# Robust Feedback Model Predictive Control of Norm-bounded Uncertain Systems

by

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*Dedicated to my family*

## Declaration of Originality

I hereby declare that this thesis is the product of my own research work. Any results and ideas from the work of other people, published or otherwise, are appropriately referenced.

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## Copyright Declaration

# Abstract

This thesis is concerned with the Robust Model Predictive Control (RMPC) of linear discrete-time systems subject to norm-bounded model-uncertainty, additive disturbances and hard constraints on the input and state. The aim is to design tractable, feedback RMPC algorithms that are based on linear matrix inequality (LMI) optimizations.

The notion of feedback is very important in the RMPC control parameterization since it enables effective disturbance/uncertainty rejection and robust constraint satisfaction. However, treating the state-feedback gain as an optimization variable leads to non-convexity and nonlinearity in the RMPC scheme for norm-bounded uncertain systems. To address this problem, we propose three distinct state-feedback RMPC algorithms which are all based on (convex) LMI optimizations. In the first scheme, the aforementioned non-convexity is avoided by adopting a sequential approach based on the principles of Dynamic Programming. In particular, the feedback RMPC controller minimizes an upper-bound on the cost-to-go at each prediction step and incorporates the state/input constraints in a non-conservative manner. In the second RMPC algorithm, new results, based on slack variables, are proposed which help to obtain convexity at the expense of only minor conservatism. In the third and final approach, convexity is achieved by re-parameterizing, online, the norm-bounded uncertainty as a polytopic (additive) disturbance. All three RMPC schemes drive the uncertain-system state to a terminal invariant set which helps to establish Lyapunov stability and recursive feasibility.

Low-complexity robust control invariant (LC-RCI) sets, when used as target sets, yield computational advantages for the associated RMPC schemes. A convex algorithm for the simultaneous computation of LC-RCI sets and the corresponding controller for norm-bounded uncertain systems is also presented. In this regard, two novel results to separate bilinear terms without conservatism are proposed. The results being general in nature also have application in other control areas. The computed LC-RCI sets are shown to have substantially improved volume as compared to other schemes in the literature.

Finally, an output-feedback RMPC algorithm is also derived for norm-bounded uncertain systems. The proposed formulation uses a moving window of the past input/output data to generate (tight) bounds on the current state. These bounds are then used to compute an output-feedback RMPC control law using LMI optimizations. An output-feedback LC-RCI set is also designed, and serves as the terminal set in the algorithm.

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# Chapter 1

# Introduction

## 1.1 Motivation

This thesis is concerned with the robust control of linear systems subject to model-uncertainty, additive disturbances/noise and constraints on the input and state. Such a problem is motivated by the fact that most real-life processes involve constraints as well as uncertain dynamics, both of which need to be taken into account within the control system design.

It is well known that, in most cases, optimal performance of a process generally requires its operation to be closer to the constraint boundaries [64]. Such constraints arise naturally within most processes. For example, actuators are subject to saturation which therefore limits the amount of force that they can apply; valves are restricted by their maximum opening area which limits the flow rates; and key system parameters (such as temperature and pressure etc) are often required to remain within their critical limits. These constraints usually define a safe region of operation for the process and their violation can often result in plant instability. On the other hand, some additional process constraints may also be imposed by the designers/operators to obtain a desirable level of performance. For example, the system states such as displacement and velocity might be restricted between certain levels and motor actuator movements may be subject to rate constraints to avoid excessive mechanical wear and tear. Any prospective control system is therefore required to satisfy all the aforementioned process constraints whilst delivering optimal performance. In this regard, a particularly suitable control algorithm is known as Model Predictive Control [22].

Model Predictive Control (MPC), also known as Receding Horizon Control, refers to a family of control schemes in which the current control action is obtained by solving

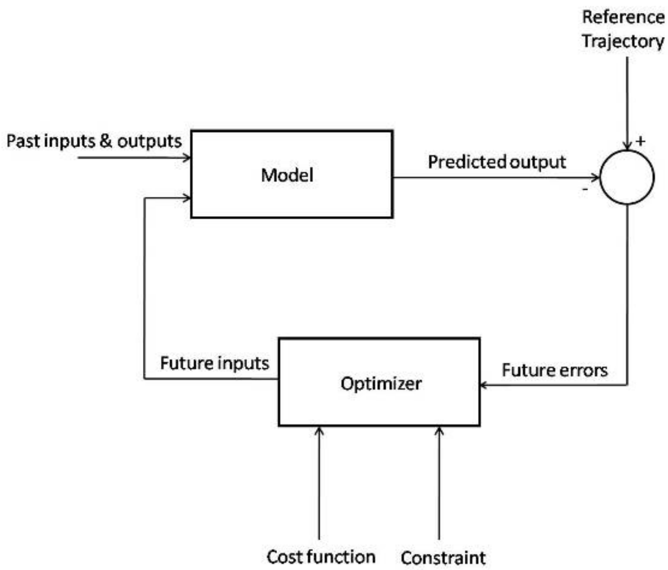


Figure 1.1: Structure of MPC scheme.

online, a constrained finite horizon optimal control problem [64, 66]. This optimization problem yields an optimal control sequence. Then, the first control in this sequence is applied to the plant and the optimization is repeated at the next time step in a receding horizon fashion. Figure 1.1 shows the basic structure of an MPC scheme [22].

MPC has been widely implemented as an advanced control technique within the process industry [28, 73, 82]. This is primarily due to the fact that, unlike many traditional control scheme (such as PID control), it explicitly takes account of process constraints within its formulation. This enables the MPC algorithm to operate the plants closer to their constraint boundaries (without violation) which, as mentioned above, results in optimal performance and therefore increased profits. Other advantages of MPC include its ability to handle multivariable, non-minimal phase and unstable processes, as well as the relatively easy tuning of the controller.

In addition to constraint handling, another important consideration in the design of any control system is robustness against any model-uncertainty and noise that maybe present within the dynamics. In particular, a control algorithm is said to be robust if it guarantees stability and maintains a prescribed level of system performance despite the presence of model-uncertainty and disturbances [9]. To highlight the detrimental effects of model-uncertainty that may be present within the system dynamics, let us investigate a simple example from [5].

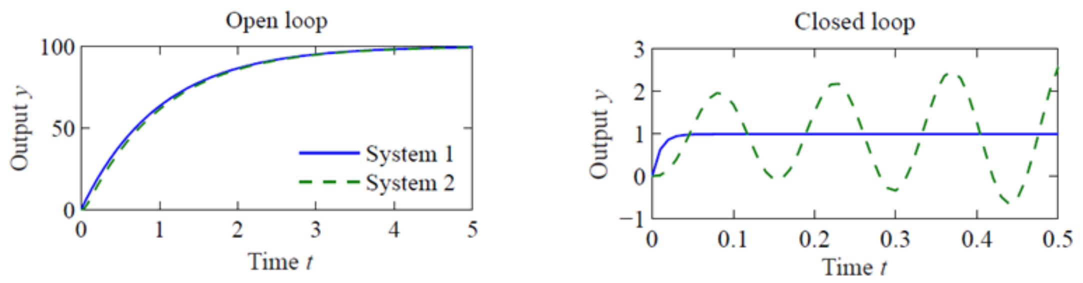


Figure 1.2: Open-loop response (left) and closed-loop response of $P_1(s)$ and $P_2(s)$.

Consider a system $P_2(s)$ and its model representation denoted by $P_1(s)$, where

$$P_2(s) = \frac{100}{(s+1)(0.025s+1)^2}\,, \qquad P_1(s) = \frac{100}{(s+1)} \tag{1.1}$$

In (1.1), both system and the model representing it are stable, with the model appropriately capturing the dominant system pole. Furthermore, the open-loop responses of both $P_1(s)$ and $P_2(s)$ are almost identical (left plot in Figure 1.2), which can further make the case for using $P_1(s)$ as an appropriate model for $P_2(s)$ in the control design. However, when we look at the closed-loop response (right plot in Figure 1.2), we see that the model's output converges to a reference value while the actual system becomes unstable. This shows the effect of ignoring the uncertainty (unmodeled dynamics) within model $P_1(s)$ and thus emphasizes the importance of explicitly handling the same within a robust control framework.

In the context of predictive control, it is important to note that MPC uses a mathematical model to predict and optimize the plant's future behaviour (see Figure 1.1). This, coupled with the fact that MPC operates the process closer to the constraint boundaries, means that any neglected disturbances/uncertainties can easily drive the system into an infeasible or unstable region. This motivates research into a class of predictive control algorithms known as Robust Model Predictive Control (RMPC) schemes [57].

RMPC schemes have received considerable amount of attention within the literature. However, as we discuss in Section 1.2 below, there is a dearth of convex feedback RMPC algorithms for systems that are subject to both 'norm-bounded' model-uncertainties and additive disturbances within their dynamics, as well as constraints on their states and input. In the light of the above discussion, this is clearly an important problem for efficient process control and hence forms the subject of research in this thesis.

## 1.2 Literature Review

In this section, we present a review of the literature on state-feedback and output-feedback RMPC schemes as well as robust control invariant sets.

### 1.2.1 State-feedback RMPC

RMPC schemes generally consider the control problem in a *worst-case* setting. That is, the control design guards against the most detrimental realization of uncertainty and disturbances in order to guarantee robust constraint satisfaction and stability.

Most of the robust predictive control schemes proposed in the literature can broadly be classified into the following two categories (or their suitable combinations/variations): open-loop MPC and feedback MPC. Open-loop schemes consider future control input profile as a function of the current state only which, though computationally efficient, is generally too conservative and often causes infeasibility [66]. On the other hand, feedback RMPC schemes consider future control inputs as linear/nonlinear function of future predicted states and have the advantage of mitigating the effect of disturbances.

To demonstrate the advantages of feedback in RMPC, we consider the following first order system example from [91]:

$$x_{k+1} = x_k + u_k + w_k \tag{1.2}$$

where disturbance $w_k \in \mathcal{W} := \Big\{w \in \mathbb{R} : -1 \leq w \leq 1\Big\}$. The state constraints are given by $x_k \in \mathcal{X} := \Big\{x \in \mathbb{R} : -1.2 \leq x \leq 2\Big\}$, $\forall k$, and the control input $u_k$ is unconstrained. The prediction horizon is taken to be $N = 3$. At time $k$, the min-max open-loop MPC scheme considers a control sequence $U_k^* := \{u_k^*,\ u_{k+1}^*,\ u_{k+2}^*\}$ based (only) on the current state $x_k$, which is required to satisfy the state constraints whilst guarding against the worst possible disturbance. By considering the two extreme disturbance profiles, $w = +1$ and $w = -1$, throughout the horizon, the terminal predicted state at time $k$, denoted by $x_{k+3|k}$, can either be given by

$$x_{k+3|k}^{+1} = x_k + u_k^* + u_{k+1}^* + u_{k+2}^* + 3 \quad \text{or} \quad x_{k+3|k}^{-1} = x_k + u_k^* + u_{k+1}^* + u_{k+2}^* - 3$$

Note that since $x_{k+3|k}^{+1} - x_{k+3|k}^{-1} = 6 > 2 > -1.2$, therefore there cannot exist a single control sequence $U_k^*$ such that both $x_{k+3|k}^{+1} \in \mathcal{X}$ and $x_{k+3|k}^{-1} \in \mathcal{X}$. Hence, the open-loop MPC problem becomes infeasible for the example in (1.2).

Let us now consider a feedback RMPC scheme where the future control inputs are functions of future states such that the sequence $U_k^* := \{u_k^*(x_k),\ u_{k+1}^*(x_{k+1}),\ u_{k+2}^*(x_{k+2})\}$. Then, it can be verified that by simply setting $u_k = -x_k$, we obtain $x_{k+3|k} = w_{k+2} \in \mathcal{X}$. Therefore, an admissible control sequence does exist in this case and hence the feedback RMPC problem is feasible. This example serves to show that disturbances/uncertainties can only be effectively taken into account, in a minimally conservative manner, by optimizing over feedback control policies within the so-called *feedback RMPC schemes*.

In the literature, a large number of feedback RMPC schemes deal with the control of constrained linear, discrete time system that are subject (only) to disturbances. That is:

$$\begin{aligned} x_{k+1} &= Ax_k + B_u u_k + B_w w_k \\ x_k \in \mathcal{X}&, \quad u_k \in \mathcal{U}, \quad w_k \in \mathcal{W} \end{aligned} \tag{1.3}$$

It is clear that within this class of RMPC algorithms, nonlinear feedback control remains the least conservative choice due to its generality. In particular, the control law is considered as $u_k = f_k(x_0, \cdots, x_k)$, where $f_k$ is potentially a nonlinear function of all the available states. However, computation of these nonlinear feedback policies is often very difficult as there is no tractable method of parameterizing such a control law in the online optimization problem [47]. To remedy this, several contributions in the literature employ Dynamic Programming techniques [13] to obtain an algorithm which results in a piecewise affine state-feedback control law, see e.g [10, 32, 52, 69, 85]. However, in most cases, the algorithm complexity grows exponentially with the problem data. This, therefore, restricts the applicability of such schemes to problems of a smaller size. An alternative approach is based on the computation of extreme disturbance profiles and assigning a different control sequence to each of these profile with a certain causality constraint imposed, see e.g. [91]. Though this results in minimal conservatism, however, the resulting RMPC scheme generally has a prohibitively high computational burden stemming from the combinatorial nature of the online optimization. Due to these reasons, many RMPC schemes are based on a linear feedback control law, which we discuss next.

In the literature, RMPC algorithms with linear feedback have been considered both in the finite as well as infinite horizon context. Typically, infinite horizon MPC schemes, such as [55, 77], have desirable stability properties associated with them. However, the control law is generally restricted to a constant state-feedback throughout the horizon, i.e. $u_k = Kx_k$, which can render the control algorithm conservative. On the other hand, in the context of finite horizon (linear) state-feedback RMPC schemes, the aim is to parameterize the control law as

$$u_k = v_k + \sum_{i=0}^{k} K_{k,i}\, x_i \ , \quad \forall k \in \{0, 1, \cdots, N-1\} \tag{1.4}$$

where $N > 0$ is the prediction horizon, and matrices $K_{k,i}$ and $v_k$ are both considered as decision variables in the online optimization. However, it is easy to verify that this leads to sequences of predicted states and inputs which are nonlinear, non-convex functions of the control gains ($K_{k,i}$, $v_k$) [46].

To make the linear feedback RMPC problem tractable, many scheme in the literature fix the feedback gain $K$ offline and optimize online with respect to control-perturbations $v_k$, see e.g. [7, 27, 51, 59]. However, there is no optimal method for computing the feedback gain offline and hence the resulting scheme can be conservative depending upon this choice of $K$.

More recently, algorithms based on Youla parameterization (sometimes also called Q-parameterization) [108], to obtain convexity in feedback RMPC schemes with control law (1.4) have also been proposed. For example, the scheme in [92] considers the use of such methods for RMPC control of systems with stochastic disturbances. A special case of Youla parameterization is the disturbance-feedback control structure, which was first proposed in the context of RMPC in [61, 103]. These results were extended in [46], where the authors showed that for systems of the form given in (1.3), under suitable assumptions, linear state-feedback (1.4) is equivalent to the following disturbance-feedback control parameterization:

$$u_k = m_k + \sum_{i=0}^{k-1} F_{k,i}\, w_i \ , \quad \forall k \in \{0, 1, \cdots, N-1\} \tag{1.5}$$

where the disturbance $w_i \in \mathcal{W}$, $\forall i$, and where matrices $F_{k,i}$, $m_k$ are treated as variables. Here note that, since full state feedback is assumed, the past disturbances can simply be computed by taking the difference between the predicted state and actual state. For instance, assuming $B_w$ as identity in (1.3) gives:

$$w_k = x_{k+1} - Ax_k - B_u u_k \ , \quad \forall k \in \{0, 1, \cdots, N-1\} \tag{1.6}$$

In [46], it is shown that for systems (1.3) with polytopic disturbance sets $\mathcal{W}$ and certain assumptions on the cost function, the equivalence between (1.4) and (1.5) implies that the optimal linear feedback RMPC control law can be computed through convex optimization

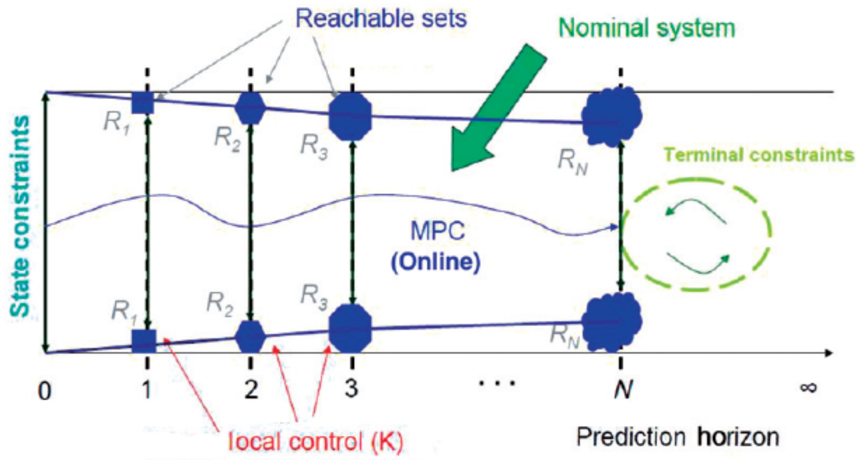


Figure 1.3: Basic structure of Tube-based MPC schemes [43]

problems. These results were extended in [45] to take account of more general convex disturbances sets, e.g. ellipsoidal sets.

Another important approach for linear feedback RMPC control of system (1.3) is the so-called tube-based MPC [58, 67]. Figure 1.3 shows the basic idea behind such schemes. Tube-based MPC typically considers a control law of the form

$$u_k = K(x_k - \bar{x}_k) + g_k \tag{1.7}$$

where $\bar{x}_k$ and $g_k$ respectively denote that state and control input for the corresponding nominal system of (1.3), which is given by:

$$\begin{aligned} \bar{x}_{k+1} &= A\bar{x}_k + B_u g_k \\ \bar{x}_k \in \bar{\mathcal{X}}_k&, \quad g_k \in \bar{\mathcal{U}}_k \end{aligned} \tag{1.8}$$

As shown in Figure 1.3, a key element of the tube-based MPC algorithm is the so-called reachable sets. These sets represent the (smallest) region which contains the state of the closed-loop uncertain system for any trajectory emanating from the origin [27]. In the context of tube-based MPC, reachable sets - along with the corresponding (local) control law $K$ - are used to take account of the mismatch between the actual system state and nominal system state (i.e. $x_k - \bar{x}_k$), which arises due to the disturbances. These sets and the control law $K$ may be computed online, at each prediction step, to take account of the disturbance, see e.g. [42]. Alternatively, a so-called robust invariant set, containing the sequence of reachable sets, maybe computed offline, see e.g. [58, 70]. The decision on which approach to adopt is based on the trade-off between conservatism and computational complexity. For example, the advantage of computing the sets offline lies in the reduced online computational burden. However, this comes at the cost of an increased level of conservatism within the tube-based MPC scheme since the feasible region is excessively reduced.

Once the reachable/invariant sets, call them $\mathcal{R}_i$, and local controller $K$ have been computed, the original state constraint set $\mathcal{X}$ in (1.3) is tightened down the prediction horizon for the nominal system (1.8), as shown in Figure 1.3. That is:

$$\bar{\mathcal{X}}_{k+i} = \mathcal{X} \ominus \mathcal{R}_{k+i}, \quad \forall i \in \{0, 1, \cdots, N\} \tag{1.9}$$

where $\ominus$ denotes the Pontryagin set difference (see Section 1.5). A similar procedure can be adopted for the input constraints $\bar{\mathcal{U}}_k$. Then, for the nominal system with tightened constraints (1.8), the tube-based MPC scheme solves a Quadratic Program (QP) online to compute $g_k$. The resulting control law (1.7) is then applied to system (1.3) and the cycle is repeated at the next time step.

The merit of tube-based MPC schemes such as [43, 58, 67] is the low computational complexity since only a simple QP is solved online for the nominal system and feasibility of the uncertain system (1.3) is guaranteed through constraint tightening procedure (1.9).

Despite their many advantages, the aforementioned feedback RMPC schemes cannot easily be extended to systems that are subject to both (norm-bounded) model-uncertainty and additive disturbances, which we consider in this thesis. That is, systems of the form:

$$\begin{gathered} x_{k+1} = (A + B_p \Delta C_q) x_k + (B_u + B_p \Delta D_{qu}) u_k + B_w w_k \\ x_k \in \mathcal{X}_k,\ u_k \in \mathcal{U}_k,\ w_k \in \mathcal{W}, \quad \Delta \in \boldsymbol{\Delta} := \{diag(\Delta_1, \cdots, \Delta_r) : \Delta_i \in \mathbb{R}^{qi \times qi}\ , \|\Delta\| \leq 1\} \end{gathered} \tag{1.10}$$

For instance, in the control parameterization (1.5), the past disturbances can no longer be computed using (1.6) due to model-uncertainty in $A$ and $B_u$ matrices. Furthermore, despite some recent results on tube-based MPC approaches for time-varying systems [42], the computation of reachable sets, along with the local control law, becomes particularly complex for norm-bounded uncertain system (1.10). This in turn is likely to add a degree of conservatism and computational complexity to the resulting tube-based algorithm.

In the literature, RMPC schemes specifically for systems with model-uncertainty have traditionally focused on linear dynamics with polytopic uncertainties. This is partly due to the fact that such a multi-model uncertainty structure fits well within the MPC framework. Having said that, however, there are a few classes of feedback RMPC schemes in the literature which deal with norm-bounded uncertain systems. For example, an infinite-horizon RMPC scheme for systems of the form given in (1.10) - but without the additive disturbances - is proposed in [55]. This algorithm has desirable stability properties but, as mentioned above, the proposed control law $u_k = Kx_k$ with $K$ fixed for the entire horizon can lead to excessive conservatism. An extension to this work was presented in [24], where the control law is considered as $u_k = Kx_k + c_k$. The feedback gain $K$ is computed offline and online optimization yields the control perturbation $c_k$ which minimizes an upper bound on the worst-case (infinite horizon) quadratic cost. The advantage of this scheme is that the number of inequality constraints grow only linearly with the control horizon $N$. However, non-convexity is avoided at the expense of conservatism through the offline choice of $K$. It is this non-convexity - associated with a variable $K$ formulation - within RMPC schemes which will be investigated in the thesis.

### 1.2.2 Output-feedback RMPC

As discussed above, MPC algorithms typically require full state information to compute the control law. However, in many processes, only noisy output measurement are available. The predictive control algorithms for uncertain systems which use only these measured outputs within their formulation are called output-feedback RMPC schemes.

Traditionally, many output-feedback control schemes have been based on the concept of certainty equivalence [71]. In particular, an estimate of the state is computed through an observer, the dynamics of which are sufficiently faster than those of the control loop. This state estimate is then used within the feedback control law. Similar ideas, with consideration of the stability properties, have been employed in the context of output-feedback RMPC, see e.g. [38], [82] and the references therein. In order to enlarge the region of attraction and reduce the conservatism associated with such predictive control schemes while still satisfying the constraints, it is important to characterize, and explicitly take account of, the estimation error. In this regard, output-feedback MPC algorithms based on (error) set-membership estimation [15, 89] have been proposed in [8] and [25].

A substantial body of output-feedback predictive control literature deals with linear systems subject to process disturbances and measurement noise. That is:

$$\begin{gathered} x_{k+1} = Ax_k + Bu_k + w_k, \quad y_k = Cx_k + v_k \\ x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad w_k \in \mathcal{W}, \quad v_k \in \mathcal{V} \end{gathered} \tag{1.11}$$

where the pair $(A, B)$ is assumed to be stabilizable and $(A, C)$ detectable. For instance, the output-feedback RMPC scheme for (1.11) given in [60] employs a Luenberger-type observer:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k) \tag{1.12}$$

where $\hat{x}$ denotes the state-estimate and $L$ is the observer-gain. The control law is then considered to be of the form:

$$u_k = F\hat{x}_k + c_k, \quad \forall k \in \{0, 1, \cdots, N-1\} \tag{1.13}$$

The stabilizing feedback gain $F$ and observer gain $L$ are both computed offline, whereas the control perturbation $c_k$ is optimized online. The bounds on the state-estimation error ($e_k = x_k - \hat{x}_k$) are computed through an invariant set for the augmented state-vector, which is composed of estimated state $\hat{x}$ and error $e$. It is these invariant sets which guarantee stability of the augmented system. Other variations of the above approach have been proposed in, for example [87] and [25]. Note that all these schemes can essentially be considered as the output-feedback versions of the (state-feedback) algorithms given in [7, 27, 59] - which we have discussed in Section 1.2.1 above.

For system (1.11), an output-feedback extension of the aforementioned tube-based MPC algorithm has also been proposed in [68]. The control law is similar to (1.7) with the state replaced by its estimate $\hat{x}$ from observer (1.12). That is:

$$u_k = K\tilde{x}_k + g_k \tag{1.14}$$

with $\tilde{x}_k := \hat{x}_k - \bar{x}_k$ and where $\bar{x}_k$ and $g_k$ represent the state and control input of the nominal system (1.8). The estimation error $e_k$ and vector $\tilde{x}_k$ are bounded by their respective invariant sets, namely $\bar{X}$ and $\tilde{X}$, which are computed with the control law $K$. Much the same way as in the state-feedback case, the idea is to solve the nominal MPC problem with tightened state/input constraints:

$$\bar{\mathcal{X}} = \mathcal{X} \ominus (\bar{X} \oplus \tilde{X}), \quad \bar{\mathcal{U}} = \mathcal{U} \ominus K\tilde{X} \tag{1.15}$$

where $\oplus$ denotes the Minkowski sum (see Section 1.5). Robust stability of the tube-based scheme [68] is ensured and the online computational burden remains almost the same as that associated with the nominal MPC problem.

Extension of the 'disturbance-feedback' control parameterization (1.5) to the output-feedback case has also been proposed in [44]. Within this work, it is first shown that the control law

$$u_k = g_k + \sum_{i=0}^{k-1} K_{k,i}\, y_i \ , \quad \forall k \in \{0, 1, \cdots, N-1\} \tag{1.16}$$

yields non-convexity in the output-feedback RMPC algorithm for system (1.11). Then, to remedy this, the following control parameterization based on the output error feedback is considered

$$u_k = c_k + \sum_{i=0}^{k-1} M_{k,i}(y_i - C\hat{x}_i) = c_k + \sum_{i=0}^{k-1} M_{k,i}(Ce_i + v_i) \tag{1.17}$$

where $\hat{x}_i$ is the state-estimate computed from the observer (1.12). Such a control law, based on the difference between measurement and predicted output, yields convexity in the RMPC algorithm [44] and is in fact shown to be equivalent to the control paramterization given in (1.16).

The aforementioned algorithms all consider systems of the form in (1.11). However, some contributions on output-feedback RMPC control of norm-bounded uncertain systems (1.10) have also been recently proposed. For instance, the scheme in [62] extends the results of [60] to linear systems with unstructured norm-bounded uncertainties (but without disturbances). The algorithm considers suitable robust stability conditions and computes online, the control perturbations $c_k$ which minimize an upper-bound on the nominal cost. Similarly, the scheme in [36] extends the state-feedback algorithm of [24] to uncertain systems (1.10). Here the feedback gain and observer are designed offline, using bilinear matrix inequalities, to stabilize the augmented system. Then, the control perturbation $c_k$ is computed online as a solution to linear matrix inequalities.

With the exception of [44], all the above mentioned schemes choose the control feedback gain $K$ offline (to avoid non-convexity) and the disadvantage of doing so, namely potential conservatism, has already been discussed in Section 1.2.1. Another particularly important design parameter in the output-feedback RMPC schemes is the observer gain $L$. It is clear that the level of (state) estimation error is directly dependent on the observer dynamics (and hence on gain $L$). Therefore, the fact that all the above schemes choose a stabilizing gain $L$ offline is a potential source of algorithm infeasibilty/instability since the control performance is heavily dependent on the estimated state. A possible solution to these issues in the context of norm-bounded uncertain systems will also be proposed within this thesis.

### 1.2.3 Robust Control Invariant Sets

Robust Control Invariant (RCI) sets define a bounded state-space region to which the system state can be confined, for all possible disturbances/uncertainties, through the application of a (state-feedback) control law $K$. For system (1.3), RCI sets can formally be defined as follows [16]:

**Definition 1.1.** $\mathcal{Z} \subset \mathbb{R}^n$ *is a Robust Control Invariant set for system (1.3) if there exists a control law* $u = Kx \in \mathcal{U}$, *such that* $(A + B_u K)\mathcal{Z} \oplus B_w \mathcal{W} \subseteq \mathcal{Z}$.

Thus, if the initial state $x_0$ belongs to the set $\mathcal{Z}$, then all subsequent states will be kept within this set by the control law $u = Kx$ [91]. It follows that the set $\mathcal{Z}$ characterizes the evolution of system (1.3) for all possible disturbances $w_k \in \mathcal{W}$ [54].

RCI sets are of great significance in the robustness analysis and synthesis of controllers for uncertain systems. These sets play a fundamental role in establishing the stability and recursive feasibility of RMPC schemes, see e.g. [66, 100] and the references therein. As discussed in the previous sections, invariant sets form an important part of tube-based MPC scheme [58, 70], as well as various output-feedback RMPC schemes [60, 68]. Furthermore, they also serve as suitable target sets in robust time-optimal control schemes [14, 41, 65].

Invariant set computation has been the subject of extensive research over the past few decades [16, 17]. The two invariant set structures most often considered in the literature are ellipsoidal and polytopic [54]. For these set structures, the problem of computing both the maximal and the minimal invariant set (or their suitable approximations) is important.

Ellipsoidal RCI sets are generally given in the form [95]:

$$\mathcal{Z} := \Big\{ x \in \mathbb{R}^n : x^T Q x \leq 1 \Big\} \tag{1.18}$$

where the matrix variable $Q = Q^T \succ 0$. As discussed in [80], the incorporation of ellipsoidal invariant terminal sets (1.18) within linear MPC scheme transforms the optimization problem into a Semidefinite program (as opposed to the standard QP). This, in turn results in an increased online computational burden. Therefore, in the context of MPC, polytopic RCI target sets generally represent a more viable option. These are typically characterized as:

$$\mathcal{Z} := \Big\{ x \in \mathbb{R}^n : Gx \leq p \Big\} \tag{1.19}$$

where $G$ is a matrix of appropriate dimensions and vector $p > 0$.

For sets of the form (1.19), important results have been reported in [104], including necessary and sufficient conditions for invariance using Farkas Lemma [50]. With regards to optimality, the minimal robust invariant set for system (1.3), call it $\mathcal{Z}_\infty$, under the given control law $u = Kx$ is characterized by [54]:

$$\mathcal{Z}_\infty = \oplus_{i=0}^{\infty}(A + B_u K)^i B_w \mathcal{W}$$

Since $\mathcal{Z}_\infty$ involves Minkowski's sum of infinite many terms, it is generally intractable to compute unless the system dynamics are nilpotent [65], i.e. $(A + B_u K)^i = 0, \ \forall i > \alpha$. As a result much of the research has been focused on computing (outer) invariant approximations of $\mathcal{Z}_\infty$ (and similarly the inner approximations of the maximal invariant set). We briefly discuss these next.

Many schemes in the literature consider the problem of computing robust invariant sets for a fixed control law [40, 80]. An algorithm to compute arbitrarily close (outer) invariant approximations to $\mathcal{Z}_\infty$, for a fixed $K$, has been proposed in [83]. While in [34, 49], methods are derived to compute the (linear) control laws which render a fixed set invariant. Clearly, a better approach for optimizing the size of the invariant set is to simultaneously consider both the control law $K$ and RCI set $\mathcal{Z}$ as variables of optimization. In this regard, [84] proposes an optimization problem which yields both RCI set and the (set-valued) feedback control law for systems with additive disturbances. However, as the authors point out in their conclusion [84], it is not straightforward to extend these results to the case of systems which are subject to both model uncertainty and disturbances, such as (1.10).

RCI set computation algorithms for linear systems with 'polytopic' uncertainty have been proposed in [6, 18]. In these contributions, the idea is to compute an initial RCI set and then iteratively enlarge its volume to yield an (inner) approximation to the maximal robust invariant set, along with the corresponding control law. In [23], a scheme is presented for computing the maximal feasible invariant low-complexity polytope for nonlinear systems.

It is worth mentioning here that none of the aforementioned invariant set algorithms can be directly applied to systems which contain both norm-bounded model-uncertainty and additive disturbances. Clearly, the efficient computation of an RCI set, and the corresponding control law, for systems (1.10) is an important problem in the context of robust predictive control. Therefore, this problem will be considered within the thesis. Set invariance for systems (1.10) under output-feedback will also be investigated.

## 1.3 Thesis Organization and Highlights

In this section, we provide a brief description as well as contributions of each of the following chapters.

**Chapter 2: Theoretical Background**

Convex optimization problems are of great significance in the design and synthesis of robust predictive control laws. Therefore, in this chapter, we present some basic concepts from optimization theory including convex sets and functions, linear matrix inequalities (LMI), semidefinite programs and relaxations. We also discuss the S-procedure which is an effective technique to re-formulate non-convex optimizations into (convex) LMI problems, and is of key importance to the developments of the following chapters.

**Chapter 3: Robust Feedback MPC for Systems with Parametric Uncertainty**

In this chapter, feedback RMPC control of linear systems subject to parametric model-uncertainties, polytopic (additive) disturbances and constraints is considered. In particular, a dual-mode control scheme that consists of an outer as well as an inner controller is presented. The outer (RMPC) controller consists of a state-feedback gain and a control-perturbation, both of which are explicitly considered as decision variables in the online optimization. The non-convexity associated with such a parameterization (see Section 1.2.1) is avoided by adopting a sequential approach based on the principles of Dynamic Programming. The RMPC controller minimizes an upper-bound on the cost-to-go at each prediction step and is responsible for steering the uncertain system-state to a designed terminal invariant set. The (hyper-rectangle) terminal RCI set and corresponding (inner) controller are both simultaneously computed in one step as solutions to an LMI optimization. To improve robustness, the disturbance is negatively weighted in the cost function (as in $\mathcal{H}_\infty$-MPC). Furthermore, conditions are derived on the terminal cost so as to guarantee Lyapunov stability of the closed-loop system. Effectiveness of the proposed RMPC scheme is illustrated through numerical examples from the literature, including a paper making process. The work in this chapter is mostly based on [100].

**Chapter 4: Low-complexity Invariant Sets for Uncertain Systems**

RCI terminal sets play an important role in establishing stability and recursive feasibility of RMPC algorithms. As discussed in [18], low-complexity RCI (LC-RCI) sets hold several computational advantages (for the associated RMPC scheme) as compared to ellipsoidal and more general polytopic invariant sets. Therefore, in this chapter, we derive an algorithm for the efficient computation of LC-RCI sets, along with the corresponding control law, for systems (1.10), which are subject to (general) norm-bounded model-uncertainty and additive disturbances. We first show that this problem is nonlinear and non-convex (including bilinear and triple product terms in the formulation) due to the presence of model-uncertainty as well as the fact the both the set and controller are being considered as decision variables. To remedy this, we propose two new results to separate the bilinear terms in the diagonal and non-diagonal matrix entries, respectively, without introducing extra conservatism. Both results are general in nature and thus have potential application in other important control problems, such as Lyapunov stability. The volume of the invariant set is maximized/minimized through iteratively solving a convex/LMI optimization. The volume enlargement/reduction (for maximal/minimal RCI sets) and recursive feasibility properties of the iterative procedure are also guaranteed. Numerical examples show improvement over the results obtained in [18] and [98]. The formulation in this chapter is mostly based on [101].

**Chapter 5: State-feedback Parameterizations in RMPC of Uncertain Systems**

Two state-feedback RMPC schemes for norm-bounded uncertain systems (1.10) are presented in this chapter. The RMPC control law is of the form (1.4), consisting of a lower-triangular feedback matrix (to ensure causality) and control-perturbation, both to which are considered as optimization variables. Unlike the formulation in Chapter 3, in which non-convexity is circumvented by adopting a sequential approach, this chapter considers a ‘stacked’ formulation (as in standard MPC) and presents new results to remove the associated non-convexity.

An initial formulation for uncertain systems shows that the RMPC problem is highly nonlinear and non-convex in the feedback gains (though it is convex for systems with only disturbances). Therefore, in the first approach, we re-cast the disturbance as an uncertainty to concentrate the nonlinearities. Then, we extend the results in [35] to propose a new theorem, using slack-variables, which enables the convexification of the

RMPC problem at the expense of only minor conservatism in the formulation.

In the second approach, uncertainty is re-parameterized as a disturbance, through the online computation of polytopic bounds. Then, the S-procedure is used to derive an LMI problem for the computation of RMPC control law.

Stability and recursive feasibility of both schemes is ensured through the incorporation of a suitable RCI terminal set (Chapter 4). Finally, numerical examples from the literature are used to demonstrate the advantages of the RMPC schemes. The algorithms in this chapter are based on [99].

**Chapter 6: Output-feedback RMPC for Norm-bounded Uncertain Systems**

In this chapter, we extend the results of Chapter 5 to propose an output-feedback RMPC scheme for norm-bounded uncertain systems (1.10) with only noisy output measurements available. The algorithm considers two sub-problems, namely the estimation of the current state, and computation of the output-feedback RMPC control law.

Unlike most schemes in the predictive control literature which employ a fixed gain observer (see Section 1.2.2), we use a moving window of the past input/output data, in a manner reminiscent of moving horizon estimation, to compute upper- and lower-bounds on the current state. For systems with uncertainty, these bounds are computed (online) through LMI optimization, and are shown to be tight under certain conditions. It is also shown that for systems with only disturbances (and no uncertainty), tight bounds can be computed by solving a linear program.

The (current) state bounds are subsequently used within the RMPC control scheme to compute the output-feedback gain and perturbation online through LMI optimizations. We also propose a convex problem for the computation of an 'output-feedback' RCI terminal set, and corresponding control law, by extending the results of Chapter 4. Numerical examples from the literature highlight the output-feedback control performance as well as the accuracy of the computed state-bounds. The work in this chapter is mostly based on [96].

**Chapter 7: Conclusions**

This chapter provides a summary of the main contributions of the thesis and suggests some future research directions.

## 1.4 Publications

Most of the results in this thesis are based on the following publications:

### Journal Publications

- Furqan Tahir and Imad M Jaimoukha. Causal state-feedback parameterizations in robust model predictive control. *Automatica*, 49(9):2675-2682, 2013.
- Furqan Tahir and Imad M Jaimoukha. Robust feedback model predictive control of constrained uncertain systems. *Journal of Process Control*, 23(2):189–200, 2013.
- Furqan Tahir and Imad M. Jaimoukha. Low-complexity polytopic invariant sets for linear systems subject to norm-bounded uncertainty. *IEEE Transactions on Automatic Control*, Accepted for publication, 2014.
- Furqan Tahir and Imad M. Jaimoukha. Robust output-feedback model predictive control using input/output data. *In Preparation.*

### Conference Publication

- Furqan Tahir and Imad M Jaimoukha. Robust positively invariant sets for linear systems subject to model-uncertainty and disturbances. In *Proceedings of the 4th IFAC Nonlinear Model Predictive Control Conference*, 2012.

## 1.5 Notation

The notation we use in this thesis is fairly standard. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the space of $n$-dimensional (column) vectors whose entries are in $\mathbb{R}$, $\mathbb{R}^{n\times m}$ denotes the space of all $n \times m$ matrices whose entries are in $\mathbb{R}$ and $\mathbb{D}^{n\times n}$ denotes the space of diagonal matrices in $\mathbb{R}^{n\times n}$. For $A \in \mathbb{R}^{n\times m}$, $A^T$ denotes the transpose of $A$. If $A \in \mathbb{R}^{n\times n}$ is symmetric, $\underline{\lambda}(A)$ denotes the smallest eigenvalue of $A$ and we write $A \succeq 0$ if $\underline{\lambda}(A) \geq 0$ and $A \succ 0$ if $\underline{\lambda}(A) > 0$. Analogous definitions apply to $\overline{\lambda}(A)$, $A \preceq 0$ and $A \prec 0$. We define the (spectral) norm of $A \in \mathbb{R}^{n\times m}$ as $\|A\| = \sqrt{\overline{\lambda}(AA^T)}$. For $x, y \in \mathbb{R}^n$, $x < y$ (and similarly $\leq$, $>$ and $\geq$) is interpreted element-wise. Given two sets $\mathcal{M}$ and $\mathcal{V}$, such that $\mathcal{M} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{M} \oplus \mathcal{V} := \{m + v | m \in \mathcal{M}, v \in \mathcal{V}\}$ and the Pontryagin difference is defined by $\mathcal{M} \sim \mathcal{V} := \{m | m + v \in \mathcal{M}, \forall v \in \mathcal{V}\}$. If $\boldsymbol{\mathcal{U}} \in \mathbb{R}^{n\times m}$ is a set, then $\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{U}}$ denotes the unit ball of $\boldsymbol{\mathcal{U}}$. Notation $I_q$ denotes a $q \times q$ identity matrix; the subscript is omitted when it can be inferred from the context. Furthermore, $e_i$ denotes the $i$th column of an appropriate identity matrix. Let $z \in \mathbb{R}^n$ and denote the $i$-th element of $z$ by $z_i$. Then, $\mathrm{diag}(z)$ is the diagonal matrix whose $(i, i)$ entry is $z_i$. For square matrices $A_1, \ldots, A_m$, $\mathrm{diag}(A_1, \ldots, A_m)$ denotes a block diagonal matrix whose $i$-th diagonal block is $A_i$. Finally, for matrices $A$ and $B$, $A \otimes B$ denotes the Kronecker product.

# Chapter 2

# Theoretical Background

In this chapter, we present some background material that is relevant in the context of Robust MPC formulations. In particular, a few basic concepts from optimization theory are briefly discussed, including convex optimization problems, semidefinite programming, linear matrix inequalities and the S-procedure.

## 2.1 Convex Optimization Problems

As discussed in Chapter 1, MPC is an optimization-based control technique. In particular, an optimization problem is solved online, at each sampling instant, to compute the optimal control sequence. Therefore, it is essential that the formulated optimization problem is such that it can be solved in an efficient manner - within the sampling interval. One such class of problems are the convex optimization problems [20].

Recall that convex optimization problems are of the general form:

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && g_i(x) \le d_i \ , \ i = 1, \cdots, m \end{aligned} \tag{2.1}$$

where $g_i(x) \le d_i$ represents convex constraints and $f(x)$ is the convex cost function to be minimized. These two components of optimization (2.1) are quite significant and we briefly discuss each of them below.

Convex sets can be defined as follows [20]:

**Definition 2.1.** *A set $C$ is convex if, for any $x_1$, $x_2 \in C$, and $\alpha$ such that $0 \le \alpha \le 1$, the following relation holds*

$$\alpha x_1 + (1 - \alpha)x_2 \in C \tag{2.2}$$

Similarly, a convex function can be defined as follows.

**Definition 2.2.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if the domain of $f$ is a convex set and if for every pair of points $x_1$, $x_2$ in the domain of $f$, and $\alpha$ such that $0 \leq \alpha \leq 1$, the following inequality is satisfied:*

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2) \tag{2.3}$$

Recall that in the context of (2.1), the advantage of minimizing a convex function subject to convex constraints is that any local minimum of the problem is also a global minimum. Furthermore, for strictly convex functions (i.e. functions for which inequality (2.3) is strict), the minimum (if it exists) is unique. Algorithms, such as interior point methods [11], exploit these properties and are thus able to solve convex problems in an efficient, fast and reliable manner.

Convex optimization methods also play an important role in solving non-convex problems. Algorithms for solving non-convex and nonlinear optimization problems are generally inefficient. One approach to solving such problems is to consider local optimization methods which yield a locally optimal solution. However, these methods require an initial solution of the decision variables as a starting point, which is a critical factor in the algorithm convergence. In such cases, an approximate convex formulation can be obtained for the original non-convex problem (see Section 2.3). Then, the solution of the (approximate) convex problem, which is easily computed, can be used as the initial condition for the local optimization.

Convex optimization subsumes a large class of problems. For example, an important type of problems are the so-called Linear Programs (LP). These are of the form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \ , \ i = 1, \cdots, m \end{array} \tag{2.4}$$

where the vectors $c$, $a_i \in \mathbb{R}^n$ and scalars $b_i \in \mathbb{R}^n$. Note that the cost function and constraints in (2.4) are both linear and, therefore, convex. Another key class of optimization problems are the convex Quadratic Programs (QP), which can be written as [20]:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} x^T Q x + c^T x + d \\ \text{subject to} & Gx \leq f \\ & Px = r \end{array} \tag{2.5}$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric, positive semidefinite matrix and $G$, $P$, which represent affine problem constraints, are matrices of appropriate dimensions.

In the context of robust optimization and RMPC formulations, a particularly important class of convex optimization problems are the so-called semidefinite programs, which we discuss next.

## 2.2 Semidefinite Programs

Semidefinite programming has attracted substantial research interest over the past few decades [30]. This is because semidefinite programs (SDPs) have extensive application in system and control theory as well as other fields such as combinatorial and robust optimization. Also, importantly, there exist efficient algorithms to solve SDPs, for instance interior point methods [2].

SDPs are convex optimization problems which involve the minimization of a linear function subject to a constraint that requires a symmetric matrix - which is affine in the decision variables - to be positive semidefinite. In particular, an SDP can be written as:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && F(x) \succeq 0 \ , \end{aligned} \tag{2.6}$$

with

$$F(x) := F_0 + \sum_{i=1}^{n} x_i F_i \tag{2.7}$$

where $x \in \mathbb{R}^n$ is the decision variable, with $x_i$ denoting the $i$th entry of the vector, and symmetric matrices $F_0$ , $F_i \in \mathbb{R}^{m \times m}$ , $\forall i$, are given. Note that for the case when all the matrices $F_0 \cdots F_n$ are diagonal, the constraint in (2.6) becomes equivalent to $m$ linear inequalities. Hence, in this case, the SDP problem simply reduces to a linear program of the form given in (2.4).

The constraint in (2.6) is more generally known as a Linear Matrix Inequality and we briefly discuss these next.

### 2.2.1 Linear Matrix Inequalities

Linear Matrix Inequality (LMI) techniques play an important role in the formulation of various problems within system and control theory [19]. For instance, one of the most widely used LMI conditions is the Lyapunov inequality for establishing stability of linear continuous-time systems, which is given by:

$$A^T P + PA \prec 0$$

and for discrete-time systems, it becomes

$$A^T PA + P \prec 0$$

The robust predictive control algorithms proposed in this thesis are also mostly based on LMI constraints, which are formally defined as:

$$F(x) := F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0 \tag{2.8}$$

Note that the symmetric matrix $F(x)$ is affine in variable $x \in \mathbb{R}^n$ and is required to be positive semidefinite, i.e. $y^T F(x) y \succeq 0, \ \forall y$. Furthermore, (2.8) represents a convex constraint on $x$. Strict inequalities (i.e. positive definite or negative definite) or negative semidefinite inequalities can also be defined analogously.

In certain cases, optimization problems involve multiple LMI constraints, for instance:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && F^k(x) \succeq 0 \ , \ k = 1, \cdots, p \end{aligned} \tag{2.9}$$

with

$$F^k(x) := F_0^k + \sum_{i=1}^{n} x_i F_i^k, \ i = 1, \cdots, n \tag{2.10}$$

Such problems can be readily transformed to an SDP of standard form (2.6), as follows:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && \mathcal{L}(x) := \text{diag}(F^1(x), \ F^2(x), \cdots, \ F^p(x)) \succeq 0 \end{aligned} \tag{2.11}$$

Finally, an important LMI result, which will be used extensively in the development throughout this thesis is known as the Schur complement [21]. This is a result to represent convex nonlinear matrix inequalities in the form of LMIs without any conservatism, and is given by the following lemma [21]:

**Lemma 2.1.** *Define matrices $A = A^T$, $C = C^T$ and $B$ of appropriate dimensions and let*

$$\mathcal{L} := \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

*Then, for $C \succ 0$, the matrix $\mathcal{L} \succeq 0$ if and only if $A - BC^{-1}B^T \succeq 0$. Similarly, for $A \succ 0$, the matrix $\mathcal{L} \succeq 0$ if and only if $C - B^T A^{-1} B \succeq 0$. Furthermore, the following three statements are also equivalent*

*(i)* $\mathcal{L} \succ 0$,

*(ii)* $A \succ 0$ *and* $C - B^T A^{-1} B \succ 0$

*(ii)* $C \succ 0$ *and* $A - BC^{-1}B^T \succ 0$

Analogous results hold for the case when $\mathcal{L}(x)$ is negative definite or semidefinite.

## 2.3 Semidefinite Relaxations

In various fields of engineering, such as robust control design, communications and signal processing, one often encounters many important optimization problems that are computationally intractable (for example nonlinear non-convex problems). For such optimizations, it is generally very difficult to compute the (global) solution, that is if one even exists [74]. In these cases, semidefinite relaxation provides a useful technique to obtain an (approximate) convex formulation for the original non-convex optimization problem, in the form of an SDP (2.6), see e.g. [63, 102]. The solution of the SDP generally serves as a good approximation to the actual optimal solution for the non-convex problem. In fact, under certain conditions, semidefinite relaxation does not introduce any conservatism and hence, the SDP solution corresponds exactly to the optimal solution.

As we will show in this thesis, feedback RMPC formulations for uncertain systems of the form (1.10) also result in optimization problems which are nonlinear and non-convex in the decision variables (the control gain $K$). Therefore, we propose to obtain convexity through the application of semidefinite relaxation techniques to derive RMPC algorithms based on SDP problems. Such an approach has the advantage that the resulting SDPs are solved very efficiently using interior point methods [19]. This, therefore means that the proposed RMPC control law can easily be computed online in polynomial time [55].

### 2.3.1 The S-Procedure

The S-procedure is a technique that is used to relax nonlinear, non-convex optimizations and obtain their SDP approximations [21, 37]. It has found great application in many problem areas within control theory. The S-procedure can formally be defined as follows [21, Page 23].

**Lemma 2.2.** *Let $F_0, \cdots, F_p$ be quadratic functions of the variable $x \in \mathbb{R}^n$ such that:*

$$F_i := x^T T_i x + 2u_i^T x + v_i, \quad i = 0, \cdots, p, \tag{2.12}$$

*where $T_i = T_i^T$. Then, the following condition*

$$F_0(x) \geq 0 \text{ for all } x \text{ such that } F_i(x) \geq 0, \quad i = 1, \cdots, p \tag{2.13}$$

*holds if there exist $\tau_1 \geq 0, \cdots, \tau_p \geq 0$ such that*

$$\begin{bmatrix} T_0 & u_0 \\ u_0^T & v_0 \end{bmatrix} - \sum_{i=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix} \succeq 0. \tag{2.14}$$

*Furthermore, when $p = 1$, the converse also holds provided that there exists an $x_1$ such that $F_1(x_1) > 0$.*

**Remark 2.1.** *If the functions $F_i$, $i = 0, \cdots, p$, are convex in $x$, then (2.13) and (2.14) become equivalent. This is the so-called Farkas' Theorem [81]. Furthermore, if the function $F_i$ are affine, then the equivalence of (2.13) and (2.14) is known as the Farkas' Lemma [50].*

### 2.3.2 An Example Problem

In this section, let us consider an example problem so as to clarify the application of the S-procedure.

Let us first define the objective function $J := e_1^T[(A + B_u K)x + B_w w]$, where $x \in \mathcal{X} := \{x \in \mathbb{R}^n : -d \leq x \leq d\}$, $w \in \mathcal{W} := \{w \in \mathbb{R}^{n_w} : -v \leq w \leq v\}$, $e_1$ denotes the first column of an $n \times n$ identity matrix and $d$, $v$ are known vectors. Suppose we wish to compute a matrix $K$, if it exists, such that

$$e_1^T[(A + B_u K)x + B_w w] - \gamma \leq 0, \quad \forall x \in \mathcal{X}, \ w \in \mathcal{W} \tag{2.15}$$

Note that the feasibility problem would require finding a $K$ for a given $\gamma$, whereas the optimization problem consists of computing a $K$ that minimizes $\gamma$. However, in both cases, it is clear that (2.15) requires nonlinear optimization techniques. To remedy this, we now use the S-procedure to obtain an *equivalent* SDP formulation for the above problem (see also Remark 2.2).

**Theorem 2.1.** *There exists $K$ and $\gamma$ such that (2.15) is satisfied if and only if there exist diagonal positive semidefinite matrices $D_x \in \mathbb{R}^{n\times n}$ and $D_w \in \mathbb{R}^{n_w\times n_w}$ as solutions to the following SDP:*

$$\begin{aligned} &\text{minimize} \quad \gamma \\ &\text{subject to} \quad \mathcal{L}(\gamma, K, D_x, D_w) := \begin{bmatrix} D_x & 0 & -\frac{1}{2}(A+B_uK)^T e_1 \\ \star & D_w & -\frac{1}{2}B_w^T e_1 \\ \star & \star & \gamma - d^T D_x d - v^T D_w v \end{bmatrix} \succeq 0 \end{aligned} \tag{2.16}$$

**Proof.** As given in (2.15), we consider a variable $\gamma \in \mathbb{R}$ such that

$$e_1^T[(A+B_uK)x + B_w w] - \gamma \leq 0 \tag{2.17}$$

Then, for any $D_x \in \mathbb{R}^{n\times n}$ and $D_w \in \mathbb{R}^{n_w\times n_w}$, the left hand side of inequality (2.17) can be written as

$$\begin{aligned} &e_1^T[(A+B_uK)x + B_w w] - \gamma = -(d-x)^T D_x(x+d) - (v-w)^T D_w(v+w) \\ &- [-(d-x)^T D_x(x+d) - (v-w)^T D_w(v+w) - e_1^T(A+B_uK)x - e_1^T B_w w + \gamma] \end{aligned}$$

Representing the square-bracket terms of the above equation in a matrix form yields:

$$\begin{aligned} e_1^T[(A+B_uK)x + B_w w] - \gamma = &\overbrace{-(d-x)^T D_x(x+d)}^{J_x} \overbrace{-(v-w)^T D_w(v+w)}^{J_w} \\ &- [x^T \quad w^T \quad 1]\mathcal{L}(\gamma, K, D_x, D_w)\begin{bmatrix} x \\ w \\ 1 \end{bmatrix} \end{aligned} \tag{2.18}$$

where $\mathcal{L}(\gamma, K, D_x, D_w)$ is the LMI defined in (2.16).

Notice that $J_x \leq 0$ and $J_w \leq 0$ for any diagonal, positive semidefinite matrices $D_x$ and $D_w$. Then, using the S-procedure (Farkas' Theorem) [81], it follows that the existence of such $D_x$ and $D_w$ such that $\mathcal{L}(\gamma, K, D_x, D_w) \succeq 0$, is a necessary and sufficient condition for (2.17). Therefore, the SDP problem in (2.16) follows. □

**Remark 2.2.** *It is worth mentioning here that in order to simplify the presentation, we skipped a step, in Theorem 2.1, of defining the functions* $F_0$ *and* $F_i$ *to represent (2.15) in the form (2.13), which allows the use of Lemma 2.2, along with Remark 2.1, to arrive at (2.16) that corresponds to (2.14). Note that the diagonal entries of* $D_x$ *and* $D_w$ *simply correspond to the* $\tau_i$ *in (2.14).*

The S-procedure (Farkas' Theorem) used in Theorem 2.1 does not introduce any gap/conservatism within the formulation. However note that, in comparison to (2.15), the SDP (2.16) can be solved much more efficiently. We would like to mention that throughout the thesis, SDP problems such as (2.16), will be referred to as LMI (optimization) problems.

Similar or appropriately modified versions of the procedure given in Theorem 2.1, in conjunction with other techniques such as slack-variable identities, will be employed in the thesis to help overcome non-convexity associated with robust optimization formulations. In addition, we will also be making use of the following version of the Farkas' Lemma [50] (see also Remark 2.1).

**Lemma 2.3.** *Let* $c \in \mathbb{R}^n,\ d \in \mathbb{R},\ A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$. *Suppose there exists* $\hat{y}$ *such that* $A\hat{y} < b$. *Then the following two statements are equivalent:*

*(1)* $c^T y \leq d\ \forall y$ *such that* $Ay \leq b$.

*(2)* $\exists \mu \in \mathbb{R}^m$ *such that* $\mu \geq 0,\ A^T\mu = c$ *and* $b^T\mu \leq d$.

# Chapter 3

# Robust Feedback MPC for Systems with Parametric Uncertainty

## 3.1 Introduction

In this chapter, we consider the RMPC control of linear, discrete-time systems subject to (scalar) parametric model-uncertainties and bounded disturbances along with hard constraints on the input and state. The proposed algorithm consists of an outer (RMPC) controller which is responsible for steering the uncertain system state to a designed terminal invariant set. Once the state is in this set, the inner (terminal) controller takes over and maintains it within the set despite the action of persistent uncertainty and disturbances.

As discussed in Section 1.2, the notion of feedback within the RMPC control law is important since it provides an effective method of mitigating the effect of uncertainty/disturbances whilst maintaining control feasibility. However, in order to avoid nonlinearity and non-convexity in the formulation, many RMPC schemes from the literature design the feedback gain offline and perform online optimization with respect to the control-perturbations [7, 27, 59]. Such an approach can be conservative depending upon the offline feedback design. Therefore, in this chapter, we propose an RMPC controller that consists of a state-feedback component as well as a control-perturbation, both of which are explicitly considered as decision variables in the online optimization at each time step. The nonlinearity and non-convexity associated with such a control structure is circumvented by adopting a sequential approach in the formulation which is based, in part, on the principles of Dynamic Programming [13].

This chapter extends the preliminary results of [97] to design a unified RMPC framework which handles both additive disturbances and (parametric) model-uncertainties simultaneously. In order to improve disturbance rejection in the formulation, we consider an $\mathcal{H}_2/\mathcal{H}_\infty$-based cost function, which is minimized by the outer (RMPC) controller. The overall algorithm is based on a sequence of low-dimensional LMI optimizations which helps to reduce the online computational burden as compared to nonlinear feedback MPC schemes such as [91] and [106].

Recursive feasibility of the proposed RMPC algorithm is ensured through the incorporation of a terminal invariant set, which - along with its corresponding control law - is computed in one step as solution to an LMI optimization problem. Furthermore, conditions to guarantee Lyapunov stability of the closed-loop system are also derived [66]. Finally, the applicability of the algorithm is illustrated through numerical examples taken from the literature. The results in this chapter are primarily based on [100].

## 3.2 RMPC Problem

In this section, we provide a description of the system and constraints followed by the cost function. We also derive an upper bound on the cost function which is minimized by the RMPC controller.

### 3.2.1 System Description and Constraints

We consider a linear, discrete-time uncertain system of the form:

$$x_{k+1} = (A + \delta_\alpha A_\delta)x_k + (B_u + \delta_\beta B_\delta)u_k + B_w w_k \tag{3.1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathbb{R}^{n_w}$ are the state, input and bounded disturbance vectors at prediction step $k$; $A$ is the system matrix and $B_u$ and $B_w$ are the input and disturbance distribution matrices, respectively. Here $\delta_\alpha$ and $\delta_\beta$, together with $A_\delta$ and $B_\delta$, represent the (parametric) model-uncertainty in the system. We assume that the pair $(A,\, B_u)$ is stabilizable and the state $x_k$ is measured. The prediction step $k$ belongs to the time set $T_N = \{0, 1, \cdots, N-1\}$, where $N > 0$ is the prediction horizon. We consider polytopic disturbance is of the form

$$w_k \in \mathcal{W} := \Big\{ w \in \mathbb{R}^{n_w} : -v \leq w \leq v \Big\}$$

where $v > 0$. The model-uncertainty is characterized by:

$$\delta_\alpha \in \mathcal{D}_\alpha := \Big\{\delta \in \mathbb{R} : |\delta| \le \alpha\Big\}, \quad \delta_\beta \in \mathcal{D}_\beta := \Big\{\delta \in \mathbb{R} : |\delta| \le \beta\Big\}.$$

**Remark 3.1.** *In the context of process control, an uncertainty description of the form given in (3.1) frequently arises as a result of imprecise system-parameter values or various simplifying approximations, for example model-order reduction. For practical examples of such systems, see [56, 78] and the references therein.*

Due to the presence of uncertainty and disturbances, the system (3.1) cannot be controlled to the origin. The uncertain-system state can, at best, be confined to an RCI set $\mathcal{Z}$ [16]. To promote such convergence as well as to establish stability of the proposed scheme, we include in our formulation, the terminal state constraint $x_N \in \mathcal{Z}$ together with other hard constraints on the input and state. All these are summarized below:

$$x_k \in \mathcal{X}_k := \Big\{x \in \mathbb{R}^n : \underline{x}_k \le Cx \le \overline{x}_k\Big\}, \quad \forall k \in T_I := \{1, 2, \cdots, N-1\} \tag{3.2}$$

$$x_N \in \mathcal{Z} := \Big\{x \in \mathbb{R}^n : -z \le C_f\, x \le z\Big\} \tag{3.3}$$

$$u_k \in \mathcal{U}_k := \Big\{u \in \mathbb{R}^{n_u} : \underline{u}_k \le u \le \overline{u}_k\Big\}, \forall k \in T_N \tag{3.4}$$

where RCI set polytope $z > 0$ and the matrices $C_f,\ C \in \mathbb{R}^{n_y \times n}$ - assumed to have a full row rank - can be chosen to represent polytopic constraints on individual states and/or their linear combinations (e.g. outputs).

The RMPC controller we consider in this chapter has the form: $u_k = F_k x_k + m_k$. Note that the control structure consists of both a state-feedback component ($F_k$) as well as an open-loop component ($m_k$).

**Remark 3.2.** *Similar to tube-based MPC [58], a constraint tightening approach can also be adopted in this algorithm to enhance robustness and convergence, see e.g. [88]. One possible method of selecting a tightening set is also given in Remark 3.12.*

**Remark 3.3.** *Many RMPC schemes incorporate the idea of a terminal invariant set [59, 91]. However, it is often difficult to compute suitable invariant sets as it generally requires iterative computations [12, 18, 83]. Furthermore, as discussed in Section 1.2.3, many existing algorithms compute invariant sets for a fixed control law, see e.g. [40, 80]. In our scheme, despite being a conservative structure, we have chosen the RCI set (3.3)* *to be a hyper-rectangle. This is because, as shown in Section 3.3, such a structure enables us to efficiently compute the optimal invariant set and the corresponding inner controller (simultaneously) in one step by solving a single LMI optimization problem. A formulation for more general RCI set structures is also presented in Chapter 4.*

### 3.2.2 Cost Function

We consider the following cost function:

$$J(x_0, u, w, \delta_\alpha, \delta_\beta) := x_N^T P_N x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k - \gamma^2 w_k^T w_k - \epsilon^2 \tag{3.5}$$

where $x_0$ is the given current state, $P_N = P_N^T \succ 0$, $Q = Q^T \succ 0$ and $R = R^T \succ 0$ are known matrices and where

$$u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{n_u N}, \quad w = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix} \in \mathbb{R}^{n_w N}$$

Here, $\gamma^2$ and $\epsilon^2$ are known positive constants used to regulate/constrain the effect of disturbances and open-loop control component $m_k$ (see Remarks 3.5 and 3.6 below).

By inserting the outer controller structure $u_k = F_k x_k + m_k$ into (3.5), we obtain:

$$J(x_0, F, m, w, \delta_\alpha, \delta_\beta) := x_N^T P_N x_N + \sum_{k=0}^{N-1} g(x_k, F_k, m_k, w_k) \tag{3.6}$$

where $g(x_k, F_k, m_k, w_k)$ is the stage cost at each prediction step and is defined as:

$$g(x_k, F_k, m_k, w_k) := x_k^T (Q + F_k^T R F_k) x_k + m_k^T R m_k + 2 m_k^T R F_k x_k - \gamma^2 w_k^T w_k - \epsilon^2;$$

and $F = \begin{bmatrix} F_0^T & F_1^T & \cdots & F_{N-1}^T \end{bmatrix}^T \in \mathbb{R}^{n_u N \times n}$ and $m = \begin{bmatrix} m_0^T & m_1^T & \cdots & m_{N-1}^T \end{bmatrix}^T \in \mathbb{R}^{n_u N}$ are the stacked feedback gain matrix and the control-perturbation vector, respectively.

**Remark 3.4.** *A number of RMPC schemes in the literature consider a stage cost which is positive outside $\mathcal{Z}$ and zero within it (see e.g. [26, 90, 91]). This approach renders the cost function discontinuous. The proposed scheme, on the other hand, penalizes the terminal state $x_N$ which keeps the cost function continuous with respect to the state. Also, importantly, this enables us to derive conditions (in Section 3.4.4) under which terminal weighting matrix $P_N$ can be chosen to guarantee stability of the RMPC scheme [66].*

**Remark 3.5.** *Note that in the cost function (3.6), the disturbance is negatively weighted through the introduction of the constant $\gamma^2$. Predictive control schemes involving such a term in their cost are known as $\mathcal{H}_\infty$-MPC algorithms, see e.g. [53, 66, 69] and the references therein. In this framework, $\gamma^2$ represents a prescribed $\mathcal{H}_\infty$ disturbance rejection measure to improve the robustness of the algorithm. The impact of $\gamma^2$ will be further clarified through numerical examples in Section 3.5.*

Bearing in mind that the system response is due to $x_0$ and $w_k$, the design specifications can be summarized as follows:

For a prescribed disturbance rejection measure $\gamma \succ 0$, find an admissible $F$ and $m$ (i.e. ones that satisfy the constraints (3.2)-(3.4) which achieve the following requirements:

(S1) $J(0, F, 0, w, \delta_\alpha, \delta_\beta) \leq 0, \quad \forall w, \forall \delta_\alpha, \delta_\beta.$

(S2) $J^\star(x_0) := \min_{F,\, m} \ \max_{w,\, \delta_\alpha,\, \delta_\beta} \ J(x_0, F, m, w, \delta_\alpha, \delta_\beta).$

In view of the above specifications, we will assume (and, in Section 3.4.2, derive sufficient conditions for) the existence of matrices $P_k = P_k^T \succ 0$ such that $\forall k \in T_N$ [55]:

$$g(x_k, F_k, m_k, w_k) \leq x_k^T P_k x_k - x_{k+1}^T P_{k+1} x_{k+1}, \ \forall w_k \in \mathcal{W}, \ \delta_\alpha \in \mathcal{D}_\alpha, \ \delta_\beta \in \mathcal{D}_\beta \tag{3.7}$$

Summing the inequality in (3.7) for all $k \in T_N$ and subsequently adding the terminal cost to both sides yields the following upper bound on the cost function:

$$J(x_0, F, m, w, \delta_\alpha, \delta_\beta) \leq x_0^T P_0 x_0 \ , \quad \forall w, \ \forall \delta_\alpha, \forall \delta_\beta. \tag{3.8}$$

It follows immediately that the requirement in (3.7) is sufficient for design specification (S1). In view of design specification (S2), the proposed outer controller is chosen so as to minimize the upper bound (3.8) on the cost function.

**Remark 3.6.** *Note that (3.8) can be written as: $x_N^T P_N x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \leq N\epsilon^2 + \gamma^2 w^T w + x_0^T P_0 x_0$. From this, we see that the conventional MPC cost (left hand side of the inequality) has three upper-bound components, namely: $x_0^T P_0 x_0$ due to the initial state, $\gamma^2 w^T w$ due to the disturbance and $N\epsilon^2$ due to $m_k$. In particular, $\epsilon^2$ is used to regulate the influence of the open-loop component $m_k$ in the overall control input ($u_k$). So, for example, choosing a small $\epsilon^2$ will nullify the effect of $m_k$. This relationship will be further clarified in Section 3.4.2.*

**Remark 3.7.** *The design specification (S1), i.e. energy of the system driven only by the disturbance should be less or equal to zero, is a typical requirement in $\mathcal{H}_\infty$ design. On the* *other hand, design specification (S2) and the minimization of the cost upper bound $x_0^T P_0 x_0$ represent the $\mathcal{H}_2$ component of the problem and emphasizes performance. Therefore, cost function (3.6) shapes the considered RMPC problem into a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ framework.*

## 3.3 RCI Set Formulation

As discussed in Section 1.2.3, RCI sets play a fundamental role in the RMPC control of uncertain systems [79]. An RCI set for system (3.1) can be defined as follows [16]:

**Definition 3.1.** *$\mathcal{Z} \subset \mathbb{R}^n$ is a Robust Control Invariant (RCI) set of system (3.1) if there exists a control law $u = Kx$, such that $(A + \mathcal{D}_\alpha A_\delta)\mathcal{Z} \oplus (B_u + \mathcal{D}_\beta B_\delta)K\mathcal{Z} \oplus B_w \mathcal{W} \subseteq \mathcal{Z}$.*

We will now derive an LMI optimization problem to compute the largest/smallest perimeter hyper-rectangle invariant set (3.3) - which is a subset of the maximal/minimal RCI set [79, 95] - along with the corresponding inner controller $K$, subject to the following state and input constraints

$$z \in \mathcal{X}_f := \{z \in \mathbb{R}^{n_y} : Ez \leq f\}, \quad u = Kx \in \mathcal{U}_f := \left\{u \in \mathbb{R}^{n_u} : \underline{u}_f \leq u \leq \overline{u}_f\right\} \tag{3.9}$$

where $E \in \mathbb{R}^{n_z \times n_y}$, $\mathcal{X}_f \subset \mathcal{X}_k,\ \forall k$, and $\mathcal{U}_f \subset \mathcal{U}_k,\ \forall k$.

**Theorem 3.1.** *There exists a constraint admissible RCI set $\mathcal{Z}$ and controller $K$ if there exist $\hat{K} \in \mathbb{R}^{n_u \times n_y}$, diagonal matrix $Z_d \succ 0$ and vectors $\overline{\rho}_x^j, \underline{\rho}_x^j, \mu_x^i, \mu_p^i \in \mathbb{R}^{n_y}$, $\mu_q^i \in \mathbb{R}^{n_y}$ and $\mu_w^i \in \mathbb{R}^{n_w}$, $i \in \mathcal{N}_y := \{1, \cdots, n_y\}$, $j \in \mathcal{N}_u := \{1, \cdots, n_u\}$, such that the following linear inequality constraints are satisfied:*

$$\begin{aligned}
&\overline{\rho}_x^j \geq 0,\ \overline{\rho}_x^j + \hat{K}^T e_{uj} \geq 0, \\
&e_{uj}^T \overline{u}_f - 2e^T \overline{\rho}_x^j - e^T \hat{K}^T e_{uj} \geq 0, && \text{(3.10a)}\\
&\underline{\rho}_x^j \geq 0,\ \underline{\rho}_x^j - \hat{K}^T e_{uj} \geq 0, \\
&- e_{uj}^T \underline{u}_f - 2e^T \underline{\rho}_x^j + e^T \hat{K}^T e_{uj} \geq 0, && \text{(3.10b)}\\
&\mu_w^i \geq 0,\ \mu_w^i + B_w^T C_f^T e_i \geq 0, \\
&\mu_x^i \geq 0,\ \mu_x^i + (AC_L^T Z_d + B_u \hat{K})^T C_f^T e_i \geq 0, \\
&\mu_p^i \geq 0,\ \mu_p^i + (A_\delta C_L^T Z_d)^T C_f^T e_i \geq 0, \\
&\mu_q^i \geq 0,\ \mu_q^i + (B_\delta \hat{K})^T C_f^T e_i \geq 0, \\
&e_i^T Z_d e - v^T (2\mu_w^i + B_w^T C_f^T e_i) - e^T (2\mu_x^i + (AC_L^T Z_d + B_u \hat{K})^T C_f^T e_i) \cdots \\
&\qquad - \alpha e^T (2\mu_p^i + (A_\delta C_L^T Z_d)^T C_f^T e_i) - \beta e^T (2\mu_q^i + (B_\delta \hat{K})^T C_f^T e_i) \geq 0, && \text{(3.10c)}\\
&EZ_d e - f \leq 0 && \text{(3.10d)}
\end{aligned}$$

*where* $C_L := (C_f C_f^T)^{-1} C_f$, $e_i$ *denotes the* $i$*th column of the* $n_y \times n_y$ *identity matrix,* $e_{uj}$ *denotes the* $j$*th column of the* $n_u \times n_u$ *identity matrix and* $e$ *is the* $n_y$*-dimensional vector of ones. If the inequalities (3.10) are satisfied, then RCI set polytope* $z = Z_d e$ *and controller* $K$ *is a solution to* $\hat{K} = K C_L^T Z_d$.

**Proof.** The input constraints in (3.9) can be written as:

$$e_{uj}^T K x \;\leq\; e_{uj}^T \overline{u}_f \;, \tag{3.11}$$
$$e_{uj}^T K x \;\geq\; e_{uj}^T \underline{u}_f \;, \tag{3.12}$$

for all $x \in \mathcal{Z}$, where $j \in \mathcal{N}_u$.
By applying Lemma 2.3, it can be shown that (3.11) is satisfied if and only if there exist $\rho_x^j,\ \overline{\rho}_x^j \in \mathbb{R}^{n_y}$ such that

$$\rho_x^j \geq 0 \;,\; \overline{\rho}_x^j \geq 0, \tag{3.13a}$$
$$C_f^T \rho_x^j = C_f^T \overline{\rho}_x^j + K^T e_{uj} \;, \tag{3.13b}$$
$$e_{uj}^T \overline{u}_f - z^T \overline{\rho}_x^j - z^T \rho_x^j \geq 0. \tag{3.13c}$$

Pre-multiplying (3.13b) by $C_L := (C_f C_f^T)^{-1} C_f$ and subsequently eliminating $\rho_x^j$ from (3.13), yields the following equivalent conditions

$$\begin{aligned} &\overline{\rho}_x^j \geq 0 \;,\; \overline{\rho}_x^j + C_L K^T e_{uj} \geq 0, \\ &e_{uj}^T \overline{u}_f - 2 z^T \overline{\rho}_x^j - z^T C_L K^T e_{uj} \geq 0. \end{aligned} \tag{3.14}$$

Note that (3.14) is nonlinear in $z$ and $K$. By defining $Z_d := \mathrm{diag}(z) \succ 0$ so that $z = Z_d e$, pre-multiplying the first and second inequality by $Z_d$ and introducing the re-definitions $\overline{\rho}_x^j := Z_d \overline{\rho}_x^j$ and $\hat{K} := K C_L^T Z_d$ results in the inequalities (3.10a), which are linear in all the variables.

Analogous to the above procedure, using Lemma 2.3 on (3.12) followed by the linearization results in (3.10b).

Now, since the sets $\mathcal{Z}$, $\mathcal{D}_\alpha$, $\mathcal{D}_\beta$ and $\mathcal{W}$ are symmetric, the invariance constraint in Definition 3.1 can be written as:

$$e_i^T C_f [(A + B_u K) x + A_\delta\, p + B_\delta K\, q + B_w w] \leq e_i^T z, \tag{3.15}$$

for all $x \in \mathcal{Z},\ w \in \mathcal{W},\ p \in \mathcal{P},\ q \in \mathcal{Q},\ i \in \mathcal{N}_y$, where $p := \delta_\alpha x$, $q := \delta_\beta x$ and $\mathcal{P}$ and $\mathcal{Q}$ are defined by

$$\mathcal{P} := \Big\{ p \in \mathbb{R}^n : -\alpha z \le C_f p \le \alpha z \Big\},$$
$$\mathcal{Q} := \Big\{ q \in \mathbb{R}^n : -\beta z \le C_f q \le \beta z \Big\}.$$

It follows from Lemma 2.3 that (3.15) is satisfied if and only if there exist $\overline{\mu}_w^i, \mu_w^i \in \mathbb{R}^{n_w}$ and $\overline{\mu}_x^i, \mu_x^i, \overline{\mu}_p^i, \mu_p^i, \overline{\mu}_q^i, \mu_q^i \in \mathbb{R}^{n_y}$ such that

$$\mu_x^i \ge 0,\ \overline{\mu}_x^i \ge 0,\ \mu_p^i \ge 0,\ \overline{\mu}_p^i \ge 0,\ \mu_q^i \ge 0,\ \overline{\mu}_q^i \ge 0 \tag{3.16a}$$
$$C_f^T \overline{\mu}_x^i = C_f^T \mu_x^i + (A + B_u K)^T C_f^T e_i \ , \tag{3.16b}$$
$$C_f^T \overline{\mu}_p^i = C_f^T \mu_p^i + (A_\delta)^T C_f^T e_i \ , \tag{3.16c}$$
$$C_f^T \overline{\mu}_q^i = C_f^T \mu_q^i + (B_\delta K)^T C_f^T e_i \tag{3.16d}$$
$$\mu_w^i \ge 0 \ ,\ \overline{\mu}_w^i \ge 0 \ ,\ \overline{\mu}_w^i = \mu_w^i + B_w^T C_f^T e_i, \tag{3.16e}$$
$$e_i^T z - v^T(\mu_w^i + \overline{\mu}_w^i) - z^T(\mu_x^i + \overline{\mu}_x^i) - \alpha z^T(\mu_p^i + \overline{\mu}_p^i) - \beta z^T(\mu_q^i + \overline{\mu}_q^i) \ge 0. \tag{3.16f}$$

Pre-multiplying (3.16b)-(3.16d) by $C_L$ and subsequently eliminating $\overline{\mu}_x^i$, $\overline{\mu}_p^i$, $\overline{\mu}_q^i$, $\overline{\mu}_w^i$ from the above inequalities yields:

$$\begin{aligned}
&\mu_x^i \ge 0 \ ,\ \mu_x^i + C_L(A + B_u K)^T C_f^T e_i \ge 0 \ ,\\
&\mu_p^i \ge 0,\ \mu_p^i + C_L(A_\delta)^T C_f^T e_i \ge 0 \ ,\\
&\mu_q^i \ge 0,\ \mu_q^i + C_L(B_\delta K)^T C_f^T e_i \ge 0 \ ,\\
&\mu_w^i \ge 0 \ ,\ \mu_w^i + B_w^T C_f^T e_i \ge 0 \ ,\\
&e_i^T z - v^T(2\mu_w^i + B_w^T C_f^T e_i) - z^T(2\mu_x^i + C_L(A + B_u K)^T C_f^T e_i) \cdots\\
&\qquad - \alpha z^T(2\mu_p^i + C_L(A_\delta)^T C_f^T e_i) - \beta z^T(2\mu_q^i + C_L(B_\delta K)^T C_f^T e_i) \ge 0.
\end{aligned} \tag{3.17}$$

Pre-multiplying the first six inequalities in (3.17) by $Z_d$ and using the re-definitions $\mu_x^i := Z_d \mu_x^i$, $\mu_p^i := Z_d \mu_p^i$, $\mu_q^i := Z_d \mu_q^i$ and $\hat{K} := K C_L^T Z_d$ yields the nine inequalities in (3.10c). Finally, using $z = Z_d e$, state constraint in (3.9) can be rewritten as (3.10d). □

Let us define the following linear inequalities:

$$\zeta - \sum_{i=1}^{n_y} e_i^T Z_d e_i \le 0 \tag{3.18a}$$
$$\zeta - \sum_{i=1}^{n_y} e_i^T Z_d e_i \ge 0. \tag{3.18b}$$

Then, using Theorem 3.1, it can be verified that an inner approximation to the largest hyper-rectangle RCI set $\mathcal{Z}$ (perimeter-wise) and corresponding $K$ can be obtained by solving the following LMI optimization problem:

$$\begin{aligned}\bar{\zeta}_o = \max\{\, \zeta : (3.10a, 3.10b, 3.10c, 3.10d, 3.18a) \text{ are satisfied for some} \\ \overline{\rho}_x^j,\ \underline{\rho}_x^j,\ \mu_x^i,\ \mu_p^i,\ \mu_q^i \in \mathbb{R}^{n_y}, \mu_w^i \in \mathbb{R}^{n_w} \text{ and } \ i \in \mathcal{N}_y,\ j \in \mathcal{N}_u\}.\end{aligned} \tag{3.19}$$

Similarly, an outer approximation to the smallest hyper-rectangle RCI set $\mathcal{Z}$ (perimeter-wise) and corresponding $K$ can be obtained by solving the following LMI problem:

$$\begin{aligned}\bar{\zeta}_o = \min\{\, \zeta : (3.10a, 3.10b, 3.10c, 3.10d, 3.18b) \text{ are satisfied for some} \\ \overline{\rho}_x^j,\ \underline{\rho}_x^j,\ \mu_x^i,\ \mu_p^i,\ \mu_q^i \in \mathbb{R}^{n_y}, \mu_w^i \in \mathbb{R}^{n_w} \text{ and } \ i \in \mathcal{N}_y,\ j \in \mathcal{N}_u\}.\end{aligned} \tag{3.20}$$

**Remark 3.8.** *For systems subject to only disturbances without model-uncertainty (i.e. $\delta_\alpha = \delta_\beta = 0$), the conditions in Theorem 3.1 become both necessary and sufficient for the existence of set $\mathcal{Z}$ and $K$. This is because for such systems, (3.15) is equivalent to the invariance constraint in Definition 3.1. Therefore, $\zeta_o$, the optimal $\mathcal{Z}$ and $K$ can be computed exactly.*

**Remark 3.9.** *Solving LMI problems (3.19) and (3.20) yields the optimal $Z_d$ ($:= \mathrm{diag}(z)$) and $\hat{K}$. Note that all possible solutions ($K$) to equation $\hat{K} = KC_L^T Z_d$ ensure control invariance. For instance, one possible choice of the control law is: $K = \hat{K} Z_d^{-1} C_f$.*

**Remark 3.10.** *For a given $K$, the RCI set $\mathcal{Z}$ can be computed through a Linear Program. Let $y = [z^T\, \mu_w^{1T} \cdots \mu_w^{n_yT}\, \mu_x^{1T} \cdots \mu_x^{n_yT}\ \mu_p^{1T} \cdots \mu_p^{n_yT}\ \mu_q^{1T} \cdots \mu_q^{n_yT}\, \overline{\rho}_x^{1T} \cdots \overline{\rho}_x^{n_uT}\, \underline{\rho}_x^{1T} \cdots \underline{\rho}_x^{n_uT}]^T$ and define a column vector $c$ whose first $n_y$ elements are 1 while the rest are 0 i.e. $c := [1, \ldots, 1, 0, \ldots, 0]^T$. Now, for the case with a fixed $K$, all the conditions in Theorem 3.1 can be linearized (e.g. by re-defining $\overline{\rho}_x^j := Z_d \overline{\rho}_x^j$ in (3.14) and similarly in other conditions). Hence, the optimization in (3.19) and (3.20) can be transformed into a simple Linear Program with cost function $c^T y$ subject to constraints of the form $Ty \le g$.*

## 3.4 RMPC Controller

In this section, we derive the (outer) RMPC controller which is responsible for steering the system state towards the terminal invariant set $\mathcal{Z}$.

Recall that the proposed outer controller structure ($u_k = F_k x_k + m_k$) results in nonlinearities and non-convexity if the problem is formulated in the standard way [46].

Therefore, in order to avoid such issues, we formulate the RMPC problem in a more sequential manner. In some sense, the proposed approach is reminiscent of our RCI set formulation since the state constraints (3.2 – 3.3) and input constraints (3.4) can respectively be written as (with a slight abuse of notation):

$$[A + \mathcal{D}_\alpha A_\delta \oplus (B_u + \mathcal{D}_\beta B_\delta)F_k]\mathcal{X}_k \oplus (B_u + \mathcal{D}_\beta B_\delta)m_k + B_w\mathcal{W} \subseteq \mathcal{X}_{k+1} \tag{3.21}$$

$$F_k\mathcal{X}_k + m_k \subseteq \mathcal{U}_k \tag{3.22}$$

for all $k \in T_N$, where $\mathcal{X}_0$ is simply the initial state $x_0$ and $\mathcal{X}_N$ is the RCI set $\mathcal{Z}$.

### 3.4.1 LMI Conditions for the Constraints

Our approach is to derive necessary and sufficient LMI conditions for the robust satisfaction of state and input constraints, (3.21) and (3.22) respectively. Subsequently, we will incorporate the cost function in the algorithm.

**Theorem 3.2.** *Let all definitions and variables be as defined above. Then, there exists an admissible feedback-gain matrix ($F$) and a perturbation vector ($m$), satisfying (3.21 - 3.22), if and only if there exist positive-definite matrices* $\overline{D}_x^i, \underline{D}_x^i \in \mathbb{D}^{n_y \times n_y}$, $\overline{D}_w^i, \underline{D}_w^i \in \mathbb{D}^{n_w \times n_w}$, $\overline{D}_\alpha^i, \underline{D}_\alpha^i, \overline{D}_\beta^i, \underline{D}_\beta^i \in \mathbb{R}$, $\overline{\mu}_w^i, \underline{\mu}_w^i \in \mathbb{R}^{n_w}$, $\overline{\mu}_\alpha^i, \underline{\mu}_\alpha^i, \overline{\mu}_\beta^i, \underline{\mu}_\beta^i \in \mathbb{R}$, $i \in \mathcal{N}_y := \{1, \cdots, n_y\}$, $\overline{\rho}_{kx}^j, \underline{\rho}_{kx}^j \in \mathbb{R}^{n_y}$, $j \in \mathcal{N}_u := \{1, \cdots, n_u\}$, *as solutions to the following LMIs:*

$$\begin{bmatrix} C^T\overline{D}_x^iC & 0 & -\frac{1}{2}A_\delta^TC^Te_i & -\frac{1}{2}F_k^TB_\delta^TC^Te_i & -\frac{1}{2}C^T\overline{D}_x^i(\overline{x}_k + \underline{x}_k) - \frac{1}{2}A_k^TC^Te_i \\ \star & \overline{D}_w^i & 0 & 0 & -\frac{1}{2}B_w^TC^Te_i \\ \star & \star & \overline{D}_\alpha^i & 0 & 0 \\ \star & \star & \star & \overline{D}_\beta^i & -\frac{1}{2}m_k^TB_\delta^TC^Te_i \\ \star & \star & \star & \star & L(\overline{x}_{k+1}, \overline{D}_x^i, \overline{D}_\alpha^i, \overline{D}_\beta^i, \overline{D}_w^i, e_i) \end{bmatrix} \succ 0 \tag{3.23}$$

$$\begin{bmatrix} C^T\underline{D}_x^iC & 0 & \frac{1}{2}A_\delta^TC^Te_i & \frac{1}{2}F_k^TB_\delta^TC^Te_i & -\frac{1}{2}C^T\overline{D}_x^i(\overline{x}_k + \underline{x}_k) + \frac{1}{2}A_k^TC^Te_i \\ \star & \underline{D}_w^i & 0 & 0 & \frac{1}{2}B_w^TC^Te_i \\ \star & \star & \underline{D}_\alpha^i & 0 & 0 \\ \star & \star & \star & \underline{D}_\beta^i & \frac{1}{2}m_k^TB_\delta^TC^Te_i \\ \star & \star & \star & \star & L(\underline{x}_{k+1}, \underline{D}_x^i, \underline{D}_\alpha^i, \underline{D}_\beta^i, \underline{D}_w^i, -e_i) \end{bmatrix} \succ 0 \tag{3.24}$$

*where* $L(\hat{x}, \hat{D}_x, \hat{D}_\alpha, \hat{D}_\beta, \hat{D}_w, e_i) := e_i^T(\hat{x} - CB_u m_k) + \overline{x}_k^T \hat{D}_x \underline{x}_k - \alpha \hat{D}_\alpha \alpha - \beta \hat{D}_\beta \beta - v^T \hat{D}_w v.$

$$\begin{aligned} &\overline{\rho}_{kx}^j \geq 0,\ \overline{\rho}_{kx}^j + C_R F_k^T e_{uj} \geq 0, \\ &e_{uj}^T(\overline{u}_k - m_k) - (\overline{x}_k - \underline{x}_k)^T \overline{\rho}_{kx}^j - \overline{x}_k^T C_R F_k^T e_{uj} \geq 0, \end{aligned} \tag{3.25}$$

$$\begin{aligned} &\underline{\rho}_{kx}^j \geq 0,\ \underline{\rho}_{kx}^j - C_R F_k^T e_{uj} \geq 0, \\ &- e_{uj}^T(\underline{u}_k - m_k) - (\overline{x}_k - \underline{x}_k)^T \underline{\rho}_{kx}^j + \overline{x}_k^T C_R F_k^T e_{uj} \geq 0, \end{aligned} \tag{3.26}$$

$\forall i \in \mathcal{N}_y,\ \forall j \in \mathcal{N}_u,\ \forall k \in T_I,$

$$\begin{aligned} &\overline{\mu}_\alpha^i \geq 0\ ,\ \overline{\mu}_\alpha^i + x_0^T A_\delta^T C^T e_i\ \geq 0, \\ &\overline{\mu}_\beta^i \geq 0\ ,\ \overline{\mu}_\beta^i + (F_0 x_0 + m_0)^T B_\delta^T C^T e_i\ \geq 0, \\ &\overline{\mu}_w^i \geq 0\ ,\ \overline{\mu}_w^i + B_w^T C^T e_i \geq 0, \\ &e_i^T(\overline{x}_1 - CA_0 x_0 - CB_u m_0) - v^T(2\overline{\mu}_w^i + B_w^T C^T e_i) \cdots \\ &- \alpha(2\overline{\mu}_\alpha^i + x_0^T A_\delta^T C^T e_i) - \beta(2\overline{\mu}_\beta^i + (F_0 x_0 + m_0)^T B_\delta^T C^T e_i) \geq 0 \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\underline{\mu}_\alpha^i \geq 0\ ,\ \underline{\mu}_\alpha^i - x_0^T A_\delta^T C^T e_i\ \geq 0, \\ &\underline{\mu}_\beta^i \geq 0\ ,\ \underline{\mu}_\beta^i - (F_0 x_0 + m_0)^T B_\delta^T C^T e_i\ \geq 0, \\ &\underline{\mu}_w^i \geq 0\ ,\ \underline{\mu}_w^i - B_w^T C^T e_i \geq 0, \\ &- e_i^T(\underline{x}_1 - CA_0 x_0 - CB_u m_0) - v^T(2\underline{\mu}_w^i - B_w^T C^T e_i) \cdots \\ &- \alpha(2\underline{\mu}_\alpha^i - x_0^T A_\delta^T C^T e_i) - \beta(2\underline{\mu}_\beta^i - (F_0 x_0 + m_0)^T B_\delta^T C^T e_i) \geq 0 \end{aligned} \tag{3.28}$$

$$\begin{aligned} e_{uj}^T(F_0 x_0 + m_0 - \overline{u}_0) &\ \leq\ 0, \\ e_{uj}^T(F_0 x_0 + m_0 - \underline{u}_0) &\ \geq\ 0, \end{aligned} \tag{3.29}$$

$\forall j \in \mathcal{N}_u,\ \forall i \in \mathcal{N}_y,\ k = 0,$
*where* $C_R := (CC^T)^{-1}C,\ A_k := A + B_u F_k,\ A_0 := A + B_u F_0$ *and where* $\overline{x}_N = -\underline{x}_N = z$ *.*

**Proof.** The state constraints in (3.21) can be written as:

$$e_i^T(Cx_{k+1} - \overline{x}_{k+1}) \leq 0 \tag{3.30}$$

$$-e_i^T(Cx_{k+1} - \underline{x}_{k+1}) \leq 0 \tag{3.31}$$

$\forall k \in T_I, \forall x_k \in \mathcal{X}_k,\ \forall w_k \in \mathcal{W},\ \forall \delta_\alpha \in \mathcal{D}_\alpha,\ \forall \delta_\beta \in \mathcal{D}_\beta,$ where:

$$x_{k+1} = \underbrace{(A + \delta_\alpha A_\delta)x_k + (B_u + \delta_\beta B_\delta)(F_k x_k + m_k) + B_w w_k}_{f_k(x_k, F_k, m_k, w_k, \delta_\alpha, \delta_\beta)}. \tag{3.32}$$

It can be verified that:

$$\begin{aligned} e_i^T(Cx_{k+1} - \overline{x}_{k+1}) = &-(v - w_k)^T \overline{D}_w^i (w_k + v) - (\beta - \delta_\beta)^T \overline{D}_\beta^i (\delta_\beta + \beta) - (\alpha - \delta_\alpha)^T \overline{D}_\alpha^i (\delta_\alpha + \alpha) \\ &- (\overline{x}_k - Cx_k)^T \overline{D}_x^i (Cx_k - \underline{x}_k) - y_k^T \overline{L}_i(\overline{D}_x^i, \overline{D}_w^i, \overline{D}_\alpha^i, \overline{D}_\beta^i, F_k, m_k) y_k \end{aligned}$$

for all $i \in \mathcal{N}_y$, where $\overline{D}_x^i, \overline{D}_w^i, \overline{D}_\alpha^i, \overline{D}_\beta^i$ are positive-definite diagonal matrices, vector $y_k := [x_k^T\ w_k^T\ \delta_\alpha^T\ \delta_\beta^T\ 1]^T$ and $\overline{L}_i(\overline{D}_x^i, \overline{D}_w^i, \overline{D}_\Delta^i, F_k, m_k)$ is the matrix given in LMI (3.23). Then, through an application of the S-Procedure (Farkas' Theorem) [81], it follows that (3.23) is necessary and sufficient for (3.30), (see also Section 2.3). Analogous to the above method, it can be verified that LMI (3.24) is a necessary and sufficient condition for (3.31).

Now for $k = 0$, with $x_0$ known, the state constraints become:

$$\begin{aligned} &e_i^T C[\delta_\alpha A_\delta x_0 + \delta_\beta B_\delta (F_0 x_0 + m_0) + B_w w_0] \leq e_i^T(\overline{x}_1 - CA_0 x_0 - CB_u m_0) \\ &e_i^T C[\delta_\alpha A_\delta x_0 + \delta_\beta B_\delta (F_0 x_0 + m_0) + B_w w_0] \geq e_i^T(\underline{x}_1 - CA_0 x_0 - CB_u m_0) \end{aligned} \tag{3.33}$$

Using Lemma 2.3 on the first inequality in (3.33) yields:

$$\begin{aligned} &\mu_\alpha^i \geq 0\ ,\ \overline{\mu}_\alpha^i \geq 0\ ,\ \mu_\alpha^i = \overline{\mu}_\alpha^i + x_0^T A_\delta^T C^T e_i, \\ &\mu_\beta^i \geq 0\ ,\ \overline{\mu}_\beta^i \geq 0\ ,\ \mu_\beta^i = \overline{\mu}_\beta^i + (F_0 x_0 + m_0)^T B_\delta^T C^T e_i, \\ &\mu_w^i \geq 0\ ,\ \overline{\mu}_w^i \geq 0\ ,\ \mu_w^i = \overline{\mu}_w^i + B_w^T C^T e_i, \\ &e_i^T(\overline{x}_1 - CA_0 x_0 - CB_u m_0) - v^T(\mu_w^i + \overline{\mu}_w^i) - \alpha(\mu_\alpha^i + \overline{\mu}_\alpha^i) - \beta(\mu_\beta^i + \overline{\mu}_\beta^i) \geq 0 \end{aligned}$$

Eliminating $\mu_w^i$, $\mu_\alpha^i$ and $\mu_\beta^i$ from the above yields the inequalities (3.27). A similar treatment on the second inequality in (3.33) yields the conditions in (3.28). Therefore, inequalities (3.27), (3.28) both become necessary and sufficient for the satisfaction of the state constraints at $k = 0$.

Now, for $k \in T_I$, the input constraints in (3.22) can be written as:

$$e_{uj}^T(F_k x_k + m_k)\ \leq\ e_{uj}^T \overline{u}_k\ ,\ \forall j \in \mathcal{N}_u,\ \forall x_k \in \mathcal{X}_k \tag{3.35}$$

$$e_{uj}^T(F_k x_k + m_k)\ \geq\ e_{uj}^T \underline{u}_k\ ,\ \forall j \in \mathcal{N}_u,\ \forall x_k \in \mathcal{X}_k. \tag{3.36}$$

By applying Lemma 2.3, it can be shown that (3.35) is satisfied if and only if there exist $\rho^j_{kx}$, $\overline{\rho}^j_{kx} \in \mathbb{R}^{n_y}$ such that

$$\rho^j_{kx} \geq 0,\ \overline{\rho}^j_{kx} \geq 0, \tag{3.37a}$$

$$C^T \rho^j_{kx} = C^T \overline{\rho}^j_{kx} + F_k^T e_{uj}, \tag{3.37b}$$

$$e_{uj}^T(\overline{u}_k - m_k) + \underline{x}_k^T \overline{\rho}^j_{kx} - \overline{x}_k^T \rho^j_{kx} \geq 0. \tag{3.37c}$$

Pre-multiplying (3.37b) by $C_R := (CC^T)^{-1}C$ and subsequently eliminating $\rho^j_{kx}$ from (3.37) yields the inequalities (3.25). Analogous to the above procedure, using Lemma 2.3, it can be shown that (3.36) is satisfied if and only if there exist a solution $\underline{\rho}^j_{kx} \in \mathbb{R}^{n_y}$ to the inequalities (3.26). Finally, for $k = 0$, with $x_0$ known, it can easily be verified that inequalities (3.35) and (3.36) reduce to (3.29). □

**Remark 3.11.** *It is worth mentioning here that the LMIs in Theorem 3.2 do not introduce any gap (conservatism) in the formulation. Therefore, the state and input constraints are incorporated in a non-conservative manner (see also Section 3.5). Furthermore, these LMIs have a low dimension which makes the online optimization tractable.*

**Remark 3.12.** *Note that, in Theorem 3.2, both $\overline{x}_{k+1}$ and $\underline{x}_{k+1}$ appear linearly and thus can be treated as variables. This suggests another method of tightening the constraints - in an a-priori manner - to yield a feasible control policy. For example, the LMIs in Theorem 3.2 can be solved (sequentially) to minimize the objective $\|\overline{x}_{k+1} - \underline{x}_{k+1}\|^2$, resulting in an optimal constraint-tightening procedure.*

### 3.4.2 Incorporation of the Cost Function

To compute an upper-bound on cost (3.6), we first derive a sufficient condition for (3.7).

**Theorem 3.3.** *Given the matrix $P_{k+1} = P_{k+1}^T \succ 0$ and a (constant) user-specified bounding on $m_k$, call it $\epsilon^2$, there exists a $P_k = P_k^T \succ 0$ satisfying (3.7) if the LMI:*

$$\begin{bmatrix} \lambda^2 I & 0 & A_\delta & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 I & B_\delta F_k & 0 & B_\delta m_k & 0 & 0 & 0 \\ \star & \star & P_k - Q & 0 & 0 & F_k^T R & A_k^T P_{k+1} & 0 \\ \star & \star & \star & \gamma^2 I & 0 & 0 & B_w^T P_{k+1} & 0 \\ \star & \star & \star & \star & \epsilon^2 & m_k^T R & m_k^T B_u^T P_{k+1} & 0 \\ \star & \star & \star & \star & \star & R & 0 & 0 \\ \star & \star & \star & \star & \star & \star & P_{k+1} & \overline{\lambda} P_{k+1} \\ \star & \star & \star & \star & \star & \star & \star & \overline{\lambda} I \end{bmatrix} \succ 0 \tag{3.38}$$

*has a feasible solution, where* $A_k := (A + B_u F_k)$*, and* $\overline{\lambda} := \lambda^2(\alpha^2 + \beta^2)$*.*

*Furthermore, with* $P_k$ *obtained as above, the quadratic function* $\overline{J}(x_k, F_k, m_k) := x_k^T P_k x_k$ *represents an upper bound on the cost-to-go at each prediction step* $k$*.*

**Proof.** Recall that the cost function we propose is:

$$J(x_0, F, m, w, \delta_\alpha, \delta_\beta) := x_N^T P_N x_N + \sum_{k=0}^{N-1} g(x_k, F_k, m_k, w_k)$$

where $g(x_k, F_k, m_k, w_k) := x_k^T(Q + F_k^T R F_k)x_k + m_k^T R m_k + m_k^T R F_k x_k + x_k^T F_k^T R m_k - \gamma^2 w_k^T w_k - \epsilon^2$. Here $\epsilon^2 > 0$ is a designer-specified upper bound on control-perturbation $m_k$. In particular, $m_k^T R m_k \leq \epsilon^2$, so that a non-zero $\epsilon^2$ corresponds to a control-perturbation component in $u_k$.

Then, by using system dynamics (3.1), inequality (3.7) can be written as:

$$s_k^T \hat{L}_k(F_k, m_k, P_k) s_k \geq 0, \qquad \forall w_k,\ \forall \delta_\alpha,\ \forall \delta_\beta$$

for all $k \in T_N$, where $s_k = [x_k^T \ \ w_k^T \ \ 1]^T \in \mathbb{R}^{n+n_w+1}$ and $\hat{L}_k(F_k, m_k, P_k) :=$

$$\begin{bmatrix} P_k - Q - F_k^T R F_k - \overline{A}_k^T P_{k+1} \overline{A}_k & -\overline{A}_k^T P_{k+1} B_w & \hat{L}_{1,3} \\ \star & \gamma^2 I - B_w^T P_{k+1} B_w & -B_w^T P_{k+1} \overline{B}_k \\ \star & \star & \epsilon^2 - \overline{B}_k^T P_{k+1} \overline{B}_k - m_k^T R m_k \end{bmatrix}$$

where $\overline{A}_k := (A_k + \delta_\alpha A_\delta + \delta_\beta B_\delta F_k)$, $\overline{B}_k := (B_u + \delta_\beta B_\delta) m_k$, and $\hat{L}_{1,3} = -\overline{A}_k^T P_{k+1} \overline{B}_k - F_k^T R m_k$.

It follows that a sufficient condition for (3.7) is:

$$\hat{L}_k(F_k, m_k, P_k) \succeq 0 \tag{3.39}$$

Application of the Schur complement on $\hat{L}_k(F_k, m_k, P_k)$, followed by a pre- and post-multiplication with the matrix $\mathrm{diag}(I, I, I, R, P_{k+1})$ and a subsequent rearrangement shows that (3.39) is equivalent to:

$$L + E\delta V^T + V\delta^T E^T \succeq 0, \qquad \forall \delta,$$

where $\delta := [\delta_\alpha \quad \delta_\beta]^T$, $V := [0 \ \ 0 \ \ 0 \ \ 0 \ \ P_{k+1}]^T$ and

$$L := \begin{bmatrix} P_k - Q & 0 & 0 & F_k^T R & A_k^T P_{k+1} \\ \star & \gamma^2 I & 0 & 0 & B_w^T P_{k+1} \\ \star & \star & \epsilon^2 & m_k^T R & m_k^T B_u^T P_{k+1} \\ \star & \star & \star & R & 0 \\ \star & \star & \star & \star & P_{k+1} \end{bmatrix}, E := \begin{bmatrix} A_\delta^T & F_k^T B_\delta^T \\ 0 & 0 \\ 0 & m_k^T B_\delta^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Using the S-procedure [94] on above inequality yields:

$$L + E\delta V^T + V\delta^T E^T = \overbrace{V(\lambda^2(\alpha^2+\beta^2) - \lambda^2\delta^T\delta)V^T}^{\succeq 0} + p^T \overline{L}_k(F_k, m_k, P_k, \lambda)p \ \succeq 0$$

where $p := [V\delta^T \quad I]^T$, $0 < \lambda \in \mathbb{R}$ is a variable and:

$$\overline{L}_k(F_k, m_k, P_k, \lambda) := \begin{bmatrix} \lambda^2 I & E^T \\ \star & L - \lambda^2(\alpha^2+\beta^2)VV^T \end{bmatrix}.$$

It follows that $\overline{L}_k(F_k, m_k, P_k, \lambda) \succeq 0$ is sufficient for inequality (3.7). Applying the Schur complement argument on $\overline{L}_k(F_k, m_k, P_k, \lambda)$ yields LMI (3.38).

To prove the second part of Theorem 3.3, let $g_k(x_k, F_k, m_k, w_k)$ and $J_k(x_k, F_k, m_k)$ denote the stage cost and the cost-to-go (respectively) at each prediction step $k$. Furthermore, let $J_k^\star(x_k)$ denote the optimal cost-to-go. Then, for our algorithm, at a given (absolute) time, initially at prediction step $k = N-1$ with $P_N$ known, we have the following:

$$J_N^\star(x_N) = J_N(x_N) = g_N(x_N) = x_N^T P_N x_N.$$

Then, iterating backwards, at each $k$, with $P_{k+1}$ known, we have

$$J_k(x_k, F_k, m_k) := \max_{w_k, \delta_\alpha, \delta_\beta} \quad g(x_k, F_k, m_k, w_k) + J_{k+1}^\star(f_k(x_k, F_k, m_k, w_k, \delta_\alpha, \delta_\beta)).$$

where $f_k(x_k, F_k, m_k, w_k, \delta_\alpha, \delta_\beta)$ is defined in (3.32).

Finally, using (3.7) - which is ensured by (3.38) - shows that:

$$J_k(x_k, F_k, m_k) \leq x_k^T P_k x_k =: \overline{J}_k(x_k, F_k, m_k). \qquad \square$$

It follows from the proof of Theorem 3.3 that an upper bound on the optimal cost function is given by:

$$\overline{J}_0^\star(x_0) = \min_{\substack{P_k \succ 0,\ \forall k \\ G_k \succeq 0,\ \forall k}} x_0^T P_0 x_0 \tag{3.40}$$

where $P_N$ is given and $G_k \succeq 0$ represents all the (applicable) necessary and sufficient conditions derived above for the state/input constraints (3.23)-(3.29), as well as inequality (3.38). Note that optimization problem (3.40) requires the constraint (3.38) to be satisfied for all $k$ and this renders the problem nonlinear and non-convex due to the (3,7) and (5,7) entries of (3.38). In order to avoid such non-convexity, we minimize an upper bound on the cost-to-go $\overline{J}_k(x_k, F_k, m_k)$ in a sequential manner, as given in the following theorem.

**Theorem 3.4.** *An upper bound on the optimal cost-to-go $\overline{J}_k^\star(x_k)$, call it $\hat{J}_k^\star$, can be computed, for all $k \in T_I$, as follows:*

$$\hat{J}_k^\star \quad = \quad \min \quad \mu_{0k}$$

*subject to:*

$$P_k \succ 0, \quad G_k \succeq 0, \quad D_k \succeq 0,$$

$$\mathcal{L}_k(P_k, D_k) := \begin{bmatrix} C^T D_k C - P_k & -\frac{1}{2} C^T D_k(\overline{x}_k + \underline{x}_k) \\ \star & \mu_{0k} + \overline{x}_k^T D_k \underline{x}_k \end{bmatrix} \succeq 0. \tag{3.41}$$

*Furthermore, for $k = 0$, $\overline{J}_0^\star(x_0)$ can be computed by minimizing $x_0^T P_0 x_0$ subject to the inequality constraints:*

$$P_0 \succ 0, \quad G_0 \succeq 0. \tag{3.42}$$

**Proof.** We consider the following min-max problem as the main optimization in the proposed algorithm, for all $k \in T_I$:

$$\overline{J}_k^\star(x_k) \ \leq \ \min_{\substack{P_k \succ 0 \\ G_k \succeq 0}} \ \max_{x_k \in \mathcal{X}_k} \ x_k^T P_k x_k. \tag{3.43}$$

Using the S-Procedure, it can be verified that:

$$x_k^T P_k x_k = \overbrace{-(\overline{x}_k - C x_k)^T D_k (C x_k - \underline{x}_k)}^{\leq 0} - c^T \mathcal{L}_k(P_k, D_k) c + \mu_{0k}$$

for all $k \in T_I$, where the diagonal matrix $D_k \succeq 0$, $c := [x_k^T \ \ 1]^T$ and $\mathcal{L}_k(P_k, D_k)$ is as defined in (3.41). Therefore, we have:

$$D_k \succeq 0,\ \mathcal{L}_k(P_k, D_k) \succeq 0 \Rightarrow x_k^T P_k x_k \leq \mu_{0k}. \tag{3.44}$$

Using (3.44) in the maximization of (3.43) yields the required minimization problem for the computation of $\hat{J}_k^\star(x_k)$. Furthermore, for $k = 0$, with $x_0$ known, $\overline{J}_0^\star(x_0)$ can be computed by minimizing $x_0^T P_0 x_0$ subject to (3.42). □

### 3.4.3 Overall RMPC Scheme

The optimizations for the computation of the feedback gain matrix $F$ and perturbation vector $m$ can be summarized as follows:
At each absolute time step, call it $t$, iterating backwards starting from prediction step $k = N - 1$ all the way down to $k = 1$, for each $k$, we solve the following problem to compute $F_k$, $m_k$ and $P_k$:

$$\begin{aligned}\hat{J}_k^\star = \min\{\ & \mu_{0k} : (3.23,\ 3.24,\ 3.25,\ 3.26,\ 3.38,\ 3.41) \text{ are satisfied for some diagonal} \\ & \text{positive definite matrices } D_k, \overline{D}_x^i, \underline{D}_x^i, \overline{D}_\alpha^i, \underline{D}_\alpha^i, \overline{D}_\beta^i, \underline{D}_\beta^i, \overline{D}_w^i, \underline{D}_w^i, \text{ and} \\ & P_k = P_k^T \succ 0,\ \overline{\rho}_{kx}^j, \underline{\rho}_{kx}^j \in \mathbb{R}^{n_y}, i \in \mathbb{N}_y,\ j \in \mathbb{N}_u\}.\end{aligned} \tag{3.45}$$

Then, for $k = 0$, with $P_1$ computed in the previous step, the following problem is solved to compute $F_0$, $m_0$ and $P_0$:

$$\begin{aligned}\overline{J}_0^\star(x_0) = \min\{\ & x_0^T P_0 x_0 : (3.27,\ 3.28,\ 3.29,\ 3.38,\ 3.42) \text{ are satisfied for} \\ & \text{some } P_0 = P_0^T \succ 0,\ \overline{\mu}_w^i, \underline{\mu}_w^i \in \mathbb{R}^{n_w}, \overline{\mu}_\alpha^i, \underline{\mu}_\alpha^i, \overline{\mu}_\beta^i, \underline{\mu}_\beta^i \in \mathbb{R}, \\ & i \in \mathbb{N}_y,\ j \in \mathbb{N}_u\}.\end{aligned} \tag{3.46}$$

Therefore, the overall RMPC algorithm can be summarized as follows.

**Algorithm 3.1.** *Robust Feedback MPC.*
*Data:* $x_t$.
*Algorithm: If* $x_t \in Z$, *set* $u_t = Kx_t$. *Otherwise, compute* $F$ *and* $m$ *by solving (3.45) and (3.46) sequentially and set* $u_t = F_0 x_t + m_0$.

**Remark 3.13.** *Algorithm 3.1 is based, in part, on the principles of Dynamic Programming [13] since it involves computing* $F_k$, $m_k$ *by minimizing the worst-case upper bound on the cost-to-go at each* $k$*. Note also that (3.45) and (3.46) are (low-dimensional) LMI optimization problems and thus the control law can be computed in polynomial time [55].*

### 3.4.4 Conditions for Stability of the RMPC Scheme

The stability of MPC schemes has been widely investigated over the past few decades (see e.g. [66] for an excellent survey). In the literature, a number of techniques have been employed to establish stability. However, the most common one involves the use of terminal (invariant) sets and a suitable terminal cost. In particular, the conditions for stability of the RMPC scheme can be summarized as follows [66]:

**C1:** The stage cost $g(x,u,w) \geq \delta(\|x\|^2)$ for all feasible states $x$, for all $w$ and for some $\delta > 0$.

**C2:** The terminal set $\mathcal{Z}$ is robust positively invariant for the system under the control law $\kappa_z(x)$, i.e. $f(x,\kappa_z(x),w) \in \mathcal{Z}$, $\forall x \in \mathcal{Z}$, $\forall w \in \mathcal{W}$, $\forall \delta_\alpha \in \mathcal{D}_\alpha$, $\forall \delta_\beta \in \mathcal{D}_\beta$, where $f(x,\kappa_z(x),w) := (A+\delta_\alpha A_\delta)x + (B_u + \delta_\beta B_\delta)\kappa_z(x) + B_w w$

**C3:** The terminal control law $\kappa_z(x)$ is such that $\kappa_z(x) \in \mathcal{U}_f \subset \mathcal{U}_k$, $\forall k$, $\forall x \in \mathcal{Z}$. Furthermore $\mathcal{Z} \subset \mathcal{X}_k$, $\forall k$, and the sets $\mathcal{U}_f$ and $\mathcal{Z}$ contain the origin in their interior.

**C4:** The terminal cost function $g_N(x)$ is such that $g_N(0) = 0$, $g_N(x) \geq 0$ ,$\forall x \in \mathcal{Z}$ and satisfies:

$$g_N(f(x,\kappa_z(x),w)) - g_N(x) \leq -g(x,\kappa_z(x),w) \tag{3.47}$$

$\forall x \in \mathcal{Z}$, $\forall \delta_\alpha \in \mathcal{D}_\alpha$, $\forall \delta_\beta \in \mathcal{D}_\beta$ , and for all admissible $w$.

We now present a theorem to ensure conditions **C1-C4** for the proposed RMPC scheme. For simplicity, as well as the fact that above conditions assume state-feedback, we will consider $\epsilon^2$ (and therefore $m_k$) to be zero.

**Theorem 3.5.** *Assume that all admissible disturbances (i.e. $w \in \mathcal{W}$) also belong to the set $\mathbb{W}$ where (see Remark 3.14):*

$$\mathbb{W} := \left\{ w \in \mathbb{R}^{n_w} : \ \|w\|^2 \leq \sigma^2(x^T Q x + u^T R u) \right\} \tag{3.48}$$

*and where $\sigma < \frac{1}{\gamma}$. Furthermore, suppose the terminal weighting $P_N = P_N^T$ is chosen as the solution to the LMI:*

$$\begin{bmatrix} \lambda_\alpha^2 I & 0 & 0 & P_N & -P_N B_w \\ \star & \lambda_\beta^2 I & 0 & P_N & -P_N B_w \\ \star & \star & L_{3,3} & A_K^T P_N & -A_K^T P_N B_w \\ \star & \star & \star & P_N & 0 \\ \star & \star & \star & \star & \gamma^2 I - B_w^T P_N B_w \end{bmatrix} \succeq 0 \tag{3.49}$$

*where* $L_{3,3} := P_N - Q - K^T RK - \beta^2\lambda_\beta^2 K^T B_\delta^T B_\delta K - \alpha^2\lambda_\alpha^2 A_\delta^T A_\delta$*, with* $K$ *as the computed inner controller (i.e.* $\kappa_z(x) = Kx$*) and* $A_K := (A + B_u K)$*.*
*Then, the conditions* ***C1-C4*** *are satisfied.*

**Proof.** Using the definition of set $\mathbb{W}$, the proposed stage-cost can be written as:

$$\begin{aligned} g(x,u,w) &= x^T Qx + u^T Ru - \gamma^2 w^T w \\ &\geq (1-\gamma^2\sigma^2)(x^T Qx + u^T Ru) \\ &:= \mu(x^T Qx + u^T Ru) \\ &\geq \delta(x^T Qx) \end{aligned}$$

where $\mu > 0$ and $\delta > 0$ if $\sigma < \frac{1}{\gamma}$. It follows that the assumption $w \in \mathbb{W}$ is sufficient to guarantee condition **C1**.
Now with the inner controller and RCI set already computed (using Theorem 3.1), inserting the system dynamics (3.1) into inequality (3.47) yields:

$$h^T \hat{L}_s(P_N, \delta_\alpha, \delta_\beta) h \geq 0$$

where $h = [x^T \quad w^T]^T \in \mathbb{R}^{n+n_w}$ and

$$\hat{L}_s(P_N, \delta_\alpha, \delta_\beta) := \begin{bmatrix} P_N - Q - K^T RK - A_{\delta K}^T P_N A_{\delta K} & -A_{\delta K}^T P_N B_w \\ \star & \gamma^2 I - B_w^T P_N B_w \end{bmatrix}$$

where $A_{\delta K} := (A + B_u K) + (\delta_\alpha A_\delta + \delta_\beta B_\delta K)$. Therefore, $\hat{L}_s(P_N, \delta_\alpha, \delta_\beta) \succeq 0$ $, \forall \delta_\alpha, \delta_\beta$ is sufficient for (3.47). In order to obtain convexity in $P_N$, we will first deal with $\delta_\beta$ followed by $\delta_\alpha$. An application of the Schur complement argument on $\hat{L}_s(P_N, \delta_\alpha, \delta_\beta) \succeq 0$ and a subsequent rearrangement yields the following sufficient condition:

$$L + E\delta_\beta F^T + F\delta_\beta^T E^T \succeq 0, \quad \forall \delta_\beta \tag{3.50}$$

where $E := [0 \quad P_N \quad -P_N B_w]^T$, $F := [B_\delta K \quad 0 \quad 0]^T$, and

$$L := \begin{bmatrix} P_N - Q - K^T RK & (A_K^T + \delta_\alpha A_\delta^T) P_N & -(A_K^T + \delta_\alpha A_\delta^T) P_N B_w \\ \star & P_N & 0 \\ \star & \star & \gamma^2 I - B_w^T P_N B_w \end{bmatrix}.$$

Much the same way as in the proof of Theorem 3.3, applying the S-procedure on inequality

(3.50), and a subsequent rearrangement yields the following condition:

$$L + E\delta_\alpha F^T + F\delta_\alpha^T E^T \succeq 0, \quad \forall \delta_\alpha \tag{3.51}$$

where $E := [0 \ \ 0 \ \ P_N \ \ -P_N B_w]^T$, $F := [0 \ \ A_\delta \ \ 0 \ \ 0]^T$ and

$$L := \begin{bmatrix} \lambda_\beta^2 I & 0 & P_N & -P_N B_w \\ \star & L_{2,2} & A_K^T P_N & -A_K^T P_N B_w \\ \star & \star & P_N & 0 \\ \star & \star & \star & \gamma^2 I - B_w^T P_N B_w \end{bmatrix}$$

where $L_{2,2} := P_N - Q - K^T RK - \beta^2 \lambda_\beta^2 K^T B_\delta^T B_\delta K$. Finally, applying the S-procedure on (3.51) yields (3.49).

Thus, a $P_N$ satisfying LMI (3.49) guarantees that the corresponding terminal cost satisfies **C4**. Finally, we note that **C2** and **C3** hold due to the RCI set formulation of Section 3.3. □

It follows from [66] that satisfaction of conditions **C1**-**C4**, by the proposed scheme, is sufficient to ensure the Lyapunov stability of the closed-loop uncertain system.

**Remark 3.14.** *The condition (3.48), though restrictive, has been used in a number of publications, e.g. in the context of $\mathcal{H}_\infty$-MPC [66, sec. 4.7]. A similar condition ( $\|w_k\|^2 \leq \|x_k\|^2$) has been used in [55] as well as for the proofs given in [105]. An alternative, less restrictive, assumption which may instead be used to guarantee condition **C1** is also given in [100, App. A].*

**Remark 3.15.** *Recursive feasibility of the proposed scheme is ensured through the incorporation of the (constraint admissible) invariant terminal set $\mathcal{Z}$. In particular note that, under the conditions given in **C1-C4**, the optimal control sequence computed at time $t$ can be shifted and subsequently appended with the terminal control law $\kappa_z(x)$ to yield the sequence: $\{u(t+1|t), \cdots, u(t+N|t), \kappa_z(x)\}$ which remains feasible at next time step $t+1$. See [66] for further details.*

## 3.5 Numerical Examples

In this section, we present three examples to demonstrate the effectiveness of the proposed algorithm.

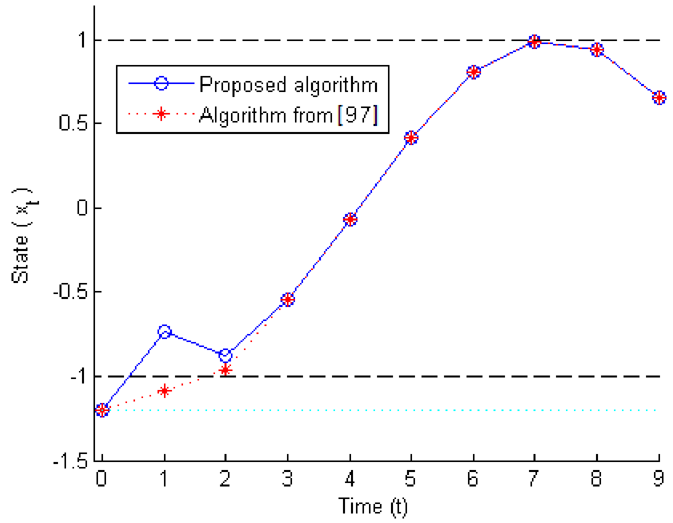


Figure 3.1: Simulation results for Example 1 with $w_t = -\cos(t/2)$.

### 3.5.1 Example 1

Firstly, we consider an example taken from [91]:

$$x_{k+1} = x_k + u_k + w_k. \tag{3.52}$$

The disturbance is constrained as: $-1 \leq w_k \leq 1$. The state constraints are defined by $\overline{x}_k = 2$ and $\underline{x}_k = -1.2$. No input constraints are imposed. The initial state $x_0 = \underline{x}$ and the horizon $N = 3$. For the cost, we have $Q = 1$, $R = 0.1$, $P_N = I$, $\epsilon^2 = 0.5$ and disturbance rejection parameter $\gamma^2 = 10$. The optimal RCI set and controller, computed using the algorithm in Section 3.3, are given by $\mathcal{Z} := \Big\{x \in \mathbb{R} : -1 \leq x \leq 1\Big\}$ and $K = -1$, respectively. Figure 3.1 shows the results using the proposed RMPC scheme as well as the algorithm from [97]. Note that both algorithms are able to steer the disturbed system state to the invariant set (shown by black dashed lines). Once in the set, the computed inner controller keep the system state within $\mathcal{Z}$ for all possible disturbances. Note, however, that the proposed scheme controls the state to the RCI set in one step as compared to two online iterations required by the algorithm in [97]. This improvement can be attributed to the introduction of the control-perturbation $m_k$ which provides an extra degree of freedom in the proposed scheme. This results in fewer number of online iterations and a less conservative algorithm.

As explained in Section 1.2, Example 1 is found to be infeasible with open-loop min-

| Scheme | Iterations required for convergence to $\mathcal{Z}$ | Average computational time per iteration (s) |
|---|---|---|
| Open-loop min-max MPC | Infeasible | - |
| Min-max feedback MPC [91] | 1 | 0.70 |
| Algorithm from [97] | 2 | 0.36 |
| Proposed Algorithm | 1 | 0.37 |

Table 3.1: Comparison of various RMPC schemes for Example 1

max MPC scheme due to its conservative nature. The min-max feedback MPC scheme in [91] does yield feasibility though at the expense of large online computational burden (due to its combinatorial nature of optimization). Table 3.1 compares the computational load and RCI set convergence of all the above schemes (running on an Intel® 2.4GHz PC with MATLAB® version 7.12). We can see that the proposed scheme approximately halves the computational time whilst still providing the fastest possible RCI set convergence, i.e. in one step.

### 3.5.2 Example 2

We now consider an uncertain version of the unstable process from [97]. In particular, we have:

$$x_{k+1} = (A + \delta_\alpha A_\delta)x_k + (B_u + \delta_\beta B_\delta)u_k + B_w w_k$$

$$\text{with} \quad A = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_w \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Furthermore, $A_\delta = A$ and $B_\delta = B_u$. For the uncertainty, we have polytopes $\alpha = 0.15$, $\beta = 0.10$ (which corresponds to 15% uncertainty in each entry of $A$ and 10% for $B_u$). The constraints on disturbance, input and state are given by: $-0.3 \leq w_k \leq 0.3$, $\overline{u}_k = -\underline{u}_k = 12 \; \forall k$, and $\overline{x}_k = -\underline{x}_k = [8 \;\; 8]^T$, respectively. We adopt a constraint tightening approach. Moreover, we set the initial state $x_0 = \overline{x}_k$ and the prediction horizon $N = 6$. For the cost, we have $\gamma^2 = 6$, $\epsilon^2 = 0$ and penalties $Q = qI$, $R = rI$, with the ratio $q/r = 0.2$. Computing the RCI set and the inner controller with the input constraint $\overline{u}_f = -\underline{u}_f = 1.5$ and state constraints: $z \leq [1.6 \;\; 0.9]^T$, yields the the following $\mathcal{Z}$ and $K$:

$$\mathcal{Z} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -1.596 \\ -0.881 \end{bmatrix} \leq x \leq \begin{bmatrix} 1.596 \\ 0.881 \end{bmatrix} \right\}, \quad K = -\begin{bmatrix} 0.499 & 0.798 \end{bmatrix}.$$

Moreover, computing the stabilizing terminal weight through LMI (3.49) yields:

$$P_N = \begin{bmatrix} 1.2545 & -0.4484 \\ -0.4484 & 2.0977 \end{bmatrix}.$$

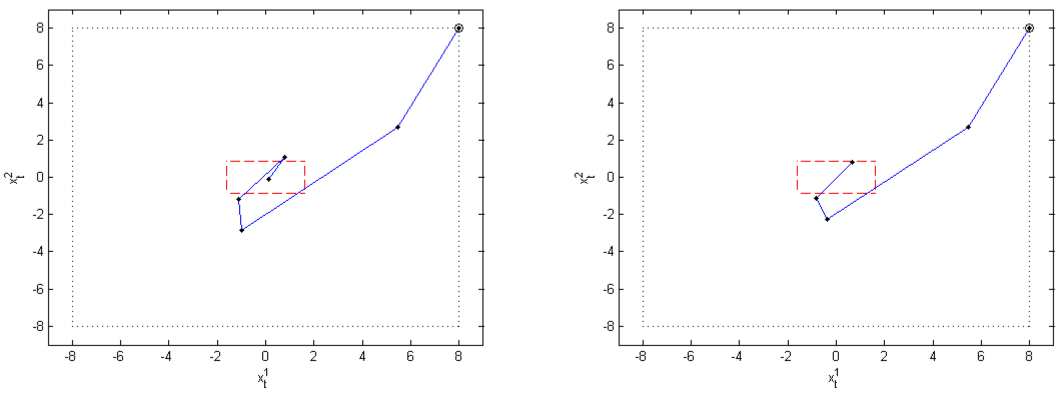


Figure 3.2: State trajectories for Example 2 with $\gamma^2 = 6$ (left) and $\gamma^2 = 1$ respectively

Figure 3.2 shows the state trajectory for $\gamma^2 = 6$ (left) and $\gamma^2 = 1$ respectively, both with worst-case oscillating uncertainties $\delta_\alpha = (-1)^{t+1}\alpha$, $\delta_\beta = (-1)^{t+1}\beta$ and disturbance $w_t = (-1)^t v$, $\forall t$. We see that even with initial state on the constraint boundary (black dotted line) and worst case uncertainty and disturbances, the RMPC controller is able to steer the system state towards the RCI set (red dashed set) in both cases. However, as expected with $\gamma^2 = 1$, the controller offers improved disturbance rejection and steers the state to RCI set in fewer steps such that $x_4 \in \mathcal{Z}$. The computed input sequences for $\gamma^2 = 6$ and 1 are respectively given in Figure 3.3 (left) with the control constraint clearly active at $t = 0$ (this illustrates the non-conservative manner in which constraints have been incorporated within the formulation, see Remark 3.11). Finally, Figure 3.3 also shows the decreasing cost upper-bound $\overline{J}_0^\star(x_0)$ which is approaching zero with each iteration.

### 3.5.3 Example 3

We consider the problem of controlling the composition (amount of pulp fibers in aqueous suspension) and liquid level in a Paper-Making process [107]. Figure 3.4 shows the schematic of a Paper machine headbox. The process states are $x^T = [H_1 \ H_2 \ N_1 \ N_2]$, where $H_1$ and $H_2$ are the liquid levels in feed tank and headbox, respectively, and $N_1$ and $N_2$ are the compositions in the feed tank and headbox, respectively. The control input is given by $u^T = [G_p \ G_w]$, where $G_p$ is the flowrate of stock entering the feed tank and $G_w$ is the recycled white water flow rate. All variables are normalized such that they are

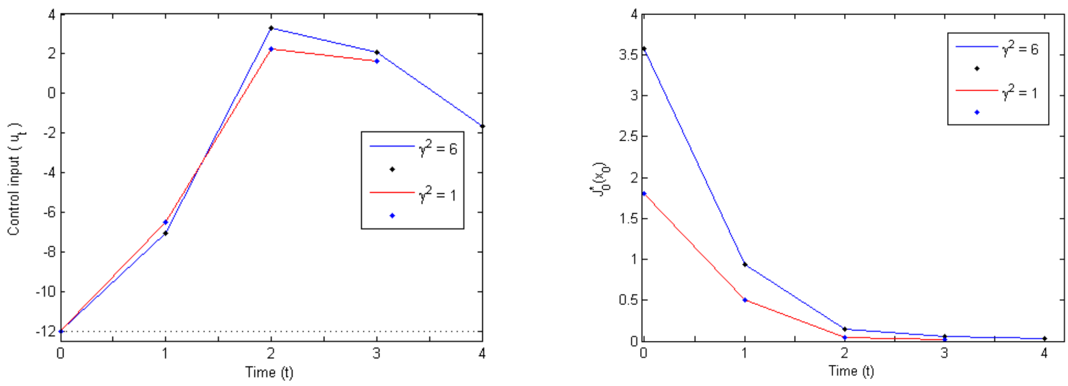


Figure 3.3: Control input trajectory (left) and cost upper-bound graphs, respectively, at each iteration for Example 2

zero at nominal steady state.

We consider both model-uncertainty and additive disturbance in the process-dynamics, due, for example, to the consistency/composition of white water. Moreover, we assume all states as measured. The control objective is to regulate the liquid levels and compositions despite the presence of persistent uncertainties/disturbances. The process dynamics, discretized using a sampling time of 2 minutes [107], are given by (3.1) where:

$$A = \begin{bmatrix} 0.0211 & 0 & 0 & 0 \\ 0.1062 & 0.4266 & 0 & 0 \\ 0 & 0 & 0.2837 & 0 \\ 0.1012 & -0.6688 & 0.2893 & 0.4266 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.6462 & 0.6462 \\ 0.2800 & 0.2800 \\ 1.5237 & -0.7391 \\ 0.9929 & 0.1507 \end{bmatrix}, \quad B_w \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

with $A_\delta = |A|$, $B_\delta = |B_u|$ and uncertainty polytopes are given by $\alpha = 0.20$ and $\beta = 0.10$. The disturbance is represented by the set: $-0.1 \leq w_k \leq 0.1$.

There are constraints on liquid levels so that headboxes never run dry or overfill. These are given by: $-3 \leq H_1, H_2 \leq 3$ and the compositions are constrained such that: $-5 \leq N_1, N_2 \leq 5$. We consider restrictive input constraints given by: $-1.5 \leq u_1, u_2 \leq 1.5$ and the initial state to be on the constraint boundary i.e. $x_0^T = [3,\ 3,\ 5,\ 5]$.
The cost parameters are $Q = 0.1I$, $R = 0.01I$, $P_N = 0.1I$, $\epsilon^2 = 0.5$ and the prediction horizon $N = 10$. Since we require to regulate the states tightly around zero (despite uncertainty), we impose the following constraints on the target RCI set: The absolute liquid levels $|H_1|$, $|H_2|$ should be less or equal to 0.6 and the absolute composition levels

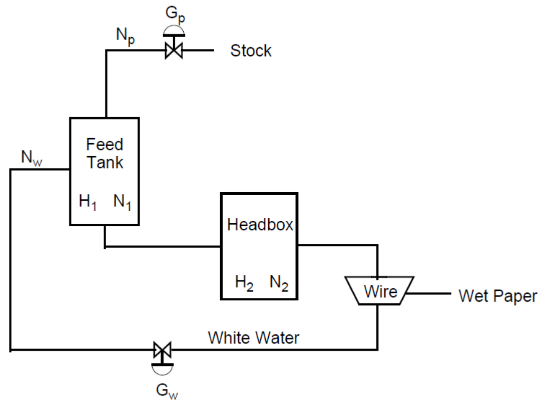


Figure 3.4: Schematic of Paper Machine Headbox Control Problem.

$|N_1|$, $|N_2|$ should be less or equal to 0.7. We also impose (inner) controller constraints: $\overline{u}_f = -\underline{u}_f = [0.5,\ 0.5]^T$. The resulting RCI set and controller are given by:

$$\mathcal{Z} = \left\{ x \in \mathbb{R}^4 : \begin{bmatrix} -0.536 \\ -0.279 \\ -0.70 \\ -0.70 \end{bmatrix} \leq x \leq \begin{bmatrix} 0.536 \\ 0.279 \\ 0.70 \\ 0.70 \end{bmatrix} \right\},\ K = -\begin{bmatrix} 0.078 & -0.335 & 0.180 & 0.142 \\ 0.160 & 0.335 & -0.180 & -0.142 \end{bmatrix}$$

The simulation results (including state-trajectory, control input and cost upper-bound) with persistent (worst-case) uncertainties $\delta_\alpha = +\alpha$, $\delta_\beta = +\beta$ and disturbance $w_t = +v$ (for $\gamma^2 = 10$ and 3) are given in Table 3.2. We note that despite persistent uncertainty, the proposed RMPC controller is able to steer the process-states from the upper constraint boundary to the invariant set $\mathcal{Z}$. Note that, with $\gamma^2 = 3$, the state converges to the RCI set in three iterations as opposed to five iterations required with $\gamma^2 = 10$. However, as can be seen from the table, the improved disturbance rejection for $\gamma^2 = 3$ comes at a cost of larger control requirement with the input constraint active at $t = 0$. Table 3.2 also shows the corresponding cost upper-bound reducing at each iteration.

| | | Time (t) | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 | 5 |
| $\gamma^2 = 10$ | $x_t$ | $\begin{bmatrix}3\\3\\5\\5\end{bmatrix}$ | $\begin{bmatrix}-1.344\\1.36\\-0.104\\1.484\end{bmatrix}$ | $\begin{bmatrix}-0.016\\0.589\\0.267\\-0.037\end{bmatrix}$ | $\begin{bmatrix}0.06\\0.382\\0.237\\-0.149\end{bmatrix}$ | $\begin{bmatrix}0.067\\0.288\\0.198\\-0.105\end{bmatrix}$ | $\begin{bmatrix}0.062\\0.239\\0.168\\-0.055\end{bmatrix}$ |
| | $u_t$ | $\begin{bmatrix}-1.421\\-0.716\end{bmatrix}$ | $\begin{bmatrix}0.054\\-0.169\end{bmatrix}$ | $\begin{bmatrix}0.004\\-0.060\end{bmatrix}$ | $\begin{bmatrix}-0.006\\-0.042\end{bmatrix}$ | $\begin{bmatrix}-0.016\\-0.040\end{bmatrix}$ | - |
| | $\overline{J}_0^\star$ | 11.223 | 0.973 | 0.155 | 0.076 | 0.044 | - |
| $\gamma^2 = 3$ | $x_t$ | $\begin{bmatrix}3\\3\\5\\5\end{bmatrix}$ | $\begin{bmatrix}-1.611\\1.244\\-0.038\\1.348\end{bmatrix}$ | $\begin{bmatrix}-0.209\\0.415\\0.173\\-0.212\end{bmatrix}$ | $\begin{bmatrix}0.068\\0.275\\0.216\\-0.189\end{bmatrix}$ | - | - |
| | $u_t$ | $\begin{bmatrix}-1.50\\-1.013\end{bmatrix}$ | $\begin{bmatrix}-0.070\\-0.307\end{bmatrix}$ | $\begin{bmatrix}0.014\\-0.051\end{bmatrix}$ | - | - | - |
| | $\overline{J}_0^\star$ | 21.451 | 1.275 | 0.146 | - | - | - |

Table 3.2: Simulation results for Example 3, including state-trajectory, control input and cost upper-bound at each time step

## 3.6 Summary

In this chapter, we have proposed a new algorithm for the Robust Model Predictive Control of linear discrete-time systems subject to bounded disturbances, (parametric) model-uncertainties and hard constraints on the input and state.

The proposed scheme consists of an outer controller, incorporating a state-feedback structure, which is responsible for steering the system state to a designed invariant set. Once in the set, the inner controller takes over and maintains the state within the RCI set despite persistent uncertainties and disturbances.

The novel features of the algorithm can be summarized as follows: **1)** The outer controller consists of a state-feedback part ($F_k$) and a control-perturbation ($m_k$), where both these components are explicitly considered as decision variables in the online optimization. The nonlinearities typically associated with such a feedback parameterization have been avoided by adopting a sequential approach in the formulation based, in-part, on the principles of Dynamic Programming; **2)** There is no requirement for any initial/offline computation of a feasible feedback control law; **3)** The state/input constraints are incorporated within the formulation in a non-conservative manner; **4)** The (terminal) RCI set and corresponding controller are both computed in one step by solving a single LMI problem; **5)** The algorithm consists of a series of low-dimensional LMI optimization problems, which makes the scheme suitable for online implementation.

As is typical in $\mathcal{H}_\infty$-MPC, the disturbance is negatively weighted in the proposed cost function through the incorporation of $\gamma^2$. This, as has been illustrated through numerical examples, improves the robustness against disturbances. Finally, conditions have been provided under which the RMPC algorithm is recursively feasible and ensures Lyapunov stability of the closed-loop uncertain system.

In this chapter, a hyper-rectangle RCI set structure has been considered due to its computational advantages (see Remark 3.3). However, depending on the disturbance/uncertainty, this remains a conservative choice. Therefore, in the next chapter, we investigate the efficient computation of more general polytopic RCI sets, along with their corresponding control law, for uncertain systems.

# Chapter 4

# Low-complexity Invariant Sets for Uncertain Systems

## 4.1 Introduction

In this chapter, we propose an algorithm to compute low complexity RCI (LC-RCI) sets, along with the corresponding state-feedback gain, for linear discrete-time systems subject to norm-bounded uncertainty, additive disturbances and state/input constraints.

As discussed in Section 1.2, RCI sets form an essential part of most RMPC schemes. With that in mind, the main motivation of this chapter is to compute, for uncertain systems, such simple polytopic RCI sets (along with the corresponding feedback gain) which can readily be incorporated within robust predictive control schemes without significantly increasing their online computational complexity. In this regard, LC-RCI sets hold several (computational) advantages over ellipsoidal and more general polytopic invariant sets. As mentioned in [18], use of ellipsoidal target sets for (linear) MPC leads to an online algorithm based on Semidefinite program (as opposed to traditional Quadratic Program) which in turn results in a significant increase in the online complexity. As a result, polytopic invariant target sets are generally preferred. However, general (maximal) polytopic RCI sets are usually described by a large number of inequalities which again leads to an increase in the computational complexity of online QP problem for MPC - particularly for higher order systems. On the other hand, LC-RCI sets are defined by only $2n$ inequalities - where $n$ is the order of the system - which makes them particularly suitable for incorporation within the overall control scheme.

In order to obtain the largest/smallest volume RCI set, an obvious approach is to consider both the set and control law $K$ as decision variables of optimization (see Section 1.2.3). However, this leads to non-convexity and nonlinearity in the formulation. Furthermore, many of the existing schemes from the literature cannot directly be applied to norm-bounded uncertain systems. An exception to this is the algorithm in [98] which proposes a method to compute hyper-rectangle RCI sets, and $K$ for system (1.10). However, hyper-rectangle set structure is generally a conservative choice for most systems.

Here, we propose an efficient algorithm based on convex/LMI optimizations, for the computation of LC-RCI sets and $K$ for systems of the form in (1.10). Using a slack variable approach [3, 31], we give general results to convexify the original nonlinear and non-convex problem whilst introducing only minor conservatism within the formulation. The algorithm can compute approximations to both the maximal as well as minimal volume polytopic invariant sets. An initial, constraint admissible LC-RCI set and corresponding $K$ are computed through a convex/LMI problem. Then, the volume of this set is iteratively optimized. Through numerical examples from the literature, we show that the initial and final maximal RCI sets computed by the proposed algorithm are larger than those obtained using the scheme in [18]. Furthermore, we show that for the special case when the RCI set is characterized as a hyper-rectangle, the proposed algorithm can yield, in one step, invariant sets which are larger/smaller than those computed using the scheme in [98]. Formulation of this chapter is mostly based on the results given in [101].

To deal with uncertainty in this work, we will use the following lemma [35].

**Lemma 4.1.** *Let $R = R^T, F, E, H$ be real matrices of appropriate dimensions and define*

$$\begin{aligned}\mathbf{\Delta} := \{&diag(\delta_1 I_{q_1}, \cdots, \delta_l I_{q_l}, \Delta_{l+1}, \cdots, \Delta_{l+r}) : \\ &\delta_i \in \mathbb{R}, |\delta_i| \leq 1\ , \Delta_i \in \mathbb{R}^{q_i \times q_i}\ , \|\Delta\| \leq 1\}\end{aligned} \tag{4.1}$$

*where $\mathfrak{B}\mathbf{\Delta}$ represents the unit ball of $\mathbf{\Delta}$.*

*Then, we have the inequality $R + F\Delta(I - H\Delta)^{-1}E + E^T(I - \Delta^T H^T)^{-1}\Delta^T F^T \succ 0$ and $\det(I - H\Delta) \neq 0$ for all $\Delta \in \mathfrak{B}\mathbf{\Delta}$, if there exist $(S, G) \in \widehat{\mathbf{\Psi}}$ such that*

$$\begin{bmatrix} R & E^T + FG^T & FS \\ E + GF^T & S + HG^T + GH^T & HS \\ SF^T & SH^T & S \end{bmatrix} \succ 0$$

We also employ the S-procedure [81]. As discussed in Section 2.3.1, this is a family of procedures used to derive sufficient (occasionally necessary and sufficient) LMI conditions for the non-negativity or non-positivity of a quadratic function on a set described by quadratic inequality constraints.

## 4.2 LC-RCI Set Problem

In this section, we first give a description of the system and constraints. Subsequently, we derive the conditions for invariance and highlight the inherent nonlinearities associated with the problem.

### 4.2.1 System Description and Constraints

We consider the following linear, discrete-time uncertain system, see e.g. [55]

$$x_{k+1} = Ax_k + B_u u_k + B_w w_k + B_p p_k, \tag{4.2a}$$

$$p_k = \Delta q_k, \tag{4.2b}$$

$$q_k = C_q x_k + D_{qu} u_k, \tag{4.2c}$$

where $x_k$, $u_k$, $w_k$, $p_k$ are the state, input, bounded disturbance and uncertainty vectors (respectively) at step $k$; $A$ is the system matrix and $B_u$, $B_w$ and $B_p$ are the input, disturbance and uncertainty distribution matrices, respectively. We assume that the pair $(A, B_u)$ is stabilizable and the state $x_k$ is measured. The polytopic disturbance is of the form:

$$w_k \in \mathcal{W} := \left\{ w \in \mathbb{R}^{n_w} : -v \leq w \leq v \right\}. \tag{4.3}$$

Furthermore, norm-bounded model uncertainty $\Delta \in \mathcal{B}\boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is defined in (4.1).

**Remark 4.1.** *We consider only symmetric disturbances here simply for the sake of clarity of exposition. The formulation below can also easily accommodate non-symmetric disturbances. Furthermore, note that we allow uncertainty (block diagonal, with repeated and/or full blocks) in all parts of the system dynamics since (4.2) can be re-written in the same form as in (1.10). That is:* $x_{k+1} = (A + B_p\Delta C_q)x_k + (B_u + B_p\Delta D_{qu})u_k + B_w w_k$.

We consider the polytopic LC-RCI set of the form [18]:

$$\mathcal{Z} := \left\{ x \in \mathbb{R}^n : -d \leq Cx \leq d \right\} \tag{4.4}$$

where $d \in \mathbb{R}^n$ is a vector of ones and $C \in \mathbb{R}^{n\times n}$ is a square matrix of full rank. The RCI set (4.4) is required to satisfy the following polyhedral state and input constraints:

$$x \in \mathcal{X} := \left\{ x \in \mathbb{R}^n : Tx \leq \overline{x} \right\} \tag{4.5}$$

$$u \in \mathcal{U} := \left\{ u \in \mathbb{R}^{n_u} : Nu \leq \overline{u} \right\} \tag{4.6}$$

with given matrices $T \in \mathbb{R}^{n_x \times n}$, $N \in \mathbb{R}^{n_c \times n_u}$ and vectors $0 < \overline{x} \in \mathbb{R}^{n_x}$, $0 < \overline{u} \in \mathbb{R}^{n_c}$.

An RCI set for system (4.2) can be defined as follows [16]:

**Definition 4.1.** *The set $\mathcal{Z} \subset \mathbb{R}^n$ is an RCI set for system (4.2) if there exists a control law $u = Kx \in \mathcal{U}$ such that:*

$$(A + B_p \mathbf{\Delta} C_q)\mathcal{Z} \oplus (B_u + B_p \mathbf{\Delta} D_{qu})K\mathcal{Z} \oplus B_w \mathcal{W} \subseteq \mathcal{Z}. \tag{4.7}$$

*where $\oplus$ denotes the Minkowski sum.*

**Remark 4.2.** *As discussed in [29], a nonlinear control law is generally the least conservative choice for optimizing the size of corresponding RCI sets. However, in addition to the computational intractabilities associated with formulating the RCI set problem in this way, such a choice is also likely to add complexity to the associated RMPC scheme. This is the reason why we focus on invariance under a linear control law, see also [18, 83].*

### 4.2.2 RCI Set Formulation

In this section, we will first derive sufficient conditions for the existence of an admissible invariant set of the form in (4.4). Subsequently, we analyze these conditions and discuss the associated nonlinearities.

**Theorem 4.1.** *Let all variables be as defined above. Then, there exists an admissible RCI set $\mathcal{Z}$ and controller $K$, i.e. ones satisfying the constraints (4.5)-(4.7), if there exist $(S_i, G_i) \in \widehat{\mathbf{\Psi}}$, and diagonal, positive semidefinite matrices $D^m$, $m \in \mathcal{N}_x := \{1, \cdots, n_x\}$, $D_u^j$, $j \in \mathcal{N}_u := \{1, \cdots, n_c\}$, and $D_x^i, D_w^i$, $i \in \mathcal{N}_n := \{1, \cdots, n\}$ as solutions to following matrix inequalities, $\forall m \in \mathcal{N}_x$, $\forall j \in \mathcal{N}_u$, $\forall i \in \mathcal{N}_n$:*

$$\begin{bmatrix} C^T D_x^i C & \star & \star & \star & \star \\ 0 & D_w^i & \star & \star & \star \\ -\frac{1}{2} e_i^T C(A + B_u K) & -\frac{1}{2} e_i^T C B_w & e_i^T d - d^T D_x^i d - v^T D_w^i v & \star & \star \\ (C_q + D_{qu} K) & 0 & -\frac{1}{2} G_i B_p^T C^T e_i & S_i & \star \\ 0 & 0 & -\frac{1}{2} S_i B_p^T C^T e_i & 0 & S_i \end{bmatrix} \succ 0 \tag{4.8}$$

$$\begin{bmatrix} C^T D_u^j C & -\frac{1}{2} K^T N^T e_j \\ \star & e_j^T \overline{u} - d^T D_u^j d \end{bmatrix} \succ 0 \tag{4.9}$$

$$\begin{bmatrix} C^T D^m C & -\frac{1}{2} T^T e_m \\ \star & e_m^T \overline{x} - d^T D^m d \end{bmatrix} \succ 0 \tag{4.10}$$

**Proof.** The invariance constraint in (4.7) can simply be written as:

$$e_i^T C[(A_K + B_p \Delta C_{qK})x + B_w w] \leq e_i^T d,\ \forall i \in \mathcal{N}_n := \{1, \cdots, n\} \tag{4.11}$$

$\forall x \in \mathcal{Z}, \forall w \in \mathcal{W}, \forall \Delta \in \mathbf{\mathcal{B}\Delta}$, where $A_K := A + B_u K$, $C_{qK} := C_q + D_{qu} K$. Here note that (4.11) automatically guarantees the corresponding lower inequality due to the symmetric nature of the sets $\mathcal{Z}$ and $\mathcal{W}$.

It can be verified that, for any $D_x^i$ and $D_w^i$:

$$\begin{aligned} e_i^T C[(A_K + B_p \Delta C_{qK})x + B_w w] - e_i^T d = -(d - Cx)^T D_x^i (Cx + d) \\ - (v - w)^T D_w^i (w + v) - y^T \mathcal{L}_i(C, K, D_x^i, D_w^i) y \end{aligned}$$

where $y^T := [x^T \ w^T \ 1]$, and

$$\mathcal{L}_i(C, K, D_x^i, D_w^i, \Delta) := \begin{bmatrix} C^T D_x^i C & 0 & -\frac{1}{2}(A_K + B_p \Delta C_{qK})^T C^T e_i \\ \star & D_w^i & -\frac{1}{2} B_w^T C^T e_i \\ \star & \star & e_i^T d - d^T D_x^i d - v^T D_w^i v \end{bmatrix} \tag{4.12}$$

Using the S-procedure (Farkas' Theorem) [81], it follows that the existence of diagonal, positive semidefinite matrices $D_x^i$ and $D_w^i$ such that $\mathcal{L}_i(C, K, D_x^i, D_w^i, \Delta) \succ 0$, $\forall i \in \mathcal{N}_n$, $\forall \Delta \in \mathbf{\mathcal{B}\Delta}$, is necessary and sufficient for invariance. It is easy to verify that this condition can be re-written in the form, $\forall i \in \mathcal{N}_n$:

$$R_i + F_i \Delta (I - H\Delta)^{-1} E + E^T (I - \Delta^T H^T)^{-1} \Delta^T F_i^T \succ 0, \tag{4.13}$$

where

$$\left[\begin{array}{c|c} R_i & F_i \\ \hline E & H \end{array}\right] := \left[\begin{array}{cccc} C^T D_x^i C & 0 & -\frac{1}{2} A_K^T C^T e_i & 0 \\ 0 & D_w^i & -\frac{1}{2} B_w^T C^T e_i & 0 \\ -\frac{1}{2} e_i^T C A_K & -\frac{1}{2} e_i^T C B_w & e_i^T d - d^T D_x^i d - v^T D_w^i v & -\frac{1}{2} e_i^T C B_p \\ \hline C_{qK} & 0 & 0 & 0 \end{array}\right]$$

Finally, an application of Lemma 4.1 on (4.13) yields the invariance condition in (4.8).

Now the input constraints in (4.6) are given by:

$$e_j^T NKx \leq e_j^T \overline{u}\ , \quad \forall x \in \mathcal{Z},\ \forall j \in \mathcal{N}_u$$

It can be verified that, for any $D_u^j$, $j \in \mathcal{N}_u$

$$e_j^T NKx - e_j^T \overline{u} = -(d - Cx)^T D_u^j (Cx + d) - y^T \mathcal{L}_u^j (K, C, D_u^j) y$$

where $y^T := [x^T\ 1]$ and $\mathcal{L}_u^j(K, C, D_u^j)$ is the matrix defined in the inequality (4.9). Using the S-procedure, it follows that the existence of diagonal, positive semidefinite matrices $D_u^j$ such that $\mathcal{L}_u^j(K, C, D_u^j) \succ 0$, $\forall j \in \mathcal{N}_u$, is necessary and sufficient for the satisfaction of input constraints and this is given in (4.9). Analogously, using the S-procedure on (4.5), it can be verified that the inequality in (4.10) is a necessary and sufficient condition for state constraints (4.5). □

Note that the problem of computing a feasible RCI set and control law is highly nonlinear in variables $C$ and $K$ - it is in fact not even bilinear. From Theorem 4.1, we see that the main source of nonlinearity is due to terms of the form $C^T D^i C$ and $\frac{1}{2} e_i^T C B_z X$ where $z$ stands for $p$ or $u$ and $X$ stands for $K$, $G_i$ or $S_i$. The problem is further complicated by the fact that decision variable matrix $C$ is not 'exposed' from either side in the $\frac{1}{2} e_i^T C B_z X$ terms which prevents the use of any congruence transformation techniques for linearization. We remedy this situation in the next section and propose an algorithm to compute $C$ and $K$ through a convex/LMI optimization problem.

**Remark 4.3.** *Note that the conditions in Theorem 4.1 become linear when the RCI set (4.4) is considered to be a hyper-rectangle, i.e.* $C = \Lambda := \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \succ 0$. *To see this, apply congruence transformation* $\mathrm{diag}(C^{-T}, I, I, I, I)$ *on (4.8), followed by multiplication with* $\lambda_i^{-1}$. *Then, noting that* $e_i^T C = \lambda_i e_i^T$, *applying the congruence transformation* $\mathrm{diag}(I, I, I, \lambda_i I, \lambda_i I)$ *and subsequently introducing the re-definitions* $\hat{K} := KC^{-1}$, $D_w^i := \lambda_i^{-1} D_w^i$, $D_x^i := \lambda_i^{-1} D_x^i$, $G_i := \lambda_i G_i$, *and* $S_i := \lambda_i S_i$ *renders (4.8) linear in variables* $\hat{K}$ *and* $C^{-1} (= \Lambda^{-1})$. *Constraint conditions in (4.9) and (4.10) can similarly be linearized by respectively applying the congruence* $\mathrm{diag}(C^{-T}, I)$ *and using the above re-definitions.*

**Remark 4.4.** *It is worth mentioning here that the Farkas' theorem (S-procedure) used in Theorem 4.1 is lossless. Furthermore, there is no gap in Lemma 4.1 for the case of unstructured uncertainties [35]. Therefore, conditions (4.8)-(4.10) become both necessary and sufficient for the existence of (constraint admissible) LC-RCI sets for systems subject* *to additive disturbances and unstructured uncertainties. Note that for such systems, (4.8)-(4.10) become necessary and sufficient LMI conditions to compute a $K$ that renders a given set $C$ invariant, which is also a problem treated in literature (see e.g. [49]).*

## 4.3 The Proposed Algorithm

In this section, we first propose general results - based on slack-variables - which allow us to remove the aforementioned nonlinearities in the RCI set problem. A cost function is then incorporated in the formulation to optimize the set volume through convex/LMI problems.

### 4.3.1 Linearization Procedure for the RCI Set Problem

As part of our main result, we now propose the following two theorems. Theorem 4.2 enables us to 'expose' $C$ and separate it from the other variables $K$, $S_i$ and $G_i$ (in the matrix inequalities of Theorem 4.1) without introducing any conservatism/approximations. Theorem 4.3 uses slack-variables to give necessary and sufficient conditions for separating bilinear terms of the form $XY + Y^T X^T$. These results allow to linearize the RCI set problem in Theorem 4.4.

**Theorem 4.2.** *Let $R = R^T, Z = Z^T$, $A$ and $B$ denote matrix variables of appropriate dimensions. Then, the following three statements are equivalent:*

*(i)* $\mathcal{L} := \begin{bmatrix} R & AB \\ \star & Z \end{bmatrix} \succ 0.$

*(ii)* $Z \succ 0,\ \mathcal{L}_0 := R - ABZ^{-1}B^T A^T \succ 0.$

*(iii)* $\exists\ X = X^T$ *such that*

$$\mathcal{L}_1 := \begin{bmatrix} R & A \\ \star & X^{-1} \end{bmatrix} \succ 0, \qquad \mathcal{L}_2 := \begin{bmatrix} X & B \\ \star & Z \end{bmatrix} \succ 0.$$

**Proof.** Note first that (i)$\Leftrightarrow$(ii) follows from a Schur complement argument. Therefore, we now prove (ii)$\Leftrightarrow$(iii) below.

- (ii)$\Rightarrow$(iii): Suppose (ii) is satisfied. Then, there exist scalars $\mu > 0$ and $\epsilon > 0$ such that $\mathcal{L}_0 \succ \mu I$ and $\mu I - \epsilon AA^T \succ 0$. Let $X = BZ^{-1}B^T + \epsilon I$. Then

$$X - BZ^{-1}B^T = \epsilon I \succ 0 \quad \Rightarrow \quad \mathcal{L}_2 \succ 0.$$

Furthermore, for this choice of $X, \epsilon$ and $\mu$, we have

$$R - AXA^T = R - ABZ^{-1}B^TA^T - A\epsilon A^T \succ \mu I - \epsilon AA^T \succ 0$$

and therefore $\mathcal{L}_1 \succ 0$.

- (iii)⇒(ii): Assume (iii) is satisfied for some $X$. Then, using Schur complement argument, we have

$$R - AXA^T \succ 0, \qquad X - BZ^{-1}B^T \succ 0. \tag{4.14}$$

It follows from (4.14) that

$$\mathcal{L}_0 = (R - AXA^T) + A(X - BZ^{-1}B^T)A^T \succ 0$$

and therefore (ii) is satisfied.

□

**Theorem 4.3.** *The Bilinear Matrix Inequality (BMI)*

$$L := Z + XY + Y^TX^T \succ 0 \tag{4.15}$$

*is satisfied if and only if there exist matrix variables, of appropriate dimensions, $Q = Q^T \succ 0$, $P = P^T \succ 0$, $G_1$, $G_2$, $F$, and $H$ as solutions to the following inequalities:*

$$\begin{bmatrix} P & Y \\ \star & Q \end{bmatrix} \succ 0 \tag{4.16}$$

$$\begin{bmatrix} Z + Q + XPX^T & F - XG_1 & H - XG_2 \\ \star & G_1 + G_1^T - P & F^T + G_2 - Y \\ \star & \star & H^T + H - Q \end{bmatrix} \succ 0 \tag{4.17}$$

**Proof.** Denote the matrix in (4.16) by $M$. Then, a manipulation shows that:

$$XY + Y^TX^T = Q + XPX^T - V^TMV$$

where $V^T := [-X \quad I]$. Replacing the above expression in (4.15), subsequently taking a Schur complement and then performing congruence transformation with $\text{diag}(I, M_o^T)$, where $M_o := \begin{bmatrix} G_1 & G_2 \\ F & H \end{bmatrix}$, yields:

$$\begin{bmatrix} Z + Q + XPX^T & V^T M_o \\ \star & M_o^T M^{-1} M_o \end{bmatrix} \succ 0 \tag{4.18}$$

Now, to deal with terms of the form $M_o^T M^{-1} M_o$, we use the following slack-variable identity:

$$M_o^T M^{-1} M_o = M_o + M_o^T - M + (M_o - M)^T M^{-1} (M_o - M) \tag{4.19}$$

Replacing, without loss of generality, the (2,2) entry of (4.18) by the first three terms on the right hand side in (4.19) yields inequality (4.17). □

**Remark 4.5.** *Theorem 4.2 allows us to separate the variables $A$ and $B$, in the $(1,2)$ entry, without any approximation. Similarly, Theorem 4.3 provides a result to separate the variables $X$ and $Y$ in the $(1,1)$ entry without any conservatism. Note that both these results are quite general in nature and hence have potential applications in other important control problems, for instance Lyapunov stability.*

**Remark 4.6.** *Results to separate $X$ and $Y$ have also been proposed in [3, 76]. However, they yield terms of the form $TZ$, where $Z$ is defined in (4.15) and $T$ is a variable. Such terms become problematic in the considered RCI set formulation. Therefore, Theorem 4.3 ensures that $Z$ is kept separate in order to obtain linearity.*

We now propose a theorem to compute a feasible RCI set $\mathcal{Z}$ and $K$ through LMIs.

**Theorem 4.4.** *Let all variables be as above. Then, there exists an initial feasible $\mathcal{Z}$ of the form in (4.4) and $K$, i.e. satisfying (4.5)-(4.7), if, for a given positive $\rho \in \mathbb{R}$, there exist matrix variables $(S_i, G_i) \in \widehat{\boldsymbol{\Psi}}$, $X_i = X_i^T$, $P_i = P_i^T$, $\Lambda := \text{diag}(\lambda_1, \cdots, \lambda_n) \succ 0$, $Q_i = Q_i^T$, $H_i$, $F_i$, and $Z_i$ of appropriate dimensions and diagonal, positive semidefinite matrices $D^m$, $m \in \mathcal{N}_x$, $D_u^j$, $j \in \mathcal{N}_u$, and $D_x^i$, $D_w^i$, $i \in \mathcal{N}_n$, as solutions to the following LMIs, $\forall i \in \mathcal{N}_n$:*

$$\begin{bmatrix} P_i & Z_i \\ \star & Q_i \end{bmatrix} \succ 0 \tag{4.20}$$

$$\begin{bmatrix} Q_i - X_i^{-1} & F_i - C^{-1} & H_i - C^{-1} & Z_i^T e_i \\ \star & 2\Lambda - P_i & \Lambda + F_i^T - Z_i & 0 \\ \star & \star & H_i^T + H_i - Q_i & 0 \\ \star & \star & \star & l_i \end{bmatrix} \succ 0 \tag{4.21}$$

$$\begin{bmatrix} D_x^i & 0 & \rho(AC^{-1}+B_u\hat{K})^T & \rho(C_qC^{-1}+D_{qu}\hat{K})^T \\ \star & D_w^i & \rho B_w^T & 0 \\ \star & \star & X_i^{-1}-B_pS_iB_p^T & B_pG_i^T \\ \star & \star & \star & S_i \end{bmatrix} \succ 0, \tag{4.22}$$

$$\begin{bmatrix} D_u^j & -\frac{1}{2}\hat{K}^TN^Te_{uj} \\ \star & e_j^T\overline{u}-d^TD_u^jd \end{bmatrix} \succ 0 \tag{4.23}$$

$$\begin{bmatrix} D^m & -\frac{1}{2}C^{-T}T^Te_m \\ \star & e_m^T\overline{x}-d^TD^md \end{bmatrix} \succ 0 \tag{4.24}$$

*where* $l_i := 4(\rho\lambda_i e_i^T d - v^T D_w^i v - d^T D_x^i d)$, *and* $K := \hat{K}C$.

**Proof.** By applying a congruence transformation and subsequently taking a Schur complement, the (nonlinear) invariance condition (4.8) can be written as:

$$R_i - A_iC^Te_ir_i^{-1}e_i^TCA_i^T \succ 0, \quad \forall i \in \mathbb{N}_n \tag{4.25}$$

where $r_i := 4(e_i^Td - d^TD_x^id - v^TD_w^iv)$ and

$$\left[\begin{array}{c|c} R_i & A_i \end{array}\right] := \left[\begin{array}{cccc|c} S_i & 0 & C_q+D_{qu}K & 0 & G_iB_p^T \\ 0 & S_i & 0 & 0 & S_iB_p^T \\ (C_q+D_{qu}K)^T & 0 & C^TD_x^iC & 0 & (A+B_uK)^T \\ 0 & 0 & 0 & D_w^i & B_w^T \end{array}\right]$$

Applying Theorem 4.2 on (4.25) verifies that (4.8) is satisfied if and only if, $\forall i \in \mathbb{N}_n$, there exist $X_i = X_i^T$ such that

$$\begin{bmatrix} S_i & 0 & C_q+D_{qu}K & 0 & G_iB_p^T \\ \star & S_i & 0 & 0 & S_iB_p^T \\ \star & \star & C^TD_x^iC & 0 & (A+B_uK)^T \\ \star & \star & \star & D_w^i & B_w^T \\ \star & \star & \star & \star & X_i^{-1} \end{bmatrix} \succ 0, \quad \begin{bmatrix} X_i & C^Te_i \\ \star & r_i \end{bmatrix} \succ 0 \tag{4.26}$$

Using Schur complement argument on the first inequality in (4.26), followed by the congruence transformation $\mathrm{diag}(I, C^{-T}, I, I)$ and a subsequent rearrangement yields the

LMI (with $\hat{K} := KC^{-1}$):

$$\mathcal{L}_1^i := \begin{bmatrix} D_x^i & 0 & C^{-T}A^T + \hat{K}^T B_u^T & C^{-T}C_q^T + \hat{K}^T D_{qu}^T \\ \star & D_w^i & B_w^T & 0 \\ \star & \star & X_i^{-1} - B_p S_i B_p^T & B_p G_i^T \\ \star & \star & \star & S_i \end{bmatrix} \succ 0 \tag{4.27}$$

Similarly, using the congruence transformation $\text{diag}(C^{-T}, I)$ on the second inequality in (4.26) yields, $\forall i \in \mathcal{N}_n$:

$$\mathcal{L}_2^i := \begin{bmatrix} C^{-T}X_iC^{-1} & e_i \\ \star & 4(e_i^T d - d^T D_x^i d - v^T D_w^i v) \end{bmatrix} \succ 0 \tag{4.28}$$

It follows that the sufficient conditions (necessary and sufficient in the case of unstructured uncertainty) for the invariance constraint (4.7) can now be given by:

$$\mathcal{L}_1^i \succ 0 \quad , \quad \mathcal{L}_2^i \succ 0 \ , \quad \forall i \in \mathcal{N}_n. \tag{4.29}$$

where $\mathcal{L}_1^i$, and $\mathcal{L}_2^i$ are defined above. Note here that (4.29)$\Leftrightarrow$(4.8).

First we deal with $\mathcal{L}_2^i$. Multiplying (4.28) by $\lambda_i \rho^{-1}$, for a given $\rho$ (see Section 4.4.1) and where $\lambda_i = e_i^T \Lambda e_i$, followed by a congruence transformation with $\text{diag}(I, \rho I)$ yields

$$\begin{bmatrix} \lambda_i \rho^{-1} C^{-T}X_iC^{-1} & \lambda_i e_i \\ \star & 4\lambda_i \rho(e_i^T d - d^T D_x^i d - v^T D_w^i v) \end{bmatrix} \succ 0 \ , \quad \forall i \in \mathcal{N}_n \tag{4.30}$$

Using the redefinitions $X_i^{-1} := \rho \lambda_i^{-1} X_i^{-1}$, $D_w^i := \rho \lambda_i D_w^i$, $D_x^i := \rho \lambda_i D_x^i$ in (4.30), recognizing that $\lambda_i e_i = \Lambda e_i$ in the (1,2) entry and subsequently performing a congruence transformation $\text{diag}(Z_i^T \Lambda^{-1}, I)$ yields

$$\begin{bmatrix} Z_i^T \Lambda^{-1} C^{-T} X_i C^{-1} \Lambda^{-1} Z_i & Z_i^T e_i \\ \star & 4(\rho \lambda_i e_i^T d - d^T D_x^i d - v^T D_w^i v) \end{bmatrix} \succ 0 \tag{4.31}$$

Now using slack-variable identity (4.19) on the (1,1) entry of (4.31) gives the following condition, which is equivalent to (4.28):

$$\begin{bmatrix} C^{-1}\Lambda^{-1}Z_i + Z_i^T \Lambda^{-1} C^{-T} - X_i^{-1} & Z_i^T e_i \\ \star & 4(\rho \lambda_i e_i^T d - d^T D_x^i d - v^T D_w^i v) \end{bmatrix} \succ 0 \tag{4.32}$$

Then, applying Theorem 4.3 on (1,1) entry of (4.32) with matrix $M_o := \begin{bmatrix} \Lambda & \Lambda \\ F_i & H_i \end{bmatrix}$, subsequently ignoring the positive term $C^{-1}\Lambda^{-1}P_i\Lambda^{-1}C^{-T}$ yields the LMIs in (4.20) and (4.21).

Now we consider $\mathcal{L}_1^i$. Multiplying (4.27) by $\rho\lambda_i^{-1}$, followed by a congruence transformation with $\text{diag}(\lambda_i I, \lambda_i I, I, I)$, $\forall i \in \mathbb{N}_n$, and using the redefinitions $S_i := \rho\lambda_i^{-1}S_i$, $G_i := \rho\lambda_i^{-1}G_i$ along with those for $X_i$, $D_w^i$ and $D_x^i$ (above) yields LMI (4.22).

Finally, for the input and state constraints, LMIs (4.23) and (4.24) are obtained by applying the congruence transformation $\text{diag}(C^{-T}, I)$ on (4.9) and (4.10), respectively. □

**Remark 4.7.** *Note that the use of Theorem 4.2 on (4.25) has removed most of the nonlinearites highlighted in the last paragraph of Section 4.2.2. In particular, $C$ has been 'exposed' from one side, and separated from variables $K$, $S_i$ and $G_i$ in (4.29). Furthermore, no additional conservatism has been introduced in (4.29) in comparison with (4.8). Finally, note that the nonlinearity in the $(1,1)$ entry of (4.28) has been overcome by using Theorem 4.3 and incorporating extra degrees of freedom $\lambda_i$ to obtain an improved initial solution through LMI optimization.*

**Remark 4.8.** *The parameter $\rho$ has been introduced in Theorem 4.4 to provide a further degree of freedom in the algorithm for computing the initial RCI set. Here an obvious choice could be to simply set $\rho = 1$. However, as we show in Section 4.4.1, other values of $\rho$ can result in a significantly improved initial set.*

**Remark 4.9.** *It is worth mentioning that the conditions in Theorem 4.1 remain valid even for non-square $C$. However, a convex re-reformulation of these conditions for the general (non-square) case is likely to be considerably different from that presented in Theorem 4.4, and thus forms part of the future work.*

### 4.3.2 Cost Function Incorporation

We now incorporate a cost function into the proposed algorithm to optimize the set-volume. The aim is to compute the largest/smallest volume constraint-admissible RCI set (herein known as maximal/minimal volume RCI set approximations). The volume of $\mathcal{Z}$ in (4.4) is proportional to $|\det(C^{-1})|$ [23]. Therefore, in the theorem below, we now derive upper/lower bounds on this determinant without making any assumptions regarding its sign (i.e. positivity or negativity).

**Theorem 4.5.** *Consider matrix variables* $\underline{W} = \underline{W}^T \succ 0$ *and* $\overline{W} = \overline{W}^T \succ 0$ *such that (without loss of generality):*

$$\underline{W} \preceq C^{-1}C^{-T} \preceq \overline{W} \tag{4.33}$$

*Then, a necessary and sufficient LMI condition for inequality* $C^{-1}C^{-T} \preceq \overline{W}$ *is given by:*

$$\begin{bmatrix} \overline{W} & C^{-1} \\ \star & I \end{bmatrix} \succeq 0 \tag{4.34}$$

*Furthermore,* $\underline{W} \preceq C^{-1}C^{-T}$ *if there exists a* $\hat{\lambda} > 0$ *such that:*

$$\begin{bmatrix} \hat{\lambda} I & \hat{\lambda} I & 0 \\ \star & C^{-T} + C^{-1} & \underline{W}^{\frac{1}{2}} \\ \star & \star & \hat{\lambda} I \end{bmatrix} \succeq 0 \tag{4.35}$$

**Proof.** Note first that applying a Schur complement argument on the matrix inequality $\overline{W} - C^{-1}C^{-T} \succeq 0$, yields (4.34).

Let us now consider the other inequality in (4.33), namely:

$$C^{-1}C^{-T} - \underline{W} \succeq 0 \tag{4.36}$$

Pre- and post-multiplying (4.36) by $C$ and $C^T$, respectively, followed by a Schur complement argument and a subsequent multiplication of the matrix by the scalar $\hat{\lambda} > 0$ yields:

$$\begin{bmatrix} \hat{\lambda} I & \hat{\lambda} I \\ \star & \hat{\lambda} C^{-T}\underline{W}^{-1}C^{-1} \end{bmatrix} \succeq 0 \tag{4.37}$$

To deal with the nonlinearity in (4.37), we consider the following identity

$$\hat{\lambda} C^{-T}\underline{W}^{-1}C^{-1} = C^{-T} + C^{-1} - \hat{\lambda}^{-1}\underline{W} + (C^{-1} - \hat{\lambda}^{-1}\underline{W})^T \hat{\lambda} \underline{W}^{-1}(C^{-1} - \hat{\lambda}^{-1}\underline{W}) \tag{4.38}$$

Replacing the (2,2) entry of (4.37) by the first three terms on the right hand side in (4.38) followed by a Schur complement yields (4.35) as a sufficient condition for (4.36). □

**Remark 4.10.** *Note that unlike the scheme in [18], we do not require* $\det(C^{-1})$ *to be positive since (4.33) implies that* $\det(\underline{W}) \leq \det(C^{-1})^2 \leq \det(\overline{W})$.

It follows that the computation of initial (inner) approximation of the maximal volume RCI set $\mathcal{Z}$ and corresponding gain $K$ can now be given by the convex optimization problem:

$$\overline{\phi} = \max\{\, log(det(\underline{W}^{\frac{1}{2}}) : (4.20-4.24), (4.35) \text{ are satisfied for all variables defined in Theorems 4.4 and 4.5}\}. \quad (4.39)$$

Now for the minimal volume case, note that the function $S_m = log(det(\overline{W}))$ is concave. Therefore, to compute initial (outer) approximation of the minimal volume RCI set and $K$, we minimize an upper-bound on $S_m$ by choosing $trace(\overline{W})$ as the cost (arithmetic mean-geometric mean inequality, see e.g. [39]). The LMI problem then becomes:

$$\underline{\phi} = \min\{\, trace(\overline{W}) : (4.20-4.24), (4.34) \text{ are satisfied for all variables defined in Theorems 4.4 and 4.5}\}. \quad (4.40)$$

We now propose the following theorem to update the (computed) initial solution to the RCI set as well as controller $K$.

**Theorem 4.6.** *Let $C = C_o$, $\underline{W} = \underline{W}_o$, $\overline{W} = \overline{W}_o$ and $X_i = X_i^o$, $\forall i$, be solutions to the optimization problem in (4.39) or (4.40). Then, these solutions (along with $K$) can be updated iteratively by solving (4.39) or (4.40), with $\rho = 1$, where (4.20)-(4.21) are replaced by LMI*

$$\begin{bmatrix} \mathcal{L}_{11} & e_i\lambda_i \\ \star & 4(\lambda_i e_i^T d - v^T D_w^i v - d^T D_x^i d) \end{bmatrix} \succ 0 \quad (4.41)$$

*with $\mathcal{L}_{11} := C^{-T}X_i^o C_o^{-1} + C_o^{-T}X_i^o C^{-1} - C_o^{-T}X_i^o X_i^{-1}X_i^o C_o^{-1}$. Furthermore, (2,2) and (2,3) entries of (4.35) are respectively replaced by*

$$C^{-T}\underline{W}_o^{-1}C_o^{-1} + C_o^{-T}\underline{W}_o^{-1}C^{-1}, \qquad C_o^{-T}\underline{W}_o^{-1}\underline{W}^{\frac{1}{2}} \quad (4.42)$$

**Proof.** In the proof of Theorem 4.4, (4.20)-(4.21) are used to ensure (4.28). Once the initial/previous solutions $C_o$ and $X_i^o$ are available, we proceed as follows.

Consider the following identity based on a slack-variable approach (see Remark 4.11):

$$\begin{aligned} C^{-T}X_iC^{-1} =& (C^{-1} - \lambda_i^{-1}X_i^{-1}X_i^oC_o^{-1})^T X_i (C^{-1} - \lambda_i^{-1}X_i^{-1}X_i^oC_o^{-1}) \\ &+ \lambda_i^{-1}C^{-T}X_i^oC_o^{-1} + \lambda_i^{-1}C_o^{-T}X_i^oC^{-1} - \lambda_i^{-2}C_o^{-T}X_i^oX_i^{-1}X_i^oC_o^{-1} \end{aligned} \quad (4.43)$$

Replacing the (1,1) entry of $\mathcal{L}_2^i$ in (4.28) by the last three terms on the right hand side in (4.43) and subsequently multiplying the resulting matrix by $\lambda_i$, followed by the redefinitions $X_i^{-1} := \lambda_i^{-1} X_i^{-1}$, $D_w^i := \lambda_i D_w^i$, and $D_x^i := \lambda_i D_x^i$ yields (4.41). Furthermore, in the proof of Theorem 4.5, using

$$\begin{aligned}\hat{\lambda} C^{-T} \underline{W}^{-1} C^{-1} =& C^{-T} \underline{W}_o^{-1} C_o^{-1} + C_o^{-T} \underline{W}_o^{-1} C^{-1} - C_o^{-T} \underline{W}_o^{-1} \hat{\lambda}^{-1} \underline{W}\, \underline{W}_o^{-1} C_o^{-1} \\ &+ (C^{-1} - \hat{\lambda}^{-1} \underline{W}\, \underline{W}_o^{-1} C_o^{-1})^T \hat{\lambda} \underline{W}^{-1} (C^{-1} - \hat{\lambda}^{-1} \underline{W}\, \underline{W}_o^{-1} C_o^{-1}) \end{aligned} \quad (4.44)$$

in place of (4.38), gives (4.35) with its (2,2) and (2,3) entries respectively replaced by the terms in (4.42). □

The overall algorithm can now be summarized as follows.

***Algorithm 4.1****: Computation of maximal/minimal volume RCI set approximations*

(1) **Initial solution**: Compute initial approximations $C$, $K$, $\overline{W}$, $\underline{W}$ and $X_i$ $\forall i$, to the maximal/minimal volume RCI set by solving (4.39) or (4.40).

(2) **Update solution**: Set $C_o = C$, $\overline{W}_o = \overline{W}$, $\underline{W}_o = \underline{W}$ and $X_i^o = X_i, \forall i$, and compute $C$, $K$, $\overline{W}$, $\underline{W}$, $X_i$ by solving modified versions of (4.39)/(4.40) as given in Theorem 4.6.

(3) **Iterate**: Loop back to step (2) until there is no further improvement in the volume of the computed RCI set.

**Remark 4.11.** *The identity (4.43) has been designed specifically to ensure recursive feasibility and iteratively optimize $\mathcal{Z}$ since setting $X_i$ and $C$ equal to $X_i^o$, and $C_o$ shows that the previous iteration solutions are feasible for the next one. Therefore, volume of RCI set $C$ would be greater or equal (less or equal for the case of minimal RCI set) to that of previous set $C_o$.*

### 4.3.3 Set Inclusion Conditions

Set inclusion is of fundamental importance in the algorithms for the computation of invariant sets. Let $\mathcal{Z}_k$ (defined by $C_k$) denote the RCI set computed at iteration $k$. Then, set inclusion requires that $\mathcal{Z}_k \subseteq \mathcal{Z}_{k+1}$ for maximal volume RCI sets and $\mathcal{Z}_{k+1} \subseteq \mathcal{Z}_k$ for minimal sets. We now derive the conditions for these inclusions in the following theorem.

**Theorem 4.7.** *Let all variables be as defined above. Then, at iteration $k+1$, we have $\mathcal{Z}_{k+1} \subseteq \mathcal{Z}_k$ if and only if there exist diagonal matrices $\underline{D}_s^i \succ 0$, $\forall i \in \mathcal{N}_n$, such that*

$$\begin{bmatrix} \underline{D}_s^i & -\frac{1}{2} C_{k+1}^{-T} C_k^T e_i \\ \star & e_i^T d - d^T \underline{D}_s^i d \end{bmatrix} \succ 0 \quad (4.45)$$

*Furthermore, $\mathcal{Z}_k \subseteq \mathcal{Z}_{k+1}$ if there exist diagonal matrices $\bar{D}_s^i \succ 0$, $\forall i \in \mathcal{N}_n$, such that*

$$\begin{bmatrix} C_{k+1}^{-T} + C_{k+1}^{-1} & C_k^{-1} & -\frac{1}{2}e_i \\ \star & \bar{D}_s^i & 0 \\ \star & \star & e_i^T d - d^T \bar{D}_s^i d \end{bmatrix} \succ 0 \tag{4.46}$$

**Proof.** Let us first consider $\mathcal{Z}_{k+1} \subseteq \mathcal{Z}_k$. At iteration $k+1$ with $C_k$ known, this inclusion can be written as:

$$e_i^T(C_k x - d) \leq 0 \;\; \forall x \;\; s.t. \; -d \leq C_{k+1} x \leq d \tag{4.47}$$

Using the S-procedure (Farkas' Theorem), we have, $\forall i \in \mathcal{N}_n$:

$$e_i^T(C_k x - d) = -(d - C_{k+1}x)^T \underline{D}_s^i (C_{k+1}x + d) - y^T \underline{\mathcal{L}}_i(C_{k+1}, \underline{D}_s^i) y$$

where $y^T := [x^T \; 1]$ and

$$\underline{\mathcal{L}}_i(C_{k+1}, \underline{D}_s^i) := \begin{bmatrix} C_{k+1}^T \underline{D}_s^i C_{k+1} & -\frac{1}{2}C_k^T e_i \\ \star & e_i^T d - d^T \underline{D}_s^i d \end{bmatrix}$$

It follows that $\underline{\mathcal{L}}_i(C_{k+1}, \underline{D}_s^i) \succ 0$ is a necessary and sufficient condition for (4.47). Finally, using the congruence transformation $\text{diag}(C_{k+1}^{-T}, I)$ on $\underline{\mathcal{L}}_i(C_{k+1}, \underline{D}_s^i)$ yields (4.45).

Next we consider the inclusion $\mathcal{Z}_k \subseteq \mathcal{Z}_{k+1}$ which can be written as:

$$e_i^T(C_{k+1} x - d) \leq 0 \;\; \forall x \;\; s.t. \; -d \leq C_k x \leq d \tag{4.48}$$

Using the S-procedure (Farkas' Theorem), we have $\forall i \in \mathcal{N}_n$:

$$e_i^T(C_{k+1} x - d) = -(d - C_k x)^T \bar{D}_s^i (C_k x + d) - y^T \bar{\mathcal{L}}_i(C_{k+1}, \bar{D}_s^i) y$$

where $y^T := [x^T \; 1]$ and

$$\bar{\mathcal{L}}_i(C_{k+1}, \bar{D}_s^i) := \begin{bmatrix} C_k^T \bar{D}_s^i C_k & -\frac{1}{2}C_{k+1}^T e_i \\ \star & e_i^T d - d^T \bar{D}_s^i d \end{bmatrix}$$

It follows that $\bar{\mathcal{L}}_i(C_{k+1}, \bar{D}_s^i) \succ 0$ is a necessary and sufficient condition for (4.48). Since we require $C_{k+1}^{-1}$ in Algorithm 4.1, therefore, effecting congruence transformation

$\mathrm{diag}(C_{k+1}^{-1}, I)$ on the $\bar{\mathcal{L}}_i(C_{k+1}, \bar{D}_s^i)$ yields:

$$\hat{\mathcal{L}}_i(C_{k+1}, \bar{D}_s^i) := \begin{bmatrix} C_{k+1}^{-T} C_k^T \bar{D}_s^i C_k C_{k+1}^{-1} & -\frac{1}{2} e_i \\ \star & e_i^T d - d^T \bar{D}_s^i d \end{bmatrix} \succ 0 \ , \quad \forall i \in \mathbb{N}_n$$

Now let us consider the identity:

$$\begin{aligned} C_{k+1}^{-T} C_k^T \bar{D}_s^i C_k C_{k+1}^{-1} =& (C_{k+1}^{-1} - C_k^{-1} (\bar{D}_s^i)^{-1} C_k^{-T})^T C_k^T \bar{D}_s^i C_k (C_{k+1}^{-1} - C_k^{-1} (\bar{D}_s^i)^{-1} C_k^{-T}) \\ & + C_{k+1}^{-T} + C_{k+1}^{-1} - C_k^{-1} (\bar{D}_s^i)^{-1} C_k^{-T} \end{aligned}$$

Substituting the last three terms on the right hand side of above identity into the $(1, 1)$ entry of $\hat{\mathcal{L}}_i(C_{k+1}^{-1}, D_s^i)$ followed by a Schur complement yields LMI (4.46). □

## 4.4 Numerical Examples

We now consider two examples from the literature to highlight the effectiveness of the algorithm.

### 4.4.1 Example 1

This example illustrates that the proposed algorithm can result in a larger volume approximation to the maximal RCI set $\mathcal{Z}$ as compared to the algorithm in [18]. We deal with the constrained, uncertain DC electric motor system (with independent excitation) considered in [18]. In particular, the continuous-time system is given by:

$$A = \begin{bmatrix} -0.07 & -0.86(1 + q_1) \\ 0.06(1 + q_1) & -q_2 \end{bmatrix} \ , \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.49}$$

where the uncertainty in parameters $q_1$ and $q_2$ is given by:

$$Q = \{(q_1, q_2)| - \bar{q}_1 \le q_1 \le \bar{q}_1, \ \underline{q}_2 \le q_2 \le \bar{q}_2\} \tag{4.50}$$

where $\bar{q}_1 = 0.2$, $\underline{q}_2 = 0.0085$ and $\bar{q}_2 = 0.5$. System is discretized by Euler discretization method using a sampling time of $T_s = 0.1s$, and then re-cast into the form (4.2) with

$$A = \begin{bmatrix} 0.993 & -0.086 \\ 0.006 & 1 - 0.1q_2^0 \end{bmatrix} \ , \quad B_u = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \ , \quad B_p = \begin{bmatrix} -0.086 & 0 & 0 \\ 0 & 0.006 & -0.1 \end{bmatrix} \ , \quad C_q = \begin{bmatrix} 0 & \bar{q}_1 \\ \bar{q}_1 & 0 \\ 0 & q_2^1 \end{bmatrix}$$

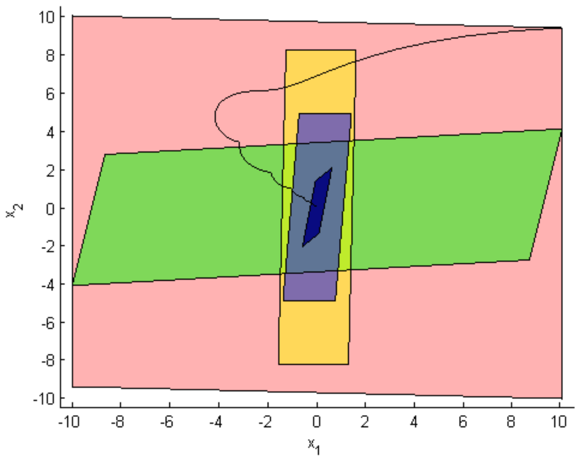


Figure 4.1: Maximal volume RCI set for Example 1

where $q_2^0 = 0.5(\bar{q}_2 + \underline{q}_2)$, $q_2^1 = 0.5(\bar{q}_2 - \underline{q}_2)$ and $\mathbf{\Delta} := \{diag(\delta_1 I_2, \delta_2) : \ \delta_i \in \mathbb{R}, \ |\delta_i| \leq 1\}$. Note that $D_{qu} = 0$ in this example. The state and input constraints are respectively given by:

$$\begin{bmatrix} -10 & -10 \end{bmatrix}^T \leq x_k \leq \begin{bmatrix} 10 & 10 \end{bmatrix}^T, \quad -10 \leq u_k \leq 10 \tag{4.51}$$

In order to obtain the initial (constraint-admissible) RCI set, we solve problem (4.39). Figure 4.1 shows the simulation results. The computed initial RCI set (with $\rho = 1$), shown in purple, and the corresponding controller are given by:

$$C = \begin{bmatrix} 0.9359 & -0.0632 \\ 0.0013 & 0.2054 \end{bmatrix}, \quad K = \begin{bmatrix} -9.3586 & 0.6315 \end{bmatrix}.$$

Following the iterative procedure specified in Algorithm 4.1, the final RCI set, shown in pink, and the computed controller are given by:

$$C = \begin{bmatrix} 0.1000 & 0.0000 \\ 0.0032 & 0.1032 \end{bmatrix}, \quad K = \begin{bmatrix} -0.9898 & -0.0109 \end{bmatrix} \tag{4.52}$$

For comparison, Figure 4.1 also shows the initial RCI set (in black/dark blue) as well as the final RCI set (in green) computed using the iterative scheme in [18]. Note that our proposed algorithm is able to yield substantially larger-volumes for both initial as well as the final (constraint-admissible) RCI sets. The figure also shows the state-trajectory of the system (black curved line) converging around the origin, despite persistent uncertainty, through the application of computed control law $K$.

To highlight the effect of $\rho$, Figure 4.1 also shows, in yellow, the initial RCI set computed using $\rho = 0.08$. Note that even with this (different) initial condition, the algorithm still converges to the same final RCI set above (pink) - though in fewer iterations.

Finally, note that maximal RCI set must be a subset of the hyper-rectangle $C_S = \text{diag}(0.1, 0.1)$, defined by the state constraints in (4.51). We would like to mention that a computation using Theorem 4.1 verifies that the hyper-rectangle $C_S$ is in fact not a feasible set for this example. This therefore shows that the computed final RCI set given in (4.52) must indeed be very close to the actual maximal LC-RCI set.

### 4.4.2 Example 2

This example illustrates that the proposed algorithm can compute improved approximations to both the maximal as well as the minimal hyper-rectangle sets, in one-step, as compared to the algorithm in [98]. We consider the uncertain version of the double-integrator system (see e.g. [83]) which is known to naturally have a hyper-rectangle RCI set structure. In particular the dynamics are as follows [98]:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_p = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$$

with $C_q = A$ and $D_{qu} = B_u$. The disturbance satisfies:

$$\begin{bmatrix} -0.5 & -0.5 \end{bmatrix}^T \leq w_k \leq \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T$$

and model-uncertainty $\mathbf{\Delta} := \{diag(\delta_1, \delta_2) : \; \delta_i \in \mathbb{R}, \; |\delta_i| \leq 1\}$. Moreover, we consider input constraints $u_k \in \mathcal{U} := \Big\{ u \in \mathbb{R} : -3 \leq u \leq 3 \Big\}$. Using Remark 4.3 and Theorem 4.5, the minimal volume RCI set approximation and controller are obtained (in one step) as

$$C^{-1} = \text{diag}(0.5, 1.1), \quad K = [-1 \;\; -1]$$

Similarly the maximal volume RCI set and controller are given by

$$C^{-1} = \text{diag}(4.32, 1.87), \quad K = [-0.26 \;\; -1]$$

The minimal and maximal invariant sets computed using the algorithm in [98] are respectively given by $C^{-1} = \text{diag}(0.5, 1.3)$ and $C^{-1} = \text{diag}(3.27, 2.03)$. Hence, the proposed algorithm yields better volume approximations to minimal/maximal RCI sets.

## 4.5 Summary

We have proposed an algorithm - based on convex/LMI optimizations - for the computation of low-complexity polyhedral RCI sets, along with the corresponding controller, for linear, discrete-time systems subject to bounded disturbances, norm-bounded model uncertainties and hard constraints on the input and state.

The main contribution of the chapter is that the proposed formulation removes the inherent problem-nonlinearities, including BMIs and triple product terms of the form $C^T X_i C$, at the expense of only minor conservatism. To this end, new results have been proposed in Theorems 4.2 and 4.3 which, being general in nature, also have applications in other important problem areas [3], e.g. Lyapunov stability of continuous-time systems.

The effectiveness of the scheme has been illustrated by numerical examples which show that the algorithm yields an initial as well as final approximation to the maximal LC-RCI set, which improve the results obtained using the scheme in [18]. We have also shown that the algorithm can compute hyper-rectangle RCI sets along with the corresponding controller in 'one-step'. Through a numerical example, it has been demonstrated that the formulation results in better approximations to the minimal as well as the maximal hyper-rectangle RCI sets in comparison to the scheme in [98].

The main reason for our study of LC-RCI sets is their particular suitability for incorporation within RMPC schemes. In the next chapter, we propose such a state-feedback RMPC algorithm for the type of norm-bounded uncertain systems considered in this chapter. While an extension of this scheme to the output-feedback case will be presented in Chapter 6.

# Chapter 5

# State-feedback RMPC for Norm-bounded Uncertain Systems

## 5.1 Introduction

In this chapter, we address the problem of nonlinearity and non-convexity typically associated with state-feedback parameterizations in the Robust Model Predictive Control (RMPC) of uncertain systems.

A state-feedback RMPC scheme for linear systems with (scalar) parametric uncertainties was presented in Chapter 3. In that algorithm, an approach based on Dynamic Programming [13] was adopted to avoid non-convexity in the formulation. In particular, an upper bound on the cost-to-go at each prediction step was minimized to compute the corresponding control gains in a sequential manner. We now focus our attention on a 'stacked' formulation of RMPC for norm-bounded uncertain systems, where the control gains throughout the prediction horizon are computed all at once in a non-sequential manner. As discussed in Section 1.2.1, such parameterizations lead to sequences of predicted states and inputs which are nonlinear, non-convex functions of the control gains. Therefore, many schemes in the literature compute the feedback gain offline, see e.g. [7, 24, 27, 51, 59], which can potentially be conservative.

In this chapter, the aim is to explicitly consider both feedback gain and control perturbation as decision variables in the online optimization and obtain convexity at the expense of only minor conservatism within the formulation. Moreover, a general (non-square matrix) norm-bounded uncertainty structure is considered within the dynamic model to capture a large class of uncertain systems. These type of model-uncertainties may arise, for example, due to imprecise system parameters that lie in a given interval (such as anywhere within a circle of radius $r$).

We propose two tractable methods of computing, online, an RMPC controller - that consists of both a causal, state-feedback gain and a perturbation component - for linear, discrete-time systems involving bounded disturbances and (norm-bounded) model-uncertainties along with hard constraints on the input and state. The first approach consists of re-casting the additive disturbance as an uncertainty, followed by use of the S-procedure and slack variable identities to obtain convexity. In the second approach - which can be considered to be a 'dual' of the first - we propose to re-parameterize the model-uncertainty such that it can be treated in a manner similar to the additive disturbance, which in turn helps to obtain convexity. Both approaches enable the online computation of optimal state-feedback gain and perturbation sequence through an LMI optimization problem. The aim of the RMPC controller is to steer the uncertain system state to an RCI terminal set (which can be designed using, for example, the algorithm in Chapter 4). The formulation in this chapter is based on the results given in [99].

To handle non-square $\Delta$, we consider a modified version of Lemma 4.1 as follows:

**Lemma 5.1.** *Let $\boldsymbol{\Delta} \subseteq \mathbb{R}^{p\times q}$ be a linear subspace and define*

$$\begin{aligned}\boldsymbol{\Psi} = \{(S,T,G) \in \mathbb{R}^{p\times p}\times\mathbb{R}^{q\times q}\times\mathbb{R}^{q\times p} : S = S^T \succ 0, T = T^T \succ 0,\\ S\Delta = \Delta T, \Delta G + G^T\Delta^T = 0,\ \forall \Delta \in \mathcal{B}\boldsymbol{\Delta}\}\end{aligned}$$

*where $\mathcal{B}\boldsymbol{\Delta}$ represents the unit ball of $\boldsymbol{\Delta}$.*

*Let $R = R^T, F, E, H$ be matrices of appropriate dimensions. Then, $det(I - H\Delta) \neq 0$ and the matrix inequality $R + F\Delta(I - H\Delta)^{-1}E + E^T(I - \Delta^T H^T)^{-1}\Delta^T F^T \succ 0$ for every $\Delta \in \mathcal{B}\boldsymbol{\Delta}$ if there exists a triple $(S,T,G) \in \Psi$ such that*

$$\begin{bmatrix} R & E^T + FG^T & FS \\ \star & T + HG^T + GH^T & HS \\ \star & \star & S \end{bmatrix} \succ 0 \tag{5.1}$$

**Remark 5.1.** *Although our development will be for general norm-bounded structured sets, an example is*

$$\boldsymbol{\Delta} := \{diag(\delta_1 I_{q1}, \cdots, \delta_l I_{ql}, \Delta_{l+1}, \cdots, \Delta_{l+r}) : \delta_i \in \mathbb{R}, |\delta_i| \leq 1\ , \Delta_i \in \mathbb{R}^{qi\times qi}\ , \|\Delta\| \leq 1\} \tag{5.2}$$

*where* $n_p = n_q = \sum_{i=1}^{l+r} q_i$. *This includes both repeated scalar and full diagonal blocks. In this particular case*

$$\begin{aligned}
\mathbf{\Psi} &= \{(S,T,G) \in \mathbb{R}^{n_p\times n_p} \times \mathbb{R}^{n_p\times n_p} \times \mathbb{R}^{n_p\times n_p} : S = T \succ 0, S \in \mathbf{\Sigma},\ G \in \mathbf{\Gamma}\} \\
\mathbf{\Sigma} &= \{diag(S_1,\cdots,S_l,\lambda_1 I_{q_{l+1}},\cdots,\lambda_s I_{q_{l+f}}) : \lambda_j \in \mathbb{R}, S_i = S_i^T \in \mathbb{R}^{q_i\times q_i}\} \\
\mathbf{\Gamma} &= \{diag(G_1,\cdots,G_l,0_{q_{l+1}},\cdots,0_{q_{l+f}}) : G_i = -G_i^T \in \mathbb{R}^{q_i\times q_i}\}
\end{aligned}$$

## 5.2 Robust MPC Problem

In this section, we give a description of the system and constraints followed by the cost function. We also derive an algebraic formulation of the causal RMPC problem and discuss the nature of nonlinearities.

### 5.2.1 System Description

We consider the following linear discrete-time uncertain system [55]:

$$\begin{aligned}
\begin{bmatrix} x_{k+1} \\ q_k \\ f_k \\ z_k \end{bmatrix} &= \begin{matrix} n \\ n_q \\ n_f \\ n_z \end{matrix} \overset{\begin{matrix} n & n_u & n_w & n_p \end{matrix}}{\begin{bmatrix} A & B_u & B_w & B_p \\ C_q & D_{qu} & D_{qw} & 0 \\ C_f & D_{fu} & D_{fw} & D_{fp} \\ C_z & D_{zu} & D_{zw} & D_{zp} \end{bmatrix}} \begin{bmatrix} x_k \\ u_k \\ w_k \\ p_k \end{bmatrix}, \ p_k = \Delta q_k,\ \Delta \in \mathcal{B}\mathbf{\Delta} \\
\begin{bmatrix} q_N \\ f_N \\ z_N \end{bmatrix} &= \begin{bmatrix} \hat{C}_q & 0 \\ \hat{C}_f & \hat{D}_{fp} \\ \hat{C}_z & \hat{D}_{zp} \end{bmatrix} \begin{bmatrix} x_N \\ p_N \end{bmatrix}, \qquad p_N = \Delta q_N
\end{aligned} \tag{5.3}$$

where $x_k$, $u_k$, $w_k$, $p_k$ are the state, input, bounded disturbance and uncertainty vectors (respectively) at prediction step $k$; $A$ is the state matrix and $B_u$, $B_w$ and $B_p$ are the input, disturbance and uncertainty distribution matrices, respectively. We assume that the pair $(A,B_u)$ is stabilizable. The state $x_k$ is assumed measured and prediction step $k$ belongs to the time set $T_N = \{0, 1, \cdots, N-1\}$, where $N > 0$ is the prediction horizon. The polytopic disturbance is of the form

$$w_k \in \mathcal{W}_k := \left\{ w \in \mathbb{R}^{n_w} : -d_k \leq w \leq d_k \right\}. \tag{5.4}$$

Furthermore, we consider a norm-bounded structured uncertainty $\Delta \in \mathcal{B}\mathbf{\Delta}$ where $\mathbf{\Delta} \subseteq \mathbb{R}^{n_p\times n_q}$ is a structured subspace. Note that we allow uncertainties in all the problem data in (5.3).

It is required to find $u_k,\ k \in T_N$, such that the future constrained outputs and terminal constrained output satisfy $f_k \le \bar{f}_k,\ f_N \le \bar{f}_N,\ \forall k \in T_N$, and minimize the cost function given by

$$J = \sum_{k=0}^{N} (z_k - \overline{z}_k)^T (z_k - \overline{z}_k) \tag{5.5}$$

where $\overline{z}_k$, which may represent a reference trajectory, is given. Note that $f_k$ may be chosen to represent polytopic constraints on the state, output and input.

### 5.2.2 Algebraic Formulation

Let $\xi$ stand for $f,\ \overline{f},\ p,\ q,\ z$ or $\overline{z}$, and $\zeta$ stand for $u,\ w,\ \underline{w}$ or $\overline{w}$ and define

$$x = \begin{bmatrix} x_1^T & \cdots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{N_n}, \quad \xi = \begin{bmatrix} \xi_0^T & \cdots & \xi_N^T \end{bmatrix}^T \in \mathbb{R}^{N_\xi}, \quad \zeta = \begin{bmatrix} \zeta_0^T & \cdots & \zeta_{N-1}^T \end{bmatrix}^T \in \mathbb{R}^{N_\zeta}$$
$$\boldsymbol{\mathcal{W}} = \{ w \in \mathbb{R}^{N_w} : \underline{w} \le w \le \bar{w} \}, \quad \widehat{\boldsymbol{\Delta}} = \{ diag(\Delta, \ldots, \Delta) : \Delta \in \boldsymbol{\Delta} \}$$

where $N_n = N \times n$, $N_\xi = n_\xi \times (N+1)$ and $N_\zeta = n_\zeta \times N$. Then, by iterating the system dynamics (5.3), it can be verified that:

$$\begin{bmatrix} x \\ q \\ f \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{A}x_0 & \mathcal{B}_w & \mathcal{B}_p & \mathcal{B}_u \\ \mathcal{C}_q x_0 & \mathcal{D}_{qw} & \mathcal{D}_{qp} & \mathcal{D}_{qu} \\ \mathcal{C}_f x_0 & \mathcal{D}_{fw} & \mathcal{D}_{fp} & \mathcal{D}_{fu} \\ \mathcal{C}_z x_0 & \mathcal{D}_{zw} & \mathcal{D}_{zp} & \mathcal{D}_{zu} \end{bmatrix} \begin{bmatrix} 1 \\ w \\ p \\ u \end{bmatrix} \tag{5.6}$$

where $p = \Delta q$, with $\Delta \in \boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}$ and

$$\mathcal{A} = \begin{bmatrix} A \\ \vdots \\ A^{N-1} \\ A^N \end{bmatrix}, \qquad \mathcal{C}_\alpha = \begin{bmatrix} C_\alpha \\ C_\alpha A \\ \vdots \\ C_\alpha A^{N-1} \\ \hat{C}_\alpha A^N \end{bmatrix}, \qquad \mathcal{B}_\beta = \begin{bmatrix} B_\beta & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-2}B_\beta & A^{N-3}B_\beta & \cdots & 0 \\ A^{N-1}B_\beta & A^{N-2}B_\beta & \cdots & B_\beta \end{bmatrix},$$

$$\mathcal{B}_p = \begin{bmatrix} B_p & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-2}B_p & A^{N-3}B_p & \cdots & 0 & 0 \\ A^{N-1}B_p & A^{N-2}B_p & \cdots & B_p & 0 \end{bmatrix}, \quad \mathcal{D}_{\alpha\beta} = \begin{bmatrix} D_{\alpha\beta} & 0 & \cdots & 0 \\ C_\alpha B_\beta & D_{\alpha\beta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_\alpha A^{N-2}B_\beta & C_\alpha A^{N-3}B_\beta & \cdots & D_{\alpha\beta} \\ \hat{C}_\alpha A^{N-1}B_\beta & \hat{C}_\alpha A^{N-2}B_\beta & \cdots & \hat{C}_\alpha B_\beta \end{bmatrix},$$

$$\mathcal{D}_{\alpha p} = \begin{bmatrix} D_{\alpha p} & 0 & \cdots & 0 & 0 \\ C_\alpha B_p & D_{\alpha p} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_\alpha A^{N-2} B_p & C_\alpha A^{N-3} B_p & \cdots & D_{\alpha p} & 0 \\ \hat{C}_\alpha A^{N-1} B_p & \hat{C}_\alpha A^{N-2} B_p & \cdots & \hat{C}_\alpha B_p & \hat{D}_{\alpha p} \end{bmatrix}$$

where $\alpha$ stands for $q$, $f$ or $z$, while $\beta$ stands for $u$ or $w$ and where $D_{qp} = 0$.

As mentioned above, we consider a causal state-feedback structure on the RMPC controller (that is, input $u_i$ depends only on $x_j, j = 0, \ldots, i$), see e.g. [92]. Therefore, we set

$$u = K_0 x_0 + K x + v \tag{5.7}$$

where, with $K_{i,j} \in \mathbb{R}^{n_u \times n}$, $v_i \in \mathbb{R}^{n_u}$ $\forall\, i, j$,

$$K_0 = \begin{bmatrix} K_{0,0} \\ K_{1,0} \\ \vdots \\ K_{N-2,0} \\ K_{N-1,0} \end{bmatrix}, \; K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ K_{1,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{N-2,1} & K_{N-2,2} & \cdots & 0 & 0 \\ K_{N-1,1} & K_{N-1,2} & \cdots & K_{N-1,N-1} & 0 \end{bmatrix}, \; v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{bmatrix}. \tag{5.8}$$

The causality is captured by the lower block triangular structure of $[K_0 \;\; K]$ and $v$ represents the control-perturbation sequence. We denote the structure of $K_0$, $K$, and $v$ in (5.8) as $\mathcal{K}_0$, $\mathcal{K}$ and $\upsilon$, respectively.

**Remark 5.2.** *The separation of the control law (5.7) into three terms is convenient since the first term depends on the (known) initial state $x_0$, the third term is free while the second term depends on the predicted states $x$, and represents the feedback component of the RMPC control law. As we shall see below, it is this term that causes the nonlinearity and affords potential extra degrees of freedom to satisfy the constraints for all possible disturbances and uncertainties. The control division in (5.7) also makes the design problem flexible, in that the designer may choose to use any combination of these terms. Note that if it is decided to use the open-loop control $v$, then we can simply absorb the first term into $v$, since $x_0$ is known.*

Substituting the equation for $x$ in (5.6) into (5.7) yields the following expression for $u$, which is affine in $(\hat{K}_0, \hat{K}, \hat{v})$.

$$u = \hat{K}_0 x_0 + \hat{K}(\mathcal{B}_w w + \mathcal{B}_p p) + \hat{v}, \quad \begin{bmatrix} \hat{K}_0 & \hat{K} & \hat{v} \end{bmatrix} := (I - K\mathcal{B}_u)^{-1} \left[K_0 \;\; K \;\; v + K\mathcal{A}x_0\right] \tag{5.9}$$

Note that there is a one-to-one mapping between $K$ and $\hat{K}$. Furthermore, the new variables $\hat{K}_0$, $\hat{K}$ and $\hat{v}$ have the same structure as $K_0$, $K$ and $v$ in (5.8), which in turn can be recovered as

$$\begin{bmatrix} K_0 & K & v \end{bmatrix} := (I + \hat{K}\mathcal{B}_u)^{-1} \begin{bmatrix} \hat{K}_0 & \hat{K} & \hat{v} - \hat{K}\mathcal{A}x_0 \end{bmatrix}$$

By using (5.9), eliminating $u$ from (5.6), gives

$$\begin{aligned} \begin{bmatrix} q \\ f \\ z - \overline{z} \end{bmatrix} &= \begin{bmatrix} \mathcal{D}_{qw} + \mathcal{D}_{qu}\hat{K}\mathcal{B}_w & \mathcal{D}_{qp} + \mathcal{D}_{qu}\hat{K}\mathcal{B}_p & \mathcal{D}_{qu}\hat{v} + (\mathcal{C}_q + \mathcal{D}_{qu}\hat{K}_0)x_0 \\ \mathcal{D}_{fw} + \mathcal{D}_{fu}\hat{K}\mathcal{B}_w & \mathcal{D}_{fp} + \mathcal{D}_{fu}\hat{K}\mathcal{B}_p & \mathcal{D}_{fu}\hat{v} + (\mathcal{C}_f + \mathcal{D}_{fu}\hat{K}_0)x_0 \\ \mathcal{D}_{zw} + \mathcal{D}_{zu}\hat{K}\mathcal{B}_w & \mathcal{D}_{zp} + \mathcal{D}_{zu}\hat{K}\mathcal{B}_p & \mathcal{D}_{zu}\hat{v} + (\mathcal{C}_z + \mathcal{D}_{zu}\hat{K}_0)x_0 - \overline{z} \end{bmatrix} \begin{bmatrix} w \\ p \\ 1 \end{bmatrix} \\ &=: \begin{bmatrix} \mathcal{D}_{qw}^{\hat{K}} & \mathcal{D}_{qp}^{\hat{K}} & \mathcal{D}_{q0}^{\hat{v},\hat{K}_0} \\ \mathcal{D}_{fw}^{\hat{K}} & \mathcal{D}_{fp}^{\hat{K}} & \mathcal{D}_{f0}^{\hat{v},\hat{K}_0} \\ \mathcal{D}_{zw}^{\hat{K}} & \mathcal{D}_{zp}^{\hat{K}} & \mathcal{D}_{z0}^{\hat{v},\hat{K}_0} \end{bmatrix} \begin{bmatrix} w \\ p \\ 1 \end{bmatrix} \end{aligned} \tag{5.10}$$

Finally, using $p = \Delta q$ to eliminate $p$ from (5.10) yields the following constraint and cost function signals

$$\begin{aligned} \begin{bmatrix} f \\ z - \overline{z} \end{bmatrix} &= \begin{bmatrix} \mathcal{D}_{fp}^{\hat{K}}\Delta(I - \mathcal{D}_{qp}^{\hat{K}}\Delta)^{-1}\mathcal{D}_{qw}^{\hat{K}} + \mathcal{D}_{fw}^{\hat{K}} & \mathcal{D}_{fp}^{\hat{K}}\Delta(I - \mathcal{D}_{qp}^{\hat{K}}\Delta)^{-1}\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} + \mathcal{D}_{f0}^{\hat{K}_0,\hat{v}} \\ \mathcal{D}_{zp}^{\hat{K}}\Delta(I - \mathcal{D}_{qp}^{\hat{K}}\Delta)^{-1}\mathcal{D}_{qw}^{\hat{K}} + \mathcal{D}_{zw}^{\hat{K}} & \mathcal{D}_{zp}^{\hat{K}}\Delta(I - \mathcal{D}_{qp}^{\hat{K}}\Delta)^{-1}\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} + \mathcal{D}_{z0}^{\hat{K}_0,\hat{v}} \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix} \\ &:= \begin{bmatrix} \mathcal{D}_{fw}^{\hat{K},\Delta} & \mathcal{D}_{f0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \\ \mathcal{D}_{zw}^{\hat{K},\Delta} & \mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix} \end{aligned} \tag{5.11}$$

### 5.2.3 An Initial RMPC Formulation

In order to formulate the RMPC problem, we use (5.11) to respectively write the constraint and cost function as

$$f(\hat{K}_0, \hat{K}, \hat{v}, w, \Delta) = \mathcal{D}_{fw}^{\hat{K},\Delta} w + \mathcal{D}_{f0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \tag{5.12}$$

$$f_0(\hat{K}_0, \hat{K}, \hat{v}, w, \Delta) = \begin{bmatrix} w^T & 1 \end{bmatrix} \begin{bmatrix} (\mathcal{D}_{zw}^{\hat{K},\Delta})^T \\ (\mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta})^T \end{bmatrix} \begin{bmatrix} \mathcal{D}_{zw}^{\hat{K},\Delta} & \mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix} \tag{5.13}$$

It follows from (5.12) that the constraint set can be written as

$$\mathcal{U} = \{(\hat{K}_0, \hat{K}, \hat{v}) : e_i^T(\mathcal{D}_{fw}^{\hat{K},\Delta} w + \mathcal{D}_{f0}^{\hat{K}_0,\hat{K},\hat{v},\Delta}) \le e_i^T \bar{f},\ \forall i \in \mathcal{N}_f,\ \forall w \in \boldsymbol{\mathcal{W}},\ \forall \Delta \in \boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}\}. \tag{5.14}$$

We now use Lemmas 2.3 and 5.1 to derive sufficient conditions (necessary and sufficient in some cases, see Remark 5.4) for $(\hat{K}_0, \hat{K}, \hat{v}) \in \mathcal{U}$ and an upper bound $\overline{f}_0$ on cost (5.13).

**Theorem 5.1.** *Let $\mathbb{D}^m$ denote the set of all real $m \times m$ diagonal matrices and let $\mathbb{D}^m_+ := \{D \in \mathbb{D}^m : D \succeq 0\}$. Furthermore, define*

$$\widehat{\boldsymbol{\Psi}} = \{(S,T,G) \in \mathbb{R}^{N_p \times N_p} \times \mathbb{R}^{N_q \times N_q} \times \mathbb{R}^{N_q \times N_p} : S = S^T \succ 0, T = T^T \succ 0, \\ S\Delta = \Delta T, \Delta G + G^T \Delta^T = 0, \ \forall \Delta \in \mathcal{B}\widehat{\boldsymbol{\Delta}}\}$$

*Then, $(\hat{K}_0, \hat{K}, \hat{v}) \in \mathcal{U}$ and $f_0(\hat{K}_0, \hat{K}, \hat{v}, w, \Delta) \le \overline{f}_0$, for all $\Delta \in \mathcal{B}\widehat{\boldsymbol{\Delta}}$, if there exist solutions $(S,T,G) \in \widehat{\boldsymbol{\Psi}}$, $(S_{ij}, T_{ij}, G_{ij}) \in \widehat{\boldsymbol{\Psi}}$, $(S_i, T_i, G_i) \in \widehat{\boldsymbol{\Psi}}$, $\mu^i_w \in \mathbb{R}^{N_w}$, and $D_w \in \mathbb{D}^{N_w}_+$, $\forall j \in \mathcal{N}_w := \{1, \cdots, N_w\}$, $\forall i \in \mathcal{N}_f$, to the following matrix inequalities:*

$$\begin{bmatrix} I & \mathcal{D}^{\hat{K}}_{zw} & \mathcal{D}^{\hat{K}_0,\hat{v}}_{z0} & \mathcal{D}^{\hat{K}}_{zp} G^T & \mathcal{D}^{\hat{K}}_{zp} S \\ \star & D_w & -\frac{1}{2} D_w(\overline{w} + \underline{w}) & (\mathcal{D}^{\hat{K}}_{qw})^T & 0 \\ \star & \star & \overline{f}_0 + \overline{w}^T D_w \underline{w} & (\mathcal{D}^{\hat{K}_0,\hat{v}}_{q0})^T & 0 \\ \star & \star & \star & T + \mathcal{D}^{\hat{K}}_{qp} G^T + G(\mathcal{D}^{\hat{K}}_{qp})^T & \mathcal{D}^{\hat{K}}_{qp} S \\ \star & \star & \star & \star & S \end{bmatrix} \succeq 0 \tag{5.15}$$

$$\mu^i_w \ge 0, \quad \begin{bmatrix} e_j^T \mu^i_w + e_j^T (\mathcal{D}^{\hat{K}}_{fw})^T e_i & e_j^T (\mathcal{D}^{\hat{K}}_{qw})^T + \frac{1}{2} e_i^T \mathcal{D}^{\hat{K}}_{fp} G^T_{ij} & \frac{1}{2} e_i^T \mathcal{D}^{\hat{K}}_{fp} S_{ij} \\ \star & T_{ij} + \mathcal{D}^{\hat{K}}_{qp} G^T_{ij} + G_{ij} (\mathcal{D}^{\hat{K}}_{qp})^T & \mathcal{D}^{\hat{K}}_{qp} S_{ij} \\ \star & \star & S_{ij} \end{bmatrix} \succeq 0 \tag{5.16}$$

$$\begin{bmatrix} e_i^T (\overline{f} - \mathcal{D}^{\hat{K}_0,\hat{v}}_{f0} - \mathcal{D}^{\hat{K}}_{fw} \overline{w}) + (\underline{w} - \overline{w})^T \mu^i_w & -(\mathcal{D}^{\hat{K}_0,\hat{v}}_{q0} + \mathcal{D}^{\hat{K}}_{qw} \overline{w})^T + \frac{1}{2} e_i^T \mathcal{D}^{\hat{K}}_{fp} G^T_i & \frac{1}{2} e_i^T \mathcal{D}^{\hat{K}}_{fp} S_i \\ \star & T_i + \mathcal{D}^{\hat{K}}_{qp} G^T_i + G_i (\mathcal{D}^{\hat{K}}_{qp})^T & \mathcal{D}^{\hat{K}}_{qp} S_i \\ \star & \star & S_i \end{bmatrix} \succeq 0 \tag{5.17}$$

*where $e_j$ ($e_i$) denotes the $j$th ($i$th) column of the $N_w \times N_w$ ($N_f \times N_f$) identity matrix.*

**Proof.** First, we consider the constraints. By applying Lemma 2.3, it can be shown that, for a given $\Delta$, the constraint in (5.14) is satisfied if and only if there exist $\hat{\mu}^i_w, \mu^i_w \in \mathbb{R}^{N_w}$ such that, $\forall i \in \mathcal{N}_f$,

$$\hat{\mu}^i_w \ge 0, \quad \mu^i_w \ge 0, \quad \hat{\mu}^i_w = \mu^i_w + (\mathcal{D}^{\hat{K},\Delta}_{fw})^T e_i, \quad \overline{w}^T \hat{\mu}^i_w - \underline{w}^T \mu^i_w - e_i^T (\overline{f} - \mathcal{D}^{\hat{K}_0,\hat{K},\hat{v},\Delta}_{f0}) \le 0.$$

By eliminating $\hat{\mu}_w^i$ from above, we obtain the following equivalent conditions:

$$\mu_w^i \geq 0, \quad \mu_w^i + (\mathcal{D}_{fw}^{\hat{K},\Delta})^T e_i \geq 0, \quad \overline{w}^T(\mathcal{D}_{fw}^{\hat{K},\Delta})^T e_i + (\overline{w} - \underline{w})^T \mu_w^i - e_i^T(\overline{f} - \mathcal{D}_{f0}^{\hat{K}_0,\hat{K},\hat{v},\Delta}) \leq 0 \quad (5.18)$$

Now the second inequality in (5.18) can be rearranged into the form:

$$R_{ij} + F_i\Delta(I - H\Delta)^{-1}E_j + E_j^T(I - \Delta^T H^T)^{-1}\Delta^T F_i^T \succ 0 \ \ , \ \forall j \in \mathcal{N}_w, \ \forall i \in \mathcal{N}_f, \quad (5.19)$$

with

$$\begin{bmatrix} R_{ij} & F_i \\ E_j & H \end{bmatrix} := \begin{bmatrix} e_j^T \mu_w^i + e_j^T(\mathcal{D}_{fw}^{\hat{K}})^T e_i & \frac{1}{2} e_i^T \mathcal{D}_{fp}^{\hat{K}} \\ \mathcal{D}_{qw}^{\hat{K}} e_j & \mathcal{D}_{qp}^{\hat{K}} \end{bmatrix}.$$

Finally, applying Lemma 5.1 on (5.19) yields the inequalities in (5.16). Analogously, it is easy to verify that the third inequality in (5.18) can be re-arranged into the form:

$$R_i + F_i\Delta(I - H\Delta)^{-1}E + E^T(I - \Delta^T H^T)^{-1}\Delta^T F_i^T \succ 0 \ \ , \ \forall i \in \mathcal{N}_f, \quad (5.20)$$

with $F_i$ and $H$ are defined below (5.19) and

$$\begin{bmatrix} R_i \\ E \end{bmatrix} := \begin{bmatrix} e_i^T(\overline{f} - \mathcal{D}_{f0}^{\hat{K}_0,\hat{v}} - \mathcal{D}_{fw}^{\hat{K}}\overline{w}) + (\underline{w} - \overline{w})^T \mu_w^i \\ -(\mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} + \mathcal{D}_{qw}^{\hat{K}}\overline{w}) \end{bmatrix}$$

Using Lemma 5.1 on (5.20) yields (5.17).

Next, we consider the cost function. Using the S-procedure, it can be shown that

$$f_0(\hat{K}_0, \hat{K}, \hat{v}, w, \Delta) - \overline{f}_0 = -(\overline{w} - w)^T D_w (w - \underline{w}) - \begin{bmatrix} w^T & 1 \end{bmatrix} \mathcal{L}_0(D_w, \overline{f}_0, \hat{K}_0, \hat{K}, \hat{v}, \Delta) \begin{bmatrix} w \\ 1 \end{bmatrix} \quad (5.21)$$

where $D_w \in \mathbb{D}_+^{N_w}$ and

$$\mathcal{L}_0(D_w, \overline{f}_0, \hat{K}_0, \hat{K}, \hat{v}, \Delta) = \begin{bmatrix} D_w & -\frac{1}{2} D_w(\overline{w} + \underline{w}) \\ \star & \overline{f}_0 + \overline{w}^T D_w \underline{w} \end{bmatrix} - \begin{bmatrix} \star \\ \star \end{bmatrix} \begin{bmatrix} \mathcal{D}_{zw}^{\hat{K},\Delta} & \mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \end{bmatrix}$$

Therefore, we have

$$D_w \succeq 0, \ \mathcal{L}_0(D_w, \overline{f}_0, \hat{K}_0, \hat{K}, \hat{v}, \Delta) \succeq 0 \Rightarrow f_0(\hat{K}_0, \hat{K}, \hat{v}, w, \Delta) \leq \overline{f}_0 \ , \quad \forall w \in \boldsymbol{\mathcal{W}}. \quad (5.22)$$

By effecting a Schur complement, $\mathcal{L}_0(D_w, \overline{f}_0, \hat{K}_0, \hat{K}, \hat{v}, \Delta) \succeq 0$ if and only if

$$\hat{\mathcal{L}}_0(D_w, \overline{f}_0, \hat{K}_0, \hat{K}, \hat{v}, \Delta) := \begin{bmatrix} I & \mathcal{D}_{zw}^{\hat{K},\Delta} & \mathcal{D}_{z0}^{\hat{K}_0,\hat{K},\hat{v},\Delta} \\ \star & D_w & -\frac{1}{2}D_w(\overline{w}+\underline{w}) \\ \star & \star & \overline{f}_0 + \overline{w}^T D_w \underline{w} \end{bmatrix} \succeq 0.$$

The matrix inequality $\hat{\mathcal{L}}_0(D_w, \overline{f}_0, \hat{K}_0, \hat{K}, \hat{v}, \Delta) \succeq 0$ can be re-arranged in the form

$$R_0 + F_0\Delta(I - H\Delta)^{-1}E_0 + E_0^T(I - \Delta^T H^T)^{-1}\Delta^T F_0^T \succ 0, \tag{5.23}$$

$$\left[\begin{array}{c|c} R_0 & F_0 \\ \hline E_0 & H \end{array}\right] := \left[\begin{array}{ccc|c} I & \mathcal{D}_{zw}^{\hat{K}} & \mathcal{D}_{z0}^{\hat{K}_0,\hat{v}} & \mathcal{D}_{zp}^{\hat{K}} \\ (\mathcal{D}_{zw}^{\hat{K}})^T & D_w & -\frac{1}{2}D_w(\overline{w}+\underline{w}) & 0 \\ \mathcal{D}_{z0}^{\hat{K}_0,\hat{v}} & -\frac{1}{2}(\overline{w}+\underline{w})^T D_w & \overline{f}_0 + \overline{w}^T D_w \underline{w} & 0 \\ \hline 0 & \mathcal{D}_{qw}^{\hat{K}} & \mathcal{D}_{q0}^{\hat{K}_0,\hat{v}} & \mathcal{D}_{qp}^{\hat{K}} \end{array}\right].$$

An application of Lemma 5.1 on inequality (5.23) yields (5.15). □

**Remark 5.3.** *The linear subspace* $\widehat{\boldsymbol{\Delta}}$ *can be defined in terms of* $\boldsymbol{\Delta}$ *as*

$$\begin{aligned}\widehat{\boldsymbol{\Psi}} = \; & \{(S,T,G) \in \mathbb{R}^{N_p\times N_p} \times \mathbb{R}^{N_p\times N_p} \times \mathbb{R}^{N_p\times N_p} : S = S^T \succ 0, T = T^T \succ 0, \\ & S = \{S_{ij}\}_{i,j=1}^{N+1} : S_{ij} \in \mathbb{R}^{n_p\times n_p}, T = \{T_{ij}\}_{i,j=1}^{N+1} : T_{ij} \in \mathbb{R}^{n_q\times n_q}, S_{ij}\Delta = \Delta T_{ij}, S_{ij}^T\Delta = \Delta T_{ij}^T, \\ & G = \{G_{ij}\}_{i,j=1}^{N+1} : G_{ij} \in \mathbb{R}^{n_q\times n_p}, \Delta G_{ij} + G_{ji}^T\Delta^T = 0, \; \forall \Delta \in \boldsymbol{\mathcal{B}\Delta}\}\end{aligned}$$

It follows from Theorem 5.1 that the RMPC problem can now be given by:

$$\phi = \min\{\overline{f}_0 : (5.15 - 5.17)\,\text{are satisfied for all variables defined in Theorem 5.1}\} \tag{5.24}$$

Note that the RMPC problem (5.24) is nonlinear and non-convex. In particular, conditions (5.15)-(5.17) are nonlinear in $\hat{K}$ while being linear in $\hat{K}_0$ and $\hat{v}$. Furthermore, the terms involving $\hat{K}$ are diffused throughout these matrix inequalities. This, therefore, make the RMPC problem intractable, unless $\hat{K}$ is fixed/computed offline. We now propose the first of our two approaches to remedy this non-convexity and hence transform the causal RMPC problem into an LMI optimization.

**Remark 5.4.** *Note that Lemma 5.1 introduces no gap in the case of unstructured* $\Delta$*. Moreover, the two statements in Lemma 2.3 are equivalent. Therefore, the conditions in (5.16)-(5.17) become both necessary and sufficient for systems (5.3) with unstructured uncertainties.*

**Remark 5.5.** *It is worth mentioning that problem (5.24) becomes convex when the system is subject only to disturbances (but no uncertainties). In this case,* $R_{ij} < 0$ *in (5.19) and* $R_i < 0$ *in (5.20),* $\forall i, j$*, become necessary and sufficient linear conditions for constraint (5.14). Similarly* $R_0 \succ 0$ *in (5.23) gives an upper-bound on the cost. Hence, for such systems, the RMPC control law can be computed through LMI optimizations.*

## 5.3 A Convexification Procedure for the RMPC Problem - Approach 1

The terms that include $\hat{K}$ in optimization (5.24) have the form $\hat{K}\mathcal{B}_w$ and $\hat{K}\mathcal{B}_p X$ where $X$ stands for $S_{ij},\ S_i,\ G_{ij},\ G_i,\ i \in \mathcal{N}_f,\ j \in \mathcal{N}_w$. Thus any linearization procedure must deal with the following issues:

1. As mentioned above, the fact that $\hat{K}$ occurs in the two forms $\hat{K}\mathcal{B}_w$ and $\hat{K}\mathcal{B}_p X$ is a complication. In order to deal with such nonlinearity as well as to concentrate the terms involving $\hat{K}$ in one place, we first re-cast the disturbance as an uncertainty in Section 5.3.1. This enables us to handle all the disturbances/uncertainties in a unified framework.

2. The resulting problem will still have terms of the form $\hat{K}\mathcal{B}_p X$. One way of dealing with this is to set $X = S_{ij} = S_i = S$ and $0 = G_{ij} = G_i = G,\ i \in \mathcal{N}_f,\ j \in \mathcal{N}_w$. While this has the merit of simplicity, it may be prohibitively conservative since we forego many degrees of freedom. This issue is dealt with in Section 5.3.2 where we generalize Lemma 5.1 by introducing slack variables that will allow us to keep only one term in the form $\hat{K}\mathcal{B}_p S_0$, for a free $S_0$ and for all the matrix inequalities, without excessive loss of the degrees of freedom.

3. Even with one nonlinearity in the form $\hat{K}\mathcal{B}_p S_0$, we have a double complication in that $\hat{K}$ has a lower block-triangular structure and $\mathcal{B}_p$ "separates" $\hat{K}$ and $S_0$. These issues will be dealt with in Section 5.3.3 where, at the expense of introducing minor conservatism, we propose general restrictions on the structure of $S_0$ that will allow us to treat $\bar{K}(:= \hat{K}\mathcal{B}_p S_0)$ as a decision variable of optimization, thus linearizing the problem, and at the same time allowing us to extract the desired variable $\hat{K}$ from $\bar{K}$.

### 5.3.1 Combining the Disturbance and Uncertainty Models

In this section, we embed the disturbance and uncertainty into a unified framework by re-parameterizing the disturbance set $\mathcal{W}$. Thus far, we have kept the the disturbances and model-uncertainties separate in order to emphasize the linearity of the RMPC problem (5.24) for systems with only (additive) disturbances and to exploit the tight necessary and sufficient conditions afforded by Farkas' lemma (Remark 5.5).

Note that the disturbance set in (5.4) can be written as:

$$w_k \in \mathcal{W}_k := \left\{ w = \Delta^w d_k : \Delta^w \in \boldsymbol{\Delta}^w \subseteq \mathbb{R}^{n_w \times \hat{n}_w} \right\} \tag{5.25}$$

where $\boldsymbol{\Delta}^w$ is a structured set with $\|\Delta^w\| \leq 1, \ \forall \Delta^w \in \boldsymbol{\Delta}^w$.

**Remark 5.6.** *It is worth mentioning that the disturbance structure in (5.25) can handle norm-bounded disturbances as well (by choosing $\Delta^w$ to be non-square). The structure can also readily handle non-symmetric disturbances through the introduction of an offset term, although for simplicity this is not carried out here.*

The state dynamics can now be written as:

$$x_{k+1} = Ax_k + B_u u_k + \underbrace{[B_p \;\; B_w]}_{B_p} \overbrace{\underbrace{\begin{bmatrix} \Delta & 0 \\ 0 & \Delta^w \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} q_k \\ d_k \end{bmatrix}}_{q_k}}^{p_k} \tag{5.26}$$

Using the re-definitions for $B_p$, $q_k$, $\Delta$ and $p_k$, we can re-write the stacked system dynamics as

$$\begin{bmatrix} x \\ q \\ f \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{A}x_0 & \mathcal{B}_p & \mathcal{B}_u \\ \mathcal{C}_q x_0 + d & \mathcal{D}_{qp} & \mathcal{D}_{qu} \\ \mathcal{C}_f x_0 & \mathcal{D}_{fp} & \mathcal{D}_{fu} \\ \mathcal{C}_z x_0 & \mathcal{D}_{zp} & \mathcal{D}_{zu} \end{bmatrix} \begin{bmatrix} 1 \\ p \\ u \end{bmatrix}, \quad d = \begin{bmatrix} \begin{bmatrix} 0 \\ d_0 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 0 \\ d_{N-1} \end{bmatrix} \end{bmatrix} \tag{5.27}$$

with all other matrices appropriately re-defined. Eliminating $u$ and $p$ as before gives the constraint and cost signals

$$\begin{bmatrix} f \\ z - \overline{z} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{fp}^{\hat{K}} \Delta (I - \mathcal{D}_{qp}^{\hat{K}} \Delta)^{-1} \hat{\mathcal{D}}_{q0}^{\hat{K}_0, \hat{v}} + \hat{\mathcal{D}}_{f0}^{\hat{K}_0, \hat{v}} \\ \mathcal{D}_{zp}^{\hat{K}} \Delta (I - \mathcal{D}_{qp}^{\hat{K}} \Delta)^{-1} \hat{\mathcal{D}}_{q0}^{\hat{K}_0, \hat{v}} + \hat{\mathcal{D}}_{z0}^{\hat{K}_0, \hat{v}} \end{bmatrix} =: \begin{bmatrix} \hat{\mathcal{D}}_{f0}^{\hat{K}_0, \hat{K}, \hat{v}, \Delta} \\ \hat{\mathcal{D}}_{z0}^{\hat{K}_0, \hat{K}, \hat{v}, \Delta} \end{bmatrix} \tag{5.28}$$

where

$$\begin{bmatrix} \hat{\mathcal{D}}_{q0}^{\hat{K}_0,\hat{v}} \\ \hat{\mathcal{D}}_{f0}^{\hat{K}_0,\hat{v}} \\ \hat{\mathcal{D}}_{z0}^{\hat{K}_0,\hat{v}} \end{bmatrix} = \begin{bmatrix} \mathcal{C}_q x_0 + d + \mathcal{D}_{qu}(\hat{v} + \hat{K}_0 x_0) \\ \mathcal{C}_f x_0 + \mathcal{D}_{fu}(\hat{v} + \hat{K}_0 x_0) \\ \mathcal{C}_z x_0 - \overline{z} + \mathcal{D}_{zu}(\hat{v} + \hat{K}_0 x_0) \end{bmatrix} \tag{5.29}$$

For this unified case, the conditions in (5.15)-(5.17) can now be given as follows.

**Theorem 5.2.** *With everything as defined above, $(\hat{K}_0, \hat{K}, \hat{v}) \in \boldsymbol{\mathfrak{U}}$ and $f_0(\hat{K}_0, \hat{K}, \hat{v}, \Delta) \leq \overline{f}_0$, for all $\Delta \in \mathcal{B}\widehat{\boldsymbol{\Delta}}$, if there exist solutions $(S, T, G)$, $(S_i, T_i, G_i) \in \widehat{\boldsymbol{\Psi}}$ for all $i \in \mathcal{N}_f$ to the following matrix inequalities*

$$\begin{bmatrix} e_i^T(\overline{f} - \hat{\mathcal{D}}_{f0}^{\hat{K}_0,\hat{v}}) & -(\hat{\mathcal{D}}_{q0}^{\hat{K}_0,\hat{v}})^T + \frac{1}{2}e_i^T \mathcal{D}_{fp}^{\hat{K}} G_i^T & \frac{1}{2}e_i^T \mathcal{D}_{fp}^{\hat{K}} S_i \\ \star & T_i + \mathcal{D}_{qp}^{\hat{K}} G_i^T + G_i(\mathcal{D}_{qp}^{\hat{K}})^T & \mathcal{D}_{qp}^{\hat{K}} S_i \\ \star & \star & S_i \end{bmatrix} \succeq 0 \tag{5.30}$$

$$\begin{bmatrix} I & \hat{\mathcal{D}}_{z0}^{\hat{K}_0,\hat{v}} & \mathcal{D}_{zp}^{\hat{K}} G^T & \mathcal{D}_{zp}^{\hat{K}} S \\ \star & \overline{f}_0 & (\hat{\mathcal{D}}_{q0}^{\hat{K}_0,\hat{v}})^T & 0 \\ \star & \star & T + \mathcal{D}_{qp}^{\hat{K}} G^T + G(\mathcal{D}_{qp}^{\hat{K}})^T & \mathcal{D}_{qp}^{\hat{K}} S \\ \star & \star & \star & S \end{bmatrix} \succeq 0 \tag{5.31}$$

**Proof.** The proof is similar to the one for Theorem 5.1 with the modified system and re-definitions above. □

**Remark 5.7.** *A simple procedure for linearizing the inequalities (5.30),(5.31) is to set $S = S_i = \lambda I_{N_p}$, $T = T_i = \lambda I_{N_q}$ and $G = G_i = 0$, $\forall i$, for a variable $\lambda > 0$, since $(\lambda I_{N_p}, \lambda I_{N_q}, 0) \in \widehat{\boldsymbol{\Psi}}$ for all types of uncertainty, and subsequently take $\lambda\hat{K}$ as the variable. Though this may be attractive from a computational point of view, however, the main issue is the excessive conservatism associated with such a solution which, in turn, is likely to render the problem infeasible (See also Section 5.5.1).*

### 5.3.2 An Extended S-procedure

In this section, we propose an extended version of Lemma 5.1 using an approach similar to that used in e.g. [31]. This will enable us to give equivalent necessary and sufficient conditions for (5.1) in a form that allows us to separate the terms multiplying $\hat{K}$ (thereby obtaining linearity with the introduction of only minor conservatism).

**Theorem 5.3.** *Let $R = R^T, F, E, H$ be matrices of appropriate dimensions and let $\Psi \subseteq \mathbb{R}^{p\times p}\times\mathbb{R}^{q\times q}\times\mathbb{R}^{q\times p}$ be a linear subspace. The following two statements are equivalent:*

*(i) There exist $(S,T,G) \in \Psi$ such that (5.1) is satisfied.*

*(ii) There exist $(S,T,G) \in \Psi$, $Y = Y^T \in \mathbb{R}^{q\times q}$, $S_0 \in \mathbb{R}^{p\times p}$, $G_0 \in \mathbb{R}^{q\times p}$ such that $(S_0,T,G_0) \in \Psi_0 \supseteq \Psi$ and*

$$P := \begin{bmatrix} S & -G^T \\ -G & Y \end{bmatrix} \succ 0, \quad \begin{bmatrix} R & E^T & FS_0 & -FG_0^T \\ \star & T+Y & HS_0-R_0 & -HG_0^T+Y_0 \\ \star & \star & S_0+S_0^T-S & -G_0^T-R_0^T+G^T \\ \star & \star & \star & Y_0+Y_0^T-Y \end{bmatrix} \succ 0 \quad (5.32)$$

Furthermore, if $S_0$, $G_0$, $R_0$ or $Y_0$ are constrained then $(ii) \to (i)$.

**Proof.** Note first that, for any $Y = Y^T \in \mathbb{R}^{q\times q}$, we have:

$$(5.1) \Leftrightarrow \begin{bmatrix} R & E^T \\ E & T+Y \end{bmatrix} - \begin{bmatrix} F & 0 \\ H & I \end{bmatrix} \begin{bmatrix} S & -G^T \\ -G & Y \end{bmatrix} \begin{bmatrix} F^T & H^T \\ 0 & I \end{bmatrix} \succ 0. \quad (5.33)$$

- $(ii) \to (i)$: Taking a Schur complement on (5.33) implies that

$$(5.1) \Leftrightarrow \quad \mathcal{L}_1(S,T,G,Y) := \left[\begin{array}{c|c} \begin{bmatrix} R & E^T \\ \star & T+Y \end{bmatrix} & \begin{bmatrix} F & 0 \\ H & I \end{bmatrix} \\ \hline \star & P^{-1} \end{array}\right] \succ 0. \quad (5.34)$$

  Denote the second matrix in (5.32) by $\mathcal{L}_2(S,T,G,Y,S_0,G_0,R_0,Y_0)$ and let

$$P_0 = \begin{bmatrix} S_0 & -G_0^T \\ -R_0 & Y_0 \end{bmatrix} \in \mathbb{R}^{2m\times 2m}$$

  Now the following identity can be verified:

$$P_0^T P^{-1} P_0 = P_0^T + P_0 - P + (P_0^T - P)P^{-1}(P_0 - P) \quad (5.35)$$

  Effecting the congruence transformation $Q^T\mathcal{L}_1(S,T,G,Y)Q$ with $Q = \mathrm{diag}(I, P_0)$, followed by the use of identity (5.35) shows that

$$\mathcal{L}_2(S,T,G,Y,S_0,G_0,R_0,Y_0) + \begin{bmatrix} 0 \\ I \end{bmatrix} (P_0^T - P)P^{-1}(P_0 - P) \begin{bmatrix} 0 & I \end{bmatrix} \succ 0 \Rightarrow (5.1) \quad (5.36)$$

since the last term in (5.35) is nonnegative. This implies statement $(i)$ from (5.32).

- $(i) \rightarrow (ii)$: Since $S \succ 0$, there exists $Y$ such that $P \succ 0$, e.g. we can take any $Y \succ GS^{-1}G^T$. Therefore, (5.34) is satisfied. Now let $P_0 = P$ so that $(S_0, T, G_0) \in \Psi_0$. Then, (5.32) is satisfied from (5.34).

□

**Remark 5.8.** *In a manner similar to identity (4.19), Theorem 5.3 introduces slack variables which provide extra degrees of freedom. As we show below, this will allow for a less conservative change of variables to overcome nonlinearity/non-convexity in the proposed Causal RMPC scheme. Moreover, since Theorem 5.3 is also general in nature, it has potential applications in other problem areas which use Lemma 5.1 (see e.g. [35, 94]).*

### 5.3.3 Extraction of $\hat{K}$ and Final Linearized RMPC Problem

Following the application of Theorem 5.3 on the inequalities in Theorem 5.2, there would still remain nonlinear terms of the form $\hat{K}\mathcal{B}_p S_0$ and $\hat{K}\mathcal{B}_p G_0^T$. Therefore, the question is how to restrict $S_0$ and $G_0$ so that the resulting inequalities are linear and we can extract $\hat{K}$ (since it is structured) without introducing excessive conservatism. While the best choice will depend on the detailed structure and properties of $\mathcal{B}_p$, in this section we propose a general procedure.

Recall from the proof of Theorem 5.3 that necessity $(i \rightarrow ii)$ requires setting $S_0 = S$ and $G_0 = G$. This suggests keeping the structures of $S_0$ and $G_0$ as close as possible to those of $S$ and $G$, respectively.

Using the fine structure of $\mathcal{B}_p$ and $\hat{K}$ in (5.6) and (5.8), respectively, we can write

$$\hat{K}\mathcal{B}_p = \begin{bmatrix} 0_{n_u \times (N-1)n_p} & 0_{n_u \times 2n_p} \\ \hat{K}_2 \overline{\mathcal{A}}(I_{N-1} \otimes B_p) & 0_{(N-1)n_u \times (N-1)n_p} \end{bmatrix}$$

where $\otimes$ denotes the Kronecker product and

$$\hat{K}_2 = \begin{bmatrix} \hat{K}_{1,1} & 0 & \cdots & 0 \\ \hat{K}_{2,1} & \hat{K}_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{K}_{N-1,1} & \hat{K}_{N-1,2} & \cdots & \hat{K}_{N-1,N-1} \end{bmatrix}, \quad \overline{\mathcal{A}} = \begin{bmatrix} I_n & 0 & \cdots & 0 \\ A & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-2} & A^{N-1} & \cdots & I_n \end{bmatrix}$$

This suggests the following structures for $S_0$ and $G_0$ respectively:

$$\mathbb{S}_0(\Lambda) := \begin{bmatrix} \Lambda \otimes I_{n_p} & 0_{(N-1)n_p \times 2n_p} \\ \mathbb{R}^{2n_p \times (N-1)n_p} & \mathbb{R}^{2n_p \times 2n_p} \end{bmatrix}, \tag{5.37}$$

$$\mathbb{G}_0 := \begin{bmatrix} 0_{(N-1)n_p \times (N-1)n_p} & \mathbb{R}^{(N-1)n_p \times 2n_p} \\ 0_{2n_p \times (N-1)n_p} & \mathbb{R}^{2n_p \times 2n_p} \end{bmatrix}. \tag{5.38}$$

where $\Lambda \in \mathbb{R}^{(N-1)\times(N-1)}$ is lower triangular. Note that

$$\hat{K}_2 \overline{\mathcal{A}} (I_{N-1} \otimes B_p)(\Lambda \otimes I_{n_p}) = \overbrace{\hat{K}_2 \overline{\mathcal{A}} (\Lambda \otimes I_n)}^{\tilde{K}_2} (I_{N-1} \otimes B_p). \tag{5.39}$$

The other blocks in $S_0$ (i.e. $S^0_{21}$ and $S^0_{22}$) and $G_0$ (i.e. $G^0_{12}$ and $G^0_{22}$) are free. With this choice, it can be verified that

$$\hat{K} \mathcal{B}_p S_0 = \tilde{K}(I_N \otimes B_p), \qquad \hat{K} \mathcal{B}_p G_0^T = 0, \qquad \tilde{K} = \begin{bmatrix} 0_{n_u \times (N-1)n} & 0_{n_u \times n} \\ \tilde{K}_2 & 0_{(N-1)n_u \times n} \end{bmatrix} \tag{5.40}$$

where $\tilde{K}$ has exactly the same structure as $\hat{K}$ since $\overline{\mathcal{A}}$ and $\Lambda$ are lower block triangular. Furthermore, $\hat{K}$ can be recovered from $\tilde{K}$ and $\Lambda$ using (5.39) and (5.40).

**Theorem 5.4.** *Let everything be as defined above. Then,* $f_0(\hat{K}_0, \hat{K}, \hat{v}, \Delta) \leq \overline{f}_0$ *and* $(\hat{K}_0, \hat{K}, \hat{v}) \in \boldsymbol{\mathcal{U}}$*, for all* $\Delta \in \boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}$*, if there exists a lower triangular* $\Lambda_0 \in \mathbb{R}^{(N-1)\times(N-1)}$ *and solutions* $(S, T, G)$*,* $(S_i, T_i, G_i) \in \widehat{\boldsymbol{\Psi}}$*,* $Y = Y^T$*,* $Y_i = Y_i^T$*,* $Y_0 \in \mathbb{R}^{N_q \times N_q}$*,* $S_0 \in \mathbb{S}_0(\Lambda_0)$*,* $G_0 \in \mathbb{G}_0$*,* $R_0 \in \mathbb{R}^{N_q \times N_p}$*,* $\forall i \in \mathcal{N}_f$*, to the following LMIs:*

$$\begin{bmatrix} S & -G^T \\ \star & Y \end{bmatrix} \succ 0, \quad \begin{bmatrix} S_i & -G_i^T \\ \star & Y_i \end{bmatrix} \succ 0 \tag{5.41}$$

$$\begin{bmatrix} I & \hat{\mathcal{D}}_{z0}^{\hat{K}_0,\hat{v}} & 0 & \mathcal{D}_{zp} S_0 + \mathcal{D}_{zu} \tilde{K} \overline{\mathcal{B}}_p & -\mathcal{D}_{zp} G_0^T \\ \star & \overline{f}_0 & (\hat{\mathcal{D}}_{q0}^{\hat{K}_0,\hat{v}})^T & 0 & 0 \\ \star & \star & T + Y & \mathcal{D}_{qp} S_0 + \mathcal{D}_{qu} \tilde{K} \overline{\mathcal{B}}_p - R_0 & -\mathcal{D}_{qp} G_0^T + Y_0 \\ \star & \star & \star & S_0^T + S_0 - S & -G_0^T - R_0^T + G^T \\ \star & \star & \star & \star & Y_0 + Y_0^T - Y \end{bmatrix} \succ 0 \tag{5.42}$$

$$\begin{bmatrix} e_i^T(\overline{f} - \hat{\mathcal{D}}_{f0}^{\hat{K}_0,\hat{v}}) & -(\hat{\mathcal{D}}_{q0}^{\hat{K}_0,\hat{v}})^T & \frac{1}{2}e_i^T(\mathcal{D}_{fp}S_0 + \mathcal{D}_{fu}\tilde{K}\overline{\mathcal{B}}_p) & -\frac{1}{2}e_i^T\mathcal{D}_{fp}G_{0,i}^T \\ \star & T_i + Y_i & \mathcal{D}_{qp}S_0 + \mathcal{D}_{qu}\tilde{K}\overline{\mathcal{B}}_p - R_0 & -\mathcal{D}_{qp}G_0^T + Y_0 \\ \star & \star & S_0^T + S_0 - S_i & -G_0^T - R_0^T + G_i^T \\ \star & \star & \star & Y_0 + Y_0^T - Y_i \end{bmatrix} \succ 0 \quad (5.43)$$

*where* $\hat{\mathcal{D}}_{q0}^{\hat{K}_0,\hat{v}}$, $\hat{\mathcal{D}}_{f0}^{\hat{K}_0,\hat{v}}$ *and* $\hat{\mathcal{D}}_{z0}^{\hat{K}_0,\hat{v}}$ *are defined in (5.29),* $\overline{\mathcal{B}}_p := (I_N \otimes B_p)$*, and where* $\hat{K}$ *can be recovered from* $\tilde{K}$ *and* $\Lambda$ *using (5.39) and (5.40).*

**Proof.** The LMIs (5.43) and (5.42) along with (5.41) result from the application of Theorem 5.3 on inequalities (5.30) and (5.31), respectively, and the use of the definitions given above. □

It follows that the optimal RMPC control law $(\hat{K}_0, \hat{K}, \hat{v})$ can now be computed by solving the (convex) problem of minimizing $\overline{f}_0$ subject to the LMI constraints of Theorem 5.4. Note that the conservatism introduced due to the use of $S_0$, $G_0$, $R_0$ and $Y_0$ for all the matrix inequalities is potentially much less than that introduced for the case when the same set of variables is used for all $i \in \mathcal{N}_f$, i.e. $(S_i, T_i, G_i) = (\lambda I_{N_p}, \lambda I_{N_q}, 0)$ (see Remark 5.7). This is also illustrated through a numerical example (Section 5.5).

### 5.3.4 Minimally Violating the Constraints When No Feasible Solution Exists

Suppose that no feasible solution $(\hat{K}_0, \hat{K}, \hat{v}) \in \mathcal{U}$ exists. Then, as an alternative to not supplying a control signal, we propose a simple procedure for minimally relaxing the constraints to obtain feasibility. Let $\overline{\overline{f}}$ denote upper bounds on the constraints for which no control is preferable to their breach. Denote the LMI (5.43) by $\mathcal{L}(\bar{f})$ and let

$$\hat{\mathcal{U}} = \{(\hat{K}_0, \hat{K}, \hat{v}) \in \mathcal{K}_0 \times \mathcal{K} \times v : e_i^T f(\hat{K}_0, \hat{K}, \hat{v}, w, \Delta) \le e_i^T \hat{f}, \forall i \in \mathcal{N}_f, \forall w \in \boldsymbol{\mathcal{W}}, \forall \Delta \in \boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}\} \quad (5.44)$$

Then a possible alternative is given by the following result.

**Theorem 5.5.** *With everything as defined above,* $(\hat{K}_0, \hat{K}, \hat{v}) \in \hat{\mathcal{U}}$ *for all* $\Delta \in \boldsymbol{\mathcal{B}}\widehat{\boldsymbol{\Delta}}$ *if there exists a lower triangular* $\Lambda_0 \in \mathbb{R}^{(N-1)\times(N-1)}$ *and solutions* $(S_i, T_i, G_i) \in \widehat{\boldsymbol{\Psi}}$, $Y_i = Y_i^T$, $Y_0 \in \mathbb{R}^{N_q \times N_q}$, $S_0 \in \mathbb{S}_0(\Lambda_0)$, $G_0, \in \mathbb{G}_0$, $R_0 \in \mathbb{R}^{N_q \times N_p}$, $\forall i \in \mathcal{N}_f$ *to the following matrix inequalities:*

$$\begin{bmatrix} S_i & -G_i^T \\ \star & Y_i \end{bmatrix} \succ 0, \qquad \mathcal{L}(\hat{f}) \succ 0 \quad (5.45)$$

It follows that to minimize the constraint relaxation while obtaining feasibility, we can solve the following LMI optimization problem

$$\min_{\substack{\hat{f} \leq \bar{\bar{f}} \\ \text{(5.45) are satisfied}}} \quad \|\hat{f} - \bar{f}\|^2$$

## 5.4 Causal RMPC - Approach 2

In this section, we formulate our second, computationally less demanding, solution to overcome the non-convexity in the Causal RMPC problem. This scheme can be considered as a ‘dual’ of Approach 1 (Section 5.3) in that it involves the re-parameterization of the uncertainty set as a polytopic set similar to the (additive) disturbance. It is motivated by the fact that, as discussed in Remark 5.5, the RMPC problem becomes linear and convex when the system is subject only to polytopic disturbances. It is also inspired from some of the Stochastic MPC schemes which, in the interest of tractability, compute bounds on stochastic disturbances and therefore approximate chance constraints with hard constraints (see e.g. [75] and the references therein). In particular, we propose to compute hard bounds on uncertainty which helps to convexify the RMPC problem and enables the computation of optimal $\hat{K}_0$, $\hat{K}$ and $\hat{v}$ through an LMI optimization.

Throughout this section, in the interest of clarity of exposition, we will make the following notational simplifications. Instead of $f$, we will consider the constraints on state and the input separately. Moreover, a conventional combination of state/input penalty will be considered in the cost function and, without loss of generality, only the regulation problem will be formulated [55]. Finally, we will consider the disturbance model to be uncertainty-free, i.e. $D_{qw} = 0$ (see Remark 5.10).

In order to ensure stability and recursive feasibility of the RMPC algorithm, we consider terminal state constraint $x_N \in Z$, where $Z$ is an RCI set, together with other hard constraints on the input and state. All these are summarized below.

$$x_k \in \mathcal{X}_k := \left\{ x \in \mathbb{R}^n : \underline{x}_k \leq Cx \leq \overline{x}_k \right\}, \quad \forall k \in T_I := \{1, 2, \cdots, N-1\} \tag{5.46}$$

$$x_N \in Z := \left\{ x \in \mathbb{R}^n : \underline{x}_N \leq C_N x \leq \overline{x}_N \right\} \tag{5.47}$$

$$u_k \in \mathcal{U}_k := \left\{ u \in \mathbb{R}^{n_u} : \underline{u}_k \leq u \leq \overline{u}_k \right\}, \forall k \in T_N. \tag{5.48}$$

**Remark 5.9.** *The algorithm proposed in Chapter 4 can be used to compute a low com-*

*plexity RCI set for system (5.3).*

### 5.4.1 Uncertainty Re-parameterization

We first propose to re-parameterize the norm-bounded uncertainty in the form of a polytopic set (theorem below). Subsequently, in Section 5.4.2, the re-parameterized uncertainty is combined with the disturbance and the RMPC scheme is formulated.

**Theorem 5.6.** *Let everything be as defined above and consider uncertainty* $\Delta \in \mathcal{B}\boldsymbol{\Delta}$ *where* $\boldsymbol{\Delta} := \{diag(\delta_1, \cdots, \delta_n) : \delta_i \in \mathbb{R}, |\delta_i| \leq 1\}$*. Then, the uncertainty vector* $p_k$ *- in (5.3) - is such that* $e_i^T p_k \leq e_i^T \overline{p}_k$ *for all* $i \in \mathcal{N}_p := \{1, \cdots, n_p\}$ *and* $k \in T_N$ *if and only if there exist* $\overline{D}_{xk}^i \in \mathbb{D}_+^m$*,* $\overline{D}_{uk}^i \in \mathbb{D}_+^{n_u}$ *and* $\overline{D}_{\Delta k}^i \in \mathbb{D}_+$ *such that,* $\forall i \in \mathcal{N}_p$

$$\mathcal{L}_k^i(\overline{D}_{xk}^i, \overline{D}_{uk}^i, \overline{D}_{\Delta k}^i, \overline{p}_k, e_i) :=$$

$$\begin{bmatrix} C^T \overline{D}_{xk}^i C & 0 & -\frac{1}{2} C_q^T e_i & -\frac{1}{2} C^T \overline{D}_{xk}^i (\overline{x}_k + \underline{x}_k) \\ \star & \overline{D}_{uk}^i & -\frac{1}{2} D_{qu}^T e_i & -\frac{1}{2} \overline{D}_{uk}^i (\overline{u}_k + \underline{u}_k) \\ \star & \star & \overline{D}_{\Delta k}^i & 0 \\ \star & \star & \star & e_i^T \overline{p}_k + \overline{x}_k^T \overline{D}_{xk}^i \underline{x}_k + \overline{u}_k^T \overline{D}_{uk}^i \underline{u}_k - \overline{D}_{\Delta k}^i \end{bmatrix} \succeq 0 \ , \forall k \in T_I \tag{5.49}$$

$$\mathcal{L}_N^i(\overline{D}_{xk}^i, \overline{D}_{\Delta k}^i, \overline{p}_k, e_i) :=$$

$$\begin{bmatrix} C_N^T \overline{D}_{xk}^i C_N & -\frac{1}{2} \hat{C}_q^T e_i & -\frac{1}{2} C^T \overline{D}_{xk}^i (\overline{x}_k + \underline{x}_k) \\ \star & \overline{D}_{\Delta k}^i & 0 \\ \star & \star & e_i^T \overline{p}_k + \overline{x}_k^T \overline{D}_{xk}^i \underline{x}_k - \overline{D}_{\Delta k}^i \end{bmatrix} \succeq 0, \quad k = N \tag{5.50}$$

$$\mathcal{L}_0^i(\overline{D}_{uk}^i, \overline{D}_{\Delta k}^i, \overline{p}_k, e_i) :=$$

$$\begin{bmatrix} \overline{D}_{uk}^i & -\frac{1}{2} D_{qu}^T e_i & -\frac{1}{2} \overline{D}_{uk}^i (\overline{u}_k + \underline{u}_k) \\ \star & \overline{D}_{\Delta k}^i & -\frac{1}{2} e_i^T C_q x_0 \\ \star & \star & e_i^T \overline{p}_k + \overline{u}_k^T \overline{D}_{uk}^i \underline{u}_k - \overline{D}_{\Delta k}^i \end{bmatrix} \succeq 0, \quad k = 0 \tag{5.51}$$

*where* $e_i$ *denotes the ith column of the* $n_p \times n_p$ *identity matrix.*

*Similarly,* $e_i^T p_k \geq e_i^T \underline{p}_k$, $\forall i \in \mathcal{N}_p$ *and* $k \in T_N$ *if and only if there exist* $\underline{D}^i_{xk} \in \mathbb{D}^m_+$, $\underline{D}^i_{uk} \in \mathbb{D}^{n_u}_+$, $\underline{D}^i_{\Delta k} \in \mathbb{D}_+$ *such that*

$$\mathcal{L}^i_k(\underline{D}^i_{xk}, \underline{D}^i_{uk}, \underline{D}^i_{\Delta k}, \underline{p}_k, -e_i) \succeq 0, \quad \forall k \in T_I \tag{5.52}$$

$$\mathcal{L}^i_N(\underline{D}^i_{xk}, \underline{D}^i_{\Delta k}, \underline{p}_k, -e_i) \succeq 0, \quad k = N \tag{5.53}$$

$$\mathcal{L}^i_0(\underline{D}^i_{uk}, \underline{D}^i_{\Delta k}, \underline{p}_k, -e_i) \succeq 0, \quad k = 0 \tag{5.54}$$

**Proof.** Using the definition of $p_k$ in (5.3) (with $D_{qw} = 0$) and the S-procedure, it can be shown that for all $k \in T_I$

$$\begin{aligned} e_i^T p_k = e_i^T \overline{p}_k &- (\overline{x}_k - Cx_k)^T \overline{D}^i_{xk}(Cx_k - \underline{x}_k) - (\overline{u}_k - u_k)^T \overline{D}^i_{uk}(u_k - \underline{u}_k) \\ &- (1 - \delta_i)^T \overline{D}^i_{\Delta k}(\delta_i + 1) - y_k^T \mathcal{L}^i_k(\overline{D}^i_{xk}, \overline{D}^i_{uk}, \overline{D}^i_{\Delta k}, \overline{p}^i_k) y_k \end{aligned}$$

where $y_k^T := [\; x_k^T \quad u_k^T \quad \delta_i^T \quad 1 \;]$, $\overline{D}^i_{xk} \in \mathbb{D}^n_+$, $\overline{D}^i_{uk} \in \mathbb{D}^{n_u}_+$, $\overline{D}^i_{\Delta k} \in \mathbb{D}_+$ and the matrix $\mathcal{L}^i_k(\overline{D}^i_{xk}, \overline{D}^i_{uk}, \overline{D}^i_{\Delta k}, \overline{p}^i_k, e_i)$ is given in (5.49). Thus, we have $\forall i \in \mathcal{N}_p$, $\forall k \in T_I$

$$\overline{D}^i_{xk} \succeq 0,\; \overline{D}^i_{uk} \succeq 0,\; \overline{D}^i_{\Delta k} \succeq 0,\; \mathcal{L}^i_k(\overline{D}^i_{xk}, \overline{D}^i_{uk}, \overline{D}^i_{\Delta k}, \overline{p}^i_k) \succeq 0 \Leftrightarrow e_i^T p_k \leq e_i^T \overline{p}_k$$

The LMIs (5.50) and (5.51) can analogously be derived for $k = N$ and $k = 0$ (respectively). Finally, to compute the lower bounds on uncertainty (i.e. $-e_i^T p_k \leq -e_i^T \underline{p}_k$), we repeat the above procedure with $e_i$ replaced by $-e_i$ which yields the LMIs (5.52)-(5.54). □

Now let us define the vectors

$$\overline{p} := \begin{bmatrix} \overline{p}^\star_0 \\ \overline{p}^\star_1 \\ \vdots \\ \overline{p}^\star_N \end{bmatrix}, \qquad \underline{p} := \begin{bmatrix} \underline{p}^\star_0 \\ \underline{p}^\star_1 \\ \vdots \\ \underline{p}^\star_N \end{bmatrix} \tag{5.55}$$

Then, it follows from Theorem 5.6 that the model uncertainty can be re-parameterized as:

$$p \in \mathcal{P} := \left\{ p \in \mathbb{R}^{N_p} : \underline{p} \leq p \leq \overline{p} \right\} \tag{5.56}$$

where, for each $k \in T_N$, we can compute the bounds through the following optimizations:

$$\begin{aligned} e_i^T \overline{p}^\star_k := \min\{\, e_i^T \overline{p}_k : (5.49)/(5.50)/(5.51),\ \text{is satisfied for corresponding } k, \\ \text{for a } \overline{D}^i_{xk} \in \mathbb{D}^m_+, \overline{D}^i_{uk} \in \mathbb{D}^{n_u}_+,\ \overline{D}^i_{\Delta k} \in \mathbb{D}_+, i \in \mathcal{N}_p \} \end{aligned} \tag{5.57}$$

$$-e_i^T \underline{p}_k^\star := \min\{ - e_i^T \underline{p}_k : (5.52)/(5.53)/(5.54), \text{ is satisfied for corresponding } k,$$
$$\text{for a } \underline{D}_{xk}^i \in \mathbb{D}_+^m, \underline{D}_{uk}^i \in \mathbb{D}_+^{n_u},\ \underline{D}_{\Delta k}^i \in \mathbb{D}_+, i \in \mathcal{N}_p\} \tag{5.58}$$

**Remark 5.10.** *The above formulation can be modified to cater to the case of $D_{qw} \neq 0$ by relaxing over $w_k$ (along with the relaxations for $x_k$ and $u_k$ in Theorem 5.6). Furthermore, a diagonal structure on $\boldsymbol{\Delta}$ has been assumed only for the sake of clarity of exposition. The re-parameterization can also be achieved for the case of general uncertainty given in (5.2). In particular, for full block elements, we have: $p_k^{iT} p_k^i \leq q_k^{iT} q_k^i$ where $(.)^i$ denotes the $i$th element of the vector. Hence, the polytopic bounds can be computed, in a manner similar to Theorem 5.6, by relaxing the following optimization problems:*

$$\underline{p}_k^i \ \leq \min_{x_k \in \mathcal{X}_k,\ u_k \in \mathcal{U}_k} q_k^{iT} q_k^i \ \ \leq \max_{x_k \in \mathcal{X}_k,\ u_k \in \mathcal{U}_k} q_k^{iT} q_k^i \leq\ \overline{p}_k^i \ \ , \ \ \forall i \in \mathcal{N}_p \ , \ \forall k \in \{0, 1, \cdots, N\}$$

### 5.4.2 Control Law Computation

In this subsection, we first combine the re-parameterized uncertainty with the disturbance and then derive conditions (on $\hat{K}_0$, $\hat{K}$, $\hat{v}$) for the satisfaction of constraints and minimization of the cost function.

Using (5.56), let us introduce the re-definitions:

$$B_w := [B_w \ \ B_p], \qquad \overbrace{[-d_k^T \ \ \underline{p}_k^T]}^{\underline{w}_k^T} \leq \overbrace{[w_k^T \ \ p_k^T]}^{w_k^T} \leq \overbrace{[d_k^T \ \ \overline{p}_k^T]}^{\overline{w}_k^T}.$$

Therefore, it can be verified that the stacked state-dynamics in (5.27) can now be written as:

$$x = \mathcal{A} x_0 + \mathcal{B}_u u + \mathcal{B}_w w \tag{5.59}$$

where $w \in \boldsymbol{\mathcal{W}} := \{w \in \mathbb{R}^{N_w} : \underline{w} \leq w \leq \overline{w}\}$ and all matrices/vectors are appropriately re-defined.

**Theorem 5.7.** *Define the cost function*

$$J(x_0, u, w) := x_N^T P_N x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \tag{5.60}$$

*and let $\mathcal{A}^{\hat{K}_0} := \mathcal{A} + \mathcal{B}_u \hat{K}_0, \underline{\overline{w}} := \frac{1}{2}(\overline{w} + \underline{w}), \widetilde{C} := \text{diag}(I_{N-1} \otimes C, C_N), \widetilde{R} := I_N \otimes R,\ \hat{K}_{\mathcal{B}} := (I + \mathcal{B}_u \hat{K}),\ \widetilde{Q} := \text{diag}(I_{N-1} \otimes Q, P_N),\ \overline{x} = [\overline{x}_1^T \ \cdots \ \overline{x}_N^T]^T,\ \overline{u} = [\overline{u}_0^T \ \cdots \ \overline{u}_{N-1}^T]^T$ (and analogously for $\underline{x}$, $\underline{u}$). Then, there exist feasible $\hat{K}_0$, $\hat{K}$ and $\hat{v}$ satisfying constraints*

*(5.46)-(5.48) and such that* $J(x_0,u,w)\leq\overline{f}_c$, *for all* $w\in\mathcal{W}$, *if there exist diagonal matrices* $D_w, \overline{D}^i_{wx}, \underline{D}^i_{wx}$, $i\in\mathcal{N}_x:=\{1,\cdots,mN\}$, $\overline{D}^j_{wu}, \underline{D}^j_{wu}$, $j\in\mathcal{N}_u:=\{1,\cdots,N_u\}$ *as solutions to the following LMIs*

$$\begin{bmatrix} D_w & \star & \star & \star \\ -\overline{w}^T D_w & \overline{f}_c+\overline{w}^T D_w \underline{w}-x_0^T Q x_0 & \star & \star \\ \hat{K}_{\mathcal{B}}\mathcal{B}_w & \mathcal{B}_u\hat{v}+\mathcal{A}^{\hat{K}_0}x_0 & \widetilde{Q}^{-1} & \star \\ \hat{K}\mathcal{B}_w & \hat{K}_0 x_0+\hat{v} & 0 & \widetilde{R}^{-1} \end{bmatrix}\succeq 0 \tag{5.61}$$

$$\mathcal{L}^i_x(\overline{D}^i_{wx},\hat{K},\hat{K}_0,\hat{v},\overline{x},e_i):= \begin{bmatrix} \overline{D}^i_{wx} & -\overline{D}^i_{wx}\overline{w}-\frac{1}{2}\mathcal{B}_w^T\hat{K}_{\mathcal{B}}^T\widetilde{C}^T e_i \\ \star & e_i^T(\overline{x}-\widetilde{C}\mathcal{A}^{\hat{K}_0}x_0-\widetilde{C}\mathcal{B}_u\hat{v})+\overline{w}^T\overline{D}^i_{wx}\underline{w} \end{bmatrix}\succeq 0 \tag{5.62}$$

$$\mathcal{L}^j_u(\overline{D}^j_{wu},\hat{K},\hat{K}_0,\hat{v},\overline{u},e_j):= \begin{bmatrix} \overline{D}^j_{wu} & -\overline{D}^j_{wu}\overline{w}-\frac{1}{2}\mathcal{B}_w^T\hat{K}^T e_j \\ \star & e_j^T(\overline{u}-\hat{K}_0 x_0-\hat{v})+\overline{w}^T\overline{D}^j_{wu}\underline{w} \end{bmatrix}\succeq 0, \tag{5.63}$$

$$\mathcal{L}^i_x(\underline{D}^i_{wx},\hat{K},\hat{K}_0,\hat{v},\underline{x},-e_i)\succeq 0, \tag{5.64}$$

$$\mathcal{L}^j_u(\underline{D}^j_{wu},\hat{K},\hat{K}_0,\hat{v},\underline{u},-e_j)\succeq 0. \tag{5.65}$$

**Proof.** Using (5.59) and (5.9), the upper state constraints (5.46)-(5.47) can be written as, $\forall w\in\mathcal{W}$:

$$e_i^T\widetilde{C}\hat{K}_{\mathcal{B}}\mathcal{B}_w w\leq e_i^T(\overline{x}-\widetilde{C}(\mathcal{A}+\mathcal{B}_u\hat{K}_0)x_0-\widetilde{C}\mathcal{B}_u\hat{v}).$$

Using the S-procedure, it can be shown that

$$e_i^T\widetilde{C}\hat{K}_{\mathcal{B}}\mathcal{B}_w w - e_i^T(\overline{x}-\widetilde{C}(\mathcal{A}+\mathcal{B}_u\hat{K}_0)x_0-\widetilde{C}\mathcal{B}_u\hat{v}) = \\ -(\overline{w}-w)^T\overline{D}^i_{wx}(w-\underline{w})-y^T\mathcal{L}^i_x(\overline{D}^i_{wx},\hat{K},\hat{K}_0,\hat{v},\overline{x},e_i)y$$

where $y^T:=[\ w^T \quad 1\ ]$, $\overline{D}^i_{wx}$, $\underline{D}^i_{wx}$, $\overline{D}^j_{wu}$, $\underline{D}^j_{wu}$ are diagonal, positive semidefinite matrices and the matrix $\mathcal{L}^i_x(\overline{D}^i_{wx},\hat{K},\hat{K}_0,\hat{v},\overline{x},e_i)$ is defined in (5.62). It follows that (5.62) is a necessary and sufficient condition for upper state constraints.

Similarly, through application of the S-procedure, it can be shown that (5.64), (5.63) and (5.65) are necessary and sufficient for lower state and upper/lower input constraints, respectively. Now, the cost function (5.60) can be written as:

$$J(x_0, u, w) = y^T X_c^T \widetilde{Q} X_c y + y^T U_c^T \widetilde{R} U_c y + x_0^T Q x_0 \tag{5.66}$$

where matrix $X_c := [\ \hat{K}_{\mathcal{B}} \mathcal{B}_w \quad (\mathcal{B}_u \hat{v} + (\mathcal{A} + \mathcal{B}_u \hat{K}_0) x_0)\ ]$, $U_c := [\ \hat{K} \mathcal{B}_w \quad \hat{K}_0 x_0 + \hat{v}\ ]$ and $y^T := [\, w^T \quad 1\, ]$. In a manner similar to above, using the S-procedure on (5.66) followed by a Schur complement argument yields LMI (5.61). □

It follows from Theorem 5.7 that the problem for computing an RMPC controller (i.e. $\hat{K}_0$, $\hat{K}$, $\hat{v}$) which satisfies state and input constraints and minimizes the cost function can be summarized as follows

$$\begin{aligned} \overline{\phi} = \min\{\, \overline{f}_c : & (5.61) - (5.65) \text{ are satisfied for diagonal} \\ & D_w, \overline{D}_{wx}^i, \underline{D}_{wx}^i, \overline{D}_{wu}^j, \underline{D}_{wu}^j, j \in \mathcal{N}_u, i \in \mathcal{N}_x\}. \end{aligned} \tag{5.67}$$

Note that problem (5.67) is linear in the variables $(\hat{K}_0, \hat{K}, \hat{v})$ due to the uncertainty re-parameterization of Section 5.4.1. We can now summarize the Approach 2 RMPC algorithm as follows:

***Algorithm 5.1****: Causal RMPC controller - Approach 2*

(1) Read the current state $x_t$.

(2) Compute polytopic bounds on the uncertainty through LMI problems (5.57) and (5.58).

(3) Compute $K_0$, $K$, $v$ by solving the LMI problem (5.67).

(4) Apply the first control.

(5) If the new state $x_{t+1} \in Z$, apply the terminal control law $\kappa_Z(x)$ for all time, else loop back to (1).

**Remark 5.11.** *The incorporation of an RCI terminal set helps ensure recursive feasibility of the above RMPC algorithms under certain conditions (see Remark 3.15). Stability of the proposed schemes can also be guaranteed in the same way as in Chapter 3. In particular, using the S-procedure, conditions on matrix $P_N$ can readily be derived to ensure that the terminal cost $x_N^T P_N x_N$ is a Lyapunov function over the invariant set $Z$. The formulation is very similar to that in Section 3.4.4, and is therefore not included here.*

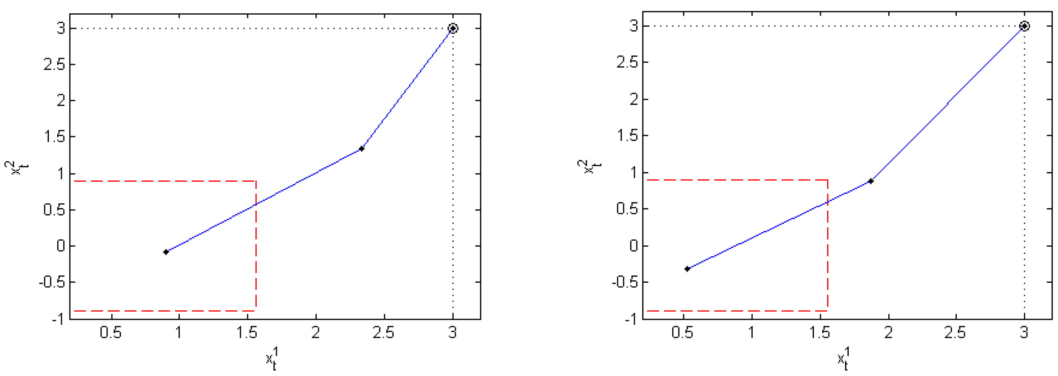


Figure 5.1: Results for Approach 1 (left) and Approach 2 with $w_t = \overline{w}cos(t)$ and $\Delta =$ diag$(1,1)$, $\forall t$

## 5.5 Numerical Examples

We now give two examples from the literature to illustrate the effectiveness of the proposed algorithms.

### 5.5.1 Example 1

We consider an uncertain version of the unstable process from [97, 100]. In particular, we have the system in (5.3) with:

$$A = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 1 \end{bmatrix}, B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, B_p = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$D_{qu} = B_u, \quad C_q = \hat{C}_q = A$$

Furthermore, system has uncertainty of the form: $\mathbf{\Delta} := \{\text{diag}(\delta_1, \delta_2) : \delta_i \in \mathbb{R}, |\delta_i| \leq 1\}$, and the disturbance set is taken to be $\mathcal{W} := \{w \in \mathbb{R}^{n_w} : -1 \leq w \leq 1\}$. The prediction horizon $N = 4$ and the parameters in the cost function (5.60) are $Q = I$, $R = I$, and $P_N = I$. The constraints on the input and state are given by: $\overline{u}_k = -\underline{u}_k = 3.8 \; \forall k$, and $\overline{x} = -\underline{x} = [3 \;\; 3]^T$, respectively. Moreover, we set the initial state $x_0 = \overline{x}$. Computing a hyper-rectangle RCI set and the corresponding controller with input constraints $-0.95 \leq u_k \leq 0.95$ and state constraints $|x_N| \leq [1.6 \;\; 1]^T$, yields (5.47) with $\overline{x}_N = -\underline{x}_N = [1.55 \;\; 0.89]^T$, $C_N = I$ along with the terminal control law $\kappa_Z(x) = -[0.34 \;\; 0.46]x$.

First of all, applying the proposed algorithm in the open-loop mode (by setting the feedback gain $K$ to zero in (5.7)) results in problem infeasibility. Moreover, the feedback RMPC problem given by Theorem 5.2, with convexity obtained using the procedure in Remark 5.7, on the above example also gives infeasibility due to the conservative nature of linearization. Now applying the two proposed schemes - as given by Theorem 5.4 and problem (5.67), respectively - give the simulation results shown in Figure 5.1. We note that even with the initial state on the constraint boundary and persistent worst-case uncertainty and disturbances, both the algorithms are able to steer the system state to RCI set (shown by red rectangle).

### 5.5.2 Example 2

We consider the coupled spring-mass system example from [55]. The mechanical system, shown in Figure 5.2, is unstable and has uncertainty in the spring constant value $k$ such that $k_{min} \leq k \leq k_{max}$. The system has four states: $x_1$ and $x_2$ are the positions of mass 1 and 2 respectively, and $x_3$ and $x_4$ are their respective velocities. The discrete-time dynamics, sampled at 0.1s, are [55]:

$$A = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1k_n & 0.1k_n & 1 & 0 \\ 0.1k_n & -0.1k_n & 0 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ 0.1 \end{bmatrix}$$

$$C_q = \begin{bmatrix} k_{dev} & -k_{dev} & 0 & 0 \end{bmatrix}, \; D_{qu} = 0$$

where $\delta = \frac{k-k_n}{k_{dev}}$, $k_n = \frac{1}{2}(k_{max} + k_{min})$, and $k_{dev} = \frac{1}{2}(k_{max} - k_{min})$. The spring constant is known to vary anywhere between $k_{min} = 0.5$ and $k_{max} = 10$. For the cost, we have $Q = 5$, $R = 1$ and prediction horizon $N = 6$.

The control objective is to make the output (state $x_2$) track a unit step whilst providing robustness against persistent variation in spring constant $k$ and respecting the input constraint: $-1 \leq u_k \leq 1$. Figure 5.3 shows the simulation results when the system is

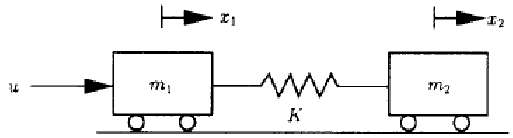


Figure 5.2: Coupled spring-mass system

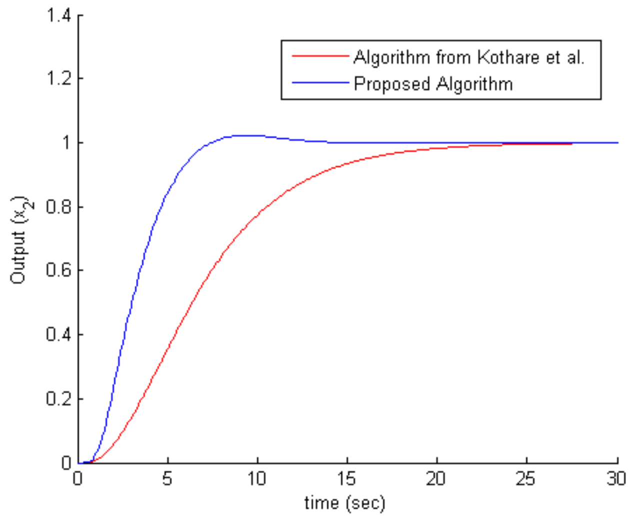


Figure 5.3: Output step-tracking results for Example 2

subjected to a sinusoidal uncertainty in the spring constant. We see that the proposed RMPC controller is able to first steer, and then maintain the system-output at the desired set-point despite the presence of a persistent uncertainty. The 5% settling time for the output, with the proposed algorithm, is approximately 6.3 sec. For comparison, Figure 5.3 also shows the response of the infinite horizon RMPC controller, proposed in [55], for the same example (red line). Although this algorithm also yields output tracking, however, the response is considerably slower with a 5% settling time of approximately 16.1 sec. Figure 5.4 also shows a comparitively faster response in control input for the proposed algorithm.

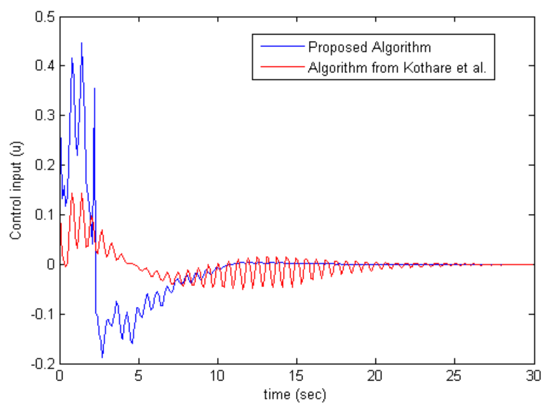


Figure 5.4: Control input for Example 2

## 5.6 Summary

In this chapter, we have presented two algorithms for the feedback Robust Model Predictive Control of linear, discrete-time systems subject to bounded disturbances, norm-bounded model-uncertainties and hard constraints on the input and state. The proposed schemes design a predictive control law - consisting of a (causal) state-feedback gain as well as a control perturbation - which is responsible for steering the uncertain system state to a terminal invariant set.

As shown in Section 5.2, despite the use of $Q$-parameterization-like methods, the RMPC problem remains nonlinear and non-convex in feedback gain $K$ due to the presence of model-uncertainty. To obtain tractability, we have proposed two methods. In the first method, the disturbance is re-cast as an uncertainty and a slack variable approach is employed which helps to remove the nonlinearity through a 'less-conservative' change of variables. In this regard, a new result to separate matrix variables $F$ and $H$ from $S$ and $G$ in inequality (5.1) has been presented (Theorem 5.3). Being general in nature, this results also has potential applications in other problem areas (see e.g. [35, 94]).

The second method involves the online re-parameterization of the uncertainty (in a manner reminiscent of a few stochastic MPC schemes) as an additive (polytopic) disturbance which subsequently leads to convexity. Both schemes allow for the online computation - through LMI problems - of an optimal control law ($K_0$,$K$,$v$) which satisfies constraints and minimizes a cost function. Moreover, the presented algorithms do not require any offline computation or initial estimates of the feedback gain $K$. Finally, the effectiveness of the proposed schemes, including the reduced conservatism of the final algorithm (Theorem 5.4) in comparison with the approach given in Remark 5.7, has been demonstrated through numerical examples from the literature.

So far we have considered control design under full-state availability. In many processes, however, only a noisy output measurement is available. Therefore, in the next chapter, we consider the output-feedback RMPC control of norm-bounded uncertain systems. In particular, we extend the results of this chapter as well as Chapter 4, to formulate a robust predictive control algorithm based only on the input/output data measurements.

# Chapter 6

# Output-feedback RMPC for Norm-bounded Uncertain Systems

## 6.1 Introduction

In this chapter, we consider the output-feedback RMPC control of constrained, linear discrete-time systems subject to norm-bounded model uncertainties, additive disturbances and measurement noise.

In the literature, most output-feedback MPC algorithms for linear systems - with additive disturbances - employ a fixed stable linear observer, such as Luenberger observer, to compute an estimate of the state which is subsequently used within the control scheme (see e.g. [87], [25], [60] [68], [44]). The (state) estimation error is generally bounded by an invariant set and is considered as an additional source of disturbance within the system. One of the major advantages of schemes such as [68] and [44] is that their online computational complexity is similar to that of (full-state) nominal MPC schemes. Output-feedback algorithms which employ observers, and are based on LMI/BMI optimization, have been proposed for systems that are subject to norm-bounded/polytopic uncertainty (see e.g. [105], [36]).

As discussed in Section 1.2.2, the choice of observer gain has a clear impact on the estimation error bounds and, therefore, on the overall control algorithm. However, in most of the aforementioned schemes, the observer is simply designed offline (to ensure stability). Moreover, the control feedback gain $K$ is also fixed. Both of these factors can potentially add to the conservatism of the corresponding predictive control algorithm.

An alternative approach to the use of observers is known as Moving Horizon Estimation (MHE) [72]. MHE techniques consider a moving - but fixed size - input/output data window, and compute state-estimates by solving an optimization problem that minimizes the difference between actual measurements and predicted outputs [86]. The fact that the estimation problem uses recent input/output data makes MHE schemes suitable for uncertain systems [1]. MHE has also recently been used in the context of output-feedback RMPC. For example, in [93], a MHE approach is combined with a tube-based RMPC scheme for system with disturbances and measurement noise. However, no such algorithms for the case of norm-bounded uncertain systems are given in the literature.

In this work, we extend the results of Chapter 5 by designing an output-feedback RMPC scheme for systems with both state/output disturbance as well as norm-bounded uncertainty. Instead of employing an observer with a fixed gain, we use the past input/output data window, in a manner similar to MHE, to compute (tight) bounds on the current state which are then used within the output-feedback control algorithm. Furthermore, to reduce conservatism, the feedback gain ($K$) and control perturbation ($v$) are both explicitly considered as decision variables in the online optimization. The associated nonlinearity is removed by using Theorem 5.3 to yield an algorithm based on LMI optimizations.

A novel feature of this algorithm is that we also extend the results of Chapter 4 to the output-feedback case. In particular, a convex problem is derived for the computation of an output-feedback RCI set, along with the corresponding control law, for norm-bounded uncertain systems. This set serves as the terminal constraint set and, under certain conditions, helps to ensure the recursive feasibility and stability of the overall control scheme.

The results in this chapter are based on the algorithm in [96].

## 6.2 Output-feedback RMPC Problem

In this section, we first provide a system description including control dynamics, constraints and cost function. Then, we derive the output-feedback RMPC problem. Note that the formulation in this section mirrors that for the state-feedback case in Sections 5.2 and 5.3.

### 6.2.1 System Description

We consider the same system as in (5.3), with the addition of an output signal $y_k$. In particular, we have

$$\begin{bmatrix} x_{k+1} \\ q_k \\ y_k \\ f_k \\ z_k \end{bmatrix} = \begin{bmatrix} A & B_u & B_w & B_p \\ C_q & D_{qu} & 0 & 0 \\ C_y & 0 & D_{yw} & 0 \\ C_f & D_{fu} & D_{fw} & D_{fp} \\ C_z & D_{zu} & D_{zw} & D_{zp} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ w_k \\ p_k \end{bmatrix}, \ p_k = \Delta q_k,$$
$$\begin{bmatrix} q_N \\ f_N \\ z_N \end{bmatrix} = \begin{bmatrix} \hat{C}_q & 0 \\ \hat{C}_f & \hat{D}_{fp} \\ \hat{C}_z & \hat{D}_{zp} \end{bmatrix} \begin{bmatrix} x_N \\ p_N \end{bmatrix}, \qquad p_N = \Delta q_N \tag{6.1}$$

with $\Delta \in \boldsymbol{\mathcal{B}\Delta}$ and $k \in \mathcal{N} := \{0, 1, \cdots, N-1\}$, where $N$ denotes the control horizon. Furthermore, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $w_k \in \mathbb{R}^{n_w}$, $f_k \in \mathbb{R}^{n_f}$, $z_k \in \mathbb{R}^{n_z}$ are the state, input, output, disturbance, constrained signal, and cost signal, respectively, at prediction step $k$. Here $p_k \in \mathbb{R}^{n_p}$ and $q_k \in \mathbb{R}^{n_q}$ represent the uncertainty vectors and all other symbols denote the appropriate distribution matrices. Only the noisy output $y_k$ is measured and we assume that the pair $(A, C_y)$ is detectable and $(A, B_u)$ stabilizable. Furthermore, bounds on the initial state are given a-priori such that (see also Section 6.3):

$$x_0 \in \mathcal{X}_0 := \left\{ x \in \mathbb{R}^n : \underline{x}_0 \leq x \leq \overline{x}_0 \right\} \tag{6.2}$$

Finally, the (unmeasured) additive disturbances belong to the set:

$$w_k \in \mathcal{W} := \left\{ w \in \mathbb{R}^{n_w} : -r \leq w \leq r \right\} \tag{6.3}$$

It is required to find $u_k$, for all $k \in \mathcal{N}$, such that the future constrained outputs satisfy $f_k \leq \bar{f}_k$, $f_N \leq \bar{f}_N$ for all $w_k \in \mathcal{W}$ and $\Delta \in \boldsymbol{\mathcal{B}\Delta}$ , and the cost function

$$J = \max_{w_k \in \mathcal{W}_k,\ \Delta \in \boldsymbol{\mathcal{B}\Delta}} \sum_{k=0}^{N} (z_k - \overline{z}_k)^T Q_k (z_k - \overline{z}_k) \tag{6.4}$$

is minimized, where $\overline{z}_k$ represents a given reference trajectory.

**Remark 6.1.** *As in Chapter 5, the terminal constraint $f_N \leq \bar{f}_N$ will involve an invariant set to help ensure the recursive feasibility and stability of the control scheme, under certain conditions. However, in this chapter, the set will be invariant under an output-feedback* *control law (see Section 6.4).*

**Remark 6.2.** *For the sake of clarity of exposition, we have combined both the state-disturbance ($\eta_k$) and measurement noise ($\nu_k$) into a single vector in (6.1), namely $w_k := [\eta_k^T \ \ \nu_k^T]^T$.*

### 6.2.2 Algebraic Formulation

Iterating the system dynamics in (6.1), we obtain

$$\begin{bmatrix} q \\ y \\ f \\ z \end{bmatrix} = \begin{bmatrix} \bar{C}_q & \bar{D}_{qw} & \bar{D}_{qp} & \bar{D}_{qu} \\ \bar{C}_y & \bar{D}_{yw} & \bar{D}_{yp} & \bar{D}_{yu} \\ \bar{C}_f & \bar{D}_{fw} & \bar{D}_{fp} & \bar{D}_{fu} \\ \bar{C}_z & \bar{D}_{zw} & \bar{D}_{zp} & \bar{D}_{zu} \end{bmatrix} \begin{bmatrix} x_0 \\ w \\ p \\ u \end{bmatrix}, \tag{6.5}$$

where $u = [u_0^T \cdots u_{N-1}^T]^T$, $w = [w_0^T \cdots w_{N-1}^T]^T$, $y = [y_0^T \cdots y_{N-1}^T]^T$, $f = [f_0^T \cdots f_N^T]^T$, $q = [q_0^T \cdots q_N^T]^T$, $p = [p_0^T \cdots p_N^T]^T$ with $p = \Delta q$, $z = [z_0^T \cdots z_N^T]^T$ and all matrices in (6.5) can easily be derived through iteration.

By defining a vector $d = [x_0^T \ \ w^T]^T$ such that

$$\begin{bmatrix} \underline{x}_0 \\ -re \end{bmatrix} =: \underline{d} \leq d \leq \overline{d} := \begin{bmatrix} \overline{x}_0 \\ re \end{bmatrix}, \tag{6.6}$$

where $e$ is a vector of ones, equation (6.5) can be written as:

$$\begin{bmatrix} q \\ y \\ f \\ z \end{bmatrix} = \begin{bmatrix} D_{qd} & \bar{D}_{qp} & \bar{D}_{qu} \\ D_{yd} & \bar{D}_{yp} & \bar{D}_{yu} \\ D_{fd} & \bar{D}_{fp} & \bar{D}_{fu} \\ D_{zd} & \bar{D}_{zp} & \bar{D}_{zu} \end{bmatrix} \begin{bmatrix} d \\ p \\ u \end{bmatrix} \tag{6.7}$$

with $D_{gd} := [\bar{C}_g \ \ \bar{D}_{gw}]$, where $g$ stands for $q$, $y$, $f$ and $z$ above.

A manipulation shows that $d$ can be re-written as:

$$d = \Delta^d \hat{d} + d_o \tag{6.8}$$

where $\Delta^d := \text{diag}(\Delta^x, \Delta^w)$ with $||\Delta^d|| \leq 1$, and

$$\hat{d} := \begin{bmatrix} \frac{1}{2}(\overline{x}_0 - \underline{x}_0) \\ re \end{bmatrix}, \quad d_o := \begin{bmatrix} \frac{1}{2}(\overline{x}_0 + \underline{x}_0) \\ 0 \end{bmatrix}$$

### 6.2.3 Output-feedback RMPC

We consider an output-feedback RMPC control law of the form:

$$u = Ky + v \tag{6.9}$$

where, to ensure causality (i.e. $u_i$ depends only on $y_j$, $j = 0, \ldots, i$), we impose that $K \in \mathcal{K}$ where $\mathcal{K}$ is the set of real lower block triangular matrices of appropriate dimensions. Substituting the equation for $y$ in (6.7) into (6.9) yields the following expression for $u$

$$u = \hat{K} D_{yd} d + \hat{K} \bar{D}_{yp} p + \bar{v} \tag{6.10}$$

where $\hat{K} := K(I - \bar{D}_{yu}K)^{-1} \in \mathcal{K}$ and $\bar{v} := (I - K\bar{D}_{yu})^{-1}v$. Note that $u$ is affine in the new variables $(\hat{K}, \bar{v})$ and $\hat{K} \in \mathcal{K}$ since $K \in \mathcal{K}$. As in Section 5.2.2, the original control variables $(K, v)$ can easily be recovered from the new variables as follows:

$$[K \quad v\,] := (I + \hat{K}\bar{D}_{yu})^{-1}[\hat{K} \quad \bar{v}\,]$$

The aim of the rest of this section is to obtain a representation of vectors $y$, $f$ and $z$ in terms of the (new) decision variables $\hat{K}$ and $\bar{v}$. To this end, by using the control structure in (6.10), we can eliminate $u$ from (6.7) to yield

$$\begin{aligned}\begin{bmatrix} y \\ q \\ f \\ z - \bar{z} \end{bmatrix} &= \left[\begin{array}{cc|c} (I+\bar{D}_{yu}\hat{K})\bar{D}_{yp} & (I+\bar{D}_{yu}\hat{K})D_{yd} & \bar{D}_{yu}\bar{v} \\ \bar{D}_{qp}+\bar{D}_{qu}\hat{K}\bar{D}_{yp} & D_{qd}+\bar{D}_{qu}\hat{K}D_{yd} & \bar{D}_{qu}\bar{v} \\ \bar{D}_{fp}+\bar{D}_{fu}\hat{K}\bar{D}_{yp} & D_{fd}+\bar{D}_{fu}\hat{K}D_{yd} & \bar{D}_{fu}\bar{v} \\ \bar{D}_{zp}+\bar{D}_{zu}\hat{K}\bar{D}_{yp} & D_{zd}+\bar{D}_{zu}\hat{K}D_{yd} & \bar{D}_{zu}\bar{v}-\bar{z} \end{array}\right] \left[\begin{array}{c} p \\ d \\ \hline 1 \end{array}\right] := \left[\begin{array}{cc|c} D^{\hat{K}}_{yp} & D^{\hat{K}}_{yd} & D^{\bar{v}}_{y} \\ D^{\hat{K}}_{qp} & D^{\hat{K}}_{qd} & D^{\bar{v}}_{q} \\ D^{\hat{K}}_{fp} & D^{\hat{K}}_{fd} & D^{\bar{v}}_{f} \\ D^{\hat{K}}_{zp} & D^{\hat{K}}_{zd} & D^{\bar{v}}_{z} \end{array}\right] \left[\begin{array}{c} p \\ d \\ \hline 1 \end{array}\right] \\ &:= \begin{bmatrix} D^{\hat{K}}_{y\hat{p}} & D^{\bar{v}}_{y} \\ D^{\hat{K}}_{q\hat{p}} & D^{\bar{v}}_{q} \\ D^{\hat{K}}_{f\hat{p}} & D^{\bar{v}}_{f} \\ D^{\hat{K}}_{z\hat{p}} & D^{\bar{v}}_{z} \end{bmatrix} \begin{bmatrix} \hat{p} \\ 1 \end{bmatrix} \end{aligned} \tag{6.11}$$

with $\hat{p} := [p^T,\ d^T]^T$ such that

$$\hat{p} = \hat{\Delta}\hat{q} + q_o \tag{6.12}$$

where, using (6.8), $\hat{\Delta} := \mathrm{diag}(\Delta, \Delta^d) \in \mathbf{\mathcal{B}}\hat{\mathbf{\Delta}}$, $q_o := [0,\ d_o^T]^T$ and

$$\hat{q} := \begin{bmatrix} q \\ \hat{d} \end{bmatrix} = \underbrace{\begin{bmatrix} D^{\hat{K}}_{qp} & D^{\hat{K}}_{qd} \\ 0 & 0 \end{bmatrix}}_{D^{\hat{K}}_{\hat{q}\hat{p}}} \hat{p} + \underbrace{\begin{bmatrix} D^{\bar{v}}_{q} \\ \hat{d} \end{bmatrix}}_{D^{\bar{v}}_{\hat{q}}} \tag{6.13}$$

For convenience, we also define

$$D_{\hat{q}\hat{p}}^{\hat{K}} = \hat{D}_q + \bar{D}_{\hat{q}u}\hat{K}\hat{C}_y \tag{6.14}$$

$$D_{f\hat{p}}^{\hat{K}} = \hat{D}_f + \bar{D}_{fu}\hat{K}\hat{C}_y \tag{6.15}$$

$$D_{z\hat{p}}^{\hat{K}} = \hat{D}_z + \bar{D}_{zu}\hat{K}\hat{C}_y \tag{6.16}$$

where $\hat{D}_f := [\bar{D}_{fp} \quad D_{fd}]$, $\hat{C}_y := [\bar{D}_{yp} \quad D_{yd}]$, $\hat{D}_z := [\bar{D}_{zp} \quad D_{zd}]$ and

$$\hat{D}_q := \begin{bmatrix} \bar{D}_{qp} & D_{qd} \\ 0 & 0 \end{bmatrix}, \quad \bar{D}_{\hat{q}u} := \begin{bmatrix} \bar{D}_{qu} \\ 0 \end{bmatrix} \tag{6.17}$$

Inserting $\hat{q}$ from (6.13) into (6.12) and simplifying yields:

$$\hat{p} = (I - \hat{\Delta}D_{\hat{q}\hat{p}}^{\hat{K}})^{-1}\hat{\Delta}(D_{\hat{q}}^{\bar{v}} + D_{\hat{q}\hat{p}}^{\hat{K}}q_o) + q_o \tag{6.18}$$

Then, using (6.18) to eliminate $\hat{p}$ from (6.11) gives

$$\begin{bmatrix} y \\ f \\ z - \bar{z} \end{bmatrix} = \begin{bmatrix} D_{y\hat{p}}^{\hat{K}}\hat{\Delta}(I - D_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}(D_{\hat{q}}^{\bar{v}} + D_{\hat{q}\hat{p}}^{\hat{K}}q_o) + D_{y\hat{p}}^{\hat{K}}q_o + D_y^{\bar{v}} \\ D_{f\hat{p}}^{\hat{K}}\hat{\Delta}(I - D_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}(D_{\hat{q}}^{\bar{v}} + D_{\hat{q}\hat{p}}^{\hat{K}}q_o) + D_{f\hat{p}}^{\hat{K}}q_o + D_f^{\bar{v}} \\ D_{z\hat{p}}^{\hat{K}}\hat{\Delta}(I - D_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}(D_{\hat{q}}^{\bar{v}} + D_{\hat{q}\hat{p}}^{\hat{K}}q_o) + D_{z\hat{p}}^{\hat{K}}q_o + D_z^{\bar{v}} \end{bmatrix} \tag{6.19}$$

Now define $\hat{v} := \bar{v} + \hat{K}D_{yd}d_o$ and let $\alpha$ denote $y$, $f$, $z$. Then, it can be verified that

$$D_{\alpha\hat{p}}^{\hat{K}}q_o + D_\alpha^{\bar{v}} = D_{\alpha d}d_o + \bar{D}_{\alpha u}\hat{v} - \bar{\alpha} := D_\alpha^{\hat{v}} \tag{6.20}$$

where the $\bar{\alpha}$ term in (6.20) is only included in the definition for $\alpha = z$, i.e. $D_z^{\hat{v}}$, and this reference trajectory $\bar{z} := [\bar{z}_0^T, \cdots, \bar{z}_N^T]^T$ is given. Furthermore, we define

$$D_{\hat{q}}^{\bar{v}} + D_{\hat{q}\hat{p}}^{\hat{K}}q_o = \begin{bmatrix} D_{qd}d_o + \bar{D}_{qu}\hat{v} \\ \hat{d} \end{bmatrix} := D_{\hat{q}}^{\hat{v}} \tag{6.21}$$

Finally, using the redefinitions in (6.20) and (6.21), we can re-write (6.19) as

$$\begin{bmatrix} y \\ f \\ z - \bar{z} \end{bmatrix} = \begin{bmatrix} D_{y\hat{p}}^{\hat{K}}\hat{\Delta}(I - D_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}D_{\hat{q}}^{\hat{v}} + D_y^{\hat{v}} \\ D_{f\hat{p}}^{\hat{K}}\hat{\Delta}(I - D_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}D_{\hat{q}}^{\hat{v}} + D_f^{\hat{v}} \\ D_{z\hat{p}}^{\hat{K}}\hat{\Delta}(I - D_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}D_{\hat{q}}^{\hat{v}} + D_z^{\hat{v}} \end{bmatrix} := \begin{bmatrix} D_y^{\hat{K},\hat{v},\hat{\Delta}} \\ D_f^{\hat{K},\hat{v},\hat{\Delta}} \\ D_z^{\hat{K},\hat{v},\hat{\Delta}} \end{bmatrix} \tag{6.22}$$

### 6.2.4 Sufficient Conditions for the Constraints and Cost

In this section, we derive sufficient conditions for the satisfaction of the constraints as well as an upper bound on the cost function. Note that using (6.22), the cost function in (6.4) can be written as

$$J(\hat{K},\hat{v},\hat{\Delta}) = (D^{\hat{K}}_{z\hat{p}}\hat{\Delta}(I-D^{\hat{K}}_{\hat{q}\hat{p}}\hat{\Delta})^{-1}D^{\hat{v}}_{\hat{q}}+D^{\hat{v}}_{z})^T Q(D^{\hat{K}}_{z\hat{p}}\hat{\Delta}(I-D^{\hat{K}}_{\hat{q}\hat{p}}\hat{\Delta})^{-1}D^{\hat{v}}_{\hat{q}}+D^{\hat{v}}_{z}) \quad (6.23)$$

where $Q := \operatorname{diag}(Q_0,\ldots,Q_N)$. Similarly, the constraint set can be written as:

$$\mathfrak{U} = \{(\hat{K},\hat{v})\colon e_i^T(D^{\hat{K}}_{f\hat{p}}\hat{\Delta}(I-D^{\hat{K}}_{\hat{q}\hat{p}}\hat{\Delta})^{-1}D^{\hat{v}}_{\hat{q}}+D^{\hat{v}}_{f}) \le e_i^T\bar{f}, \forall i\in\mathcal{N}_f, \forall\hat{\Delta}\}. \quad (6.24)$$

with $\mathcal{N}_f = \{1,\ldots,(N+1)n_f\}$.

The following theorem uses Lemma 4.1 to derive sufficient conditions for $(\hat{K},\hat{v})\in\mathfrak{U}$ (necessary and sufficient in the case of unstructured uncertainties) and an upper bound, call it $\gamma^2$, on the cost function in (6.23).

**Theorem 6.1.** *Let all variables be as defined above. Then, $J(\hat{K},\hat{v},\hat{\Delta}) \le \gamma^2$ and $(\hat{K},\hat{v}) \in \mathfrak{U}$ for all $\hat{\Delta}\in\mathcal{B}\widehat{\boldsymbol{\Delta}}$, if there exist solutions $(S,G),(S_i,G_i)\in\widehat{\boldsymbol{\Psi}}$, $\forall i\in\mathcal{N}_f$, to the following matrix inequalities*

$$\begin{bmatrix} \gamma^2 & (D^{\hat{v}}_{z})^T & (D^{\hat{v}}_{\hat{q}})^T & 0 \\ \star & Q^{-1} & D^{\hat{K}}_{z\hat{p}}G^T & D^{\hat{K}}_{z\hat{p}}S \\ \star & \star & S+D^{\hat{K}}_{\hat{q}\hat{p}}G^T+G(D^{\hat{K}}_{\hat{q}\hat{p}})^T & D^{\hat{K}}_{\hat{q}\hat{p}}S \\ \star & \star & \star & S \end{bmatrix} \succ 0 \quad (6.25)$$

$$\begin{bmatrix} e_i^T(\bar{f}-D^{\hat{v}}_{f}) & (D^{\hat{v}}_{\hat{q}})^T-\frac{1}{2}e_i^T D^{\hat{K}}_{f\hat{p}}G_i^T & -\frac{1}{2}e_i^T D^{\hat{K}}_{f\hat{p}}S_i \\ \star & S_i+D^{\hat{K}}_{\hat{q}\hat{p}}G_i^T+G_i(D^{\hat{K}}_{\hat{q}\hat{p}})^T & D^{\hat{K}}_{\hat{q}\hat{p}}S_i \\ \star & \star & S_i \end{bmatrix} \succ 0 \quad (6.26)$$

**Proof.** The constraints in (6.24) can be written as, $\forall i\in\mathcal{N}_f$,

$$e_i^T\bar{f} - e_i^T f = e_i^T\bar{f} - e_i^T D^{\hat{K}}_{f\hat{p}}\hat{\Delta}(I-D^{\hat{K}}_{\hat{q}\hat{p}}\hat{\Delta})^{-1}D^{\hat{v}}_{\hat{q}} - e_i^T D^{\hat{v}}_{f} \ge 0 \quad (6.27)$$

Through re-arrangement, (6.27) can be written in the form:

$$R_i+F_i\Delta(I-H\Delta)^{-1}E+E^T(I-\Delta^T H^T)^{-1}\Delta^T F_i^T \succ 0, \quad \forall i\in\mathcal{N}_f \quad (6.28)$$

$$\left[\begin{array}{c|c} R_i & F_i \\ \hline E & H \end{array}\right] := \left[\begin{array}{c|c} e_i^T(\bar{f} - D_f^{\hat{v}}) & -\frac{1}{2}e_i^T D_{f\hat{p}}^{\hat{K}} \\ \hline D_{\hat{q}}^{\hat{v}} & D_{\hat{q}\hat{p}}^{\hat{K}} \end{array}\right].$$

Using Lemma 4.1 on (6.28) yields the matrix inequality (6.26).

Next, we consider the cost function. Let $\gamma^2$ be an upper bound on the cost such that

$$J(\hat{K}, \hat{v}, \hat{\Delta}) = (z - \bar{z})^T Q(z - \bar{z}) \leq \gamma^2. \tag{6.29}$$

Taking the Schur complement in inequality (6.29) yields

$$\begin{bmatrix} \gamma^2 & (z - \bar{z})^T \\ \star & Q^{-1} \end{bmatrix} \succ 0 \tag{6.30}$$

Using the definitions in (6.22), it is easy to verify that (6.30) can be re-arranged into the form

$$R_0 + F_0\Delta(I - H\Delta)^{-1}E_0 + E_0^T(I - \Delta^T H^T)^{-1}\Delta^T F_0^T \succ 0, \tag{6.31}$$

$$\left[\begin{array}{c|c} R_0 & F_0 \\ \hline E_0 & H \end{array}\right] := \left[\begin{array}{cc|c} \gamma^2 & (D_z^{\hat{v}})^T & 0 \\ D_z^{\hat{v}} & Q^{-1} & D_{z\hat{p}}^{\hat{K}} \\ \hline D_{\hat{q}}^{\hat{v}} & 0 & D_{\hat{q}\hat{p}}^{\hat{K}} \end{array}\right].$$

Finally, an application of Lemma 4.1 on (6.31) yields (6.25). □

It follows from Theorem 6.1 that the output-feedback RMPC problem can be given by:

$$\begin{aligned} \bar{\phi} = \min\{\gamma^2 : (\hat{K}, \hat{v}) \in (\mathcal{K}, v), (S, G), (S_i, G_i) \in \widehat{\boldsymbol{\Psi}}, \\ i \in \mathcal{N}_f \text{ s.t. } (6.25), (6.26) \text{ are satisfied}\}. \end{aligned} \tag{6.32}$$

By considering the definitions (6.14)-(6.16), it can be verified that problem (6.32) is highly nonlinear and non-convex in $\hat{K}$ due to terms of the form $\bar{D}_{\zeta u}\hat{K}\hat{C}_y X$ where $\zeta$ stands for $f$, $\hat{q}$ and $z$ and $X$ stands for $S$, $S_i$, $G$, $G_i$, $i \in \mathcal{N}_f$. Here, note that optimization (6.32) becomes convex for a fixed $K$. However, as discussed already, this introduces a degree of conservatism depending on the offline choice of $K$. To remedy this, we now use Theorem 5.3 to convexify problem (6.32) at the expense of only minor conservatism within the formulation.

**Remark 6.3.** *When the system is subject only to additive disturbance (and no model-uncertainty), the matrix inequalities (6.25), (6.26) become linear. To see this, note that in such a case, $C_q$, $D_{qu}$ become zero and therefore, $D_{\hat{q}\hat{p}}^{\hat{K}}$ and $D_{\hat{q}}^{\hat{v}}$ no longer remain func-*

*tions of variables* $(\hat{K}, \hat{v})$. *In addition, the variables* $G$, $G_i$ *become zero since* $\Delta$ *is now purely diagonal. Then, effecting the congruence transformation* $\mathrm{diag}(I, I, S^{-1}, S^{-1})$ *on (6.25), and considering* $S^{-1}$ *as a variable, renders (6.25) linear in* $(\hat{K}, \hat{v})$. *A similar procedure can be adopted to linearize (6.26). Therefore, the output-feedback RMPC problem for systems with additive disturbances becomes convex. Furthermore, the formulation incorporates the constraints in a non-conservative manner.*

**Theorem 6.2.** *Let everything be as defined above. Then,* $J(\hat{K}, \hat{v}, \hat{\Delta}) \leq \gamma^2$ *and* $(\hat{K}, \hat{v}) \in \mathfrak{U}$ *for all* $\hat{\Delta} \in \mathfrak{B}\widehat{\boldsymbol{\Delta}}$ *if there exist solutions* $(S, G)$, $(S_i, G_i) \in \widehat{\boldsymbol{\Psi}}$, $M = M^T$, $M_i = M_i^T$, $M_0$, $S_0 \in \mathbb{S}_0$, $R_0$, $\forall i \in \mathcal{N}_f$ *to following LMIs:*

$$\begin{bmatrix} S & \star \\ -G & M \end{bmatrix} \succ 0, \quad \begin{bmatrix} S_i & \star \\ -G_i & M_i \end{bmatrix} \succ 0 \tag{6.33}$$

$$\begin{bmatrix} \gamma^2 & (D_z^{\hat{v}})^T & (D_{\hat{q}}^{\hat{v}})^T & 0 & 0 \\ \star & Q^{-1} & 0 & \hat{D}_z S_0 + \bar{D}_{zu}\bar{K} & 0 \\ \star & \star & S+M & \hat{D}_q S_0 + \hat{D}_{qu}\bar{K} - R_0 & M_0 \\ \star & \star & \star & S_0 + S_0^T - S & G^T - R_0^T \\ \star & \star & \star & \star & M_0 + M_0^T - M \end{bmatrix} \succ 0 \tag{6.34}$$

$$\begin{bmatrix} e_i^T(\overline{f} - D_f^{\hat{v}}) & (D_{\hat{q}}^{\hat{v}})^T & -\frac{1}{2}e_i^T(\hat{D}_f S_0 + \bar{D}_{fu}\bar{K}) & 0 \\ \star & S_i + M_i & \hat{D}_q S_0 + \hat{D}_{qu}\bar{K} - R_0 & M_0 \\ \star & \star & S_0 + S_0^T - S_i & G_i^T - R_0^T \\ \star & \star & \star & M_0 + M_0^T - M_i \end{bmatrix} \succ 0 \tag{6.35}$$

*where* $\bar{K} := \hat{K}\hat{C}_y S_0$.

**Proof.** The LMIs (6.33)-(6.35) follow from the application of Theorem 5.3 on (6.25) and (6.26), respectively, with $G_0 = 0$ □

It follows that the output-feedback RMPC problem can now be given by the following LMI optimization:

$$\begin{aligned} \overline{\overline{\phi}} = \min\{\, \gamma^2 : (6.33) - (6.35) \text{ are satisfied for } (\hat{K}, \hat{v}) \in (\mathcal{K}, v), \\ (S_0, G_0) \in \widehat{\boldsymbol{\Psi}}_0, (S, G), (S_i, G_i) \in \widehat{\boldsymbol{\Psi}}, i \in \mathcal{N}_f\}. \end{aligned} \tag{6.36}$$

**Remark 6.4.** *In Theorem 6.2, we have chosen* $G_0 = 0$ *purely for the sake of clarity of exposition. By studying the finer structure of* $\hat{K}$ *and* $\hat{C}_y$, *extra degrees of freedom can be* *incorporated within the structure of matrix* $G_0$ *such that* $\hat{K}\hat{C}_y G_0 = 0$ *with* $G_0 \neq 0$. *The analysis is similar to that in the state-feedback case (Section 5.3.3) and therefore we do not go into the details here.*

## 6.3 Updation of the State Bounds

The basic idea in the proposed output-feedback RMPC algorithm is to apply the first control in the sequence computed through LMI problem (6.36) and then, at the next step, obtain bounds on the current state (by considering the past input/output data in a moving window framework) before solving (6.36) again. The control scheme thus requires lower and upper bounds on the current state at time $k$, namely $\underline{x}_k$ and $\overline{x}_k$. Therefore, in this section, we formulate an optimization problem which uses the past $\tilde{N}$ inputs and outputs (as well as the current output $y_k$) to compute $\underline{x}_k$ and $\overline{x}_k$, where $\tilde{N} > 0$ denotes a given estimation horizon.

We start by iterating the process dynamics in (6.1) to obtain:

$$\begin{bmatrix} x_k \\ \tilde{q} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_u & \tilde{B}_w & \tilde{B}_p \\ \tilde{C}_q & \tilde{D}_{qu} & \tilde{D}_{qw} & \tilde{D}_{qp} \\ \tilde{C}_y & \tilde{D}_{yu} & \tilde{D}_{yw} & \tilde{D}_{yp} \end{bmatrix} \begin{bmatrix} x_{k-\tilde{N}} \\ \tilde{u} \\ \tilde{w} \\ \tilde{p} \end{bmatrix}, \tag{6.37}$$

where the input/output data vectors $\tilde{u} = [u_{k-\tilde{N}}^T \cdots u_{k-1}^T]^T$ and $\tilde{y} = [y_{k-\tilde{N}}^T \cdots y_k^T]^T$ are known, and $\tilde{w} = [w_{k-\tilde{N}}^T \cdots w_k^T]^T$, $\tilde{q} = [q_{k-\tilde{N}}^T \cdots q_{k-1}^T]^T$, $\tilde{p} = [p_{k-\tilde{N}}^T \cdots p_{k-1}^T]^T$ with $\tilde{p} = \tilde{\Delta}\tilde{q}$. All the matrices in (6.37) can also easily be computed through iteration.

Using the definition of $\tilde{q}$ in (6.37), the vector $\tilde{p}$ ($:= \tilde{\Delta}\tilde{q}$) can be re-arranged as:

$$\tilde{p} = \tilde{\Delta}(I - \tilde{D}_{qp}\tilde{\Delta})^{-1}(\tilde{C}_q x_{k-\tilde{N}} + \tilde{D}_{qu}\tilde{u} + \tilde{D}_{qw}\tilde{w}) \tag{6.38}$$

Then, using (6.38) to eliminate $\tilde{p}$ from (6.37) gives:

$$\begin{bmatrix} x_k \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A_d + \tilde{B}_p\overline{\Delta}\tilde{C}_d & \tilde{B}_u + \tilde{B}_p\overline{\Delta}\tilde{D}_{qu} \\ \tilde{C}_{yd} + \tilde{D}_{yp}\overline{\Delta}\tilde{C}_d & \tilde{D}_{yu} + \tilde{D}_{yp}\overline{\Delta}\tilde{D}_{qu} \end{bmatrix} \begin{bmatrix} d \\ \tilde{u} \end{bmatrix} \tag{6.39}$$

where $\overline{\Delta} := \tilde{\Delta}(I - \tilde{D}_{qp}\tilde{\Delta})^{-1}$, $A_d := [\tilde{A} \;\; \tilde{B}_w]$, $\tilde{C}_{yd} := [\tilde{C}_y \;\; \tilde{D}_{yw}]$, $\tilde{C}_d := [\tilde{C}_q \;\; \tilde{D}_{qw}]$ and $d := [x_{k-\tilde{N}}^T \;\; \tilde{w}^T]^T$ such that

$$\begin{bmatrix} \underline{x}_{k-\tilde{N}} \\ -\tilde{r}e \end{bmatrix} =: \underline{d} \leq d \leq \overline{d} := \begin{bmatrix} \overline{x}_{k-\tilde{N}} \\ \tilde{r}e \end{bmatrix}, \tag{6.40}$$

By using (6.39) and (6.40), we now derive (tight) upper- and lower-bounds on $x_k$ in the following theorem (see also Remark 6.5).

**Theorem 6.3.** *Let all variables be as defined above. Then, an upper-bound on the ith element of $x_k$, i.e. $e_i^T x_k$, $\forall i \in \mathbb{N}_n := \{1, \cdots, n\}$, can be computed by minimizing $e_i^T \overline{x}_k$ subject to the LMI:*

$$\begin{bmatrix} \overline{D}_x^i & -\frac{1}{2}\overline{D}_x^i(\overline{d}+\underline{d}) - \frac{1}{2}A_d^T e_i & \tilde{C}_{yd}^T & \tilde{C}_d^T & 0 \\ \star & e_i^T\overline{x}_k + \overline{d}^T\overline{D}_x^i\underline{d} - e_i^T\tilde{B}_u\tilde{u} & \tilde{u}^T\tilde{D}_{yu}^T - \tilde{y}^T & \tilde{u}^T\tilde{D}_{qu}^T - \frac{1}{2}e_i^T\tilde{B}_p\overline{G}_i^T & -\frac{1}{2}e_i^T\tilde{B}_p\overline{S}_i \\ \star & \star & \overline{Y}_i^{-1} & \tilde{D}_{yp}\overline{G}_i^T & \tilde{D}_{yp}\overline{S}_i \\ \star & \star & \star & \overline{S}_i + \tilde{D}_{qp}\overline{G}_i^T + \overline{G}_i\tilde{D}_{qp}^T & \tilde{D}_{qp}\overline{S}_i \\ \star & \star & \star & \star & \overline{S}_i \end{bmatrix} \succ 0 \tag{6.41}$$

*Similarly, a lower-bound on $e_i^T x_k$, $\forall i \in \mathbb{N}_n$, can be obtained by maximizing $e_i^T \underline{x}_k$ subject to the LMI:*

$$\begin{bmatrix} \underline{D}_x^i & -\frac{1}{2}\underline{D}_x^i(\overline{d}+\underline{d}) + \frac{1}{2}A_d^T e_i & \tilde{C}_{yd}^T & \tilde{C}_d^T & 0 \\ \star & -e_i^T\underline{x}_k + \overline{d}^T\underline{D}_x^i\underline{d} + e_i^T\tilde{B}_u\tilde{u} & \tilde{u}^T\tilde{D}_{yu}^T - \tilde{y}^T & \tilde{u}^T\tilde{D}_{qu}^T + \frac{1}{2}e_i^T\tilde{B}_p\underline{G}_i^T & \frac{1}{2}e_i^T\tilde{B}_p\underline{S}_i \\ \star & \star & \underline{Y}_i^{-1} & \tilde{D}_{yp}\underline{G}_i^T & \tilde{D}_{yp}\underline{S}_i \\ \star & \star & \star & \underline{S}_i + \tilde{D}_{qp}\underline{G}_i^T + \underline{G}_i\tilde{D}_{qp}^T & \tilde{D}_{qp}\underline{S}_i \\ \star & \star & \star & \star & \underline{S}_i \end{bmatrix} \succ 0 \tag{6.42}$$

**Proof.** In order to take account of the available past input/output data $(\tilde{u}, \tilde{y})$ in our formulation, we consider the following equality constraint, based on the expression for $\tilde{y}$ in (6.39):

$$y^{\tilde{\Delta}} - C_d^{\tilde{\Delta}} d = 0 \tag{6.43}$$

where $y^{\tilde{\Delta}} := \tilde{y} - (\tilde{D}_{yu} + \tilde{D}_{yp}\overline{\Delta}\tilde{D}_{qu})\tilde{u}$ and $C_d^{\tilde{\Delta}} := (\tilde{C}_{yd} + \tilde{D}_{yp}\overline{\Delta}\tilde{C}_d)$.

Now considering $\overline{x}_k$ as an upper-bound on $x_k$ in (6.39), with (6.43) incorporated, we require, $\forall i \in \mathbb{N}_n$,

$$e_i^T x_k - e_i^T\overline{x}_k = e_i^T(A_d^{\tilde{\Delta}} d + B_{\tilde{u}}^{\tilde{\Delta}}\tilde{u}) - e_i^T\overline{x}_k + (y^{\tilde{\Delta}} - C_d^{\tilde{\Delta}} d)^T\overline{Y}_i(y^{\tilde{\Delta}} - C_d^{\tilde{\Delta}} d) \leq 0 \tag{6.44}$$

where $A_d^{\tilde{\Delta}} := A_d + \tilde{B}_p\overline{\Delta}\tilde{C}_d$, $B_{\tilde{u}}^{\tilde{\Delta}} = \tilde{B}_u + \tilde{B}_p\overline{\Delta}\tilde{D}_{qu}$, and $\overline{Y}_i = \overline{Y}_i^T \succ 0, \forall i$.

It can then be verified that

$$e_i^T x_k - e_i^T \overline{x}_k = -(\overline{d} - d)^T \overline{D}_x^i (d - \underline{d}) - m^T \overline{\mathcal{L}}_i(\overline{Y}_i, \overline{D}_x^i, \tilde{\Delta}) m \leq 0, \ \forall i \in \mathbb{N}_n \tag{6.45}$$

where $m := [d^T \ 1]^T$ and

$$\overline{\mathcal{L}}_i(\overline{Y}_i, \overline{D}_x^i, \tilde{\Delta}) := \begin{bmatrix} \overline{D}_x^i & -\frac{1}{2}\overline{D}_x^i(\overline{d} + \underline{d}) - \frac{1}{2}(A_d^{\tilde{\Delta}})^T e_i \\ \star & e_i^T \overline{x}_k - e_i^T B_{\tilde{u}}^{\tilde{\Delta}} \tilde{u} + \overline{d}^T \overline{D}_x^i \underline{d} \end{bmatrix} - \begin{bmatrix} (C_d^{\tilde{\Delta}})^T \\ -(y^{\tilde{\Delta}})^T \end{bmatrix} \overline{Y}_i \begin{bmatrix} C_d^{\tilde{\Delta}} & -y^{\tilde{\Delta}} \end{bmatrix} \tag{6.46}$$

By using the S-procedure (Farkas' Theorem) [81], it follows that $\overline{\mathcal{L}}_i(\overline{Y}_i, \overline{D}_x^i, \tilde{\Delta}) \succ 0$, $\forall i \in \mathbb{N}_n$, is a necessary and sufficient condition for (6.44). Applying a Schur complement argument followed by a re-arrangement shows that this condition can be written as:

$$R_i + F_i \Delta (I - H\Delta)^{-1} E + E^T (I - \Delta^T H^T)^{-1} \Delta^T F_i^T \succ 0, \quad \forall i \in \mathbb{N}_n \tag{6.47}$$

$$\left[\begin{array}{c|c} R_i & F_i \\ \hline E & H \end{array}\right] := \left[\begin{array}{ccc|c} \overline{D}_x^i & -\frac{1}{2}\overline{D}_x^i(\overline{d} + \underline{d}) - \frac{1}{2}A_d^T e_i & \tilde{C}_{yd}^T & 0 \\ \star & e_i^T \overline{x}_k + \overline{d}^T \overline{D}_x^i \underline{d} - e_i^T \tilde{B}_u \tilde{u} & \tilde{u}^T \tilde{D}_{yu}^T - \tilde{y}^T & -\frac{1}{2} e_i^T \tilde{B}_p \\ \star & \star & \overline{Y}_i^{-1} & \tilde{D}_{yp} \\ \hline \tilde{C}_d & \tilde{D}_{qu} \tilde{u} & 0 & \tilde{D}_{qp} \end{array}\right].$$

Using Lemma 4.1 on (6.47) yields the LMI (6.41). A similar procedure can be used to derive LMI (6.42) for the lower-bound i.e. $-e_i^T x_k \leq -e_i^T \underline{x}_k, \forall i \in \mathbb{N}_n$. □

**Remark 6.5.** *Note that the S-procedure (Farkas' Theorem) used in Theorem 6.3 does not introduce any gap (conservatism) [81]. Therefore, the LMIs in (6.41) and (6.42) have no conservatism for systems with unstructured (norm-bounded) uncertainties and thus the computed state-bounds are tight.*

**Remark 6.6.** *For systems with only disturbances (i.e. no uncertainty), tight lower/upper bounds on $x_k$ can easily be computed through a simple Linear Program (LP) given by minimizing/maximizing $e_i^T(A_d d + \tilde{B}_u \tilde{u})$ subject to the constraints $\underline{d} \leq d \leq \overline{d}$ and $\tilde{C}_{yd} d = \tilde{y} - \tilde{D}_{yu}\tilde{u}$ (see also Section 6.6.1).*

## 6.4 Output-feedback RCI Set

The significance of RCI terminal sets, particularly in the context of RMPC, has already been discussed in the previous chapters. There exists a vast amount of literature for the computation of such sets in the case when all states are measured (see e.g. [18], [98] and the references therein). However, relatively few contributions have been made for the case when only noisy output measurements are available (see e.g. [33], [4], [48]). To the best of our knowledge, there are no algorithms in the literature for the computation of these so called output-feedback RCI sets for systems subject to both norm-bounded uncertainty and disturbances. Therefore, in this section, we extend the (state-feedback based) results of Chapter 4 to derive an algorithm for computation of low-complexity output-feedback RCI (OF-RCI) sets, along with the feedback gain $F$, for system (6.1). We focus on low-complexity invariant sets since, as highlighted in [18], they hold significant advantages over ellipsoidal and more general polytopic sets with regards to the computational complexity of the overall predictive control scheme. These sets are given by:

$$\mathcal{Z} := \Big\{x \in \mathbb{R}^n : -d \leq Cx \leq d\Big\} \tag{6.48}$$

where $C \in \mathbb{R}^{n \times n}$ is a square matrix such that $\det(C) \neq 0$. The RCI set (6.48) is required to satisfy polyhedral state and input constraints of the form:

$$x \in \mathcal{X} := \Big\{x \in \mathbb{R}^n : Tx \leq \overline{x}\Big\} \tag{6.49}$$

$$u \in \mathcal{U} := \Big\{u \in \mathbb{R}^{n_u} : Gu \leq \overline{u}\Big\} \tag{6.50}$$

with given matrices $T \in \mathbb{R}^{n_x \times n}$, $G \in \mathbb{R}^{n_c \times n_u}$ and vectors $0 < \overline{x} \in \mathbb{R}^{n_x}$, $0 < \overline{u} \in \mathbb{R}^{n_c}$.

An OF-RCI set can be defined as follows [33]:

**Definition 6.1.** *The set $\mathcal{Z} \in \mathcal{X}$ is an OF-RCI set for system (6.1) if there exists a control law $u = Fy \in \mathcal{U}$ such that:*

$$(A + B_p\mathbf{\Delta}C_q)\mathcal{Z} \oplus (B_u + B_p\mathbf{\Delta}D_{qu})FC_y\mathcal{Z} \oplus (B_u + B_p\mathbf{\Delta}D_{qu})FD_{yw}\mathcal{W} \oplus B_w\mathcal{W} \subseteq \mathcal{Z}. \tag{6.51}$$

where output $y = C_yx + D_{yw}w$ and $\oplus$ denotes the Minkowski sum.

We now propose the following theorem to compute the OF-RCI set $\mathcal{Z}$ and controller $F$ using convex optimizations.

**Theorem 6.4.** *Let us define variables $\lambda \in \mathbb{R}$, $(S_i, G_i) \in \widehat{\mathbf{\Psi}}$, $\hat{C} := \lambda C$, $\hat{F} := \lambda F$, $X_i = X_i^T \succ 0$, $M_u = M_u^T \succ 0$, $\mathcal{N}_n := \{1,\cdots,n\}$, $\mathcal{N}_{n_c} := \{1,\cdots,n_c\}$, $\mathcal{N}_{n_x} := \{1,\cdots,n_x\}$, and diagonal, positive definite matrices $D^m$, $D_u^j$, $D_w^j$, $D_x^i$, $D_w^i$. Then, given initial conditions $\lambda_o$, $\hat{C}_o$, $X_{oi}$, $M_{uo}$, $D_o^m$, $D_{uo}^j$, $D_{wo}^j$, $D_{xo}^i$ and $D_{wo}^i$ (see Remark 6.8), a maximal-volume constraint admissible OF-RCI set approximation can be computed by maximizing*

$\log(\det(M_u^{-1}))$ *subject to the LMIs*

$$\begin{bmatrix} \lambda M_{uo}\lambda_o + \lambda_o M_{uo}\lambda - \lambda_o M_{uo} M_u^{-1} M_{uo}\lambda_o & \hat{C}^T \\ \star & I \end{bmatrix} \succ 0, \tag{6.52}$$

$$\begin{bmatrix} S_i & 0 & \lambda C_q + D_{qu}\hat{F}C_y & D_{qu}\hat{F}D_{yw} & G_i B_p^T \\ \star & S_i & 0 & 0 & S_i B_p^T \\ \star & \star & L_i(3,3) & 0 & \lambda A^T + C_y^T \hat{F} B_u^T \\ \star & \star & \star & L_i(4,4) & \lambda B_w^T + D_{yw}^T \hat{F} B_u^T \\ \star & \star & \star & \star & X_i^{-1} \end{bmatrix} \succ 0, \quad \forall i \in \mathbb{N}_n, \tag{6.53}$$

$$\begin{bmatrix} \lambda X_{oi}\lambda_o + \lambda_o X_{oi}\lambda - \lambda_o X_{oi} X_i^{-1} X_{oi}\lambda_o & \hat{C}^T e_i & 0 & 0 \\ \star & 4e_i^T d & 2d^T & 2r^T \\ \star & \star & (D_x^i)^{-1} & 0 \\ \star & \star & \star & (D_w^i)^{-1} \end{bmatrix} \succ 0, \quad \forall i \in \mathbb{N}_n, \tag{6.54}$$

$$\begin{bmatrix} L_j(1,1) & 0 & -\frac{1}{2}C_y^T \hat{F}^T G^T e_{uj} & 0 & 0 \\ \star & L_j(2,2) & -\frac{1}{2}D_{yw}^T \hat{F}^T G^T e_{uj} & 0 & 0 \\ \star & \star & e_{uj}^T \overline{u} & d^T & r^T \\ \star & \star & \star & (D_u^j)^{-1} & 0 \\ \star & \star & \star & \star & (D_w^j)^{-1} \end{bmatrix} \succ 0, \quad \forall j \in \mathbb{N}_{n_c} \tag{6.55}$$

$$\begin{bmatrix} \hat{C}^T D_o^m \hat{C}_o + \hat{C}_o^T D_o^m \hat{C} - \hat{C}_o^T D_o^m (D^m)^{-1} D_o^m \hat{C}_o & -\frac{1}{2}\lambda T^T e_m \\ \star & e_m^T \overline{x} \end{bmatrix} \succ 0, \quad \forall m \in \mathbb{N}_{n_x} \tag{6.56}$$

*where*

$$\begin{aligned} L_i(3,3) &:= \hat{C}^T D_{xo}^i \hat{C}_o + \hat{C}_o^T D_{xo}^i \hat{C} - \hat{C}_o^T D_{xo}^i (D_x^i)^{-1} D_{xo}^i \hat{C}_o \\ L_i(4,4) &:= \lambda D_{wo}^i \lambda_o + \lambda_o D_{wo}^i \lambda - \lambda_o D_{wo}^i (D_w^i)^{-1} D_{wo}^i \lambda_o \\ L_j(1,1) &:= \hat{C}^T D_{uo}^j \hat{C}_o + \hat{C}_o^T D_{uo}^j \hat{C} - \hat{C}_o^T D_{uo}^j (D_u^j)^{-1} D_{uo}^j \hat{C}_o \\ L_j(2,2) &:= \lambda D_{wo}^j \lambda_o + \lambda_o D_{wo}^j \lambda - \lambda_o D_{wo}^j (D_w^j)^{-1} D_{wo}^j \lambda_o. \end{aligned}$$

**Proof.** Since $\mathcal{Z}$ and $\mathcal{W}$ are both symmetric, therefore the invariance constraint (6.51) can simply be written as:

$$e_i^T C[(A_\Delta + B_\Delta F C_y)x + (B_\Delta F D_{yw} + B_w)w] \leq e_i^T d, \quad \forall i \in \mathcal{N}_n, \ \forall x \in \mathcal{Z} \tag{6.57}$$

where $A_\Delta := A + B_p \Delta C_q$ and $B_\Delta := B_u + B_p \Delta D_{qu}$.
Now it can be verified that

$$\begin{aligned} e_i^T C[(A_\Delta + B_\Delta F C_y)x + (B_\Delta F D_{yw} + B_w)w] - e_i^T d = -(d-Cx)^T D_x^i (Cx+d) \\ - (r-w)^T D_w^i (w+r) - y^T \mathcal{L}_i(C, F, D_x^i, D_w^i, \Delta) y \end{aligned}$$

where $y^T := [x^T \ w^T \ 1]$, and $\mathcal{L}_i(C, F, D_x^i, D_w^i, \Delta) :=$

$$\begin{bmatrix} C^T D_x^i C & 0 & -\frac{1}{2}(A + B_p \Delta C_q + B_u F C_y + B_p \Delta D_{qu} F C_y)^T C^T e_i \\ \star & D_w^i & -\frac{1}{2}(B_u F D_{yw} + B_p \Delta D_{qu} F D_{yw} + B_w)^T C^T e_i \\ \star & \star & e_i^T d - d^T D_x^i d - r^T D_w^i r \end{bmatrix} \tag{6.58}$$

Using the S-procedure (Farkas' Theorem), it follows that $\mathcal{L}_i(C, F, D_x^i, D_w^i, \Delta) \succ 0$, $\forall i \in \mathcal{N}_n$, is a necessary and sufficient condition for invariance (6.51). This condition can be re-arranged into the form:

$$R_i + F_i \Delta (I - H\Delta)^{-1} E + E^T (I - \Delta^T H^T)^{-1} \Delta^T F_i^T \succ 0, \quad \forall i \in \mathcal{N}_n \tag{6.59}$$

$$\left[\begin{array}{c|c} R_i & F_i \\ \hline E & H \end{array}\right] := \left[\begin{array}{cccc} C^T D_x^i C & 0 & -\frac{1}{2}(A + B_u F C_y)^T C^T e_i & 0 \\ \star & D_w^i & -\frac{1}{2}(B_u F D_{yw} + B_w)^T C^T e_i & 0 \\ \star & \star & e_i^T d - d^T D_x^i d - r^T D_w^i r & -\frac{1}{2} e_i^T C B_p \\ \hline C_q + D_{qu} F C_y & D_{qu} F D_{yw} & 0 & 0 \end{array}\right]$$

Using Lemma 4.1 on (6.59) followed by the congruence transformation $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ yields the following invariance condition:

$$\begin{bmatrix} S_i & 0 & C_q + D_{qu} F C_y & D_{qu} F D_{yw} & -\frac{1}{2} G_i B_p^T C^T e_i \\ \star & S_i & 0 & 0 & -\frac{1}{2} S_i B_p^T C^T e_i \\ \star & \star & C^T D_x^i C & 0 & -\frac{1}{2}(A + B_u F C_y)^T C^T e_i \\ \star & \star & \star & D_w^i & -\frac{1}{2}(B_w + B_u F D_{yw})^T C^T e_i \\ \star & \star & \star & \star & e_i^T d - d^T D_x^i d - r^T D_w^i r \end{bmatrix} \succ 0, \ \forall i \in \mathcal{N}_n \tag{6.60}$$

Inequality (6.60) is highly nonlinear and non-convex due to terms of the form $C^T D^i C$

and $\frac{1}{2}e_i^T C B_z X$ where $z$ stands for $p$ or $u$ and $X$ stands for $F$, $G_i$ or $S_i$. To separate $C$ from $F$, $S_i$ and $G_i$, we first re-write (6.60) in the form:

$$R_i - A_i C^T e_i r_i^{-1} e_i^T C A_i^T \succ 0, \quad \forall i \in \mathbb{N}_n \tag{6.61}$$

where $r_i := 4(e_i^T d - d^T D_x^i d - r^T D_w^i r)$ and $\begin{bmatrix} R_i \mid A_i \end{bmatrix} :=$

$$\left[\begin{array}{cccc|c} S_i & 0 & C_q + D_{qu} F C_y & D_{qu} F D_{yw} & G_i B_p^T \\ 0 & S_i & 0 & 0 & S_i B_p^T \\ C_q^T + C_y^T F^T D_{qu}^T & 0 & C^T D_x^i C & 0 & (A + B_u F C_y)^T \\ D_{yw}^T F^T D_{qu}^T & 0 & 0 & D_w^i & (B_w + B_u F D_{yw})^T \end{array}\right]$$

Then, by using Theorem 4.2 on (6.61), it can be verified that (6.60) is satisfied if and only if

$$\begin{bmatrix} S_i & \star & \star & \star & \star \\ 0 & S_i & \star & \star & \star \\ C_q^T + C_y^T F^T D_{qu}^T & 0 & C^T D_x^i C & \star & \star \\ D_{yw}^T F^T D_{qu}^T & 0 & 0 & D_w^i & \star \\ B_p G_i^T & B_p S_i & A + B_u F C_y & B_w + B_u F D_{yw} & X_i^{-1} \end{bmatrix} \succ 0 \tag{6.62a}$$

$$\begin{bmatrix} X_i & \star \\ e_i^T C & 4(e_i^T d - d^T D_x^i d - r^T D_w^i r) \end{bmatrix} \succ 0 \tag{6.62b}$$

Now to deal with triple product terms of the form $C^T D^i C$, we propose the following identity based on slack-variables:

$$V^T M V = V^T M_o V_o + V_o^T M_o V - V_o^T M_o M^{-1} M_o V_o + (V - M^{-1} M_o V_o)^T M (V - M^{-1} M_o V_o) \tag{6.63}$$

where $M = M^T \succ 0$ and matrices $M_o$ and $V_o$ are known. It follows that $V^T M V \succ 0$ can be replaced by the first three terms on the right hand side in (6.63) without loss of generality (since the last positive term simply becomes zero by setting $V = M^{-1} M_o V_o$). Then, applying congruence transformation $\text{diag}(I, I, \lambda I, \lambda I, I)$ on (6.62a), using the re-definitions $\hat{C} := \lambda C$, $\hat{F} := \lambda F$, and subsequently using identity (6.63) in the $(3, 3)$ and $(4, 4)$ entries of the resulting matrix, respectively, yields LMI (6.53). Similarly, congruence transformation $\text{diag}(\lambda I, I)$ on the inequality in (6.62b), followed by the use of identity (6.63) on the $(1, 1)$ entry and a Schur complement argument gives LMI (6.54).

We now consider input constraint (6.50) which can be written in the form:

$$e_{uj}^T GF(C_y x + D_{yw} w) \leq e_{uj}^T \overline{u} \ , \quad \forall x \in \mathcal{Z}, \ \forall w \in \mathcal{W}, \ \forall j \in \mathcal{N}_{n_c}$$

It can be verified that, $\forall j \in \mathcal{N}_{n_c}$,

$$\begin{aligned} e_{uj}^T GF(C_y x + D_{yw} w) - e_{uj}^T \overline{u} =& - (d - Cx)^T D_u^j (Cx + d) - (r - w)^T D_w^i (w + r) \\ & + y^T \mathcal{L}_u^j (F, C, D_u^j, D_w^j) y \end{aligned}$$

where $y^T := [x^T \ w^T \ 1]$ and $\mathcal{L}_u^j(F, C, D_u^j) :=$

$$\begin{bmatrix} C^T D_u^j C & 0 & -\frac{1}{2} C_y^T F^T G^T e_{uj} \\ \star & D_w^j & -\frac{1}{2} D_{yw}^T F^T G^T e_{uj} \\ \star & \star & e_{uj}^T \overline{u} - d^T D_u^j d - r^T D_w^j r \end{bmatrix} \tag{6.64}$$

It follows from Farkas' Theorem that $\mathcal{L}_u^j(F, C, D_u^j) \succ 0$, $\forall j \in \mathcal{N}_{n_c}$, is a necessary and sufficient condition for the input constraint (6.50). Applying congruence transformation $\text{diag}(\lambda I, \lambda I, I)$ on the above condition, followed by the respective use of identity (6.63) on the $(1, 1)$ and $(2, 2)$ entries of the resulting matrix and a Schur complement argument yields LMI (6.55). Following a similar procedure to above, we can obtain LMI (6.56) for the state constraints in (6.49).

Having derived the conditions for invariance and state/input constraints, we now incorporate a cost function in the formulation. As discussed in Section 4.3.2, the volume of $\mathcal{Z}$ is inversely proportional to $|\det(C)|$. Therefore, in order to maximize the OF-RCI set volume, we need to minimize $|\det(C)|$ which is a non-convex problem. To remedy this, we consider a matrix variable $M_u = M_u^T \succ 0$ such that $\det(C)^2 \leq \det(M_u)$. In particular,

$$M_u - C^T C \succ 0 \tag{6.65}$$

Then, using a Schur complement argument on (6.65), followed by the congruence transformation $\text{diag}(\lambda I, I)$ and a subsequent use of identity (6.63) on the $(1, 1)$ entry of the resulting matrix yields the LMI (6.52). Hence, maximizing the (convex) objective $\log(\det(M_u^{-1}))$ optimizes the set-volume. □

The algorithm to compute OF-RCI set and corresponding controller $F$ can now be summarized as follows.

***Algorithm 6.1***: *Maximal volume OF-RCI set computation*

(1) Set the initial conditions $\lambda_o$, $\hat{C}_o$, $X_{oi}$, $M_{uo}$, $D_o^m$, $D_{uo}^j$, $D_{wo}^j$, $D_{xo}^i$ and $D_{wo}^i$ equal to identity in Theorem 6.4 and compute an initial OF-RCI set $(C, d)$ and controller $F$ (see Remark 6.8).

(2) Update all the initial conditions with the optimization problem solutions obtained from the previous step.

(3) Solve the convex optimization problem in Theorem 6.4 to compute OF-RCI set $(C, d)$ and controller $F$ as well as all the other matrix variables.

(4) Loop back to step (2) until there is no further improvement in the volume of $\mathcal{Z}$

**Remark 6.7.** *Similar to the formulation in Section 4.3, the identity (6.63) ensures recursive feasibility of the iterative Algorithm 6.1. In particular, by setting $V = V_o$ and $M = M_o$ in (6.63), it can be verified that previous solution of Theorem 6.4 remains feasible at the next iteration. Therefore, volume of the new invariant set $\mathcal{Z}$ is greater than or equal to that of the set from previous iteration.*

**Remark 6.8.** *The fact that $C$, $d$ and $F$ are all simultaneously considered as decision variables as well as the introduction of $\lambda$ in Theorem 6.4 helps to minimize the conservatism in Algorithm 6.1 (the formulation in Chapter 4 considers vector $d$ to be fixed). It is also worth mentioning here that instead of setting all initial conditions to identity at the beginning, a more elaborate formulation to compute the initial RCI set can be readily derived (as was the case in Chapter 4). However, for the sake of brevity, we do not include it here.*

## 6.5 Overall Output-feedback RMPC Algorithm

In this section, we describe the implementation of the algorithm and give a summary of the overall output-feedback RMPC scheme.

The proposed scheme relies on the availability of the past input/output data, at current time step $t$, to compute bounds on the state $x_t$, which are then used in output-feedback RMPC problem (6.36). However, no past data is available at the beginning (i.e. $t = 0$). In this case, we proceed as follows. We use the a-priori bounds on $x_0$ at $t = 0$ to compute $u_0$. Then, as more data comes in, estimation horizon $\tilde{N}$ is incremented until it reaches the designer prescribed value (call it $\tilde{N}_0$) - during this period all (available) past input/output data is used to compute the current state bounds. Once the desired amount of past data (corresponding to $\tilde{N}_0$) has been gathered, $\tilde{N}$ is fixed at $\tilde{N}_0$ and a moving window framework is used as discussed in Section 6.3. The overall scheme can therefore be summarized as follows.

***Algorithm 6.2****: Output-feedback RMPC scheme*

(1) Initially at $t = 0$, given $y_0$ and a-priori bounds on $x_0$, solve (6.36) to compute $u_t$.

(2) Update the vectors $\tilde{u}$, $\tilde{y}$ with the newly available input/output data from the previous step.

(3) If $\tilde{N} < \tilde{N}_0$, increment $\tilde{N}$, else fix $\tilde{N} = \tilde{N}_0$. Then, using vectors $\tilde{u}$ and $\tilde{y}$ solve the LMI/LP problem in Theorem 6.3 to compute bounds on the current state $x_t$.

(4) Using the state bounds from step (3), check if the state is inside the OF-RCI set $\mathcal{Z}$. If so, apply terminal (invariant set) control law $F$ forever. Otherwise, solve (6.36) to compute $u_t$ and loop back to step (2).

**Remark 6.9.** *Note that whilst a large value of $\tilde{N}_0$ means a more accurate computation of the state-bounds (due to greater amount of data being considered in the fixed-size moving window), it can be computationally expensive (particularly in the presence of model-uncertainty which leads to an LMI problem instead of an LP) since the estimation problem is solved online at every time step. Hence, the choice of $\tilde{N}_0$ is problem-dependent and should be made in a way so as to find a balance between the conflicting requirements of computational complexity versus state-bound accuracy.*

**Remark 6.10.** *By imposing that the terminal control law is constraint admissible, recursive feasibility of the overall output-feedback RMPC algorithm can be ensured in the usual way. In this case, it can be shown that the control sequence computed at time $t$ can be shifted and appended with the terminal control law $F$ to yield the sequence $\{u(t+1|t), \cdots, u(t+N-1|t), F\}$ which remains feasible at the next time step $t+1$. Moreover, under certain conditions, it should be possible to establish stability of the proposed scheme in a manner similar to the state-feedback case (Section 3.4.4). Though this requires careful further consideration.*

## 6.6 Numerical Examples

We now apply the proposed algorithm to two examples from the literature in order to illustrate its effectiveness.

### 6.6.1 Example 1

We first consider the double integrator example from [68]. In particular, we have

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \eta_k, \quad y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + \nu_k$$

The state-disturbance and measurement noise respectively belong to the sets:

$$\eta_k \in \mathcal{E} := \left\{ \eta \in \mathbb{R}^2 : -0.1 \le \eta \le 0.1 \right\}, \quad \nu_k \in \mathcal{V} := \left\{ \nu \in \mathbb{R} : -0.05 \le \nu \le 0.05 \right\}$$

The input constraints are given by: $-3 \le u_k \le 3$, and we consider (tightened) state constraints given by $[-12 \quad -12]^T \le x \le [3 \ \ 3]^T$. We have the cost signal $z_k := [x_k \ u_k]^T$ and cost weighting $Q_k = \mathrm{diag}(0.3, 0.3, 0.01)$, $\forall k$. The OF-RCI set and the corresponding controller, computed through a single iteration of Algorithm 6.1, are given by:

$$\mathcal{Z} := \left\{ x \in \mathbb{R}^n : \begin{bmatrix} -0.3802 \\ -0.4177 \end{bmatrix} \le \begin{bmatrix} 0.4269 & 0.0198 \\ -0.0368 & 0.2954 \end{bmatrix} x \le \begin{bmatrix} 0.3802 \\ 0.4177 \end{bmatrix} \right\}, \quad F = -0.7209 \ , \tag{6.66}$$

and this $\mathcal{Z}$ is used as the terminal constraint set. We consider the control and estimation horizons to be $N = 10$ and $\tilde{N}_0 = 3$. Finally, to remain consistent with [68] which considers $x_0 = [-3; -8]$, we set initial state bounds as: $[-3.02; -8.02] \le x_0 \le [-2.98; -7.98]$.

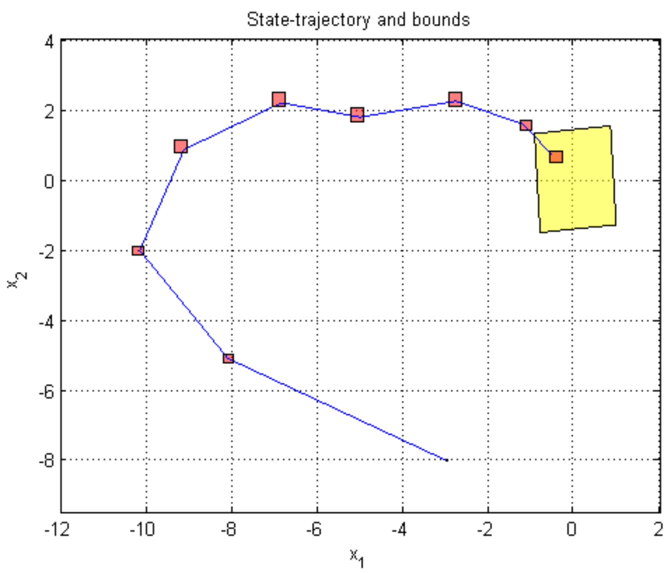


Figure 6.1: State bounds trajectory for Example 1, with $\eta_t$ and $\nu_t$ uniformly distributed

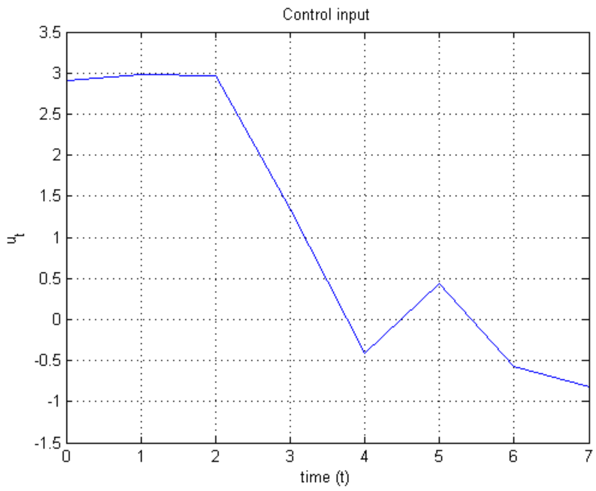


Figure 6.2: Control input for Example 1, with $\eta_t$ and $\nu_t$ uniformly distributed

Since the system is subject to disturbances only, we use the LMI conditions given in Remark 6.3. Figure 6.1 shows the simulation results with random process disturbance $\eta_t$ as well as measurement noise $\nu_t$ (both uniformly distributed). The red rectangles represent the state-bounds which, for this uncertainty-free system, are computed through the Linear Program given in Remark 6.6. Furthermore, the yellow polytope represents OF-RCI set in (6.66). The state, which of course is unavailable in the algorithm, is shown in Figure 6.1 (blue line) simply for reference purposes. We see that, despite the action of persistent disturbance and noise, the proposed output-feedback RMPC algorithm yields convergence to the RCI set - at which point the terminal control law $F$ takes over. The figure also shows that the state estimation bounds, computed using the results in Section 6.3, are accurate (they are in fact tight in this case) and produce good regulation performance. The computed control input, shown in Figure 6.2 is also constraint admissible. Note here that the control input is in fact on the constraint boundary at $t = 1$, which verifies that constraints have indeed been incorporated in the formulation in a non-conservative manner.

### 6.6.2 Example 2

We re-consider the control of a paper-making process from Chapter 3 [107]. The system, shown in Figure 6.3, consists of process states $x = [H_1 \; H_2 \; N_1 \; N_2]^T$, where $H_1$ and $N_1$ denote liquid level and composition of the feed tank, respectively, and $H_2$ and $N_2$ denote liquid level and composition of the headbox, respectively. The control input vector is

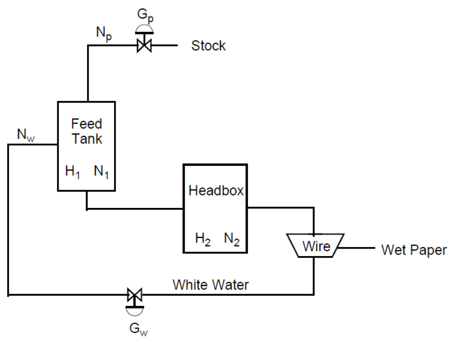


Figure 6.3: Schematic of Paper Machine Headbox

given by $u = [G_p\ G_w]^T$, where $G_p$ is the flow rate of stock entering the feed tank and $G_w$ is the recycled white water flow rate. All variables are normalized (i.e. they are zero in steady state) and only noisy measurements of $H_2$ and $N_2$ are available.

The consistency and composition of white water is a source of uncertainty within the dynamics, particularly in the state $N_1$ and input $G_w$. Moreover, disturbance $\eta_t$ affects all four states and $\nu_t$ denotes the output measurement noise. The discrete-time dynamics, sampled at 2 minutes [107], are given by (6.1) with:

$$A = \begin{bmatrix} 0.0211 & 0 & 0 & 0 \\ 0.1062 & 0.4266 & 0 & 0 \\ 0 & 0 & 0.2837 & 0 \\ 0.1012 & -0.6688 & 0.2893 & 0.4266 \end{bmatrix}, B_u = \begin{bmatrix} 0.6462 & 0.6462 \\ 0.2800 & 0.2800 \\ 1.5237 & -0.7391 \\ 0.9929 & 0.1507 \end{bmatrix}, B_w = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_q = \begin{bmatrix} 0 & 0 & 0.2 & 0 \end{bmatrix}, D_{qu} = \begin{bmatrix} 0 & 0.2 \end{bmatrix}, C_y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D_{yw} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

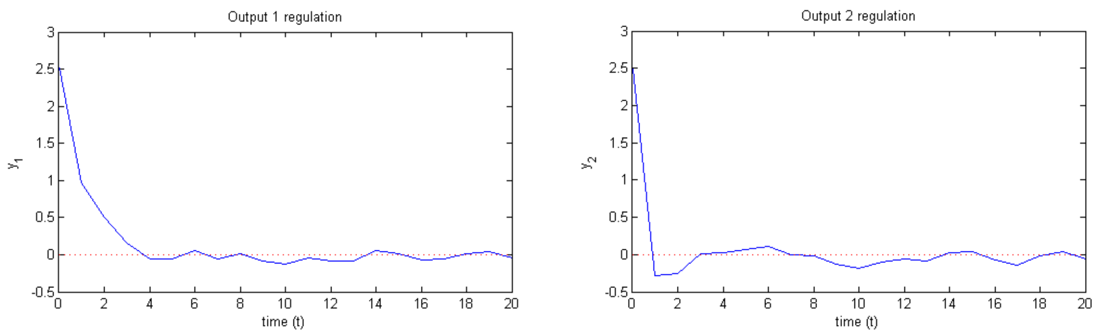


Figure 6.4: Output-regulation results for Example 2 with $\eta_t$ and $\nu_t$ uniformly distributed and worst-case uncertainty

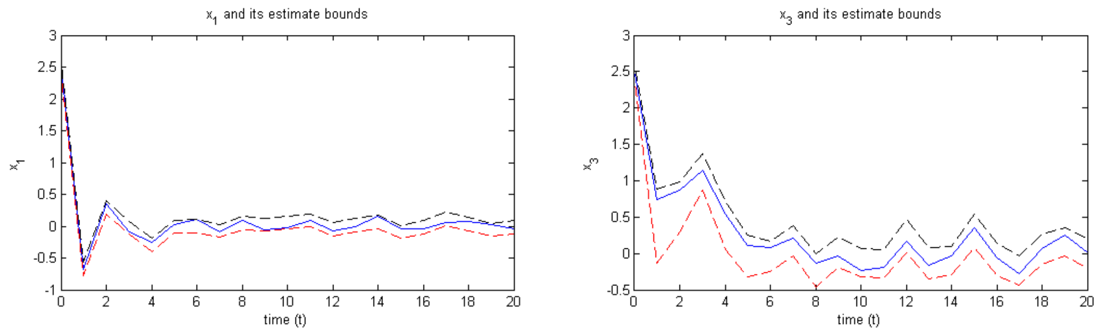


Figure 6.5: States $x_1$ and $x_3$, with the computed upper- and lower-bounds

The process disturbance and output measurement noise are respectively characterized by the sets:

$$\eta_k \in \mathcal{E} := \left\{ \eta \in \mathbb{R} : -0.1 \leq \eta \leq 0.1 \right\}, \quad \nu_k \in \mathcal{V} := \left\{ \nu \in \mathbb{R} : -0.05 \leq \nu \leq 0.05 \right\}$$

The control objective is to regulate both outputs subject to physical system constraints: $-3 \leq H_1, H_2 \leq 3$, $-5 \leq N_1, N_2 \leq 5$, and $-1.5 \leq u_1, u_2 \leq 1.5$. For the cost, we consider the parameters: $N = 3$, $\tilde{N}_0 = 4$, and $Q = 2I$.

Figures 6.4, 6.5 and 6.6 show the simulation results. From Figure 6.4, we see that the proposed algorithm is able to produce good output-regulation in both $y_1$ and $y_2$ despite persist worst-case uncertainty and randomly distributed disturbances.

Figure 6.5 shows the upper- and lower-bounds for the unmeasured states $x_1$ and $x_3$. For comparison, the actual states are also included in these figures (solid blue lines). We note that the computed bounds in fact touch the state $x_1$ at some places which verifies

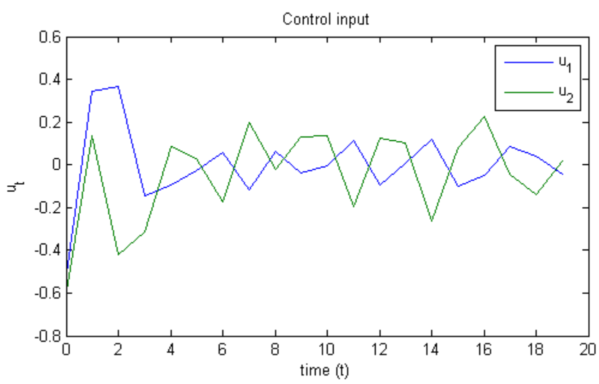


Figure 6.6: Computed control input for Example 2

their accuracy. The control inputs, shown in Figure 6.6, also satisfy the constraints and oscillate around the origin due to persistent random disturbances and uncertainty.

## 6.7 Summary

In this chapter, we have extended the results of Chapters 4 and 5 to the output-feedback case. In particular, an algorithm for the output-feedback RMPC Control of constrained, linear discrete-time systems subject to norm-bounded model-uncertainties, disturbances and measurement noise has been proposed.

The novelty lies in the fact that the algorithm computes, online, both the output-feedback gain and a control perturbation through an LMI optimization. Moreover, unlike most output-feedback MPC schemes from the literature which use a fixed (linear) state observer, the proposed algorithm uses a past input/output data window - in a manner similar to Moving Horizon Estimation - to compute (tight) bounds on the current state. These bounds are then used within the output-feedback control scheme in place of the actual (unmeasured) state.

A new algorithm to simultaneously compute, offline, an output-feedback RCI set and terminal control law has also been presented. The volume of the RCI set is enlarged iteratively through convex optimizations. Incorporation of such an RCI set as a target set helps to ensure recursive feasibility of the RMPC algorithm. Finally, the effectiveness of the proposed scheme has been demonstrated through numerical examples taken from the literature.

# Chapter 7

# Conclusions

In this chapter, we summarize the main contributions of the thesis and also suggest some future research directions.

## 7.1 Main Contributions

The focus of this research has been on the development of efficient algorithms - based on convex/LMI optimizations - for robust control of constrained, norm-bounded uncertain systems (6.1). In this regard, the main contributions of the thesis are summarized below:

**Robust Control Invariant Sets**

- An algorithm for the computation of (hyper-rectangle) RCI sets, and the corresponding control law, for linear systems subject to parametric uncertainties and disturbances has been presented in Chapter 3. The algorithm computes both the RCI set and controller $K$ in *one step* through LMI optimizations. Furthermore, for a given $K$, the invariant set can be computed through a simple linear program. It is also shown that for systems with only disturbances, the conditions in Theorem 3.1 are necessary and sufficient and hence, in this case, the optimal invariant set and $K$ can be computed *exactly.*

- Computation of low complexity RCI (LC-RCI) sets, which - as target sets - hold significant advantages for the associated RMPC schemes, has been investigated in Chapter 4. Due to the presence of (norm-bounded) model uncertainty as well as the fact that both the invariant set and $K$ are being considered as decision variables, the problem becomes non-convex with nonlinear terms of the form $C^TXC$ and $XAY$

(where $X$, $Y$ and $C$ are variables). To deal with this, we have proposed new results in Theorems 4.2 and 4.3 which separate bilinear terms in the diagonal/non-diagonal matrix entries without introducing any conservatism. Furthermore, these results being general in nature have potential applications in other problem areas such as Lyapunov stability. Application of these theorems yields an algorithm based of convex/LMI optimizations. To deal with triple product terms, identity (4.43) is proposed which also ensures set-volume optimization and recursive feasibility of the iterative algorithm. It is also shown that for uncertain systems, maximal/minimal volume hyper-rectangle RCI sets can be computed in one-step.

- A relatively new area of set-invariance under output-feedback has been studied in Chapter 6. In particular, results from Chapter 4 have been extended for the simultaneous computation of an output-feedback LC-RCI set and the corresponding control law for uncertain systems using convex optimization. To incorporate extra degrees of freedom, the algorithm in Theorem 6.4 also considers vector $d$ (in (6.48)) as a variable, whilst still retaining key algorithm properties from Chapter 4, such as recursive feasibility and volume enlargement.

**State-feedback RMPC**

- A feedback RMPC algorithm for constrained systems with parametric uncertainty and disturbances has been proposed in Chapter 3. The scheme considers both the state-feedback gain and control perturbation as decision variables in the online optimization. The non-convexity associated with such a parameterization is avoided by adopting a sequential approach based on Dynamic Programming. The RMPC controller minimizes an upper-bound on the cost-to-go at each prediction step and incorporates state/input constraints in a *non-conservative* manner. The proposed cost function includes a negatively weighted disturbance term which helps to improves robustness (as in $\mathcal{H}_\infty$ control). Furthermore, conditions for the Lyapunov stability of the closed-loop uncertain system have also been derived.
- Two novel methods to obtain convexity in the state-feedback RMPC problem for norm-bounded uncertain systems (5.3) have been proposed in Chapter 5. Unlike the sequential approach of Chapter 3, here the state-feedback gain matrix - which has a lower block triangular structure for causality - and control perturbation sequence

are computed simultaneously in one-step. It is shown through initial formulation that the problem is convex for disturbed systems but becomes nonlinear and non-convex (in $K$) in the presence of model-uncertainty. Then, in the first approach, a new result that uses slack-variables to extend Lemma 5.1 is proposed in Theorem 5.3. This result is quite general in nature and enables the RMPC control law to be computed through LMI optimizations.

- The second RMPC approach in Chapter 5 re-parameterizes, online, the norm-bounded uncertainty as a polytopic disturbance without introducing extra conservatism. This again results in a tractable RMPC scheme based on LMI optimizations. An RCI set - which can be designed using the results from Chapter 4 - is considered as a target set with the corresponding terminal control law. This helps to ensure recursive feasibility and stability of both the algorithms in the standard way.

**Output-feedback RMPC**

- An output-feedback RMPC scheme has been formulated in Chapter 6 for norm-bounded uncertain systems (6.1) for which only noisy output measurements are available. The formulation consists of first computing the current state bounds and then using this information to generate an output-feedback predictive control law. The novelty lies in the fact that unlike most schemes in the literature which employ a *fixed* linear observer, we use a moving window of the past input/output data, in a manner reminiscent of moving horizon estimation. Upper and lower-bounds on the current state are computed online through LMI optimizations, with the bounds being tight in the case of unstructured uncertainties. Moreover, it is also shown that for systems with only disturbances, tight bounds on the current state can be computed using simple linear programs.
- The current-state bounds are subsequently used to compute an output-feedback RMPC control law for system (6.1). The control formulation - which is originally nonlinear and non-convex in output-feedback gain $K$ - is rendered convex through the use of Theorem 5.3. The RMPC control law minimizes a cost function and is responsible for driving the (unmeasured) state to a designed output-feedback LC-RCI set. Finally, in the case that the current-state bounds belong to this invariant set, then the (corresponding) terminal control law is applied for all times.

## 7.2 Future Research Directions

We now outline a few potential future research directions.

- The focus of this research has been on the computation of LC-RCI sets since they hold computational advantages - as terminal sets - for the associated RMPC schemes. However, it would be useful to extend the results of Chapter 4 to more general RCI sets (i.e. considering a non-square $C$ in (4.4)). In this regard, note that conditions (4.8)-(4.10) also hold for a non-square $C$. However, the challenge is to convexify these conditions in a minimally conservative manner.

- Throughout this thesis, we have considered $\Delta$ as a norm-bounded model-uncertainty. In theory, it should be possible to extend the results to formulate fault-tolerant RMPC schemes. In that case $\Delta$ can be considered as a binary variable. So $\Delta = 0$ could correspond to system faults such as an actuator failure or loss of signals. Research in this direction could yield some interesting results.

- The estimation procedure in Section 6.3 computes bounds (on the current state $x_k$) which are of the form:

  $$\underline{x}_k \leq x_k \leq \overline{x}_k$$

  However, improved results can potentially be obtained if we parameterize the bounds in the form:

  $$-d \leq C_k x_k \leq d$$

  where $d$ is a vector of ones and matrix variable $C_k \in \mathbb{R}^{n \times n}$ is of full rank. In principle, the formulation should follow the same path as in Chapter 4 for LC-RCI sets. However, the computational complexity of the estimation problem will require careful consideration since the bounds need to be computed online at each time step for output-feedback RMPC (Chapter 6).

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