential utility function. Applying dynamic programming and using some of the results from Chapter 2, we obtained the utility indifference price and marginal utility indifference price of a European option. In the presence of transaction costs, the utility indifference price and marginal utility indifference price of the option are shown to depend on the price of the underlying risky asset as well as on the investor's holdings in the risky asset. Moreover, we identified the regions where the option prices do not vary with the investor's holdings in the risky asset. Numerical results are presented for the case of a European call option, where the underlying risky asset is assumed to follow a binomial price process. We examined how one is able to determine the bid and ask price of the option. In the utility indifference pricing approach, the bid price is given by the minimum utility indifference buying price over the investor's holdings in the risky asset. On the other hand, the ask price corresponds to the maximum of the utility indifference selling price. It is observed that the bid-ask spread widens as the investor becomes more risk averse. Moreover, this approach also provides one with a natural definition of the hedging strategy. In marginal utility indifference pricing, the bid and ask price is given by the minimum and maximum marginal utility indifference price over the investor's holdings in the risky asset, respectively. In this case, we observed that the bid-ask spread is not influenced by the investor's level of risk aversion.

There are a number of possible extensions to the work that has been presented in this thesis. In the limit of small transaction costs, we applied perturbation analysis to obtain approximate solutions to the portfolio selection problem. It might also be useful to apply perturbation analysis to derive approximate solutions to our option pricing model since it is based on the utility maximisation approach.

In the portfolio selection and option pricing models with transaction costs, we considered the case of a single risky asset. It will be interesting to analyse the case of multiple risky assets, which allows for a richer level of interaction via the correlation between these assets. In the context of option pricing, the case of multiple risky assets potentially enables one to obtain the value of a basket option.

Instead of proportional transaction costs, one might choose to formulate a model with a more general structure of transaction costs. A general costs structure could include constant costs, proportional costs or a combination of both constant and proportional costs.

In the option pricing model, we focused on the pricing and hedging of a European option with a payoff that depends on the price of the risky asset at the expiration date. A possible area of research is to investigate the pricing and hedging of a European path dependent option (such as a Barrier option or an Asian option) by adopting the utility maximisation approach. It will also be interesting to consider the valuation of an American option, which may be exercised at any time before and up to its expiration date.

Appendix A

Notations and Definitions in Perturbation Analysis

In this appendix, we introduce the notations and definitions of order symbols, asymptotic sequences and asymptotic expansions (as found in Nayfeh (2004)). Perturbation analysis is widely used to approximate the solutions of problems when one is not able to obtain exact analytical solutions. In perturbation analysis, the solution to a problem is usually represented by the first few terms of an asymptotic expansion. The expansion may be carried out in terms of a (small or large) parameter that appears naturally in the equation or in terms of a (small or large) coordinate.

In perturbation analysis, one frequently has to compare the order of magnitude of two functions in a certain limit. For instance, suppose that the behaviour of a function $f(\varepsilon)$ is compared to a function $g(\varepsilon)$ in the limit $\varepsilon \to 0$. It is convenient to employ the order symbols of O or o.

One says that $f(\varepsilon)$ is of the order $g(\varepsilon)$ and writes

$$f(\varepsilon) = O\left[g(\varepsilon)\right] \quad \text{as} \quad \varepsilon \to 0 \tag{A.1}$$

if there exists a $k > 0$ independent of ε and an $\varepsilon_0 > 0$ such that

$$|f(\varepsilon)| \leq k\,|g(\varepsilon)| \quad \text{for all} \quad |\varepsilon| \leq \varepsilon_0. \tag{A.2}$$

Alternatively, this condition can be replaced by

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| < \infty. \tag{A.3}$$

Moreover, one says that $f(\varepsilon)$ is much smaller than $g(\varepsilon)$ and writes

$$f(\varepsilon) = o\left[g(\varepsilon)\right] \quad \text{as} \quad \varepsilon \to 0 \tag{A.4}$$

if for every $\delta > 0$ independent of ε, there exists an ε_0 such that

$$|f(\varepsilon)| \leq \delta\, |g(\varepsilon)| \quad \text{for} \quad |\varepsilon| \leq \varepsilon_0. \tag{A.5}$$

Alternatively, this condition can be replaced by

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0. \tag{A.6}$$

A sequence of functions $\{\phi_n(\varepsilon)\}$ $(n = 0, 1, 2, \ldots)$ is called an asymptotic sequence if

$$\phi_{n+1}(\varepsilon) = o\left[\phi_n(\varepsilon)\right] \quad \text{as} \quad \varepsilon \to 0. \tag{A.7}$$

An example of an asymptotic sequence is $\{\varepsilon^n\}$. In terms of asymptotic sequences, one can define asymptotic expansions. Given the expression $\sum_{n=0}^{N-1} a_n \phi_n(\varepsilon)$ where a_n is independent of ε and $\{\phi_n(\varepsilon)\}$ is an asymptotic sequence, one says that this is an asymptotic expansion of $f(\varepsilon)$ to N terms if

$$f(\varepsilon) = \sum_{n=0}^{N-1} a_n \phi_n(\varepsilon) + O\left[\phi_N(\varepsilon)\right] \quad \text{as} \quad \varepsilon \to 0. \tag{A.8}$$

Appendix B

Perturbation Analysis with Exponential Utility Function

B.1 Remainder Term of Suboptimal Value Function

In this appendix, an estimate of the remainder term in the approximation of the suboptimal value function $\widehat{V}^{(B)}_{N-k}$ in Section 2.4.1 is presented and shown to be bounded. In order to estimate the remainder term, it is noted that to obtain Equation (2.89) from Equation (2.81), the expression

$$\begin{aligned} &\int_0^{\widetilde{s}_{N-k}} e^{\kappa r_{N-1}\cdots r_{N-k+1}\lambda_{N-k+1}\widetilde{a}_{N-k}(\widetilde{s}_{N-k}-s_{N-k})} \\ &\times e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &+\int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}\mu_{N-k+1}\widetilde{a}_{N-k}(\widetilde{s}_{N-k}-s_{N-k})} \\ &\times e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \end{aligned} \tag{B.1}$$

is expanded as a power series in terms of λ_{N-k+1} and μ_{N-k+1}. Applying Taylor's Theorem and collecting together terms of the same order, the above expression is written as

$$\mathbb{E}_{N-k}\left[e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}}\right] + \zeta_{N-k} + \eta_{N-k} + R_{N-k}, \tag{B.2}$$

where ζ_{N-k} is given by Equation (2.90), η_{N-k} by Equation (2.91) and R_{N-k} by the following equation:

$$\begin{aligned} R_{N-k} &= \frac{1}{6}(\kappa r_{N-1}\cdots r_{N-k+1}\lambda_{N-k+1})^3 \\ &\times \int_0^{\widetilde{s}_{N-k}} (\widetilde{s}_{N-k}-s_{N-k})^3 \widetilde{a}_{N-k}^3 e^{\kappa r_{N-1}\cdots r_{N-k+1}\epsilon_{N-k+1}^{(B)}\widetilde{a}_{N-k}(\widetilde{s}_{N-k}-s_{N-k})} \\ &\times e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &- \frac{1}{6}(\kappa r_{N-1}\cdots r_{N-k+1}\mu_{N-k+1})^3 \\ &\times \int_{\widetilde{s}_{N-k}}^{\infty} (\widetilde{s}_{N-k}-s_{N-k})^3 \widetilde{a}_{N-k}^3 e^{-\kappa r_{N-1}\cdots r_{N-k+1}\epsilon_{N-k+1}^{(S)}\widetilde{a}_{N-k}(\widetilde{s}_{N-k}-s_{N-k})} \\ &\times e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k}, \end{aligned} \tag{B.3}$$

where $0 < \epsilon_{N-k+1}^{(B)} < \lambda_{N-k+1}$ and $0 < \epsilon_{N-k+1}^{(S)} < \mu_{N-k+1}$. Recalling that $\widetilde{F}_{N-k} = (s_{N-k} - r_{N-k})\widetilde{a}_{N-k}$, each of the two integrals can be shown to be bounded by

$$\begin{aligned} &\left|\int_0^{\widetilde{s}_{N-k}} (\widetilde{s}_{N-k}-s_{N-k})^3 \widetilde{a}_{N-k}^3 e^{\kappa r_{N-1}\cdots r_{N-k+1}\epsilon_{N-k+1}^{(B)}\widetilde{a}_{N-k}(\widetilde{s}_{N-k}-s_{N-k})} \right. \\ &\left. \times e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k}\right| \\ &< (\widetilde{s}_{N-k}\widetilde{a}_{N-k})^3 e^{\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{a}_{N-k}(r_{N-k}+\lambda_{N-k+1}\widetilde{s}_{N-k})} \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} &\left|\int_{\widetilde{s}_{N-k}}^{\infty} (\widetilde{s}_{N-k}-s_{N-k})^3 \widetilde{a}_{N-k}^3 e^{-\kappa r_{N-1}\cdots r_{N-k+1}\epsilon_{N-k+1}^{(S)}\widetilde{a}_{N-k}(\widetilde{s}_{N-k}-s_{N-k})} \right. \\ &\left. \times e^{-\kappa r_{N-1}\cdots r_{N-k+1}\widetilde{F}_{N-k}} p(s_{N-k})\,\mathrm{d}s_{N-k}\right| \\ &< \left[\frac{3}{\kappa r_{N-1}\cdots r_{N-k+1}(1-\mu_{N-k+1})}\right]^3 e^{\kappa r_{N-1}\cdots r_{N-k+1} r_{N-k}\widetilde{a}_{N-k}}. \end{aligned} \tag{B.5}$$

Therefore, by the triangle inequality, the remainder term R_{N-k} is bounded by

$$
\begin{aligned}
|R_{N-k}| &< \frac{1}{6} e^{\kappa r_{N-1}\cdots r_{N-k+1} r_{N-k} \widetilde{a}_{N-k}} \\
&\quad \times \Bigg\{ (\kappa r_{N-1}\cdots r_{N-k+1}\lambda_{N-k+1}\widetilde{s}_{N-k}\widetilde{a}_{N-k})^3 e^{\kappa r_{N-1}\cdots r_{N-k+1}\lambda_{N-k+1}\widetilde{s}_{N-k}\widetilde{a}_{N-k}} \\
&\quad + \left(\frac{3\mu_{N-k+1}}{1-\mu_{N-k+1}}\right)^3 \Bigg\}. \qquad \text{(B.6)}
\end{aligned}
$$

B.2 Preliminary Estimate of Optimal Value Function

In this appendix, we obtain a preliminary estimate of the optimal value function V_{N-k} in the buy, sell and no-transaction regions in Section 2.4.2 via the following propositions.

Proposition B.2.1. *If*

(i) $V^{(B)}_{N-k+1} - V^{(N)}_{N-k+1} = O(\varepsilon^2)$ *at* $s_{N-k} = \theta^-_{N-k}$, *where* θ^-_{N-k} *lies between* $\widetilde{s}_{N-k}$ *and* $s^-_{N-k} = \dfrac{a^-_{N-k+1}}{a^-_{N-k}}$, *and*

(ii) $V^{(S)}_{N-k+1} - V^{(N)}_{N-k+1} = O(\varepsilon^2)$ *at* $s_{N-k} = \theta^+_{N-k}$, *where* θ^+_{N-k} *lies between* $\widetilde{s}_{N-k}$ *and* $s^+_{N-k} = \dfrac{a^+_{N-k+1}}{a^-_{N-k}}$, *then*

$$
\begin{aligned}
V^{(B)}_{N-k} &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1} F^{(B)}_{N-k}} V^{(B)}_{N-k+1} p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1} F^{(B)}_{N-k}} V^{(S)}_{N-k+1} p(s_{N-k})\,\mathrm{d}s_{N-k} + O(\varepsilon^3). \qquad \text{(B.7)}
\end{aligned}
$$

Proof. Recall that $V^{(B)}_{N-k}$ is given by Equation (2.47) for $k = 2, \ldots, N$, which can be rewritten as

$$\begin{aligned} V_{N-k}^{(B)} &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(B)}} V_{N-k+1}^{(B)} p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &+ \int_{\widetilde{s}_{N-k}}^{s_{N-k}^{-}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(B)}} \left[V_{N-k+1}^{(B)} - V_{N-k+1}^{(N)}\right] p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &+ \int_{\widetilde{s}_{N-k}}^{s_{N-k}^{+}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(B)}} \left[V_{N-k+1}^{(N)} - V_{N-k+1}^{(S)}\right] p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &+ \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(B)}} V_{N-k+1}^{(S)} p(s_{N-k})\,\mathrm{d}s_{N-k}, \end{aligned} \tag{B.8}$$

where $s_{N-k}^{\pm} = \dfrac{a_{N-k+1}^{\pm}}{a_{N-k}^{-}}$. Since $a_{N-k}^{\pm} = \widetilde{a}_{N-k} + \omega_{N-k}^{\pm}$ and $\omega_{N-k}^{\pm} = O(\varepsilon)$, $s_{N-k}^{\pm}$ can be approximated by

$$s_{N-k}^{\pm} = \widetilde{s}_{N-k} + \phi_{N-k}^{\pm} + O(\varepsilon^2), \tag{B.9}$$

where

$$\phi_{N-k}^{\pm} = \widetilde{s}_{N-k}\left[\frac{\omega_{N-k+1}^{\pm}}{\widetilde{a}_{N-k+1}} - \frac{\omega_{N-k}^{-}}{\widetilde{a}_{N-k}}\right] \tag{B.10}$$

is of $O(\varepsilon)$. Applying the Mean Value Theorem for the second and third integrals,

$$\begin{aligned} V_{N-k}^{(B)} &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(B)}} V_{N-k+1}^{(B)} p(s_{N-k})\,\mathrm{d}s_{N-k} \\ &+ \left\{\left[s_{N-k}^{-} - \widetilde{s}_{N-k}\right]\left[V_{N-k+1}^{(B)} - V_{N-k+1}^{(N)}\right] e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(B)}} p(s_{N-k})\right\}\Big|_{s_{N-k}=\theta_{N-k}^{-}} \\ &+ \left\{\left[s_{N-k}^{+} - \widetilde{s}_{N-k}\right]\left[V_{N-k+1}^{(N)} - V_{N-k+1}^{(S)}\right] e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(B)}} p(s_{N-k})\right\}\Big|_{s_{N-k}=\theta_{N-k}^{+}} \\ &+ \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F_{N-k}^{(B)}} V_{N-k+1}^{(S)} p(s_{N-k})\,\mathrm{d}s_{N-k}, \end{aligned} \tag{B.11}$$

where θ_{N-k}^{-} lies between $\widetilde{s}_{N-k}$ and s_{N-k}^{-}, and θ_{N-k}^{+} lies between $\widetilde{s}_{N-k}$ and s_{N-k}^{+}. Applying the assumptions from Proposition B.2.1(i), B.2.1(ii) and Equation (B.9) leads to the required result (Equation (B.7)). □

Similar results are obtained for the optimal value functions in the sell and no-transaction regions at time period $N-k$, which are stated as follow. The proofs, which will not be provided, are similar to the proof of Proposition B.2.1.

Proposition B.2.2. *If*

(i) $V^{(B)}_{N-k+1} - V^{(N)}_{N-k+1} = O(\varepsilon^2)$ *at* $s_{N-k} = \theta^-_{N-k}$*, where* θ^-_{N-k} *lies between* $\widetilde{s}_{N-k}$ *and* $s^-_{N-k} = \dfrac{a^-_{N-k+1}}{a^+_{N-k}}$*, and*

(ii) $V^{(S)}_{N-k+1} - V^{(N)}_{N-k+1} = O(\varepsilon^2)$ *at* $s_{N-k} = \theta^+_{N-k}$*, where* θ^+_{N-k} *lies between* $\widetilde{s}_{N-k}$ *and* $s^+_{N-k} = \dfrac{a^+_{N-k+1}}{a^+_{N-k}}$*, then*

$$
\begin{aligned}
V^{(S)}_{N-k} &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}V^{(B)}_{N-k+1}}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(S)}_{N-k}V^{(S)}_{N-k+1}}p(s_{N-k})\,\mathrm{d}s_{N-k} + O(\varepsilon^3).
\end{aligned} \tag{B.12}
$$

Proposition B.2.3. *If*

(i) $V^{(B)}_{N-k+1} - V^{(N)}_{N-k+1} = O(\varepsilon^2)$ *at* $s_{N-k} = \theta^-_{N-k}$*, where* θ^-_{N-k} *lies between* $\widetilde{s}_{N-k}$ *and* $s^-_{N-k} = \dfrac{a^-_{N-k+1}}{a_{N-k}}$*, and*

(ii) $V^{(S)}_{N-k+1} - V^{(N)}_{N-k+1} = O(\varepsilon^2)$ *at* $s_{N-k} = \theta^+_{N-k}$*, where* θ^+_{N-k} *lies between* $\widetilde{s}_{N-k}$ *and* $s^+_{N-k} = \dfrac{a^+_{N-k+1}}{a_{N-k}}$*, then*

$$
\begin{aligned}
V^{(N)}_{N-k} &= \int_0^{\widetilde{s}_{N-k}} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}V^{(B)}_{N-k+1}}p(s_{N-k})\,\mathrm{d}s_{N-k} \\
&\quad + \int_{\widetilde{s}_{N-k}}^{\infty} e^{-\kappa r_{N-1}\cdots r_{N-k+1}F^{(N)}_{N-k}V^{(S)}_{N-k+1}}p(s_{N-k})\,\mathrm{d}s_{N-k} + O(\varepsilon^3).
\end{aligned} \tag{B.13}
$$

Appendix C

Perturbation Analysis with Power Utility Function

C.1 Remainder Term of Suboptimal Value Function

In this appendix, we derive a bound on the remainder term in the approximation of the suboptimal value function $\widehat{V}_{N-1}^{(B)}$ in Section 3.4.2. Applying Taylor's Theorem to Equation (3.61),

$$\begin{aligned}
\widehat{V}_{N-1}^{(B)} &= \frac{1}{\gamma}\mathbb{E}_{N-1}\Bigg[\widetilde{F}_{N-1}^{\gamma} - \gamma\widetilde{F}_{N-1}^{\gamma-1}\varepsilon\bar{\lambda}_{N-1}r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right) \\
&\quad + \frac{1}{2}\gamma\left(\gamma-1\right)\widetilde{F}_{N-1}^{\gamma-2}\varepsilon^{2}\bar{\lambda}_{N-1}^{2}r_{N-1}^{2}\left(\widetilde{A}_{N-1} - A_{N-1}\right)^{2} \\
&\quad - \frac{1}{6}\gamma\left(\gamma-1\right)\left(\gamma-2\right)\varepsilon^{3}\bar{\lambda}_{N-1}^{3}r_{N-1}^{3}\left(\widetilde{A}_{N-1} - A_{N-1}\right)^{3} \\
&\quad \times\left\{\widetilde{F}_{N-1} - \xi\bar{\lambda}_{N-1}r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma-3}\Bigg],
\end{aligned} \tag{C.1}$$

where $0 < \xi < \varepsilon$. Therefore, the absolute value of the remainder term is given by

$$\begin{aligned}
\left|\widehat{R}_{N-1}^{(B)}\right| &= \frac{1}{6}\left(\gamma-1\right)\left(\gamma-2\right)\varepsilon^{3}\bar{\lambda}_{N-1}^{3}r_{N-1}^{3}\left(\widetilde{A}_{N-1} - A_{N-1}\right)^{3} \\
&\quad \times\mathbb{E}_{N-1}\left[\left\{\widetilde{F}_{N-1} - \xi\bar{\lambda}_{N-1}r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma-3}\right].
\end{aligned} \tag{C.2}$$

Since $\widetilde{F}_{N-1} = r_{N-1} + (s_{N-1} - r_{N-1})\,\widetilde{A}_{N-1}$, we have

$$\begin{aligned}
&\mathbb{E}_{N-1}\left[\left\{\widetilde{F}_{N-1} - \xi\bar{\lambda}_{N-1} r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma-3}\right] \\
&= \int_0^\infty \left\{r_{N-1} + (s_{N-1} - r_{N-1})\,\widetilde{A}_{N-1} - \xi\bar{\lambda}_{N-1} r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma-3} \\
&\times p(s_{N-1})\,\mathrm{d}s_{N-1}.
\end{aligned} \tag{C.3}$$

For $\gamma < 1$, note that the expression

$$\left\{r_{N-1} + (s_{N-1} - r_{N-1})\,\widetilde{A}_{N-1} - \xi\bar{\lambda}_{N-1} r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma-3} \tag{C.4}$$

is a decreasing function of s_{N-1} and has a maximum at $s_{N-1} = 0$. Hence,

$$\begin{aligned}
&\mathbb{E}_{N-1}\left[\left\{\widetilde{F}_{N-1} - \xi\bar{\lambda}_{N-1} r_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma-3}\right] \\
&< r_{N-1}^{\gamma-3}\left\{\left(1 - \widetilde{A}_{N-1}\right) - \xi\bar{\lambda}_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma-3}.
\end{aligned} \tag{C.5}$$

Since $0 < \xi < \varepsilon$, we obtain a bound for

$$\begin{aligned}
\left|\widehat{R}_{N-1}^{(B)}\right| \;&<\; \frac{1}{6}(\gamma-1)(\gamma-2)\,\varepsilon^3\bar{\lambda}_{N-1}^3 r_{N-1}^{\gamma}\left(\widetilde{A}_{N-1} - A_{N-1}\right)^3 \\
&\quad\times\left\{\left(1 - \widetilde{A}_{N-1}\right) - \varepsilon\bar{\lambda}_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right)\right\}^{\gamma-3}.
\end{aligned} \tag{C.6}$$

Note that this bound is finite since we have assumed that

$$0 < \left(1 - \widetilde{A}_{N-1}\right) - \varepsilon\bar{\lambda}_{N-1}\left(\widetilde{A}_{N-1} - A_{N-1}\right) < 1. \tag{C.7}$$

C.2 Approximation of Optimal Value Function

In this appendix, we present the details of the perturbation analysis that derives the correction and approximation of the optimal value function V_{N-2} in Section 3.4.3.

In the *buy region* at time period $N-2$, recall that $V_{N-2}^{(B)}$ is given by Equation (3.32).

We observe that

$$s^-_{N-2} = \frac{r_{N-2}A^-_{N-1}\left\{\left(1-A^-_{N-2}\right)-\lambda_{N-2}\left(A^-_{N-2}-A_{N-2}\right)\right\}}{A^-_{N-2}\left(1-A^-_{N-1}\right)} = \widetilde{s}_{N-2}+O(\varepsilon) \quad \text{(C.8)}$$

and

$$s^+_{N-2} = \frac{r_{N-2}A^+_{N-1}\left\{\left(1-A^-_{N-2}\right)-\lambda_{N-2}\left(A^-_{N-2}-A_{N-2}\right)\right\}}{A^-_{N-2}\left(1-A^+_{N-1}\right)} = \widetilde{s}_{N-2}+O(\varepsilon). \quad \text{(C.9)}$$

Therefore, $\widetilde{s}_{N-2}$ is the leading order term of s^-_{N-2} and s^+_{N-2}. which motivates us to rewrite $V^{(B)}_{N-2}$ in terms of integrals delineated by $\widetilde{s}_{N-2}$. Thus, we have

$$\begin{aligned} V^{(B)}_{N-2} &= \int_0^{\widetilde{s}_{N-2}} F^{(B)\gamma}_{N-2}V^{(B)}_{N-1}p(s_{N-2})\,\mathrm{d}s_{N-2} \\ &\quad+\int_{\widetilde{s}_{N-2}}^{\infty} F^{(B)\gamma}_{N-2}V^{(S)}_{N-1}p(s_{N-2})\,\mathrm{d}s_{N-2} \\ &\quad+\int_{s^-_{N-2}}^{\widetilde{s}_{N-2}} F^{(B)\gamma}_{N-2}\left\{V^{(N)}_{N-1}-V^{(B)}_{N-1}\right\}p(s_{N-2})\,\mathrm{d}s_{N-2} \\ &\quad+\int_{\widetilde{s}_{N-2}}^{s^+_{N-2}} F^{(B)\gamma}_{N-2}\left\{V^{(N)}_{N-1}-V^{(S)}_{N-1}\right\}p(s_{N-2})\,\mathrm{d}s_{N-2}. \end{aligned} \quad \text{(C.10)}$$

Applying the Mean Value Theorem to the third integral of the above equation,

$$\begin{aligned} &\int_{s^-_{N-2}}^{\widetilde{s}_{N-2}} F^{(B)\gamma}_{N-2}\left\{V^{(N)}_{N-1}-V^{(B)}_{N-1}\right\}p(s_{N-2})\,\mathrm{d}s_{N-2} \\ &= \left(\widetilde{s}_{N-2}-s^-_{N-2}\right)F^{(B)\gamma}_{N-2}\left\{V^{(N)}_{N-1}-V^{(B)}_{N-1}\right\}p(s_{N-2}) \end{aligned} \quad \text{(C.11)}$$

evaluated at a point $s_{N-2}\in\left(s^-_{N-2},\widetilde{s}_{N-2}\right)$, that is, $s_{N-2}=s^-_{N-2}+O(\varepsilon)$. Recall that when $s_{N-2}=s^-_{N-2}$, we have $A_{N-1}=A^-_{N-1}$ by definition. Therefore, when $s_{N-2}=s^-_{N-2}+O(\varepsilon)$, we have $A_{N-1}=A^-_{N-1}+O(\varepsilon)=\widetilde{A}_{N-1}+O(\varepsilon)$. This implies that the term $V^{(N)}_{N-1}-V^{(B)}_{N-1}=\varepsilon\bar{\lambda}_{N-1}\left(\widetilde{A}_{N-1}-A_{N-1}\right)\mathbb{E}_{N-1}\left[\widetilde{F}^{\gamma}_{N-1}\right]+O(\varepsilon^2)=O(\varepsilon^2)$. Since $\widetilde{s}_{N-2}-s^-_{N-2}=O(\varepsilon)$, we thus conclude that Equation (C.11) is of $O(\varepsilon^3)$. Similarly, by applying the Mean Value Theorem to the fourth integral of Equation (C.10), we can also show that it is of $O(\varepsilon^3)$. In conclusion, Equation (C.10) can be simplified to

$$\begin{aligned} V_{N-2}^{(B)} &= \int_0^{\widetilde{s}_{N-2}} F_{N-2}^{(B)\gamma} V_{N-1}^{(B)} p(s_{N-2}) \, \mathrm{d}s_{N-2} \\ &\quad + \int_{\widetilde{s}_{N-2}}^{\infty} F_{N-2}^{(B)\gamma} V_{N-1}^{(S)} p(s_{N-2}) \, \mathrm{d}s_{N-2} + O(\varepsilon^3). \end{aligned} \tag{C.12}$$

Expressing $V_{N-2}^{(B)}$ in this form allows us to perturb it about the suboptimal value function $\widehat{V}_{N-2}^{(B)}$, which is also in terms of integrals delineated by $\widetilde{s}_{N-2}$. Using Equation (3.78) and the approximation of the optimal value function from time period $N-1$, express $V_{N-2}^{(B)}$ as

$$\begin{aligned} V_{N-2}^{(B)} &= \int_0^{\widetilde{s}_{N-2}} \left\{ \widehat{F}_{N-2}^{(B)} + \varepsilon \omega_{N-2}^{-} \left(s_{N-2} - r_{N-2} \right) - \varepsilon^2 \bar{\lambda}_{N-2} \omega_{N-2}^{-} r_{N-2} \right\}^{\gamma} \\ &\quad \left\{ \widehat{V}_{N-1}^{(B)} + \delta_{N-1}^{(B)} \right\} p(s_{N-2}) \, \mathrm{d}s_{N-2} \\ &\quad + \int_{\widetilde{s}_{N-2}}^{\infty} \left\{ \widehat{F}_{N-2}^{(B)} + \varepsilon \omega_{N-2}^{-} \left(s_{N-2} - r_{N-2} \right) - \varepsilon^2 \bar{\lambda}_{N-2} \omega_{N-2}^{-} r_{N-2} \right\}^{\gamma} \\ &\quad \left\{ \widehat{V}_{N-1}^{(S)} + \delta_{N-1}^{(S)} \right\} p(s_{N-2}) \, \mathrm{d}s_{N-2} + O(\varepsilon^3). \end{aligned} \tag{C.13}$$

Here, we note that $\widehat{V}_{N-1}^{(B)}$, $\widehat{V}_{N-1}^{(S)}$, $\delta_{N-1}^{(B)}$ and $\delta_{N-1}^{(S)}$ are functions of $A_{N-1} = \dfrac{s_{N-2} A_{N-2}^{-}}{F_{N-2}^{(B)}}$ since the investor buys to reach the optimal buy boundary A_{N-2}^{-}.

In addition, recall that the suboptimal value function given by Equation (3.68) is

$$\widehat{V}_{N-2}^{(B)} = \int_0^{\widetilde{s}_{N-2}} \widehat{F}_{N-2}^{(B)\gamma} \widehat{V}_{N-1}^{(B)} p(s_{N-2}) \, \mathrm{d}s_{N-2} + \int_{\widetilde{s}_{N-2}}^{\infty} \widehat{F}_{N-2}^{(B)\gamma} \widehat{V}_{N-1}^{(S)} p(s_{N-2}) \, \mathrm{d}s_{N-2} + O(\varepsilon^3). \tag{C.14}$$

In this case, $\widehat{V}_{N-1}^{(B)}$ and $\widehat{V}_{N-1}^{(S)}$ are functions of $A_{N-1} = \dfrac{s_{N-2} \widetilde{A}_{N-2}}{\widehat{F}_{N-2}^{(B)}}$ since the investor buys to reach the Merton proportion $\widetilde{A}_{N-2}$. Adopting a similar approach as the derivation of the correction $\delta_{N-1}^{(B)}$ at time period $N-1$, we first perturb the optimal value function about the suboptimal value function, followed by a perturbation about the no transaction costs solution. Subtracting Equation (C.14) from Equation (C.13), expanding in powers of ε and simplifying with Equations (3.43), (3.46) and (3.47), it can be shown after much algebra that

$$\begin{aligned}\delta^{(B)}_{N-2} &= \widetilde{V}_{N-2}\varepsilon^2\Bigg\{\left[\bar{\lambda}_{N-2}\omega^{-}_{N-2}\widetilde{A}_{N-2}\left(\widetilde{A}_{N-2}-A_{N-2}\right)+\frac{1}{2}\omega^{-2}_{N-2}\right]\phi_{N-2}\\ &\quad -\bar{\lambda}_{N-2}\omega^{-}_{N-2}\gamma+\omega^{-}_{N-2}\psi_{N-2}+\theta_{N-2}\Bigg\}+O(\varepsilon^3), \qquad \text{(C.15)}\end{aligned}$$

where ϕ_{N-2}, ψ_{N-2} and θ_{N-2} are defined in Equations (3.97), (3.98) and (3.99) respectively.

In the *sell region*, the correction term can be immediately deduced from that in the buy region by a change of variables from $\bar{\lambda}_{N-2}$ to $-\bar{\mu}_{N-2}$ and from ω^{-}_{N-2} to ω^{+}_{N-2} to give us

$$\begin{aligned}\delta^{(S)}_{N-2} &= \widetilde{V}_{N-2}\varepsilon^2\Bigg\{\left[\bar{\mu}_{N-2}\omega^{+}_{N-2}\widetilde{A}_{N-2}\left(A_{N-2}-\widetilde{A}_{N-2}\right)+\frac{1}{2}\omega^{+2}_{N-2}\right]\phi_{N-2}\\ &\quad +\bar{\mu}_{N-2}\omega^{+}_{N-2}\gamma+\omega^{+}_{N-2}\psi_{N-2}+\theta_{N-2}\Bigg\}+O(\varepsilon^3). \qquad \text{(C.16)}\end{aligned}$$

In the *no-transaction region*, recall that $V^{(N)}_{N-2}$ is given by Equation (3.38). Following the same line of argument as in the buy region, we can show that similar to Equation (C.12),

$$\begin{aligned}V^{(N)}_{N-2} &= \int_0^{\widetilde{s}_{N-2}} F^{(N)\gamma}_{N-2}V^{(B)}_{N-1}p(s_{N-2})\,\mathrm{d}s_{N-2}\\ &\quad +\int_{\widetilde{s}_{N-2}}^{\infty} F^{(N)\gamma}_{N-2}V^{(S)}_{N-1}p(s_{N-2})\,\mathrm{d}s_{N-2}+O(\varepsilon^3). \qquad \text{(C.17)}\end{aligned}$$

Using Equation (3.80) and the approximation of the optimal value function from time period $N-1$,

$$\begin{aligned}V^{(N)}_{N-2} &= \int_0^{\widetilde{s}_{N-2}} \left\{\widetilde{F}_{N-2}+\varepsilon\omega_{N-2}\left(s_{N-2}-r_{N-2}\right)\right\}^{\gamma}\\ &\quad \left\{\widehat{V}^{(B)}_{N-1}+\delta^{(B)}_{N-1}\right\}p(s_{N-2})\,\mathrm{d}s_{N-2}\\ &\quad +\int_{\widetilde{s}_{N-2}}^{\infty} \left\{\widetilde{F}_{N-2}+\varepsilon\omega_{N-2}\left(s_{N-2}-r_{N-2}\right)\right\}^{\gamma}\\ &\quad \left\{\widehat{V}^{(S)}_{N-1}+\delta^{(S)}_{N-1}\right\}p(s_{N-2})\,\mathrm{d}s_{N-2}+O(\varepsilon^3), \qquad \text{(C.18)}\end{aligned}$$

where $\widehat{V}^{(B)}_{N-1}$, $\widehat{V}^{(S)}_{N-1}$, $\delta^{(B)}_{N-1}$ and $\delta^{(S)}_{N-1}$ are functions of $A_{N-1}=\dfrac{s_{N-2}A_{N-2}}{F^{(N)}_{N-2}}$ since the investor does not transact in this region where $A^{-}_{N-2} \leq A_{N-2} \leq A^{+}_{N-2}$. Let us denote

$$V^{(N)}_{N-2} = \widehat{V}^{(N)}_{N-2} + \delta^{(N)}_{N-2}, \tag{C.19}$$

where

$$\begin{aligned} \widehat{V}^{(N)}_{N-2} &= \int_0^{\widetilde{s}_{N-2}} \left\{ \widetilde{F}_{N-2} + \varepsilon\omega_{N-2}\left(s_{N-2} - r_{N-2}\right) \right\}^\gamma \widehat{V}^{(B)}_{N-1} p(s_{N-2})\,\mathrm{d}s_{N-2} \\ &\quad + \int_{\widetilde{s}_{N-2}}^{\infty} \left\{ \widetilde{F}_{N-2} + \varepsilon\omega_{N-2}\left(s_{N-2} - r_{N-2}\right) \right\}^\gamma \widehat{V}^{(S)}_{N-1} p(s_{N-2})\,\mathrm{d}s_{N-2} \end{aligned} \tag{C.20}$$

and

$$\begin{aligned} \delta^{(N)}_{N-2} &= \int_0^{\widetilde{s}_{N-2}} \left\{ \widetilde{F}_{N-2} + \varepsilon\omega_{N-2}\left(s_{N-2} - r_{N-2}\right) \right\}^\gamma \delta^{(B)}_{N-1} p(s_{N-2})\,\mathrm{d}s_{N-2} \\ &\quad + \int_{\widetilde{s}_{N-2}}^{\infty} \left\{ \widetilde{F}_{N-2} + \varepsilon\omega_{N-2}\left(s_{N-2} - r_{N-2}\right) \right\}^\gamma \delta^{(S)}_{N-1} p(s_{N-2})\,\mathrm{d}s_{N-2}. \end{aligned} \tag{C.21}$$

Expanding in powers of ε and simplifying with Equation (3.43), it can be shown after some algebra that

$$\begin{aligned} \widehat{V}^{(N)}_{N-2} &= \widetilde{V}_{N-2} \bigg\{ 1 + \varepsilon^2\omega_{N-2}\psi_{N-2} + \frac{1}{2}\varepsilon^2\omega^2_{N-2}\phi_{N-2} \\ &\quad + \varepsilon\zeta_{N-2} + \frac{1}{2}\varepsilon^2\eta_{N-2}\alpha_{N-1} \bigg\} + O(\varepsilon^3) \end{aligned} \tag{C.22}$$

and

$$\delta^{(N)}_{N-2} = \widetilde{V}_{N-2}\varepsilon^2\theta_{N-2} + O(\varepsilon^3). \tag{C.23}$$

Recall that α_{N-1}, ζ_{N-2}, η_{N-2}, ϕ_{N-2}, ψ_{N-2} and θ_{N-2} are defined in Equations (3.63), (3.73), (3.74), (3.97), (3.98) and (3.99) respectively. Thus, this concludes our derivation of $\delta^{(B)}_{N-2}$, $\delta^{(S)}_{N-2}$ and $V^{(N)}_{N-2}$ at time period $N-2$.

References

Akian, M., Menaldi, J. L., and Sulem, A. (1996). On an investment-consumption model with transaction costs. *SIAM Journal on Control and Optimization*, 34(1):329–364.

Andersen, E. D. and Damgaard, A. (1999). Utility based option pricing with proportional transaction costs and diversification problems: an interior-point optimization approach. *Applied Numerical Mathematics*, 29(3):395–422.

Arrow, K. (1971). *Essays in the Theory of Risk-Bearing*. North-Holland, Amsterdam.

Atkinson, C. and Alexandropoulos, C. A. (2006). Pricing a european basket option in the presence of proportional transaction costs. *Applied Mathematical Finance*, 13(3):191–214.

Atkinson, C. and Quek, G. (2012). Dynamic portfolio optimization in discrete-time with transaction costs. *Forthcoming in Applied Mathematical Finance*, 0(0):1–34.

Atkinson, C. and Storey, E. (2010). Building an optimal portfolio in discrete time in the presence of transaction costs. *Applied Mathematical Finance*, 17(4):323–357.

Atkinson, C. and Wilmott, P. (1995). Portfolio management with transaction costs: An asymptotic analysis of the morton and pliska model. *Mathematical Finance*, 5(4):357–367.

Barles, G. and Soner, H. M. (1998). Option pricing with transaction costs and a nonlinear black-scholes equation. *Finance and Stochastics*, 2:369–397.

Bellman, R. (1957). *Dynamic Programming*. Princeton University Press, Princeton, New Jersey. Sixth Printing, 1972.

Bensaid, B., Lesne, J.-P., Pags, H., and Scheinkman, J. (1992). Derivative asset pricing with transaction costs. *Mathematical Finance*, 2(2):63–86.

Bertsekas, D. P. (2005). *Dynamic Programming and Optimal Control*, volume 1. Athena Scientific, Belmont, Massachusetts, 3rd edition.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *The Journal of Political Economy*, 81(3):637–654.

Bobryk, R. V. and Stettner, L. (1999). Discrete time portfolio selection with proportional transaction costs. *Probability and Mathematical Statistics*, 19(2):235–248.

Boyle, P. P. and Vorst, T. (1992). Option replication in discrete time with transaction costs. *The Journal of Finance*, 47(1):271–293.

Carmona, R., editor (2009). *Indifference Pricing: Theory and Applications*. Princeton University Press, Princeton, New Jersey.

Clewlow, L. and Hodges, S. (1997). Optimal delta-hedging under transactions costs. *Journal of Economic Dynamics and Control*, 21(8-9):1353–1376.

Constantinides, G. M. and Zariphopoulou, T. (1999). Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences. *Finance and Stochastics*, 3:345–369.

Cox, J. C., Ross, S. A., and Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, 7(3):229–263.

Davis, M. H. A. (1997). *Mathematics of Derivative Securities*, chapter Option Pricing in Incomplete Markets, pages 216–226. Cambridge University Press.

Davis, M. H. A. and Clark, J. M. C. (1994). A note on super-replicating strategies. *Philosophical Transactions: Physical Sciences and Engineering*, 347(1684):485–494.

Davis, M. H. A. and Norman, A. R. (1990). Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713.

Davis, M. H. A., Panas, V. G., and Zariphopoulou, T. (1993). European option pricing with transaction costs. *SIAM Journal on Control and Optimization*, 31(2):470–493.

Denis, E. (2010). Approximate hedging of contingent claims under transaction costs for general pay-offs. *Applied Mathematical Finance*, 17(6):491–518.

Edirsinghe, C., Naik, V., and Uppal, R. (1993). Optimal replication of options with transactions costs and trading restrictions. *The Journal of Financial and Quantitative Analysis*, 28(1):117–138.

Gerhold, S., Muhle-Karbe, J., and Schachermayer, W. (2011). Asymptotics and duality for the davis and norman problem. *Forthcoming in Stochastics An International Journal of Probability and Stochastic Processes*, 0(0):1–17.

Hodges, S. D. and Neuberger, A. (1989). Optimal replication of contingent claim under transactions costs. *Review of Futures Markets*, 8:222–239.

Janecek, K. and Shreve, S. (2004). Asymptotic analysis for optimal investment and consumption with transaction costs. *Finance and Stochastics*, 8:181–206.

Kabanov, Y. M. and Safarian, M. M. (1997). On leland's strategy of option pricing with transactions costs. *Finance and Stochastics*, 1:239–250.

Kamrad, B. and Ritchken, P. (1991). Multinomial approximating models for options with k state variables. *Management Science*, 37(12):1640–1652.

Leland, H. E. (1985). Option pricing and replication with transactions costs. *The Journal of Finance*, 40(5):1283–1301.

Magill, M. J. P. and Constantinides, G. M. (1976). Portfolio selection with transactions costs. *Journal of Economic Theory*, 13(2):245–263.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1):77–91.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413.

Merton, R. C. (1990). *Continuous-Time Finance*. Blackwell Publishers Ltd, Oxford, UK. Reprinted 2001. Chapter 14, Section 14.2.

Mokkhavesa, S. and Atkinson, C. (2002). Perturbation solution of optimal portfolio theory with transaction costs for any utility function. *IMA Journal of Management Mathematics*, 13(2):131–151.

Monoyios, M. (2004). Option pricing with transaction costs using a markov chain approximation. *Journal of Economic Dynamics and Control*, 28(5):889–913.

Morton, A. J. and Pliska, S. R. (1995). Optimal portfolio management with fixed transaction costs. *Mathematical Finance*, 5(4):337–356.

Mossin, J. (1968). Optimal multiperiod portfolio policies. *The Journal of Business*, 41(2):215–229.

Nayfeh, A. H. (2004). *Perturbation Methods*. Wiley-VCH, Weinheim.

Palmer, K. (2001). A note on the boyle-vorst discrete-time option pricing model with transactions costs. *Mathematical Finance*, 11(3):357–363.

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica*, 32(1/2):122–136.

Roux, A., Tokarz, K., and Zastawniak, T. (2008). Options under proportional transaction costs: An algorithmic approach to pricing and hedging. *Acta Applicandae Mathematicae*, 103:201–219.

Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51(3):239–246.

Sass, J. (2005). Portfolio optimization under transaction costs in the crr model. *Mathematical Methods of Operations Research*, 61(2):239–259.

Shreve, S. E. and Soner, H. M. (1994). Optimal investment and consumption with transaction costs. *The Annals of Applied Probability*, 4(3):609–692.

Soner, H. M., Shreve, S. E., and Cvitanic, J. (1995). There is no nontrivial hedging portfolio for option pricing with transaction costs. *The Annals of Applied Probability*, 5(2):327–355.

Taksar, M., Klass, M. J., and Assaf, D. (1988). A diffusion model for optimal portfolio selection in the presence of brokerage fees. *Mathematics of Operations Research*, 13(2):277–294.

Whalley, A. E. and Wilmott, P. (1997). An asymptotic analysis of an optimal hedging model for option pricing with transaction costs. *Mathematical Finance*, 7(3):307–324.

Zakamouline, V. I. (2006). European option pricing and hedging with both fixed and proportional transaction costs. *Journal of Economic Dynamics and Control*, 30(1):1–25.