Monopoles in Higher Dimensions

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Para a minha familia e amigos.

Declaration of Originality

I declare that this thesis is my own work and that all else is appropriately referenced.

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Abstract

The Bogomolnyi equation is a PDE for a connection and a Higgs field on a bundle over a 3 dimensional Riemannian manifold. Possible extensions of this PDE to higher dimensions preserving the ellipticity modulo gauge transformations require some extra structure, which is available both in 6 dimensional Calabi-Yau manifolds and 7 dimensional G_2 manifolds. These extensions are known as higher dimensional monopole equations and Donaldson and Segal, [DS11], proposed that "counting" solutions (monopoles) may give invariants of certain noncompact Calabi-Yau or G_2 manifolds. In this thesis this possibility is investigated and examples of monopoles are constructed on certain Calabi-Yau and G_2 manifolds. Moreover, this thesis also develops a Fredholm setup and a moduli theory for monopoles on asymptotically conical manifolds.

Introduction

In [DT98], Donaldson and Thomas propose generalizing some of the gauge theoretical constructions from 3 and 4 dimensions to some higher dimensional situations. This generalization requires extra structure, satisfying some integrability conditions. For example, the 4 dimensional instanton equations find a parallel in an 8 dimensional $Spin(7)$ manifold and the resulting equation is known as the $Spin(7)$ instanton equation. Later, in [DS11] Donaldson and Segal explored more possibilities for these new higher dimensional gauge theories, in particular mimicking the monopole (Bogomolnyi) equation in 3 dimensions. In the same way as the Bogomolnyi equation arises from dimensional reduction of the instanton equations in 4 dimensions, there are higher dimensional monopole equations arising from dimensional reduction of the $Spin(7)$ instanton equations. These can be written in real 6 dimensional Calabi-Yau and 7 dimensional G_2 manifolds, being most interesting when the underlying manifold is noncompact. Calabi-Yau 6-manifolds and G_2 manifolds occupy a special place in Berger's theorem [Ber55]: Their holonomy is contained in $SU(3) \subset SO(6)$ and in $G_2 \subset SO(7)$ respectively, which are Ricci flat holonomy groups. Moreover, both Calabi-Yau and G_2 manifolds come equipped with calibrations, as in [HL82], and an interesting but hard problem is to understand the existence of calibrated submanifolds and their moduli. For example, the Hodge Conjecture in a Kähler manifold (X, ω) can be interpreted as an existence problem for cycles calibrated with respect to $\frac{\omega^k}{k!}$, for some $k \in \mathbb{N}$. The Hodge conjecture holds for $(1,1)$ classes and then, on a Kähler 4 manifold, Gromov-Witten theory studies the moduli of ω calibrated cycles.
Special Lagrangian and coassociative submanifolds in a Calabi-Yau 3-folds or G_2 manifold respectively, are codimension 3 calibrated cycles. McLean showed in [McL98], that given a compact special Lagrangian (resp. coassociative) submanifold N, there is a smooth local moduli space of deformations of dimension b_1 (resp. b^2_-). There are some conjectural theories due to Dominic Joyce [Joy02], [Joy12], attempting to define invariants of both Calabi-Yau 3-folds and G_2 manifolds by “counting" rigid special Lagrangian and coassociative submanifolds respectively. In [DS11] it is suggested that, in both Calabi-Yau and G_2 manifolds, there may exist an invariant counting monopoles, and this may be easier to define and related to the conjectural invariants counting rigid codimension 3 calibrated cycles. The main goal of the thesis is to tackle these problems, first by giving concrete existence results for monopoles in special manifolds suitable to test ideas and second by studying the analytic properties of the monopole equation.

Chapter 1 introduces Calabi-Yau and G_2 manifolds, as well as the notion of finite mass

monopoles in Asymptotically Conical (AC) manifolds. Chapter 2 studies the original 3 dimensional monopole equations, more precisely it focuses on spherically symmetric monopoles in $\mathbb{R}^3$, equipped with spherically symmetric metrics.
Chapter 3 defines complex monopoles and also a special kind of these called Calabi-Yau monopoles. In this case one needs to consider complex monopoles in order to obtain an elliptic problem and perhaps it is more appropriate to compare these with solutions to Hitchin's equations, instead of the Bogomolny equation. However, after some preliminary results one proves a proposition giving conditions under which complex monopoles do reduce to Calabi-Yau monopoles. Then, certain examples of AC Calabi-Yau manifolds are given, in which the study of the complex monopole equation is an interesting Fredholm problem. Some of these examples do contain special Lagrangian submanifolds, which makes it even more interesting to study monopoles. In one of these cases, namely the Stenzel metric on $T^*\mathbb{S}^3$, the symmetries allow for using ODE methods to explore invariant Calabi-Yau monopoles. These Calabi-Yau monopoles are constructed, their moduli studied as well as the relation to the zero section, which is a rigid special Lagragian. A vanishing theorem for complex monopoles on some AC Calabi-Yau manifolds which have no compact special Lagrangian submanifolds is also given at the end of the chapter.
The G_2 monopole equation is studied in chapter 4. In fact, there are only 3 known examples of AC G_2 manifolds, these are all cohomogeneity 1 and the underlying manifolds $\Lambda^2_-(\mathbb{S}^4)$, $\Lambda^2_-(\mathbb{CP}^2)$, $\mathcal{S}(\mathbb{S}^3)$, are respectively the total space of anti-self-dual 2 forms on the round $\mathbb{S}^4$, $\mathbb{CP}^2$ with the Fubini-Study metric and the spinor bundle of $\mathbb{S}^3$. These were in fact the first examples of complete G_2 holonomy metrics and were first constructed in [BS89]. In the first two examples the zero section is a compact coassociative submanifold, while in the third case these do not exist. Using the symmetries, ODE methods are employed to construct invariant monopoles in the first two examples and to study their moduli. Also, regarding the last example $\mathcal{S}(\mathbb{S}^3)$, where there are no compact coassociative submanifolds, a vanishing theorem for monopoles is given.
Finally, chapter 5 gives an analytical setting in which finite mass (complex) monopoles in an AC manifold are a good Fredholm problem. More specifically, one introduces function spaces in which the deformation operator associated with the monopole equation (complex monopole equation in the case of Calabi-Yau manifolds) is shown to be Fredholm. Then, one uses this result in order to define the moduli space of monopoles as the zero set of a Fredholm section of a vector bundle over a Banach manifold.

There remain many open questions. The central one is whether monopoles can indeed be used to define an invariant of these AC manifolds, and in case this is possible, how to do it? There are 3 main problems towards such a definition: 1. Computing the index of the deformation operator. Standard techniques can probably be successfully applied to this problem and the author is currently addressing this in joint work with Mark Stern. 2. Establishing the smoothness of the moduli space. The second part of the results stated in propositions 3.1.9 and 4.1.2, in the cases of Calabi-Yau and G_2 manifolds respectively, can be interpreted as intermediate steps in that direction. 3. The compactness problem, which is probably a very hard one, and there is little hope of establishing concrete general results in the near future. Despite this, the possibility that this can be carried out

in special classes of examples must not be discarded.
Still in the AC setting, there is one other interesting problem worth mentioning, and this addresses the question of relating monopoles with codimension 3 calibrated cycles. In fact, it may be possible to use known analytical techniques to construct a large mass monopole transverse to certain rigid codimension 3 calibrated submanifolds. The author is currently investigating this possibility in joint work with Thomas Walpuski. Moreover, it is worth mentioning that the analogous construction in 3 dimensions can be done. In fact, in [Oli13] starting with some points in a 3 dimensional manifold, it is shown to be possible to construct large mass multimonopoles with monopoles located close to the given points.

There are a number of interesting directions for monopoles that can be pursued outside the AC world. In fact for any other kind of asymptotic behavior the definition of an invariant should go along different lines. In these cases a good Fredholm problem is lacking and in general monopoles are expected to have moduli. For example, it is possible to prove that for $X = \mathbb{R}^3 \times \mathbb{T}^3$ (resp. $X = \mathbb{R}^3 \times \mathbb{T}^4$) with the Calabi-Yau (resp. G_2) structure where each torus slice is special Lagrangian with phase zero (resp. coassociative) the pullback of any 3 dimensional monopole in $\mathbb{R}^3$ gives rise to a Calabi-Yau (resp. G_2) monopole on X with the given structure. It is then interesting to understand finite mass monopoles in other classes of manifolds (with other asymptotic behaviors) and find a Fredholm setup in which to fit these monopoles.
Moreover, one can also consider an even more ambitious program, to extend the theory to compact Calabi-Yau and G_2 manifolds. This could be done by introducing singularities, i.e. to allow the monopoles to have Dirac type singularities along calibrated codimension 3 cycles. Similar ideas do successfully extend 3 dimensional monopoles to compact 3 manifolds, see [Pau98].

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Contents

Chapter 1

Calabi-Yau, G_2 Manifolds and Gauge Theory

This chapter introduces the reader to the central objects in the thesis and states some of their properties for future reference. Besides defining Calabi-Yau and G_2 manifolds, sections 1.1 and 1.2 will study some of their properties, such as their Dirac operators, whose Weitzenböck formulae, are useful in studying the linearized monopole equation (or the complex monopole equation in the Calabi-Yau case). These two sections also introduce asymptotically conical Calabi-Yau and G_2 manifolds respectively.

Later, section 1.3 introduces a unified setup for dealing with monopoles on 3 manifolds, Calabi-Yau 3 folds and G_2 manifolds. Using it some relevant energies are introduced and used to study the relation of monopoles with the volume growth of the underlying manifold. The upshot is proposition 1.3.9 which gives conditions under which there is a vanishing theorem for monopoles. It is also proved that under the conditions of this vanishing result, monopoles are reducible and determined by flat connections, Hermitian Yang Mills connections and reducible G_2 instantons respectively for 3 manifolds, Calabi-Yau 3-folds and G_2 manifolds.

Section 1.4 sets up the problem for monopoles on Asymptotically Conical (AC) manifolds. Definition 1.4.1 introduces finite mass monopoles, and the subsequent results study some of their properties. Namely, proposition 1.4.6 studies the data determined by the asymptotics of finite mass monopoles, which is then abstracted to produce definition 1.4.7. This last section also proves a vanishing result for finite mass monopoles on AC manifolds; this is stated in proposition 1.4.9 (and corollary 1.4.11 for the special case $G = SU(2)$). The whole setup of sections 1.3 and 1.4 is unified for all three cases of monopoles on 3 manifolds, Calabi-Yau 3 folds and G_2 manifolds. In chapter 3 the setup for complex monopoles on Calabi-Yau 3 folds is slightly different, but the same kind of techniques will apply to the situation there.

1.1 Calabi-Yau Manifolds

This thesis only deals with Calabi-Yau 3 folds, i.e. with real dimension 6, but the general definition in any real dimension $n = 2m$ is given

Definition 1.1.1. *A Calabi-Yau manifold (X, ω, Ω) is a Kähler manifold (X, ω) with trivial canonical bundle and a choice of holomorphic volume form $\Omega \in \Omega^{m,0}(X, \mathbb{C})$ trivializing K_X and satisfying*

$$\frac{\omega^m}{m!} = (-1)^{\frac{m(m-1)}{2}} \left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega}. \tag{1.1.1}$$

According to this definition Calabi-Yau manifolds with real dimension $n = 2m$ have holonomy contained in $SU(m)$. Some authors require the holonomy to be exactly $SU(m)$ and here these will be called irreducible Calabi-Yau manifolds.

1.1.1 The Dirac Operator

Calabi-Yau manifolds are $Spin$ and $\mathcal{S}^+ = \Omega^{0,odd}(X, \mathbb{C})$ and $\mathcal{S}^- = \Omega^{0,ev}(X, \mathbb{C})$ are the vector bundles respectively associated with the positive and negative Spin representation. Let E be a vector bundle with connection A and denote by $\mathcal{S}_E^{\pm}$ the twisted bundle $\mathcal{S}^{\pm} \otimes E$, which comes equipped with the connection induced from A and the Spin connection. This gives rise to the twisted Dirac operator whose first component is

$$\mathcal{D}_A = \overline{\partial}_A + \overline{\partial}_A^* : \Omega^{0,odd}(X, E) \to \Omega^{0,ev}(X, E). \tag{1.1.2}$$

The other component will be denoted by $\mathcal{D}_A^*$ as it is the formal L^2 adjoint of $\mathcal{D}_A$. The goal of this section is to obtain some Weitzenböck type formulae, which will be useful in studying the linearisation to the monopole equations.

Proposition 1.1.2. *Let $(a, w) \in (\Omega^{0,1} \oplus \Omega^{0,3})(X, E)$ and $(\phi, b) \in (\Omega^0 \oplus \Omega^{0,2})(X, E)$, then*

$$\mathcal{D}_A^* \mathcal{D}_A(a, w) = \Delta_{\overline{\partial}_A}(a, w) + \Big(*[F_A^{2,0} \wedge *w], [F_A^{0,2} \wedge a] \Big) \tag{1.1.3}$$

$$\mathcal{D}_A \mathcal{D}_A^*(\phi, b) = \Delta_{\overline{\partial}_A}(\phi, b) + \Big(- * [F_A^{2,0} \wedge *b], [F_A^{0,2}, \phi] \Big). \tag{1.1.4}$$

Proof. To prove the first of these recall that since the dimension 6 is even $\overline{\partial}_A^* = - * \partial_A *$ and that $*^2 = (-1)^k$ on k forms. Then one can compute

$$\begin{aligned} \mathcal{D}_A^* \mathcal{D}_A(a, w) &= \mathcal{D}_A^* \Big(\overline{\partial}_A^* a, \overline{\partial}_A a + \overline{\partial}_A^* w \Big) = \Big(\overline{\partial}_A \overline{\partial}_A^* a + \overline{\partial}_A^* \overline{\partial}_A a + \overline{\partial}_A^* \overline{\partial}_A^* w, \overline{\partial}_A \overline{\partial}_A a + \overline{\partial}_A \overline{\partial}_A^* w \Big) \\ &= \Big(\Delta_{\overline{\partial}_A} a + *[F_A^{2,0} \wedge *w], \Delta_{\overline{\partial}_A} w + [F_A^{0,2} \wedge a] \Big). \end{aligned}$$

And the result follows for the first case. The second formula follows from a similar computation. □

Lemma 1.1.3. *(Twisted Kähler Identities) Let V be a complex vector bundle over X, equipped a unitary connection A. Then,*

$$\overline{\partial}_A^* = -i[\Lambda, \partial_A] \ , \ \partial_A^* = i[\Lambda, \overline{\partial}_A], \tag{1.1.5}$$

and these imply that $\Delta_{\overline{\partial}_A} - \Delta_{\partial_A} = -i\Lambda \circ [F_A^{1,1} \wedge \cdot]$

Proof. See page 240 in [Huy05]. □

Corollary 1.1.4. *In the setup of proposition 1.1.2, the following Weitzenböck formulae hold*

$$\mathcal{D}_A^*\mathcal{D}_A = \frac{1}{2}\nabla_A^*\nabla_A + W_{odd}\ ,\ \mathcal{D}_A\mathcal{D}_A^* = \frac{1}{2}\nabla_A^*\nabla_A + W_{ev} \tag{1.1.6}$$

Where, $W_{ev,odd} \in \Omega^0(X, \mathrm{End}(\Lambda^{0,ev,odd}))$ are the endomorphisms respectively defined by

$$\begin{aligned} W_{odd}(a,w) &= \left(*[F_A^{2,0} \wedge *w] + \left[\frac{i}{2}\Lambda F_A^{1,1}, a\right] - i\Lambda[F_A^{1,1} \wedge a], [F_A^{0,2} \wedge a] + \left[\frac{i}{2}\Lambda F_A^{1,1}, w\right]\right) \\ W_{ev}(\phi,b) &= \left(- * [F_A^{2,0} \wedge *b] - \left[\frac{i}{2}\Lambda F_A^{1,1}, \phi\right], [F_A^{0,2}, \phi] + \left[\frac{i}{2}\Lambda F_A^{1,1}, b\right] - i\Lambda[F_A^{1,1} \wedge b]\right). \end{aligned}$$

Proof. Compute the first by using formula 1.1.3, then for $(a,w) \in \Omega^{0,odd}(X,E)$,

$$\mathcal{D}_A^*\mathcal{D}_A(a,w) = \Delta_{\overline{\partial}_A}(a,w) + \left(*[F_A^{2,0} \wedge *w], [F_A^{0,2} \wedge a]\right).$$

Now to compute the Laplacian $\Delta_{\overline{\partial}_A}$ use lemma 1.1.3. For $a \in \Omega^{0,1}(X,E)$ this gives $\Delta_{\overline{\partial}_A} a = \Delta_{\partial_A} a - i\Lambda[F_A^{1,1} \wedge a]$, while for $w \in \Omega^{0,3}(X,E)$ the formula gives $\Delta_{\overline{\partial}_A} w = \Delta_{\partial_A} w - i\Lambda[F_A^{1,1} \wedge w]$, but this last term vanishes as w is of type $(0,3)$, so $\Delta_{\overline{\partial}_A} w = \Delta_{\partial_A} w$. Putting these two together

$$\mathcal{D}_A^*\mathcal{D}_A(a,w) = \Delta_{\partial_A}(a,w) + \left(-i\Lambda[F_A^{1,1} \wedge a] + *[F_A^{2,0} \wedge *w], [F_A^{0,2} \wedge a]\right). \tag{1.1.7}$$

Now use lemma 1.1.3 for (a,ϕ) viewed as a section of $V = \Lambda_{\mathbb{C}}^{0,odd} \otimes E$ equipped with the twist of the Spin connection with A. This gives

$$\Delta_{\overline{\partial}_A}(a,w) = \Delta_{\partial_A}(a,w) - [i\Lambda F_{\Lambda_{\mathbb{C}}^{0,odd}\otimes E}, (a,w)].$$

The terms involving the curvature of the Spin connection on $\Lambda_{\mathbb{C}}^{0,odd}$ are respectively the Ricci curvature on the $\Lambda_{\mathbb{C}}^{0,1}$ component and the scalar curvature in the $\Lambda_{\mathbb{C}}^{0,3}$ component. Both these vanish since (X,ω,Ω) is a Calabi-Yau manifold. So the only remaining terms are those involving F_A, i.e. the curvature of the connection A on E. So that $\Delta_{\overline{\partial}_A}(a,w) = \Delta_{\partial_A}(a,w) - ([i\Lambda F_A, a], [i\Lambda F_A, w])$. For $(a,w) \in \Omega^0(X, \Lambda_{\mathbb{C}}^{0,odd} \otimes E)$ and using ∇_A to denote the twisted connection from both the Spin connection and A one can write

$$\nabla_A^*\nabla_A(a,w) \quad = \quad \Delta_A(a,w) = \Delta_{\overline{\partial}_A}(a,w) + \Delta_{\partial_A}(a,w) = 2\Delta_{\partial_A}(a,w) - ([i\Lambda F_A, a], [i\Lambda F_A, w]).$$

Passing the last term to the left hand side and diving by 2 gives

$$\Delta_{\partial_A}(a,w) = \frac{1}{2}\nabla_A^*\nabla_A(a,w) + \frac{1}{2}([i\Lambda F_A, a], [i\Lambda F_A, w]).$$

To conclude the computation one needs to notice that since a and w are of type $(0,q)$ for some q the ∂-Laplacian Δ_{∂_A} is the same if we view (a,w) as an element of $\Omega^0(X, \Lambda_{\mathbb{C}}^{0,odd} \otimes E)$ or as an element of $\Omega^{0,odd}(X,E)$. So one can directly substitute the last formula above into equation 1.1.7 and this gives the desired result. The other Weitzenböck formula is a similar computation. □

1.1.2 Asymptotically Conical (AC) Calabi-Yau Manifolds

Definition 1.1.5. *A Riemannian manifold (X^n, g) is called asymptotically conical (AC) with rate $\nu < 0$ if there is a compact set $K \subset X$, a Riemannian manifold (Σ^{n-1}, g_Σ) and a diffeomorphism $\varphi : (1, \infty) \times \Sigma \to X \backslash K$, such that: the metric $g_C = dr^2 + r^2 g_\Sigma$ on $(1, \infty) \times \Sigma$ satisfies $|\nabla^j (\varphi^* g - g_C)|_C = O(r^{\nu - j})$, for all $j \in \mathbb{N}_0$. Here ∇ is the Levi Civita connection of g_C. A radius function will be any positive function $\rho : X \to \mathbb{R}_+$, such that in $X \backslash K$, $\rho = r \circ \varphi^{-1}$.*

If a metric cone $C(\Sigma) = (\mathbb{R}^+ \times \Sigma, g_C)$ is Ricci flat and Kähler with Kähler form ω_C associated with g_C, then its link (Σ, g_Σ) is said to be Sasaki-Einstein, see [Spa10] for a survey of Sasaki-Einstein geometry. Moreover, one must suppose $C(\Sigma)$ has trivial canonical bundle with a trivialization Ω_C. In fact, the $C(\Sigma)$'s that appear as the asymptotic cones of smooth AC Calabi Yau manifolds X can be supposed to be of this form (up to working on a covering $\tilde{X}$ of X), see [CH13b].

Definition 1.1.6. *A noncompact, complete Calabi-Yau manifold (X, ω, Ω) is an asymptotically conical Calabi-Yau manifold, if it is AC to a complex cone $(C(\Sigma), g_\Sigma)$ over a Sasaki-Einstein manifold (Σ, g_Σ), such that the cone $(C(\Sigma), \omega_C, \Omega_C)$ has its canonical bundle trivialized by Ω_C and $|\nabla^j (\varphi^* \Omega - \Omega_C)|_C = O(r^{\lambda - j})$, for some $\lambda < 0$ and all $j \in \mathbb{N}_0$.*

This definition requires that both the metric and the complex structure are asymptotic to those on the model cone $(C(\Sigma), \omega_C, \Omega_C)$. One can regard the problem of existence of AC Calabi-Yau manifolds as follows. If (X, Ω) is complex with trivial canonical bundle K_X and its complex structure is asymptotic to the one on a model cone, are there Ricci flat, Kähler metrics which are asymptotic to a Ricci flat Kähler metric on the cone? The next result (by Ronan Conlon and Hans Joachim Hein in [CH13a]), summarizes what is known regarding existence and uniqueness theorems for AC Calabi-Yau manifolds.

Theorem 1.1.7. *(R. Conlon, H.J. Hein, theorem 2.4 in [CH13a]) Let (X, Ω) be a noncompact complex manifold with trivial canonical bundle trivialized by Ω. Suppose that there is a Sasaki-Einstein manifold (Σ, g_Σ), a compact set $K \subset X$, a diffeomorphism $\varphi : (1, \infty) \times \Sigma \to X \backslash K$ and a trivialization of the canonical bundle Ω_C of $C(\Sigma)$, such that*

$$|\nabla^j \left(\varphi^* \Omega - \Omega_C \right)|_C = O(r^{\lambda - j}),$$

for some $\lambda < 0$ and all $j \in \mathbb{N}_0$. Then, for each class $\mathfrak{k} \in H^2_\mu(X, \mathbb{Z})$ with $\mu < 0$ and $a \in \mathbb{R}^+$, there is a rate $\nu < 0$ with $\nu \geq \max\{-n, \lambda, \mu\}$ and a unique Ricci flat Kähler metric $\omega_a \in \mathfrak{k}$, with

$$|\nabla^j \left(\varphi^* \omega_a - a \omega_C \right)|_C = O(r^{\nu - j}),$$

for all $j \in \mathbb{N}_0$.

Remark 1.1.8. *In the above proposition $H^2_\mu(X, \mathbb{Z})$ represents the μ almost compactly supported Kähler classes, as in [CH13a]. These are those classes which on $X \backslash K$ can be represented by a Kähler form of rate μ.*

Below some facts about Sasaki-Einstein geometry are collected, see [Spa10] and [Con09] for a survey. The fact that Σ has a Sasaki-Einstein structure is equivalent to the cone $C(\Sigma)$ having a Ricci-flat, Kähler metric. Hence, given a Calabi-Yau cone $(C(\Sigma), \omega_C, \Omega_C)$, by embedding Σ into it as $\{1\} \times \Sigma$ the Sasaki-Einstein geometry of Σ can be completely recovered from the cone as follows.

The vector field $r\partial_r$ is known as the Euler vector field and using that the cone is Kähler it can be shown that $r\partial_r$ is real holomorphic. If J denotes the complex structure on the cone, we may define $\xi = J(r\partial_r)$, this restricts to $\{1\} \times \Sigma$ as a unit length, Killing vector field known as the Reeb field. The flow of ξ foliates Σ and Sasaki-Einstein manifolds can be classified according to whether the leaves of this foliation are compact or noncompact. In the first case the orbits are geodesic circles and the flow of ξ integrates to a $\mathbb{S}^1$-action on Σ. If this action is free the Sasaki-Einstein manifold is said to be regular, if not it is said to be quasi-regular. In the case where the orbits are noncompact the Sasaki-Einstein structure is said to be irregular. Before proceeding into the contact geometry of Σ it is worth noticing that the fact that the cone $(C(\Sigma), g_C = dr^2 + r^2 g_\Sigma)$ is Ricci-flat implies that g_Σ is Einstein (with fixed Einstein constant).

It is possible to define the contact form $\eta \in \Omega^1(\Sigma, \mathbb{R})$ as the unique 1 form on Σ such that $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. This extends homogeneously to the cone as $\eta = i(\overline{\partial} - \partial)\log(r)$ and can be used to write $\omega_C = \frac{i}{2}\partial\overline{\partial}r^2 = \frac{1}{2}d(r^2\eta)$. The horizontal distribution $\ker\eta$ is transverse to the Reeb foliation, and is preserved by the complex structure J. Moreover, $(J|_{\ker\eta}, \omega_T = \frac{1}{2}d\eta)$ equip $\ker\eta$ with a transverse Kähler structure compatible with $g_\Sigma|_{\ker\eta}$.

One other construction which mimics Kähler geometry ones is the basic de Rham cohomology, see [Con09] and [EKA90]. The basic de Rham complex Ω^*_B consists of those differential forms $\alpha \in \Omega^*$ which satisfy $i_\xi\alpha = L_\xi\alpha = 0$. The restriction of the usual exterior differential $d_B = d|_{\Omega^*_B}$ preserves the basic de Rham complex and one can define its cohomology $H^*_B(\Sigma)$, called Basic cohomology. Moreover, one can also define basic (p,q)-forms and split $d_B = \partial_B + \overline{\partial}_B$. These satisfy the basic Kähler identities (see Lemme $3.4.4$ in [EKA90]) which as in the Kähler case can be used to construct a basic version of Hodge theory. The main consequence of these results is that the splitting into (p,q) forms passes to basic cohomology [EKA90]

$$H^k_B(\Sigma, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_B(\Sigma, \mathbb{C}).$$

At this point I remark that the similarities with Kähler geometry may not continue indefinitely, see [Con09] for further details along this line.

Example 1. *Subsection 3.2.1 in chapter 3 mentions AC Calabi-Yau manifolds, whose asymptotic cone is regular, i.e. their link is a regular Sasaki-Einstein manifold. It is always the case that a regular Sasaki-Einstein Σ manifold is the total space of a $\mathbb{S}^1$-bundle over a Fano surface D, equipped with a Kähler-Einstein metric. Associated with the $\mathbb{S}^1$-bundle one can construct a complex line bundle $L \to D$ and regard the contact form η as a connection which equips L with a holomorphic structure, as the curvature $d\eta$ is of type $(1,1)$. In particular, denoting by $\pi : \Sigma \to D$*

the bundle projection one can write

$$g_\Sigma = \eta \otimes \eta + \pi^* g_D,$$

where g_D is a Kähler-Einstein metric on D. Moreover, in this regular case the basic cohomology of the Sasaki-Einstein is just the pullback of the cohomology of D, i.e. $H^*_B(\Sigma, \mathbb{C}) = \pi^* H^*(D, \mathbb{C})$, see section 2 of [Spa10] for a survey.
The standard example of a regular Sasaki-Einstein manifold is $\mathbb{S}^5$ with the round metric, in which case $D = \mathbb{CP}^2$ with the Fubini-Study metric. One other example is when Σ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^2$ and $D = \mathbb{CP}^1 \times \mathbb{CP}^1$ with the product Fubini-Study metric; we shall get back to this example in subsection 3.3.1 of chapter 3.
The last class of examples finishes the list of all simply connected Sasaki-Einstein manifolds. Let $k \in [3, 8]$ and let $D_k = Bl_k\mathbb{CP}^2$, i.e. the blow up of $\mathbb{CP}^2$ at k points in general position. It is a result of Tian and Siu that, there is a unique Kähler-Einstein metric on $Bl_k\mathbb{CP}^2$. In this case Σ_k is diffeomorphic to $\sharp_k \mathbb{S}^2 \times \mathbb{S}^3$ and it admits a unique regular Sasaki-Einstein metric compatible with the unique Kähler-Einstein metric on $Bl_k\mathbb{CP}^2$.

1.2 G_2 Manifolds

Definition 1.2.1. *A 3 form ϕ on manifold X^7 determines a G_2 structure if at each point $x \in X^7$ the $GL(7, \mathbb{R})$ orbit of ϕ_x is open in Λ^3_x.*

The stabilizer of ϕ_x is the Lie group G_2. It is compact and preserves a Riemannian metric g_x for which there is an orthonormal frame $\{e_i\}_{i=1}^7$, such that

$$\phi_x = e^{123} + e^{145} - e^{167} + e^{246} - e^{275} + e^{347} - e^{356}.$$

Hence, a G_2 structure reduces the structure group of the frame bundle to G_2 and determines a Riemannian metric g on X. In this case ϕ and g are said to be compatible.

Definition 1.2.2. *A G_2 manifold (X, ϕ) is a 7-manifold X^7 equipped with a compatible G_2 structure ϕ, such that*

$$d\phi = d\psi = 0,$$

where $\psi = *\phi$ and $*$ is the Hodge-$*$ operator given by the metric g determined above.

Theorem 1.2.3. *(Fernández and Gray [FG82]) For a Riemannian manifold (X^7, g) equipped with a compatible 3 form ϕ, the following are equivalent*

1. $\nabla \phi = 0$,
2. $d\phi = d^*\phi = 0$,
3. *The holonomy of g is contained in G_2.*

Proposition 1.2.4. *Let (X, ϕ) be a G_2 manifold. Then the exterior bundle splits orthogonally as $\Lambda^1 = \Lambda^1_7$, $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$ and $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where the subscript indicates the rank of the irreducible component*

$$\begin{aligned}
\Lambda^2_7 &= \{\iota_V \phi, V \in \Gamma(TX)\} = \{\omega \mid *(\omega \wedge \phi) = 2\omega\} \\
\Lambda^2_{14} &= \{\omega \mid \omega \wedge \psi = 0\} = \{\omega \mid *(\omega \wedge \phi) = -\omega\} \\
\Lambda^3_1 &= \langle \phi \rangle \\
\Lambda^3_7 &= \{\iota_V \psi, V \in \Gamma(TX)\} \\
\Lambda^3_{27} &= \{\omega \mid \omega \wedge \psi = 0 \ \textit{ and } \ \omega \wedge \phi = 0\}.
\end{aligned}$$

Moreover if β is a 2 form and π_7, π_{14} denote the respective projections on the irreducible components, then the following algebraic identities hold

$$*(*(\beta \wedge \psi) \wedge \psi) = 3\pi_7(\beta), \tag{1.2.1}$$

$$*(\beta \wedge \phi) = 2\pi_7(\beta) - \pi_{14}(\beta). \tag{1.2.2}$$

1.2.1 The Dirac Operator

In Theorem 3.1 of [Gra69] Alfred Gray showed that a 7 manifold X^7 is $Spin$ if and only if it admits a G_2 structure. Hence a G_2 manifold (X, ϕ) is always $Spin$ and let $\mathcal{S} = \mathbb{R} \oplus T^*X$ denote the vector bundle associated with the standard irreducible $Spin(7)$ representation. Clifford multiplication $\gamma : T^*X \to \mathrm{End}(\mathcal{S})$ is given by

$$\gamma(b)(\phi, a) = (g(b, a), *(b \wedge a \wedge \psi) - b\phi),$$

for a, b 1-forms and ϕ a function. Let E be a vector bundle with connection A over X and denote by $\mathcal{S}_E = \mathcal{S} \otimes E$ the twisted bundle equipped with the connection ∇_A obtained from the Spin connection and A. Then one can define a twisted Dirac operator $\mathcal{D}_A$, which having in mind that $\Omega^0(X, \mathcal{S}_E) \cong \Omega^0(X, E) \oplus \Omega^1(X, E)$ can be written as

$$\mathcal{D}_A(\phi, a) = (-\nabla_A^* a, *(d_A a \wedge \psi) - \nabla_A \phi), \tag{1.2.3}$$

for $(\phi, a) \in \Omega^0(X, E) \oplus \Omega^1(X, E)$.

Proposition 1.2.5. *The Dirac operator $\mathcal{D}_A$ is formally self adjoint and for $(\phi, a) \in \Omega^0(X, E) \oplus \Omega^1(X, E)$, the following Weitzenböck type formula holds*

$$\mathcal{D}_A^2(\phi, a) = \nabla_A^* \nabla_A(\phi, a) + R^W(\phi, a),$$

with $R^W(\phi, a) = (*[F_A \wedge \psi \wedge a], *[*F_A \wedge a] - *[F_A \wedge \psi, \phi])$.

Proof. One can compute $\mathcal{D}_A^2$ using formula 1.2.3

$$\mathcal{D}_A^2(\phi, a) = (\Delta_A \phi, \Delta_A a) + (*[F_A \wedge \psi \wedge a], - * [F_A \wedge \psi, \phi]), \tag{1.2.4}$$

and then use the more standard Weitzenböck formula $\Delta_A a = \nabla_A^* \nabla_A a + *[*F \wedge a]$, where there is no term involving the Ricci curvature, as (X, ϕ) is a G_2 manifold, hence Ricci flat. □

1.2.2 AC G_2 Manifolds

This subsection starts by describing the geometric structures on the Riemannian 6-dimensional manifolds (Σ, g_Σ) that arise as the links of Riemannian G_2 cones. The first result is a lemma which describes the algebraic structures that reduce the structure group of the tangent bundle $T\Sigma$ to $SU(3)$.

Lemma 1.2.6. *Let Σ^6 be a 6 dimensional manifold, then the forms $(\omega, \Omega_1) \in \Omega^2 \oplus \Omega^3(\Sigma, \mathbb{R})$, determine an $SU(3)$ structure on Σ if:*

- *The $GL(6, \mathbb{R})$ orbit of Ω_1 is open, with stabilizer a covering of $SL(3, \mathbb{C})$;*
- *The following compatibility relations hold*

$$\omega \wedge \Omega_1 = \omega \wedge \Omega_2 = 0 \, , \; \frac{\omega^3}{3!} = \frac{1}{4} \Omega_1 \wedge \Omega_2. \tag{1.2.5}$$

  where $\Omega_2 = J\Omega_1$ and J denotes the almost complex structure determined by Ω_1

- *and $h(\cdot, \cdot) = \omega(\cdot, J\cdot)$ determines on Σ a Riemannian metric, i.e. h is positive definite.*

Proof. See page 3 in [CS02]. □

Proposition 1.2.7. *The Riemannian cone $(C(\Sigma), g_C = dr^2 + r^2 g_\Sigma)$, with the G_2 structure*

$$\phi = r^2 dr \wedge \omega + r^3 \Omega_1 \, , \; \psi = r^4 \frac{\omega^2}{2} - r^3 dr \wedge \Omega_2, \tag{1.2.6}$$

has holonomy in G_2 if and only if (Σ^6, g_Σ) is nearly Kähler, i.e. the forms $(\omega, \Omega_1, \Omega_2)$ satisfy

$$d\Omega_2 = -2\omega^2 \; , \; d\omega = 3\Omega_1. \tag{1.2.7}$$

Proof. From theorem 1.2.3, g_C has holonomy contained in G_2 if and only if $d\phi = d\psi = 0$. Since

$$\begin{aligned} d\phi &= r^2 dr \wedge (3\Omega_1 - d\omega) + r^3 d\Omega_1 \\ d\psi &= r^4 d\left(\frac{\omega^2}{2}\right) + r^3 dr \wedge \left(d\Omega_2 + 2\omega^2\right), \end{aligned}$$

one concludes that this holds if and only if (Σ, g_Σ) is nearly Kähler, i.e. the equations 1.2.7 for the forms $(\omega, \Omega_1, \Omega_2)$ hold. □

Definition 1.2.8. *A G_2 manifold (X, g) is Asymptotically Conical (AC) with rate $\nu < 0$ if there is a compact set $K \subset X$, a compact nearly Kähler 6-manifold (Σ, g_Σ) and a diffeomorphism $\varphi : (1, \infty) \times \Sigma \to X \backslash K$, such that on $(1, \infty) \times \Sigma$, the metric $g_C = dr^2 + r^2 g_\Sigma$ and its Levi Civita connection ∇ satisfy*

$$|\nabla^j \left(\varphi^* g - g_C\right)|_C = O(r^{\nu - j}),$$

for all $j \in \mathbb{N}_0$. A radius function will be any positive function $\rho : X \to \mathbb{R}_+$, such that in $X\backslash K$, $\rho = r \circ \varphi^{-1}$.

Example 2. *There are only 3 known examples of complete, AC and irreducible G_2 manifolds, these are known as the Bryant-Salamon manifolds [BS89], see also [GPP90]. The reference [KL12] studies the moduli spaces of AC G_2 manifolds, with a fixed asymptotic cone $C(\Sigma)$, and these examples are shown to be rigid. Further ahead, in chapter 4 of this thesis, these examples will be examined in more detail.*

1. *Let $\Lambda^2_-(M)$ be the total space of the bundle of anti-self-dual 2-forms over (M^4, g), where (M^4, g) denotes either the round $\mathbb{S}^4$ or the Fubini-Study $\mathbb{CP}^2$. Then, $\Lambda^2_-(M)$ admits a complete AC G_2 metric with rate $\nu = -4$ asymptotic to the cone over $\mathbb{CP}^3$ or $\mathbb{F}_3$ (the manifold of flags in $\mathbb{C}^3$), for $M = \mathbb{S}^4$ or $\mathbb{CP}^2$ respectively.*

2. *$\mathcal{S}(\mathbb{S}^3)$, the spinor bundle over the round $\mathbb{S}^3$ admits a complete AC G_2 metric with rate $\nu = -3$ which is asymptotic to the cone over $\mathbb{S}^3 \times \mathbb{S}^3$.*

The links of the cones which these are asymptotic to are (apart from $\mathbb{S}^6$) the only known examples of nearly Kähler manifolds. In fact all three are homogeneous: $\mathbb{CP}^3 = Sp(2)/U(1) \times SU(2)$, $\mathbb{F}_3 = SU(3)/T^2$ and $\mathbb{S}^3 \times \mathbb{S}^3 = SU(2) \times SU(2)$.

1.3 The Monopole Equation

Let (X^n, g, Θ) be an n-dimensional Riemannian manifold, together with $\Theta \in \Omega^{n-3}(X, \mathbb{R})$ a differential form (in examples it will be a calibration, i.e. closed, with comass $\sup_{\xi \in \Lambda^{n-3}TX\backslash\{0\}} \frac{|\Theta(\xi)|}{|\xi|} = 1$). Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and $P \to X$ a principal G bundle over X. Denote by $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$ the bundle with fibre $\mathfrak{g}$ associated with the adjoint representation and equip it with an Ad invariant metric $\langle \cdot, \cdot \rangle$. Let ∇_A be a connection on P and $\Phi \in \Omega^0(X, \mathfrak{g}_P)$ an Higgs Field, i.e. a section of $\mathfrak{g}_P$. This section studies the properties of pairs (∇_A, Φ) satisfying

$$* \nabla_A \Phi = F_A \wedge \Theta, \tag{1.3.1}$$

where $*$ is the Hodge-$*$ operator acting on the form components. Moreover, notice that if Θ is closed, the Bianchi identity implies that a solution to 1.3.1 satisfies $\Delta_A \Phi = 0$. The examples of most interest and which this thesis restricts attention are the following

Example 3. *In this thesis the data (X, g, Θ) will always be one of the following cases*

1. *(X^3, g) a 3 dimensional Riemannian manifold, take $\Theta = 1$, then equation 1.3.1 is the Bogomolnyi equation $*\nabla_A \Phi = F_A$, which will be studied in chapter 2.*

2. *(X^6, g) a Calabi-Yau 3-fold, take $\Theta = \Omega_1$ the real part of the holomorphic volume form $\Omega \in \Omega^{3,0}(X, \mathbb{C})$. Then, chapter 3 studies complex Calabi-Yau monopoles (definition 3.1.1) and a particular case, called just Calabi-Yau monopoles; these solve $*\nabla_A \Phi = F_A \wedge \Omega_1$ and it will also be imposed that $\Lambda F_A = 0$.*

3. *(X^7, g) a G_2 manifold and $\Theta = \psi \in \Omega^4(X, \mathbb{R})$ the calibrating 4-form. Then, equation 1.3.1 is the G_2 monopole equation $*\nabla_A \Phi = F_A \wedge \psi$, which will be studied in chapter 4.*

The first example above, i.e. dimension 3 is the so-called Bogomolnyi equation which has been the subject of intense research over the last 30 years, both by mathematicians and physicists. The last have appeared in some scattered places both in the mathematics and physics literature. The first occurrence that the author has been able to track down is in [War84], where the $Spin(7)$ and G_2 instanton equations on $\mathbb{R}^8$ and $\mathbb{R}^7$ are written down. The monopole equations in items 2 and 3 of example 3 arise by dimensional reduction of these, see [DS11].

1.3.1 Energy Identities

This section contains some energy identities which will be further refined and used to study monopoles on AC Calabi-Yau and G_2 manifolds.

Definition 1.3.1. *Let $U \subset X$ be precompact. The Yang-Mills-Higgs energy E_U and the Intermediate Energy E_U^I of a configuration (∇_A, Φ) on U are defined by*

$$E_U = \frac{1}{2} \int_U |\nabla_A \Phi|^2 + |F_A|^2 \ , \ E_U^I = \frac{1}{2} \int_U |\nabla_A \Phi|^2 + |F_A \wedge \Theta|^2 \tag{1.3.2}$$

The YMH Energy and the Intermediate energy agree for $n = 3$, i.e. case 1 in example 3. However, both in the Calabi-Yau and G_2 case the intermediate energy just measures the L^2 norm of some of the components of the curvature, namely those in $\mathrm{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2})$ and Λ^2_7 respectively.

Proposition 1.3.2. *Let (∇_A, Φ) be a configuration i.e. a connection and Higgs field on P. The Euler Lagrange equations for the Yang-Mills-Higgs functional are*

$$d_A^* F_A = [d_A \Phi, \Phi] \ , \ \Delta_A \Phi = 0. \tag{1.3.3}$$

Moreover, if Θ is a calibration the Euler Lagrange equations for the Intermediate Energy are

$$d_A^* \pi(F_A) = [d_A \Phi, \Phi] \ , \ \Delta_A \Phi = 0. \tag{1.3.4}$$

where $\pi(F_A) = *(*(F_A \wedge \Theta) \wedge \Theta)$.

Proof. If (a, ϕ) is a compactly supported variation, the boundary terms in the integration by parts of $\delta E_{(d_A, \tilde{\Phi})}(a, \phi) = \frac{d}{dt}|_{t=0} E(d_A + ta, \Phi + t\phi)$ can be ignored. Then, Stokes' theorem and the Bianchi identity give

$$\begin{aligned}
\delta E_{(d_A, \tilde{\Phi})}(a, \phi) &= \int_X \langle d_A a \wedge *F_A \rangle + \langle (d_A \phi + [a, \Phi]) \wedge *d_A \Phi \rangle \\
&= \int_X \langle a \wedge d_A * F_A \rangle - \langle \phi, d_A * d_A \Phi \rangle + \langle a \wedge [\Phi, *d_A \Phi] \rangle \\
&= \int_X \langle a \wedge (d_A * F_A + [\Phi, *d_A \Phi]) \rangle - \langle \phi, d_A * d_A \Phi \rangle.
\end{aligned}$$

So the critical points of such a functional are precisely the solutions to the second order equations $d_A^* F_A = [d_A \Phi, \Phi]$ and $\Delta_{d_A} \Phi = 0$. The computations for the variation of the Intermediate energy are similar and will be omitted. □

These Euler Lagrange equations are second order equations for (A, Φ) while the monopole equations are first order. In fact for the asymptotic behavior to be studied in this thesis, it will be shown that monopoles are minimizers of the intermediate energy.

Example 4. *In the cases from example 3*

1. *If $n = 3$, then the YMH and the Intermediate Energies are equal and so are the associated Euler Lagrange equations.*

2. *If $n = 6$, the complex structure gives the splitting $\Omega^2 = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$, then $\pi(F_A) = -2(F_A^{2,0} + F_A^{0,2})$. So the Intermediate Energy just measures the L^2 norm of $F_A^{0,2}$ and its Euler Lagrange equations are $\Delta_{d_A} \Phi = 0$ and $\partial_A^* F_A^{2,0} = -\frac{1}{2} [\partial_A \Phi, \Phi]$.*

3. *If $n = 7$, the G_2 structure gives the splitting $\Omega^2 = \Omega^2_{14} \oplus \Omega^2_7$ and $\pi(F_A) = 3\pi_7(F_A)$. The Intermediate Energy just measures the L^2 norm of $\pi_7(F_A)$, i.e. the component of F_A which lies in Ω^2_7, and the Euler Lagrange equations are $\Delta_{d_A} \Phi = 0$ and $d_A^* \pi_7(F_A) = \frac{1}{3} [d_A \Phi, \Phi]$.*

The following differential and consequent integral relations are very useful

Lemma 1.3.3. *(Green's first identity) Let $\phi, \psi \in \Omega^0(\mathfrak{g}_P)$ and $U \subseteq X$ precompact with smooth boundary, then*

$$\langle \phi, \Delta_A \psi \rangle = d^* \langle \phi, \nabla_A \psi \rangle + \langle \nabla_A \phi, \nabla_A \psi \rangle. \tag{1.3.5}$$

$$\int_{\partial \overline{U}} * \langle \phi, \nabla_A \psi \rangle = \int_U \langle \nabla_A \phi, \nabla_A \psi \rangle - \langle \phi, \Delta_A \psi \rangle * 1. \tag{1.3.6}$$

Proof. Since on 1 forms $d^* = - * d *$ one has $d^* \langle \phi, \nabla_A \psi \rangle = - * d \langle \phi, * \nabla_A \psi \rangle$. By the Leibniz rule this is $- * \langle \nabla_A \phi \wedge * \nabla_A \psi \rangle - \langle \phi, * d_A * \nabla_A \psi \rangle$. The second term is $\Delta_A \psi = d_A^* \nabla_A \psi$ and this gives the differential relation in the statement. Integrating over U gives

$$\int_U - *^2 \, d * \langle \phi, \nabla_A \psi \rangle \quad = \quad \int_U - *^2 \langle \nabla_A \phi \wedge * \nabla_A \psi \rangle + * \langle \phi, \Delta_A \psi \rangle.$$

Using $*^2 = 1$ and Stokes' theorem on the left hand side gives the integral relation. □

Proposition 1.3.4. *Let Θ be a calibration, $U \subset X$ precompact with smooth boundary, and (∇_A, Φ) a configuration. Then,*

$$E^I_U(A, \Phi) = \int_{\partial \overline{U}} \langle \Phi, F_A \rangle \wedge \Theta + \frac{1}{2} \| F_A \wedge \Theta - * \nabla_A \Phi \|^2_{L^2(U)}. \tag{1.3.7}$$

Moreover, for those (∇_A, Φ) satisfying equation 1.3.1

$$E^I_U(A, \Phi) = \frac{1}{2} \| F_A \wedge \Theta \|^2_{L^2(U)} + \frac{1}{2} \| \nabla_A \Phi \|^2_{L^2(U)} = \int_{\partial \overline{U}} \langle \Phi, F_A \rangle \wedge \Theta. \tag{1.3.8}$$

In particular, if X is compact, then $E^I_X < \infty$ and $\nabla_A \Phi = F_A \wedge \Theta = 0$ and the connection A is reducible.

Proof. The proof amounts to compute

$$\|F_A \wedge \Theta - *\nabla_A \Phi\|^2_{L^2(U)} = \|F_A \wedge \Theta\|^2_{L^2(U)} + \|\nabla_A \Phi\|^2_{L^2(U)} - 2\langle F_A \wedge \Theta, *\nabla_A \Phi\rangle_{L^2(U)}.$$

The first two terms are $2E^I_U$ and the last one is given by the integral

$$\langle F_A \wedge \Theta, *\nabla_A \Phi\rangle_{L^2(U)} = \int_U \langle \nabla_A \Phi \wedge F_A\rangle \wedge \Theta = \int_{\partial \overline{U}} \langle \Phi, F_A\rangle \wedge \Theta,$$

where one uses $*^2 \nabla_A \Phi = -\nabla_A \Phi$, the Bianchi identity $d_A F_A = 0$ and the fact that Θ is closed as it is a calibration, in order to ignore the other term in the integration by parts. $\square$

The argument used in the proof of proposition 1.3.4 will be extended for certain classes of noncompact manifolds.

1.3.2 Volume Growth and Boundary Data

Definition 1.3.5. *Let (X^n, g) be a real n dimensional, complete, noncompact, Riemannian manifold and $a \geq 0$. One says g has strict volume growth r^a if for all $p \in X$ there is $R_p \in \mathbb{R}^+$ and positive constants $A_1 \leq A_2$, such that for all $s \geq t \geq R_p$*

$$A_1(s^a - t^a) \leq Vol(B_s(p)) - Vol(B_t(p)) \leq A_2(s^a - t^a), \tag{1.3.9}$$

where $B_r(p)$ is the geodesic ball with center $p \in X$ and radius r.

Remark 1.3.6. *Since both Calabi-Yau and G_2 manifolds are Ricci flat, it follows from Bishop's Volume comparison that $a \in [1, 7]$ ($a \geq 1$ follows from trick due to Yau). Moreover, Cheeger-Gromoll's splitting theorem implies that if (X, g) is an irreducible Calabi-Yau or G_2 manifold (i.e. g has full holonomy $SU(3)$ or G_2 respectively), then it has only one end.*

Having this in mind, from now on assume that $a \in [1, 7]$ and there is a compact set $K \subset X$ and an $(n-1)$-dimensional manifold Σ, such that $X \backslash K \cong (R, +\infty)_\rho \times \Sigma$, for some large R and $\rho : X \to \mathbb{R}^+$ a smooth approximation to $dist(p, \cdot)$, such that $|\nabla \rho|$ is very close to one. Then, the inequality in 1.3.9 holds for $s \geq t \geq R$ and $B_r(p)$ replaced by $\rho^{-1}[0, r)$.

Lemma 1.3.7. *(X, g) as above has strict volume growth r^a if and only if there are positive constants $A'_1 < A'_2$ such that $A'_1 r^{a-1} \leq Vol(\rho^{-1}(r)) \leq A'_2 r^{a-1}$.*

Proof. The first direction follows from setting $t = r$ and $s = r + \varepsilon$ and differentiating the inequality in 1.3.9 having in mind that $Vol(\rho^{-1}(r+\varepsilon)) - Vol(\rho^{-1}(r)) = \int_r^{r+\varepsilon} Vol(\rho^{-1}(u))du$, and using the fundamental theorem of calculus. The reverse direction follows in the same way from integration. $\square$

Definition 1.3.8. *Let $P \to X$ be a principal G-bundle and ∇_A a connection on P. If the holonomy group H of ∇_A is isomorphic to a proper subgroup of G (i.e. $H \subsetneq G$), then the connection is said to be reducible. Moreover, a connection is said to be irreducible if it is not reducible.*

The main interest of this thesis is to study irreducible monopoles and so we shall now focus on these. Let P be a principal G-bundle and ∇_A a connection on P as above. Given a vector space V and a representation $\rho : G \to GL(V)$ with no trivial subfactors (i.e. there is no subspace $W \subset V$ where G acts trivially), one can construct the associated vector bundle $\mathbb{V} = P \times_{(G,\rho)} V$, which comes equipped with an associated connection, also denoted ∇_A. Then, if $s \neq 0$ is ∇_A parallel section of $\mathbb{V}$, the holonomy group H of ∇_A must preserve s and so be a proper subgroup of G. For a solution to 1.3.1 with irreducible connection, then $\nabla_A \Phi \neq 0$ and so $\|\nabla_A \Phi\|^2_{L^2(\rho^{-1}(0,r))}$ needs to be positive for some $r > 0$. If Θ is a calibration, then one can use the formula $\|\nabla_A \Phi\|^2_{L^2(\rho^{-1}(0,r))} = \int_{\rho^{-1}(r)} \langle \Phi, F_A \rangle \wedge \Theta$, from proposition 1.3.4 and conclude that $\langle \Phi, F_A \wedge \Theta \rangle$ can not decay too fast. This argument proves.

Proposition 1.3.9. *Let (X,g) be complete, noncompact with strict volume growth r^a and (∇_A, Φ) a solution to equation 1.3.1. Suppose that*

$$\lim_{r \to \infty} \langle \Phi, F_A \wedge \Theta \rangle r^{a-1} = 0. \tag{1.3.10}$$

Then $\nabla_A \Phi = F_A \wedge \Theta = 0$. In particular, if (X, g) is asymptotically cylindrical and $\langle \Phi, F_A \wedge \Theta \rangle \to 0$ as $r \to \infty$, or (X, g) is asymptotically conical and for r large enough $|\langle \Phi, F_A \wedge \Theta \rangle| \leq cst.r^{-(n-1)-\varepsilon}$, for some $\varepsilon > 0$, the result applies.

We now introduce two notions in order to name the special case where monopoles also satisfy $\nabla_A \Phi = 0$.

Definition 1.3.10. *Let (X^6, ω, Ω) be a Calabi-Yau manifold. A connection A on a bundle P is said to be Hermitian Yang Mills (HYM) if $F_A^{2,0} = 0$ and $\Lambda F_A = 0$.*

Definition 1.3.11. *Let (X^7, ϕ) be a G_2 manifold. A connection A on a bundle P is said to be a G_2-instanton if $F_A \wedge \psi = 0$.*

From definition 1.3.8 and the discussion immediately below, if $\nabla_A \Phi = 0$ the connection ∇_A is reducible and the equations in example 3 respectively give: 1. flat connections on a 3 manifold, i.e. $F_A = 0$. 2. On Calabi-Yau manifolds ∇_A is an HYM connection, and 3. In G_2 manifolds the connection ∇_A is a G_2-instanton.

In general the rough conclusion that follows from proposition 1.3.9 is as follows. The faster the volume of (X, g) grows, the less strict the decay conditions need to be for $\nabla_A \Phi \neq 0$. For example: while for an asymptotically conical manifold it is enough to suppose that $\langle \Phi, F_A \wedge \Theta \rangle$ decays at most at rate $r^{-(n-1)}$, for an asymptotically cylindrical one $\langle \Phi, F_A \wedge \Theta \rangle$ cannot decay. Proposition 1.3.9 is analogous to proposition 1.3.4, but for noncompact manifolds. In the rest of the thesis there will be further analogous results which combine this reasoning with more detailed information on the asymptotic behavior of monopoles, in order to obtain other vanishing results.

1.4 Monopoles on Asymptotically Conical (AC) Manifolds

Let (X, g) be an AC manifold as in definition 1.1.5 and equipped with $\Theta \in \Omega^{n-3}(X, \mathbb{R})$ as in the previous section. As X is AC, we require that the closed form Θ is asymptotic to a form Θ_C on the cone $C(\Sigma)$, i.e. for all $j \in \mathbb{N}_0$, $|\nabla^j(\varphi^*\Theta - \Theta_C)|_C = O(r^{\nu - j})$, for some $\nu < 0$. Suppose there are differential forms $\theta_1 \in \Omega^{n-4}(\Sigma)$ and $\theta_2 \in \Omega^{n-3}(\Sigma)$ such that

$$\Theta_C = r^{n-4} dr \wedge \theta_1 + r^{n-3}\theta_2.$$

In the cases of interest, listed in example 3, one has

Example 5. *1. $n = 3$ case: $\Theta = 1$ and so $\theta_1 = 0$ and $\theta_2 = 1$*

2. $n = 6$ case: Recall that the link of a Calabi-Yau cone is Sasaki-Einstein. In this case $\Theta = \Omega_1$, i.e. the real part of a holomorphic volume form Ω. Since Ω is asymptotic to $\Omega_C = r^2(r\eta - idr) \wedge \Omega_T$, where η is the contact form on the link and $\Omega_T \in \Omega_B^{(2,0)}(\Sigma, \mathbb{C})$ pulled back to the cone and

$$\Theta_C = \mathrm{Re}(\Omega_C) = r^2 dr \wedge \mathrm{Im}(\Omega_T) + r^3 \eta \wedge \mathrm{Re}(\Omega_T).$$

As a side remark, note that since Ω_C is holomorphic on the cone Ω_T satisfies $\overline{\partial}\Omega_T = \frac{3i}{2}(\eta - id\log(r)) \wedge \Omega_T$.

3. $n = 7$ case: Recall that the link (Σ, g_Σ) has a nearly Kähler structure $(\omega, \Omega_1, \Omega_2)$ and

$$\Theta_C = \psi_C = -r^3 dr \wedge \Omega_2 + r^4 \frac{\omega^2}{2}.$$

1.4.1 Finite Mass Monopoles

This subsection defines finite mass monopoles and studies their asymptotics: see propositions 1.4.5 and 1.4.5. Let $P \to X$ be a G bundle and suppose there is another G bundle $P_\infty \to \Sigma$ together with an isomorphism of principal bundles $\varphi^* P_{X \setminus K} \cong \pi^* P_\infty$, such that the connection ∇_A is asymptotic to a connection ∇_∞ on P_∞, i.e. $\varphi^*\nabla_A = \pi^*\nabla_\infty + a$, with $|\rho^j \nabla_\infty^j a| = O(r^{-1-\varepsilon})$ for some $\varepsilon > 0$.

Definition 1.4.1. *Under the hypothesis above, a monopole (A, Φ) is said to have finite mass if*

$$\lim_{\rho \to \infty} |\Phi| = m,$$

is finite at each end of X. The constant $m \in \mathbb{R}_0^+$ is called the mass of the monopole.

As monopoles always satisfy equation 1.3.1, in the rest of this section some consequences of that equation and the finite mass assumption are studied.

Lemma 1.4.2. *Let $n > 2$ and (X^n, g) be an AC manifold and (A, Φ) a finite mass, irreducible monopole. Denote by μ the smallest number such that $|\nabla_A \Phi| = O(\rho^{\mu - 1})$ outside a compact set. Then $\mu \geq -(n-2)$.*

Proof. Let $B_r = \rho^{-1}(0,r)$, then since (A,Φ) is a monopole $\nabla_A \Phi = F_A \wedge \Theta$, and the integration by parts in proposition 1.3.4 gives the boundary term

$$\|\nabla_A \Phi\|^2_{L^2(B_r)} = \lim_{r\to\infty} \int_{\partial \overline{B_r}} \langle \Phi, F_A \wedge \Theta \rangle \leq \lim_{r\to\infty} \int_{\partial \overline{B_r}} |\langle \Phi, *\nabla_A \Phi \rangle|.$$

Since $\nabla_A \Phi = O(\rho^{\mu-1})$ the term on the right is estimated as $|\langle \Phi, *\nabla_A \Phi\rangle| \leq \rho^{\mu-1}|\Phi| \leq m\rho^{\mu-1}$ and the volume of the cross sections grows like r^{n-1}, so the limit above becomes $\|\nabla_A \Phi\|^2_{L^2} \leq cst. \lim_{r\to\infty} r^{(n-1)+(\mu-1)}$. So if $\mu < -(n-2)$, one concludes that $\nabla_A \Phi = 0$ and the monopole would be reducible which is a contradiction. $\square$

Lemma 1.4.3. *Let (X,g) be AC and $f : X \to \mathbb{R}^+$ be a smooth positive function, $\Delta f \geq 0$, such that $|\nabla^j f| = O(\rho^{\mu - j})$ for all $j \in \mathbb{N}$ and all $\mu < 0$, for which $f = O(\rho^\mu)$. Also suppose there is such a $\mu_0 < 0$ with $f = O(\rho^{\mu_0})$ and that there is a constant $c > 0$ with the property that $\max_{\rho^{-1}(r)} f \leq c \min_{\rho^{-1}(r)} f$. Then, for all sufficiently large $R > 0$, there are $c_2 \geq c_1 > 0$, such that*

$$\frac{c_1}{\rho^{n-2}} \leq f \leq \frac{c_2}{\rho^{n-2}}.$$

Proof. The first step is to prove the lower bound, which follows by a comparison argument. The first thing to notice is that since (X,g) is AC, there is a compact set $\overline{B}_R \subset X$ and a harmonic function G on $X \backslash \overline{B}_R$ with rate $-(n-2)$, i.e. $G = O(r^{-(n-2)})$. By possibly replacing G by εG, for small $\varepsilon > 0$, one can suppose that $\inf_{\partial \overline{B}_R} f > \sup_{\partial \overline{B}_R} G$. Then, combining this with the fact that both f, G tend to 0 at the ends of X and f is superharmonic, one concludes that $f > G$, on $X \backslash K$.

To prove the upper bound, let Δ^C and Δ^Σ denote the Laplacian on the cone and on the link respectively. Since by hypothesis $|\nabla^j f| = O(\rho^{\mu-j})$ for all $j \in \mathbb{N}_0$, the inequality $\Delta f \geq 0$ turns into

$$\Delta^C f + O(r^{\mu-2-\varepsilon}) \geq 0,$$

where $\varepsilon > 0$. Expand this using separation of variables

$$-\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta^\Sigma f \geq O(r^{\mu-2-\varepsilon}).$$

Now, for each $r \in \mathbb{R}^+$ integrate this over $\{r\} \times \partial\Sigma$ with respect to the constant volume form $dvol_{g_\Sigma}$ and let

$$F(r) = \frac{1}{r^{n-1}} \int_{\{r\}\times\Sigma} f dvol_{g|_{\{r\}\times\Sigma}} = \int_{\{1\}\times\Sigma} f \circ s_r dvol_{g_\Sigma},$$

where $s_r(x) = rx$ is the scaling map on the cone. The integration of the term $\Delta^\Sigma f$ vanishes since $\int_{\{1\}\times\Sigma} \Delta^\Sigma (f \circ s_r) dvol_{g_\Sigma} = 0$ and so one obtains

$$\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial F}{\partial r}\right) \leq O(r^{\mu-2-\varepsilon}).$$

Then there is a constant C, such that $r^{n-1}\frac{\partial F}{\partial r} \leq C + O(r^{n-1+\mu-\varepsilon})$. Integrating this gives that for large r one has $F(r) \leq c_2 r^{-(n-2)} + O(r^{\mu-\varepsilon})$ for $c_2 = -\frac{C}{n-2}$. The hypothesis that for large r, $\max_{\rho^{-1}(r)} f \leq c \min_{\rho^{-1}(r)} f$ implies that a similar inequality holds for f.
Suppose that $\mu - \varepsilon > -(n-2)$, then $f = O(\rho^{\mu-\varepsilon})$ and going through the same arguments again one proves that $f(r) \leq c_2 r^{n-2} + O(r^{\mu-2\varepsilon})$. So one can iterate this procedure k times until one obtains $f(r) \leq c_2 r^{n-2} + O(r^{\mu-k\varepsilon})$ with $\mu - k\varepsilon < -(n-2)$. Moreover, one must also have $c_2 > 0$ or otherwise one would get a contradiction with the lower bound $f \geq c_1 \rho^{-(n-2)}$ proved in the beginning. □

Proposition 1.4.4. *Let (X,g) be AC and (A,Φ) be a finite mass irreducible monopole, and let $m \in \mathbb{R}^+$ denote its mass. Then, there are positive constants c_1, c_2, such that on $X \backslash K$*

$$m^2 - \frac{c_1}{\rho^{n-2}} \leq |\Phi|^2 \leq m^2 - \frac{c_2}{\rho^{n-2}}. \tag{1.4.1}$$

Moreover, $|\nabla_A \Phi| \in L^2$ and there is an ∇_∞ parallel Higgs Field Φ_∞ over Σ such that $\lim_{\rho\to+\infty} \Phi = \Phi_\infty$.

Proof. Consider the function

$$w = m^2 - |\Phi|^2,$$

which satisfies $\lim_{\rho\to\infty} w = 0$, $\Delta_A \Phi = 0$ and so $\Delta w = -\Delta|\Phi|^2 = 2|\nabla_A\Phi|^2$. The problem reduces to the setup of lemma 1.4.3 for the function w and the inequality 1.4.1 follows as a corollary to this.
To prove that $|\nabla_A\Phi| \in L^2$, let χ_R be a smooth bump function which is 1 in B_R and vanishes on $X \backslash B_{2R}$, (here $B_r = \rho^{-1}[0,r]$). Since (X,g) is AC the derivatives of the distance function ρ are uniformly bounded and $|\nabla^2 \chi_R| \leq cR^{-2}$ for some constant $c > 0$. Multiplying the identity $2|\nabla_A\Phi|^2 = \Delta w$ by χ_R and integrating gives

$$2\int_X \chi_R |\nabla_A\Phi|^2 = \int_X \chi_R \Delta w.$$

The left hand side is greater or equal than $\|\nabla_A\Phi\|^2_{L^2(B_R)}$ and one can integrate the left hand side by parts $\int_X \chi_R \Delta w = \int_X \Delta\chi_R w$. Since $|\nabla^2\chi_R| \leq cR^{-2}$ and is supported in $B_{2R}\backslash B_R$, while $0 \leq w \leq c\rho^{-(n-2)}$, one concludes that

$$\|\nabla_A\Phi\|^2_{L^2(B_R)} \leq \frac{c}{R^2}\int_{B_{2R}\backslash B_R} \rho^{-(n-2)} \leq \frac{C}{2R^2}\int_R^{2R} \rho d\rho \leq C.$$

This gives a bound on the L^2 norm of $|\nabla_A\Phi|$ over any B_R which is independent of R and so proves that $|\nabla_A\Phi| \in L^2$. The existence of Φ_∞ follows from the fact that (A,Φ) solves the monopole equation and $\nabla_A\Phi \in L^2$. Then, $|\rho^{j-1}\nabla_A^j\Phi| \in L^2$ for all $j \in \mathbb{N}_0$ and one can apply proposition A.0.17 in the Appendix A, which gives the existence of Φ_∞. □

Proposition 1.4.5. *Let (X,g) be AC and (A,Φ) a finite mass, irreducible monopole. Let $a = \varphi^*\nabla_A - \pi^*\nabla_\infty$ be as in the discussion preceding definitions 1.4.1 and assume $[a,\Phi_\infty] =$*

$O(\rho^{-(n-1)-\varepsilon'})$, for some $\varepsilon' > 0$. Then, there is a section Ψ_∞ of $\mathfrak{g}_{P_\infty}$, such that $\langle \Phi_\infty, \Psi_\infty \rangle \neq 0$ and pulling back Φ_∞, Ψ_∞ to $X \backslash K$

$$\Phi = \Phi_\infty + \frac{\Psi_\infty}{\rho^{n-2}} + O(\rho^{-(n-2)-\varepsilon'}),$$

for some $\varepsilon' > 0$.

Proof. On $X\backslash K$ one can write $\nabla_A = \nabla_\infty + a$ and $a = O(\rho^{-1-\varepsilon})$. Then

$$\Delta_A \Phi_\infty = \Delta_\infty \Phi_\infty + [d^*_\infty a, \Phi_\infty] - 2 * [a \wedge * \nabla_\infty \Phi_\infty] - *[a \wedge [*a, \Phi_\infty]],$$

and the first and third term vanish. The fact that (A, Φ) is a monopole and $[a, \Phi_\infty] = O(\rho^{-(n-1)-\varepsilon'})$ guarantees that $[d^*_\infty a, \Phi_\infty] = d^*_\infty [a, \Phi_\infty] = O(\rho^{-n-\varepsilon'})$, hence the second and fourth terms have rate $O(\rho^{-(n+\varepsilon')})$. Write $\Phi = \Phi_\infty + \phi$ with $|\nabla^i \phi| = O(\rho^{\mu - i})$, for all i and some $\mu < 0$. Then using the computation above $0 = \Delta_A \Phi = \Delta_A \Phi_\infty + \Delta_A \phi = \Delta_A \phi + O(\rho^{-n-\varepsilon'})$. Denote by Δ^C_∞ the ∇_∞ connection Laplacian on the cone and pull back the equation $\Delta_A \Phi = 0$ to the cone. This gives

$$\Delta^C_\infty \phi + O(r^{\mu-2-\varepsilon}) = O(r^{-n-\varepsilon'}),$$

for some $\varepsilon > 0$. The strategy for solving this is to use separation of variables. Write $\phi = \sum_{\lambda \in Spec(\Delta^\Sigma_\infty)} \phi_\lambda f_\lambda$, where $\Delta^{\partial X}_\infty f_\lambda = \lambda f_\lambda$ are the eigenfunctions for the ∇_∞ Laplacian on the link Σ. Then one obtains the following set of ODE's

$$\ddot{\phi}_\lambda + \frac{n-1}{r} \dot{\phi}_\lambda - \frac{\lambda}{r^2} \phi_\lambda = O(r^{\max\{\mu-2-\varepsilon', -n-\varepsilon'\}}).$$

Up to a harmonic function on the cone, these can be solved for all $\lambda \in Spec(\Delta^\Sigma_\infty)$, with the solutions ϕ_λ having rate $\max\{\mu - \varepsilon', -(n+2) - \varepsilon'\}$. If one takes the rate μ to be optimal then one must have $\mu = -(n-2)$. The irreducibility condition implies proposition 1.4.4 whose statement can be written as

$$-\frac{c_1}{\rho^{n-2}} \leq \langle \Phi_\infty, \phi \rangle + |\phi|^2 \leq -\frac{c_2}{\rho^{n-2}}.$$

Then, since $|\phi|^2$ is positive one concludes that $\langle \Phi_\infty, \phi \rangle \neq 0$ and decays at rate $-(n-2)$. Define Ψ_∞ to be the leading term in $\rho^{(n-2)}\phi$, i.e. such that $\phi = \rho^{-(n-2)} \Psi_\infty + O(\rho^{-(n-2)-\varepsilon'})$. $\square$

Proposition 1.4.6. *Let (X, g) be AC and (A, Φ) a finite mass, irreducible monopole under the hypothesis of proposition 1.4.5 and $\nabla_\infty, \Phi_\infty, \Psi_\infty$ the data determined by (A, Φ) on Σ, then*

1. *If $n = 3$ one has $F_\infty = \Psi_\infty dvol_\Sigma$, with $\nabla_\infty \Psi_\infty = 0$ and $\langle \Phi_\infty, \Psi_\infty \rangle \neq 0$.*

2. *If $n > 3$, then $F_\infty \wedge \theta_1 = F_\infty \wedge \theta_2 = 0$.*

Proof. As usual, in the notation, the pullbacks by φ used to identify objects on $X\backslash K$ with objects on the cone $(1, +\infty) \times \Sigma$ are omitted. Since (A, Φ) is a monopole one must have $F_A \wedge \Theta = *\nabla_A \Phi$, writing $\Theta = \Theta_C + (\Theta - \Theta_C)$ and recalling that $|\Theta - \Theta_C| = O(r^\nu)$. This, together with prop 1.4.5

gives

$$\begin{aligned} F_A \wedge \Theta &= r^{n-4} F_A \wedge dr \wedge \theta_1 + r^{n-3} F_A \wedge \theta_2 + O(r^{\nu-2}) \\ *\nabla_A \Phi &= (n-2)\frac{\Psi_\infty}{r^{n-1}} r^{n-1} dvol_\Sigma + r^{n-3} dr \wedge ([*_\Sigma a, \Phi_\infty] + *_\Sigma \nabla_\infty \Psi_\infty) + O(r^{-(n-1)-\varepsilon}). \end{aligned}$$

Equating both sides gives the following equations

$$\begin{aligned} F_A \wedge \theta_1 &= r\left([*_\Sigma a, \Phi_\infty] + *_\Sigma \nabla_\infty \Psi_\infty\right) + O(r^{-(n-2)+\nu}) + O(r^{-(n-2)-(n-3)-\varepsilon}) \\ F_A \wedge \theta_2 &= (n-2) r^{-(n-3)} \Psi_\infty dvol_\Sigma + O(r^{-(n-1)-\varepsilon}). \end{aligned}$$

Having in mind that k-homogeneous q-forms on the cone have rate $O(r^{k-q})$ (Lemma 1.6 in [CH13a]), the left hand sides are respectively $O(r^{-(n-2)})$ and $O(r^{-(n-1)})$. For $n = 3$ one has $\theta_1 = 0$, $\theta_2 = 1$, and since $a = O(r^{-1-\varepsilon})$ gives

$$F_\infty \wedge \theta_2 = \Psi_\infty dvol_\Sigma \ , \ \nabla_\infty \Psi_\infty = 0.$$

The case $n > 3$ immediately gives $F_\infty \wedge \theta_2 = 0$ by comparing decay rates. Moreover, now $\theta_1 \neq 0$ and one needs to notice that $\nu < 0$. Moreover the hypothesis of proposition 1.4.5 also gives $r[*_\Sigma a, \Phi_\infty] = O(r^{-(n-2)-\varepsilon'})$, so that $F_\infty \wedge \theta_1 = 0$ as well. □

1.4.2 Boundary Data For Finite Mass Monopoles

Based on propositions 1.4.5 and 1.4.6 this subsection abstracts, in definition 1.4.7, the boundary conditions determined by finite mass, irreducible monopoles on AC manifolds. Then, one goes on to prove some more detailed vanishing results stated in proposition 1.4.9 and corollary 1.4.11 in the particular case of $G = SU(2)$.

Definition 1.4.7. *The* ***boundary data*** *of a monopole is defined to be a G bundle P_∞ over Σ, a reducible connection ∇_∞ on P_∞ and a ∇_∞-parallel Higgs Field Φ_∞. Moreover, in the case $n = 3$, one further assumes there is Ψ_∞ such that $\langle \Phi_\infty, \Psi_\infty \rangle \neq 0$ and*

$$F_\infty = \Psi_\infty dvol_\Sigma \ , \ \nabla_\infty \Psi_\infty = 0,$$

while for $n > 3$ one assumes that

$$F_\infty \wedge \theta_1 = F_\infty \wedge \theta_2 = 0.$$

Remark 1.4.8. *Propositions 1.4.5 and 1.4.6 prove that such boundary data is precisely the one determined by an irreducible, finite mass monopole (A, Φ) with $[a, \Phi_\infty] = O(\rho^{-(n-1)-\varepsilon'})$, for some $\varepsilon' > 0$. In other words, given such a monopole (A, Φ) with $(\nabla_\infty, \Phi_\infty)$ the connection and Higgs field to which it is asymptotic. Then, these do satisfy the required conditions to be the boundary data of a monopole as in definition 1.4.7.*

Example 6. *1. Case $n = 3$: The data is given by the two parallel Higgs Fields Φ_∞, Ψ_∞ such that $\langle \Phi_\infty, \Psi_\infty \rangle \neq 0$ and a connection ∇_∞ with curvature $F_\infty = \Psi_\infty dvol_\Sigma$. Moreover, the fact that $\nabla_\infty \Psi_\infty = 0$ can also be stated as $d_\infty^* F_\infty = 0$. So one can equivalently consider a connection with Yang-Mills curvature F_∞ and a parallel Higgs Field Φ_∞, such that $\langle \Phi_\infty, F_\infty \rangle \neq 0$. Also notice that since $\nabla_\infty \Phi_\infty = 0$, this immediately implies that $0 = d_\infty \nabla_\infty \Phi_\infty = [F_\infty, \Phi_\infty]$, i.e. $[\Psi_\infty, \Phi_\infty] = 0$.*

2. *Case $n = 6$: Here Σ comes equipped with a Sasaki-Einstein structure, and recalling example 5, the connection ∇_∞ must be such that $F_\infty^{0,2} = 0$ and $\iota_\xi F_\infty = 0$, where ξ denotes the Reeb vector field. Proposition 3.1.28 in chapter 3 proves that (complex) Calabi-Yau monopoles have an even more restrictive asymptotic behavior. In this case ∇_∞ must be a basic HYM connection, see definition 3.1.29, together with remark 3.1.30.*

3. *Case $n = 7$: In this case Σ has a nearly Kähler structure $(\omega, \Omega_1, \Omega_2)$ and the connection ∇_∞ must be HYM with respect to it, i.e. $F \wedge \Omega_1 = F \wedge \omega^2 = 0$.*

Proposition 1.4.9. *Let (X, g) be AC and (A, Φ) a finite mass monopole with $|A - A_\infty| = O(\rho^{-(n-2)-\varepsilon'})$, for some $\varepsilon' > 0$. Denote by $[i^*\Theta] \in H^{n-3}(\Sigma, \mathbb{R})$ the cohomology class obtained by restricting $[\Theta] \in H^{n-3}(X, \mathbb{R})$ to any cross section along the end $X \backslash K$. Then,*

$$E_X^I = \int_\Sigma [\langle \Phi_\infty, F_\infty \rangle] \cup [i^*\Theta].$$

In particular, if $[\langle \Phi_\infty, F_\infty \rangle] \cup [i^*\Theta] = 0 \in H^{n-1}(\Sigma, \mathbb{R})$ or (X, g) has rate $\nu < -(n-3)$, then $\nabla_A \Phi = 0$, so A is reducible and $F_A \wedge \Theta = 0$.

Proof. Since (A, Φ) has finite mass proposition 1.4.4 guarantees $|\nabla_A \Phi| \in L^2$. Moreover, it is a monopole and so $E_X^I = \|\nabla_A \Phi\|_{L^2(X)}^2$. Hence, the sequence $E_{B_r}^I$, is bounded, monotone and increasing, the limit as $r \to \infty$ exists and

$$E_X^I = \lim_{r \to \infty} E_{B_r}^I = \lim_{r \to \infty} \int_{\partial \overline{B}_r} \langle \Phi, F_A \rangle \wedge i_r^* \Theta,$$

where the formula 1.3.4 for the intermediate energy was used and $i_r : \partial \overline{B}_r \hookrightarrow X$ denotes the inclusion. Write $\Phi = \Phi_\infty + \phi$, $\nabla_A = \nabla_\infty + a$ and $\Theta = \Theta_C + \eta$ with $a = O(\rho^{-(n-2)-\varepsilon'})$ and $\eta = O(\rho^\nu)$. Then, using proposition 1.4.6

$$\langle \Phi, F_A \rangle \wedge \Theta = \langle \Phi_\infty, F_\infty \rangle \wedge \eta + O(\rho^{-(n-1)-\varepsilon'}),$$

for some $\varepsilon' > 0$. If one supposes $\nu < -(n-3)$ the first item is also $O(\rho^{-(n-1)-\varepsilon'})$ and so the limit above vanishes and $E_X^I = 0$, i.e. $F_A \wedge \Theta = \nabla_A \Phi = 0$ and the connection is reducible. $\square$

Remark 1.4.10. *The connection A_∞ is reducible to a subgroup $H \subset G$, hence induced from an H-principal bundle Q_∞. One can then extend Φ_∞ to $P_\infty = Q_\infty \times_H G$ in a G-equivariant way. Fixing a point $p \in P_\infty$, one can identify H with a subgroup of the centralizer of $\mu = \Phi_\infty(p)$, and the curvature $F_\infty \in \Omega^2(P_\infty, \mathfrak{h})$, i.e. takes values in the Lie algebra of H. Taking* *a Cartan decomposition starting with* $\mu \in \mathfrak{t}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha(\mu)=0} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha(\mu)\neq 0} \mathfrak{g}_\alpha$, *with* $\mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha(\mu)=0} \mathfrak{g}_\alpha$. *Then* H *acts on* $\bigoplus_{\alpha(\mu)\neq 0} \mathfrak{g}_\alpha$ *via the adjoint action, since if* $\alpha(\mu) = 0$ *and* $\beta(\mu) \neq 0$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ *and* $(\alpha + \beta)(\mu) \neq 0$. *One can then consider the associated vector bundle* $E = Q_\infty \times_H \bigoplus_{\alpha(\mu)\neq 0} \mathfrak{g}_\alpha$, *equipped with the connection induced by* A_∞. *Its curvature* $\tilde{F}_\infty$ *acts on a section* Ψ *as* $\tilde{F}_\infty(\Psi) = [F_\infty, \Psi]$ *and* $\langle \Phi_\infty, F_\infty \rangle$ *denotes a combination of curvature components. The Bianchi identity and* $\nabla_\infty \Phi_\infty = 0$ *imply that* $\langle \Phi_\infty, F_\infty \rangle$ *is a closed 2-form and so determines a cohomology class in* Σ.

Corollary 1.4.11. *Assume the hypothesis of proposition 1.4.9 and that* $G = SU(2)$. *Then* P_∞ *and* A_∞ *are reducible to a complex line bundle* $L \to \Sigma$ *and*

$$\|\nabla_A \Phi\|^2_{L^2} \quad = \quad 4\pi m \langle c_1(L) \cup [i^*\Theta], [\Sigma] \rangle, \tag{1.4.2}$$

where $m = |\Phi_\infty| \in \mathbb{R}$ *is the mass. Moreover, if* $c_1(L) \cup [i^*\Theta] = 0$ *or* (X, g) *has rate* $\nu < -(n-3)$, *then* $\nabla_A \Phi = 0$, *so it is reducible and* $F_A \wedge \Theta = 0$.

Proof. If $G = SU(2)$, then it follows from $SU(2)$ representation theory that $\mathfrak{g}_{P_\infty} \otimes \mathbb{C} = \underline{\mathbb{C}} \oplus L_\alpha \oplus L_{-\alpha}$, and $c_1(L_\alpha) = -c_1(L_{-\alpha})$, i.e. $L_{-\alpha} \cong L_\alpha^*$. Alternatively, one constructs the bundle $E = P_\infty \times_{SU(2)} \mathbb{C}^2$ associated with the standard representation. This splits into eigenspaces for Φ_∞ as $E = L \oplus L^*$, where $L^2 \cong L_\alpha$ and since $\nabla_{A_\infty} \Phi_\infty = 0$ the connection ∇_{A_∞} is reducible to a connection on L. In the end, one obtains

$$\Phi_\infty = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} \;,\; F_{A_\infty} = \begin{pmatrix} F_L & 0 \\ 0 & -F_L \end{pmatrix}, \tag{1.4.3}$$

with $F_L \in -2\pi i c_1(L) \in H^2(\Sigma, -2\pi i \mathbb{Z})$ and $|\Phi_\infty| = m$, so

$$\begin{aligned} \|\nabla_A \Phi\|^2_{L^2} &= \lim_{r\to\infty} \int_{\partial B_r} \langle \Phi, F_A \rangle \wedge \Theta = 2i \lim_{r\to\infty} \int_{\partial B_r} F_L \wedge \Theta \\ &= 4\pi m \langle c_1(L) \cup \Theta, [\Sigma] \rangle. \end{aligned}$$

□

Example 7. *In the case of* $\mathbb{R}^3$, *finite mass monopoles have finite energy and* $|F_A|^2$ *is integrable, i.e. the curvature is in* L^2. *Let* (∇_A, Φ) *be a charge* k *and mass* m *monopole on* $\mathbb{R}^3$. *This has finite energy* $E = 2\pi m k$. *The formula from corollary 1.4.11 reads*

$$4\pi m k = \int_{\mathbb{S}^2} \langle F_\infty, \Phi_\infty \rangle,$$

in this case. In fact, $F_\infty = k \frac{\Phi_\infty}{|\Phi_\infty|} dvol_{\mathbb{S}^2}$ *and so* $[F_L] = c_1(H^{2k}) = k c_1(H^2)$, *where* H *denotes the Hopf line bundle over* $\mathbb{S}^2$.

Chapter 2

Monopoles in 3 Dimensions

This chapter focuses on the study of the usual monopole equation in 3 dimensions, also known as the Bogomolny equation. It starts off in section 2.1 with some preliminaries on the Bogomolnyi equation. Namely the study of the linearized operator in subsection 2.1.1 is essential to show it satisfies the necessary conditions to fit in the setup of chapter 5. Section 2.2, studies spherically symmetric monopoles on $\mathbb{R}^3$ equipped with a spherically symmetric metric g. The main theorem 2.2.1 of that section completely classifies these invariant monopoles under some conditions on g. Roughly, these monopoles are shown to have finite mass, which is shown to completely classify them. Then one studies the large mass limit and proves that in a small ball around the origin, these large mass monopoles approach (after rescaling) a BPS monopole (the unique mass 1 and spherically symmetric monopole for the Euclidean metric). Outside such a ball and also in the large mass limit, one proves that symmetric monopoles on $(\mathbb{R}^3, g)$ converge uniformly on compact sets in $\mathbb{R}^3 \backslash \{0\}$ to a reducible Abelian monopole (which shall be called g-Dirac monopoles by analogy with the Euclidean Dirac monopoles).

2.1 Preliminaries

Let (X^3, g) be a 3 manifold and $P \to X$ a $G = SU(2)$-bundle. Denote the adjoint bundle of P by $\mathfrak{su}(P) = P \times_{ad} \mathfrak{su}(2)$ and refer to its sections as Higgs fields. Recall that a pair (A, Φ) made of an $SU(2)$ connection A on P and an Higgs field Φ is said to be a monopole if it satisfies the Bogomolny equations

$$\nabla_A \Phi = *F_A, \tag{2.1.1}$$

where F_A is the curvature of A and $*$ is the Hodge operator of the metric g. For 3 dimensional monopoles there is a vast literature, see [JT80] and [AH88] for the case of monopoles in the Euclidean $\mathbb{R}^3$. Moreover, the results of the first chapter 1 give a detailed study of the boundary conditions and energy identities and this chapter will refer to these.

2.1.1 Linearized Operator

Let (∇_A, Φ) be a connection and Higgs field, which need not satisfy the Bogomolny equations. For such a configuration the quantity $e_0 = *F_{A_0} - \nabla_A \Phi$ may be nonzero. The linearized Bogomolny equation fits into a sequence

$$\Omega^0(\mathfrak{su}(P)) \xrightarrow{d_1} \Omega^1(\mathfrak{su}(P)) \oplus \Omega^0(\mathfrak{su}(P)) \xrightarrow{d_2} \Omega^1(\mathfrak{su}(P)), \tag{2.1.2}$$

with $d_1 \xi = (-\nabla_A \xi, -[\Phi, \xi])$ and

$$d_2(a, \phi) \quad = \quad *d_A a - \nabla_A \phi - [a, \Phi]. \tag{2.1.3}$$

Their formal adjoints are given by $d_1^*(a, \phi) = -\nabla_A^* a + [\Phi, \phi]$ and $d_2^* a = (*d_A a + [a, \Phi], -\nabla_A^* a)$. If (A, Φ) is a monopole then the sequence in 2.1.2 is actually an elliptic complex and so the operator $D = d_2 \oplus d_1^*$ acting on sections of $(\Lambda^1 \oplus \Lambda^0)(\mathfrak{su}(P))$ is elliptic. Its formal adjoint is $D^* = d_2^* \oplus d_1$ and these can be written as

$$D = \begin{pmatrix} *d_A & -\nabla_A \\ -d_A^* & 0 \end{pmatrix} + [\Phi, .] \ \ , \ \ D^* = D - 2[\Phi, .].$$

Lemma 2.1.1. *(Standard Weitzenböck) Let ∇_A be a connection and $u \in \Omega^1(\mathfrak{su}(2)) \oplus \Omega^0(\mathfrak{su}(2))$, then*

$$\Delta_A u = \nabla_A^* \nabla_A u + F^W(u) + \mathrm{Ric}^W(u). \tag{2.1.4}$$

Where $F^W(a, \phi) = (*[*F_A \wedge a], 0)$ and $\mathrm{Ric}^W(a, \phi) = (\mathrm{Ric}(a), 0)$.

Lemma 2.1.2. *(Monopole Weitzenböck) Let (∇_A, Φ) be a connection and an Higgs Field. Let $u \in \Omega^1(\mathfrak{su}(2)) \oplus \Omega^0(\mathfrak{su}(2))$, then*

$$\begin{aligned} DD^* u &= \nabla_A^* \nabla_A u - [[u, \Phi], \Phi] + \mathrm{Ric}^W(u) + \varepsilon_0^W(u) & (2.1.5) \\ D^* D u &= DD^* u + 2(\nabla_A \Phi)^W(u). & (2.1.6) \end{aligned}$$

Where $b^W(a, \phi) = (*[a \wedge b] - [b, \phi], [\langle b, a \rangle])$ and b is either $\varepsilon_0 = *F_A - \nabla_A \Phi$, Ric or $(\varepsilon_0 + 2 d_A \Phi)$.

If $(A, \Phi) = (A_0 + a, \Phi_0 + \phi)$ for suitable $u = (a, \phi) \in \Omega^1(\mathfrak{su}(P)) \oplus \Omega^0(\mathfrak{su}(P))$ is a monopole, then

$$\varepsilon_0 + D(u) + Q(u, u) = 0, \tag{2.1.7}$$

where the operator D is as above and $Q(u, u) = \begin{pmatrix} *[a \wedge a] - [a, \phi] \\ 0 \end{pmatrix}$.

2.1.2 Some Further Analytical Remarks

There is a scale invariance in the Bogomolny equation which is inherited from the conformal invariance of the ASD equations in 4 dimensions. The precise result is

Proposition 2.1.3. *Let (∇_A, Φ) be a monopole on (M^3, g), where M^3 is a Riemannian 3 manifold. Then $(\nabla_A, \delta^{-1}\Phi)$ is a monopole for $(M^3, \tilde{g} = \delta^2 g)$.*

Proof. In general, if ω is a k form and $\tilde{*}$ the Hodge operator for the metric $\tilde{g}$, then $\tilde{*}\omega = \delta^{n-2k} * \omega$ $(n = 3)$. This implies that $\tilde{*}F_A = \delta^{-1} * F_A = \delta^{-1}\nabla_A\Phi$, and the result follows. □

Recall from definition 1.3.1 that in the 3 dimensional case both the Energy and the Intermediate Energy are equal and defined on a precompact set $U \subset X$ as

$$E_U = \frac{1}{2}\int_U |\nabla_A\Phi|^2 + |F_A|^2. \tag{2.1.8}$$

Proposition 1.3.2 computes its Euler Lagrange equations $d_A^* F_A = [d_A\Phi, \Phi]$, $\Delta_{d_A}\Phi = 0$, which one can check monopoles do satisfy.

Proposition 2.1.4. *Let (A, Φ) be a monopole, then the following hold*

1. *$|\Phi|^2$ is subharmonic and so has no local maxima, in fact $\Delta|\Phi|^2 = -2|\nabla_A\Phi|^2$. Moreover, one can also compute that $\Delta|\Phi| = |\Phi|^{-1}\left(|\nabla|\Phi||^2 - |\nabla_A\Phi|^2\right)$, which is ≤ 0 by Kato's inequality.*

2. *The energy over a precompact set U with smooth boundary is given by the flux $\int_{\partial\overline{U}}\langle\Phi, F_A\rangle$.*

Now let (X, g) be an AC 3 manifold as in definition 1.1.5, with asymptotic cone $C(\Sigma)$. The next two results will be used later in this chapter for the construction of monopoles on AC 3 manifolds.

Lemma 2.1.5. *Let ∇_A be a metric compatible connection on a Hermitian vector bundle E over an AC manifold (X^3, g). Then, for all $\alpha \in [1, 3]$, there is a constant $c_K(\alpha) > 0$, such that*

$$\left(\int_X |\rho^{\frac{1}{2}} u|^{2\alpha} \frac{dvol_g}{\rho^3}\right)^{\frac{1}{2\alpha}} \leq c_K(\alpha)\left(\int_X |\nabla_A u|^2\right)^{\frac{1}{2}}$$

for all smooth and compactly supported section u. In particular for $\alpha = 3, 1$ one has respectively $\|u\|_{L_6}^2 \leq c_K\|\nabla_A u\|_{L_2}^2$ and $\|\rho^{-1}u\|_{L_2}^2 \leq c_K\|\nabla_A u\|_{L_2}^2$.

Proof. Kato's inequality $|\nabla|u|| \leq |\nabla_A u|$, holds pointwise for all irreducible Hermitian connections. The proof follows from combining this with corollary 1.3 in [Hei11]. □

Lemma 2.1.6. *In the conditions of lemma 2.1.5. Let u be a section such that $\nabla_A u \in L^2$, then there is a covariant constant limit $u|_\Sigma \in \Gamma(\Sigma, E|_\Sigma)$. Moreover, on the cone $C(\Sigma_i)$ over each end there is an inequality*

$$\||u| - u_{\Sigma_i}\|_{L^{2\alpha}_{0,-\frac{1}{2}}} \leq \|\nabla_A u\|_{L^2}.$$

Proof. This lemma is a particular case of propositions A.0.16 and A.0.17 in the Appendix A. □

2.2 Symmetric Monopoles on $\mathbb{R}^3$

Let g be a spherically symmetric metric on $\mathbb{R}^3$. Then, on $\mathbb{R}^3 \backslash \{0\} = \mathbb{R}^+ \times \mathbb{S}^2$, one can write

$$g = dr^2 + h^2(r) g_{\mathbb{S}^2}, \tag{2.2.1}$$

with $h(r) = r + h_3 r^3 + ...$, in order for the metric to be smooth and have bounded curvature at $r = 0$. This section studies spherically symmetric monopoles on the trivial $SU(2)$ bundle over $(\mathbb{R}^3, g)$. Under suitable conditions on h spherically symmetric solutions are constructed and these solve a system of nonlinear first order ODE's for two real valued functions a, ϕ. These ODE's have a singularity at $r = 0$ and are given by

$$\dot{\phi} = \frac{1}{2h^2}(a^2 - 1) \tag{2.2.2}$$
$$\dot{a} = 2\phi a. \tag{2.2.3}$$

together with the conditions $a(0) = 1, \phi(0) = 0$ and that a grows at most polynomially in r, i.e. $\lim_{r\to+\infty} r^{-k} a = 0$, for some $k \in \mathbb{Z}$. The first two of these are necessary and sufficient to guarantee the solution extends over $r = 0$ (they guarantee the curvature and the Higgs field are bounded) [SS84]. To understand the third condition, recall that there is a unique spherically symmetric connection ∇_∞ on the Höpf bundle over the $\mathbb{S}^2$. Then, one must require that over the 2 sphere at infinity, the connection is asymptotic to the reducible connection induced by ∇_∞. Using any metric with polynomial volume growth (the Euclidean metric for example) in order to compare connections certainly implies the condition that a must grow at most polynomially. In fact, for the applications in the current thesis, the metric g itself has polynomial volume growth and requiring that the connection is asymptotic to ∇_∞ with respect to g does imply that there is $k \in \mathbb{Z}$ such that $\lim_{r\to+\infty} r^{-k} a = 0$.

Notice that in case ϕ does not explode at a finite r, then $\mathrm{sign}(a)$ is preserved by the evolution. As changing a by $-a$ keeps the equations invariant there is no loss in restricting to the case $a > 0$. All the results of this section can be interpreted as properties of this system of ODE's and that is in fact the relevant point of view for the applications in the current thesis. The moduli space $\mathcal{M}_{inv}$ of spherically invariant monopoles on $(\mathbb{R}^3, g)$ modulo the action of the spherically symmetric gauge transformations is defined by

$$\mathcal{M}_{inv} = \left\{ (a, \phi) \;\middle|\; \text{solving 2.2.2 with } a(0) = 1, \phi(0) = 0 \text{ and } \exists_{k\in\mathbb{Z}} \lim_{r\to+\infty} r^{-k} a = 0 \right\}. \tag{2.2.4}$$

The metric g will be called non-parabolic if its Green's function G is bounded above, then it is uniquely defined by

$$G(r) = -\int \frac{1}{2h^2(r)} dr \; , \; \lim_{r\to\infty} G = 0.$$

It will be shown that spherically invariant solutions to the Bogomolnyi equations actually have bounded Higgs field Φ and a well defined mass

$$m(A,\Phi) = \lim_{r\to\infty} |\Phi(r)|.$$

Recall proposition 2.1.3, which contains a very important scaling property of the Bogomolny equations and denote by $s_\delta(x) = \delta x$ the scaling map on $\mathbb{R}^3$. This can be used to map a monopole (A,Φ) for the metric g into a monopole $s_\delta^*(A,\delta\Phi)$ for the metric $\delta^{-2}s_\delta^* g$. In the case where $g = g_E$ is the Euclidean metric there is a unique mass 1 and charge 1 monopole known as the BPS monopole [PS75], this is spherically symmetric and denoted by (A^{BPS},Φ^{BPS}). Moreover, the Euclidean metric is scale invariant and so from (A^{BPS},Φ^{BPS}) one can construct a whole family of monopoles $(A_m^{BPS},\Phi_m^{BPS}) = s_m^*(A,m\Phi)$, related by scaling and parametrized by their mass $m \in \mathbb{R}^+$. The solutions constructed in this chapter are modeled on these and the main result is

Theorem 2.2.1. *Let g be spherically symmetric, real analytic and non-parabolic. Then, $\mathcal{M}_{inv}$ is nonempty and consists of real analytic monopoles. Moreover, the following hold:*

1. *For all monopoles in $\mathcal{M}_{inv}$, the Higgs field is bounded and $\Phi^{-1}(0) = 0$ is the origin in $\mathbb{R}^3$. Moreover, the mass is well defined and gives a bijection*

$$m : \mathcal{M}_{inv} \to \mathbb{R}^+.$$

2. *Let $\{(A_\lambda,\Phi_\lambda)\}_{\lambda\in[\Lambda,+\infty)} \in \mathcal{M}_{inv}$ a sequence of monopoles with mass λ converging to $+\infty$. Then, for all $R > 0$ there is a sequence $\eta(\lambda,R)$ converging to 0 as λ converges to $+\infty$, such that the rescaled monopole*

$$s_\eta^*(A_\lambda,\eta\Phi_\lambda)$$

converges uniformly with all derivatives to the BPS monopole (A^{BPS},Φ^{BPS}) in the ball of radius R in $(\mathbb{R}^3,g_E)$.

3. *Let $\{(A_\lambda,\Phi_\lambda)\}_{\lambda\in[\Lambda,+\infty)}$ be the sequence above. Then the translated sequence*

$$\left(A_\lambda, \Phi_\lambda - \lambda\frac{\Phi_\lambda}{|\Phi_\lambda|}\right),$$

converges uniformly with all derivatives on $(\mathbb{R}^3\backslash\{0\},g)$ to a reducible monopole made of two copies of the g-Dirac monopole $(A^D,\Phi^D = G)$ with zero mass.

Remark 2.2.2. *The above statement is not at all surprising and in fact it is possible to prove that if in the complement of some ball $h^2(r) \geq cr^{1+\varepsilon}$, for some $c,\varepsilon > 0$ (g is non-parabolic in this case). Then, there is a spherically symmetric finite energy solution to the Yang-Mills-Higgs equations $d_A^* F_A = [\nabla_A\Phi,\Phi]$, $\Delta_A\Phi = 0$ in $(\mathbb{R}^3,g)$ with bounded Higgs field. This can be achieved by direct minimization of the spherically invariant Yang-Mills-Higgs functional on $(\mathbb{R}^3,g)$.*

The proof of theorem 2.2.1 occupies the rest of this chapter, which is organized in the following way. In section 2.2.1 the reduction to an ODE of the Bogomolny equations in $(\mathbb{R}^3,g)$ is outlined and an explicit formula for the BPS monopole with the Euclidean metric is given. For general spherically symmetric metrics g, the solutions to the ODE's 2.2.2 are not known. Besides these ODE's being nonlinear, there is a singularity at the origin, $r = 0$. The initial conditions one would like to give at $r = 0$ do not satisfy the Lipschitz hypothesis required by the standard existence and uniqueness theorem for ODE's. It is then convenient to go back to elliptic PDE theory and obtain a solution on the ball $B_\delta(0)$ which can be used to give initial conditions at the Lipschitz point $r = \delta$. Instead of solving the monopole equations for the metric g in the ball B_δ, use the scale invariance of the Bogomolny equations stated in proposition 2.1.3 in order to solve the equations for the metric $g_\delta = \delta^{-2} s_\delta^* g$ on its unit ball. Then, one obtains

Proposition 2.2.3. *For each $m \in \mathbb{R}^+$, there is $\Delta(m) > 0$, such that for each $\delta \leq \Delta(m)$ there is a spherically symmetric, real analytic monopole $(\tilde{A}_m^\delta, \tilde{\Phi}_m^\delta)$ for g_δ in $B_1(0)$.*

This is basically proposition 2.2.8 in 2.2.2. Then given a monopole $(\tilde{A}_m^\delta, \tilde{\Phi}_m^\delta)$ for g_δ in $B_1(0)$, proposition 2.1.3 gives that $(A_m^\delta, \Phi_m^\delta) = s_{\delta^{-1}}^*(\tilde{A}_m^\delta, \delta^{-1}\tilde{\Phi}_m^\delta)$ is a monopole for g on $B_\delta(0)$. A first step towards the proof of the first item in theorem 2.2.1 is achieved by applying the ODE analysis in section 2.2.3 to the solutions constructed on $B_\delta(0)$ which provide initial conditions for the ODE's at $r = \delta$. This analysis gives,

Proposition 2.2.4. *There is a one parameter family of spherically symmetric monopoles on $(\mathbb{R}^3, g)$. Moreover, these can all be obtained by extending the monopoles $(A_m^\delta, \Phi_m^\delta)$ on $(B_\delta(0), g)$ for (m, δ) such that $m \in \mathbb{R}^+$ and $0 < \delta \leq \Delta(m)$.*

Proof. In Lemma 2.2.12 a Taylor expansion for solutions of the ODE is obtained. It gives a recursive formula which depends only on 1 parameter $\dot{\phi}(0)$. The lemma does not address the question of convergence and there are basically 3 different possibilities.

1. Case $\dot{\phi}(0) = 0$, is the easiest one. In this case there is indeed a unique solution given by $a = 1$ and $\phi = 0$ and recovers back the flat connection. In terms of the notation in lemma 2.2.12 note that this corresponds to $v = 0$.

2. Case $\dot{\phi}(0) > 0$, for which there are no solutions, as proved in section 2.2.3, corollary 2.2.16 in terms of the function $v = \log(a^2)$ defined in the beginning of section 2.2.3.

3. Case $\dot{\phi}(0) < 0$, this is the case for which the PDE analysis shows existence of solutions. If one can find in the 2 parameter family constructed by the analysis a solution for each value of $\dot{\phi}(0) < 0$. Then, lemma 2.2.12 gives uniqueness of solutions for each value of $\dot{\phi}(0) < 0$ and makes of this a genuine global coordinate for $\mathcal{M}_{inv}$.

To proceed one shows that the PDE construction of the solutions $(A_m^\delta, \Phi_m^\delta)$ for (m, δ) with $m \in \mathbb{R}^+$ and $0 < \delta \leq \Delta(m)$ does indeed give configurations with all negative values of $\dot{\phi}(0)$. This is the reason why one uses two parameters in the construction of monopoles, i.e. with the two parameters (m, δ) it is easier to tune the properties of the monopole constructed than with only one parameter. Estimate 2.2.23 in lemma 2.2.13 gives bounds on $\dot{\phi} \in [I(m, \delta), J(m, \delta)]$. Then, lemma 2.2.14 gives two sequences of (m_n, δ_n). The first makes the lower bound $I_n = I(m_n, \delta_n)$

get as close to zero as one wants, while the second one makes the upper bound $J_n = J(m_n, \delta_n)$ get as close to $-\infty$ as one wants. The fact that all intermediate values are obtained follows from continuity. $\square$

Proposition 2.2.5. *Let $R > 0$, then there is a sequence δ converging to zero, such that the monopole $s^*_{\frac{\delta}{R}}\left(A^\delta_R, \frac{\delta}{R}\Phi^\delta_R\right)$ converges uniformly with all derivatives to (A^{BPS}, Φ^{BPS}) on the Euclidean ball $B_R(0)$.*

Proof. One needs to prove that for all $\varepsilon > 0$, there is δ, such that

$$\|s^*_{\frac{\delta}{R}}\left(A^\delta_R, \frac{\delta}{R}\Phi^\delta_R\right) - (A^{BPS}, \Phi^{BPS})\|_{C^\infty(B_R)} \leq \varepsilon.$$

In a first step one can consider $s^*_\delta\left(A^\delta_R, \delta\Phi^\delta_R\right) = \left(\tilde{A}^\delta_R, \tilde{\Phi}^\delta_R\right)$, then the estimate in proposition 2.2.8 gives that for all $\varepsilon > 0$, there is $\Delta(R, \varepsilon)$, such that for $\delta \leq \Delta(R, \varepsilon)$

$$\|s^*_\delta\left(A^\delta_R, \delta\Phi^\delta_R\right) - (A^{BPS}_R, \Phi^{BPS}_R)\|_{C^\infty(B_1)} \leq \varepsilon,$$

for the norm induced by the Euclidean metric. Since the Euclidean metric is invariant by scaling and $(A^{BPS}_R, \Phi^{BPS}_R) = s^*_R(A^{BPS}, R\Phi^{BPS})$ one can scale everything by R^{-1} and obtain the desired result for $\delta = \Delta(R, \varepsilon)$. $\square$

The next proposition will finish the proof of both the first and second items in theorem 2.2.1. The first item will be immediate from the statement and for the second item one needs to combine the statement with the previous proposition 2.2.5, in order to match those monopoles with the large mass limit.

Proposition 2.2.6. *For all monopoles in $\mathcal{M}_{inv}$, the mass is well defined and gives a bijection*

$$m : \mathcal{M}_{inv} \to \mathbb{R}^+.$$

Moreover, fix $R > 0$ and let $\delta \to 0$, the sequence of monopoles $\left(A^\delta_R, \Phi^\delta_R\right)$ previously constructed has mass $m(\delta) \to +\infty$.

Proof. One already knows that $\mathcal{M}_{inv} \cong \mathbb{R}^+$ corresponding to each value of $-\dot{\phi}(0)$ and this can be used to topologise $\mathcal{M}_{inv}$ as a 1 dimensional manifold. The next step one needs to take care is in showing that the map m is surjective. From proposition 4.3.19 and its corollary 2.2.20 one knows that for all $0 < \varepsilon < \varepsilon_0$, $m > 0$ and $\delta \leq \Delta(m, \varepsilon)$ there are bounds $m(A^\delta_m, \Phi^\delta_m) \in [\Phi_-(m, \varepsilon), \Phi_+(m, \varepsilon)]$, given by

$$\Phi_-(m, \varepsilon) = \frac{1}{\delta}\left(m\coth(m) - 1 - 2\varepsilon\right)\ ,\ \Phi_+(m, \varepsilon) = \frac{1}{\delta}\left(m\coth(m) + 2\varepsilon + G(\delta)\right) + 2G(\delta).$$

Take both m, ε converging to zero in the same way as in the first sequence in lemma 2.2.14 with

$\varepsilon_n = m_n^\alpha$, with $\alpha < 1$. Then, it is straightforward to check that

$$\lim_{m_n \to 0} |\Phi_+(m_n, \varepsilon_n)| = 0.$$

The other extreme can be made using the second sequence in lemma 2.2.14, this keeps m fixed but sends $\varepsilon \to 0$, moreover the choice of $\delta_n \leq \Delta(m, \varepsilon_n)$ is such that $\frac{\varepsilon_n}{\delta_n}$ still converges to 0. Then

$$\lim_{\varepsilon_n \to 0} |\Phi_-(m, \varepsilon_n)| = +\infty,$$

which gives the surjectivity of of the mass onto the positive real line. This second sequence also establishes that the mass of the monopoles $\left(A_R^\delta, \Phi_R^\delta\right)$ diverges. Just take $m = R$ fixed and δ converging to zero as it was just done. The last step is to show that the derivative of the map m is everywhere injective. As m is a map between 1 dimensional manifolds, this together with the surjectivity proved above imply the mass is actually a diffeomorphism. Let $(A, \Phi) \in \mathcal{M}_{inv}$, then any $v \in T_{(A,\Phi)}\mathcal{M}_{inv} \subset \Omega^1 \oplus \Omega^0(\mathbb{R}^3, \mathfrak{su}(2))$ is represented by two functions (b, ψ) of r solving the linearized monopole ODE's. This mean that $b(0) = \psi(0) = 0$ and they solve $\dot{\psi} = \frac{ab}{h^2}$, $\dot{a} = 2\phi b + 2a\psi$. Differentiating the first of these equations and using the second to substitute for b gives a second order ODE for ψ

$$\ddot{\psi} + \left(2\partial_r \left(\log(h)\right) - 4\phi\right) \dot{\psi} - 2a^2\psi = 0. \tag{2.2.5}$$

Solutions to this satisfy a maximum principle

- If ψ has a maximum at M, then $\ddot{\psi}(M) \leq 0$ and $\dot{\psi}(M) = 0$ and so $\psi(M) \leq 0$,
- If ψ has a minimum at m, then $\ddot{\psi}(m) \geq 0$ and $\dot{\psi}(m) = 0$ and so $\psi(m) \geq 0$.

The derivative of the mass is

$$dm(v) = 2\psi(\infty) : \mathbb{R} \to \mathbb{R}.$$

If v is in the kernel of dm, then $\psi(\infty) = 0$. The argument using these maximum principles is as follows. If $\psi(0) = 0$, one concludes that ψ must have a positive maximum or a negative minimum. Both of these hypothesis are impossible due to the maximum principle unless if $\psi = 0$ and hence also $b = 0$, which gives $v = 0$. $\square$

The last item which remains to be shown is that in the large mass limit after bubbling a BPS monopole at 0, one is left with a g-Dirac monopole on the exterior.

Proposition 2.2.7. *Let $\{(A_\lambda, \Phi_\lambda)\}_{\lambda \in [\Lambda, +\infty)}$ be a sequence of monopoles with mass $\lambda \to \infty$. Then the translated monopole sequence*

$$\left(A_\lambda, \Phi_\lambda - \lambda \frac{\Phi_\lambda}{|\Phi_\lambda|}\right),$$

converges uniformly with all derivatives to direct sum of two g-Dirac monopoles $(A^D, \Phi^D = G)$ with mass 0, on $(\mathbb{R}^3 \backslash \{0\}, g)$.

Proof. Working in a fixed gauge to this amounts to prove that given $R > 0$ and $\varepsilon > 0$, there is a λ such that $\|(\Phi_\lambda + \lambda/2T_1) - GT_1\|_{C^\infty[R,+\infty)} \leq \varepsilon$. For this one needs to study the function

$$u = \left(-\frac{\lambda}{2} + G\right) - \phi_\lambda,$$

where ϕ_λ is the scalar such that $\Phi_\lambda = \phi_\lambda T_1$. Then $\dot{u} = -\frac{1}{2h^2} - \frac{1}{2h^2}(a_\lambda^2 - 1) = -\frac{a_\lambda^2}{2h^2}$, which shows that $\dot{u} < 0$. This, together with $\lim_{r\to\infty} u = 0$ can be integrated to give

$$u(r) \leq G(r) \sup_{t\in[R,+\infty)} a_\lambda^2(t).$$

Moreover, G is bounded in $[R, +\infty)$ and a_λ^2 is decreasing, so that $a_\lambda^2 \leq a_\lambda^2(R)$. Now it is time to pick δ, m such that $a_\lambda = a_m^\delta$. This may be done with $\delta(\lambda), m(\lambda)$ as in the second sequence in lemma 2.2.14, but such that such that $m(\lambda)$ also converges to ∞ (see the proof of lemma 2.2.14). Then $\delta(\lambda)$ converges to 0 and as a_λ^2 is decreasing $a_\lambda^2 \leq a_\lambda^2(\delta) \sim me^{-m}$ by the estimates in lemma 2.2.13, which converges to 0. $\square$

2.2.1 The $SU(2)$ Invariant Bogomolny Equations

As $\mathbb{R}^3\backslash 0 \cong \mathbb{R}_+ \times \mathbb{S}^2$, one pulls back the homogeneous bundle

$$P_k = SU(2) \times_{\lambda_k} SU(2),$$

from $\mathbb{S}^2 \cong SU(2)/U(1)$. Where $\lambda_k : U(1) \to SU(2)$ is the isotropy homomorphism given by taking $\lambda_k(e^{i\alpha}) = \mathrm{diag}(e^{ik\alpha}, e^{-ik\alpha})$, for $k \in \mathbb{Z}$. Let T_1, T_2, T_3 be a basis of $\mathfrak{su}(2)$, such that $[T_i, T_j] = 2\varepsilon_{ijk}T_k$, and $\omega_1, \omega_2, \omega_3$ the dual coframe. Let $\mathfrak{h} = T_1$ and $\mathfrak{m} = \langle T_2, T_3\rangle$, this splitting equips the Höpf bundle $SU(2) \to \mathbb{S}^2$ with an $SU(2)$ invariant connection whose horizontal space is $\mathfrak{m}$. This induces a connection in each P_k known as the canonical invariant connection. It is encoded by the 1-form $A_k^c = kT_1 \otimes \omega^1 \in \Omega^1(SU(2), \mathfrak{su}(2))$. By Wang's theorem B.0.21, other invariant connections differ from it by morphisms of $U(1)$-representations $(\mathfrak{m}, Ad) \to (\langle T_2, T_3\rangle, Ad \circ \lambda_k)$. Invoking Schur's lemma these vanish for all $k \neq \pm 1$, and are isomorphisms for $k = \pm 1$. Suppose $k = 1$, then

$$A = A^c + a(r)(T_2 \otimes \omega^2 + T_3 \otimes \omega^3),$$

with $a : \mathbb{R}_+ \to \mathbb{R}$. The curvature of such a connection is given by $F_A = 2(a^2 - 1)T_1 \otimes \omega^{23} + \dot{a}\left(T_2 \otimes dr \wedge \omega^2 + T_3 \otimes dr \wedge \omega^3.\right)$. For each $r \in \mathbb{R}^+$ an invariant Higgs field $\Phi(r) \in \Omega^0(\{r\} \times SU(2), \mathfrak{su}(2))$ must be a constant in the trivial component of the $U(1)$ representation $(\mathfrak{su}(2), Ad \circ \lambda)$, i.e. $\Phi = \phi(r)\, T_1$, with $\phi : \mathbb{R}_+ \to \mathbb{R}$. Its covariant derivative $\nabla_A \Phi$ with respect to the connection A is $\nabla_A \Phi = \dot{\phi} T_1 \otimes dr + 2a\phi\left(T_2 \otimes \omega^3 - T_3 \otimes \omega^2\right)$. The metric 2.2.1 on $\mathbb{R}^+ \times \mathbb{S}^2$ can then be written as $g = dr^2 + 4h^2(r)(\omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3)$ and is invariant under the $SU(2)$ action, i.e. spherically symmetric. The Bogomolny equation $*\nabla_A \Phi = F_A$ turns into the ODE's 2.2.2 and 2.2.3 and explicit solutions to these are known in two different cases.

First, and most important here is the Euclidean case $h(r) = r$. Some special solutions are the flat connection $|a| = 1$ and $\phi = 0$ and the Dirac monopole with $a = 0$ and $\phi = m - \frac{1}{2r}$, for $m \in \mathbb{R}$. For $a \neq 0$, the general solution to the ODE's is

$$\phi^{BPS}_{C,D} = \frac{1}{2}\left(\frac{1}{r} - \frac{C}{\tanh(Cr+D)}\right) \ , \ a^{BPS}_{C,D} = \frac{Cr}{\sinh(Cr+D)}. \tag{2.2.6}$$

The solutions with $D = 0$ and $C = m < \infty$ are the only ones that extend over the origin, giving rise to irreducible monopoles on $\mathbb{R}^3$. These are the so called BPS monopole $(a^{BPS}_m, \phi^{BPS}_m)$ and first appeared in [PS75]. For small r

$$\phi^{BPS}_m(r) = -\frac{m^2 r}{6} + \frac{m^4 r^3}{90} + \ldots \ , \ a^{BPS}_m(r) = 1 - \frac{m^2 r^2}{6} + \frac{7m^4 r^4}{360} - \ldots$$

while for large r

$$\phi^{BPS}_m(r) = -\frac{1}{2}\left(m - \frac{1}{r}\right) + O(e^{-mr}) \ , \ a^{BPS}_m(r) = O(2re^{-mr}).$$

In the hyperbolic case $h(r) = \sinh(r)$ and there is also a one parameter family of monopoles parametrized their mass $m \in \mathbb{R}^+$ and given by

$$\phi_m(r) = \frac{1}{2}\left(\frac{1}{\tanh(r)} - \frac{m+1}{\tanh((m+1)r)}\right) \ , \ a_m(r) = \frac{(m+1)\sinh(r)}{\sinh((m+1)r)}. \tag{2.2.7}$$

In both cases the parameter m is the asymptotic value of the Higgs field at ∞, i.e. the mass of the monopole.

2.2.2 PDE Analysis

The metric $g_\delta = \delta^{-2} s_\delta^* g$ on its unit ball can be written as

$$g_\delta = dt^2 + h_\delta^2(t) g_{\mathbb{S}^2}$$

where $t \in (0,1)$ is the geodesic coordinate of the new metric (i.e. $\delta t = r \circ s_\delta$) and $h_\delta^2(t) = t^2 + \delta^2 G_\delta(t)$, with G_δ an analytic function such that $\frac{G_\delta(t)}{t^4}$ can be bounded independently of δ. This changes the problem of solving the equations in a small δ ball to that of solving the equations in a unit ball but with a varying metric g_δ, which is a spherically symmetric perturbation in δ from the Euclidean one. So one needs to solve $*_\delta F_A - \nabla_A \Phi = 0$, where $*_\delta$ is the g_δ-Hodge operator. For each $m \in \mathbb{R}^+$ consider the mass m Euclidean BPS monopoles [PS75], $(A^{BPS}_m, \Phi^{BPS}_m)$. Their error term

$$\varepsilon^\delta_m = *_\delta F_{A^{BPS}_m} - \nabla_{A^{BPS}_m} \Phi^{BPS}_m = O((\delta m)^2),$$

is small and vanishes for $\delta = 0$, where the metric is Euclidean. The idea is to use these as approximate solutions and search for a solution of the form $(A^\delta_m, \Phi^\delta_m) = (A^{BPS}_m, \Phi^{BPS}_m) + (b, \psi)$, with $v = (b, \psi)$ a section of $\left(\Lambda^1 \oplus \Lambda^0\right) \otimes \mathfrak{su}(2)$). The Bogomolny equation looks like a first order quasilinear PDE and

$$P(u) = \varepsilon_\delta^m + d_2(v) + Q(v,v) = 0, \tag{2.2.8}$$

where $Q(v,v) = *[b \wedge b] - [b, \psi]$ is a quadratic 0 order term and the linearized Bogomolnyi equation as in formula 2.1.3. Search for a solution of the form $v = d_2^* u$, then the new problem is to solve $P(d_2^* u) = 0$, and a first step to do this is to find an inverse for $d_2 d_2^*$. This can be achieved by further requiring a boundary condition giving rise to an elliptic problem,

$$\varepsilon_C^\delta + d_2 d_2^*(u) + Q(d_2^* u, d_2^* u) = 0, \tag{2.2.9}$$

$$u|_{\partial B_1(0)} = 0. \tag{2.2.10}$$

The claim is that the Dirichlet boundary allows inverting $d_2 d_2^*$. This follows from a Weitzenböck formula, which at $\delta = 0$ is

$$d_2 d_2^* u = \nabla^*_{A_m^{BPS}} \nabla_{A_m^{BPS}} u - \left[[u, \Phi_m^{BPS}] \Phi_m^{BPS} \right],$$

acting on $\mathfrak{su}(2)$ valued 1 forms. Then $d_2 d_2^*$ at $\delta = 0$, together with the boundary condition $u|_{\partial B_1} = 0$ is an elliptic, positive and self adjoint operator. As it is self adjoint it has index 0 and the boundary condition and positivity show it has zero kernel. So at $\delta = 0$, the unique solution is $u = 0$ and the linearisation of $P(d_2^* u)$ is $d_2 d_2^*$ which has a bounded inverse

$$L : C^{k,\alpha} \to C^{k+2,\alpha}.$$

The Implicit Function Theorem applies and for each $m \in \mathbb{R}^+$ there is $\Delta(m)$, such that for all $\delta < \Delta(m)$, there is a small solution u_m^δ of 2.2.9. Since ε_m^δ and the metric are analytic, elliptic regularity guarantees that u_m^δ is itself analytic, see sections 5.8 and 6.7 of [Mor08]. This result can be improved to come together with useful estimates which are stated in the following

Proposition 2.2.8. *Let $m > 0$, then for all positive ε, there is $\Delta(m, \varepsilon) > 0$, such that for $\delta \leq \Delta(m, \varepsilon)$, the solution u_m^δ is the unique one satisfying*

$$\| d_2^* u_m^\delta \|_{C^\infty} \leq \varepsilon. \tag{2.2.11}$$

Moreover, u_m^δ is real analytic and for a bound in the C^1 norm it is sufficient to take $\Delta(m, \varepsilon) = \frac{1}{m} \min \left\{ \sqrt{\frac{\varepsilon}{\|d_2^*\| \|L\|}} \frac{1}{\|d_2^*\| \|L\|} \right\}$, where $\|d_2^*\|, \|L\|$ denote the norms of the operators $d_2^* : C^{1,\alpha} \to C^{0,\alpha}$ and $L : C^{0,\alpha} \to C^{2,\alpha}$.

To prove proposition 2.2.8 one uses an alternative formulation to the Implicit Function Theorem via interpreting 2.2.9 as a fixed point equation and making use of the following lemma. It is proved by using the contraction mapping principle and keeping track of the norms in the iterations converging to the solution, see lemma 7.2.23 in [DK90].

Lemma 2.2.9. *Let B be a Banach space and $q : B \to B$ a smooth map such that for all $u, v \in B$*

$$\|q(u) - q(v)\| \leq k \left(\|u\| + \|v\| \right) \|u - v\|,$$

for some fixed constant k (i.e. independent of u and v). Then, if $\|v\| \leq \frac{1}{10k}$ there is a unique solution u to the equation

$$u + q(u) = v, \tag{2.2.12}$$

which satisfies the bound $\|u\| \leq 2\|v\|$.

This is applied to prove proposition 2.2.8 as follows. Let B be the space of $C^{2,\alpha}$ sections of $\Lambda^1 B_1(0)$ vanishing at the boundary and apply L to the left of $P(d_2^* u) = 0$, this equation is now the form of 2.2.12

$$u + LQ(d_2^* u, d_2^* u) = -L\varepsilon_m^{\delta},$$

and $q(u) = LQ(d_2^* u, d_2^* u)$ does satisfy the hypothesis of lemma 2.2.9 as shown below

$$\begin{aligned}
\|q(u) - q(v)\|_{C^{2,\alpha}} &= \|LQ(d_2^* u, d_2^* u) - LQ(d_2^* v, d_2^* v)\|_{C^{2,\alpha}} = \|LQ(d_2^*(u+v), d_2^*(u-v)\|_{C^{2,\alpha}} \\
&\leq cst.\|L\|\|d_2^*(u+v)\|_{C^{0,\alpha}}\|d_2^*(u-v)\|_{C^{0,\alpha}} \\
&\leq cst.\|L\|\|d_2^*\|^2 \left(\|u\|_{C^{2,\alpha}} + \|v\|_{C^{2,\alpha}}\right)\|u-v\|_{C^{2,\alpha}}
\end{aligned}$$

So that $k = cst.\|L\|\|d_2^*\|^2$. Then, the lemma applies for $\|L\varepsilon_m^{\delta}\|_{C^{0,\alpha}} \leq cst.k^{-1}$, since $\|L\varepsilon_m^{\delta}\|_{C^{2,\alpha}} \leq \|L\|\|\varepsilon_m^{\delta}\|_{C^{0,\alpha}}$ it is enough to guarantee that

$$\|\varepsilon_m^{\delta}\|_{C^{0,\alpha}} \leq cst.(\|L\|\|d_2^*\|)^{-2}, \tag{2.2.13}$$

and in this case there is a unique solution u_m^{δ} satisfying the estimate $\|u_m^{\delta}\|_{C^{2,\alpha}} \leq cst.\|L\varepsilon_m^{\delta}\|_{C^{2,\alpha}}$. Proposition 2.2.8 is proven by showing that given $\varepsilon > 0$ it is possible to make $\|d_2^* u_m^{\delta}\|_{C^{1,\alpha}} \leq \varepsilon$. Since

$$\|d_2^* u_m^{\delta}\|_{C^{1,\alpha}} \leq \|d_2^*\|\|u_m^{\delta}\|_{C^{2,\alpha}} \leq cst.\|d_2^*\|\|L\|\|\varepsilon_m^{\delta}\|_{C^{0,\alpha}},$$

it is enough to make $\delta \leq \delta(m, \varepsilon)$ small enough so that $\|\varepsilon_m^{\delta}\|_{C^{0,\alpha}} \leq \varepsilon\|d_2^*\|^{-1}\|L\|^{-1}$. Having in mind that one still needs to guarantee the estimate 2.2.13 holds, one concludes that $\|\varepsilon_m^{\delta}\|_{C^{0,\alpha}}$ needs to be small enough so that

$$\|\varepsilon_m^{\delta}\|_{C^{0,\alpha}} \leq cst.\min\{\|L\|^{-1}\|d_2^*\|^{-1}\varepsilon, \|d_2^*\|^{-2}\|L\|^{-2}\}. \tag{2.2.14}$$

Lemma 2.2.10. *The estimate $\|\varepsilon_m^{\delta}\|_{C^{0,\alpha}} \leq cst.m^2\delta^2$ holds.*

Proof. For $\delta \neq 0$, the error term does not vanish and is given by

$$\varepsilon_0 = *F_{A_0^m} - \nabla_{A_0^m}\Phi_0^C = \frac{a_m^2 - 1}{2t^2}\left(\frac{t^2}{h_\delta^2} - 1\right) T_1 \otimes dt. \tag{2.2.15}$$

Moreover, the point-wise norm of the above quantity is

$$|\varepsilon_0| \leq \frac{1 - a_m^2(t)}{2t^2}\delta^2 \frac{|G_\delta(t)|}{t^2} + o(\delta^4) \leq \delta^2 \sup_{t\in[0,1]}\left(|\dot{\phi}_m|\frac{|G_\delta(t)|}{t^2}\right).$$

Since as remarked at the beginning of this subsection $\frac{|G(t)|}{t^4}$ can be bounded independently of δ on can just use the explicit formula for ϕ_m and compute $\sup_{t\in[0,1]} |\dot{\phi_m}| = \frac{m^2}{6}$ the result follows. In fact, it is easy to see that this also holds for the C^1-norm and so for all $C^{0,\alpha}$ norms with $\alpha < 1$. $\square$

Putting this together with equation 2.2.14 finally gives that is is enough to set $\delta \leq \Delta(m,\varepsilon)$, with

$$\Delta(m,\varepsilon) = \frac{1}{m} \min\left\{ \sqrt{\frac{\varepsilon}{\|d_2^*\| \|L\|}}, \frac{1}{\|d_2^*\| \|L\|} \right\}, \tag{2.2.16}$$

in order to obtain

$$\|d_2^* u_m^\delta\|_{C^{1,\alpha}} \leq \varepsilon.$$

Improving this to a C^∞ bound can be made by standard bootstrapping arguments in elliptic PDE theory. Notice that all the coefficients of the PDE are real analytic as the BPS monopole is real analytic and so is the metric by assumption. Then it follows by the regularity theory for elliptic PDE's, sections 5.8 and 6.7 of [Mor08], that the solution u_m^δ is real analytic. This finishes the proof of proposition 2.2.8.

The solution to the monopole equations on $B_1(0)$ for the metric g_δ obtained is $\left(A_m^{BPS}, \Phi_m^{BPS}\right) + d_2^* u_m^\delta$. Denote by $(d_2^* u_m^\delta)_i$ the component of $d_2^* u_C^\delta$ in Λ^i, then proposition 2.1.3 gives the monopole on $B_\delta(0)$ for the metric g, given by

$$\begin{aligned} \left(A_m^\delta, \Phi_m^\delta\right) &= \left(s_{\delta^{-1}}^*(A_m^{BPS} + (d_2^* u_m^\delta)_1), \delta^{-1} s_{\delta^{-1}}^*(\Phi_m^{BPS} + (d_2^* u_m^\delta)_0)\right) \\ &= \left(A_{\delta^{-1}m}^{BPS} + s_{\delta^{-1}}^*(d_2^* u_m^\delta)_1, \Phi_{\delta^{-1}m}^{BPS} + \delta^{-1} s_{\delta^{-1}}^*(d_2^* u_m^\delta)_0\right) \end{aligned} \tag{2.2.17}$$

Rescaling the estimate 2.2.11 gives

Lemma 2.2.11. *Let m and ε be positive, then for $\delta \leq \Delta(m,\varepsilon)$, the monopole $\left(A_m^\delta, \Phi_m^\delta\right)$ for g in B_δ is such that*

$$\|A_m^\delta - A_{\delta^{-1}m}^{BPS}\|_{C^\infty(B_\delta)} + \|\Phi_m^\delta - \Phi_{\delta^{-1}m}^{BPS}\|_{C^\infty(B_\delta)} \leq \delta^{-1}\varepsilon, \tag{2.2.18}$$

where the norms are measured in the metric g. In particular, there is $\varepsilon_0(m) = \frac{1}{\|d_2^*\| \|L\|} > 0$, such that for all $\varepsilon \leq \varepsilon_0(m)$ and $\delta = \Delta(m,\varepsilon)$

$$\|A_m^\delta - A_{\delta^{-1}m}^{BPS}\|_{C^\infty(B_\delta)} + \|\Phi_m^\delta - \Phi_{\delta^{-1}m}^{BPS}\|_{C^\infty(B_\delta)} \leq m\sqrt{\frac{\varepsilon}{\varepsilon_0}}, \tag{2.2.19}$$

and once again the norms are measured using the metric g.

Proof. Denote by (B_r, g) the radius r ball centred at zero where the distance r is measured with respect to the metric g. Then, as sets $(B_\delta, g) = (B_1, g_\delta)$, moreover the norm of a 1 form ω gets scaled according to $|\omega|_g = \delta^{-1} |\omega|_{\delta^{-2}g}$

$$\|A_m^\delta - A_{\delta^{-1}m}^{BPS}\|_{C^\infty(B_\delta,g)} = \delta^{-1} \|s_{\delta^{-1}}^*(d_2^* u_m^\delta)_1\|_{C^\infty(B_\delta, s_{\delta^{-1}} g_\delta)} \leq \delta^{-1} \|(d_2^* u_m^\delta)_1\|_{C^\infty(B_1,g_\delta)} \leq \delta^{-1}\varepsilon.$$

In the same way for Φ_m^δ one computes

$$\|\Phi_m^\delta - \Phi_{\delta^{-1}m}^{BPS}\|_{C^\infty(B_\delta,g)} = \quad \delta^{-1}\|(d_2^* u_m^\delta)_0\|_{C^\infty(B_\delta,g)} \leq \delta^{-1}\|(d_2^* u_m^\delta)_0\|_{C^\infty(B_1,g_\delta)} \leq \delta^{-1}\varepsilon.$$

the second statement follows directly from inserting the formula 2.2.16 and ε_0 is determined by $\varepsilon_0(m) = \frac{1}{\|d_2^*\|\|L\|}$ in order to make the first term in 2.2.16 smaller than the second. □

2.2.3 ODE Analysis

Recall the monopole ODE's 2.2.2 and 2.2.3 and define $v = 2\log(a)$ (note that this implies $\dot{v} = 4\phi$) and write the equations 2.2.3 as a second order ODE for v

$$\ddot{v} = \frac{2}{h^2}\left(e^v - 1\right). \tag{2.2.20}$$

The first result in this section gives conditions on the existence of a formal power series solution to equation 2.2.20. Before the statement, recall that one is interested in solutions of 2.2.3 satisfying $a(0) = 1, \phi(0) = 0$ and $\lim_{r\to\infty} r^{-k}a(r) = 0$, for some $k \in \mathbb{Z}$. Translated into v, these are the conditions that $v(0) = \dot{v}(0) = 0$ and $\lim_{r\to\infty} r^{-k}e^{v(r)} = 0$, for some $k \in \mathbb{Z}$.

Lemma 2.2.12. *Let h be analytic and $b \in \mathbb{R}$. Write $h^2(r) = r^2\varphi(r)$ with $\varphi(r)$ analytic such that its expansion can be written as $\varphi(r) = \sum_{i\geq 0}\varphi_i r^i$, with $\varphi_0 = 1$. Then, there is a unique formal power series solution $v = \sum_{i\geq 0} v_i r^i$ to the equation 2.2.20 such that $v(0) = \dot{v}(0) = 0$ and $\ddot{v}(0) = b \in \mathbb{R}$. It is determined by $v_0 = v_1 = 0$, $v_2 = b$ and*

$$v_{i+2} = \frac{2}{i(i+3)}\left(\left(\sum_{k\geq 2}\frac{1}{k!}\sum_{l_1+\ldots+l_k=i+2} v_{l_1}\ldots v_{l_k}\right) + \sum_{j<i}\varphi_{i-j}\left(\sum_{k\geq 1}\frac{1}{k!}\sum_{l_1+\ldots+l_k=j+2} v_{l_1}\ldots v_{l_k}\right)\right), \tag{2.2.21}$$

for all $i + 2 \geq 3$.

Proof. Substituting into the equation shows that the recurrence relation formally satisfies equation 2.2.20. It remains to check that the recurrence relation is completely determined by setting $v_0 = v_1 = 0$ and $v_2 = b \in \mathbb{R}$. This, as well, can be directly checked from equation 2.2.21. To do this notice that the first term

$$\sum_{k\geq 2}\frac{1}{k!}\sum_{l_1+\ldots+l_k=i+2} v_{l_1}\ldots v_{l_k},$$

contains no terms in v_{i+2}, since $k \geq 2$ and so one must have at least two v_l's. Since $v_0 = 0$, each $l \geq 1$, which is the same as saying that each $l \leq i + 1$. As for the second term

$$\sum_{j<i}\varphi_{i-j}\left(\sum_{k\geq 1}\frac{1}{k!}\sum_{l_1+\ldots+l_k=j+2} v_{l_1}\ldots v_{l_k}\right),$$

it just contains terms in $j + 2 < i + 2$. □

The monopoles from the last section give a family of solutions $(A_m^\delta, \Phi_m^\delta)$ on $r \leq \delta$ depending on two parameters $m \in \mathbb{R}^+$ and $\delta \leq \Delta(m)$. These can be used to give initial conditions for the ODE's at $r = \delta$. The estimates from lemma 2.2.11, can be used to obtain estimates to these initial conditions as follows.

Lemma 2.2.13. *Let $m \in \mathbb{R}^+$ and $\varepsilon > 0$, then for all $\delta \leq \Delta(m, \varepsilon)$ the monopole $(A_m^\delta, \Phi_m^\delta)$ constructed in the previous section has its fields satisfying*

$$|\phi_m^\delta(\delta) - \frac{1}{2\delta}(1 - m\coth(m))| \quad \leq \quad \delta^{-1}\varepsilon, \tag{2.2.22}$$

and $|a_m^\delta(\delta) - \frac{m}{\sinh(m)}| \leq \delta^{-1}\varepsilon$. Moreover, the following estimate also holds

$$\dot{\phi}_m^\delta(0) \quad \in \quad [I(m,\delta), J(m,\delta)], \tag{2.2.23}$$

with $I(m,\delta) = -\frac{1}{6}\frac{m^2}{\delta^2} - \varepsilon\delta^{-1}$ and $J(m,\delta) = -\frac{1}{2}\frac{m^2}{\delta^2}\left(m^{-2} - \sinh^{-2}(m)\right) + \varepsilon\delta^{-1}$.

Proof. The estimates from lemma 2.2.11 guarantee that

$$\sup_{r \leq \delta}\left(|\phi_m^\delta - \phi_{\delta^{-1}m}^{BPS}| + |h^{-1}\left(a_m^\delta - a_{\delta^{-1}m}^{BPS}\right)|\right) \leq \delta^{-1}\varepsilon.$$

Using the explicit formulas $\phi_{\delta^{-1}m}^{BPS}(\delta) = \frac{1}{2\delta}(1 - m\coth(m))$ and $a_{\delta^{-1}m}^{BPS}(\delta) = \frac{m}{\sinh(m)}$, one obtains the desired bounds on the values of the fields at δ. Since lemma 2.2.11 actually gives C^1 estimates one also has $\sup_{r\leq\delta}|\dot{\phi}_m^\delta - \dot{\phi}_{\delta^{-1}m}^{BPS}| \leq \delta^{-1}\varepsilon$ and once again the explicit formula for $\dot{\phi}_{\delta^{-1}m}^{BPS}$ gives the result in the statement. In order to obtain the bounds stated one must notice that $\dot{\phi}_{\delta^{-1}m}^{BPS}$ is increasing, so one bounds below by $\dot{\phi}_{\delta^{-1}m}^{BPS}(0)$ and above by $\dot{\phi}_{\delta^{-1}m}^{BPS}(\delta)$. ∎

The following lemma contains two sequences of values (m_n, ε_n) inducing sequences of values (m_n, δ_n) which can be used to show that the PDE constructed monopoles are actually all monopoles as done in proposition 2.2.4 and that there are monopoles with all values of mass $m \in \mathbb{R}^+$ as done in proposition 2.2.6.

Lemma 2.2.14. *Let I, J be the quantities provided by the previous lemma, then:*

1. *There are sequences (m_n, ε_n) and $\delta_n \leq \Delta(m_n, \varepsilon_n)$, such that $I_n = I(m_n, \delta_n) \to 0$. Moreover, for this sequence of (m_n, ε_n) and δ_n, the quantity*

$$\Phi_+(n) = \frac{1}{\delta_n}(m_n\coth(m_n) - 1 + 2\varepsilon_n) + 2G(\delta_n),$$

   also converges to zero.

2. *There are other sequences (m_n, ε_n) and $\delta_n \leq \Delta(m_n, \varepsilon_n)$, such that $J_n = J(m_n, \delta_n) = \rightarrow -\infty$. For these sequences of (m_n, ε_n) and δ_n, the quantity*

$$\Phi_-(n) = \frac{1}{\delta_n}(m_n\coth(m_n) - 1 - 2\varepsilon_n),$$

   converges to $+\infty$.

Proof. 1. We shall first fix a sequence $m_n \to 0$. Then, $m_n \coth(m_n) - 1 = O(m_n^2)$ and notice that to prove the statement it is enough to show that one can take the sequences to be such that both $\frac{m_n}{\delta_n}$ and $\frac{\varepsilon_n}{\delta_n}$ converge to 0, while δ_n can be taken arbitrarily large, so that $G(\delta_n) \to 0$. To achieve this we shall first take $m_n \to 0$ as remarked before, and $\varepsilon_n = m_n^a$, for some positive $a < 1$, then $\varepsilon_n \leq \sqrt{\varepsilon_n}$ and the formula for $\Delta(m_n, \varepsilon_n)$ in proposition 2.2.8 is

$$\Delta(m_n, \varepsilon_n) \geq \frac{\varepsilon_n}{m_n} \min\left\{\frac{1}{\sqrt{\|(d_2)_n^*\| \|L_n\|}}, \frac{1}{\|(d_2)_n^*\| \|L_n\|}\right\}. \tag{2.2.24}$$

As $\|(d_2)_n^*\| \|L_n\|$ is uniformly bounded above and below for any sequence $m_n \to 0$, we can take $\delta_n = C\frac{\varepsilon_n}{m_n} = Cm_n^{a-1}$, for some $C > 0$. In this way we do have δ_n getting arbitrarily large and

$$\frac{m_n}{\delta_n} = C^{-1} m_n^{2-a}, \quad \frac{\varepsilon_n}{\delta_n} = C^{-1} m_n,$$

which do converge to zero as m_n does.

2. One can take $m_n = m > 0$ constant and ε_n to be a sequence converging to zero, in this way the inequality 2.2.24 still holds and it is enough to set $\delta_n = Cm^{-1}\varepsilon_n$, where $C > 0$ is constant. By substitution in J_n one obtains $J_n = -k_1\varepsilon_n^{-1} + k_2\sqrt{\varepsilon_n}$, for some positive real constants k_1, k_2 and this converges to $-\infty$ as $\varepsilon_n \to 0$.
To check that $\Phi_-(n) \to +\infty$, notice that by increasing n, ε_n can be taken arbitrarily small and so $m\coth(m) - 1 - 2\varepsilon_n$ is greater than a positive constant C'. Since $\delta_n = Cm^{-1}\varepsilon_n$ is converging to zero we see that

$$\Phi_-(n) \geq \frac{Cm}{C'}\frac{1}{\varepsilon_n} \to +\infty.$$

□

Lemma 2.2.15. *Let v be a solution of 2.2.20. Suppose v has a minimum at m, or a maximum at M, then $v(m) \geq 0$ and $v(M) \leq 0$. Moreover, if v satisfies initial conditions $v(\delta) < 0$, $\dot{v}(\delta) < 0$ (resp. $v(\delta) > 0$, $\dot{v}(\delta) > 0$), then $v < 0$ (resp. $v > 0$) in (δ, ∞).*

Proof. Let m be the point at which the minimum is achieved, then $\ddot{v}(m) \geq 0$ and so

$$\frac{2}{h^2}(e^v - 1) \geq 0 \quad \Longrightarrow \quad v \geq 0.$$

In the same way at a maximum M, $\ddot{v}(M) \leq 0$ and this gives $\frac{2}{h^2}(e^v - 1) \leq 0$, which implies $v \leq 0$. For the second part assume that $v(\delta), \dot{v}(\delta) < 0$, then one needs to prove that $v < 0$, for all $t \geq \delta$. Suppose not, then let $x > \delta$ be the smallest possible such that $v = 0$. Since $v(\delta), \dot{v}(\delta) < 0$ there must be a minimum $m \in (\delta, x)$. Applying the maximum principles just proved to conclude that $v(m) \geq 0$ and this contradicts the minimality of x. □

Corollary 2.2.16. *There are no solutions to the ODE 2.2.20 with $v(0) = \dot{v}(0) = 0$ and $\lim_{r\to\infty} r^{-k}e^v = 0$ for some $k \in \mathbb{Z}$, such that $\ddot{v}(0) = b > 0$.*

Proof. Since $v(0) = \dot{v}(0) = 0$ and $\ddot{v}(0) = b > 0$, there is $\delta > 0$ such that $v(\delta), \dot{v}(\delta)$ are both positive. Then, by lemma 2.2.15, $v > 0$ in $(\delta, +\infty)$. Using the equation $\ddot{v} = \frac{2}{h^2}(e^v - 1)$ we see that $\ddot{v} > 0$ in $(\delta, +\infty)$. Integrating this gives that

$$v(r) \geq v(\delta) + \dot{v}(\delta)(r - \delta),$$

for all $r \geq \delta$. Then $r^{-k}e^v \geq r^{-k}e^{v(\delta)+\dot{v}(\delta)(r-\delta)}$, and since $\dot{v}(\delta)$ is positive, for all $k \in \mathbb{Z}$ this diverges as $r \to +\infty$. □

Lemma 2.2.17. *Let $u, v, a : (\delta, \infty) \to \mathbb{R}$ be differentiable $u < 0$, such that*

$$\ddot{v} - av \geq 0 \ , \ \ddot{u} - au = 0.$$

If $u(\delta) = v(\delta)$ and $\dot{u}(\delta) = \dot{v}(\delta)$, then $v(r) \geq u(r)$ for all $r \geq \delta$.

Proof. Define $f = \frac{v}{u}$, since by assumption $u < 0$ it is enough to prove that $f \leq 1$, for $r \geq \delta$ and that $f \geq 1$ for $r \leq \delta$. Moreover, since $f(\delta) = 1$ it is enough to prove that $\dot{f} \leq 0$, i.e. that $\dot{v}u - v\dot{u} \leq 0$. Once again, our hypothesis dictate that at $r = \delta$ this expression vanishes and so it is enough to show that its derivative $\ddot{v}u - v\ddot{u}$ is nonpositive. Substituting $\ddot{u} = au$ and $\ddot{v} \geq av$ gives that indeed $\ddot{v}u - v\ddot{u} \leq 0$. □

Proposition 2.2.18. *Let v be a solution of 2.2.20 on (δ, ∞), with the initial conditions $v(\delta) = -k_2 < 0$ and $\dot{v}(\delta) = -k_1 < 0$, for some positive constants k_1, k_2. Then, for $t \geq \delta$*

$$v_b(r) \leq v(t) \leq v_u(r),$$

where $v_b(r) = -k_2 - k_1(r - \delta) - 2\int_\delta^r \int_\delta^s h^{-2}(s')ds'ds$, and $v_u(t)$ solves $\ddot{v_u} - \frac{2}{h^2}v_u = 0$ with the initial conditions $v_u(\delta) = -k_2$, $\dot{v}_u(\delta) = -k_1$.

Proof. Since the function $F(v) = e^v$ is convex it lies above all its tangents, then $\ddot{v} = \frac{2}{h^2}(e^v - 1) \geq \frac{2}{h^2}v$. The second step is using lemma 2.2.17 with $a = \frac{2}{h^2}$ and $u = v_b$ to obtain the lower bound. The upper bound comes from integrating $\ddot{v} \geq -\frac{2}{h^2}$, which holds since e^v is positive. □

Insert $a^2 = e^v$ into the first monopole ODE in 2.2.3, then

$$\dot{\phi} = \frac{1}{2h^2}(e^v - 1).$$

The above bounds on v can be used to estimate the values of the Higgs field. However, in the following application a crude approach to these bounds will be given. Since $\ddot{v}(0) < 0$, the maximum principle from lemma 2.2.15 guarantees $v \leq 0$ for all r. Moreover, the standard existence and uniqueness theorem applies locally at $r = \delta$ and the estimates in 2.2.23 show this extends to the right. Moreover, this can be applied to compute

Proposition 2.2.19. *Let (a, ϕ) be a solution to the monopole ODE's 2.2.3, then for all $t \in (\delta, \infty)$*

$$\phi(\delta) \geq \phi(r) \geq \phi(\delta) - \int_\delta^r \frac{1}{2h^2(t)}dt.$$

So, if the Green's function $G(r) = -\int \frac{1}{2h^2(r)} dr$ *is bounded at* ∞*, then so is the Higgs field.*

This together with the fact that $\dot{\phi}(r) \to 0$ as $r \to \infty$ allows the conclusion that the limit $\phi(\infty) = \lim_{r\to\infty} \phi(r)$, exists and is finite. As an application one obtains

Corollary 2.2.20. *Let* g *be a spherically symmetric metric and* $(A, \Phi) \in \mathcal{M}_{inv}$ *an invariant monopole on* $(\mathbb{R}^3, g)$*. The norm of the Higgs field is dominated by the Green's function G. Moreover, if G is bounded at infinity then the mass* $m(A, \Phi)$ *exists and is finite. Let* $m \in \mathbb{R}^+$ *and* $\varepsilon > 0$*, then for* $\delta \leq \Delta(m, \varepsilon)$*, the monopole* $(A^\delta_m, \Phi^\delta_m)$ *satisfies*

$$m(A^\delta_m, \Phi^\delta_m) \in \left[\frac{1}{\delta} \left(m \coth(m) - 1 - 2\varepsilon \right), \frac{1}{\delta} \left(m \coth(m) + 2\varepsilon \right) + 2G(\delta) \right].$$

Chapter 3

Monopoles on Calabi-Yau 3 Folds

This chapter is organized as follows, in section 3.1 one defines complex monopoles and also a particular case of these which shall be called just Calabi-Yau monopoles. For complex monopoles, one goes to study the associated linearized operator, which fits into an elliptic complex. This is done in subsection 3.1.2 and is a necessary step in order to use the results of chapter 5. Subsection 3.1.3 defines the relevant energies for complex monopoles and deduces some integral identities. These will be used later in subsection 3.1.4 for AC Calabi-Yau manifolds to compute the relevant energies and to prove proposition 3.1.26 which is a vanishing theorem for complex monopoles. This gives conditions under which all complex monopoles reduce to Calabi-Yau monopoles. This subsection also gives existence results such as proposition 3.1.31 for the boundary data determined by the asymptotics of complex monopoles.

In section 3.2.1 a promising source of examples to study these monopoles and their interaction with special Lagrangian geometry is explored. For one of these, the Stenzel metric on $T^*\mathbb{S}^3$, Calabi-Yau monopoles are actually found. In the other cases one sets up the problem of studying complex monopoles, for which the results of chapter 5 give a nice Fredholm setup. Also in this case proposition 3.1.26 applies and gives conditions under which these complex monopoles are actually Calabi-Yau monopoles.

Section 3.3 proves theorem 3.3.1 regarding Calabi-Yau monopoles for the Stenzel metric. It proves that there is a class of Calabi-Yau monopoles called invariant monopoles which are parametrized by their mass. In this setting, the large mass limit is studied. It is is shown that in the limit where the mass goes to infinite, there is a BPS monopole bubbling off along the transverse directions to the zero section (which is special Lagrangian). This leaves behind a reducible monopole on its complement (which will be called a Dirac monopole).

3.1 The Equations

Let (X^6, ω, Ω) be a noncompact Calabi-Yau manifold, G a compact semisimple Lie group with Lie algebra $\mathfrak{g}$ and $P \to X$ a principal G bundle. Denote by $\mathfrak{g}_P = P \times_{(Ad,G)} \mathfrak{g}$ the adjoint bundle and $\mathfrak{g}_P^{\mathbb{C}}$ its complexification. Equip the first of these with an Ad-invariant metric and the second one with the respective Hermitian metric.

Definition 3.1.1. *Let A be a G connection and $\Phi = \Phi_1 + i\Phi_2 \in \Omega^0(X, \mathfrak{g}_P^{\mathbb{C}})$ a complex Higgs Field, with $\Phi_1, \Phi_2 \in \Omega^0(X, \mathfrak{g}_P)$. The pair (A, Φ) is called a complex monopole if*

$$* \partial_A \Phi = \frac{1}{2} F_A \wedge \Omega, \tag{3.1.1}$$

$$\Lambda F_A = \frac{i}{2}[\Phi, \overline{\Phi}], \tag{3.1.2}$$

where $\Lambda\beta = *(\beta \wedge \frac{\omega^2}{2})$ for $\beta \in \Omega^2(X, \mathbb{C})$ and $*$ is the $\mathbb{C}$-linear extension of the Hodge $*$ operator.

Definition 3.1.2. *A complex monopole (A, Φ) is called a Calabi-Yau monopole if $\Phi = \Phi_1$, i.e. $\Phi_2 = 0$, these satisfy*

$$* \nabla_A \Phi = F_A \wedge \Omega_1, \tag{3.1.3}$$

$$\Lambda F_A = 0. \tag{3.1.4}$$

3.1.1 Rewriting the Equations

Proposition 3.1.3. *The following are equivalent:*

1. *(A, Φ) is a complex monopole, i.e. a solution to 3.1.1 and 3.1.2.*
2. *The pair (A, Φ) satisfies*

$$F_A + *(F_A \wedge \omega) = *(d_A\Phi_1 \wedge \Omega_1) + *(d_A\Phi_2 \wedge \Omega_2) + [\Phi_1, \Phi_2]\omega. \tag{3.1.5}$$

3. *The pair (A, Φ) satisfies*

$$* d_A\Phi_1 = F_A \wedge \Omega_1 - d_A\Phi_2 \wedge \frac{\omega^2}{2}, \tag{3.1.6}$$

$$F_A \wedge \frac{\omega^2}{2} = [\Phi_1, \Phi_2]\frac{\omega^3}{3!}. \tag{3.1.7}$$

Moreover, one can also rewrite the first equation as $*d_A\Phi_2 = F_A \wedge \Omega_2 + d_A\Phi_1 \wedge \frac{\omega^2}{2}$.

4. *The pair (A, u) with $u = -\frac{i}{4}\Phi\overline{\Omega} \in \Omega^{0,3}(X, \mathfrak{g}_P^{\mathbb{C}})$ is a solution to*

$$F_A^{0,2} = -\overline{\partial}_A^* u, \tag{3.1.8}$$

$$\Lambda F_A = *[u \wedge \overline{u}], \tag{3.1.9}$$

Proof. The proof will outline the equivalence of all equation in items $2, 3, 4$ with the equations 3.1.1 and 3.1.2.
$(1 \Leftrightarrow 2)$: Setting $\Phi = \Phi_1 + i\Phi_2$ gives $\frac{i}{2}[\Phi, \overline{\Phi}] = [\Phi_1, \Phi_2]$. Next, it follows from linear algebra that $F_A + *(F_A \wedge \omega) = \Lambda F_A \omega + 2(F_A^{2,0} + F_A^{0,2})$, hence the component along the Kähler form gives back equation 3.1.2. To recover equation 3.1.1 take the wedge of equation 3.1.5 with Ω and use that $*(d_A\Phi \wedge \Omega) \wedge \Omega = 0$ and $*(d_A\Phi \wedge \overline{\Omega}) \wedge \Omega = 8 * \partial_A\Phi$.

$(1 \Leftrightarrow 3)$: Taking the Hodge $*$ of the second equation and using that $*(F_A \wedge \frac{\omega^2}{2}) = \Lambda F_A$ one obtains equation 3.1.2. The equation 3.1.1 is obtained by taking the Hodge $*$ of the first equation and using the fact that $*\left(d_A\Phi_2 \wedge \frac{\omega^2}{2}\right) = -Id_A\Phi_2$, where I denotes the complex structure.

$(1 \Leftrightarrow 4)$: This case is a bit more involved. Start with the first complex monopole equation 3.1.8, replace $u = -\frac{i}{4}\Phi\overline{\Omega}$ to obtain

$$F^{0,2} \quad = \quad -\overline{\partial}_A^* \left(\frac{-i}{4}\Phi\overline{\Omega}\right) = -\frac{i}{4} * \partial_A \left(\Phi * \overline{\Omega}\right) = \frac{1}{4} * \left(\partial_A \Phi \wedge \overline{\Omega}\right).$$

Where one uses that $\overline{\partial}_A^* = - * \partial_A *$, $*\overline{\Omega} = i\overline{\Omega}$ and $\overline{\partial}\Omega = 0$. The next step is to wedge this with Ω and take the resulting equation is $F \wedge \Omega = \frac{1}{4} * \left(\partial_A\Phi \wedge \overline{\Omega}\right) \wedge \Omega = 2 * \partial_A \Phi$. To unwind the right hand side it was needed to use the fact that the projection $\Omega^1 \to \Omega^{1,0}$ can be written as $a^{1,0} = -\frac{1}{8} * (*(a \wedge \overline{\Omega}) \wedge \Omega)$, for $a \in \Omega^1$. This finishes the proof that the equations 3.1.8 and 3.1.1 are equivalent. Regarding the second equations, start with 3.1.2 and replace $u = -\frac{i}{4}\Phi\Omega$, then after using that $*(\Omega \wedge \overline{\Omega}) = -8i$, equation 3.1.9 pops up. ☐

There is a very useful vanishing result stated below as lemma 3.1.5. This will be used in the proof of propositions 3.1.9 and 3.1.23 and its proof requires the following extension of Stokes' theorem to complete Riemannian manifolds.

Theorem 3.1.4. *([Gaf54]) Let (M^n, g) be an orientable and complete Riemannian manifold and $\alpha \in \Omega^{n-1}(X, \mathbb{R})$ be such that $\gamma, d\gamma \in L^1$, then, $\int_M d\gamma = 0$.*

Lemma 3.1.5. *Let (X, ω, Ω) be a complete Calabi-Yau manifold, (A, u) a complex monopole on $P \to X$, i.e. a solution to 3.1.8 and 3.1.9. Then, if $\phi \in \Omega^0(X, \mathfrak{g}_P^{\mathbb{C}})$ is bounded and such that $\overline{\partial}_A\phi = [\overline{u}, \phi] = 0$ while $\partial_A\phi, [u, \phi] \in L^2$ and $\langle \partial_A\phi, \phi\rangle_{\mathbb{C}} \in L^1$, then in fact also $\partial_A\phi = [u, \phi] = 0$ and so $d_A\phi = 0$.*

Proof. Let $\langle \cdot, \cdot \rangle_{\mathbb{C}} = \langle \cdot, \bar{\cdot} \rangle$ be the Hermitian extension of the inner product and differentiate $\langle \partial_A\phi, \phi\rangle_{\mathbb{C}}$. This gives

$$\partial^* \langle \partial_A\phi, \phi\rangle_{\mathbb{C}} \quad = \quad \langle \Delta_{\partial_A}\phi, \phi\rangle_{\mathbb{C}} - |\partial_A\phi|^2. \tag{3.1.10}$$

Moreover, since by hypothesis ϕ is holomorphic, $\Delta_{\partial_A}\phi = \Delta_{\partial_A}\phi - \Delta_{\overline{\partial}_A}\phi = [i\Lambda F_A^{1,1}, \phi]$. This is a straightforward application of the twisted Kähler identities stated in lemma 1.1.3. Inserting in this the equation $i\Lambda F = i*[u\wedge\overline{u}]$ and $iu = *u$, gives $\Delta_{\partial_A}\phi = [*[*u\wedge\overline{u}], \phi] = [[\overline{u}, \phi]\wedge *u] - [[\phi, *u]\wedge\overline{u}]$. So replacing this back into equation 3.1.10, integrating and using theorem 3.1.4 gives

$$0 = \|[\overline{u}, \phi]\|_{L^2}^2 - \|[u, \phi]\|_{L^2}^2 - \|\partial_A\phi\|_{L^2}^2.$$

The first of these vanishes by hypothesis and hence so do the other two terms. ☐

3.1.2 Linearized Operator

Recall that a complex monopole is a pair consisting of a connection A on P and a complexified Higgs field $u \in \Omega^{0,3}(X, \mathfrak{g}_P^{\mathbb{C}})$ satisfying the equations

$$F_A^{0,2} + \overline{\partial}_A^* u = 0 \ , \ i\Lambda F_A - i * [u \wedge \overline{u}] = 0. \tag{3.1.11}$$

Remark 3.1.6. *Further below proposition 3.1.26 proves that under certain conditions complex monopoles (A, Φ) are solutions to $*\nabla_A \Phi = F_A \wedge \Omega_1$ and $\Lambda F_A = 0$ with $\Phi \in \Omega^0(X, \mathfrak{g}_P)$, i.e. are Calabi-Yau monopoles. In fact, the Calabi-Yau monopole equation is overdetermined. However in a Calabi-Yau one may have hope that solutions exist, since the complex structure is integrable. Instead of working with these, for the deformation theory it is convenient to consider the more general complex monopole equations, as these are elliptic modulo gauge transformations.*

Use the identification $\Lambda^1 \cong \Lambda^{0,1}_{\mathbb{C}}$ to view deformations of the connection as $(0,1)$ forms. Then, at a complex monopole the linearized complex monopole equation gives a map d_2 from $\Omega^{0,odd}(X, \mathfrak{g}_P^{\mathbb{C}})$ to $\Omega^{0,2}(X, \mathfrak{g}_P^{\mathbb{C}}) \oplus i\Omega^0(X, \mathfrak{g}_P)$. This together with the linearization of the action by gauge transformations gives an elliptic complex

$$\Omega^0(X, \mathfrak{g}_P) \xrightarrow{d_1} \Omega^{0,1}(X, \mathfrak{g}_P^{\mathbb{C}}) \oplus \Omega^{0,3}(X, \mathfrak{g}_P^{\mathbb{C}}) \xrightarrow{d_2} \Omega^{0,2}(X, \mathfrak{g}_P^{\mathbb{C}}) \oplus i\Omega^0(X, \mathfrak{g}_P), \tag{3.1.12}$$

where

$$\begin{aligned} d_1\zeta &= (\overline{\partial}_A \zeta, [u, \overline{\zeta}]) & (3.1.13)\\ d_2(a, w) &= \left(\overline{\partial}_A a + \overline{\partial}_A^* w - i * [u \wedge \overline{a}], 2i\,\mathrm{Im}(\overline{\partial}_A^* a + i * [u \wedge \overline{w}])\right). & (3.1.14) \end{aligned}$$

Lemma 3.1.7. *If (A, u) is a complex monopole, the sequence 3.1.12 is a complex.*

Proof. This is an immediate consequence of the gauge invariance of the complex monopole equations. However a full computation of $d_2 \circ d_1$ is given below. So take $\zeta \in \Omega^0(X, \mathfrak{g}_P)$ and show that $d_2 d_1 \zeta = d_2(\overline{\partial}_A \zeta, [u, \overline{\zeta}])$ vanishes. The first component c_1 of this equation is

$$\begin{aligned} c_1 &= \overline{\partial}_A \overline{\partial}_A \zeta + \overline{\partial}_A^* [u, \overline{\zeta}] - i * [u \wedge \overline{\overline{\partial}_A \zeta}] \\ &= [F_A^{0,2} + \overline{\partial}_A^* u, \zeta] + i * [u \wedge \partial_A \overline{\zeta}] - i * [u \wedge \partial_A \overline{\zeta}] \end{aligned} \tag{3.1.15}$$

the last two terms annihilate each other and the first one vanishes since (A, u) is a complex monopole and $\zeta = \overline{\zeta} \in \Omega^0(X, \mathfrak{g}_P)$. The second component of $d_2 d_1 \zeta$ is

$$\begin{aligned} c_2 &= \overline{\partial}_A^* \overline{\partial}_A \zeta - \partial_A^* \partial_A \zeta + i * [u \wedge \overline{[u, \zeta]}]) + i * [[u, \zeta] \wedge \overline{u}] \\ &= [i\Lambda F_A, \zeta] - i * [[u \wedge \overline{u}], \zeta], \end{aligned} \tag{3.1.16}$$

and this also vanishes for a complex monopole. Moreover, one must remark that the computation above makes of use the twisted Kähler identities stated in lemma 1.1.3 and the graded Jacobi Identity, which reads $[a \wedge [b \wedge c]] + (-1)^{i(j+k)}[b \wedge [c \wedge a]] + (-1)^{k(i+j)}[c \wedge [a \wedge b]] = 0$, for $\mathfrak{g}_P$

valued forms a, b, c of degree i, j, k respectively. $\square$

So if (A, u) is a complex monopole then an elliptic operator can be made out of the complex 3.1.12. To do this notice that $d_1^*(a, w) = \operatorname{Re}\left(\overline{\partial}_A^* a + i * [u \wedge \overline{w}]\right)$, so one divides the second equation in d_2 by 2 and takes

$$D : d_2 \oplus d_1^* : \underbrace{\Omega^{0,1}(X, \mathfrak{g}_P^{\mathbb{C}}) \oplus \Omega^{0,3}(X, \mathfrak{g}_P^{\mathbb{C}})}_{\cong \Omega^{0,odd}(X, \mathfrak{g}_P^{\mathbb{C}})} \to \underbrace{\Omega^{0,2}(X, \mathfrak{g}_P^{\mathbb{C}}) \oplus \left(i\Omega^0(X, \mathfrak{g}_P) \oplus \Omega^0(X, \mathfrak{g}_P)\right)}_{\cong \Omega^{0,ev}(X, \mathfrak{g}_P^{\mathbb{C}})},$$

given by

$$D(a, w) = \left(\overline{\partial}_A^* a, \overline{\partial}_A a + \overline{\partial}_A^* w\right) + \left(i * [u \wedge \overline{w}], -i * [u \wedge \overline{a}]\right). \tag{3.1.17}$$

The first of these terms is just the Dirac operator

$$D_A = \overline{\partial}_A + \overline{\partial}_A^* : \Omega^{0,odd}(X, \mathfrak{g}_P^{\mathbb{C}}) \to \Omega^{0,ev}(X, \mathfrak{g}_P^{\mathbb{C}}),$$

which is a $\mathbb{C}$-linear operator. The second one q defines a section of $\operatorname{End}(\Lambda^{0,odd} \otimes \mathfrak{g}_P^{\mathbb{C}}, \Lambda^{0,ev} \otimes \mathfrak{g}_P^{\mathbb{C}})$ and is $\mathbb{C}$-antilinear.

Remark 3.1.8. *One must notice that $D = D_A + q$ is the sum of a $\mathbb{C}$-linear and a $\mathbb{C}$-antilinear term respectively. Hence* $\ker D$ *is not a vector space over $\mathbb{C}$, but just a vector space over $\mathbb{R}$.*

Let $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ be as usual the Hermitian extention of the L^2 inner product on $\Lambda^* X \otimes \mathfrak{g}_P^{\mathbb{C}}$. Denote by D_A^* the formal adjoint of the Dirac operator and by q^+ the antiadjoint of q, i.e. such that $\langle D_A s_1, s_2 \rangle_{\mathbb{C}} = \langle s_1, D_A^* s_2 \rangle_{\mathbb{C}}$ and $\langle q(s_1), s_2 \rangle_{\mathbb{C}} = \overline{\langle s_1, q^+(s_2) \rangle_{\mathbb{C}}}$ for all s_1, s_2. The next result computes a Weitzenböck type formula for the operator D.

Proposition 3.1.9. *Let (X, g) be a complete Calabi-Yau manifold and (A, u) a pair on $P \to X$. With the notation $D^* = D_A^* + q^+$, then*

$$DD^*(\phi, b) = \left(\Delta_{\overline{\partial}_A} + W + qq^+\right)(\phi, b),$$

where $W(\phi, b) = \left(-\overline{\langle [F_A^{0,2}, b] \rangle_{\mathbb{C}}} - \langle [\overline{\partial}_A^* u, b] \rangle_{\mathbb{C}}, \frac{1}{4} * \left(*[*\overline{b} \wedge \partial_A \Phi] \wedge \overline{\Omega}\right) + [F_A^{0,2}, \phi] + [\overline{\partial}_A^* u, \overline{\phi}]\right)$ *and* $qq^+(\phi, b) = -\left(*[u \wedge *[\overline{u}, \phi]], *[u \wedge *[\overline{u} \wedge b]]\right)$. *In particular, if (A, u) is an irreducible complex monopole and $\phi = \overline{\phi}$, i.e. it is real, then* $W(\phi, b) = \left(0, \frac{1}{4} * \left(*[*\overline{b} \wedge \partial_A \Phi] \wedge \overline{\Omega}\right)\right)$. *Moreover, if $(\phi, b) \in \ker(D^*) \cap L^2$, then $\phi = 0$ while b satisfies $\overline{\partial}_A b = 0$ and $\overline{\partial}_A^* b + i * [\overline{b} \wedge u] = 0$.*

Proof. The proof will just give some intermediate steps of the computation leading to the formula above. First one computes $q^+(\phi, b) = (i * [u \wedge \overline{b}], -i * [u, \overline{\phi}])$, which after combined with q, gives $qq^+(\phi, b) = -\left(*[u \wedge *[\overline{u}, \phi]], *[u \wedge *[\overline{u} \wedge b]]\right)$. Next, one computes $DD^* \cdot = D_A D_A^* \cdot + q(D_A^* \cdot) + D_A(q^+ \cdot) + qq^+ \cdot$ and here one uses the Weitzenböck formula in proposition 1.1.2 for $D_A D_A^*$ and the computation above for qq^+. Regarding the other terms, these are

$$\begin{aligned} qD_A^*(\phi,b) &= \left(i*[u\wedge\partial_A\overline{b}], -i*[u\wedge\partial_A\overline{\phi}] - i*[u\wedge\partial_A^*\overline{b}]\right), \\ D_Aq^+(\phi,b) &= \left(i\overline{\partial}_A^* * [u\wedge\overline{b}], -i*\partial_A[u,\overline{\phi}] + i\overline{\partial}_A * [u\wedge\overline{b}]\right). \end{aligned}$$

The first entry in the second line can be expanded using the Leibniz rule giving two terms, one of them kills the first term in the first line, while the other one is $-*[\overline{\partial}_A^*u\wedge *\overline{b}]$. In the second entry in the second line one can compute $-i*\partial_A[u,\overline{\phi}] = i*[u\wedge\partial_A\overline{\phi}] + [\overline{\partial}_A^*u,\overline{\phi}]$, the first of which kills the respective term in the first line. Then, expand $i\overline{\partial}_A*[u\wedge\overline{b}]$ in two terms, one of them kills the second term in the second entry of the first line and the other one is $\frac{1}{4}*\left(*[*\overline{b}\wedge\partial_A\Phi]\wedge\overline{\Omega}\right)$. Summing these with the zero order terms appearing in the Weitzenböck formula for $D_AD_A^*$ in proposition 1.1.2 gives that the only term left is precisely $W(\phi,b)$. The second assertion follows from using the using the first complex monopole equation $F_A^{0,2}+\overline{\partial}_A^*u=0$ twice, with ϕ real, i.e. $\overline{\phi}=\phi$. Moreover, if $(\phi,b)\in\ker(D^*)\cap L^2$, then in particular $(\phi,b)\in\ker(DD^*)\cap L^2$. Taking the inner product of $(\phi,0)$ with $DD^*(\phi,b)$ and using the formula just proved and theorem 3.1.4 to integrate by parts, gives $\|\overline{\partial}_A\phi\|_{L^2}^2+\|[\overline{u},\phi]\|_{L^2}^2=0$ and so ϕ commutes with $\overline{u}$ and is holomorphic. Then lemma 3.1.5 proves that under such conditions, ϕ also commutes with u and $\partial_A\phi=0$, hence $\nabla_A\phi=0$ and ϕ is covariant constant. This, together with the assumption that A is irreducible implies that $\phi=0$. □

Proposition 3.1.10. *Under the conditions of proposition 3.1.9, then*

$$D^*D(a,w) = \left(\Delta_{\overline{\partial}_A} + \tilde{W}_1 + \tilde{W}_2 + q^+q\right)(a,w),$$

where $\tilde{W}_1(a,w) = (i*[\partial_Au\wedge\overline{a}], 2[u,\partial_A^*\overline{a}])$, $\tilde{W}_2(a,w) = (-\langle[\nabla_A^{0,1}u,\overline{w}]\rangle + *[F_A^{2,0}\wedge *w], [F_A^{0,2}\wedge a] - \frac{i}{4}*[\overline{\partial}_A\Phi\wedge *\overline{a}]\wedge\overline{\Omega})$ *and* $q^+q(a,w) = -(*[u\wedge *[\overline{u}\wedge a]], *[u,*[\overline{u}\wedge w]])$.

Proof. The proof is a computation, similar to the one of proposition 3.1.9 of which the main intermediate steps will be given. First one computes $D^*D\cdot = D_A^*D_A\cdot + D_A^*q\cdot + q^+D_A\cdot + q^+q\cdot$, then for the first term one uses the Weitzenböck formula in proposition 1.1.2 and the computation of q^+q is straightforward and gives the last term in the formula in the statement. Next one needs to compute the two terms in the middle which are

$$\begin{aligned} q^+D_A(a,w) &= \left(i*[u\wedge\partial_A\overline{a}] + i*[u\wedge\partial_A^*\overline{w}], [u,\partial_A^*\overline{a}]\right), \\ D_A^*q(a,w) &= \left(i*\partial_A[u\wedge\overline{a}] + i\overline{\partial}_A*[u\wedge\overline{w}], -i\overline{\partial}_A*[u\wedge\overline{a}]\right). \end{aligned}$$

Expanding the terms in the second line using the Leibniz rule gives: In the first term in the first entry $i*\partial_A[u\wedge\overline{a}] = -i*[u\wedge\partial_A\overline{a}] + i*[\partial_Au\wedge\overline{a}]$, the first of which kills the first term in the first line. Next is the term $i\overline{\partial}_A*[u\wedge *\overline{w}] = -\overline{\partial}_A*[u\wedge *\overline{w}] = -\langle[\nabla_A^{0,1}u,\overline{w}]\rangle - \langle[u,\nabla_A^{0,1}\overline{w}]\rangle$ and the second of these kills the corresponding term in the first line since $-\langle[u,\nabla_A^{0,1}\overline{w}]\rangle = *[u\wedge *\overline{\partial_Aw}] = -*[u\wedge\partial_A^*\overline{w}]$. Finally, the last term in the second line gives, after a tedious computation $-\frac{i}{4}*[\overline{\partial}_A\Phi\wedge *\overline{a}]\wedge\overline{\Omega} + [u,\partial_A^*\overline{a}]$ and this second term adds with the last term in the first line. Putting these together with the zero order terms appearing in $D_A^* D_A$ gives the formulas in the statement for $\tilde{W}_1$ and $\tilde{W}_2$. $\square$

3.1.3 Energy Identities

Proposition 3.1.11. *Let (A, Φ) be a pair on P and $U \subset X$ precompact with smooth boundary ∂U. Then*

$$\begin{aligned} \|\Lambda F_A\|^2_{L^2(U)} + \|[\Phi_1, \Phi_2]\|^2_{L^2(U)} &= \|\Lambda F_A - [\Phi_1, \Phi_2]\|^2_{L^2(U)} \\ &\quad -2\int_{\partial U} \langle \Phi_1, *Id_A\Phi_2\rangle + 2\int_U \Lambda(d_A\Phi_1 \wedge d_A\Phi_2), \end{aligned}$$

where ω denotes the dual of the Kähler form and I is acting by pullback.

Proof. Start by working out the first term in the right hand side

$$\|\Lambda F_A - [\Phi_1, \Phi_2]\|^2_{L^2(U)} = \|\Lambda F_A\|^2_{L^2(U)} + \|[\Phi_1, \Phi_2]\|^2_{L^2(U)} - 2\langle \Lambda F_A, [\Phi_1, \Phi_2]\rangle_{L^2(U)}.$$

And so, one just needs to identify the mixed term with the integrals in the second line of the statement. This is done as follows

$$\begin{aligned} \langle \Lambda F_A, [\Phi_1, \Phi_2]\rangle_{L^2(U)} &= \int_U \langle F_A, [\Phi_1, \Phi_2]\rangle \wedge \frac{\omega^2}{2} = -\int_U \langle [F_A, \Phi_2], \Phi_1\rangle \wedge \frac{\omega^2}{2} \\ &= -\int_U \langle d_A^2\Phi_2, \Phi_1\rangle \wedge \frac{\omega^2}{2} \\ &= -\int_U d\left(\langle d_A\Phi_2, \Phi_1\rangle \wedge \frac{\omega^2}{2}\right) + \langle d_A\Phi_2 \wedge d_A\Phi_1\rangle \wedge \frac{\omega^2}{2} \\ &= -\int_{\partial U} \langle d_A\Phi_2, \Phi_1\rangle \wedge \frac{\omega^2}{2} - \int_U \langle d_A\Phi_2 \wedge d_A\Phi_1\rangle \wedge \frac{\omega^2}{2}, \end{aligned}$$

where in the second line one uses the Ad-invariance of the inner product and the definition of curvature. The result then follows from the fact that $*(d_A\Phi_2 \wedge \frac{\omega^2}{2}) = -Id_A\Phi_2$ (with I acting by pullback) and that $*\left(\langle d_A\Phi_2 \wedge d_A\Phi_1\rangle \wedge \frac{\omega^2}{2}\right) = -g(Id_A\Phi_2, d_A\Phi_1) = \omega(d_A\Phi_2, d_A\Phi_1)$, or $\Lambda(d_A\Phi_2 \wedge d_A\Phi_1)$ in the previous notation. Apply these to the last term, the add it to the first equation. $\square$

Proposition 3.1.12. *Let (A, Φ) be a complex monopole and $U \subset X$ precompact with smooth boundary ∂U. Write $\Phi = \Phi_1 + i\Phi_2$, then for both $i = 1, 2$*

$$\|\nabla_A \Phi_i\|^2_{L^2(U)} = -\int_U \Lambda(d_A\Phi_1 \wedge d_A\Phi_2) + \int_{\partial U} \langle \Phi_i, F\rangle \wedge \Omega_i.$$

Proof. We prove only the case $i = 2$ as the case $i = 1$ follows from a similar computation. Write $|d_A\Phi_2|^2 = \langle d_A\Phi_2 \wedge *d_A\Phi_2\rangle$ and use the equation in item 3 of proposition 3.1.3 to replace $*d_A\Phi_2$.

This gives

$$\begin{aligned}\|\nabla_A \Phi_2\|^2_{L^2(U)} &= \int_U \langle d_A \Phi_2 \wedge \left(F \wedge \Omega_2 + d_A \Phi_1 \wedge \frac{\omega^2}{2} \right) \rangle \\ &= + \int_U d\left(\langle \Phi_2, F \rangle \wedge \Omega_2 \right) - \int_U \langle d_A \Phi_1 \wedge d_A \Phi_2 \rangle \wedge \frac{\omega^2}{2},\end{aligned}$$

where one used the Bianchi identity $d_A F_A = 0$ and the closedness of Ω_1. Then the result follows from Stokes' theorem. The second identity follows from a similar computation. □

Corollary 3.1.13. *Let (A, Φ) be a complex monopole and $U \subset X$ precompact with smooth boundary ∂U. Then*

$$\|[\Phi_1, \Phi_2]\|^2_{L^2(U)} + \|\nabla_A \Phi_i\|^2_{L^2(U)} = - \int_{\partial U} \langle \Phi_1, *I d_A \Phi_2 \rangle + \int_{\partial U} \langle \Phi_i, F \rangle \wedge \Omega_i.$$

Lemma 3.1.14. *Let $(A, \Phi = \Phi_1 + i\Phi_2)$ be a complex monopole, then*

$$\Delta \frac{1}{2} |\Phi_i|^2 = -|[\Phi_1, \Phi_2]|^2 - |\nabla_A \Phi_i|^2.$$

In particular $|\Phi_1|^2$ and $|\Phi_2|^2$ are subharmonic and so is $|\Phi|^2 = |\Phi_1|^2 + |\Phi_2|^2$.

Proof. The proof follows from $\Delta \frac{1}{2} |\Phi_1|^2 = \langle \Phi_1, \Delta_A \Phi_1 \rangle - |\nabla_A \Phi_1|^2$ and the computation of the first of these terms. Using both the complex monopole equations as in the third item of proposition 3.1.3, the definition of curvature and the Bianchi identity, gives

$$\begin{aligned}\Delta_A \Phi_1 &= - * d_A * d_A \Phi_1 = - * d_A \left(F_A \wedge \Omega_1 - d_A \Phi_2 \wedge \frac{\omega^2}{2} \right) \\ &= *[F_A, \Phi_2] \wedge \frac{\omega^2}{2} = [[\Phi_1, \Phi_2], \Phi_2].\end{aligned}$$

Then the Ad-invariance of the metric gives $\langle \Delta_A \Phi_1, \Phi_1 \rangle = -|[\Phi_1, \Phi_2]|^2$ which gives the equation in the statement for $\Delta |\Phi_1|^2$. Regarding the equation for $\Delta |\Phi_2|^2$, a computation along the same lines gives $\Delta_A \Phi_2 = -[[\Phi_1, \Phi_2], \Phi_1]$ and the result in the statement then follows from the Ad invariance of the metric. □

As in the preliminary case analyzed in section 1.3.1 there are two relevant energies in play. One of them is an analogue of definition 1.3.1.

Definition 3.1.15. *The Yang-Mills-Higgs (YMH) energy E_U and the intermediate energy E^I_U of a pair (A, Φ) over precompact set $U \subset X$ with smooth boundary ∂U are respectively defined by*

$$E_U(A, \Phi) = \frac{1}{2} \|F_A\|^2_{L^2(U)} + \frac{1}{2} \|\nabla_A \Phi\|^2_{L^2(U)}, \tag{3.1.18}$$

$$E^I_U(A, \Phi) = \frac{1}{2} \|\frac{1}{2} F_A \wedge \Omega\|^2_{L^2(U)} + \frac{1}{2} \|\partial_A \Phi\|^2_{L^2(U)}. \tag{3.1.19}$$

The intermediate energy does not measure the full L^2 norm of the curvature, so there may be complex monopoles with infinite YMH energy but finite intermediate energy and this is indeed the case for the complex monopoles to be constructed.

Proposition 3.1.16. *Let $U \subset X$ be precompact with smooth boundary ∂U as above, such that the intermediate energy of the pair (A, Φ) on U is finite, i.e. $E_U^I(A,\Phi) < \infty$, then*

$$E_U^I(A,\Phi) = \frac{1}{2}\|\frac{1}{2} * (F_A \wedge \Omega) - \partial_A \Phi\|^2_{L^2(U)} + \frac{1}{2}\int_{\partial U} \langle \Phi, F_A \wedge \Omega \rangle.$$

In particular, if (A, Φ) is a complex monopole, then

$$E_U^I(A,\Phi) \;=\; \frac{1}{2}\int_{\partial U} \langle \Phi_1, F_A \rangle \wedge \Omega_1 + \frac{1}{2}\int_{\partial U} \langle \Phi_2, F_A \rangle \wedge \Omega_2, \tag{3.1.20}$$

which is just another way of writing the boundary integral in the first formula. Moreover, $\int_{\partial U} \langle \Phi_2, F_A \rangle \wedge \Omega_1 = \int_{\partial U} \langle \Phi_1, F_A \rangle \wedge \Omega_2$.

Proof. Start by computing $\|\frac{1}{2} * (F_A \wedge \Omega) - \partial_A \Phi\|^2_{L^2} = \|\frac{1}{2} F_A \wedge \Omega\|^2_{L^2} + \|\partial_A \Phi\|^2_{L^2} - \langle \partial_A \Phi, *(F_A \wedge \Omega)\rangle_{L^2}$. The first two terms give twice the Intermediate energy, i.e $2E_U^I(A,\Phi) \in \mathbb{R}$, so the last term must also be real. Then integrating it by parts and using Stokes' theorem, $d\Omega = 0$ and $d_A F_A = 0$, gives

$$\langle \partial_A \Phi, *(F_A \wedge \Omega)\rangle_{L^2} \;=\; \int_U \langle \partial_A \Phi, F_A \wedge \Omega \rangle = \int_{\partial U} \langle \Phi, F_A \wedge \Omega \rangle,$$

where the Bianchi identity and the closedness of Ω have been used. Dividing by 2 and rearranging gives the result in the statement. The rest of the statement follows from noticing that for a complex monopole $\frac{1}{2} * (F_A \wedge \Omega) - \partial_A \Phi = 0$ and expanding the boundary integral. The last identity follows from expanding $0 = \langle \nabla_A \Phi_1, \nabla_A \Phi_2 \rangle - \langle \nabla_A \Phi_2, \nabla_A \Phi_1 \rangle$ using the complex monopole equations and integrating by parts. $\square$

Corollary 3.1.17. *Suppose X is compact and (A, Φ) a complex monopole, then $F_A^{0,2} = \Lambda F_A = 0$ and $\nabla_A \Phi = [\Phi, \overline{\Phi}] = 0$, i.e. A is a reducible Hermitian Yang Mills connection, with an explicit reduction Φ.*

Proof. This is an immediate consequence of proposition 3.1.16 and corollary 3.1.13, by integrating over X. $\square$

Proposition 3.1.18. *Let the pair (∇_A, Φ) be real, i.e. the Higgs field is such that $\Phi = \Phi_1$. Then the YMH energy of the pair (A, Φ) on a precompact set $U \subset X$ with smooth boundary ∂U is given by*

$$\begin{aligned} E_U \;=\; & \frac{3}{2}\|\Lambda F_A \omega\|^2_{L^2(U)} + \frac{1}{2}\| * (F_A \wedge \Omega_1) - \nabla_A \Phi\|^2_{L^2(U)} \\ & + \int_{\partial U} \langle \Phi, F_A \rangle \wedge \Omega_1 - \frac{1}{2}\int_U F_A \wedge F_A \wedge \omega. \end{aligned}$$

Proof. The proof follows from splitting the curvature into orthogonal components

$$F = (F^{2,0} + F^{0,2}) + F_P^{1,1},$$

where $F_P^{1,1} = \frac{\Lambda F}{3}\omega$. It follows from linear algebra for the two forms that $(F^{2,0} + F^{0,2}) = -\frac{1}{2} * (*(F \wedge \Omega_1) \wedge \Omega_1)$ and $F_P^{1,1} = \frac{2}{3}F - \frac{1}{3} * (F \wedge \omega) - \frac{1}{3}(F^{2,0} + F^{0,2})$. The first of these formulas gives $\|F^{2,0} + F^{0,2}\|^2_{L^2} = \frac{1}{2}\|F \wedge \Omega_1\|^2_{L^2}$. Summing the various orthogonal components and solving for $\|F\|^2_{L^2}$ one concludes that

$$\|F\|^2_{L^2} = 3\|\Lambda F\omega\|^2_{L^2} + \|F \wedge \Omega_1\|^2_{L^2} - \int F \wedge F \wedge \omega. \tag{3.1.21}$$

To finally compute $E = \frac{1}{2}\|F\|^2_{L^2} + \frac{1}{2}\|\nabla_A \Phi\|^2_{L^2}$ sum $\frac{1}{2}\|\nabla_A \Phi\|^2_{L^2}$ with half of equation 3.1.21. Using proposition 3.1.16 to substitute for the term $\|F \wedge \Omega_1\|^2_{L^2}$, gives

$$\begin{aligned} E &= \frac{3}{2}\|\Lambda F_A \omega\|^2_{L^2(U)} + \frac{1}{2}\|F \wedge \Omega_1\|^2_{L^2} + \frac{1}{2}\|\nabla_A \Phi\|^2_{L^2(U)} - \frac{1}{2}\int_U F_A \wedge F_A \wedge \omega \\ &= \frac{3}{2}\|\Lambda F_A\, \omega\|^2_{L^2(U)} + \frac{1}{2}\| * (F_A \wedge \Omega_1) - \nabla_A \Phi\|^2_{L^2(U)} \\ &\quad + \int_{\partial U} \langle \Phi, F_A \rangle \wedge \Omega_1 - \frac{1}{2}\int_U F_A \wedge F_A \wedge \Omega_1. \end{aligned}$$

□

3.1.4 Monopoles on AC Calabi-Yau Manifolds

Let (X, ω, Ω) be an AC Calabi-Yau manifold as in section 1.1.2 and $P \to X$ a principal G-bundle. This section studies asymptotic conditions for irreducible complex monopoles on P analogous to the discussion in section 1.4.1. In particular, the boundary integrals $\lim_{r\to\infty} \int_{\partial B_r} \langle \Phi_1, *Id_A\Phi_2 \rangle$ and $\lim_{r\to\infty} \int_{\partial B_r} \langle \Phi_i, F \rangle \wedge \Omega_i$ appearing in the propositions in the previous section will be convergent. Then, in the spirit of proposition 1.4.9 and corollary 1.4.11 it is possible to obtain further results regarding the energy, such as proposition 3.1.23 and corollary 3.1.25. This subsection also gives proposition 3.1.26, which gives conditions under which complex monopoles end up being real and satisfy the equations in definition 3.1.2.

The first thing to be done is to adapt the definition of finite mass monopoles (A, Φ) as in definition 1.4.1, to complex monopoles. Suppose there is $K \subset X$ compact such that on the end $X \backslash K$, there is a bundle isomorphism

$$P|_{X\backslash K} \cong \varphi^* \pi^* P_\infty, \tag{3.1.22}$$

where φ is the diffeomorphism in definition 1.2.8, P_∞ is a G bundle over Σ and $\pi : (1, +\infty) \times \Sigma \to \Sigma$ is the projection on the second factor.

Definition 3.1.19. *A pair (A, Φ) is a finite mass complex (resp. Calabi-Yau) monopole on P, if it is a complex (resp. Calabi-Yau) monopole and there is $m \in \mathbb{R}$ such that $\lim_{\rho\to\infty} |\Phi| = m$ and a connection A_∞ on P_∞, such that after the identification 3.1.22, A is asymptotic to A_∞ on P_∞. i.e. there is $\varepsilon > 0$, such that $|A - A_\infty| = O(\rho^{-1-\varepsilon})$, outside K and using the isomorphism 3.1.22 to* *pullback the connection A_∞ to a connection on $P|_{X\backslash K}$.*

For a complex monopole the Higgs Field $\Phi \in \Omega^0(X, \mathfrak{g}_P^{\mathbb{C}})$ is a section of the complexified adjoint bundle and the results in section 1.4.1 do not apply directly to these. Nevertheless, the techniques used there, do extend in order to be applied to this complexified situation.

Proposition 3.1.20. *Let (X, ω, Ω) be AC and (A, Φ) be a finite mass, irreducible complex monopole. Then $\nabla_A \Phi, [\Phi, \overline{\Phi}] \in L^2$ and there is an A_∞-parallel Higgs Field $\Phi_\infty \in \Omega^0(\Sigma, \mathfrak{g}_{P_\infty}^{\mathbb{C}})$ such that Φ converges to Φ_∞. Moreover, there are positive constants c_1, c_2, such that on $X \backslash K$*

$$m^2 - \frac{c_1}{\rho^4} \leq |\Phi|^2 \leq m^2 - \frac{c_2}{\rho^4}. \tag{3.1.23}$$

Proof. Then, lemma 3.1.14 proves that $|\Phi|$ is subharmonic and the argument used in lemma 1.4.3 to prove proposition 1.4.4 applies to prove the inequality 3.1.23. The fact that $\nabla_A \Phi, [\Phi, \overline{\Phi}] \in L^2$ follows from applying the proof of the analogous fact in proposition 1.4.4. However, due to lemma 1.4.3, in this case $\Delta |\Phi|^2 = -\frac{1}{2} |[\Phi, \overline{\Phi}]|^2 - \frac{1}{2} |\nabla_A \Phi|^2$ and so one obtains instead that $|[\Phi, \overline{\Phi}]|^2 + |\nabla_A \Phi|^2 \in L^2$. The existence of Φ_∞ as in the statement follows from the fact that $\nabla_A \Phi \in L^2$ and applying proposition A.0.17 in the Appendix A. □

Remark 3.1.21. *If both Φ_1, Φ_2 converge respectively to $\Phi_{\infty,1}, \Phi_{\infty,2} \in \Omega^0(\Sigma, \mathfrak{g}_P)$ with these being A_∞-parallel. Then $|\Phi_{\infty,i}| = m_i$ is constant for $i = 1, 2$ and $[\Phi_{\infty,1}, \Phi_{\infty,2}] = 0$ hence one can use the fact that both $|\Phi_i|$'s are subharmonic by lemma 3.1.14 in order to get an inequality as in 3.1.23 for both of these.*

Below, the consequences of the finite mass assumption will continue to be explored. It will be useful to introduce some cohomology classes of the cross section Σ of the asymptotic cone $C(\Sigma)$. It will be obvious from the definition that these depend on the complex structure of the Calabi-Yau (X, ω, Ω) and are well defined by homotopy invariance.

Definition 3.1.22. *Let $[i^* \Omega_j] \in H^3(\Sigma, \mathbb{R})$ for $j = 1, 2$ denote the cohomology classes obtained from the restriction of $[\Omega_j] \in H^3(X, \mathbb{R})$ to any cross section $\varphi(\{r\} \times \Sigma)$ over the end of X.*

Proposition 3.1.23. *Let (X, ω, Ω) be AC and $(A, \Phi \neq 0)$ a finite mass, irreducible complex monopole with $|A - A_\infty| = O(\rho^{-4-\varepsilon'})$, for some $\varepsilon' > 0$, then*

$$E_U^I = \int_\Sigma \langle \Phi_{\infty,1}, F_\infty \rangle \cup [i^* \Omega_1] + \int_\Sigma \langle \Phi_{\infty,2}, F_\infty \rangle \cup [i^* \Omega_2].$$

In particular, if the complex structure decays at rate $\lambda < -3$ or the cohomology classes $[i^* \Omega_i]$ both vanish, then $F_A^{0,2} = \Lambda F_A = 0$ and $\nabla_A \Phi = 0$, so A is reducible.

Proof. Under the finite mass hypothesis $\nabla_A \Phi \in L^2$ by proposition 3.1.20. Then, if (A, Φ) is a complex monopole one can use equation 3.1.20 in proposition 3.1.16 over very large balls B_r centered at $p \in X$ to give

$$E_{B_r}^I (A, \Phi) = \frac{1}{2} \int_{\partial B_r} \langle \Phi_1, F_A \rangle \wedge \Omega_1 + \frac{1}{2} \int_{\partial B_r} \langle \Phi_2, F_A \rangle \wedge \Omega_2. \tag{3.1.24}$$

Then, one can regard these boundary integrals $E^I_{B_r}$ as a monotone increasing function of r which is bounded above by E^I_X. Hence it does converge, the limit is E^I_X and to conclude it is given by the formula in the statement expand $\Phi = \Phi_\infty + O(\rho^{-4})$ and $F = F_\infty + O(\rho^{-5-\varepsilon'})$, then for $i = 1, 2$

$$\langle \Phi_i, F_A \rangle \wedge \Omega_i = \langle \Phi_{\infty,i}, F_\infty \rangle \wedge \Omega_i + O(\rho^{-5-\varepsilon'}).$$

So, when one takes the limit as $r \to \infty$ of the integrals in the right hand side of 3.1.24 the higher order terms vanish and one is left with the result in the statement. In the case where $\lambda < -3$, one can write $\Omega_i = (\Omega_C)_i + \eta$ with $\eta = O(\rho^\lambda)$, which gives $\langle \Phi_{\infty,i}, F_\infty \rangle \wedge \Omega_i = \langle \Phi_{\infty,i}, F_\infty \rangle \wedge \eta = O(\rho^{-2+\lambda})$ and so the limit of 3.1.24 vanishes and $E^I_X = 0$. This implies that $F \wedge \Omega = \partial \Phi = 0$, moreover taking the complex conjugate of this second one has $\overline{\partial}_A \overline{\Phi} = 0$. Using the form of the complex monopole equation in the fourth item of proposition 3.1.3 gives $[\overline{\Phi}, \overline{\Phi}] = \frac{i}{4}[\overline{\Phi}, \overline{\Phi}] = 0$ and so one can appeal to lemma 3.1.5 to conclude that also $\partial_A \overline{\Phi} = [\overline{\Phi}, \Phi] = 0$. Hence $d_A \Phi = 0$ and so A is reducible, moreover the second complex monopole equation gives $i\Lambda F_A = [\Phi, \overline{\Phi}] = 0$. The same holds if the classes $[i^*\Omega_i]$ vanish. $\square$

Remark 3.1.24. *For $G = SU(2)$ one is led to a similar problem as the one in corollary 1.4.11 and H is either $\{1\}$ or $U(1)$ and the connection A_∞ is induced by a connection on a circle bundle Q_∞. The decomposition $\mathfrak{su}_{\mathbb{C}}(2) = \mathfrak{u}_{\mathbb{C}}(1) \oplus \mathbb{C}_\alpha \oplus \mathbb{C}_{-\alpha}$ gives that $E = L_\alpha \oplus L_{-\alpha}$, and $\mathfrak{su}(2)$ representation theory shows $L_\alpha \cong L^2$, where $L = Q_\infty \times_{U(1)} \mathbb{C}$ is the line bundle associated with the standard $U(1)$ representation. Then, L has a connection induced by A_∞ and $c_1(L) = \frac{1}{2\pi}\left[\frac{1}{2|\Phi_{\infty,1}|}\langle \Phi_{\infty,1}, F_\infty \rangle\right]$. The energy formula in proposition 3.1.23 shows that*

Corollary 3.1.25. *Let (X, ω, Ω) be AC, $G = SU(2)$ and (A, Φ) an irreducible, finite mass, complex monopole with $m_i = |\Phi_{\infty,i}|$ for $i = 1, 2$ and $|A - A_\infty| = O(\rho^{-4-\varepsilon'})$ with $\varepsilon' > 0$*

$$E^I_X = 4\pi m_1 \langle c_1(L) \cup [i^*\Omega_1], [\Sigma] \rangle + 4\pi m_2 \langle c_1(L) \cup [i^*\Omega_2], [\Sigma] \rangle.$$

In particular, if L is trivial or the complex structure has rate $\lambda < -3$ or both $[i^*\Omega_i] = 0$, then $E^I_X = 0$ and so $F^{0,2}_A = \lambda F_A = 0$ and also $\nabla_A \Phi = 0$ so A is reducible.

Proposition 3.1.26. *Let (X, ω, Ω) be AC, $G = SU(2)$ and (A, Φ) a finite mass complex monopole asymptotic to (A_∞, Φ_∞) with $|A - A_\infty|, |\langle \Phi_1, \nabla^A_{I\rho\partial_\rho} \Phi_2 \rangle| = O(\rho^{-4-\varepsilon'})$, for $\varepsilon' > 0$ and A_∞ induced from a connection on a line bundle L as in proposition 3.1.31 such that $c_1(L) \cup [i^*\Omega_2] = 0$. Then, $\nabla_A \Phi_2 = [\Phi_1, \Phi_2] = 0$. In particular if A is irreducible, then $\Phi_2 = 0$, i.e. $\Phi = \Phi_1 \in \Omega^0(X, \mathfrak{g}_P)$ is a real Higgs field, the equations reduce to*

$$\begin{aligned} * \nabla_A \Phi &= F_A \wedge \Omega_1 \\ \Lambda F_A &= 0, \end{aligned}$$

i.e. (A, Φ) is a Calabi-Yau monopole as in definition 3.1.2.

Proof. From proposition 3.1.23 the finite mass condition implies that $\nabla_A \Phi_2, [\Phi_1, \Phi_2] \in L^2$. Define $f(r) = \|[\Phi_1, \Phi_2]\|^2_{L^2(B_r)} + \|\nabla_A \Phi_2\|^2_{L^2(B_r)}$, then corollary 3.1.13 can be used to give the integration by parts

$$f(r) = -\int_{\partial B_r} \langle \Phi_1, *Id_A\Phi_2\rangle + \int_{\partial B_r} \langle \Phi_2, F\rangle \wedge \Omega_2, \tag{3.1.25}$$

Using the hypothesis that there is $\varepsilon' > 0$ such that $|\langle \Phi_1, \nabla^A_{Ir\partial_r}\Phi_2\rangle| = O(r^{-4-\varepsilon'})$, the higher order part of the first term in 3.1.25 is given by $\int_\Sigma \langle \Phi_1, \nabla^A_{I\rho\partial_\rho}\Phi_2\rangle r^4\eta \wedge (d\eta)^2 = O(r^{-\varepsilon'})$ and so vanishes in the limit $r \to \infty$. The second boundary integral converges by the assumption that $\langle \Phi_\infty, A - A_\infty\rangle = O(r^{-4-\varepsilon'})$ and so $f(r)$ is monotone, increasing and bounded above by the sum of the L^2 norms of $\nabla_A\Phi_2$ and $[\Phi_1, \Phi_2]$

$$\begin{aligned} \frac{1}{2}\|[\Phi_1,\Phi_2]\|^2_{L^2} + \|\nabla_A\Phi_2\|^2_{L^2} &= \int_\Sigma \langle \Phi_{\infty,2}, F_\infty\rangle \wedge [i^*\Omega_2] \\ &= 4\pi|\Phi_{\infty,2}|\langle c_1(L) \cup [i^*\Omega_2], [\Sigma]\rangle. \end{aligned}$$

Since by assumption, the cohomology class $c_1(L) \cup [i^*\Omega_2] = 0$, the quantity above vanishes implying that $\nabla_A\Phi_2 = [\Phi_1, \Phi_2] = 0$. ☐

Remark 3.1.27. *The author believes the boundary condition $c_1(L)\cup[i^*\Omega_2] = 0$ above, is necessary in order to relate Calabi-Yau monopoles with phase 0 special Lagrangian submanifolds. This will be more clear after definitions 3.2.2 and 3.2.3. Regarding the condition $|\langle \Phi_1, \nabla^A_{I\rho\partial_\rho}\Phi_2\rangle| = O(\rho^{-4-\varepsilon'})$, it is possible that this is a consequence of the other assumptions, namely $|A - A_\infty| = O(\rho^{-4-\varepsilon})$ and (A, Φ) being a complex monopole.*

The rest of this section analyses the boundary problem that $(\nabla_\infty, \Phi_\infty)$ must satisfy. It is useful to recall some Sasaki-Einstein geometry and the reader may consult section 1.1.2 (and the references therein), where some facts are collected.

Proposition 3.1.28. *Let (X, ω, Ω) be AC and (A, Φ) is a finite mass, irreducible complex monopole, then $F_A \wedge \Omega \in L^2$ and the connection A_∞ on P_∞ is such that $\nabla_\infty\Phi_\infty = 0$ and*

$$\Lambda_T F_\infty = F^{0,2}_\infty = 0\ ,\ \iota_\xi F_\infty = 0,$$

where ξ denotes the Reeb vector field of the contact structure η on Σ, Λ_T the dual of the transverse Kähler form $\omega_T = \frac{d\eta}{2}$ and $F^{0,2}_\infty$ is the $(0,2)$ component of F_∞ with respect to the transverse complex structure on the horizontal distribution.

Proof. Under the finite mass hypothesis $\nabla_A\Phi \in L^2$ by proposition 3.1.20 and so is $F_A \wedge \Omega$. On the cone the highest order term of $F_A \wedge \Omega$ is $F_\infty \wedge \Omega_C$ which in general is $O(\rho^{-2})$ and so must vanish so that $F_A \wedge \Omega \in L^2$. One can write $\Omega_C = -ir^2 dr \wedge \Omega_T + r^3\eta \wedge \Omega_T$, where Ω_T is a basic $(2,0)$-form, see example 5 in section 1.4.1. And so the condition that $F_\infty \wedge \Omega_C$ on the cone can be translated into $F_\infty \wedge \Omega_T = 0$, $F_\infty \wedge \eta \wedge \Omega_T = 0$, over Σ. These equations imply $F^{0,2}_\infty = 0$ and the first one also implies $\iota_\xi F_\infty = 0$. The last thing to prove is that $\Lambda_T F_\infty = 0$ and recall that for a finite mass, proposition 3.1.20 implies $\Lambda F_A = [\Phi, \overline{\Phi}] \in L^2$ and so the higher order terms $[\Phi_\infty, \overline{\Phi_\infty}], \Lambda_T F_\infty$ vanish. ☐

Definition 3.1.29. *A complex vector bundle E on Σ is said to be basic holomorphic if it is equipped with an operator $\overline{\partial}_E : \Omega_B^{(p,q)}(\Sigma, E) \to \Omega_B^{(p,q+1)}(\Sigma, E)$, such that $\overline{\partial}_E^2 = 0$ and which satisfies the Leibniz rule $\overline{\partial}_E(fs) = \overline{\partial} f \wedge s + f \overline{\partial}_E s$ for all $f \in \Omega_B^0(\Sigma, \mathbb{C})$ and $s \in \Omega_B^{(p,q)}(\Sigma, E)$.*

Remark 3.1.30. *A connection ∇ on E is said to be basic if $\nabla(\Omega_B^0(\Sigma, E)) \subset \Omega_B^1(\Sigma, E)$; in this case its curvature F_∇ is a basic form. Given such a ∇ on E one can define $\overline{\partial}_\nabla = d_\nabla^{0,1}$ and this equips E with a basic holomorphic structure if and only if $F_\nabla^{0,2} = 0$. Moreover, ∇ is called a basic Hermitian Yang Mills (HYM) connection if it further satisfies $\Lambda_T F_\nabla = 0$. Proposition 3.1.28 states that A_∞ is a basic HYM connection.*

The following result gives necessary and sufficient conditions for the existence of boundary conditions (A_∞, Φ_∞) for $G = SU(2)$.

Proposition 3.1.31. *Let $L \to \Sigma$ be a basic holomorphic line bundle on Σ such that $c_1(L) \cup [\omega_T] = 0$. Then, there is a basic HYM connection on L, i.e. its curvature satisfies $F^{0,2} = \Lambda_T F = 0$, where Λ_T is the contraction with the transverse Kähler form $\omega_T = \frac{d\eta}{2}$.*

Proof. Equip L with an hermitian metric h, then there is a unique basic Chern connection which is compatible with both the holomorphic structure and the metric. The fact that $F^{0,2} = 0$ is obvious from the compatibility of the Chern connection with the holomorphic structure. That $F^{2,0} = \overline{F^{0,2}} = 0$ is a consequence of the compatibility with the hermitian metric h. Moreover, locally its curvature can be written as a basic $(1,1)$ form

$$F = i\overline{\partial}_B \partial_B \log(h). \tag{3.1.26}$$

Hodge theory for basic forms gives $\Omega_B^0(X, \mathbb{R}) = \mathbb{R} \oplus \operatorname{im}(\partial_B^* \partial_B)$ and since by hypothesis $c_1(L) \cup [\omega_T] = 0$, $\Lambda_T F = \partial_B^* \partial_B f$, for some real valued basic function f. Change the metric h on L to a metric $h' = he^{-f}$. The claim is that the curvature F' of the Chern connection of this new hermitian metric has the right properties. In fact, $F'^{2,0} = F'^{0,2} = 0$ still hold in the same way. Moreover, using the local formula 3.1.26, $F' = F - i\overline{\partial}_B \partial_B f$. Using the basic Kähler identity $i[\Lambda_T, \overline{\partial}_B] = \partial_B^*$

$$\begin{aligned} \Lambda_T F' &= \partial_B^* \partial_B f - i\Lambda_T \overline{\partial}_B \partial_B f \\ &= \partial_B^* \partial_B f - \partial_B^* \partial_B f = 0. \end{aligned}$$

□

Remark 3.1.32. *Recall that if Σ is a regular Sasaki-Einstein manifold, then it is the total space of an $\mathbb{S}^1$ bundle on a Fano surface D with a Kähler-Einstein metric. The Sasaki structure can then be viewed as a connection on this bundle whose curvature is a Kähler form on D, in fact $d\eta = 2\omega_T$. Then, the basic cohomology is the pullback to Σ of the cohomology of D. So L is the pullback of a holomorphic line bundle on D with $c_1(L) \cup c_1(\Sigma) = 0$, and the connection from 3.1.31 is the Chern connection of a suitable hermitian metric on L.*

3.2 Examples

3.2.1 Monopoles on Affine Smoothings

This section sets up the problem for studying Calabi-Yau monopoles on the AC Calabi-Yau manifolds described in section 5 of [CH13a]. In view of proposition 5.1 of that reference one can consider a compact Fano 3-fold X_c of index $k+1$ and D a smooth anticanonical divisor in X_c, such that $K_{X_c} = -(k+1)[D]$, for some $k \in \mathbb{N}$. Then, $X = X_c \backslash D$ is a smoothing of $C = (\frac{1}{k} K_D)^{\times}$, the blow down of the zero section in $\frac{1}{k} K_D$. In fact D is the orbit space of the $\mathbb{C}^*$-action on the Calabi-Yau cone C. Moreover, C can be $\mathbb{C}^*$ equivariantly embedded in some $\mathbb{C}^N$, for a weighted action on the latter, as shown in [van11].

Let C be a complete intersection Calabi Yau cone in $\mathbb{C}^N$. We shall consider smoothings X by adding lower order terms to the equations defining C. Hence, also X will be a complete intersection affine manifold. The cohomology of such an X is supported in the middle dimension, 3 in this case, in fact they are homotopy equivalent to a bouquet of $\mathbb{S}^3$'s [CH13a]. These examples are asymptotic to a cone over a regular Sasaki-Einstein manifold $\pi_D : \Sigma \to D$, which is the total space of an $\mathbb{S}^1$ bundle over a Fano surface D with a Kähler-Einstein metric g_D. The Weitzenböck formula for 1-forms shows that since (D, g_D) and (Σ, g_Σ) have positive Ricci $H^1(D) = H^1(\Sigma) = 0$. In fact, we shall suppose that the cone C has trivial canonical bundle and $\pi_1(D) = \pi_1(\Sigma) = 0$. Moreover $H^2(\Sigma) \cong H^{1,1}_{pr}(D)$ as Kodaira vanishing implies $H^{2,0}(D) = 0$.

Definition 3.2.1. *Let $H^*_{cs}(X, \mathbb{Z})$ denote the compactly supported cohomology of X. A class $P \in H^3_{cs}(X, \mathbb{Z})$ is said to be a special Lagrangian (SL) class if $P \cup [\Omega_2] = 0 \in H^6_{cs}(X, \mathbb{Z})$ and $P \cup [\omega] = 0 \in H^5_{cs}(X, \mathbb{Z})$. Moreover, if $P \in \ker(H^3_{cs}(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}))$ then it is said to be a monopole-SL class.*

Remark 3.2.2. *The definition above makes sense for any Calabi-Yau manifold. In fact, in the cases to be considered here the condition $P \cup [\omega] = 0$ is immediate as $H^5_{cs}(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^* = 0$.*

Definition 3.2.3. *A class $\alpha \in H^2(\Sigma, \mathbb{Z})$ is said to be a monopole class if $\alpha \cup [i^*\Omega_2] = 0$.*

Remark 3.2.4. *Take the long exact sequence for compactly supported cohomology and recall that $H^2(X, \mathbb{Z}) = 0$*

$$0 \to H^2(\Sigma, \mathbb{Z}) \to H^3_{cs}(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}) \to ...$$

Hence the image of the map $H^2(\Sigma, \mathbb{Z}) \to H^3_{cs}(X, \mathbb{Z})$ is exactly the kernel of $H^3_{cs}(X, \mathbb{Z}) \to H^3(X, \mathbb{Z})$ and so identifies the image of the monopole classes with the monopole-SL classes.

Remark 3.2.5. *Alternatively one could have considered the exact sequence for the pair (X_c, X), which together with the Thom isomorphism $H^*(X_c, X) \cong H^{*-2}(D)$ gives*

$$0 \to H^2(X_c, \mathbb{Z}) \to H^2(D, \mathbb{Z}) \xrightarrow{i} H^3_{cs}(X, \mathbb{Z}) \to H^3(X_c, \mathbb{Z}) \to 0.$$

Since by Kodaira vanishing $H^{2,0}(D)$ vanishes, $H^2(\Sigma, \mathbb{Z}) \cong H^{1,1}_{pr}(D, \mathbb{Z})$ and one can give an alternative definition of monopole classes as those $\alpha \in H^2(D, \mathbb{Z})$ which are primitive of type

$(1,1)$ *and* $\pi_D^*\alpha \cup [i^*\Omega_2] = 0$. *Their image* $i(\alpha)$ *corresponds to those classes in the kernel of the map* $H^3_{cs}(X,\mathbb{Z}) \to H^3(X_c,\mathbb{Z})$ *such that* $i(\alpha) \cup [\Omega_2] = 0$ *and these could have been used as an alternative definition of monopole-SL class.*

Let $\pi_D^*(\alpha) \in H^2(\Sigma,\mathbb{Z})$ be a monopole class and $\mathcal{L} \to D$ a line bundle with $c_1(\mathcal{L}) = \alpha$. Then, from proposition 3.1.31 there is an HYM connection on $\mathcal{L}$, or equivalently a basic HYM on $L = \pi_D^*\mathcal{L} \to \Sigma$, the pullback of L to Σ via π_D. Take two copies of this connection to obtain a reducible connection A_∞ on $L \oplus L^{-1}$ over Σ. Let P be an $SU(2)$ bundle over X_c such that for $\mathcal{E}$ the rank 2 complex vector bundle associated with the standard representation, one has

$$\mathcal{E}|_D \cong \mathcal{L} \oplus \mathcal{L}^{-1}. \tag{3.2.1}$$

Then one searches for finite mass complex monopoles (as in definition 3.1.19) on $E = \mathcal{E}|_X$. Indeed the work to be developed later in chapter 5 gives a Fredholm setup for this problem and proposition 3.1.26 shows that for rate $\varepsilon > 3$ these complex monopoles are actually Calabi-Yau monopoles and so satisfy $*\nabla_A\Phi = F_A \wedge \Omega_1$ and $\Lambda F_A = 0$.

Example 8. *Take* C *to be the ordinary double point* $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ *in* $\mathbb{C}^4$. *In this case* $D = \mathbb{P}^1 \times \mathbb{P}^1$ *and* C *can be smoothed out by adding a zero order term to the equation.* X *is diffeomorphic to* $T^*\mathbb{S}^3$ *and can be equipped with a Calabi-Yau metric known as the Stenzel metric* [Ste93]. *The zero section is a special Lagrangian* $\mathbb{S}^3$ *and its class in* $H^3_{cs}(X,\mathbb{Z})$ *lies in the image of a monopole class. Moreover, the Stenzel metric is cohomogeneity* 1 *and so this is a particularly interesting example for studying Calabi-Yau monopoles and their interaction with the special Lagrangian submanifold, via ODE methods. This will be done in the next section 3.3, whose upshot is theorem 3.3.1.*

Example 9. *Take* C *to be given by the cubic singularity* $z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$. *Consider the deformations which can be written as*

$$X = \Big\{(z_1,z_2,z_3,z_4) \in \mathbb{C}^4 \mid \sum_{i=1}^4 z_i^3 + \sum_{1\leq i\leq j\leq 4} t_{ij} z_i z_j + \sum_{i=1}^4 t_i z_i = \varepsilon\Big\},$$

for $(t_{ij}, t_i, \varepsilon) \in \mathbb{C}$. *Each of these is diffeomorphic to a bouquet of* 16 *spheres* [GH78]. *AC Calabi-Yau metrics are constructed in* [CH13a], *which have rate* -3 *in general and* -6 *in the case where all the* $t_{ij} = 0$. *In this example* $D = Bl_6\mathbb{P}^2$, $\Sigma = \sharp_6\mathbb{S}^2 \times \mathbb{S}^3$ *and so* $H^2(\Sigma,\mathbb{Z}) \cong \mathbb{Z}^6$. *Those classes* $\pi_D^*\alpha \in H^2(\Sigma,\mathbb{Z})$ *such that* $\pi_D^*\alpha \cup [i^*\Omega_2] = 0$ *are the monopole classes which certainly exist and form an Abelian group isomorphic to* $\mathbb{Z}^6$ *or* $\mathbb{Z}^5$ *according to whether* $[i^*\Omega_2]$ *vanishes or not. For each of these classes proposition 3.1.31 gives the asymptotic basic HYM connection* A_∞ *on a line bundle* L *over* Σ *such that* $c_1(L) = \pi_D^*\alpha$. *Then, given a mass* $m \in \mathbb{R}^+$, *chapter 5 gives a good Fredholm setup for studying mass* m *Calabi-Yau monopoles with connection asymptotic to* A_∞.

Example 10. *Take* C *to be given by the intersection of two quadrics in* $\mathbb{C}^5$, *given by* $\sum_{i=1}^5 z_i^2 =$

$\sum_{i=1}^5 \lambda_i z_i^2 = 0$, *and the* λ_i*'s all distinct. Consider the deformations of* C *which can be written as*

$$X = \Big\{ (z_1 ..., z_5) \in \mathbb{C}^5 \mid \sum_{i=1}^5 z_i^2 + \sum_{1=1}^5 t_i z_i = \varepsilon_1 \ , \ \sum_{i=1}^5 \lambda_i z_i^2 = \varepsilon_2 \Big\},$$

for $(t_i, \varepsilon_1, \varepsilon_2) \in \mathbb{C}^7$*. Each of these is diffeomorphic to a bouquet of* 9 *spheres* [GH78]*. AC Calabi-Yau metrics are constructed in* [CH13a]*, these have rate* -3 *in general and* -6 *in the case where all* $t_i = 0$*. In this example* $D = Bl_5\mathbb{P}^2$ *is the intersection of the two quadrics in* $\mathbb{P}^4$*, so* $\Sigma = \sharp_5 \mathbb{S}^2 \times \mathbb{S}^3$ *and* $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^5$*. Once again there are monopole classes and they form an Abelian group isomorphic to* $\mathbb{Z}^5$ *or* $\mathbb{Z}^4$ *according to whether* $[i^*\Omega_2]$ *vanishes or not. For each of these classes proposition 3.1.31 gives the asymptotic basic HYM connection* A_∞ *on a line bundle* L *over* Σ*. Then, given a mass* $m \in \mathbb{R}^+$*, chapter 5 gives a good Fredholm setup for studying mass* m *complex monopoles with connection asymptotic to* A_∞*.*

This example is also promising for studying the relation between monopoles and special Lagrangian submanifolds. For the statement of the next result suppose with no loss of generality that all the λ_i *are real and* $\lambda_i < \lambda_j$ *if* $i < j$*.*

Proposition 3.2.6. *Let* $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$*, such that* $\lambda_2 > \frac{\varepsilon_2}{\varepsilon_1} > \lambda_1$*. Then, for all sufficiently small* t_i*'s, there are two special Lagrangian* 3 *spheres in* X*.*

Proof. Consider the antiholomorphic involution $h : z_i \mapsto \overline{z}_i$ and let $(\omega, \Omega = \Omega_1 + i\Omega_2)$ denote respectively the Kähler form and the holomorphic volume form of the Calabi Yau structure. Since the complex structure on X is induced from that on $\mathbb{C}^5$, $h^*\omega = -\omega$ and $h^*\Omega_2 = -\Omega_2$, hence its fixed points cut out special Lagrangian submanifolds in X. In order to ease the computation suppose the t_i's vanish, the general case follows from the implicit function theorem. Define real coordinates by $z_i = x_i + iy_i$, the fixed points of h are such that all $y_i = 0$ and

$$\sum_{i=1}^5 x_i^2 = \varepsilon_1 \ , \ \sum_{i=1}^5 \lambda_i x_i^2 = \varepsilon_2.$$

Both of these are 4 spheres inside $\mathbb{R}^5$, in fact the one on the left is a round sphere, while the one on the right is an ellipsoid for general λ_i. Next one needs to show that under the conditions in the statement they do intersect and the intersections are diffeomorphic to $\mathbb{S}^3$. Assume with no loss of generality that $\lambda_1 = \min_i\{\lambda_i\}$ and replace $x_1^2 = \varepsilon_1 - \sum_{i=2}^5 x_i^2$ in the second equation. This gives

$$\sum_{i=2}^5 (\lambda_i - \lambda_1) x_i^2 = \varepsilon_2 - \lambda_1 \varepsilon_1 > 0,$$

and so defines a 3 sphere in $\mathbb{R}^4_{(x_2,...,x_5)}$. Moreover, if $\lambda_2\varepsilon_1 > \varepsilon_2$, then all (x_2, x_3, x_4, x_5) in the 3 spheres defined by $\sum_{i=2}^5 (\lambda_i - \lambda_1) x_i^2 = \varepsilon_2 - \lambda_1\varepsilon_1$ are such that $\sum_{i=2}^5 x_i^2 < \varepsilon_1$. So there are two distinct disconnected branches of the square root in the first equation $x_1 = \pm\sqrt{\varepsilon_1 - \sum_{i=2}^5 x_i^2}$ and each of these gives rise to a special Lagrangian 3 sphere. $\square$

3.2.2 Monopoles on Crepant Resolutions

Given a Calabi-Yau cone (C, ω_C, Ω_C), then $C \cup \{0\}$ can be identified with an affine variety (Theorem 3.1 in [van11]), equivariantly embedded in some $\mathbb{C}^N_{w_1,\ldots,w_N}$ with respect to a $\mathbb{C}^*$ action for some weights $(w_1, ..., w_N)$. In many cases there is a resolution $\pi : X \to C$ which is crepant, i.e. X has trivial canonical bundle equipped with a nonvanishing holomorphic section $\Omega = \pi^*\Omega_C$. There are many examples of AC Calabi-Yau 3 folds obtained from crepant resolutions, see section 4 in [CH13a], where the main examples are reviewed and some new ones given. These include, for example, Calabi's explicit metric on $K_{\mathbb{P}^2}$ [Cal79], the small resolution of the ordinary double point $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, Joyce's ALE examples [Joy00], Van Coevering's examples in [van10], the cohomogeneity 1 examples associated with flag manifolds in [CH13a] and others. In this class of examples, there are no compact special Lagrangian submanifolds because $H_3(X) \cong H_3(E)$, where E denotes the exceptional locus and $H_3(E) = 0$. So, the following vanishing result is a promising motivation for the conjectural relation between monopoles and special Lagrangian submanifolds.

Proposition 3.2.7. *Let X be a crepant resolution of a Calabi-Yau cone with complex dimension 3, then there are no irreducible, finite mass complex monopoles (A, Φ) as in definition 3.1.19 on X such that $|A - A_\infty| = O(\rho^{-4-\delta})$ for some $\delta > 0$.*

Proof. Recall the definition 3.1.19 of finite mass complex monopoles and suppose (A, Φ). The hypothesis say that there is A_∞ as in definition 3.1.19 such that $|A - A_\infty| = O(\rho^{-4-\delta'})$ and $\delta > 0$. Using this together with the fact that away from the exceptional locus X is biholomorphic to the cone, i.e. the complex structure approaches the conical one at rate $\lambda = -\infty < -3$, one can use proposition 3.1.23 to conclude that A is reducible. $\square$

3.3 Calabi-Yau Monopoles on $T^*\mathbb{S}^3$

This section analyzes example 8 from section 3.2.1 regarding the existence of Calabi-Yau monopoles and proves theorem 3.3.1 below. The Stenzel metric will be discussed in detail in section 3.3.1, moreover it will be showed to be of cohomogeneity 1, i.e. there is a Lie group acting by isometries with codimension 1 principal orbits. In the presence of such a Lie group action there is a notion of homogeneous bundle, i.e. a bundle where the previous action lifts via bundle automorphisms to the total space. Let E be a rank 2 complex vector bundle associated with a homogeneous principal bundle P with structure group $SU(2)$, then there is a notion of invariant connection and invariant Higgs field and it makes sense to define the moduli space of invariant Calabi-Yau monopoles on P, $\mathcal{M}_{inv}(P)$. This is defined as the set of those (A, Φ) on P as in definition 3.1.19, which are invariant and solve the Calabi-Yau monopole equations, up to the action of the invariant gauge transformations.

Theorem 3.3.1. *There is a homogeneous $SU(2)$ bundle P over $T^*\mathbb{S}^3$, such that the space of invariant Calabi-Yau monopoles $\mathcal{M}_{inv}(P)$ is non empty and the following hold:*

1. *For all Calabi-Yau monopoles in $\mathcal{M}_{inv}(P)$, the Higgs field Φ is bounded, the mass is well*

defined and gives a bijection

$$m : \mathcal{M}_{inv}(P) \to \mathbb{R}^+.$$

2. *Let* $R > 0$*, and* $\{(A_\lambda, \Phi_\lambda)\}_{\lambda \in [\Lambda, +\infty)} \in \mathcal{M}_{inv}(P)$ *be a sequence of Calabi-Yau monopoles with mass* λ *converging to* $+\infty$*. Then there is a null sequence* $\eta(\lambda, R)$ *such that the restriction to each fibre* $T_x\mathbb{S}^3$ *for* $x \in \mathbb{S}^3$ *of the rescaled Calabi-Yau monopole*
$$\exp_\eta^*(A_\lambda, \eta\Phi_\lambda)$$
converges uniformly to the BPS monopole (A^{BPS}, Φ^{BPS}) *in the ball of radius* R *in* $(\mathbb{R}^3, g_E)$*.*

3. *Let* $\{(A_\lambda, \Phi_\lambda)\}_{\lambda \in [\Lambda, +\infty)} \subset \mathcal{M}_{inv}(P)$ *be a sequence of Calabi-Yau monopoles with mass* $m(A_\lambda, \Phi_\lambda) = \lambda$ *converging to* ∞*. Then the translated Calabi-Yau monopole sequence*
$$\left(A_\lambda, \Phi_\lambda - \lambda \frac{\Phi_\lambda}{|\Phi_\lambda|}\right),$$
converges uniformly with all derivatives to a zero mass Dirac Calabi-Yau monopole on $T^*\mathbb{S}^3 \backslash \mathbb{S}^3$*, i.e. a reducible, singular Calabi-Yau monopole.*

The proof of this theorem occupies this whole section and it is organized as follows. Subsection 3.3.1 explicitly obtains the Stenzel metric on $T^*\mathbb{S}^3$. Subsection 3.3.2 constructs homogeneous bundles and studies invariant connections and Higgs fields on them. Using these as input, the Calabi-Yau monopole equations are then reduced to the ODE's in proposition 3.3.16. The solutions to these equations are studied in subsections 3.3.3, 3.3.4 and 3.3.5, where these are solved first for the cone and then for the Stenzel metric. The proof of theorem 3.3.1 requires rewriting the equations; this is done at the end of subsection 3.3.5 with the discussion after lemma 3.3.25. This lemma is the last one in a sequence of rearrangements of the equations, which reduce the relevant ODE's to the ones analyzed in chapter 2 for spherically symmetric Calabi-Yau monopoles in $\mathbb{R}^3$ equipped with a certain spherically symmetric metric. This subsection finishes with one other solution to the equations giving an explicit formula for an $SU(2)$-irreducible Hermitian Yang Mills (HYM) connection, which to the author's knowledge was previously unknown.

3.3.1 Stenzel's Ricci Flat Metric

This subsection begins with an informal discussion of the Conifold and its deformations. Later the Stenzel's Calabi-Yau structure [Ste93] will be computed explicitly and shown to be asymptotic to the Conifold one. Moreover, the uniqueness of Stenzel's Calabi-Yau structure was recently shown in [CH13a]

The Conifold and its Deformations

The ordinary double point in $\mathbb{C}^4$ gives rise to a Calabi-Yau cone (C, ω_C, Ω_C), known in the physics literature as the Conifold [Cd90]. It is a Ricci flat Kähler cone $\left(C = \mathbb{R}^+ \times \Sigma, g_0 = d\rho^2 + \rho^2 g_\Sigma\right)$, whose link (Σ, g_Σ) is a regular Sasaki-Einstein manifold. Topologically $\Sigma \cong \mathbb{S}^3 \times \mathbb{S}^2$ is the total space of a $U(1)$-bundle over $D = \mathbb{P}^1 \times \mathbb{P}^1$ with the product Fubini-Study Kähler structure ω_D. Let η be the contact structure on Σ, so $g_\Sigma = \pi_D^* g_D + \eta \otimes \eta$, where g_D is the product round metric. The curvature of the connection η is $d\eta = 2\pi_D^* \omega_D$, so in $H^{1,1}(D, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $c_1 = \frac{1}{2\pi}[d\eta] = \frac{1}{\pi}[\omega_D]$ represents the first Chern class of the associated complex line bundle. Since Σ is simply connected and $c_1(-\frac{1}{2}K_{\mathbb{P}^1 \times \mathbb{P}^1}) = (1,1)$, one concludes that Σ is the total space of the unit circle bundle in $-\frac{1}{2}K_{\mathbb{P}^1 \times \mathbb{P}^1}$. The complex structure J_C on the cone C is the one given by viewing it as the ordinary double point in $\mathbb{C}^4$. It matches the one in D along the transverse directions and rotates $\rho\partial_\rho$ to the Reeb vector field ξ. This makes (C, g_C, J_C) a Ricci flat Kähler cone with a global Kähler potential ρ^2, so $\omega_C = \frac{1}{2}d(\rho^2\eta) = \frac{i}{2}\partial\overline{\partial}\rho^2$. The smoothings,

$$X_\varepsilon = \left\{ F(z_1, z_2, z_3, z_3, z_4) = z_1^2 + z_2^2 + z_3^2 + z_4^2 = \varepsilon^2 \right\} \subseteq \mathbb{C}^4,$$

for $\varepsilon \in \mathbb{R}^+$, make it nonsingular at the expense of changing the complex structure. Topologically these are $T^*\mathbb{S}^3$ and one obtains a complex 1-parameter family of complex structures on $T^*\mathbb{S}^3$. To see that $X_\varepsilon \cong T^*\mathbb{S}^3$, restrict to each X_ε the function $r^2 = \sum_{i=1}^4 |z_i|^2$ taking values into $[\varepsilon^2, +\infty)$ and introduce the coordinates $(x_i, y_i) \in \mathbb{R}^4 \times \mathbb{R}^4 \cong \mathbb{C}^4$, via $z_i = x_i + iy_i$. Then the real and imaginary parts of the quadratic equation for X_ε are respectively

$$|x|^2 = R_+^2 = \frac{r^2 + \varepsilon^2}{2} \ , \ |y|^2 = R_-^2 = \frac{r^2 - \varepsilon^2}{2} \ , \ x \cdot y = 0. \tag{3.3.1}$$

This shows that the map that to $(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4$ associates $\left(\frac{x}{R_+}, y\right) \in \mathbb{S}^3 \times \mathbb{R}^4 \subset \mathbb{R}^4 \times \mathbb{R}^4$, restricts to $X_\varepsilon \subset \mathbb{C}^4$ as a diffeomorphism onto $T\mathbb{S}^3 \subset \mathbb{R}^4 \times \mathbb{R}^4$. Moreover, the level sets of r are either $\Sigma = \mathbb{S}^3 \times \mathbb{S}^2$ for $r \neq \varepsilon$, or the zero section $\mathbb{S}^3$ for $r = \varepsilon$.
Regarding symmetries, $SO(4)$ acts on $\mathbb{C}^4$ by matrix multiplication preserving F and r and so acts on X_ε. The action is transitive on each level set of r. In fact Stenzel's Calabi-Yau structure, is invariant under this $SO(4)$ action. This symmetry allows for the reduction of the Monge-Ampère equation to an ODE. For the purpose of constructing the metric it is irrelevant whether one considers an $SO(4)$-action or its lift to a $Spin(4)$-action. However, regarding the existence of interesting invariant connections it is convenient to work with the $Spin(4)$-action instead.

Stenzel's Ricci Flat Metric

Identify the Lie algebra $\mathfrak{so}(4)$ with the skewsymmetric matrices. Then, let $X_1 = C_{12}, X_2 = C_{13}, X_3 = C_{14}, X_4 = C_{23}, X_5 = C_{24}, X_6 = C_{34}$, where C_{ij} denotes the matrix whose (i, j) and (j, i) entries are respectively $1, -1$ and all other vanish. These satisfy the relations $[C_{ij}, C_{ik}] = -C_{jk}$ and $[C_{ij}, C_{kl}] = 0$ if i, j, k, l are all distinct. Let $p = (R_+, iR_-, 0, 0) \in X_\varepsilon \subset \mathbb{C}^4$, with R_+, R_- defined as in equation 3.3.1, then at p the isotropy subgroup is generated by exponentiating X_6 and this is

$$H_p = \left\{ \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \mid A \in SO(2) \right\} \subseteq SO(4). \tag{3.3.2}$$

One fixes a lift of $SO(4)$ to $Spin(4)$, such that the isotropy subgroup $H_p \subset SO(4)$ lifts to $H \cong U(1)$ in $Spin(4) = SU(2) \times SU(2)$, with

$$H \cong \left\{ \gamma(t) = \left(\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \right) \mid t \in \mathbb{R} \right\} \cong U(1). \tag{3.3.3}$$

and $\frac{d\gamma}{dt}\Big|_{t=0} = -2X_6$. Using the basis for $\mathfrak{spin}(4) = \mathfrak{so}(4)$ given by the $\{X_i\}_{i=1}^6$ and its dual basis $\{\theta_i\}_{i=1}^6$, the Maurer Cartan form on $Spin(4)$ is $\theta = \sum_{i=1}^6 \theta_i X_i$ and the 1-form

$$-\frac{i}{2}\theta^6 \in \Omega^1(Spin(4), i\mathbb{R}) \tag{3.3.4}$$

equips the bundle $Spin(4) \to \Sigma = Spin(4)/U(1)$ with a connection. This is the canonical invariant connection in the language of [KN63]. The tangent space to the $Spin(4)$-orbits can be identified with an Ad invariant complement to the isotropy algebra $\mathfrak{h} = \langle X_6 \rangle$. Fix the one given by defining $\mathfrak{m}$ to be the span of $\{X_i\}_{i=1}^5$, then

$$\mathfrak{spin}(4) = \mathfrak{h} \oplus \mathfrak{m},$$

and extending $\mathfrak{m}$ as a left invariant distribution in $Spin(4)$ gives another point of view on the canonical invariant connection. Moreover, one can further decompose $\mathfrak{m}$ into irreducible representations of $H = U(1)$ as

$$\mathfrak{m} = \langle X_1 \rangle \oplus \langle X_2, X_3 \rangle \oplus \langle X_4, X_5 \rangle, \tag{3.3.5}$$

where $\langle X_1 \rangle$ is the trivial representation and $\langle X_2, X_3 \rangle \cong \langle X_4, X_5 \rangle \cong \mathbb{C}$ with the standard weight one representation. One can check that at p, $\langle X_4, X_5 \rangle$ is the tangent space to the fibres of the sphere bundle inside $T^*\mathbb{S}^3 \to \mathbb{S}^3$ (using the round metric on $\mathbb{S}^3$), while $\langle X_1 \rangle \oplus \langle X_2, X_3 \rangle$ projects surjectively onto the tangent space to the base $\mathbb{S}^3$.

Proposition 3.3.2. *There is a $Spin(4)$-invariant Ricci flat Kähler metric on $T^*\mathbb{S}^3$ with Kähler form*

$$\omega = \dot{\mathcal{G}} dr \wedge \theta^1 + \mathcal{G}(\theta^{24} + \theta^{35}), \tag{3.3.6}$$

where $\mathcal{G} = \sqrt{r^4 - \varepsilon^4}\frac{\mathcal{F}'}{2}$, $\dot{\mathcal{G}} = \frac{d\mathcal{G}}{dr}$ and $\mathcal{F}(r^2)$ is the (global) Kähler potential, which satisfies

$$\mathcal{F}'(r^2(t)) = \frac{1}{\sinh(t)} \left(\frac{3}{4\varepsilon^2} \right)^{\frac{1}{3}} (\sinh(2t) - 2t)^{\frac{1}{3}}, \tag{3.3.7}$$

where $t \in [0, +\infty]$ is the coordinate implicitly determined by $r^2 = \varepsilon^2 \cosh(t)$.

Proof. Since $b_2(T^*\mathbb{S}^3) = 0$ any Kähler metric has a global Kähler potential $\mathcal{F}(r^2)$. The proof splits into 3 steps:

1) Find a ($SO(4)$-invariant) formula for the Kähler form in terms of $\mathcal{F}(r^2)$. To do this expand

the formula for the Kähler form $\frac{i}{2}\partial\overline{\partial}\mathcal{F}(r^2)$ in terms of the Kähler potential

$$\omega_C = \frac{i}{2}\mathcal{F}'\partial\overline{\partial}(r^2) + \frac{i}{2}\mathcal{F}''\partial(r^2) \wedge \overline{\partial}(r^2). \tag{3.3.8}$$

The first term is $\partial\overline{\partial}(r^2) = \sum_i dz^i \wedge d\overline{z}^i$ and for the second

$$\begin{aligned} \partial r^2 \overline{\partial} r^2 &= (d - \overline{\partial})r^2 \wedge (d - \partial)r^2 = -2r dr \wedge \partial r^2 - 2r\overline{\partial} r^2 \wedge dr - \partial r^2 \overline{\partial} r^2 \\ &= 2r dr \wedge (\overline{\partial} - \partial)r^2 - \partial r^2 \overline{\partial} r^2. \end{aligned}$$

Pass the last term to the left hand side and get $\partial r^2 \overline{\partial} r^2 = r dr \wedge (\overline{\partial} - \partial) r^2$, substituting this back in equation 3.3.8 so that $\omega_C = \frac{i}{2}\mathcal{F}'\partial\overline{\partial} r^2 + i\mathcal{F}'' r dr \wedge (\overline{\partial} - \partial) r^2$. At $p = (R_+, iR_-, 0, 0) \in X_\varepsilon \subset \mathbb{C}^4$ one may write

$$\begin{aligned} dz^1 &= \frac{r}{2R_+} dr + iR_- \theta^1 & \qquad dz^2 &= -R_+\theta^1 + \frac{ir}{2R_-} dr, \\ dz^3 &= -R_+\theta^2 - iR_-\theta^4 & \qquad dz^4 &= -R_+\theta^3 - iR_-\theta^5. \end{aligned}$$

and notice that the forms on the right hand side extend to $SO(4)$-invariant forms outside the zero section. With these relations one computes $(\overline{\partial} - \partial) r^2 = \sum_i z^i d\overline{z}^i - \overline{z}^i dz^i = 2i(R_- dx_2 - R_+ dy_1) = -4iR_- R_+ \theta^1$. The same can be done for the terms $dz^i \wedge d\overline{z}^i$ and one discovers that

$$\omega_C = \frac{r}{\sqrt{r^4 - \varepsilon^4}} \left(r^2 \mathcal{F}' + (r^4 - \varepsilon^4)\mathcal{F}'' \right) dr \wedge \theta^1 + \sqrt{r^4 - \varepsilon^4}\frac{\mathcal{F}'}{2}(\theta^2 \wedge \theta^4 + \theta^3 \wedge \theta^5),$$

which in terms of $\mathcal{G}$ is the Kähler form in the statement, for a (yet) unknown $\mathcal{F}(r^2)$.

2) Find a formula for the holomorphic volume form. This is done on the chart $\{z^i \frac{\partial F}{\partial z^i} \neq 0\}$, where recall $F = \sum_i z_i^2$. There, it is given by $\Omega = \left(\frac{\partial F}{\partial z^i}\right)^{-1} dz^1 \wedge ...\hat{dz^i}... \wedge dz^4$ and one can compute it at p, since $z_1 \neq 0$ there. Writing the result in terms of the $SO(4)$ invariant forms

$$\begin{aligned} \mathrm{Re}(\Omega) &= -\left(R_+^2\theta^{123} - R_-^2\theta^{145}\right) - \frac{r}{2} dr \wedge \left(\theta^{25} - \theta^{25}\right), \\ \mathrm{Im}(\Omega) &= \frac{r}{2}\left(\frac{R_+}{R_-} dr \wedge \theta^{23} - \frac{R_-}{R_+} dr \wedge \theta^{45}\right) + R_+R_-(\theta^{134} - \theta^{125}). \end{aligned} \tag{3.3.9}$$

3) Use the formulas computed in the previous steps to reduce the Monge-Ampère equation to an ODE and solve it. This is done by combining $\frac{\omega^3}{3!} = -\frac{i}{8}\Omega \wedge \overline{\Omega}$ with the formulas for ω and Ω obtained in the first two steps. Since $\frac{i}{8}\Omega \wedge \overline{\Omega} = -\frac{rR_+R_-}{2} dr \wedge \theta^{12345}$ and $\frac{\omega^3}{3!} = -\dot{\mathcal{G}}\mathcal{G}^2 dr \wedge \theta^{12345}$ the ODE is $2\dot{\mathcal{G}}\mathcal{G}^2 = rR_+R_-$, or in terms of the Kähler potential $\mathcal{F}$

$$r^2(\mathcal{F}')^3 + \frac{r^4 - \varepsilon^4}{3}\frac{d}{dr^2}(\mathcal{F}')^3 = 1. \tag{3.3.10}$$

Change variables to t such that $r^2 = \varepsilon^2\cosh(t)$, then $\varepsilon^4\sinh^2(t) = r^4 - \varepsilon^4$ and $\frac{d}{dr^2} = \frac{1}{\varepsilon^2\sinh(t)}\frac{d}{dt}$.

Substituting this into 3.3.10, the ODE turns out to be

$$\varepsilon^2 \cosh(t)(\mathcal{F}')^3 + \frac{\varepsilon^2 \sinh(t)}{3}\frac{d}{dt}(\mathcal{F}')^3 = 1, \tag{3.3.11}$$

which can be solved by introducing an integrating factor, giving the formula in the statement for the solution. □

Remark 3.3.3. *In some computations to be carried out further ahead it will be useful to recall the ODE 3.3.10 in the form* $2\dot{\mathcal{G}}\mathcal{G}^2 = rR_+R_-$.

For completeness, the complex structure can also be worked out explicitly in terms of the invariant forms. This can be read out of the formulas relating the $dz_i's$ with the θ^i's and this gives $I\theta^1 = \frac{r}{2R_-R_+}dr$, $Idr = -\frac{2R_+R_-}{r}\theta^1$, $I\theta^2 = -\frac{R_-}{R_+}\theta^4$, $I\theta^4 = \frac{R_+}{R_-}\theta^2$, $I\theta^3 = -\frac{R_-}{R_+}\theta^5$ and $I\theta^5 = \frac{R_+}{R_-}\theta^3$. These, together with the equation 3.3.6 for the Kähler form, give the following expression for the metric

$$g = \dot{\mathcal{G}}\frac{r}{2R_-R_+}dr^2 + \dot{\mathcal{G}}\frac{2R_+R_-}{r}\theta_1^2 + \mathcal{G}\frac{R_+}{R_-}\left(\theta_2^2+\theta_3^2\right) + \mathcal{G}\frac{R_-}{R_+}\left(\theta_4^2+\theta_5^2\right). \tag{3.3.12}$$

Definition 3.3.4. *For each ε define the radial function given by*

$$\rho(r) = \int_\varepsilon^r \frac{l}{2\mathcal{G}}dl = \int_\varepsilon^r \frac{l}{\sqrt{l^4-\varepsilon^4}}\frac{1}{\mathcal{F}'(l^2)}dl. \tag{3.3.13}$$

The function ρ just defined is the length through a geodesic orthogonal to the principal orbits and for $\varepsilon = 0$ it agrees with the geodesic distance to the apex of the cone. Next one defines a function which captures the volume growth of the level sets of ρ. The volume form for the induced metric is given by $\mathcal{G}\frac{R_-}{R_+}\mathcal{G}\frac{R_+}{R_-}\sqrt{\dot{\mathcal{G}}\frac{2R_+R_-}{r}}dr\wedge\theta^{1\ldots5} = (R_+R_-)^2\mathcal{F}'dr\wedge\theta^{1\ldots5}$.

Definition 3.3.5. *Define the radial function* $h^2(\rho) = \frac{1}{\varepsilon^2}(R_+R_-)^2\mathcal{F}'$.

Remark 3.3.6. *For the Conifold, which corresponds to $\varepsilon = 0$ one already knows the Kähler potential is ρ^2. Moreover, in this case the $SO(4)$ invariant Monge-Ampère equation 3.3.10 is*

$$r^2(\mathcal{F}')^3 + \frac{r^4}{3}\frac{d}{dr^2}(\mathcal{F}')^3 = 1. \tag{3.3.14}$$

The Kähler potential $\mathcal{F}$ is given by $\mathcal{F} = \left(\frac{3}{2}\right)^{\frac{4}{3}} r^{\frac{4}{3}}$ and so one concludes that the geodesic distance to the apex of the cone is $\rho = \left(\frac{3}{2}\right)^{\frac{2}{3}} r^{\frac{2}{3}}$. This can be used to rewrite the Ricci Flat Kähler metric 3.3.12 on the conifold C as

$$g = d\rho^2 + \rho^2\left(\left(\frac{2}{3}\theta_1\right)^2 + \left(\frac{\theta_2}{\sqrt{3}}\right)^2 + \left(\frac{\theta_3}{\sqrt{3}}\right)^2 + \left(\frac{\theta_4}{\sqrt{3}}\right)^2 + \left(\frac{\theta_5}{\sqrt{3}}\right)^2\right). \tag{3.3.15}$$

3.3.2 The Calabi-Yau Monopole Equations

Recall that $X_\varepsilon \backslash r^{-1}(\varepsilon) \cong (\varepsilon;\infty)\times\Sigma$, where $\Sigma = Spin(4)/U(1)$ is homogeneous and r is the coordinate on the $(\varepsilon;\infty)$ component. This section describes homogeneous bundles having invariant connections and invariant Higgs Fields. Then, these are used to compute the Calabi-Yau monopole equations and reduce them to ODE's. Background material on homogeneous bundles and invariant connections can be found for example in section 2 of chapter X in [KN63], or in Appendix B.

Homogeneous $SU(2)$ Bundle

Recall that given a Lie group G, a principal G bundle P over $\Sigma = Spin(4)/U(1)$ is said to be $Spin(4)$-homogeneous (or just homogeneous) if there is a lift of the $Spin(4)$ action on Σ to its total space, which commutes with the right G action on P. In particular, $Spin(4) \to \Sigma$ is itself a homogeneous $U(1)$-bundle. In general homogeneous $SU(2)$ principal bundles over Σ are determined by their isotropy homomorphisms $\lambda_l : U(1) \to SU(2)$ and are constructed via

$$P_{\lambda_l} = Spin(4) \times_{(U(1),\lambda_l)} SU(2), \tag{3.3.16}$$

where the possible group homomorphisms λ_l are parametrized by $l \in \mathbb{Z}$ and given by

$$\lambda_l(\theta) = \begin{pmatrix} e^{il\theta} & 0 \\ 0 & e^{-il\theta} \end{pmatrix}.$$

By construction the P_{λ_l} are reducible to $Spin(4)$ and each connection on the latter extends to a reducible connection on P_{λ_l} (see [KN63]). The goal is to find invariant connections on P_l which are not reducible to connections on $Spin(4)$ and it will be seen in proposition 4.2.1, that this is not possible for all but one l, which is $l = 1$.

Remark 3.3.7. *Let $E_l = P_{\lambda_l} \times_{(SU(2),c)} \mathbb{C}^2$, or equivalently $P_{\lambda_1} \times_{(SU(2),c^{\otimes l})} \mathbb{C}^2$, where c denotes the standard representation of $SU(2)$ on $\mathbb{C}^2$. As the P_l's are reducible,*

$$E_l = Spin(4) \times_{c \circ \lambda_l} \mathbb{C}^2 = L^l \oplus L^{-l},$$

splits as a sum of complex line bundles L^l associated with $Spin(4)$ from the degree l representation of $U(1)$ on $\mathbb{C}$. As Σ is topologically $\mathbb{S}^2 \times \mathbb{S}^3$, the bundles E_l are trivial and so do extend over $T^*\mathbb{S}^3$, i.e. when the zero section is glued back in. However, the splitting above only holds outside the zero section in $T^*\mathbb{S}^3$, as the bundle L itself does not extend.

Recall the canonical invariant connection $-\frac{i}{2}\theta^6 \in \Omega^1(Spin(4), i\mathbb{R})$ on $Spin(4) \to \Sigma$ defined in equation 3.3.4. This is a $U(1)$ connection and the next step is to extend it to a reducible connection on each P_{λ_l}.

Definition 3.3.8. *Let T_1, T_2, T_3 be a basis for $\mathfrak{su}(2)$ such that $[T_i, T_j] = 2\varepsilon_{ijk}T_k$. Then, the canonical invariant connection on P_{λ_l} is*

$$A_c^l = -\frac{l\theta^6}{2} \otimes T_1 \in \Omega^1(Spin(4), \mathfrak{su}(2)). \tag{3.3.17}$$

Lemma 3.3.9. *The curvature of the canonical invariant connection A_c^l is*

$$F_c^l = -\frac{l}{2}\left(\theta^{23} + \theta^{45}\right) \otimes T_1. \tag{3.3.18}$$

Proof. This follows from the Maurer-Cartan relation $d\theta^6 = \theta^{23} + \theta^{45}$, the other ones are $d\theta^1 = \theta^{24} + \theta^{35}$, $d\theta^3 = -\theta^{15} - \theta^{26}$, $d\theta^5 = \theta^{13} - \theta^{46}$, $d\theta^2 = -\theta^{14} + \theta^{36}$. □

In the same way one computes $c_1(L) = \frac{1}{4\pi}\left[\theta^{23} + \theta^{45}\right]$, and this can be compared this with the transverse Kähler structure. The vector field X_1 is the infinitesimal generator of a free $\mathbb{S}^1$-action on Σ and this is precisely the flow of the Reeb field. The contact form equips the bundle $\Sigma \to D$ with a connection which needs to be proportional to θ^1, and one can read from 3.3.15 that $\omega_D = \frac{1}{3}\left(\theta^{24} + \theta^{35}\right)$. Moreover, since $\omega_D = \frac{d\eta}{2}$, one discovers from the Maurer Cartan relations that $\eta = -\frac{2}{3}\theta^1$, as expected from 3.3.15 and so $c_1(\Sigma) = 2c_1(D) = \frac{1}{3\pi}\left[\theta^{24} + \theta^{35}\right]$.

Remark 3.3.10. *In fact L is the pull back of a holomorphic line bundle $\mathcal{L}$ over D. Moreover, $-i\frac{\theta^6}{2}$ is then a Hermitian Yang Mills connection on $\mathcal{L} \to D$ and in the case of the Conifold C it does lift to a reducible Calabi-Yau monopole. In fact one wants to construct Calabi-Yau monopoles whose connection A is asymptotic to $A_\infty = A_c^l$.*

Invariant Connections and Higgs Fields

The problem of finding invariant connections on P_l is an application of Wang's theorem, for which the reader is referred to [KN63] or Appendix B in this thesis.

Proposition 3.3.11. *Let $A^l \in \Omega^1(Spin(4), \mathfrak{su}(2))$ be the connection 1 form of an invariant connection on P_l. Then it is left-invariant and can be written as*

$$A^l = A_c^l + (A - A_c) \tag{3.3.19}$$

where $(A - A_c) \in \mathfrak{m}^* \otimes \mathfrak{su}(2)$, extended as a left-invariant 1-form with values in $\mathfrak{su}(2)$ is given by $A - A_c = A_1\theta^1 \otimes T_1$ if $l \neq 1$, while if $l = 1$

$$\begin{aligned} A - A_c \;=\; & A_1\theta^1 \otimes T_1 \\ & + \left(A_2\theta^2 - A_3\theta^3 + A_4\theta^4 - A_5\theta^5\right) \otimes T_2 \\ & + \left(A_3\theta^2 + A_2\theta^3 + A_5\theta^4 + A_4\theta^5\right) \otimes T_3, \end{aligned}$$

and $A_1, A_2, A_3, A_4, A_5 \in \mathbb{R}$.

Proof. By Wang's theorem [KN63], invariant connections are given by morphisms of $U(1)$ representations

$$\Lambda_l : (\mathfrak{m}, \mathrm{Ad}) \longrightarrow (\mathfrak{su}(2), \mathrm{Ad} \circ \lambda_l).$$

Then by extending Λ_l as a left invariant $\mathfrak{su}(2)$-valued 1-form in $Spin(4)$ one obtains an invariant connection $A = A_c^l + \Lambda_l$ on P_l (notice that $\Lambda_l = 0$ gives the canonical invariant connection). Let c be the standard, weight 1, $U(1)$ representation on $\mathbb{C} \cong \mathbb{R}^2$. Split the representations above into irreducibles $\mathfrak{m} \cong \mathbb{R} \oplus c \oplus c$, and $\mathfrak{su}(2) \cong \mathbb{R} \oplus c^{\otimes l}$, where in the first of these $c \oplus c \cong \langle X_2, X_3 \rangle \oplus \langle X_4, X_5 \rangle$, from equation 3.3.5. Then, Schur's lemma states that Λ should restrict to each piece as an isomorphism or as 0. So for $l \neq 1$, $\lambda_l = A_1 T_1 \oplus 0$, while for $l = 1$, $\Lambda_1 = A_1 T_1 \oplus 1_1 \oplus 1_2$, where $A_1 \in \mathbb{R}$ and 1_1 and 1_2 are isomorphisms matching the c components in both sides. Using the basis of $\mathfrak{m}$ given by the X_i's as in section 3.3.1 and the basis for $\mathfrak{su}(2)$ given by the T_i's as in definition 3.3.8, $1_1, 1_2$ can be written

$$\begin{aligned} 1_1 &= \left(A_2\theta^2 - A_3\theta^3\right) \otimes T_2 + \left(A_3\theta^2 + A_2\theta^3\right) \otimes T_3 \\ 1_2 &= \left(A_4\theta^4 - A_5\theta^5\right) \otimes T_2 + \left(A_5\theta^4 + A_4\theta^5\right) \otimes T_3, \end{aligned}$$

with $A_1, A_2, A_3, A_4, A_5 \in \mathbb{R}$. Rearranging gives the result in the statement. □

Proposition 3.3.12. *For all $l \in \mathbb{Z}$, there are invariant Higgs fields Φ and these are of the form $\Phi = \phi\, T_1$, with $\phi \in \mathbb{R}$.*

Proof. The adjoint bundle is constructed via $\mathfrak{g}_{P_l} \times_{(SU(2),\mathrm{Ad})} \mathfrak{su}(2)$ and unwinding the construction of P in equation 3.3.16, gives

$$\mathfrak{g}_{P_l} = Spin(4) \times_{U(1),\mathrm{Ad}\circ\lambda_l} \mathfrak{su}(2).$$

So, think of Higgs fields (sections of $\mathfrak{g}_{P_l}$) as functions in $Spin(4)$ with values in $\mathfrak{su}(2)$ which are equivariant for the $U(1)$ right-action on $Spin(4)$ and $\mathrm{Ad} \circ \lambda_l$-action on $\mathfrak{su}(2)$ via $\mathrm{Ad} \circ \lambda_l$. For $Spin(4)$-invariant Higgs fields, these functions must be constant. So the previous equivariance condition reduces to the statement that such a constant must be fixed by the $\mathrm{Ad} \circ \lambda_l$-action, i.e. it must lie in a irreducible component given by the trivial representation. There is only one such and is the direction singled out by T_1. □

Then a $Spin(4)$-invariant pair (A, Φ) on the pull back of P_l to $(\varepsilon, +\infty) \times \Sigma$ can be written as

$$A = dr \otimes A_r(r) + A_\Sigma(r) \ \ , \ \ \Phi = \phi(r) \otimes T_1,$$

with A_Σ 1-parameter family as in proposition 3.3.11 and A_r, Φ 1 parameter families as in proposition 3.3.12, parametrized by $r \in (\varepsilon, \infty)$. Moreover, one can always get rid of the radial component in A via a gauge transformation g that only depends on the r-direction, for this one needs to solve $(g \cdot A)(\partial_r) = 0$. This equation can be written as $g^{-1}\frac{\partial g}{\partial r} + g^{-1} A_r g = 0$, and so amounts to solving an ODE for g. This can always be solved with the condition $\lim_{r\to\infty} g(r) = 1_{SU(2)}$, the solution is unique and so there is no loss in assuming that $A_r = 0$.

Remark 3.3.13. *For the proof of theorem 3.3.1 one must consider invariant gauge transformations. The gauge-fixing above uses an invariant gauge transformation such that $\lim_{r\to\infty} g(r) = 1_{SU(2)}$, which is a usual requirement in monopole problems, but not here. So one can still use a gauge transformation g' which must not depend on r and be invariant, i.e. g must be a constant is the subgroup $Z_{U(1)}(SU(2)) = U(1) \subset SU(2)$ of those elements which are centralized by $U(1)$. These do not affect the radial gauge fixing above, they preserve A_c^l and act by conjugation as*

$g(A^l - A^l_c)g^{-1}$ and so one can get rid of one the A_i's. The choice of such a gauge will be postponed to a later stage, where a particular choice will ease the computations.

Lemma 3.3.14. *For $l \neq 1$, the curvature of an invariant connection A on P_l is given by*

$$F^l = \left(-\frac{l}{2}(\theta^{23} + \theta^{45}) + \dot{A}_1 dr \wedge \theta^1 + A_1(\theta^{24} + \theta^{35}) \right) \otimes T_1, \tag{3.3.20}$$

in particular the connection is always reducible for $l \neq 1$. For $l = 1$, the curvature is

$$\begin{aligned} F_A \ &= \ \left(\left(2(A_2^2 + A_3^2) - \frac{1}{2} \right) \theta^{23} + \left(2(A_4^2 + A_5^2) - \frac{1}{2} \right) \theta^{45} \right) \otimes T_1 \\ & \quad + \left(2(A_2A_4 + A_5A_3)(\theta^{25} - \theta^{34}) + (A_1 + 2(A_2A_5 - A_4A_3)) \, (\theta^{24} + \theta^{35}) \right) \otimes T_1 \\ & \quad + (A_4 - 2A_1A_3)(T_2 \otimes \theta^{12} + T_3 \otimes \theta^{13}) + (A_5 + 2A_1A_2)(T_3 \otimes \theta^{12} - T_2 \otimes \theta^{13}) \\ & \quad - (A_2 + 2A_1A_5)(T_2 \otimes \theta^{14} + T_3 \otimes \theta^{15}) - (A_3 - 2A_1A_4)(T_3 \otimes \theta^{14} - T_2 \otimes \theta^{15}) \\ & \quad + dr \wedge \frac{\partial}{\partial r}(A - A_c) \end{aligned} \tag{3.3.21}$$

Proof. The curvature of an invariant connection $A = A^l_c + (A - A^l_c)$ is given by

$$F_A = F^l_c + d_{A^l_c}(A - A_c) + \frac{1}{2}\left[\left(A - A^l_c \right) \wedge \left(A - A^l_c \right) \right], \tag{3.3.22}$$

where F^l_c is the curvature of the canonical invariant connection, computed in equation 3.3.18, and $d_{A^l_c}\left(A - A^l_c\right)$ is the covariant derivative of $A - A^l_c$ with respect to A^l_c. The statement that the connection is reducible follows from the Ambrose-Singer theorem, since the curvature always takes value in the $\mathfrak{u}(1) \subset \mathfrak{su}(2)$ generated by T_1.

For $l \neq 1$, the third therm in 3.3.22 is $A_1^2\theta^1 \wedge \theta^1 \otimes [T_1, T_1]$ and so vanishes. One is left with the computations of the second term, for which the Bianchi identity $d_{A^l_c}F^l_c = 0$ can be used to conclude $d_{A^l_c}T_1 = 0$ and so

$$\begin{aligned} d_{A^l_c}(A - A^l_c) \ &= \ d(A - A^l_c) + [A^l_c \wedge (A - A^l_c)] \\ &= \ \dot{A}_1 dr \wedge \theta^1 \otimes T_1 + A_1 T_1 \otimes (\theta^{24} + \theta^{35}). \end{aligned}$$

The case $l = 1$ is more involved. Using the Maurer-Cartan relations, the second term in 3.3.22 $I_2 = d_{A_c}(A - A_c) = d(A - A_c) + [A_c \wedge (A - A_c)]$ is

$$\begin{aligned} d_{A_c}(A - A_c) \ &= \ dr \wedge \frac{\partial}{\partial r}(A - A_c) + A_1 T_1 \otimes (\theta^{24} + \theta^{35}) \\ & \quad - (A_2T_2 + A_3T_3) \otimes \theta^{14} + (A_3T_2 - A_2T_3) \otimes \theta^{15} \\ & \quad + (A_4T_2 + A_5T_3) \otimes \theta^{12} + (-A_5T_2 + A_4T_3) \otimes \theta^{13}, \end{aligned}$$

where the vertical terms (i.e. those in $\mathfrak{h}$) from the exterior derivative have canceled with the ones coming from $[A_c \wedge (A - A_c)]$. The last term $I_3 = \frac{1}{2}[(A - A_c) \wedge (A - A_c)]$ is given by

$$
\begin{aligned}
I_3 &= A_1\theta^1 \wedge (A_2\theta_2 - A_3\theta^3 + A_4\theta^4 - A_5\theta^5) \otimes [T_1, T_2] \\
&\quad + A_1\theta^1 \wedge (A_3\theta_2 + A_2\theta^3 + A_5\theta^4 + A_4\theta^5) \otimes [T_1, T_3] \\
&\quad + \left(A_2\theta_2 - A_3\theta^3 + A_4\theta^4 - A_5\theta^5\right) \wedge \left(A_3\theta_2 + A_2\theta^3 + A_5\theta^4 + A_4\theta^5\right) \otimes [T_2, T_3] \\
&= 2A_1(A_2T_3 - A_3T_2) \otimes \theta^{12} + 2A_1(A_4T_3 - A_5T_2) \otimes \theta^{14} \\
&\quad - 2A_1(A_2T_2 + A_3T_3) \otimes \theta^{13} - 2A_1(A_4T_2 + A_5T_3) \otimes \theta^{15} \\
&\quad + 2(A_2A_5 - A_4A_3)T_1 \otimes (\theta^{24} + \theta^{35}) + 2(A_2A_4 + A_5A_3)T_1 \otimes (\theta^{25} - \theta^{34}) \\
&\quad 2(A_2^2 + A_3^2)T_1 \otimes \theta^{23} + 2(A_4^2 + A_5^2)T_1 \otimes \theta^{45}.
\end{aligned}
$$

□

Lemma 3.3.15. *Let $\Phi \in \Omega^0(T^*\mathbb{S}^3, \mathfrak{g}_{P_l})$ be an invariant Higgs field and A^l an invariant connection on P_l. Then, if $l \neq 1$, $\nabla_{A^l}\Phi = \dot{\phi}dr \otimes T_1$, while for $l = 1$*

$$
\begin{aligned}
\nabla_{A^1}\Phi &= \dot{\phi}dr \otimes T_1 \\
&\quad + 2\phi A_2\left(T_2 \otimes \theta^3 - T_3 \otimes \theta^2\right) + 2\phi A_3\left(T_2 \otimes \theta^2 + T_3 \otimes \theta^3\right) \\
&\quad - 2\phi A_4(T_3 \otimes \theta^4 - T_2 \otimes \theta^5) + 2\phi A_5(T_2 \otimes \theta^4 + T_3 \otimes \theta^5).
\end{aligned}
$$

Proof. This follows from computing $\nabla_{A^l}\Phi = \nabla_{A^l}(\phi T_1) = d\phi \otimes T_1 + \phi\nabla_{A^l}T_1$. The first term is just $\dot{\phi}dr \otimes T_1$, while for the second term one uses that $A^l = A_c^l + (A^l - A_c^l)$, then $\nabla_{A^l}T_1 = d_{A_c^l}T_1 + [A^l - A_c^l, T_1]$, i.e.

$$\nabla_{A^l}\Phi = \dot{\phi}dr \otimes T_1 + \phi\left(d_{A_c^l}T_1 + [A^l - A_c^l, T_1]\right).$$

Again, the Bianchi identity $d_{A_c^l}F_{A_c^l} = 0$ for A_c^l gives $d_{A_c^l}T_1 = 0$ and one is left with the remaining terms. In the case $l \neq 1$ these vanish and $\nabla_{A^l}\Phi = \dot{\phi}dr \otimes T_1$, while for $l = 1$ one has

$$[A^l - A_c^l, T_1] = 2(A_3\theta^2 + A_2\theta^3 + A_5\theta^4 + A_4\theta^5) \otimes T_2 - 2(A_2\theta^2 - A_3\theta^3 + A_4\theta^4 - A_5\theta^5) \otimes T_3.$$

The result follows. □

Reduction to ODE's

This section uses the results from the previous section to reduce the Calabi-Yau monopole equations for invariant connections and Higgs fields to ODE's. The two cases $l = 1$ and $l \neq 1$ are presented separately and the case $l = 1$ ends up being the more important one. Recall from the third item of proposition 3.1.3, namely equations 3.1.6 and 3.1.7 with $\Phi_2 = 0$, that the Calabi-Yau monopole equations are

$$d_A\Phi_1 \wedge \frac{\omega^2}{2} + F_A \wedge \Omega_2 = 0\ ,\ F_A \wedge \frac{\omega^2}{2} = 0.$$

Proposition 3.3.16. *Up to the action of a constant gauge transformation, $Spin(4)$ invariant Calabi-Yau monopoles on $P_l \to T^*\mathbb{S}^3 \backslash \mathbb{S}^3$ are in correspondence with solutions to the following set of ODE's. For $l \neq 1$,*

$$\begin{aligned}
\dot{A}_1 &= -2\frac{\dot{\mathcal{G}}}{\mathcal{G}} A_1 \\
\dot{\phi} &= \frac{l}{4}\frac{r}{\mathcal{G}^2}\left(\frac{R_-}{R_+} - \frac{R_+}{R_-}\right).
\end{aligned}$$

While for $l = 1$, the fields must satisfy the constraint $A_2A_4 + A_3A_5 = 0$ and solve

$$\begin{aligned}
\dot{A}_1 &= -2\frac{\dot{\mathcal{G}}}{\mathcal{G}}\left(A_1 + 2(A_2A_5 - A_4A_3)\right) \\
\dot{\phi} &= \frac{1}{\mathcal{G}^2}\left(\frac{r}{4}\frac{R_-}{R_+}\left(1 - 4(A_2^2 + A_3^2)\right) - \frac{r}{4}\frac{R_+}{R_-}\left(1 - 4(A_4^2 + A_5^2)\right)\right) \\
\dot{A}_2 &= -\frac{r}{2}\frac{1}{R_-^2}\left(A_2 + 2A_1A_5\right) - \frac{r}{\mathcal{G}}\phi A_2 \\
\dot{A}_3 &= -\frac{r}{2}\frac{1}{R_-^2}\left(A_3 - 2A_1A_4\right) - \frac{r}{\mathcal{G}}\phi A_3 \\
\dot{A}_4 &= -\frac{r}{2}\frac{1}{R_+^2}\left(A_4 - 2A_1A_3\right) + \frac{r}{\mathcal{G}}\phi A_4 \\
\dot{A}_5 &= -\frac{r}{2}\frac{1}{R_+^2}\left(A_5 + 2A_1A_2\right) + \frac{r}{\mathcal{G}}\phi A_5,
\end{aligned}$$

with $\phi, A_i : (\varepsilon, \infty) \to \mathbb{R}$, for $i = 1,2,3,4,5$, $R_+ = \sqrt{\frac{r^2+\varepsilon^2}{2}}$, $R_- = \sqrt{\frac{r^2-\varepsilon^2}{2}}$ and $\mathcal{G} = \sqrt{R_+R_-}\mathcal{F}'(r^2)$, where $\mathcal{F}$ is the Kähler potential for the Stenzel metric and $\mathcal{F}'$ its derivative.

Proof. We use the formulae 3.3.6 and 3.3.9, together with those computed in the previous section to evaluate the quantities, $\nabla_A\Phi \wedge \frac{\omega^2}{2}$, $F_A \wedge \omega^2$ and $F_A \wedge \Omega_2$.

$$\begin{aligned}
\nabla_A\Phi \wedge \frac{\omega^2}{2} &= -\mathcal{G}^2\dot{\phi}\, T_1 \otimes dr \wedge \theta^{2345} \\
&\quad +2\mathcal{G}\dot{\mathcal{G}}\phi\left((A_3T_2 - A_2T_3) \otimes dr \wedge \theta^{1235} - (A_2T_2 + A_3T_3) \otimes dr \wedge \theta^{1234}\right) \\
&\quad -2\mathcal{G}\dot{\mathcal{G}}\phi\left((A_5T_2 - A_4T_3) \otimes dr \wedge \theta^{1345} - (A_4T_2 + A_5T_3) \otimes dr \wedge \theta^{1245}\right).
\end{aligned}$$

$$\begin{aligned}
F_A \wedge \omega^2 &= -2\mathcal{G}^2 dr \wedge \frac{\partial}{\partial r}(A - A_c) \wedge \theta^{2345} \\
&\quad -4\mathcal{G}\dot{\mathcal{G}}\left(A_1 + 2(A_2A_5 - A_4A_3)\right) T_1 \otimes dr \wedge \theta^{12345} \\
&= 2\mathcal{G}\left(\mathcal{G}\dot{A}_1 + 2\dot{\mathcal{G}}\left(A_1 + 2(A_2A_5 - A_4A_3)\right)\right) T_1 \otimes dr \wedge \theta^{12345}.
\end{aligned}$$

The computation of $F_A \wedge \Omega_2$ is long, but the outcome is

$$
\begin{aligned}
& F_A \wedge \Omega_2 = 4R_+R_-(A_2A_4 + A_3A_5)T_1 \otimes \theta^{12345} \\
& + \left(\frac{r}{4}\frac{R_-}{R_+}\left(1 - 4(A_2^2 + A_3^2)\right) - \frac{r}{4}\frac{R_+}{R_-}\left(1 - 4(A_4^2 + A_5^2)\right) \right) T_1 \otimes dr \wedge \theta^{2345} \\
& -R_-R_+ \left(\dot{A}_2T_2 + \dot{A}_3T_3 + \frac{r}{2R_-^2}\left((A_2 + 2A_1A_5)T_2 + (A_3 - 2A_1A_4)T_3\right) \right) \otimes dr \wedge \theta^{1234} \\
& -R_-R_+ \left(-\dot{A}_3T_2 + \dot{A}_2T_3 + \frac{r}{2R_-^2}\left((A_2 + 2A_1A_5)T_3 - (A_3 - 2A_1A_4)T_2\right) \right) \otimes dr \wedge \theta^{1235} \\
& -R_-R_+ \left(\dot{A}_4T_1 + \dot{A}_5j + \frac{r}{2R_+^2}\left((A_4 - 2A_1A_3)T_2 + (A_5 + 2A_1A_2)T_2\right) \right) \otimes dr \wedge \theta^{1245} \\
& -R_-R_+ \left(-\dot{A}_5T_2 + \dot{A}_4T_3 + \frac{r}{2R_+^2}\left((A_4 - 2A_1A_3)T_3 - (A_5 + 2A_1A_2)T_2\right) \right) \otimes dr \wedge \theta^{1345}
\end{aligned}
$$

Matching all these computations in $-\nabla_A\Phi \wedge \frac{\omega^2}{2} = F \wedge \Omega_2$ gives the constraint $4R_+R_-(A_2A_4 + A_3A_5)$ and

$$
\begin{aligned}
\mathcal{G}^2\dot{\phi} &= \left(\frac{r}{4}\frac{R_-}{R_+}\left(1 - 4(A_2^2 + A_3^2)\right) - \frac{r}{4}\frac{R_+}{R_-}\left(1 - 4(A_4^2 + A_5^2)\right) \right) \\
2\mathcal{G}\dot{\mathcal{G}}\phi(A_2T_2 + A_3T_3) &= -R_-R_+\left(\dot{A}_2T_2 + \dot{A}_3T_3\right) - \frac{r}{2}\frac{R_+}{R_-}\left((A_2 + 2A_1A_5)T_2 + (A_3 - 2A_1A_4)T_3\right) \\
-2\mathcal{G}\dot{\mathcal{G}}\phi(A_3T_2 - A_2T_3) &= -R_-R_+\left(-\dot{A}_3T_2 + \dot{A}_2T_3\right) - \frac{r}{2}\frac{R_+}{R_-}\left((A_2 + 2A_1A_5)T_3 - (A_3 - 2A_1A_4)T_2\right) \\
-2\mathcal{G}\dot{\mathcal{G}}\phi(A_4T_2 + A_5T_3) &= -R_-R_+\left(\dot{A}_4T_2 + \dot{A}_5T_3\right) - \frac{r}{2}\frac{R_-}{R_+}\left((A_4 - 2A_1A_3)T_2 + (A_5 + 2A_1A_2)T_3\right) \\
2\mathcal{G}\dot{\mathcal{G}}\phi(A_5T_2 - A_4T_3) &= -R_-R_+\left(-\dot{A}_5T_2 + \dot{A}_4T_3\right) - \frac{r}{2}\frac{R_-}{R_+}\left((A_4 - 2A_1A_3)T_3 - (A_5 + 2A_1A_2)T_2\right).
\end{aligned}
$$

From these equations and using $\frac{2\mathcal{G}\dot{\mathcal{G}}}{R_+R_-} = \frac{r}{\mathcal{G}}$, which is the ODE for the Ricci flatness of the metric gives the statement. □

Remark 3.3.17. *Recall that the Calabi-Yau monopole equations are overdetermined. In this specific example this can be directly seen from the ODE's in the statement of the previous proposition. In fact, for $l = 1$ one sees that there are 6 ODE's for 6 real valued functions, but they are constrained to satisfy the identity $A_2A_4 + A_3A_5 = 0$. Since the complex structure is integrable it is expected that the evolution encoded in the 6 ODE's does preserve this constraint. In fact this will be shown later in lemma 3.3.23.*

3.3.3 Calabi-Yau Monopoles on the Cone

This subsection studies Calabi-Yau monopoles on the Conifold. The most important point is the existence of an Abelian Calabi-Yau monopole given by the canonical invariant connection. This is the pull back from $L^l \to D = \mathbb{P}^1 \times \mathbb{P}^1$ of a HYM connection, which, recall, is the model for the asymptotic behavior of finite mass Calabi-Yau monopoles. Since $c_1(L^l) \in H^{1,1}(D, \mathbb{Z})$ is in the kernel of $\cdot \cup [\omega_D]$ proposition 3.1.31 gives its existence, but here an explicit formula for the connection is given. One also has $\mathfrak{g}_P \cong i\mathbb{R} \oplus L^{2l}$ and using this decomposition let $\Phi = \phi \oplus 0$, with ϕ constant. Then (A_c^l, Φ) are Calabi-Yau monopoles on the Conifold and provide good asymptotic conditions for finite mass Calabi-Yau monopoles on $T^*\mathbb{S}^3$. In the system of ODE's this corresponds to taking ϕ constant and all the A_i's to be zero. After writing the equations on the cone it will be trivial to see that this is indeed a solution. In fact a slightly more general result, proposition 3.3.18, classifying all "constant" mass Calabi-Yau monopoles on the Conifold is obtained. Recall that the radius function on the cone is $\rho = \left(\frac{3r}{2}\right)^{\frac{2}{3}}$ and $\mathcal{F} = \rho^2$ and one has the relations

$$\mathcal{F}'(r^2) = \left(\frac{3}{2}\right)^{\frac{1}{3}} r^{-\frac{2}{3}} = \frac{3}{2}\frac{1}{\rho} \ , \ \mathcal{G} = \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{1}{3}} r^{\frac{4}{3}} = \frac{\rho^2}{2} \ , \ \dot{\mathcal{G}} = \left(\frac{2}{3}\right)^{\frac{2}{3}} r^{\frac{1}{3}} = \frac{2}{3}\sqrt{\rho}.$$

Substitute these in the equations, then for $l = 1$ these turn into

$$\begin{aligned}
\dot{A}_1 &= -\frac{8}{3r}\left(A_1 + 2(A_2A_5 - A_4A_3)\right) \\
\dot{\phi} &= 4\left(\frac{2}{3}\right)^{\frac{2}{3}} r^{-\frac{5}{3}}\left((A_4^2 + A_5^2) - (A_2^2 + A_3^2)\right)
\end{aligned}$$

together with the constraint $A_2A_4 + A_3A_5 = 0$ and

$$\begin{aligned}
\dot{A}_2 &= -\frac{1}{r}\left(A_2 + 2A_1A_5\right) - 2\left(\frac{2}{3r}\right)^{\frac{1}{3}}\phi A_2 & \dot{A}_3 &= -\frac{1}{r}\left(A_3 - 2A_1A_4\right) - 2\left(\frac{2}{3r}\right)^{\frac{1}{3}}\phi A_3, \\
\dot{A}_4 &= -\frac{1}{r}\left(A_4 - 2A_1A_3\right) + 2\left(\frac{2}{3r}\right)^{\frac{1}{3}}\phi A_4 & \dot{A}_5 &= -\frac{1}{r}\left(A_5 + 2A_1A_2\right) + 2\left(\frac{2}{3r}\right)^{\frac{1}{3}}\phi A_5.
\end{aligned}$$

The following rescaling simplifies the equations and is a good preview of what will be done later for $T^*\mathbb{S}^3$. Define the fields B_i via

$$B_2 = rA_2 \ , \ B_3 = rA_3 \ , \ B_4 = rA_4 \ , \ B_5 = rA_5.$$

Use $\dot{A}_i + \frac{1}{r}A_i = \frac{1}{r}\dot{B}_i$, and change coordinates to ρ via $\frac{d}{dr} = \left(\frac{2}{3r}\right)^{\frac{1}{3}}\frac{d}{d\rho}$ to obtain

$$\begin{aligned}
\frac{dA_1}{d\rho} &= -\frac{4}{\rho}A_1 + \frac{18}{\rho^4}(B_2B_5 - B_4B_3), \\
\frac{d\phi}{d\rho} &= \frac{3^3}{2\rho^5}\left((B_4^2 + B_5^2) - (B_2^2 + B_3^2)\right),
\end{aligned}$$

together with the constraint $B_2B_4 + B_3B_5 = 0$ and

$$\begin{aligned}
\frac{dB_2}{d\rho} &= -\frac{3}{\rho}A_1B_5 - 2\phi B_2 & \frac{dB_3}{d\rho} &= +\frac{3}{\rho}A_1B_4 - 2\phi B_3, \\
\frac{dB_4}{d\rho} &= +\frac{3}{\rho}A_1B_3 + 2\phi B_4 & \frac{dB_5}{d\rho} &= -\frac{3}{\rho}A_1B_2 + 2\phi B_5.
\end{aligned}$$

Proposition 3.3.18. *For all l and in radial gauge, any $Spin(4)$ invariant Calabi-Yau monopole on*

P_l over the Conifold with $|\Phi| \neq 0$ constant is given by

$$A^l = A^l_c + C\rho^{-4}\theta^1 \otimes T_1 \ , \ \Phi = mT_1, \tag{3.3.23}$$

with $C \in \mathbb{R}$ and $m \in \mathbb{R}\backslash\{0\}$. In particular, the canonical invariant connection A^l_c is obtain by $C = 0$.

Proof. If $|\Phi|$ is constant, then $\phi = m \in \mathbb{R}$ and in a first case focus in the more involved case $l = 1$. Make use of the extra gauge freedom and use $g \in U(1) \subset SU(2)$ to change the connection from $A - A^l_c$ to $g(A - A^l_c)g^{-1}$. This rotates $A_2T_2 + A_3T_3$ and $A_4T_2 + A_5T_3$ simultaneously. Hence, there is no loss of generality in supposing that $A_2 = 0$, i.e. $B_2 = 0$. Then, the constraint turns into $B_3B_5 = 0$, while the third equation is $A_1B_5 = 0$, then either $A_1 = B_3 = 0$ or $B_5 = 0$. In the following these two cases are analyzed.
First the case $A_1 = B_3 = 0$, then in fact $A_2T_2 + A_3T_3 = 0$ and so the gauge freedom is still available to set $B_4 = 0$. Since $\phi = m$ the equation for $\frac{d\phi}{d\rho} = 0$ gives $B_5 = 0$ as well. So in this case $\Phi = mT_1$ and the connection is the canonical invariant one.
For the case where $B_5 = 0$, the second equation gives $B_4^2 = B_3^2$, i.e. $B_3 = \pm B_4$. If one defines $B_1 = \rho^4 A_1$, the remaining equations are

$$\frac{dB_1}{d\rho} = \mp 18 B_4^2 \tag{3.3.24}$$

$$\frac{d(B_4^2)}{d\rho} = \pm\frac{3}{\rho^5}B_1B_4^2 - 4mB_4^2 \tag{3.3.25}$$

$$\frac{d(B_4^2)}{d\rho} = \pm\frac{3}{\rho^5}B_1B_4^2 + 4mB_4^2. \tag{3.3.26}$$

Since $m \neq 0$ by hypothesis, the last two ODE's are compatible only in the case $B_4 = 0$ and so also $B_3 = 0$. One is left with solving the first equation which now says that B_1 is constant. The Calabi-Yau monopole to which this corresponds is given by the connection $A = A^1_c + \frac{C}{\rho^4}\theta^1 \otimes T_1$ and the Higgs field $\Phi = mT_1$. Hence its is reducible and the connection is HYM and for $C = 0$ is the canonical invariant one.

One must now discuss what happens when $l \neq 1$. If that is the case, then immediately $B_2 = B_3 = B_4 = B_5 = 0$ and the only equation is $\frac{dA_1}{d\rho} = -\frac{4}{\rho}A_1$. This can be integrated to give the Calabi-Yau monopole in equation 3.3.23, which was obtained before for $l = 1$. They do decay to the canonical invariant connection. However, this decay is at a polynomial rate, more specifically $|A - A^l_c| = O(\rho^{-5})$, which is due to the (unique) component which is "parallel" to the Higgs field. So if one imposes that the connection must decay faster than this rate the canonical invariant connection is the unique solution (setting $C = 0$). □

Remark 3.3.19. *All these Calabi-Yau monopoles are reducible and their connections are Hermitian Yang Mills (HYM) on the Conifold. The canonical invariant connection, obtained from $C = 0$, is the unique one which is pulled back from the link. For $C \neq 0$ the connections differ from this one by $C\rho^{-4}\theta^1 = Id\left(\frac{3C}{8}\rho^{-4}\right)$, which is a harmonic 1-form on the cone. In fact, notice that given*

an Abelian Calabi-Yau monopole (A^0, Φ^0) and a harmonic 1-form a, then $(A^0 + a, \Phi^0)$ is also a Calabi-Yau monopole.
Also, notice that it is also possible to solve the equations with $m = 0$. Following the proof above the equations reduce to $\frac{dB_1}{d\rho} = \mp 18 B_4^2$ and $\frac{d(B_4^2)}{d\rho} = \pm \frac{3}{\rho^5} B_1 B_4^2$. Integrating these gives rise to an $SU(2)$-irreducible HYM connection on the cone, which is not pulled back from $D = \mathbb{P}^1 \times \mathbb{P}^1$.

3.3.4 Reducible Calabi-Yau Monopoles in $T^*\mathbb{S}^3$

For reducible Calabi-Yau monopoles one must put all $A_i = 0$, for $i \geq 2$. Then, only the first two equations in proposition 3.3.16 survive. For $l \neq 1$, the first of them $\frac{dA_1}{d\rho} = -2\frac{\dot{\mathcal{G}}}{\mathcal{G}} A_1$, can be readily integrated to give $A_1(r) = \frac{C}{\mathcal{G}^2}$, where $C \in \mathbb{R}$ is a constant. Regarding the second equation, using the function $h^2 = \frac{1}{\varepsilon^2} R_+ R_- \mathcal{G}$ and the radial coordinate ρ gives $\frac{d\phi}{d\rho} = -\frac{l}{2h^2}$. This can be integrated to

$$\phi_l(\rho) = m - \int \frac{l}{2h^2(\rho)} d\rho,$$

with $m \in \mathbb{R}$. This diverges at $\rho = 0$, i.e. the zero section. Notice that such solutions also exist for $l = 1$ and by analogy with 3 dimensions are called Dirac Calabi-Yau monopoles.

Definition 3.3.20. *Let (X, ω, Ω) be a noncompact Calabi-Yau manifold and $N \subset X$ a special Lagrangian submanifold. A Dirac Calabi-Yau monopole is a Calabi-Yau monopole on a line bundle defined on the complement of N. N will be called the singular set of the Calabi-Yau monopole.*

Proposition 3.3.21. *For all $l \in \mathbb{Z}$ and $C, m \in \mathbb{R}$, the connections and Higgs fields*

$$A = A_c^l + \frac{C}{\mathcal{G}} \theta^1 \quad , \quad \phi = m - \int \frac{l}{2h^2(\rho)} d\rho,$$

are Dirac Calabi-Yau monopoles on $L^{\otimes l}$ for the Stenzel metric, with the zero section as singular set.

Their curvature is

$$F^l = -\frac{l}{2}(\theta^{23} + \theta^{45}) - 2C \frac{\dot{\mathcal{G}}}{\mathcal{G}^3} dr \wedge \theta^1 + \frac{C}{\mathcal{G}^2}(\theta^{24} + \theta^{35}) \tag{3.3.27}$$

Moreover, from the Appendix C one knows that $h(\rho) = \rho + O(\rho^3)$ for $\rho \ll 1$ and $h(\rho) = O(\rho^{5/2})$ for $\rho \gg 1$ and so

$$\phi(\rho) = \begin{cases} \frac{1}{\rho} + O(\rho^0) & \text{if } \rho \ll 1 \\ m + \frac{cl}{\rho^4} + O(\rho^{-4-\varepsilon}) & \text{if } \rho \gg 1, \end{cases} \tag{3.3.28}$$

where $c > 0$ is a constant independent of l and only depending on $Vol_{g_\Sigma}(\Sigma)$ and $\varepsilon > 0$. In fact ϕ is harmonic on $T^*\mathbb{S}^3 \backslash \mathbb{S}^3$ for the Stenzel metric. This can be checked explicitly using the formula 3.3.12 for Stenzel's metric. Since $*\Delta\phi = d * d\phi$, one computes

$$* \Delta\phi = d * d \int_0^\rho \frac{1}{2h^2(s)} ds = d \left(\frac{1}{2h^2(\rho)} \frac{\partial \rho}{\partial r} * dr \right) = d\varepsilon^2 = 0.$$

3.3.5 Irreducible Calabi-Yau Monopoles in $T^*\mathbb{S}^3$

This subsection reduces the system of ODE's in proposition 3.3.16 to simpler ones and uses it to prove the main theorem 3.3.1. This is done in a series of steps: first proposition 3.3.22 rescales the fields A_i and changes coordinate to ρ in order to rewrite the ODE's. Then lemma 3.3.23 rewrites the equations once again and shows the constraint $A_2A_4 + A_3A_5 = 0$ is preserved by the evolution encoded in the other equations. At the end of the subsection theorem 3.3.1 is proven and this requires splitting into 3 cases. One of these cases requires using lemma 3.3.25, which is stated and proved just before. This lemma reduces that case to the problem of solving a certain initial value problem. That problem is precisely the one parameterizing spherically symmetric Bogomolnyi monopoles in $(\mathbb{R}^3, dr^2 + h^2(r)g_{\mathbb{S}^2})$, and this has already been done in chapter 2. The rest of the proof consists of using the results in the first part of chapter 2, namely theorem 2.2.1.

Proposition 3.3.22. *Let the rescaled fields B_i be defined via $B_1 = \mathcal{G}^2 A_1$, $B_2 = R_- A_2$, $B_3 = R_- A_3$, $B_4 = R_+ A_4$, $B_5 = R_+ A_5$. Then, in terms of the distance function ρ, defined in 3.3.13, and using $h^2(\rho) = \frac{1}{\varepsilon^2} R_+ R_- \mathcal{G}$ the ODE's in proposition 3.3.16 are given by the constraint $B_2B_4 + B_3B_5 = 0$ and*

$$\begin{aligned}
\frac{d\phi}{d\rho} &= -\frac{1}{2h^2(\rho)}\left(1 - \frac{4}{\varepsilon^2}\left((B_4^2 + B_5^2) - (B_2^2 + B_3^2)\right)\right) \\
\frac{dB_1}{d\rho} &= -4\left(B_2B_5 - B_4B_3\right) \\
\frac{dB_2}{d\rho} &= -\frac{2}{\varepsilon^2 h^2} B_1B_5 - 2\phi B_2 \\
\frac{dB_3}{d\rho} &= \frac{2}{\varepsilon^2 h^2} B_1B_4 - 2\phi B_3 \\
\frac{dB_4}{d\rho} &= \frac{2}{\varepsilon^2 h^2} B_1B_3 + 2\phi B_4 \\
\frac{dB_5}{d\rho} &= -\frac{2}{\varepsilon^2 h^2} B_1B_2 + 2\phi B_5.
\end{aligned}$$

Proof. The constraint $B_2B_4 + B_3B_5 = 0$ is immediate from $A_2A_4 + A_3A_5 = 0$. Inserting the rescaled fields into the equation for $\dot{\phi}$ in proposition 3.3.16 and rearranging gives

$$\dot{\phi} = -\frac{r}{R_-R_+\mathcal{G}^2}\frac{\varepsilon^2}{4}\left(1 - \frac{4}{\varepsilon^2}\left((B_4^2 + B_5^2) - (B_2^2 + B_3^2)\right)\right)$$

Next use $\frac{d}{dr} = \frac{r}{2\mathcal{G}}\frac{d}{d\rho}$ to change coordinates to ρ and $h^2 = \frac{1}{\varepsilon^2}R_+R_-\mathcal{G}$ to obtain the equation in the statement for $\frac{d\phi}{d\rho}$.

To analyze the other equations use $\dot{R_+} = \frac{r}{2R_+}$ and $\dot{R_-} = \frac{r}{2R_-}$, which gives $\dot{B}_i = R_-\left(\dot{A}_i + \frac{r}{2R_-^2}A_i\right)$, for $i = 2, 3$ and $\dot{B}_j = R_+\left(\dot{A}_j + \frac{r}{2R_+^2}A_j\right)$ for $j = 4, 5$. Inserting the equations in proposition 3.3.16 into these, gives $\dot{B}_2 = -\frac{r}{R_+R_-}A_1B_5 - \frac{r}{\mathcal{G}}\phi B_2$, $\dot{B}_3 = \frac{r}{R_+R_-}A_1B_4 - \frac{r}{\mathcal{G}}\phi B_3$, $\dot{B}_4 = \frac{r}{R_+R_-}A_1B_3 + \frac{r}{\mathcal{G}}\phi B_4$ and $\dot{B}_5 = -\frac{r}{R_+R_-}A_1B_2 + \frac{r}{\mathcal{G}}\phi B_5$. Changing coordinates to ρ

again, these equations turn into

$$\begin{aligned} \frac{dB_2}{d\rho} &= -\frac{2\mathcal{G}}{R_- R_+} A_1 B_5 - 2\phi B_2 \quad , \quad \frac{dB_3}{d\rho} = \frac{2\mathcal{G}}{R_- R_+} A_1 B_4 - 2\phi B_3 \\ \frac{dB_4}{d\rho} &= \frac{2\mathcal{G}}{R_- R_+} A_1 B_3 + 2\phi B_4 \quad , \quad \frac{dB_5}{d\rho} = -\frac{2\mathcal{G}}{R_- R_+} A_1 B_2 + 2\phi B_5, \end{aligned} \tag{3.3.29}$$

and now changing from A_1 to $B_1 = \mathcal{G} A_1$ and using $h^2 = \frac{1}{\varepsilon^2} R_+ R_- \mathcal{G}$, gives the equations in the statement. To obtain the remaining equation multiply the equation containing $\dot{A}_1$ in proposition 3.3.16 by $\frac{2\mathcal{G}}{r}$ in order to ease the coordinate change. This gives

$$\frac{dA_1}{d\rho} = -\frac{4\dot{\mathcal{G}}}{r} A_1 - \frac{2\dot{\mathcal{G}}}{r R_+ R_-} 4 \left(B_2 B_5 - B_4 B_3 \right).$$

Multiply this equation by $\mathcal{G}^2$ and pass the terms having A_1 to the same side, then this term of the equation turns into $\mathcal{G}^2 \frac{dA_1}{d\rho} + \frac{4\mathcal{G}^2}{r} \frac{r}{2\mathcal{G}} \frac{d\mathcal{G}}{d\rho} A_1 = \mathcal{G}^2 \frac{dA_1}{d\rho} + 2\mathcal{G} \frac{d\mathcal{G}}{d\rho} A_1$, which is precisely $\frac{d}{d\rho} \left(\mathcal{G}^2 A_1 \right)$ and replaced back into the equation gives

$$\frac{dB_1}{d\rho} = -\frac{2\mathcal{G}^2 \dot{\mathcal{G}}}{r R_+ R_-} 4 \left(B_2 B_5 - B_4 B_3 \right).$$

Next recall that the reduction to ODE of the Monge-Ampère equation is $2\mathcal{G}^2 \dot{\mathcal{G}} = r R_+ R_-$ as alluded to in remark 3.3.3. Hence this equation also turns into the one in the statement. □

Lemma 3.3.23. *Let $f_1, f_2 : X \to \mathbb{C}$ be given by $f_1 = B_2 + iB_3$, $f_2 = B_4 + iB_5$ and denote their phases by χ_1, χ_2 respectively. The constraint in theorem 3.3.22 is $\mathrm{Re}(f_1 \overline{f_2}) = 0$ and if initially satisfied, is preserved by the other equations which are*

$$\begin{aligned} \frac{d\phi}{d\rho} &= -\frac{1}{2h^2(s)} \left(1 - \frac{4}{\varepsilon^2} \left(|f_2|^2 - |f_1|^2 \right) \right) \\ \frac{dB_1}{d\rho} &= 4Im(f_1 \overline{f_2}) \\ \frac{df_1}{d\rho} &= \frac{2i}{\varepsilon^2 h^2} B_1 f_2 - 2\phi f_1 \\ \frac{df_2}{d\rho} &= -\frac{2i}{\varepsilon^2 h^2} B_1 f_1 + 2\phi f_2. \end{aligned}$$

Moreover, the phases χ_1, χ_2 are constant and if $f_1 f_2 \neq 0$, then $\chi_2 - \chi_1 = \frac{\pi}{2} + \pi k$, for some $k \in \mathbb{Z}$.

Proof. The evolution equation for B_1 and the constraint are obtained by using $Re(f_1 \overline{f_2}) = B_2 B_4 + B_3 B_5$ and $-\mathrm{Im}(f_1 \overline{f_2}) = B_2 B_5 - B_3 B_4$. The other equations follow from computing

$$\begin{aligned} \frac{df_1}{d\rho} &= \frac{2}{\varepsilon^2 h^2} A_1 (-B_5 + iB_4) - 2\phi (B_2 + iB_3) \\ &= \frac{2i}{\varepsilon^2 h^2} B_1 f_2 - 2\phi f_1, \end{aligned}$$

and similarly for f_2. To obtain the first equation, just notice $\frac{4}{\varepsilon^2} \left((B_4^2 + B_5^2) - (B_2^2 + B_3^2) \right) =$

$\frac{4}{\varepsilon^2}\left(|f_2|^2 - |f_1|^2\right)$. The proof that the constraint $\mathrm{Re}(f_1\overline{f_2}) = 0$ is preserved by the motion and the statement regarding the phases is a direct application of lemma 3.3.24 below. □

Lemma 3.3.24. *Let $A_1(r), A_2(r), B_1(r), B_2(r)$ be real valued functions and $f(r), g(r)$ complex valued functions, such that $\mathrm{Re}(f\overline{g}) = 0$ at $r = r_0 \in \mathbb{R}$. Suppose f and g are subject to the following ODE's*

$$\dot{g} = A_1 g + iB_1 f \ , \ \dot{f} = A_2 f + iB_2 g.$$

If $\mathrm{Re}(f\overline{g}) = 0$ at $r = r_0 \in \mathbb{R}$, then $\mathrm{Re}(f\overline{g}) = 0$ for all $r \in \mathbb{R}$ and both phases χ_1, χ_2 of f, g are constant. Moreover, for $fg \neq 0$ these satisfy $\chi_2 - \chi_1 = \frac{\pi}{2} + \pi k$, for some $k \in \mathbb{Z}$.

Proof. The fact that $Re(f\overline{g}) = 0$ is preserved by the flow follows from computing

$$\begin{aligned} \frac{d}{dr}(f\overline{g}) &= \dot{f}\overline{g} + f\dot{\overline{g}} = (A_2 f + iB_2 g)\overline{g} + f(A_1\overline{g} - iB_1\overline{f}) \\ &= (A_1 + A_2)f\overline{g} + i(B_2|g|^2 - B_1|f|^2). \end{aligned}$$

So $\frac{d}{dr}\mathrm{Re}(f\overline{g}) = (A_1 + A_2)\mathrm{Re}(f\overline{g})$, so that in general $\mathrm{Re}(f\overline{g}) = ke^{\int A_1 + A_2}$ and if at r_0 this vanishes then $\mathrm{Re}(f\overline{g}) = 0$ always. If both $f, g \neq 0$ and $0 = \mathrm{Re}(f\overline{g}) = r_1 r_2 \mathrm{Re}(e^{i(\chi_1 - \chi_2)})$, then one needs $e^{i(\chi_1 - \chi_2)}$ to be purely imaginary, i.e. $\chi_2 - \chi_1 = \frac{\pi}{2} + \pi k$ for some $k \in \mathbb{Z}$. To see that also each phase is constant let $f = r_1 e^{i\chi_1}$ and $g = r_2 e^{i\chi_2}$, then the second equation is

$$\dot{r_1} e^{i\chi_1} + \dot{\chi}_1 e^{i\left(\chi_1 + \frac{\pi}{2}\right)} = A_2 r_1 e^{i\chi_1} + B_2 r_2 e^{i\left(\frac{\pi}{2} + \chi_1 \pm \frac{\pi}{2}\right)} = \left(A_2 r_1 \pm B_2 r_2\right) e^{i\chi_1}.$$

So as a result one has $\dot{\chi}_1 = 0$ and since the phase difference is constant also $\dot{\chi}_2 = 0$. □

The next result will be central in the proof of the main theorem. During that proof one needs to handle the equations in proposition 3.3.23. To do this, it will be useful to split into the cases $f_1 f_2 = 0$ and $f_1 f_2 \neq 0$. In the second case $f_1 f_2 \neq 0$ and so as stated in lemma 3.3.24, the phases χ_1, χ_2 are constant and $\chi_1 - \chi_2 = \frac{\pi}{2} + \pi k$. One can then use an invariant constant gauge transformation, in order to have $\chi_1 = \frac{\pi}{2}, \chi_2 = -\pi k$, which gives $f_1 = iB_3$ and $f_2 = (-1)^k B_4$. One must remark that the initial conditions in equation 3.3.34 in the statement, are those which are required for the connection to extend over the zero section.

Lemma 3.3.25. *Let (ϕ, B_1, B_3, B_4) a be solution to the equations*

$$\frac{d\phi}{d\rho} = -\frac{1}{2h^2(s)}\left(1 - \frac{4}{\varepsilon^2}\left(B_4^2 - B_3^2\right)\right) \tag{3.3.30}$$

$$\frac{dB_1}{d\rho} = 4(-1)^k B_3 B_4 \tag{3.3.31}$$

$$\frac{dB_3}{d\rho} = 2\frac{(-1)^k}{\varepsilon^2 h^2} B_1 B_4 - 2\phi B_3 \tag{3.3.32}$$

$$\frac{dB_4}{d\rho} = 2\frac{(-1)^k}{\varepsilon^2 h^2} B_1 B_3 + 2\phi B_4, \tag{3.3.33}$$

such that for $\rho \ll 1$

$$B_1(\rho) = O(\rho^3) \ , \ B_3(\rho) = O(\rho) \ , \ B_4(\rho) = \frac{\varepsilon}{2} + O(\rho^2). \tag{3.3.34}$$

Then $B_1 = B_3 = 0$, $B_4 = \frac{2}{\varepsilon} a$ and (a, ϕ) must satisfy the equations

$$\frac{d\phi}{d\rho} = -\frac{1}{2h^2(\rho)}\left(1 - a^2\right) \tag{3.3.35}$$

$$\frac{da}{d\rho} = 2\phi a, \tag{3.3.36}$$

subject to the conditions that $a(0) = 1$ and $\phi(0) = 0$.

Proof. One must find all the possible solutions ϕ, B_1, B_3, B_4 to the system in the statement constrained so that 3.3.34 holds. Notice that a possible solution is given by taking $B_1 = B_3 = 0$, $B_4 = \frac{2}{\varepsilon} a$ and (a, ϕ) solving the system 3.3.35, 3.3.36 with the conditions that $a(0) = 1$ and $\phi(0) = 0$. These conditions together with the equations do guarantee 3.3.34. The proof is then reduced to showing that these are all the solutions. To do this use equations 3.3.31, 3.3.32 and 3.3.33 and compute

$$\begin{aligned}
\frac{d^2 B_1}{d\rho^2} &= 4(-1)^k \left(\frac{dB_3}{d\rho} B_4 + B_3 \frac{dB_4}{d\rho} \right) \\
&= 4(-1)^k \left(2\frac{(-1)^k}{\varepsilon^2 h^2} B_1 \left(B_4^2 + B_3^2\right) + 2\phi(B_4 B_3 - B_3 B_4) \right) \\
&= \frac{2u}{h^2} B_1,
\end{aligned}$$

where $u = \frac{4}{\varepsilon^2}\left(B_3^2 + B_4^2\right)$. This can be used to show that $B_1 = 0$ as follows. Recall from the lemma C.1.1 in Appendix C that for $\rho \ll 1$, $h^2(\rho) = \rho^2 \psi(\rho)$, where $\psi(\rho)$ is real analytic with $\psi(0) = 1$. Then the solutions must be real analytic and one can write

$$\frac{2u}{h^2} = \rho^{-2} \sum_{j=0}^{+\infty} \varphi_j \rho^j \ , \ B_1(\rho) = \sum_{k=0}^{+\infty} b_k \rho^k,$$

for some φ_j, with $\varphi_0 \neq 0$ and b_k. Recall the hypothesis that $B_1(\rho) = O(\rho^3)$, this implies $b_0 = b_1 = b_2 = 0$. Inserting the series above into $\frac{d^2 B_1}{d\rho^2} = \frac{2u}{h^2} B_1$, just using that $b_0 = b_1 = 0$ and rearranging gives

$$\sum_{i=0}^{+\infty} (i+2)(i+1) b_{i+2} \rho^i = \sum_{i=0}^{+\infty} \left(\sum_{0 \leq j \leq i} \varphi_j b_{i-j+2} \right) \rho^i,$$

so one can use this to get the recurrence relation

$$b_{i+2} = \frac{1}{(i+1)(i+2) - \varphi_0} \sum_{0 < j \leq i} \varphi_j b_{i+2-j},$$

with $b_0 = b_1 = 0$. This recurrence relation is completely determined by b_2, which vanishes by hypothesis ($B_1(\rho) = O(\rho^3)$). Hence, all the b_i's vanish by the recurrence relation above and so $B_1 = 0$.
We now use the fact that $B_1 = 0$ to finish the proof. First, notice that from $B_1 = 0$ it follows from equation 3.3.31 that $B_3 B_4 = 0$. So one must have $B_3 = 0$ as $B_4 = 0$ would contradict the hypothesis that $B_4(0) = \frac{\varepsilon}{2}$, which then reduces the system to the one in the statement. The initial conditions $\phi(0) = 0$ and $a(0) = 1$ together with the equations do guarantee that 3.3.34 holds because 3.3.36 implies that $\dot{a}(0) = 2a(0)\phi(0) = 0$. □

As an application of the results in this section and in the first part of chapter 2, one can now prove the main theorem 3.3.1 regarding Calabi-Yau monopoles for the Stenzel metric in $T^*\mathbb{S}^3$.

Proof of the main theorem 3.3.1

Start from the equations as stated in lemma 3.3.23, then the phases χ_1, χ_2 are constant and $\mathrm{Re}(f_1\overline{f_2}) = \mathrm{Re}(|f_1||f_2|e^{i(\chi_1 - \chi_2)})$ vanishes if and only if either $|f_1| = 0$, or $|f_2| = 0$, or $\chi_1 - \chi_2 = \frac{\pi}{2} + \pi k$ for some $k \in \mathbb{Z}$. Before proceeding with the case splitting, notice that for the connection to be asymptotic to the canonical invariant connection (which is HYM on the cone) one must have all A_i's converging to 0. This implies that the B_i's must grow at most at a polynomial rate. Moreover, recall from remark 3.3.13 that one can still use an invariant constant gauge transformation, i.e. $g \in U(1) \subset SU(2)$ which rotates $A - A_c^1$ to $g(A - A_c^1)g^{-1}$. This rotates the phases χ_1, χ_2 simultaneously and will be used in different ways in each of the different cases below

1. If $f_1 = 0$, the equations imply χ_2 is constant and so a constant gauge transformation can be used to make $\chi_2 = 0$ so that $f_2 = B_4$ is real. Then, the equations from lemma 3.3.23 give that $B_1 B_4 = 0$, $\frac{dB_1}{d\rho} = 0$ and

$$\frac{d\phi}{d\rho} = \frac{1}{2h^2}\left(\frac{4}{\varepsilon^2}B_4^2 - 1\right), \quad \frac{dB_4}{d\rho} = 2\phi B_4.$$

The conditions that the connection which a possible solution encodes extends over the zero section are studied in the Appendix C. It is shown in lemma C.2.2 that for the connection to extend one needs $B_1(\rho) = O(\rho^3)$, $B_3(\rho) = O(\rho)$ and $B_4(\rho) = \frac{\varepsilon}{2} + O(\rho^2)$, for $\rho \ll 1$. From the equations one knows that B_1 must be constant and so vanish in order to satisfy the initial condition. Setting $a = \frac{2}{\varepsilon}B_4$, the equations reduce to

$$\frac{d\phi}{d\rho} = \frac{1}{2h^2}\left(a^2 - 1\right), \quad \frac{da}{d\rho} = 2\phi a$$

Together with the conditions that $a(0) = 1$ and $\phi(0) = 0$, which do imply (using the second equation) $a(\rho) = 1 + O(\rho^2)$ and so $B_4(\rho) = \frac{\varepsilon}{2} + O(\rho^2)$. Notice that this is the system analyzed in chapter 2 for invariant monopoles in $\mathbb{R}^3$ equipped with the metric $d\rho^2 + h^2(\rho)g_{\mathbb{S}^2}$.

2. The case $|f_2| = 0$ is excluded as the condition that $B_4^2(0) = \frac{\varepsilon}{2}$ can not be satisfied and the connection would not extend smoothly through the zero section.

3. The last case is when $f_1 f_2 \neq 0$ and $\chi_1 - \chi_2 = \frac{\pi}{2} + \pi k$ and the phases are constant. As above, one can then use an invariant constant gauge transformation, to make $\chi_1 = \frac{\pi}{2}, \chi_2 = -\pi k$, which gives $f_1 = iB_3$ and $f_2 = (-1)^k B_4$. The Calabi-Yau monopole equations are

$$\begin{aligned}
\frac{d\phi}{d\rho} &= -\frac{1}{2h^2(s)}\left(1 - \frac{4}{\varepsilon^2}\left(B_4^2 - B_3^2\right)\right) \\
\frac{dB_1}{d\rho} &= 4(-1)^k B_3 B_4 \\
\frac{dB_3}{d\rho} &= 2\frac{(-1)^k}{\varepsilon^2 h^2} B_1 B_4 - 2\phi B_3 \\
\frac{dB_4}{d\rho} &= 2\frac{(-1)^k}{\varepsilon^2 h^2} B_1 B_3 + 2\phi B_4,
\end{aligned}$$

subject to the conditions so that the connection extends smoothly over the zero section as shown in lemma C.2.2 in the Appendix C. This is precisely the system analyzed in lemma 3.3.25 and once again the problem has been reduced to the one analyzed in chapter 2.

The solution to the problem will now be obtained by invoking theorem 2.2.1 in chapter 2. The first item in the statement says that any solution (a, ϕ) has a well-defined finite limit

$$\lim_{\rho \to \infty} \phi(\rho) \in \mathbb{R}^-,$$

Moreover, for each value of $m \in \mathbb{R}^-$ there is one and only one solution. Hence, such value parametrizes the moduli space of invariant Calabi-Yau monopoles and this proves the first item in theorem 3.3.1.

For the proof of the second and third statements, a preliminary digression is needed. Let (a_m, ϕ_m) give the solution to the system given by equations 3.3.35, 3.3.36, with the initial conditions $\phi(0) = 0$, $a(0) = 1$ and ϕ_m converging to $m \in \mathbb{R}^-$. This corresponds to the Calabi-Yau monopole with $B_1 = B_2 = B_3 = B_5 = 0$, $B_4 = \frac{\varepsilon}{2} a_m$ and $\phi = \phi_m$, which can be written

$$A_m = A_c^1 + \frac{\varepsilon}{2}\frac{a_m}{R_+}\left(\theta^4 \otimes T_2 + \theta^5 \otimes T_3\right) \quad , \quad \Phi_m = \phi_m T_1. \tag{3.3.37}$$

The results in the second and third item of theorem 2.2.1 do not directly apply to these, instead they apply for monopoles on the $\mathbb{R}^3$ fibres normal to the zero section equipped with the spherically symmetric metric $h = d\rho^2 + h^2(\rho) g_{\mathbb{S}^2}$. These 3-dimensional monopoles on the fibres can be written

$$\tilde{A}_m = A_c^1 + \frac{a_m}{2}\left(\theta^4 \otimes T_2 + \theta^5 \otimes T_3\right) \quad , \quad \tilde{\Phi}_m = \phi_m T_1. \tag{3.3.38}$$

However, it will be possible to use the results for these in order to prove the corresponding statement for the genuine Calabi-Yau monopole 3.3.37. The two Higgs fields are the same $\tilde{\Phi}_\lambda = \Phi_\lambda$ so focus on the connections. For the proof of the second item one needs to show that for all $R, \delta > 0$ there are m and $\eta(R, \delta, m) > 0$ such that $\| s_\eta^* A_m - A^{BPS} \|_{C^0(B_R)} \leq \delta$. Let $s_\eta = \exp_\eta$ be the exponential in the fibre directions and expand

$$\|s_\eta^* A_m - A^{BPS}\|_{C^0(B_R)} \leq \|s_\eta^* \tilde{A}_m - A^{BPS}\|_{C^0(B_R)} + \|s_\eta^* \tilde{A}_m - s_\eta^* A_m\|_{C^0(B_R)}$$

and use the corresponding statement in second item of theorem 2.2.1. This guarantees the first term can be made as small as one wishes, i.e. there is $\eta' > 0$ such that the first term is less than $\frac{\delta}{2}$. Regarding the second term

$$\begin{aligned}
\|s_\eta^* \tilde{A}_m - s_\eta^* A_m\|_{C^0(B_R)} &= \|\tilde{A}_m - A_m\|_{C^0(B_{\eta R})} \\
&\leq \sup_{\rho \leq \eta R} \left| \left(a_m \left(1 - \frac{\varepsilon}{R_+} \right) \right) |\theta_4|_{g_E} \right| \\
&\leq \sup_{\rho \leq \eta R} \left| (a_m \left(\frac{\rho^2}{2\varepsilon^{-4/3}} \right) \frac{1}{\rho} \right| \leq \frac{\eta R}{4\varepsilon^{4/3}},
\end{aligned}$$

where in the last line one uses the fact that $R_+ = \varepsilon + \frac{1}{2\varepsilon^{1/3}}\rho^2 + \ldots$. Hence the estimate

$$\|s_\eta^* A_\lambda - A^{BPS}\|_{C^0(B_R)} \leq \delta,$$

follows by making η equal to the minimum of η' and $\delta \frac{2\varepsilon^{4/3}}{R}$.
Notice that $A_m - A_c^1$ and $\tilde{A}_m - A_c^1$ differ by a factor of $\frac{\varepsilon}{R_+}$. Since, this is bounded and independent of m, the third item statement of theorem 3.3.1 follows directly from applying the third item in theorem 2.2.1.

Remark 3.3.26. *In the same gauge used so far, the curvature of A_m is*

$$\begin{aligned}
F_{A_m} &= \left(\left(\left(\frac{\varepsilon a_m}{R_+} - 1 \right)^2 - \theta^{23} \right) \theta^{45} \right) \otimes \frac{T_1}{2} + \frac{\varepsilon a_m}{2R_+} \left(\theta^{12} \otimes T_2 + \theta^{13} \otimes T_3 \right) \\
&+ \frac{d}{dr} \left(\frac{\varepsilon a_m}{2R_+} \right) \left(dr \wedge \theta^4 \otimes T_2 + dr \wedge \theta^5 \otimes T_3 \right).
\end{aligned}$$

Since the functions a_m decay exponentially with ρ, the connection A_m is exponentially asymptotic to the canonical invariant connection A_c^1.

Remark 3.3.27. *Following the case splitting in the proof there were some cases whose analysis were excluded as they did not satisfy the necessary conditions for the connection to extend over the zero section (see lemma C.2.2 in the Appendix C). However in some cases Calabi-Yau monopoles with singularities are possible*

1. *In the first case with $f_1 = 0$ one can also take $f_2 = 0$ in order to solve the equations. Then, B_1 is constant, $\frac{d\phi}{d\rho} = -\frac{1}{2h^2}$ and the only solutions are reducible to one of the Dirac Calabi-Yau monopoles in proposition 3.3.21, i.e. $A = A_c^1 + \frac{C}{\mathcal{G}^2}\theta^1 \otimes T_1$ and $\Phi = \left(m - \int \frac{1}{2h^2(\rho)} d\rho \right) \otimes T_1$.*

2. *In the case $f_1 \neq 0$ but $|f_2| = 0$, and using the gauge in which $f_1 = iB_3$, the system in 3.3.23*

reduces to $B_1 B_3 = \frac{dB_1}{d\rho} = 0$ *and*

$$\frac{d\phi}{d\rho} = -\frac{1}{2h^2(s)}\left(1 + B_3^2\right) \ , \ \frac{dB_3}{d\rho} = -2\phi B_3.$$

So B_1 is constant and either $B_3 = 0$ or $B_1 = 0$. If $B_3 = 0$ the unique solutions are the Dirac Calabi-Yau monopole from the previous case. If $B_1 = 0$, then there are no smooth solutions as well since $1 + B_3^2 > 0$ and $h(\rho) = O(\rho)$ for $\rho \ll 1$, also the Higgs field is unbounded at the zero section. So any possible solution will give rise to irreducible Calabi-Yau monopoles with a Dirac type singularity at the zero section.

3.3.6 Explicit Hermitian Yang Mills $SU(2)$ Connection

Theorem 3.3.28. *There is an irreducible Hermitian Yang Mills connection on $P_1 \to T^*\mathbb{S}^3$ for Stenzel's Calabi-Yau structure. In the same gauge used before, it is given by*

$$A = A_c^1 + \frac{\varepsilon}{2R_+}\left(\theta^4 \otimes T_2 + \theta^5 \otimes T_3\right), \tag{3.3.39}$$

and its curvature by

$$\begin{aligned} F_A &= -\frac{1}{2}\left(\theta^{23} + \frac{R_-^2}{R_+^2}\theta^{45}\right) \otimes T_1 + \frac{\varepsilon}{2R_+}\left(T_2 \otimes \theta^{12} + T_3 \otimes \theta^{13}\right) \\ &\quad -\frac{\varepsilon}{4}\frac{r}{R_+^3}\left(T_2 \otimes dr \wedge \theta^4 + T_3 \otimes dr \wedge \theta^5\right). \end{aligned}$$

Proof. This solution is obtained by setting $a = 1$ and $\phi = 0$, i.e. $B_1 = B_3 = 0$ and $B_4 = \frac{\varepsilon}{2}$. These satisfy the conditions from lemma C.2.2 in the Appendix C, so the resulting connection extends over the zero section, is irreducible and HYM. For this solution $A_4 = \frac{\varepsilon}{2R_+}$ and $\dot{A}_4 = -\frac{\varepsilon}{4}\frac{r}{R_+^3}$, so using the formula 3.3.21 one can compute the curvature as in the statement. □

Remark 3.3.29. *$A \to A_c^1$ as $\rho \to \infty$, i.e. this HYM connection is asymptotic to the canonical invariant connection, which recall is the pullback of a reducible HYM connection on a line bundle over $D = \mathbb{P}^1 \times \mathbb{P}^1$.*

Chapter 4

Monopoles on G_2 Manifolds

The goal of this chapter is to construct and study monopoles on G_2 manifolds and it is organized as follows. In section 4.1 one studies the G_2 monopole equation 4.1.1. Namely it is shown that these fit into an elliptic complex which is encompassed by the setup of chapter 5. All the analysis developed in section 1.3 holds for this specific case, in particular for the energies defined in 1.3.1 the identities in proposition 4.1.4 are obtained. Then in section 4.1.3 monopoles on all examples of known AC G_2 manifolds are studied. On $\Lambda^2_-(\mathbb{P}^2)$ and $\Lambda^2_-(\mathbb{S}^4)$ the zero sections are the only coassociative submanifolds and theorem 4.1.9 shows that up to gauge there is also only one invariant monopole for each fixed mass. Moreover, for large mass these monopoles concentrate on the respective coassociative submanifold. This is proved in sections 4.2 and 4.3 respectively for $\Lambda^2_-(\mathbb{S}^4)$ and $\Lambda^2_-(\mathbb{P}^2)$. Moreover, in the case of $\mathcal{S}(\mathbb{S}^3)$ there are no compact coassociative cycles and an application of proposition 1.4.9 gives a vanishing theorem for monopoles, stated in proposition 4.1.10.

4.1 The Equations

Let Y be a G_2 holonomy manifold, then $\Theta = \psi \in \Omega^4(X, \mathbb{R})$ and $*\psi = \phi \in \Omega^3(X, \mathbb{R})$ are both parallel and hence closed. In this case the monopole equation is

$$F_A \wedge \psi = *\nabla_A \Phi. \tag{4.1.1}$$

4.1.1 Linearised Operator

The linearisation of the monopole equation $-\nabla_A \Phi + *(F_A \wedge \psi) = 0$ at a configuration (A, Φ) gives a linear map

$$\begin{aligned} d_2 : \Omega^0(X, \mathfrak{g}_P) \oplus \Omega^1(X, \mathfrak{g}_P) \quad &\rightarrow \quad \Omega^0(X, \mathfrak{g}_P) \\ (\phi, a) \quad &\mapsto \quad *(d_A a \wedge \psi) - (\nabla_A \phi + [a, \Phi]) \, . \end{aligned}$$

Moreover, the infinitesimal action of the gauge group at (A, Φ) gives rise to $d_1 : \Omega^0(X, \mathfrak{g}_P) \rightarrow \Omega^0(X, \mathfrak{g}_P) \oplus \Omega^1(X, \mathfrak{g}_P)$ and maps ξ to $(-\nabla_A \xi, [\xi, \Phi])$. These two maps together give rise to a sequence

$$0 \to \Omega^0(X, \mathfrak{g}_P) \xrightarrow{d_1} \Omega^0(X, \mathfrak{g}_P) \oplus \Omega^0(X, \mathfrak{g}_P) \xrightarrow{d_2} \Omega^1(X, \mathfrak{g}_P) \to 0. \tag{4.1.2}$$

Lemma 4.1.1. *If (A, Φ) is a monopole, then the sequence 4.1.2 is a complex.*

Proof. One just needs to compute $d_2 d_1 \xi$ and show that this vanishes

$$d_2 d_1 \xi = - * (d_A^2 \xi \wedge \psi) - d_A[\xi, \Phi] + [d_A \xi, \Phi] = -[*(F_A \wedge \psi) - \nabla_A \Phi, \xi].$$

And this vanishes indeed if (A, Φ) is a monopole. □

The formal adjoint of d_1 is $d_1^*(a, \phi) = -\nabla_A^* a + [\Phi, \phi]$ and can be put together with d_2 to construct an operator

$$D = d_1^* \oplus d_2 : \Omega^0(X, \mathfrak{g}_P) \oplus \Omega^1(X, \mathfrak{g}_P) \to \Omega^0(X, \mathfrak{g}_P) \oplus \Omega^1(X, \mathfrak{g}_P),$$

which in view of lemma 4.1.1 is elliptic if (A, Φ) is a monopole. This operator is given by

$$D(\phi, a) = (-\nabla_A^* a, *(d_A a \wedge \psi) - \nabla_A \phi) + ([\Phi, a], [\Phi, \phi]) \tag{4.1.3}$$

The first of these is just the twisted Dirac operator $\mathcal{D}_A$ defined in equation 1.2.3 acting on $\mathcal{S}_{\mathfrak{g}_P}$ and the second defines an endomorphism $q \in \Omega^0(X, \mathrm{End}(\mathcal{S}_{\mathfrak{g}_P}))$. The following result gives a Weitzenböck formula for the elliptic operator D

Proposition 4.1.2. *There are Weitzenböck formulas*

$$\begin{aligned} D^* D(\phi, a) &= \Delta_A (\phi, a) + G_+ (\phi, a) - q^2(\phi, a) \\ DD^*(\phi, a) &= \Delta_A (\phi, a) + G_- (\phi, a) - q^2(\phi, a), \end{aligned}$$

where $G_{\pm}(\phi, a) = ([(F_A \wedge \psi \pm *\nabla_A \Phi) \wedge a], -*[(F_A \wedge \psi \pm *\nabla_A \Phi), \phi] \pm *[\nabla_A \Phi \wedge \psi \wedge a])$ and $q^2(\phi, a) = [\Phi[\Phi, (a, \phi)]]$.
Suppose that (A, Φ) is a monopole, if $(\phi, a) \in \ker D^*$ is bounded and $d_A \phi \in L^2$, then actually $d_A \phi = [\Phi, \phi] = 0$. In particular if A is irreducible, then $\phi = 0$.

Proof. In the computation $D^* D(\phi, a) = \mathcal{D}_A^2 (\phi, a) + \mathcal{D}_A (q(\phi, a)) - q (\mathcal{D}_A(\phi, a)) - q^2(\phi, a)$, one can use equation 1.2.4 to replace $\mathcal{D}_A^2 = \Delta_A + W$ where $W(\phi, a) = (*[F_A \wedge \psi \wedge a], -*[F_A \wedge \psi, \phi])$ is zero order and involves only the curvature terms. Then, one needs to compute the term $I(\phi, a) = \mathcal{D}_A (q(\phi, a)) - q (\mathcal{D}_A(\phi, a))$, this is given by

$$(-d_A^*[\Phi, a], *(d_A[\Phi, a] \wedge \psi) - \nabla_A[\Phi, \phi]) - ([\Phi, d_A^* a], [\Phi, *(d_A a \wedge \psi)] - [\Phi, \nabla_A \phi]) \tag{4.1.4}$$

and one can use the Leibniz rule to work out the terms in the first summand. The first of these is $-d_A^*[\Phi, a] = *[\nabla_A \phi \wedge *a] + *[\Phi, d_A * a] = *[\nabla_A \phi \wedge *a] - [\Phi, d_A^* a]$. The second one is

$$* (d_A[\Phi, a] \wedge \psi) - \nabla_A[\Phi, \phi] \quad = \quad * ([\nabla_A \Phi \wedge a] \wedge \psi + [\Phi, d_A a] \wedge \psi) - [\nabla_A \Phi, \phi] - [\Phi, \nabla_A \phi]$$

Then, one just needs to identify which of these terms gets annihilated against the second term in equation 4.1.4 and summing with W gives G_+. For the other Weitzenböck formula for DD^* one proceeds in the same way but now one as $G_- = W - I$ which gives the difference. To prove the second assertion regarding $(\phi, a) \in \ker D^*$ one uses the formula just proved which implies $\Delta_A \phi - [\Phi, [\Phi, \phi]] = 0$. Using the hypothesis, one can integrate $\langle \phi, \Delta_A \phi - [\Phi, [\Phi, \phi]] \rangle$ and using Gaffney's extension of the Stokes' theorem to complete Riemannian manifolds one obtains

$$\|d_A \phi\|_{L^2}^2 + \|[\Phi, \phi]\|_{L^2}^2 = 0,$$

and so $d_A \phi = [\Phi, \phi] = 0$. □

Remark 4.1.3. *The more usual Weitzenböck formula $\Delta_A a = \nabla_A^* \nabla_A a + *[*F \wedge a]$ can be used to write the statement above in a slightly different way.*

4.1.2 Energy Identities

In the case of a G_2 manifold the setup in 1.3 fits perfectly since as described in point 3. of example 3 the equation 1.3.1 for $\Theta = \psi$ is precisely the G_2 monopole equation $*\nabla_A \Phi = F_A \wedge \psi$. In particular, all the energy identities in section 1.3.1 make sense, namely definition 1.3.1 gives respectively

$$E_U = \frac{1}{2} \int_U |\nabla_A \Phi|^2 + |F_A|^2 \ , \ E_U^I = \frac{1}{2} \int_U |\nabla_A \Phi|^2 + |F_A \wedge \psi|^2. \tag{4.1.5}$$

for the YMH energy and Intermediate energy of a configuration (A, Φ) respectively. Moreover, one can see that monopoles do satisfy the Euler Lagrange equations for E_U^I derived in proposition 1.3.2 and stated in example 4. Then, proposition 1.3.4 gives

Proposition 4.1.4. *Let $U \subset X$ be precompact with smooth boundary. If (∇_A, Φ) is a configuration with finite Intermediate Energy on U, then*

$$E_U^I = \int_{\partial U} \langle \Phi, F_A \rangle \wedge \psi + \frac{1}{2} \|F_A \wedge \psi - *\nabla_A \Phi\|_{L^2(U)}^2. \tag{4.1.6}$$

Moreover, if the energy on U is also finite, then

$$E_U = -\frac{1}{2} \int_U \langle F_A \wedge F_A \rangle \wedge \phi + \int_{\partial U} \langle \Phi, F_A \rangle \wedge \psi + \frac{1}{2} \|F_A \wedge \psi - *\nabla_A \Phi\|_{L^2(U)}^2. \tag{4.1.7}$$

In particular, if X is compact and (A, Φ) smooth with $\Phi \neq 0$, then $\nabla_A \Phi = 0$ and A is a reducible G_2 instanton with energy $E = -\frac{1}{2} \int_X \langle F_A \wedge F_A \rangle \wedge \phi$.

Proof. The first identity 4.1.6 is proved in proposition 1.3.4 as for the second one, let $\beta \in \Omega^2$, then one can write $|\beta|^2 = |\pi_7(\beta)|^2 + |\pi_{14}(\beta)|^2$ and rearranging this as the sum of$|\pi_{14}(\beta)|^2 - 2|\pi_7(\beta)|^2$ with $3|\pi_7(\beta)|^2$. Then using equations 1.2.1 and 1.2.2

$$|\beta|^2 dvol_X = -\beta \wedge *(\beta \wedge \phi) + \beta \wedge *(*(\beta \wedge \psi) \wedge \psi) = -\beta \wedge \beta \wedge \phi + \beta \wedge \psi \wedge *(\beta \wedge \psi).$$

Using this for β the 2 form part of the curvature gives

$$\begin{aligned} E_U &= \frac{1}{2}\|F_A\|^2_{L^2(U)} + \frac{1}{2}\|\nabla_A\Phi\|^2_{L^2(U)} \\ &= -\frac{1}{2}\int_U \langle F_A \wedge F_A\rangle \wedge \phi + \frac{1}{2}\|F_A \wedge \psi\|^2_{L^2(U)} + \frac{1}{2}\|\nabla_A\Phi\|^2_{L^2(U)}. \end{aligned} \tag{4.1.8}$$

Then replacing the last two terms by the Intermediate energy and using identity 4.1.6 gives the last one, formula 4.1.7. In the case where X is compact one can take $U = X$ and this energy identity gives that $\nabla_A\Phi = 0$ and $F_A \wedge \psi = 0$. Then A is a G_2 instanton and since $\Phi \neq 0$ and $\nabla_A\Phi = 0$, it is reducible. The computation of the energy is reduced to the first term in equation 4.1.7. □

4.1.3 Monopoles on AC G_2 Manifolds

In what follows (X, ϕ) will always be an AC G_2 manifold, as in section 1.2.2, recall that in this case it is asymptotic to a metric cone whose cross section is a nearly Kähler 6 manifold (Σ, g_Σ). As there are only three known examples, see example 2, one may suppose (at the time of writing) that (X, ϕ) is one of these.

Definition 4.1.5. *Let $H^*_{cs}(X)$ denote the compactly supported cohomology of X. A class $P \in H^3_{cs}(X, \mathbb{Z})$ is said to be a coassociative class if $P \cup [\phi] = 0 \in H^6_{cs}(X, \mathbb{R})$. Moreover, if $P \in \ker(H^3_{cs}(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}))$ then it is said to be monopole-coassociative class.*

Definition 4.1.6. *Define the monopole classes as the set of equivalence classes $H^2(\Sigma, \mathbb{Z})/i^*H^2(X, \mathbb{Z})$.*

Remark 4.1.7. *Take the long exact sequence for compactly supported cohomology*

$$...H^2(X, \mathbb{Z}) \xrightarrow{i^*} H^2(\Sigma, \mathbb{Z}) \xrightarrow{j} H^3_{cs}(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}) \to ... \tag{4.1.9}$$

As in the case of Calabi-Yau manifolds, the monopole classes are exactly the ones that map to monopole-coassociative classes. There is no need to force the monopole classes β to satisfy $\beta \cup [\phi] = 0 \in H^5(X, \mathbb{Z})$, since for a nearly Kähler manifold $b^1(\Sigma) = 0$.

Now one considers the setup for finite mass monopoles which in this case adapts with no change from section 1.4, then keeping in mind proposition 1.4.6, the third point in example 6 and corollary 1.4.11 one can suppose the situation is as follows. Given a monopole class $\alpha \in H^2(\Sigma, \mathbb{R})$ one considers a complex line bundle L over Σ with $c_1(L) = \alpha$ and denote by Q_∞ the underlying principal $U(1)$ bundle. Let L be equipped with an HYM connection A_∞, i.e. such that

$$F_\infty \wedge \omega^2 = F_\infty \wedge \Omega_2 = 0, \tag{4.1.10}$$

for the nearly Kähler structure $(\omega, \Omega_1, \Omega_2)$ on Σ. This induces a reducible connection on a $G = SO(3), SU(2)$ bundle P_∞ over Σ, which we still call A_∞. One can now consider the problem of finding finite mass monopoles (A, Φ) on a G bundle $P \to X$ asymptotic to these.

Corollary 4.1.8. *Let (A, Φ) be a finite mass $m \in \mathbb{R}^+$ monopole on P as above and $|A - A_\infty| = O(\rho^{-5-\varepsilon'})$ for some $\varepsilon' > 0$. Let $[i^*\psi] \in H^4(\Sigma, \mathbb{R})$ be the restriction of $[\psi] \in H^4(X, \mathbb{R})$ to any cross section of the end of X, then*

$$\|\nabla_A \Phi\|^2_{L^2} \quad = \quad 4\pi m \langle \alpha \cup [i^*\psi], [\Sigma] \rangle. \tag{4.1.11}$$

In particular, if $\alpha \cup [i^*\psi] = 0$ or (X, g) has rate $\nu < -4$, then A is a G_2 instanton and $\nabla_A \Phi = 0$, so it is reducible.

Recall that there are 3 known examples of AC G_2 manifolds, see example 2.

Example 11. *In the two first examples, which are $\Lambda^2_- M$ ($M = \mathbb{CP}^2, \mathbb{S}^4$), the zero section is a compact coassociative submanifold and these determine a coassociative class $P \in H^4_{cs}(\Lambda^2_- M, \mathbb{R})$. Moreover, in both these cases $b^2_-(\mathbb{CP}^2) = b^2_-(\mathbb{S}^4) = 0$ and so due to McLean's work [McL98], these coassociatives are rigid. Recall the long exact sequence 4.1.9 with $\Sigma = \mathbb{F}_3, \mathbb{CP}^3$ for $M = \mathbb{CP}^2, \mathbb{S}^4$. In the next section, homogeneous principal bundles P over $\Lambda^2_-(M)$ are constructed, on these ODE methods will be used to study invariant monopoles and their moduli space $\mathcal{M}_{inv}(\Lambda^2_-(M), P)$. Here $\mathcal{M}_{inv}$ denotes the irreducible, invariant monopoles $(A, \Phi \neq 0)$ up to the action of the invariant gauge transformations. The main result of the next two sections is*

Theorem 4.1.9. *For $M = \mathbb{S}^4, \mathbb{P}^2$ there are respectively a $SU(2), SO(3)$ bundle P which is invariant under the action of a compact Lie group acting with cohomogeneity 1 on $\Lambda^2_-(M)$, such that the space $\mathcal{M}_{inv}(\Lambda^2_-(M), P)$ of invariant irreducible monopoles on P are non empty and the following hold:*

1. *For all monopoles in $\mathcal{M}_{inv}$, the Higgs field Φ vanishes at the zero section M, is bounded, the mass is well defined and gives a bijection*

$$m : \mathcal{M}_{inv}(\Lambda^2_-(M), P) \to \mathbb{R}^+.$$

2. *Let $R > 0$, and $\{(A_\lambda, \Phi_\lambda)\} \in \mathcal{M}_{inv}(\Lambda^2_-(M), P)$ a sequence of monopoles with mass λ converging to $+\infty$. Then there is a null sequence $\eta(\lambda, R)$ such that the restriction to each fibre $\Lambda^2_-(M)_x$ for $x \in M$ of the rescaled monopole*

$$\exp^*_\eta(A_\lambda, \eta\Phi_\lambda)$$

converges uniformly to the BPS monopole (A^{BPS}, Φ^{BPS}) in the ball of radius R in $(\mathbb{R}^3, g_E)$.

3. *Let $\{(A_\lambda, \Phi_\lambda)\}_{\lambda \in [\Lambda, +\infty)} \subset \mathcal{M}_{inv}$ be a sequence of monopoles with mass $m(A_\lambda, \Phi_\lambda) = \lambda$ converging to ∞. . Then the translated monopole sequence*

$$\left(A_\lambda, \Phi_\lambda - \lambda \frac{\Phi_\lambda}{|\Phi_\lambda|} \right),$$

converges uniformly with all derivatives to a reducible, singular monopole on $\Lambda^2_-(M)$ with zero mass and which is smooth on $\Lambda^2_-(M) \backslash M$.

Besides this, the next section also contains explicit formulas for irreducible G_2 instantons in $SU(2)$ and $SO(3)$ bundles on $\Lambda^2_-(\mathbb{S}^4)$ and $\Lambda^2_-(\mathbb{P}^2)$ respectively. Moreover, the ODE's for $SU(3)$ monopoles on $\Lambda^2_-(\mathbb{P}^2)$ are also obtained and from these two families of irreducible G_2 instantons with structure group $SU(3)$ are obtained explicitly.

Example 12. *In the case of $\mathcal{S}(\mathbb{S}^3)$, there are no compact coassociative cycles. In fact $H^3_{cs}(\mathcal{S}(\mathbb{S}^3)) = H^2(\mathbb{S}^3 \times \mathbb{S}^3) = 0$, where $\Sigma = \mathbb{S}^3 \times \mathbb{S}^3$, and so there are no coassociative classes or monopole classes at all. Moreover, the corollary 4.1.8 of proposition 1.4.9 can be invoked to state*

Proposition 4.1.10. *There are no finite mass $m \neq 0$, irreducible monopoles (A, Φ) on $\mathcal{S}(\mathbb{S}^3)$ with $|A - A_\infty| = O(\rho^{-5-\varepsilon'})$ for some $\varepsilon' > 0$.*

Proof. Suppose there is a G bundle P over $\mathcal{S}(\mathbb{S}^3)$ equipped with (A, Φ) a finite mass $m \neq 0$, irreducible monopole on P. Let (A_∞, Φ_∞) be the connection and Higgs field on $P_\infty \to \mathbb{S}^3 \times \mathbb{S}^3$ determined by (A, Φ). The connection A_∞ is HYM according to proposition 1.4.6 and the third bullet in example 6. Then, the second item in proposition 1.4.4 implies the Intermediate Energy is finite and as in the proof of proposition 1.4.9 given by the limit

$$E^I_X = \lim_{r\to\infty} E^I_{B_r} = \lim_{r\to\infty} \int_{\partial B_r} \langle \Phi, F_A \rangle \wedge i_r^* \psi = \langle [\langle \Phi_\infty, F_\infty \rangle] \cup [i^* \psi], \mathbb{S}^3 \times \mathbb{S}^3] \rangle.$$

This vanishes because both $[\langle \Phi_\infty, F_\infty \rangle] = 0$ and $[i^*\psi] = 0$ as $\mathbb{S}^3 \times \mathbb{S}^3$ has vanishing second and hence fourth cohomology groups. $\square$

One must also remark that in this case there are G_2 instantons and these have been recently been constructed by Andrew Clarke in [Cla14].

4.2 Monopoles on $\Lambda^2_-(\mathbb{S}^4)$

Let $\mathbb{S}^4 \subset \mathbb{R}^5$ be the round sphere. Its isometry group is $SO(5)$ whose universal cover is $K = Spin(5)$ and so there is a $Spin(5)$ action on $\mathbb{S}^4$. This action lifts to $\Lambda^2_-(\mathbb{S}^4)$ as $A \cdot \Omega_x = (A^{-1})^* \Omega_x$, for $A \in Spin(5)$ and $\Omega_x \in \Lambda^2_-(\mathbb{S}^4)$ an anti self dual 2 form on the tangent space to the point $x \in \mathbb{S}^4$. Let $Spin(4) \subset Spin(5)$ be the isotropy of the action at $x \in \mathbb{S}^4$, which then acts on the fibre over x as follows. Split $Spin(4) = SU_1(2) \times SU_2(2)$ and identify each $SU(2)$ with the unit quaternions. Let $\eta_x \in T^*_x \mathbb{S}^4$, so η_x gives an identification $T_x \mathbb{S}^4 \cong \mathbb{H}$. The action of $(p, q) \in Spin(4)$ by pullback on $\eta_x \in T^*_x \mathbb{S}^4$ is given by $(p, q)\eta_x = p \eta_x \overline{q}$, In the same way $\Lambda^2_-(\mathbb{S}^4)_x$ gets identified with the purely imaginary quaternions and the action is $(p, q)\Omega_x = q \Omega_x \overline{q}$. The conclusion is that away from the zero section, the isotropy of the $Spin(5)$ action is $H = SU_1(2) \times U_2(1)$. The action is isometric, so the principal orbits

$$Spin(5)/SU_1(2) \times U_2(1) \cong \mathbb{CP}^3,$$

are the level sets of the norm function $r = |\cdot|$ in $\Lambda^2_-(\mathbb{S}^4)$ induced by the round metric on $\mathbb{S}^4$. Let $\mathfrak{h}$ denote the Lie algebra of $H = SU_1(2) \times U_2(1)$; Then, given a reductive decomposition

$$\mathfrak{spin}(5) = \mathfrak{h} \oplus \mathfrak{m},$$

the bundle $Spin(5)$ over $\mathbb{CP}^3$ gets equipped with a connection whose horizontal space is $\mathfrak{m}$. To give such a splitting write $\mathfrak{spin}(5) = \mathfrak{m}_1 \oplus \mathfrak{su}_1(2) \oplus \mathfrak{su}_2(2)$ and introduce a basis for the dual $\mathfrak{spin}(5)^*$ such that

$$\mathfrak{m}_1^* = \langle e^1, e^2, e^3, e^4 \rangle \ , \ \mathfrak{su}_1^*(2) = \langle \eta^1, \eta^2, \eta^3 \rangle \ , \ \mathfrak{su}_2^*(2) = \langle \omega^1, \omega^2, \omega^3 \rangle, \tag{4.2.1}$$

and the η^i, ω^i form standard dual basis for $\mathfrak{su}(2)$. Using the notation $e^{12} = e^1 \wedge e^2$, define the 2 forms

$$\begin{aligned} \Omega_1 &= e^{12} - e^{34} \ , \quad \Omega_2 = e^{13} - e^{42} \quad , \ \Omega_3 = e^{14} - e^{23} \\ \overline{\Omega}_1 &= e^{12} + e^{34} \ , \quad \overline{\Omega}_2 = e^{13} + e^{42} \quad , \ \overline{\Omega}_3 = e^{14} + e^{23}, \end{aligned} \tag{4.2.2}$$

The Maurer Cartan relations encode the Lie algebra structure

$$d\omega^1 = -2\omega^{23} + \frac{1}{2}\Omega_1 \ , \quad d\omega^2 = -2\omega^{31} + \tfrac{1}{2}\Omega_2 \quad , \ d\omega^3 = -2\omega^{12} + \frac{1}{2}\Omega_3, \tag{4.2.3}$$

$$d\eta^1 = -2\eta^{23} - \frac{1}{2}\overline{\Omega}_1 \ , \quad d\eta^2 = -2\eta^{31} - \tfrac{1}{2}\overline{\Omega}_2 \quad , \ d\eta^3 = -2\eta^{12} - \frac{1}{2}\overline{\Omega}_3. \tag{4.2.4}$$

The ones involving the de's are less important for what follows, but need to be used to compute

$$d\Omega_i = \varepsilon_{ijk} \left(2\Omega_j \wedge \omega^k - 2\Omega_k \wedge \omega^j \right), \tag{4.2.5}$$

for $i \in \{1, 2, 3\}$. Take the reductive decomposition $\mathfrak{spin}(5) = \mathfrak{h} \oplus \mathfrak{m}$ with

$$\mathfrak{m}^* = \mathfrak{m}_1 \oplus \mathfrak{m}_2 = \mathfrak{m}_1 \oplus \mathbb{R}\langle \omega^2, \omega^3 \rangle \tag{4.2.6}$$

$$\mathfrak{h}^* = \mathfrak{su}_1(2) \oplus \mathbb{R}\langle \omega^1 \rangle. \tag{4.2.7}$$

The sphere bundle of Λ^2_- is the twistor fibration $\pi : \mathbb{CP}^3 \to \mathbb{S}^4$ and at each point $p \in \mathbb{CP}^3$ there are non-canonical identifications $\mathfrak{m} \cong T_p\mathbb{CP}^3$ and $\mathfrak{m}_1 \cong T_{\pi(p)}\mathbb{S}^4$. The 2 forms Ω_i give a basis for the anti-self-dual 2 forms at $\pi(p)$, while the $\overline{\Omega}_i$ for the self-dual ones.

4.2.1 The Bryant-Salamon G_2 Metric

As seen above $\Lambda^2_-(\mathbb{S}^4) \backslash \mathbb{S}^4 \cong \mathbb{CP}^3 \times \mathbb{R}^+$, where each $\mathbb{CP}^3$ is a principal orbit of the $K = Spin(5)$ action. One may write the metric on $\Lambda^2_-(\mathbb{S}^4) \backslash \mathbb{S}^4$ from a family of $Spin(5)$ invariant metrics on $\mathbb{P}^3$ and by letting the coordinate $\rho \in \mathbb{R}^+$ be the length through a geodesic intersecting the principal orbits orthogonally. As remarked at the beginning of section B in the Appendix B, a $Spin(5)$ invariant metric on $\mathbb{CP}^3$ is determined by the splitting of $\mathfrak{m}$ into $\mathfrak{h}$ irreducible pieces. In the current situation one can write

$$\tilde{g} = d\rho \otimes d\rho + a^2(\rho)\left(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3\right) + b^2(\rho)\left(\sum_{i=1}^{4} e^i \otimes e^i\right), \tag{4.2.8}$$

where a, b are suitable real valued functions, which shall be chosen to make this metric have G_2 holonomy. The associative and coassociate calibrations are respectively

$$\phi = d\rho \wedge \left(a^2\omega^{23} + b^2\Omega_1\right) + ab^2\left(\omega^3 \wedge \Omega_2 - \omega^2 \wedge \Omega_3\right) \tag{4.2.9}$$
$$\psi = b^4 e^{1234} - a^2b^2\omega^{23} \wedge \Omega_1 - ab^2 d\rho \wedge \left(\omega^2 \wedge \Omega_2 + \omega^3 \wedge \Omega_3\right). \tag{4.2.10}$$

The condition that the holonomy be in G_2 is equivalent to the closedness of these both. Using $d\Theta_1 = 4d\omega^{23}$ and $d\left(\omega^2 \wedge \Omega_2 + \omega^3 \wedge \Omega_3\right) = -2e^{1234} + 4\omega^{23} \wedge \Omega_1$, this reduces to the following set of ODE's

$$\frac{d}{d\rho}(ab^2) = \frac{a^2}{2} + 2b^2\ ,\ \frac{d}{d\rho}(b^2) = a\ ,\ \frac{d}{d\rho}(a^2b^2) = 4ab^2. \tag{4.2.11}$$

These are solved by setting $a(s) = 2sf(s^2)$ and $b(s) = g(s^2)$, where the functions f, g and the coordinate s are given by

$$\rho(s) = \int_0^s f ds\ ,\ f(s^2) = (1+s^2)^{-\frac{1}{4}}\ ,\ g(s^2) = \sqrt{2}(1+s^2)^{\frac{1}{4}}. \tag{4.2.12}$$

These will be referred as f, g but this should be understood as $f(s^2), g(s^2)$. The notation here is to be matched with the original reference [BS89], see also [GPP90]. For future reference, rewrite the G_2 structure in terms of these as

$$\tilde{g} = f^2 ds \otimes ds + 4s^2f^2\left(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3\right) + g^2(e^a \otimes e^a) \tag{4.2.13}$$
$$\psi = g^4 e^{1234} - 4s^2f^2g^2\omega^{23} \wedge \Omega_1 - 2sf^2g^2 ds \wedge \left(\omega^2 \wedge \Omega_2 + \omega^3 \wedge \Omega_3\right). \tag{4.2.14}$$

This is shown in [BS89] to have full G_2 holonomy. For large s, $\rho(s) \sim 2\sqrt{s}$, $a(\rho) \sim \rho$ and $b(\rho) \sim \frac{\rho}{\sqrt{2}}$, so that the G_2 structure converges to the conical metric over the nearly Kähler $\mathbb{CP}^3$

$$\tilde{g}_C = d\rho \otimes d\rho + \rho^2\left(\omega^2 \otimes \omega^2 + \omega^2 \otimes \omega^2\right) + \rho^2\left(\sum_{i=1}^{4} \frac{e^i}{\sqrt{2}} \otimes \frac{e^i}{\sqrt{2}}\right) \tag{4.2.15}$$
$$\phi_C = \rho^2 d\rho \wedge \left(\omega^{23} + \frac{\Omega_1}{2}\right) + \frac{\rho^3}{2}\left(\omega^3 \wedge \Omega_2 - \omega^2 \wedge \Omega_3\right) \tag{4.2.16}$$
$$\psi_C = \rho^4\left(\frac{e^{1234}}{4} - \omega^{23} \wedge \frac{\Omega_1}{2}\right) - \frac{\rho^3}{2} d\rho \wedge \left(\omega^2 \wedge \Omega_2 + \omega^3 \wedge \Omega_3\right). \tag{4.2.17}$$

4.2.2 G_2 Monopoles on $\Lambda^2_-(\mathbb{S}^4)$

Let $l \in \mathbb{Z}$ be an integer and $\lambda_l : SU_1(2) \times U_2(1) \to SU(2)$ be the group homomorphism $\lambda_l(g, \theta) = diag\left(e^{il\theta}, e^{-il\theta}\right)$ and $P_l = Spin(5) \times_{\lambda_l, SU_1(2) \times U_2(1)} SU(2)$, the family of homogeneous bundles determined by these λ_l.

Lemma 4.2.1. *1. For each $l \in \mathbb{Z}$, the canonical invariant connection is given by $A_c = l\omega^1 \otimes T_1$, where T_1, T_2, T_3 is a standard basis for $\mathfrak{su}(2)$.*

2. *Let $A \in \Omega^1(Spin(5), \mathfrak{su}(2))$ be an invariant connection on P_l, then $A = A_c + (A - A_c)$ and $A - A_c = 0$ for $l \neq 1$. For $l = 1$ this is (up to an invariant gauge transformation)*

$$A - A_c \quad = \quad a\left(T_2 \otimes \omega^2 + T_3 \otimes \omega^3\right), \tag{4.2.18}$$

with $a \in \mathbb{R}$ a constant.

3. *Let Φ be an invariant Higgs field of P_1, i.e. a section of the adjoint bundle $\mathfrak{g}_{P_1}$ invariant with respect to the $Spin(5)$ action, then $\Phi = \phi T_1$, for some constant $\phi \in \mathbb{R}$.*

Proof. 1. The proof of the first assertion amounts to compute the derivative of the isotropy homomorphism λ_l, this is $d\lambda = l\omega^1 \otimes T_1$.

2. The second assertion is an application of Wang's theorem B.0.21. Invariant connections correspond to morphisms of $SU_1(2) \times U_2(1)$ representations

$$\Lambda_l : (\mathfrak{m}, Ad) \to (\mathfrak{su}(2), Ad \circ \lambda_l)\,.$$

Decompose these into irreducible factors $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and $\mathfrak{su}(2) = \mathbb{R}\langle T_1 \rangle \oplus \mathbb{R}\langle T_2, T_3 \rangle$, where on $\mathbb{R}\langle T_1 \rangle$ the representation is trivial and $(\mathbb{R}\langle T_2, T_3 \rangle, Ad \circ \lambda_l) \cong (\mathfrak{m}_2, Ad)$ as representations, if and only if $l = 1$. Then, Schur's lemma gives $\Lambda|_{\mathfrak{m}_1} = 0$, while $\Lambda|_{\mathfrak{m}_2}$ vanishes for $l \neq 1$ and is an isomorphism for $l = 1$. Invariant gauge transformations g are constants in the subgroup of $SU(2)$ centralized by $\lambda_l(SU_1(2) \times U_2(1)) = U(1)$, the maximal torus in $SU(2)$. This is obviously its own centralizer and so g must lie in the maximal torus which acts on $\mathbb{R}\langle T_2, T_3 \rangle$ by rotations. So up to such a rotation Λ_1 can be picked to look like 4.2.18.

3. To prove the third item, recall from the Appendix B that $Ad(P) = P \times_{(SU(2),Ad)} \mathfrak{su}(2)$ which is $Spin(5) \times_{(SU_1(2) \times U_2(1), Ad\circ\lambda)} \mathfrak{su}(2)$ and Φ must be constant with values in a trivial component of $(\mathfrak{su}(2), Ad \circ \lambda)$ as an $SU_1(2) \times U_2(1)$ representation.

$\square$

Remark 4.2.2. *The bundles P_l are reducible to $\mathbb{S}^1$ bundles associated with the degree l homomorphism of $\mathbb{S}^1$. Moreover, the canonical invariant connection is also reducible and induced from the canonical invariant connection on this bundle.*

The same discussion as the one preceding remark 3.3.13 applies and pulling back the bundle P_1 to $\Lambda^2_-\mathbb{S}^4 \backslash \mathbb{S}^4$ one can suppose that an invariant connection is in radial gauge. However, the invariant data is now determined by a, ϕ which are constant along each principal orbit and so functions of ρ. From now on the dot $\cdot$ denotes differentiation with respect to s.

Lemma 4.2.3. *1. The curvature of the connection $A = A_c + (A - A_c)$ is $F_A = F^H + F^V$, where F^H and F^V are respectively given by*

$$F^H = \frac{1}{2}T_1 \otimes \Omega_1 + \frac{a}{2}(T_2 \otimes \Omega_2 + T_3 \otimes \Omega_3) \tag{4.2.19}$$

$$F^V = -2\left(1 - a^2\right) T_1 \otimes \omega^{23} + \dot{a}\left(T_2 \otimes ds \wedge \omega^2 + T_3 \otimes ds \wedge \omega^3\right). \tag{4.2.20}$$

2. The covariant derivative of the invariant Higgs field $\Phi = \phi T_1$ is given by

$$\nabla_A \Phi = \dot{\phi} T_1 \otimes ds + 2\phi a\left(T_2 \otimes \omega^3 - T_3 \otimes \omega^2\right)$$

Proof. The curvature of the invariant connection is computed as $F_A = F_c + d_{A_c}(A - A_c) + \frac{1}{2}[(A - A_c) \wedge (A - A_c)]$. Making use of the Maurer Cartan relations 4.2.3, these terms are

$$F_c = \left(-2\omega^{23} + \frac{1}{2}\Omega_1\right) \otimes T_1. \tag{4.2.21}$$

$$\begin{aligned}
d_{A_c}(A - A_c) &= ds \wedge \frac{d}{ds}(A - A_c) + d(A - A_c) + [A_c \wedge (A - A_c)] \\
&= \dot{a}T_2 \otimes ds \wedge \omega^2 + \dot{a}T_3 \otimes ds \wedge \omega^3 + aT_2 \otimes \left(-2\omega^{31} + \frac{1}{2}\Omega_2\right) \\
&\quad + aT_3 \otimes \left(-2\omega^{12} + \frac{1}{2}\Omega_3\right) + a\underbrace{[T_1, T_2]}_{=2T_3} \otimes \omega^{12} + a\underbrace{[T_1, T_3]}_{-2T_2} \otimes \omega^{13} \\
&= \dot{a} \otimes ds \wedge \omega^2 + \dot{a} \otimes ds \wedge \omega^3 + \frac{a}{2} \otimes \Omega_2 + \frac{a}{2} \otimes \Omega_3 \\
\frac{1}{2}\left[(A - A_c)^2\right] &= a^2 [T_2, T_3] \otimes \omega^{23} = 2a^2 T_1 \otimes \omega^{23}.
\end{aligned}$$

Summing all of these one can write $F_A = F^H + F^V$, where each of these is as in the statement. To compute is the covariant derivative of the Higgs field, write $\nabla_A \Phi = \dot{\phi} T_1 \otimes ds + \phi \nabla_\theta T_1 + \phi\left[(A - A_c), T_1\right]$ and using the Bianchi identity for A_c, $\nabla_{A_c} T_1 = 0$ and so

$$\nabla_A \Phi = \dot{\phi} T_1 \otimes ds + 2\phi a\left(T_2 \otimes \omega^3 - T_3 \otimes \omega^2\right).$$

□

Proposition 4.2.4. *Let $h^2(\rho) = 2s^2(\rho) f^{-2}(s^2(\rho))$ and consider the following set of ODE's for (b, ϕ)*

$$\frac{d\phi}{d\rho} = \frac{1}{2h^2}\left(b^2 - 1\right) \tag{4.2.22}$$

$$\frac{db}{d\rho} = 2\phi b. \tag{4.2.23}$$

Then, the moduli space of invariant Monopoles $\mathcal{M}_{inv}(\Lambda^2_-(\mathbb{S}^4), P)$ can be identified with those pairs $(b = f^{-2}a, \phi)$ which solve 4.2.22 and 4.2.23 with $b(0) = 1, \dot{b}(0) = 0$ and $\lim_{\rho \to +\infty} f^2(s^2)b = 0$.

Proof. To compute the monopole equation $F_A \wedge \psi = *\nabla_A \Phi$ one uses lemma 4.2.3. The left hand side is

$$\begin{aligned} F_A \wedge \psi &= F^V \wedge g^4 e^{1234} + F^H \wedge \left(-4s^2 f^2 g^2 \omega^{23} \wedge \Omega_1 + 2sf^2 g^2 ds \wedge \left(\omega^2 \wedge \Omega_2 + \omega^3 \wedge \Omega_3\right)\right) \\ &= \left(-2g^4\left(1-a^2\right) T_1 \otimes \omega^{23} + g^4 \dot{a}\left(T_2 \otimes ds \wedge \omega^2 + T_3 \otimes ds \wedge \omega^3\right)\right) \wedge e^{1234} \\ &\quad + \left(4s^2 g^2 f^2\, T_1 \otimes \omega^{23} + 2sg^2 f^2 a\left(T_2 \otimes ds \wedge \omega^2 + T_3 \otimes ds \wedge \omega^3\right)\right) \wedge e^{1234} \\ &= g^4 \left(2\left(2s^2 \frac{f^2}{g^2} - \left(1-a^2\right)\right) T_1 \otimes \omega^{23}\right) \wedge e^{1234} \\ &\quad + g^4 \left(2s\frac{f^2}{g^2} a + \dot{a}\right)\left(T_2 \otimes ds \wedge \omega^2 + T_3 \otimes ds \wedge \omega^3\right) \wedge e^{1234}. \end{aligned}$$

While for the right hand side of the equation, i.e. $*\nabla_A \Phi$, it is given by

$$* \nabla_A \Phi = f g^4 \left(4s\dot{\phi} T_1 \otimes \omega^{23} + 2\phi a\left(T_2 \otimes ds \wedge \omega^2 + T_3 ds \wedge \omega^3\right)\right) \wedge e^{1234}. \tag{4.2.24}$$

Hence the monopole equation reduces to the following set of ODE's

$$\frac{d\phi}{ds} = -\frac{1}{2s^2 f}\left(1-a^2\right) + \frac{f}{g^2}\,,\ \frac{da}{ds} = -2s\frac{f^2}{g^2} a + 2f\phi a. \tag{4.2.25}$$

Which in terms of $\rho(s) = \int_0^s dl f(l) = \int_0^s dl \left(1+l^2\right)^{-\frac{1}{4}}$ are

$$\frac{d\phi}{d\rho} = \frac{1}{2s^2 f^2}\left(1-a^2\right) + \frac{1}{g^2}\,,\ \frac{da}{d\rho} = -2s\frac{f}{g^2} a + 2\phi a.$$

Define $b(\rho) = f^{-2}(s^2(\rho))a(\rho)$ as in the statement, then the second ODE in 4.2.26 is equivalent to $\frac{db}{d\rho} = 2\phi b$. What is left to show is that substituting a by b in the first equation in 4.2.26 gives rise to the remaining equation for ϕ. Notice that $\frac{1}{g^2} - \frac{1}{2s^2 f^2} = -\frac{1}{2s^2} f^2$, and factor this term out in the following way

$$\frac{d\phi}{d\rho} = \frac{1}{g^2} - \frac{1}{2s^2 f^2}\left(1-a^2\right) = -\frac{1}{2s^2} f^2 - \frac{1}{2s^2 f^2} a^2 = -\frac{1}{2s^2} f^2 \left(1 - f^{-4} a^2\right).$$

Replacing the term $f^{-4}a^2$ by b^2 gives the equation in the statement. Then $\mathcal{M}_{inv}$, it is identified with the solutions to the ODE's that give rise to a connection and Higgs field extending over the zero section. This is the same as requiring the curvature to be bounded at $\rho = 0$, which from formula 4.2.19 holds if and only if $a(0) = 1$ and $\dot{a}(0) = 0$. The ODE's imply that if these two hold then also $\phi(0) = 0$ and $\dot{\phi}(0)$ is finite and so the Higgs field extends as well. So the conditions $a(0) = 1$ and $\dot{a}(0) = 0$ are necessary and sufficient to guarantee the monopole extends over the zero section. Moreover, as defined in section 1.4.1, see also section 4.1.3, the connection of a monopole is asymptotic to the pullback of an HYM connection on the nearly Kähler $\mathbb{CP}_3$. In this case, this is the canonical invariant connection A^c and so $\lim_{\rho \to +\infty} a = 0$, i.e. $\lim_{\rho \to +\infty} f^2(s^2) b = 0$. □

Remark 4.2.5. *During the proof above a rescaling from the field a to the field b was done. This made the ODE look more familiar. It is precisely the same as the one obtained for invariant monopoles on $\mathbb{R}^3$ with a spherically symmetric metric $g = dr^2 + h^2(r) g_{\mathbb{S}^2}$.*

Reducible Monopoles

There are solutions to the equations in proposition 4.2.4 by setting $b = a = 0$ and letting ϕ solve $\frac{d\phi}{d\rho} = \frac{1}{2h^2}$. This is analogous to the Dirac monopole in the $\mathbb{R}^3\backslash\{0\}$ case, as the connection $A = A_c$ is the canonical invariant connection which is reducible and Φ is unbounded and has singularities at the zero section.

Remark 4.2.6. *The canonical invariant connection A_c is pulled back from an HYM connection on a complex line bundle L over $\mathbb{CP}^3$ with $c_1(L) = [-2\omega^{23} + \frac{1}{2}\Omega_1] \in H^2(\mathbb{CP}^3, \mathbb{R}) \cap H^2(\mathbb{CP}^3, \mathbb{Z})$ a monopole class.*

As it was done in the case of Calabi-Yau manifolds, see definition 4.2.7 one can define

Definition 4.2.7. *Let (X, ϕ) be a noncompact G_2 manifold and $P \subset X$ a coassociative submanifold. A Dirac monopole is an Abelian monopole on a line bundle defined on the complement of P. Moreover, P will also be called the singular set of the Dirac monopole.*

Define the Green's function G, to be the unique function on $\Lambda^2_-(\mathbb{S}^4)\backslash\mathbb{S}^4$, such that $dG = \frac{1}{2h^2}d\rho$ and $\lim_{\rho\to\infty} G(\rho) = 0$. One can check it is harmonic on $\Lambda^2_-(\mathbb{S}^4)\backslash\mathbb{S}^4$, since $*\Delta = d * d$

$$* \Delta G = *d\left(\frac{\partial G}{\partial \rho} * d\rho\right) = *d\left(4s^2\frac{\partial G}{\partial \rho}f^2 g^4 \omega^{23} \wedge e^{1234}\right) = 0,$$

since $\frac{\partial G}{\partial \rho} = \frac{1}{2h^2} = \frac{f^2}{2s^2}$ and $g^2 = 2f^{-2}$ and so the quantity inside the parenthesis is constant on $\Lambda^2_-(\mathbb{S}^4)\backslash\mathbb{S}^4$. The upshot of this section is

Proposition 4.2.8. *The solution to the monopole equations $(A^D, \Phi^D_m) = (A_c, G - m)$, is a mass m Dirac monopole on $\Lambda^2_-(\mathbb{S}^4)$ with the zero section as its singular set.*

Irreducible Monopoles

The general strategy to solve the ODE's to which proposition 4.2.4 reduced the initial problem is via remark 4.2.5 and the work in chapter 2. This gives an existence theorem for monopoles on $\Lambda^2_-(\mathbb{S}^4)$ parametrized by their mass and modeled on transverse BPS monopoles on a small neighborhood of the zero section the $\mathbb{R}^3$ fibers. The precise statement is theorem 4.1.9 which is proved below

Proof. (of theorem 4.1.9) One needs to find the solutions of the ODE's in proposition 4.2.4 giving rise to Monopoles extending over the zero section, i.e. such that $b(0) = 1$ and $\dot{b}(0) = 0$. This together with the ODE's then implies that $\phi(0) = 0$ and $\frac{d\phi}{d\rho}$ is bounded. Theorem 2.2.1 in chapter 2, gives the solutions (b_m, ϕ_m) to the ODE's which are unique by fixing $\lim_{\rho\to\infty} = -\frac{m}{2} \in \mathbb{R}^-$. From these solutions one obtains $a_m = f^2 b_m$ and ϕ_m which give rise to the monopole

$$(A_m, \Phi_m) = \left(A_c + f^2 b_m \left(T_2 \otimes \omega^2 + T_3 \otimes \omega^3\right), \phi_m T_3\right)$$

The fact that the mass function is well defined and a bijection is a direct consequence from theorem 2.2.1 in chapter 2, which basically asserts the previously claimed uniqueness of the solutions to the

ODE's.
The results in the last two items refers to the bubbling behavior, which can be proven by using the corresponding one for monopoles in $\mathbb{R}^3$ and stated in Theorem 2.2.1. Those results are proved in propositions 2.2.5 and 2.2.7 and based on the estimates provided by lemma 2.2.11. One must note that the result one wants to prove does not follow immediately from those ones. The reason is the following: The results from theorem 2.2.1 are for a family of monopoles

$$\left(\tilde{A}_\lambda, \tilde{\Phi}_\lambda\right) = \left(A_c + b_\lambda \left(T_2 \otimes \omega^2 + T_3 \otimes \omega^3\right), \phi_\lambda T_3\right).$$

on $\mathbb{R}^3 \cong (\Lambda^2_-)_x$ equipped with the metric $g|_{(\Lambda^2_-)_x}$. These need to be re-proven for a monopole on $\Lambda^2_-(\mathbb{S}^4)$ restricted to a fibre, which differs from $(\tilde{A}_\lambda, \tilde{\Phi}_\lambda)$ by rescaling the fields as

$$(A_\lambda, \Phi_\lambda)|_{(\Lambda^2_-)_x} = \left(A_c + f^2 b_\lambda \left(T_2 \otimes \omega^2 + T_3 \otimes \omega^3\right), \phi_\lambda T_3\right).$$

Let $exp_\eta = s_\eta$, since the Higgs field is unchanged $\tilde{\Phi}_\lambda = \Phi_\lambda$ one just needs to check that for all $\varepsilon > 0$ there is λ and $\eta(R, \varepsilon, \lambda)$ making $\|s_\eta^* A_\lambda - A^{BPS}\|_{C^0(B_R)} \leq \varepsilon$. Proceed as follows

$$\|s_\eta^* A_\lambda - A^{BPS}\|_{C^0(B_R)} \leq \|s_\eta^* \tilde{A}_\lambda - A^{BPS}\|_{C^0(B_R)} + \|s_\eta^* \tilde{A}_\lambda - s_\eta^* A_\lambda\|_{C^0(B_R)},$$

as already remarked the first of these can be made arbitrarily small due to proposition 2.2.5. So there is η' which makes the first term less than $\frac{\varepsilon}{2}$, as for the second term $\|s_\eta^* \tilde{A}_\lambda - s_\eta^* A_\lambda\|_{C^0(B_R)} = \|\tilde{A}_\lambda - A_\lambda\|_{C^0(B_{\eta R})}$ and so

$$\|s_\eta^* \tilde{A}_\lambda - s_\eta^* A_\lambda\|_{C^0(B_R)} = \sup_{s \leq \eta R} \left| \left(b_\lambda (1 - f^2)\right) |\omega_2|_{g_E} \right| \leq \sup_{s \leq \delta} \left| \frac{1}{s} \left(b_\lambda (1 - f^2)\right) \right| \leq \frac{\eta R}{2},$$

where in the last line one uses the fact that $f = (1 + s^2)^{\frac{1}{4}}$. The conclusion is that estimate $\|s_\eta^* A_\lambda - A^{BPS}\|_{C^0(B_R)} \leq \varepsilon$ follows by making η equal to the minimum of η' and $\frac{\varepsilon}{R}$. The last item in the statement needs no further check and follows directly from proposition 2.2.7 in chapter 2. $\square$

Remark 4.2.9.

- *It is straightforward to check that the connection of these monopoles converges to the canonical invariant connection A_c, which recall from remark 4.2.6 is the pullback of an HYM connection on a line bundle $L \to \mathbb{CP}^3$ with $c_1(L)$ a monopole class.*

- *The energy of these monopoles is not finite (as they are asymptotic to a nonflat connection on $\mathbb{CP}^3$). However, the Intermediate energy is indeed finite and the formula 4.1.6 in proposition 4.1.4 can be used to compute*

$$E^I(A_m, \Phi_m) = \lim_{\rho \to \infty} 2\phi_m(\rho) \int_{\mathbb{P}^3} 2\omega_{23} \wedge 4e_{1234} = 4\pi m \langle [\mathbb{P}^3], c_1(L) \cup [i^* \psi] \rangle. \quad (4.2.26)$$

Moreover, recall that inside the cohomology ring of $\Lambda^2_-(\mathbb{P}^2)$ the class $[\psi]$ is dual to the zero section $\mathbb{S}^4$. As for $m \in \mathbb{R}^+$, it denotes the mass of the monopole (A_m, Φ_m).

4.2.3 G_2 Instantons

There is one further solution to the ODE's in proposition 4.2.4 obtained by setting $\phi = 0$ and $b = 1$, which gives $a = f^2$. This is not contained in $\mathcal{M}_{inv}$, since $\Phi = 0$ in this case, in fact this solution gives rise to an irreducible G_2 instanton, the solution is explicit and shall be stated below.

Theorem 4.2.10. *The connection on $SU(2)$ bundle $P \to \Lambda^2_-(\mathbb{S}^4)$ given by $A = A_c + f^2(s^2)\left(T_2 \otimes \omega^2 + T_3 \otimes \omega^3\right)$ is an irreducible G_2 instanton. Its curvature is given by*

$$\begin{aligned} F_A &= \left(\frac{\Omega_1}{2} - \frac{2s^2}{1+s^2}\omega^{23}\right) \otimes T_1 + \frac{1}{2\sqrt{1+s^2}}\left(\Omega_2 \otimes T_2 + \Omega_3 \otimes T_3\right) \\ &\quad - \frac{s}{1+s^2}\left(ds \wedge \omega^2 \otimes T_2 + ds \wedge \omega^3 \otimes T_3\right). \end{aligned}$$

Remark 4.2.11. *As for the monopoles from the last section and these G_2 instantons also converge to the canonical invariant connection, see remarks 4.2.6 and 4.2.9. However, the convergence is much slower in the case of the instantons.*

Next one considers the $Spin$ bundle over $\mathbb{S}^4$, it may be equipped with a self dual connection. Lifting this to $\Lambda^2_-(\mathbb{S}^4)$ also gives rise to a G_2 instanton.

Proposition 4.2.12. *The Spin connection θ on $\mathbb{S}^4$ is a G_2 instanton with curvature*

$$F_\theta = -\frac{1}{2}\overline{\Omega}_1 \otimes T_1 - 2\eta^{31} + \frac{1}{2}\overline{\Omega}_2 \otimes T_2 - \frac{1}{2}\overline{\Omega}_3 \otimes T_3.$$

Proof. The lift of the positive $Spin$ bundle, denoted by Q is constructed by choosing the isotropy homomorphism $\lambda : SU_1(2) \times U_2(1) \to SU(2)$, given by $\lambda(g, e^{i\theta}) = g$, for $(g, e^{i\theta}) \in SU_1(2) \times U_2(1)$. The canonical invariant connection $\theta \in \Omega^1(Spin(5), \mathfrak{su}(2))$ is given by extending the projection on $\mathfrak{su}_1(2)$ as a left invariant 1 form. Let T_1, T_2, T_3 denote a basis for $\mathfrak{su}(2)$ such that $[T_i, T_j] = 2\varepsilon_{ijk}T_k$. Then $\theta = \eta^1 \otimes T_1 + \eta^2 \otimes T_2 + \eta^3 \otimes T_3$. Using the Maurer Cartan relations 4.2.3 to compute the curvature $F_\theta = d\theta + \frac{1}{2}[\theta \wedge \theta]$, gives

$$\begin{aligned} F_\theta &= 2\eta^{23} \otimes T_1 + 2\eta^{31} \otimes T_2 + 2\eta^{12} \otimes T_3 \\ &\quad - \left(2\eta^{23} + \frac{1}{2}\overline{\Omega}_1\right) \otimes T_1 - \left(2\eta^{31} + \frac{1}{2}\overline{\Omega}_2\right) \otimes T_2 - \left(2\eta^{12} + \frac{1}{2}\overline{\Omega}_3\right) \otimes T_3 \end{aligned}$$

$\square$

In fact one can check that A^c is the unique invariant connection on Q and $\Phi = 0$ the unique invariant Higgs field. The first of these claims follows from an application of Wang's theorem B.0.21, which identifies other invariant connections with morphisms of reps $\Lambda : (\mathfrak{m}, Ad) \to (\mathfrak{su}(2), Ad \circ \lambda)$. The left hand side splits into irreducibles as $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where $\mathfrak{m}_1$ is irreducible and $\mathfrak{m}_2$ is trivial. Since the right hand side is irreducible not isomorphic to $\mathfrak{m}_1$ (they have different dimensions), Schur's lemma gives $\Lambda = 0$ as the only possibility.
Regarding invariant Higgs Fields Φ, these must be constant for each ρ and have values in the trivial component of the representation $(\mathfrak{su}(2), Ad \circ \lambda)$, which is irreducible and nontrivial.

4.3 Monopoles on $\Lambda^2_-(\mathbb{P}^2)$

The unit tangent bundle in $\Lambda^2_-(\mathbb{P}^2)$, i.e. the twistor space of $\mathbb{P}^2$, is the manifold of flags in $\mathbb{C}^3$. One may write

$$\mathbb{F}_3 = \{(x,\xi) \in \mathbb{P}^2 \times (\mathbb{P}^2)^* \mid \xi(x) = 0\},$$

i.e. x is a line in the hyperplane ξ. Then, there are three natural projections to $\mathbb{P}^2$, given by $\pi_1(x,\xi) = x$, $\pi_2(x,\xi) = \xi \cap x^\perp$ and $\pi_3(x,\xi) = \xi^\perp$, where $x^\perp, \xi^\perp$ denote the duals using the standard Hermitian product in $\mathbb{C}^3$. The fibrations π_1 and π_3 are holomorphic while π_2 is the twistor fibration.

The standard action of $SU(3)$ on $\mathbb{C}^3$ descends to a transitive action on $\mathbb{F}_3$ with isotropy the maximal torus $T^2 \subset SU(3)$, i.e.

$$\mathbb{F}_3 = SU(3)/T^2.$$

Moreover, $SU(3)$ also acts on the different $\mathbb{P}^2$'s making the respective projections equivariant. The isotropy of this action on each $\mathbb{P}^2$ is a different subgroup $H \cong S(U(1) \times U(2))$ of $SU(3)$, and are all conjugate by σ an element of order 3 in the Weyl group of $SU(3)$, i.e. $\pi_1 \circ \sigma^2 = \pi_2 \circ \sigma = \pi_3$. (Recall that the Weyl group is the residual action on $SU(3)/T^2$, descending from the action of $SU(3)$ on itself by conjugation.) The standard Hermitian structure gives an isomorphism $\mathbb{P}^2 \times (\mathbb{P}^2)^* \cong \mathbb{P}^2 \times \mathbb{P}^2$. Let $([x_1,x_2,x_3],[\xi_1,\xi_2,\xi_3]) \in \mathbb{P}^2 \times \mathbb{P}^2$ be homogeneous coordinates, then $\mathbb{F}_3 \subset \mathbb{P}^2 \times \mathbb{P}^2$ is given by the points such that $x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0$. At the point $(x,\xi) = ([1,0,0],[0,1,0])$, the isotropy is a fixed T^2 subgroup of $SU(3)$ given by

$$T^2 = \left\{ i(e^{i\alpha_1}, e^{i\alpha_2}) = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{-i(\alpha_1+\alpha_2)} \end{pmatrix},\ (\alpha_1,\alpha_2) \in [0,2\pi]^2 \right\}, \tag{4.3.1}$$

and this identification will be used throughout. Identify $\mathfrak{su}(3)$ with the anti-Hermitian matrices. Denote by C_{ij} the matrix with all entries vanishing but ±1 on the (i,j) and (j,i) positions respectively, and let D_{ij} the matrix with all entries vanishing but the (i,j) and (j,i) equal to i. Moreover, let $X_1 = diag(i,0,-i)$ and $X_2 = diag(0,i,-i)$, these generate the Lie algebra $\mathfrak{t}^2$ of the isotropy subgroup T^2. Then, the decomposition of $\mathfrak{su}(3)$ into $\mathfrak{t}^2$ irreducibles (the root space decomposition) is

$$\mathfrak{su}(3) = \mathfrak{t}^2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3, \tag{4.3.2}$$

where $\mathfrak{t}^2 = \langle X_1, X_2\rangle$, $\mathfrak{m}_1 = \langle C_{13}, D_{13}\rangle$, $\mathfrak{m}_2 = \langle C_{12}, D_{12}\rangle$, $\mathfrak{m}_3 = \langle C_{23}, D_{23}\rangle$. The splitting $\mathfrak{su}(3) = \mathfrak{t}^2 \oplus \mathfrak{m}$, with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, equips the bundle $SU(3) \to \mathbb{F}_3$ with a connection whose horizontal space is $\mathfrak{m}$. In particular $\pi_2(x,\xi) = [0:0:1]$ and $\mathbb{P}^2 = \pi_2(\mathbb{F}_3)$ is identified with $\mathbb{P}^2 \cong SU(3)/S(U(2) \times U(1))$ for an explicit subgroup $S(U(2) \times U(1))$. Under this identification $\mathfrak{m}_1 \oplus \mathfrak{m}_3$ is the horizontal space of a connection on $\pi_2 : \mathbb{F}_3 \to \mathbb{P}^2$. Then the tangent space to the fibres of the twistor projection π_2 gives a distribution which is $\mathfrak{m}_2$. Define left invariant one forms on $SU(3)$, such that

$$(\mathfrak{t}^2)^* = \langle \theta_1, \theta_2 \rangle, \ \mathfrak{m}_1^* = \langle e_3, e_4 \rangle, \ \mathfrak{m}_2^* = \langle \nu_1, \nu_2 \rangle, \ \mathfrak{m}_3^* = \langle e_1, e_2 \rangle,$$

dual to the respective vectors above. One then defines the anti self dual forms Ω_i as given in 4.2.2 and define the 3 forms

$$\gamma = (\Omega_2 \wedge \nu_2 - \Omega_3 \wedge \nu_1) \ , \ \delta = -\Omega_3 \wedge \nu_2 - \Omega_2 \wedge \nu_1$$

The Maurer Cartan relations are

$$\begin{aligned} d\theta^1 &= -2e_{34} - 2\nu_{12} \quad , \quad d\theta^2 = -2e_{12} + 2\nu_{12} \\ d\nu_1 &= \left(-\theta^2 + \theta^1\right) \wedge \nu_2 + \Omega_2 \quad , \quad d\nu_2 = -\left(-\theta^2 + \theta^1\right) \wedge \nu_1 + \Omega_3 \\ de_1 &= \left(2\theta^2 + \theta^1\right) \wedge e_2 - \nu_1 e_3 - \nu_2 e_4 \quad , \quad de_2 = -\left(2\theta^2 + \theta^1\right) \wedge e_1 - \nu_1 e_4 - \nu_2 e_3 \\ de_3 &= \left(\theta^2 + 2\theta^1\right) \wedge e_4 + \nu_1 e_1 - \nu_2 e_2 \quad , \quad de_4 = -\left(\theta^2 + 2\theta^1\right) \wedge e_3 + \nu_1 e_2 + \nu_2 e_1. \end{aligned} \tag{4.3.3}$$

These can in turn be used to compute

$$d\delta = 4\left(e_{1234} - \nu_{12} \wedge \Omega_1\right) \ , \ d\gamma = 0,$$

and in fact $\gamma = de_{12} = d\nu_{12} = -de_{34}$ is exact.

4.3.1 The Bryant-Salamon G_2 Metric

Using the fact that $\Lambda^2_-(\mathbb{P}^2)\backslash\mathbb{P}^2 \cong \mathbb{R}^+ \times \mathbb{F}_3$ and each $\mathbb{F}_3$ slice is a principal orbit for the $SU(3)$ action, this section reduces the equations of G_2 holonomy with $SU(3)$ symmetry to ODE's on $\mathbb{R}^+$. Integrating these, one constructs the Bryant Salamon metric on $\Lambda^2_-(\mathbb{P}^2)$. The notation tries to match up with the original reference [BS89] and also with [CGLP02]. Let $\rho \in \mathbb{R}^+$ be the distance along a geodesic emanating from the zero section and intersecting the principal orbits of the $SU(3)$ action orthogonally. The adjoint action of T^2 on $\mathfrak{m}$ decomposes into irreducible components as $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ (the root space decomposition after complexification) and any invariant metric can be written as

$$\tilde{g} = d\rho^2 + a^2(\rho)\left(e_1^2 + e_2^2\right) + b^2(\rho)\left(e_3^2 + e_4^2\right) + c^2(\rho)\left(\nu_1^2 + \nu_2^2\right),$$

for some positive functions a, b, c. The 3 form ϕ and $\psi = *\phi$ defining the G_2 structure are given by

$$\begin{aligned} \phi &= d\rho \wedge \left(-a^2 e_{34} + b^2 e_{12} + c^2 \nu_{12}\right) + abc\,\gamma \\ \psi &= -b^2c^2\, e_{12} \wedge \nu_{12} + a^2c^2\, e_{34} \wedge \nu_{12} + a^2b^2\, e_{1234} + abc\, d\rho \wedge \delta. \end{aligned}$$

By theorem 1.2.3, the metric g has holonomy reduced to a subgroup of G_2 if and only if $d\phi = d\psi = 0$. Since γ is closed and $d\delta = 4(e_{1234} - \nu_{12} \wedge \Omega_1)$, the equations reduce to the following

ODE's

$$4abc = \frac{d}{d\rho}(a^2b^2) = \frac{d}{d\rho}(a^2c^2) = \frac{d}{d\rho}(b^2c^2) \ , \ \frac{d}{d\rho}(abc) = a^2+b^2+c^2. \tag{4.3.4}$$

Recall from section 4.2.1, equation 4.2.12, the definition of the following implicit functions of ρ

$$\rho(s) = \int_0^s f ds \ , \ f(s) = (1+s^2)^{-\frac{1}{4}} \ , \ g(s) = \sqrt{2}(1+s^2)^{\frac{1}{4}}. \tag{4.3.5}$$

Then as already done for $\Lambda^2_-(\mathbb{S}^4)$, one can regard s as a radial coordinate. Moreover, the solution to the ODE's 4.3.4, which gives the Bryant Salamon G_2 structure is given by setting $a(\rho) = b(\rho) = 2f^{-1}(s(\rho))$ and $c(\rho) = 2s(\rho)f(s(\rho))$. The G_2 structure obtained is

$$\begin{aligned}
\tilde{g} &= f^2ds^2 + 4s^2f^2\left(\nu_1^2+\nu_2^2\right) + 2g^2\left(e_1^2+e_2^2+e_3^2+e_4^2\right) && (4.3.6)\\
\phi &= 4s^2f^3ds\wedge\nu_{12} + 4fg^2ds\wedge\Omega_1 - 2sf^2g^2\left(\nu_1\wedge\Omega_2+\nu_2\wedge\Omega_3\right) && (4.3.7)\\
\psi &= 4g^4e_{1234} - 8s^2f^2g^2\Omega_1\wedge\nu_{12} + 2sf^2g^2ds\wedge\left(\Omega_2\wedge\nu_2-\Omega_3\wedge\nu_3\right). && (4.3.8)
\end{aligned}$$

It converges for large ρ to the Riemannian cone over the nearly Kähler $\mathbb{F}^3$. To check this use $\frac{d\rho}{dr} = \frac{1}{2\sqrt{2r}}(r+c)^{-\frac{1}{4}} \sim \frac{1}{2\sqrt{2}}r^{-\frac{3}{4}}$, i.e. $\rho(r) \sim \sqrt{2}r^{\frac{1}{4}} = \sqrt{s}$ and

$$\begin{aligned}
\tilde{g}_C &= d\rho^2 + \rho^2\left(4e_1^2+4e_2^2+4e_3^2+4e_4^2+4\nu_1^2+4\nu_2^2\right)\\
\phi_C &= \rho^2d\rho\wedge\left(\Omega_1+\nu_{12}\right) - \rho^3\left(\nu_1\wedge\Omega_2+\nu_2\wedge\Omega_3\right)\\
\psi_C &= \rho^4\left(\left(\sigma_{12}-\Sigma_{12}\right)\wedge\nu_{12}+\sigma_{12}\wedge\Sigma_{12}\right) + \rho^3d\rho\wedge\left(\Omega_2\wedge\nu_2-\Omega_3\wedge\nu_3\right).
\end{aligned}$$

4.3.2 G_2 Monopoles

This section will use the $SU(3)$ symmetry to construct G_2 monopoles and G_2 instantons on $\Lambda^2_-(\mathbb{P}^2)$. The strategy for the construction of the invariant data (homogeneous bundle with invariant connections and Higgs Fields) is as follows (see Appendix B for further details). Given an isotropy homomorphism $\lambda : T^2 \to G$, one constructs homogeneous principal G-bundles via $P_\lambda = SU(3)\times_{(T^2,\lambda)} G$ on $\mathbb{F}_3 \cong SU(3)/T^2$. The invariant connections are determined by their left-invariant connection 1-form $A \in \Omega^1(SU(3),\mathfrak{g})$. Once a complement $\mathfrak{m}$ to $\mathfrak{t}^2$ has been chosen, Wang's theorem B.0.21 parametrizes invariant connections in terms of morphisms of T^2 representations $\Lambda : \mathfrak{m} \to \mathfrak{g}$. The decomposition of $\mathfrak{m}$ into irreducible components is

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

where each component is labeled by a positive root. Then by Schur's lemma $\Lambda|_{m_i}$ will either vanish or map $\mathfrak{m}_i$ into an isomorphic representation inside $\mathfrak{g}$. In the same way, invariant Higgs fields, i.e. invariant sections of the adjoint bundle $\mathfrak{g}_{P_\lambda} = P_\lambda \times_{Ad} \mathfrak{g}$, i.e. $SU(3)\times_{Ad\circ\lambda}\mathfrak{g}$, correspond to vectors in the trivial components of the T^2-representation $Ad\circ\lambda$ on $\mathfrak{g}$.

$G = \mathbb{S}^1$ Bundles

For gauge group $G = \mathbb{S}^1$, the possible isotropy homomorphisms are given by the weights

$$\lambda_{n,l}(e^{i\alpha_1}, e^{i\alpha_2}) = e^{i(n\alpha_1 + l\alpha_2)}, \tag{4.3.9}$$

and so parametrized by two integers $(n, l) \in \mathbb{Z}^2$. Moreover, since none of the root spaces is a trivial representation of $\mathbb{S}^1$ and the $Ad \circ \lambda_{n,l}$ action on $\mathfrak{u}(1)$ is trivial the canonical invariant connection

$$A^c_{n,l} = n\theta^1 + l\theta^2,$$

is the unique invariant connection. The Maurer Cartan relations for $SU(3)$, in 4.3.3, give $F_{n,l} = -2n(e_{34} + \nu_{12}) + 2l(\nu_{12} - e_{12})$, which one rearranges to

$$F^c_{n,l} = -2ne_{34} - 2le_{12} + 2(l - n)\nu_{12}, \tag{4.3.10}$$

is a closed, T^2-invariant, horizontal 2-form in $SU(3)$ and descends to a closed 2-form on $\mathbb{F}_3 = SU(3)/T^2$. Particular cases are $d\theta^1 = F_{1,0}$ and $d\theta^2 = F_{0,1}$, hence their classes generate $H^2(\mathbb{F}_3, \mathbb{R})$. It is a consequence of the next lemma that $[d\theta_1], [d\theta_2]$ also generate $H^2(\mathbb{F}_3, \mathbb{Z})$ seen as a lattice inside $H^2(\mathbb{F}_3, \mathbb{R})$.

Lemma 4.3.1. *$H^2(\mathbb{F}_3, \mathbb{Z}) \cong H^1(\mathbb{T}^2, \mathbb{Z})$ is the lattice generated by the roots. Let $\mathcal{O}(1)$ denote the canonical line bundle of $\mathbb{P}^2$, then $c_1(\pi_1^*\mathcal{O}(1)) = [F_{1,0}]$, $c_1(\pi_2^*\mathcal{O}(1)) = [F_{-1,-1}]$ and $c_1(\pi_3^*\mathcal{O}(1)) = [F_{0,1}]$.*

Proof. The first assertion is a consequence of Serre's spectral sequence and the fact that $SU(3)$ is 2-connected, so $H^2(\mathbb{F}_3, \mathbb{Z}) \cong H^1(\mathbb{T}^2, \mathbb{Z})$. This identification can be made explicit by noticing that the integral weights can be taken as generators and also as giving rise to the isotropy homomorphisms generating the group of complex line bundles. Then, given $\alpha \in H^1(T^2, \mathbb{Z})$, its exponential gives the isotropy homomorphism of the line bundle $L_\alpha = SU(3) \times_{T^2, e^\alpha} \mathbb{C}$ whose first Chern class is $[d\alpha] \in H^2(\mathbb{F}_3, \mathbb{R}) \cap H^2(\mathbb{F}_3, \mathbb{Z})$. Notice that in this case α is actually the canonical connection of the underlying $\mathbb{S}^1$ bundle and $d\alpha$ its curvature. Since π_1 is holomorphic, π_2 is real and π_3 antiholomorhic

$$\pi_1^*\mathcal{K}_{\mathbb{P}^2} \cong (\overline{\mathfrak{m}_2^{\mathbb{C}}})^* \otimes (\overline{\mathfrak{m}_1^{\mathbb{C}}})^* \ , \ \pi_2^*\mathcal{K}_{\mathbb{P}^2} \cong (\mathfrak{m}_1^{\mathbb{C}})^* \otimes (\mathfrak{m}_3^{\mathbb{C}})^* \ , \ \pi_3^*\mathcal{K}_{\mathbb{P}^2} \cong (\mathfrak{m}_2^{\mathbb{C}})^* \otimes (\overline{\mathfrak{m}_3^{\mathbb{C}}})^*,$$

these are the complex line bundles determined from the isotropy homomorphisms $e^{\alpha_i} : T^2 \to \mathbb{S}^1$ with

$$\begin{aligned}
\alpha_1 &= -(2\theta^1 + \theta^2) - (\theta^1 - \theta^2) = -3\theta^1 \\
\alpha_2 &= (2\theta^1 + \theta^2)(\theta^1 + 2\theta^2) = 3(\theta^1 + \theta^2) \\
\alpha_3 &= +(\theta^1 - \theta^2) - (\theta^1 + 2\theta^2) = -3\theta^2.
\end{aligned}$$

Since $\mathcal{K}_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, the statement follows and $c_1(\pi_i^*\mathcal{O}_{\mathbb{P}^2}(-1))$ generate the integral second homology the statement follows. $\square$

Lemma 4.3.2. *$F_{1,1}$ generates a subgroup of $H^2(\mathbb{P}^2,\mathbb{Z})$ corresponding to the first Chern classes of the line bundles pulled back from $\mathbb{P}^2$ via π_2.*

Proof. This is a consequence of the previous lemma. Alternatively the base of the twistor fibration π_2 is $\mathbb{P}^2 = SU(3)/S(U(2)\times U(1))$ so the bundles that are pull back from the base must have an isotropy homomorphisms $\lambda_{n,l} : T^2 \to \mathbb{S}^1$, which factors via $T^2 \hookrightarrow S(U(2)\times U(1)) \to \mathbb{S}^1$. For the choices made before this is a fixed subgroup $S(U(2)\times U(1))$ of $SU(3)$ for which the aforementioned homomorphisms are precisely the ones with $n = l$. In fact, these are the only cases for which the curvature $F_{n,n}$ of the canonical invariant connection stays bounded close to the zero section. $\square$

Since $\mathbb{S}^1$ is Abelian an invariant Higgs Φ is just a real valued function of the radial coordinate ρ and to compute the monopole equations one needs

$$\begin{aligned} F_{n,l}\wedge\psi &= \left(8g^4(l-n) - 16s^2f^2g^2(l-n)\right) e_{1235}\wedge\nu_{12} \\ &= 8(l-n)g^2\left(g^2-2s^2f^2\right) e_{1234}\wedge\nu_{12} \\ &= 32(l-n)e_{1234}\wedge\nu_{12}, \end{aligned}$$

where it is useful to use $g^2 = 2f^{-2}$. Moreover, $d\Phi = \frac{d\Phi}{d\rho}d\rho$ and so $*d\Phi = 64s^2f^{-2}\frac{d\Phi}{d\rho}e_{1234}\wedge\nu_{12}$. The monopole equation can then be written as an ODE for Φ. For each (n,l) and a given mass it has a unique solution obtained by solving

$$d\Phi^m_{n,l} = \frac{l-n}{2h^2(\rho)}d\rho \ , \ \lim_{\rho\to\infty}\Phi^m_{n,l} = m. \tag{4.3.11}$$

Moreover, the connection associated with this is the canonical invariant one $A^c_{n,l}$. This monopole does not extend over the zero section unless $l = n$ in which case Φ is constant and so for $n \neq l$ gives a Dirac type monopole, see definition 4.2.7

Proposition 4.3.3. *For $n \neq l$ the monopole $(A^c_{n,l}, \Phi^m_{n,l})$ is a Dirac monopole on $\Lambda^2_-(\mathbb{P}^2)$ with singular set the zero section. For $n = l$ the connection $A^c_{n,n}$ is a G_2-instanton obtained by lifting a self dual connection on $\mathcal{O}_{\mathbb{P}^2}(-n)$ via π_2, their curvature is $F_{n,n} = -2n\,(e_{12}+e_{34})$.*

The Higgs field is then a harmonic function, which in the case $n \neq l$ is non constant and unbounded at the zero section. For large ρ one uses 4.2.12, $s \sim \frac{\rho^2}{4}$ and $h^2(\rho) = s^2\sqrt{1+s^2}$ to conclude that $h^2(\rho) \sim \frac{\rho^6}{64}$. Plugging this back in equation 4.3.11 gives $\Phi_{n,l} \sim -\frac{32}{5}(l-n)\rho^{-5}$, i.e. $\Phi_{n,l}$ decays like the Green's function for the cone metric.

Remark 4.3.4. *These invariant connections are Hermitian Yang Mills type connections on line bundles over the nearly Kähler $\mathbb{F}_3$ pulled back to the cone.*

$G = SO(3)$ Bundles

The possible isotropy homomorphisms $\lambda_{n,l} : T^2 \to SO(3)$ are also parametrized by two integers $(n, l) \in \mathbb{Z}^2$. These are constructed by using 4.3.9 from the previous example and letting the image $\mathbb{S}^1$ be the maximal torus in $SO(3)$. Associated with each $\lambda_{n,l}$ is the principal $SO(3)$ bundle $P_{n,l} = SU(3) \times_{T^2, \lambda_{n,l}} SO(3)$. These are reducible and one can also construct there reducible connections induced by the canonical invariant ones on the respective $\mathbb{S}^1$ bundles. Let T_1, T_2, T_3 be an orthonormal basis of $\mathfrak{so}(3)$, such that $[T_i, T_j] = 2\varepsilon_{ijk} T_k$ and fix $\frac{T_1}{2}$ as the generator of the maximal torus. The canonical invariant connection on $P_{n,l}$ is then $A^c_{n,l} = \left(n\theta^1 + l\theta^2\right) \otimes \frac{T_1}{2}$, with curvature

$$F^c_{n,l} = (-ne_{34} - le_{12} + (l-n)\nu_{12}) \otimes T_1. \tag{4.3.12}$$

Other invariant connections are given by morphisms of T^2 representations

$$\Lambda : (\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3, Ad) \to (\mathfrak{so}(3), Ad \circ \lambda_{n,l}).$$

Let $L_{n,l}$ denote the real two dimensional representation of T^2, where the first $\mathbb{S}^1$ acts by rotations with degree n and the second $\mathbb{S}^1$ acts by rotations with degree l (this is the same as the complex representation of T^2 induced with weight $(n,l) \in \mathbb{Z}^2$, i.e. by exponentiating $n\theta^1 + l\theta^2 \in (\mathfrak{t}^2)^*$). Identifying the corresponding representations

$$\Lambda : (2,1) \oplus (1,-1) \oplus (1,2) \to (0,0) \oplus (n,l).$$

These are irreducible and it follows from Schur's lemma, that Λ must vanish unless (n,l) is one of $(2,1)$, $(1,2)$, $(1,-1)$. In each of these cases $\Lambda|_{m_i}$ is either 0 or an isomorphism for the corresponding (n,l). Up to invariant gauge transformations such an isomorphism is determined by a constant. Then, it is possible to make Λ be one of the following

$$A_{2,1} = \left(2\theta^1 + \theta^2\right) \otimes \frac{T_1}{2} + a\left(\sigma_1 \otimes T_2 + \sigma_2 \otimes T_3\right) \tag{4.3.13}$$
$$A_{1,-1} = \left(\theta^1 - \theta^2\right) \otimes \frac{T_1}{2} + a\left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right) \tag{4.3.14}$$
$$A_{1,2} = \left(\theta^1 + 2\theta^2\right) \otimes \frac{T_1}{2} + a\left(\Sigma_1 \otimes T_2 + \Sigma_2 \otimes T_3\right), \tag{4.3.15}$$

with $a \in \mathbb{R}$ a function of the radial coordinate ρ. Invariant Higgs fields $\Phi = \Phi(\rho)$ must have values in the components corresponding to the trivial T^2 representation, i.e. $\Phi \in (0,0)$ and one writes

$$\Phi = \phi T_1, \tag{4.3.16}$$

with $\phi \in \mathbb{R}$ a function of the radial coordinate ρ.

Lemma 4.3.5. *The above $SO(3)$ bundles $P_{n,l}$ for $(n,l) = (2,1), (1,-1), (1,2)$ extend over the zero section giving rise to a bundle over $\Lambda^2_-(\mathbb{CP}^2)$ if and only if $(n,l) = (1,-1)$.*

Proof. One needs to show that only when $(n,l) = (1,-1)$ the bundle $E_{n,l} = P_{n,l} \times_{SO(3)} \mathbb{R}^3$ associated via the standard representation is trivial along the fibres of the projection $\pi_2 : \mathbb{F}_3 \to \mathbb{CP}^2$.

Equivalently one can show that only for $(n,l) = (1,-1)$, is the bundle $E_{n,l} \to \mathbb{F}_3$ isomorphic to a bundle pulled back from $\mathbb{CP}^2$ via π_2. To do this notice that $w_1(E_{n,l}) = 0$ for all (n,l), so it is enough to show that $w_2(E_{n,l}), p_1(E_{n,l})$, are pulled back from $H^*(\mathbb{P}^2, \mathbb{Z}_2)$ via π_2^* only for $(n,l) = (1,-1)$. At this point it is convenient to work with $U(2)$ bundles to compute the characteristic classes. Consider the group homomorphism $\tilde{\lambda}_{n,l} : T^2 \to U(2)$ given by

$$\tilde{\lambda}_{n,l}(\alpha_1, \alpha_2) = \left(e^{i\frac{n\alpha_1 + l\alpha_2}{2}}, \begin{pmatrix} e^{i\frac{n\alpha_1 + l\alpha_2}{2}} & 0 \\ 0 & e^{-i\frac{n\alpha_1 + l\alpha_2}{2}} \end{pmatrix} \right) \in (U(1) \times SU(2))/\mathbb{Z}_2 \cong U(2).$$

It has the property that after composed with the map $U(2) \to SO(3)$ given by

$$A \mapsto \mathrm{diag}(\det(A)^{-1/2}, \det(A)^{-1/2})A,$$

it agrees with $\lambda_{n,l}$. Define $W_{n,l}$ as the rank-2 complex vector bundle associated via the canonical $U(2)$ representation with $SU(3) \times_{(T^2, \tilde{\lambda}_{n,l})} U(2)$. Then, $\underline{\mathbb{R}} \oplus E_{n,l} \cong \mathfrak{g}_{W_{n,l}}$ and regarding characteristic classes

$$w_2(E_{n,l}) = c_1(W_{n,l}) \mod 2\ ,\ p_1(E_{n,l}) = c_1(W_{n,l})^2 - 4c_2(W_{n,l}).$$

The canonical invariant connection of such a bundle is $\tilde{A}^c_{n,l} = (n\theta^1 + l\theta^2) \otimes \mathrm{diag}(i, 0)$, and its curvature is given by $\tilde{F}^c_{n,l} = (nd\theta^1 + ld\theta^2) \otimes \mathrm{diag}(i, 0) \in \Omega^2(\mathbb{F}_3, \mathfrak{u}(2))$. Using $c_1(W_{n,l}) = i[\mathrm{tr}(\tilde{F}^c_{n,l})]$ and $c_2(W_{n,l}) = \frac{1}{2}(\mathrm{tr}(\tilde{F}^c_{n,l} \wedge \tilde{F}^c_{n,l}) - \mathrm{tr}(\tilde{F}^c_{n,l})^2)$ and inserting the formula above for the curvature gives

$$c_1(W_{n,l}) = -[nd\theta^1 + ld\theta^2]\ ,\ c_2(W_{n,l}) = 0.$$

First focus on $w_2(E_{n,l})$, from lemma 4.3.2 the only classes in $H^2(\mathbb{F}_3, \mathbb{Z})$ which are pulled back from $\mathbb{P}^2$ via π_2 are those for which $n = l$. So one can write

$$w_2(E_{n,l}) = [nd\theta^1 + ld\theta^2] = l[d\theta^1 + ld\theta^2] + (n-l)[d\theta^1],$$

and this equals $l[d\theta^1 + d\theta^2] \in H^2(\mathbb{F}_3, \mathbb{Z}_2)$ if and only if $n - l$ is even. Then $(n,l) = (1,-1)$ is the only case in $(n,l) = \{(2,1), (1,-1), (1,2)\}$ for which this holds. Next one needs to check that $p_1(E_{1,-1}) = c_1(E_{1,-1})^2$ is also the pull back of a class via π_2. To do this one computes $p_1(E_{1,-1}) = [-2e_{1234} - 4\nu_{12} \wedge \Omega_1]$ and using the fact that $d\delta = 4(e_{1234} - \nu_{12} \wedge \Omega_1)$ one concludes that $[4\nu_{12} \wedge \Omega_1] = [4e_{1234}]$ and so

$$p_1(E_{1,-1}) = [-8e_{1234}],$$

which is indeed the pullback via π_2 of a multiple of the fundamental class of $\mathbb{P}^2$. And so $P_{1,-1}$ does extend over the zero section while the other two cases do not. □

Having in mind this proposition focus for now on the case $(n,l) = (1,-1)$. The curvature of the invariant connection $A_{1,-1}$ is computed via

$$F_{1,-1} = F^c_{1,-1} + d_{A^c_{1,-1}} a\left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right) + \frac{a^2}{2}\left[\left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right) \wedge \left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right)\right].$$

Denote these by I_1, I_2, I_3 respectively, then first term $I_1 = F^c_{1,-1} = (\Omega_1 - 2\nu_{12}) \otimes T_1$, is the curvature of the invariant connection. Use the Maurer Cartan relations 4.3.3 to compute the other terms and the dot $\cdot$ to denote differentiation with respect to s, then

$$\begin{aligned}
I_2 &= \dot{a}\left(ds \wedge \nu_1 \otimes T_2 + ds \wedge \nu_2 \otimes T_3\right) + a\left[\left(\theta^1 - \theta^2\right) \otimes \frac{T_1}{2} \wedge \left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right)\right] \\
&\quad + a\left(d\nu_1 \otimes T_2 + d\nu_2 \otimes T_3\right) \\
&= \dot{a}\left(ds \wedge \nu_1 \otimes T_2 + ds \wedge \nu_2 \otimes T_3\right) + a(\theta^1 - \theta^2) \wedge \left(\nu_1 \otimes T_3 - \nu_2 \otimes T_2\right) \\
&\quad a\left(\left(\theta^1 - \theta^2\right) \wedge \nu_2 + \Omega_2\right) \otimes T_2 - a\left(-\left(\theta^1 - \theta^2\right) \wedge \nu_1 - \Omega_3\right) \otimes T_3 \\
&= \left(\dot{a} ds \wedge \nu_1 + a\Omega_2\right) \otimes T_2 + \left(\dot{a} ds \wedge \nu_2 + a\Omega_3\right) \otimes T_3,
\end{aligned}$$

while

$$I_3 = \frac{a^2}{2}\left(\nu_{12} \otimes [T_2, T_3] + \nu_{21} \otimes [T_3, T_2]\right) = 2a^2 \nu_{12} \otimes T_1.$$

Put all these together and obtain

$$F_{1,-1} = \left(2(a^2 - 1)\nu_{12} + \Omega_1\right) \otimes T_1 + \left(\dot{a} ds \wedge \nu_1 + a\Omega_2\right) \otimes T_2 + \left(\dot{a} ds \wedge \nu_2 + a\Omega_3\right) \otimes T_3.$$

The computation of $F_{A_{1,-1}} \wedge \psi$ requires the G_2 structure as computed in section 4.3.1. It is useful to recall that $2g^2 = 4f^{-2}$, which helps in computing

$$\begin{aligned}
F_{A_{1,-1}} \wedge \psi &= 16 f^{-4} \dot{a}\left(ds \wedge \nu_1 \otimes T_2 + ds \wedge \nu_2 \otimes T_3\right) \wedge e_{1234} \\
&\quad + \left(32 f^{-4}(a^2 - 1) + 32 s^2\right) \sigma_{12} e_{1234} \otimes T_1 + 16 sa\left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right) \wedge e_{1234} \\
&= 32\left(f^{-4}a^2 - 1\right) e_{1234}\nu_{12} \otimes T_1 + 16\left(f^{-4}\dot{a} + sa\right) ds e_{1234} \wedge \left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right).
\end{aligned} \tag{4.3.17}$$

The other ingredient of the equations is the covariant derivative of the Higgs field $\Phi = \phi T_1$. The Bianchi identity for the connection $A^c_{1,-2}$ gives $d_{A^c_{1,-1}} T_1 = 0$ and so inserting this into $\nabla_{A_{1,-1}} \Phi$ gives

$$\begin{aligned}
\nabla_{A_{1,-1}} \Phi &= \nabla_{A^c_{1,-1}} \Phi + \left[a(\nu_1 \otimes T_2 + \nu_2 \otimes T_3), \phi T_1\right] \\
&= \dot{\phi}\, ds \otimes T_3 + 2a\phi\left(T_2 \otimes \nu_2 - T_3 \otimes \nu_1\right),
\end{aligned}$$

and

$$\begin{aligned}
* \nabla_{A_{1,-1}} \Phi &= 64 s^2 f^{-1} \dot{\phi} e_{1234} \wedge \nu_{12} \otimes T_1 + 2a\phi\left(T_2 \otimes *\nu_2 - T_3 \otimes *\nu_1\right) \\
&= 64 s^2 f^{-1} \dot{\phi}\, e_{1234}\nu_{12} \otimes T_1 + 32 f^{-3} a\phi\, ds e_{1234} \wedge \left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right).
\end{aligned} \tag{4.3.18}$$

Equating both sides of the monopole equation, i.e. equation 4.3.17 on the left hand side with equation 4.3.18 on the right gives the following set of ODE's

$$64 s^2 f^{-1}\dot{\phi} = 32(f^{-4}a^2 - 1) \tag{4.3.19}$$
$$16\left(f^{-4}\dot{a} + sa\right) = 32 f^{-3} a\phi. \tag{4.3.20}$$

Proposition 4.3.6. *As a set $\mathcal{M}_{inv}(\Lambda^2_-(\mathbb{CP}^2), P_{1,-1})$ is given by those connections and Higgs fields as in equations 4.3.14 and 4.3.16 such that $(\phi, b = f^{-2}(s^2)a)$ satisfy the ODE's*

$$\frac{d\phi}{d\rho} = \frac{1}{2h^2}\left(b^2 - 1\right) \tag{4.3.21}$$
$$\frac{db}{d\rho} = 2b\phi. \tag{4.3.22}$$

with $h^2(\rho) = s^2(\rho) f^{-2}(s^2(\rho)) = s^2(\rho)\sqrt{s^2(\rho)+1}$ and $b(0) = 1, \dot{b}(0) = 0$ and $\lim_{\rho\to+\infty} f^2(s^2)b = 0$.

Proof. This amounts to substitute $b = f^{-2}(s^2)a$ and change coordinates from s to ρ in equations 4.3.19 and 4.3.20. The first equation follows immediately and the second one from noticing that $f^{-2}\frac{da}{d\rho} + sfa = \frac{db}{d\rho}$. The initial conditions on b follow from the requirements that the connection and Higgs field extend over the zero section. This requires the curvature of the connection and the Higgs field to be bounded, which requires $\dot{a}(0) = 0$ and $a(0) = 1$ for the first and $\dot{\phi}(s)$ to be bounded as $s \to 0$. Since $f(0) = 1$ and $\dot{f}(0) = 0$ the conditions on a end up being equivalent to $b(0) = 1$ and $\dot{b}(0) = 0$. From the first ODE and the fact that $h^2(s) \sim s^2$ for small s it follows that these conditions are also sufficient. Recall that for a finite mass monopole as defined in section 1.4.1, the connection is asymptotic to the pullback of an HYM connection on the nearly Kähler $\mathbb{F}_3$. In this case, it must be to $A^c_{1,-1}$ and so $\lim_{\rho\to+\infty} a = 0$, i.e. $\lim_{\rho\to+\infty} f^2(s^2)b = 0$. □

Remark 4.3.7. *The equations in proposition 4.3.6 are the same as the ones in proposition 4.2.4. Hence, the problem has been reduced to the one of solving the ODE's for a spherically symmetric monopole in $\mathbb{R}^3$ (with a non-Euclidean metric though). Moreover, one can check that $h(\rho) \geq \rho$, is real analytic and as already remarked before behaves like: for small ρ, $h(\rho) = \rho + o(\rho^3)$ and for large ρ it grows as ρ^3.*

With this remark one can use the results in chapter 2 to prove theorem 4.1.9 with $P = P_{1,-1}$. The rest of the proof is done in exactly the same way as the the corresponding one for $\Lambda^2_-(\mathbb{S}^4)$ in section 4.2.2 and shall be omitted. The work in chapter 2 gives for each $m \in \mathbb{R}^+$ a unique solution (ϕ_m, b_m) to the equations in proposition 4.3.6 such that $\lim_{\rho\to\infty} |\phi_m(\rho) \otimes T_1| = m$. In the gauge used before this monopole can be written

$$(A_m, \Phi_m) = \left(A^c_{1,-1} + f^2 b_m \left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right), \phi_m T_1\right),$$

and the curvature of the connection is given by

$$\begin{aligned} F_A &= \left(2\left(f^4 b_m^2 - 1\right)\nu_{12} + \Omega_1\right)\otimes T_1 + f^2 b_m \left(T_2 \otimes \Omega_2 + T_3 \otimes \Omega_3\right) \\ &+ \frac{d}{d\rho}\left(f^2 b_m\right)\left(T_2 \otimes d\rho \wedge \nu_1 + T_3 \otimes d\rho \wedge \nu_3\right). \end{aligned}$$

Remark 4.3.8. • *These monopoles converge to the canonical invariant connection $A^c_{1,-1}$. This is reducible to a HYM connection on $L_{1,-1}$ over the nearly Kähler $\mathbb{F}_3$. Their curvature is given by $F^c_{1,-1}(\infty) = (2\nu_{12} + \Omega_1) \otimes T_1$, compare with equation 4.3.10 and see remark 4.3.4.*

• *The energy of these monopoles is not finite (as they are asymptotic to a nonflat connection on $\mathbb{F}_3$). However, the Intermediate energy is indeed finite and the formula 4.1.6 in proposition 4.1.4 can be used to compute*

$$E^I(A_m, \Phi_m) = \lim_{\rho\to\infty} 2\phi_m(\rho)\int_{\mathbb{F}^3} 16 e_{1234} \wedge 2\nu_{12} = 4\pi m \langle [\mathbb{F}_3], c_1(L_{1,-1}) \cup [i^*\psi]\rangle.$$

The next result regards the bundles $P_{1,2}$ as well as $P_{2,1}$. Recall from lemma 4.3.5 that these do not extend over the zero section and so are defined on $\Lambda^2_-(\mathbb{P}^2)\backslash\mathbb{P}^2$. However the monopole equations can still be integrated to give monopoles on the complement of the zero section and in the following result these solutions are shown not to extend directly from the ODE's.

Proposition 4.3.9. *There is no smooth invariant monopole on the bundles $P_{2,1}$ and $P_{1,2}$ which extends over the zero section.*

Proof. Start with the case $(n, l) = (2, 1)$, most of the computations are similar to the ones above and so will be omitted. In the case at hand, there are no solutions to the monopole ODE's that can be extended to the zero section as the bundle itself does not extend over the zero section as shown in lemma 4.3.5. The curvature and covariant derivative of the Higgs field are respectively given by

$$\begin{aligned} F_{2,1} &= -\dot{a}\left(ds \wedge e_3 \otimes T_2 + ds \wedge e_4 \otimes T_3\right) + \left(-2(a^2-1)\Omega_1 - \nu_{12}\right)\otimes T_1 \\ &+ a\left(\nu_2 \wedge e_2 - \nu_1 \wedge e_1\right)\otimes T_2 - a\left(\nu_1 \wedge e_2 + \nu_2 \wedge e_1\right)\otimes T_3 \\ \nabla_{A_{2,1}}\Phi &= \dot{\phi}\, ds \otimes T_1 - 2a\phi\left(T_2 \otimes e_3 - T_3 \otimes e_4\right), \end{aligned}$$

Equating $*\nabla_{A_{2,1}}\Phi = F_{2,1} \wedge \psi$ gives the following equations

$$\begin{aligned} \frac{d\phi}{d\rho} &= -\frac{1}{2h^2}\left(b^2+1\right) \\ \frac{db}{d\rho} &= -2b\phi, \end{aligned}$$

where $b = sa$ and as in the previous section $h^2(\rho) = s^2(\rho)\sqrt{s^2(\rho)+1}$. These equations will never give bounded solutions. In fact notice that since $1 + b^2 > 0$ and $h(0) = 0$, so $\dot{\phi}$ can not be bounded as $\rho \to 0$. The case $(n,l)=(1,2)$ is similar

$$\begin{aligned}
F_{1,2} &= \dot{a}\,(ds\wedge e_1\otimes T_2 + ds\wedge e_2\otimes T_3) + \big(2(a^2-1)\Omega_1+\nu_{12}\big)\otimes T_1 \\
&\quad -a\,(\nu_1\wedge e_3+\nu_2\wedge e_4)\otimes T_2 - a\,(\nu_1\wedge e_4-\nu_2\wedge e_3)\otimes T_3 \\
\nabla_{A_{1,2}}\Phi &= \dot{\phi}\,ds\otimes T_1 + 2a\phi\,(T_2\otimes e_2 - T_3\otimes e_1)\,,
\end{aligned}$$

and the monopole equations for these with $b=sa$ and $h^2(\rho)=s^2(\rho)\sqrt{s^2(\rho)+1}$ as before, are

$$\begin{aligned}
\frac{d\phi}{d\rho} &= \frac{1}{2h^2}\left(1+b^2\right) \\
\frac{db}{d\rho} &= 2b\phi.
\end{aligned}$$

Once again as it was the case for $(n,l)=(1,2)$, there is no hope of finding smooth solutions in this case, as $\dot{\phi}$ is unbounded at the zero section. □

$G=SU(3)$ Bundles

For gauge group $G=SU(3)$, the possible isotropy homomorphisms $\lambda : T^2\to SU(3)$ are parametrized by automorphisms of T^2 by identifying the image T^2 with the maximal torus in $SU(3)$. These depend on four integers $(n_{11},n_{12},n_{21},n_{22})\in\mathbb{Z}^4$ each corresponding to the degree of a different map $\pi_i\circ\lambda\circ i_j : \mathbb{S}^1\to\mathbb{S}^1$. Explicitly, such an homomorphism is given by

$$\lambda(e^{i\alpha_1},e^{i\alpha_2}) = i(e^{i(n_{11}\alpha_1+n_{12}\alpha_2)}, e^{i(n_{21}\alpha_1+n_{22}\alpha_2)}),$$

where $i : T^2\hookrightarrow SU(3)$ is a fixed embedding of the maximal torus (as in 4.3.1). For each of these homomorphisms one obtains a bundle $P_\lambda = SU(3)\times_\lambda SU(3)$. The reductive decomposition 4.3.2 equips each of these with a canonical invariant connection $A^c_\lambda = (n_{11}X_1+n_{21}X_2)\otimes\theta^1 + (n_{12}X_1+n_{22}X_2)\otimes\theta^2$, whose curvature is represented by the horizontal form

$$\begin{aligned}
F^c_\lambda &= -2\,(n_{11}X_1+n_{21}X_2)\otimes(e_{34}+\nu_{12}) + 2\,(n_{12}X_1+n_{22}X_2)\otimes(\nu_{12}-e_{12}) \\
&= -2\,(n_{11}X_1+n_{21}X_2)\otimes e_{34} - 2\,(n_{12}X_1+n_{22}X_2)\otimes e_{12} \\
&\quad +2\,((n_{12}-n_{11})X_1+(n_{21}+n_{22})X_2)\otimes\nu_{12}. \qquad (4.3.23)
\end{aligned}$$

Other invariant connections are given by morphisms of T^2 representations $\Lambda : (\mathfrak{m}, Ad)\to(\mathfrak{su}(3), Ad\circ\lambda)$. The following lemma is a tautology which will be helpful in decomposing the right hand side into irreducible components

Lemma 4.3.10. *Let* $\exp(ih) : T^n\to\mathbb{C}^* = GL(\mathbb{C})$ *be an irreducible* $\mathfrak{t}^n_{\mathbb{C}}$ *representation with weight vector* $dh\in(\mathfrak{t}^n)^*$*, and* $\lambda : T^n\to T^n$ *is a group homomorphism, then* $\exp(ih)\circ\lambda = \exp(i\lambda^* h)$.

Since as T^2 representations $(\mathfrak{m}_{\mathbb{C}}, Ad) = (2,1)\oplus(1,-1)\oplus(1,2)$, and $(n,l)\in\mathbb{Z}^2$ denotes the representation $e^{i(n\alpha_1+l\alpha_2)}$ and $\mathfrak{su}_{\mathbb{C}}(3) = \mathfrak{t}^2_{\mathbb{C}}\oplus\mathfrak{m}_{\mathbb{C}})$ the lemma splits the $(\mathfrak{su}_{\mathbb{C}}(3), Ad\circ\lambda)$

representation in the right hand side, as

$$\mathbb{C} \oplus \mathbb{C} \oplus (2n_{11} + n_{21}, 2n_{12} + n_{22}) \oplus (n_{11} - n_{21}, n_{12} - n_{22}) \oplus (n_{11} + 2n_{21}, n_{12} + 2n_{22}).$$

Restrict to the special case where $n_{11} = n_{22} = 1$ and $n_{21} = n_{12} = 0$. Pick an invariant connection given by $\Lambda : \mathfrak{m} \to \mathfrak{m}$ an isomorphism of representations. For each $\rho \in \mathbb{R}^+$, these depend on $a_1, a_2, a_3 \in \mathbb{R}$, each corresponding to a scaling factor associated with a an isomorphism between the corresponding irreducible components and induce the connection 1 forms given by

$$\begin{aligned} A_\lambda &= X_1 \otimes \theta^1 + X_2 \otimes \theta^2 - a_1 \left(C_{13} \otimes e_3 + D_{13} \otimes e_4 \right) \\ &\quad + a_2 \left(C_{12} \otimes \nu_1 + D_{12} \otimes \nu_2 \right) + a_3 \left(C_{23} \otimes e_1 + D_{23} \otimes e_2 \right). \end{aligned} \tag{4.3.24}$$

After a computation which is omitted the curvature is

$$\begin{aligned} F_\lambda &= -\frac{da_1}{d\rho} \left(C_{13} \otimes d\rho \wedge e_3 + D_{13} \otimes d\rho \wedge e_4 \right) + \frac{da_2}{d\rho} \left(C_{12} \otimes d\rho \wedge \nu_1 + D_{12} \otimes d\rho \wedge \nu_2 \right) \\ &\quad + \frac{da_3}{d\rho} \left(C_{23} \otimes d\rho \wedge e_1 + D_{23} \otimes d\rho \wedge e_2 \right) \\ &\quad + X_1 \otimes \left(2(a_1^2 - 1) e_{34} + 2(a_2^2 - 1) \nu_{12} \right) + X_2 \otimes \left(2(a_3^2 - 1) e_{12} + 2(1 - a_2^2) \nu_{12} \right) \\ &\quad + (a_1 - a_2 a_3) \left(C_{13} \otimes (-\nu_1 e_1 + \nu_2 e_2) - D_{13} \otimes (\nu_1 e_2 + \nu_2 e_1) \right) \\ &\quad + (a_2 - a_1 a_3) \left(C_{12} \otimes \Omega_2 + D_{12} \otimes \Omega_3 \right) \\ &\quad + (a_3 - a_1 a_2) \left(C_{23} \otimes (\nu_1 e_1 + \nu_2 e_2) + D_{23} \otimes (\nu_1 e_2 - \nu_2 e_1) \right). \end{aligned} \tag{4.3.25}$$

Remark 4.3.11. *The connection extends over to a connection on the whole $\Lambda^2_-(\mathbb{P}^2)$ if and only if its curvature 4.3.26 is bounded. This is equivalent to the statement that $a_2^2(0) = 1$, $\dot{a}_2(0) = 0$ and $a_1(0) = a_2(0)a_3(0)$. For example, the special cases where $a_2 = -1, a_2 = a_3 = \pm 1$ and $a_2 = 1, a_1 = -a_3 = \pm 1$, can be easily checked (using the formula above) to give rise to flat connections and these do extend over the zero section.*

The invariant Higgs field $\Phi \in \Omega^0(SU(3), \mathfrak{su}(3))$ must have values in $\mathfrak{t}^2 \subset \mathfrak{su}(3)$, so can be written $\Phi = \phi_1 X_1 + \phi_2 X_2$, where ϕ_1, ϕ_2 are functions of the radial coordinate. After a short computation

$$\begin{aligned} \nabla_A \Phi &= \frac{d\phi_1}{d\rho} d\rho \otimes X_1 + \frac{d\phi_1}{d\rho} d\rho \otimes X_2 + a_1 (2\phi_1 + \phi_2) \left(D_{13} \otimes e_3 - C_{13} \otimes e_4 \right) \\ &\quad - a_2 (\phi_1 - \phi_2) \left(D_{12} \otimes \nu_1 - C_{12} \otimes \nu_2 \right) - a_3 (\phi_1 + 2\phi_2) \left(D_{23} \otimes e_1 - C_{23} \otimes e_2 \right). \end{aligned}$$

Omitting some more computations the monopole equation $F_A \wedge \psi = *\nabla_A \Phi$ gives rise to the following set of ODE's

$$\begin{aligned}
\frac{d\phi_1}{d\rho} &= \frac{1}{2h^2}\left(f^{-4}a_2^2 - s^2a_1^2 - 1\right)\\
\frac{d\phi_2}{d\rho} &= \frac{1}{2h^2}\left(s^2a_3^2 - f^{-4}a_2^2 + 1\right)\\
s\frac{da_1}{d\rho} + f^{-1}\left(a_1 - a_2a_3\right) &= -sa_1(2\phi_1 + \phi_2)\\
f^{-2}\frac{da_2}{d\rho} + sf\left(a_2 - a_1a_3\right) &= f^{-2}a_2(\phi_1 - \phi_2)\\
s\frac{da_3}{d\rho} + f^{-1}\left(a_3 - a_1a_2\right) &= sa_3(\phi_1 + 2\phi_2),
\end{aligned}$$

where $h^2(\rho) = s^2(\rho)f^{-2}(s(\rho)) = s^2(\rho)\sqrt{s^2(\rho)+1}$. Introduce the rescaled fields $b_1 = sa_1$, $b_2 = f^{-2}a_2$, $b_3 = sa_3$. Then the ODE's above can be written as

$$\begin{aligned}
\frac{d\phi_1}{d\rho} &= \frac{1}{2h^2}\left(b_2^2 - b_1^2 - 1\right)\\
\frac{d\phi_2}{d\rho} &= \frac{1}{2h^2}\left(b_3^2 - b_2^2 + 1\right)\\
\frac{db_1}{d\rho} &= \frac{f}{s}b_2b_3 - b_1(2\phi_1 + \phi_2)\\
\frac{db_2}{d\rho} &= \frac{f}{s}b_1b_3 + b_2(\phi_1 - \phi_2)\\
\frac{db_3}{d\rho} &= \frac{f}{s}b_1b_2 + b_3(\phi_1 + 2\phi_2).
\end{aligned}$$

Theorem 4.3.12. *There is a* 1*-parameter family of solutions to the system of equations above, parametrized by their mass* $m \in \mathbb{R}^+$*. Moreover, such a solution gives rise to a smooth* G_2*-monopole, which in the previous gauge is given by the Higgs field* $\Phi = \phi_m(X_1 - X_2)$ *and the connection* $A_m = X_1 \otimes \theta^1 + X_2 \otimes \theta^2 + f^2a_m\left(C_{12}\otimes\nu_1 + D_{12}\otimes\nu_2\right)$*, whose curvature is*

$$\begin{aligned}
F_m &= \left(-2e_{34} + 2(f^4a_m^2 - 1)\nu_{12}\right)\otimes X_1 + \left(-2e_{12} + 2(1 - f^4a_m^2)\nu_{12}\right)\otimes X_2\\
&\quad + \left(f^4a_m^2\Omega_2 + \frac{d}{d\rho}\left(f^4a_m^2\right)d\rho\wedge\nu_1\right)\otimes C_{12} + \left(f^4a_m^2\Omega_3 + \frac{d}{d\rho}\left(f^4a_m^2\right)d\rho\wedge\nu_2\right)\otimes D_{12}.
\end{aligned}$$

Proof. The particular solutions stated above follow from an ansatz that reduces the system to the same ODE's that have been obtained in all the other cases (i.e. the ones for spherically symmetric monopoles in $(\mathbb{R}^3, d\rho^2 + h^2(\rho)g_{\mathbb{S}^2})$). Set $b_1 = b_3 = 0$, then the third and fifth equations are trivially satisfied. The other equations are

$$\begin{aligned}
\frac{d\phi_1}{d\rho} &= -\frac{d\phi_2}{d\rho} = \frac{1}{2h^2}\left(b_2^2 - 1\right)\\
\frac{db_2}{d\rho} &= b_2(\phi_1 - \phi_2).
\end{aligned}$$

If it is further supposed that $\phi_1 = -\phi_2 = \phi$, one obtains

$$\begin{aligned}\frac{d\phi}{d\rho} &= \frac{1}{2h^2}\left(b_2^2 - 1\right) \\ \frac{db_2}{d\rho} &= 2\phi b_2,\end{aligned}$$

and the existence result from chapter 2 can be applied, to give a family of solutions. These are parametrized by $m \in \mathbb{R}^+$ and given by $(\phi, b) = (\phi_m, a_m)$, where (ϕ_m, a_m) is the solution provided by theorem 2.2.1. One then computes the formula in the statement for their curvature, which is bounded at $\rho = 0$ and so extends to a solution on $\Lambda^2_-(\mathbb{P}^2)$, see remark 4.3.11. $\square$

The Ambrose-Singer theorem identifies the Lie algebra of the holonomy group with the values of the curvature. This allows the conclusion that the holonomy of the monopoles above is contained in a $S(U(1) \times SU(2))$ subgroup of $SU(3)$.

4.3.3 G_2 Instantons

This subsection constructs G_2 instantons on bundles over $\Lambda^2_-(\mathbb{P}^2)$ equipped with the Bryant-Salamon G_2 structure. One must remark that G_2 instantons for the the other Bryant-Salamon metrics also exist, as constructed in subsection 4.2.3 for $\Lambda^2_-(\mathbb{S}^4)$ and by Andrew Clarke in [Cla14] for $\mathcal{S}(\mathbb{S}^3)$.

$G = \mathbb{S}^1$ Bundles

In the case $n = l$, lemma 4.3.1 states that the bundle is $P_{n,n} = \pi_2^* \mathcal{O}_{\mathbb{P}^2}(-n)$. The bundles $\mathcal{O}_{\mathbb{P}^2}(-n)$ are self-dual and one can check that the canonical invariant connection associated with these will give rise to a G_2-instanton on $\Lambda^2_-(\mathbb{P}^2)$. This is stated in proposition 4.3.3.

$G = SO(3)$ Bundles

Irreducible G_2-instanton in the bundle $P_{1,-1}$ can be obtained by solving the ODE's in proposition 4.3.6 for $\phi = 0$. This implies $b^2 = 1$, i.e. $b = \pm 1$, $a = \pm f^2(s^2)$ and $\frac{da}{ds} = \mp s f^6(s^2)$, the solution is a smooth irreducible G_2-instanton on $P_{1,-1} \to \Lambda^2_-(\mathbb{P}^2)$.

Theorem 4.3.13. *The connection on $P_{1,-1}$ over $\Lambda^2_-(\mathbb{P}^2)$ given by $A = A^c_{1,-1} + f^2(s^2)\left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right)$ is an irreducible G_2-instanton with curvature*

$$\begin{aligned}F_{1,-1} &= \frac{2s^2}{s^2+1}\nu_{12} \otimes T_1 + \Omega_1 \otimes T_1 \pm \frac{1}{\sqrt{s^2+1}}\left(\Omega_2 \otimes T_2 + \Omega_3 \otimes T_3\right) \\ &\quad \mp \frac{s}{(1+s^2)^{\frac{3}{2}}}\left(ds \wedge \nu_1 \otimes T_2 + ds \wedge \nu_2 \otimes T_3\right).\end{aligned}$$

This instanton converges to the canonical invariant connection, which recall is the pullback to the cone of a reducible HYM connection on $\mathbb{F}_3$ equipped with its standard nearly Kähler structure. Its curvature is $F_{1,-1}(\infty) = (2\nu_{12} + \Omega_1) \otimes T_1$.

$G = SU(3)$ Bundles

To obtain irreducible G_2 instantons, one solves the system of ODE's 4.3.12.

Theorem 4.3.14. *There are two families of irreducible G_2 instantons parametrized by $c \geq 0$. These are respectively given by*

$$A_\lambda = X_1 \otimes \theta^1 + X_2 \otimes \theta^2 - \frac{u_c(s)}{\sqrt{1+s^2}} \left(C_{12} \otimes \nu_1 + D_{12} \otimes \nu_2\right) \tag{4.3.26}$$
$$\mp \frac{\sqrt{u_c^2(s) - 1}}{s} \left(C_{13} \otimes e_3 + D_{13} \otimes e_4 - C_{23} \otimes e_1 - D_{23} \otimes e_2\right) \tag{4.3.27}$$

and

$$A_\lambda = X_1 \otimes \theta^1 + X_2 \otimes \theta^2 + \frac{u_c(s)}{\sqrt{1+s^2}} \left(C_{12} \otimes \nu_1 + D_{12} \otimes \nu_2\right) \tag{4.3.28}$$
$$\mp \frac{\sqrt{u_c^2(s) - 1}}{s} \left(C_{13} \otimes e_3 + D_{13} \otimes e_4 + C_{23} \otimes e_1 + D_{23} \otimes e_2\right), \tag{4.3.29}$$

where

$$u_c(s) = 1 - 2c \frac{s^2}{s^2(1+c) + 2\left(\sqrt{1+s^2} + 1\right)}. \tag{4.3.30}$$

In particular, the case $c = -1$, recovers the flat connections alluded to in remark 4.3.11.

Proof. For $\Phi = 0$ one has to set $\phi_1 = \phi_2 = 0$ in the system of equations above. This gives the equations

$$\begin{aligned} 1 &= b_2^2 - b_3^2 = b_2^2 - b_1^2 \\ \frac{db_i}{d\rho} &= \frac{f}{s} b_j b_k, \end{aligned}$$

for $i, j, k \in 1, 2, 3$ and $i \neq j \neq k$. In order to guarantee that the connection extends over the zero section, its curvature must be bounded and from remark 4.3.11 together with the definitions of the $b_i's$ one must have $b_2(0)^2 = a_2(0)^2 = 1$, $b_1(0) = b_2(0) = 0$ and $\dot{b}_3(0) = (-1)^k \dot{b}_1(0)$, where $a_2(0) = (-1)^k$. Moreover, from the equations above $\frac{db_1^2}{d\rho} = \frac{db_2^2}{d\rho} = \frac{db_3^2}{d\rho} = \frac{f}{s} b_1 b_2 b_3$ and so the three last equations are indeed compatible with the constraints imposed by the first two ones. These also imply that $b_1 = \pm b_3 = (-1)^k b_3$, and the system gets reduced to solve

$$\begin{aligned} b_2^2 - b_1^2 &= 1 \\ \frac{db_1}{d\rho} &= \frac{f}{s} (-1)^k b_2 b_1 \\ \frac{db_2}{d\rho} &= \frac{f}{s} (-1)^k b_1^2. \end{aligned}$$

Inserting the first equation (the constraint) into the last one and using $\frac{d}{d\rho} = f^{-1} \frac{d}{ds}$ gives the following nonlinear singular ODE

$$\frac{db_2}{ds} = (-1)^k \frac{f^2}{s}(b_2^2 - 1). \tag{4.3.31}$$

For k even there is a 1-parameter family of solutions given by $b_2(s) = -u_c(s)$, for all $c \geq -1$ and $b_1(s) = b_3(s) = \pm\sqrt{u_c^2(s) - 1}$. In the same way for k odd, there is a 1-parameter given by $b_2(s) = u_c(s)$ for all $c \geq 0$ and so $b_1(s) = \pm\sqrt{u_c^2(s) - 1}$. These give rise to the connections on the statement and to check the connections extend one needs to show that $\frac{\sqrt{u_c^2(s)-1}}{s}$ is bounded at $s = 0$ which is indeed the case. □

Chapter 5

Moduli Spaces via Analysis

This chapter constructs an analytical setting in which to define the moduli spaces of finite mass monopoles on AC manifolds. The results hold in all three cases of interest, namely 3 dimensions, Calabi-Yau 3-folds and G_2-manifolds. The chapter is organized into two main sections. The first one 5.1 is focused on analyzing the linearization of the monopole equation, namely it defines Banach spaces of sections on which the gauge fixed linearized operator is shown to be Fredholm. Then section 5.2 combines this linear theory with Sobolev and multiplication properties of the relevant Banach spaces in order to handle the nonlinearities of the (complex) monopole equation. The main result of the first part is theorem 5.1.18 and that of the second part is theorem 5.2.3. This is then reinterpreted in theorem 5.2.15 as saying that the moduli space of (complex) monopoles is the zero locus of a Fredholm section of a suitable bundle over a Banach manifold.

5.1 Linear Analysis for Monopoles

Let (X^n, g) be an asymptotically conical manifold with n odd (even) and (A, Φ) either a finite mass (resp. complex) monopole on $P \to X$ as in definition 1.4.1 (resp. 3.1.19). It is shown in sections 2.1.1, 3.1.2 and 4.1.1 that for 3 manifolds, Calabi-Yau 3 folds and G_2-manifolds respectively, the gauge fixed elliptic operator $D = d_1^* \oplus d_2$ associated with the (resp. complex in the Calabi-Yau case) monopole equation is as follows. Denote by $\mathcal{S}$ the vector bundle associated with the standard $Spin(n)$ representation and equip it with the standard spin connection (induced by the Levi Civita one on TX). Equip the vector bundle $E = \mathfrak{g}_P \to X$ with the connection induced by A and $\mathcal{S}_E = \mathcal{S} \otimes E$ equipped with the connection induced from both A and the spin connection. To ease notation also denote this connection by A, and by $\mathcal{D}_A$ its Dirac operator and let $q = \mathrm{ad}_\Phi \in \Omega^0(X, \mathrm{End}(E))$ denote the induced endomorphism. Then, as computed in the sections alluded to above $D = \mathcal{D}_A + q$ and the goal of this section is to prove theorem 5.1.18 below, which one can write as

Theorem 5.1.1. *There are Banach spaces of sections of $\mathcal{S}_E$ denoted by $H^p_{k,\alpha}$ as in definition 5.1.16 below (for $p \geq 2$), and a discrete set $\mathcal{K}(\mathcal{D}_A) \subset \mathbb{R}$ such that the operator*

$$D = \mathcal{D}_A + q : H^p_{k+1,\alpha+1} \to H^p_{k,\alpha} \tag{5.1.1}$$

is Fredholm for all $\alpha \geq -n/2$, such that $\alpha \notin \mathcal{K}(\mathcal{D}_A)$ and $p \geq 2$.

The strategy to prove this result is to study the relevant model situations. First one studies the operator D in a model metric cone $(\mathbb{R}^+_r \times \Sigma, g = dr^2 + r^2 g_\Sigma)$, where (A, Φ) coincide with the pull back to the cone of the boundary data (A_∞, Φ_∞). The connection A respects the eigenspace decomposition of q, and so one can split $\mathcal{S}_E|_\Sigma = \ker(q) \oplus \ker(q)^\perp$. This splitting gives rise to two distinguished cases: on $\ker(q)$, $D = \mathcal{D}_A$ is the usual Dirac operator and on $\ker(q)^\perp$, q is invertible and such operators are known as Callias operators due to the work in [Cal78]. Both these cases are analyzed separately on a model cone in subsections 5.1.2 and 5.1.3 respectively, where one constructs suitable Banach spaces in which parametrices exists. In subsection 5.1.4, one constructs a mixed Banach space of sections, which allows to construct a parametrix in the general case where both $\ker(q), \ker(q)^\perp$ are nonzero. The operator D is elliptic and the usual parametrix construction gives inverses on small Euclidean patches. Also in subsection 5.1.4 these two kinds of model parametrices are matched to prove that on an asymptotically conical manifold, the overall operator is Fredholm as claimed in theorem 5.1.15. In the final subsection 5.1.5 one extends the previous Banach spaces of sections to depend on an exponent p which in the case $p = 2$ gives back the previous ones. Then, one proves that the Fredholm property of D extends from $p = 2$ to $p > 2$; this is stated in theorem 5.1.18 and will be used in section 5.2 to deal with the nonlinear theory.

5.1.1 The Model Conical Operators

On the metric cone $(C, g_C) = (\mathbb{R}^+_r \times \Sigma, dr^2 + r^2 g_\Sigma)$ denote by $P^n \to C$ (resp. $P^{n-1} \to \Sigma$) the principal $SO(n)$ (resp. $SO(n-1)$) frame bundle of (C, g_C) (resp. (Σ, g_Σ)). For both $i = n, n+1$, let $\tilde{P}^i \to P^i$ be the lifts to the $Spin(i)$ bundle and $\mathcal{S}_i = \tilde{P}^i \times_{\rho_i} S_i$ the vector bundle associated with the standard $Spin(i)$ representation, $\rho_i : Spin(i) \to U(S_i)$. The Clifford Algebra splits as $Cl^i = (Cl^i)^0 \oplus (Cl^i)^1$ in even and odd elements. Then, $Spin(i)$ lies in $(Cl^i)^0$ and is generated by those elements of the form $v \cdot w$, where $\|v\| = \|w\| = 1$. This permits to see the $Spin(i)$ representations above as being induced by restricting to $Spin(i)$ a representation of the Clifford algebra. This is the key point for comparing ρ_{n-1} with ρ_n via the algebra isomorphism between Cl^{n-1} and $(Cl^n)^0$ given by

$$e_i \in Cl^{n-1} \mapsto e_i \cdot e_0 \in (Cl^n)^0, \tag{5.1.2}$$

where $\{e_i\}_{i=1}^{n-1}$ is an orthonormal frame of $\mathbb{R}^{n-1}$ and is extended to an orthonormal frame of $\mathbb{R}^n$ by adding e_0.

Remark 5.1.2. *There are now to cases to distinguish,*

- *If $n-1$ is even and $(\rho_{n-1}, S^{n-1} = S^+_{n-1} \oplus S^-_{n-1})$ is the direct sum of the two irreducible spin representations, then the $Spin(n)$ representation obtained via Cl^{n-1} is the unique irreducible one. Or conversely, if ρ_n is the unique irreducible $Spin(n)$ respresentation, then the induced representation of $Spin(n-1)$ via the isomorphism of algebras 5.1.2 is the direct sum of the two irreducible ones. This is the relevant setup for the deformation operator of the G_2 monopole equation.*

- *If $n-1$ is odd and ρ_{n-1} is the unique irreducible $Spin(n-1)$ representation, then it induces one of the half $Spin(n)$ representations S_n^+. Conversely, if ρ_n is the half $Spin(n)$ representation S_n^+, then the 5.1.2 induced $Spin(n-1)$ representation is the unique irreducible one. This is the relevant situation for Calabi-Yau monopoles.*

Let $\pi_\Sigma : C \to \Sigma$ is the projection on the second factor, then by remark 5.1.2, $\mathcal{S}^n \cong \pi_\Sigma^*(\mathcal{S}^{n-1})$. Parallel transport along the radial direction constructs a map $P' : \Omega^0(\Sigma, \mathcal{S}^{n-1}) \to \Omega^0(C, \mathcal{S}^n)$, which for $\sigma \in \Omega^0(\Sigma, \mathcal{S}^{n-1})$ yields $P'\sigma \in \Omega^0(C, \mathcal{S}^n)$, solving the initial value problem $\nabla_{\frac{\partial}{\partial r}}(P'\sigma) = 0$, $(P'\sigma)|_{\{1\}\times S} = \sigma$. This extends to an isomorphism $P : \Omega^0(\mathbb{R}^+, \Omega^0(\Sigma, \mathcal{S}^{n-1}) \xrightarrow{\sim} \Omega^0(C, \mathcal{S}^n)$, which identifies sections of $\mathcal{S}^n$ over C with 1 parameter families of sections of $\mathcal{S}^{n-1}$ over $\Sigma \cong \{1\} \times \Sigma$ in such that $P^{-1}\nabla_{\frac{\partial}{\partial r}} Ps = \frac{\partial s}{\partial r}$ for $s \in \Omega^0(\mathbb{R}^+, \Omega^0(\Sigma, \mathcal{S}^{n-1}))$.

Lemma 5.1.3. *Let $\partial\!\!\!/$ denote the spin Dirac operator on $\Omega^0(\Sigma, \mathcal{S}^{n-1})$ and $\{e_i\}_{i=0}^{n-1}$ orthonormal. Then, for $s \in \Omega^0(C, \mathcal{S}^n)$*

$$\mathcal{D}(s) \quad = \quad e_0 \cdot \left(\nabla_{\frac{\partial}{\partial r}} s + \frac{1}{r} \left(P \partial\!\!\!/ P^{-1} s + \frac{n}{2}(s) \right) \right). \tag{5.1.3}$$

Proof. This follows from a lengthy but straightforward computation using the formula for the $Spin$ connection the second fundamental form of the cross sections of the cone which are $\frac{n}{2r}$ times the identity. Details of this computation are given for example in [Ang90]. □

Let $E \to \Sigma$ be a vector bundle with connection A which is pulled back to the cone. Construct the bundle $\mathcal{S}_E = \mathcal{S}^{n-1} \otimes E$, equipped with the twisted connection ∇_A and the twisted Dirac operator $\mathcal{D}_A$. Also let $q \in \Omega^0(C, \mathrm{End}(\mathcal{S}_E))$ be skew symmetric and such that $\nabla_A(q) = 0$ for the connection on the endomorphism bundle. Using lemma 5.1.3 the operator D acting on sections of $\mathcal{S}^n \otimes \pi_\Sigma^* E \cong \pi_\Sigma^* \mathcal{S}_E$ is equivalent to an operator on $\Omega^0(\mathbb{R}^+, \Omega^0(\Sigma, \mathcal{S}_E))$ is

$$D(s) = e_0 \cdot \left(\nabla^A_{\frac{\partial}{\partial r}} s + \frac{1}{r} \left(P \partial\!\!\!/ P^{-1} s + \frac{n}{2} s \right) \right) + q(s). \tag{5.1.4}$$

The goal now is to use good Banach Spaces, which ensure the existence of suitable parametrices for this model operator. Then, patch this together with the parametrices given by standard elliptic theory over open bounded sets to give global parametrices for operators on asymptotically conical manifolds.

5.1.2 The Dirac Operator ($q = 0$)

Back to the setup where (X, g) is an asymptotically conical manifold, this subsection gives the Fredholm property in the case where $q = 0$, i.e. $\mathcal{S}_E = \mathcal{S}_E^{||}$ and so $D = \mathcal{D}_A$ is the Dirac operator. Suitable Banach spaces where the Fredholm property for the Dirac Operator holds exist and this is reviewed in this subsection. Let ρ be a radius function as in definition 1.1.5, $\alpha \in \mathbb{R}$ and $p, k \in \mathbb{N}_1$. Denote the Lockhart-McOwen [LM85] weighted norm by $\| \cdot \|_{L^p_{k,\alpha}}$, this is given by on a smooth compactly supported $f \in \Gamma(X, \mathcal{S}_E)$

$$\|f\|_{L^p_{k,\alpha}} = \|\nabla_A f\|_{L^p_{k-1,\alpha-1}} + \|f\|_{L^p_{0,\alpha}}, \tag{5.1.5}$$

and $\|f\|^p_{L^p_{0,\alpha}} = \int_X |\rho^{-\alpha} f|^p \rho^{-n} dvol_X$.

Definition 5.1.4. *The Lockhart-McOwen* [LM85] *Sobolev spaces $L^p_{k,\alpha}$ with weight $\alpha \in \mathbb{R}$ are the completion of the smooth compactly supported functions in the norm 5.1.5. Moreover, one will further require the radius function ρ to be such that $\rho \in (0,1]$ inside a compact set K' with smooth boundary, such that $\partial K' = \rho^{-1}(1)$ and K' contains K, then on $X \backslash K'$ one takes $\rho = r \circ \varphi$, with φ as in definition 1.1.5.*

The next result states that the twisted Dirac operator $\mathcal{D}_A$ on an asymptotically conical Spin manifold is Fredholm for the Lockhart McOwen weighted Sobolev spaces. This is a standard result, as $\mathcal{D}_A$ is an asymptotically conical operator [Mar02]. Alternatively this theorem follows by translating all the setup into the cylindrical setting and using the results in [Don02] or in [LM85]. In fact the results in [Don02] also prove that the model operator on a cone admits a right inverse in this case.

Theorem 5.1.5. *Let (X, g) be asymptotically conical, then there is a discrete set of weights $\mathcal{K}(\mathcal{D}_A)$ such that for all $\alpha \notin \mathcal{K}(\mathcal{D}_A)$ and $k \in \mathbb{N}$, the Dirac operator $\mathcal{D}_A : L^2_{k+1,\alpha+1} \rightarrow L^2_{k,\alpha}$ is Fredholm. Moreover,*

$$L^2_{k,\alpha} = \mathcal{D}_A(L^2_{k+1,\alpha+1}) \oplus W_\alpha, \tag{5.1.6}$$

with $W_\alpha \cong \ker(\mathcal{D}^*_A)_{-\alpha-n}$ and in the case where $\ker(\mathcal{D}^*_A)_{-\alpha-n} \subset L^2_{k,\alpha}$, i.e. $\alpha \geq -\frac{n}{2}$ equality holds.

5.1.3 The Conical Callias Operator (q invertible).

This subsection focuses on the case where q is pointwise invertible along the ends $X \backslash K$ and bounded below. Such a case is worked out in [Ang90] and [Kot10] where a formula for the index in a quite general setup is given. Here a proof of the Fredholm property is given and the treatment given is motivated by [Tau83] and [Don02]. The idea is to start and study the model situation on a cone and then extend this to the AC setting. Before proceeding recall the relation to monopoles, when restricted to the component $\mathcal{S}_E^{\perp}$. The operators $D, D^* : \Omega^0(X \backslash K, \mathcal{S}_E^{\perp}) \rightarrow \Omega^0(X \backslash K, \mathcal{S}_E^{\perp})$ associated with the (complex in the Calabi-Yau case) monopole equations, satisfy certain Weitzenböck formulas, see lemma 2.1.2 in the 3 dimensional case, propositions 3.1.9 and 3.1.10 for the Calabi-Yau case and finally proposition 4.1.2 for the G_2 case. In all cases one can write D^*D and DD^* as $\nabla_A^* \nabla_A + W + q^* q$, where W is a zeroth order differential operator which for finite mass (complex) monopoles decays along the ends. In fact, for finite mass (complex) monopoles, along $X \backslash K$ the configuration (A, Φ) is modeled on (A_∞, Φ_∞) and so $F_A, \nabla_A \Phi$ appearing in W do decay with rate smaller or equal to -2. In fact, the results proven below in corollary 5.1.10 and proposition 5.1.11 will hold under slightly more general assumptions. They just require the F_A and $\nabla_A\Phi$ to decay, i.e. they assume the existence of a smooth function $\varepsilon > 0$, such that

$$\lim_{r\to+\infty} |\rho^j \nabla^j \varepsilon(r)| = 0 \tag{5.1.7}$$

and $|\rho^j \nabla_A^j F_A|, |\rho^j \nabla_A^j \Phi| \leq \varepsilon^2$ for all $j \in \mathbb{N}_0$.

The Model Cone

Proposition 5.1.6. *Let $C = (1, +\infty) \times \Sigma$ equipped with the cone metric $g = dr^2 + r^2 g_\Sigma$ and D as before with q constant and bounded by bellow, i.e. $\nabla_A q(f) = q(\nabla_A f)$ and $|q(f)|^2 \geq c|f|^2$ for some constant $c > 0$ and all $f \in \Omega^0(C, \mathcal{S}_E)$. Suppose there is a Weitzenböck formula*

$$D^* D = \nabla_A^* \nabla_A + W + q^* q,$$

with W decaying as r goes to ∞, i.e. there is a function $\varepsilon(r) > 0$ as in equation 5.1.7 such that $|W(f)| \leq \varepsilon^2(r)|f|$. Then, the following inequality holds

$$\|f\|^2_{L^2_1} \leq c_1 \|Df\|^2_{L^2} + c_2 \|\varepsilon(r) f\|^2_{L^2}, \tag{5.1.8}$$

for some positive constants c_1, c_2 and all f compactly supported in C.

Proof. For compactly supported f one can integrate by parts in $\|Df\|^2_{L^2}$ and use the Weitzenböck formula in the hypothesis

$$\begin{aligned} \|Df\|^2_{L^2} &= \langle D^* D f, f\rangle_{L^2} = \|\nabla_A f\|^2_{L^2} + \langle W(f), f\rangle_{L^2} + \|q(f)\|^2_{L^2} \\ &\geq \|\nabla_A f\|^2_{L^2} - \|\varepsilon(r) f\|^2_{L^2} + c\|f\|^2_{L^2}. \end{aligned}$$

Now one passes the term $-\|\varepsilon(r)f\|^2_{L^2}$ to the other side and this gives the inequality

$$\|\nabla_A f\|^2_{L^2} + c\|f\|^2_{L^2} \leq \|Df\|^2_{L^2} + c'\|\varepsilon(r) f\|^2_{L^2},$$

which after suitably rearranging the constants gives the inequality 5.1.8, which one is trying to prove. ☐

From the Cone to Asymptotically Conical

We return to the case where X is asymptotically conical and q bounded by below. The following lemmata will prove the Fredholm property for an operator which is globally like this, i.e. in the case $\mathcal{S}_E^\perp$ extends over the whole X.

Lemma 5.1.7. *Let $\varepsilon : X \to \mathbb{R}^+$ be smooth and such that $\lim_{\rho\to\infty} \varepsilon(\rho) = 0$. Then the embedding $L^2_1 \hookrightarrow L^2_\varepsilon$ is compact. Where in the right hand side L^2_ε denotes the completion of the smooth compactly supported sections in the norm $\|f\|_{L^2_\varepsilon} = \|\varepsilon f\|_{L^2}$.*

Proof. Let $\{f_i\} \subset L^2_1$ be a sequence with $\|f_i\|^2_{L^2_1} = 1$ one needs to prove that there is a subsequence which has a limit in L^2_ε. To do this notice that since $\|f_i\|^2_{L^2_1} = 1$, there is a subsequence with a weak limit in L^2_1, denote this by f and notice that $\|f\|^2_{L^2_1} \leq 1$. The claim is that this subsequence converges to f strongly in L^2_ε. To see this denote by $B_R = \rho^{-1}[1, R)$ and compute

$$\begin{aligned} \|\varepsilon(f_i - f)\|^2_{L^2} &= \|\varepsilon(f_i - f)\|^2_{L^2(B_R)} + \|\varepsilon(f_i - f)\|^2_{L^2(X\backslash B_R)} \\ &\leq c_1\|f_i - f\|^2_{L^2(B_R)} + \varepsilon^2(R)\|f_i - f\|^2_{L^2(X\backslash B_R)} \\ &\leq c_1\|f_i - f\|^2_{L^2(B_R)} + 4\varepsilon^2(R). \end{aligned} \tag{5.1.9}$$

Where in the last inequality one uses that

$$\|f_i - f\|^2_{L^2(X\backslash B_R)} \leq \|f_i - f\|^2_{L^2_1(X\backslash B_R)} \leq 2\|f_i\|^2_{L^2_1(X\backslash B_R)} + 2\|f\|^2_{L^2_1(X\backslash B_R)} \leq 4.$$

The second term in equation 5.1.9 is $4\varepsilon^2(R)$ and can be made as small as one wishes by making R big. Regarding the first one $\|f_i - f\|^2_{L^2(B_R)}$, since the embedding $L^2_1(B_R) \hookrightarrow L^2(B_R)$ is compact, f_i does converge strongly to f in $L^2(B_R)$ and the term $\|f_i - f\|^2_{L^2(B_R)}$ can also be made arbitrarily small by letting i get big. □

Lemma 5.1.8. *There is a positive constant C such that for all $f \in L^2_1$*

$$\|f\|^2_{L^2_1} \leq C \left(\|Df\|^2_{L^2} + \|f\|^2_{L^2_\varepsilon} \right). \tag{5.1.10}$$

Proof. There are two cases to distinguish, the interior of X and its ends. Let R be big and $B_R = \rho^{-1}(0, R)$, then by the ellipticity of D, for compactly supported f inside B_{R+1} there is $R' > R$ such that

$$\begin{aligned} \|f\|^2_{L^2_1(B_{R+1})} &\leq c_1\|Df\|^2_{L^2(B_{R'})} + c_2\|f\|^2_{L^2(B_{R'})} \\ &\leq c_1\|Df\|^2_{L^2(B_{R'})} + c_2\varepsilon(R')^{-1}\|\varepsilon f\|^2_{L^2(B_{R'})}, \end{aligned} \tag{5.1.11}$$

for some constants $c_1, c_2 > 0$ which do depend on R, R' but independent of f. At the ends of X, i.e. on $X\backslash B_R$, pull back all the data to the cone via a quasi isometry, then there is an operator D_C on the cone satisfying the hypothesis in proposition 5.1.6, such that $D - D_C = O(\rho^{-1-\varepsilon})$ for some $\varepsilon > 0$. So from proposition 5.1.6

$$\|f\|^2_{L^2_1(X\backslash B_R)} \leq c_1'\|Df\|^2_{L^2(X\backslash B_R)} + c_2'\|\varepsilon f\|^2_{L^2(X\backslash B_R)},$$

for some constants $c_1', c_2' > 0$. The last step is to put this together with the interior inequality 5.1.11 let φ_R be a function supported on B_{R+1} which equals 1 on B_R, then

$$\begin{aligned} \|f\|^2_{L^2_1} &= \|f\|^2_{L^2_1(B_R)} + \|f\|^2_{L^2_1(X\backslash B_R)} \leq \|\varphi_{R+1}f\|^2_{L^2_1(B_{R+1})} + \|(1-\varphi_R)f\|^2_{L^2_1(X\backslash B_R)} \\ &\leq 2(c_1 + c_1')\|Df\|^2_{L^2} + 2(c_2\varepsilon(R')^{-1} + c_2')\|\varepsilon f\|^2_{L^2}, \end{aligned}$$

which is the inequality one is trying to prove. □

Corollary 5.1.9. *The AC operator $D : L^2_1 \to L^2$ has closed range and finite dimensional kernel.*

Proof. To prove that the kernel is finite dimensional one proves that the unit ball in the kernel is compact. So let $\{f_i\} \subset \ker(D)$ be a sequence with $\|f_i\|^2_{L^2_1} = 1$. From lemma 5.1.7, the embedding $L^2_1 \hookrightarrow L^2_\varepsilon$ is compact and so there is a subsequence f_i, which converges strongly in L^2_ε to some $f \in \ker(D) \cap L^2_\varepsilon$. But then, the inequality 5.1.10 gives $\|f_i - f\|^2_{L^2_1} \leq c_2 \|\varepsilon(f_i - f)\|^2_{L^2} \to 0$, and so f_i does converge to f in L^2_1. Next one needs to prove that the image is closed, for that it is enough to prove that there is a constant $c > 0$, such that for all $f \in (\ker D)^\perp \cap L^2_1$

$$\|Df\|_{L^2} \geq c \|f\|^2_{L^2_1}. \tag{5.1.12}$$

Suppose not, then there is a sequence $\{f_i\} \subset (\ker D)^\perp \cap L^2_1$ with $\|Df_i\|^2_{L^2} \to 0$ and $\|f_i\|^2_{L^2_1} = 1$. There is a weak limit $f \in L^2_1$ such that $Df = 0$ and from lemma 5.1.7, the limit f is strong in L^2_ε. In fact $f = 0$ since by assumption it is the limit of the f_i's which are in the orthogonal complement to the kernel. Then the inequality 5.1.10 gives $1 = \|f_i\|^2_{L^2_1} \leq \|Df_i\|^2_{L^2} + \|\varepsilon f_i\|^2_{L^2}$, as the first term in the right hand side vanishes, while the second one converges to zero this is a contradiction. □

Corollary 5.1.10. *Let $D : \Omega^0(X, \mathcal{S}_E) \to \Omega^0(X, \mathcal{S}_E)$ be such that on $X \backslash K$ it is modeled on a conical operator D_C as in proposition 5.1.6. Then, $D : L^2_1 \to L^2$ is a Fredholm operator.*

Proof. Corollary 5.1.9 gives that the kernel is finite dimensional and the image is closed, so it is enough to prove that the cokernel is finite dimensional as well. As $cokD \cong \ker D^* \cap L^2$ one just needs to prove that this later one is finite dimensional. Since D^* is also modeled on an operator as in the hypothesis of proposition 5.1.6, it satisfies an inequality as in equation 5.1.10. Using such an inequality, one concludes that there is a constant $c_2 > 0$ with the meaning for all $f \in \ker D^* \cap L^2$, $\|f\|_{L^2_1} \leq c_2 \|\varepsilon f\|_{L^2} \leq c_2 \|f\|_{L^2}$ and so $\ker D^* \cap L^2 \hookrightarrow L^2_1$ and since by proposition 5.1.9 applied to D^* the kernel of D^* in L^2_1 is finite dimensional. □

Proposition 5.1.11. *Let D be as before and $k \in \mathbb{N}_0$, then $D : L^2_{k+1} \to L^2_k$ is a Fredholm operator.*

Proof. If one can prove an inequality of the form

$$\|f\|^2_{L^2_{k+1}} \leq c_1 \|Df\|^2_{L^2_k} + c_2 \|\varepsilon'(r) f\|^2_{L^2_k}, \tag{5.1.13}$$

for both D and D^* and some ε' as in equation 5.1.7. Then by repeating all the steps done before with L^2 replaced by L^2_k and L^2_1 replaced by L^2_{k+1} the proposition follows. Before, starting with the proof of inequality 5.1.13, notice that the operator D can be extended to act on sections of $T^*X \otimes \mathcal{S}_E$. Then, the Weitzenböck formulas for D^*D and DD^* have a further contribution coming from the Riemannian curvature, which actually vanishes in the Ricci flat case. In general, the manifold is AC and this algebraic term decays and it can be bounded from above by a function as in equation 5.1.7, so one can assume these Weitzenböck formulae are as in proposition 5.1.6. To establish the inequality, notice that $\|f\|^2_{L^2_{k+1}} \leq \|f\|^2_{L^2_1} + \|\nabla_A f\|^2_{L^2_k}$ and arguing by induction one can assume 5.1.13 to be true for k replaced by $j < k$, hence

$$\|f\|^2_{L^2_{k+1}} \leq \left(\|Df\|^2_{L^2_1} + \|D\nabla_A f\|^2_{L^2_{k-1}} \right) + \left(\|\varepsilon f\|^2_{L^2_1} + \|\varepsilon \nabla_A f\|^2_{L^2_{k-1}} \right). \tag{5.1.14}$$

Notice that $\varepsilon \nabla_A f = \nabla_A(\varepsilon f) - \nabla \varepsilon \otimes f$. Moreover, since ε satisfies equation 5.1.7, there is some other function ε_1 still decaying as in equation 5.1.7 and so that $|\varepsilon| + |\nabla \varepsilon| \leq \varepsilon_1$. So one can bound by above the terms in the second bracket by $\|\varepsilon_1 f\|^2_{L^2_k}$. To bound from above the terms in the first bracket in 5.1.14, let $\{e_i\}$ be an orthonormal frame at $p \in X$ such that $\nabla e_i = 0$ at p. Then at p

$$\begin{aligned} D(\nabla^A_j f) &= \mathcal{D}_A \nabla^A_j f + q(\nabla^A_j f) = \sum_i e_i \nabla^A_i \nabla^A_j f + q(\nabla^A_j f) \\ &= \sum_i \left(\nabla^A_j (e_i \nabla^A_i f) + e_i F_A(e_i, e_j)(f) \right) + \nabla^A_j (q(f)) - (\nabla^A_j q)(f) \\ &= \nabla^A_j (Df) + \sum_i e_i F_A(e_i, e_j)(f) - (\nabla^A_j q)(f). \end{aligned}$$

Recalling the model situation, one has $\nabla^A q = 0$ and F_A bounded above by some ε_2 as in equation 5.1.7. From this it follows immediately that

$$\|D \nabla_A f\|^2_{L^2_{k-1}} \leq c(\|\nabla_A D f\|^2_{L^2_{k-1}} + \|\varepsilon_2 f\|^2_{L^2_{k-1}}),$$

which together with the previous bound $\|\varepsilon f\|^2_{L^2_1} + \|\varepsilon \nabla_A f\|^2_{L^2_{k-1}} \leq \|\varepsilon_1 f\|^2_{L^2_k}$, gives the inequality 5.1.14 for any $\varepsilon' \geq \varepsilon_1 + \varepsilon_2$. $\square$

5.1.4 The general case

This subsection puts together the Banach spaces of the two previous ones in order to measure the components of the splitting $\mathcal{S}_E = \mathcal{S}_E^{\|} \oplus \mathcal{S}_E^{\perp}$ in an appropriate way. For future reference given $s \in \Omega^0(C, \mathcal{S}_E)$, denote the components of s in each of these by $s^{\|}, s^{\perp}$ respectively. This subsection starts by studying the model conical situation on which one constructs model parametrices. The usual strategy of patching parametrices will then be used to deduce the Fredholm property in the AC case; this is stated as theorem 5.1.15, which is the version $p = 2$ of theorem 5.1.1.

The Model Cone

In the model situation, the configuration (A, Φ) is pulled back from the cross section, i.e. from (A_∞, Φ_∞) and recall that in all cases $\nabla_\infty \Phi_\infty = 0$. So, in the cone $C = (1, +\infty) \times \Sigma$ the operator $q = \mathrm{ad}_{\Phi_\infty}$ is constant, i.e. $\nabla_A q(s) = q(\nabla_A s)$ for all $s \in \Omega^0(C, \mathcal{S}_E)$, so q does preserve the splitting $\mathcal{S}_E = \mathcal{S}_E^{\|} \oplus \mathcal{S}_E^{\perp}$. The existence of a model parametrix in this more general situation will follow from patching together parametrices for each component, which exist by sections 5.1.2 and 5.1.3 respectively. To do this, one requires the definition a suitable mixed Banach space of sections of $\mathcal{S}_E$ over the cone.

Definition 5.1.12. *In the setup above define the norm*

$$\|s\|^2_{H_{k,\alpha}} = \|s^{\|}\|^2_{L^2_{k,\alpha}} + \|s^{\perp}\|^2_{L^2_k},$$

for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}^+$. Define the spaces $H_{k,\alpha}$ as the completion of the smooth compactly supported sections in the norm $\| \cdot \|_{H_{k,\alpha}}$.

On each component of the decomposition $\mathcal{S}_E = \mathcal{S}_E^{||} \oplus \mathcal{S}_E^{\perp}$, the operator $D : H_{k+1,\alpha+1} \to H_{k,\alpha}$ restricts as the operator studied in sections 5.1.2 and 5.1.3. Since the direct sum of Fredholm operators is Fredholm, one has

Corollary 5.1.13. *For $\alpha \in \mathbb{R}$ there is a discrete set $\mathcal{K}(D)$, such that if $\alpha \in \mathbb{R} \backslash \mathcal{K}(D)$ the operator $D : H_{1,\alpha+1} \to H_{0,\alpha}$, admits parametrices $P_L, P_R : H_{k,\alpha} \to H_{k+1,\alpha+1}$, such that*

$$DP_R = I + S_R \ , \ P_L D = I + S_L,$$

with $S_R : H_{k+1,\alpha+1} \to H_{k+1,\alpha+1}$ and $S_L : H_{k,\alpha} \to H_{k,\alpha}$ compact operators.

From the Cone to Asymptotically Conical

Let (X, g) be AC and $K \subset X$ such that along the conical end $X \backslash K$ the operator D is modeled by an operator as analysed in the previous subsection 5.1.4. Below the function spaces from definition 5.1.12 will be adapted to the AC setting and then used to prove the main theorem 5.1.1. The strategy is the usual one of matching the model parametrices over $X \backslash K$ obtained in corollary 5.1.13 with the ones for the model constant coefficient operators obtained over sufficiently small interior balls covering the compact piece K.

Definition 5.1.14. *Let ρ be the radius function from definition 5.1.4, $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}^+$. Define*

$$\|s\|^2_{H_{k,\alpha}} = \|s\|^2_{L^2_k(K)} + \|s^{||}\|^2_{L^2_{k,\alpha}(X\backslash K)} + \|s^{\perp}\|^2_{L^2_k(X\backslash K)},$$

and the spaces $H_{k,\alpha}$ as the completion of the smooth compactly supported sections in this norm.

Theorem 5.1.15. *Let D be as above, $k \in \mathbb{N}, \alpha \in \mathbb{R}$. Then, there is a discrete set $\mathcal{K}(D) \subset \mathbb{R}$ such that for $\alpha \notin \mathcal{K}(D)$, the operator $D : H_{k+1,\alpha+1} \to H_{k,\alpha}$, is a Fredholm operator.*

Proof. This follows from a standard procedure, which constructs global parametrices by gluing those obtained for the model operators. This will be illustrated below, in the construction of a global right parametrix Q_R.
Let $U = X \backslash K$ and $K \subset \cup_{i \in I} V_i$, with $|I| < \infty$ form an open cover of X, such that there are local right inverses Q_i to the operator D, defined on some slightly larger open sets U_i containing V_i. Moreover, suppose K is big enough, so that on U, the operator D is modeled on some conical operator D^C as in section 5.1.4. Let $\beta, \{\beta_i\}_{i \in I}$ be a partition of unity subordinate to this cover. First, notice that one can change the operator D over U so that it is exactly conical as D^C. In fact this amounts to subtract to D the operator $K(s) = \beta(Ds - D^C(\beta s)))$, which is a compact operator $K : H_{k+1,\alpha+1} \to H_{k,\alpha}$, and the Fredholm property is not affected by perturbations by compact operators. Then there is a parametrix P_R constructed for D^C in section 5.1.4 and this must be now glued with the local inverses Q_i. Define the candidate for a global parametrix as $Q_R = \sqrt{\beta} P_R \sqrt{\beta} + \sum_{i \in I} \sqrt{\beta_i} Q_i \sqrt{\beta_i}$ and notice that even though the P_R and the Q_i's are not

globally defined the expression above is. To check that Q_R is indeed a parametrix, compute

$$\begin{aligned} DQ_R(s) &= \sigma(d\sqrt{\beta})P_R\sqrt{\beta}s + \sum_{i\in I}\sigma(d\sqrt{\beta_i})Q_i\sqrt{\beta_i}s \\ &\quad +\sqrt{\beta}DP_R\sqrt{\beta}s + \sum_{i\in I}\sqrt{\beta_i}DQ_i\sqrt{\beta_i}s, \end{aligned}$$

where σ denotes the higher order symbol of D. The term in the first line is a compact operator $K' : H_{k,\alpha} \to H_{k,\alpha}$. This follows from the fact that it is supported on a compact set where the derivatives of the β's are non vanishing. Moreover, over this compact set, by elliptic regularity one can control the L^2 norms of the derivatives of $P_R s$ and $Q_i s$ in terms of the L^2 norms of s. For the term in the second line one can use $DP_R = I + S_R$ over U and $DQ_i = I$ over V_i to obtain

$$\begin{aligned} DQ_R(s) &= K'(s) + \sqrt{\beta}(I + S_R)\sqrt{\beta}s + \sum_{i\in I}\beta_i s \\ &= s + K'(s) + \sqrt{\beta}S_R\sqrt{\beta}s. \end{aligned}$$

Moreover since the last term is supported on the conical end where it agrees with S_R, which is a compact operator on these function spaces the operator $K_1 + \sqrt{\beta}S_R\sqrt{\beta}s$ is compact and this proves that Q_R is a right parametrix for D. $\square$

5.1.5 From $p = 2$ to $p > 2$.

The goal of this section is to extend the previous results, i.e. the statement regarding the Fredholmness of the operator D from the case when $p = 2$ to $p > 2$. The upshot is theorem 5.1.18, which contains the main result of the section and was announced in theorem 5.1.1. The relevant function spaces for the general situation are the ones in definition 5.1.14 but constructed with $p > 2$.

Definition 5.1.16. *For $\alpha \in \mathbb{R}$, $k \in \mathbb{N}_1$ and $p \geq 2$ define the spaces $H^p_{k,\alpha}$ to be the completion of the smooth compactly supported sections in the norm $\|\cdot\|_{H^p_{k,\alpha}}$ given by*

$$\|s\|^p_{H^p_{k,\alpha}} = \|s\|^p_{L^p_k(K)} + \|s^{\|}\|^p_{L^p_{k,\alpha}(X\backslash K)} + \|s^{\perp}\|^p_{L^p_k(X\backslash K)},$$

where $K \subset X$ is a large compact set outside of which the splitting $\mathcal{S}_E = \mathcal{S}_E^{\|} \oplus \mathcal{S}_E^{\perp}$ is well defined.

Remark 5.1.17. *Notice that $H^2_{k,\alpha} = H_{k,\alpha}$ in the notation from the previous section. Moreover, recall these $L^p_{k,\alpha}$ spaces are weighted with a distance function ρ as in definition 5.1.4.*

Theorem 5.1.18. *In the conditions of theorem 5.1.15 and $p \geq 2$, there is a discrete set $\mathcal{K}(D) \subset \mathbb{R}$ such that for $\alpha \notin \mathcal{K}(D)$ and $\alpha \geq -n/2$*

$$D : H^p_{k+1,\alpha+1} \to H^p_{k,\alpha},$$

is a Fredholm operator.

To prove this, i.e. that the Fredholm property extends for the operator

$$D : H^p_{k+1,\alpha+1} \to H^p_{k,\alpha}, \tag{5.1.15}$$

it is enough to fix some parametrices P_R, P_L obtained for $p = 2$ and show these extend to bounded operators with S_R, S_L compact operators when regarded as operators on the spaces with $p > 2$.

Proposition 5.1.19. *Let $\alpha \geq -n/2$, $\alpha \notin \mathcal{K}(D)$ and P_R, P_L be the parametrices for D obtained for $p = 2$ by inverting $D|_{(\ker(D)\cap H_{0,\alpha+1})^{\perp_{L^2}}} : (\ker(D) \cap H_{0,\alpha+1})^{\perp_{L^2}} \to (\ker(D^*) \cap H_{0,-n-\alpha})^{\perp_{L^2}}$. These extend to bounded operators*

$$P_R, P_L : H^p_{0,\alpha} \to H^p_{1,\alpha+1},$$

such that $DP_R = I + S_R$ and $P_L D = I + S_L$ with $S_R : H^p_{0,\alpha} \to H^p_{0,\alpha}$ and $S_L : H^p_{1,\alpha+1} \to H^p_{1,\alpha+1}$ compact operators.

Proof. Notice that the operator $D|_{(\ker(D)\cap H_{0,\alpha+1})^{\perp_{L^2}}} : (\ker(D) \cap H_{0,\alpha+1})^{\perp_{L^2}} \to (\ker(D^*) \cap H_{0,-n-\alpha})^{\perp_{L^2}}$ is well defined as long as $D(H_{1,-n-\alpha-1}) \subset H_{0,\alpha}$, which is true for $\alpha \geq -n/2 - 1$ and this is guaranteed by the hypothesis that $\alpha \geq -n/2$. Start by proving the last assertion, namely that the extensions of S_R, S_L are compact. The parametrix P_L in the statement is obtained by constructing a left inverse to $D|_{(\ker(D)\cap H_{0,\alpha+1})^{\perp_{L^2}}}$, then S_L is minus the projection onto $\ker(D) \cap H_{0,\alpha+1}$, which is finite dimensional as D is Fredholm for $p = 2$ due to theorem 5.1.15. In the same way, P_R is obtained by constructing a right inverse to D as an operator onto $(\ker(D^*) \cap H_{0,-n-\alpha})^{\perp_{L^2}}$ and so S_R is minus the projection onto $\ker(D^*) \cap H_{0,-n-\alpha}$, which is the cokernel in the case $\alpha \geq -n/2$ and so finite dimensional as D is Fredholm for $p = 2$.

Next, one turns to the proof that the parametrices P_R, P_L do extend to bounded operators from $H^p_{0,\alpha}$ to $H^p_{0,\alpha+1}$. The two important models to have in attention in order to set this up are

1. There is a big compact set $\rho^{-1}[0, R] \subset X$, over which the spaces $H^p_{k,\alpha}$ can be taken to agree with the usual L^p_k ones. Equip $\rho^{-1}[0, R]$ with a finite open cover $\{V_i\}_{i\in I}$, where the standard Calderon-Zygmund inequalities hold. These are

$$\begin{aligned} \|\nabla_A g\|^p_{L^p(V_i)} &\leq C(\|Dg\|^p_{L^p(V_i')} + \|g\|^p_{L^p(V_i')}) \\ \| g\|^p_{L^p(V_i')} &\leq C(\|Dg\|^p_{L^p(V_i'')} + \|g\|^p_{L^2(V_i'')}), \end{aligned}$$

where $V_i' \supset V_i$ and $V_i'' \supset V_i'$ are slightly larger open sets and $C > 0$ is a generic constant, to be possibly actualized at each stage. The reason why we chose to arrange them in this way is that these can now be combined into

$$\|g\|^p_{L^p_1(V_i)} \leq C(\|Dg\|^p_{L^p(V_i'')} + \|g\|^p_{L^2(V_i'')}).$$

Then, by inserting $g = P_R f$ into the inequality above and using that $DP_R = I + S_R$, gives

$$\begin{aligned}\|P_R f\|^p_{L^p(V_i)} &\leq C\left(\|DP_R f\|^p_{L^p(V_i'')} + \|P_R f\|^p_{L^2(V_i'')}\right)\\ &\leq C\left(\|f\|^p_{L^p(V_i'')} + \|S_R f\|^p_{L^p(V_i'')} + \|P_R f\|^p_{L^2(V_i'')}\right).\end{aligned}$$

Then the fact that P_R is bounded for $p = 2$ and S_R is compact and hence bounded for $p \geq 2$ combine to further give $\|P_R f\|^p_{L^p(V_i)} \leq C\|f\|^p_{L^p(V_i'')}$.

2. On the noncompact end $\rho^{-1}(R, +\infty)$, D is modeled on a conical operator D_C as in section 5.1.4. The rest of the proof requires lemmas 5.1.20 and 5.1.24 below. For now assume these hold, then from lemma 5.1.24 one can use the alternative $H^p_{1,\alpha+1}$ norm

$$\|g\|^p_{H^p_{1,\alpha+1}} = \|Dg\|^p_{H^p_{0,\alpha}} + \|g\|^p_{H^p_{0,\alpha+1}}.$$

Insert into this $g = P_R f$ with $f \in H^p_{0,\alpha}$ and use $DP_R = I + S_R$, gives

$$\|P_R f\|^p_{H^p_{1,\alpha+1}} = \|f + S_R f\|^p_{H^p_{0,\alpha}} + \|P_R f\|^p_{H^p_{0,\alpha+1}}.$$

By using the generalized Young inequality and the fact that $S_R : H^p_{0,\alpha} \to H^p_{0,\alpha}$ is compact , the first term can be bounded above by $c\|f\|^p_{H^p_{0,\alpha}}$, for some $c > 0$. As for the second term, it is guaranteed by lemma 5.1.24 that it is no greater than $c\|f\|^p_{H^p_{0,\alpha}}$, for some other constant $c > 0$. This shows that the model parametrix $P_R : H^p_{0,\alpha} \to H^p_{1,\alpha+1}$ is bounded.

Then by combining the two pieces above finishes the proof of proposition 5.1.19. □

The rest of this section focuses on proving lemmas 5.1.20 and 5.1.24.

Lemma 5.1.20. *The norm $H^p_{k+1,\alpha+1}$ is equivalent to the norm $\|\cdot\|$ defined by*

$$\|f\|^p = \|Df\|^p_{H^p_{k,\alpha}} + \|f\|^p_{H^p_{0,\alpha+1}}.$$

Proof. The result follows from induction and the general step is not more difficult than the case $k = 1$. In this case it is enough to show that $\|f\|^p$ can be bounded from above and below by $\|f\|^p_{H^p_{1,\alpha+1}}$.

1. To prove the upper bound, use $Df = \mathcal{D}_A f + q(f)$ and the generalized version of Young's inequality

$$\|f\|^p \leq c_1(\|\mathcal{D}_A f\|^p_{H^p_{0,\alpha}} + \|q(f)\|^p_{H^p_{0,\alpha}}) + \|f\|^p_{H^p_{0,\alpha+1}}. \tag{5.1.16}$$

Using $|\mathcal{D}_A(f)| \leq c_2|\nabla_A f|$ one can bound the first term above by $c_2^p c_1\|\nabla_A f\|^p_{H^p_{0,\alpha}}$. For the second term, use that $|q(f)| \leq c_3|f^\perp|$ and $\|f^\perp\|^p_{H^p_{0,\alpha}} = \|f^\perp\|^p_{H^p_{0,\alpha+1}}$, i.e. the weights do not affect the $f^\perp$ component. These two facts combine to bound the second term as $\|q(f)\|^p_{H^p_{0,\alpha}} \leq c_3^p\|f^\perp\|^p_{H^p_{0,\alpha+1}}$, which can be further bounded by $c_3^p\|f\|^p_{H^p_{0,\alpha+1}}$. Inserting these bounds back into 5.1.16 gives

$$\|f\|^p \leq C(\|\nabla_A f\|^p_{H^p_{0,\alpha}} + \|f\|^p_{H^p_{0,\alpha+1}}) = C\|f\|^p_{H^p_{1,\alpha+1}},$$

where $C = \max\{c_2^p c_1, 1 + c_1 c_3^p\} > 0$.
2. To prove the lower bound on $\|f\|^p$, one needs to establish an inequality as

$$\|f\|^p_{H^p_{1,\alpha+1}} \leq C'(\|Df\|^p_{H^p_{0,\alpha}} + \|f\|^p_{H^p_{0,\alpha+1}}), \tag{5.1.17}$$

for some $C' > 0$. To do this, it is convenient to split the proof into cases, i.e. to prove the result independently for the $\mathcal{S}_E^{\|}$ and $\mathcal{S}_E^{\perp}$ components.
For $f \in \mathcal{S}_E^{\|}$, $Df = \mathcal{D}_A f$ and $\|f\|_{H^p_{k,\alpha}} = \|f\|_{L^p_{k,\alpha}}$, i.e. the $H^p_{k,\alpha}$ norm is the Lockhart-McOwen one from definition 5.1.4. As $\mathcal{D}_A$ is an elliptic asymptotically conical operator, there is an inequality

$$\|f\|^p_{L^p_{1,\alpha+1}} \leq c_5(\|\mathcal{D}_A f\|^p_{L^p_{0,\alpha}} + \|f\|^p_{L^p_{0,\alpha+1}}), \tag{5.1.18}$$

which follows immediately from a change of coordinates into Lockhart and McOwen's asymptotically cylindrical setting in [LM85].
For $f \in \mathcal{S}_E^{\perp}$, $\|f\|_{H^p_{k,\alpha}} = \|f\|_{L^p_k}$, i.e. the $H^p_{k,\alpha}$ norm agrees with the usual L^p_k one. To bound $\|\nabla_A f\|^p_{L^p}$ by above one can use the fact that $L^p = L^p_{0,-n/p}$ to rewrite $\|\nabla_A f\|^p_{L^p} \leq \|\nabla_A f\|^p_{L^p} + \|r^{-1} f\|^p_{L^p} = \|f\|^p_{L^p_{1,-n/p+1}}$. Then, using the weighted inequality in equation 5.1.18, for the case $\alpha = -n/p$, gives

$$\begin{aligned} \|\nabla_A f\|^p_{L^p} &\leq c_5\left(\|\mathcal{D}_A f\|^p_{L^p} + \|r^{-1} f\|^p_{L^p}\right) \leq c_5 c_6\left(\|Df\|^p_{L^p} + \|f\|^p_{L^p} + \|r^{-1} f\|^p_{L^p}\right) \\ &\leq 2c_5 c_6\left(\|Df\|^p_{L^p} + \|f\|^p_{L^p}\right), \end{aligned}$$

where in the second inequality in the first line one uses $\mathcal{D}_A f = Df - q(f)$ and the fact that q is bounded. The inequality 5.1.17 is now immediate from summing these two components and choosing C' as the biggest constant. ∎

It will be useful in the analysis to be carried out to introduce a mixed norm

Definition 5.1.21. *Define the intermediate norm $\|\cdot\|_{H^{(p,2)}_{0,\alpha}}$ by*

$$\|f\|^p_{H^{(p,2)}_{0,\alpha}} = \int_1^{+\infty} \left(r^{-\alpha p - n}\|f^{\|}\|^p_{L^2(\rho^{-1}(r))} + \|f^{\perp}\|^p_{L^2(\rho^{-1}(r))}\right) r^{-(n-1)\frac{p-2}{2}} dr,$$

where the L^2 norms on the right hand side are with respect to the induced metric on $\rho^{-1}(r) \cong \Sigma$.

Lemma 5.1.22. *Let $p \geq 2$ and $\alpha \in \mathbb{R}$, then there is a constant $c > 0$, such that for $f \in H^p_{0,\alpha}$,*

$$\|f\|_{H^{(p,2)}_{0,\alpha}} \leq c\|f\|_{H^p_{0,\alpha}}.$$

Proof. The proof follows from the observation that for $p \geq 2$ and over compact sets, the L^p norm is stronger than the L^2 norm. In fact over a radius 1 ball $B_1 \subset \mathbb{R}^k$ there is a constant $c' > 0$ such that $\|f\|_{L^2(B_1)} \leq c'\|f\|_{L^p(B_1)}$, then by scaling $\|f\|_{L^2(B_r)} \leq c' r^{k\frac{p-2}{2p}}\|f\|_{L^p(B_r)}$ for all $r \in \mathbb{R}$. Applying this scaling behavior of the L^p norms, there is $c > 0$ such that $\|f\|^p_{L^2(\rho^{-1}(r))} \leq c^p r^{(n-1)\frac{p-2}{2}}\|f\|^p_{L^p(\rho^{-1}(r))}$. Inserting this into the definition of the $H^{(p,2)}_{0,\alpha}$ norm above gives an upper bound with respect to the $H^p_{0,\alpha}$ norm. $\square$

Lemma 5.1.23. *Let $p \geq 2$ and $\alpha \in \mathbb{R}$, there is a constant $c' > 0$, such that for all $f \in H^p_{0,\alpha}$ one has $\|P_R f\|_{H^{(p,2)}_{0,\alpha+1}} \leq c' \|f\|_{H^{(p,2)}_{0,\alpha}}$. Moreover, combining this with lemma 5.1.22 and possibly changing the constant c' gives*

$$\|P_R f\|_{H^{(p,2)}_{0,\alpha+1}} \leq c' \|f\|_{H^p_{0,\alpha}}.$$

Proof. It is enough to prove the first inequality since as asserted in the statement the second one follows from combining the first one with lemma 5.1.22. Recall that P_R is a bounded operator for $p = 2$, i.e. $H_{0,\alpha}$ to $H_{1,\alpha+1}$. To proceed with the proof it is convenient to split the problem between the $\mathcal{S}^{\|}_E$ and $\mathcal{S}^{\perp}_E$ components.

1. For $f \in \mathcal{S}^{\|}_E$, the $H^p_{k,\alpha}$ norm is the standard Lockhart-McOwen one $L^p_{k,\alpha}$. Then, by changing coordinates to $t = \log(r)$, the statement that P_R is bounded from $H_{0,\alpha} = L^2_{0,\alpha}$ into $H_{0,\alpha+1} = L^2_{0,\alpha+1}$ gives

$$\int_{\log(R)}^{+\infty} \|e^{-t} P_R f\|^2_{L^2(\Sigma, g_\Sigma)} e^{-2\alpha t} dt \leq C \int_{\log(R)}^{+\infty} \|f\|^2_{L^2(\Sigma, g_\Sigma)} e^{-2\alpha t} dt,$$

for some $C > 0$ and where $L^2(\Sigma, g_\Sigma)$ denotes the L^2 norm on the cross section Σ with respect to the fixed metric g_Σ. Equivalently, this statement can be formulated as saying that for all $T > \log(R)$, the assignment $e^{-\alpha t} f \mapsto e^{-(\alpha+1)T}(P_R f)(T)$ gives rise to a bounded map

$$M_\alpha(T) : L^2((\log(R), +\infty), L^2(\Sigma, g_\Sigma)) \to L^2(\Sigma, g_\Sigma),$$

and the operator norm of this family is integrable, with integral no greater than C. Still in the cylindrical setting, the fact that $f \in H^{(p,2)}_{0,\alpha}$ means that $e^{-\alpha t - (n-1)\frac{p-2}{2p} t} f(t) \in L^p((\log(R), \infty), L^2(\Sigma, g_\Sigma))$. Hence, one can use the fact that the family $M_\alpha(\cdot)$ has integrable operator norm and the map $L^1 \times L^p \hookrightarrow L^p$ along $(\log(R), +\infty) \times L^2(\Sigma, g_\Sigma)$ to prove that

$$\|e^{-(n-1)\frac{p-2}{2p} T} (M_\alpha e^{-\alpha t} f(t))(T)\|^p_{L^p} \leq \|M_\alpha(T)\|^p_{L^1} \|e^{-(n-1)\frac{p-2}{2p} t} e^{-\alpha t} f(t)\|^p_{L^p}.$$

Since $\|M_\alpha(T)\|_{L^1} < C < \infty$, changing coordinates back to the asymptotically conical setting this statement is equivalent to

$$\|P_R f\|^2_{H^{(p,2)}_{0,\alpha+1}} \leq C \|f\|^2_{H^{(p,2)}_{0,\alpha}}$$

and proves that $P_R : H^{(p,2)}_{0,\alpha} \to H^{(p,2)}_{0,\alpha+1}$ is bounded for those components in $\mathcal{S}^{\|}_E$.

2. For $f \in \mathcal{S}^{\perp}_E$, the $H^p_{k,\alpha}$ norm is the standard L^p_k one. The statement that P_R is bounded from and into L^2 can equivalently be stated in the cylindrical setting, as follows. Using the measure

$e^{nt}dt$ on $(\log(R), +\infty)$, and all $T > \log(R)$, the assignment $f \mapsto (P_R f)(T)$ gives a bounded map

$$P_R(T) : L^2((\log(R), +\infty), L^2(\Sigma, g_\Sigma)) \to L^2(\Sigma, g_\Sigma),$$

and this T-parametrized family has integrable operator norm. Then, given $f \in H^{(p,2)}_{0,\alpha}$, in the cylindrical setting this means that $e^{-(n-1)\frac{p-2}{2p}t} f(t)$ is in $L^p((\log(R), \infty), L^2(\Sigma, g_\Sigma))$, using the measure $e^{nt}dt$ on $(\log(R), +\infty)$. Proceeding as before and combining the map $L^1 \times L^p \hookrightarrow L^p$ with the fact that the family $P_R(T)$ has integrable operator norm gives

$$\|e^{-(n-1)\frac{p-2}{2p}T}(P_R f(t))(T)\|^p_{L^p} \leq \|P_R(T)\|^p_{L^1} \|e^{-(n-1)\frac{p-2}{2p}t} e^{-\alpha t} f(t)\|^p_{L^p},$$

with $\|P_R(T)\|^p_{L^1} = C' < +\infty$. Back to the conical world this statement gets translated into

$$\|P_R f\|^2_{H^{(p,2)}_{0,\alpha+1}} \leq C' \|f\|^2_{H^{(p,2)}_{0,\alpha}},$$

proving the statement for those components in $\mathcal{S}_E^\perp$. Then by putting together both cases 1. and 2. proves the complete statement. $\square$

Lemma 5.1.24. *There is a constant $c' > 0$, such that for all $f \in H^p_{0,\alpha}$*

$$\|P_R f\|_{H^p_{0,\alpha+1}} \leq c' \|f\|_{H^p_{0,\alpha}}.$$

Proof. Recall the $H^p_{0,\alpha}$ norm in definition 5.1.16, in what follows it will be useful to rewrite it as a sum

$$\begin{aligned} \|g\|^p_{H^p_{0,\alpha}(U)} &= \int_1^{+\infty} \left(r^{-\alpha p - n} \|g^{\|}\|^p_{L^p(\rho^{-1}(r))} + \|g^\perp\|^p_{L^p(\rho^{-1}(r))} \right) dr \\ &\cong \sum_{k\geq 0} \left(R^{-k(\alpha p + n)} \|g^{\|}\|^p_{L^p(C_k)} + \|g^\perp\|^p_{L^p(C_k)} \right), \end{aligned} \tag{5.1.19}$$

where $\cong$ above denotes an equivalence of norms (which is straightforward to check) and $C_k = (R^k, R^{k+1}) \times \Sigma$ equipped with the conical metric $g_C = dr^2 + r^2 g_\Sigma = r^2(\frac{dr^2}{r^2} + g_\Sigma)$. Notice that the conical annulus C_{k+1} is obtained from C_k by scaling with a factor of $R > 1$. As usual, in what follows it will be convenient to separate into components.

1. First, one focuses on the components in $\mathcal{S}_E^{\|}$. Over the bounded annulus C_1, the standard Calderon-Zygmund inequalities give $\|g\|^p_{L^p(C_1)} \leq c(\|Dg\|^p_{L^p(C_1')} + \|g\|^p_{L^2(C_1')})$, where $C_1' \supset C_1$ is a slightly larger annulus in the cone. This inequality is not scale invariant and scaling it gives

$$\|g\|^p_{L^p(C_k)} \leq c \left(R^{kp} \|Dg\|^p_{L^p(C_k')} + R^{-nk\frac{p-2}{2}} \|g\|^p_{L^2(C_k')} \right),$$

and in this component $D = \mathcal{D}_A$. Moreover, since $p > 2$, $R^{-nk\frac{p-2}{2}} \leq R^{-(n-1)k\frac{p-2}{2}}$ and $\|g\|^p_{L^2(C_k)} \leq c \int_{R^k}^{R^{k+1}} \|g\|^p_{L^2(\rho^{-1}(r))} dr$. Then by inserting these into the norm 5.1.19, gives for

$g \in \mathcal{S}_E^{\|}$

$$\begin{aligned}\|g\|^p_{H^p_{0,\alpha+1}} &\leq c_1 \sum_{k\geq 0} R^{-k((\alpha+1)p+n)} R^{pk} \|Dg^{\|}\|^p_{L^p(C_k)} \\ &\quad + c_2 \int_R^{+\infty} r^{-(\alpha+1)p-n} \|g\|^p_{L^2(\rho^{-1}(r))} r^{-(n-1)\frac{p-2}{2}} dr \\ &\leq C(\|Dg\|^p_{H^p_{0,\alpha}} + \|g\|^p_{H^{(p,2)}_{0,\alpha+1}}).\end{aligned}$$

Insert into this inequality $g = P_R f$, then by using $DP_R = I + S_R$, the fact that S_R is bounded from and into $H^p_{0,\alpha}$ and lemma 5.1.23, give

$$\begin{aligned}\|P_R f\|^p_{H^p_{0,\alpha+1}} &\leq c\left(\|f + S_R f\|^p_{H^p_{0,\alpha+1}} + \|P_R f\|^p_{H^{(p,2)}_{0,\alpha+1}}\right) \\ &\leq C\|f\|^p_{H^p_{0,\alpha+1}}.\end{aligned}$$

2. Next, one turns to those components in $\mathcal{S}_E^{\perp}$, recall that for these the map $q \in \Omega^0(\mathrm{End}(\mathcal{S}_E))$ is bounded below, i.e. $|q(g)| \geq c|g|$, for some $c > 0$ and all $f \in \mathcal{S}_E^{\perp}$. Then over any C_k, the inequality

$$\|g\|^p_{L^p(C_k)} \leq \|q(g)\|^p_{L^p(C_k)} \leq \|Dg\|^p_{L^p(C_k)} + \|\mathcal{D}_A g\|^p_{L^p(C_k)}. \tag{5.1.20}$$

Moreover, rescaling the fact that $\mathcal{D}_A : L^p_1(C_1) \to L^p(C_1)$ is bounded and the standard Calderon-Zygmund inequality gives $\|\mathcal{D}_A g\|^p_{L^p(C_k)} \leq c_1(\|\nabla_A g\|^p_{L^p(C'_k)} + R^{-pk}\|g\|^p_{L^p(C'_k)})$ and $\|\nabla_A g\|^p_{L^p(C'_k)} \leq c_1(\|Dg\|^p_{L^p(C''_k)} + R^{-nk\frac{p-2}{2}}\|g\|^p_{L^2(C''_k)})$, where $C'_k \supset C_k$ and $C''_k \supset C_k$ to denote slightly larger annulus. Then by combining these gives

$$\|\mathcal{D}_A g\|^p_{L^p(C_k)} \leq C(\|Dg\|^p_{L^p(C''_k)} + R^{-nk\frac{p-2}{2}}\|g\|^p_{L^2(C''_k)} + R^{-pk}\|g\|^p_{L^p(C'_k)}),$$

and inserting this back into equation 5.1.20 gives for $R \gg 1$

$$\|g\|^p_{L^p(C_k)} \leq C(\|Dg\|^p_{L^p(C''_k)} + R^{-nk\frac{p-2}{2}}\|g\|^p_{L^2(C_k)}). \tag{5.1.21}$$

Moreover since $p > 2$ also in this case $R^{-nk\frac{p-2}{2}} \leq R^{-(n-1)k\frac{p-2}{2}}$ and one can dominate the second term in the right above by $c\int_{R^k}^{R^{k+1}} \|g\|^p_{L^2(\rho^{-1}(r))} r^{-(n-1)k\frac{p-2}{2}} dr$, which is for components in $\mathcal{S}_E^{\perp}$ the $H^{(p,2)}_{0,\alpha+1}$ norm. Then, inserting equation 5.1.21 into the norm in equation 5.1.19 for $g \in \mathcal{S}_E^{\perp}$ gives

$$\begin{aligned}\|g\|^p_{H^p_{0,\alpha+1}} &\leq C\sum_{k\geq 0} \|Dg\|^p_{L^p(C''_k)} + C\int_R^{+\infty} \|g\|^p_{L^2(\rho^{-1}(r))} r^{-(n-1)k\frac{p-2}{2}} dr \\ &\leq C(\|Dg\|^p_{H^p_{0,\alpha}} + \|g\|^p_{H^{(p,2)}_{0,\alpha+1}}),\end{aligned}$$

and notice that the weights α here are irrelevant but are introduced in order to use the appropriate notation. Then, following a similar strategy as in the previous case let $g = P_R f$ in the inequality above. Then using $DP_R = I + S_R$, that S_R is bounded on $H^p_{0,\alpha}$ and lemma 5.1.23 gives

$\|P_R f\|^p_{H^p_{0,\alpha+1}} \leq \|f\|^p_{H^p_{0,\alpha}}$. The general result follows immediately from combining $\square$

Remark 5.1.25. *Recall that restricted to the components in $\mathcal{S}_E^{\|}$, $D = \mathcal{D}_A$ is the Dirac operator and $H^p_{k,\alpha} = L^p_{k,\alpha}$ are the Lockhart-McOwen spaces. The results obtained in this section, when restricted to these components also follow from standard Lockhart-McOwen theory and this could have been used instead. In fact, this section relies partially on these results, when in the proof of lemma 5.1.20 the inequality in equation 5.1.18 is used. However, such an inequality follows from scaling the standard Calderon-Zygmund one $\|\nabla_A g\|^p_{L^p(C_1)} \leq C(\|Dg\|^p_{L^p(C_1')} + \|g\|^p_{L^p(C_1')})$, from the annulus C_1 to all the annuli C_k, in a similar fashion to what was done in the proof of lemma 5.1.24.*

5.2 The Moduli Theory

This section studies the properties of the moduli spaces of finite mass, irreducible monopoles (resp. complex monopoles) on an asymptotically conical manifold (X, g), which is either a 3 dimensional manifold or a G_2 manifold (resp. a Calabi-Yau 3 fold). The main result is theorem 5.2.3 which shows the setup from the previous section extends to the nonlinear case. Namely that the (complex) monopole equations give rise to a Fredholm map between the Banach spaces in definition 5.1.16 from the previous section.

5.2.1 Moduli of Finite Mass (complex) Monopoles

Recall the boundary conditions for a finite mass monopole; let $P_\infty \to \Sigma$ be the asymptotic bundle and fix a framing

$$\eta : \varphi^* P|_{X \setminus K} \to \pi^* P_\infty, \tag{5.2.1}$$

together with a pair $(\nabla_\infty, \Phi_\infty)$ as in definitions 1.4.1 and 3.1.19 for monopoles and complex monopoles respectively. Here φ is diffeomorphism from definition 1.2.8 and $\pi : C \to \Sigma$ denotes the projection to the second factor. Moreover, also recall that $\nabla_\infty \Phi_\infty = 0$ and ∇_∞ satisfies the conditions summarized in definition 1.4.7 and the examples following it (or proposition 3.1.28 in the case of complex monopoles). Denote by $[(\nabla_\infty, \Phi_\infty)]$ the gauge equivalence class of this pair and define

$$\begin{aligned} \Gamma_\infty &= \{ g \in \mathrm{Aut}(P_\infty) \mid g \cdot (\nabla_\infty, \Phi_\infty) = (\nabla_\infty, \Phi_\infty) \}, \\ \gamma_\infty &= \{ \xi \in \Gamma(\mathfrak{g}_{P_\infty}) \mid \nabla_\infty \xi = [\xi, \Phi_\infty] = 0 \}. \end{aligned}$$

Then Γ_∞ are the gauge transformations of P_∞ which preserve the boundary data and γ_∞ its Lie algebra. There are two possible approaches to setting up the moduli theory:

1. Consider pairs (A, Φ) on P such that there are representatives $(\nabla'_\infty, \Phi'_\infty) \in [(\nabla_\infty, \Phi_\infty)]$, with (A, Φ) asymptotic to $(\nabla'_\infty, \Phi'_\infty)$. Take these modulo the action of the gauge group $\mathcal{G}$ of continuous gauge transformations, which have a limit $g_\infty = \lim_{\rho \to \infty} g(\rho) \in \mathcal{G}_\infty$.

2. Fix the representative $(\nabla_\infty, \Phi_\infty) \in [(\nabla_\infty, \Phi_\infty)]$ and consider pairs (A, Φ) asymptotic to this representative modulo the action of $\Gamma \subset \mathcal{G}$. Where Γ is defined to be those gauge transformations $g \in \mathcal{G}$ such that $g_\infty \in \Gamma_\infty$ and so preserves the asymptotic conditions.

The automorphism group of the boundary data Γ_∞ is isomorphic to a subgroup $H \subset G$. An explicit subgroup can be taken by fixing a point $p \in P_\infty$ and setting $H = C(\Phi_\infty(p))$, i.e. the centralizer. It is also usefull to consider a slightly larger moduli space which fibers over these ones with fibre Γ_∞. Recall that the gauge group $\mathcal{G}$ comes equipped with an evaluation map $ev : \mathcal{G} \to \mathcal{G}_\infty$ by taking the limit at infinity. Using the framing 5.2.1, $\Gamma = ev^{-1}(\Gamma_\infty)$ and one can define $\mathcal{G}(0) = \ker(ev)$. Then, consider the moduli space of configurations to be those pairs (A, Φ) which are asymptotic to $(\nabla_\infty, \Phi_\infty)$ modulo the action of $\mathcal{G}(0)$. Any implementation of this idea gives a moduli space of configurations, which fibers over the previous ones with fiber H.

Remark 5.2.1. *There is also one other way of constructing such a moduli space which comes with the framing η incorporated in the definition at the expense of considering a slightly larger gauge group. Consider triples (A, Φ, η) of configurations and a framing η modulo the action of Γ. Here Γ acts on the framing in a nontrivial way and this is what accounts for increasing the gauge group from $\mathcal{G}(0)$ to Γ.*

Example 13. *Let $G = SU(2)$, then P_∞ is reducible and since $\nabla_\infty \Phi_\infty = 0$ so is the connection. Then H is either $\{1\}$ or $U(1)$ standard facts of representation theory give a splitting $\mathfrak{g}_{P_\infty} \cong \underline{\mathbb{R}} \oplus L^2$, where L is a line bundle over Σ. Moreover, if $H = U(1)$, then L must be nontrivial. In fact one must suppose that is the case, otherwise assuming $\nabla_A \Phi \in L^2$ would give via corollary 1.4.11 that $\nabla_A \Phi = 0$ and the (complex) monopole would be reducible. Then, Γ_∞ is the subgroup $\mathcal{G}_\infty$ consisting on automorphisms of P_∞ preserving Φ_∞ and the connection ∇_∞*

- *If $g \in \mathrm{Aut}(P_\infty)$ and $g \cdot \Phi_\infty = \Phi_\infty$, then one can write $g = e^{if\Phi_\infty}$, for some $f \in C^\infty(\Sigma, \mathbb{R}/\mathbb{Z})$. Moreover, if g is further supposed to preserve the connection then it must be constant, this gives an isomorphism $\Gamma_\infty \cong \mathbb{S}^1$.*
- *If $\xi \in \mathfrak{g}_{P_\infty}$ and $[\xi, \Phi_\infty] = 0$, then $\xi = f\Phi_\infty$ for $f \in C^\infty(\Sigma, \mathbb{R})$ and if $\nabla_\infty \xi = 0$ then f must be constant. This gives an isomorphism $\gamma_\infty \cong \mathbb{R}$.*

Let (∇_0, Φ_0) be a connection and an Higgs Field on P which as $\rho \to \infty$ converge to the pullbacks of $(\nabla_\infty, \Phi_\infty)$ via the framing η fixed before in 5.2.1. Then, on $X \backslash K$ the adjoint action of Φ_0 gives an endomorphism $\mathrm{ad}_{\Phi_0} = [\Phi_0, \cdot] \in \mathrm{End}(\mathfrak{g}_P|_{X\backslash K})$ and this defines a splitting

$$\mathfrak{g}_P|_{X\backslash K} \cong V^{\parallel} \oplus V^{\perp}, \tag{5.2.2}$$

where $V^\perp = \mathrm{im}(\mathrm{ad}_{\Phi_0})$ and $V^\parallel = \ker(\mathrm{ad}_{\Phi_0})$. So one can uniquely split sections $\eta \in \Omega^k(X\backslash K, \mathfrak{g}_P)$ as $\eta = \eta^\parallel + \eta^\perp$, for $\eta^\parallel \in \Omega^k(X\backslash K, V^\parallel)$ and $\eta^\perp \in \Omega^k(X\backslash K, V^\perp)$.

Remark 5.2.2. *Recall that the boundary data determine a reduction of P_∞ and ∇_∞ to an $H \subset G$ bundle $Q \to \Sigma$ equipped with an H connection which will also be denoted ∇_∞. Then $P_\infty =$*

$Q \times_H G$ and $\mathfrak{g}_{P_\infty} = P_\infty \times_{G,Ad} \mathfrak{g} = Q \times_{H,Ad} \mathfrak{g}$. Split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H which acts via the adjoint action on the complement $\mathfrak{m}$. Then, write

$$\mathfrak{g}_{P_\infty} = Q \times_{H,Ad} \mathfrak{h} \oplus Q \times_{H,Ad} \mathfrak{m}$$

and these are respectively $V^{\|}, V^{\perp}$ in the splitting 5.2.2. Digging a bit further one can let Φ_∞ be an H equivariant map from Q to $\mathfrak{g}$, which constant along ∇_∞ parallel path. Extend it G equivariantly to P_∞, let $p \in P_\infty$ and define $H = C(\Phi_\infty(p))$, i.e. the centralizer of $\Phi_\infty(p) = m \in \mathfrak{g}$. One can choose a set of positive roots R_+ and a fundamental Weyl chamber so that $\Phi_\infty(p)$ lies in its closure. Introduce the notation $\mathfrak{g}_\alpha^{\mathbb{R}} = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}$, then

$$\mathfrak{g} = (\mathfrak{t} \oplus \bigoplus_{\alpha(m)=0} \mathfrak{g}_\alpha^{\mathbb{R}}) \oplus \bigoplus_{\alpha(m)\neq 0} \mathfrak{g}_\alpha^{\mathbb{R}},$$

with $\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha(m)=0} \mathfrak{g}_\alpha^{\mathbb{R}}$ and $\mathfrak{m} = \bigoplus_{\alpha(m)\neq 0} \mathfrak{g}_\alpha^{\mathbb{R}}$.

The rest of this chapter develops a moduli theory for the monopole (resp. complex monopole) equation. The general setup will be a familiar one in gauge theory, but there are many technicalities involved. However, at this stage it is already possible to state a result which will be one of the main ingredients of that larger moduli theory. In this result, the linear theory from the previous section 5.1 is shown to generalize to the nonlinear (complex) monopole equations. Suppose (A_0, Φ_0) is a (complex) monopole, i.e. a solution to equations 2.1.1 or 4.1.1 for the 3 dimensional case or the G_2 case respectively and a solution of equations 3.1.1 and 3.1.2 in the Calabi-Yau case. Let $\Lambda_{\mathfrak{g}}^* = \Lambda^* \otimes \mathfrak{g}_P$ and $\tilde{\Lambda}_{\mathfrak{g}} = \Lambda^1 \otimes \mathfrak{g}_P$ in 3 dimensions or G_2 manifolds, while for Calabi-Yau manifolds these denote $\Lambda_{\mathfrak{g}}^* = \Lambda^* \otimes \mathfrak{g}_P^{\mathbb{C}}$ and $\tilde{\Lambda}_{\mathfrak{g}} = (\Lambda^0 \otimes i\mathfrak{g}_P) \oplus (\Lambda^1 \otimes \mathfrak{g}_P^{\mathbb{C}})$. Then, in each of these cases the (complex) monopole equation for the pair $(A, \Phi) = (A_0, \Phi_0) + (a, \phi)$ with $(a, \phi) \in \Omega^0(X, \Lambda_{\mathfrak{g}}^0 \oplus \Lambda_{\mathfrak{g}}^1)$ defines a map

$$mon : \Omega^0(X, \Lambda_{\mathfrak{g}}^0 \oplus \Lambda_{\mathfrak{g}}^1) \to \Omega^0(X, \Lambda_{\mathfrak{g}}^*). \tag{5.2.3}$$

Moreover, it is straightforward to see that this map can be written as

$$mon(a, \phi) = d_2(a, \phi) + q((a, \phi), (a, \phi)),$$

where $q(\cdot, \cdot)$ is multilinear, so the overall equation has quadratic nonlinearities. Moreover, d_2 above denotes the linearized (monopole) equation as computed in sections 2.1.1, 3.1.2 and 4.1.1 for each case. To each of these equations one can add the gauge fixing condition $d_1^*(a, \phi) = 0$, where $d_1^* : \Omega^0(X, \Lambda_{\mathfrak{g}}^0 \oplus \Lambda_{\mathfrak{g}}^1) \to \Omega^0(X, \Lambda_{\mathfrak{g}}^0)$ is also computed in sections 2.1.1, 3.1.2 and 4.1.1 respectively in 3 dimensions, Calabi-Yau and G_2 manifolds. These two can be combined in the gauge fixed monopole equation for $(A, \Phi) = (A_0, \Phi_0) + (a, \phi)$

$$Mon(a, \phi) = mon(a, \phi) + d_1^*(a, \phi), \tag{5.2.4}$$

and maps $\Omega^0(X, \Lambda^0_{\mathfrak{g}} \oplus \Lambda^1_{\mathfrak{g}})$ to itself. The following result can be stated in more generality, but for concreteness we shall restrict to $G = SU(2)$.

Theorem 5.2.3. *Let $G = SU(2)$, $p \in [n/2, n)$ and $\alpha = -n/p + 1 \notin \mathcal{K}(D)$, then the map Mon defined in equation 5.2.4 gives rise to a nonlinear Fredholm map*

$$Mon : H^p_{1,\alpha} \to H^p_{0,\alpha-1}.$$

Proof. Since $Mon = mon + d_1^* = d_1^* \oplus d_2 + q$ and $d_1^* \oplus d_2 = D$ the linear operator analyzed in section 5.1 it follows that $D : H^p_{1,\alpha} \to H^p_{0,\alpha-1}$ is well defined and Fredholm by the theorem 5.1.18 or 5.1.1.

Next one needs to check that the nonlinear term $q(\cdot,\cdot) : H^p_{1,\alpha} \times H^p_{1,\alpha} \to H^p_{0,\alpha-1}$ is well defined. This term is multilinear, i.e. the overall equation has quadratic nonlinearities, and so $q((a,\phi),(a,\phi))$ is a sum of the terms $[a \wedge a]$, $[a, \phi]$, $[\phi, \phi]$.

First one proves a particular case, which is when $p = n/2$, then $\alpha = 1 - 2 = -1$. Let $\xi, \chi \in H^{n/2}_{1,-1}$ be either a or ϕ. Then $\chi^{\|}, \xi^{\|} \in L^{n/2}_{1,-1}$ and the weighted Sobolev embedding, ([LM85], or Theorem 4.17 in [Mar02]) guarantees that $L^{n/2}_{1,-1} \hookrightarrow L^n_{0,-1}$. Moreover $\chi^\perp, \xi^\perp \in L^{n/2}_1$ and it is immediate to check from the definition of the weighted norms that $L^{n/2}_1 = L^{n/2}_{0,-2} \cap L^{n/2}_{1,-1}$, which once again lies in $L^n_{0,-1}$ from the weighted Sobolev embedding. So, one concludes that $\chi, \xi \in L^n_{0,-1}$.

Since by hypothesis $G = SU(2)$, $[\mathfrak{g}^{\|}_P, \mathfrak{g}^{\|}_P] = 0$, $[\mathfrak{g}^{\|}_P, \mathfrak{g}^{\perp}_P] \subset \mathfrak{g}^{\perp}_P$ and $[\mathfrak{g}^{\perp}_P, \mathfrak{g}^{\perp}_P] \subset \mathfrak{g}^{\|}_P$. Then $[\chi^{\|}, \xi^{\|}] = 0$, and

$$[\chi, \xi] = [\chi^\perp, \xi^\perp] + ([\chi^{\|}, \xi^\perp] + [\chi^\perp, \xi^{\|}]),$$

where the first term lies in $\mathfrak{g}^{\|}_P$, while the second and the third lie in $\mathfrak{g}^{\perp}_P$. So, in order to prove that $[\chi, \xi] \in H^{n/2}_{0,-2}$, it is enough to prove that $[\chi^{\|}, \xi^\perp] \in L^{n/2}$ and $[\chi^\perp, \xi^\perp] \in L^{n/2}_{0,-2} = L^{n/2}$. This is indeed true, as $\chi, \xi \in L^n$ and

$$\|\xi\chi\|_{L^{n/2}} \leq C \|\xi\|_{L^n} \|\chi\|_{L^n}.$$

The general case for $p \in [n/2, n)$ and $\alpha = -n/p + 1$, follows from applying the multiplication map in corollary 5.2.8 below and so $q((a,\phi),(a,\phi)) \in H^p_{0,\alpha-1}$ and the result follows. □

5.2.2 Sobolev Embeddings and Multiplication Maps

Denote by $L^p_{k,\alpha}$ the weighted spaces defined in 5.1.4 using the pair (A_0, Φ_0) as in the previous subsection. The moduli theory requires some important properties of these spaces which are important in handling the nonlinearities. The most relevant of these properties is the one stated in corollary 5.2.6 below, but its proof will require lemmas 5.2.4, 5.2.5 and 5.2.6.

Lemma 5.2.4. *(Weighted Hölder Inequality) Let $\beta, \gamma \in \mathbb{R}$ and $\frac{1}{r} + \frac{1}{s} = \frac{1}{q}$, then the multiplication property $L^r_{0,\beta} \times L^s_{0,\gamma} \hookrightarrow L^q_{0,\gamma+\beta}$ holds. In particular, if $\gamma \leq 0$, then $L^r_{0,\beta} \times L^s_{0,\gamma} \hookrightarrow L^q_{0,\beta}$.*

Proof. Let $f \in L^r_{0,\beta}, g \in L^s_{0,\gamma}$, then using the definition of the weighted norms, rearranging the exponents and the usual Hölder inequality shows

$$\begin{aligned}\|fg\|_{L^q_{0,\gamma+\beta}} &= \|\rho^{-\frac{n}{q}-\gamma-\beta}fg\|_{L^q} = \|\left(\rho^{-\frac{n}{r}-\beta}f\right)\left(\rho^{-\frac{n}{s}-\gamma}g\right)\|_{L^q} \\ &\leq \|\rho^{-\frac{n}{r}-\beta}f\|_{L^r}\|\rho^{-\frac{n}{s}-\gamma}g\|_{L^q} \\ &= \|f\|_{L^r_{0,\beta}}\|g\|_{L^s_{0,\gamma}}.\end{aligned}$$

This shows the first statement, which in the particular case where $\gamma \leq 0$, then $L^q_{0,\gamma+\beta} \hookrightarrow L^q_{0,\beta}$. □

Lemma 5.2.5. *Let $p_2 > p_1$ and $\gamma_2 > \gamma_1$, then for all $s \in [p_1, p_2]$ and $\gamma \geq \max_{i=1,2}\{\frac{n}{p_i} - \frac{n}{s} + \gamma_i\}$, there is an inclusion $L^{p_1}_{0,\gamma_1} \cap L^{p_2}_{0,\gamma_2} \hookrightarrow L^s_{0,\gamma}$.*

Proof. First one notices that since $p_1 < s < p_2$, then for all $g \in L^{p_1} \cap L^{p_2}$, it holds that $\|g\|_{L^s} \leq c(\|g\|_{L^{p_1}} + \|g\|_{L^{p_2}})$, for some $c > 0$. Let $f \in L^s_{0,\gamma}$, then

$$\begin{aligned}\|f\|_{L^s_{0,\gamma}} &= \|\rho^{-\gamma-\frac{n}{s}}f\|_{L^s} \leq c(\|\rho^{-\gamma-\frac{n}{s}}f\|_{L^{p_1}} + \|\rho^{-\gamma-\frac{n}{s}}f\|_{L^{p_2}}) \\ &= c(\|f\|_{L^{p_1}_{0,\gamma+\frac{n}{s}-\frac{n}{p_1}}} + \|f\|_{L^{p_2}_{0,\gamma+\frac{n}{s}-\frac{n}{p_2}}}).\end{aligned}$$

Since $\gamma \geq \max_i\{\frac{n}{p_i} - \frac{n}{s} + \gamma_i\}$, one has $\gamma + \frac{n}{s} - \frac{n}{p_i} \geq \gamma_i$ for $i = 1, 2$ and so $\|f\|_{L^s_{0,\gamma}} \leq c(\|f\|_{L^{p_1}_{0,\gamma_1}} + \|f\|_{L^{p_2}_{0,\gamma_2}})$. □

Lemma 5.2.6. *Let $\beta \in \mathbb{R}$, $p \in \left[\frac{n}{2}, n\right]$ and $k \in \mathbb{N}_1$. Then, the following hold*

- $L^p_k = \bigcap_{i=0}^k L^p_{i,-\frac{n}{p}+i}$
- $L^p_{k+1,\beta} \hookrightarrow L^q_{k,\beta}$, *for* $q = \frac{np}{n-p}$.
- $L^p_{k+1,loc} \hookrightarrow C^{k-1}_{loc}$ *and* $L^p_{k+1,\beta} \hookrightarrow C^{k-1}_{\beta}$,
- *Suppose ∇_∞ is H irreducible, i.e. it induces an irreducible connection on $V^\perp$. Let $\xi \in \Omega^*(X, \mathfrak{g}_P)$ with $\nabla_0\xi \in L^p_{k,\beta}$, then $\xi^\perp \in L^p_{k+1,\beta+1}$. In particular if $\beta \leq 1$, then $\lim_{\rho\to\infty}\xi = \xi_\infty$ exists and $\xi_\infty \in \gamma_\infty$.*

Proof. The first bullet is an immediate consequence of the definition in equation 5.1.5 of the $L^p_{k,\beta}$ norms. The case $k = 0$ amounts to $\|f\|_{L^p_{0,-n/p}} = \|\rho^{-n/p+n/p}f\|_{L^p}$ and the general case follows from an induction argument, where the general step is not more difficult than the case $k = 1$ and so the proof sticks to this one. Write for the norm in the right hand side

$$\begin{aligned}\|f\|^p &= \|f\|^p_{L^p_{0,-n/p}} + \|f\|^p_{L^p_{1,-n/p+1}} = \|f\|^p_{L^p} + \|r^{-1}f\|^p_{L^p} + \|\nabla f\|^p_{L^p} \\ &= \|f\|^p_{L^p_1} + \|r^{-1}f\|^p_{L^p}.\end{aligned}$$

Then, one can bound this from above by $2\|f\|^p_{L^p_1}$ and from below by $\|f\|^p_{L^p_1}$ and so the two norms are equivalent.

The next two bullets are particular instances of the standard weighted Sobolev embedding theorems (Theorem 4.17 in [Mar02]). To apply them one just needs to check that $1 - \frac{n}{p} > -\frac{n}{q}$ and

$k+1-\frac{n}{p}\geq k-1$.
The third statement is a direct consequence of proposition A.0.16 in the Appendix A. □

Corollary 5.2.7. *In the hypothesis of lemma 5.2.6 and ∇_∞ being H-irreducible. Let $\xi\in\Omega^k(X,\mathfrak{g}_P)$ with $\rho\nabla_0\xi\in H^p_{k,\beta+1}$, then $\xi^\perp\in L^p_{k+1}$ and in case $\beta<-1+\frac{1}{p}$, then $\lim_{\rho\to\infty}\xi=\xi_\infty$ exists and $\xi_\infty\in\gamma_\infty$.*

Proof. Since $\rho\nabla_0\xi\in H^p_{k,\beta+1}$, one knows $\rho\nabla_0\xi^\perp\in L^p_{k+1}$ and the same proof as that of the beginning of proposition A.0.16 shows that $\xi^\perp\in L^p_{k+1}$ and converges to 0 as $\rho\to\infty$. The other component follows from the fact that $\rho\nabla_0\xi^\parallel\in L^p_{k,\beta+1}$ is equivalent to $\nabla_0\xi^\parallel\in L^p_{k,\beta}$. Then, one can repeat the final part of the proof of proposition A.0.16 and notice that the argument there using Hölder's inequality works for $\beta<-1+\frac{1}{p}$. □

The following is the main result of this subsection and is an application of the previous lemmata.

Corollary 5.2.8. *Let $G=SU(2)$ and $p\in[n/2,n)$ and $\alpha=1-n/p$, then the Lie bracket $[\cdot,\cdot]$ gives rise to a continuous multiplication map*

$$[\cdot,\cdot]:H^p_{1,\alpha}\times H^p_{1,\alpha}\hookrightarrow H^p_{0,\alpha-1}.$$

Proof. Let $\chi,\xi\in H^p_{1,\alpha}$ and $q=\frac{np}{n-p}$, then by definition $\chi^\parallel,\xi^\parallel\in L^p_{1,\alpha}$, which using the embedding in the second bullet of lemma 5.2.6 lies in $L^q_{0,\alpha}$. In the same way, the definition of the $H^p_{1,\alpha}$ space gives $\chi^\perp,\xi^\perp\in L^p_1=L^p_{0,-n/p}\cap L^p_{1,-n/p+1}$ by the first bullet in lemma 5.2.6. Moreover, using the second bullet in this lemma again one knows that $L^p_{1,-n/p+1}\subset L^q_{0,-n/p+1}$. In conclusion,

$$\chi^\parallel,\xi^\parallel\in L^p_{0,\alpha}\cap L^q_{0,\alpha}\ ,\ \chi^\perp,\xi^\perp\in L^p_{0,-n/p}\cap L^q_{0,-n/p+1}.$$

For $G=SU(2)$, $[\mathfrak{g}_P^\parallel,\mathfrak{g}_P^\parallel]=0$, $[\mathfrak{g}_P^\parallel,\mathfrak{g}_P^\perp]\subset\mathfrak{g}_P^\perp$ and $[\mathfrak{g}_P^\perp,\mathfrak{g}_P^\perp]\subset\mathfrak{g}_P^\parallel$. So the term $[\chi^\parallel,\xi^\parallel]$ vanishes and $[\chi,\xi]=[\chi^\perp,\xi^\perp]+([\chi^\parallel,\xi^\perp]+[\chi^\perp,\xi^\parallel])$, where the first term lies in $\mathfrak{g}_P^\parallel$ and both the second and the third lie in $\mathfrak{g}_P^\perp$. So it is enough to show that $[\chi^\perp,\xi^\perp]\in L^p_{0,\alpha-1}$ and $[\chi^\parallel,\xi^\perp],[\chi^\perp,\xi^\parallel]\in L^p=L^p_{0,-n/p}$.
First one analyses the term $[\chi^\perp,\xi^\perp]$, by using twice lemma 5.2.4 in the form $L^p_{0,-n/p}\times L^p_{0,-n/p}\subset L^{p/2}_{0,-2n/p}$ and $L^q_{0,-n/p+1}\times L^q_{0,-n/p+1}\subset L^{q/2}_{0,-2n/p+2}$. Then, $[\chi^\perp,\xi^\perp]\in L^{p/2}_{0,-2n/p}\cap L^{q/2}_{0,-2n/p+2}$ and using lemma 5.2.5 with $p_1=p/2$, $\gamma_1=-2n/p$, $p_2=q/2$, $\gamma_2=-2n/p+2$ and $s=p$ gives that $[\chi^\perp,\xi^\perp]\in L^p_{0,\alpha-1}$ for all α such that

$$\alpha-1\geq\max\{\frac{2n}{p}-\frac{n}{p}-\frac{2n}{p},\frac{2n}{q}-\frac{n}{p}-\frac{2n}{p}+2\}=-\frac{n}{p}.$$

Next, one turns to the other terms and apply again lemma 5.2.4 twice, now in the form $L^q_{0,\alpha}\times L^q_{0,-n/p+1}\subset L^{q/2}_{0,\alpha-n/p+1}$ and $L^q_{0,\alpha}\times L^p_{0,-n/p}\subset L^{\frac{np}{2n-p}}_{0,\alpha-n/p}$. Then $[\chi^\parallel,\xi^\perp],[\chi^\perp,\xi^\parallel]\in L^{q/2}_{0,\alpha-n/p+1}\cap L^{\frac{np}{2n-p}}_{0,\alpha-n/p}$ and now use lemma 5.2.5 with $p_1=np/(2n-p)$, $\gamma_1=\alpha-n/p$, $p_2=q/2$, $\gamma_2=$

$\alpha - n/p + 1$ and $s = p$, which gives that $[\chi^{\|}, \xi^{\perp}], [\chi^{\perp}, \xi^{\|}] \in L^p = L^p_{0,-n/p}$ for all α such that

$$\max\{\frac{2n-p}{np} - \frac{n}{p} + \alpha - \frac{n}{p}, \frac{2n}{q} - \frac{n}{p} + \alpha - \frac{n}{p} + 1\} = -\frac{n}{p} \leq -\frac{n}{q}.$$

Since $p \geq n/2$ this is equivalent to $\alpha \leq -n/p+1$. So in the end the result holds for $\alpha = -n/p+1$. One must remark that the condition $p \geq n/2$ is required for the Sobolev embeddings in lemma 5.2.6 to hold and the condition that $p < n$ is required in order for $p_1 = np/(2n-p) < p$ and lemma 5.2.5 to apply in the second case above. $\square$

5.2.3 Moduli of Configurations

This subsection gives a first step towards implementing the ideas in subsection 5.2.1 using the $H^p_{k,\alpha}$ spaces from definition 5.1.16. Namely it defines and constructs moduli spaces of configurations (A, Φ). The upshot is theorem 5.2.14 which gives the moduli space of configurations the structure of a smooth Banach manifold. Then the boundary conditions defined by a finite mass monopole are preserved in

$$\mathcal{A}^p_{k,\alpha} = \{\nabla_A = \nabla_0 + a \ \mid \ a \in H^p_{k,\alpha}\} \ , \ \mathcal{H}^p_{k,\alpha} = \{\Phi = \Phi_0 + \phi \ \mid \ \phi \in H^p_{k,\alpha}\}.$$

Let $\mathcal{C}^p_{k,\alpha} = \mathcal{A}^p_{k,\alpha} \times \mathcal{H}^p_{k,\alpha}$ denote the space of configurations. The topology induced in these spaces will in principle depend on the background configuration (∇_0, Φ_0) and on p, k, α. Recall the gauge group $\mathcal{G}$ of continuous gauge transformations with a limit in $\mathcal{G}_\infty$ (the gauge transformations which preserve the boundary data (A_∞, Φ_∞)). Explicitly expanding around the background configuration (∇_0, Φ_0) a gauge transformation g acts on a configuration $(\nabla_0 + a, \Phi_0 + \phi)$ via

$$\left(\nabla_0 + g\nabla_0 g^{-1} + gag^{-1}, \Phi_0 + (g\Phi_0 g^{-1} - \Phi_0) + g\phi g^{-1}\right). \tag{5.2.5}$$

Two configurations in $\mathcal{C}^p_{k,\alpha}$ shall be considered equivalent if related by a continuous $g \in \mathcal{G} \cap L^p_{k+1,loc}$. To view this equivalence relation as generated by the action of a Banach Lie Group, let

$$\mathcal{G}^p_{k,\alpha} = \{g \in L^p_{k+1,loc} \ \mid \ \rho\nabla_0 g \in H^p_{k,\alpha+1}\} \ , \ L(\mathcal{G})^p_{k,\alpha} = \{\xi \in \Omega^0(X, \mathfrak{g}_P) \ \mid \ \rho\nabla_0\xi \in H^p_{k,\alpha+1}\}.$$

The pointwise exponential defines a map $\exp : L(\mathcal{G})^p_{k,\alpha} \to \mathcal{G}^p_{k,\alpha}$. For $\varepsilon > 0$ define

$$V_\varepsilon = \{\xi \in L(\mathcal{G})^p_{k,\alpha} \ \mid \ \|\rho\nabla_0\xi\|_{H^p_{k,\alpha+1}} \leq \varepsilon\},$$

and let the topology on $\mathcal{G}^p_{k,\alpha}$ be generated by the image under the exponential of the open sets $V_\varepsilon \subset L(\mathcal{G})^p_{k,\alpha}$ together with their translations.

Proposition 5.2.9. *Let $p \in [\frac{n}{2}, n)$, $\alpha = -n/p + 1$, then the following hold*

1. *With the topology defined above $\mathcal{G}^p_{1,\alpha}$ is a Banach Lie group with Lie algebra $L(\mathcal{G})^p_{1,\alpha}$.*
2. *If one further supposes that $p < \frac{n+1}{2}$, then there is a surjective evaluation homomorphism $ev : \mathcal{G}^p_{1,\alpha} \to \Gamma_\infty$, with derivative $dev : L(\mathcal{G})^p_{1,\alpha} \to \gamma_\infty$.*

3. *$\mathcal{G}^p_{1,\alpha}$ acts smoothly in $\mathcal{C}^p_{1,\alpha}$.*

Proof. Start by noticing that if $g \in \mathcal{G}^p_{1,\alpha}$, then $g \in L^p_{2,loc}$ and since one is working in a range where $p \geq n/2$, the third bullet in lemma 5.2.6 applies and $g \in C^0_{loc}$, i.e. these gauge transformations are continuous.

1. First prove that indeed pointwise multiplication and inversion are well defined on $\mathcal{G}^p_{1,\alpha}$. Then by construction of the topology above it will be a Lie group whose Lie Algebra is $L(\mathcal{G})^p_{1,\alpha}$.

   (a) To prove multiplication is well defined let $g, h \in \mathcal{G}^p_{1,\alpha}$, so $\rho\nabla_0 g, \rho\nabla_0 h \in H^p_{1,\alpha+1}$ and one needs to show that

   $$\rho\nabla_0(gh) = \rho(\nabla_0 g)h + \rho g\nabla_0 h \in H^p_{1,\alpha+1},$$

   for all $l \leq k$. The gauge transformations are continuous and $\rho\nabla_0 h \in H^p_{1,\alpha+1}$, so it follows that $\rho g\nabla_0 h \in H^p_{1,\alpha+1}$ and the same applies for $\rho(\nabla_0 g)h$. Alternatively one uses the Sobolev embedding in the second bullet of lemma 5.2.6, which gives

   $$\rho\nabla_0 h^{\parallel}, \rho\nabla_0 g^{\parallel} \in L^p_{1,\alpha+1} \subset L^q_{0,\alpha+1} \ , \ \rho\nabla_0 h^{\perp}, \rho\nabla_0 g^{\perp} \in L^p_{1,-n/p} \subset L^q_{0,-n/p},$$

   i.e. since $\alpha = 1 - n/p$, $\rho\nabla_0 h, \rho\nabla_0 g \in L^p_{0,-n/p+1} \cap L^q_{0,-n/p}$. Then, the multiplication map in lemma 5.2.4 and lemma 5.2.5 do guarantee that $\rho\nabla_0 g\nabla_0 h \in L^p \subset H^p_{0,\alpha}$.

   (b) To prove $g^{-1} \in \mathcal{G}^p_{1,\alpha}$ notice that $\nabla_0 g^{-1} = -g^{-1}(\nabla_0 g)g^{-1}$. Then proceeding as before, separating terms and using $g, g^{-1} \in C^0_{loc}$ and lemmas 5.2.4 and 5.2.5 one shows $\rho\nabla_0 g^{-1} \in H^p_{1,\alpha+1}$.

2. Let $g \in \mathcal{G}^p_{1,\alpha}$, then $\rho\nabla_0 g \in H^p_{1,\alpha+1}$, i.e. $(\nabla_0 g)^{\parallel} \in L^p_{1,\alpha} = L^p_{0,1-n/p}$ and $(\nabla_0 g)^{\perp} \in L^p_{0,-n/p-1}$. Next, using the arguments in the proof of proposition A.0.16 one can show that $(\nabla_0 g)^{\perp} \to 0$ always, but this does not hold for the other component. However, the last part of the argument in proposition A.0.16 can be used and is repeated here. Notice that $\nabla_0 g \in L^p_{0,-n/p+1}$, then Hölder's inequality gives

   $$\int_1^{+\infty} \left|\frac{\partial g}{\partial \rho}\right| \leq \int_1^{+\infty} |\rho^{n/p-1}\nabla_0 g|^p \int_1^{+\infty} \rho^{\frac{p}{p-1}(1-n/p)}.$$

   The first integral is bounded above by $\|\nabla_0 g\|^p_{L^p_{0,n/p+1}}$ and the second is finite if and only if $p < \frac{n+1}{2}$. Hence in this case this proves there is $g_\infty \in \mathcal{G}_\infty$ such that $g \to g_\infty$ and $\nabla_\infty g_\infty = 0$ (i.e. $g_\infty \in \Gamma_\infty$). Using a bump function it is straightforward to check that the evaluation maps given by taking the limit are surjective.

3. To check the action of $\mathcal{G}^p_{1,\alpha}$ on $\mathcal{C}^p_{1,\alpha}$ is well defined, one needs to prove that $g\nabla_0 g^{-1} + gag^{-1}$ and $(g\Phi_0 g^{-1} - \Phi_0) + g\phi g^{-1}$ are in $H^p_{1,\alpha}$. For the terms gag^{-1}, $g\phi g^{-1}$ and $g\nabla_0 g^{-1} = -(\nabla_0 g)g^{-1}$ notice that $(a, \phi) \in H^p_{1,\alpha}$, $g \in C^0$ as it is in $L^p_{2,loc}$ and $\rho\nabla_0 g \in H^p_{1,\alpha+1}$. Then, repeating the arguments in the proof of the first item proves that these are $H^p_{1,\alpha}$. One is now

left with analyzing $(g\Phi_0 g^{-1} - \Phi_0)$, for which one requires again the second item, namely that if $\rho\nabla_0 g, \rho\nabla_0\xi \in H^p_{1,\alpha+1}$, then g, ξ converge to limits $g_\infty \in \Gamma_\infty$ and $\xi_\infty \in \gamma_\infty$. Moreover, using the decomposition of $\mathfrak{g}_P$ in $X\backslash K$ one has $\xi^\perp \in L^p_2$ by corollary 5.2.7. Then, let $g = e^\xi$ and so

$$g\Phi_0 g^{-1} - \Phi_0 = [\xi, \Phi_0] + \frac{1}{2!}[\xi, [\xi, \Phi_0]] + ...$$

and the multiplication maps in lemma 5.2.6, used in the same way as before, show that the higher order terms are in $H^p_{1,\alpha}$ if and only if the first order one $[\xi, \Phi_0] \in H^p_{1,\alpha}$. Away from K, one can write $[\xi, \Phi_0] = [\xi^\perp, \Phi_0] \in V^\perp$ and since Φ_0 is smooth and bounded and $\xi^\perp \in L^p_2$ it is indeed true that $[\xi, \Phi_0] \in H^p_{1,\alpha}$. The convergence of the series above is an immediate from the fact that $|[\xi, \Phi_0]| \leq |\xi^\perp|$ which converges to 0 as $\rho \to \infty$. Then, this must bounded in $C^0(X\backslash K)$ and the term $\frac{1}{k!}$ in the series guarantees the convergence.
To prove the converse result namely that if (A, Φ) and $g \cdot (A, \Phi)$ both in $\mathcal{C}^p_{k,\alpha}$ are related by an $L^p_{k+1,loc}$ gauge transformation $g = e^\xi$, then actually $e^\xi \in \mathcal{G}^p_{k,\alpha}$ one rewinds the previous arguments. First, the fact that $[\xi, \Phi] = [\xi^\perp, \Phi_0] + ... \in L^p_2 \subset L^p_k$ implies $\rho\nabla_0\xi^\perp \in L^p_k$. Second, the fact that $g^{-1}\nabla_0 g = \nabla_0\xi \in H^p_{k,\alpha}$ implies that $\rho\nabla_0\xi^\parallel \in L^p_{1,\alpha+1}$. Put these two together to conclude that $\rho\nabla_0\xi \in H^p_{1,\alpha+1}$ and so $g \in \mathcal{G}^p_{1,\alpha}$.

□

Due to the second item in this proposition, one can use the Lie group homomorphism ev to define

$$\mathcal{G}^p_{k,\alpha}(0) = \ker(ev), \tag{5.2.6}$$

as a Banach Lie subgroup of $\mathcal{G}^p_{k,\alpha}$. This consists of gauge transformations which converge to the identity as $\rho \to \infty$. For $p \in [n/2, n)$ and $\alpha = 1 - n/p$ its Lie Algebra is the Lie subalgebra of $L(\mathcal{G}^p_{k,\alpha})$ consisting of those sections which decay, i.e $Lie(\mathcal{G}^p_{k,\alpha}(0)) = H^p_{k+1,\alpha+1}(X, \mathfrak{g}_P)$.

Proposition 5.2.10. *Let $\beta \in \mathbb{R}$ and $(\nabla_A, \Phi) \in \mathcal{C}^p_{k,\beta}$ and d^*_A the formal L^2 adjoint of the operator d_A and for all β extend d_A, d^*_A to operators*

$$d_A, d^{*,\alpha}_A : L^p_{k+1,\beta+1}(X, \mathfrak{g}_P) \to L^p_{k,\beta}(X, T^*X \otimes \mathfrak{g}_P).$$

Then, the following holds

1. *For $\beta \neq -1$, there is a constant $c > 0$ and an inequality $\|d_A\eta\|_{L^p_{0,\beta}} \geq c\|\eta\|_{L^p_{0,\beta+1}}$, and so a decomposition*

$$L^p_{k,\beta}(X, T^*X \otimes \mathfrak{g}_P) \quad = \quad \ker(d^*_A) \cap L^p_{k,\beta} \oplus \mathrm{im}(\nabla_A). \tag{5.2.7}$$

2. *On $X\backslash K$, there is a constant $c > 0$ and a pointwise inequality $|[\Phi, \eta]| \geq c|\eta^\perp|$.*

Proof. For all p, k, β the map $\rho^{-\beta} : L^p_{k,\beta} \to L^p_{k,0}$ is a Banach Space isomorphism. Conjugation with it gives then an equivalence of linear operators

$$\begin{array}{ccc} L^p_{k+1,\beta+1} & \xrightarrow{d_A} & L^p_{k,\beta} \\ \downarrow & & \downarrow \\ L^p_{k+1,0} & \xrightarrow{d^\alpha_A} & L^p_{k,-1} \end{array} , \tag{5.2.8}$$

with $d^\beta_A = \rho^{-(\alpha+1)} d_A \rho^{\alpha+1} = (\alpha+1)\frac{d\rho}{\rho} + d_A$. In what follows the proof will restrict to the case $p = 2$ for simplicity as in this case it is easy to complete squares. As K is compact and d_A is irreducible on K, one can combine Kato's and Poincaré's inequalities to achieve $\|d_A f\|_{L^2(K)} \geq c_1 \|f\|_{L^2(K)}$, for some $c_1 > 0$ and all f compactly supported in the interior of K. Moreover, as ρ is bounded on K, this holds equally well for $d^\beta_A = d_A$. Then one needs to prove a similar inequality for a section η which is supported on the conical end $\rho^{-1}[R,\infty) = X\backslash K$ one writes $\eta = \eta^\parallel + \eta^\perp \in L^2_{1,0}$ and splitting $d^\alpha_A \eta$ into orthogonal components to compute $\|d^\beta_A \eta\|^2_{L^2_{0,-1}(X\backslash K)} = \int_R^{+\infty} \frac{d\rho}{\rho} \int_\Sigma |\rho d^\beta_A \eta|^2 dvol_\Sigma$, gives

$$\|d^\beta_A \eta\|^2_{L^2_{0,-1}(X\backslash K)} = \int_R^\infty d\rho \int_\Sigma \left(\rho \Big| \frac{\partial \eta}{\partial \rho} + \frac{\beta+1}{\rho}\eta \Big|^2 + \rho |\nabla_0 \eta^\parallel|^2 + \rho |\nabla_0 \eta^\perp|^2 \right) dvol_\Sigma$$

In computing a lower bound for this one can ignore the term $\rho|\nabla_0 \eta^\parallel|^2$ and the term $\rho|\frac{\partial\eta}{\partial\rho}|^2$ which appears when one expands the square, as both these two terms are positive. Also, when one expand the square there is a mixed term appearing, however as this is $2(\beta+1)\langle \eta, \frac{\partial\eta}{\partial\rho}\rangle = (\beta+1)\frac{\partial|\eta|^2}{\partial\rho}$ and since η is compactly supported on $X\backslash K$, one can integrate by parts and this term vanishes. One is left with

$$\|d^\beta_A \eta\|^2_{L^2_{0,-1}(X\backslash K)} = \int_1^\infty d\rho \int_\Sigma \left(\frac{(\beta+1)^2}{\rho} |\eta|^2 + \rho |\nabla_0 \eta^\perp|^2 \right) dvol_\Sigma.$$

To handle this let Σ_ρ denote $\varphi(\{\rho\} \times \Sigma)$, then the irreducibility of the connection ∇_∞ on $V^\perp$, gives a Poincaré type inequality, which after scaling is $\|\nabla_\infty \eta^\perp\|^2_{L^2(\Sigma_\rho)} \geq c_2 \rho^{-2} \|\eta^\perp\|^2_{L^2(\Sigma_\rho)}$ for some constant $c_2 > 0$. Moreover, as the connection ∇_0 is asymptotic to ∇_∞ one can assume the same inequality holds for ∇_0 for very big ρ and inserting it above gives

$$\begin{aligned} \|d^\beta_A \eta\|^2_{L^2_{0,-1}(X\backslash K)} &\geq \int_1^\infty d\rho \int_\Sigma \left(\frac{(\beta+1)^2}{\rho} |\eta^\parallel|^2 + \frac{c_2 + (\beta+1)^2}{\rho} |\eta^\perp|^2 \right) dvol_\Sigma \\ &\geq (1+\beta)^2 \|\eta\|^2_{L^2_{0,0}(X\backslash K)}. \end{aligned}$$

Combining this with the similar inequality one has on K, gives the inequality in the first item of the statement. It is a corollary of such a Poincaré type inequality that d^β_A has closed image and the decomposition in the theorem follows. Recall that the operator d^β_A above is equivalent to $d_A : L^2_{1,\beta+1} \to L^2_{0,\beta}$, so this one has closed image. Then the same is true for $d_A : L^p_{k+1,\beta+1} \to L^p_{k,\beta}$, which gives the decomposition 5.2.7. Using the weighted inner product $\langle \cdot, \cdot \rangle_{L^2_{0,\beta}}$ one can identify a copy of cokernel of d_A with the orthogonal complement, i.e. the kernel of the adjoint $d^{*,\beta}_A = \rho^{2(\beta+1)+n} d^*_A \rho^{-2\beta-n} = (2\beta+n)\iota_{\rho\frac{\partial}{\partial\rho}} + d^*_A$.

Regarding the second item, since Φ_0 is asymptotic to Φ_∞, outside K there is the decomposition $\eta = \eta^{\|} + \eta^{\perp}$ and by definition $[\Phi, \eta^{\|}] = 0$ while it is pointwise bounded by below on the $V^{\perp}$ component. Since the cross sections are compact one can find a constant $c > 0$ which always works. $\square$

Remark 5.2.11. *The proof above gives a bound* $\|d_A \eta\|_{L^p_{0,\beta}} \geq c \|\eta\|_{L^p_{0,\beta+1}}$ *with an explicit constant* $c = |1+\beta|$. *For* $\beta = -n/2$ *this gives back the well known Hardy inequality*

$$\|d_A \eta\|^2_{L^2} \geq \left(\frac{n-2}{2} \right)^2 \|\rho^{-1} \eta\|^2_{L^2}.$$

Actually this gives the best possible constant on any asymptotically Euclidean manifold.

Corollary 5.2.12. *For* $\beta \neq -1$, *the operator*

$$\begin{aligned} L : H^p_{k+1,\beta+1}(X, \mathfrak{g}_P) &\to H^p_{k,\beta}(X, (\Lambda^0 \oplus \Lambda^1) \otimes \mathfrak{g}_P). && (5.2.9) \\ \xi &\mapsto (-\nabla_A \xi, [\xi, \Phi]), && (5.2.10) \end{aligned}$$

has closed image. Using the notation $H^p_{k,\beta}$ *for the right hand side in 5.2.9, there is an orthogonal decomposition*

$$H^p_{k,\beta} = \ker(L^*) \oplus \mathrm{im}(L). \tag{5.2.11}$$

Where $L_1^*(a, \phi) = -\nabla_A^* a + [\Phi, \phi]$.

Proof. This proof copies the one above and goes by using the inequalities in the first and second item of the previous proposition 5.2.10, as $\|L_1(\xi)\|_{H^2_{0,\beta}} = \|d_A \xi\|^2_{H^2_{0,\beta}} + \|[\Phi, \xi]\|^2_{H^2_{0,\alpha}}$. This shows that L_1 has closed image and the result follows as in the proof of the theorem above. $\square$

Definition 5.2.13. *A configuration* (A, Φ) *is said to be irreducible if* $\ker(L) = 0$.

Theorem 5.2.14. *Let* $p \in [n/2, n)$ *and* $\alpha = 1 - n/p$. *There are Banach manifolds* $\tilde{\mathcal{B}}^p_{1,\alpha} = \mathcal{C}^p_{1,\alpha}/\mathcal{G}^p_{1,\alpha}(0)$ *and* $\mathcal{B}^p_{1,\alpha} = \mathcal{C}^p_{1,\alpha}/\mathcal{G}^p_{1,\alpha}$, *such that*

$$\mathcal{B}^p_{1,\alpha} = \tilde{\mathcal{B}}^p_{1,\alpha}/\Gamma_\infty.$$

Moreover, the subset obtained as the image of the irreducible configurations ${\mathcal{B}^*}^p_{1,\alpha} \subset \mathcal{B}^p_{1,\alpha}$ *is a smooth Banach manifold.*

Proof. To prove that $\tilde{\mathcal{B}}^p_{1,\alpha} = \mathcal{C}^p_{1,\alpha}/\mathcal{G}^p_{1,\alpha}(0)$ is a Banach manifold one constructs local slices to the action of $\mathcal{G}^p_{1,\alpha}(0)$ using the Inverse Function Theorem. Then these slices can be used as charts for $\tilde{\mathcal{B}}^p_{1,\alpha}$. Let $\varepsilon > 0$ and define the slice candidates as

$$T_{(\nabla_A, \Phi),\varepsilon} = \{(a, \phi) \in H^p_{1,\alpha} \mid \nabla_A^* a - [\Phi_0, \phi] = 0 \, , \ \|(a, \phi)\|_{H^p_{1,\alpha}} < \varepsilon \}.$$

Then, in order to prove that these are actual slices one needs to show that the map

$$h : T_{(\nabla_A, \Phi),\varepsilon} \times \mathcal{G}^p_{1,\alpha}(0) \to \mathcal{C}^p_{1,\alpha},$$

which for $g = e^{\xi}$ sufficiently close to the identity, sends $((a, \phi), g)$ to the gauge equivalent configuration $g \cdot (\nabla_A + a, \Phi + \phi)$ is an isomorphism onto an open set around (A, Φ). This is proved using the Inverse Function Theorem, so one needs to check that the derivative

$$\begin{aligned} dh = id \oplus L : \left(\ker(L^*) \cap H^p_{1,\alpha}\right) \oplus H^p_{1,\alpha} &\to H^p_{1,\alpha} \\ ((a, \phi), \xi) &\mapsto (-\nabla_A \xi + a, [\xi, \Phi] + \phi), \end{aligned}$$

is an isomorphism. But this is a direct consequence of corollary 5.2.12. There is still the extra action of Γ_∞ on $\mathcal{C}^p_{1,\alpha}$ and one can quotient out by its action to obtain the full quotient $\mathcal{B}^p_{1,\alpha} = \mathcal{C}^p_{1,\alpha}/\mathcal{G}^p_{1,\alpha}$. Moreover, away from reducible configurations the action of $\mathcal{G}^p_{1,\alpha}$ is free and $\tilde{\mathcal{B}}^{*p}_{1,\alpha}$ is smooth. □

5.2.4 Moduli of Monopoles

This subsection uses $p \in [n/2, n)$ and $\alpha = 1 - n/p \notin \mathcal{K}(D)$, then theorem 5.2.14 holds. Moreover, that statement can also be made in a more general setup where one need not restrict to the case $G = SU(2)$. The goal is to show that the moduli space of monopoles either in the G_2 case, in the Calabi-Yau case or in a 3 manifold can always be obtained as a quotient of the zero set of a Γ_∞-invariant Fredholm section of a bundle $\mathcal{F}^p_{1,\alpha}$, where $\mathcal{F}^p_{1,\alpha}$ is the bundle over $\tilde{\mathcal{B}}^p_{1,\alpha}$

$$\mathcal{F}^p_{1,\alpha} = \mathcal{C}^p_{1,\alpha} \times_{\mathcal{G}^p_{1,\alpha}(0)} H^p_{0,\alpha-1}(X, \Lambda^* X \otimes \mathfrak{g}_P). \tag{5.2.12}$$

Notice that sections of this bundle are in one-to-one correspondence with $\mathcal{G}^p_{1,\alpha}(0)$-equivariant maps from $\mathcal{C}^p_{1,\alpha} \to H^p_{0,\alpha-1}(X, \Lambda^* X \otimes \mathfrak{g}_P)$. Moreover, in each case (3 dimensions, Calabi-Yau and G_2 manifolds) the map mon defined in equation 5.2.3 is invariant by the action of the gauge transformations $\mathcal{G}^p_{1,\alpha} \supset \mathcal{G}^p_{1,\alpha}(0)$. For this p, α theorem 5.2.3 holds and proves

Theorem 5.2.15. *Let $G = SU(2)$ and $p \in [n/2, n)$ such that $\alpha = 1 - n/p \notin \mathcal{K}(D)$. Then, there is a Γ_∞-invariant Fredholm section mon of the bundle $\mathcal{F}^p_{1,\alpha} \to \tilde{\mathcal{B}}^p_{1,\alpha}$ such that the moduli space of (complex) monopoles is $mon^{-1}(0)/\Gamma_\infty \subset \mathcal{B}^p_{1,\alpha}$.*

Proof. Due to theorem 5.2.3, the monopole equation can be written as the zero set of mon, which due to the gauge invariance is a section of $\mathcal{F}^p_{1,\alpha} \to \tilde{\mathcal{B}}^p_{1,\alpha}$. Locally one can define $s^{-1}(0)$ inside $\ker(L^*) \subset H^p_{k,\alpha}$ by using the local slices for $\mathcal{B}^p_{k,\alpha}$ constructed in the proof of theorem 5.2.14 and intersecting such configurations with the ones satisfying the monopole equation. This is precisely the same as the zero set of the map $Mon = mon + d_2$ to which theorem 5.2.3 refers to. Recall that linearization of the (complex) monopole equation gives d_2 in each case as computed in sections 2.1.1, 3.1.2 and 3.1.2. Then coupling this with $d_1^* = L^*$ the operator $D = d_2 \oplus d_1^* : H^p_{1,\alpha} \to H^p_{0,\alpha-1}$ which is the linearization of Mon is shown to be Fredholm in theorem 5.1.18 in section 5.1. □

Appendix A

Decay Estimates

Let (X^n, g) be an AC manifold of dimension $n > 2$ with asymptotic cone $C(\Sigma^{n-1})$ as in definition 1.1.5 and $V \to X$ a vector bundle equipped with a connection ∇. Suppose these have fixed asymptotics, i.e outside a compact set $K \subset X$, $V|_{X \backslash K} \cong \varphi^* \pi^* V_\infty$ and $\nabla = \varphi^* \pi^* \nabla_\infty + a$ with $|\nabla^j a| = O(\rho^{-1-j-\varepsilon})$ for all $j \in \mathbb{N}_0$ and some $\varepsilon > 0$.

Proposition A.0.16. *Let $p \geq n/2$, $\beta \leq -1$ and $\xi \in \Omega^0(X, V)$ with $\nabla \xi \in L^p_{1,\beta}$. Then $\lim_{\rho \to \infty} \xi(\rho)$ exists and is equal to a ∇_∞ parallel and continuous section ξ_∞ of V_∞.*

Proof. Let $V_\infty^{\|} \subset V_\infty$ be a maximal vector subbundle, generated by the ∇_∞ parallel sections, i.e. there is $l \in \mathbb{N}_0$ which is maximal such that there is an isomorphism $(V_\infty, \nabla_\infty) \cong (\underline{\mathbb{R}}^l \oplus V_\infty^\perp, d \oplus \nabla_\infty^\perp)$ of vector bundles with connection. Write $V_\infty = V_\infty^{\|} \oplus V_\infty^\perp$, then $\nabla_\infty^\perp$ and so ∇_∞ is irreducible on $V_\infty^\perp$. Using the fixed asymptotic behavior of (V, ∇) one can suppose a similar decomposition holds for V and from now on the notation is according to this. So on $X \backslash K$ one writes

$$\nabla \xi = \frac{\partial \xi^{\|}}{\partial \rho} \otimes d\rho + \nabla \xi^{\|} + \frac{\partial \xi^\perp}{\partial \rho} \otimes d\rho + \nabla \xi^\perp,$$

and as the summands are linearly independent as sections of $\Lambda^1 \otimes V$, each of them has its norm bounded by that of $\nabla \xi$. Since ∇ is irreducible on the $V^\perp$ component, there is a Poincaré type inequality on the level set $\Sigma_1 = \rho^{-1}(1)$ of ρ, which can be written as $\|\xi^\perp\|_{L^p(\Sigma_1)} \leq c \|\nabla \xi^\perp\|_{L^p(\Sigma_1)}$, for some $c > 0$. Scaling this inequality gives

$$\|\xi^\perp\|_{L^p(\Sigma_r)} \leq cr \|\nabla \xi^\perp\|_{L^p(\Sigma_r)} \leq cr \|\nabla \xi\|_{L^p(\Sigma_r)},$$

on each $\Sigma_r = \rho^{-1}(r)$. This together with the hypothesis that $\nabla \xi \in L^p_{k,\beta}$ shows that

$$\int_1^{+\infty} r^{-(\beta+1)p} \|\xi^\perp\|^p_{L^p(\Sigma_r)} \frac{dr}{r^n} \leq \int_1^{+\infty} cr^{-\beta p} \|\nabla_0 \xi\|^p_{L^p(\Sigma_r)} \frac{dr}{r^n} < \infty.$$

Scaling the metric on $(1, +\infty)_r \times \Sigma$ to the cylindrical metric $r^{-2} g = dt^2 + g_\Sigma$, where $t = \log(r)$. This implies that as $t \to \infty$, all three $e^{-tp(\beta+1)} \xi^\perp$, $e^{-tp(\beta+1)} \nabla \xi^\perp$ and $e^{-tp(\beta+1)} \nabla \nabla \xi^\perp$ converge in the L^p to zero, over the intervals $(t, t+1) \times \Sigma$, equipped with the cylindrical metric $dt^2 + g_\Sigma$.

Since $-(\beta+1)>0$, one concludes that as $t\to\infty$, ξ converges to zero in L^p_2 over these intervals equipped with the fixed cylindrical metric. Using, the Sobolev embedding $L^p_2\hookrightarrow C^0$, which holds for $p\geq n/2$, one concludes that $\xi^\perp$ converges uniformly to 0.
For the other component, i.e. $\xi^\parallel$ one has $|\frac{\partial\xi^\parallel}{\partial\rho}|\leq|\nabla\xi|$ and using this together with the Hölder inequality into

$$\int_1^{+\infty}\Big|\frac{\partial\xi^\parallel}{\partial\rho}\Big|d\rho\leq\int_1^{+\infty}\rho^{-\beta p}|\nabla\xi|^p d\rho\int_1^{+\infty}\rho^{\beta p'}d\rho,$$

where $p'=p/(p-1)$ is the conjugate exponent. The first integral is bounded above by $\|\nabla\xi\|^p_{L^p_{0,\beta}}$. The second one is $\int_1^{+\infty}\rho^{\frac{\beta p}{p-1}}d\rho$ and since $\beta\leq-1<1/p-1=(1-p)/p$ one concludes this integral is finite. It follows that there is a limit ξ_∞ to which $\xi^\parallel$ converges. □

Proposition A.0.17. *Let $k>\frac{n}{2}$ and $\xi\in\Omega^0(X,V)$ with $\rho^{j-1}\nabla^j\xi\in L^2$ for all $1\leq j\leq k$. Then ξ converges to a ∇_∞ parallel section ξ_∞ of V.*

Proof. The same proof as above works by replacing the Sobolev embedding $L^p_2\hookrightarrow C^0$ over Σ^{n-1} by the Sobolev embedding $L^2_k\hookrightarrow C^0$ which holds for $k>\frac{n}{2}$. □

The following propositions and their proofs have been explained to me by Mark Stern. They can be used to estimate the rate of decay of sections in the kernel of some elliptic operators.

Proposition A.0.18. *Let X^n be an AC manifold with $n>2$ and D an operator acting on sections of a vector bundle V equipped with a connection ∇. Suppose D satisfies a Weitzenböck type formula*

$$D^*D=\nabla^*\nabla+W,$$

with $W=O(\rho^{-2-2\delta})$ for some $\delta>0$. Then if $f\in\ker(D)\cap L^2$ it is smooth and $\rho^{\frac{n}{2}-2-\varepsilon}f\in L^2$, for all $\varepsilon>0$.

Proof. Let $L>0$ be large and to be fixed at the of the proof, $R>L$ and ψ a function smoothly interpolating between ρ^β for $\rho\leq R$ and $(R+1)^\beta$ for $\rho\geq R+1$. The goal is to show it is possible to obtain an R independent bound on $\|\rho^{-1}\psi f\|_{L^2(\rho^{-1}(L,+\infty))}$ for all $f\in\ker(D)\cap L^2$ and $\beta<\frac{n-2}{2}$.
Since ψ is bounded and $f\in L^2$ we have $\psi f\in L^2$ and one can compute

$$\begin{aligned}\langle D^*Df,\psi^2f\rangle_{L^2}&=\langle\nabla^*\nabla f+W(f),\psi^2f\rangle_{L^2}\\&=\|\psi\nabla f\|_{L^2}+2\langle\psi\nabla f,d\psi\otimes f\rangle_{L^2}+\langle W(f),\psi^2f\rangle_{L^2}.\end{aligned}$$

Using the identity

$$\|\psi\nabla f\|^2_{L^2}+2\langle\psi\nabla f,d\psi\otimes f\rangle_{L^2}=\|\nabla(\psi f)\|^2_{L^2}-\|d\psi\otimes f\|^2_{L^2},$$

to replace in the expression $0=\langle D^*Df,\psi^2f\rangle_{L^2}$, for $f\in\ker(D^*D)$, gives

$$0=\|\nabla(\psi f)\|^2_{L^2}-\|d\psi\otimes f\|^2_{L^2}+\langle W(f),\psi^2f\rangle_{L^2}.$$

Now, pass the last term to the left hand side and use the hypothesis that $W = O(\rho^{-2-2\delta})$. Then, there is a constant $c > 0$ independent of f and R, such that $\|\nabla(\psi f)\|_{L^2}^2 - \|d\psi \otimes f\|_{L^2}^2 \leq \|\rho^{-1-\delta}\psi f\|_{L^2}^2$. Nest we split the integration into the regions $\rho^{-1}[0, L]$ and $\rho^{-1}(L, +\infty)$, then we can write

$$\|\nabla(\psi f)\|_{L^2(L,+\infty)}^2 - \|d\psi \otimes f\|_{L^2(L,+\infty)}^2 \quad \leq \quad C(L) + \frac{1}{L^{2\delta}}\|\rho^{-1-\delta}\psi f\|_{L^2(L,+\infty)}^2, \text{(A.0.1)}$$

where $C(L) > 0$ is some constant independent of R. Now one can use the Hardy type inequality $\|\nabla(\psi f)\|_{L^2(L,+\infty)} \geq \frac{n-2}{2}\|\rho^{-1}\psi f\|_{L^2(L,+\infty)} + \frac{1}{L^{\delta'}}\|\rho^{-1}\psi f\|_{L^2(L,+\infty)}$, for some $\delta' > 0$. Which, together with the fact that $|d\psi| \leq |\beta|\rho^{-1}\psi$ transform the inequality A.0.1 above into

$$\left(\left(\frac{n-2}{2}\right)^2 - \beta^2 - \frac{1}{L^{2\delta}} - \frac{1}{L^{\delta'}}\right)\|\rho^{-1}\psi f\|_{L^2(L,+\infty)}^2 \quad \leq \quad C(L).$$

Now, notice that for all $\beta < \frac{n}{2} - 1$ it is possible to chose L sufficiently large so that the left hand side is greater than zero. Moreover since $C(L)$ does not depend on R, the inequality above holds for all $R > L$ giving the R independent bound we were looking for. □

Remark A.0.19. *The decay estimates from the previous proposition are optimal in the case where the cross section is a sphere.*

Similar techniques to those employed in the proof of proposition A.0.18 show

Proposition A.0.20. *Let X^n, $n > 2$ be an asymptotically conical manifold and D an operator acting on sections of a vector bundle V equipped with a connection ∇. Suppose D satisfies a Weitzenböck type formula*

$$D^*D = \nabla^*\nabla + W + q^*q,$$

with $W = O(\rho^{-\delta})$ for some $\delta > 0$ and $|q(f)|^2 \geq c^2|f|^2$ for $c > 0$ and all f supported outside a compact set $K \subset X$. Then if $f \in L^2$, in fact $e^{(c-\varepsilon)\rho}f \in L^2$, for all $\varepsilon > 0$.

Appendix B

Homogeneous Bundles and Invariant Connections

This section contains standard material on bundles over homogeneous spaces and Wang's theorem classifying invariant connections on these, the main reference is [KN63].
Let K be a connected Lie group, $H \subset K$ a normal subgroup, then K acts transitively on the homogeneous space $X = K/H$ with isotropy H. Denote by $\mathfrak{h} \subset \mathfrak{k}$ the Lie algebras of H and K respectively and suppose there is an $H - Ad$ complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{k}$, i.e. $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$ for all $h \in H$. It is a standard result that there is a one-to-one correspondence between K-invariant metrics on X and metrics on $\mathfrak{m}$ invariant under the adjoint H action.

Let $\pi : P \to X$ be a principal G-bundle. As usual K acts on the left on X and G on the right on P. The bundle P is said to be Homogeneous if there is a lift of the left action of K on X to the total space of P which commutes with the right G action on P. Suppose such a lift is given and let H be the isotropy subgroup at $x \in X$, then it acts on the fibre $\pi^{-1}(x)$. As this action commutes with the transitive right G action and gives rise to the **isotropy homomorphism** $\lambda : H \to G$ λ, which can be used to construct back the bundle P via $P = K \times_{(H,\lambda)} G$.
Let (V, η) be a G representation, where V is a vector space and $\eta : G \to GL(V)$, construct the associated bundle $E = P \times_{G,\eta} V$ with fibre V. The lift of the K action to P naturally gives a K action on E and there is an isomorphism of homogeneous bundles

$$E \cong K \times_{H,\eta\circ\lambda} V. \tag{B.0.1}$$

A section $s_E \in \Gamma(E)$ is said to be an **invariant section** under the K action on E if once regarded as an H-equivariant map $s_E : K \to V$ it is actually constant. Hence, $\eta \circ \lambda : H \to GL(V)$ can be used to decompose V into irreducibles and the H-equivariant condition restricts s_E to take values in the trivial components of V. A slight modification of the above paragraph in order to obtain invariant section of more general bundles can be stated. In particular, gauge transformations can be regarded as sections of the bundle $c(P) = P \times_{c,G} G$, where $c(g_1)g_2 = g_1 g_2 g_1^{-1}$ is the action by conjugation. And under the isomorphism $c(P) \cong K \times_{c\circ\lambda,H} G$ the K-invariant Gauge transformations correspond to those constant $g \in \Omega^0(K, G)$ with values in the subgroup of G

centralized by $\lambda(H)$.
One turns now to the definition of invariant connections on the principal bundle $P = K \times_{H,\lambda} G$. These are given by a left invariant connection 1 form $A \in \Omega^1(K, \mathfrak{g})$ and classified by Wang's theorem. The reductive decomposition $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$ equips the bundle $K \to X = K/H$ with a K-invariant connection whose horizontal spaces are the left translates of $\mathfrak{m}$. This is known as the canonical invariant connection and it's connection 1 form is the left invariant translate of $A = \pi_{\mathfrak{h}}$, where $\pi_{\mathfrak{h}}$ is the projection $\mathfrak{m} \oplus \mathfrak{h} \to \mathfrak{h}$. One can now state Wang's theorem (see [KN63] volume II., theorem 11.5).

Theorem B.0.21. *(Wang* [Wan58]*) Let $P = K \times_{H,\lambda} G$ be a principal homogeneous G-bundle. Then K-invariant connections A on P are in one to one correspondence with morphisms of H representations*

$$\Lambda : (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{g}, \mathrm{Ad} \circ \lambda). \tag{B.0.2}$$

The upshot is that the left invariant 1-form A at the identity $1 \in K$ is given by $d\lambda \oplus \Lambda : \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m} \to \mathfrak{g}$. Moreover, the H-equivariant condition implies that the component $\Lambda : \mathfrak{m} \to \mathfrak{g}$ is a morphism of H representations. In this case the canonical invariant connection is given by taking $\Lambda = 0$, so that $A = d\lambda \circ \pi_{\mathfrak{h}}$.

Appendix C

Appendix to Monopoles on $T^*\mathbb{S}^3$

This is an appendix to section 3.3. It will be used to study the function $h(\rho)$ and the conditions that ensure a given connection and Higgs field to extend over the zero section.

C.1 The function $h(\rho)$.

Studying the function $h(\rho)$ is a necessary step in order to use the results of chapter 2 in order to solve the ODE's in lemma 3.3.25 to which the problem was reduced to at the end of section 3.3.5. One starts with some preliminary explicit formulas. In terms of r

$$\mathcal{F}'(r^2) = \left(\frac{3}{2}\right)^{\frac{1}{3}} \frac{\varepsilon^{-\frac{2}{3}}}{\sqrt{\frac{r^4}{\varepsilon^4} - 1}} k^{\frac{1}{3}}\left(\frac{r^2}{\varepsilon^2}\right) \ , \ \mathcal{G}(r) = \left(\frac{3\varepsilon^4}{2^4}\right)^{\frac{1}{3}} k^{\frac{1}{3}}\left(\frac{r^2}{\varepsilon^2}\right). \tag{C.1.1}$$

where $k : (1, \infty) \to \mathbb{R}$ is the function defined by $k(x) = x\sqrt{x^2 - 1} - \log(\sqrt{x^2 - 1} + x)$. To write ρ in terms of r and using this function, insert C.1.1 into equation 3.3.13, one has

$$\rho(r) \quad = \quad \left(\frac{2}{3\varepsilon^4}\right)^{\frac{1}{3}} \int_{\varepsilon}^{r} l k^{-\frac{1}{3}}\left(\frac{l^2}{\varepsilon^2}\right) dl = \left(\frac{\varepsilon^2}{12}\right)^{\frac{1}{3}} \int_{1}^{\frac{r^2}{\varepsilon^2}} k^{-\frac{1}{3}}(l)\, dl. \tag{C.1.2}$$

In order to see how the function $h^2(\rho) = \frac{1}{\varepsilon^2} R_+ R_- \mathcal{G}$ in terms r, it is useful to use k

$$h^2(\rho(r)) \quad = \quad \left(\frac{3\varepsilon^4}{2^7}\right)^{\frac{1}{3}} \sqrt{\frac{r^4}{\varepsilon^4} - 1} k^{\frac{1}{3}}\left(\frac{r^2}{\varepsilon^2}\right). \tag{C.1.3}$$

Lemma C.1.1. *The function $h(\rho)$ behaves for $\rho \ll 1$ as $h(\rho) = \rho + O(\rho^3)$ and for $\rho \gg 1$ one has $h(\rho) = O(\rho^{5/2})$.*

Proof. Regarding the function $k : (1, \infty) \to \mathbb{R}$, for x close to 1 one has the following expansions in terms of $\sqrt{x-1}$

$$k^{\frac{1}{3}}(x) = \frac{2^{\frac{5}{6}}}{3^{\frac{1}{3}}}\sqrt{x-1} + \frac{(x-1)^{3/2}}{10(2)^{\frac{1}{6}}(3)^{\frac{1}{3}}} \ , \ k^{-\frac{1}{3}}(x) = \frac{3^{\frac{1}{3}}}{2^{\frac{5}{6}}}\frac{1}{\sqrt{x-1}} - \frac{3^{\frac{1}{3}}}{20(2)^{\frac{5}{6}}}\sqrt{x-1} + ...$$

Inserting these expressions on h^2 and ρ, one has that for $\rho \ll 1$

$$\begin{aligned}\rho(r) &= \frac{\varepsilon^{\frac{2}{3}}}{\sqrt{2}}\left(\sqrt{\frac{r^2}{\varepsilon^2}-1}-\frac{1}{60}\left(\frac{r^2}{\varepsilon^2}-1\right)^{\frac{3}{2}}+...\right)\\ h^2(r) &= \frac{\varepsilon^{\frac{4}{3}}}{2}\left(\left(\frac{r^2}{\varepsilon^2}-1\right)+\frac{1}{20}\left(\frac{r^2}{\varepsilon^2}-1\right)^2+...\right),\end{aligned}$$

hence, for small ρ, $h(\rho) \sim \rho + O(\rho^3)$. To get the behavior for large ρ, it is convenient to introduce one further coordinate given by $x = \cosh(t)$ for $t \in (0, \infty)$ since $x \in (1, \infty)$. Inverting this gives $t = \log(\sqrt{x^2-1}+x)$ and replacing it on k shows that $h(\rho(t)) \sim \varepsilon^{2/3} e^{\frac{t}{2}} e^{\frac{t}{3}} = \varepsilon^{2/3} e^{\frac{5t}{6}}$, while $\rho(t) \sim \varepsilon^{2/3} \int e^t e^{-\frac{2t}{3}} = \varepsilon^{2/3} e^{\frac{t}{3}}$ and the result follows. □

C.2 Extending the Connection

Studying the conditions that ensure a given connection and Higgs field to extend over the zero section is a necessary step for the proof of the main theorem 3.3.1, which appears at the end of 3.3.5. These conditions give rise to initial conditions at $\rho = 0$ (the zero section) for the ODE's. These are the initial conditions that where stated in the hypothesis of lemma 3.3.25, which reduces the problem to that of solving the ODE's analyzed in the first part of chapter 2.
It follows from formula 3.3.12 for Stenzel's metric that the 1-forms defined by

$$\omega_1 = \sqrt{2\frac{R_+R_-}{r}\frac{d\mathcal{G}}{dr}}\theta_1 \ , \ \omega_{2,3} = \sqrt{\frac{R_+}{R_-}}\mathcal{G}\theta_{2,3} \ , \ \omega_{4,5} = \sqrt{\frac{R_-}{R_+}}\mathcal{G}\theta_{4,5},$$

have constant norm equal to 1 and so are bounded. For a connection to extend it is a necessary condition that the curvature remains bounded.

Lemma C.2.1. *Let $l = 1$ and A an invariant connection parametrized by the fields A_i. Let the B_i's be the rescaled fields introduced in the statement of proposition 3.3.22. Fix a gauge such that $B_2 = 0$ and suppose as well that $B_5 = 0$. Then, the curvature of the invariant connection can be written in this frame as*

$$\begin{aligned}F_A &= \left(I_4\omega^{23}+I_4\omega^{45}+I_1\omega^1+I_8(\omega^{24}+\omega^{35})\right)\otimes T_1\\ &\quad +I_2\left(T_3\otimes d\rho\wedge\omega^2 - T_2\otimes d\rho\wedge\omega^3\right)+I_3\left(T_2\otimes d\rho\wedge\omega^4+T_3\otimes d\rho\wedge\omega^5\right)\\ &\quad +I_6\left(T_2\otimes\omega^{12}+T_3\otimes\omega^{13}\right)+I_7(T_2\otimes\omega^{15}-T_3\otimes\omega^{14}),\end{aligned}$$

where

$$I_1 = \frac{1}{\varepsilon^2 h^2(\rho)} \left(\frac{dB_1}{d\rho} - \frac{2\dot{\mathcal{G}}}{r} B_1 \right) \quad , \quad I_8 = \frac{1}{\mathcal{G}} \left(\frac{B_1}{\mathcal{G}^2} - 2\frac{B_3 B_4}{R_+ R_-} \right)$$

$$I_2 = \frac{1}{\varepsilon h(\rho)} \left(\frac{dB_3}{d\rho} - \frac{\mathcal{G}}{r R_-^2} B_3 \right) \quad , \quad I_3 = \frac{1}{\varepsilon h(\rho)} \left(\frac{dB_4}{d\rho} - \frac{\mathcal{G}}{r R_+^2} B_4 \right)$$

$$I_4 = \frac{1}{\varepsilon^2 h^2(\rho)} \left(4B_3^2 - R_-^2 \right) \quad , \quad I_5 = \frac{1}{\varepsilon^2 h^2(\rho)} \left(4B_4^2 - R_+^2 \right)$$

$$I_6 = \left(\frac{B_4}{R_+} - 2\frac{B_1 B_3}{\mathcal{G}^2 R_-} \right) \frac{1}{R_+} \sqrt{\frac{\mathcal{G}}{R_+ R_-}} \quad , \quad I_7 = \left(\frac{B_3}{R_-} - 2\frac{B_1 B_4}{\mathcal{G}^2 R_+} \right) \frac{1}{R_-} \sqrt{\frac{\mathcal{G}}{R_+ R_-}}.$$

Proof. It follows from lemma 5.2.4 that the curvature can be written as

$$\begin{aligned} F_A \;=\; & \left(\left(2A_3^2 - \frac{1}{2} \right) \theta^{23} + \left(2A_4^2 - \frac{1}{2} \right) \theta^{45} + \frac{dA_1}{d\rho} d\rho \wedge \theta^1 + (A_1 - 2A_4 A_3)(\theta^{24} + \theta^{35}) \right) \otimes T_1 \\ & + \frac{dA_3}{d\rho} \left(T_3 \otimes d\rho \wedge \theta^2 - T_2 \otimes d\rho \wedge \theta^3 \right) + \frac{dA_4}{d\rho} \left(T_2 \otimes d\rho \wedge \theta^4 + T_3 \otimes d\rho \wedge \theta^5 \right) \\ & + (A_4 - 2A_1 A_3) \left(T_2 \otimes \theta^{12} + T_3 \otimes \theta^{13} \right) + (A_3 - 2A_1 A_4)) \left(T_2 \otimes \theta^{15} - T_3 \otimes \theta^{14} \right). \end{aligned}$$

Using the definition of the B_i's in terms of the A_i's, the definition of the bounded forms ω_i and the relations between $\rho, h, \mathcal{G}, R_+, R_-$ this turns into the formula in the statement. □

Lemma C.2.2. *The invariant connection A from lemma C.2.1 extends over the zero section if and only if, for $\rho \ll 1$*

$$B_1(\rho) = O(\rho^3) \;\; , \;\; B_3(\rho) = O(\rho^2) \;\; , \;\; B_4(\rho) = \frac{\varepsilon}{2} + O(\rho^2).$$

Proof. The connection extends over the zero section if and only if the curvature does remain bounded. Since the forms ω_i are bounded, one concludes from lemma C.2.1 that this will be the case if and only if the I_i's are bounded for $\rho \ll 1$. The fact that I_5 needs to stay bounded implies that

$$\left(4B_4(\rho)^2 - R_+(\rho)^2 \right) = O(h^2(\rho)) = O(\rho^2).$$

Since $R_+^2 = \frac{\varepsilon^2}{2} \left(\frac{r^2}{\varepsilon^2} + 1 \right) = \varepsilon^2 + \frac{\varepsilon^2}{2} \left(\frac{r^2}{\varepsilon^2} - 1 \right) = \varepsilon^2 + O(\rho^2)$, then from the above one must have

$$B_4(\rho) = \frac{\varepsilon^2}{4} + O(\rho^2),$$

and this gives the result in the statement. In the same way one can proceed to analyze I_4, which gives $4B_3^2 - R_-^2 = O(\rho^2)$, but since $R_-^2 = O(\rho^2)$, one concludes that $B_3^2 = O(\rho^2)$ and so $B_3 = O(\rho)$. This is again the result in the statement and the only thing left to do is to compute the estimate on B_1. From $B_4(\rho) = \frac{\varepsilon}{2} + O(\rho^2)$ and $B_3(\rho) = O(\rho)$. In fact inserting these into I_8 together with $\mathcal{G} = O(\rho)$ and $R_- = O(\rho)$, gives that

$$\rho^{-2} B_1 = O\left(\frac{B_3 B_4}{R_+ R_-} \right) = O(1),$$

from what it is straightforward to get $B_1(\rho) = O(\rho^2)$. So far, one has just used the boundedness of I_4, I_5, I_8 and obtained that

$$B_1(\rho) = O(\rho^2) \ , \ B_3(\rho) = O(\rho) \ , \ B_4(\rho) = \frac{\varepsilon}{2} + O(\rho^2). \tag{C.2.1}$$

One must analyze the behavior of the other I_i's. Writing $B_1 = b_1\rho^2$, $B_3 = b_3\rho$ and $B_4 = \frac{\varepsilon}{2} + b_4\rho^2$ one can see that the boundedness of I_1, I_3, I_6 are guaranteed just by the estimates in lemma C.2.1, while the boundedness of I_7, I_8, I_2 require respectively

$$b_3 = 2\sqrt{2}\varepsilon^{-\frac{7}{3}} b_1 b_4 \ , \ b_1 = 2\sqrt{2} b_3 b_4 \ , \ b_3 = 0.$$

Combining these implies that $b_1 = b_3 = 0$ and the result follows. □

Remark C.2.3. *Moreover, a posteriori to lemma 3.3.25, bounded invariant connections satisfying the Calabi Yau monopole equations, are known to satisfy a Bogomolny equation when restricted to the fibres of $T^*\mathbb{S}^3 \to \mathbb{S}^3$. Hence, by the main theorem of* [SS84] *the condition that the curvature remains bounded is also a sufficient one for an invariant Calabi-Yau monopole to extend.*

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