Modelling Animal Spirits in Financial Markets

A thesis presented for the degree of Doctor of Philosophy of Imperial College London and the Diploma of Imperial College

by

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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To Ada and the kids.

Abstract

The term 'animal spirits' was introduced by Keynes to describe the entrepreneur's often irrational optimism and drive to act as opposed to basing decisions on formal analysis. This PhD thesis provides an analysis, both theoretical and empirical, of this phenomenon in the financial markets from several points of view. In the first chapter we show that the pricing kernel in the economy may be represented in a probabilistic form, as a solution to a stochastic filtering problem. The noise in the associated information process may contain drift term that is impossible to estimate from current market prices of assets. This drift can be associated with 'animal spirits' driving the market. The second chapter is explicitly devoted to 'animal spirits': it introduces a factor based risk-management model for an illiquid project. We show that behavioural factors together with the collateralization mechanism often employed by banks not only increase the risk for the banking system, but also introduce anomalies during high-volatility crisis periods. In the third chapter we apply Hidden Markov Models to estimate animal spirits from historic asset prices. We argue that an arbitrary addition of a stress scenario to the model can greatly improve risk estimation. The last chapter deals with optimal investment problem in a model with behavioural factors. This may be linked to the pricing kernel discussion from the first chapter by the marginal utility maximisation approach to pricing derivatives.

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Table of contents

Abstract					
1	Intro	oduction	9		
	1.1	Background	11		
		1.1.1 Animal spirits	11		
		1.1.2 Confidence indices	13		
		1.1.3 Financial bubbles	15		
		1.1.4 Valuation of financial derivatives	18		
		1.1.5 Structural models of credit risk	20		
	1.2	Organisation	21		
2	Nois	e, risk premium and bubble	23		
	2.1	Pricing kernel	25		
	2.2	Probabilistic representation of the pricing kernel	28		
	2.3	Back to the market measure	33		
	2.4	Indeterminacy of the risk premium	35		
	2.5	Information-based interpretation	38		
	2.6	Anomalous price dynamics	40		
	2.7	Relation to the risk-neutral measure	43		
	2.8	Stochastic volatility	44		
	2.9	Discussion	45		
3	Tam	ing animal spirits	48		
	3.1	Introduction	48		
	3.2	The Project Finance Model	49		
		3.2.1 Financial Market	50		
		3.2.2 Project Finance	52		
	3.3	A simply computable example	54		
		3.3.1 Model specification	54		
		3.3.2 Results	56		
		3.3.3 Computations for the general case	62		

4	Esti	mating animal spirits	64
	4.1	Hidden Markov Models	67
	4.2	Continuous models	68
		4.2.1 Geometric Brownian Motion	69
		4.2.2 Geometric Ornstein-Uhlenbeck	69
		4.2.3 Mapping parameters in multidimensional case	70
	4.3	Data	71
	4.4	Estimation results	72
	4.5	Risk management and animal spirits	78
	4.6	Summary	81
	4.A	Appendix: Estimation	82
		4.A.1 The Baum-Welch algorithm	82
		4.A.2 Scaling	84
		4.A.3 The Viterbi Algorithm	87
		4.A.4 Markov observations	88
	4.B	Appendix: Model choice and forecasting	89
		4.B.1 Model choice	89
		4.B.2 Forecasting	90
		4.B.3 Multivariate model with lags	91
		4.B.4 Verification of forecast distribution	93
5	Anir	nal control	95
	5.1	Introduction	95
	5.2	Market	96
	5.3	Criterion	
	5.4	Investment	
	5.5	Optimal portfolio	
		5.5.1 General case	
		5.5.2 Independent case	
	5.A	Appendix: Measure change	
		5.A.1 Measure change for Markov chains	
		5.A.2 Measure change defined for regime-switching compound Poisson	
		processes	117
		5.A.3 Risk-adjusted stochastic exponentials	
	5.B	Appendix: Proof of Proposition 5.5.1	
Re	eferen	ces	131

Chapter 1

Introduction

When the recent credit crunch unfolded, many leading economists were trying to understand its origins. In several countries a major factor contributing to the current economic crisis was massive borrowing to fund investment projects on the basis of, in retrospect, grossly optimistic valuations. It seems that 'behavioural' factors—Keynes' 'animal spirits' or Greenspan's 'irrational exuberance'—played an important role in the build-up of the bubble preceding the crisis. Understanding these factor is crucial for portfolio management, investment and hedging, and this is the main aim of this PhD thesis.

In this work we assume behavioural factors to be given exogenously, resulting from the market players being not fully rational, having limited information or computational capacities, being influenced by 'animal spirits' or characterised by 'irrational exuberance', etc. We don't work with equilibrium models and we don't try to understand where these factors are originating from. We assume that behavioural factors impact the parameters of price processes of relevant assets, in particular the expected returns. We show how they tie with risk-neutral valuation, hedging, risk management and optimal investment. In many ways we are trying to explain phenomena similar to financial bubbles, e.g. asset prices rising quicker than expected in some rational models, and then crashing down. Our approach is different from the one presented in financial literature on bubbles, however. Firstly, the financial bubble literature concentrates on comparing the market price of the asset to the so called fundamental value, whereas we adopt a more dynamic approach where we look at the rate of change of the prices, including the drift. Moreover, we assume that

Chapter 1. Introduction

the animal spirits exist in the market measure, and the financial bubbles are defined in the risk neutral measure.

Behavioural factors may impact the drift and volatility of financial assets. They are real world phenomena and hence take place in the market probability measure \mathbb{P} . Because of the no-arbitrage principle, by definition in the risk neutral \mathbb{Q} measure all assets have the same drift, equal to the short rate of interest. Therefore, as we show in Chapter 2, it is not possible to estimate the real-world drift from market prices of derivatives, because it is impossible to estimate the behavioural effects in the drift of the assets. On the other hand, in models driven by Brownian motion, the volatility of traded assets is the same in both the market and pricing measures. Note however, the latter property is model dependent—it is not the case in discrete-time models or jump models. This fact may be linked with an observed discrepancy between implied and realized volatilities.

We believe 'animal spirits' had a profound effect on the property market, before and during the crisis. Initially investors were over optimistic and drove the prices to record levels, only to stumble down when the bubble burst. In Chapter 3 we show that on top of the behavioural factors, the standard practises in the banking industry of marking all the assets to market made the crisis even worse.

Behavioural factors may be estimated from historic prices, as opposed to current prices of financial derivatives. Of course, in the noise-dominated financial data, it is not possible to precisely estimated all the parameter values. It is, however, possible to pretty closely determine the market regimes. In Chapter 4 we show that the most likely regimes estimated from historic property prices and consumer confidence index coincide with the intuition. We also introduce a straight-forward procedure for risk calculation and addition of stress scenarios in these models.

Finally, 'animal spirits' are critical for market investors and portfolio managers. In Chapter 5 we propose a regime-switching jump-diffusion model for asset prices and solve the optimal portfolio selection problem for investors maximising the risk-sensitive criterion.

This work builds up and is closely related to a wide array of existing literature, which we summarise in the next section. We start by showing the origin of the terms 'animal spirits' and 'irrational exuberance', then we discuss confidence indices, financial bubbles,

Chapter 1. Introduction

the theory of financial derivative valuation and structural models of credit risk. In the last section of this chapter we outline the contents of the thesis.

1.1 Background

The Sections 1.1.1, 1.1.2, parts of 1.1.4 and 1.1.5 of this chapter are reprinted from the paper Andruszkiewicz et al. (2013), with kind permission from Springer Science+Business Media B.V.

1.1.1 Animal spirits.

John Maynard Keynes is not always recognised as a founding father of behavioural finance but, as in so many areas, the great man got there first. Indeed, in his Nobel Prize lecture, George Akelof (2003) states that 'Keynes' General Theory was the greatest contribution to behavioural economics before the present era'. The key to Keynes' thinking can be found in the General Theory (Keynes, 2007, page 161):

[A] large proportion of our positive activities depend on spontaneous optimism rather than on a mathematical expectation, whether moral or hedonistic or economic. Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as a result of animal spirits*—of a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities.

A similar concept—irrational exuberance—was introduced by the then-Federal Reserve Board Chairman, Alan Greenspan (1996) in his speech at the Annual Dinner and Francis Boyer Lecture of The American Enterprise Institute for Public Policy Research, Washington, D.C:

^{*}The phrase is an allusion to the classical term *spiritus animalis* conveying the idea of animation, not atavism!

1.1 Background

Clearly, sustained low inflation implies less uncertainty about the future, and lower risk premiums imply higher prices of stocks and other earning assets. We can see that in the inverse relationship exhibited by price/earnings ratios and the rate of inflation in the past. But how do we know when irrational exuberance has unduly escalated asset values, which then become subject to unexpected and prolonged contractions as they have in Japan over the past decade?

This term was later picked-up by other economists and journalists, including Shiller (2000), who used it as the title of his book.

While nobody, surely, could disagree with the basic point, there is a mixed message here in that Keynes appears to be warning us off probabilistic and statistical analysis, and indeed he was quite sceptical about it, as reported by Akerlof and Shiller (2009, page 16). This point of view was in fact prevalent at the time. Frank Knight, whose seminal work substantiated the distinction between risk and uncertainty, noted in Knight (1921):

It is a world of change in which we live, and a world of uncertainty. We live only by knowing something about the future; while the problems of life, or of conduct at least, arise from the fact that we know so little. This is as true of business as of other spheres of activity. The essence of the situation is action according to opinion, of greater or less foundation and value, neither entire ignorance nor complete and perfect information, but partial knowledge. If we are to understand the workings of the economic system we must examine the meaning and significance of uncertainty; and to this end some inquiry into the nature and function of knowledge itself is necessary.

Knight categorizes 'probabilities' into

- 1. *A priori probability*, a probability that can be computed exactly and objectively because the exact nature and structure of the underlying experiment is known;
- 2. Statistical probability, an empirical probability;
- 3. Estimates

Concerning estimates, Knight writes:

It is this third type of probability or uncertainty which has been neglected in economic theory, and which we propose to put in its rightful place. As we have repeatedly pointed out, an uncertainty which can by any method be reduced to an objective, quantitatively determinate probability, can be reduced to complete certainty by grouping cases. [...] The present and more important task is to follow out the consequences of that higher form of uncertainty not susceptible to measurement and hence to elimination. It is this true uncertainty which by preventing the theoretically perfect outworking of the tendencies of competition gives the characteristic form of 'enterprise' to economic organization as a whole and accounts for the peculiar income of the entrepreneur.

This message is markedly different from the ideas promoted by standard financial economics starting in the 1950s: uncertainty appears as irrelevant because it can be diversified away or hedged against. The implication is that rational decision makers should rely on *a priori* probabilities when known or on statistical probabilities to form their opinions. Decision makers who are not rational, such as noise traders, will be arbitraged out of the economy.

The flourishing field of behavioural finance has demonstrated convincingly that the view held by standard finance theory is not tenable *stricto sensu*. Noise traders are alive and well and arbitrage is fraught with difficulties (see for example Shleifer (2000) for a discussion). Although most of the ideas and tests performed by behavioural finance theorist focus on pure decision theory and on psychology, behavioural finance does not exclude quantitative models and methods. In fact the scope for quantitative methods is much greater now than it was in the 1920s and 1930s, or even in the 1950s and 1960s thanks to the creation of confidence indices designed to gauge 'animal spirits,' the development of a theory of financial economics and advances in computational technology.

1.1.2 Confidence indices

Confidence indices are the closest readily available data that tries to capture factors related to animal spirits. Data on confidence indices is now widely available, see for example Markit (2011). There are two varieties, the *consumer confidence index* and the *purchasing managers' index* (PMI). Both are based on surveys, and represent respectively the propensity of consumers to go out and spend, and the propensity of businesses to invest. In the United States, consumer confidence is measured by the Conference Board and by the University of Michigan. The main difference between the two surveys is in the time horizon: while the Conference Board polls households on their expectations over the next six months, the University of Michigan looks at expectations over the coming year. On the other hand, purchasing manager expectations are assessed regionally: the Chicago PMI is widely regarded as the most representative of nationwide sentiment.

There are a number of empirical studies—see Akerlof and Shiller (2009), footnote 9, page 179— aimed at testing whether confidence actually 'causes' economic growth (interpreted in the sense of 'Granger causality' Granger (1969)). These include Matsusaka and Sbordone (1995) who produce quite convincing evidence that this link exists[†].

Following in the footsteps of Matsusaka and Sbordone, Howrey (2001) investigates the predictive power of the University of Michigan consumer confidence index over the period 1961 to 1999. He finds that the consumer confidence index is a statistically significant predictor of the future rate of growth of real GDP and of recessions. He also finds that consumer confidence provides a good point estimate of future consumer spending, albeit with a large standard error. In the case of Japan, Utaka (2003) finds that consumer confidence has a short term impact on GDP growth, but no short term effect.

If confidence is a good predictor of macroeconomic trends and cycles, could it also have an impact on asset prices? In an event study, Rigobon and Sack (2008) test the impact of unexpected changes in 13 macroeconomic data series including the Chicago PMI and consumer confidence on eurodollar futures contracts, treasury yields and the S&P 500. They find that surprises in the Chicago PMI and in consumer confidence have a statistically significant impact on the rate of six-month and 12-months eurodollar futures contracts and on the yields of 2-year and 10-year Treasuries, but not on the S&P 500.

Still, sentiment by and large plays a significant role in the behaviour of stock markets, as evidenced for example by Baker and Wurgler (2006). It is therefore natural to investigate

[†]Akerlof and Shiller (2009) somehow understate the reach of Matsusaka and Sbordone's argument.

Chapter 1. Introduction

specifically the relation between consumer confidence and stock market returns. Jansen and Nahuis (2003) study the relationship between stock market developments and consumer confidence in eleven European countries over the period 1986-2001. Although consumer confidence is positively correlated with stock market returns in nine countries, they did not find statistical evidence that consumer confidence Granger-causes stock market returns. To the contrary, stock market returns appears to Granger-cause consumer confidence over a short horizon of two weeks to one month. This result is intriguing, especially when we consider studies on leading economic indicators. Hertzberg and Beckman (1989) find that consumer confidence has a lead time of 14 months with respect to economic peaks while the S&P has a shorter lead time of 8.5 months. The gap is narrower for economic troughs: 4.5 months for consumer confidence versus 4 months for the S&P 500.

Fisher and Statman (2000, 2003) find that statistically significant increases in the bullishness of individual investors follow increases in consumer confidence. Over the period 1989 to 2002, large improvements in consumer confidence appear to have be followed by high returns on the S&P 500 index, NASDAQ index and among small caps. Lemmon and Portniaguina (2006) find that consumer confidence is useful in forecasting the returns on small stocks. Their view is that consumer confidence reflects not only current and expected fundamentals but also excessive sentiment such as overoptimism and pessimism. As a result of excessive optimism (pessimism), investors will overvalue (undervalue) small stocks relative to large stocks.

The relation between investor or entrepreneur sentiment and asset market is both important and complex, and more research is needed to understand their connection. This is particularly true for real estate, for which the literature linking confidence and real estate prices is scarcer.

1.1.3 Financial bubbles

Financial bubbles are a common subject of research in economics, possibly because they are often commented on in popular press. Some economists argue that bubbles are a key ingredient of economic cycles, and hence impact the performance of whole economy. Usually bubbles are defined as the difference between the market price of an asset and the

fundamental value, which in turn is defined as discounted expectation of all the cashflows (dividends) associated with with the asset.

Economic perspective

Most of the models on bubbles published in economic journals are in discrete time setting for easier tractability. In the economic literature authors always take expectations in the real probability measure, because the agents optimize consumption and investment rather than construct replicating portfolios. There are two types of expectations used in economic models: adaptive (where the agents' expectations are based on current and historic levels) and rational ("self-fulfilling", agents' expectations are aligned with true distribution of future events). One can show that bubbles can't exist with adaptive expectations, hence they can be associated with rational expectations, see Camerer (1989) for details.

Rational bubbles must grow on average at a rate consistent with other securities: the short rate if traders are risk-neutral or a higher rate if they are risk-averse. Blanchard and Watson (1982) show that if the bubble has a positive probability of bursting then in periods before it bursts it needs to grow faster than other securities, to offer a sort of risk premium for the investor. They show that the bubbles may exist even though rational traders know that the probability they will burst tends to one as the time goes to infinity.

Tirole (1982, 1985) shows that neither restrictions on short selling nor heterogeneous information among traders affect the possibility of bubbles. However, certain market limits are enough to rule out bubbles. If an asset has a finite time horizon in discrete setting then by backward induction no rational bubbles can exist. With known wealth constraints the rational bubble needs to stop growing at some point, so the same argument applies. There can be no bubbles if there are only finite many traders with rational trading strategies—at some date each trader will retire to spend all his gains on consumption. Other traders will be left with a negative sum game, so no-one will buy the asset at a bubble price. This argument doesn't hold if there is an infinite number of traders (e.g. overlapping generations).

Harrison and Kreps (1978) point out that traders usually take into account the resell value of an asset rather than the dividend flow when making decisions. They also note that the intrinsic value is not well defined if traders have heterogeneous opinions based on the

Chapter 1. Introduction

same information. In this case the possibility of resale of the asset drives the price higher hence causing a bubble.

Risk-neutral approach

Jarrow et al. (2007, 2010) provided the most comprehensive analysis of bubbles in the financial setting. Unlike economic equilibrium setting described above, financial models don't impose as much structure on the economy. In particular, financial models don't assume that investors are trading optimally or that the market clears.

The authors define the fundamental value as the risk neutral expectation of future dividends (they assume zero short rate for simplicity of exposition), treating the fundamental price almost as a derivative contract based on the dividend stream:

$$S_t^* = \mathbb{E}^{\mathbb{Q}} \left[\int_t^\infty D_s \mathrm{d}s \, \middle| \, \mathcal{F}_t \right] \tag{1.1}$$

The bubble is defined as usual as $\beta = S - S^*$. In this setting the bubble can either be a true martingale (uniformly integrable or not) or a strict local martingale. The most interesting is the strict local martingale case, because it corresponds to the situation when the underlying asset has a finite bounded time horizon. If this type of bubble exists, it is not possible to create an arbitrage portfolio and benefit from it because we require that all trading strategies are bounded from below (admissibility condition).

Jarrow et al. (2007) show that in complete markets it is impossible for bubbles to start existence. Either they exist at the start of the model or there are no bubbles. Moreover, they notice that under the most commonly used no-arbitrage condition, *No Free Lunch with Vanishing Risk* (NFLVR), bubbles can cause the put-call parity to fail. And because put-call parity almost always holds in reality, they decided to introduce a stronger no-arbitrage condition: Merton's *no dominance*. However, under this condition bubbles can't exist at all.

The situation is different in incomplete markets. Jarrow et al. (2010) show that bubbles can be created if the market "decides" to change the chosen risk-neutral probability measure (at some random time). They provide a suitable extension to standard NFLVR theory so

that it accommodates a change of risk neutral measure. Also, because there is no guarantee that replicating strategies exist, the *no dominance* condition doesn't rule out bubbles in incomplete markets.

In this work we will not deal with the concept of the financial bubble directly. We try to explain the same phenomena using behavioural approach by introducing animal spirits. We attribute the rises and falls of asset prices observed in the market to the optimism of investors and mismanagement of risk by the banks. We show that such a price process behaviour can be observed in the market measure \mathbb{P} , without ever generating a bubble in Jarrow et al. (2007) sense.

1.1.4 Valuation of financial derivatives

Financial derivative is a financial instrument whose value depends on other, more basic, underlying variables. Usually the underlying variables are traded securities such as stocks and bonds. However, contracts with underlying which is impossible to trade, such as the temperature, are becoming more and more popular.

The theory of derivative valuation hinges upon the principle of no arbitrage and replication. It formalizes the natural assumption that in a liquid market it should not be possible to make a profit with zero net investment and without bearing any risk. This idea was first introduced in the famous works of Black and Scholes (1973) and Merton (1973). They showed that the no arbitrage assumption is enough to uniquely determine the price of European options in the geometric Brownian motion setting and that it is possible to construct a dynamic self-financing portfolio that has the same value as the European option at maturity.

Harrison and Kreps (1979); Harrison and Pliska (1981); Kreps (1981) linked the no arbitrage pricing approach with stochastic calculus and martingale theory. Their main contribution was *the fundamental theorem of asset pricing*: there is no arbitrage in the economy if and only if there exists a probability measure \mathbb{Q} , equivalent to the market measure \mathbb{P} , such that all asset prices discounted using the bank account are local martingales. The financial market is *complete* if and only if the probability measure \mathbb{Q} is unique. This new probability measure is usually called the *risk-neutral measure* or *equivalent martingale measure*. Once the \mathbb{Q} measure is established contingent claim valuation boils down to calculation of

Chapter 1. Introduction

a conditional expectation with respect to this measure:

$$H_t = B_t \mathbb{E}^{\mathbb{Q}} \left[\left. \frac{H_T}{B_T} \right| \mathcal{F}_t \right]$$
(1.2)

where B_t is the value of the bank account and H_T is the contingent claim maturing at time T. Delbaen and Schachermayer (1994, 1998) further formalized this result for more general semi-martingale models.

Equation (1.2) shows that C(K) is the expected value of the option payoff expressed in units of the 'savings account' B_t . It turns out that using the savings account as numéraire is an arbitrary choice. The modern view, stated explicitly in Geman et al. (1995) and clearly expounded by Hunt and Kennedy (2004) for example, is to think in terms of *numéraire pairs* (N, \mathbb{Q}) , where N_t is a tradable asset with strictly positive price, conventionally normalized to $N_0 = 1$, and \mathbb{Q} is a measure such that for any traded asset S the price ratio S_t/N_t is a martingale. A key point is that if one fixes the measure \mathbb{Q} and searches for an asset price process $N^{\mathbb{Q}}$ such that $(N^{\mathbb{Q}}, \mathbb{Q})$ is a numéraire pair, then there is a unique solution, namely that $N^{\mathbb{Q}}$ is the *growth-optimal* portfolio[‡] when the asset prices are governed by the probability law \mathbb{Q} . The growth-optimal portfolio maximizes, over investment strategies h, the expected log-utility $\mathbb{E}_{\mathbb{Q}}[\log V_T^h]$ at some fixed time T, where V_T^h is the value of the investment portfolio at time T using strategy h, starting conventionally at $V_0^h = 1$. If $(N^{\mathbb{Q}}, \mathbb{Q})$ is a numéraire pair then using the inequality $\log x \leq x - 1$ and the numéraire property of $N^{\mathbb{Q}}$ we have

$$\mathbb{E}_Q \log V_T^h - \mathbb{E}_Q \log N_T^{\mathbb{Q}} = \mathbb{E}_Q[\log(V_T^h/N_T^{\mathbb{Q}})] \le \mathbb{E}_Q[V_T^h/N_T^{\mathbb{Q}}] - 1 = 0.$$

Thus $N_T^{\mathbb{Q}}$ maximises logarithmic utility under \mathbb{Q} .

J.B. Long (1990) first realized the significance of this fact, namely that there is nothing stopping us choosing $\mathbb{Q} = \mathbb{P}$, the real-world 'statistical' measure governing asset prices, and then $N^{\mathbb{P}}$ is the optimal investment portfolio for an investor with logarithmic utility, which is easily computed in many cases. This approach has the decisive advantage that *all*

[‡]See MacLean et al. (2011) for a comprehensive account of investment based on the growth-optimal or 'Kelly' criterion

1.1 Background

modelling is carried out under the statistical measure. It is the basis for Platen's 'benchmark approach' to financial valuation, see Platen and Heath (2006). Karatzas and Kardaras (2007) discuss the relation between the existence of the numeraire portfolio and noarbitrage conditions in the market. In our case we want to include econometric factors such as confidence indices, GDP growth etc. in our modelling framework. If we use the benchmark approach then econometric models for these quantities, estimated using historical data, can be plugged right into our model without worrying about the distinction between real-world and risk-neutral measures since these two things are now one and the same.

An alternative for valuation in the \mathbb{P} measure is the *pricing kernel* (also known as *state price deflator, stochastic discount factor* among other names) approach. The pricing kernel, denoted here by $\{\pi_t\}_{t\geq 0}$, is a positive supermartingale with the property that if S_T is the price at time T of an asset that pays no dividend, then the price at time t of the asset is given by the following conditional expectation under the market measure:

$$S_t = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T S_T]. \tag{1.3}$$

In particular, if $S_T = 1$, then (1.3) gives the pricing formula for the discount bond: $P_{tT} = \mathbb{E}_t[\pi_T]/\pi_t$. Because both the pricing kernel as well as the value of the numeraire portfolio are strictly positive we have the obvious relationship:

$$\pi_t = \frac{1}{N_t} \tag{1.4}$$

As opposed to the log-optimal portfolio, the pricing kernel is not unique in incomplete markets. Long's numeraire portfolio in diffusion environment corresponds to the Minimal Martingale Measure introduced by Föllmer and Schweizer (1991), see Černý (1999). This in turn corresponds to a certain choice of pricing kernel, and hence of the risk-neutral measure.

1.1.5 Structural models of credit risk

In Chapter 3 we shall be considering investment funded by collateralized loans, where the investor may default if he is unable to post sufficient additional margin in case of a

Chapter 1. Introduction

fall in the value of the collateral. In our model, the time at which this happens will be a stopping time of some filtration. This is true of all models in the modern theory of credit risk, although in this theory, as is seen in textbooks such as Lando (2004), there are two distinct classes of model, 'reduced form' and 'structural form'. The latter, which contains our model, is ultimately derived from early work by Robert Merton (1974) in which the default risk on corporate debt is represented as a put option on the value of the firm. Modelling firm value accurately is not an easy thing to do (it is not the same thing as market capitalization), and later modellers such as Hull and White (2001) or Longstaff and Schwartz (1995) have concentrated on stylized models in which default occurs at the first hitting time of a possibly time-varying boundary by some stochastic process, where parameters specifying the process and/or the boundary are calibrated from market credit default swap quotes. Our model is in the same vein mathematically, but because we model explicitly the collateral value and the evolution of the margin account we return to a Mertonlike picture where the credit model has economic as well as mathematical content.

1.2 Organisation

In this thesis we explore different aspects of dealing with behavioural factors in mathematical finance. In Chapter 2 we look at 'animal spirits' from information-theoretic perspective: we show that the pricing kernel (state price density) has the form of the solution to an auxiliary filtering problem, and animal spirits can be interpreted as the drift part of the noise structure in the information process. We conclude the chapter by showing that the drift caused by anomalies in market price of risk is not visible in market prices of derivatives. In Chapter 3 we propose a risk model that illustrates how property investors' optimism, together with margin-call procedure employed by the money-lending banks, could possibly trigger the crisis and cause high losses to the banks. Chapter 4 is devoted to analysis of historic data. Using the example of house prices in the USA we discuss ways of estimating animal spirits and employ a hidden Markov model. It is used for forecasting and calculating the value at risk. We introduce an efficient method of improving the risk estimates by introducing an auxiliary stress scenario. Finally the last chapter, 5, solves the problem of optimal investment in a market driven by behavioural factors modelled as finite state Markov chains. A novel feature introduced in this chapter is the joint model for jumps in the asset and the factor processes.

Please note that the individual chapters form independent research projects, with different approach and technical toolset. Hence they may be read independently and in any order, and different notation is used in every chapter—most appropriate in the given setting.

Chapter 2

Noise, risk premium and bubble

The market risk premium is one of the main factors that drive the return of any given portfolio of assets. As we explain in detail later in the chapter, in our opinion it is closely linked to 'animal spirits' in the market. It is a key quantity for hedge funds, pension funds, and numerous other investors. The risk premium can make investments grow smoothly or jump up and down widely, often in an unpredictable manner. In spite of the important role it plays in asset allocation, however, the risk premium is in the traditional sense notoriously difficult to estimate from observed price processes of various risky assets (see, e.g., Rogers 2001). Is it possible then to estimate the risk premium from current prices of financial derivatives?

In a Brownian-driven market, if $\{S_t\}$ denotes the price process of a risky asset and H(s) is the payout function of a European contingent claim expiring at T, then the current price of this derivative is given by the expectation of the cash flow $H(S_T)$, suitably discounted, in the risk-neutral measure. Because asset price processes in the risk-neutral measure are independent of the market risk premium, one might be tempted to conclude therefore that derivative prices are likewise entirely independent of the risk premium. Indeed, in the case of the Black-Scholes-Merton model where all relevant parameters remain constant in time, the risk premium drops out of derivative pricing formulae. Notwithstanding this example, it is worth bearing in mind that the choice of the pricing measure does depend on the choice of the risk premium. Thus, derivative prices in general will depend implicitly on the risk premium, often in a nonlinear way. It follows that calibration of the market risk premium

from option prices is feasible within a given modelling framework (Brody et al. 2012).

The main purpose of this chapter is to address the question to what extent it is possible, at least in principle, to determine the risk premium, if the totality of arbitrage-free market prices for various derivatives were available. We shall find that the market risk premium consists of two components in an additive manner (for models based on Brownian filtrations): The first of the two, which we might call a 'systematic' component, depends explicitly on the term structure of the market, while the second, which we might call an 'idiosyncratic' component, is independent of the term structure of the market, and thus can be identified as *pure noise*. We show that the systematic component can in principle be determined from current market data, whereas the idiosyncratic noise component is strictly 'hidden' and thus cannot be inferred from derivative prices. Therefore, the risk premium can be backed out from market data only up to an indeterminable additive noise.

Although the noise component cannot be inferred directly, it nevertheless has an impact on the dynamics of asset prices under the physical measure, even though it does not carry information concerning the 'true' state of affairs. Hence a spontaneous creation of superfluous noise can move the price of an asset in an essentially arbitrary direction. In particular, because the risk premium, and hence its noise component, is a vectorial quantity, the direction of the noise vector can at times lie close to the directions of volatility vectors of the share prices of a particular industrial sector, leading to the creation of a 'bubble' for that sector by pushing up these share prices. When a more reliable information concerning the state of that sector is unveiled, the direction of the risk premium vector is likely to change so as to generate a negative component in the excess rate of return. This can be exacerbated by an increase in the magnitude of asset volatilities due to information revelation, thus leading to a 'burst'. Such a scenario need not be confined to a particular financial sector; the apparent existence of the so-called 'equity premium puzzle' over a specified period can likewise be attributed to the prevailing noise that points in the general direction of the equity market volatility, but not in the direction of the bond market volatility.

The formulation that we shall develop in what follows thus entails the an element of phenomenology for characterising the mechanism of anomalous price dynamics in an intuitive manner; as such, it is not meant to 'explain' the cause of the creation of anomalous price movements; nor does it address the predictability of these events. Indeed, the presence of the noise might well be interpreted as what Keynes described 'animal spirits' or what Greenspan referred to as an 'irrational exuberance', see Section 1.1.1 for background. In any event, since our characterisation of the phenomena commonly known as bubbles is radically different from those adopted in the literature, we are able to circumvent the analysis based on subtle distinctions between strict local martingales and true martingales. We also provide a heuristic argument why the hidden noise might have the tendency of creating equity premium.

2.1 Pricing kernel

For definiteness, we shall be adopting the pricing kernel approach (see, e.g., Cochrane 2005, Björk 2009). We model the financial market on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Here \mathbb{P} denotes the 'physical' probability measure, and $\{\mathcal{F}_t\}$ is assumed to be generated by a multi-dimensional Brownian motion. Expectation under \mathbb{P} is denoted $\mathbb{E}[-]$, and for the conditional expectation with respect to \mathcal{F}_t we write $\mathbb{E}_t[-]$. Two other probability measures enter the ensuring discussion; these are the risk-neutral measure \mathbb{Q} and an auxiliary measure \mathbb{Q} to be described below. Expectations in these measures will be written $\mathbb{E}^{\mathbb{Q}}[-]$ and $\mathbb{E}^{\mathbb{Q}}[-]$, respectively.

We assume that the market is free of arbitrage opportunities, and that there is an established pricing kernel; whereas market completeness is not assumed. These assumptions imply the existence of a unique preferred risk-neutral measure \mathbb{Q} . The pricing kernel, denoted here by $\{\pi_t\}_{t\geq 0}$, is a positive supermartingale with the property that if S_T is the price at time T of an asset that pays no dividend, then the price at time t of the asset is given by

$$S_t = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T S_T].$$
(2.1)

In particular, if $S_T = 1$, then (2.1) gives the pricing formula for the discount bond: $P_{tT} = \mathbb{E}_t[\pi_T]/\pi_t$ (cf. Constantinides 1992).

We begin by discussing properties of the pricing kernel that are relevant to our analysis here. In addition to being a positive right-continuous supermartingale, the pricing kernel fulfils the condition that $\mathbb{E}[\pi_t] \to 0$ as $t \to \infty$. A positive supermartingale possessing this property is known as a *supermartingale potential* (Doob 1984). In fact, any strictly positive class-D potential (cf. Meyer 1966, Protter 2005), i.e. a potential that can be expressed in the form $\mathbb{E}_t[A_{\infty}] - A_t$ in terms of an integrable increasing process $\{A_t\}$, such that $\{A_t\}$ can be expressed in the form

$$A_t = \int_0^t a_s \mathrm{d}s \tag{2.2}$$

for some adapted positive process $\{a_t\}$, determines a pricing kernel $\{\pi_t\}$. It suffices to choose the process $\{a_t\}$ to model the pricing kernel. This leads to the potential approach of Rogers (1997) to model term structure dynamics. A substitution shows that

$$\pi_t = \int_t^\infty \mathbb{E}_t[a_u] \mathrm{d}u. \tag{2.3}$$

The representation (2.3) takes the form of that of Flesaker and Hughston (Flesaker & Hughston 1996, 1997; Rutkowski 1997; Jin & Glasserman 2001), if we make the following identification. First, writing $\rho_0(T) = -\partial_T P_{0T}$, where P_{0T} is the initial discount function, we see that the processes $\{M_t(u)\}_{t\geq 0, u\geq t}$ defined by

$$M_t(u) = \frac{\mathbb{E}_t[a_u]}{\rho_0(u)} \tag{2.4}$$

is a one-parameter family of positive martingales, i.e. for each fixed $u \ge t$, $\{M_t(u)\}$ is a martingale. This follows on account of the martingale property of the conditional expectation $\mathbb{E}_t[a_u]$, and the fact that $\{a_t\}$ is positive. In terms of these positive martingales, the pricing kernel can be expressed in the Flesaker-Hughston form:

$$\pi_t = \int_t^\infty \rho_0(u) M_t(u) \mathrm{d}u.$$
(2.5)

From the martingale representation theorem we deduce that the dynamical equations satisfied by the positive martingale family $\{M_t(u)\}$ take the form:

$$\mathbf{d}M_t(u) = M_t(u)v_t(u)\mathbf{d}\xi_t,\tag{2.6}$$

where $\{v_t(u)\}$ is a family of adapted processes and $\{\xi_t\}$ is a standard Brownian motion under the \mathbb{P} measure. (Here and in what follows, for simplicity of notation we shall write $v_t(u)d\xi_t$, $\lambda_t d\xi_t$, and so on, to denote vector inner products.) We observe therefore that modelling of the pricing kernel is equivalent to modelling of the one-parameter family of volatility processes $\{v_t(u)\}$. Note that we require $\{M_t(u)\}$ be a strict martingale. For this purpose it suffices that $\{v_t(u)\}$ satisfies, for each u, the Novikov condition. In fact, throughout the paper we shall make a stronger assumption that the volatility processes $\{v_t(u)\}$ are *bounded*. The advantage of this assumption is that it simplifies calculations without losing the generality of economic considerations, thus avoids 'pathological' situations.

On account of (2.5) and (2.6) we deduce, by an application of Ito's lemma, that the dynamical equation satisfied by the pricing kernel takes the form

$$\frac{\mathrm{d}\pi_t}{\pi_t} = -r_t \mathrm{d}t - \lambda_t \mathrm{d}\xi_t, \qquad (2.7)$$

where

$$r_{t} = \frac{\rho_{0}(t)M_{t}(t)}{\int_{t}^{\infty}\rho_{0}(u)M_{t}(u)\mathrm{d}u}$$
(2.8)

is the short rate, and

$$\lambda_t = -\frac{\int_t^\infty \rho_0(u)v_t(u)M_t(u)\mathrm{d}u}{\int_t^\infty \rho_0(u)M_t(u)\mathrm{d}u}$$
(2.9)

is the market risk premium. The fact that the drift of $\{\pi_t\}$ can be identified with the short rate can be seen by applying the martingale condition (2.1) on the money market account $\{B_t\}$ satisfying $dB_t = r_t B_t dt$. That is, the drift of $\{\pi_t B_t\}$ vanishes if and only if the drift of $\{\pi_t\}$ is $\{-r_t\}$. Similarly, let us write $\{\mu_t\}$ for the drift of a risky asset $\{S_t\}$ that pays no dividend, and $\{-\lambda_t\}$ for the volatility of $\{\pi_t\}$. Then the martingale condition on $\{\pi_t S_t\}$ implies that $\mu_t = r_t + \lambda_t \sigma_t$, which shows that $\{\lambda_t\}$ indeed expresses the excess rate of return above the risk-free rate in unit of volatility.

An advantage of working with the pricing kernel is that once a model is chosen for the volatility processes $\{v_t(u)\}$ of the martingale family, we are able not only to price a wide range of derivatives via the pricing formula $\mathbb{E}[\pi_T H_T]$, where H_T is the payout of a derivative, but also to obtain a model for the interest rate term structure. Furthermore, a model for $\{v_t(u)\}$, which can be calibrated by use of market data for derivative prices, *implies* a process for the risk premium $\{\lambda_t\}$ according to the prescription (2.9), and this in turn can be used for asset allocation purposes. This is the sense in which derivative prices can be used to calibrate the risk premium, within any modelling framework (cf. Brody *et al.* 2012). With this in mind, the issues that we would like to address here are: (a) the ambiguity associated with the determination of the risk premium from market data in the Brownian setup; and (b) the identification of the origin of this ambiguity. For these purposes, it is useful to examine the probabilistic characterisation of the pricing kernel, within the term structure density approach of Brody and Hughston (2001).

2.2 Probabilistic representation of the pricing kernel

To proceed, we shall make the following observation that (i) the positivity of nominal interest, and (ii) the requirement that a bond with infinite maturity must have vanishing value, imply that $\rho_0(T) = -\partial_T P_{0T}$ defines a probability density function on the positive half-line (Brody and Hughston 2001, 2002). More generally, the positivity of the martingale family $\{M_t(u)\}$ implies that $\{\rho_t(u)\}$ defined by

$$\rho_t(u) = \frac{\rho_0(u)M_t(u)}{\int_0^\infty \rho_0(u)M_t(u)du}$$
(2.10)

is a measure-valued process, i.e. $\rho_t(u) \ge 0$ for all t and all u; and

$$\int_0^\infty \rho_t(u) \mathrm{d}u = 1 \tag{2.11}$$

for all $t \ge 0$. The measure-valued process thus introduced suggests the existence of a random variable X whose conditional density under some probability measure is given by (2.10). Furthermore, an application of Ito's lemma on (2.10) shows that

$$\frac{\mathrm{d}\rho_t(u)}{\rho_t(u)} = \left(v_t(u) - \hat{v}_t\right) \left(\mathrm{d}\xi_t - \hat{v}_t\mathrm{d}t\right),\tag{2.12}$$

where

$$\hat{v}_t = \int_0^\infty v_t(u)\rho_t(u)\mathrm{d}u \tag{2.13}$$

can be thought of as the conditional expectation of $v_t(X)$. Observe that the dynamical equation (2.12) takes the form of a Kushner equation, thus indicates the existence of an *auxiliary* filtering problem. More precisely, we have:

Proposition 2.2.1 For every admissible model of the pricing kernel, the associated term structure density process $\{\rho_t(u)\}$ is the solution to an auxiliary filtering problem.

That this indeed is the case will be shown as follows. Let $(\Omega, \mathcal{F}, \tilde{\mathbb{Q}})$ be a probability space, upon which X is a positive random variable with density $\rho_0(u)$, and $\{\beta_t\}$ is an $\tilde{\mathbb{Q}}$ -Brownian motion, independent of X. Take a time interval $[0, \tau]$, with τ fixed, and let the information (observation) process $\{\xi_t\}_{0 \le t \le \tau}$ be given by

$$\xi_t = \int_0^t v_s(X) \mathrm{d}s + \beta_t, \qquad (2.14)$$

where $\{v_t(u)\}$ is bounded. For the moment let us assume, further, that $\{v_t(u)\}$ is deterministic, for simplicity of exposition. (A more general case is discussed in the Remark at the end of this section, and also in Section 2.8.) Now define the process $\{\Lambda_t\}_{0 \le t \le \tau}$ over $[0, \tau]$ by the expression

$$\Lambda_t = \exp\left(\int_0^t v_s(X) d\xi_s - \frac{1}{2} \int_0^t v_s^2(X) ds\right).$$
 (2.15)

We use $\{\Lambda_t\}$ to introduce a measure change, which will be defined on the measurable space $(\Omega, \mathcal{G}_{\tau})$, where $\mathcal{G}_{\tau} \subset \mathcal{F}$ is the sigma-subalgebra generated jointly by $\{\beta_t\}_{0 \leq t \leq \tau}$ and the value of X. The new measure \mathbb{P} on $(\Omega, \mathcal{G}_{\tau})$ is defined by the property that for any set $A \in \mathcal{G}_{\tau}$ we have the relation

$$\mathbb{P}(A) = \mathbb{E}^{\mathbb{Q}}\left[\Lambda_{\tau}^{-1}\mathbb{1}\left\{\omega \in A\right\}\right].$$
(2.16)

With this setup, we have the following:

Lemma 2.2.2 (i) On the probability space $(\Omega, \mathcal{G}_{\tau}, \mathbb{P})$ the process $\{\xi_t\}$ defined by (2.14) is a Brownian motion; (ii) $\{\xi_t\}$ is independent of X; (iii) the \mathbb{P} -density of the random variable X is given by $\rho_0(u)$; and (vi) for all $t \in [0, \tau]$ the expectation $\hat{X}_t = \mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t]$ of Xconditional on $\mathcal{F}_t = \boldsymbol{\sigma}(\{\xi_s\}_{0 \le s \le t})$ is given by the generalised Bayes formula:

$$\hat{X}_{t} = \frac{\mathbb{E}^{\mathbb{P}}[X\Lambda_{t}|\mathcal{F}_{t}]}{\mathbb{E}^{\mathbb{P}}[\Lambda_{t}|\mathcal{F}_{t}]}.$$
(2.17)

Statement (i) follows as a direct consequence of Girsanov's theorem. Statements (ii) and (iii) can be verified by showing that the joint characteristic function of X and $\{\xi_t\}$ factorises under the \mathbb{P} measure, and that it takes the desired form. For all real a, b we find, by use of the measure change $\mathbb{P} \to \tilde{\mathbb{Q}}$ involving $\{\Lambda_{\tau}^{-1}\}$:

$$\mathbb{E}^{\mathbb{P}}\left[\mathrm{e}^{\mathrm{i}a\xi_{t}+\mathrm{i}bX}\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}a(\int_{0}^{t} v_{s}(X)\mathrm{d}s+\beta_{t})+\mathrm{i}bX} \mathrm{e}^{-\int_{0}^{t} v_{s}(X)\mathrm{d}\beta_{s}-\frac{1}{2}\int_{0}^{t} v_{s}(X)^{2}\mathrm{d}s}\right]$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_{0}^{t}\left[\mathrm{i}a-v_{s}(X)\right]\mathrm{d}\beta_{s}-\frac{1}{2}\int_{0}^{t}\left[\mathrm{i}a-v_{s}(X)\right]^{2}\mathrm{d}s} \mathrm{e}^{-\frac{1}{2}\int_{0}^{t}a^{2}\mathrm{d}s+\mathrm{i}bX}\right]$$
$$= \mathrm{e}^{-\frac{1}{2}a^{2}t}\int_{0}^{\infty}\mathrm{e}^{\mathrm{i}bx}\rho_{0}(x)\mathrm{d}x, \qquad (2.18)$$

where we have made use of the statement (i), and the martingale property satisfied by the first exponential in the second line. As regards the conditional expectation of X, by change of measure using Λ_{τ} we obtain

$$\hat{X}_t = \frac{\mathbb{E}^{\mathbb{P}}[X\Lambda_\tau | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[\Lambda_\tau | \mathcal{F}_t]},\tag{2.19}$$

but since X and $\{\xi_t\}$ are \mathbb{P} -independent we have $\mathbb{E}^{\mathbb{P}}[X\Lambda_{\tau}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[X\Lambda_t|\mathcal{F}_t]$ and (2.17) follows. The expression (2.17) in fact is an example of the Kallianpur-Striebel formula familiar in the theory of nonlinear filtering (Wonham 1965, Kallianpur & Striebel 1968, Liptser & Shiryaev 2001). In particular, it follows at once that the conditional density of X

takes the form

$$\rho_t(u) = \frac{\rho_0(u) \exp\left(\int_0^t v_s(u) d\xi_s - \frac{1}{2} \int_0^t v_s^2(u) ds\right)}{\int_0^\infty \rho_0(u) \exp\left(\int_0^t v_s(u) d\xi_s - \frac{1}{2} \int_0^t v_s^2(u) ds\right) du},$$
(2.20)

which agrees with (2.10). This establishes Proposition 2.2.1 in the case where $\{v_s(u)\}$ is deterministic.

Leaving aside the financial interpretation of the measure $\tilde{\mathbb{Q}}$ for the moment (which we shall discuss in Section 2.7 in more detail; for now it suffices to note that $\tilde{\mathbb{Q}}$ is $\{X > t\}$ conditionally the risk-neutral measure \mathbb{Q}), the important consequence of Proposition 2.2.1
is that for every admissible term structure model characterised by the positive martingale
family $\{M_t(u)\}$ with deterministic volatility structure $\{v_t(u)\}$, there exists an ambient filtering problem in an auxiliary probability space that can be identified as an extension of the
risk-neutral measure.

The above-specified filtering problem leads to the following probabilistic interpretation for the pricing kernel. Writing

$$N_{t} = \int_{0}^{\infty} \rho_{0}(u) M_{t}(u) du$$

=
$$\int_{0}^{\infty} \rho_{0}(u) \exp\left(\int_{0}^{t} v_{s}(u) d\xi_{s} - \frac{1}{2} \int_{0}^{t} v_{s}^{2}(u) ds\right) du \qquad (2.21)$$

for the normalisation of the conditional density $\{\rho_t(u)\}\)$, we see that the pricing kernel is given by the 'unnormalised' conditional probability that X > t:

$$\pi_t = N_t \mathbb{Q}_t(X > t), \tag{2.22}$$

where for simplicity we have written $\tilde{\mathbb{Q}}_t(-) = \tilde{\mathbb{Q}}(-|\mathcal{F}_t)$ for the conditional probability. Further, the price of a discount bond admits a probabilistic representation in the $\tilde{\mathbb{Q}}$ measure:

$$P_{tT} = \frac{\mathbb{Q}_t(X > T)}{\tilde{\mathbb{Q}}_t(X > t)}.$$
(2.23)

This formula shows that the price process of a discount bond is given by the ratio of the $(\tilde{\mathbb{Q}}, \mathcal{F}_t)$ -conditional probability that the positive random variable X taking values greater than T and that of X taking values greater than t. By use of the Bayes formula, and the fact that the set $\{X > t\}$ contains the set $\{X > T\}$, we deduce the following:

Proposition 2.2.3 The prices of discount bonds $\{P_{tT}\}$ can be expressed in the form

$$P_{tT} = \tilde{\mathbb{Q}}_t(X > T | X > t), \qquad (2.24)$$

where $\tilde{\mathbb{Q}}_t(-) = \tilde{\mathbb{Q}}(-|\mathcal{F}_t)$ and $\mathcal{F}_t = \boldsymbol{\sigma}(\{\xi_s\}_{0 \le s \le t})$.

The expression (2.24) is essentially the representation obtained by Brody & Friedman (2009) for the discount bond using the information-based approach to interest rate modelling. It is worth remarking that the random variable X has the dimension of time. In Brody & Friedman (2009), X was interpreted as the arrival time of liquidity crisis, in the narrow sense of a cash demand. Hence, under this interpretation, (2.24) shows that the bond price at t is the probability that the timing of the occurrence of a cash demand is beyond T, given that it has not yet occurred at t, and given the noisy information (2.14) concerning the value of X, in a suitably risk-adjusted measure $\tilde{\mathbb{Q}}$.

Remark. In the foregoing analysis we have assumed that the volatility $\{v_t(u)\}$ of the martingale family is deterministic. Once the desired result on the existence of the 'hidden' filtering problem is established in the deterministic context, however, we may turn the argument around by starting with the filtering equation (2.14) and then generate a model for the pricing kernel. This, in turn, shows how the volatility structure $\{v_t(u)\}$ can be made random: Provided that the solution to the nonlinear filtering problem exists and is expressible in terms of the generalised Bayes formula (2.17), then $\{v_t(u)\}$ can depend on the history of $\{\xi_t\}$ in a general way. In Section 2.8 an explicit example of such a model will be constructed for illustrative purposes. It is also worth remarking that a more generic case often studied in the literature of nonlinear filtering concerns the situation in which the signal X is itself a random process; often taken to be a diffusion process (see, e.g., Liptser & Shiryaev 2001). We shall comment further on this situation in Section 2.8.

2.3 Back to the market measure

The normalisation $\{N_t\}$ can be used to effect a measure change $\tilde{\mathbb{Q}} \to \mathbb{P}$. To see this, note first that the process $\{W_t\}$ defined by

$$W_t = \xi_t - \int_0^t \mathbb{E}_s^{\tilde{\mathbb{Q}}}[v_s(X)] \mathrm{d}s$$
(2.25)

is a $\tilde{\mathbb{Q}}$ -Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$ generated by the information process (2.14). In fact, this is just the innovations representation for the filtering problem posed above. Thus, the Brownian property can be verified by checking that $\{W_t\}$ satisfies the martingale condition:

$$\mathbb{E}_{t}^{\tilde{\mathbb{Q}}}[W_{T}] = \mathbb{E}_{t}^{\tilde{\mathbb{Q}}}\left[\int_{0}^{T} v_{s}(X) \mathrm{d}s + \beta_{T} - \int_{0}^{T} \mathbb{E}_{s}^{\tilde{\mathbb{Q}}}[v_{s}(X)] \mathrm{d}s\right] \\
= \mathbb{E}_{t}^{\tilde{\mathbb{Q}}}\left[\int_{0}^{t} v_{s}(X) \mathrm{d}s + \beta_{t} - \int_{0}^{t} \mathbb{E}_{s}^{\tilde{\mathbb{Q}}}[v_{s}(X)] \mathrm{d}s\right] \\
+ \mathbb{E}_{t}^{\tilde{\mathbb{Q}}}\left[\int_{t}^{T} v_{s}(X) \mathrm{d}s + (\beta_{T} - \beta_{t}) - \int_{t}^{T} \mathbb{E}_{s}^{\tilde{\mathbb{Q}}}[v_{s}(X)] \mathrm{d}s\right] \\
= W_{t},$$
(2.26)

where we have made use of the martingale property $\mathbb{E}_t^{\tilde{\mathbb{Q}}}[\mathbb{E}_s^{\tilde{\mathbb{Q}}}[v_s(X)]] = \mathbb{E}_t^{\tilde{\mathbb{Q}}}[v_s(X)]$ of the conditional expectation for s > t, and the tower property of conditional expectation to deduce that $\mathbb{E}_t^{\tilde{\mathbb{Q}}}[\beta_T] = \mathbb{E}_t^{\tilde{\mathbb{Q}}}[\beta_t]$. Along with $d[W]_t = dt$, Lévy's characterisation shows that $\{W_t\}$ is an $\tilde{\mathbb{Q}}$ -Brownian motion.

On the other hand, from (2.21) and the martingale representation

$$M_t(u) = 1 + \int_0^t v_s(u) M_s(u) d\xi_s$$
(2.27)

for the positive martingale family $\{M_t(u)\}\$ we find

$$N_{t} = 1 + \int_{0}^{\infty} \left(\rho_{0}(u) \int_{0}^{t} v_{s}(u) M_{s}(u) d\xi_{s} \right) du$$

= $1 + \int_{0}^{t} \left(\int_{0}^{\infty} \rho_{0}(u) v_{s}(u) M_{s}(u) du \right) d\xi_{s},$ (2.28)

where we have made use of the fact that $\{v_t(u)\}$ is bounded and thus stochastic version of Fubini's theorem applies to interchange the limits. Upon differentiation, and recalling the definition (2.13), we obtain

$$\frac{\mathrm{d}N_t}{N_t} = \hat{v}_t \mathrm{d}\xi_t,\tag{2.29}$$

from which it follows that

$$N_t = \exp\left(\int_0^t \hat{v}_s \mathrm{d}\xi_s - \frac{1}{2}\int_0^t \hat{v}_s^2 \mathrm{d}s\right)$$
(2.30)

is a positive martingale over any finite time interval, satisfying $N_0 = 1$. Hence $\{N_t\}$ can be used as the likelihood process to change the probability measure. Specifically, for any bounded \mathcal{F}_t -measurable random variable Z_t we have

$$\mathbb{E}_{s}^{\tilde{\mathbb{Q}}}\left[Z_{t}\right] = \frac{1}{N_{s}}\mathbb{E}_{t}^{\mathbb{P}}\left[N_{t}Z_{t}\right] \quad \text{and} \quad \mathbb{E}_{s}^{\mathbb{P}}\left[Z_{t}\right] = N_{s}\mathbb{E}_{t}^{\tilde{\mathbb{Q}}}\left[\frac{1}{N_{t}}Z_{t}\right].$$
(2.31)

In particular, (2.25) and (2.30) shows that $\{\xi_t\}$ is a Brownian motion under the \mathbb{P} measure, thus providing an alternative derivation for Proposition 2.2.2. It should be noted that while on the space (Ω, \mathcal{G}_t) it is the process $\{\Lambda_t\}$ that defines the measure change from \mathbb{P} to $\tilde{\mathbb{Q}}$, on (Ω, \mathcal{F}_t) it is $N_t = \mathbb{E}^{\tilde{\mathbb{Q}}}[\Lambda_t | \mathcal{F}_t]$ that defines the relevant measure change.

We remark that the conditional probability $\tilde{\mathbb{Q}}_t(X > t)$ appearing in (2.22) can be interpreted as representing the pricing kernel in the $\tilde{\mathbb{Q}}$ measure. Specifically, writing Π_t for $\tilde{\mathbb{Q}}_t(X > t)$, we deduce from (2.20) that

$$\Pi_{t} = \frac{\int_{t}^{\infty} \rho_{0}(u) \exp\left(\int_{0}^{t} v_{s}(u) \mathrm{d}\xi_{s} - \frac{1}{2} \int_{0}^{t} v_{s}^{2}(u) \mathrm{d}s\right) \mathrm{d}u}{\int_{0}^{\infty} \rho_{0}(u) \exp\left(\int_{0}^{t} v_{s}(u) \mathrm{d}\xi_{s} - \frac{1}{2} \int_{0}^{t} v_{s}^{2}(u) \mathrm{d}s\right) \mathrm{d}u}.$$
(2.32)

A short calculation making use of (2.25) shows that the \mathbb{Q} -pricing kernel (2.32) can be expressed manifestly in the Flesaker-Hughston representation (2.5):

$$\Pi_t = \int_t^\infty \rho_0(x) G_t(x) \mathrm{d}x, \qquad (2.33)$$

where $\{G_t(x)\}$ is a one-parameter family of positive $\tilde{\mathbb{Q}}$ -martingales:

$$G_t(x) = \exp\left(\int_0^t \tilde{v}_s(x) \mathrm{d}W_t - \frac{1}{2} \int_0^t \tilde{v}_s(x)^2 \mathrm{d}t\right), \qquad (2.34)$$

and where $\tilde{v}_t(x) = v_t(x) - \mathbb{E}_t^{\mathbb{Q}}[v_t(X)]$. By Ito's formula, (2.7), (2.9) and (2.29), dynamical equation satisfied by the \mathbb{Q} -pricing kernel therefore reads

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = -r_t \mathrm{d}t - (\hat{v}_t + \lambda_t) \mathrm{d}W_t, \qquad (2.35)$$

where $\hat{v}_t = \mathbb{E}_t^{\mathbb{Q}}[v_t(X)]$ and $\lambda_t = -\mathbb{E}_t^{\mathbb{Q}}[v_t(X)|X > t]$.

2.4 Indeterminacy of the risk premium

Returning to the \mathbb{P} -measure, we recall that once a parametric model for the martingale volatility $\{v_t(x)\}$ is chosen, then prices of derivatives will in general depend on this model choice. Hence $\{v_t(x)\}$ can be calibrated from derivative prices. The initial term structure density $\rho_0(u)$, on the other hand, can be calibrated from the initial yield curve. By substituting these ingredients in (2.9) we thus obtain a *market implied risk premium*, subject of course to the model choice. Evidently, any tractable model is unlikely to fit all derivative prices. One can nevertheless ask whether it is possible to fix $\{v_t(x)\}$ in a hypothetical situation where one has access to the totality of liquidly-traded derivative prices and an unlimited computational resource, i.e. whether it is possible in principle to fix $\{v_t(x)\}$ unambiguously under the assumption that asset drift processes cannot be estimated easily. Perhaps not surprisingly, as one would have expected from the Black-Scholes-Merton theory, the answer is negative. A more precise statement is as follows:

Proposition 2.4.1 In the Brownian-motion driven market, any addition to the volatility of the Flesaker-Hughston martingale family $\{M_t(u)\}$ that is independent of X and has no parametric dependence on 'u' will not affect current price levels. Furthermore, riskpremium vector can be estimated from derivative prices only up to an additive process.

2.4 Indeterminacy of the risk premium

Remark. We emphasised here the fact that the stated result is applicable strictly within the Brownian setup. This is on account of the recent observation that within the geometric models for risky assets driven by general Lévy processes, Brownian motion family is the *only* one for which the excess rate of return is linear in the risk aversion (Brody *et al* 2012a). It seems reasonable to conjecture that an analogous result holds in the general Lévy context; however, the indeterminacy of the risk aversion factor will not be of an additive nature when asset prices entail jumps.

The main technical difficulty to extend the model to the Lévy case is the lack of wellestablished filtering theory in this setting, in particular the form of the innovations process. A first step would be to use a Poisson or compound Poisson model. This remains an open topic for future research.

To verify Proposition 2.4.1, suppose that the volatility of the Flesaker-Hughston martingale family is decomposed in the form

$$v_t(u) = \phi_t(u) - \alpha_t, \tag{2.36}$$

where the vector process $\{\alpha_t\}$ is independent of X, and has no parametric dependence on u. The minus sign here is purely a matter of convention. We assume that $\{\phi_t(u)\}$ is bounded, and hence so is $\{\alpha_t\}$. Then writing

$$L_t = \exp\left(-\int_0^t \alpha_s \mathrm{d}\xi_s - \frac{1}{2}\int_0^t \alpha_s^2 \mathrm{d}s\right),\tag{2.37}$$

we find that the pricing kernel takes the form

$$\pi_t = L_t \int_t^\infty \rho_0(u) \,\mathrm{e}^{\int_0^t \phi_s(u) \mathrm{d}\xi_s - \frac{1}{2} \int_0^t \phi_s^2(u) \mathrm{d}s + \int_0^t \phi_s(u) \alpha_s \mathrm{d}s} \mathrm{d}u.$$
(2.38)

Since $\{L_t\}$ is a unit-initialised positive martingale on any fixed time interval, we can define a new probability measure \mathbb{P}^{α} according to the prescription

$$\frac{\mathrm{d}\mathbb{P}^{\alpha}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_{t}} = L_{t}.$$
(2.39)

It follows that the price at time t of a contingent claim, with payout $H_T = h(S_T)$ at T > t, is given by

$$H_{t} = \mathbb{E}_{t} \left[\frac{\pi_{T}}{\pi_{t}} H_{T} \right]$$

$$= \mathbb{E}_{t} \left[\frac{L_{T}}{L_{t}} \frac{\int_{T}^{\infty} \rho_{0}(u) e^{\int_{0}^{T} \phi_{s}(u) d\xi_{s} - \frac{1}{2} \int_{0}^{T} \phi_{s}^{2}(u) ds + \int_{0}^{T} \phi_{s}(u) \alpha_{s} ds} du}{\int_{t}^{\infty} \rho_{0}(u) e^{\int_{0}^{t} \phi_{s}(u) d\xi_{s} - \frac{1}{2} \int_{0}^{T} \phi_{s}^{2}(u) ds + \int_{0}^{t} \phi_{s}(u) \alpha_{s} ds} du} H_{T} \right]$$

$$= \mathbb{E}_{t}^{\alpha} \left[\frac{\int_{T}^{\infty} \rho_{0}(u) e^{\int_{0}^{T} \phi_{s}(u) d\xi_{s} - \frac{1}{2} \int_{0}^{T} \phi_{s}^{2}(u) ds + \int_{0}^{T} \phi_{s}(u) \alpha_{s} ds} du}{\int_{t}^{\infty} \rho_{0}(u) e^{\int_{0}^{t} \phi_{s}(u) d\xi_{s} - \frac{1}{2} \int_{0}^{t} \phi_{s}^{2}(u) ds + \int_{0}^{t} \phi_{s}(u) \alpha_{s} ds} du} H_{T} \right]. \quad (2.40)$$

Evidently, under the new measure \mathbb{P}^{α} , the process $\{\xi_t^{\alpha}\}$ defined by

$$\xi_t^{\alpha} = \xi_t + \int_0^t \alpha_s \mathrm{d}s \tag{2.41}$$

is a standard Brownian motion. Substituting (2.41) in (2.40) we deduce that

$$H_{t} = \mathbb{E}_{t}^{\alpha} \left[\frac{\int_{T}^{\infty} \rho_{0}(u) e^{\int_{0}^{T} \phi_{s}(u) d\xi_{s}^{\alpha} - \frac{1}{2} \int_{0}^{T} \phi_{s}^{2}(u) ds} du}{\int_{t}^{\infty} \rho_{0}(u) e^{\int_{0}^{t} \phi_{s}(u) d\xi_{s}^{\alpha} - \frac{1}{2} \int_{0}^{t} \phi_{s}^{2}(u) ds} du} H_{T} \right].$$
(2.42)

The pricing formula above is identical to the pricing formula under the \mathbb{P} measure had $\{\alpha_t\}$ been identically zero in the first place, on account of the following observation. The price of the underlying asset at time T can be expressed in the form

$$S_T = S_0 \exp\left(\int_0^T \left(r_s + \lambda_s \sigma_s - \frac{1}{2}\sigma_s^2\right) \mathrm{d}s + \int_0^T \sigma_s \mathrm{d}\xi_s\right),\tag{2.43}$$

where $\{\sigma_t\}$ is the volatility of $\{S_t\}$. Now if the volatility of the martingale family $\{M_t(u)\}$ takes the form (2.36), then the risk premium can be expressed as

$$\lambda_t = \lambda_t^{\alpha} + \alpha_t, \tag{2.44}$$

where $\{\lambda_t^{\alpha}\}$ is the risk premium in the \mathbb{P}^{α} measure:

$$\lambda_t^{\alpha} = -\frac{\int_t^{\infty} \rho_0(u)\phi_t(u)\mathrm{e}^{\int_0^t \phi_s(u)\mathrm{d}\xi_s^{\alpha} - \frac{1}{2}\int_0^t \phi_s^2(u)\mathrm{d}s}\mathrm{d}u}{\int_t^{\infty} \rho_0(u)\mathrm{e}^{\int_0^t \phi_s(u)\mathrm{d}\xi_s^{\alpha} - \frac{1}{2}\int_0^t \phi_s^2(u)\mathrm{d}s}\mathrm{d}u}.$$
(2.45)

Substituting (2.41) and (2.44) in (2.43), we obtain

$$S_T = S_0 \exp\left(\int_0^T \left(r_s + \lambda_s^\alpha \sigma_s - \frac{1}{2}\sigma_s^2\right) \mathrm{d}s + \int_0^T \sigma_s \mathrm{d}\xi_s^\alpha\right).$$
(2.46)

We thus find that the parametric form of S_T , and hence of any contingent claim H_T , under the \mathbb{P} -measure with $\alpha_t = 0$, is the same as that under the \mathbb{P}^{α} -measure with $\alpha_t \neq 0$. It follows that any *addition* of terms in the martingale volatility $\{v_t(u)\}$ that is independent of the parameter u does not alter quoted market values of derivatives, even though such a shift in the martingale volatility will affect distributions of future prices. In particular, it is not possible to ascertain from current market prices whether \mathbb{P}^{α} or \mathbb{P} is the true real-world probability measure.

This result shows that the risk premium vector $\{\lambda_t\}$ can be determined from market prices of derivatives only up to an additive vectorial term $\{\alpha_t\}$; thus establishing the claim. This freedom, however, is not arbitrary; it can only arise from a constant (i.e. independent of the parameter u) addition to the volatility of the martingale family in the form of (2.36).

Remark. Note that because it is impossible to estimate α from the prices of traded securities, it is not even possible to estimate the sign of the idiosyncratic component (i.e. the 'mood' of the market). Hedge fund managers use different sources of information for this purpose, including historic performance, news, expert opinions, etc. This is however outside of the scope of this research.

2.5 Information-based interpretation

The ambiguity in the determination of the risk premium can be given an interpretation from the viewpoint of information-based asset pricing theory of Brody *et al.* (2007). In the information-based pricing framework one models the market filtration directly in the form of an information process concerning market factors relevant to the cash flows of a given asset. Our objective here, which extends the previous work of Brody and Friedman (2009), is to analyse the model (2.14) for the information process that determines the pricing kernel.

The interpretation of the information process (2.14) is as follows. Market participants are concerned with the realised value of the random variable X, which, in a certain re-

stricted sense can be interpreted as the timing of a serious liquidity crisis. In reality, market participants observe price processes, or equivalently the underlying Brownian motion family $\{\xi_t\}$. As indicated above, under the physical \mathbb{P} -measure the random variables X and ξ_t are independent. However, market participants 'perceive' information with certain risk adjustments characterised by the density martingale $\{N_t\}$ of (2.30). In this risk-adjusted measure, the path $\{\xi_t\}$ represents the aggregate of noisy information for the value of X in the form of (2.14). The 'signal' concerning the value of X, in particular, is revealed to the market through the structure function $\{v_t(u)\}$, which in turn determines the volatility structure of the pricing kernel, and hence the risk premium.

Suppose that the structure function $\{v_t(u)\}$ takes the form (2.36), where $\{\alpha_t\}$ is independent of X. Then because (2.14) represents the information process for the random variable X, the constant $\{\alpha_t\}$ combines with the 'noise' term $\{\beta_t\}$. In other words, the choice of $\{\alpha_t\}$ is entirely equivalent to the choice of noise; the Brownian noise is replaced by a drifted Brownian noise. This change of noise composition does not affect current asset prices, and therefore is not directly detectable from market data, even though asset-price drifts are modified, in general in an unidentifiable manner. Note that the point of view that the indeterminacy of the asset price drifts is caused by noise has been put forward heuristically by Black (1986); our observation thus formalises this argument more precisely.

It is worth remarking briefly the observation made in Brody and Friedman (2009) concerning the form of the structure function $\{v_t(u)\}$ in the absence of the noise drift $\{\alpha_t\}$. Since small values of X imply imminent liquidity crisis, in an ideal market the signalto-noise ratio of the information process (2.14) should be large for small values of X, as compared to large values of X. In other words, under normal market conditions we expect the signal magnitude $|v_t(u)|$ be decreasing in u for every t. Conversely, if $|v_t(u)|$ is increasing in u, then the excess rate of return above the short rate for discount bonds, i.e. the inner product of the risk premium and the discount bond volatility, can be shown to be negative, yielding negative excess rate of return due to the inverted form of the structure function $\{v_t(u)\}$.

2.6 Anomalous price dynamics

The fact that current asset prices are unaffected by changes in the structure of the noise term does not imply that $\{\alpha_t\}$ can be ignored altogether. Indeed, (2.44) shows that the existence of such a component does shift the risk premium. Since the drift of an asset with volatility $\{\sigma_t\}$ is given in the \mathbb{P} -measure by $r_t + \lambda_t \sigma_t$, the noise-induced drift $\alpha_t \sigma_t$ can generate various anomalous price dynamics under the physical \mathbb{P} -measure.

As an example, let us consider the case of an anomalous price growth, commonly called a bubble. In the large vector space of asset volatilities, it is inevitable that volatility vectors form clusters consisting of different sectors or industries. This is because, by definition, a given sector of companies share similar risk exposures. Now if an anomalous noise component { α_t } at some point in time emerges to point in the direction of one of these volatility clusters, then this can cause a sharp rise in the share prices of firms from that sector. Since the noise vector { α_t } carries no real economic information, this can be identified as a bubble, where prices of a set of assets grow sharply, and independently of the 'true' state of affairs, without seriously affecting price processes of other assets. Similarly, at a later time, the magnitude of { α_t } can diminish. In particular, more reliable information concerning the true state of affairs may be revealed, which in turn leads to an increase in the magnitudes of volatilities on the one hand, while on the other hand the risk premium vector can point in a direction such that the inner product $\lambda_t \sigma_t$ takes a large negative value; thus leading to a bubble 'burst'.

In the finance and economics literature, there exists a substantial work on the study of various aspects of financial bubbles (see, e.g., Camerer 1989 for an early review). It is important to note that our characterisation of a bubble is motivated by an information-based perspective and is assumed to be caused by behavioural factors. Commonly used definition of a bubble, on the other hand, is given by the difference between the current price and the expected discounted future cash flows in the risk-neutral measure (cf. Tirole 1985, Heston *et al.* 2007). Under this definition, discounted asset prices in the risk-neutral measure can be modelled by use of strict local martingales (Cox and Hobson 2005, Jarrow *et al.* 2007, 2010), within the arbitrage-free pricing framework.

While this formulation of a bubble leads to the unravelling of many interesting mathe-

matical subtleties underlying fundamental theorems of asset pricing, from both behavioural as well as an information-theoretic viewpoint the plausibility of such a definition for a bubble seems questionable. In particular, a mathematical definition of a financial bubble that involves no reference to the \mathbb{P} measure seems restrictive; a bubble, after all, is a phenomenon seen under the \mathbb{P} measure. The pricing kernel approach, on the other hand, is based on a stronger assumption that if $\{S_t\}$ represents the price process of a liquidly traded asset, then $\{\pi_t S_t\}$ must be a true \mathbb{P} -martingale. As such, the discounted \mathbb{Q} -expectation of future asset price necessarily agrees with the current value, or else there are arbitrage opportunities.

The conventional definition of a financial bubble in terms of the inequality

$$S_t > \pi_t^{-1} \mathbb{E}_t [\pi_T S_T] \tag{2.47}$$

is sometimes justified heuristically by the fact that some traders, when they are under the impression that there is a bubble and thus traded prices are above the 'fundamental' values, will nevertheless participate in the apparent bubble with the view that they can withdraw from their positions before the crunch (see, e.g., Camerer 1989 and references cited therein). This example and other similar ones are often used in support of the argument that some traders are willing to purchase stocks even when they know that the price level is above its fundamental value. The shortcomings in such an argument are that (a) the role of market filtration is not adequately taken into account; and that (b) the fact that such a stock purchase is equivalent to the purchase of an American option is overlooked (see Harrison and Kreps 1978 for a discussion related to the second point). A more plausible characterisation of a bubble participation seems to be as follows. Given the information $\{\mathcal{F}_t\}$ (and perhaps following the 'animal spirits' of the market), a trader infers that the asset prices will continue to grow for a while. Hence, subject to the filtration, the best estimate of the future cash flow for this trader, with a suitable risk-adjustment, is given by $\sup_{\tau} \pi_t^{-1} \mathbb{E}_t[\pi_{\tau} S_{\tau}]$, where τ is a stopping time when the stock is sold. If this expectation agrees with the current price level, then a transaction occurs. Conversely, it seems implausible that a transaction takes place if the best estimate by a rational trader of a discounted cash flow is lower than the current price level.

2.6 Anomalous price dynamics

The view we put forward here is that phenomena commonly called bubbles in an asset ought to be identified with an anomaly in the rate of return of that asset, and not with an anomaly in the price level itself. Here, a precise definition of an 'anomaly' in the drift is essentially what we have described above, namely, the existence of an additive term in the volatility of the martingale family $\{M_t(u)\}$ that is constant in the parameter u. Based on this definition, it is admissible that price processes behave in a manner that does not always reflect what one might perceive as the true state of affairs, had one possessed better information concerning the true worth of the assets. Put the matter differently, decisions concerning transactions that ultimately lead to price dynamics are made in accordance with the unfolding of information. Since this information is necessarily noisy, the best filters chosen by market participants will inevitably deviate from true values of assets being priced. If the noise structure changes, then it is only reasonable that the dynamical aspects of these deviations will likewise change. In particular, the increment at time t of the innovations representation—that characterises the arrival of 'real' information over the small time interval [t, t + dt]—is given by

$$\mathbf{d}W_t = \mathbf{d}\xi_t - \hat{\phi}_t \mathbf{d}t + \alpha_t \mathbf{d}t, \qquad (2.48)$$

where $\hat{\phi}_t = \mathbb{E}_t^{\mathbb{Q}}[\phi_t(X)]$, and this illustrates in which way the existence of a nonzero noise drift $\{\alpha_t\}$ affects the dynamics.

Our characterisation of anomalous price dynamics is not confined to the consideration of financial bubbles. Again, in the large vector space of asset volatilities, it seems plausible that equity market volatilities and fixed-income volatilities generally lie on distinct subspaces. If the noise vector $\{\alpha_t\}$ has a tendency to lie in the direction of equity-volatility subspace, then this naturally leads to an excess growth in the equity market, explaining the phenomena of the so-called equity premium puzzle, where over time the rate of return associated with the equity market considerably exceeds that of the bond market (see, e.g., Kocherlakota 1996 for a review).

2.7 Relation to the risk-neutral measure

We have established the relation between the auxiliary probability measure $\tilde{\mathbb{Q}}$ and the physical measure \mathbb{P} . The relation between the latter and the risk-neutral measure \mathbb{Q} , on the other, involves the risk premium process $\{\lambda_t\}$. To recapitulate these two relations, we have

$$dW_t = d\xi_t - \hat{v}_t dt \quad \text{and} \quad dW_t^* = d\xi_t + \lambda_t dt, \qquad (2.49)$$

where

$$\hat{v}_{t} = \frac{\int_{0}^{\infty} \rho_{0}(u) v_{t}(u) M_{t}(u) du}{\int_{0}^{\infty} \rho_{0}(u) M_{t}(u) du} \quad \text{and} \quad \lambda_{t} = -\frac{\int_{t}^{\infty} \rho_{0}(u) v_{t}(u) M_{t}(u) du}{\int_{t}^{\infty} \rho_{0}(u) M_{t}(u) du},$$
(2.50)

and where we let $\{W_t^*\}$ denote the Q-Brownian motion. By combining the two relations in (2.49) we deduce at once that the measure-change density martingale is given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\tilde{\mathbb{Q}}}\Big|_{\mathcal{F}_t} = \exp\left(-\int_0^t (\hat{v}_s + \lambda_s) \mathrm{d}W_s - \frac{1}{2}\int_0^t (\hat{v}_s + \lambda_s)^2 \mathrm{d}s\right),\tag{2.51}$$

which determines the general relation between \mathbb{Q} and \mathbb{Q} .

As indicated above, a closer inspection on (2.50), however, shows that

$$\hat{v}_t = \mathbb{E}_t^{\tilde{\mathbb{Q}}}[v_t(X)] \quad \text{and} \quad \lambda_t = -\mathbb{E}_t^{\tilde{\mathbb{Q}}}[v_t(X)|X>t].$$
 (2.52)

In other words, under the restriction X > t, we have, conditionally, $\hat{v}_t + \lambda_t = 0$. These observations lead to the following conclusion:

Proposition 2.7.1 The auxiliary measure $\tilde{\mathbb{Q}}$, whose existence is ensured by the absence of arbitrage and the existence of the pricing kernel, is conditionally identical to the risk-neutral measure \mathbb{Q} , where the conditioning is on the event X > t.

Putting the matter differently, by restricting to the event X > t, we can think of the auxiliary measure $\tilde{\mathbb{Q}}$, upon which the filtering problem is defined, indeed as the risk-adjusted measure \mathbb{Q} .

2.8 Stochastic volatility

It is important to bear in mind the fact that a deterministic volatility structure $\{v_t(u)\}$ of the martingale family $\{M_t(u)\}$ does not imply deterministic volatilities for asset prices. On the contrary, even for an elementary discount bond, the associated volatility process arising from a deterministic $\{v_t(u)\}$ is highly stochastic. Hence when we speak about a 'stochastic volatility' we have in mind the volatility for the martingale family $\{M_t(u)\}$, whereas the stochasticity for asset prices is presumed in the foregoing material. Thus, from the viewpoint of practical implementation, it probably suffices to restrict attention to deterministic volatility structures, since deterministic volatilities for $\{M_t(u)\}$ give rise to a range of sophisticated stochastic volatility models for asset prices. Indeed, it is shown in Brody *et al.* (2012) that even in the very restricted case of a single factor model with the time-independent volatility $v_t(u) = e^{-\sigma u}$ that depends only on one model parameter σ , it is possible to calibrate caplet prices across different maturities reasonably accurately.

It is nevertheless of interest to enquire in which way stochastic volatility models arise in the auxiliary information process. There appear to be three distinct ways in which stochasticities arise: (i) when the deterministic volatility structure $\{v_t(u)\}$ is augmented by an additive stochastic process that has not parametric dependence on u; (ii) when the volatility structure $\{v_t(u)\}$ depends on the information process $\{\xi_t\}$; and (iii) when the signal Xis elevated to a stochastic process $\{X_t\}$.

Let us begin by considering the case (i) in which $\{v_t(x)\}$ admits the decomposition (2.36) and where $\{\phi_t(x)\}$ is deterministic and $\{\alpha_t\}$ is a Gaussian process so that the noise term $n_t \equiv \int_0^t \alpha_s ds + \beta_t$ is an $\{\mathcal{F}_t^\beta\}$ -measurable Gaussian process. Then an application of the martingale representation theorem shows that $\{n_t\}$ admits a decomposition of the form

$$n_t = \int_0^t b_s \mathrm{d}s + \int_0^t \gamma_s \mathrm{d}\beta_s, \qquad (2.53)$$

where $\{b_s\}$ and $\{\gamma_s\}$ are deterministic. A short calculation then shows that an auxiliary information process

$$\xi_t = \int_0^t \phi_s(X) \mathrm{d}s + n_t \tag{2.54}$$

in the $\tilde{\mathbb{Q}}$ measure indeed exists, with the property that the scaled information process $\int_0^t \gamma_s^{-1} d\xi_s$ determines the market Brownian motion and that $\{b_t\}$ plays the role similar to that of a deterministic $\{\alpha_t\}$ in the previous analysis, and hence is not determinable form current market prices. This example shows how one can model the random rise and fall of anomalous price dynamics.

Alternatively, in case (ii) the structure function $\{v_t(x)\}$ can depend in a general way on the history of the information process up to time t. In this case, we obtain a generic stochastic volatility model for the martingale family. Provided that the structure function is sufficiently well behaved so that relevant stochastic integrals exist, the auxiliary information process can be seen to exist in the \mathbb{Q} measure. To illustrate this, consider an elementary 'toy model' for which information process takes the form of an Ornstein-Uhlenbeck process:

$$\xi_t = e^{\sigma\phi(X)t} \int_0^t e^{-\sigma\phi(X)s} d\beta_s, \qquad (2.55)$$

where σ is a parameter, X and $\{\beta_t\}$ are independent, and $\phi(u)$ is an invertible function. Such an information process corresponds to a stochastic volatility model for which the volatility process is given by a linear function of the P-Brownian motion: $v_t(u) = \sigma \phi(u)\xi_t$.

The case (iii), the analysis of which is of considerable interests although it goes outside the scope of the present paper, is to consider a more general situation often considered in the literature of nonlinear filtering (cf. Liptser & Shiryaev 2001), namely, the unobserved 'signal' X is elevated from a fixed random variable to a random process. This case leads to generic unhedgeable stochastic volatility models.

2.9 Discussion

The main results of this chapter are as follows: We have derived the existence of an auxiliary filtering problem underlying arbitrage-free modelling of the pricing kernel; the solution to which determines the volatility structure of the positive martingale family $\{M_t(u)\}$ appearing in the Flesaker-Hughston representation for the pricing kernel. We have demonstrated that the structure of the ambient information process fully characterises the risk premium process $\{\lambda_t\}$. We have shown, under the Brownian-filtration setup, that $\{\lambda_t\}$ admits a canonical decomposition into two terms in an additive manner; the systematic term that can be calibrated from current market data for derivative prices, and the idiosyncratic term that cannot be estimated, and thus can be identified as pure noise.

It is worth emphasising that these results hold irrespective of our choice of interpretation. Nevertheless, our characterisation of anomalous price dynamics seems sufficiently compelling, for, such phenomena are ultimately observed under the physical measure \mathbb{P} . One might ask what causes the evolution of the noise drift $\{\alpha_t\}$. This is an interesting econometric question that, however, goes beyond the scope of the present investigation. It suffices to remark that the random variable X that constitutes the signal component of the ambient information process has units of time, i.e. [X] = [time], and thus is ultimately linked to the term structure of financial markets. One possible explanation of the excess equity premium therefore is that fixed-income market intrinsically embodies more information concerning the term structure as compared to the equity market, and this imbalance is manifested in the form of an additional drift in the noise component pointing generally towards the direction of equity volatility vectors.

The indeterminacy of the risk premium, of course, could have been anticipated from the elementary geometric Brownian motion model, for, in this model derivative prices are indeed independent of the risk premium. This follows from the fact that in the geometric Brownian motion model, the structure function is constant: $v_t(u) = -\lambda$. Hence in this model, which gives rise to the well-known Black-Scholes option pricing formula, the underlying asset price dynamics in the physical \mathbb{P} measure necessarily grows anomalously (assuming as usual that $\lambda > 0$), without any reference to investor liquidity preferences.

We conclude by quoting a line from Wiener: "That particular branch of sociology which is known as economics ..." is just a branch of "sciences of communication" (Jerison & Stroock 1997). Indeed, the point of view that we advocate here, is that price processes in financial markets should be regarded as emergent phenomena, based on markets acting as filters to identify fair prices, given the available information and agents driven by 'animal spirits'. We hope therefore that the observations made in the present paper will lead to a new line of research with an emphasis on the interplay involving information and communication theory, mathematical finance, and behavioural economics. The authors thank Mark Davis, Robyn Friedman, Matheus Grasselli, Lane Hughston, Andrea Macrina, and Bernhard Meister for comments and stimulating discussions.

Chapter 3

Taming animal spirits

In this chapter and the remainder of the thesis we assume that the 'animal spirits' are given, defined as factor processes driving the parameters of the model for asset prices. In this chapter we assume that the factors have the form of a diffusion, in the following two chapters we assume they are Markov chains.

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3.1 Introduction

The goal of this chapter is to initiate a quantitative theory of credit risk relevant to scenarios that contributed to the credit crunch of 2008. We have in mind specifically the experience of Ireland and Spain, in which banks funded massive investments in property developments on the basis of heroically optimistic valuations of the return on these investments. At the World Economic Forum, Davos (2012), Enda Kenny, Taoiseach of Ireland, noted that 'Ireland's problems stem from a kind of madness that led to the country borrowing \$60 billion at unrealistically high rates'. It is obvious that any explanation must include behavioural factors.

In this chapter we study a project finance problem involving two parties,

• Bank (B), which borrows from other commercial banks or a central bank, at funding rate r^f and lends to entrepreneurs;

• Entrepreneur (E) who borrows funds from B at a contract rate r_c in order to finance a project that will deliver a product of value G at time T.

E's loan will be paid off (with interest) in a bullet repayment at T. The investment project is the collateral for the loan, but of course its value is uncertain until T. B will insist that an over-collateralization ratio $\kappa > 1$ be maintained at all times $t \in [0, T]$, based on B's current assessment of the value G, and will insist on margin payments should this ratio be breached. This is what makes the loan so risky: E has to use other capital (assumed to be invested in the financial markets) to make margin payments, and if this capital is insufficient the loan will be foreclosed and the project sold off at a 'fire sale' price—some fraction of its pre-default assessed value at time t.

Clearly, the key question here is how B assesses the value of the project. E is, as Keynes says in the quote in Section 1.1.1, an optimist, but B should take a rational view. We assume B abides by the principles of market-consistent valuation, i.e. uses a model such that no arbitrage would be introduced if the project were traded at the model price in addition to existing traded asset in the market. This principle allows (see Section 1.1.2) a wide range of estimates, and we assume that B's valuation is affected by 'confidence' as represented by published business or consumer confidence indices.

In our analysis we find ourselves at the intersection of five lines of thought, namely (i) 'animal spirits', (ii) confidence indices, (iii) market-consistent valuation, (iv) the numéraire portfolio, and (v) structural models of credit risk. In Section 1.1 we give some background information on these topics that informs the models we construct and analyse in subsequent sections. Our project finance model is introduced in Section 3.2, and results for a simply computable example are described in Section 3.3. This section also discusses the computational requirements for larger-scale problems.

3.2 The Project Finance Model

We now proceed to a formal specification of our model.

3.2.1 Financial Market

We consider a simple model in which tradable asset prices $S_i(t), i = 1, ..., m$ satisfy SDEs of the form

$$dS_i(t) = S_i(t)\mu_i(X(t))dt + S_i(t)\sigma_i(X(t))dW_t, \quad i = 1, ..., m,$$
(3.1)

$$dX(t) = \alpha(X(t))dt + \Lambda(X(t))dW_t, \quad X(0) = x$$
(3.2)

for $t \in [0, T]$, where W_t is P-Brownian motion in \mathbb{R}^{n+m} and X(t) is an *n*-dimensional factor process on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. We assume that α , Λ are Lipschitz continuous so that a unique strong solution of (3.2) exists. $S_i(t)$ is then given explicitly by

$$S_i(t) = S_i(0) \exp\left(\int_0^t (\mu_i - \frac{1}{2}|\sigma_i|^2) ds + \int_0^t \sigma_i dW_s\right).$$
 (3.3)

The short rate of interest available to E is r(X(t)) for some given function $r(\cdot)$. We will describe the components of the factor process X(t) below. The integrability conditions on μ_i, σ_i are such that the integrals in (3.3) are well defined and $\mathbb{E}[S_i(t)] < \infty$ for all i, t. The main point is that X(t) includes confidence indices.

The factor process X is in general multidimensional and may contain both financial and non-financial components. The diffusion parameters of the stock price processes μ and σ are functions of the factors, and they should depend on the financial components, but not the non-financial components. The non-financial components may drive other parts of the economy, e.g. property prices.

The log-optimal portfolio for this model is (omitting the X-dependence)

$$dY(t) = Y(t) \left(r + h_* \Sigma \Sigma' h'_* \right) dt + Y(t) h_* \Sigma dW, \tag{3.4}$$

where $h_*(t) = (\mu - \mathbf{1} r)'(\Sigma\Sigma')^{-1}$ is the optimal asset allocation (the allocation to the money market account being $h_*^0 = 1 - h_* \mathbf{1}$). Here μ [Σ] is the vector [matrix] with rows μ_i [σ_i] and $\mathbf{1}$ is the vector with all entries equal to 1. We assume that, for some $\epsilon > 0$, $s'\Sigma\Sigma'(x)s \ge \epsilon |s|^2$ for all $(s, x) \in \mathbb{R}^{m+n}$, which is equivalent to saying that there are no

redundant assets. Equation (3.4) can be expressed as

$$dY(t) = Y(t)(r + \beta^2)dt + Y(t)\beta \, dB_t,$$
(3.5)

where B_t is the scalar Brownian motion

$$B_t = \int_0^t \frac{h_* \Sigma(s)}{|h_* \Sigma(s)|} dW_s$$

and $\beta(t) = |h_*(t)\Sigma(t)|$. With initial endowment Y(0) = 1, equation (3.5) has explicit solution

$$Y(t) = \exp\left(\int_0^t (r + \frac{1}{2}\beta^2)ds + \int_0^t \beta \, dB_s\right).$$
 (3.6)

We are going to use the log-optimal portfolio Y(t) given by (3.5) for two different purposes:

(a) It is assumed that entrepreneur E is a log-optimal ("Kelly") investor, so that his surplus wealth (capital not invested in the project) is just $x_0Y(t)$ if his initial surplus wealth is x_0 . (This will hold up to the time of B's first margin call; see below.)

(b) Y(t) is the numéraire asset, so the risk-neutral value at t of an \mathcal{F}_T -measurable payment H paid at T is

$$H_t = Y(t)\mathbb{E}\left[\frac{H}{Y(T)}\middle| \mathcal{F}_t\right].$$
(3.7)

This is the valuation formula used by B.

A discrete-time formulation. Most of the econometric data we consider, such as confidence indices, is posted monthly. Let us suppose that T is an integer number n of months, and denote by $0 = t_0, t_1, \ldots, t_{n-1}$ the first day of each monthly period. If $X_i(\cdot)$, the *i*th component of the factor process $X(\cdot)$, is an econometric variable based on monthly data then we simply define $X_i(t) = X_i(t_k)$ for $t \in [t_k, t_{k+1})$. If every component of $X(\cdot)$ is obtained from discrete data in this way then equation (3.6) has piecewise-constant coefficients, and the solution $Y(t_k)$ can be expressed as

$$Y(t_k) = \prod_{j=1}^k U_j, \qquad U_j = \exp\left((r_j + \frac{1}{2}\beta_j^2)\delta_j + \beta_j\sqrt{\delta_j}Z_j\right)$$
(3.8)

where $\delta_j = t_j - t_{j-1}$, $r_j = r(X(t_{j-1}))$, $\beta_j = \beta(X(t_{j-1}))$, and the Z_j are independent N(0, 1) random variables.

Let $\{\mathcal{G}_k, k = 0, ..., n\}$ be the discrete filtration where \mathcal{G}_0 is the trivial σ -field and $\mathcal{G}_k = \sigma\{B(t_j) - B(t_{j-1}), j = 1, ..., k\}$ for k = 1, ..., n. If H is a \mathcal{G}_n -measurable random variable then we see from (3.7) and (3.8) that the value at time t_k is

$$H_{t_k} = \mathbb{E}\left[\left.\left(\prod_{j=k+1}^n U_j^{-1}\right) H\right| \mathcal{G}_k\right].$$
(3.9)

3.2.2 Project Finance

The project finance valuation problem was informally described in Section 3.1. The entrepreneur E has initial capital x and can, for a payment A, invest in a venture which, at time T = n months, will yield a reward G(X(T)) as above. G is a function that will be specified below, but it is a function of X(T) only and hence is \mathcal{G}_T -measurable. He invests a of his own money and borrows b = A - a from a bank at a term rate of interest r_c (expressed for convenience in continuously-compounding terms), repayable by a bullet payment at T. Thus effectively his initial capital is reduced to $x_0 = x - a$ while the eventual reward is $G(X(T)) - e^{r_c T}b$. The capital x_0 is invested in the Kelly portfolio described above.

The value of the project at an intermediate time t_k is deemed by the bank B to be the market-consistent value $G_k(X(t_k))$ given by (3.9) as

$$G_k = \mathbb{E}\left[\left.\left(\prod_{j=k+1}^n U_j^{-1}\right)G(X(T))\right|\mathcal{G}_k\right].$$
(3.10)

The loan is collateralized by the value of the project, and B stipulates over-collateralization with factors $\kappa > \kappa' > 1$, checked at monthly intervals. Thus $G_0 \ge \kappa b$ and in any subsequent verification time k we have

$$G_k(X(t)) \ge \kappa' b e^{r_c t_k}.$$

Defining $H_k(x) = e^{-r_c t_k} G_k(x)$ this is equivalent to $H_k \ge \kappa' b$. Let

$$\begin{aligned} \theta_1 &= \min\{k : H_k \le \kappa' b\}, \quad \tau_1 = t_{\theta_1}, \\ b_1 &= \frac{H_{\theta_1}}{\kappa}. \end{aligned}$$

At τ_1 , E is contractually obliged to provide additional collateral to restore the collateral level to κ by paying off the amount $d_1 = e^{r_c \tau_1}(b - b_1)$ of the loan. Since the project is illiquid, he can only do this from his investment portfolio, and hence this experiences a jump of $-d_1$. In general, we define for j = 2, 3, ...

$$\theta_j = \min\{k : \theta_{j-1} < k < n, H_k \le \kappa' b_{j-1}\}, \quad \tau_j = t_{\theta_j},$$

$$b_j = \frac{H_{\theta_j}}{\kappa}$$

giving a jump in V of $-d_j = -e^{r_c \tau_j} (b_{j-1} - b_j)$. The entrepreneur's market investment portfolio evolves as follows:

$$V(t) = V(0) + \int_0^t (r + \beta^2) V(s) ds + \int_0^t V(s) \beta dB_s - \sum_{\tau_j \le t} d_j.$$

Let $\tau^* = \min\{\tau_j : V(\tau_j) < 0\}$, with $\tau^* = +\infty$ if there is no such t_j , and let $\theta^* = j$ when $\tau^* = t_j$. If $\tau^* < T$ the entrepreneur is insolvent at τ^* and the project must be liquidated at 'fire sale' value $F(\tau^*) = \phi(\tau^*)G_{\theta^*}(X(\tau^*))$, where ϕ is an increasing function of time with values in [0, 1[. The bank receives $F(\tau^*) + V(\tau^*-)$.

The market-consistent value of the loan to the bank is therefore

$$MCV = \mathbb{E}\left[\sum_{j} \frac{d_{j}}{Y(\tau_{j})} \mathbf{1}_{(\tau_{j} < \tau^{*} \wedge T)} + \frac{b_{n} \wedge (V(T) + G(X(T)))}{Y(T)} \mathbf{1}_{(\tau^{*} > T)} + \frac{F(\tau^{*}) + V(\tau^{*})}{Y(\tau^{*})} \mathbf{1}_{(\tau^{*} < T)}\right].$$
(3.11)

We thus have a credit risk model. The bank loses value because of early partial repayment of the loan together with the risk of actual default. Define $p_k = (1 + r_k^f/12)^{-1}$ where r_k^f is the Bank's (annualized) funding cost for the kth month, and $p_{0,k} = \prod_{l=1}^k p_l$. Then, recalling that the initial loan amount is *b*, the Bank's P&L along one sample path, discounted to time 0, is

$$\Xi(\omega) = \sum_{j} p_{0,\theta_{j}} d_{j} \mathbf{1}_{(\tau_{j} < \tau^{*} \wedge T)} + p_{0,n} [b_{n} \wedge (V(T) + G(T))] \mathbf{1}_{(\tau^{*} > T)} + p_{0,\theta^{*}} [F(\tau^{*}) + V(\tau^{*})] \mathbf{1}_{(\tau^{*} < T)} - b.$$

The Bank will be interested in the *expected profit* $\mathbf{e} = \mathbb{E}[\Xi]$, the *value at risk* VaR = $\mathbf{e} - \mathbf{q}$, where \mathbf{q} is the (say) 5% quantile of the P&L distribution, and the *expected shortfall*

$$\mathrm{CVaR} = \mathbf{e} - \mathbb{E}[\Xi | \Xi < \mathbf{q}] = \mathbf{e} - \frac{1}{\mathbf{q}} \mathbb{E}\left[\Xi \mathbf{1}_{(\Xi < \mathbf{q})}\right].$$

3.3 A simply computable example

In this section we demonstrate the computations required in a simple example where the factor process is a scalar Ornstein-Uhlenbeck process. This is a stylized model intended mainly to illustrate the computational process. We do not attempt to connect the factor variable to econometric data, an entirely separate matter.

3.3.1 Model specification

Recall that the numéraire asset is Y_t satisfying

$$dY_t = (r(X_t) + \beta^2(X_t))Y_t dt + \beta(X_t)Y_t dB_t, \qquad Y_0 = 1.$$
(3.12)

In this example we suppose that X_t is scalar, $\beta(x) = b_0 + b_1 x$ and $r(x) = r_0 + r_1 x$. X_t is the mean-reverting Gaussian process

$$dX_t = -\alpha X_t dt + \gamma dW_t, \qquad X_0 = x_0, \tag{3.13}$$

where $\alpha, \gamma > 0$ are constant and W_t is a Brownian motion with $\mathbb{E}[dW \, dB] = \rho \, dt$. The project value at completion is defined by $G(X_T) = e^{\eta + \xi X_T}$. This is analogous to conventional modelling of commodity prices as exponentials of mean-reverting processes. Note

that ξ represents, up to a constant, the volatility of project value.

The Bank's valuation of the project at t < T is

$$G(t, X_t) = \mathbb{E}\left[\left.\frac{Y_t}{Y_T} e^{\eta + \xi X_T} \right| \mathcal{F}_t\right].$$
(3.14)

Proposition 3.3.1 Let $c_0 = \gamma \rho b_0$, $c_1 = \alpha + \gamma \rho b_1$, $d = r_1/c_1$. Then

$$G(t,x) = \exp(v_0(t) + v_1(t)x)$$
(3.15)

where

$$v_{1}(t) = (\xi + d)e^{-c_{1}(T-t)} - d,$$

$$v_{0}(t) = \eta - \left(r_{0} - c_{0}d - \frac{1}{2}\gamma^{2}d^{2}\right)(T-t) - (\xi + d)(c_{0} + \gamma^{2}d)\left[\frac{1}{c_{1}}(1 - e^{-c_{1}(T-t)})\right]$$

$$+ \frac{1}{2}\gamma^{2}(\xi + d)^{2}\left[\frac{1}{2c_{1}}(1 - e^{-2c_{1}(T-t)})\right].$$
(3.16)
(3.16)
(3.16)
(3.17)

PROOF. The result follows from the fact that (3.12),(3.13) is an affine factor model Duffie et al. (2000). We outline the steps, which can be completed by routine—if tedious— computations.

(i) The risk-neutral measure \mathbb{Q} with money-market account as numéraire is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \beta \, dB - \frac{1}{2}\int_0^T \beta^2 dt\right).$$

If we express W_t in (3.13) as $W_t = \rho B_t + \sqrt{1 - \rho^2} W_t^0$, where B, W^0 are \mathbb{P} -independent Brownian motions, then $d\tilde{B} = dB + \beta dt$, W_t^0 and $dW_t^1 = \rho d\tilde{B}_t + \sqrt{1 - \rho^2} dW^0$ are \mathbb{Q} -independent Brownian motions and X_t satisfies

$$dX_t = -(\gamma \rho b_0 + (\alpha + \gamma \rho b_1) X_t) dt + \gamma dW_t^1$$
$$= -(c_0 + c_1 X_t) dt + \gamma dW_t^1.$$

(ii) The project value is expressed under the measure \mathbb{Q} as $G(t, X_t)$ where

$$G(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x} \left[e^{-\int_{t}^{T} r(s)ds} e^{\eta + \xi X_{T}} \right]$$

(iii) By the Feynman-Kac formula, v(t, x) satisfies the backward equation

$$\frac{\partial G}{\partial t} - (c_0 + c_1 x) \frac{\partial G}{\partial x} + \frac{1}{2} \gamma^2 \frac{\partial^2 G}{\partial x^2} - (r_0 + r_1 x) G = 0, \quad G(T, x) = e^{\eta + \xi x}.$$
 (3.18)

(iv) The PDE (3.18) has solution (3.15) where v_1 , v_0 are given respectively by (3.16), (3.17). Indeed, one can check that a solution of the form (3.15) satisfies (3.18) if v_1 satisfies the ODE

$$\frac{d}{dt}v_1(t) = c_1v_1(t) + r_1, \quad v_1(T) = \xi,$$

whose solution is (3.16). v_0 is then given by direct integration of the following expression involving v_1 :

$$\frac{d}{dt}v_0(t) = c_0 v_1(t) - \frac{1}{2}\gamma^2 \left(v_1(t)\right)^2 + r_0, \quad v_0(T) = \eta,$$

Working this out gives (3.17).

With Proposition 3.3.1 in hand, we can estimate the project value MCV of (3.11), and the VaR and CVaR, by Monte Carlo simulation. Note that simulation is exact, because of the discrete-time formulation, in that the result ultimately depends only on a finite vector of N(0, 1) random variables.

3.3.2 Results

We consider a project with initial cost A = \$12 (or \$12,000,000) and an entrepreneur with initial cash of $x_0 = \$10$. We assume that the project price is "fair", in the sense that it coincides with the risk-neutral value at time 0. The parameters of the model are presented in Table 3.1. Note that η can be determined given the initial price of the project and the other parameters.

The entrepreneur can be considered rather respectable—he has almost enough cash to finance the entire project without resorting to loans. Given the overcollateralization requirement imposed by the bank, the entrepreneur may choose to borrow between \$2 and

r_0	0.03	α	0.90
r_1	0.01	γ	0.80
b_0	0.01	ρ	0.70
b_1	0.01	r_c	0.05
T	5	dt	1/12
κ	1.2	κ'	1.1
ξ	1	η	1.948

Table 3.1: Parameter Values

\$10, and he invests the surplus of between \$8 and \$0 respectively in the market. Figure 3.1 shows CVaR of the loan from the bank's perspective, plotted against the notional of the loan. All the values are presented as the fraction of the initial value of the loan.

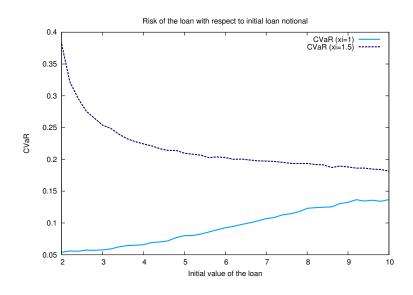


Figure 3.1: CVaR for different initial loan values, expressed as a fraction of the initial value of the loan.

The immediate thing we notice in Figure 3.1 is that the risk behaves in a completely different way for different levels of project volatility ξ . Under normal market conditions ($\xi = 1$, which gives the project a similar volatility as the stock market) the bank prefers, from the risk management perspective, lower loans. This is perfectly intuitive, because the initial value of the collateral is the same for loans with different notionals and equal to A =\$12. The lower-value loans hardly ever default—around 1.1% of them compared to 6.1% for loans with value \$10.

In the case of high volatility ($\xi = 1.5$, when the market for the project is possibly in crisis or distress), the situation becomes very different. Despite the lower-value loans having a lower probability of default—9% for \$3 loans comparing to 14.5% for \$10 loans, the higher-value loans have smaller risk. The key to understanding this seeming paradox is to notice that loans with higher values leave the entrepreneur with more liquid assets. These assets are used to cover the margin payments, so the bank can recover large proportion of the loan before default, whereas in the case of minimal loan first margin call immediately causes insolvency. Moreover, the entrepreneur invests his liquid assets in the market. The extra leverage causes more volatility and defaults, but also increases his average return, and hence the amount of money the bank can recover.

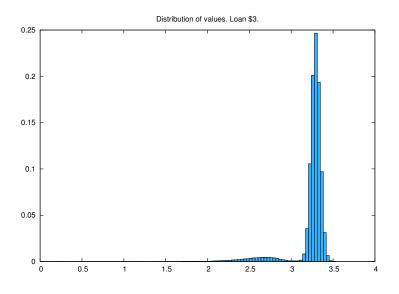


Figure 3.2: Distribution of values for loan with initial value \$3 and $\xi = 1.5$. The average is 3.26 and CVaR = 0.77.

Figures 3.2 and 3.3 show the distribution of the value of the loan with notional of \$3 and \$10 respectively. In both cases the graph has two distinct peaks: the one with values above the notional corresponds to no-default scenarios, whereas the other one contains values after default. The values are dispersed because of the effect of stochastic discount rates, random early repayments and—for the relevant cases—different default times. We immediately notice that in the \$3 notional case more mass is in the no-default peak comparing to the \$10 case, but the returns if default occurs are proportionately much lower. Figures

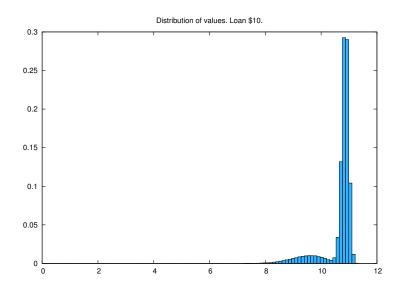


Figure 3.3: Distribution of values for loan with initial value \$10 and $\xi = 1.5$. The average is 10.74 and CVaR 1.82.

3.4 and 3.5 stress this point even more. They depict the distribution of the outstanding loan value for the cases that ended up in default (taken just before the default time and discounted suitably). In these figures we can clearly see that in most cases almost half of the \$10 loan is repaid early, whereas in a considerable percentage of cases the first margin call made the \$3 loan default. Note that these defaults are much more costly for the bank (in terms of percentage of the initial loan value).

For all considered ξ and loan values the defaults happen mostly just before the maturity of the project, see for example Figure 3.6.

The entrepreneur always prefers to borrow more, so that he has more leverage and more potential to earn money. His losses are only limited to his initial capital $x_0 = \$10$, and—as is apparent in Figure 3.7—the potential gains are very high. For the initial loan value of \$6, even in the normal risk circumstances ($\xi = 1$) the entrepreneur has a positive probability to earn more than ten times his initial investment. In high risk case this goes up to almost thirty times his initial investment. Despite the single most probable outcome in both cases is a loss (a default in high risk case), the entrepreneur has a positive profit expectation of \$3.57 in the low risk case and \$4.53 in the high risk case. This kind of highly asymmetric payoff characteristics foster the entrepreneurs drive to invest and provide examples of others who

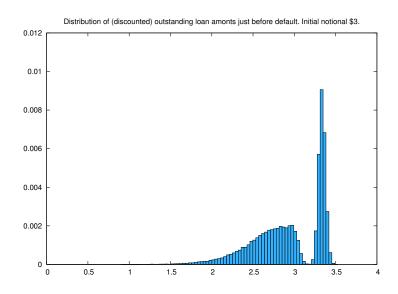


Figure 3.4: Distribution of (discounted) outstanding loan just before default for the initial loan value \$3 and $\xi = 1.5$. This graph contains only cases that ended up in default. Total default probability is 0.08.

succeeded in a spectacular way—even though most of them failed. Being an "optimist" in Keynes' sense is perfectly rational in this model.

When agreeing to the loan amount in normal market circumstances the entrepreneur will prefer to borrow as much as possible, but the risk-optimizing bank will prefer to lend much less—because the risk increases with the notional of the loan. Figure 3.8 shows that this statement is only true in the case of ξ being around unity. Based on this parameter we can distinguish three market regimes: very low risk regime with $\xi < 0.5$, normal market circumstances ($0.5 < \xi < 1.2$) and high risk market ($\xi > 1.5$). In the first case the project is bound to succeeded and the bank only faces interest rate risk. Hence the risk doesn't depend on the value of the loan. As discussed, in the normal case the bank prefers to have more collateral compared to the amount of the loan. In the high risk case, however, things change dramatically. It becomes less risky for the bank to offer maximum loans to the entrepreneurs. For the economy it may have severe consequences: on the one hand these loans have much higher probability of default—which can be further aggravated by contagion effects, and on the other the bank starts having big items on its balance sheet. Even one default of these big loans could deplete the bank's Tier 1 capital and cause its collapse.

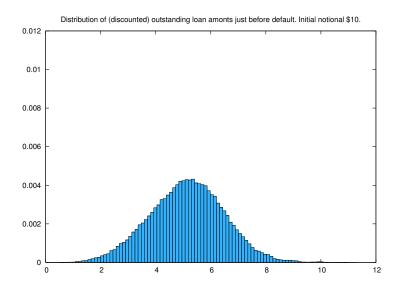


Figure 3.5: Distribution of (discounted) outstanding loan just before default for the initial loan value \$10 and $\xi = 1.5$. This graph contains only cases that ended up in default. Total default probability is 0.13.

Although not explicit in here, the ξ parameter can be assumed to be linked to the agents' perception of the state of the market, with higher ξ meaning more volatility and uncertainty. Then a crisis would mean a shift from the lower risk regime to a higher one, with all the economic implications.

The mechanism of margin payments is effective way to minimize the risk for the bank only for large loans in the volatile case, but somewhat surprisingly not in other cases. In all cases introduction of forced early repayments increases the number of defaults between 2.5 and 6 times and in normal market conditions it increases the bank's risk as well. In particular, if a bank decides to give a "safe" loan of \$3 under normal volatility conditions then the margin payments are inefficient from the very start. But if a crisis begins and volatility rises, it gets much worse. Looking at Figure 3.1 the risk will increase more than threefold and the mechanism of early repayments becomes even greater burden for the bank and the economy.

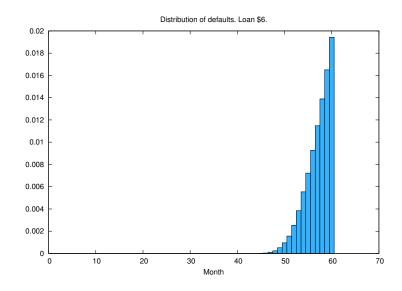


Figure 3.6: Distribution of default times for loan with initial value \$6 and $\xi = 1.5$. Total default probability is 0.10.

3.3.3 Computations for the general case

The above computation is in two stages: solve the backward equation (3.18) to determine the project valuation function G of (3.14), then simulate (forwards) to determine the value of the project and the risk parameters. In our example the first stage is easy because the backward equation has the closed-form solution (3.15). However, when we have a general factor model with, say, 5 or 6 factors, a numerical method will be required, and the dimensionality stretches standard finite-difference methods up to, or beyond, their normal limits. The best candidates seem to be stochastic mesh methods, see Glasserman (2003), and specifically the basis function approach originally devised by Longstaff and Schwartz (2001) for American options. While computationally intensive these methods have the advantage that the same set of sample paths used for the forward simulation is also used to solve the backward equation. They are becoming the methods of choice for large-scale credit risk calculations, see Cesari et al. (2010).

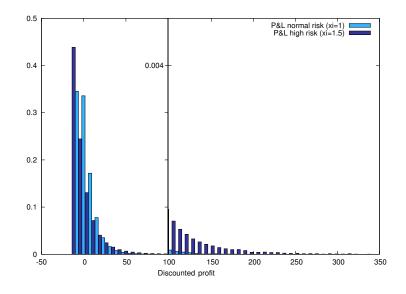


Figure 3.7: Distribution of P&L of the entrepreneur with the initial loan value \$6. In case of default P&L is set to -\$10, otherwise it is the difference between the discounted value of the project and the stock account at maturity minus the outstanding loan, and the initial capital of the entrepreneur $x_0 = \$10$. The average profit in the case $\xi = 1$ is \$3.57 and in the case $\xi = 1.5$ it is \$4.53. The right-hand panel has a 100 times smaller scale.

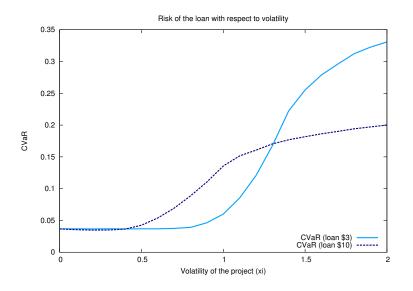


Figure 3.8: CVaR for different values of project volatility (ξ), expressed as a fraction of the initial loan amount.

Chapter 4

Estimating animal spirits

In this chapter work with historic data. We propose a way to estimate the 'animal spirits' in the real estate market, use the model for forecasting and to compute Value-at-Risk (VaR). We also introduce a procedure in which we add stress scenarios to the model to make the risk estimates more conservative.

A natural approach to quantifying animal spirits would be to use confidence indices. Figure 4.1 depicts log-returns of the Shiller house price index and the University of Michi-

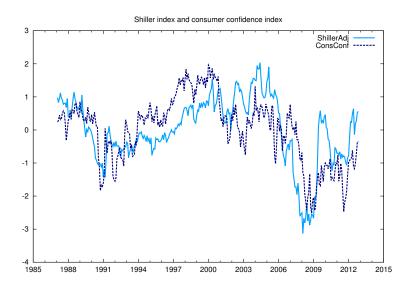


Figure 4.1: Log-returns of the Shiller property index (seasonally adjusted and normalized), University of Michigan consumer sentiment index.

gan consumer sentiment index. Even by looking at the graph it is clear that there exists

a relationship between these two quantities. There is not enough data to find evidence of causality in the sense of Granger (1969) in either direction though—which might suggest the existence of another factor, causing both of these data series.

In this chapter we propose to use Hidden Markov Models (HMMs) to estimate this unobservable factor, which is assumed to be a "state of the world", impacting both house prices and consumer confidence. Hidden Markov Models were initially developed in the field of communications, in particular for speech recognition. Recently, this approach was successfully adopted in finance, where the interpretation of the hidden state process is that it is the current market 'regime', and this class of models is often referred to as 'regime-switching'. Some early discussions include Hamilton (1988) and Pagan and Schwert (1990). It was used by Hamilton (1989) to explain US GNP, Rogers and Zhang (2011) use it to reproduce several well known stylized facts about asset returns. Haidinger and Warnung (2012) analyse risk measures in the setting of Rogers and Zhang (2011). Kritzman et al. (2012) use it for stock prices to come up with a trading strategy, they also include the Matlab source code used for the estimation. Giampieri et al. (2005) analyse defaults of companies. This approach is also used in the financial computing and artificial intelligence fields, see e.g. Rao and Hong (2010) or Hassan and Nath (2005).

HMMs consist of two (possibly multidimensional) processes: observable state (property price index and consumer confidence index in our case) and a finite state Markov chain that is unobservable (hence hidden), but which impacts the distribution of the observable process. A natural interpretation of the hidden state is that it defines market regimes, or 'animal spirits' if we look at it from the behavioural perspective. An attractive feature of HMMs is that the Viterbi algorithm makes it possible to efficiently calculate the most likely path of the hidden state process, which gives a really intuitive way to verify the model.

Dempster et al. (2012) use a hidden factor model in the commodity market. As opposed to HMMs, the hidden factors in their model are continuous processes. The most interesting feature of the cited paper is that the authors use standard structural Vector Auto-Regressive analysis of the most likely path of the hidden factors against published economic data. This in turn might be used for prediction of the hidden factors and hence prediction of the underlying commodity prices.

In this chapter we use HMMs to estimate 'animal spirits' from the price process itself.

Instead of using the most likely path of the hidden process in a regression model however, we introduce a multidimensional model with both house price process, which is our main point of interest, as well as the consumer confidence index as observable time series. We find that a model with three hidden states captures the data well, with one of the states corresponding to the worst part of the crisis. The consumer confidence index implied very similar path of the state process as the Shiller index, however it did not improve forecasting or VaR calculation.

Finally, we use the distribution forecast implied by the model for VaR calculation. We introduce a procedure of augmenting the model with an auxiliary state that corresponds to a crisis. Later in the chapter we show that this procedure provides a much more conservative risk estimate by running back-testing on the period during property crisis and that without this state the model completely doesn't capture the negative returns on house prices observed in the crisis.

Our estimation algorithm is based on Rabiner (1989), Rabiner and Juang (1993) and improvements from Rahimi (2000). See also Zucchini and MacDonald (2009) for a comprehensive treatment of the subject. We extend the standard algorithm to cater for meanreverting processes, show how to estimate parameters of correlated multi-dimensional continuoustime stochastic processes and provide a procedure for forecasting. Elliott et al. (1995) approach the subject using techniques from stochastic filtering. The use the filters also for maximum-likelihood parameter estimation in the expectation-maximisation (EM) iterative framework. Their approach is different from the one used here, but the resulting estimation procedure is very similar.

Section 4.1 contains basic discussion on hidden Markov models—all technical details can be found in appendices 4.A.1 and 4.B.1. Section 4.2 shows how to translate discrete-time HMMs into continuous-time stochastic regime-switching models. 4.3 lists the sources of data used for the estimation. Sections 4.4 and 4.5 contain estimation and VaR calculation results. Finally, section 4.6 concludes the chapter.

4.1 Hidden Markov Models

Let O_t denote the log-returns of the relevant asset price at time t. The time index is discrete and corresponds to the market observation times. In our case we assume that O_t is a continuous random variable with a density function.

The main idea behind HMMs is that there exists a corresponding time series $\{q_t\}$, which is not directly observable (hence hidden), and which denotes the 'state' of the system at every time t. In the class of models we are looking into $\{q_t\}$ is modelled as a finite-state Markov chain with initial distribution π and transition matrix A. The parameters of the distribution of O_t depend on the state variable q_t , e.g. if we assume that O_t is normally distributed then the mean and variance of the distribution are functions of q_t .

For simplicity of exposition we assume that q_t takes values in \mathbb{N} , the set of positive integers and the parameters of the distribution of O_t given q_t are in the q_t -th row of the matrix θ . So the whole model $\lambda = (\pi, A, \theta)$ consists of the initial distribution π of the state variable, its transition matrix A and the mapping θ of the distribution parameters of the observable process given the hidden state.

To estimate the parameters of the model we shall adopt the maximum likelihood method, so we find parameter values $\hat{\lambda}$ that maximise the likelihood function of the observed series:

$$\mathcal{L}(\lambda; O) = \mathbb{P}\left[O|\lambda\right],\tag{4.1}$$

where in our case the observations are continuous random variables, and hence $\mathbb{P}[O|\lambda]$ denotes the density function:

$$\mathbb{P}\left[O|\lambda\right] \equiv \mathbb{P}\left[O \in (x, x + \mathrm{d}x)|\lambda\right] \tag{4.2}$$

Direct maximisation is in this case very difficult and hence we will implement a version of Expectation-Maximisation (EM) algorithm, first introduced by Dempster et al. (1977). The algorithm starts from some user-defined initial guess for the parameter values and then improves them iteratively. It was proved that it always converges monotonically to a local maximum. A single iteration consists of two steps:

Expectation First, given the parameter values from previous step λ^{i-1} we calculate the

quantity:

$$Q(\lambda, \lambda^{i-1}) = \mathbb{E} \left[\log \mathbb{P} \left[O, Q | \lambda \right] | O, \lambda^{i-1} \right]$$

=
$$\sum_{q} \log \mathbb{P} \left[O, q | \lambda \right] \mathbb{P} \left[O, q | \lambda^{i-1} \right]$$
(4.3)

as a function of λ .

Maximisation Next, we find $\lambda^i = \arg \max_{\lambda} Q(\lambda, \lambda^{i-1})$, which will be used in the next iteration.

We iterate over these steps until we reach convergence to a local optimum. Note that the quantity that is being maximised (4.3) is different from a logarithm of the likelihood (4.1). Dempster et al. (1977) proved that the parameter values that maximise (4.3) also maximise (4.1), hence the algorithm gives the required estimate.

In particular we add certain simplifying assumptions, so that we can employ a slightly modified version of the Baum-Welch algorithm, which is specifically optimized in the case when the observations O_t are conditionally Markov. All the technical detail of the Baum-Welch algorithm can be found in the appendix 4.A. Appendix 4.B contains the discussion on model choice and our methodology of distribution-forecasting and verifying the model.

4.2 Continuous models

Most models in mathematical-finance are set in continuous time, mainly for easier tractability. However, real world data is only available at discrete time points (monthly in case of the Shiller index) and hence the statistics literature deals with discrete time series. To bridge these two fields easily and without introducing additional error terms, we only work with price processes that can be discretized exactly. The state process is assumed to only change value on the observation dates, so the continuous-time version has right-continuous piecewise-constant paths.

4.2.1 Geometric Brownian Motion

The Geometric Brownian Motion (GBM):

$$\mathbf{d}S_t = \mu S_t \mathbf{d}t + \sigma S_t \mathbf{d}W_t \tag{4.4}$$

is a very popular model for asset returns in finance, mainly due to its tractability. It is widely accepted that it doesn't capture the behaviour of the market very well. It can be vastly improved though if we introduce hidden state and make the drift and volatility dependent on the state (see e.g. Kritzman et al. (2012)). It is well known that in the discretized world the log-returns $O_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$ have a Gaussian distribution with mean $m(q_t) =$ $[\mu(q_t) - \frac{1}{2}\sigma^2(q_t)]\Delta t$ and variance $var(q_t) = \sigma^2(q_t)\Delta t$ (both state-dependent as discussed above), where Δt denotes the time between observations. For simplicity we estimate the parameters of q_t in two stages. First we estimate the mean $m(q_t)$ and variance $var(q_t)$, using the algorithm described above. Then we can easily retrieve the original parameters:

$$\mu_i^* = \frac{m^*(i) + \frac{1}{2} \operatorname{var}(i)}{dt}$$

$$\sigma_i^* = \sqrt{\frac{\operatorname{var}(i)}{dt}}.$$
(4.5)

4.2.2 Geometric Ornstein-Uhlenbeck

Now we assume that the price process is given by:

$$S_t = e^{X_t},\tag{4.6}$$

where X_t is a state-dependent Ornstein-Uhlenbeck process with dynamics:

$$\mathbf{d}X_t = \alpha(\beta - X_t)\mathbf{d}t + \sigma\mathbf{d}W_t,\tag{4.7}$$

and where the parameters α , β and σ depend on the hidden state q_t . According to Glasserman (2003, pp. 108-111) the exact discretization of X_t is given by:

$$X_t = e^{-\alpha\Delta t} X_{t-1} + \beta (1 - e^{-\alpha\Delta t}) + \sigma \sqrt{\frac{1}{2\alpha} (1 - e^{-2\alpha\Delta t})} Z_t, \qquad (4.8)$$

where Z_t is a standard normal variable for every t. In this case it is easier to assume that we observe the log of the price process rather than log returns as previously, so let $O_t = X_t$. At each time t the random variable O_t is normally distributed with mean $RO_{t-1} + B$ and variance C^2 , where:

$$R = e^{-\alpha\Delta t}$$

$$B = \beta(1 - e^{-\alpha\Delta t})$$

$$C = \sigma \sqrt{\frac{1}{2\alpha}(1 - e^{-2\alpha\Delta t})}$$
(4.9)

We can use the algorithm for Markov observations detailed in Appendix 4.A.4 for estimation. The original parameters can be recovered as follows:

$$\alpha = \frac{-\log R}{\Delta t}$$

$$\beta = \frac{B}{1-R}$$

$$\sigma = \frac{C}{\sqrt{\frac{1}{2\alpha}(1-R^2)}}$$
(4.10)

4.2.3 Mapping parameters in multidimensional case

One of the outputs of the Baum-Welch algorithm is the distribution parameters of the observable process in every state q_t :

$$O_t \sim \mathcal{N}(m(q_t, O_{t-1}), \Sigma(q_t)). \tag{4.11}$$

To recover the parameters of the original SDEs we first observe that each individual component of O_t can be written as:

$$O_t^i = m^i(q_t, O_{t-1}) + \phi^i(q_t) Z_t^i, \tag{4.12}$$

where $\phi^i(q_t) = \sqrt{\Sigma_{ii}(q_t)}$, and Z^i are standard normal random variables, with correlation matrix:

$$\rho(q_t) = \{\rho_{ij}(q_t)\}, \quad \rho_{ij}(q_t) = \frac{\sum_{ij}(q_t)}{\phi^i(q_t)\phi^j(q_t)}.$$
(4.13)

We can now recover parameters of each individual single-dimensional SDEs from (4.12). $\rho(q_t)$ is the correlation matrix between the Brownian motions driving these individual SDEs.

To recover the covariance matrix $\Sigma(q)$ from $\phi(q)$ and $\rho(q)$ we immediately notice that

$$\Sigma(q) = \operatorname{diag}(\phi(q))\rho(q)\operatorname{diag}(\phi(q)). \tag{4.14}$$

4.3 Data

For estimation we use the Standard & Poor's (S&P)/Case-Shiller 10-City Composite Home Price Index (ticker SPCS10) as the indicator of property prices in the United States. Announcements of index levels are made at 09:00 AM Eastern Time, on the last Tuesday of each month. Historic index values are available on the S&P website: http://eu. spindices.com/indices/real-estate/sp-case-shiller-10-city-composite-Before the estimation we calculate the seasonally adjusted version of the index using X-12-ARIMA Seasonal Adjustment Program (see http://www.census.gov/srd/www/ x12a/). Specifically we use the implementation provided as part of the open-source Gretl package (http://gretl.sourceforge.net/).

To capture consumer confidence we use the University of Michigan Consumer Sentiment Index. The preliminary index releases are scheduled around the middle of the relevant month, and the final release is around the end of the month. Exact release dates are available on the index web page. The historic press releases are available from the index page: http://thomsonreuters.com/products_services/financial/financial_ products/a-z/umichigan_surveys_of_consumers/. The historical values can also be downloaded in a more convenient format from FRED: http://research.stlouisfed.org/fred2/series/UMCSENT.

We also tried the Conference Board consumer confidence index, but the estimation results were inferior to the University of Michigan Consumer Sentiment Index, and hence are not presented here.

4.4 Estimation results

The Shiller index is characterised by a significantly lower volatility compared to stock indices, and most of the volatility has strongly seasonal characteristics. This is mostly due to low liquidity, long transaction times and high transaction costs. The distribution of

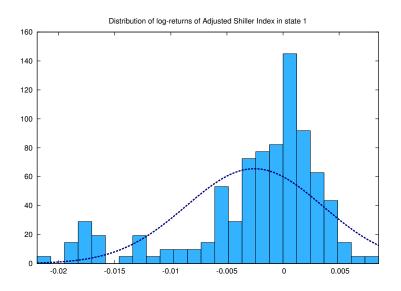


Figure 4.2: A histogram of log-returns of seasonally-adjusted Shiller index and a fitted normal distribution density.

log-returns of the Shiller index is highly non-Gaussian and multi-modal, as can be seen in Figure 4.2. The fitted normal distribution does not capture the underlying data well. Moreover, unlike the stock markets, using a fat-tailed distribution is not going to help in this case either. However, employing a hidden Markov model proves to be fruitful.

First, we estimate a HMM with an underlying geometric Brownian motion model and with two hidden states The most likely path of the hidden state process, obtained using

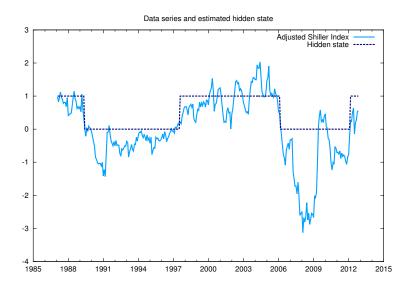


Figure 4.3: Log-returns of the Shiller property index (seasonally adjusted and normalized) and an estimated most likely path of the hidden state process with two states.

State	μ	σ	Interpretation
0	-0.0304282	0.0210861	Decline/Stagnation
1	0.112926	0.0114603	Growth

Table 4.1: Estimated parameters of the Shiller index (seasonally adjusted) for every value of the hidden state process. The model used is a geometric Brownian motion with two regimes.

the Viterbi algorithm described in the appendix 4.A.1, is in Figure 4.3. The estimated parameters of the process are in Table 4.1 and the transition matrix between states is given by:

$$A = \begin{bmatrix} 0.9854 & 0.0146\\ 0.0175 & 0.9825 \end{bmatrix}$$
(4.15)

The estimation procedure is able to identify different regimes surprisingly well. Figure 4.4 contains the histograms of log-returns corresponding to the two states. Whereas the observations corresponding to the growth regime are matching the normal distribution reasonably well, given the limited number of data points, the histogram corresponding to the weak market conditions is clearly skewed by the extreme losses observed during the crisis. We decided to include an extra state and as seen in Figure 4.5 the estimation procedure used it for the observations corresponding to the market crash. Figure 4.6 contains the relevant

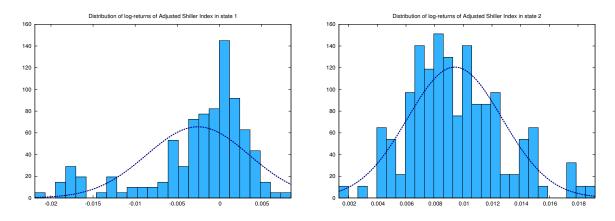


Figure 4.4: Histograms of log-returns of seasonally-adjusted Shiller index corresponding to most likely states and a fitted normal distribution density.

State	μ	σ	Interpretation
0	-0.176807	0.0139859	Crisis
1	-0.00798775	0.00962791	Stagnation
2	0.110411	0.0116515	Growth

Table 4.2: Estimated parameters of the Shiller index (seasonally adjusted) for every value of the hidden state process. The model used is a geometric Brownian motion with three regimes.

histograms. Note that the first state corresponds to very few observations. The estimated parameters of the regime-switching geometric Brownian motion process are given in Table 4.2. The transition matrix of the hidden factor process is given by:

$$A = \begin{bmatrix} 0.9609 & 0 & 0.0391 \\ 0.0075 & 0.9775 & 0.015 \\ 0 & 0.0202 & 0.9798 \end{bmatrix},$$
(4.16)

and the third state is estimated as the initial state.* Log-likelihood for this model is 1318.23, which is higher than 1212.1 and 1065.47 for a model with two hidden states and a simple geometric Brownian motion respectively. The Akaike information criterion (AIC) has the value of -2608.45, as opposed to -2410.2 and -2126.95 respectively. The Bayesian in-

^{*}It is always the case that the initial distribution degenerates to a single state. Intuitively this is caused by the fact that the observation data includes only one realisation of the initial state. Hence we can only estimate the most likely realisation of that state, as opposed to the actual distribution.

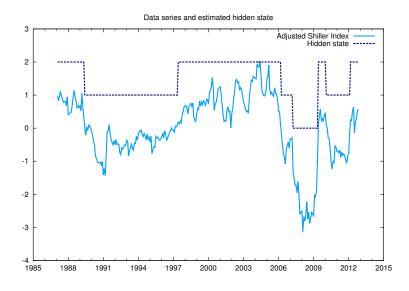


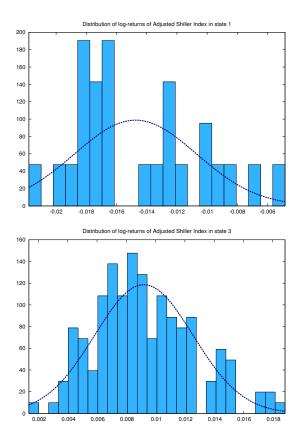
Figure 4.5: Log-returns of the Shiller property index (seasonally adjusted and normalized) and an estimated most likely path of the hidden state process with three states.

formation criterion (BIC) with value -2556.09 also confirms the model choice, as opposed to the values of -2384.02 and -2119.47 for the other models respectively. The most likely path of the hidden process had 26 observations in the first state, 133 in the second state and 151 in the third state.

Number of	State	Number of Skewness		Excess	Jarque-Berra
states	number	observations	Skewness	Kurtosis	statistic
1	0	310	-0.7569	0.6344	34.79819
2	0	170	-1.354	1.2474	62.96567
	1	140	0.4922	0.0833	5.69323
	0	26	0.5674	-0.6295	1.824378
3	1	133	-0.5046	-0.3809	6.448114
	2	151	0.5101	-0.0129	6.549464

Table 4.3: Statistics for log-returns of the Shiller index in models with varying number of states.

Table 4.3 summarises the basic statistics for the most likely observations corresponding to all the cases described above, including the Jarque-Bera test statistic. Even though the null hypothesis that the values are normally distributed is rejected in every case, the value of the statistic is much lower in the model with three states. We believe it is unlikely that standard distributions will fit this data well, because the Shiller index is artificially con-



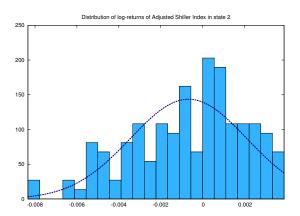


Figure 4.6: Histograms of log-returns of seasonally-adjusted Shiller index corresponding to most likely states and a fitted normal distribution density.

structed and then manipulated further to remove seasonality. Despite these shortcomings, we get promising results in the next section.

It is well known that exact estimation of parameters of financial assets, the drift in particular, is virtually impossible, see e.g. Rogers (2001). Hence, and also because of limited available data, our estimation procedure produces different parameter values for different estimation windows. However, the estimation of the most likely states is very stable, irrespective of the choice of historical data set. In particular, the states assigned to the observations before the crisis remain virtually the same if we include the crisis period in the estimation or not.

Note that the 'a posteriori' changes of states in Figure 4.5 may be linked to specific events. The slowdown that started in 1989 may be linked with the Financial Institutions Reform, Recovery and Enforcement Act (FIRREA) and the resulting closing of hundreds of

Chapter 4. Estimating animal spirits

insolvent thrifts. Later the market improved after The Taxpayer Relief Act of 1997, which introduced breaks on capital gains on the sale of a home and encouraged people to buy more expensive first homes, as well as invest in second homes and investment properties. In 2006 we see the factor process jumping down, marking the beginning of the Credit Crunch. At that time many sub-prime lenders are declaring bankruptcy. The sharp fall to the crisis state in 2007 corresponds to the deepening of the crisis. In August that year most of the global banks find sub-prime mortgage backed securities on their balance sheets, declare losses and cut down on lending. The short bounce-back of 2010 is reflected in the state process jumping to the growth state for a few months, which is followed by stagnation in the market. Finally in 2013 the market starts to recover again.

We also estimated a geometric Ornstein-Uhlenbeck model (known as the Longstaff-Schwartz model in the commodity pricing literature), but the mean-reversion coefficient in the growth state was negative—suggesting that the Shiller index doesn't have any meanreversion properties and hence the model is inadequate. The model was also estimated for joint Shiller property price index and consumer confidence index, using a geometric Brownian motion model for the former data series, and a geometric Ornstein-Uhlenbeck model for the latter. Obtained results were very similar to the single-dimensional case,

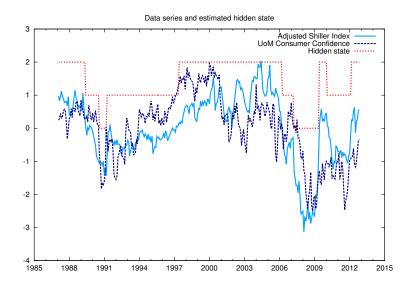


Figure 4.7: Log-returns of the Shiller property index (seasonally adjusted and normalized), University of Michigan consumer sentiment index and an estimated most likely path of the hidden state process with three states.

which suggests that the consumer confidence index doesn't add more information over what is already contained in property prices. The most likely path is presented in Figure 4.7. The estimated parameters of the regime-switching geometric Brownian motion process

State	μ	σ	α	β	σ_{cons}	Interpretation
0	-0.14999	0.0180475	2.29584	4.21205	0.292439	Crisis
1	-0.00374117	0.00843199	0.71729	4.44809	0.164324	Stagnation
2	0.11024	0.0116677	1.00023	4.52742	0.149742	Growth

Table 4.4: Estimated parameters of the seasonally adjusted Shiller index (μ and σ) and University of Michigan consumer confidence index (α , β and σ_{cons}) for every value of the hidden state process. The model used is a geometric Brownian motion and exponential Ornstein-Uhlenbeck with three regimes.

corresponding to the Shiller index and the parameters of exponential Ornstein-Uhlenbeck process for consumer confidence are given in Table 4.4. The transition matrix of the hidden factor process is given by:

$$A = \begin{vmatrix} 0.9406 & 0.0318 & 0.0276 \\ 0.0163 & 0.9671 & 0.0166 \\ 0 & 0.0201 & 0.9799 \end{vmatrix},$$
(4.17)

The joint model didn't improve value at risk estimation.

In the financial markets it may be possible to estimate 'animal spirits' from the put option markets (the implied volatility of put options becomes higher in markets in distress, when the market participants are pessimistic). However, liquid option prices are not available in the property market analysed in this chapter.

4.5 **Risk management and animal spirits**

One of the main purposes of this chapter is to propose a model for value at risk for house prices. Back-testing is done for the period from July 2006 to November 2012, starting just before the credit crunch. First we employ the forecasting method described in the appendix 4.B. We estimate the model for every period in the back-testing range, and use the parameters for one-period ahead forecast. Using the forecast distribution, we calculate

Chapter 4. Estimating animal spirits

the 95% VaR level[†]. For all the relevant periods we calculate the proportion of observations that were below the corresponding VaR level. For a perfect forecast we would expect this ratio to be 5%. Unfortunately, just by looking at the data, it is clear that purely statistical methods are bound to fail, because the series before the crisis is in no way representative for the crash period. Indeed, more than 22% of all the observations fall in the VaR region, which shows that our risk was grossly underestimated. The results would be much better if we knew the true distribution of the data series before the crisis happened. If, instead of estimating the parameters at every period, we use the parameters estimated for the whole series (of course in reality they would not be known at that time), we get the proportion of observations down to a much more reasonable 7.8%.

We can achieve similar results without 'cheating'—it is enough to introduce stress scenarios to the data set. To get conservative risk estimates, one should assume that the house prices might fall—even though they haven't in the historic data series. To illustrate that this simple idea works in our context, we proceed as follows. We estimate parameters of a hidden Markov model with two hidden states in June 2006. The resulting model is given

State	μ	σ	Interpretation
1	0.0002445	0.0094117	Stagnation
2	0.11767	0.0109299	Growth

Table 4.5: Estimated parameters of the Shiller index (seasonally adjusted) just before the benchmarking period.

in Table 4.5, with the transition matrix of the hidden Markov chain given by:

$$A = \begin{bmatrix} 0.9900 & 0.0100\\ 0.0154 & 0.9846 \end{bmatrix}, \tag{4.18}$$

Next we introduce an arbitrary 'crisis' state using the following heuristics:

- The mean of the log-returns in the new state is the negative of the double mean of the existing growth state, with variance the same as the existing decline state
- The crisis starts only when the system is in the decline state, with probability 0.03

[†]The 95% level was chosen because of the limited number of observations.

• The crisis ends with probability 0.05, and the system moves directly into the growth state to capture the 'bounce-back' effect

State	μ	σ	Interpretation
1	-0.2351763	0.0094117	New Crisis state
2	0.0002445	0.0094117	Stagnation
3	0.11767	0.0109299	Growth

Table 4.6: Estimated parameters of the Shiller index (seasonally adjusted) with an auxiliary state added for a house price crash.

The model parameters after applying the simple heuristics described above are given in Table 4.6, and the transition matrix is given by:

$$A = \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.03 & 0.9600 & 0.0100 \\ 0 & 0.0154 & 0.9846 \end{bmatrix},$$
 (4.19)

For simplicity of calculations, we keep these parameters throughout the benchmarking period. This simple inclusion of a 'crisis' state allows us to get the proportion of observations in the VaR region down to 7.8%. The need of an auxiliary state is even more apparent in

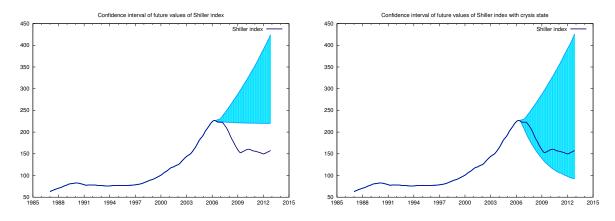


Figure 4.8: Estimation of 5% and 95% quantiles of future distribution of Shiller index starting from July 2006. The left panel uses 2-state model estimated in June 2006, the right panel uses the same model with an auxiliary crisis state added.

Figure 4.8, which shows the 5% to 95% range of possible future paths of the Shiller index

Chapter 4. Estimating animal spirits

starting from July 2006. The left graph was generated using the two state model estimated in June 2006, whereas the right graph was generated using the model with a crisis state added using the procedure described above. The most striking feature is that the original model doesn't account for a fall in house prices at all, which caused the risk estimates to be grossly overoptimistic. This situation exemplifies A. Greenspan's 'irrational exuberance'.

Of course, there is no reason to believe that this new model with a crisis state reflects the true distribution of the house prices, it provides a way to obtain a conservative risk estimates though. The key idea is to identify the stress scenario that could happen, but is not reflected in historic data, and assign a positive probability to it. This is a very intuitive procedure in hidden Markov models, because the stress scenario corresponds in a natural way to a new state of the hidden factor process. Note also that if the system is in a growth state, then the probability of a crash is very low—hence the model is not over-conservative in the periods of high returns.

4.6 Summary

In this chapter we summarised the Baum-Welch algorithm for estimating parameters of Hidden Markov Models. We extended the standard algorithm to cater for two most popular models in finance: geometric Brownian motion and Ornstein-Uhlenbeck type, both in single and multiple dimensions. Moreover, we introduced formulas for density forecasts and risk calculation. We estimated the model for Shiller Home Price Index and consumer confidence index, and calculated the most likely paths of the 'animal-spirits' hidden state process. We used the model for VaR calculation.

Finally, using the example of property price crash, we highlighted the importance of including auxiliary stress scenarios in risk calculations, especially in markets driven by 'animal spirits' or 'irrational exuberance'. Conservative risk managers should always account for negative sentiment and bubble burst. HMMs prove to be perfect models for inclusion of stress scenarios. The procedure is intuitive, the extended model captures the downside risk, and yet is not overly pessimistic during good times.

4.A Appendix: Estimation

4.A.1 The Baum-Welch algorithm

First let us assume that the observations are conditionally (given the hidden process) independent of each other. This case satisfies the assumptions of the Baum-Welch algorithm, which is a fast-performing special case of the more general EM method. The following discussion follows Rabiner and Juang (1993). For simplicity we consider a single-dimensional case, but the discussion carries over to the multidimensional case. We can write the densities $\mathbb{P}[O, q | \lambda]$ as:

$$\mathbb{P}[O,q|\lambda] = \pi_{q_1} f_{q_1}(O_1) \prod_{t=2}^T a_{q_{t-1}q_t} f_{q_t}(O_t), \qquad (4.20)$$

where $a_{q_{t-1}q_t}$ is the transition probability from the state q_{t-1} to q_t and $f_{q_t}(O_t) = f(O_t; q_t)$ is the density of the observable process given that the system is in the state q_t . The Q function (4.3) simplifies to a much more computation-friendly formula:

$$Q((\pi, A, \theta), \lambda) = \sum_{i=1}^{N} \mathbb{P} [O, q_{1} = i | \lambda] \log \pi_{i}$$

+ $\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=2}^{T} \mathbb{P} [O, q_{t-1} = i, q_{t} = j | \lambda] \log a_{ij}$ (4.21)
+ $\sum_{i=1}^{N} \sum_{t=1}^{T} \mathbb{P} [O, q_{t} = i | \lambda] \log f_{i}(O_{t}),$

where the only dependence on θ is in the f_i functions. Because of the separability of Q with respect to different parameters, it is easy to calculate the maximum analytically using the constraints:

$$\sum_{j=1}^{N} \pi_j = 1$$

$$\sum_{j=1}^{N} a_{ij} = 1, \quad \text{for every } i$$
(4.22)

and the normal density function for f. The maximised values of parameters are given by:

$$\pi_{i}^{*} = \frac{\mathbb{P}\left[O, q_{1} = i \mid \lambda\right]}{\mathbb{P}\left[O \mid \lambda\right]}$$

$$a_{ij}^{*} = \frac{\sum_{t=2}^{T} \mathbb{P}\left[O, q_{t-1} = i, q_{t} = j \mid \lambda\right]}{\sum_{t=2}^{T} \mathbb{P}\left[O, q_{t-1} = i \mid \lambda\right]}$$

$$m^{*}(i) = \frac{\sum_{t=1}^{T} \mathbb{P}\left[O, q_{t} = i \mid \lambda\right] O_{t}}{\sum_{t=1}^{T} \mathbb{P}\left[O, q_{t} = i \mid \lambda\right]}$$

$$\operatorname{var}^{*}(i) = \frac{\sum_{t=1}^{T} \mathbb{P}\left[O, q_{t} = i \mid \lambda\right]}{\sum_{t=1}^{T} \mathbb{P}\left[O, q_{t} = i \mid \lambda\right]}$$

$$(4.23)$$

To be able to efficiently evaluate the equations above, we use the Forward-Backward procedure introduced by Baum and Welch. First define the forward variables:

$$\alpha_t(i) = \mathbb{P}\left[O_1 O_2 \dots O_t, q_t = i | \lambda\right], \tag{4.24}$$

which can be efficiently calculated inductively, using the dynamic-programming ideas. Initial value is given by:

$$\alpha_1(i) = \pi_i f_i(O_1) \tag{4.25}$$

and the induction step by:

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] f_j(O_{t+1}).$$
(4.26)

Note that, because we might end up in any state at the final time T, we have:

$$\mathbb{P}\left[O|\lambda\right] = \sum_{i=1}^{N} \alpha_T(i).$$
(4.27)

Analogously we define the backward variables:

$$\beta_t(i) = \mathbb{P}\left[O_{t+1}O_{t+2}\dots O_T | q_t = i, \lambda\right],\tag{4.28}$$

The final values are arbitrarily defined to be unity:

$$\beta_T(i) = 1. \tag{4.29}$$

and backward induction step is given by:

$$\beta_t(i) = \sum_{j=1}^N a_{ij} f_j(O_{t+1}) \beta_{t+1}(j).$$
(4.30)

By looking at the probabilistic definitions of α and β and using the Bayes formula we get:

$$\gamma_t(i) = \mathbb{P}\left[O|q_t = i, \lambda\right]$$

= $\frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^N \alpha_t(j)\beta_t(j)}.$ (4.31)

Analogously we have the formula for the last missing ingredient:

$$\xi_t(i,j) = \mathbb{P}\left[q_t = i, q_{t+1} = j | O, \lambda\right]$$

=
$$\frac{\alpha_t(i)a_{ij}f_j(O_{t+1})\beta_{t+1}(j)}{\mathbb{P}\left[O | \lambda\right]}.$$
(4.32)

It is convenient to express the estimates (4.23) in terms of γ and ξ :

$$\pi_{i}^{*} = \gamma_{1}(i)$$

$$a_{ij}^{*} = \frac{\sum_{t=1}^{T-1} \xi_{t}(i, j)}{\sum_{t=1}^{T-1} \gamma_{t}(i)}$$

$$m^{*}(i) = \frac{\sum_{t=1}^{T} \gamma_{t}(i)O_{t}}{\sum_{t=1}^{T} \gamma_{t}(i)}$$

$$var^{*}(i) = \frac{\sum_{t=1}^{T} \gamma_{t}(i)(O_{t} - m^{*}(i))^{2}}{\sum_{t=1}^{T} \gamma_{t}(i)}$$
(4.33)

4.A.2 Scaling

Unfortunately the induction described above is not numerically stable, because the terms α_t go to zero exponentially fast as t increases and get out of the scope of double-precision

numbers used in computer systems. To avoid this we introduce scaled versions of these coefficients, following Rabiner and Juang (1993) and Rahimi (2000)[‡]:

$$\bar{\alpha}_{1}(i) = \alpha_{1}(i)$$

$$n_{t} = \sum_{j=1}^{N} \bar{\alpha}_{t}(j)$$

$$\hat{\alpha}_{t}(i) = \frac{\bar{\alpha}_{t}(i)}{n_{t}}$$

$$\bar{\alpha}_{t+1}(j) = \left[\sum_{i=1}^{N} \hat{\alpha}_{t}(i)a_{ij}\right] f_{j}(O_{t+1})$$
(4.34)

It is easy to notice that the scaled values are given by:

$$\hat{\alpha}_t(i) = \frac{\alpha_t(i)}{N_t},\tag{4.35}$$

where

$$N_t = \prod_{\tau=1}^t n_t \tag{4.36}$$

By using the equation above for $\hat{\alpha}_t(i)$ and the definition for induction (4.34) we can write:

$$\hat{\alpha}_{t}(i) = \frac{\sum_{j=1}^{N} \alpha_{t-1}(j) a_{ji} f_{i}(O_{t}) / N_{t}}{\sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_{t-1}(j) a_{jk} f_{k}(O_{t}) / N_{t}} = \frac{\alpha_{t}(i)}{\sum_{k=1}^{N} \alpha_{t}(k)},$$
(4.37)

which shows that our induction is indeed producing scaled variables as indicated above. We use the same normalisation factors to calculate scaled backward variables:

$$\hat{\beta}_{T}(i) = \frac{1}{n_{T}}$$

$$\hat{\beta}_{t}(i) = \frac{1}{n_{t}} \sum_{j=1}^{N} a_{ij} \hat{\beta}_{t+1}(j) f_{j}(O_{t+1})$$
(4.38)

[‡]Note that there is a typing mistake in one of the formulas in the latter reference, this was corrected in our calculations.

And analogously we get:

$$\hat{\beta}_t(i) = \frac{\beta_t(i)}{M_t},\tag{4.39}$$

where

$$M_t = \prod_{\tau=t}^T n_t \tag{4.40}$$

These can be now directly used to calculate the needed probabilities. First because $\hat{\alpha}$ are normalized, the identity holds:

$$\sum_{i=1}^{N} \hat{\alpha}_T(i) = \frac{\sum_{i=1}^{N} \alpha_T(i)}{N_T} = 1,$$
(4.41)

thus by (4.27) we get $\mathbb{P}[O|\lambda] = N_T$. As already mentioned, this quantity is outside of the range of double numbers, but we can efficiently calculate the logarithm (the log-likelihood function):

$$\log \mathbb{P}\left[O|\lambda\right] = \sum_{t=1}^{T} \log n_t \tag{4.42}$$

To calculate ξ_t we will substitute α_t for $\hat{\alpha}_t N_t$ and β_t for $\hat{\beta}_t M_t$ in (4.32):

$$\begin{aligned} \xi_t(i,j) &= \frac{\alpha_t(i)a_{ij}f_j(O_{t+1})\beta_{t+1}(j)}{\mathbb{P}\left[O|\lambda\right]} \\ &= \frac{\hat{\alpha}_t(i)a_{ij}f_j(O_{t+1})\hat{\beta}_{t+1}(j)N_tM_{t+1}}{\mathbb{P}\left[O|\lambda\right]} \\ &= \frac{\hat{\alpha}_t(i)a_{ij}f_j(O_{t+1})\hat{\beta}_{t+1}(j)N_T}{\mathbb{P}\left[O|\lambda\right]} \\ &= \hat{\alpha}_t(i)a_{ij}f_j(O_{t+1})\hat{\beta}_{t+1}(j), \end{aligned}$$
(4.43)

where we used the fact that $N_t M_{t+1} = N_T$ and that $\mathbb{P}[O|\lambda] = N_T$. The γ coefficients may be computed analogously:

$$\gamma_{t}(i) = \frac{\alpha_{t}(i)\beta_{t}(i)}{\mathbb{P}\left[O|\lambda\right]}$$

$$= \frac{\hat{\alpha}_{t}(i)\hat{\beta}_{t}(i)N_{t}M_{t}}{\mathbb{P}\left[O|\lambda\right]}$$

$$= \hat{\alpha}_{t}(i)\hat{\beta}_{t}(i)n_{t},$$
(4.44)

because $N_t M_t = n_t N_T = n_t \mathbb{P}[O|\lambda]$. We can plug these values to calculate the parameter estimates according to (4.33), just as in the unscaled case.

4.A.3 The Viterbi Algorithm

Once we have the optimal parameter values, we would like to find the most likely path of the hidden state process. This can be achieved using the Viterbi algorithm, introduced in Viterbi (1967). We follow Rabiner and Juang (1993) and present a version that directly optimizes log-likelihood as opposed to likelihood, which makes it both numerically stable and more efficient. For performance reasons, the algorithm first pre-computes logarithms of the relevant parameters:

$$\tilde{\pi}_{i} = \log \pi_{i}$$

$$\tilde{f}_{i}(O_{t}) = \log f_{i}(O_{t}) \qquad (4.45)$$

$$\tilde{a}_{ij} = \log a_{ij}$$

Next define:

$$\delta_t(i) = \max_{q_1 q_2 \dots q_{t-1}} \log \mathbb{P} \left[q_1 q_2 \dots q_{t-1}, q_t = i, O_1 O_2 \dots O_t | \lambda \right],$$
(4.46)

that is, $\delta_t(i)$ denotes the highest joint log-likelihood along a single path up to time t that ends in the state i. This quantity may be efficiently computed by the following induction:

$$\delta_{1}(i) = \tilde{\pi}_{i} + f_{i}(O_{1})$$

$$\delta_{t}(j) = \max_{1 \le i \le N} [\delta_{t-1}(i) + \tilde{a}_{ij}] + \tilde{f}_{j}(O_{t}).$$
(4.47)

We also need another set of variables ψ to track which previous state was chosen in the maximisation above, at every time and state j:

$$\psi_{1}(i) = 0$$

$$\psi_{t}(j) = \underset{1 \le i \le N}{\arg \max} [\delta_{t-1}(i) + \tilde{a}_{ij}].$$
(4.48)

Having calculated these numbers, the maximum joint log-likelihood of hidden and observable paths is given by:

$$P^* = \max_{1 \le i \le N} [\delta_t(i)] \tag{4.49}$$

and the last element of the most likely hidden path is given by:

$$q_T^* = \underset{1 \le i \le N}{\arg \max} [\delta_t(i)].$$
(4.50)

To calculate the most likely hidden state at earlier times we use the backtracking method:

$$q_t^* = \psi_{t+1}(q_{t+1}^*). \tag{4.51}$$

4.A.4 Markov observations

We can extend the model by assuming that, given the hidden process value, the observations depend also on the previous observations, i.e. observations are conditionally Markov. For simplicity we start the estimation from time t = 2, so that O_{t-1} is always well defined. The equation for total likelihood accounts for the dependence in the density function:

$$\mathbb{P}\left[O,q\,|\,\lambda\right] = \pi_{q_2} f_{q_2}(O_2|O_1) \prod_{t=3}^T a_{q_{t-1}q_t} f_{q_t}(O_t|O_{t-1}),\tag{4.52}$$

hence the simplified equation for the Q function has the form:

$$Q((\pi, A, \theta), \lambda) = \sum_{i=1}^{N} \mathbb{P} [O, q_{2} = i | \lambda] \log \pi_{i}$$

+
$$\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=3}^{T} \mathbb{P} [O, q_{t-1} = i, q_{t} = j | \lambda] \log a_{ij} \qquad (4.53)$$

+
$$\sum_{i=1}^{N} \sum_{t=2}^{T} \mathbb{P} [O, q_{t} = i | \lambda] \log f_{i}(O_{t} | O_{t-1}).$$

Optimisation yields exactly the same formulas for π^* and a^* as for the conditionally independent case (with the time index starting at 2). If the observable process O_t is assumed to have a Gaussian distribution with mean $R(q_t)O_{t-1} + B(q_t)$ and variance var, then the

optimal parameters of the observable process are given by:

$$R^{*}(i) = \frac{\left(\sum_{t=2}^{T} P_{ti}O_{t-1}O_{t}\right)\left(\sum_{t=2}^{T} P_{ti}\right) - \left(\sum_{t=2}^{T} P_{ti}O_{t}\right)\left(\sum_{t=2}^{T} P_{ti}O_{t-1}\right)}{\left(\sum_{t=2}^{T} P_{ti}O_{t-1}\right)\left(\sum_{t=2}^{T} P_{ti}\right) - \left(\sum_{t=2}^{T} P_{ti}O_{t-1}\right)^{2}}$$

$$B^{*}(i) = \frac{\left(\sum_{t=2}^{T} P_{ti}O_{t}\right)\left(\sum_{t=2}^{T} P_{ti}O_{t-1}^{2}\right) - \left(\sum_{t=2}^{T} P_{ti}O_{t-1}O_{t}\right)\left(\sum_{t=2}^{T} P_{ti}O_{t-1}\right)}{\left(\sum_{t=2}^{T} P_{ti}\right)\left(\sum_{t=2}^{T} P_{ti}O_{t-1}^{2}\right) - \left(\sum_{t=2}^{T} P_{ti}O_{t-1}\right)^{2}}$$

$$var^{*}(i) = \frac{\sum_{t=2}^{T} P_{ti}(O_{t} - R^{*}(i)O_{t-1} - B^{*}(i))^{2}}{\sum_{t=2}^{T} P_{ti}}$$
(4.54)

where we introduced the notation $P_{ti} = \mathbb{P}[O, q_t = i | \lambda]$. The only modification needed to calculate all the probabilities by induction is to define the backward variables as:

$$\beta_t(i) = \mathbb{P}\left[O_{t+1}O_{t+2}\dots O_T | q_t = i, O_t, \lambda\right],\tag{4.55}$$

The induction is again similar to the basic case, and the variables γ and ξ are calculated in the same way. We can apply the same scaling approach as previously, and after obtaining optimal estimates the Viterbi algorithm to calculate the most likely hidden path.

4.B Appendix: Model choice and forecasting

4.B.1 Model choice

A problem that arises naturally when estimating a class of models to data is that of model selection, i.e. deciding which model fits best. A natural measure of model fit is the like-lihood value. However, likelihood always increases with the number of parameters of the model, eventually leading to over-fitting.

Zucchini and MacDonald (2009) propose to use two standard measures of discrepancy ('lack of fit' of the model) to overcome these issues. Akaike information criterion is defined by:

$$AIC = -2\log L + 2p, \tag{4.56}$$

where L denotes the likelihood of the model and p is the number of parameters. The Bayesian information criterion is given by:

$$BIC = -2\log L + p\log T, \qquad (4.57)$$

where L and p are as defined above, and T is the number of observations. The best model minimizes the chosen criterion in the relevant class of models. Note that both of these criteria favour high log-likelihood and penalize the number of parameters. BIC often chooses models with smaller number of parameters than AIC, i.e. HMMs with smaller number of states.

4.B.2 Forecasting

One of the benefits of Hidden Markov Models is that they can be easily used for density forecasting. Given the history of the process up to time T, the distribution of the observable at time T + 1 is given by:

$$\mathbb{P}[O_{T+1} = o | O_1 \dots O_T] = \frac{\mathbb{P}[O_1 \dots O_T, O_{T+1} = o]}{\mathbb{P}[O_1 \dots O_T]} \\ = \frac{\sum_{i=1}^N \alpha_T(i) \sum_{j=1}^N a_{ij} f_j(o)}{\sum_{i=1}^N \alpha_T(i)},$$
(4.58)

where $\alpha_T(i)$ is defined as in (4.24). Note that in our case the observable process is continuous, so we need to interpret the equation above as the density function. It is easy to notice that the forecast distribution is a mixture of the base distributions (Gaussian in our case):

$$\mathbb{P}[O_{T+1} = o | O_1 \dots O_T] = \sum_{j=1}^N \psi_T(j) f_j(o), \qquad (4.59)$$

with

$$\psi_T(j) = \frac{\sum_{i=1}^N \alpha_T(i) a_{ij}}{\sum_{i=1}^N \alpha_T(i)}$$
(4.60)

The forecast distribution of the hidden state is:

$$\mathbb{P}\left[Q_{T+1} = q | O_1 \dots O_T\right] = \frac{\mathbb{P}\left[O_1 \dots O_T, Q_{T+1} = q\right]}{\mathbb{P}\left[O_1 \dots O_T\right]}$$

$$= \frac{\sum_{i=1}^N \alpha_T(i) a_{iq}}{\sum_{i=1}^N \alpha_T(i)}.$$
(4.61)

Note that in all the formulas above we can substitute α_T with the scaled version $\hat{\alpha}_T$, because the normalization factors cancel out.

4.B.3 Multivariate model with lags

It is often observed that the variable of interest is correlated with a lagged version of a different variable. This happens if there is a true inter-temporal relation between these variables, but also when the lag is introduced by measurement and publication policies. We can harness this relationship in a way inspired by VAR models.

Assume that X_t is the variable of interest and Y_t is another observable variable, such that a lagged version of Y is correlated with X_t . We can use the measures of fit of the model introduced in section 4.B.1 to verify that this is indeed the case and determine the optimal lag. For simplicity of exposition, in this section we assume that the optimal lag is one period, hence the observable process at time t is composed of both X_t and Y_{t-1} :

$$O_t = [X_t Y_{t-1}]'. (4.62)$$

To forecast X_{T+1} at time T we can use the available observation Y_T in a natural way:

$$\mathbb{P}\left[X_{T+1} = x | X_1 \dots X_T, Y_0 \dots Y_T\right] = \frac{\mathbb{P}\left[X_{T+1} = x, X_1 \dots X_T, Y_0 \dots Y_T\right]}{\mathbb{P}\left[X_1 \dots X_T, Y_0 \dots Y_T\right]} \\
= \frac{\sum_{i=1}^N \alpha_T(i) \sum_{j=1}^N a_{ij} f_j(x, Y_T)}{\sum_{i=1}^N \alpha_T(i) \sum_{j=1}^N a_{ij} f_j^Y(Y_T)},$$
(4.63)

where $f_j^Y(Y_T)$ is the marginal distribution of Y_T . By applying the definition of conditional

probability $f_l(x, Y_T) = f_l(x|Y_T)f_l^Y(Y_T)$ we can transform the equation above to arrive at:

$$\mathbb{P}\left[X_{T+1} = x | X_1 \dots X_T, Y_0 \dots Y_T\right] = \sum_{l=1}^N \psi_{T+1}^l(Y_T) f_l(x|Y_T), \quad (4.64)$$

where

$$\psi_{T+1}^{l}(Y_{T}) = \frac{\sum_{i=1}^{N} \alpha_{T}(i) a_{il} f_{l}^{Y}(Y_{T})}{\sum_{i=1}^{N} \alpha_{T}(i) \sum_{j=1}^{N} a_{ij} f_{j}^{Y}(Y_{T})}$$
(4.65)

The forecast is thus a mixture of distributions. Note that this approach automatically takes into account the most likely distribution of the hidden variable and the correlation between X and Y in all the hidden states to produce the forecast.

In particular, if the mean of all the conditional distributions $f_l(\cdot|Y_T)$ is known, then the expected value of X_T is just the weighted sum of expected values in all the states. Moreover, if X and Y are jointly-Gaussian in every state *i* with parameters:

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_i^X \\ \mu_i^Y \end{bmatrix}, \begin{bmatrix} \Sigma_i^X & \Sigma_i^{XY} \\ \Sigma_i^{YX} & \Sigma_i^Y \end{bmatrix} \right), \qquad (4.66)$$

then Y is normally distributed $Y \sim \mathcal{N}(\mu_i^Y, \Sigma_i^Y)$, and X, conditionally given Y, is also normally distributed with

$$X_{T+1}|Y_T = y \sim \mathcal{N}\left(\mu_i^X + \Sigma_i^{XY}(\Sigma_i^Y)^{-1}(y - \mu_i^Y), \Sigma_i^X - \Sigma_i^{XY}(\Sigma_i^Y)^{-1}\Sigma_i^{YX}\right).$$
(4.67)

As a result the forecast of X is given by a mixture of Gaussians. In the case where both X and Y are single-dimensional the formula simplifies to:

$$X_{T+1}|Y_T = y \sim \mathcal{N}\left(\mu_i^X + \frac{\sigma_i^X}{\sigma_i^Y}\rho(y - \mu_i^Y), (1 - \rho^2)(\sigma_i^X)^2\right),$$
(4.68)

where σ denotes the variance of the corresponding random variable and ρ denotes the correlation between X and Y.

4.B.4 Verification of forecast distribution

Of course it is impossible to verify a single forecast distribution after we get the actual realisation—a single point is not enough to do that. We can, however, verify that the model produces consistent density forecast if we look at a series of distributions and realisations. The idea is to apply the forecast cumulative distribution function at every time t to the actual realisation at that time. The resulting random variable is uniformly distributed, provided that the model is correct. Moreover, these uniform random variables will be independent of each other. This approach was first introduced by Diebold et al. (1998), and can be summarised in the following proposition:

Proposition 4.B.1 Assume that at each point in time $T_0 \le t \le T$ we are given a distribution forecast for the value of process X:

$$F_t(x) = \mathbb{P}\left[X_t < x | \mathcal{G}_{t-1}\right],\tag{4.69}$$

where for simplicity we assume that F_t is a continuous and strictly increasing function (as it is the case for mixture Gaussian distribution) and \mathcal{G}_{t-1} is the sigma algebra representing the information available at time t - 1. Let $Y_t = F_t(X_t)$ for every t. Then the random variables Y_t are uniformly distributed on (0, 1) and are mutually independent.

Proof First, note that the inverse of the distribution function F_t^{-1} is well defined for every time t. We can use direct computation to show the conditional distribution of Y_t :

$$F_t^{Y_t}(y) = \mathbb{P}\left[Y_t < y | \mathcal{G}_{t-1}\right]$$

= $\mathbb{P}\left[F_t(X_t) < y | \mathcal{G}_{t-1}\right]$
= $\mathbb{P}\left[X_t < F_t^{-1}(y) | \mathcal{G}_{t-1}\right]$
= $F_t(F_t^{-1}(y))$
= $y,$ (4.70)

which indeed is a distribution function of a uniform random variable on (0, 1). To show the

unconditional distribution, we need the tower property of conditional expectations:

$$F^{Y_t}(y) = \mathbb{E} \left[\mathbf{1} \left\{ Y_t < y \right\} \right]$$

= $\mathbb{E} \left[\mathbb{P} \left[Y_t < y | \mathcal{G}_{t-1} \right] \right]$
= $\mathbb{E} \left[y \right]$
= $y.$ (4.71)

Another application of the tower property is needed to show the independence of Y_{t_1} and Y_{t_2} for $t_1 < t_2$:

$$\mathbb{P} [Y_{t_1} < y_1, Y_{t_2} < y_2] = \mathbb{E} [\mathbf{1} \{Y_{t_1} < y_1\} \mathbf{1} \{Y_{t_2} < y_2\}]$$

= $\mathbb{E} [\mathbf{1} \{Y_{t_1} < y_1\} \mathbb{E} [\mathbf{1} \{Y_{t_2} < y_2\} | \mathcal{G}_{t_2-1}]]$
= $\mathbb{E} [\mathbf{1} \{Y_{t_1} < y_1\} y_2]$ (4.72)
= $y_2 \mathbb{E} [\mathbf{1} \{Y_{t_1} < y_1\}]$
= $y_1 y_2$,

which ends the proof.

Note that the filtration $\{\mathcal{G}_t\}_{t\geq 0}$ might be generated by the history of the process X alone, but also might be enlarged e.g. by a lagged version of another process, as proposed in above. Note also that this result doesn't depend on the specific structure of hidden Markov models.

We use the proposition above to calculate the proportion of observations that fall below the estimated value at risk. Note, that $Y_t < \alpha$ implies that $X_t < \text{VaR}_{1-\alpha}$. Because of the independence of Y, if the model is correct than the proportion of Y that fall below a fixed level α should be close to α , as the number of observations goes to infinity.

Chapter 5

Animal control

5.1 Introduction

Recently, especially after the latest credit crisis, many hedge funds and portfolio managers have been taking interest in improved modelling of asset returns in the market. They are using the distribution of returns to derive an optimal trading strategy that optimises the trade-off between high expected returns and volatility.

This is the problem we tackle in this chapter. First we introduce a regime-switching model for the market. We assume that in every state the price process follows a jump diffusion. The change of regime might or might not correspond to a jump in the asset value. This formulation permits fat-tailed distribution of returns in every state. Jump diffusion models were introduced to finance by Merton (1976). Recently there also appeared a few papers dealing with regime-switching jump diffusions models, e.g. Elliott et al. (2007) studies option pricing and Zhao (2010) studies portfolio selection. To our best knowledge however, this is the first chapter that explicitly models simultaneous jumps in asset prices and changes of regime.

We believe that the regimes in the market are directly linked with the Keynes' 'animal spirits', see Section 1.1.1 for background. In the times of boom, the market participants are optimistic and don't follow rational calculation of expected returns. They have the urge to invest, which often over-inflates the prices. Then when the bubble bursts, the opposite mechanism takes place.

5.2 Market

After defining the market model we optimise the risk-sensitive criterion in the given setting, see Section 5.3 for details. Bielecki and Pliska (2003) showed that this can be viewed as a continuous-time equivalent of mean-variance analysis introduced by Markowitz, which is still highly popular in the market. Maximisation of a risk-sensitive criterion in factor models has been studied by many authors. One of the most general versions can be found in Davis and Lleo (2013), where the authors include jumps in both the assets as well as factors. Unlike in the model presented here however, in their paper the assets are not allowed to jump at the same time as the factors, which we find limiting in reality. Also, the authors of the cited paper assume that the factor processes have a non-degenerate diffusion part, and hence their model is not of a regime-switching type.

The closest results to ours are in Frey and Wunderlich (2013), where a standard regimeswitching model is assumed. The jumps in the assets are not taken into account however, instead the authors assume that the regimes are not observable and include stochastic filtering in their analysis. We assume that the investor knows which state the system is currently in—in fact we believe this information is part of the investors view of the market—together with all the other parameter values of the model. The question how to determine these values is outside of the scope of this article—and indeed this is the very skill that allows the investors to generate *alpha*. It remains an open topic for future research.

5.2 Market

We would like to propose a factor-based model for asset prices. The factor process $\{x_t\}$ is a finite state Markov chain, with states in $\mathcal{N} = \{1, \ldots, N\}$ and generator Q in the real-world probability measure \mathbb{P} . It is convenient to identify the process x_t with a process $X_t \in \mathbb{R}^N$ where $X_t = e_k$, the k-th unit coordinate vector, when $x_t = k$. Note that the jumps of X_t arrive according to the state-dependent Poisson process $\{\Lambda_t\}$ with intensity $\lambda(X_t)$ at time t, where from the theory of Markov chains the jump intensity is defined by the generator matrix: $\lambda(i) = -Q_{ii}$ for every i.

First let $\mathfrak{F} = \{f : \mathbb{R}^m \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}\}$ be the class of density functions of asset jump sizes that satisfy the following conditions:

(i) $f(\cdot, i, j)$ is a density function for every $i, j \in \mathcal{N}, i \neq j$.

Chapter 5. Animal control

- (ii) f(z, i, j) = 0 for every $i, j \in \mathcal{N}, i \neq j$ and $z \notin \mathcal{Z} \subseteq [z_{min}, z_{max}]^m$, where $z_{min} > -1$ and $z_{max} < \infty$.
- (iii) $\sum_{i,j;i\neq j} \int_{\mathbb{R}^m} |z| f(z;i,j) \mathrm{d} z < \infty$
- (iv) For any $Z_b \in \partial \mathcal{Z}$, $\lim_{z \to Z_b} f(z; i, j) \ge \epsilon$, for all i, j and some $\epsilon > 0$

The last condition guarantees that jumps near the boundary of \mathcal{Z} can indeed happen with positive probability, and is needed in the proof of Proposition 5.5.5 below. Define a process family $M_t \in \mathbb{R}^m$ to be:

$$M_{t} = \sum_{T_{i} < t} Z_{i} - \int_{0}^{t} \sum_{j \neq X_{s-}} \int_{\mathbb{R}^{m}} zf(z; X_{s-}, j)Q(X_{s-}, j) dz ds$$
(5.1)

where the jump times T_i coincide with jumps in process X and random variables Z_i are conditionally independent of $\{\Lambda\}$ and each other, and have an m-dimensional distribution with density $f(\cdot; X_{t-}, X_t) \in \mathfrak{F}$, depending on the state before and after the jump. Note that $\{M\}$ is a martingale family in the filtration $\mathcal{F}_t^{M,X} = \sigma(\{M_s\}_{0 \le s \le t}, \{X_s\}_{0 \le s \le t})$, generated by both M and X. Denote the expected value of Z if X jumps from state i to j as:

$$\xi(i,j) = \int_{\mathbb{R}^m} z f(z;i,j) \mathrm{d}z \tag{5.2}$$

and define centred jumps as:

$$Y_t = Z_t - \xi(X_{t-}, X_t).$$
(5.3)

The quantities above are well defined because of the integrability assumptions in the definition of \mathfrak{F} . Then M_t may be written as:

$$M_t = \sum_{T_i < t} Y_i + \sum_{T_i < t} \xi(X_{T_i}, X_{T_i}) - \int_0^t \sum_{j \neq X_{s-}} \xi(X_{s-}, j) Q(X_{s-}, j) \mathrm{d}s$$
(5.4)

Note that M_t is a Piecewise Deterministic Process (PDP), see Davis (1993) for an in-depth discussion.

Assets There are *m* risky assets in the market, given by:

$$\frac{\mathrm{d}S_t^i}{S_{t-}^i} = \mu_i(t, X_t)\mathrm{d}t + \Sigma_i(t, X_t)\mathrm{d}W_t + \mathrm{d}M_t^i, \tag{5.5}$$

where M_t^i is the *i*-th coordinate of M_t . We assume that for some $\epsilon > 0$, $\Sigma(t, i)\Sigma(t, i)' > \epsilon I$ for every t, *i*. Moreover, for every $i \in \mathcal{N} u(t, i)$, interpreted as a function of time, needs to be integrable on [0, T] and $\Sigma(t, i)$ square-integrable. The solution to the SDE above is given by:

$$S_{t}^{i} = \exp\left(\int_{0}^{t} \mu_{i}(s, X_{s}) ds - \frac{1}{2} \int_{0}^{t} \Sigma_{i}(s, X_{s}) \Sigma_{i}(s, X_{s})' ds + \int_{0}^{t} \Sigma_{i}(s, X_{s}) dW_{s}\right) \\ \times \exp\left(-\int_{0}^{t} \sum_{j \neq X_{s-}} \xi_{i}(X_{s-}, j) Q(X_{s-}, j) ds\right) \prod_{0 \le s \le t} (1 + Z_{s}^{i}).$$
(5.6)

The stock prices are guaranteed positive, thanks to the assumption that $Z^i \ge z_{min} > -1$ in the definition of \mathfrak{F} . The upper bound, $Z^i \le z_{max} < \infty$, was introduced to allow shortselling of the stocks without a possibility of bankruptcy. Note that some authors work with jumps ζ defined by $\zeta = \log(1 + Z)$ instead, and as a result ζ can take any real value. E.g. in Merton (1976) jump diffusion model the ζ are Gaussian, and in Kou (2002) they are doubly-exponential. The risk free asset is assumed to grow at a rate dependent on the factor process as well^{*}:

$$\frac{\mathrm{d}S_t^0}{S_t^0} = r(t, X_t)\mathrm{d}t, \qquad S_0^0 = 1$$
(5.7)

Let $\mathcal{F}_t^S = \sigma(\{S_u\}_{0 \le u \le t})$ denote the natural filtration generated by the asset processes and let $\mathcal{F}_t^X = \sigma(\{X_u\}_{0 \le u \le t})$ be the filtration generated by the factor process X. As mentioned in the introduction, in this chapter we work in the filtration generated by both assets and the factor process:

$$\mathcal{F}_t = \sigma(\{S_u, X_u\}_{0 \le u \le t}) \tag{5.8}$$

Because the jumps in the assets correspond to the jumps in the martingale m, the filtration generated jointly by M and X is a subset of the full filtration: $\mathcal{F}_t^{M,X} \subseteq \mathcal{F}_t$. $\{M\}$ is also a

^{*}Note that this is a very simple model for stochastic interest rates.

martingale in \mathcal{F}_t .

Note that in our model the state variable X_t not only tracks the current market regime, but also drives the jumps in the asset prices. In practice, the latter jumps are expected to happen much more often than regime changes. In particular consider the case where the jumps in the assets are independent of the current regime. Let A_t be the regime at time t, and let $\{B_t\}$ be a Markov chain with two states b_1, b_2 such that $B_{t-} \neq B_t$ if and only if the asset price process has a jump at time t. Then the functions μ and Σ depend only on A:

$$\mu(t, (A_t, B_t)) = \mu(t, A_t)$$

$$\Sigma(t, (A_t, B_t)) = \Sigma(t, A_t).$$
(5.9)

By assumption, state changes only in the process B cause jumps in assets, so the jump size distribution in the case of no jump in B is given by the Dirac delta function centred at zero:

$$f(\cdot, (A_{t-}, b), (A_t, b)) = \delta_0(\cdot).$$
(5.10)

Also the jump distribution depends only on the regime, hence we have the condition:

$$f(\cdot, (A_{t-}, b_1), (A_t, b_2)) = f(\cdot, (A_{t-}, b_2), (A_t, b_1)).$$
(5.11)

The jump distribution is arbitrary otherwise. Note that this setup is just a special case of a model with multidimensional factor process discussed above, and hence is handled by our model out of the box. Figure 5.1 shows an example of the market regime process (A) and jump driver (B) together with a stock price path. The stock price jumps together with the jump driver B. Regime change can be accompanied by a jump in the assets (e.g. first regime change in Figure 5.1) or not (second regime change respectively). This example shows that the main two effects that the factor process has on the asset dynamics: defining the parameters of the model in a given regime and generating jumps in the price processes may easily be decoupled within the framework.

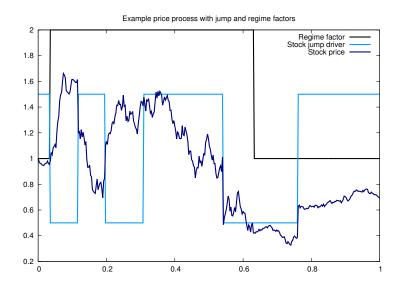


Figure 5.1: Example of a path of a stock price process together with corresponding market regime and jump driver paths.

5.3 Criterion

In this chapter we consider the risk-sensitive asset management criterion, given by:

$$J(x_0, T, h; \theta) = -\frac{1}{\theta} \log \mathbb{E}\left[e^{-\theta F(T, x_0, h)}\right],$$
(5.12)

where x_0 is the state at time 0, T is the fixed time horizon, h is the control variable, and θ is the risk sensitivity parameter, assumed to be in the range $\theta \in (1, +\infty)$. F is a given cost or reward function. The risk-sensitive criterion was first introduced in the context of portfolio optimisation by Bielecki and Pliska (1999). Following their paper, we only consider $F(T, x_0, h) = \log V(T)$, where V(T) is the value of a trading portfolio at time T, and h denotes the portfolio strategy.

This criterion has a number of desirable properties, as discussed by Bielecki and Pliska (2003). On one hand it has strong ties to Markowitz's mean-variance analysis. A Taylor expansion of the criterion J around $\theta = 0$ yields:

$$J(x_0, T, h; \theta) = \mathbb{E}\left[\log V(T)\right] - \frac{\theta}{2} \operatorname{Var}\left[\log V(T)\right] + O(\theta^2).$$
(5.13)

Chapter 5. Animal control

If we define the realised growth rate R_T by $V(T) = e^{R_T T}$, then the equation above becomes:

$$\frac{1}{T}J(x_0, T, h; \theta) = \mathbb{E}\left[R_T\right] - \frac{\theta T}{2} \operatorname{Var}\left[R_T\right] + O(\theta^2 T^2).$$
(5.14)

When maximising the risk-sensitive criterion, we effectively maximise the expected log return of the portfolio, penalized by the variance. Thus it can be interpreted as a continuoustime extension of the Markowitz model. Unlike Markowitz, we do take into account higher order terms, hidden in the $O(\theta^2)$ term. After expanding it further, we notice that the risksensitive approach penalizes high variance, negative skewness and high kurtosis, while it favours positive skewness. These are perfectly in line with the portfolio manager's preferences in the industry.

On the other hand the problem is equivalent to utility maximisation. Note that the criterion can be written as:

$$J(x_0, T, h; \theta) = -\frac{1}{\theta} \log \mathbb{E} \left[V(T)^{-\theta} \right]$$
(5.15)

so that our optimisation problem is equivalent to Merton-style maximisation of power utility function $U(V(T)) = \frac{1}{\gamma}V(T)^{\gamma}$ for $\gamma < 0$.

Note also that the maximisation problem does not depend on the initial value of the investment portfolio v_0 . If we denote $V(t) = v_0 V^1(t)$, where V^1 is the portfolio price process with unit initial value, we have:

$$J(x_0, T, h; \theta) = -\frac{1}{\theta} \log \mathbb{E} \left[e^{-\theta \log(v_0 V^1(t))} \right]$$

= $\log v_0 - \frac{1}{\theta} \log \mathbb{E} \left[e^{-\theta \log V^1(t)} \right].$ (5.16)

Without loss of generality we assume that $v_0 = 1$ in the remainder of this chapter.

Finally, thanks to a clever change of measure first introduced by Kuroda and Nagai (2002), the problem can be solved in a fairly general factor based model, with jumps in both the factors as well as in the underlying stocks, see Davis and Lleo (2013) for a general result. In this chapter we propose a considerably simpler model, with the factor process being a finite state Markov chain, and the assets following regime-dependent jump-diffusion process, see Section 5.2 for details. Unlike the other authors however, we allow for the

assets and factors to jump simultaneously. We believe that is an important feature, because in reality often a big jump (drop) is asset prices is associated with a regime change.

An interesting extension of this criterion, in line with recent work in the field of utility optimisation, would be to allow the risk-aversion parameter θ to depend on the state of the economy. The intuition is that in difficult times the investors are more risk averse, and during boom they are willing to have a higher leverage. This is however outside of the scope of this chapter.

5.4 Investment

At every point in time t the investor chooses a trading strategy h_t , where h_t is a vector and its *i*-th component denotes the proportion of the value of the portfolio invested in *i*-th asset, i = 1, ..., m. Because the risk-sensitive criterion is defined as $-\infty$ for negative portfolio values (and hence bankruptcy), we need to add assumptions on h so that the portfolio never jumps into negative territory:

$$h \in \mathcal{J}_h = \{h \in \mathbb{R}^m : h'\psi > -1 \quad \forall \psi \in \mathcal{Z}\},$$
(5.17)

where $Z \subseteq [z_{min}, z_{max}]^m$ is defined in Section 5.2. Note that this set of strategies is nonempty and bounded, because Z is bounded and greater than -1.

Definition 5.4.1 (Admissible strategy) The trading strategy $h : \Omega \times \mathbb{R} \to \mathbb{R}^m$ is in the set of admissible strategies \mathcal{H} if:

- (i) $h_t \in \mathcal{J}_h$ for every time $t \in [0, T]$
- (ii) $\{h\}$ is a predictable process with respect to the filtration \mathcal{F}_t defined in (5.8).

The trading portfolio is assumed to be self-financing, and hence all the change in portfolio value are caused by the changes in the underlying asset prices:

$$dV_t = \left(\frac{V_t h_t}{S_t}\right)' dS_t + \frac{V_t (1 - h' \mathbf{1})}{S_t^0} dS_t^0,$$
(5.18)

Chapter 5. Animal control

where $\frac{V_T h_t}{S_t}$ is the vector containing the number of units in each asset and the division is interpreted componentwise. $1 - h'\mathbf{1}$ is the proportion invested (or borrowed) in the money market account. After substituting in the previous formula we get:

$$\frac{\mathrm{d}V_t}{V_{t-}} = r(t, X_t)\mathrm{d}t + h'_t(\mu - r\mathbf{1})\mathrm{d}t + h'_t\Sigma\mathrm{d}W_t + h'_t\mathrm{d}M_t.$$
(5.19)

As remarked in section 5.3, without loss of generality assume that the portfolio has unit value at the beginning: $V_0 = 1$. Note that the process $h'_t \Sigma dW_t + h'_s dM_s$ is a local martingale, hence the solution of the SDE above is the stochastic exponential of the martingale part with the drift:

$$V_t = \exp\left(\int_0^t r_s + h'(\mu_s - \mathbf{1}) \mathrm{d}s - \frac{1}{2} \int_0^t h'_s \Sigma \Sigma' h_s \mathrm{d}s + \int_0^t h'_s \Sigma \mathrm{d}W_s + \int_0^t h'_s \mathrm{d}M_s\right)$$

$$\times \prod_{0 \le s \le t} (1 + h'_s Z_s) e^{-h'_s Z_s}.$$
(5.20)

See e.g. Protter (2005, pp.84-85) for detailed calculations. The logarithm of the value process is thus given by:

$$\log V_{t} = \int_{0}^{t} r_{s} + h'(\mu_{s} - r_{s}\mathbf{1}) ds - \frac{1}{2} \int_{0}^{t} h'_{s} \Sigma \Sigma' h_{s} ds + \int_{0}^{t} h'_{s} \Sigma dW_{s} + \int_{0}^{t} h'_{s} dM_{s}$$

$$- \sum_{0 \le s \le t} \log(1 + h'_{s}Z_{s}) - h'_{s}Z_{s}$$

$$= \int_{0}^{t} [r_{s} + h'_{s}(\mu_{s} - r_{s}\mathbf{1})] ds - \frac{1}{2} \int_{0}^{t} h'_{s} \Sigma \Sigma' h_{s} ds + \int_{0}^{t} h'_{s} \Sigma dW_{s}$$

$$- \sum_{0 \le s \le t} \log(1 + h'_{s}Z_{s}) - \int_{0}^{t} \sum_{j \ne X_{s-}} h'_{s} \xi(X_{s-}, j, s) Q(X_{s-}, j) ds.$$

(5.21)

5.5 Optimal portfolio

The investor maximises the risk-sensitive criterion given by (5.12), using strategies that ensure V > 0 at all times. This condition is equivalent to (5.17), because for every realisation of Z it guarantees:

$$\forall_t h'_t Z_t > -1 \qquad \text{a.s.} \tag{5.22}$$

Following the idea from Kuroda and Nagai (2002), we can write the term under the expectation in the criterion (5.12) as:

$$e^{-\theta \log V_T} = \exp\left(\theta \int_0^T g(t, X_{t-}, h_t) \mathrm{d}t\right) \chi_T^h,\tag{5.23}$$

where:

$$\chi_t^h = \exp\left(-\theta \int_0^t h_s' \Sigma \mathrm{d}W_s - \frac{\theta^2}{2} \int_0^t h_s' \Sigma \Sigma' h_s \mathrm{d}s\right) \prod_{0 < s \le T} (1 + h_s' Z_s)^{-\theta} \\ \times \exp\left(-\int_0^T \sum_{j \ne X_{s-}} \int_{\mathcal{Z}} [(1 + h'z)^{-\theta} - 1] f(z; X_{s-}, j) \mathrm{d}z Q(X_{s-}, j) \mathrm{d}s\right)$$
(5.24)

and

$$g(t, i, h) = \frac{1}{2}(\theta + 1)h'(t, i)\Sigma(t, i)\Sigma(t, i)'h(t, i) - r(t, i) - h'_t(\mu - r\mathbf{1}) + \sum_{j \neq i} Q(i, j) \left[\frac{1}{\theta} \int_{\mathcal{Z}} [(1 + h'z)^{-\theta} - 1]f(z; i, j)dz + h'\xi(i, j, t) \right]$$
(5.25)

Proposition 5.5.1 For any fixed trading strategy $h \in \mathcal{H}$ the stochastic process χ_t^h is a martingale with $\mathbb{E}[\chi_t^h] = 1$.

Proof Proof can be found in Appendix 5.B.

Using the proposition above and measure change theory summarised in Appendix 5.A, we use the martingale $\{\chi_t^h\}$ to change the probability measure:

$$\frac{\mathrm{d}\mathbb{P}^{h}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_{t}} = \chi_{t}^{h} \tag{5.26}$$

Under the new measure \mathbb{P}^h the risk-sensitive criterion becomes:

$$J(\theta, h) = -\frac{1}{\theta} \log \mathbb{E} \left[e^{-\theta \log V_T} \right]$$

= $-\frac{1}{\theta} \log \mathbb{E}^h \left[e^{\theta \int_0^T g(t, X_t, h_t) dt} \right]$ (5.27)

and $Q^h = [Q^h(i, j)]$ becomes a generalized generator of X with elements:

$$Q^{h}(i,j)(t) = Q(i,j) \left[\int_{\mathcal{Z}} [(1+h'z)^{-\theta}] f(z;i,j) dz \right],$$

$$Q^{h}(i,i)(t) = -\sum_{j \neq i} Q^{h}(i,j)(t).$$
(5.28)

We can write the optimal value function as:

$$v(t,i) = \sup_{h \in \mathcal{H}} -\frac{1}{\theta} \log \mathbb{E}_{t,i}^{h} \left[e^{\theta \int_{t}^{T} g(t,X_{t},h_{t})dt} \right]$$

= $-\frac{1}{\theta} \log u(t,i)$ (5.29)

where

$$u(t,i) = \inf_{h \in \mathcal{H}} \mathbb{E}^{h}_{t,i} \left[e^{\theta \int_{t}^{T} g(t,X_{t},h_{t}) \mathrm{d}t} \right].$$
(5.30)

Note that, thanks to the measure change and the normalisation of the initial investment, the value function doesn't depend on the value of the portfolio at time t, it only depends on the state of the factor process X. In the remainder of the chapter we solve for the function u, and the original value function v can be easily obtained using the formula above. Of course, by applying the measure change backwards, we also have:

$$u(t,i) = \inf_{h \in \mathcal{H}} \mathbb{E}_{t,i} \left[e^{-\theta \log(V_T/V_t)} \right]$$
(5.31)

We write u(t) for the N-vector with components u(t, i).

Proposition 5.5.2 The range of u is a compact set $\mathcal{U} \subseteq [u_{min}, u_{max}]^N$, such that $0 < u_{min} \leq u_{max} < \infty$, where u is defined in (5.30).

Proof Because the value function is defined in terms of minimisation of the expectation $\mathbb{E}\left[e^{-\theta \log(V_T/V_t)}\right]$ we can bound the value from below and above. First note that the function g defined in (5.25) is bounded from below for any t, i, h, and let $g_{min} = \inf\{g(t, i, h) : t \in [0, T], i \in \mathcal{N}, h \in \mathcal{J}_h\}$. Then

$$u(t,i) \ge e^{\theta g_{min}(T-t)} > 0.$$
 (5.32)

5.5 Optimal portfolio

The argument showing the upper bound is a bit more subtle. Note that the expectation $\mathbb{E}\left[e^{-\theta \log(V_T/V_t)}\right]$ is large when the portfolio value on average performs badly. However, the investor always has the option to put all his wealth in the money market account. This pays a guaranteed return, which is different in every regime, but it is at least $r_{min} = \inf\{r(i, t) : i \in \mathcal{N}, t \in [0, T]\}$. Thanks to the minimisation operator in the definition of u, the upper bound is given by:

$$u(t,i) = \inf_{h \in \mathcal{H}} \mathbb{E}\left[e^{-\theta \log(V_T/V_t)}\right] \le e^{-\theta r_{min}(T-t)}$$
(5.33)

Note that $g_{min} \leq \inf\{g(t, i, 0) : t \in [0, T], i \in \mathcal{N}\} = -r_{min}$, hence \mathcal{U} is not empty. The solution is defined on the finite time interval [0, T], which finishes the argument.

5.5.1 General case

We need to solve the HJB equation, which in this case is the ODE in \mathbb{R}^N :

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \inf_{h \in \mathcal{H}} \left\{ A(u,h) \right\} = 0, \tag{5.34}$$

with

$$A(u,h) = Q^{h}u + \theta \operatorname{diag}(g)u, \qquad (5.35)$$

where $Q^{h}(i)$ is the *i*th row of Q^{h} , and *u* without a parameter is interpreted as a vector in \mathbb{R}^{N} . All the functions of *X* are interpreted as corresponding vectors and $\theta > 1$. The inf operator is interpreted componentwise, that is the vector h_{i} minimises the *i*th element of A(u, h). The boundary condition is given by:

$$u(T,i) = 1 \quad \text{for every } i. \tag{5.36}$$

Theorem 5.5.3 Suppose the market is defined as in Section 5.2, admissible strategies are as in Definition 5.4.1 and $\theta > 1$. Then the HJB equation (5.34) with final condition (5.36) defined above, has a unique solution on [0, T], which coincides with the value function defined in (5.30).

Proof The theorem follows from Propositions 5.5.6 and 5.5.8 below.

Remark 5.5.4 The assumption $\theta > 1$ is required in the proof of Proposition 5.5.5 below, although we believe that the Proposition itself is valid under the minimal assumption $\theta > 0$.

Following the standard approach, we first find the optimal strategy for every time t and state i, where we take the value function u as an argument:

$$h^{*}(u,t,i) = \underset{h \in \mathcal{H}}{\arg\min} \left\{ A(u,h)(t,i) \right\},$$
(5.37)

By substituting (5.25) and (5.28) into (5.35) we get an explicit formula for the operator A:

$$\begin{aligned} A(u,h)(t,i) &= Q^{h}(i)u + \theta g(t,h,i)u(i) \\ &= \sum_{j \neq i} Q(i,j) \int_{\mathcal{Z}} (1+h'z)^{-\theta} f(z;i,j) \mathrm{d}z [u(j) + (\theta-1)u(i)] \\ &+ \frac{1}{2} u(i) \theta(\theta+1) h' \Sigma(t,i) \Sigma(t,i)' h \\ &- \theta u(i) h' [\mu + \sum_{j \neq i} Q(i,j) \xi(i,j,t) - r\mathbf{1}] \\ &- \theta u(i) [r(t,i) + \sum_{j \neq i} Q(i,j)], \end{aligned}$$
(5.38)

where h is a function of time t and state i. This operator is linear in u, and can be explicitly written as matrix multiplication A(h)u, with:

$$\begin{aligned} A(h)_{ij} &= Q(i,j) \int_{\mathcal{Z}} (1+h'z)^{-\theta} f(z;i,j) dz & \text{for } i \neq j \\ A(h)_{ii} &= \sum_{j \neq i} Q(i,j) \left[(\theta-1) \int_{\mathcal{Z}} (1+h'z)^{-\theta} f(z;i,j) dz \right] \\ &+ \theta \left[\frac{1}{2} (\theta+1) h'(t,i) \Sigma(t,i) \Sigma(t,i)' h(t,i) - h'_t (\mu + \sum_{j \neq i} Q(i,j) \xi(i,j,t) - r\mathbf{1}) \right] \\ &- \theta \left[r(t,i) + \sum_{j \neq i} Q(i,j) \right] \end{aligned}$$
(5.39)

Let:

$$\mathcal{A}(u)(t,i) = \inf_{h \in \mathcal{T}_{h}} \left\{ A(u,h)(t,i) \right\}.$$
(5.40)

Proposition 5.5.5 The operator A(h)u, as a function of h, has a unique minimum in \mathcal{J}_h for every $u \in \mathcal{U}$. Hence the operator $\mathcal{A}(u)$ is well defined.

Proof Under the assumption that $\theta > 1$, for any $u \in \mathcal{U}$ and for every index *i* and time t, A(u, h)(t, i) is a finite sum of convex continuous functions bounded from below, hence it is also continuous, convex and bounded from below. As *h* approaches any point on the boundary of the admissible set $h \to h_0 \in \mathcal{J}_h$, then by definition there exists some $Z_0 \in \partial \mathcal{Z}$ such that $(1 + h'Z_0) \to 0$ and the integral $\int_{\mathcal{Z}} (1 + h'z)^{-\theta} f(z; i, j) dz$ diverges to infinity:

$$\lim_{h \to h_0} \int_{\mathcal{Z}} (1 + h'z)^{-\theta} f(z; i, j) \mathrm{d}z = \infty.$$
(5.41)

Hence the optimal value $h^*(u)$ is well defined and is in the interior of the admissible set \mathcal{J}_h . This implies that the operator \mathcal{A} is well defined for all u.

The HJB equation (5.34) may be written as:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}(u) = 0 \tag{5.42}$$

with final condition:

$$u(T) = 1 \tag{5.43}$$

Proposition 5.5.6 (Verification theorem) If \tilde{u} is a solution to ODE (5.42) with the final condition (5.43) on some interval [t, T], then \tilde{u} is the value function (5.30) and the corresponding trading strategy $\tilde{h} \in \mathcal{H}$ is optimal.

Proof Using Ito's lemma, we get the following relationship:

$$d\left(e^{\theta\int_{0}^{t}g(s,X_{s},\tilde{h}_{s})ds}\tilde{u}(t,X_{t})\right) = e^{\theta\int_{0}^{t}g(s,X_{s},\tilde{h}_{s})ds}\tilde{u}(t,X_{t})\theta g(t,X_{t},\tilde{h}_{t})dt + e^{\theta\int_{0}^{t}g(s,X_{s},\tilde{h}_{s})ds} \left[\frac{\partial\tilde{u}}{\partial t}(t,X_{t}) + \Delta\tilde{u}(t,X_{t})\right] = e^{\theta\int_{0}^{t}g(s,X_{s},\tilde{h}_{s})ds}\tilde{u}(t,X_{t})\theta g(t,X_{t},\tilde{h}_{t})dt + e^{\theta\int_{0}^{t}g(s,X_{s},\tilde{h}_{s})ds} \left[\frac{\partial\tilde{u}}{\partial t}(t,X_{t}) + Q^{\tilde{h}}\tilde{u}(t,X_{t})\right]dt + e^{\theta\int_{0}^{t}g(s,X_{s},\tilde{h}_{s})ds} \left[\Delta\tilde{u}(t,X_{t}) - Q^{\tilde{h}}\tilde{u}(t,X_{t})dt\right],$$
(5.44)

where $Q^{\tilde{h}}$ is the generator of the process $\{X\}$ in the measure corresponding to the trading strategy \tilde{h} . After integrating over [t, T], multiplying both sides by $e^{-\theta \int_0^t g(s, X_s, \tilde{h}_s)}$ and rearranging, we get:

$$\begin{split} \tilde{u}(t, X_t) &= e^{\theta \int_t^T g(s, X_s, \tilde{h}_s) \mathrm{d}s} \tilde{u}(T, X_T) \\ &- \int_t^T e^{\theta \int_t^s g(u, X_u, \tilde{h}_u) \mathrm{d}u} \left[\frac{\partial \tilde{u}}{\partial s}(t, X_s) + A_i(\tilde{h}, \tilde{u}) \right] \mathrm{d}s \\ &- \left[\sum_{t \leq s \leq T} \Delta \tilde{f}(t, X_t) - Q^{\tilde{h}} \tilde{f}(t, X_t) \mathrm{d}t \right], \end{split}$$
(5.45)

where the operator A is defined in (5.35) and $\tilde{f}(t, X_t) = e^{\theta \int_t^s g(u, X_u, \tilde{h}_u) du} \tilde{u}(t, X_t)$. Note that the term on the last line in the equation above is a martingale. By taking a conditional expectation $\mathbb{E}_{t,i}^{\tilde{h}}[\cdot]$ on both sides, and using the final condition (5.43), the above equation simplifies to:

$$\tilde{u}(t,i) = \mathbb{E}_{t,i}^{\tilde{h}} \left[e^{\theta \int_{t}^{T} g(s,X_{s},\tilde{h}_{s})ds} \right] - \mathbb{E}_{t,i}^{\tilde{h}} \left[\int_{t}^{T} e^{\theta \int_{t}^{s} g(w,X_{w},\tilde{h}_{w})dw} \left(\frac{\partial \tilde{u}}{\partial s}(t,X_{s}) + A_{i}(\tilde{h},\tilde{u}) \right) ds \right],$$
(5.46)

where the expectation of the increment of a martingale is zero. From the definition of the ODE (5.42) the following condition holds:

$$\frac{\partial \tilde{u}}{\partial s}(t, X_s) + A_i(h, \tilde{u}) \ge 0,$$
(5.47)

with equality for the optimal strategy \tilde{h} . Hence we have:

$$\tilde{u}(t,i) \leq \mathbb{E}_{t,i}^{\tilde{h}} \left[e^{\theta \int_t^T g(s, X_s, h_s) \mathrm{d}s} \right].$$
(5.48)

with equality for the optimal strategy \tilde{h} :

$$\tilde{u}(t,i) = \mathbb{E}_{t,i}^{\tilde{h}} \left[e^{\theta \int_t^T g(s, X_s, \tilde{h}_s) \mathrm{d}s} \right],$$
(5.49)

which is equation (5.30) as required.

We also need the following lemma:

Lemma 5.5.7 (Lemma 2 from Davis (1998)) Let the operator A be given by (5.40) and let both A and A be differentiable in u. Then for any u in the domain:

$$\left. \frac{\partial \mathcal{A}(u)}{\partial u} \right|_{u=u_0} = \left. \frac{\partial A(h^*, u)}{\partial u} \right|_{u=u_0},\tag{5.50}$$

where $h^*(u_0)$ is the optimum at $u = u_0$. Moreover, the result holds even if $h^*(\cdot)$ is not differentiable.

Proof Because h^* is optimal for $u = u_0$, we have for all u:

$$\mathcal{A}(u) \le A(h^*, u), \tag{5.51}$$

with equality at $u = u_0$. The lemma follows from the fact that if the derivatives of $A(h^*, \cdot)$ and \mathcal{A} were different at u_0 , then the inequality (5.51) would fail in any neighbourhood of u_0 . Note that the proof doesn't require the differentiability of $h^*(\cdot)$.

The main results of the chapter are summarised in the following proposition:

Proposition 5.5.8 *The differential equation* (5.42) *with boundary condition* (5.43) *has a unique solution on* [0, T].

Proof The derivative of the operator $\mathcal{A}(u)$ may be calculated using Lemma 5.5.7:

$$\frac{\partial \mathcal{A}(u)}{\partial u} = \frac{\partial A(h^*, u)}{\partial u} = A(h^*), \tag{5.52}$$

where h^* is the optimal trading strategy, which depends on the argument u and by Proposition 5.5.5 it is well defined and unique in the whole domain. To show that $\mathcal{A}(u)$ is Lipschitz continuous we show that the derivative $A(h^*)$ is uniformly bounded. The lower bound of A(h) comes from the fact that every element of the matrix A(h), as a function of h, is continuous and bounded from below. To show the upper bound take any value $h_0 \in \mathcal{J}_h$ and take the constant function $h_0(u) = h_0$. Using the optimality of h^* we have:

$$\sup_{u \in \mathcal{U}} \mathcal{A}(u) = \sup_{u \in \mathcal{U}} \left\{ \inf_{h \in \mathcal{J}_h} \left\{ A(u,h)(t,i) \right\} \right\}$$

$$\leq \inf_{h \in \mathcal{J}_h} \left\{ \sup_{u \in \mathcal{U}} \left\{ A(h)u \right\} \right\}$$

$$\leq \inf_{h \in \mathcal{J}_h} \left\{ |A(h)| u_{max} \mathbf{1} \right\}$$

$$\leq |A(h_0)| u_{max} \mathbf{1},$$
(5.53)

where the absolute value in |A(h)| is interpreted componentwise and 1 is an *N*-vector with every element being unity. Hence the value of the operator \mathcal{A} is uniformly bounded for all u. Let $A_- = \sup \{x : x \leq 0 \land A(h)_{ij} \forall h \in \mathcal{J}_h; \forall i, j \in \mathcal{N}\}$ be the lower bound of all the elements of A(h) or zero if positive, and let $A_+ = \max \{|A(h_0)|u_{max}1\}$ be the biggest element of the vector bounding the operator $\mathcal{A}(u)$ from above. From Proposition 5.5.2 and from the obvious relationship $\mathcal{A}(u) = A(h^*)u$, using a straightforward combinatorial argument, we get that for any $i, j \in \mathcal{N}$:

$$A(h^*)_{ij} \le \frac{A_+}{u_{min}} + (N-1)A_-u_{max},$$
(5.54)

hence $A(h^*)$ is uniformly bounded from above.

Because $A(h^*)$ is uniformly bounded, the operator \mathcal{A} is Lipschitz continuous and the solution to the ODE (5.42) is well defined in the whole domain.

Unfortunately it is not possible to provide a closed form solution for the problem in the general case. The value function is given by an ordinary differential equation though, which may be efficiently solved numerically.

As mentioned in the introduction, in order to back-test the performance of these trading strategies using historic stock prices, one would need to come up with an algorithm to determine the state (animal spirits) at every point in time. It remains an open topic for future research.

5.5.2 Independent case

Corollary 5.5.9 (To Theorem 5.5.3) In the special case with no jumps in asset prices, i.e. when $f(\cdot; i, j) \equiv 0$ for all $i, j \in N$, the HJB equation (5.34) with final condition (5.36) defined above has a closed-form solution:

$$u(t,x) = e^{(Q-\theta \operatorname{diag}(g^*))(T-t)},$$
(5.55)

where

$$g^{*}(i) = g(t, i, h_{t}^{*}(i)) = -\frac{1}{2(\theta + 1)}(\mu - r\mathbf{1})'(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1}) - r$$
(5.56)

Proof Note that in the special case when there are no jumps in the asset prices, the measure change described above has no effect on the generator of the factor process X. The function g becomes:

$$g(t,x,h) = \frac{1}{2}(\theta+1)h'(t,x)\Sigma(t,x)\Sigma(t,x)'h(t,x) - r(t,x) - h'_t(\mu-r\mathbf{1})$$
(5.57)

Because the distribution of X does not depend on the control h, the optimum strategy can be calculated pointwise:

$$h_t^*(X_t) = \underset{h \in \mathcal{H}}{\arg\min} g(t, X_t, h).$$
(5.58)

For any $h, k \in \mathbb{R}^M$ we have:

$$g(h+k) = \frac{1}{2}(\theta+1)(h+k)'\Sigma\Sigma'(h+k) - (h+k)'(\mu-r\mathbf{1}) - r$$

$$= \frac{1}{2}(\theta+1)h'\Sigma\Sigma'h + (\theta+1)h'\Sigma\Sigma'k$$

$$+ \frac{1}{2}(\theta+1)k'\Sigma\Sigma'k - h'(\mu-r\mathbf{1}) - (\mu-r\mathbf{1})'k - r$$

$$= g(h) + [(\theta+1)h'\Sigma\Sigma' - (\mu-r\mathbf{1})']k + \frac{1}{2}(\theta+1)k'\Sigma\Sigma'k$$

As $k'\Sigma\Sigma'k = o(|k|)$, g is differentiable with $g'(h)k = (\theta + 1)h'\Sigma\Sigma'k - (\mu - r\mathbf{1})'k$, any

 $h,k\in \mathbb{R}^M,$ we have g'(h)=0 exactly for

$$h^* = \frac{1}{(\theta+1)} (\Sigma \Sigma')^{-1} (\mu - r\mathbf{1})$$
(5.59)

and as $g''(h) = (\theta + 1)\Sigma\Sigma'$ is positive, it is a minimum. The optimal value of function g is thus given by:

$$g^{*}(i) = g(t, i, h_{t}^{*}(i)) = -\frac{1}{2(\theta + 1)}(\mu - r\mathbf{1})'(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1}) - r$$
(5.60)

We can take expectation in the original probability measure in the value function:

$$u(t,i) = \mathbb{E}_{t,i} \left[e^{\theta \int_t^T g(t,X_t,h_t^*) \mathrm{d}t} \right]$$
(5.61)

The corresponding PDE has the form:

$$\frac{\partial u}{\partial t} + Qu - \theta \operatorname{diag}(g^*)u = 0, \qquad (5.62)$$

where all the functions of X are interpreted as corresponding vectors. The boundary condition is given by:

$$u(T, x) = 1 \quad \text{for every } x. \tag{5.63}$$

Hence the solution is:

$$u(t,x) = e^{(Q-\theta \operatorname{diag}(g^*))(T-t)}$$
(5.64)

The original value function can be recovered using (5.29) and is given by:

$$v(t,x) = -\frac{1}{\theta} \log u(t,x) = -\frac{1}{\theta} (Q - \theta \operatorname{diag}(g^*))(T - t)$$
 (5.65)

5.A Appendix: Measure change

In this appendix we summarise the measure change theory used in the chapter. Section 5.A.1 is a special case of the theory presented in Davis (2011) and deals with finite state Markov chains. Section 5.A.2 discusses the measure change induced by regime-switching

compound Poisson process, in particular the impact on the underlying Markov chain. Finally the last section discusses the risk-adjusted changes of measure.

5.A.1 Measure change for Markov chains

Let X_t be a Markov chain with generator matrix Q, as described in Section 5.2. For any function $\xi : \mathbb{R} \times \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ define a martingale M^{ξ} as follows:

$$M_t^{\xi} = \sum_{T_i \le t} \xi(X_{T_{i-}}, X_{T_i}, T_i) - \int_0^t \sum_{j \ne X_{s-}} \xi(X_{s-}, j, s) Q(X_{s-}, j) \mathrm{d}s,$$
(5.66)

Next, define the measure change martingale as the stochastic exponential of M^{ξ} :

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_{T}} = \mathcal{E}(M^{\xi})$$

$$= e^{M_{T}^{\xi} - \frac{1}{2}[M^{\xi}, M^{\xi}]_{T}^{c}} \prod_{0 < s \leq T} (1 + \Delta M_{s}^{\xi}) e^{-\Delta M_{s}^{\xi}}$$

$$= \prod_{0 < s \leq T} (1 + \xi(X_{s-}, X_{s}, s)) \exp\left(-\int_{0}^{t} \sum_{j \neq X_{s-}} \xi(X_{s-}, j, s) Q(X_{s-}, j) \mathrm{d}s\right),$$
(5.67)

where $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$, using arguments analogous to the proof of Proposition 5.5.1. Note that by the properties of the generator of Markov chains

$$\lambda(i) = -Q(i,i) \tag{5.68}$$

is the intensity of jumps of the process $\{\Lambda\}$. Once the factor process jumps, the probability of jump from state *i* to another state *j* is given by:

$$P_{ij} = \begin{cases} \frac{Q(i,j)}{\lambda(i)} & i \neq j \\ 0 & \text{otherwise} \end{cases}$$
(5.69)

Proposition 5.A.1 In the new probability measure defined by (5.67) the generator of X is given by:

$$\widetilde{q}_{ij} = \begin{cases} Q(i,j)(\xi(i,j,t)+1) & i \neq j \\ -\sum_{k \neq i} \widetilde{q}_{ik} & otherwise \end{cases}$$
(5.70)

Let γ and β be such that:

$$\beta(i) = \sum_{j} P_{ij}[\xi(i,j,t) + 1]$$
(5.71)

and

$$\gamma_{ij} = \frac{\xi(i, j, t) + 1}{\beta(i)}$$
(5.72)

Then, in particular, the jump intensity becomes $\tilde{\lambda} = \beta \lambda$ and the probability of jumps become:

$$\widetilde{P}_{ij} = \gamma_{ij} P_{ij}, \tag{5.73}$$

Proof Note that (5.71) and (5.72) imply that:

$$\sum_{j} \gamma_{ij} P_{ij} = 1. \tag{5.74}$$

and

$$\xi(i,j,t) = \gamma_{ij}\beta(i) - 1 \tag{5.75}$$

Hence the result is a special case of the change of measure theory presented in Davis (2011).

In our particular case the equation simplifies to:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_{T}} = \prod_{0 < s \leq T} (1 + \Delta M_{s}^{\xi}) e^{-\int_{0}^{T} \sum_{j} \xi(j, s, X_{s-}) P(X_{s-}, j)\lambda(X_{s-}) \mathrm{d}s}$$
(5.76)

Following Cont and Tankov (2012) we can express the measure change in the exponential form:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_{T}} = \exp\left(\sum_{T_{i} < T} \log(1 + \Delta M_{T_{i}}^{\xi}) - \int_{0}^{T} \sum_{j} \xi(X_{s-}, j, s) P(X_{s-}, j) \lambda(X_{s-}) \mathrm{d}s\right)$$

$$= \exp\left(\sum_{T_{i} < T} \log(\gamma(X_{T_{i-}}, X_{T_{i}}) \beta(X_{T_{i-}}))\right)$$

$$\times \exp\left(-\int_{0}^{T} \sum_{j} \xi(X_{s-}, j, s) P(X_{s-}, j) \lambda(X_{s-}) \mathrm{d}s\right)$$
(5.77)

The last integral might be simplified:

$$\sum_{j} \xi(X_{s-}, j, s) P(X_{s-}, j) \lambda(X_{s-}) = \lambda(X_{s-}) \sum_{j} (\gamma_{ij} \beta(i) - 1) P(X_{s-}, j)$$
$$= \lambda(X_{s-}) \beta(i) \sum_{j} \gamma_{ij} P(X_{s-}, j)$$
$$- \lambda(X_{s-}) \sum_{j} P(X_{s-}, j)$$
$$= \tilde{\lambda}(X_{s-}) - \lambda(X_{s-})$$
(5.78)

After substitution and application of some simple algebra, we get the final form:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_{T}} = \exp\left(\sum_{T_{i} < T} \log \gamma(X_{T_{i}-}, X_{T_{i}})\right) \times \exp\left(\sum_{T_{i} < T} \log \beta(X_{T_{i}-}) - \int_{0}^{T} \widetilde{\lambda}(X_{s-}) - \lambda(X_{s-}) \mathrm{d}s\right).$$
(5.79)

Note, that the first term is responsible for the change of measure of the distribution of jump destination, and the second term is just a Poisson process intensity change corresponding to the jump times.

5.A.2 Measure change defined for regime-switching compound Poisson processes

Now let the measure change be defined by:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_{T}} = \mathcal{E}(M_{T})$$

$$= e^{M_{T} - \frac{1}{2}[M,M]_{T}^{c}} \prod_{0 < s \leq T} (1 + \Delta M_{s})e^{-\Delta M_{s}},$$
(5.80)

where M_t is defined in (5.1) and $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$, using arguments analogous to the proof of Proposition 5.5.1 If we denote by $\mathcal{F}_T^X = \sigma(\{X_t\}_{0 \le t \le T})$ the filtration generated by the factor process up to time T, then the measure change relevant for process $\{X\}$ is given by:

$$\begin{aligned} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \Big|_{\mathcal{F}_{T}^{X}} &= \mathbb{E}\left[\left|\mathcal{E}(M_{T})\right| \mathcal{F}_{T}^{X}\right] \\ &= \exp\left(-\int_{0}^{T} \sum_{j \neq X_{s-}} \xi(X_{s-}, j)Q(X_{s-}, j)\mathrm{d}s\right) \mathbb{E}\left[\prod_{0 < s \leq T} (1+Z_{s}) \left|\mathcal{F}_{T}^{X}\right] \right] \\ &= \exp\left(-\int_{0}^{T} \sum_{j \neq X_{s-}} \xi(X_{s-}, j)Q(X_{s-}, j)\mathrm{d}s\right) \prod_{0 < s \leq T} \mathbb{E}\left[(1+Z_{s}) \left|\mathcal{F}_{T}^{X}\right]\right] \\ &= \prod_{0 < s \leq T} (1+\xi(X_{s-}, j)) \exp\left(-\int_{0}^{T} \sum_{j \neq X_{s-}} \xi(X_{s-}, j)Q(X_{s-}, j)\mathrm{d}s\right), \end{aligned}$$
(5.81)

where we used the independence property of Z_i -s to interchange the product and expectation in the third line and the definition of ξ in (5.2). Note that this measure change martingale is of the same form as used in the previous section:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t^X} = \mathbb{E}\left[\left.\mathcal{E}(M_T)\right|\mathcal{F}_t^X\right] = \mathcal{E}(M_t^\xi),\tag{5.82}$$

hence results from Proposition 5.A.1 apply.

5.A.3 Risk-adjusted stochastic exponentials

Given the factor process X and the jump sequence Z_i defined in Section 5.2, let $M_t^{h,\theta}$ be a martingale given by:

$$M_t^{h,\theta} = \sum_{T_i < t} [(1 + h'Z_i)^{-\theta} - 1] - \int_0^t \sum_{j \neq X_{s-}} \int_{\mathcal{Z}} [(1 + h'Z_i)^{-\theta} - 1]\phi(z; X_{s1}) dz ds, \quad (5.83)$$

where

$$\phi(z;i) = \lambda(i)\tilde{f}(z;i) \tag{5.84}$$

is the compensator of jumps, $\lambda(i) = \sum_{j \neq i} Q(i,j)$ is the jump intensity in state i and

$$\tilde{f}(z;i) = \frac{\sum_{j \neq i} Q(i,j) f(z;i,j)}{\lambda(i)}$$
(5.85)

is the (mixture) density of jump size in state *i*. The stochastic exponential of $M_t^{h,\theta}$ is given by:

$$\mathcal{E}(M_t^{h,\theta}) = \prod_{0 < T_i \le T} (1 + h'Z_i)^{-\theta} e^{-\int_0^t \sum_{j \ne X_{s-}} \int_{\mathcal{Z}} [(1 + h'Z_i)^{-\theta} - 1]\phi(z;X_{s1}) dz ds}.$$
(5.86)

To see the impact of a measure change defined by this stochastic exponential on the factor process X, let:

$$\xi^{\theta}(X_{s-},j) = \int_{\mathbb{R}} (1+h'z)^{-\theta} f(z;X_{s-},j) \mathrm{d}z - 1.$$
(5.87)

Using the results from the previous section, the projection to the filtration generated by the factor process is given by:

$$\mathbb{E}\left[\left.\mathcal{E}^{\theta}(M_{t})\right|F_{T}^{X}\right] = \prod_{0 < T_{i} \leq T} \left(1 + \xi^{\theta}(X_{T_{i}-}, X_{T_{i}})\right)e^{-\int_{0}^{T}\sum_{j \neq X_{s-}}\xi^{\theta}(X_{s-}, j)Q(X_{s-}, j)\mathrm{d}s}, \quad (5.88)$$

because of the independence of the jump sizes Z and the identity

$$\mathbb{E}\left[\left(1+h'Z_{i}\right)^{-\theta}\middle|F_{T}^{X}\right] = \int_{\mathbb{R}} (1+h'z)^{-\theta} f(z;X_{T_{i}-},X_{T_{i}}) \mathrm{d}z = \xi^{\theta}(X_{T_{i}-},X_{T_{i}}) + 1 \quad (5.89)$$

for $T_i \leq T$. Hence the effect of this measure change on the factor process follows from Proposition 5.A.1, where the martingale is defined by the function ξ^{θ} .

The following proposition summarises the effect of this measure change on the distribution of jumps sizes Z, it is needed in the proof of Proposition 5.5.1.

Proposition 5.A.2 *Provided that* $\mathcal{E}(M_t^{h,\theta})$ *is a martingale, let:*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(M_t^{h,\theta}) \tag{5.90}$$

with $\mathcal{E}(M_t^{h,\theta})$ defined in (5.83). Then the compensator of jumps Z in the new measure $\tilde{\mathbb{P}}$ is given by:

$$\tilde{\phi}(z;i) = (1+h'z)^{-\theta}\phi(z;i)$$
(5.91)

Proof The change of measure formula for compound Poisson processes (extended with state-dependence) is given by:

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_{t}} = \prod_{s \le t} \frac{\tilde{\phi}(Z_{s}; X_{s-})}{\phi(Z_{s}; X_{s-})} e^{(\lambda(X_{s-}) - \tilde{\lambda}(X_{s-}))t},$$
(5.92)

see e.g. Shreve (2004, pp. 498-499). Comparing to (5.86) and using calculations similar to proof of Proposition 5.A.1, we get that:

$$\frac{\tilde{\phi}(z; X_{s-})}{\phi(z; X_{s-})} = (1 + h'z)^{-\theta},$$
(5.93)

which finishes the proof.

5.B Appendix: Proof of Proposition 5.5.1

This proof in an adaptation of Klebaner and Lipster (2014) to the current setting. First note that the process χ_t^h solves the following stochastic differential equation:

$$\mathrm{d}\chi_t^h = \chi_{t-}^h \mathrm{d}M_t^\chi,\tag{5.94}$$

where

$$M_t^{\chi} = -\theta \int_0^t h'_s \Sigma dW_s + \sum_{0 < s \le T} [(1 + h'_s Z_s)^{-\theta} - 1] - \int_0^t \int_{\mathcal{Z}} [(1 + h'z)^{-\theta} - 1] \phi(z; X_{s-}) dz ds$$
(5.95)

where ϕ is defined as in (5.84). Let us define a localizing sequence of stopping times as:

$$\tau_n = \inf\{t : \chi_t \ge n\} \tag{5.96}$$

Then for every n, the stopped process $\chi_{t \wedge \tau_n}$ is bounded. The main idea behind the proof is that the uniform integrability of the family $\{\chi_{t \wedge \tau_n}\}_{n \to \infty}$ is verified by the Vallée de Poussin theorem with function $x \log x$ for x > 0. By Ito's lemma:

$$\begin{split} \chi_{t\wedge\tau_{n}}^{2} - 1 &= -2\theta \int_{0}^{t} \mathbf{1} \left\{ s \leq \tau_{n} \right\} \chi_{(s\wedge\tau_{n})-}^{2} h_{s}' \Sigma dW_{s} \\ &+ 2 \sum_{0 < s \leq T} \mathbf{1} \left\{ s \leq \tau_{n} \right\} \chi_{(s\wedge\tau_{n})-}^{2} [(1 + h_{s}'Z_{s})^{-\theta} - 1] \\ &- 2 \int_{0}^{t} \mathbf{1} \left\{ s \leq \tau_{n} \right\} \chi_{(s\wedge\tau_{n})-}^{2} \int_{\mathcal{Z}}^{2} [(1 + h_{s}'Z_{s})^{-\theta} - 1] \phi(z; X_{s-}) dz ds \\ &+ \sum_{0 < s \leq T} \mathbf{1} \left\{ s \leq \tau_{n} \right\} \chi_{(s\wedge\tau_{n})-}^{2} [(1 + h_{s}'Z_{s})^{-\theta} - 1]^{2} \\ &- \int_{0}^{t} \mathbf{1} \left\{ s \leq \tau_{n} \right\} \chi_{(s\wedge\tau_{n})-}^{2} \int_{\mathcal{Z}}^{2} [(1 + h_{s}'Z_{s})^{-\theta} - 1]^{2} \phi(z; X_{s-}) dz ds \\ &+ \theta \int_{0}^{t} \mathbf{1} \left\{ s \leq \tau_{n} \right\} \chi_{(s\wedge\tau_{n})-}^{2} h_{s}' \Sigma \Sigma' h_{s} ds \\ &+ \int_{0}^{t} \mathbf{1} \left\{ s \leq \tau_{n} \right\} \chi_{(s\wedge\tau_{n})-}^{2} \int_{\mathcal{Z}}^{2} [(1 + h_{s}'z)^{-\theta} - 1]^{2} \phi(z; X_{s-}) dz ds \end{split}$$

Note that the first five lines in the formula above form a martingale with expectation zero. Hence:

$$\mathbb{E}\left[\chi_{t\wedge\tau_{n}}^{2}-1\right] = \mathbb{E}\left[\int_{0}^{t} \mathbf{1}\left\{s \leq \tau_{n}\right\}\chi_{(s\wedge\tau_{n})-}^{2}\left(\theta h_{s}'\Sigma\Sigma'h_{s}\right.\\\left.+\int_{\mathcal{Z}}[(1+h'z)^{-\theta}-1]^{2}\phi(z;X_{s-})\mathrm{d}z\right)\mathrm{d}s\right]$$
(5.98)

Using the assumptions from Section 5.2, the following process is bounded for every state $i \in \mathcal{N}$ and time $s \leq \tau_n$:

$$\theta h'_s \Sigma(i) \Sigma(i)' h_s + \int_{\mathcal{Z}} [(1 + h'_s z)^{-\theta} - 1]^2 \phi(z; i) \mathrm{d}z \le r,$$
(5.99)

and so $\chi_{t \wedge \tau_n}$ is a square integrable martingale with $\mathbb{E} [\chi_{t \wedge \tau_n}] = 1$. We can use it to define a measure change:

$$\frac{\mathrm{d}\mathbb{P}^n}{\mathrm{d}\mathbb{P}} = \chi_{t \wedge \tau_n} \tag{5.100}$$

From Girsanov Theorem the Brownian motion in the new measure \mathbb{P}^n for all $t \leq \tau_n$ is given by:

$$\tilde{W}_t = W_t + \theta \int_0^t h'_s \Sigma \mathrm{d}s \tag{5.101}$$

and from Proposition 5.A.2, the jump compensator becomes:

$$\tilde{\phi}(z;i) = (1+h'z)^{-\theta}\phi(z;i) = \phi(z;i) + [(1+h'z)^{-\theta} - 1]\phi(z;i)$$
(5.102)

Note that $\chi_{t\wedge\tau_n}$ can be decomposed as:

$$\chi_{t\wedge\tau_n} = \exp(M_{t\wedge\tau_n} - A_{t\wedge\tau_n}),\tag{5.103}$$

where

$$M_{t \wedge \tau_n} = -\theta \int_0^t h'_s \Sigma dW_s + \sum_{0 < s \le T} [(1 + h'_s Z_s)^{-\theta} - 1] - \int_0^t \int_{\mathcal{Z}} [(1 + h'z)^{-\theta} - 1] \phi(z; X_{s-}) dz ds$$
(5.104)

and

$$A_{t \wedge \tau_n} = \frac{\theta^2}{2} \int_0^t h'_s \Sigma \Sigma' h_s \mathrm{d}s + \sum_{0 < s \le T} [(1 + h'_s Z_s)^{-\theta} - 1] - \log[(1 + h'_s Z_s)^{-\theta}].$$
(5.105)

5.B Appendix: Proof of Proposition 5.5.1

The elementary inequality $\log(x) \leq x - 1$ for all x > 0 implies that the process A_t is non-negative. Therefore $\log(\chi_{t \wedge \tau_n}) \leq M_{t \wedge \tau_n}$, and we have the bound:

$$\mathbb{E}\left[\chi_{t\wedge\tau_n}\log\chi_{t\wedge\tau_n}\right] \le \mathbb{E}\left[\chi_{t\wedge\tau_n}M_{t\wedge\tau_n}\right] = \mathbb{E}^n\left[M_{t\wedge\tau_n}\right],\tag{5.106}$$

where $\mathbb{E}^{n}[\cdot]$ denotes the expectation in the \mathbb{P}^{n} probability measure. Using (5.101) and (5.102) we can write $M_{t \wedge \tau_{n}}$ as :

$$M_{t\wedge\tau_{n}} = -\theta \int_{0}^{t} h'_{s} \Sigma d\tilde{W}_{s} + \theta \int_{0}^{t} h'_{s} \Sigma \Sigma' h_{s} ds + \sum_{0 < s \le T} [(1 + h'_{s} Z_{s})^{-\theta} - 1] - \int_{0}^{t} \int_{\mathcal{Z}} [(1 + h'z)^{-\theta} - 1] \tilde{\phi}(z; X_{s-}) dz ds + \int_{0}^{t} \int_{\mathcal{Z}} [(1 + h'z)^{-\theta} - 1]^{2} \phi(z; i) dz ds,$$
(5.107)

and so:

$$\mathbb{E}^{n}\left[M_{t\wedge\tau_{n}}\right] = \mathbb{E}^{n}\left[\int_{0}^{t}\theta h_{s}'\Sigma\Sigma'h_{s} + \int_{\mathcal{Z}}\left[(1+h'z)^{-\theta} - 1\right]^{2}\phi(z;i)\mathrm{d}z\mathrm{d}s\right]$$

$$\leq r,$$
(5.108)

by (5.99). This is a uniform bound, hence:

$$\sup_{n} \mathbb{E} \left[\chi_{t \wedge \tau_n} \log \chi_{t \wedge \tau_n} \right] < \infty.$$
(5.109)

As indicated above, the family $\{\chi_{t\wedge\tau_n}\}_{n\to\infty}$ is uniformly integrable by the Vallée de Poussin theorem, which finishes the proof.

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