A proposed framework for characterising uncertainty and variability in rock mechanics and rock engineering

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Declaration of Originality

This thesis describes work carried out in the Department of Earth Science and Engineering at Imperial College London between 2010 and 2013. I declared that the work presented in this thesis is my own, except where acknowledged.

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ABSTRACT

This thesis develops a novel understanding of the fundamental issues in characterising and propagating unpredictability in rock engineering design. This unpredictability stems from the inherent complexity and heterogeneity of fractured rock masses as engineering media. It establishes the importance of: a) recognising that unpredictability results from epistemic uncertainty (i.e. resulting from a lack of knowledge) and aleatory variability (i.e. due to inherent randomness), and; b) the means by which uncertainty and variability associated with the parameters that characterise fractured rock masses are propagated through the modelling and design process. Through a critical review of the literature, this thesis shows that in geotechnical engineering – rock mechanics and rock engineering in particular – there is a lack of recognition in the existence of epistemic uncertainty and aleatory variability, and hence inappropriate design methods are often used. To overcome this, a novel taxonomy is developed and presented that facilitates characterisation of epistemic uncertainty and aleatory variability in the context of rock mechanics and rock engineering. Using this taxonomy, a new framework is developed that gives a protocol for correctly propagating uncertainty and variability through engineering calculations. The effectiveness of the taxonomy and the framework are demonstrated through their application to simple challenge problems commonly found in rock engineering. This new taxonomy and framework will provide engineers engaged in preparing rock engineering designs an objective means of characterising unpredictability in parameters commonly used to define properties of fractured rock masses. These new tools will also provide engineers with a means of clearly understanding the true nature of unpredictability inherent in rock mechanics and rock engineering, and thus direct selection of an appropriate unpredictability model to propagate unpredictability faithfully through engineering calculations. Thus, the taxonomy and framework developed in this thesis provide practical tools to improve the safety of rock engineering designs through an improved understanding of the unpredictability concepts.

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NOTATION

General notation

A	Area of sliding block
$\text{AI}(a)$	Agreement index at a value a
c	Cohesion
E	Elastic modulus
E_r	Elastic modulus of intact rock
E_{rm}	Elastic modulus of fractured rock mass
$F\hat{o}S$	Fuzzy factor of safety
H	Height of slope
J_a	Joint alteration number
J_n	Joint set number
J_r	Joint roughness number
J_w	Joint water reduction factor
k	In-situ stress ratio (σ_h / σ_v)
L	Distribution of driving forces (loads)
m_b	Hoek-Brown material constant for jointed rock mass
m_i	Hoek-Brown material constant for intact rock
M	Distribution of reliability or performance function
P_{roof}	Roof support pressure
Q	Rock quality index for tunnelling
R	Distribution of resisting forces
s	Hoek-Brown material constant for intact rock
S_b	Spacing of rock bolts
S_s	Spacing of strand anchors

T	Tension in bolt
T_b	Yield strength of rock bolts
T_s	Yield strength of strand anchors
U	Water pressure acting on failure plane
V	Water pressure acting in tension crack
W	Weight of sliding block
z	Depth of tension crack
z_w	Height of water in tension crack
α	Acceleration coefficient
β	Reliability index
γ	Unit weight of rock
γ_w	Unit weight of water
ϕ	Angle of friction
λ	Mean discontinuity spacing
ψ_f	Angle of slope face
ψ_p	Angle of failure plane
θ	Inclination of bolt to failure plane
σ_1	Major principal stress at failure
σ_3	Minor principal stress at failure
σ_{ci}	Uniaxial compressive strength of intact rock
σ_h	In-situ horizontal stress
σ_v	In-situ vertical stress
σ_θ	Induced tangential stress around a circular underground opening
σ_{spall}	Rock spalling strength

Set and probabilistic notation

$[a,b]$	Interval bounded by a and b
$\mathbf{X}$	Set containing all possible values of variable x
$\{x\}$	A set containing all values of x

$\in$	Element of (set membership)
$\notin$	Not an element of (non-set membership)
$f(x)$	Function of x
Sup	Supremum – least upper bound
Inf	Infimum – greatest lower bound
$\hat{X}$	Fuzzy number of variable x
$\mu_X(x)$	Fuzzy membership value
X_α	α-cut of $\hat{X}$
$L(x)$	Lower bound fuzzy membership function
$R(x)$	Upper bound fuzzy membership function
$\pi(x)$	Possibility distribution of x
Π	Possibility measure
N	Necessity measure
$\left[\overline{F}, \underline{F}\right]$	A p-box bounded by lower and upper CDFs of F
$F(x)$	Cumulative density function of variable x
$\underline{F}(x)$	Lower bound cumulative density function of variable x
$\overline{F}(x)$	Upper bound cumulative density function of variable x
$f_X(x)$	Probability density function of variable x
E	An event E
S	Sample space
$\overline{E}$	Complementary event of E (i.e. not E)
$P(X)$	Probability of a value X
$f_{post}(x)$	Posterior probability distribution of variable x
$f_{prior}(x)$	Prior probability distribution of variable x
$P(h\,|\,e)$	Conditional probability of h given e
$\forall_\alpha$	Universal quantifier (for all α)
μ	Mean
σ	Standard deviation
r^2	Coefficient of variation

Abbreviations

CDF	Cumulative density function
CHILE	Continuous Homogeneous Isotropic Linear Elastic
DIANE	Discontinuous In-homogeneous Anisotropic Non-Elastic
FMF	Fuzzy membership function
FORM	First order reliability method
FoS	Factor of safety
FOSM	First order-second moment
GSI	Geological strength index
JRC	Joint roughness co-efficient
LEM	Limit equilibrium model
LoI	Level of information
LSD	Limit state design
MC	Monte-Carlo
PDF	Probability distribution function
RMR	Rock mass rating
RQD	Rock quality designation
SORM	Second order reliability method
SRF	Stress reduction factor
TFN	Triangular fuzzy number
TrFN	Trapezoidal fuzzy number
UCS	Uniaxial compressive strength

GLOSSARY OF TERMS

The following glossary presents the meaning of the terms as used throughout this thesis. Terms used in the definitions that are themselves defined elsewhere in this glossary are **_emboldened and italicised_**.

Accurate	Refers to a situation where data can be measured or assessed, without significant error, and close to the correct value of the parameter in question. Accuracy is required to attain a state of **_precise_** information. In general, accurate data can only be obtained through objective measurement of quantitative data.
Aleatory Variability	Stemming from the Latin 'alea', which means rolling of dice, aleatory variability refers to that part of **_unpredictability_** resulting from inherent randomness (see **_random_**), or natural **_variability_** in a physical system of environment. Also known as stochastic (see **_stochastic model_**) **_uncertainty_**, objective uncertainty or irreducible uncertainty, and can be modelled using a **_probability distribution function_**.
Bayesian	From Bayes's Theorem; the Bayesian approach requires an unpredictable (see **_unpredictability_**) parameter to be modelled as a **_random_** variable (i.e. with a **_probability distribution function_** that is **_precise_**) defined using prior knowledge, expert opinion and any objective information, no matter how little, which may be available. Bayes's Theorem can then be applied to update this **_'prior'_** distribution to a **_'posterior'_** distribution as further evidence or data becomes available.
Bonus-Malus	A system of reward and penalty often used by insurance companies in minimising risk of loss to the company. This system is analogous to the framework of exchangeable bets on which the subjective **_Bayesian_** approach is based.

Calculus	A method of undertaking mathematical calculations.
Degree-of-belief	A subjective assessment of probability, likelihood or level of confidence in the materialisation of an event, or a parameter or system taking on a particular value. Required when defining a probability distribution entirely through expert opinion or judgement – i.e. with no justification through objective data – when applying the ***Bayesian*** approach using ***subjectivist probability***.
Dissonance	Lack of agreement; inconsistency. Dissonance between experts refers to disagreement between their beliefs.
Epistemic Uncertainty	Derived from the Greek 'episteme', meaning knowledge – epistemic uncertainty refers to that part of ***unpredictability*** resulting from a lack of knowledge; it is both subjective in nature and influenced by preconceptions of what is considered realistic for the system in question. It has also been called ignorance, imprecision (see ***imprecise***) or reducible ***uncertainty***, and can be reduced or eliminated through additional information or knowledge, and is most appropriately modelled using non-stochastic methods (see ***stochastic model***).
Exceedence	Refers to a situation when the value of a ***parameter*** is surpassed or exceeded. For example, the probability of exceedance refers to probability of a ***parameter*** exceeding a certain value.
Exemplar	An illustrative problem serving as a typical example or excellent model.
Extrinsic	Not belonging to the essential nature or constitution of a thing. See for example, ***extrinsically epistemic.***
Extrinsically epistemic	Refers to parameters for which a ***probability distribution function*** could be determined if the data can be refined from ***imprecise*** to ***precise*** values or, if the data are precise, additional information deems the quantity sufficient to define an aleatory model (see ***aleatory variability***). Thus, when sufficient information becomes available, an ***extrinsically epistemic*** property can be treated as an aleatory property, and modelled using stochastic methods (see ***stochastic model***).

Faithfulness	Faithfulness can be seen as the pursuit for consistency with available information. That is, when characterising ***unpredictability***, one should select a suitable modelling method commensurate with the level of information available. In particular, faithfulness requires that in the absence of any objective information, a non-stochastic (see ***stochastic model***), interval-oriented, unpredictability modelling method should be used in lieu of the subjective assignment of a ***PDF*** that is ***precise*** (i.e. the ***Bayesian*** approach). The latter approach would arguably be misrepresenting the available information and in fact introduce information on probabilities of occurrence that are not actually available.
Frequentist probability	Probabilistic approach appropriate for modelling ***aleatory variability***, which assumes that an event is the result of a random process, which can be realised by repeating an experiment a large number of times and plotting the number of times each outcome occurs. The variability in the results is characterised by one of the well known ***probability distribution functions***, fitted to the data using various statistical tools and accepted on passing a number of hypothesis tests.
Imprecise	In this thesis, imprecise refers to situation where there is either an insufficient quantity of ***precise*** data, or the quality of data is neither ***precise*** nor ***accurate*** enough to objectively fit a ***probability distribution function*** to characterise the ***unpredictability*** in the ***parameter*** in question. Generally, subjectively determined parameters are considered imprecise.
Indifference	Refers to a situation where one has no objective information or ***degree-of-belief*** on which to select any particular shape of a ***probability distribution function*** except for a uniform distribution. The principle of indifference is utilised in ***Bayesian*** approach using ***subjectivist probability***.
Intrinsic	Belonging to the essential nature or constitution of a thing. See for example, ***intrinsically epistemic***.
Intrinsically epistemic	Refers to parameters that are inherently ***imprecise*** and for which, no

matter the quantity of information, the quality of data could not be improved to reach a *precise* state. For such parameters it is inappropriate to assign a precise probability distribution.

Nominal   A scale of measurement where numerals assigned to define *parameters* are used only as labels or type numbers, and words or letters would serve just as well. A classic application of the nominal scale is where numbers are assigned to identify football players.

Objective   A method of assessing data in which the values assigned to *parameters* can be justified by physical or mathematical tests undertaken on factual and quantitative data. This method of assessment reduces *dissonance* between experts.

Ordinal   A scale of measurement where numerals are used to define rank ordering in the values of the *parameters* they define. That is, the numerical information on an ordinal scale provides information only on the ordering of the measurement. Ordinal scales are commonly used in rock mass classification systems.

P-Box   Probability boxes, or p-boxes, are mathematical structures that are able to represent both *epistemic uncertainty* and *aleatory variability* through the concept of *imprecise* probability. Imprecise probability, also referred to as probability bounds, analysis combines the methods of interval analysis and classical, or *frequentist probability* theory to produce a p-box comprising two non-intersecting cumulative distribution functions that generalise an interval.

Parameter   Parameters are defined as inputs required to define mathematical models. Parameters may be used to specify *properties* of the material or system they describe. For example, a commonly used parameter to define stiffness of intact rock is the Elastic Modulus (E).

Posterior   When applying *Bayesian* updating, the *prior* probability distribution is updated, using Bayes's Theorem, as further data is obtained. The initial (i.e. prior) distribution is thus updated to the 'posterior' distribution.

Precise	Refers to situation where the data can be measured or assessed without ambiguity, vagueness and with sufficient exactness such that the value of the obtained measurement may be considered an *accurate* value of the *parameter* in question. In general, a sufficient number of objective measurements are required to obtain a state of *precise* information.
Predictable	The opposite of unpredictable. See *unpredictability*. A predictable parameter is one which may be exactly defined by a single value, e.g., the height of a rock slope can be *accurately* and *precisely* measured using surveying equipment and defined by a single value of height.
Prior	The *Bayesian* approach requires an unpredictable *parameter* to be modelled as a *random* variable (i.e. with a *probability distribution function*) defined using prior knowledge, expert opinion and any objective information, no matter how little, which may be available. This is known as the 'prior' probability distribution.
Probability Distribution Function (PDF)	A *stochastic model* used to characterise *aleatory variability*. A probability distribution function is a mathematical model defined by parameters that include its statistical moments (e.g. mean, standard deviation, etc.); well known examples include normal and uniform PDFs. A PDF can be fitted to the data using various statistical tools, and accepted on passing a number of well known, statistical hypothesis tests.
Property	A property refers to a physically observable manifestation of the behaviour of a material or system. For example, the discontinuity spacing is a physical property of a fractured rock mass; the *unpredictability* in this property is commonly defined by a negative exponential *PDF* using the parameter λ, which describes mean discontinuity spacing.
Random	Refers to an outcome or event chosen by chance; relating to, having, or being elements or events with definite probability of occurrence. Something being random implies complete *unpredictability*, except in the relative frequencies with which it occurs (see *frequentist probability*).

Robustness	Robustness refers to a characteristic of interval-oriented ***uncertainty*** modelling methods such that, so long as the intervals forming the inputs bound the true value of the ***parameters*** they represent, the output is also guaranteed to bound the true result.
Stochastic model	Over a large number of trials, ***variability*** will tend to follow some distribution – the stochastic model, which describes a system of countable events, where the events occur according to some well-defined ***random*** process defined over some domain.
Subjective	A method of assessing data that used expert opinion, induction and ones degree-of-belief in estimating or assessing the values assigned to ***parameters***. This method of assessment is used when no ***objective*** data are available, or the data are entirely qualitative in nature. Subjective assessment can lead to ***dissonance*** between experts.
Subjectivist probability	Probabilistic approach that interprets probability as a subjective measure of confidence (i.e. one's ***degree-of-belief***) in the available information. Subjectivist probability forms the basis of the ***Bayesian*** approach, which suggests that both ***aleatory variability*** and ***epistemic uncertainty*** (i.e. total ***unpredictability***) should be modelled as a ***random*** variable – i.e. using a ***probability distribution function*** that is ***precise***. This thesis demonstrates how this approach is neither ***faithful*** nor ***robust*** when the ***unpredictability*** is ***epistemic***.
Taxonomy	In this thesis, the purpose of the proposed taxonomy is to provide a means of orderly arrangement of the terms required to objectively characterise the true nature of ***unpredictability***, and present guidance on the appropriate unpredictability model with which to model and propagate the unpredictability of the parameter in question.
Uncertainty	Uncertainty represents that component of ***unpredictability*** which is due to a lack of knowledge, and thus a deficiency in the available information. It may be qualitative or quantitative in nature.
Unpredictability	Unpredictability characterises all our deficiencies and inabilities to be able to precisely predict the value of a ***parameter*** or system.

Unpredictability is due to the combination of lack of knowledge and *randomness*, i.e. the combination of *epistemic uncertainty* and *aleatory variability*.

Variability   Variability is the result of *randomness* and can be characterised by *stochastic models* and propagated using probability theory.

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Chapter 1

INTRODUCTION

The inherent complexity and heterogeneity of fractured rock masses as engineering media makes their detailed and accurate characterisation an exceptionally challenging task. Feng & Hudson (2010) identify the need for collection of sufficient site investigation data as paramount to this characterisation process and consequently producing robust engineering designs. The reality is however, on actual rock engineering projects, site investigation is usually discontinued once the (small) budget allocated to it is depleted. Consequently, the combination of rock mass complexity and a lack of information lead to both significant simplifications regarding characterisation and subjective estimation of many physical parameters. Together, these simplifications and estimations result in an element of unpredictability in the engineering properties of fractured rock masses.

In geotechnical engineering, the term uncertainty has been broadly – and, as this thesis will show, incorrectly - used throughout geotechnical engineering (Baecher & Christian, 2003; Bárdossy & Fodor, 2004; Christian, 2004) to characterise all our deficiencies and inabilities to be able to precisely predict the value of a parameter or total unpredictability of a system (Vose, 2000). However, much of the literature from various fields of science and technology recognises that unpredictability in a parameter or system results from the combined contribution of epistemic uncertainty and aleatory variability (Hoffman & Hammonds, 1994; Ferson & Ginzburg, 1996; Helton & Oberkampf, 2004; Ang & Tang, 2007), which are fundamentally different in nature. Epistemic uncertainty – derived from the Greek *'episteme'*, meaning knowledge – is due to lack of knowledge (Baecher & Christian, 2003); it is both subjective in nature and influenced by preconceptions of what is considered realistic for the system in question (Kiureghian & Ditlevsen, 2009). It has also been called ignorance, imprecision or reducible uncertainty, and can be reduced or eliminated through additional information or knowledge (Guo & Du, 2007). Aleatory variability – etymologically from the Latin *'alea'*, which means the rolling of dice – on the other hand, describes the

inherent random variability in a physical system or environment (Baecher & Christian, 2003). It has been suggested that as aleatory variability – also known as stochastic uncertainty, objective uncertainty or irreducible uncertainty (Kiureghian & Ditlevsen, 2009) – describes inherent randomness (Ferson, 2002; Dubois & Guyonnet, 2011), it can be characterised by stochastic models and handled using probabilistic methods (Dubois & Guyonnet, 2011). It is now widely recognised that uncertainty and variability are fundamentally different in nature and so cannot be modelled using the same techniques (Dubois & Prade, 1989; Hoffman & Hammonds, 1994; Ferson & Ginzburg, 1996; Guyonnet et al., 1999; Ferson, 2002; Moller & Beer, 2008; Dubois & Guyonnet, 2011).

The putative difficulty in characterising heterogeneous rock masses coupled with the, all too often, case of limited objective data with which to characterise unpredictability is perhaps one reason for traditionally handling total unpredictability using deterministic models with conservative ('lower bound' or 'worst case') values as their inputs (Christian, 2004). This approach, however, fails to address the problem of satisfactorily quantifying and consistently dealing with lack of knowledge or randomness uncertainties (Nadim, 2007), but rather introduces further uncertainty and room for disagreement amongst experts on the question, '*how conservative is conservative enough?*' The answer to which is based upon the subjective experience of the modeller or analyst. In some cases perceived 'conservatism' may still result in unsafe design assumptions (Becker, 1996). To account for these shortcomings, probabilistic approaches to analysing and quantifying uncertainty have become commonplace in rock engineering (e.g. Priest & Brown, 1983; Zhang & Einstein, 1998; Cai et al., 2000). In fact various authors have suggested that total unpredictability, i.e. both epistemic uncertainty and aleatory variability, can be handled using the Bayesian approach with associated subjective probabilistic methods (Jeffreys, 1961; Lindley, 2000; Howson, 2002). The Bayesian approach then allows one to make statements using familiar statistical terms such as 'probability of occurrence', 'mean value', 'confidence limit' and so forth. However, the appropriateness of probabilistic methods to characterise and propagate epistemic uncertainty has recently been increasingly questioned (Baudrit et al., 2006; Baudrit et al., 2007; Dubois & Guyonnet; 2011), and in fact has been shown to produce erroneous and unconservative results. For geotechnical engineering design, one of the consequences of such errors is the potential for unsafe or unstable structures.

In the context of geotechnical engineering, and rock mechanics and rock engineering in particular, it appears that the true meaning of uncertainty has not been correctly understood, and thus methods for its quantification have not been applied in an appropriate manner. This

may well result from the deficiency of a formal definition of uncertainty in the field of geotechnical engineering; rock mechanics and rock engineering in particular.

For these reasons, this thesis develops a new taxonomy that will allow the true nature of geotechnical uncertainty to be correctly addressed rather than erroneously considering all unpredictability as aleatory variability (Uzielli, 2008). By drawing on non-stochastic models developed and presented in the wider literature – which explicitly account for incomplete/imprecise information, and have thus been extensively utilised to handle epistemic uncertainty in other fields of science and engineering – this thesis develops and presents a new framework, applicable to rock mechanics and rock engineering, that directs the user to simply and objectively characterise the nature of unpredictability in a parameter or system before propagating it through the analysis and design process using the appropriate (mathematical) tools.

Applications of the new taxonomy and framework are demonstrated through three 'challenge problems' commonly encountered in rock engineering. This concept of challenge problems is adopted from their inception at the epistemic uncertainty workshop, hosted by Sandia National Laboratories, and focuses on the representation, aggregation, and propagation of mixtures of epistemic and aleatory uncertainty through simple analytical models (Ferson et al., 2002; Helton & Oberkampf, 2004; Oberkampf et al., 2004). The challenge problems presented in this thesis follow this premise. As a result, this thesis is able to show that using non-stochastic methods when the unpredictability is epistemic can reduce dissonance amongst experts and even avoid potentially erroneous results obtained by the bias outputs that result from the Bayesian approach (Klir, 1989; Klir & Yuan, 1995; Ferson & Ginzburg, 1996; Baudrit & Dubois, 2006).

The outcome of the developments presented in this thesis is that application of these new tools will harmonise designs by reducing arbitrary choices in characterising and propagating unpredictability in rock mechanics and rock engineering. This will mean that designers and policy makers will have a framework against which rock mechanics designs can be assessed and scrutinised. As such, this would mean that safety of rock mechanics designs will be greatly improved as the unpredictability concepts, currently not properly understood, will be better incorporated in to designs.

1.1 Structure of this thesis

This thesis consists of 8 chapters and various appendices.

Following this introduction, Chapter 2 presents a critical discussion on the concepts of epistemic uncertainty and aleatory variability, and the unique characteristics of each in the context of rock engineering. This discussion demonstrates the need to distinguish between epistemic uncertainty and aleatory variability with specific reference to design methods commonly used in rock engineering. This chapter confirms that aleatory variability may be handled using well known probabilistic techniques, but epistemic uncertainty requires alternative, non-probabilistic approaches. As a result, a novel taxonomy for characterising epistemic uncertainty and aleatory variability in rock mechanics and rock engineering is presented.

Chapter 3 applies the taxonomy to demonstrate the importance of selecting an appropriate unpredictability model, after assessing the available information, to propagate uncertainty or variability. To support this, the unpredictability modelling methods of interval analysis, fuzzy arithmetic, imprecise probability boxes (i.e. p-boxes) and Bayesian and classical, or frequentist, probabilistic methods are examined.

Chapter 4 presents a novel framework, in a series of three flowcharts, for characterising and propagating uncertainty or variability when undertaking design through engineering computations. The first flowchart is the overall framework, which contains two sub-charts. The first of these directs characterisation of the available data, with the second selecting an appropriate unpredictability model.

Following this, the new taxonomy and framework are applied to three challenge problems. Chapter 5 uses a planar slope instability problem to compare application of an aleatory model with a non-probabilistic approach selected by following the framework. Chapter 6 demonstrates how empirical rock mass classification systems, and the Q-system in particular, are intrinsically epistemic. In both cases, conclusions are drawn regarding the appropriate unpredictability models that should be applied. Chapter 7 examines the problem of predicting the peak strength or intact rock and jointed rock masses. This problem demonstrates how, as information becomes progressively available, epistemic uncertainty may be re-classified as aleatory variability, and probabilistic methods then applied to the calculation. These three challenge problems illustrate the strength of the new taxonomy and framework in directing selection of an appropriate unpredictability model through an assessment of the available information.

Chapter 8 draws together the conclusions reached through this research and presents proposals for further research. Finally, the thesis is supported by references and various appendices.

Chapter 2

CHARACTERISATION OF UNPREDICTABILITY

The introduction of this thesis identified the need to differentiate between uncertainty and variability as the two components that contribute to the total unpredictability within a parameter or system, especially when the available information to characterise the properties of the parameter or system is limited. This Chapter commences by presenting formal definitions of uncertainty and variability, followed by a discussion on the importance of characterising each through a quantitative and qualitative assessment of the available information. This discussion leads to the presentation of a new taxonomy for objectively characterising uncertainty and variability. Finally, this Chapter demonstrates the applicability of this new taxonomy through examples specific to rock mechanics and rock engineering. As a result, this chapter shows the effectiveness of the new taxonomy when selecting an appropriate unpredictability model if the available information is imprecise and/or sparse.

2.1 Uncertainty and variability

A review of the wider literature reveals the general acceptance that unpredictability is due to the combination of lack of knowledge and randomness (Dubois & Prade, 1988; Hoffman & Hammonds, 1994; Ferson & Ginzburg, 1996; Vose, 2000; Ferson, 2002; Baecher & Christian, 2003; Helton & Oberkampf, 2004; Christian, 2004; Ang & Tang, 2007; Moller & Beer, 2008; Dubois & Prade, 2009; Helton et al., 2010; Beer et al., 2012). In geotechnical engineering, however, the term 'uncertainty' is often universally applied to define the total unpredictability of a parameter or system, with probability theory and statistics seen as the optimal methods for its quantification (Whitman, 2000; Duncan, 2001; Bárdossy & Fodor, 2004; Christian, 2004; Uzielli, 2008). This may be a consequence of geotechnical industrialists' failure to

distinguish between variability and uncertainty as the two components that contribute to unpredictability. Indeed, to faithfully characterise unpredictability, it is essential that these terms are recognised as being applicable to specific, different characteristics (Ferson & Ginzburg, 1996; Kiureghian & Ditlevsen, 2009; Dubois & Guyonnet, 2011).

A dictionary definition of uncertainty is *"not able to be relied on; not known or definite"* (Merriam-Webster Inc., 2005), which suggests that uncertainty and knowledge are related. In turn, knowledge itself may be defined as *"what is known in a particular field or in total; facts and information"* (Merriam-Webster Inc., 2005). On a scientific level, the definition is the subject of heavy debate, even by theoretical mathematicians (Bárdossy & Fodor, 2004), and various science and technology fraternities interpret it in different ways (Ferson et al., 2002; Oberkampf et al., 2004). Zimmermann (2000) presents a generic definition of uncertainty in the context of scientific understanding as: *"Uncertainty implies that in a certain situation a person does not dispose about information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behaviour or other characteristica* [sic]*"*. These definitions infer a link between knowledge and information, from which one can conclude that uncertainty represents a lack of knowledge, and thus a deficiency in the available information, which may be qualitative or quantitative in nature.

In geotechnical engineering, a lack of knowledge – and thus uncertainty – may eventuate from a shortage of field or laboratory investigation data (i.e. incompleteness), or because the nature of the data is such that they cannot be objectively measured (e.g. degree of weathering). Such data require subjectivity or expert judgement in their estimation, which leads to dissonance, ambiguity and vagueness (Dubois & Prade, 1988; Klir & Yuan, 1995; Bárdossy & Fodor, 2004). In rock mechanics and rock engineering, many parameters are empirical in origin and not physically measureable, rather they are derived from expert opinion or imprecise correlations (e.g. rock mass classification). Similarly, other parameters are either based on an approximation, or require the analyst to make one, which Zimmermann (2000) defines as a situation with insufficient information to make a precise description. All of these situations introduce imprecision and inaccuracy (Dubois & Prade, 1988; Dubois & Prade, 1989; Walley, 1991; Bárdossy & Fodor, 2004). On this basis, any part of total unpredictability that stems from a lack of knowledge due to shortage of objective data, subjective estimation, or reliance on the beliefs of experts is termed **epistemic uncertainty**.

Variability differs from uncertainty in that it is a function of the inherent randomness of a system. The key term here is 'random', a dictionary definition of which is: *"chosen at*

random; relating to, having, or being elements or events with definite probability of occurrence." (Merriam-Webster Inc., 2005). Indeed, the statistician Sir David Cox stated: *"Variability is a phenomenon in the physical world to be measured, analysed and where appropriate explained. By contrast uncertainty is an aspect of knowledge"* (Vose, 2000). Consequently, in this thesis, the term **aleatory variability** is used to characterise those aspects of unpredictability deriving from inherent random variability related to natural fluctuations of the property in question (Dubois & Guyonnet, 2011). In the context of geotechnical engineering, aleatory variability is exemplified by the variation, within a nominally uniform material, of properties such as uniaxial compressive strength. The variability in uniaxial compressive strength can be characterised through a series of measurements obtained from laboratory test, to which a stochastic model can be fitted.

Having identified epistemic uncertainty as a function of the available information, it follows that obtaining additional knowledge – for example undertaking more field or laboratory tests – will reduce this aspect of unpredictability. If sufficient additional information to improve the state of information is obtained, it may be possible to re-characterise the uncertainty as variability. Using this concept of reducibility, the distinction between aleatory variability and epistemic uncertainty can be made through an understanding of the current level of knowledge, given the available information (Aughenbaugh & Paredis, 2006; Guo & Du, 2007; Dubois & Guyonnet, 2011), as shown in Figure 1. This figure shows how complete ignorance is one extreme of epistemic uncertainty, and that as knowledge increases so it may be possible to recognise that aleatory variability exists. Figure 2 shows how this transition from epistemic uncertainty to aleatory variability occurs as knowledge, and thus information, increases and a threshold – the state of precise information – is crossed. The state of precise information is achieved when there is sufficient data to use established statistical methods to objectively fit a precise probability distribution function to characterise it, i.e. apply an aleatory model. That is, the data can be measured with acceptable accuracy to allow a unique probability of occurrence to be assigned to each value of a variable. Once an acceptable aleatory model has been developed, additional investigation will not reduce the variability – which is inherent in the system and thus irreducible – but may increase the precision of the parameters that describe it (Christian, 2004). This aspect of reducibility is discussed in the following sections of this Chapter and also demonstrated later in Chapter 7 by the challenge problems on characterising unpredictability in estimating the strength of intact rock and fractured rock masses.
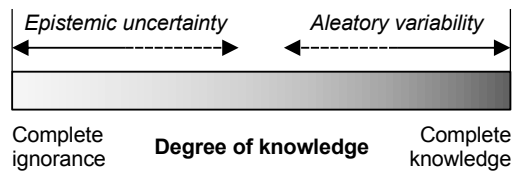
Figure 1: Total unpredictability: Uncertainty, variability and degree of knowledge (from Bedi & Harrison, 2013a).
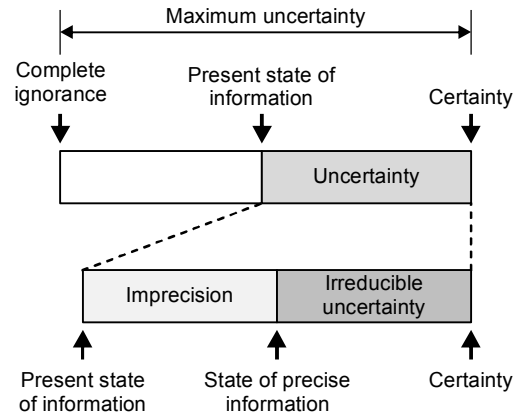
Figure 2: Uncertainty and information states (from Bedi & Harrison, 2013b).

Figure 3 illustrates how an assessment of the quantity and quality of the available information can be used to characterise the nature of unpredictability. This figure shows that aleatory variability can only be invoked once a sufficient quantity of precise data is available. It also suggests that a transition from epistemic uncertainty to aleatory variability can be achieved by gathering more (quantitative) or better (qualitative) information. However, attaining this additional information is not always possible, which presents the following corollary: that the state of information remains imprecise and the unpredictability must be characterised as epistemic uncertainty.
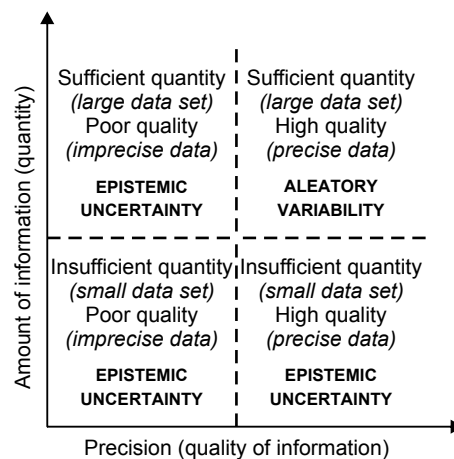
Figure 3: Uncertainty and variability as a function of quality and quantity of available information (from Bedi & Harrison, 2013a).

Through the concepts presented so far, it can be concluded that a key step when characterising unpredictability is to ascertain whether the current state of information is

precise or imprecise. Kurighien & Ditlevsen (2009) propose that it is the job of the analyst or engineer to make this distinction between aleatory variability and epistemic uncertainty before commencing on methods to propagate them through the modelling and design process. On this basis, the following discussion shall examine the circumstances that lead to a state of imprecise information, and thus introduce epistemic uncertainty. This is followed by a discussion on the nature of aleatory variability, with specific references in each case to rock mechanics and rock engineering.

2.2 Epistemic uncertainty

The archetypal problem often quoted to illustrate the nature of epistemic uncertainty is a deck of playing cards in a strategic game; after the deck of cards is shuffled, the arrangement of the cards is fixed but unknown (i.e. a lack of knowledge). The arrangement cannot be modelled stochastically, but can be discovered by examining each card in turn (i.e. increasing information). However, in games like Contract Bridge such an examination does not take place, these games use observation and induction in an attempt to obtain information about the arrangement of the cards (i.e. subjectivity) (Christian, 2004).

From an engineering perspective, as part of the design process we often rely on idealised models of reality in our analysis and predictions (e.g. assumption that the rock mass is continuous, homogeneous, isotropic, linearly elastic). These idealised models, which may be mathematical or physical models, require inputs in the form of parameters – usually obtained from laboratory or site investigation data – to define engineering properties that then govern the behaviour of the system. Both the input parameters and the models themselves are abstractions of reality (Kiureghian & Ditlevsen, 2009). Therefore, the results of analyses, estimations, or predictions obtained on the basis of such models are inaccurate; they yield some unknown degree of error and thus also contain uncertainty (Ang & Tang, 2007). It follows that epistemic uncertainty can eventuate throughout the various stages of this design process; investigation and data collection, analysis and decision-making. Sources of uncertainty that arise in the course of investigation and data collection include lack of representative sampling, insufficient quantity or errors in precise measurements, uncertainties in the description of non-measureable properties and temporal uncertainty (Bárdossy & Fodor, 2004). During the analysis phase concept and model uncertainties, or uncertainties due to subjective information (belief) and uncertainties in mathematical modelling, may arise. Lastly, uncertainty in the final design may result from decision-making based upon outputs from uncertain inputs. All of these sources of uncertainty are routine in geotechnical

engineering and constitute a lack of information, which leads to a state of imprecise information. The uncertainties stem from either a qualitative or quantitative lack of information, or the type of data available.

The following section first discusses how qualitative or quantitative lack of information leads to an imprecise state of information with specific reference to rock mechanics and rock engineering. This is followed by an examination of the types of information attributed to the means employed in measuring, or quantifying, rock engineering parameters. This discussion substantiates the earlier claim that probability theory is inappropriate for the quantification of epistemic uncertainty.

2.2.1 *Qualitative and quantitative lack of information*

According to Figure 3 (above), a quantitative lack of precise data requires that the state of knowledge be regarded as imprecise and, consequently, characterised as epistemic. This is now demonstrated with reference to an example of attempting to characterise data that can be objectively and precisely measured (e.g. standardised laboratory test results of uniaxial compressive strength) by a precise stochastic model. Figure 4a presents a set of data containing thirty samples and the distribution fitted to it. The closeness of the fit between the histogram and the distribution suggests that an aleatory model, i.e. a known stochastic function – in this case, normal – is appropriate to characterise the unpredictability. On the contrary, if presented with a limited number of precise measurements – for example, either of the two subsets ('A' or 'B') shown in Figure 4b, each limited to seven outcomes drawn from the data set – there are too few results to justify an aleatory model. This insufficiency of information requires the unpredictability to be characterised as epistemic uncertainty.

a) Data exhibiting aleatory uncertainty   b) Data exhibiting epistemic uncertainty
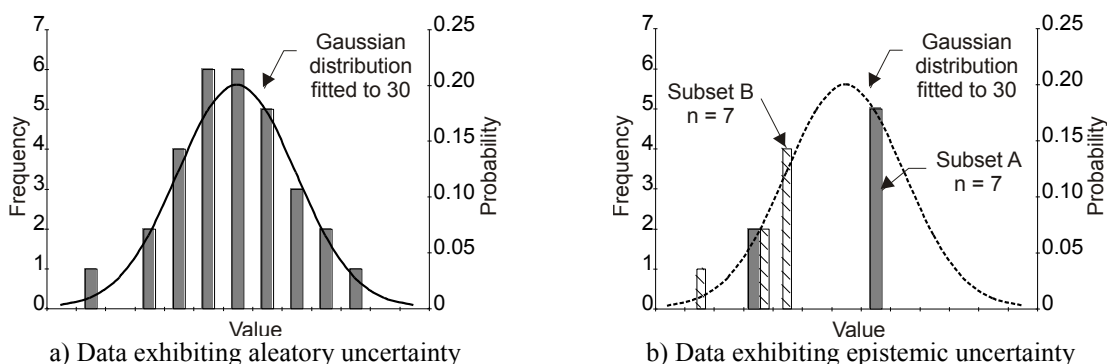
Figure 4: Appropriateness of a stochastic model to define an extrinsically epistemic data set
(from Bedi & Harrison, 2012).

Alternatively, many parameters used to characterise properties in rock mechanics are either defined qualitatively or quantified entirely subjectively through expert judgment. Consequently, their estimation requires one to make an approximation. Examples include the many parameters used within empirical rock mass classification systems such as the joint set number J_n in the Q-system (Barton et al., 1974), or the discontinuity condition rating used in the Rock Mass Rating (RMR) (Bieniawski, 1989). The empirical Geological Strength Index (GSI) (Hoek, 1994; Hoek et al., 1995) is another example. In any case, regardless of the amount of information collected or expert consultation undertaken, the subjectivity required to estimate such parameters will always result in approximate values and dissonance between experts (Klir, 1989; Tonon et al., 2000; Sonmez et al., 2003). Consequently, the state of information will always remain imprecise. Indeed, one of the originators of the GSI recognised this inherent imprecision and advised, *"Do not try to be too precise. Quoting a range from 33 to 37 is more realistic than stating that GSI = 35"* (Hoek, 2007). In these instances, imprecision results from a qualitative lack information, which may be further augmented by the use of parameters derived from approximate correlations. Examples include prediction of rock mass deformation from an estimated Q-value (Barton, et al., 1974) (Figure 5a) or the estimation of rock mass modulus from GSI (Figure 5b), both of which are derived from approximate correlation with empirical evidence. There are a multitude of such empirical correlations commonly used in rock mechanics (see Gokceoglu et al., 2003; 2004 for an extensive review); the precision of these correlations is generally unknown and in fact, as Figure 5 demonstrates, may be rather imprecise (Stille & Palmström, 2003; Palmström &
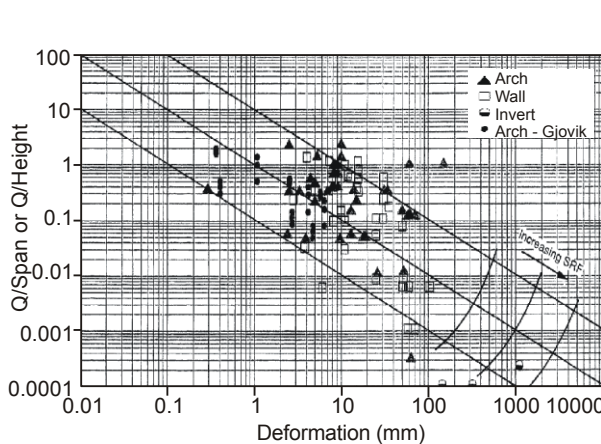
a) Deformation predicted from Q-value
(from Barton, et al., 1994)
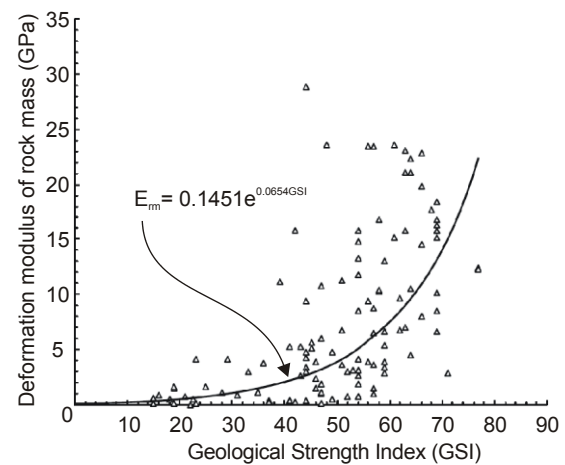
b) Relationship between GSI and deformation modulus of rock mass (after Gokceoglu et al., 2003)

Figure 5: Empirical correlation commonly used in rock engineering design.

Broch, 2006). Once again, one of the originators of the Q-system realises this limitation, which is evident through the statement: *"Q gives relatively simple correlations with parameters needed for design, due to the fact that rock masses also display a huge range of strengths, stiffnesses and degrees of stability or instability"* (Barton, 2002). Characterising the unpredictability that results from the use of such rock mass classification systems is discussed in further detail in section 2.8.1.

A final but significant example is a parameter that can be objectively measured, though the measurements are often sparse, imprecise or erratic; that parameter is k, which defines the ratio of the in-situ horizontal stress (σ_h) to the in-situ vertical stress (σ_v). In the absence of objective measurements, simple correlations based on empirical measurements are often utilised to estimate the in-situ horizontal stress from the vertical stress (see Figure 6). The vertical stress is often computed directly from the depth and density of the rock mass. Figure 6a suggests that such a relationship is valid, though there is a significant amount of scatter (variability) in the measurements. Figure 6b indicates the presence of clear bounds on the value of k, but a high degree of imprecision in intermediate values. Whilst the correlations in Figure 6b are global, site specific measurements of the parameter that defined in-situ stress ratio, k, also show a high degree of imprecision in its measurement, locally at any particular site (see e.g. Obara & Sugawara, 2003; Martin et al., 2003).

a) Correlation between vertical stress and depth   b) Correlation between in-situ stress ratio (k) and depth
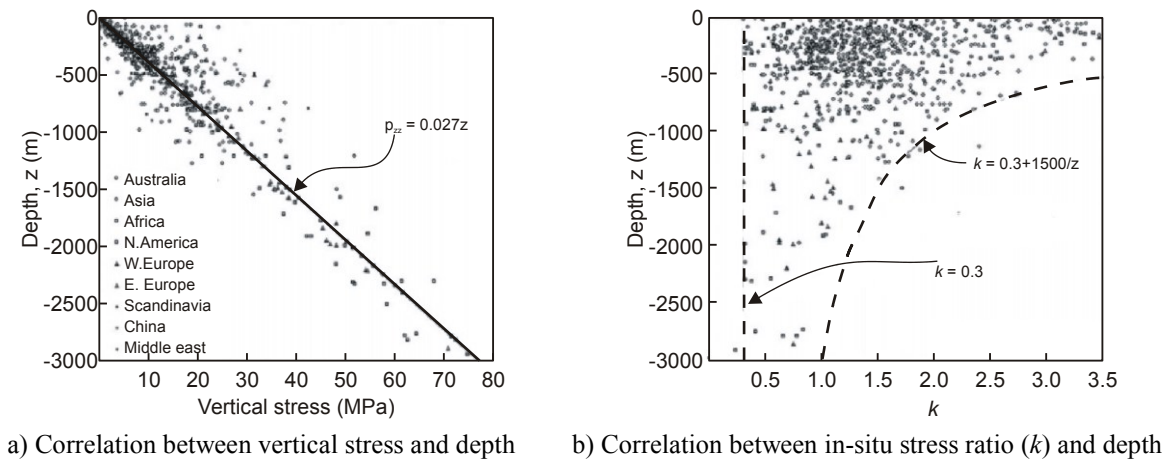
Figure 6: Various correlations between in-situ vertical and horizontal effective stress (after Brady & Brown, 2004).

From the discussion and examples presented thus far, it can be concluded that parameters used in rock engineering that fundamentally incorporate significant approximation or require subjectivity (e.g. expert judgement) in their derivation are qualitatively lacking information, and are therefore imprecise. Alternatively, a situation where the parameter in

question can be precisely measured though there is an insufficient quantity of data to fit a precise stochastic model, also constitutes imprecision. With respect to Figure 2 and Figure 3 presented earlier, it is evident that either a quantitative or qualitative lack of information means that the state of information can only fall in the region of 'imprecision' and therefore the parameter in question must be categorised as epistemic. This imprecision naturally leads us to the conclusion that a stochastic model – which incorporates a precise probability distribution as its basis – is not appropriate to characterise such epistemic uncertainty.

2.2.2 Uncertainty as a function of information type

The objective and subjective measurement of parameters used to characterise rock mass properties introduces various types of information resulting from the measurement process itself. The types of information can be broadly characterised as; numerical, linguistic, interval-valued or symbolic (Zimmermann, 2000), and each influence the state of information differently. Thus, a qualitative and quantitative assessment of each data type is required to determine whether the state of information can be characterised as imprecise or precise. In the following discussion, we explore the theory of measurement with respect to the data types that result from objective and subjective measurement. These are discussed with specific reference to rock mechanics and rock engineering.

In a seminal paper outlining the fundamentals of measurements, measurement is defined as: *"in the broadest sense, as the assignment of numerals to objects or events according to rules. The fact that numerals can be assigned under different rules leads to different kinds of scales and different kinds of measurement"* (Stevens, 1946).

Stevens (1946) advocates that measurement exists in a variety of forms and thus, scales of measurement fall in to distinct classes, which are determined both by the empirical operations invoked in the process of 'measuring' and by the mathematical properties of the scales. Stevens (1946) thus concludes that:

"the statistical manipulations that can be legitimately applied to empirical data depend upon the types of scales against which the data are observed. The type of scale achieved depends upon the character of the basic empirical operations performed. These operations are limited ordinarily by the nature of the thing being scaled and by our choice of procedures, but, once selected, the operations determine that there will eventuate only one or another of the following scales: nominal, ordinal, interval and ratio".

The scale levels with the appropriate operations for each are given in Table 1.

Table 1: Scales of measurement (after Stevens, 1946).

Scale	Basic Empirical Operations	Permissible statistics
Nominal	Determination of equality	Number of cases, mode, contingency correlation
Ordinal	Determination of greater or less	Median, percentiles
Interval	Determination of equality of intervals or differences	Mean, standard deviation, rank-order correlation, product-moment correlation
Ratio	Determination of equality of ratios	Coefficient of variation

Of the scales listed in Table 1, the nominal and ordinal scales are of particular interest to this discussion, which Stevens (1946) defines as follows: *"The nominal scale represents the most unrestricted assignment of numerals. The numerals are used only as labels or type numbers, and words or letters would serve just as well"*. An example of this is rock mass classification in terms such as 'fair', 'good', 'very good', where each class is assigned the same number; Class III, for all 'fair' rock, Class IV for all 'good' rock, and so on. Stevens (1946) defines the ordinal scale as that which *"arises from the operation of rank ordering"*. That is, the numerical information on an ordinal scale provides information only on the ordering of the measurement. A typical example is Moh's scale of mineral hardness.

When only nominal or ordinal data are available, conventional statistics such as mean and standard deviation are inappropriate. Indeed, Stevens (1946) states: *"...for these statistics imply a knowledge of something more than the relative rank order of data"*. Whilst it is acknowledged that pragmatically there may be some merit in computing such statistics, strictly speaking these computations will be in error to the extent that the successive intervals on the scale are unequal in size (Stevens, 1946).

According to Stevens (1946), a true quantitative assessment of data can only be made once one reaches an interval scale. Bárdosy & Fodor (2004) suggest that geological data may be categorised as quantitative, semi-quantitative and qualitative based on the amount of uncertainty in their measurement. By adopting Stevens' scales of measurement it is proposed that, of these scales, an aleatory model may only be applied to quantitative data that is derived from direct measurement; interval and ratio scales are included within this category. Semi-quantitative data includes imprecise interval or ratio, as well as ordinal data. An example of this type of semi-quantitative data is the empirical correlation for estimating the in-situ stress ratio, k. Data resulting from observations that are expressed linguistically should be categorised as qualitative; this group encompasses nominal data. It follows that both semi-

quantitative and qualitative data is epistemic, which needs to be analysed using alternative, non-stochastic methods

Engineers generally feel more confident when working with numbers rather than adjectives, as it is complicated to couple adjectives from different parameters when calculations are needed (Palmström & Broch, 2006). Consequently, rock mass parameters are often derived by assigning a numerical rating to a mixture of recordable observations – made in the field through visual comparison to exemplars, adjectives or descriptions – and measurements made either in the field or in the laboratory in an attempt to quantify them through some basic parameters. This process is an attempt by geotechnical engineers to map the various types of information into a numerical form, to which standard calculus may be applied. Typical examples are the rock mass classification systems mentioned in the preceding section, or the commonly applied Joint Roughness Coefficient (JRC) (Barton & Choubey, 1977) in which the joint roughness is estimated by comparing the appearance of a discontinuity surface with exemplar profiles and assigning it a numerical rating. This visual to numerical mapping of such parameters clearly implies a rank ordering of each input parameter, which by definition would declare such parameters ordinal and thus, according to Bárdossy & Fodor (2004), semi-quantitative. For example, the RMR classification (Bieniawski, 1989) assigns a numerical rating to six parameters in rank order considered 'favourable' to 'unfavourable' for tunnelling. However, a difficulty arises when one considers that, in a particular empirical scheme, the linguistic descriptions may be of nominal scale but require assignment of a numeric value for use in further calculations. It must be emphasised that assignment of a numerical value to such qualitative data does not automatically render them as increasing in scale (e.g. from nominal to ordinal).

From the discussion presented here, it can be concluded that it is important to correctly identify the scale of measurement appropriate to particular data, as this will both permit correct characterisation of the associated unpredictability and prevent application of incorrect mathematical methods in any subsequent calculations.

2.2.3 A nomenclature of epistemic uncertainty

Having considered lack of knowledge qualitatively and quantitatively, at this point two new definitions for characterising unpredictability are introduced. These definitions affirm that epistemic uncertainty can be further sub-characterised as being either **intrinsically** or **extrinsically** epistemic. The former represents parameters that are inherently imprecise and for which, no matter the quantity of information, the quality of data could not be improved to

reach a precise state. For these parameters it is inappropriate to assign a precise probability distribution, and this is validated through further discussion in section 2.5. The earlier examples of subjectively derived parameters and empirical rock mass classification systems fall in to this category. The latter represents those parameters for which such a distribution could be determined if the data could be refined from imprecise to precise values, or, if the data are precise, additional information deems the quantity sufficient to define an aleatory model. Thus, when sufficient information becomes available, an extrinsically epistemic property can be treated as an aleatory property, and modelled using stochastic methods. This definition of extrinsically epistemic uncertainty is further used in the discussion concerning stochastic methods for modelling unpredictability in section 2.6.

Many of the parameters commonly used in rock mechanics and rock engineering may either be intrinsically epistemic (i.e. the subjectivity or approximation in their measurement makes them imprecise), or extrinsically epistemic (there is a lack of information to quantify the aleatory characteristics). Specific examples of these differences are presented in section 2.8. Consequently, it is imperative to determine the cause of the unpredictability of a parameter or system prior to embarking on an analysis using a specific uncertainty model. The proposed taxonomy presented later in section 2.7 refers extensively to these new definitions of the sub-categories of epistemic uncertainty.

2.3 Aleatory variability

The introduction of this thesis put forward that aleatory variability is a result of inherent random variation related to natural processes and can be handled using stochastic methods. The often-cited, classic examples that epitomise aleatory variability are the rolling of dice, tossing of coins or sampling a particular trait (e.g. height of an individual) from a population. The outcome of each trial is the effect of chance and cannot be practically predicted. However, over a large number of trials, the variation will tend to follow some distribution – the stochastic model. The stochastic model, simply put, describes *"a system of countable events, where the events occur according to some well-defined random process"* defined over some domain (Vose, 2000; Baecher & Christian, 2003), which in geotechnical engineering is time (temporal variability, for example the variability in seasonal fluctuation of ground water level over a number of years) or space (spatial variability, for example the variation of properties such as uniaxial compressive strength with position). Based on these concepts, one can conclude that something being random implies complete unpredictability, except in the

relative frequencies with which it occurs (Baecher & Christian, 2003). That is, one cannot be sure of the true value of a parameter, rather merely best characterise it by a stochastic model.

The fundamental assumption embodied in the use of a stochastic model is that the total unpredictability of a parameter or system can be characterised by a precise probability distribution function (PDF), defined by its statistical moments (e.g. mean, standard deviation, etc.) (Walley, 1991; Walley, 1996; Sober, 2002; Ferson et al, 2003; Colyvan, 2008). A precise PDF is one for which any data value (i.e. the abscissa of the cumulative density function, CDF) can be determined with sufficient accuracy to allow a unique probability of occurrence (i.e., the ordinate of the CDF) to be assigned. In order to justify this assumption, the PDF must be objectively fitted to the data, using well-known statistical tests (e.g. Kolomogorov-Smirnoff or Chi-squared goodness-of-fit tests – see e.g. Davis, 2002; Ang & Tang, 2007). This demonstrates the objective nature of aleatory variability; characterisation of the parameter or system is not influenced by personal feelings or opinions in considering and representing facts, and so, "*aleatory variability possesses an objective reality that is independent of the level of empirical study*" (Ferson & Ginzburg, 1996). This substantiates the earlier assertion that aleatory variability cannot be reduced or eliminated by further data acquisition, i.e., it is inherent. In effect, if the type of distribution and the moments that define it are known perfectly, then the variability is known precisely. Collection of further information will not improve the calculated probability of occurrence of a value (Ferson & Ginzburg, 1996; Ferson, 2002; Baecher & Christian, 2003; Christian, 2004; Aughenbaugh & Paredis, 2006; Nadim, 2007; Moller & Beer, 2008). This idea is illustrated in Figure 7.
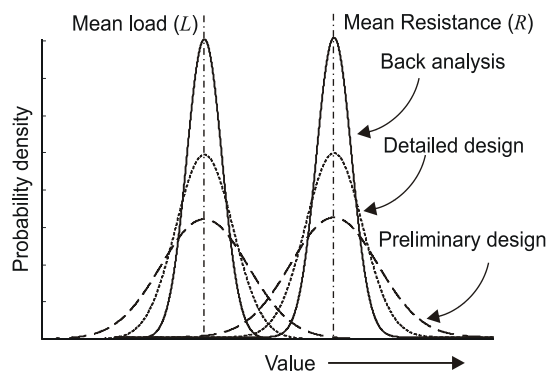
Figure 7: Updating the precision of an aleatory model with additional information. A limiting precision of variability will be reached at a given level of information (after Hoek, 1991).

The objective nature and irreducible property of aleatory variability demonstrates that it is very different from epistemic uncertainty. Consequently, it can be concluded that when

characterising unpredictability, there is a need to clearly differentiate between epistemic uncertainty and aleatory variability; the argument for this is presented in the next section.

2.4 Necessity for separating uncertainty and variability

The discussion so far has illustrated that uncertainty and variability possess very different characteristics. Epistemic uncertainty is due to a qualitative or quantitative lack of knowledge; it is subjective in nature and can be reduced by improving the level of information. On the contrary, aleatory variability is objective and requires precise information to define a stochastic model with which to characterise it. Furthermore, because it is due to randomness, it is inherent in the system and thus irreducible.

If epistemic uncertainty is characterised as if it is aleatory variability and then propagated through an analytical model, it would be impossible to see how much of the resulting unpredictability was due to uncertainty and variability, and that information is useful. If a large part of the unpredictability is known to be due to epistemic uncertainty then one knows that collecting further information that reduces epistemic uncertainty will significantly reduce unpredictability. On the contrary, as aleatory variability is the result of randomness, collecting additional data to refine the parameters that define it will not reduce unpredictability but only serve to improve the precision in the model (Vose, 2000; Christian, 2004). In general, the separation of uncertainty and variability allows us to understand what steps can be taken to reduce the unpredictability within a model and allows data collection to be focused on those aspects of the model that will benefit most from it. This is validated through one of the challenge problems presented in sections 7.1 and 7.2.

Perhaps the foremost reason for separating uncertainty and variability is that it is philosophically (Walley, 1991; Mayo, 1996; Walley, 1996; Zimmermann, 2000; Sober, 2002; Swinburne, 2002; Ferson et al., 2003; Tucker & Ferson, 2003) and mathematically (Ferson & Ginzburg, 1996; Vose, 2000; Ferson et al., 2004; Baudrit & Dubois, 2006; Rinderknecht et al., 2012) more correct. However, it has been suggested that adoption of a subjective, or Bayesian, view of probability allows epistemic uncertainty be analysed using stochastic methods (Jeffreys, 1961; Lindley, 2000; Howson, 2002; Jaynes & Bretthorst, 2003; Ang & Tang, 2007; Aven & Steen, 2010). This 'Bayesian approach' uses expert opinion to subjectively assign a precise PDF to any analysis, and although popular, the presence of much philosophical argument suggests disagreement regarding its validity. One forthright example is the statement *"many of the hypotheses of interest to science do not have objective prior probabilities"* (Sober, 2002). Therefore, it is questionable whether statistically meaningful

PDFs can be used when the state of information is imprecise (Guyonnet et al., 1999); their use would in fact introduce information on probabilities of occurrence which are not actually available. This approach of wrongly characterising imprecision using an aleatory model can significantly bias the results of any analysis in a non-conservative or inefficient manner (Ferson & Ginzburg, 1996). Indeed, there is increasing evidence which supports the argument that subjective assignment of a PDF can lead to misinformed decisions, dissonance amongst experts and even potentially erroneous results (Klir, 1989; Klir & Yuan, 1995; Tonon et al., 2000; Ferson & Ginzburg, 1996; Vose, 2000; Ferson et al., 2004; Baudrit & Dubois, 2006; Rinderknecht et al., 2012). Consequently, the literature recognises that non-stochastic characterisation methods that explicitly incorporate imprecision are required for those parameters that cannot be objectively measured (Walley, 1991; Dubois & Prade, 1988; Zimmermann, 2000; Ferson & Ginzburg, 1996; Baudrit & Dubois, 2006; Dubois, 2006; Helton et al., 2004; Dubois & Guyonnet, 2011).

The following section introduces various non-stochastic approaches that are appropriate for representing epistemic uncertainty. Following this, section 2.6 discusses the basis of classical, or frequentist, and Bayesian probabilistic methods, respectively. This discussion is thus able to show that the characteristics of epistemic uncertainty require a non-stochastic method for its characterisation. Conversely, stochastic methods are only appropriate once the very specific characteristics that define aleatory variability have been met.

2.5 Non-stochastic methods for modelling uncertainty

As epistemic uncertainty is typified by imprecision, it follows that precise probability distributions are inappropriate to characterise it. It is now widely recognised that imprecision is best represented by intervals and their generalisations, rather than precise probability distributions (Cooper et al., 1996; Ferson & Ginzburg, 1996; Baudrit & Dubois, 2005; Baudrit & Dubois, 2006; Baudrit et al., 2007; Dubois & Prade, 2009; Dubois & Guyonnet, 2011). Consequently, several interval-oriented uncertainty theories have been developed that explicitly handle imprecision. These include: interval analysis (Moore, 1966; Moore & Bierbaum, 1979), possibility theory (Dubois & Prade, 1988), which incorporates fuzzy numbers (Zadeh, 1965; Kaufmann & Gupta, 1991), and the theory of imprecise probabilities (Williamson & Downs, 1990; Walley; 1991), which uses p-boxes to represent imprecision (Tucker & Ferson, 2003). All of these are discussed in detail in Chapter 3.

In the context of uncertainty in rock engineering, Wenner & Harrison (1996) introduced the 'level of information' concept and suggested that that for any given amount of knowledge and hence uncertainty there is an optimal model that should be applied (see Figure 8) and for each modelling approach shown in Figure 8, there is a particular amount of information required. The lowest amount of information is associated with an uncertain parameter for which there is only a single value available. As more information becomes available, so higher modelling approaches can be applied.
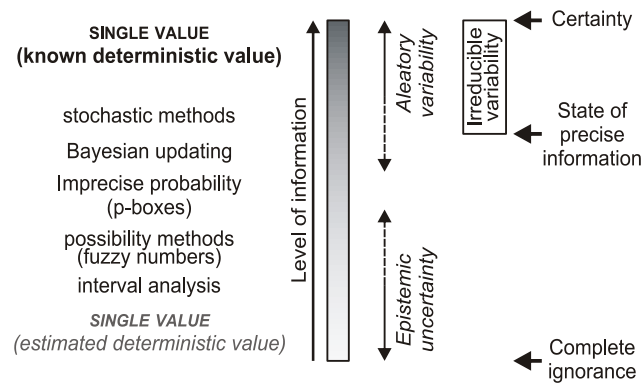
Figure 8: Uncertainty models and the level of information concept (after Wenner & Harrison, 1996; Aughenbaugh & Paredis, 2006; Guo & Du, 2007; Bedi & Harrison, 2013a).

Figure 8 also shows that only the interval-oriented methods are applicable when the state of information is imprecise. The motivation for this is that these interval-oriented theories have been developed to provide new tools to faithfully and robustly characterise and propagate imprecision (Ferson & Ginzburg, 1996; Baudrit et al., 2005; Ferson, 2002; Dubois & Guyonnet, 2011). Doing so allows a decision to be made based on an assessment of the complete unpredictability. The discussion that follows first examines the need for faithfulness and robustness in any analysis, but especially those situations where the unpredictability is dominated by epistemic uncertainty. We are thus able to draw conclusions on the necessity of applying interval-oriented uncertainty models to characterise and propagate epistemic uncertainty. The mathematical bases for these interval-oriented methods are discussed in detail in section Chapter 3.

2.5.1 Faithfulness

Dubois (2010) defines a 'faithfulness principle' that suggests, when faced with characterising epistemic uncertainty, one should select a suitable interval-oriented uncertainty model commensurate with the level of information available. This is in contrast to the Bayesian view, which purports subjective assignment of a precise PDF even in the absence of any

objective information. The latter approach would arguably be misrepresenting the available information and in fact introduce information on probabilities of occurrence that are not actually available. Thus, faithfulness can be seen as the pursuit for consistency with available information (Dubois & Guyonnet, 2011). This is demonstrated with a very simple example based on the work of Ferson & Ginzburg (1996).

Suppose we want to compute the product AB of two parameters for which the only information we have is: A lies somewhere between 2 and 4, and B somewhere between 3 and 5. If we characterise A and B as intervals and compute the product using interval analysis (discussed in detail in section 3.1), the result is another interval [6, 20]. Figure 9a shows the smallest region guaranteed to contain the cumulative distribution of the product AB, which this interval represents. Alternatively, if we were to characterise A and B as uniform probability distributions – as one would be required to when following the Bayesian doctrine – an exact solution using probabilistic convolution (Ang & Tang, 2007) or a Monte-Carlo strategy can be applied to estimate the distribution of the product AB, the result of which is shown Figure 9b. This figure clearly shows a concentration of probability near the geometric centre of the output interval. Additionally, the cumulative probability calculated from such an analysis allows one to make precise statements about the probability of occurrence of specific values. For example, based on Figure 9b, we could state that there is a 95% probability that the product of A and B will be less than 17.3. However, nowhere in the information provided to characterise A and B is it stated that there is a preference towards any value of A or B, nor is there any evidence to suggest anything about their variability. Thus, by using a precise PDF for A and B we have in fact introduced information that we never had. On the contrary, the interval analysis faithfully propagates the imprecision in the

a) Region representing the interval [6, 20], guaranteed to contain the cumulative distribution of the product AB

b) PDF of the product of two uniform distributions A and B obtained using a Monte-Carlo simulation with 5000 iterations
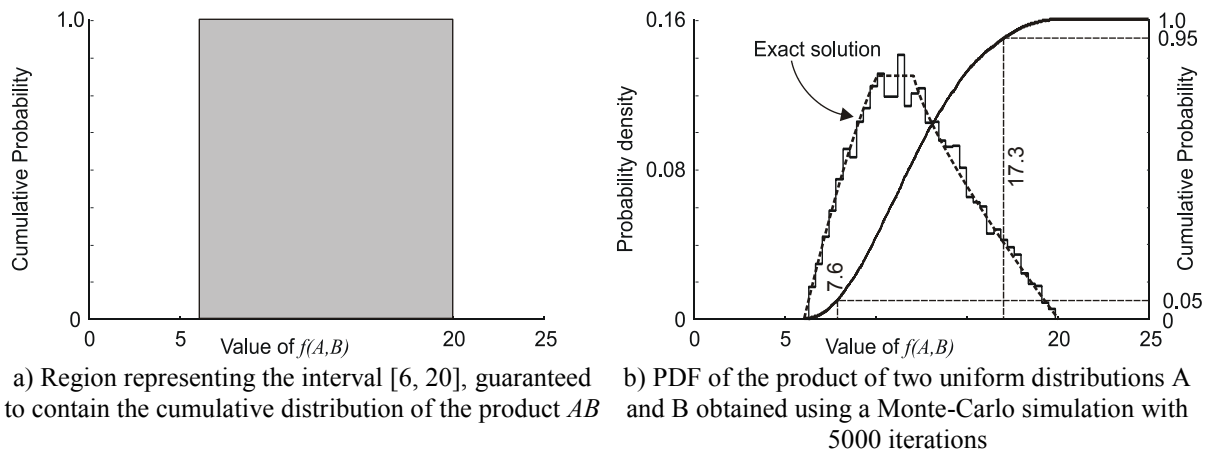
Figure 9: Comparison of interval versus probabilistic output from only bounds as an input.

output, from which we can state nothing more except that the product AB lies somewhere in the shaded region of Figure 9a.

2.5.2 Robustness

The robustness of interval-oriented uncertainty methods demands that so long as the intervals forming the inputs bound the true value of the parameters they represent, the output is also guaranteed to bound the true result (Ferson, 2002; Ferson & Hajagos, 2004). This is not necessarily the case when applying stochastic modelling techniques (Guyonnet et al., 1999; Vose, 2000). For example, when using Monte-Carlo type simulations, scenarios that combine low probability parameter values have very little chance of being randomly selected (Guyonnet et al., 1999), as is demonstrated by the following example. Let us now assume that we have a further two parameters C equal to [4,6] and D equal to [5, 7], and we wish to compute the unpredictability in AB/CD. Figure 10a shows the area that results when the inputs are represented by intervals, the bounds of which are [0.14, 1]. Figure 10b presents the results of a Monte-Carlo simulation in which the four inputs parameters are characterised by uniform random variables. In this figure, the bounds are between 0.17 and 0.85; approximately 15% and 20% from the actual upper and lower bound values, respectively. At this point, we note that an exact solution using probability convolution would correctly bound the answer. However, for all but the simplest functions of random variables the exact solutions are notoriously difficult, if not impossible, to compute. Thus, numerical methods such as Monte-Carlo simulation are all but always used (Davis, 2002; Ang & Tang, 2007).

a) Region representing the interval [0.14,1.0], guaranteed to contain the cumulative distribution of *AB/CD*

b) Monte-Carlo simulation of *AB/CD*
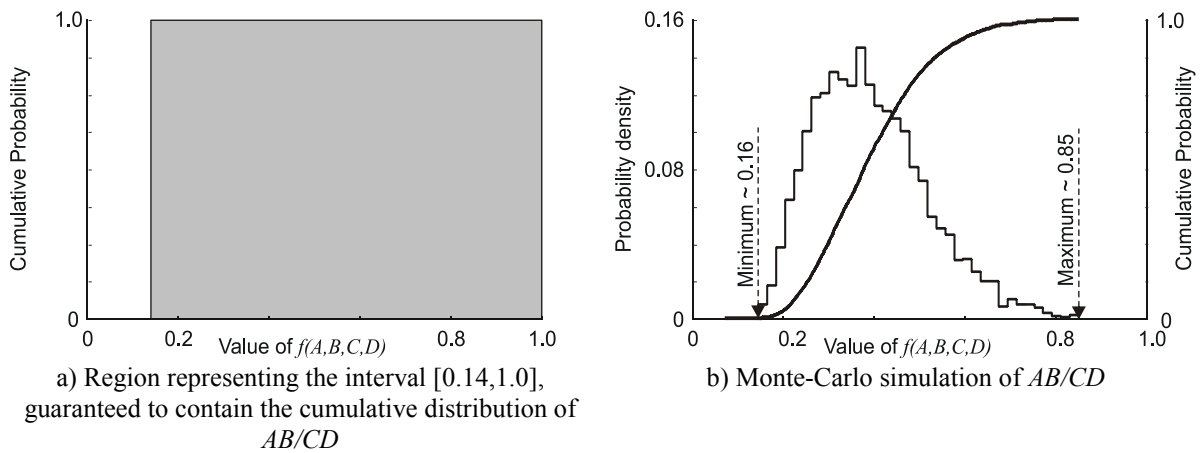
Figure 10: Comparison of interval and Monte-Carlo simulation involving further arithmetic manipulations.

The corollary of robustness is that the output intervals get wider as more arithmetic manipulations are applied, or the number of input parameters is increased. This widening of

the output can lead to difficulty in decision making (Helton et al. 2010). Consequently, interval analysis is sometimes criticised as suffering from 'hyper-conservatism' (Ferson, 2002). Conversely, the bounds of the Monte-Carlo simulation will shrink away from the bounds and towards the mean as more mathematical operations are undertaken, which can lead to unconservative or inefficient decisions (Guyonnet et al., 1999; Vose, 2000). From a risk minimisation perspective, and especially in a situation where data are scarce, the possibility of the 'worst case' has important implications to the design decisions and thus an approach which robustly reflects all possibilities seems more appropriate.

2.5.3 Decision making

The Bayesian approach requires definition of a subjective PDF prior to the analysis, which results in a precise output on which to base a decision. However, when using interval-oriented uncertainty methods, the subjective decision takes place at the end of the analysis process when no further collection of information that might reduce epistemic uncertainty is possible (Dubois & Guyonnet, 2011). Due to the imprecision in the inputs of an interval-oriented approach, the output is also imprecise and in interval form. Thus, one of the major criticisms of interval-oriented uncertainty models is the problem in decision making.

As the outputs of interval-oriented uncertainty models do not specify a single measure of (un)certainty on the selection of any one value, it may be hard to make a decision when the output is a wide interval (Helton et al. 2010). However, there is strong argument to support the notion that if a subjective decision cannot be made at the end, the level of knowledge is clearly insufficient to make a critical decision (Ferson & Ginzgburg, 1996; Beer et al., 2013). As the level of knowledge has remained unchanged from the gathering phase to the decision making stage, it follows that the level of knowledge must have been insufficient to assign a precise probability distribution in the first place, and as will be shown through an example in section 5.2 , the results of such analysis can only lead to the conclusion that further data collection is required.

In fact, a wide output from an interval-oriented uncertainty model contains vital information about unpredictability: it informs the analyst or designer about the lack of knowledge, and specifically what he or she does not know. This critical information is masked by the precise distribution that results from adopting a Bayesian approach. Indeed, Dubois (2004) recognises the importance of faithfulness and robustness in decision making, stating that wide output from interval-oriented methods allow a decision maker to " *...know when he* (or she) *actually doesn't not know enough about the phenomenon under study. It is*

better to know that you do not know, than make a wrong decision because you delusively think you know. It allows one to postpone such a wrong decision in order to start a new measurement campaign, for instance" (Dubois, 2004).

2.6 Stochastic methods for modelling variability

Having discussed the non-stochastic methods appropriate for modelling uncertainty, with respect to the Level of Information concept introduced earlier in Figure 8, the following sub-sections now reviews stochastic methods that can be applied to model variability. Whilst probability theory forms the basis for modelling unpredictability in all stochastic methods, the interpretation of probability is not universal; it can be categorised into two schools, the frequentist and subjectivist – or Bayesian – view. Here, both interpretations of probability are presented. This section also reviews the implication of modelling unpredictability in rock engineering using each of these views of probability with respect to the faithfulness principle and robustness introduced above.

2.6.1 Frequentist or classical probability

The frequentist approach is perhaps the most commonly understood notion of probability and assumes an event is the result of a random process that can be realised by repeating an experiment – in our case, perhaps a site or laboratory test – a large number of times and plotting the number of times each outcome occurs. The variability in the results is characterised by one of the well known probability distributions, fit to the data using various statistical tools and accepted on passing a number of hypothesis tests (e.g. Kolmogorov-Smirnov goodness-of-fit test) (Davis, 2002; Fellin et al., 2005; Ang & Tang, 2007). The frequentist view of probability can accordingly be seen as an objective approach. In this thesis, it is this definition of probability that is adopted for aleatory variability.

Many rock mechanics properties have been shown to follow stochastic distributions; in this thesis, such properties are defined as intrinsically aleatory. Well known examples include intact rock strength (Yamaguchi, 1970; Ruffolo & Shakoor, 2009) (see Figure 11 on next page), the modelling of discontinuity spacing (Priest & Hudson, 1976) and discontinuity orientation (Priest, 1985). Further examples are discussed in detail in section 2.8.4.

With reference to Figure 3, which previously defined unpredictability as a function of the quality and quantity of information, it can be concluded that the frequentist probability model is that which is best suited to characterise the unpredictability in rock mass parameters that can be objectively measured with sufficient precision such that the quality and quantity of
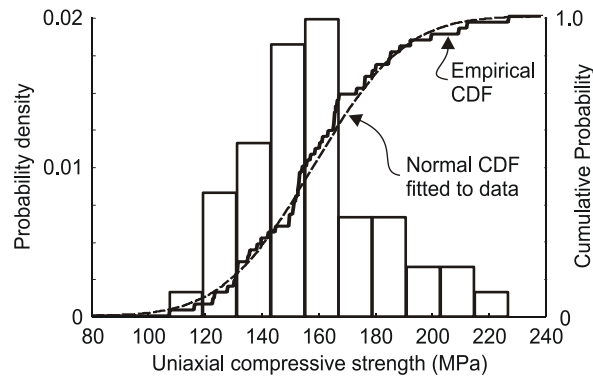
Figure 11: Normal distribution associated with uniaxial compressive strength of intact rock – Milbank granite (data from Ruffolo & Shakoor, 2009).

information is precise. However, one philosophical problem with this approach is that it is not always practical to obtain a sufficiently large data set, from which to fit a representative probability distribution. In such a case, the parameters must be classified as extrinsically epistemic (as defined previously in section 2.2.3) and characterised using alternative, appropriate means until sufficient data become available to formulate an aleatory model. A second problem in adopting the intrinsically aleatory assumption is the implication that the engineer or modeller has sufficient knowledge or data available to validate the statistical assumptions encapsulated by the definition of a probability distribution. For example, how does one fit and justify a precise PDF to characterise the unpredictability in a parameter (e.g. GSI) where the only information is two interval estimates of it, say [30,40] and [45,50], one of which has been obtained from prior experience and the other from the opinion of an expert? As was previously shown by a few examples presented in section 2.2.1 , many parameters used to quantify rock mass properties are deduced entirely in this subjective manner (e.g. JRC, GSI etc.). Evidently, the frequentist approach cannot be applied to such parameters, which were termed intrinsically epistemic (see section 2.2.3). For this reason, the degree-of-belief – or Bayesian – approach to uncertainty has been suggested as a means to amalgamate uncertainty and variability using subjective probabilities and expert judgement.

2.6.2 *Subjectivist probability: the Bayesian approach*

The Bayesian approach interprets probability as a subjective measure of confidence – one's degree-of-belief – in the available information (Davis, 2002; Fellin et al., 2005). Bayesian scholars attest that both aleatory variability and epistemic uncertainty (i.e. total unpredictability) should be handled in a probabilistic framework. The Bayesian approach requires an unpredictable parameter to be modelled as a random variable (i.e. with a precise

probability distribution) defined using prior knowledge, expert opinion and any objective information, no matter how little, which may be available. This is known as the 'prior' probability distribution. The Bayesian approach can then be applied in two ways: (1) as additional information becomes available, the prior distribution is modified formally using Bayes's Theorem (the method is detailed in section 3.5) to produce an updated, or 'posterior', probability distribution, or in the absence of any objective information; (2) the total unpredictability is defined subjectively by the prior PDF and propagated using statistical methods (e.g. Monte-Carlo simulation), the output of which is another precise PDF that provides a basis for decision making and formulating design(s) (Ang & Tang, 1984; Ang & Tang, 2007).

When using the 'Bayesian updating' approach, the priors are continually updated as further objective information becomes available, which may be during the subsequent investigation or construction phase(s) of a project. In this way, if sufficient objective information becomes available, with continued updating, the Bayesian probability model will tend to the frequentist model. This updating process is somewhat analogous to the 'observational method' (Peck, 1969) commonly employed in tunnel engineering. That is, a design is prepared based on a prior knowledge and updated as excavation progresses, and detailed information on the ground conditions becomes available through observation and/or measurement.

A recent example shows Bayesian updating being used to determine the elastic modulus (E) of a fractured rock mass in which the Venda Nova II, Portugal, hydroelectric power plant is constructed (Miranda et al., 2009). In this analysis, background field and laboratory test data suggested that various geotechnical parameters at the site could be characterised by either truncated normal or lognormal distributions (the priors), however there was no specific information on the expected distribution of E . The analysis considered the parameters that define these two 'priors' as random variables, and it was these that were updated. The updating was performed using in-situ test data obtained from large flat jack (LFJ) tests in exploration adits close to the main cavern. Figure 12 presents both priors, and updated posterior distributions of E . This figure shows the convergence of both solutions towards each other with updating based on the LFJ test data. Whilst this demonstrates the strength of the Bayesian approach, a key question in this analysis, and all similar analyses, is how to select the prior distributions. In the absence of any objective information on the frequencies of probable values, the Bayesian approach demands that 'non-informative priors' be used (Ang & Tang, 2007). The reasoning behind this can be traced back to Laplace's
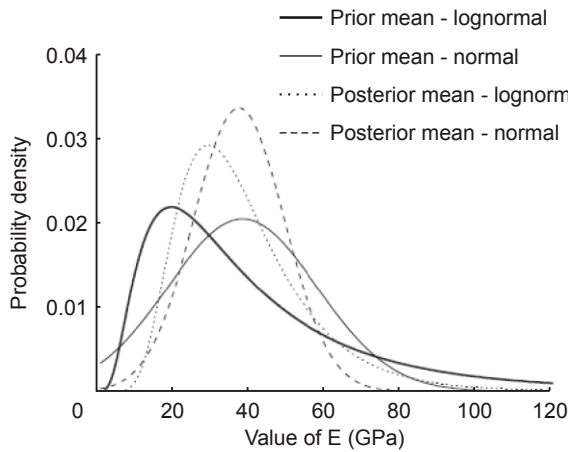
Figure 12: Prior and posterior distributions of the mean obtained from normal and lognormal priors (after Miranda et al., 2009).
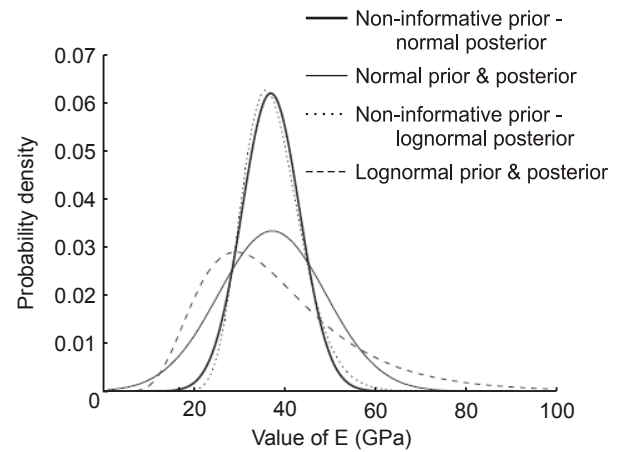
Figure 13: Posterior distributions obtained from normal, lognormal and 'non-informative' priors (after Miranda et al., 2009).

Principle of Insufficient Reason, which suggests that the unpredictability be characterised by a uniform distribution (Jeffreys, 1961; Baecher & Christian, 2003; Ang & Tang, 2007). Whilst this may seem a logical choice, it has been shown (Ferson, 1996; Ferson & Ginzburg, 1996; Ferson, 2002; Ferson & Hajagos, 2004) that the shape of the output distribution is extremely sensitive to that of the inputs. This is further demonstrated in Figure 13 for the case study of Miranda et al. (2009).

This figure shows that the means of the posterior distributions, updated using the same objective evidence but based on different priors, have similar mean values but the variances are not in close agreement. The conclusion to be drawn from this example is: to faithfully propagate information through a Bayesian analysis, the priors should be formed when there is a strong basis for such judgement, i.e. the data must be extrinsically epistemic. Verbraak (1990), in his essay 'The logic of objective Bayesianism', supports this conclusion and refutes the subjective estimation of priors, including the Laplacean approach of automatically falling back to non-informative priors in the absence of any objective information. The reasoning given is simply that these approaches assume that unpredictability of the property in question is already known to be a result of aleatory variability. This is exemplified by Verbraak (1990) in stating that the subjective Bayesian approach is often (justifiably) used in industries such as motor insurance where *"the statistics of the whole portfolio are known for certain already. The insurer then tries to particularise via a bonus-malus system according to the probable individual risk levels"*. This is analogous to the framework of exchangeable bets on which the subjective Bayesian approach is based (Dubois, 2006).

However, in rock mechanics and rock engineering, the existence of variability in a property that a parameter defines is not conclusive. For example, the variability in the condition of discontinuities cannot be defined when it is characterised using the subjective method of measurement required by the RMR classification. Given the ordinal nature of this measurement, it is questionable whether the variability in this property could ever be defined. In fact, and as will be shown through examples in section 2.8.4, definition of priors based on well known precedence can only be applied to but a few rock mass properties.

2.6.3 *Faithfulness and robustness*

Using the frequentist approach to probability, discussed previously in section 2.6.1, the variability in the objective data can be visualised by plotting a histogram, to which a PDF can then be fitted. Figure 14 shows a histogram of data to which two different PDFs have been fitted, both of which appear to adequately characterise the variability in the data. In order to establish the best fit, and thus reduce subjectivity, the choice of the PDF to define the data should be established by well known statistical goodness-of-fit tests (Davis, 2002; Fellin et al., 2005; Ang & Tang, 2007). Evidently, this objective approach of fitting an aleatory model to the available data obeys the faithfulness principle defined earlier in section 2.5.1. That is, given the same data, two observers will arrive at the same, or very similar, PDFs to characterise variability, which in turn will lead to more consistency in the results of any analyses upon which they are based. Hence, decisions based on the output of any analyses through which these are propagated will also be similar. This approach is thus considered to be both faithful and robust to the available information. However, the same cannot be immediately said when adopting the Bayesian approach and associated subjectivist view to probability. As was discussed in the preceding section, this is especially the case when a precise prior PDF is assumed without evidence to support such a hypothesis.
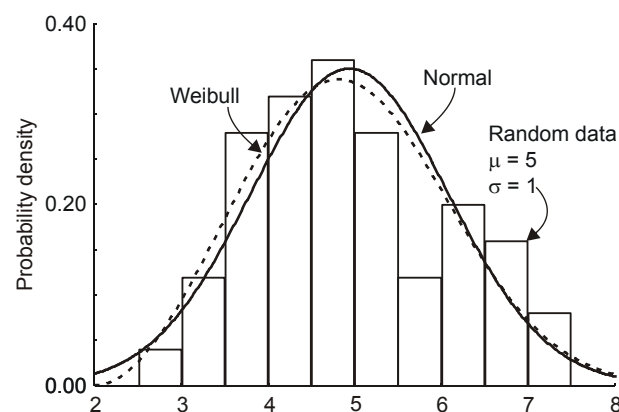
Figure 14: Two probability density functions overlain on a histogram of objective data.

Based on the example of Miranda et al. (2009) presented in the preceding section (2.6.2), it was concluded that a Bayesian updating approach would faithfully propagate unpredictability if the parameters in question are extrinsically epistemic, the prior can be objectively formed and sufficient objective data becomes available to update the priors such that the posteriors converge towards an aleatory model. However, section 2.6.2 also established that the second application of the Bayesian approach advocates that, even in the absence of any objective information, the unpredictability can be represented by a precise PDF and propagated using conventional probabilistic analysis. However, and as will be shown here by example, this latter approach does not faithfully or robustly propagate epistemic uncertainty.

Consider the following scenario: A tunnel is to be excavated in a rock mass with the support design determined using the Q-system (Barton et al., 1974), which can be calculated using: $Q = (RQD/J_n) \times (J_r/J_a) \times (J_w/SRF)$ (see section 6.1 for a further description). Field investigation in the form of geological mapping in the vicinity of the tunnel alignment has been undertaken by an expert geologist, who has returned the following description of the rock mass in question:

'The rock mass is classified as 'good' (RQD = 75-90) with one to two joint sets (J_n = 2-4) present. The joint roughness varies between discontinuous, rough, irregular and undulating (J_r = 2-4). The joint wall surfaces are tightly healed, hard, non-softening to unaltered with surface staining only (J_a = 0.75-1). Previous tunnelling experience in this rock mass indicates that the excavation may encounter minor inflow, i.e. < 5 l/m locally, to occasional medium inflow or pressure (J_w = 0.66-1). The in-situ stresses are expected to be between low and medium (SRF = 1.0-2.5).'

In accordance with the discussion presented in section 2.2.2, it is immediately apparent that the data provided are linguistic but have been mapped, by the geologist, in to numeric form using the descriptors provided by the Q-system. This subjective means of measurements and assignment of numerical ratings to observations introduces a mixture of nominal and ordinal data; the numerical ratings are semi-quantitative. Therefore, the information is both quantitatively and qualitatively insufficient to define a precise PDF; the state of knowledge is clearly imprecise and thus the unpredictability in these parameters is due to epistemic uncertainty. Most importantly, there is no information contained in the geologists' statement that would allow one to assign probabilities of occurrence for any of the

parameters. However, in keeping with the Bayesian approach – applying the principle of indifference – we adopt non-informative priors for all the input parameters and calculate the Q-value using a Monte-Carlo simulation with 5000 iterations. The output of expected Q-values is shown in Figure 15. This figure allows the following deductions to be made: 'The minimum and maximum likely values of Q are 14 and 160, respectively. The mean value is expected to be 50.' In fact, Figure 15 allows us to make much more informed statements, such as: 'there is a 95% probability that Q will be less than 91 and a 5% probability it will be less than 24'. However, the initial information does not mention anything about preference or probabilities one way or the other. In fact, given the paucity of the information, the only justifiable statement one could make would be based on calculating the interval which bounds all possible values of Q, which is [9.9, 240].

From this example, we can conclude that using a subjective Bayesian approach actually introduces information and fails to actually bound all the possible values; this goes against both faithfulness and robustness.
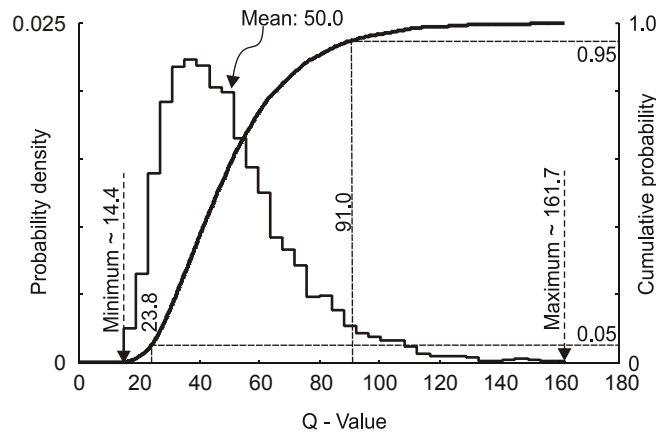
Figure 15: Output of Monte-Carlo simulation to calculate Q-value.

2.6.4 Decision making

The example in the preceding section, which used a subjectively defined precise prior PDF to characterise and propagate epistemic uncertainty, demonstrated that the output is neither robust nor faithful to the input information. Thus any decision formulated based on the bounds of this output may be un-conservative or inefficient. Secondly, a design based on statistical measures extracted from the output PDF is erroneous in the sense that it has introduced a bias towards a specific value. This bias is not because a probabilistic analysis has been adopted, rather because a precise PDF of a defined shape has been adopted to characterise epistemic uncertainty (Ferson & Ginzburg, 1996). The shape of the distribution
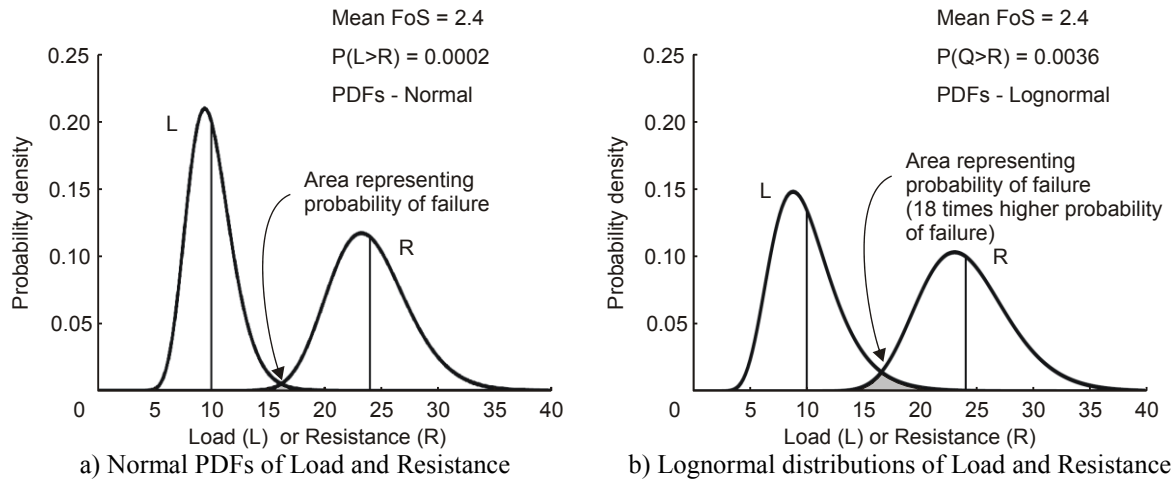
Figure 16: Effect of shape of PDF on calculated probability of failure. Both figures have the same mean factor of safety (FoS=R/L) (after Naghibi, 2010).

reflects the subjective views of the analyst defining it. As will be shown by the challenge problem in Chapter 5, the choice of the shape of the prior has a significant influence on the output.

When probability distributions are used to make decisions in engineering design, it is usually the tails that govern. Thus it is critically important to recognise that the tails of the posterior PDFs are extremely sensitive to information about the shapes and dependencies of the priors (Soundappan et al., 2004; Oberguggenberger & Fellin, 2008). As these tails give the probabilities of extreme events, ensuring the safety and efficiency of engineering structures demands a precise assessment of them. Figure 16 shows how the predicted probability of failure can vary significantly as the shape and variance of the distributions of load and resistance also vary. This figure confirms that the choice of probability distribution, even among the standard types in use, has dramatic effects on the predicted probability of failure or occurrence and consequently two experts may arrive at vastly different conclusions if the priors are not objectively determined (Verbraak, 1990; Christian et al., 1994; Sober, 2002; Fellin et al., 2005). It is the author's view that many proponents of Bayesian techniques do not pause to consider this issue, instead regarding the Bayesian approach – essentially dogmatically – as the natural way to handle epistemic uncertainty (e.g. Walley, 1991; Rinderknecht et al, 2012).

For these reasons, this thesis supports use of the Bayesian updating approach, with objectively assigned priors, to tackle problems involving extrinsically epistemic parameters (as defined earlier in 2.2.3). That is, parameters that are intrinsically aleatory however, at the time of analysis and design, insufficient quality of data is available on which to formulate an

objective probability distribution using the aforementioned statistical procedures. The prior probability distribution may be formulated from subjective information or expert opinion; however, its selection should be justified through prior evidence. As more information becomes available, the design is updated via Bayes's Theorem and at completion will converge towards an optimal output that may have been produced originally, had sufficient information initially been available to characterise the parameters and model using a frequentist approach from the outset. However, in the presence of intrinsically epistemic uncertainty, or where additional information is not likely to become available, the statistical basis of the Bayesian approach is not robust or faithful to the available information, and hence inappropriate. For example, as opposed to the frequentist view, given the same information, two experts are likely to come up with different subjective prior distributions and outputs. At this stage, the question could be raised: 'which expert should I believe?' The definitive answer to this would require objective measurements to confirm the correct distribution of the input parameters, by which juncture an expert opinion would not be required. The Bayesian answer to this is to revert to adopting a 'non-informative prior' in the absence of any objective information. However, adopting said 'prior' and propagating the analysis using standard probability calculus results in a bias (Hoffman & Hammonds, 1994; Ferson & Ginzburg, 1996; Tonon et al., 2000; Ferson, 2002), and more to the point, introduces information that was not available at the outset (refer to earlier discussion in section 2.6.2). Most fundamentally, and as was shown in section 2.5, precise probability distributions are inappropriate for intrinsically epistemic parameters which are inherently imprecise.

2.7 Proposed taxonomy

The preceding discussion showed that the total unpredictability of a parameter or system is an accumulation of its components: aleatory variability and epistemic uncertainty. Section 2.3 and 2.6 identified aleatory variability as due to the inherent random variability of a parameter or system, which may be characterised by precise stochastic models that allow the use of powerful mathematical tools – probability theory, in particular – to bear on a problem that may otherwise be difficult to address. It is objective in nature and applicable to characterise random events in the form of a frequency of occurrence in a long series of similar trials. That is, two observers, given the same evidence and enough of it, should converge to the same numerical value for this frequency of aleatory variability.

Epistemic uncertainty is subjective by definition, because it is a function of the assessor's level of knowledge (Vose, 2000). As illustrated earlier by Figure 3, a parameter or

system must be characterised as epistemic if the quality or quantity of data renders the level of information imprecise. It may be reduced through improving both the quantity (amount) and/or quality (precision) of information. If additional quantitative or qualitative information is obtained, it may become justifiable to characterise the uncertainty as variability, i.e. apply an aleatory model. Once an acceptable aleatory model has been developed, additional investigation will not reduce the variability but may increase the precision of the parameters that describe it (Christian, 2004). As stated in section 2.2.3, such uncertainty is defined by the new term extrinsically epistemic.

Many parameters used to characterise material, or other, properties in rock mechanics are defined qualitatively or quantified entirely subjectively through expert judgement. The reliance on such subjective measurements suggests that while the underlying property or process may be the result of a random process, dissonance and approximation resulting from the subjective method used to characterise the variable means – irrespective of the amount of additional information or expert consultation – the type of information will always remain imprecise. Furthermore, in rock engineering, empirical parameters are routinely used in engineering calculations. Such parameters are derived through approximate correlations with field evidence (e.g. rock mass classification systems) and thus contain fundamental approximation and imprecision in their genesis. Parameters or systems displaying this form of uncertainty are termed intrinsically epistemic. It thus logically follows that such parameters are not amenable to characterisation using stochastic models – or propagation using the associated probabilistic analysis – which are suitable only for parameters exhibiting aleatory variability.

All of these concepts presented so far and these key characteristics of epistemic uncertainty and aleatory variability are presented in the proposed taxonomy of Figure 17. The key purpose of this new taxonomy is to allow an objective means of differentiating between epistemic uncertainty and aleatory variability. The failure to do so has been the source of much confusion in geotechnical engineering. To correct this, Figure 17 is organised in to two parts that allow the reader to characterise the total unpredictability through scrutinisation of the available data, both quantitatively and qualitatively, with respect to all the concepts introduced thus far.

Figure 17a presents the new taxonomic terms: intrinsically epistemic, extrinsically epistemic and aleatory. The characteristics that define each are listed below each, with respect to quantitative and qualitative assessment of information. Figure 17a also suggests appropriate unpredictability models with respect to the level of information concept (i.e., Figure 8

introduced previously in section 2.5) for each of the three sub-classifications of unpredictability. For instance, interval arithmetic (Moore, 1966) has been suggested as the basic calculus to propagate intrinsic epistemicity when the level of knowledge is at a minimum. For situations in which the uncertainty about quantities is purely aleatory in character, probability theory is usually preferred.

TAXONOMIC TERMS		INTRINSICALLY EPISTEMIC	EXTRINSICALLY EPISTEMIC	ALEATORY
QUANTITY OF INFORMATION	*Cause*	Lack of information		Random variability
	Amount of information	No data or purely subjective data	Insufficient objective data	Sufficient objective data
QUALITY OF INFORMATION	*Type of information*	Qualitative (Linguistic, symbolic)	Semi-quantitative data (*Imprecise* numerical)	Quantitative (*Precise* numerical data)
	Type of measurement	Subjective assessments	Objective measurements	Objective measurements
	Scale of measurement	Nominal, ordinal	Ratio, interval	Ratio
UNPREDICTABILITY MODEL	*Appropriate calculus*	Intervals, fuzzy numbers	P-box, Bayesian updating	Stochastic model/Probability distribution
		EPISTEMIC UNCERTAINTY		ALEATORY VARIABILITY
		TOTAL UNPREDICTABILITY		

a) Taxonomic terms – characteristics of uncertainty and variability
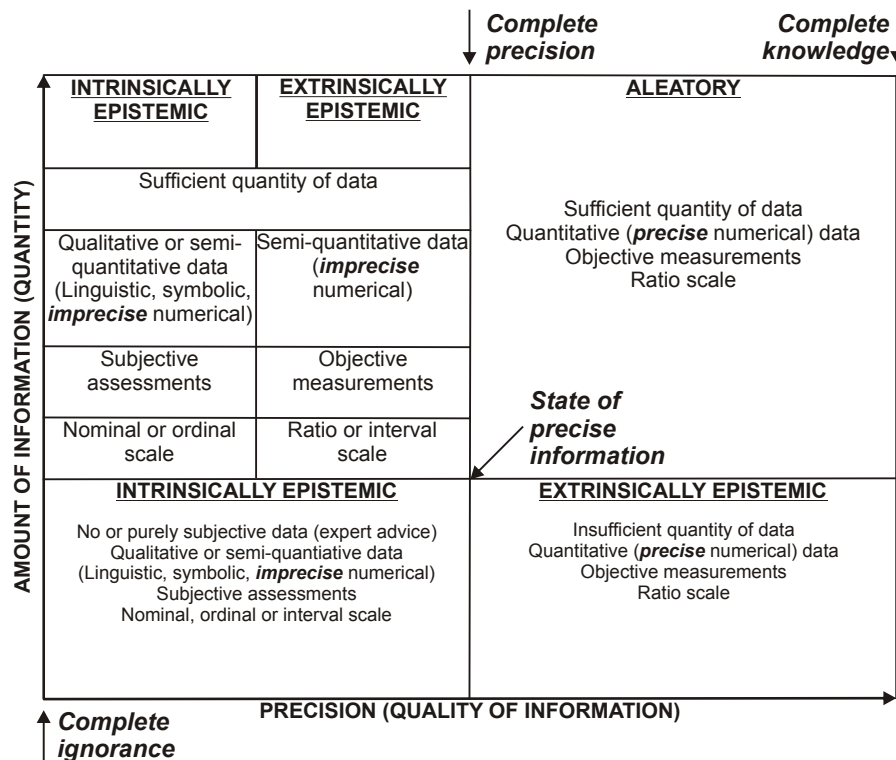
b) Taxonomy arranged with respect to quantity and quality of information

Figure 17: Proposed taxonomy.

When following this taxonomy, the first considerations are whether the cause of unpredictability is from a lack of knowledge or random variability. The next consideration is quantitative; if the data are precise, though the quantity is limited, the parameter must be classed as extrinsically epistemic until sufficient data become available with which to fit a precise probability distribution function. As one moves down the columns, a qualitative assessment of the information is undertaken; if the parameter can only be determined through subjective assessment, e.g. if the data are qualitative and thus inherently imprecise, the parameter must be classified as intrinsically epistemic. Most importantly, this table shows that to characterise a parameter or system as aleatory, very specific criteria must be fulfilled: the unpredictability stems from inherent randomness and there must be a sufficient quantity of precise data available with which to objectively fit a probability distribution function.

Following this, Figure 17b arranges the new taxonomy with respect to the quality and quantity of information axes, as first introduced by Figure 3. Figure 17b also indicates the states of information that were first depicted in Figure 2. The lower left corner, a state of zero quantity and quality of information, represents complete ignorance. Moving diagonally across, i.e., by increasing the quantity and quality of information, one reaches 'the state of precise information'. It is at this point that aleatory variability is realised. A lower quantity of information (below this point) indicates insufficient data with which to objectively fit a stochastic model to the data. To the left of this point indicates a lower quality of information, thereby resulting in imprecise data. From the state of precise information, if one obtains a greater quantity of data, with precise values, a state of complete precision may be reached. This signifies that – assuming one cannot refine the measurement process further to obtain higher quality data – further quantity of information will not further improve our estimation of the degree of variability. The top right corner of Figure 17 indicates a state of complete knowledge; the measurements are precise enough and the quantity of data is such that variability is completely eliminated. At this point a single, deterministic value of the parameter, which is completely known, can be used. Here, one has eliminated unpredictability in the parameter or system.

The next section applies the proposed taxonomy to characterise the unpredictability in parameters commonly encountered in rock mechanics and rock engineering. The examples presented in the following section, show how proposed taxonomy will allow the characterisation of unpredictability to be an objective process. This supports the conclusion introduced by the level of information concept (i.e. Figure 8 in section 2.5) that selection of an appropriate uncertainty model should be commensurate with the given level of information.

2.8 Rock mass parameters: aleatory or epistemic?

Two useful acronyms to describe rock masses are CHILE (Continuous, Homogeneous, Isotropic, Linear, and Elastic) and DIANE (Discontinuous, Inhomogeneous, Anisotropic, Non-linear Elastic) (Hudson & Harrison, 1997). The first of these is the simplifying assumption commonly adopted when undertaking design of rock engineering structures, whereas the second is the physical nature of the material in which engineering takes place. Undertaking rock engineering in CHILE rock masses would be straightforward: material properties determined through laboratory or field tests undertaken on small scale samples of the rock could be used to characterise the variability in the rock mass. However, the heterogeneity in DIANE rock masses makes it particularly difficult to undertake objective or precise measurement on samples that are representative of the rock mass as a whole. In fact, the distribution and in-situ mechanical properties of the discontinuities generally govern the behaviour of the rock mass, and it is the parameters that define these properties that cannot be captured through small scale sampling or testing. These complexities in DIANE rock masses introduce epistemic uncertainty through: measurement or interpretation errors – or inadequate data representation – during site characterisation; modelling uncertainty, as to whether the selected mathematical model is an accurate representation of reality; and, parameter uncertainty in terms of how model parameters are estimated and analysed. As shown in Figure 18, these sources of epistemic uncertainty combined with the aleatory component make up the total unpredictability of the DIANE rock mass.
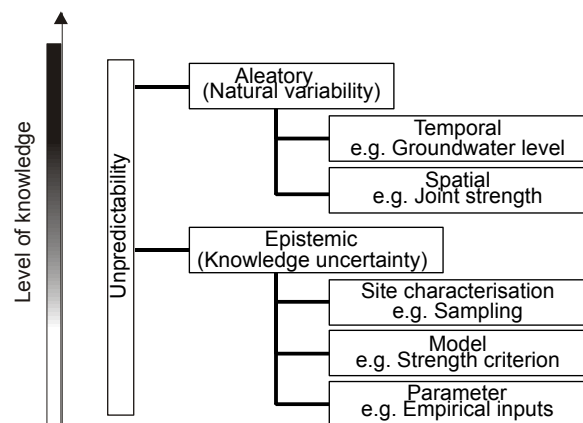
Figure 18: Sources of unpredictability (after Baecher & Christian, 2003).

Additionally, geotechnical engineers often rely on empiricism or expert judgement to determine rock mass parameters, and these may introduce subjectivity as a form of epistemic uncertainty. In rock engineering in particular, parameters required to characterise DIANE rock masses are commonly derived through subjective estimates made by geologists through

field observations using various exploration methods such as outcrop, core or tunnel mapping. Figure 19 illustrates the complexity in the characterisation, analysis and design making processes when undertaking design in DIANE rock masses.
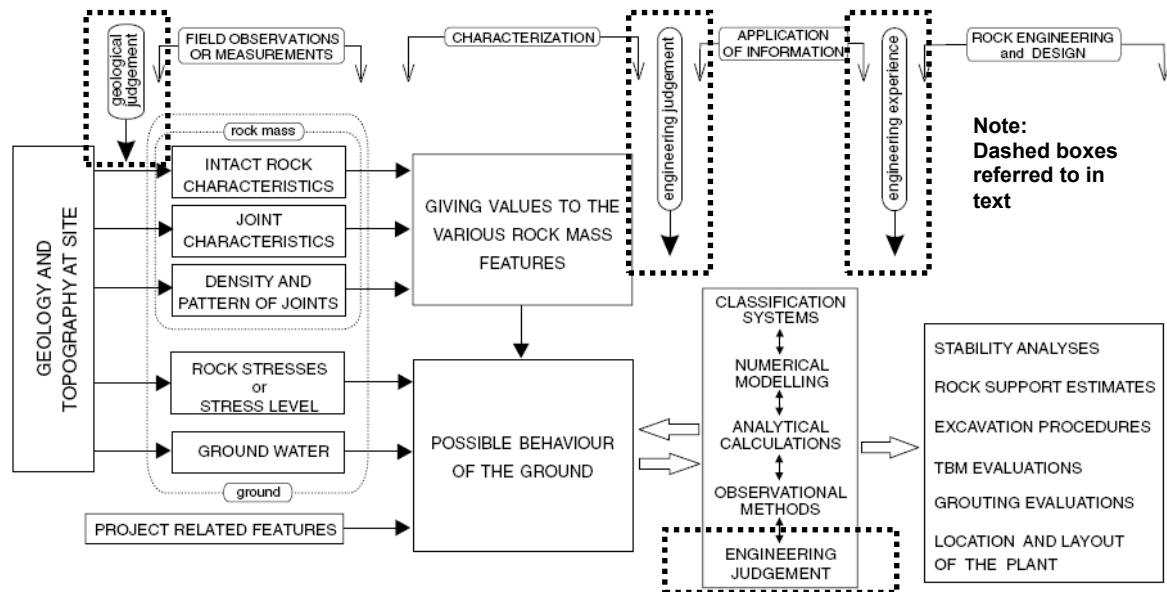
Figure 19: Stages of design process where subjective assessment is required: from geological characterisation to decision making (from Palmström & Stille, 2007) .

Of most significance, this figure demonstrates the reliance on engineering or geological judgement during various phases of the design process (dashed boxes in Figure 19). With respect to the taxonomy (Figure 17), it is this subjectivity which leads to a quantitative or qualitative lack of information. It follows then, that parameters used to characterise DIANE rock masses that require subjective determination can mean the state of information upon which a design is based is in fact imprecise.

The succeeding sub-sections discuss these sources of uncertainty with respect to the proposed taxonomy, shown previously by Figure 17 (see section 2.7). This discussion begins by investigating the nature of epistemic uncertainty in empirically derived parameters, in particular rock mass classification systems. This is followed by a discussion on epistemic uncertainty arising in the choice of parameters that are required to define strength criteria commonly used to model the strength of intact rock and jointed rock masses. The discussion continues by using examples to compare the nature of unpredictability in site characterisation data that results from the means with which the parameters are estimated. Finally, examples of rock mass parameters that have been shown to be intrinsically aleatory are presented. This discussion highlights the applicability of the taxonomy for characterising unpredictability that arises from methods commonly applied in rock mechanics and rock engineering.

2.8.1 Rock mass classification systems

The difficulty in using objective test methods to characterise DIANE rock masses has led to the development and wide use of rock mass classification systems – a compendium of which is listed in Table 2 (Note: the references shown in Table 2 have not been retrieved as part of this work) – for engineering design in fractured rock masses.

Table 2: A compendium of rock mass classification systems (from Harrison, 2010).

Scheme	Originator	Origin	Applications	Originator	Origin	Applications
Rock load	Terzaghi, 1946	USA	Tunnels with steel supports			
Stand-up time	Lauffer, 1958	Austria	Tunnelling			
NATM	Pacher et al 1964	Austria	Tunnelling			
RQD	Deere et all 1967	USA	Core logging, tunnelling			
RSR	Wickham et al 1972	USA	Tunnelling			
RMR	Bieniawski, 1973, 1979	RSA	Tunnels, mines, slopes, foundations	Serafim & Pereira, 1983	Portugal	Foundations
	Weaver, 1975	RSA	Rippability	Gonzalez de Vallejo, 1983	Spain	Tunnelling
	Laubscher, 1977	RSA	Mining	Unal, 1983	USA	Coal mine roof bolting
	Olivier, 1979	RSA	Weatherability	Romana, 1985	Spain	Slope stability
	Ghose and Raju, 1981	India	Coal mining	Newman, 1985	USA	Coal mining
	Moreno Tallon, 1982	Spain	Tunnelling	Sandbak, 1985	USA	Boreability
	Kendorski, 1983	USA	Hard rock mining	Smith, 1986	USA	Dredgeability
	Nakao et al 1983	Japan	Tunnelling	Venkateswarlu, 1986	India	Coal mining
				Robertson, 1988	Canada	Slope stability
Q-system	Barton et al 1974	Norway	Tunnels, chambers			
	Kirsten, 1982	RSA	Excavatability			
	Kirsten, 1983	RSA	Tunnelling			
Strength-size	Franklin, 1975	Canada	Tunnelling			
Basic geotechnical description	ISRM, 1981		General & communication			
Unified classification	Williamson, 1984	USA	General & communication			
Geological Strength Index	Hoek, 1995	Canada	General			

Bieniawski (1989) defines classification as *"the arrangement of objects into groups on the basis of their relationship"*. In this light, the aim of classification systems is to group similar rock mass characteristics in to classes, which can be compared against observed behaviours of the rock masses. The rock mass classes (the groups) are generally obtained by combining a series of parameters determined by assigning a numeric value to a visual observation of a particular rock mass characteristic against a given linguistic or graphical description. These numerical values are then combined into a final 'classification index' using ordinary calculus. This use of numerical indices and ordinary calculus may introduce a false perception of precision; however, with respect to the scale of measurement (see section 2.2.2), the subjective assessment against linguistic or symbolic descriptor introduces nominal or ordinal measurements. As an example, let us consider the joint alteration parameter J_a, which is one index used to calculate the Q-value (Barton et al., 1974). The linguistic descriptions used to assign numerical ratings to J_a are divided up in to three major classes: joints that exhibit 'rock wall contact', 'rock wall contact before 10cm shear' and 'no rock wall contact

when sheared'. Within each of these classes, more detailed joint descriptions are provided with the subsequent numerical rating for each.

Figure 20 presents a simplified arrangement of the joint classes, descriptions and associated range of numeric values of J_a. One can see that the descriptions encompass a range of significantly different joint conditions, none of which can be objectively measured. Instead, one must assign a rating based on judgement, with a higher rating for those joint conditions which are less favourable to stability, and lower rating to those considered favourable. Furthermore, there is a considerable overlap in numeric ranges across various joint types. Thus a numerical rating assigned to J_a is nothing more than a rank ordering, and therefore according to the scales of measurement shown previously in Table 1 (see section 2.2.2), J_a is of ordinal scale. The ordinal nature of J_a means it is not clear whether a numeric value, say 10 for example, has any precise meaning. Similarly, according to Stevens' (1946) scales of measurements, and as summarised previously in section 2.2.2, for a collection of measurements of J_a although mode and median values can be determined, a statistical mean is, strictly, invalid. It follows then, that precise probability distributions that are characterised by such statistical moments are inappropriate to characterise such rock mass classification indices.

Yet another source of imprecision resulting from subjective assessments of parameters that form the inputs to rock mass classification systems is the need for approximation. That is, different experts undertaking an assessment of the rock mass characteristics may well assign different numeric values for the parameter in question, which introduces dissonance. Additionally, the linguistic or symbolic descriptors that are used as exemplars for deriving the
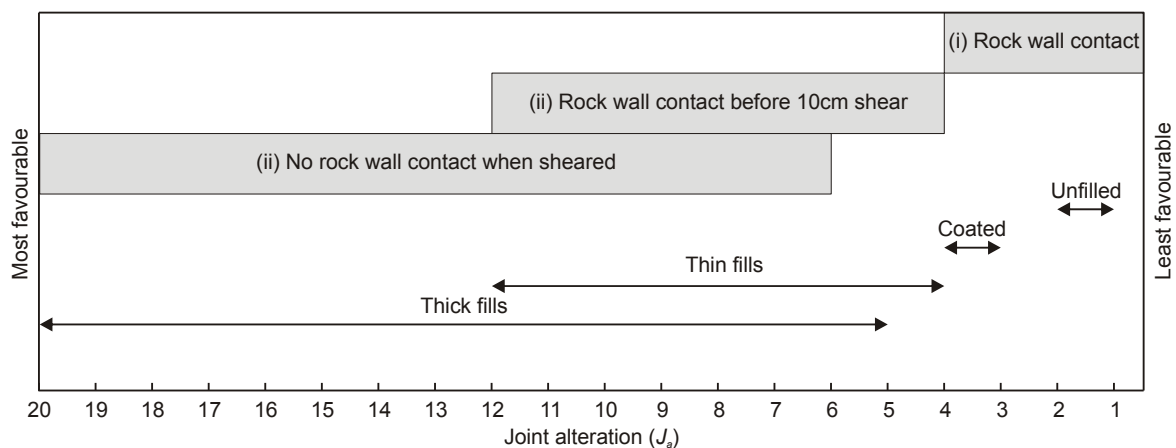
Figure 20: Simplified arrangement of descriptions associated with the numeric range of J_a (after Barton et al., 1974; Barton, 2002).

numeric values of a parameter introduce ambiguity because; a) different experts interpret this type of information in unique ways, and b) the rock mass characteristic may fit across a range of descriptions. For example, if we consider a situation where a geologist is mapping part of a tunnel to determine the joint alteration number, J_a, used in the Q-system; the geologist considers that the joints in this area 'are in contact before 10cm of shear. They contain a heavily over-consolidated clay infill less than 5mm in thickness, and montmorillonite particles that may have a high potential for swelling in the presence of water'. According to the descriptors given in the Q-system, J_a may range between 6 and 12. Alternatively, another geologist assessing the same area may have a different interpretation on the degree of overconsolidation of the clay infill or the potential for swelling, and may thus give a range of J_a between 8 and 10. In fact, this approximation means that an objective and precise measurement of the joint alteration is impossible.

With respect to the taxonomy presented earlier in Figure 17, the purely subjective assessment and assignment of numeric values against qualitative descriptions results in nominal and ordinal scales of the parameter. All these characteristics require the unpredictability in such rock mass classification systems to be characterised as intrinsically epistemic, and the parameter estimates represented by intervals (bottom left corner of Figure 17). Tonon et al. (2000) note that many rock mass classification systems, and RMR in particular, disregard this imprecision and present single measures for the basic parameter (e.g. joint spacing), which, according to the taxonomy of Figure 17, is incorrect . Indeed, Tonon et al. (2000) further note that some scholars and practitioners consider it appropriate to take imprecision into account (e.g. Barton et al., 1994; Hoek et al., 1995) by using intervals to define the basic parameters. Consequently, Tonon et al. (2000) suggest an approach where, using the RMR system as an example, each observation for the basic parameters is assigned an interval rather than precise values.

Palmström & Stille (2007) suggest that classification does not provide definitive information on the mechanical properties of the rock, but rather only a qualitative assessment of them in an attempt to facilitate a common means of understanding the behaviour. This then provides a tool for engineers to understand how various features of the DIANE rock mass can affect its overall behaviour. Often, the numerical value of the obtained empirical index is re-transformed to an adjective that describes the quality of the rock mass. For example, when using the Q-system, six input parameters describing various facets of the rock mass are combined to compute a Q-value which can then be used to linguistically classify the rock

mass. In the Q-system, values between 1 and 4 are classified as 'poor' ground. With respect to the taxonomy of Figure 17, the subjective means of assessment and type of information (qualitative) means that the unpredictability in parameters determined through rock mass classification undertaken in this manner must be characterised as epistemic.

A common feature of these systems is that they have been developed through approximate correlation of some easily observable, measureable or recordable characteristics of the rock mass with prior experience (Palmström & Stille, 2007). Thus, their use in a particular design situation is essentially a transfer of this prior knowledge, through the developed correlations, to the site/project at which they are being applied. With respect to the discussion presented in section 2.2, the approximations employed in deriving these correlations introduce imprecision of an unknown magnitude. The unpredictability is due to a lack of knowledge regarding the relationship between the measurements of the observed rock mass characteristic and the behaviour being assessed. This is one aspect that requires rock mass parameters to be regarded as intrinsically epistemic.

These aspects of parameter estimation constitute an inherent qualitative lack of information, which cannot be reduced or eliminated with additional estimates of the parameter. These parameters are intrinsically epistemic and must not be modelled and analysed using stochastic models or probabilistic methods. From this, it can be concluded that (as described in section 2.2.1) any classification scheme which requires subjective determination of parameters through comparison against published descriptors can only ever be characterised as intrinsically epistemic.

2.8.2 Parameters from empirical correlations

Similar to rock mass classification systems, various empirical relations have been developed in an attempt to capture the DIANE response of rock masses through correlations of measured rock mass behaviour against easily observable or measurable parameters. The numerous published empirical correlations commonly used in rock engineering can be separated into two categories; those that use rock mass classification indices – which were shown to be intrinsically epistemic (refer definition in section 2.2.3) – correlated against a measured property (e.g. GSI versus rock mass modulus, as shown earlier in Figure 5), and; those that correlate an objective measurement against a measured property, an example of which is rock mass modulus ratio E_{rm}/E_r derived from RQD (Deere, 1989) shown in Figure 21, where

E_{rm} is the deformation modulus of the rock mass and E_r that of the intact rock. Based on the

conclusion drawn in the previous section that unpredictability in rock mass classification systems must be characterised as intrinsically epistemic, it follows that any correlation that utilises a rock mass classification scheme will also inherit this uncertainty and thus must also be characterised as intrinsically epistemic.
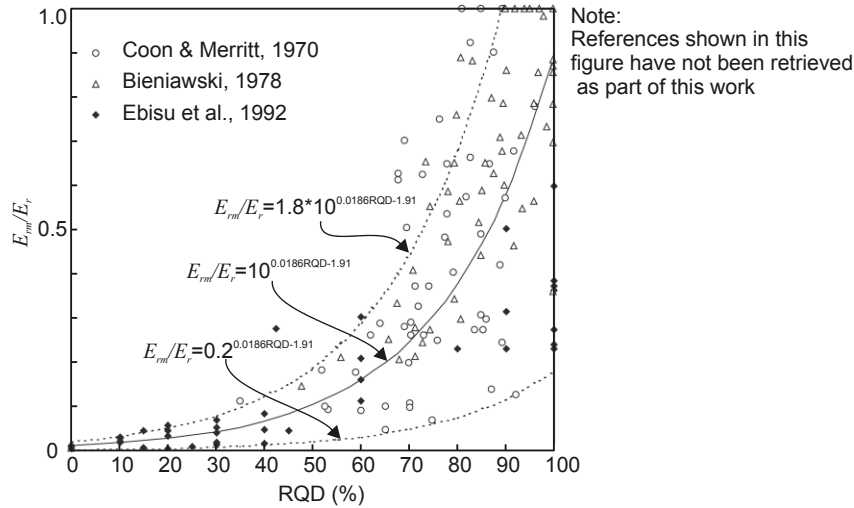
Figure 21: Correlation between deformation and RQD (after Zhang & Einstein, 2004).

The unpredictability in any empirical relation based on objective measurements is dependent on the number and quality of the employed data, which in many cases is unknown (Gokceoglu et al., 2003; Zhang & Einstein., 2004). Thus, a number of issues need to be considered when characterising the unpredictability introduced through the use of such empirical correlations. Firstly, an empirical correlation may provide a poor fit to a series of objectively measured data gathered from many different sites. This may be due to either a quantitative or qualitative lack of information.

One parameter frequently estimated from empirical correlations is the elastic modulus of the rock mass (E_{rm}). Figure 21 illustrates various empirical correlations between RQD and E_{rm}, alongside a variety of measured data. Whilst this figure suggests there may be some correlation between RQD and E_{rm}, and perhaps lower and upper bounds for it, it does not suggest that the distribution of the data between these bounds follow a stochastic model. Nonetheless, a 'mean empirical relationship' between RQD and E_{rm} has been determined using statistical fitting through ordinary least squares regression (Zhang & Einstein, 2004), with the goodness-of-fit estimated by the co-efficient of variation r^2. The coefficient of variation measures the variability of the test results around the mean – by assuming the variability is normally distributed around it – that is explained by the fitted regression model

(Davis, 2002). For example, an r^2 value of 1.0 indicates no variation around the fit regression. Conversely, an r^2 of 0 implies that errors are not normally distributed about the mean but may be explained by unknown, lurking variables or other uncertainty (Davis, 2002). The r^2 value can thus be used to test the hypothesis that the regression model, and associated statistics, can be used to define the unpredictability in the data. Low r^2 values imply that the statistical model defined by the least squares regression is inappropriate to model the unpredictability in the data. With reference to the empirical relation between RQD and E_{rm} in Figure 21, the r^2 value of 0.75 implies that 25% of the data cannot be explained by the statistically fit regression model. This is more evident at RQD values greater than about 75%. Thus, even though both RQD and E_{rm} may have been objectively measured, there appears to be a degree of imprecision in the measurement of them; the type of information obtained is imprecise numerical data. It is this lack of precision that would require this empirical correlation to be characterised as epistemic. However, as with many similar empirical correlations, additional site-specific data may significantly improve the fit of the regression model. In which case, the correlation can be considered extrinsically epistemic. In fact, Zhang et al. (2004) show how site-specific objective measurements coupled with the Bayesian updating approach may be applied to these empirical correlations.

Whilst RQD may arguably be objectively measured, various empirical relationships utilise parameters from rock mass classification systems to estimate E_{rm}. Figure 22 presents the results of a study undertaken by Gokceoglu et al. (2003) that reports the performance of a few such relationships in predicting the rock mass modulus through comparison against 57 measured values from in-situ plate loading tests. The correlations studied by Gockceoglu et al. (2003) are reported in Figure 22 but have not been retrieved as part of this current work. In this figure, the prediction error (on the abscissa) is the difference between the measured (E_M) and predicted (E_P) value of rock mass modulus at each location, expressed as a percentage of the measured value, i.e., Prediction error (%) = (E_M - E_P)/ E_M. The ordinate reflects the cumulative distribution of prediction error over the set of 57 data. In this figure, a positive prediction error indicates that the subjectively estimated value of the rock mass modulus is greater than that measured. For example, correlation 7 over-predicts 70% of the objectively measured rock mass modulus values by 100%.

This figure, which is truncated at -/+200% error, shows that the use of empirical relations to estimate rock mass modulus can result in large over-estimations of the measured,
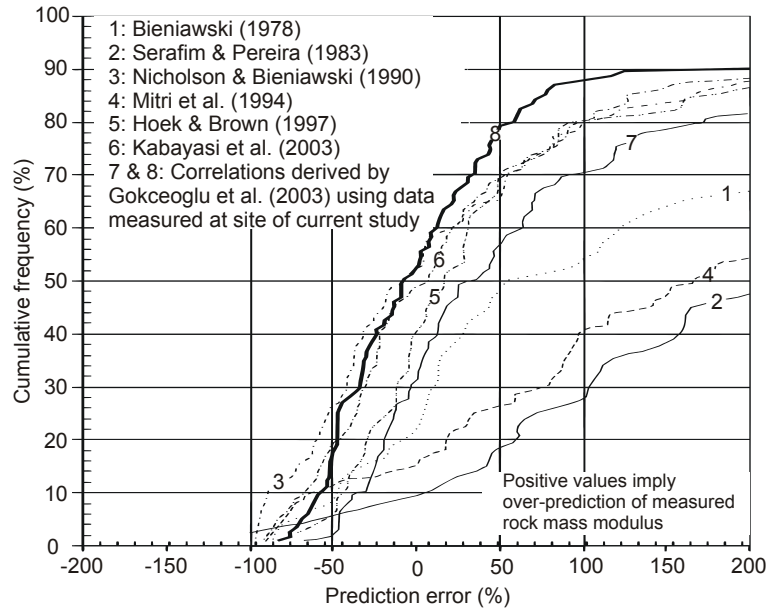
Figure 22: Prediction error of rock mass modulus using the various empirical relations against in-situ plate loading test measurements (after Gokceoglu et al., 2003).

in-situ rock mass modulus. This figure also emphasises that the degree of prediction error is highly variable between the various relationships studied by Gokceoglu et al. (2003). This may be attributed to the fact that the rock mass classification parameters are determined subjectively and this introduces a high degree of imprecision in their estimation. That is, dissonance between experts means that each estimation of the rock mass classification parameter is dependent on the perception of the expert. Furthermore, this reinforces the earlier statement that the unpredictability in estimating many rock mass classification parameters is epistemic. Thus any further analyses based on these parameters will only further propagate the uncertainty.

2.8.3 Strength of intact rock and rock masses

Various peak strength criteria have been proposed to predict the strength of both intact rock and jointed rock masses, of which the Hoek-Brown criterion (Hoek & Brown, 1980a; 1980b) is one of the most common criteria used in practical applications. The original Hoek-Brown criterion was first developed using theoretical and experimental studies (Hoek & Brown, 1980a), and is given by Equation (2.1) in terms of principal stresses.

$$\sigma_1 = \sigma_3 + \sqrt{m\sigma_{ci}\sigma_3 + s\sigma_{ci}^2} \qquad (2.1)$$

In Equation (2.1) , m and s are constants which depend upon the extent and distribution of fracturing in the rock mass, with σ_{ci} representing the uniaxial compressive strength (UCS) of

the intact rock material. In a situation where failure through the intact rock governs the design – for example, a small diameter wellbore at significant depth in massive rock – the parameter s reduces to $s_i = 1$, with m and σ_c determined through triaxial tests on samples of intact rock. Hoek & Brown (1980b) recommend that at least five triaxial tests should be carried out over a confining stress range from zero to one-half of the uniaxial compressive strength. The parameter m is then determined using a statistical fitting procedure (least squares regression), with the goodness-of-fit estimated by the co-efficient of variation r^2. Hoek & Brown (1980b) have demonstrated that for intact rock, very high r^2 values (mostly greater than 0.9 and approaching 1) are obtained when $m = m_i$, and is objectively fitted to the results of triaxial tests, which suggests that this parameter may be intrinsically aleatory. The same can be said for σ_c determined through uniaxial compressive strength tests undertaken in the laboratory.

This is verified in the following section and further demonstrated through an example presented in Chapter 7. It can thus be concluded that, if the material constants required by Equation (2.1) are determined objectively, they may be considered precise and, with a sufficient number available may be characterised as aleatory and modelled by stochastic models fit using statistical tools. Whilst the intact rock parameters for the Hoek-Brown criterion may be determined objectively in the laboratory, similar to the rock mass modulus, determining the strength of jointed rock masses by objective testing is generally impractical (Hoek, 2007). For this reason, Hoek & Brown (1988) extended the criterion to incorporate an empirical relationship between the intact rock material constants m_i and the rock mass rating (RMR) system of Bieniawski (1989) to estimate the 'broken' rock mass constant m_b. Hoek (1994) and Hoek et al. (1995) further extended the criterion to incorporate the empirical Geological Strength Index (GSI). The latter relationships are given in Equations (2.2) to (2.4).

$$\sigma_1 = \sigma_3 + \sigma_c \left(m_b \frac{\sigma_3}{\sigma_{ci}} + s \right)^{0.5} \tag{2.2}$$

$$m_b = m_i \, exp\left(\frac{GSI - 100}{28} \right) \tag{2.3}$$

$$s = exp\left(\frac{GSI - 100}{9} \right) \tag{2.4}$$

As was discussed in section 2.2.1 and expanded upon in section 2.8.1, rock mass classification systems such as RMR and GSI require subjective estimation and incorporate nominal and ordinal scales of measurement, all of which mean they are inherently imprecise. This imprecision will be perpetuated through any model, such as the Hoek-Brown criterion Equation (2.2) to (2.4), which is formulated on using such rock mass classification systems as inputs. Thus, it can be concluded that rock mass strength which is estimated using Equations (2.2) to (2.4) must be considered as epistemic. In fact, given that GSI is a purely subjective estimation and thus inherently imprecise, it follows that use of the Hoek-Brown criterion using GSI as an input requires it to be characterised as intrinsically epistemic. Consequently, it cannot be characterised by stochastic models or propagated using conventional probabilistic analyses. These concepts are demonstrated using an example presented later, in section 7.1.

2.8.4 Parameters derived from objective measurement

According to the new taxonomy developed here and presented in Figure 17, a key requirement in characterising a parameter as being aleatory is that it can be measured precisely, i.e., in a ratio or cardinal scale, using objective methods. In rock engineering, this may come in the form of laboratory test data, e.g., triaxial tests, or field tests such as the point load index for uniaxial compressive strength (UCS). However, in order to fit a probability distribution, the taxonomy also requires there be a sufficient quantity of data, otherwise the unpredictability must be regarded as epistemic uncertainty.

The uncertainty associated with small data sets is exemplified by the variability of the UCS with respect to the number of strength measurements made (Ruffolo & Shakoor, 2009). Ruffolo and Shakoor analysed five different rock types, with statistical analyses being undertaken on subsets of test specimens to determine the minimum number of strength tests required to render a reliable estimate of the average strength of the entire set of specimens. Figure 23 presents typical results for one of the rock types tested, and shows the precision of variability converging to a limiting value with increasing number of specimens. This confirms the irreducibility concept first raised in section 2.2. Furthermore, this convergence of the mean value is to be expected, in line with the central limit theorem (Davis, 2002), which applies to data that can be characterised by stochastic models. However, if we simply consider the case of very small sample sizes (e.g., five or fewer specimens), then such statistical considerations are invalid and thus strength must be considered as epistemic (i.e., similar to the concept presented in the earlier example of Figure 4). These results demonstrate that, whilst UCS may be intrinsically aleatory (resulting, for example, from variability within a

rock layer), unless sufficient data exist with which to characterise it, the use of an aleatory model may be inappropriate. In such a case, uncertainty in UCS should be treated as epistemic, and handled using an appropriate, non-stochastic, approach.
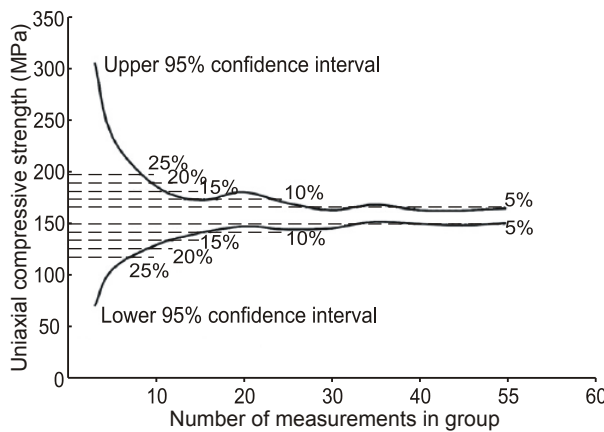
Figure 23: Confidence intervals and acceptable strength deviation of Milbank granite (from Bedi & Harrison, 2012).
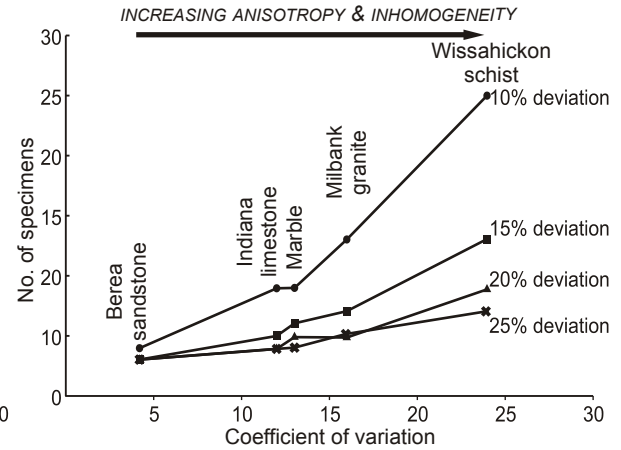
Figure 24: Minimum number of samples needed to estimate the mean unconfined compressive strength (from Bedi & Harrison, 2012).

The work of Ruffolo & Shakoor (2009) also showed that strength variability and hence the number of tests required to make adequate estimates of mean strength varies with rock type, as shown in Figure 24. In this figure, the degree of anisotropy and heterogeneity in the rock type (sandstone to schist) increases from left to right. This suggests that there may be a geological link between variability and number of samples required to reduce uncertainty, and implies that the minimum number of strength tests required may not be the same for all rock types. If true, this will have important ramifications for the codification of testing requirements in order to characterise rock strength as aleatory.

An example of objective measurements obtained from field observations is discontinuity spacing determined along a scanline. Priest & Hudson (1976) describe the application of this measurement process 'in-tunnel' by, wherever possible, setting up measuring tapes (the scanlines) of equal lengths in orthogonal directions to obtain a true three-dimensional picture of the discontinuity spacing. Figure 25 presents the histogram of measurements obtained in an experimental study, which due to limitations in the measurement process could only be measured to the nearest 0.01m, along with the negative exponential PDF fit to this data.
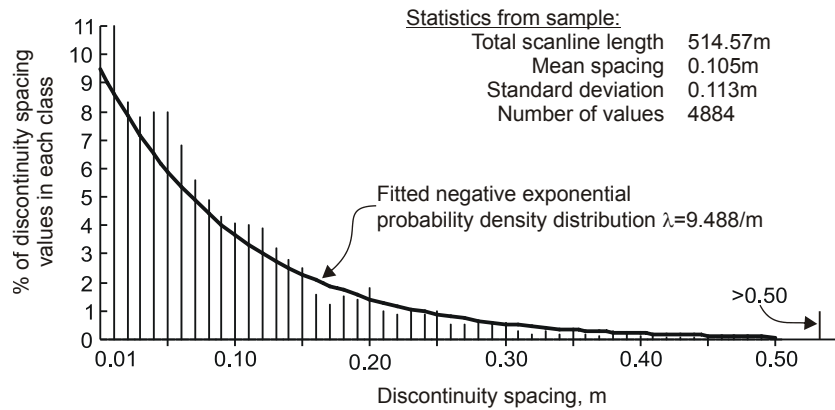
Figure 25: Distribution of discontinuity spacing measured from scanlines
(after Priest & Hudson, 1976).

Whilst the PDF appears to describe discontinuity spacing, an important aspect of precision is worth noting. With regard to the proposed taxonomy, (shown previously in Figure 17) precision implies that the measurement process is objective with sufficient accuracy to represent the phenomena being modelled. In this example, the accuracy of 0.01m is considered sufficiently small with respect to the statistics computed from the data (mean spacing and standard deviation). Thus, applying the taxonomy, it is evident that discontinuity spacing can be considered as aleatory because it is a phenomenon resulting from natural random variation of joints in the rock mass, a sufficient number of objective measurements – which produce precise numerical data – can be obtained using objective measurement techniques to which a precise PDF can be fit.

2.8.5 Field estimates of random variability

It is often the case, especially in preliminary stages of a design, that there is insufficient time or budget available to undertake objective laboratory measurements to characterise rock mass properties. Thus, we often rely on geologists, armed with standard geological field equipment (geological hammer, compass, pocket-lens and measuring tape), to undertake field measurements to assess values of various parameters used to characterise rock mass properties in lieu of laboratory tests. Using these tools, geologists can make measurements of: discontinuity parameters – dip/dip direction of joints, fracture spacing and joint roughness (e.g. JRC; Barton, 1973), intact rock strength – UCS, shear strength parameters (cohesion, c, and friction angle, ϕ) and lithological parameters such as quartz content (Raab & Brosch, 1996). With respect to the proposed taxonomy, whilst many of the parameters used to define such properties are due to random variability, it is the type of measurement (subjective) that introduces imprecision. As such, parameters estimated in this way must be characterised as

extrinsically epistemic until objective means of determining precise numerical values are employed. The following examples investigate the nature of imprecision in such parameters estimated from field observations.

Uniaxial Compressive Strength (UCS) can be estimated in the field by comparing blows from a geological hammer against subjective description of strength (Brown, 1980). Fookes (1991) compared the field estimate of the UCS for a range of sandstones and igneous intrusive rocks on a road site in Africa by an engineering geologist of some ten years' experience with the point load tests subsequently made on the same material (Figure 26). The points that lie in the shaded diagonal in this figure indicate those values where the subjective and objective estimates are sufficiently similar that the subjective estimate could be considered precise. This figure demonstrates that subjective estimates by experts can provide reasonable estimates for intrinsically aleatory properties such as UCS. However, and as Fookes (1991) acknowledges, *"it must be borne in mind that there are many exceptions to prove the rule and it must always be clearly stated in reports or in discussions when an estimation has been made"*. It is this approximation, as illustrated by the spread of the subjective estimates in the field estimation, that introduces imprecision and therefore the unpredictability should be characterised as epistemic uncertainty.
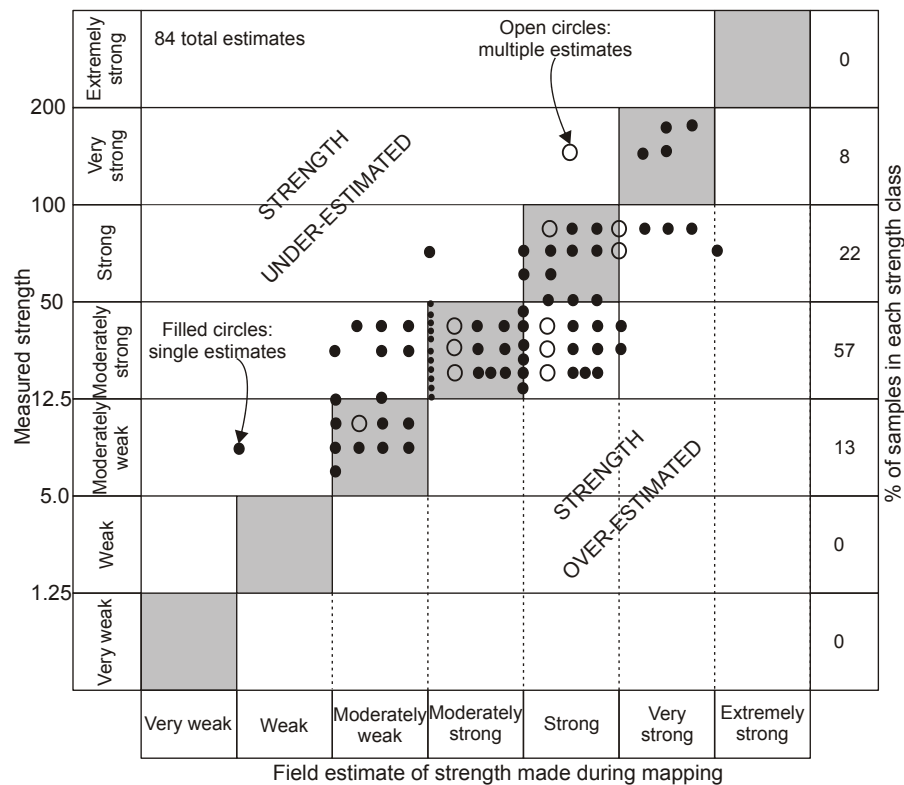
Figure 26: Comparison of field estimates of strength with measured values for the same materials (after Fookes, 1991).

A similar study undertaken by Raab & Brosch (1996) compared field estimates of various rock mass properties along a tunnel alignment against 'reference values' determined through objective laboratory measurements. One of these properties for which Raab & Brosch (1996) provide statistics and the shape of the distribution fit using the Chi-squared goodness-of-fit test, is discontinuity spacing. The field estimates of discontinuity spacing were obtained from forty-three geologists given the standard geological tools stated above, each of whom was requested to provide their 'best estimate' of discontinuity spacing. Figure 27 presents a comparison of the PDFs fit to the reference set and the set of forty three field observations for discontinuity spacing.

a) Distribution of objective measurements using scanline data

b) Distribution of 'best-guess' mode spacing from 43 field estimates
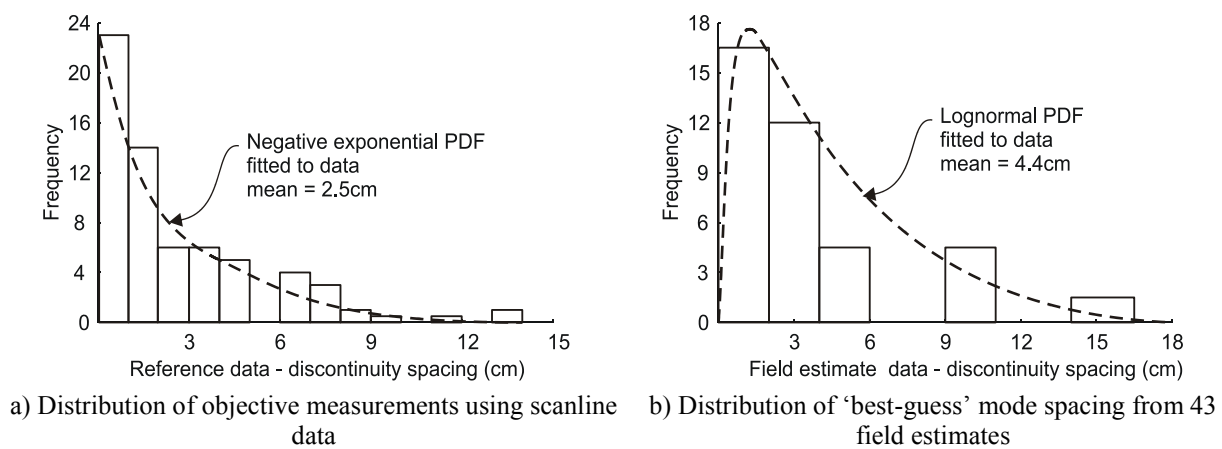
Figure 27: Comparison of discontinuity spacing estimated objectively and subjectively (after Raab & Brosch, 1996).

The conclusion to be drawn from this example is that a series of objective measurements of a property, such as discontinuity spacing – that is the result of random variability and hence intrinsically aleatory – can be used to fit an aleatory model confirmed by statistical tests. However, if the same parameters are determined through subjective field estimates, the same conclusion cannot be immediately drawn for the following reason: the subjectively determined 'best estimates' by individual experts varies considerably. So, if a single expert is employed to subjectively define a precise PDF for such parameters, his/her definition would vary from the next. In fact, according to the new taxonomy (Figure 17), the subjective estimation of an individual expert would deem the quantity of information insufficient to objectively fit an aleatory model. However, a series of subjective measurements (as in this study) constitute further information, and whilst this subjectivity requires the state of knowledge to be regarded as imprecise, the additional information obtained from the distribution of numerous subjective estimates can allow one to utilise a higher modelling method that utilises this information. Chapter 3 will present a detailed

discussion of modelling methods that are appropriate in such instances. This will be followed by an exemplar calculation in Chapter 6 that demonstrates the applicability of an appropriate modelling method where the unpredictability in the problem is epistemic, however a series of field estimations are available.

Another important conclusion of this, and other similar studies, is that in the absence of objective laboratory tests multiple experts may be consulted to estimate a 'prior' distribution for such extrinsically epistemic parameters, which can then be updated in subsequent design phases as further information becomes available. However, the assignment of priors to subjectively determined properties should be undertaken with some caution. This is exemplified by the work of Beer et al. (2002), which describes the results of an online test of the visual assessment of rock profile roughness in terms of the joint roughness coefficient (JRC) (Barton & Choubey, 1977). In this test, individuals involved in geotechnical engineering were asked to visually assess the JRC values of three surface profiles obtained from the same granite block; the results are presented in Figure 28. Through various statistical hypothesis tests, the authors concluded that the observations could not be defined by a specific stochastic function. In this example there is sufficient test data to attempt a statistical analysis. Having done so, the original authors found that the mean and standard deviation of the data fluctuated until 50 or so estimations had been made. Regardless of this, Figure 28 clearly shows that the visual estimations of JRC do not follow any specific distribution. This demonstrates that rock mass parameters derived through expert judgement may be epistemic,
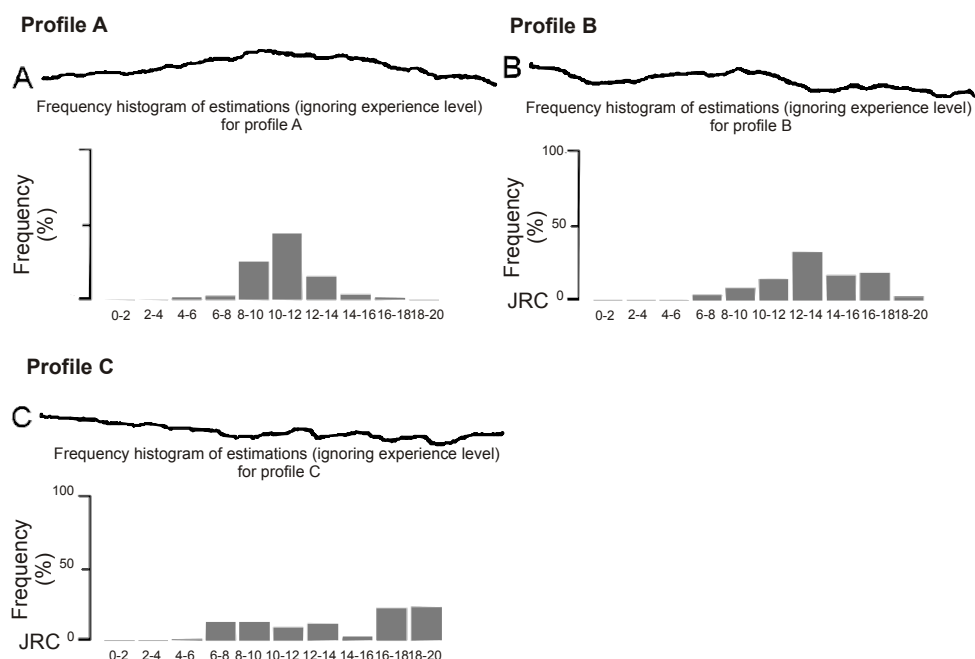
Figure 28: Epistemic uncertainty in Joint Roughness Coefficient (after Beer et al., 2002).

rather than aleatory. It is also important to recognise that, in this study, the number of participants – and thus estimates – was high (in the region of 122-125). In general this will not be the case. For example, in practice a single or small team of design engineers would agree on a value or range of values of JRC to be adopted for design. This is likely to introduce subjectivity into the characterisation process, and, unless an appropriate model is used to capture the uncertainty, may neither adequately represent the epistemic uncertainty nor provide appropriate parameter values (Crawford et al., 2006). However, if JRC had been measured objectively using the tilt-test, with repeated experiment it may perhaps follow an aleatory model.

2.9 Synopsis

Through a critical review of the wider literature, this Chapter presented formal definitions for epistemic uncertainty and aleatory variability as the two components that contribute to the total unpredictability within a parameter or system. Section 2.2 identified epistemic uncertainty as that portion of unpredictability that is due to lack of knowledge; it is both subjective in nature and influenced by preconceptions of what is considered realistic for the system in question, and can be reduced or eliminated through additional information or knowledge. This Chapter demonstrated that in order to remain faithful to the available information and propagate epistemic uncertainty robustly through any analysis, it must be modelled using non-stochastic methods. Aleatory variability, on the other hand, describes the inherent variability in a physical system or environment, it can be modelled using stochastic models and handled using probabilistic methods.

This Chapter demonstrated the importance of differentiating between epistemic uncertainty and aleatory variability by considering the precision of the information available. This discussion identified that aleatory variability can be invoked only when we have reached a state of precise information, and this requires a sufficient quantity of measurements that are precise enough to objectively fit a probability distribution to the data using statistical methods, otherwise the unpredictability must be characterised as epistemic uncertainty and modelled using non-stochastic methods.

Using these definitions, a new taxonomy has been proposed. The new taxonomy has been presented as one simple figure (Figure 17 in section 2.7) that draws together all the concepts presented in this Chapter. A key contribution of this new taxonomy is that it will allow engineers undertaking rock engineering designs to correctly and objectively identify the true nature of unpredictability. The developed taxonomy presented new definitions to sub-

categorise unpredictability in rock mechanics and rock engineering. These definitions identified that if the unpredictability is either intrinsically epistemic or aleatory, then obtaining further information will not allow re-categorisation of the type of uncertainty. However, if the data is extrinsically epistemic, collection of more information may reduce the unpredictability and allow the use of different unpredictability models

Finally, this Chapter concluded by applying the new taxonomy to characterise many parameters commonly used to define the properties of DIANE rock masses (Section 2.8), using the new taxonomic terms. This discussion identified that many parameters used to characterise DIANE rock masses are determined entirely subjectively and thus must be regarded as intrinsically epistemic and modelled using an appropriate non-stochastic method. On the contrary, this Chapter showed how parameters that can be objectively measured, such as uniaxial compressive strength, may be modelled as aleatory. The terms presented in this new taxonomy and the latter examples have assisted in developing an understanding of the mathematical methods for modelling unpredictability in rock mechanics. Chapter 3 now examines these methods for modelling unpredictability more fully.

Chapter 3

MATHEMATICAL METHODS FOR MODELLING UNPREDICTABILITY

In Section 2.5 the Level of Information (LoI) concept was introduced (see Figure 8 in section 2.5), which suggested a hierarchy of unpredictability modelling methods with respect to the available level of information. This in turn implies that the available level of information defines an upper bound for the techniques that can be used, with each technique itself being defined by the minimum amount of information it requires. Following this, the proposed taxonomy presented previously in Figure 17 (see section 2.7) listed unpredictability models considered appropriate for a given level of information. Together, these concepts demonstrate that the selection of an unpredictability model should not be arbitrary: in each case it must be based on an assessment of the nature and cause of the unpredictability, and the quality and quantity of the information available (diagram on the left of Figure 29). The diagram on the
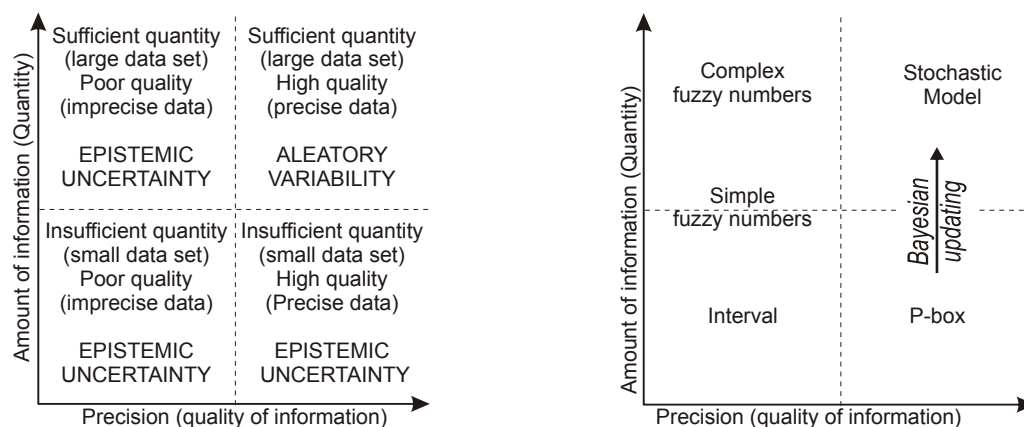
Figure 29: Appropriate uncertainty models for a given level of information (from Bedi & Harrison, 2013b).

right of Figure 29 arranges the appropriate unpredictability models with respect to these concepts. The conclusions that can be drawn from these figures are: firstly, stochastic methods can only be applied when there is a sufficient quantity of precise data. Secondly, Bayesian methods are appropriate where the measurements are precise and additional information can be obtained which will allow convergence to an aleatory model through updating, using Bayes's Theorem (Ang & Tang, 2007). Where the data are imprecise, or there is insufficient quantity of data available, alternative non-probabilistic modelling methods are required.

This Chapter describes the mathematical basis for each of the unpredictability modelling methods of Figure 29, starting with interval analysis and working through the hierarchy of modelling methods in an increasing level of information. The discussion presented in this section further demonstrates, through examples, the applicability of these unpredictability modelling methods with specific reference to rock mechanics and rock engineering problems. The mathematical definitions presented in this Chapter are applied to undertake the analyses required for the challenge problems presented in Chapter 5 to Chapter 7. The algorithms developed to analyse the challenge problems, using the methods presented in this Chapter, are provided in Appendix A to D.

3.1 Interval analysis

As intervals represent one of the lowest levels of information (Figure 8), they are practical for characterising imprecise values when little or no information is available (Ferson, 2002; Ferson et al., 2007; Dubois & Guyonnet, 2011). The available information may be objective (e.g. we are certain that the parameter has a value between some measured data) or subjective (the interval is obtained on experience or the opinion of experts) (Kaufmann & Gupta, 1991).

3.1.1 Definition and examples of an interval

Mathematically, an interval is formulated on the assumption that a set $\mathbf{X}$ of possible values for a variable x is known but with no specified uncertainty structure within the set (Moore, 1966; Moore & Bierbaum, 1979); the only information that may be inferred from an interval is that the value of x is somewhere in the set which is bounded by the values $[a,b]$ and can be expressed as:

$$\mathbf{X} = \{x \mid a \leq x \leq b\} \tag{3.1}$$

Figure 30 presents two theories of what an interval may represent. Figure 30a depicts a 'spike representation' of an interval and implies that the parameter in question is not drawn from an underlying random process – it is intrinsically epistemic. Further information could only serve to reduce the bounds of the interval. Figure 30b is referred to as a 'box representation', and suggests that the interval represents the set of absolutely all cumulative probability distribution curves between the bounds (Ferson et al., 2007). That is, the parameter in question is drawn from an underlying random process, though the current level of information is insufficient to identify the form or parameters of the aleatory model with which to characterise it; it is extrinsically epistemic.
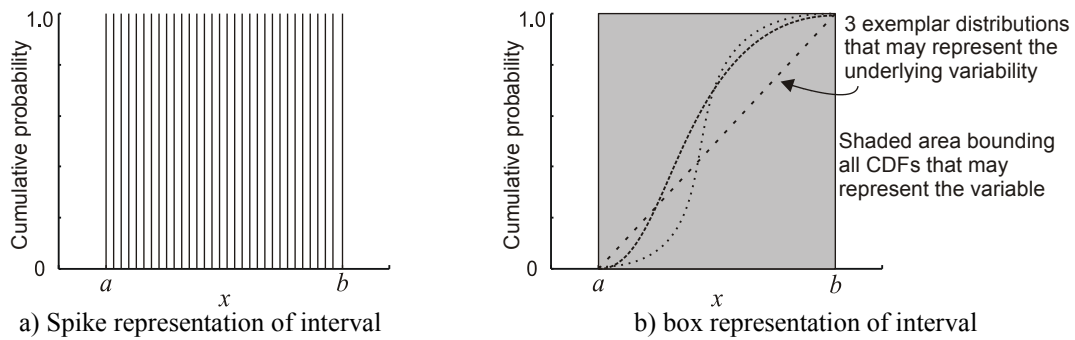
Figure 30: Alternative representation of interval numbers (after Ferson et al., 2007).

In geomechanics, 'spike-intervals' may arise in situations where parameters are determined subjectively and thus inherently imprecise. A common example is that of rock mass classification systems, one of which is the Geological Strength Index (GSI) (Hoek, 1994). The GSI provides a number which, when combined with the intact rock properties, can be used for estimating the reduction in rock mass strength for different geological conditions. The GSI is determined by comparing a linguistic description of certain rock mass attributes to a tabulated range (see Figure 31). Consider a situation where an estimation of GSI is required, however no field investigation has been undertaken. At this point, one could consult an expert for advice, who may suggest: 'Based on my previous experience in a similar rock mass, the surface condition is likely to range between 'fair' and 'good', and the structure of the rock mass from 'blocky' to 'very blocky'. With this information, one could only define an interval of GSI $= [40,75]$ (solid outline in Figure 31). If additional information were to become available, for instance field mapping of nearby outcrops, the expert may choose to refine the rock mass description to, say: 'the surface condition is likely to be 'good', and the structure of the rock mass 'blocky'. The refined interval of GSI now becomes $[55,75]$, as shown by the dashed area in Figure 31 .

Figure 31: Table for estimating GSI (after Hoek, 2007).

It is evident that the box-interval representation does not apply to subjectively determined parameters such as GSI. That is, GSI is not a measurement of a random process; it is a subjective estimation that contains imprecision and requires significant approximation. With respect to the new taxonomy (Figure 17) and Figure 3, this qualitative lack of information means that no matter how much additional expert consultation is obtained for GSI, it cannot be considered as aleatory variability. On the contrary, the box-interval analogy is appropriate for precise parameters, such as the uniaxial compressive strength (UCS) of intact rock. Say for example, at an early stage of design no test data is available and thus expert consultation is enlisted to estimate UCS. The expert advises: 'Based on my previous experience in this rock type, I estimate UCS to lie between 40 and 80MPa', i.e., the interval $[40,80]$. Published literature (Yamaguchi, 1970; Gill et al., 2005; Ruffolo & Shakoor, 2009) suggests that UCS may in fact be intrinsically aleatory, and best characterised by a truncated normal (or beta) distribution. However, at this stage, the lack of quantitative and qualitative data requires it to be classified as epistemic and characterised by an interval. With subsequent data collection, a sufficient number of precise measurements (laboratory tests in this case) may become available to objectively fit a probability distribution for UCS, which would turn out to be one of the infinite number of distributions initially encapsulated by the box-interval.

Using this example, it can be concluded that intervals are required when it is inappropriate to make statistical statements about a parameter, with the information available. In this respect, an interval differs from a uniform random variable, which implies that the values between the bounds of an interval are equally probable. Consider once again the expert's estimate of GSI characterised by the interval $[40,75]$. Given the qualitative and quantitative lack of information, it is invalid to make statistical statements such as; 'the mean value of GSI is 57.5', or 'there is a 75% probability that the GSI will be less than 86.3', both of which are implied by a uniform probability distribution function (PDF). It is evident that a uniform PDF contains a significantly greater amount of information than an interval. Consequently, and as will be shown in Chapter 7, the output of any analysis which adopts this GSI as a uniformly distributed random variable will lead to potentially invalid statements based on additional information not initially present.

3.1.2 *Mathematics of interval analysis*

Intervals can be propagated through a model using interval analysis (Moore, 1966; Moore & Bierbaum, 1979), the output of which is another interval that bounds all possible values the model may take. That is, an arithmetic operation, denoted by $\bullet$, performed on two interval numbers $x \in [a,b]$ and $y \in [c,d]$ results in the output interval:

$$[a,b] \bullet [c,d] = \{x \bullet y \mid a \le x \le b, c \le y \le d\} \tag{3.2}$$

The basic mathematical operations involving two interval numbers are given by Equations (3.3) to (3.6), however, the mathematics of intervals covers all arithmetic manipulations, including trigonometric functions and matrix operations (Moore & Bierbaum, 1979), and so the calculations routinely undertaken in rock mechanics can generally be readily tackled using interval analysis.

$$[a,b] + [c,d] = [a+c, b+d] \tag{3.3}$$

$$[a,b] - [c,d] = [a-d, b-c] \tag{3.4}$$

$$[a,b] \times [c,d] = [\min(ac,ad,bc,cd), \max(ac,ad,bc,cd)] \tag{3.5}$$

$$\begin{aligned} &if \quad 0 \notin [c,d], then \\ &[a,b]/[c,d] = [a,b] \times [1/d, 1/c] \end{aligned} \tag{3.6}$$

Similarly, all arithmetic operations on interval numbers can be applied to functions of intervals. That is, a function f of the variables $x_1, x_2, ..., x_n$ results in a set of all possible values that could be obtained from f given any combination of inputs from the sets of the respective intervals $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$, and is defined as:

$$f(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n) = [f(x_1, x_2, ..., x_n) | x_1 \in \mathbf{X}_1, x_2 \in \mathbf{X}_2, ..., x_n \in \mathbf{X}_n] \qquad (3.7)$$

To simplify computations involving multiple (n) interval functions, Dong & Shah (1987) proposed the 'vertex method', which involves performing a series of computations on the end points of each interval functions. For a model involving n intervals functions, the number of computations required is 2^n. Each computation can be represented by a vertex of an n-dimensional hypercube. For a 3-dimensional space, the cube produced using the vertex method is shown in Figure 32.
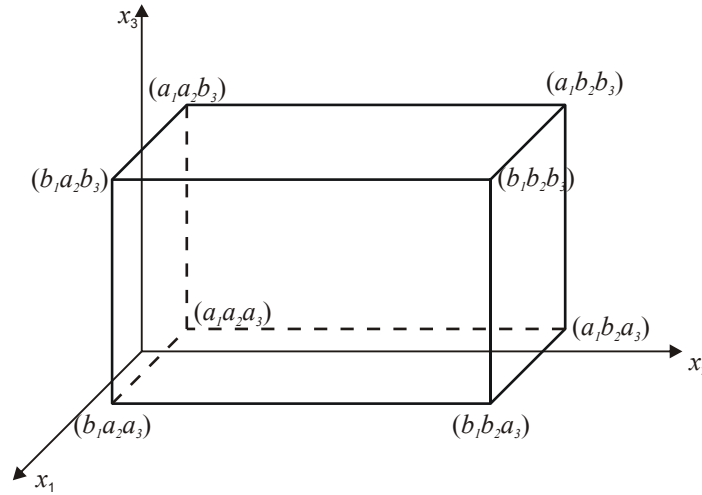
Figure 32: Vertex method of computing bounds with interval inputs (after Dong & Shah, 1987).

The output interval is then obtained from the two vertices representing the minimum and maximum values in the hypercube, as given by:

$$\mathbf{Y} = f(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n) = \left[\min_j (f(c_j)), \max_j (f(c_j)) \right], j = 1, ..., n \qquad (3.8)$$

where c_j is the ordinate of the j-th vertex.

This method of interval analysis allows computation of complex functions of intervals while faithfully and robustly propagating uncertainty (Walley, 1991; Ferson & Ginzburg, 1996; Baudrit et al., 2005; Dubois & Guyonnet, 2011).

The analyses discussed in the challenge problems presented in Chapter 5, onward, and the complementary computations presented in the appendices have been undertaken using the vertex method. More specifically, the challenge problem presented in Chapter 6 shows how the vertex method can be applied to assist in maximising the information obtained from an interval analysis.

3.2 Fuzzy numbers and Possibility theory

Fuzzy arithmetic is a specific field of fuzzy set theory (Zadeh, 1965), which uses fuzzy numbers as an extension of intervals to characterise epistemic uncertainty (Kaufmann & Gupta, 1991). Possibility theory uses fuzzy numbers in a framework that allows measures of confidence (i.e. possibility measures) to assist in decision making. The following section first defines fuzzy numbers with possibility theory discussed further in section 3.2.3.

3.2.1 Definition and examples of fuzzy numbers

If sufficient information is available that allows one to make statements about levels of preference of values within an interval, a fuzzy number ($\hat{X}$) can be constructed through a series of nested intervals that are assigned a degree of possibility through a membership value $\mu_X(x)$ between 0 and 1. These nested intervals are termed the 'α-cuts' of the fuzzy number. Figure 33 shows the fuzzy relationship between the likelihood that the quantity **X** may take on a certain value x through its membership value $\mu_X(x)$ (Kaufmann & Gupta, 1991).
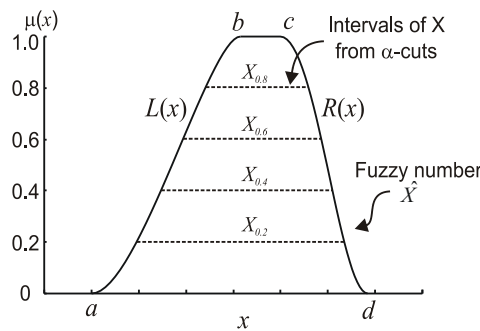
Figure 33: Fuzzy numbers as an extension of intervals.

A fuzzy number, $\hat{X}$, is defined by the quadruplet $\{a,b,c,d\}$. a and d represent the bounds, and b and c the 'core'. The membership values of $\hat{X}$ are given by:

$$\mu_X(x) = \begin{cases} L(x), & a \leq x < b \\ 1, & b \leq x \leq c \\ R(x), & c < x \leq d \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

$L(x)$ and $R(x)$ are continuous functions in the interval $[a,b]$ and $[c,d]$, respectively, and termed the fuzzy membership functions (FMFs). $L(a) = R(d) = 0$ and $L(b) = R(c) = 1$. The α-cuts of $\hat{X}$ are a 'crisp' set, defined by:

$$X_\alpha = \{x \mid \mu_X(x) \geq \alpha, x \in X, 0 < \alpha \leq 1\} \qquad (3.10)$$

Fuzzy numbers may take many shapes, though these should be justified by the available information, which may be objective or subjective. It is triangular and trapezoidal fuzzy numbers (TFN and TrFN, respectively) that are most commonly used (Dubois & Prade, 1989; Kaufmann & Gupta, 1991; Bárdossy & Fodor, 2004). Let us consider again the examples of GSI & UCS presented earlier. Let us now assume that during data collection we obtained the interval of GSI from the expert $[40,75]$ and additional mapping of an outcrop near the construction site indicated a GSI range of $[50,70]$. Based on this information, we may construct the TrFN, $\hat{GSI} = \{40,50,70,75\}$, as shown in Figure 34a. Similarly for UCS, we have the interval of UCS $= [40,80]$ from expert opinion, but we now also have a small number (say 2) of UCS tests undertaken on specimens collected from the proposed site, both of which indicate a UCS of 60MPa. Based on this data, we may represent UCS with a TFN, $\hat{UCS} = \{40,60,60,80\}$ as shown in Figure 34b. Figure 34a may be interpreted as: 'the most possible value of GSI lies in the interval [50,70] ($\mu_{GSI} = 1$). Values below 40 and above 75 are considered impossible ($\mu_{GSI} = 0$)'. Similarly, the TFN characterising UCS may be interpreted as 'the most possible value of UCS is 60MPa ($\mu_{UCS} = 1$). Values below 40 and above 80 are considered impossible ($\mu_{UCS} = 0$)'.

Both these figures now contain more information on the structure of uncertainty

a) Trapezoidal fuzzy number of GSI   b) Triangular fuzzy number of UCS
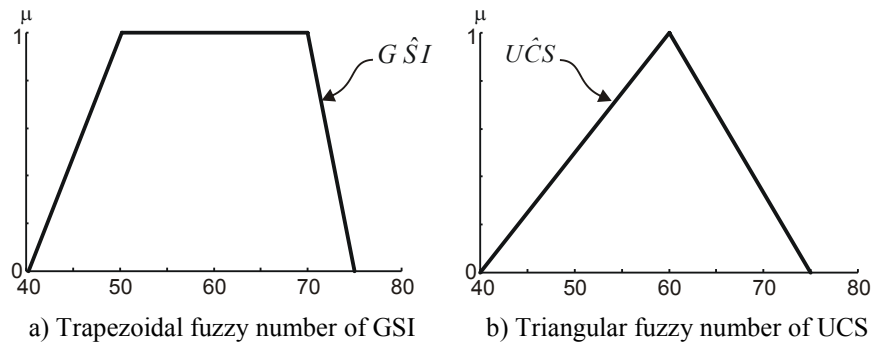
Figure 34: Type of fuzzy numbers.

between the bounds of their respective intervals; the possibility of the values that lie between them. The first and most important step, in deciding whether to progress from interval to fuzzy analysis is to determine whether the level of information is sufficient that the fuzzy membership functions (FMFs) of the uncertain parameters can be justified. It should be noted that while the fuzzy numbers shown in Figure 34 appear similar to a PDFs, fuzzy numbers are not probability distributions. A FMF is a subjective valuation, as opposed to an objective measure defined by a PDF, and contains much less information than a PDF. As such, fuzzy numbers follow their own rules of arithmetic (Kaufmann & Gupta, 1991).

3.2.2 *Mathematics of fuzzy analysis*

According to the 'extension principle' introduced by Zadeh (1975), algebraic operations on real numbers can be extended to fuzzy numbers. Using this extension principle, various authors have presented closed form solutions for arithmetic manipulations involving triangular fuzzy numbers, (e.g. Hanss, 2005; Chutia et al., 2011). The obvious limitation of many of such solutions are first that they are limited to triangular fuzzy numbers, and second that closed form solutions can become cumbersome when many arithmetic manipulations are required. To overcome this, fuzzy analysis can be undertaken by discretising the fuzzy number and applying numerical computational techniques. In fact, as fuzzy numbers can be represented by a series of nested intervals, i.e. the α-cuts of the fuzzy number, the vertex method, described above, can be extended to undertake numerical computations involving functions of multiple fuzzy numbers. Figure 35 illustrates the extension of the vertex method for computing the output of a function of n fuzzy numbers, each discretised into k number α-cuts. The number of computations required is $k \cdot 2^n$.
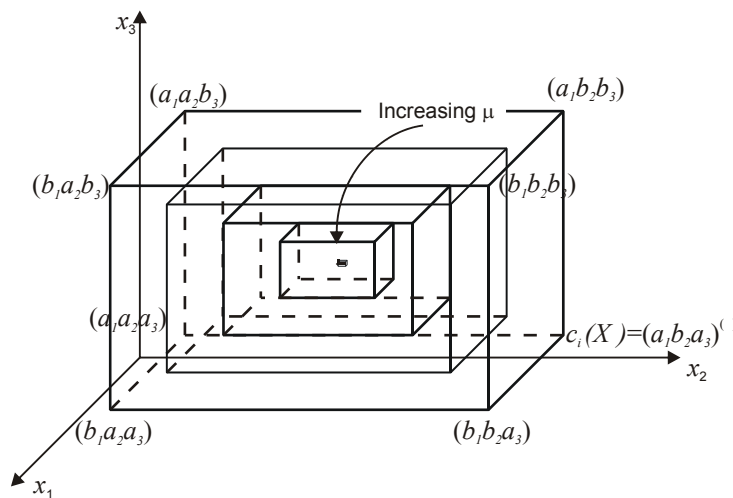
Figure 35: Vertex method of computing bounds with interval inputs (after Hanss, 2002).

Figure 36, presents a flow-chart of the implementation procedure in a function involving fuzzy and non-fuzzy parameters.
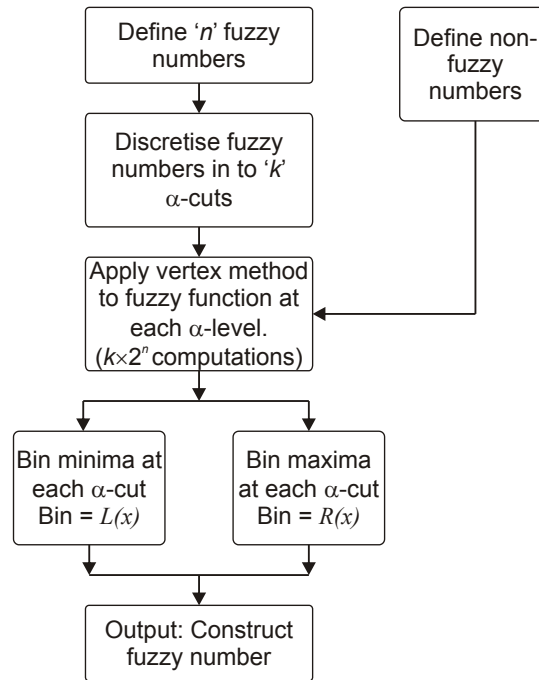
Figure 36: Vertex method applied to functions involving fuzzy and non-fuzzy numbers.

The fuzzy analyses undertaken in the challenge problems presented in Chapter 5 and Chapter 6, with the corresponding computations presented in the appendices, have been undertaken using this extended vertex method, and procedure shown in Figure 36.

3.2.3 Possibility theory

The theory of possibility (Dubois & Prade, 1988) encapsulates fuzzy numbers as possibility distribution, analogous to the way a probability distribution is associated with a random variable (Guyonnet et al., 1999; Hanss, 2005). The possibility distribution $\pi(x)$ can be effectively represented by means of a fuzzy number, $\hat{X}$, whose membership function is $\mu_X(x) = \pi(x)$ (Dubois & Prade, 1988). For the fuzzy number to be implemented in a possibilistic framework, two important properties are required; convexity and normality. A fuzzy number is convex if, and only if, the α-cuts are bounded and closed intervals (Dubois & Prade, 1988). That is, $L(x)$ is a non-decreasing function and $R(x)$ is a non-increasing function (Chutia et al., 2011). The normality condition requires that at membership value of the core, $[b,c]$, equal 1. This condition specifies that at least one value of the parameter is entirely possible.

Unlike probability theory – which defines the likelihood of an event through a single precise utility measure – the imprecision characterised by a possibility distribution results in two evaluations of the likelihood of an event: the possibility (Π) and necessity (N) measures. The possibility and necessity that the value of a parameter defined by the fuzzy number $\hat{X}$, is less than A are then given by Equation (3.11) and Equation (3.12), and depicted in Figure 37 (Baudrit & Dubois, 2006).

$$\Pi(X \leq A) = \underset{x}{\text{Sup}} \quad \min\left[\mu_X(x), \mu_A(x)\right] \tag{3.11}$$

$$N(X \leq A) = \underset{x}{\text{Inf}} \quad \max[1 - \mu_X(x), \mu_A(x)] \tag{3.12}$$

Figure 37 demonstrates the application of the possibility and necessity measures with regard to the proposition of a parameter X taking on a value A. In Figure 37a, $\Pi(X \leq A) = 1$ and $N(X \leq A) = 1$: the proposition that X is less than A is necessarily true (certain). Figure 37b shows a situation where $\Pi(X \leq A) = 1$ and $N(X \leq A) = 1 - \alpha$: the proposition at X will be less than A is entirely possible but not necessarily true, with a necessity measure of $1 - \alpha$. This implies a greater possibility of the proposition being true than not true. In Figure 37c, $\Pi(X \leq A) = \beta$ and $N(X \leq A) = 0$: The proposition that X is less than A is possibly true with a possibility measure β. That is, the value is more likely to be greater than A. Figure
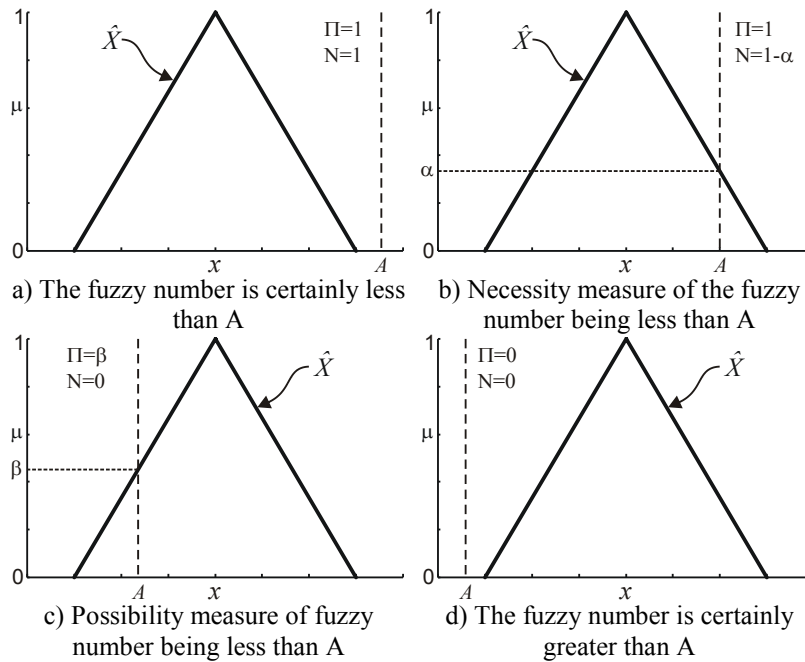
Figure 37: Fuzzy numbers and possibility theory.

37d, $\Pi(X \leq A) = 0$ and $N(X \leq A) = 0$: the proposition that X is less than A is necessarily false. That is, it is entirely possible that the value of X is not less than A.

The challenge problem in Chapter 5 shows how this concept of possibility and necessity measures has been applied in considering the stability of a rock slope when faced with epistemic uncertainty. Additionally, the challenge problem in Chapter 6 uses these possibility and necessity measures to investigate how they may assist in decision making when using rock mass classification systems.

Having discussed the possibility and necessity measures, it naturally follows that of critical importance to the output of a possibilistic analysis is the interpretation of 'possibility'. The semantics of possibility have been debated amongst theorists (Zadeh, 1980; 1982; Dubois & Prade, 1988; Dubois, 2006) with the following ideas offered to describe it; 'feasibility', referring to the solution of a problem: e.g. "it is possible to solve this problem"; 'plausibility', referring to the propensity for events to occur: "it is possible that the train arrives on time"; 'logical', describing the degree of consistency with the available information (Dubois, 2006), i.e. a possible proposition does not contradict the information. Yet another view of possibility relates to 'degree of surprise' (Baudrit & Dubois, 2005). This thesis adopts the view that a possibility distribution describes the more or less plausible values of an uncertain parameter, given the available information, which may be objective, subjective or a combination of the two (e.g. Figure 34). Indeed, Kaufmann and Gupta (1991) suggest that fuzzy numbers are well suited to characterise epistemic parameters because rather than being a measurement, they are functions that allow assignment of a subjective valuation to represent imprecise values.

3.3 P-boxes and Imprecise Probability

Probability boxes, or p-boxes, are mathematical structures that are able to represent both epistemic uncertainty and aleatory variability through the concept of imprecise probability (Williamson & Downs, 1990; Walley, 1991; Ferson et al., 2003). Imprecise probability, also referred to as probability bounds, analysis combines the methods of interval analysis and classical probability theory to produce a p-box (Ferson & Hajagos, 2004; Baudrit & Dubois, 2006) comprising two non-intersecting cumulative distribution functions (CDF) that generalise an interval.

3.3.1 Definition and examples of p-boxes

Figure 38 illustrates the concept of a p-box and imprecise probability. In Figure 38a, the upper bound CDF measures the degree of plausibility of an event (plausibility function), with the lower bound distribution used as a measure of the degree of certainty (belief function) of an event (Ferson et al., 2003; Dubois & Guyonnet, 2011). The distance between the plausibility and belief functions is a function of the imprecision in the model (Dubois & Guyonnet, 2011). Figure 38b shows how a p-box degenerates to a precise CDF when uncertainty is eliminated and only variability remains. This reducibility supports the definitions of epistemic uncertainty presented previously in section 2.2.1; the degeneration to a precise CDF may be achieved by improving the quality and/or quantity of information such that the threshold of precise information is crossed (as was illustrated by Figure 2). With respect to the new taxonomy previously presented in Figure 17 (see section 2.7), this transition from epistemic uncertainty (the p-box in Figure 38a) to aleatory variability (the precise CDF of Figure 38b) can be achieved by improving the quality and/or quantity of information.
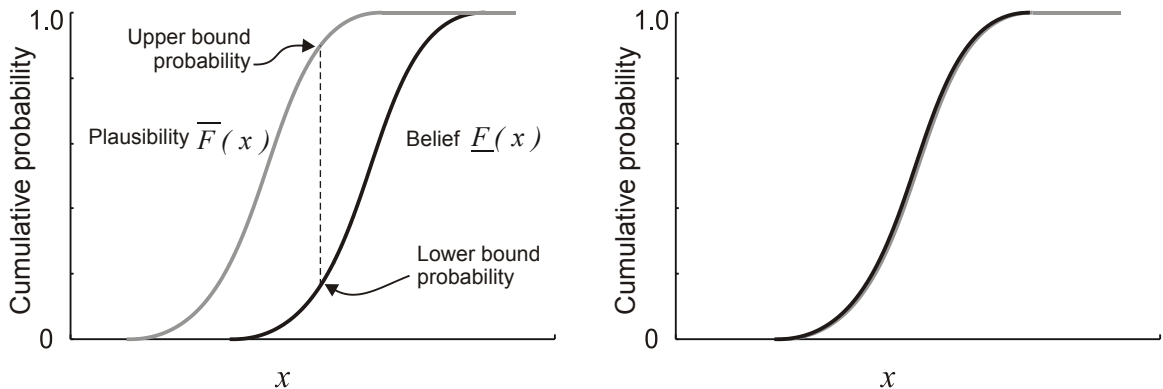
a) P-box with upper and lower bound CDF representing
imprecision

b) P-box degenerates to a precise CDF when the state of
precise information is reached

Figure 38: Imprecision represented by a p-box and degenerate p-box with no imprecision.

3.3.2 Mathematics of p-boxes

The p-box of Figure 38a represents the family of all possible probability distributions between the upper and lower bounds, and is denoted by the interval $\left[\overline{F}(x), \underline{F}(x)\right]$ of all cumulative probability functions such that $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$. That is, $\underline{F}(x)$ is the lower bound on the probability of occurrence of the imprecisely known parameter x, and an upper bound on the quantiles (i.e. the value of x). Similarly, $\overline{F}(x)$ is an upper bound on the same probability and a lower bound on the quantile (Ferson et al., 2003).

P-boxes may be employed to characterise extrinsically epistemic parameters (as defined earlier in section 2.2.3) when; the shape of the underlying distribution is known but precise values are not available with which to define its statistical moments (a parametric p-box), or; the shape of the distribution is unknown, but statistical parameters such as mean, mode or median are known (a non-parametric p-box) (Tucker & Ferson, 2003). These two approaches are demonstrated by returning to the example of UCS.

Earlier, using evidence from examples presented in published literature, it was explained how UCS could be characterised as an extrinsically epistemic property that can be characterised by a truncated normal distribution. Suppose now that on top of the information received thus far, one undertakes a few additional laboratory tests, which are insufficient in number to fit a precise PDF, but allow us to estimate intervals of the mean, say $[55,65]$, and standard deviation, say $[5,8]$. Having evidence of the underlying shape of the distribution, with this information a p-box can be obtained by computing the envelope of all normal distributions that have parameters within these intervals. These bounds are determined by convolution on the CDF of the normal distribution (F^{norm}), where the imprecise values of the moments are given by the set $\alpha \in \{(\mu, \sigma) | \mu \in [\mu_1, \mu_2], \sigma \in [\sigma_1, \sigma_2]\}$, as follows:

$$\underline{F}(x) = \min_{\alpha} F^{norm}{}_{\alpha}(x) \qquad (3.13)$$

$$\overline{F}(x) = \max_{\alpha} F^{norm}{}_{\alpha}(x) \qquad (3.14)$$

In practical terms, the bounds of the p-box are simply the lower and upper envelope of the four permutations: $[\sigma_1, \mu_1]$, $[\sigma_1, \mu_2]$, $[\sigma_2, \mu_1]$, $[\sigma_2, \mu_2]$, as shown in Figure 39a. The parametric p-boxes for other well known probability distributions can be similarly obtained. If sufficient information is obtained to define precise values for the moments of the underlying distribution, the parametric p-box will degenerate to a precise CDF, similar to the example of Figure 38.

Alternatively, if one was unaware of the underlying distribution of UCS (or any other parameter) but could provide the bounds and a statistic such the mode, mean or median value, it would be possible to construct a non-parametric p-box, as shown in Figure 39b-d. Simple mathematical expressions to generate these non-parametric p-boxes are presented in detail by Tucker & Ferson (2003).

a) parametric (normal) p-box of UCS

b) Non-parametric p-box of UCS with a mode of 50MPa

c) Non-parametric p-box of UCS with a mean of 50MPa

d) Non-parametric p-box of UCS with a median of 50MPa
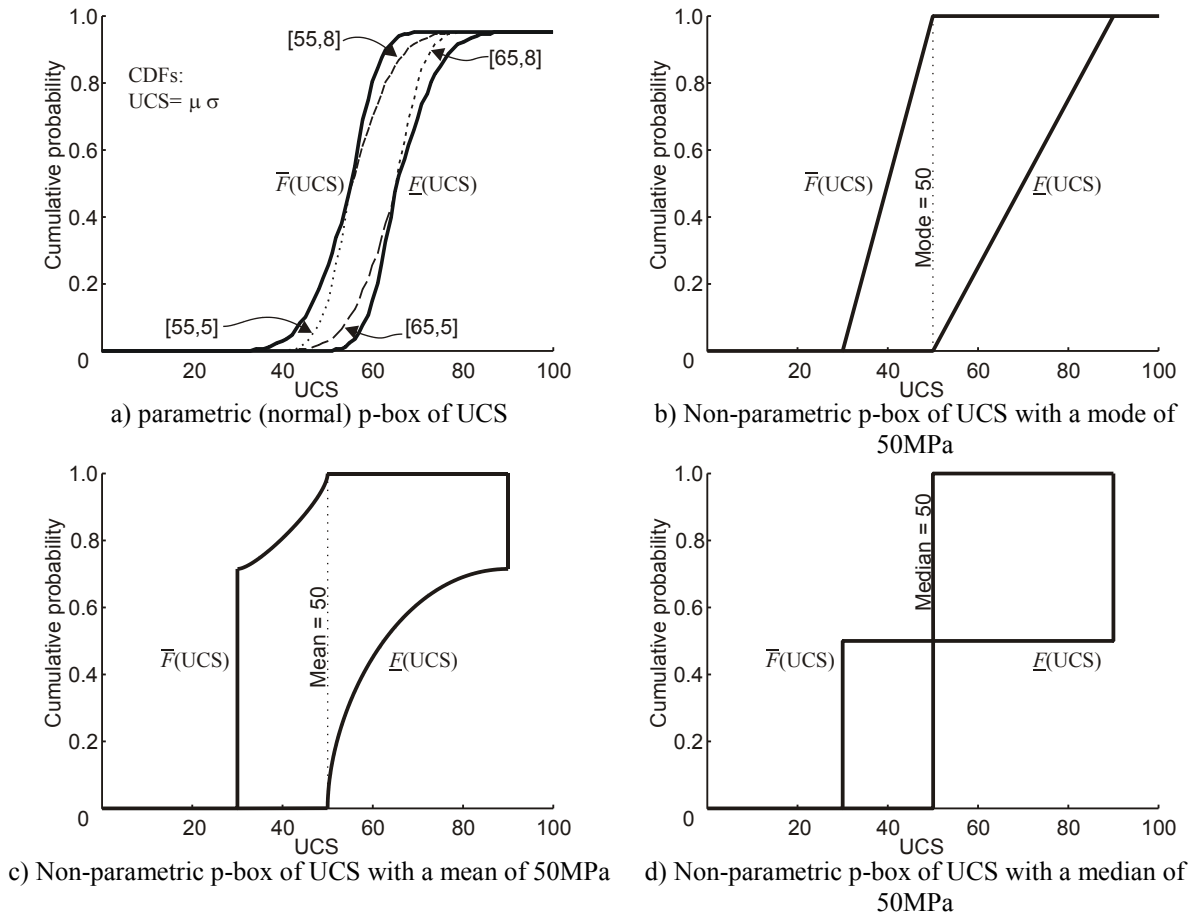
Figure 39: Parametric and non-parametric p-boxes (after Tucker & Ferson, 2003).

In Figure 39, the horizontal spans between the bounds of the p-boxes represent the interval of values at a given probability level. The vertical distance between the bounds of the p-box represents the imprecise probability for any given value. That is, the area between the bounds is proportional to the degree of imprecision (Tucker & Ferson, 2003). Consequently, only imprecise statements can be made on either the probability of occurrence or quantiles of the parameter. For example, from the p-box in Figure 39a, the following statements can be made: 'the probability that UCS is less than 60MPa is between 0.9 and 0.1', or; 'there is a 50% probability that UCS is contained in the interval $[55,65]$'. Note that this is consistent with the information available from our few precise measurements and previous knowledge on the shape of the distribution. This example demonstrates how the p-boxes follow the faithfulness principle, which was first detailed in section 2.5.1, the crux of which is that the representation model does not require one to subjectively invent a precise probability distribution when the data are in fact imprecise.

The challenge problem presented later in Chapter 7 uses the concept of parametric p-boxes presented in this section to characterise the unpredictability in UCS and propagate this

through a mathematical model. Appendix D presents the algorithms used in the challenge problem Chapter 7, which are based on the theory presented in this section.

3.4 Frequentist probability

Section 2.6 introduced the frequentist approach to probability as being that which assumes an event is the result of a random process that can be realised by repeating an experiment a large number of times and characterising the variability by a probability distribution function (PDF). This section describes this precise nature of probability theory.

3.4.1 Axioms of frequentist probability

Through a large series of trials, the variability in the objective data can be visualised by plotting a histogram, to which a PDF can then be fit. The probability distribution function contains very specific information on the probability of occurrence of the parameter it defines. This information is derived through statistics obtained from the data sampled and defined by a probability density function $f_X(x)$, which describes the relative probability that a random variable X will take on a given value x. From this, the cumulative density function (CDF) can be derived to calculate the probability that the random variable X will be less than or equal to x, as follows:

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx \tag{3.15}$$

Any function used to define the probability distribution of a random variable must satisfy the following axioms of probability theory (Ang & Tang, 2007):

For every event E in a sample space S, there is a probability

$$(i) \qquad\qquad P(E) \geq 0 \tag{3.16}$$

The probability of the certain event S, is

$$(ii) \qquad\qquad P(S) = 1 \tag{3.17}$$

For two mutually exclusive events E_1 and E_2

$$(iii) \qquad\qquad P(E_1 \cup E_2) = P(E_1) + P(E_2) \tag{3.18}$$

On the basis of these axioms of probability theory, the following condition for an event and its complement, E^c, must be satisfied:

$$P(E^c) = 1 - P(E) \qquad (3.19)$$

It is Equation (3.19) which epitomises the precise nature of probability theory. This equation implies that the probability of a specific event occurring or not occurring is certain. That is, it removes the possibility that the event could take on a range of values; it does not allow for imprecision (Colyvan, 2008). From this, one can conclude that probability distribution functions are only appropriate to define random variability when the state of information is sufficiently precise. It is on the basis of these axioms that the proposed taxonomy presented earlier in section 2.7, and illustrated in Figure 17, requires a sufficient quantity of precise (high quality) objective data to characterise aleatory variability. If these criteria are met and the state of information can be characterised as precise, probability theory offers powerful tools to propagate variability through analytical models in order to develop probabilistic representations of the response of a parameter or system.

3.4.2 Applications of frequentist probability models

One of the most widely used tools is Monte-Carlo (MC) simulation, which randomly samples values from the PDFs that define the variability of the input parameters, and enters these into the calculation to obtain a PDF of the output variable being investigated. A large number of iterations are required in order to produce an adequately representative output PDF. The theory of MC type simulation procedures has been comprehensively published and is therefore not repeated here (see e.g. Ang & Tang, 2007). Monte-Carlo simulations have been widely applied to many rock mechanics and rock engineering problems to calculate distributions of various performance measures. One example is that of predicting displacements around an underground cavern (Cai, 2011), as shown in Figure 40. However, one of the most common applications of MC simulations has been to calculate the 'probability of failure' (see Figure 41), i.e. the probability that the Factor of Safety is less than 1, of various rock engineering structures in an attempt to quantify risk and reliability. Risk is defined as the probability of occurrence of some adverse consequence (Vose, 2000; Tucker & Ferson, 2003; Aven, 2010) with reliability being the probability that a system or product will perform its intended functions within specifications over its intended design life (Booker & MacNamara, 2005).
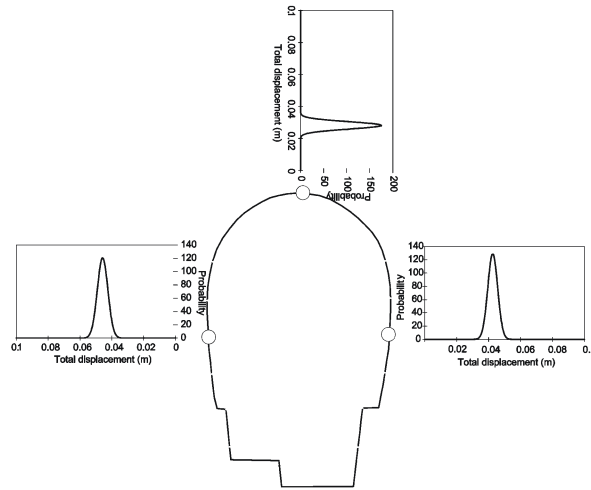
Figure 40: Distributions of boundary displacement determined using Monte Carlo simulation (from Cai, 2011).
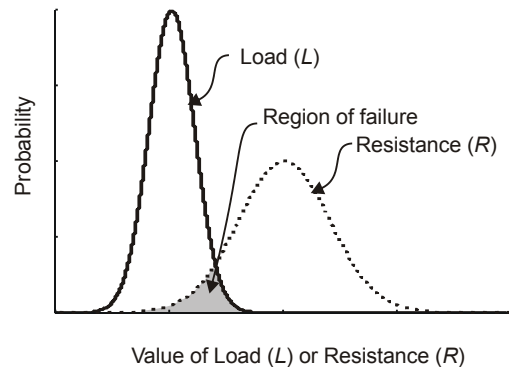
Figure 41: Probability of failure by assuming Loads (*L*) and Resistances (*R*) are aleatory.

For risk quantification studies, and as alternative to MC simulation, increased use has been made of reliability analysis in an attempt to formally quantify unpredictability and ascertain the level of risk prior to execution of the project. Commonly applied tools in reliability analysis are; first order-second moment (FOSM) approaches, first and second-order reliability methods (FORM and SORM), and event tree analysis (Einstein & Baecher, 1983; Zhao & Ono, 2001; Low, 2008). To perform a reliability analysis using these tools requires knowledge of the means and the variances (the second moments) of the input variables that form the load (*L*) and resistance (*R*) functions, which in turn are used to evaluate the performance function (*M*) that defines the safety factor (see Figure 42) (Christian, 2004; Johari et al., 2013). Using these tools, a reliability index (Figure 43) can be computed, which is a factor by which variability, quantified in terms of the standard deviations of the random variables, would need to change to bring the system to the failure condition. The advantage of these reliability methods over MC simulation is the reduced computational effort required. For instance, Low (2008) presents a simple example of FORM analysis applied to assess rock slope stability using a simple spreadsheet program (see Figure 43).

The immediate benefit of such probabilistic analysis is that one can quantify uncertainty in well known terms of risk, reliability and probabilities of failure etc. This is especially useful in decision making, where a precise output means that definitive decisions upon which they are based are easily made. For example: we will only accept a design where the probability of failure or the probability of deflections exceeding a given value is greater than 95%, or the reliability index is greater than a pre-specified value.
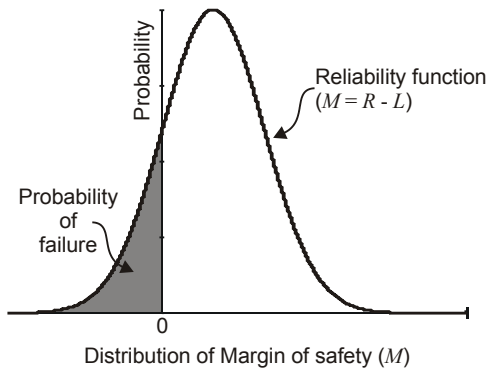
Figure 42: Distribution of margin of safety (*M=R-L*) used in reliability analysis (after Christian, 2004).
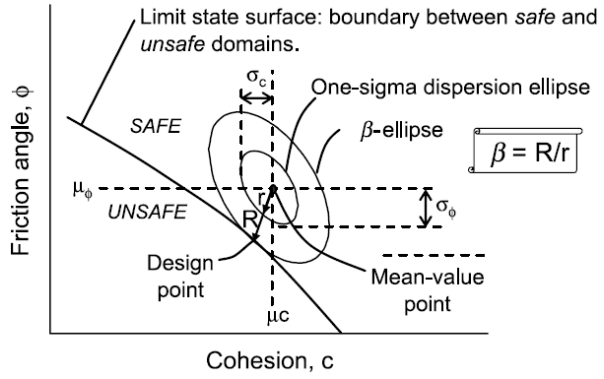
Figure 43: Graphical representation of the reliability index (from Low, 2008).

Of course, all of these risk and reliability based tools fail to differentiate between aleatory variability and epistemic uncertainty. The fundamental assumption embodied in these methods is to treat the total unpredictability as being entirely aleatory. That is, the unpredictable rock mass parameters are defined as random variables described by statistical parameters (Nadim, 2007) with the uncertainty modelled as a known stochastic distribution, i.e. $f_X(x)$. The resulting output is a precise probability distribution, which follows the axioms of probability theory, and this can then be used to predict the probability of occurrence of certain values (Becker & Moore, 2007; Ruffolo & Shakoor, 2009). However, the proposed taxonomy (i.e. Figure 17) presented in section 2.7 demonstrated that in order to characterise unpredictability as aleatory variability, a very strict set of requirements is needed; one of which is the objective nature of the type of measurement, i.e. two people observing the same data will arrive at the same conclusions. In this way, as was discussed earlier in section 2.6.3, the analysis remains both faithful and robust to the level of information present in the input data. Otherwise, and as was first shown by the example of Figure 16 (see section 2.6.4), analysis using different values for each of these parameters can result in a different FoS value, which in turn introduces subjectivity. This subjectivity goes against the criteria required by the new taxonomy Figure 17 that characterise aleatory variability.

3.5 Subjectivist probability: Bayes's Theorem

When applying a subjectivist approach to probability, in order to remain faithful to the available level of information, Section 2.6.2 supported application of the Bayesian updating approach, with objectively assigned priors, to tackle problems involving extrinsically epistemic parameters. This section briefly describes the basis of Bayes's Theorem that forms the basis of the Bayesian updating approach.

3.5.1 Definition of Bayes's Theorem

Bayes's Theorem adopts the subjectivist view of probability by using the concept of conditional probability. Conditional probability links the subjective degree of belief one has in the likelihood of a proposition, before and after accounting for objective evidence. Most simply, Bayes's Theorem uses conditional probability to determine the probability of a hypothesis h being true, given the evidence, e, and is given by (Swinburne, 2002):

$$P(h\,|\,e) = \frac{P(e\,|\,h)P(h)}{P(e)}$$
(3.20)

In Equation (3.20), $P(h)$ is the prior probability of the hypothesis, and $P(h\,|\,e)$ is the posterior probability given the probability based on the evidence, i.e. the quotient $P(e\,|\,h)\,/\,P(e)$, which is a measure of the support the evidence provides for the initial hypothesis (Ang & Tang, 1984; Tucker & Ferson, 2003; Ang & Tang, 2007).

   With respect to the new taxonomy presented in Figure 17, it is the continued gathering of evidence, and hence support for the hypothesis, that allows an objectively assigned PDF to converge to an aleatory model

3.5.2 Mathematics of subjectivist probability

Bayes's Theorem to update the prior distribution ($f_{prior}(x)$) of a parameter (x) modelled as random variable (x) using a precise PDF, to obtain an updated, or 'posterior', probability distribution ($f_{post}(x)$) is given as follows:

$$f_{post}(x) = \frac{P(e\,|\,x)f_{prior}(x)}{\int P(e\,|\,x)f_{prior}(x)dx}$$
(3.21)

In Equation (3.21), $P(e\,|\,x)$ is the conditional probability, or likelihood, of observing the experimental outcome e assuming that the value of the parameter is x (Ang & Tang, 2007).This 'Bayesian updating' process is continued as further objective information (e) becomes available. This updating process is repeated as further data become available by adopting the posterior distribution as the new prior for subsequent iterations in the updating process. The updating may be performed by data gathered in any order, and singly or in groups; the final posterior distribution obtained once all the data have been collected is the same irrespective of this. As data accumulates during sequential updates, the initial choice of the first prior has a smaller and smaller influence on the final posterior (Ang & Tang, 2007).

However, and as exemplified by the discussion presented earlier in section 2.6.2 (Figure 12 and Figure 13), experts may assign different priors based on their subjective belief in the initial hypothesis. As a result, their posteriors will likely differ. Therefore, it is paramount that sufficient objective data is collected so that the updated posteriors converge to a distribution near that which would have resulted if the data had been available to assign an aleatory model in the first instance (Tucker & Ferson, 2003).

Section 2.6.2 also concluded that application of the Bayesian approach where the priors are defined subjectively, i.e. without any objective evidence as justification, and not subsequently updated using Bayes's Theorem is neither faithful nor robust. To account for this shortcoming of the subjective Bayesian approach, robust Bayes's analysis, also called Bayesian sensitivity analysis (Berger, 1985; Insua & Ruggeri, 2000), has been proposed. In this approach, an analyst's uncertainty about which prior distribution should be used is expressed by replacing a single precise prior distribution by an entire class of prior distributions. The analysis proceeds by studying the variety of outcomes as each possible prior distribution is considered. In this approach, uncertainty about the likelihood function or even the utility function can likewise be expressed with classes of PDFs (Tucker & Ferson, 2003). This approach is closely related to probability bounds analysis discussed in section 3.3.

3.6 Hybrid analysis

Rock mechanics calculations are generally multi-parameter problems, some of which may be intrinsically epistemic, extrinsically epistemic and others aleatory. As these parameters represent varying levels of information, a framework is required with which to jointly propagate uncertainty and variability represented by any combination of the unpredictability models discussed in this Chapter. Fortunately, the theory of imprecise probability provides such a framework, with the output being in the form of a p-box. Joint propagation, or hybrid, analysis methods have been developed using formal links between intervals, possibility theory, imprecise probability and belief functions (Baudrit & Dubois, 2006). Their applications in various fields of science and technology have been published (Cooper & Ferson, 1999; Baudrit et al., 2005; Baudrit & Dubois, 2006; Baudrit et al., 2007) and extensively reviewed by Dubois & Guyonnet (2011). The following discussion presents a summary of the key concepts required to undertake hybrid analysis involving problems combining deterministic values, intervals, fuzzy numbers and probability distributions. These concepts are then used to construct the generic algorithms (see Appendix A) to propagate

unpredictability in the hybrid challenge problem presented later in Chapter 7 (implementation provided in Appendix D).

3.6.1 Hybrid interval and fuzzy analysis

As a fuzzy number is a generalisation of an interval, it follows that an interval can be represented by a fuzzy number. This is shown in Figure 44. The information contained in this figure is as follows: 'at every possibility level, the value of x lies between $[a,b]$', which is equivalent to the definition of intervals presented in section 3.1 (Equation (3.1)). Therefore, if the parameters in an analysis are a mix of intervals and fuzzy numbers, the computation may be propagated by representing the intervals as fuzzy numbers and propagating the analyses using fuzzy arithmetic, the output of which will be another fuzzy number.
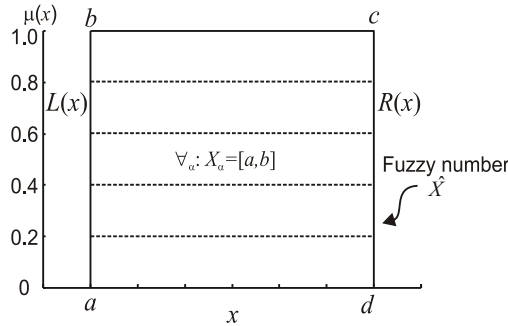
Figure 44: Fuzzy representation of an interval. Each α-cut is an interval $[a,b]$.

3.6.2 Hybrid epistemic and aleatory analysis

Analyses where the parameters are a combination of intervals, fuzzy numbers or probability distributions require the use of imprecise probabilities. As imprecise probability naturally couples interval and stochastic analysis, and as fuzzy numbers are generalisations of intervals, it follows that intervals, fuzzy numbers and probability distributions may be combined if a relationship exists between each of these. This relationship is demonstrated with reference to Figure 45 and Figure 46.

An interval, whether spike or box representation (see Figure 30 in section 3.1), can be represented by the p-box shown in Figure 45. This p-box contains the following information: there is a 0% probability that the value is less than 'a' and 100% probability that the value is less than 'b'. That is, the value must lie within $[a,b]$, with no other information about the uncertainty structure between them. Once again, this is the same level of information that was defined for intervals in section 3.1 (Equation (3.1)). For an extrinsically epistemic interval ('box-interval'), as additional data became available, one could move to represent it by a

unique p-box, and eventually a precise PDF. This is demonstrated later through the challenge problem presented in Chapter 7.

The possibility and necessity measures of a possibility distribution have been shown to be linked to the boundaries of a p-box, as shown in Figure 46 (Zadeh, 1965; Zadeh, 1995; Baudrit & Dubois, 2006; Baudrit et al., 2007). In basic terms, the relationship between possibility and probability can be understood through the following: If an event X, which takes on a value x, is impossible, $\Pi(X) = 0$, then it is also improbable and so $P(X) = 0$. Similarly, if the event X is necessary, $N(X) = 1$ (a certainty), then it is also completely probable, i.e. $P(X) = 1$. Using these definitions, Figure 46 can be plainly interpreted as: there is 0% probability that X is less than the interval represented by the minimum and most possible value, the interval $[a,c]$, and there is 100% probability that the value is less than the interval defined by the most possible and maximum value $[b,d]$. The link between Figure 45 and the non-parametric p-box of Figure 39b, above, can be seen; a p-box defined with the mode value is equivalent to a fuzzy number with the core set at the mode value.
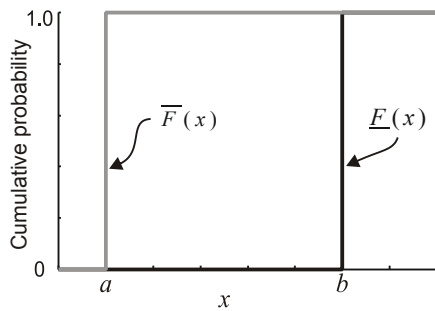
Figure 45: Interval represented as a p-box.   **Figure 46:** Possibility distribution as a p-box.
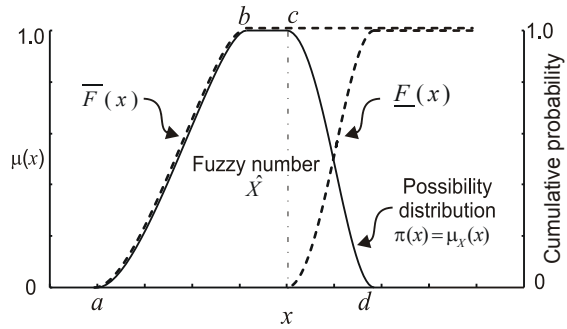
One important aspect of this possibility-probability transformation needs to be realised: while a possibility distribution can encode a family of probability distributions, it does not imply that the parameter represented by the possibility distribution is aleatory. This is because the p-box induced by the possibility distribution cannot degenerate to a precise PDF, it is inherently imprecise and the output of any analysis using this p-box will also be imprecise. Indeed the normality and convexity criterion of a possibility distribution means that any possibility distribution can be expressed as a p-box, however not any p-box can be expressed as a possibility distribution (Baudrit & Dubois, 2006). This implies that p-boxes can be used to convey additional information that a fuzzy number, and its associated possibility distributions, cannot. Based on these concepts, it can be concluded that in multi-parameter models, each of the parameters should be characterised based on the level of

information available, and if required expressed as an equivalent p-box to propagate the hybrid analysis.

As part of this thesis, using the concepts presented in this section, simple algorithms to undertake hybrid analysis that combines intervals, fuzzy numbers, p-boxes, precise PDFs and deterministic parameters have been set up using MathCAD. Verification of these algorithms has been undertaken by replicating the results of a numerical example (involving deterministic, fuzzy and precise PDFs) presented in the literature by Dubois & Guyonnet (2011). The hybrid algorithms developed alongside the verification example are presented in Appendix A. These algorithms are used later in the challenge problems presented in Chapter 5 to Chapter 7.

3.7 Synopsis

This Chapter presented a detailed discussion on the mathematical basis of the unpredictability models initially introduced in the Level of Information concept (Figure 8 in section 2.5). The definitions presented in this Chapter conclude that intervals are required when it is inappropriate to make statistical statements about a parameter, with the information available. With respect to the proposed taxonomy (Figure 17), this section also defines what an interval may represent; the first theory uses the analogy of 'spike-intervals' and implies that the parameter in question is not drawn from an underlying random process – it is intrinsically epistemic (as defined in section 2.2.3). Further information could only serve to reduce the bounds of the interval. The second theory is referred to as a 'box representation', and suggests that the parameter in question is drawn from an underlying random process, though the current level of information is insufficient to identify the form or parameters of the aleatory model with which to characterise it; it is extrinsically epistemic (also defined in section 2.2.3)

On the contrary, through an examination of the axioms of probability theory, the discussion in this Chapter has demonstrated the precise nature of probability theory. On this basis, it is concluded that that probability distribution functions are only appropriate to define random variability when the state of information is sufficiently precise, which (as was stated in section 2.2.1) requires a sufficient quantity and quality of objective data. The proposed taxonomy identifies this need for precision as one criterion that must be fulfilled in order to characterise unpredictability as aleatory variability.

This Chapter has shows how intermediate levels of information can be modelled using theories that generalise intervals. As information increases epistemic uncertainty may be characterised by a fuzzy numbers if one is able to define preferences to values between the

intervals. It is triangular and trapezoidal fuzzy numbers that are most commonly used. The fuzzy numbers may be defined subjectively but should be consistent with the available information. If one is able to further increase the level of information, p-boxes may be employed to characterise extrinsically epistemic parameters when; the shape of the underlying distribution is known but precise values are not available with which to define its statistical moments (a parametric p-box), or; the shape of the distribution is unknown, but statistical parameters such as mean, mode or median are known (a non-parametric p-box) (Tucker & Ferson, 2003). These unpredictability modelling methods may be combined using hybrid analysis, the output of which is a p-box. Verified algorithms for hybrid analysis have been developed and presented in Appendix A.

Having examined the basis of each of these unpredictability modelling methods and the level of information required to implement each, the next Chapter combines the concepts presented in the proposed taxonomy (Figure 17 in section 2.7), the level of information concept illustrated in Figure 17 and the concepts presented in this section to develop a novel framework that directs the user to objectively determine the optimal unpredictability modelling method through a review of the available information.

Chapter 4

PROPOSED FRAMEWORK FOR CHARACTERISING AND PROPAGTING UNCERTAINTY AND VARIABILITY

The concepts and discussion presented in section Chapter 2 suggested that as an initial step, it is important to recognise the distinction between epistemic uncertainty and aleatory variability when characterising a parameter or system. The proposed taxonomy, presented in Figure 17, provides a tool to assist in identifying of the nature of unpredictability through a qualitative and quantitative assessment of the available information so that a complete picture of the total unpredictability can be developed. This information forms the input to one of the unpredictability models discussed in section Chapter 3 (i.e. interval-oriented or probabilistic approaches), which processes the information in specified ways presenting an output in terms of 'measures of unpredictability' (e.g. possibility measures, probability of exceedence, etc.) or descriptions of unpredictability (e.g. probability distribution function) (Zimmermann, 2000).

In order to select the most appropriate uncertainty model, therefore, the next steps of the analyst should be to consider: the causes of uncertainty, quantity and quality of information available, type of information processing required by the respective uncertainty calculus (e.g. precise PDFs or intervals) and the language required as an output. Currently, the selection of epistemic or aleatory models seems to be undertaken at the whim of the analyst, which is incorrect. However, it is proposed that that the selection of these models should be an objective process.

Therefore, in this Chapter, a new framework is introduced that draws together all these concepts and directs the user to the most appropriate unpredictability model, through an assessment of the available level of information. This framework is one of the principal contributions of this work. Uniquely, it provides a new tool that will allow engineers engaged in rock mechanics and rock engineering to objectively characterise and propagate unpredictability in parameters that define the properties of fractured rock masses.

Figure 47 presents a conceptual layout of the overall framework, which consists of three individual flowcharts; the main-framework, a data characterisation strategy sub-chart and an unpredictability model selection sub-chart.
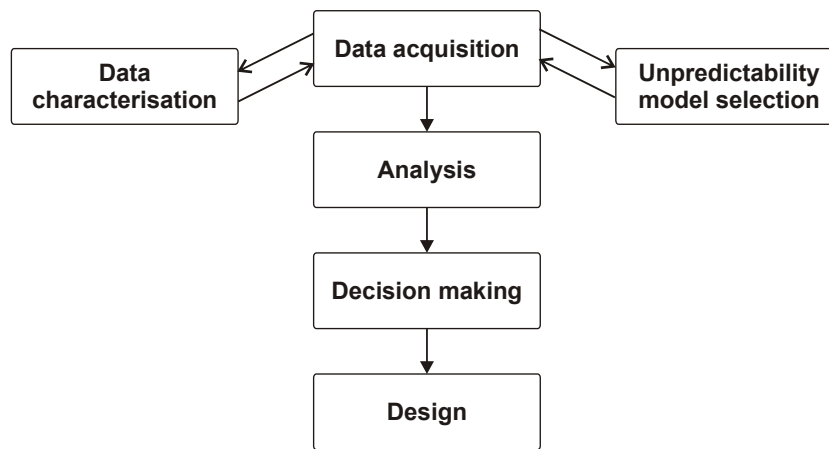
Figure 47: Conceptual outline of proposed framework.

4.1 Proposed framework

Chapter 2 and Chapter 3 discussed the range of models available for handling uncertainty and variability. Here, a new framework (Figure 48) is presented that gives a protocol for correctly characterising and propagating uncertainty and variability through engineering calculations, based on a faithful assessment of the available information. The framework is divided in to three distinct phases of the design process; data acquisition and characterisation, model propagation, and decision making. The entry point of Figure 48 is at the initial data acquisition stage, leading to a second data acquisition stage following identification of the unpredictable parameters to be used in the analytical model. This allows the second data acquisition stage to target collection of data for the epistemic parameters (as noted earlier in section 2.4). Prior to undertaking the analysis, the framework leads to a separate data characterisation and model selection strategy (discussed further in section 4.2 and 4.3, respectively), both of which influence the form of the output.

Following the analysis, the framework directs the analyst in interpreting the output. Analyses which contain epistemic parameters result in an imprecise output, i.e. an interval, fuzzy number or p-box, while the output of a Bayesian analysis produces a subjectively determined precise PDF. Both these outputs require the analyst to make a subjective assessment based on the available information in order to produce a design. If the analyst is unable to make a decision because the bounds of the output are too wide, the framework directs the user back to collect further data. On the other hand, an entirely aleatory analysis produces an objective precise PDF, which can be used to form a decision based on statistical measures (e.g. reliability index. See e.g. Baecher & Christian, 2003; Low, 2008), or probability of occurrence).

The strength of this framework is two-fold: firstly, it assists in directing investigation (which can be costly) appropriately to reduce unpredictability. Secondly, it presents a method for objectively selecting an appropriate uncertainty analysis based on the available information. The overall result is that following this framework will harmonise designs by reducing arbitrary choices in characterising and propagating unpredictability in rock mechanics and rock engineering, and thus improve the safety and efficiency in rock engineering designs.

4.2 Data characterisation strategy

The data characterisation flowchart of Figure 49 directs the selection of an appropriate theory to represent the unpredictability of a parameter. The first question divides the path between representation tools appropriate for parameters which may be aleatory and those which are intrinsically epistemic. The former of these require the state of information to be precise, which can only be achieved by a sufficient number of (objective) precise measurements (see Figure 3 in Chapter 2). If the parameter is inherently imprecise, and requires subjective estimation (e.g. GSI), the flowchart leads towards intervals and fuzzy models; i.e. the parameter is intrinsically epistemic, as defined previously in 2.2.3.

The first question in Figure 49, *'Can the data be objectively measured?'* ensures that the parameter in question is not inherently imprecise which, as we have seen, would require the use of an imprecise modelling method. After this, the sequence of questions in the data characterisation strategy sub-chart are organised in a manner that directs the user through a path starting from the highest level of information to the lowest (from right to left in Figure 49). In this way, the user may determine – through a series of 'no' answers – the true nature of

the unpredictability in the parameters and thus potentially pre-empt what further data collection may be required to improve the level of information, if necessary.

A path which requires specific consideration is that which leads to a subjective PDF via the Bayesian approach. The path leading to the Bayesian approach requires one to answer 'yes' to the first question. That is, the data *can be* – though they may not *have been* – objectively measured; they must be extrinsically epistemic. This eliminates subjective estimation of precise PDFs for intrinsically epistemic parameters. In this way, the path presented in this framework reflects our earlier assertion (see section 2.6.2) that the definition of a 'prior' PDF should be based on objective empirical evidence and updated to converge to the aleatory model as information becomes progressively available.

4.3 Uncertainty model selection strategy

Figure 50 presents the model selection strategy flowchart which directs the user to select the most appropriate unpredictability model, following characterisation of the parameters used in the analysis.

In Figure 50, the solid arrows represent the path that should be followed if all the parameters in the analysis are characterised by the same unpredictability representation tool (i.e. the bottom of the data characterisation strategy sub-chart). The dashed arrows direct the user to an unpredictability model capable of handling multiple data types. In this way, the framework leads the user to a modelling method which requires the least amount of computational effort, given the available information. Figure 50 ends by identifying the type of output expected, which then allows the user to pick up at the appropriate location in the main framework.

4.4 Synopsis

This Chapter presents one of the main contributions of this thesis: a novel framework for characterising and propagating epistemic uncertainty and aleatory variability in rock mechanics and rock engineering. This framework brings together all the concepts presented in the new taxonomy (Figure 17 introduced in section 2.7) with the Level of Information concept (Figure 8 in section 2.5), in a series of three flowcharts. These flowcharts are set out in a methodical manner, commencing with the data acquisition phase and leading the user through the data characterisation process given the available information, on to selecting an appropriate unpredictability model and thence to decision making.

The framework is presented in a series of three separate flow charts. The main framework (Figure 48) leads the user through the design process and directs further investigation/data acquisition through-out the design process, if so required. In this process, the data characterisation strategy sub-chart (Figure 49) leads the user to characterise each of the unpredictable parameters with an appropriate representation tool. These tools are the mathematical modelling methods detailed in Chapter 3. This data characterisation strategy sub-chart amalgamates the level of information concept (Figure 8) within it. Once the unpredictable parameters have been adequately characterised, the model selection strategy sub-chart (Figure 50) directs the user to apply the appropriate analytical methods detailed in Chapter 3.

This framework provides the single tool that can be applied in practice to properly characterise and propagate unpredictability in rock engineering design. In following this framework, the output will be both faithful and robust to the available information. The taxonomy of (Figure 17) may be used to supplement understanding of the framework with the concepts presented in this thesis. Using these tools, engineers will be able to tackle, in a manner that has never been done before, the problem of unpredictability in rock engineering problems. In order to demonstrate the use of these new tools, the succeeding Chapters will now embark on a series of challenge problems commonly encountered in rock engineering.
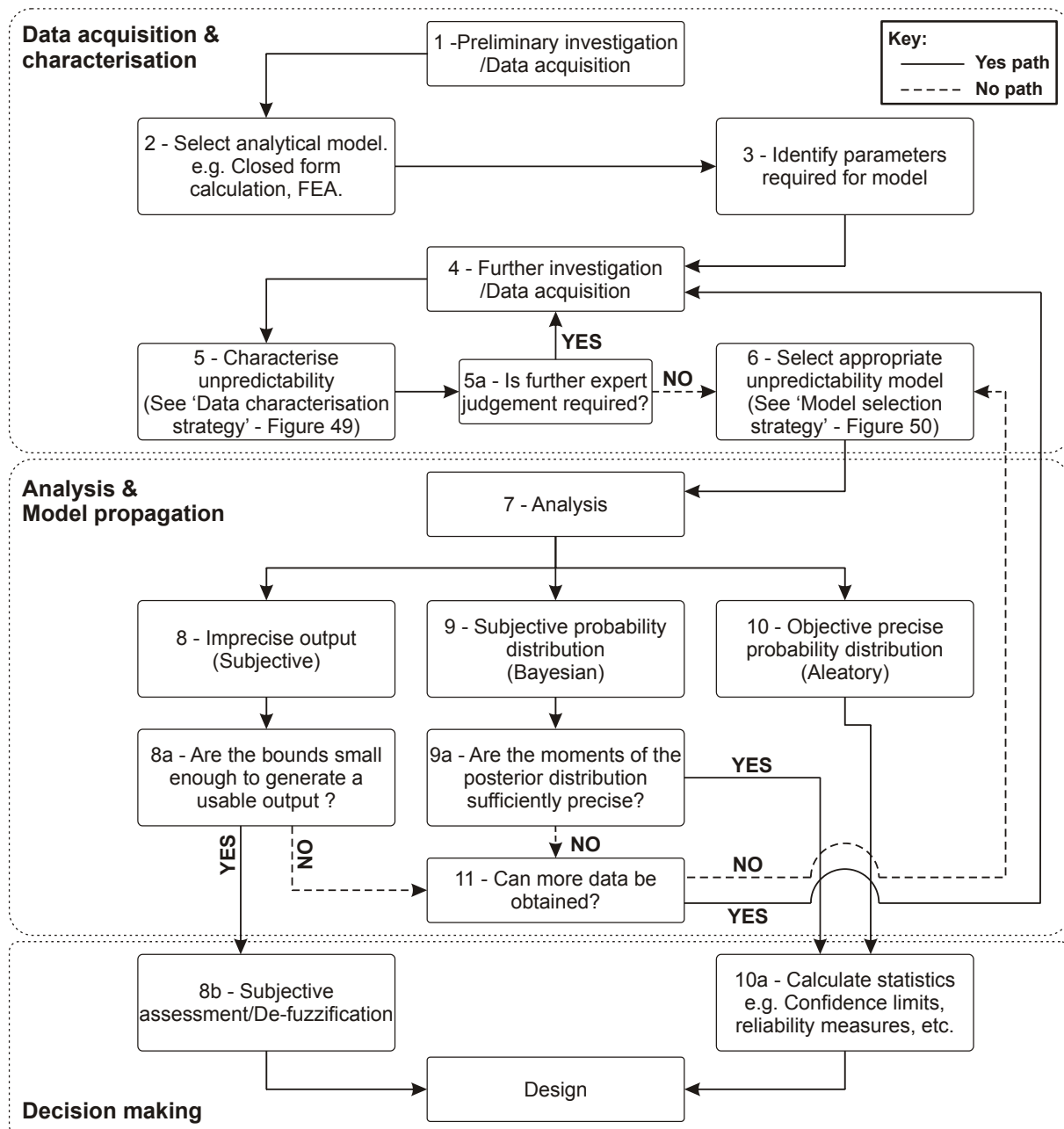
Figure 48: Proposed framework for characterising and propagating unpredictability.
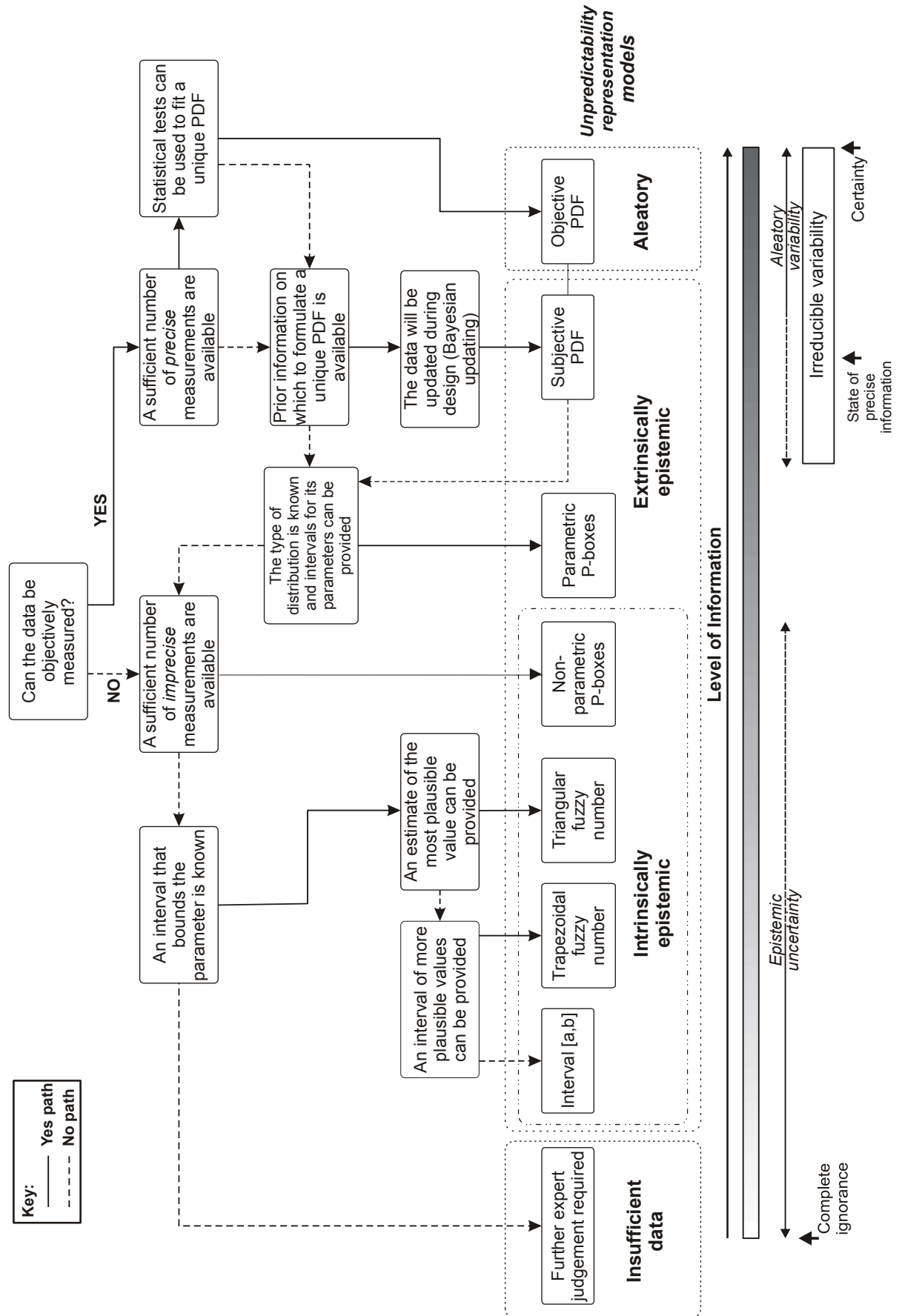
Figure 49: Data characterisation strategy sub-chart (after Aughenbaugh, & Paredis, 2006; Guo & Du, 2007; Wenner & Harrison, 1996; Dubois & Guyonnet, 2011).
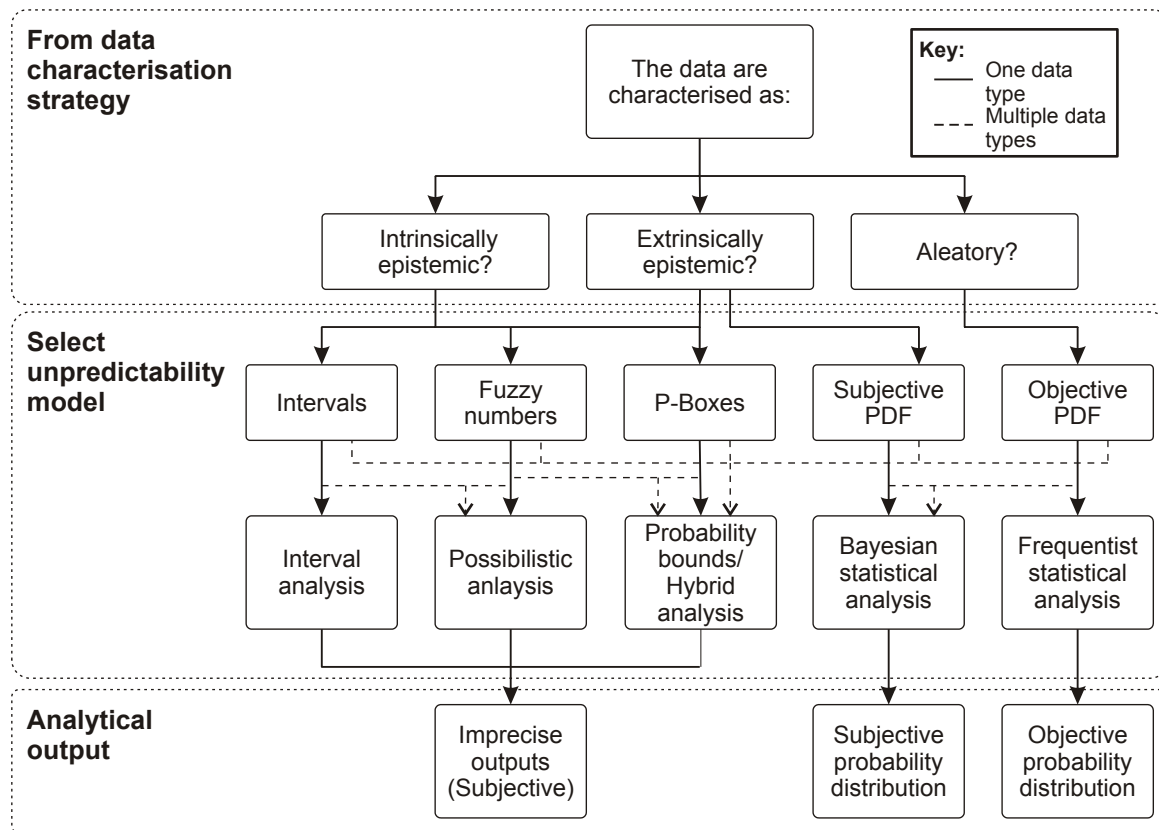
Figure 50: Model selection strategy sub-chart.

Chapter 5

CHALLENGE PROBLEM 1 – PLANAR SLOPE STABILITY

In DIANE rock masses, the stability of rock slopes is usually governed by the potential for sliding along well-defined discontinuity or fracture surfaces. Rock slope stability is often assessed using closed form, limit equilibrium models (LEMs) that compute a Factor of Safety (FoS) against sliding along one, or a series of intersecting, joint surfaces. Hoek & Bray (1974) provide a comprehensive account of the methods for calculating the Factor of Safety (FoS) for planar slopes using deterministic inputs within LEMs.

Customarily, the inputs to LEMs have been deterministic values, which lead to a deterministic FoS. Consequently, the acceptable FoS in a particular design situation has been based on the analyst's level of confidence in the input parameters as well as the perceived importance of the structure (Hoek & Bray, 1974; Hoek, 1991). In fact, Hoek (2007) states that there are *"no simple universal rules for acceptability nor are there standard factors of safety which can be used to guarantee that a rock structure will be safe and that it will perform adequately"*. One fundamental problem with the deterministic LEM approach is that the arbitrary definition of a FoS means that unpredictability, in both the input parameters and resulting FoS, is not explicitly expressed but hidden in the calculation. This makes hazard perception and the quantification of the risk of slope instability impossible. For these reasons, various unpredictability-oriented approaches have been studied and published in the literature; these include both non-probabilistic and probabilistic studies.

In many of the slope stability analyses presented in the literature, limited or no objective data was available to characterise the parameters required for the LEM. Consequently, the analyses are based on input parameters formulated subjectively through expert opinion which, in accordance with the proposed taxonomy (Figure 17), introduces

epistemic uncertainty. In line with the new framework presented in the preceding Chapter, any analysis where the unpredictability is epistemic requires a non-stochastic modelling method, commensurate with the available level of information. Therefore, any analyses using stochastic methods and subjectively assigned priors without evidence to support them, or updating them using Bayes's Theorem, are inappropriate. Through a critical review of various analyses presented in the literature, focussing on the model of planar slope stability, the following section examines the validity of the unpredictability model applied in various studies with respect to the level of information available and the concepts presented in the taxonomy (Figure 17). Following this, the discussion uses a case study to explore the effect on the FoS of slope stability calculated using probabilistic models that incorporate alternative subjectively assigned probability distributions. These alternatives mimic the opinion of multiple experts. The results are shown to strongly depend on the shape of the input distributions, and thus the expert opinion utilised. This section concludes by showing the applicability of the framework presented in Chapter 4 to select a more appropriate analytical model that is both faithful and robust given the epistemic nature of the available information.

5.1 Critical review of planar slope stability analyses

The basic planar slope stability model is shown in Figure 51 with the required input parameters explained. This 'classical approach' defines FoS as the ratio between forces resisting sliding (R) to those inducing sliding (L), as per Equation (5.1) (see Appendix B for a full definition of all parameters in this equation). A FoS of 1 is the condition of limiting equilibrium and thus a factor less than one implies instability.

Parameters:
H – Height of slope
z – Depth of tension crack
z_w – Height of water in tension crack
ψ_p – Angle of failure surface
ψ_f – Angle of slope face
c – Cohesion of failure surface
ϕ – Angle of friction of failure surface
T – Tension in bolt
θ – Angle of bolt installation
W – Weight of rock
U – Water pressure on sliding plane
V –- Water pressure in tension crack
α – acceleration co-efficient

Figure 51: Limit equilibrium model for planar slope stability
(after Hoek & Brown, 1980b; Low, 2008).

$$\text{FoS} = \frac{\sum \text{Resisting forces(R)}}{\sum \text{Driving forces(L)}} = \frac{cA + \left(W\left(\cos\psi_p - \alpha\sin\psi_p\right) - U - V\sin\psi_p + T\cos\theta\right)\tan\phi}{V\cos\psi_p + W(\sin\psi_p + \alpha\cos\psi_p) - T\sin\theta} \quad (5.1)$$

Various studies that utilise the LEM applied to the free body diagram of Figure 51 have been presented in the literature; some of these are listed in Table 3. The methods of analysis within the studies reviewed include: estimated deterministic values, interval analysis, fuzzy analysis – using both fuzzy arithmetic and fuzzy inference systems – and various probabilistic methods. The entries in Table 3 have been sorted by the parameters required in the methods of analysis, with those requiring the lowest level of information at the top to the highest at the bottom (as previously defined by level of information concept presented in Figure 8).

Table 3: List of studies on planar slope stability.

Parameters	Method of analysis	Reference
Estimated deterministic value & Interval	Analytical LEM	Hoek, 2007
Estimated deterministic value	Analytical LEM	Nilsen, 2000
Fuzzy; using rock mass classification	Fuzzy inference system	Basarir & Saiang (2012)
Fuzzy; using rock mass classification	Fuzzy inference system	Daftaribesheli et al (2011)
Fuzzy numbers	Fuzzy arithmetic	Park et al., 2012
Fuzzy numbers	Fuzzy arithmetic	Sakurai & Shimizu, 1987
Stochastic classification	Monte-Carlo	Priest & Brown, 1983
Stochastic; using Joint Roughness Coefficient	Monte-Carlo	Tamini et al. (1989)
Stochastic	Monte-Carlo	Jimenez-Rodriguez et al. (2006)
Stochastic	FORM	Low (2008)
Stochastic using GSI compared to interval	Monte-Carlo	Li et al. (2012)
Stochastic	Monte-Carlo	Park & West, 2001
Stochastic	Monte-Carlo	Hoek (2007)
Stochastic	Monte-Carlo	Park et al., 2005
Stochastic	Point Estimate Method	Park et al., 2012
Stochastic; using Joint Roughness Coefficient	Monte-Carlo	Feng & Lajtai (1998)
Stochastic	Monte-Carlo	Nilsen, 2000

5.1.1 Review of selected non-stochastic analyses

Hoek (2007) presents an investigative study of the potential instability of the Sau Mau Ping road slope adjacent to an area where slope failures had recently occurred; this is later used as a case study in section 5.2. At the time of the study, no objective information was available and so the estimates of parameters required for computing both the driving (e.g. acceleration co-efficient and depth of water in the tension crack) and resisting forces (e.g. shear strength parameters) were estimated purely through expert judgement. Given the lack of objective data, Hoek (2007) estimated intervals of the shear strength parameters (c & ϕ) using published literature. The interval of the depth of water in the tension crack was taken as the

minimum and maximum depth of the crack physically possible, with a 'worst case' estimate of the acceleration co-efficient based on local experience. With respect to the taxonomy and level of information concept (Figure 17 and Figure 8, respectively), this characterisation of the uncertain parameters as intervals is a faithful representation of the epistemic uncertainty present in the problem. Furthermore, the 'worst case' deterministic value for acceleration co-efficient indicates the lowest level of information – complete ignorance.

Using this combination of deterministic and interval valued parameters, Hoek (2007) undertook a sensitivity study – a form of interval analysis (Saltelli, 2004) – to estimate the change in FoS by varying the parameter values between these bounds. Indeed, Hoek & Londe (1974) state that sensitivity studies can provide useful information on the response of the structure to changes in significant parameters. Nilsen (2000) presents a similar sensitivity analysis using 'worst case' and 'best case' parameter combinations; the calculated FoS ranges between approximately 1.0 and 2.0. Whilst the analyses of Nilsen (2000) and Hoek (2007) are both faithful to the available information and robust, there is no means of objectively estimating the level of uncertainty in the calculated FoS, nor the likelihood of intermediate conditions (e.g., intermediate water levels or smaller accelerations). Hoek & Londe (1974) recognised the lack of precision in such analyses suggesting that, given the paucity of information to undertake the analysis, it is the responsibility of the engineer "*not to compute accurately but judge soundly*". Whilst the merit in this statement is recognised, one of the major aims of this thesis is to provide tools to assist making objective judgements when faced with such cases where the unpredictability is highly epistemic. The eventual goal of which is to reduce subjectivity and dissonance between experts.

Sakurai & Shimizu (1987) present an example in which rock mass classification, RMR in particular, is used to estimate the shear strength parameters (c & ϕ) as the inputs to the LEM (Figure 52). Sakurai & Shimizu (1987) recognise the imprecision in the estimation of RMR through their statement that "*compared with materials such as steel and concrete, the determination of a probability density function for the mechanical constants of rock masses is extremely difficult*". Furthermore, they appreciate the value of fuzzy numbers to characterise epistemic uncertainty resulting from the subjective means of estimating RMR. In their analysis, imprecise correlations between the RMR rock class – which are of a nominal scale, as previously defined in Table 1 – and the shear strength parameters are used to estimate fuzzy numbers for c and ϕ. By characterising the shear strength parameters as fuzzy numbers, Sakurai & Shimizu (1987) obtain a fuzzy FoS for a number of failure surface

angles. It should be noted that the angle of the failure surface itself could have been characterised as a fuzzy number and propagated through the analysis to obtain a single fuzzy FoS. However, it appears to not have been implemented for ease of computation.
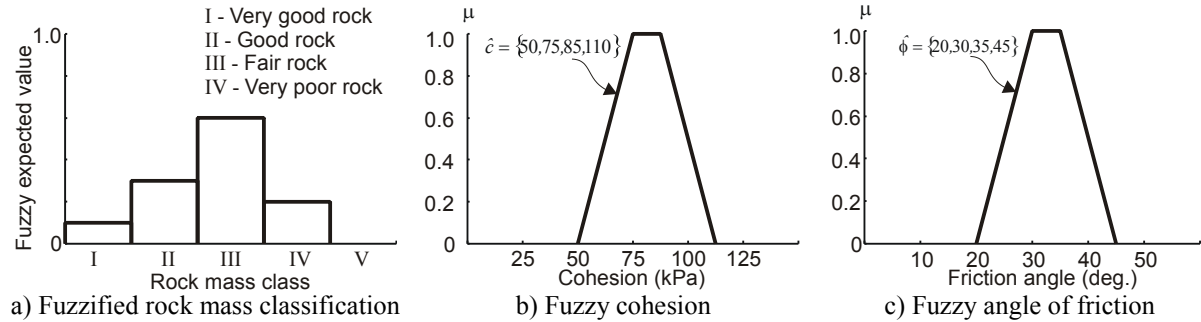
Figure 52: Fuzzy shear strength parameters computed from RMR (after Sakurai & Shimizu, 1987).

To assist in decision-making, Sakurai & Shimizu (1987) define a 'stability index', as illustrated in Figure 53, based on possibility theory (see possibility and necessity measures discussed in section 3.2) for classifying stability, which expresses the degree of plausibility on which to form a judgement on the question: 'this slope is stable'. Given the imprecision in the input information and by applying the new taxonomy of Figure 17, it is evident that the imprecise FoS and stability index calculated by Sakurai & Shimizu (1987) remains faithful to the information. In particular, the RMR input classes are of a nominal scale as are the output stability classes. From this, it can be concluded that the analysis of Sakurai & Shimizu (1987) does not introduce additional information, in the form of a precise PDF, in the computation of FoS. Further, the use of fuzzy numbers in lieu of intervals allows some measure on the uncertainty in the calculated FoS.
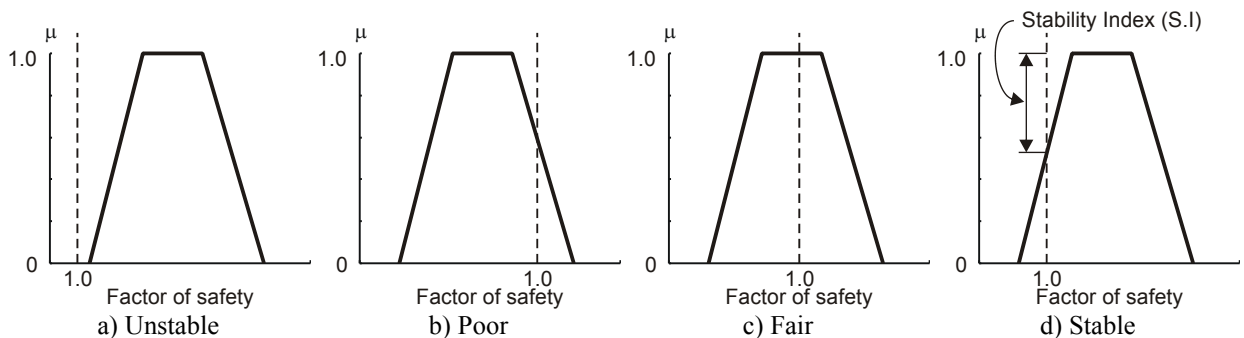
Figure 53: Proposed stability index (after Sakurai & Shimizu, 1987).

5.1.2 Review of selected stochastic analyses

As Table 3 illustrates, numerous probabilistic techniques such as Monte-Carlo simulation and Point Estimate Methods have been applied to FoS analyses using LEMs. In these studies, the

primary impetus for using probabilistic methods is the perception that probabilities of failure can be calculated to quantify the risk of slope instability; the reliability-based approaches, which include FORM and FOSM methods, offer an attractive framework in this endeavour (Jimenez-Rodriguez et al., 2006). All these methods share a commonality: requiring the unpredictable parameters to be modelled as random variables characterised by a precise probability distribution function. In accordance with the taxonomy, to substantiate the use of aleatory models and the subsequent probabilistic analyses requires a sufficient quantity of precise data. However, many of the analyses listed in Table 3 use subjectively defined PDFs. This subjectivity in defining PDFs conforms to the Bayesian approach. As discussed in section 2.6.2 and illustrated in the framework of Chapter 4, in order for this approach to remain faithful to the available information, the 'priors' must be objectively derived and updated using Bayes's Theorem. However, in many of the analyses listed in Table 3, this is not the case.

One of the first probabilistic analyses of planar slope stability was undertaken by Priest & Brown (1983), who based the resistance to sliding on shear strength parameters derived through empirical correlations between RMR and the Hoek-Brown strength criterion. As mentioned previously and discussed in detail in section 2.8.1, these correlations incorporate significant approximations and subjectivity in their estimation and are thus intrinsically epistemic. It follows that the inherent imprecision within RMR means that characterising unpredictability with precise PDFs, as described in section 2.5, is inappropriate due to the introduction of information and precision that does not exist; it is unfaithful to the available level of knowledge.

More recently, various researchers have re-analysed the Sau Mau Ping road case study using various probabilistic techniques, as listed in Table 4. Whilst each of these probabilistic methods has differences (computational effort), they all require the input parameters to be characterised as random variables and, consequently, knowledge of the statistical moments that define them. However, earlier in this discussion we identified the entirely subjective means employed by Hoek (2007) to estimate the input parameters for the LEM. Given this quantitative and qualitative lack of information, it was concluded that the deterministic and interval analysis undertaken by Hoek (2007) was faithful to the available information. Hoek (2007) also presents a study in which the epistemic uncertainty in the LEM parameters is incorrectly treated as aleatory variability and characterised using PDFs. Consequently, this introduces information and precision into the output. Furthermore, as Monte-Carlo simulation is utilised to propagate the unpredictability, the outputs are neither faithful to, nor robust with

the available level of information (refer to section 2.5). These statements are further validated by the following investigation of this case study presented in section 5.2. Consequently, any of the analyses listed in Table 4, all of which utilise the subjectively defined random variables of Hoek (2007) will suffer from the same drawbacks as Hoek's Monte-Carlo simulation. In fact, a review of these other analyses shows that the distribution of FoS calculated by the various other studies is in close agreement with the PDF determined by Hoek (2007).

Table 4: Slope stability analyses undertake on Sau Mau Ping road.

Method of analysis	Reference
Monte-Carlo simulation	Hoek (2007)
First Order Reliability Method (FORM)	Low (2008)
Response Surface Method (RSM)	Li et al. (2011)
Reliability-based Robust Geotechnical Design	Wang et al. (2013)
Jointly Distributed Random Variable (JDRV) method	Johari et al. (2013)

In contrast to the subjectively assigned PDFs used in the various probabilistic analyses of the Sau Mau Ping road slope, the study by Park & West (2001) demonstrates a far greater appreciation for the attributes required to characterise unpredictability as aleatory variability for propagation through probabilistic analysis. In their study, Park & West (2001) state that the parameters defining the orientation and geometry of discontinuities, such as length, spacing and persistence as well as the shear strength parameters, may be defined as random variables. However, unlike the studies discussed above, they recognise the objective nature of aleatory variability and recommend that the *"types of distribution functions for each random variable should be selected carefully in a probabilistic analysis. However, there is a lack of consensus on these choices, which could lead to very different analysis results"*. For this reason, Park & West (2001) use objective data obtained from measurements on a total of 280 discontinuities to objectively fit the PDFs. The measurement techniques involved the scanline method on rock outcrops and existing road cut slopes, as well as the use of a borehole method providing oriented cores. The selected PDFs are justified using Chi-square goodness-of-fit tests.

A deficiency in Park & West's study was the inability to objectively measure the shear strength parameters, and thus these were characterised by subjectively defined PDFs. The following statement exemplifies their justification for this deficiency: *"In addition, since the number of tests performed and data measured are generally insufficient for a sound statistical analysis, a certain amount of experience and good engineering judgement are always needed"*. Whilst the latter part of the statement calling for good engineering judgement is

supported, in the context of the framework presented in Chapter 4, to faithfully characterise unpredictability the imprecision in these parameters should be expressly acknowledged and the data characterised by a non-stochastic method commensurate with the available level of information and propagated using hybrid analysis. Further studies by Park et al. (2005) and Park et al. (2012) follow the same methodology, characterising the variability in many input parameters using large data sets (280 and 350 measurements, respectively), with the shear strength parameters characterised subjectively.

The examples reviewed in this section present both non-stochastic and stochastic approaches for characterising unpredictability in subjectively assigned parameters, i.e. when the unpredictability is epistemic. With respect to the new taxonomy of Figure 17, such subjective assessment of parameters means that the unpredictability cannot be characterised as aleatory and thus the stochastic approaches are inappropriate. With respect to the discussion presented earlier in section 2.5.1, these non-stochastic approaches are not faithful to the available information. This review highlights the need for the new taxonomy proposed in this thesis (i.e. Figure 17). Furthermore, it suggests that an objective means of characterising unpredictability and thus identifying the optimal modelling method is required. This objective means of characterising and propagating unpredictability is provided by both the proposed taxonomy and framework (presented earlier in Chapter 4). The following discussion now demonstrates the applicability of the proposed framework to the Sau Mau Ping road case study.

5.2 Case study: Sau Mau Ping road

Following a series of landslides in Hong Kong that were triggered by exceptionally heavy rains, which caused some loss of life and a significant amount of property damage, the stability of a rock slope on Sau Mau Ping Road in Kowloon – located immediately across the road from two blocks of apartments, each housing approximately 5,000 people – was brought into question (Hoek, 2007).

Given the critical nature of this slope, a study was required to investigate the factor of safety (FoS) of the slope under normal conditions and under conditions that could occur during an earthquake or during exceptionally heavy rains. Unfortunately – and as is often the case in geotechnical engineering – no objective data (i.e. laboratory or field observations and measurements) were available at the time of undertaking the study. Consequently, critical input parameters for the analysis had to be determined from expert judgement and previous experience (Hoek 2007). The geometry of the slope, as well as those parameters for which no

objective data were available – referred to herein as the non-deterministic parameters – are shown in Figure 54.

Random variables	PDF	Mean	St. Dev.
Sliding surface friction, ϕ (deg)	Normal	35°	5°
Sliding surface cohesion, c (t/m^2)	Normal	10	2
Tension crack depth, z (m)	Normal	14	3
Tension crack water depth, z_w (m)	Exponential	$z/2$	max = z
Horizontal seismic accln. factor, α	Exponential	0.08	max = 2α
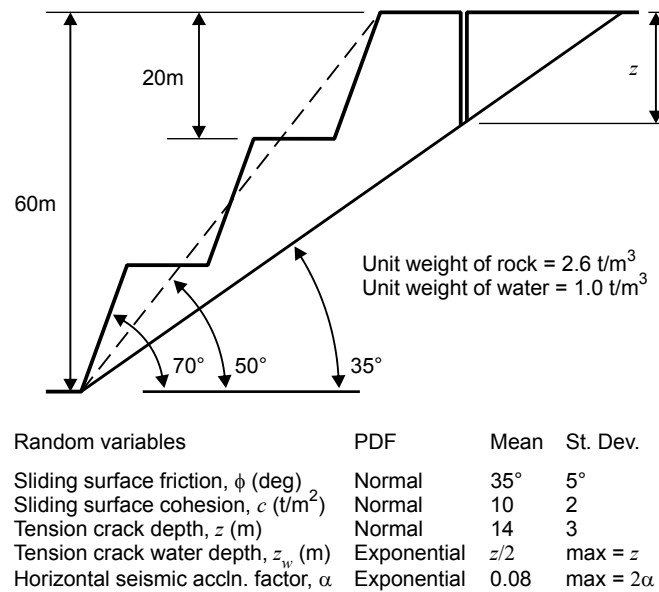
Figure 54: Geometry and non-deterministic parameters in Sau Mau Ping Road analysis (from Bedi & Harrison, 2013a).

Irrespective of the lack of objective data with which to characterise the non-deterministic parameters, various authors have presented probabilistic approaches to assess the factor of safety of the Sau Mau Ping road slope (see Table 4). In these analyses, the non-deterministic parameters are characterised as random variables that have been defined subjectively using expert judgement (Hoek, 2007). However, given the absence of objective information, the validity of these probabilistic approaches for this case study is questionable. The discussion that follows, using the concepts presented in the proposed taxonomy and framework, first presents a critical review of the basis on which the non-deterministic parameters have been characterised as random variables, and thus draws conclusions on the suitability of stochastic analysis to determine the FoS, given the level of information. This is followed by a comparison of the results from Monte-Carlo simulation based on a subjectivist (Bayesian) approach to probability and a non-probabilistic approach selected by following the framework presented in Chapter 4. This example illustrates the significant differences in design decisions that may result depending on the model adopted to characterise and propagate uncertainty and compare the results with an alternative calculation in which the non-deterministic parameters are characterised as fuzzy numbers.

5.2.1 Critical review of data characterisation with respect to the proposed framework

With respect to the proposed taxonomy as illustrated in Figure 17 (see section 2.7), the absence of objective data with which to fit a PDF for each non-deterministic parameter constitutes a quantitative lack of information. According to this figure and the new taxonomy as set out in Figure 17, the non-deterministic parameters must therefore be classified as epistemic and propagated using an appropriate, non-stochastic uncertainty model. However, and contrary to these concepts, Hoek (2007) suggests that even in the absence of objective information the non-deterministic parameters can be modelled as random variables (i.e. aleatory) defined solely from expert judgement or experience. Indeed, Hoek (2007) suggests that lack of objective data is often used as an excuse for not using probabilistic tools in geotechnical engineering. On this basis, Hoek (2007) characterises the non-deterministic parameters as random variables in order to undertake a probabilistic assessment of the factor of safety of the Sau Mau Ping slope.

Recalling the level of information concept presented in Figure 8 and the faithfulness principle previously discussed in section 2.5.1, the simple act of defining a probability of occurrence when faced with epistemically uncertain parameters introduces a significantly greater level of knowledge than is actually available. Of most importance is not the magnitude of the selected minimum, maximum or mean values, but rather the shape of the PDF chosen to define them (Ferson & Gizburg 1996). Contrary to this, Hoek (2007) reasons that properties arising from the sum of a number of random effects, none of which dominate the total, are normally distributed., and that the normal distribution is *"generally used for probabilistic studies in geotechnical engineering unless there are good reasons for selecting a different distribution"*. On this basis, Hoek (2007) suggests that, in the absence of information on the actual distribution, a normal distribution be used where the means represent the 'most likely' values and the minimum, maximum and standard deviations are arbitrarily chosen.

In the analysis presented by Hoek (2007), the shear strength parameters (c and ϕ) were modelled using truncated normal distributions with the mean and standard deviation estimated subjectively based on literature reports (see Figure 55, reproduced from Hoek & Bray, 1974) of back analysed slope failures in similar rock types. Hoek (2007) states the minimum and maximum truncation limits were arbitrarily chosen to allow for a wide range of values in the analysis. For the friction angle, these bounds represent extreme limits of a smooth slickensided surface (30°) and a fresh, rough tension fracture (70°), and for cohesive strengths the minimum and maximum values chosen were 0 and 25 tonnes/m^2 (i.e. 0 and

0.25MPa), respectively. Similarly, the lack of access to inspect the crests of the slopes for the presence of tension cracks meant the PDF defining their depth was also speculative. Finally, truncated exponential PDFs were used to define the tension crack water depth and seismic acceleration with the means defined through *"expert judgement and using very crude guidelines"*. Table 5 summarises the minimum, maximum and most likely values determined by Hoek (2007), and form the basis of the subjectively defined PDFs of Figure 56.

Table 5: Minimum, maximum and mean values used by Hoek (2007).

Parameter	Minimum	Mean	Maximum
Cohesion (c ,tonnes/m^2)	0	10	25
Angle of friction (ϕ , deg)	15	35	70
Depth of tension crack (z , m)	0	$0.5\,z_{max}$	$0.5\,z_{max}$
Depth of water in tension crack (z_w , m)	0	0	z_{max}
Accelaration coefficient (a)	0	0	0.16
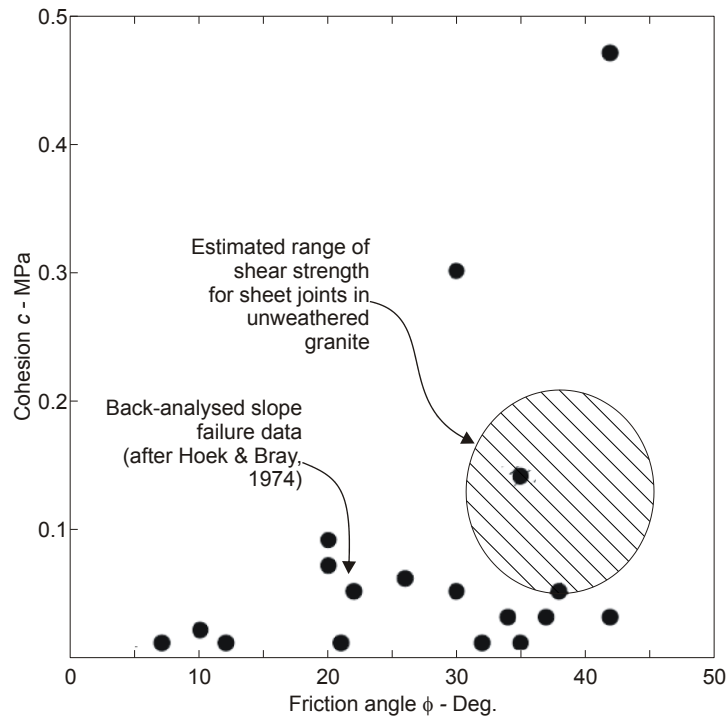
Figure 55: Empirical data of c & ϕ based on back analysis of failed slopes (after Hoek & Bray, 1974; Hoek, 2007).
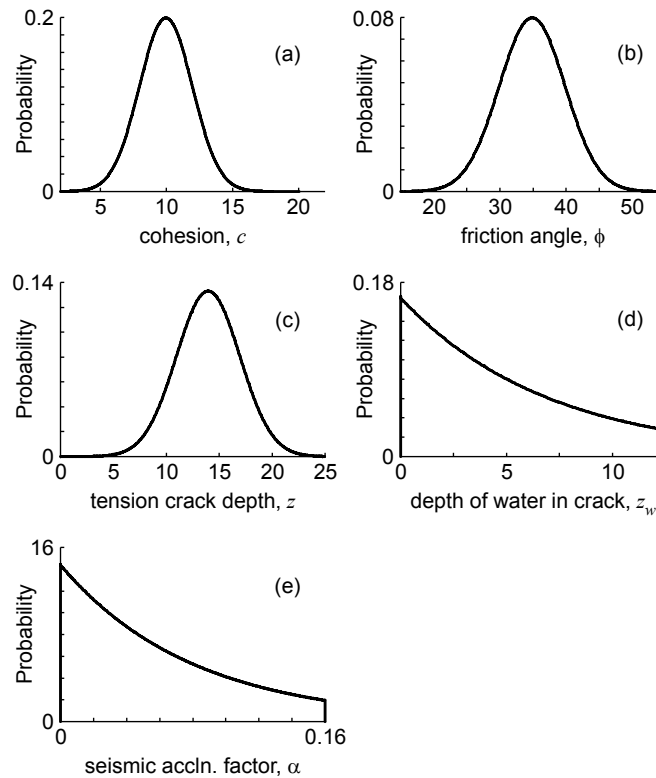
Figure 56: PDFs of non-deterministic parameters used in Monte Carlo simulation (from Bedi & Harrison, 2013a).

In this example, there is perhaps some basis (local seismological and meteorological information) for selecting exponential PDFs to define the probability of occurrence of typhoons and earthquakes; however, there is no evidence to suggest that the shear strength parameters and tension crack depth are better defined by normal distributions rather than, say, triangular or beta distributions. Consequently, dissonance between experts may well result in others opting for alternative distributions; the triangular distribution is a common choice. For these reasons, the Bayesian view contends that 'non-informative' PDFs (i.e. uniform distributions between the estimated lower and upper bounds) may be assigned in the absence of information on the shape of the distribution.

Regardless of whether non-informative or other distributions are chosen to model the non-deterministic parameters, Figure 8 (i.e. uncertainty models and the LoI concept) and Figure 29 (i.e. appropriate uncertainty models for a given level of information) coupled with the proposed taxonomy illustrated in Figure 17, indicate that an aleatory model represents the highest level of information and requires a sufficient quantity of precise data to justify the choice of PDF. In the case of Hoek's (2007) analysis, the information met neither the qualitative or quantitative criteria required to define an aleatory model. Furthermore, with respect to the proposed framework presented earlier in Chapter 4, the data characterisation

strategy sub-chart (Figure 49) suggests that the Bayesian approach is only suitable when the subjectively determined priors can be justified through prior evidence, and further data collection is planned with which to update the initial (i.e. prior) distributions to 'posterior' distributions, using Bayes's Theorem (Ang & Tang 2007). This case study offered no evidence to justify the chosen shape of all the distributions, nor was it possible to obtain further objective data. Therefore, even though the non-deterministic parameters may be the result of random processes, the level of information available does not fulfil all the key attributes required to characterise unpredictability as aleatory, as set out by the proposed taxonomy and framework. Instead, the current lack of knowledge requires that they should have been characterised as epistemic. The following exemplar calculations investigate the effect on the calculated factor of safety obtained by Monte-Carlo simulation when using various subjectively assigned PDFs.

5.2.2 Effect of subjectively assigned priors

Figure 57 compares the results of three Monte-Carlo (MC) simulations – with 5000 iterations for each run – each with a different PDF defining the non-deterministic shear strength parameters (c and ϕ) and the depth of the tension crack (z). The first MC simulation uses the PDFs shown in Figure 56, the second adopts triangular distributions that approximate the parameters of Figure 56(a-c), and the final simulation adopts the Bayesian philosophy and assigns non-informative priors to these parameters. In each simulation the minimum, maximum and mean values shown in Table 5 are used to define the PDFs.
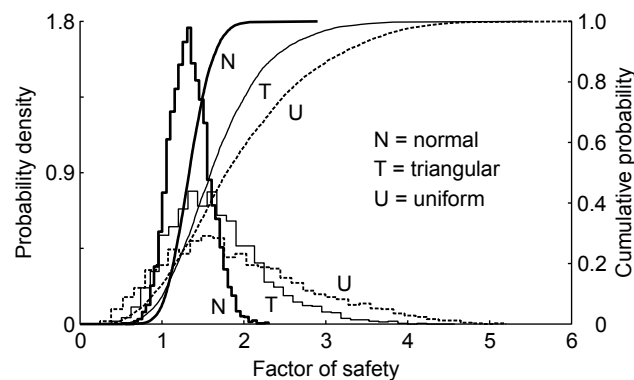
Figure 57: Comparison of three Monte-Carlo simulations (from Bedi & Harrison, 2013a).

Figure 57 demonstrates how the choice of the input PDFs has a significant influence on the output of the Monte-Carlo simulations. Most importantly, despite the input distributions having the same minimum, maximum and mean values the different shapes of

the PDFs have resulted in significant differences in the bounds and fractile values of the output.

These differences result from the varying degree of information contained purely in the shape of the selected PDF. That is, the triangular distribution contains more information on the probability of occurrence of the mean value than the uniform distribution. Likewise, the truncated uniform distribution contains even further information on the distribution of probabilities around the mean and between the bounding values; the standard deviation of 5° implies that about 68% of the friction angle values defined by the distribution will lie between 30° and 40°. This precise statement has been made on the basis of a subjectively determined, and hence imprecise, area shaded in Figure 55 that does not support such statements about probability of occurrence within the shaded region. It is thus evident that in this example, there is simply no evidence to warrant the selection of one PDF over another. In fact, this dispute on the selection of an appropriate PDF in the absence of any knowledge on the parameters which define it dates back to Laplace's principle of indifference, which itself dictates the use of uniform distributions (Ferson & Ginzburg 1996).

In this example, the results of the three MC simulations presented in Figure 57 can be considered a reflection of the subjective opinions of three different experts. The question then becomes, how do the views of each of these experts differ with respect to the FoS of the slope and what influence does this have in determining remedial measures, if any?

5.2.3 Decision making

In terms of stability of the Sau Mau Ping slope, a FoS less than 1.0 indicates that the slope is unstable; a FoS of 1.0 can be thought of as the 'limit state.' In civil engineering, a 5% probability of occurrence of the limit state is often considered as the threshold of acceptable risk. Table 6 presents a summary of various statistics for the FoS, calculated from the results of the three analyses presented in Figure 57. The final column in this table presents the calculated probability of the FoS being less than 1.0, i.e., the probability of occurrence of the limit state.

Table 6: Statistics computed from Monte-Carlo simulations.

Simulation	Type of PDF	Min FOS	Lower 5%	Mean	Max FOS	P(FoS≤1)
N	Normal	0.59	0.97	1.34	2.31	6.4%
T	Triangular	0.46	0.88	1.68	4.59	9.5%
U	Uniform	0.25	0.70	1.94	5.21	14%

The results of Simulation 'N' indicate that the probability of a FoS less than 1.0 is 6.4%. As a result of this analysis, Hoek (2007) states that this FoS represents *"a reasonable risk of failure for short term conditions and a risk of this magnitude may be acceptable in an open pit mine, with limited access of trained miners, and even on a rural road. However, in the long term, this probability of failure is not acceptable for a densely populated region such as Kowloon"*, Simulations using triangular and uniform PDFs, respectively, indicate that the probability of a FoS less than 1.0 is 9.5% and 14%. These are both substantially more than that obtained in the simulation using normal PDFs, and suggest that the slope may not be stable. Note also that there are significant differences in the other minimum and maximum calculated values. If we consider the results of Simulation 'T' and 'U' as the findings of two other experts, it is apparent that their conclusions may be vastly different to those of Hoek (2007).

These simulations demonstrate how differing views on the stability of the slope may be obtained when a subjective approach to assigning probability distributions is applied. As subjective distributions are determined by expert opinion, and the conclusion each expert reaches on the basis of these subjective inputs varies, such a situation would only serve to generate dissonance between the experts. In essence, the results of the analysis reflect a situation where the experts have agreed to disagree. Therefore, one would have to adopt the decision of the expert they deem most competent (introducing further subjectivity) or undertake objective tests to verify the assumptions of the input distributions defined by the expert; the only means of doing so would be to undertake objective measurements.

The next section applies the proposed framework and shows that given only the minimum, maximum and 'most possible' values used in this analysis, the path followed would lead each expert to the same non-stochastic approach to characterise and propagate the epistemic uncertainty in this problem.

5.3 Application of proposed framework applied to Sau Mau Ping slope stability analysis

The analysis presented in the preceding section used non-deterministic parameters that were defined as random variables (i.e. aleatory) around a common minimum, maximum and mean value. Regardless of these commonalities, the calculated performance of the slope varied with the chosen shape of the input PDFs. The discussion in section 5.2.1 concluded that for the Sau Mau Ping road slope case study, the lack of objective data required the unpredictability in the non-deterministic parameters to be characterised as epistemic uncertainty, and so a more

appropriate uncertainty model is one that uses only the minimum, maximum and 'most typical' value of any of the non-deterministic parameter. The discussion that follows demonstrates how the framework presented in section Chapter 4 leads the user to an appropriate uncertainty model that is faithful to the available information.

5.3.1 Framework paths

The first step in the proposed framework (Figure 48) is preliminary investigation and data acquisition, which is later followed by, in step 5, characterisation of the non-deterministic parameters for propagation through an appropriate unpredictability analysis. For the Sau Mau Ping road case study, the shear strength parameters were estimated through published empirical relations (Figure 55), with the remainder being estimated through expert judgement. Table 7 presents a summary of the path followed through the main framework (Figure 48) and the data characterisation strategy (Figure 49) and model selection strategy (Figure 50) sub-charts. This table presents the decision made or question answered at each box encountered in the path through the framework.

Whilst the questions and answers presented in Table 7 are straightforward, a few key stages require additional discussion. Firstly, the first question in the data characterisation strategy (box 5.1) asks the question '*Can the data be measured objectively?*' For this case study, the answer to this is of course 'yes', and thus the framework directs us towards questions that determine whether the quality or quantity of information is sufficient to characterise the parameter as aleatory (see taxonomy and figures). In this case, the insufficiency of objective data leads us away from the aleatory model towards the Bayesian updating route (box 5.3), and asked the question '*Prior information on which to formulate a precise PDF is known?*'. At this stage, knowing that the parameters in question can be objectively measured and thus may be intrinsically aleatory, we could arguably suggest that a PDF should be formulated on expert judgement. However, the earlier discussion in section 5.2.1 demonstrated the lack of evidence to support the shape of a distribution. Nonetheless, if we were confident in justifying a subjectively determined prior distribution and answered 'yes' to the above question, the next question posed would be whether we propose to gather further data to update the priors to posterior PDFs. As we do not intend to subsequently update the priors with objective data, the state of information is realised as imprecise and we are returned back towards questions leading towards an epistemic characterisation of the unpredictability in the parameter. The data characterisation strategy finally leads us to characterise the parameters using triangular fuzzy numbers.

Table 7: Framework paths – questions and answers table.

Box #	Question	Answer
1	Preliminary investigation/data acquisition	No objective data available. Parameters derived from expert judgement
2	Select analytical model	Closed form limit equilibrium model for planar slope stability (See Equation (5.1))
3	Identify parameters required for model	Non-deterministic parameters per Figure 54
4	Further investigation/data acquisition	Not available
5	Uncertainty parameters characterisation	Move to 'Data characterisation strategy' sub-chart (Figure 49)
	START DATA CHARACTERISATION STRATEGY SUB-CHART	
5.1	Can the data be objectively measured?	The non-deterministic parameters may be measured through laboratory or field measurements
5.2	A sufficient number of precise measurements are available?	No measurements available
5.3	Prior information on which to formulate a precise PDF is known?	No. But we could arguable suggest that a PDF is formulated on expert judgement, which leads to the Bayesian Updating path. However, as we do not propose to subsequently update the priors with objective data, we are returned back to the following question
5.4	The type of distribution is known and intervals for its parameters can be provided?	There is no evidence to support any particular shape of distribution
5.5	A sufficient number of imprecise measurements are available?	Refer 5.2
5.6	An interval that bounds the parameters is known?	Yes. Prior published data and expert opinion can be used to provide bounds for each parameter
5.7	An estimate of the most plausible values can be provided?	Yes. Refer 5.6. All the non-deterministic parameters can be characterised by **triangular fuzzy numbers.** We now return to the main flowchart
6	Select appropriate uncertainty model	Move to 'Uncertainty model strategy' sub-chart (Figure 50)
	START MODEL SELECTION STRATEGY SUB-CHART	
6.1	Parameter characterisation	Given that no further data collection is proposed, at this stage the data are characterised as intrinsically epistemic.
6.2	Select uncertainty model	All the parameters are fuzzy numbers and so we use possibility analysis
6.3	Analytical output	Subjective bounded output. We now return back to the main flowchart (Figure 48)
	RETURN TO MAIN FRAMEWORK FLOW CHART	
7	Analysis	We undertake a fuzzy analysis
8	Model propagation	The analysis results in a subjective bounded output
8a	Are the bounds small enough to generate a useable output	Yes. See discussion in 5.3.3
8b	Subjective assessment/defuzzification	We use the concept of agreement index (as will be discussed further in section 5.3.3, Figure 59) to obtain the agreement index from which to formulate design decision
	DESIGN	**Using the agreement index curve, we conclude that the FoS is insufficient at the required agreement level and thus mitigation measures need to be implemented. We now propose to investigate the impact on agreement index with the installation of rock bolts to improve the FoS.**

One final note is box 5.4, which asks whether the type of distribution is known and if intervals for its parameters can be provided. At this point, one could argue that the exponential distributions for depth of water in the tension crack and earthquake could be justified. However, given the approximation used by Hoek (2007) in coming up with these distributions, to remain as faithful as possible to the available information, we have chosen to answer 'no' for all the non-deterministic parameters and continue with a possibility analysis.

5.3.2 Possibility analysis

By following the framework to characterise the non-deterministic parameters using triangular fuzzy numbers (TFNs), the minimum, maximum and 'most likely' values provided by Hoek (2007) are used to construct these, as shown in Figure 58. It is important to realise that the TFNs are different to triangular PDFs, in that they contain a lower level of information. The fuzzy numbers do not define precise probabilities of occurrence for values of the parameter they characterise, but rather encode preferences of imprecise measures. The resulting fuzzy factor of safety ($F\hat{o}S$) is shown at the lower right of Figure 58. The generation of the fuzzy numbers and output fuzzy factor of safety has been computed by discretising the fuzzy numbers into $\alpha - cuts$ and applying the vertex method discussed in 3.2.2. The calculation algorithms are provided in Appendix B.
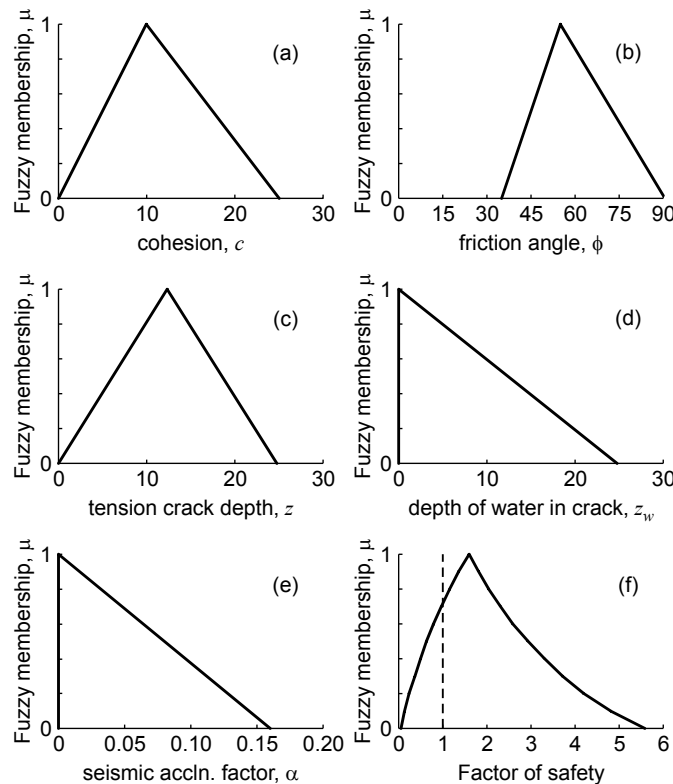
Figure 58: Fuzzy inputs and computed fuzzy factor of safety (from, Bedi & Harrison, 2013a).

In comparison with the results of the earlier analyses (see Table 6), fuzzy arithmetic ensures that the resulting fuzzy factor of safety correctly bounds the minimum and maximum possible values, however improbable they may be (Kaufmann & Gupta 1991). Thus, the fuzzy number faithfully represents the full range of uncertainty. Figure 58f also depicts the most possible factor of safety, so the result shown in Figure 58f may be expressed linguistically as 'A FoS less than 0.05 or greater than 5.58 is considered impossible. Values close to these bounds are considered least possible; the most possible FoS is 1.58. The median FoS is likely to lie in the interval [0.63, 2.93]'.

Although fuzzy analysis faithfully propagates epistemic uncertainty, the imprecise output means pragmatic decision-making is awkward. One way to overcome this issue is to use a defuzzification measure, as discussed below.

5.3.3 Decision making

Kaufmann & Gupta (1991) present a defuzzification procedure using the concept of an 'agreement index,' which is a measure of the proportion of the fuzzy number, by area, less than a certain upper bound, as shown in Figure 59. An agreement index of 0 represents a condition where every value of the fuzzy number is greater than the upper bound, and an agreement index of 1 represents the case in which every part of the fuzzy number is less than the upper bound. By using the factor of safety as the value of the upper bound, an 'agreement index' is computed by calculating the agreement index for a range of factor of safety values. Figure 60 shows a comparison between the agreement index and the cumulative probability distribution functions from the Monte-Carlo simulations.

$$\mathrm{AI}(a) = \frac{\int^a \mu(x)dx}{\int_x \mu(x)dx} \tag{5.2}$$
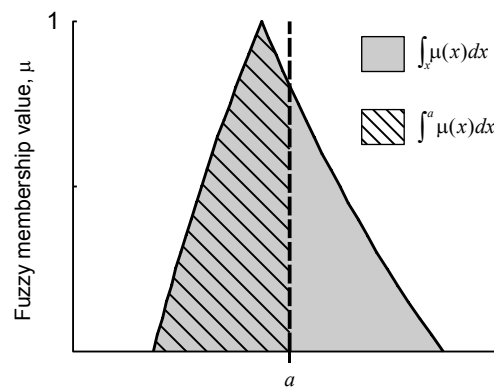
Figure 59: Defuzzification of a fuzzy number using the agreement index (after Kaufmann & Gupta, 1991; Harrison & Hudson 2010).

The agreement index is interpreted as follows: Figure 58(f) shows that the most possible FoS is 1.58. Now, using the agreement index curve in Figure 60, we can see that there is a 35% agreement that the FoS will be less than or equal to this value. Similarly, there is a 15% agreement that the FoS will be less than or equal to 1. Comparing the 5% agreement index with the lower 5% fractile value obtained from the three Monte-Carlo simulations (Table 6), the fuzzy analysis indicates a FoS of 0.55 at this level.

When comparing the agreement of the limit state with the probability of occurrence based on the Monte-Carlo simulations, the fuzzy analysis indicates more conservative conclusions. This is perhaps warranted given the little objective information on which to base critical decisions. However, it is not clear what agreement index should be deemed acceptable in terms of rock engineering designs.
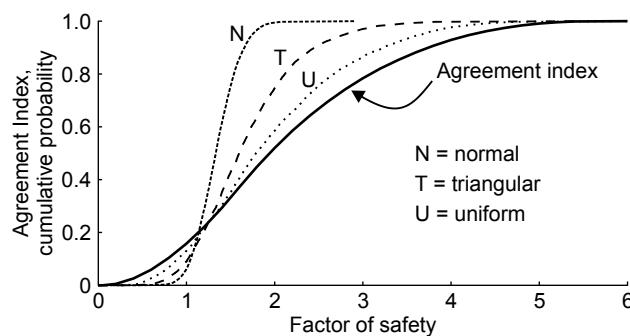
Figure 60: Defuzzification of a fuzzy number using agreement index (from Bedi & Harrison, 2013a).

As all the analyses predict potential instability of the slope, stability may be improved using various remedial measures, such as the installation of rock bolts. Using the fuzzy approach, Figure 61 compares the agreement index obtained from a further analysis with a support force of 1000 t per metre of slope (applied via rock bolts inclined 30° counter-clockwise from the normal to the sliding plane) with the in-situ condition. This figure shows that the installation of rock bolts reduces the minimum agreement index to 0.33, with a most possible value of 3.10 and a maximum of 10.34. Figure 61 also shows that the agreement index at a FoS = 1 falls from 15% to 3% with the applied support. On this basis, one can see that the proposed agreement index curve may be useful for comparing design scenarios during the decision-making stage.
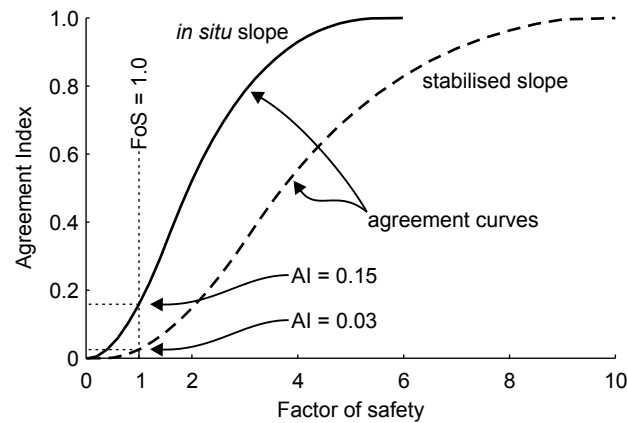
Figure 61: Agreement index for *in-situ* and stabilised slope (from Bedi & Harrison, 2013a).

5.4 Synopsis

The example presented in this Chapter demonstrates the fundamental errors that may result if subjective probabilities are applied to characterise epistemic parameters without prior objective information to support them. The example calculations presented here showed that by arbitrarily assuming a prior probability density function, one implies a greater level of information than is actually available; the increased level of information is in the definition of the shape of the PDF. The significantly different results can be taken to represent the subjective views of different experts. This example also demonstrated that the assumed prior PDF coupled with Monte-Carlo simulation has the effect of erroneously producing distribution tails that the information does not support. This may have detrimental consequences for engineering design, as it is often the extreme values represented by the tails of the distributions that govern design decisions. The Bayesian approach allows statements that presume a greater level of information than is available, thereby introducing a false sense of confidence by introducing precise statistical measures that have no real basis. Additionally, the assumed prior PDF, coupled with Monte-Carlo simulation, results in extreme combinations of parameter values disappearing from the analysis as a result of Monte-Carlo averaging. In civil engineering design, especially where critical decisions on the in-situ factor of safety are required, it is important that the engineer is able to clearly see these 'worst case' events in order to make an informed decision based on the information available.

In contrast, this example demonstrated how following the framework to characterise and propagate unpredictability leads to the selection of a non-probabilistic method commensurate with the given level of information, and allows one to use all the available information and propagate the uncertainty faithfully through the analysis of an intrinsically

epistemic system. This example has shown how non-probabilistic analyses using fuzzy mathematics are more suitable for the characterisation and propagation of epistemic uncertainty. Associated with this, the Chapter presented a new measure to defuzzify the output of such an analysis and thus assist in decision-making. As a result, it has been possible to demonstrate how a possibility analysis may give more meaningful results than subjective probability in the face of epistemic uncertainty. Most importantly, such methods will always contain the extreme events, however unlikely their occurrence may be. At the end of the modelling and risk analysis process the designer may then make a completely informed decision with regard to these unlikely events.

In this challenge problem, many of the parameters used to define slope stability may have been objectively measured however, a quantitative lack of objective data and reliance on subjectivity required them to be characterised as epistemic. The next challenge problem investigates the application of the new taxonomy and framework when faced with a qualitative lack of information; that of intrinsically epistemic rock mass classification systems.

Chapter 6

CHALLENGE PROBLEM 2 – ROCK MASS CLASSIFICATION

Section 2.8.1 identified that the difficulty in using objective test methods to characterise DIANE rock masses has led to the wide development and use of rock mass classification systems. While the simplicity of these rock mass classification systems makes them attractive to practitioners faced with limited data, the presence of numerous approximations embodied within them has raised many questions regarding their use in engineering design (Palmström & Broch, 2006; Schubert, 2012). This thesis does not continue the debate on either the fundamental assumptions made in deriving such schemes or the validity of their application. Rather, it is shown how the concepts presented in the new taxonomy (Figure 17) and framework (Chapter 4) require the unpredictability resulting from their use to be characterised as epistemic uncertainty and thus propagated using an appropriate, non-stochastic modelling method.

The discussion in section 2.8.1 showed the inherent imprecision in the parameters that form the basis for Q, and thus concluded that when using this classification system, the unpredictability must be characterised as intrinsically epistemic. With respect to the new taxonomy, Figure 29 (i.e. appropriate uncertainty models for a given level of information presented in Chapter 3) and the data characterisation strategy flowchart (Figure 49 in Chapter 4), it follows that inherently imprecise data, i.e., intrinsically epistemic parameters such as those found in empirical rock mass classification systems like Q, are best characterised by intervals or fuzzy numbers.

In recognition of the inherent imprecision embodied in rock mass classification systems, many researchers present investigations on the application of fuzzy methods when using rock mass classification systems. The previous section discussed Sakurai & Shimizu's (1987) approach to rock slope stability using a fuzzified approach to select shear strength parameters based on RMR. Nguyen (1985) established a general fuzzy set approach to rock

mass classification, which lead to studies on specific rock mass classification systems using a similar approach (e.g. Juang & Lee, 1990; Habibagahi & Katebi, 1996; Aydin, 2004). These studies all use fuzzy sets (Zadeh, 1965) to capture vagueness in the linguistic descriptors of rock mass classification systems, i.e. subjective measurement of purely qualitative data. Nguyen (1985) and Hudson & Harrison (1997) proposed using fuzzy numbers and fuzzy arithmetic (as defined in section 3.2) to characterise and propagate imprecision in the parameters used in the Q-system. Tonon et al. (2000) present a random set approach (i.e., analogous to using an imprecise p-box) to rock mass classification. Various analyses also suggest using probabilistic approaches that use random variables to characterise rock mass classification parameters (e.g. Priest & Brown, 1983; Carter & Miller, 1995).

The proposed taxonomy (Figure 17) and framework (Chapter 4) support the non-stochastic analyses of the various authors mentioned above, which recognise the intrinsically epistemic nature of rock mass classification systems. Similarly, the probabilistic analyses of rock mass classification presented in the literature are considered inappropriate. As illustrated by the new taxonomy, it is the reliance on subjective assessment of the rock mass classification parameters that requires the state of information to be regarded as imprecise and thus inappropriate to support an aleatory model. This subjectivity is, once again, captured by the first question in the data characterisation strategy of the proposed framework (Figure 49): '*Can the data be objectively measured?*'. For rock mass classification systems, and the Q-system in particular, the answer is of course 'no', and thus the data characterisation strategy requires the use of an imprecise unpredictability modelling method. On this basis, the next section examines the application of interval analysis and fuzzy numbers to a case study where the Q-system has been used to estimate the support requirements for an underground cavern.

6.1 Case study – Gjøvik Cavern support design

The Gjøvik cavern, constructed in Norway in 1994, measures a span of 60 m, a length of 90m and a height of 25m. Support requirements were principally determined using the Q-system (Barton et al., 1974). In the example that follows, actual field investigation data collected during the feasibility phases of the Gjøvik cavern project and published in Barton et al. (1994) are used to show how the taxonomy and framework may be applied to estimate support requirements and assess the feasibility of constructing the cavern, based on the Q-system.

The Q-value is estimated using Equation (6.1), with tunnel roof support pressure being estimated from a common correlation based upon analyses of case records, given by Equation (6.2) (Grimstad & Barton, 1993).

$$Q = \frac{RQD}{J_n} \frac{J_r}{J_a} \frac{J_w}{SRF} \tag{6.1}$$

$$P_{roof} = \frac{2\sqrt{J_n} Q^{1/3}}{3J_r} \tag{6.2}$$

Due to the dependence of the required roof support pressure on Q, its calculation will inherit any uncertainty in the estimation of Q. In fact, as this expression contains additional repetitions of the intrinsically epistemic parameters J_n and J_r, the resulting uncertainty may be exacerbated further.

For the Gjøvik cavern, support pressure was proposed to be provided through permanent rock reinforcement in the form of grouted rebar rock bolts, untensioned fully-grouted strand anchors and 10cm of steel fibre reinforced shotcrete (Barton et al., 1994). The spacing of the strand anchors (S_s) was proposed to be twice that required for the rock bolts (S_b). Using this proposed rock reinforcement layout, in the analysis presented here the spacing of the rock bolts is calculated using Equation (6.3), where T_s and T_b are the capacity of the anchors and bolts at yield, respectively. The shotcrete is not assumed to provide any active support. A full derivation of Equation (6.3) is provided in Appendix C.

$$S_b = \sqrt{\frac{T_b + 0.25T_s}{P_{roof}}} \tag{6.3}$$

6.1.1 Project conception: Interval analysis

At the project conception stage, it is often the case that little or no factual data are available for use in the engineering design calculation. In these situations, it is common practice to rely on precedent experience or expert judgement to determine the bounds of the parameters used in the analysis. In the case of Gjøvik, precedent experience came in the form of two smaller caverns previously constructed in the same hillside, approximately 100m from the proposed cavern (Barton et al., 1994).

In this example, we assume that at this preliminary stage, experts were able to provide only the bounds of the various Q parameters based on a qualitative assessment of the rock

mass through visual mapping of the adjacent caverns. The rock mass description assumed is as follows:

'The rock quality designation ranges between 'poor' and 'excellent' (RQD = 30-100) with one to three plus random joint sets (J_n = 2-12) present. The joint roughness varies between rough/irregular, planar to discontinuous (J_r = 1.5-4). The joint wall surfaces range from unaltered joint walls with surface staining only to those having softening or low-friction clay mineral coatings with rock wall contact (J_a = 1-4). Previous tunnelling experience in this rock mass indicates that the excavation may encounter minor inflow, i.e. < 5 l/m locally, to occasional medium inflow or pressure (J_w = 0.66-1). At this stage, the in-situ stress classification of 'medium' is considered appropriate (SRF = 1.0).'

The bounds of each of the interval parameters required to calculate the Q-value are summarised in Table 8.

Table 8: Lower and upper bound of input parameters for Q.

Parameter	Lower bound (L)	Upper Bound (U)
RQD	30	100
J_n	2	12
J_r	1.5	4
J_a	1	4
J_w	0.66	1

With respect to the proposed taxonomy (Figure 17) and framework (Chapter 4), having obtained a set of subjectively estimated bounds of the input parameters, we can only apply an interval analysis to obtain the bounds of Q. The interval operations applied to obtain the output interval of Q are those presented in section 3.1; specifically Equations (3.5) and (3.6). Applying these equations to Equation (6.1) and simplifying, the output interval of the lower and upper bounds of Q is obtained as defined by Equation (6.4), in which the superscript L denotes the lower bound and the superscript U the upper bound:

$$\left[Q^L, Q^U\right] = \left[\frac{RQD^L}{J_n^U} \frac{J_r^L}{J_a^U} \frac{J_w^L}{SRF^U}, \frac{RQD^U}{J_n^L} \frac{J_r^U}{J_a^L} \frac{J_w^{mU}}{SRF^L}\right] \tag{6.4}$$

Using the values of Table 8 within Equation (6.4), the resulting interval of Q is given by: $\left[Q^L, Q^U\right] = \left[0.62, 200\right]$. Applying this interval of Q within Equations (6.2) and (6.3), and once

again applying interval arithmetic we obtain an interval of bolt spacing $\left[S_b^L, S_b^U\right] = \left[1.1, 7.5\right]$m, and consequently a strand anchor spacing of $\left[S_s^L, S_s^U\right] = \left[2.2, 15.0\right]$m.

With the current level of information, the only statement that may be interpreted from this interval is, simply that 'Q will lie somewhere between 0.62 and 200'. When converted back to the linguistic classes given by the Q-system, this translates to: 'the rock quality will lie somewhere between poor and extremely good'. Similarly, only statements on the minimum and maximum spacing of the proposed rock reinforcement can be made.

As discussed previously in section 2.5.3, the lack of information contained in intervals leads to difficulty in decision-making as exemplified by the wide range of the calculated interval of Q and bolt spacing. In order to assist in decision-making, the previous section showed how various researchers undertook sensitivity studies using intervals (e.g. Hoek, 2007). The following discussion demonstrates how the vertex method (see section 3.1.2) can be applied to provide additional insight into the result of the interval analysis.

If we consider that the initial data contained five intervals of input parameters for Q (Table 8), with the output being a single interval of Q; this appears to represent a loss of information. Applying the vertex method, we obtain a five-dimensional hypervolume with 32 (2^5) vertices, each representing a unique combination of the five intervals that form the inputs to Q. Figure 62 presents a two-dimensional representation of the five-dimensional hypervolume, with the 32 (2^5) vertices representing unique combinations of the five intervals that form the inputs to Q. Each vertical line in this figure represents one vertex of the five-dimensional hypervolume in the interval of Q. From it we see that the hypervolume bounds Q between the values predicted by the interval solution, $Q = [0.62, 200]$; however, we can also see the concentration, or spread, of the vertices.
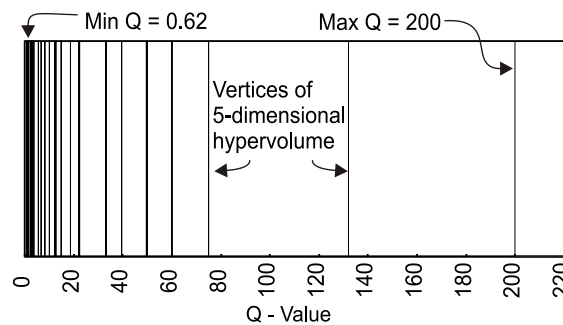
Figure 62: 2D representation of the 5D hypervolume of Q obtained from the vertex method.

The following information may be drawn from this figure: there are two vertices, and thus combinations of the input parameters, which give a Q value greater than 75. However, we cannot deduce the likelihood of occurrence of these two larger values because the input intervals contain no such information. For example, it would be unfaithful to the information to say: '2 out of 32 of the vertices result in a Q value < 75, therefore there is a 94% probability that Q will be less than 75'. However, this analysis allows one to identify the combination of parameters that result in these two higher value vertices, and thus identify the attributes of the rock mass classification leading to the higher calculated values of Q. In this case, it is the calculations that involve the lower bound value of J_n (2) and the upper bound value of J_a (4) that result in the values at these two vertices.

Having obtained an imprecise interval output from Q and reviewed the vertices of the resulting five-dimensional hypervolume, with respect to the framework (box 8a in Figure 48) the question would now be posed: *'Are the bounds small enough to generate a useable output?'*. Given the large output interval of Q, the answer to this is likely to be 'no', at which point the framework would lead one back to *'further investigation/data acquisition'*. Armed with the knowledge of the two extreme vertices, J_n and J_a, one could arguably attempt to obtain additional information to increase the level of understanding on the joint number and alteration. This information is used later in the following section (6.1.4) when we consider decision making.

Nonetheless, with the available information, the interval analysis allows us to only make statements such as 'because our knowledge is limited to only the values defining each interval, we are not able to give any estimate of what will be the most likely value of Q between the values of 0.62 and 200. Further information is required to make a more detailed assessment'. This analysis clearly shows there is too little information on which to make an engineering design decision, and the uncertainty is too large to make a subjective judgment.

6.1.2 Project conception: Comparison with the Bayesian approach

Due to the difficulty in decision-making based on interval analysis, and as previously discussed in section 2.6.2 on subjectivist probability, the Bayesian approach, which applies Laplace's principle of insufficient reason – i.e. using non-informative priors – is often utilised. With respect to the new taxonomy of Figure 17, due to the intrinsically epistemic nature of the uncertainty in rock mass classification systems, this is approach is incorrect, and as demonstrated earlier in section 2.6.3, is neither faithful nor robust. Whilst applying this

Bayesian approach is strictly incorrect, for demonstration purposes, Figure 63a presents the expected distribution of Q obtained from a Monte-Carlo analysis using 5000 simulations with 'non-informative priors' in the form of uniform PDFs, given the available information. The resulting distribution of bolt spacing is illustrated in Figure 63b.
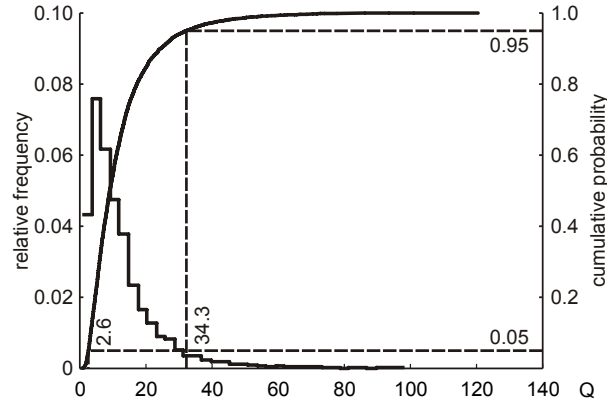
**Figure 63:** Monte-Carlo simulation of Q based on uniform prior PDFs as inputs.

In comparison to the interval analysis, the results of the Bayesian analysis allow much more informative statements to be made, such as: 'The mean value of Q is 12.5. There is a 5% probability that Q will be less than 2.6, and a 95% probability that it will be less than 34.3. The analysis predicts that Q will neither be less than 0.8 nor greater than 127'. Critically, these statements are based on the same information as used in the interval analysis, but with the addition of an assumed prior. Thus, it is clear that it is the priors that allow these statements to be made, not the underlying information. The statements are therefore unsubstantiated, and suggest the presence of more information than is actually available.

From this example, it can be concluded that adopting a Bayesian approach using uniform PDFs to characterise the unpredictability in estimating the Q-value is neither faithful nor robust. Furthermore, recalling the proposed taxonomy (Figure 17) and the scales of measurement (Stevens, 1946) detailed previously in section 2.2.2 and given that many of the parameters of Q are nominal and ordinal, it is questionable whether the calculated statistics are meaningful. Therefore, we now continue the investigation on the feasibility of the Gjøvik cavern based on results obtained from the interval analysis.

6.1.3 Additional information: mapping of adjacent caverns

The initial feasibility assessments, using interval analysis, concluded that the range in the interval was too large to make definitive decisions and so, by following the framework,

required further data to reduce the uncertainty in the model. For the Gjøvik cavern, additional data from detailed mapping of the adjacent caverns, as shown in the histograms in Figure 64a, became available later in the project. We now re-apply the framework of Chapter 4 to this additional data. One important aspect that the histograms in Figure 64a represent the distributions of the subjective assessments of the parameters that are used to estimate the Q-value. While the histograms imply that the parameters may be defined by a probability distribution, the data characterisation strategy does not allow one to reach this conclusion. The reasoning behind this lies in the earlier discussion in this Chapter; the subjective nature of the assessments as well as the approximations embodied in the estimation of Q mean that the unpredictability from its use must be characterised as intrinsically epistemic. In this way, the new framework directs one to a non-stochastic method that remains both faithful and robust to the available information.
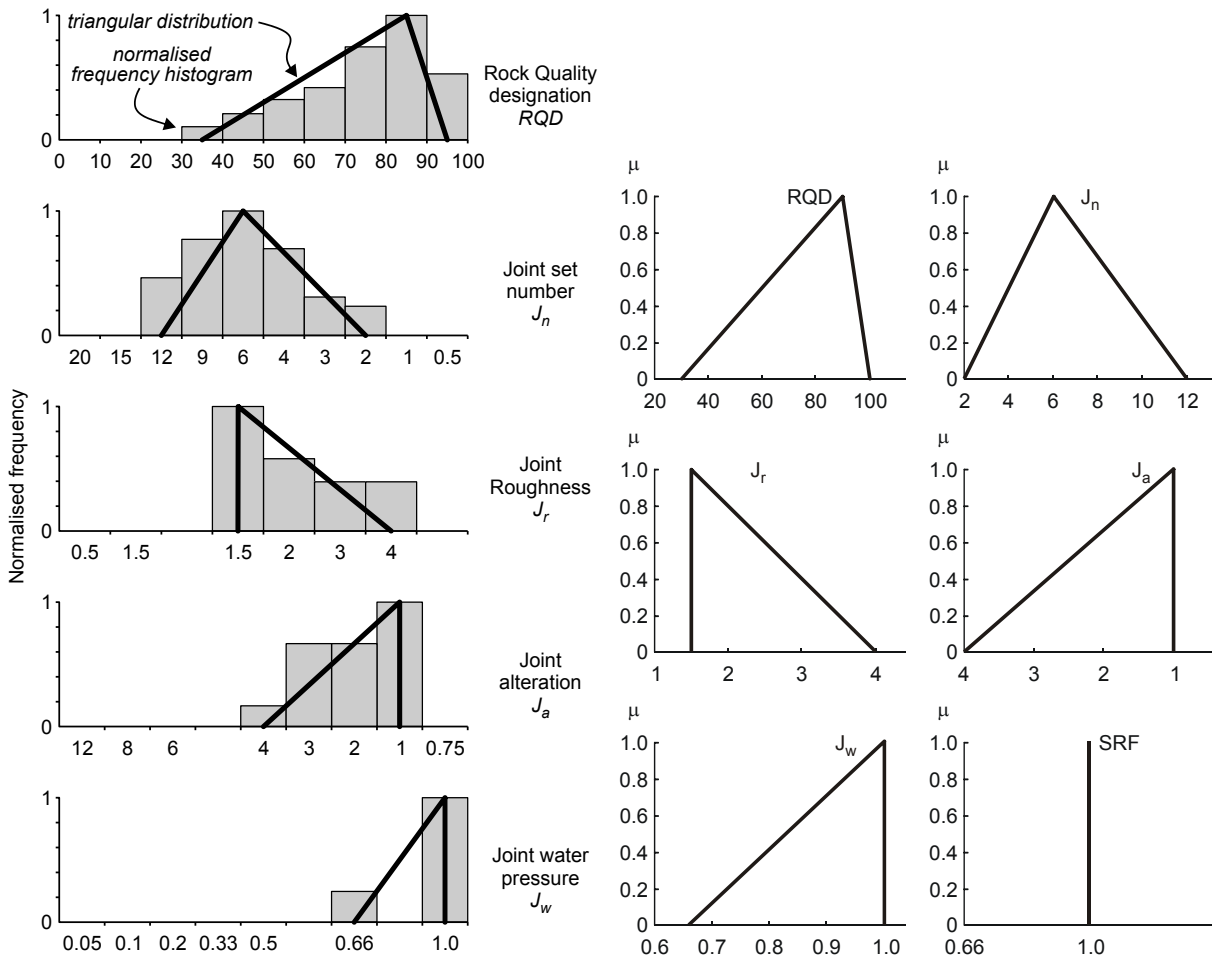
a) Normalised histograms of mapping results from existing caverns adjacent to proposed Gjøvik site

b) Fuzzy numbers defined from histograms of mapping

Figure 64: Histograms of Q-Mapping and fuzzy numbers fit to the data (from Bedi & Harrison, 2013b).

As concluded previously in section 2.8.1, the subjective means of assessing many of the parameters used in the Q-system results in nominal and ordinal input values. Recalling Stevens' (1946) scales of measurement presented in Table 1, the mode is a valid statistic that can be used from the histograms in Figure 64a. The mode values represent additional information that through application of the data characterisation strategy sub-chart (Figure 49), can be used in a new fuzzy analysis. Consequently, we have used the data from Figure 64a to define triangular fuzzy numbers (TFNs) with the mode specifying the 'most possible' ($\mu = 1$) value (see Table 9) as illustrated in Figure 64b. The algorithms used to undertake the fuzzy analysis are presented in Appendix C.

Table 9: Lower, upper bound and most typical values of input parameters for Q.

Parameter	Lower bound	'Most possible'	Upper Bound
RQD	30	90	100
J_n	2	6	12
J_r	1.5	1.5	4
J_a	1	1	4
J_w	0.66	1	1

The resulting possibility distribution for fuzzy-Q ($\hat{Q}$) is presented in Figure 65a with the resulting fuzzy bolt and strand spacing given in Figure 65b. It is clear from Figure 65 that the use of TFNs resulting from the collection of further information has given internal structure to the uncertainty in Q: the figure shows the most possible value of Q to be 30. By comparison with the interval analysis, this is a substantial reduction in uncertainty.

a) Fuzzy Q b) Fuzzy bolt and strand spacing
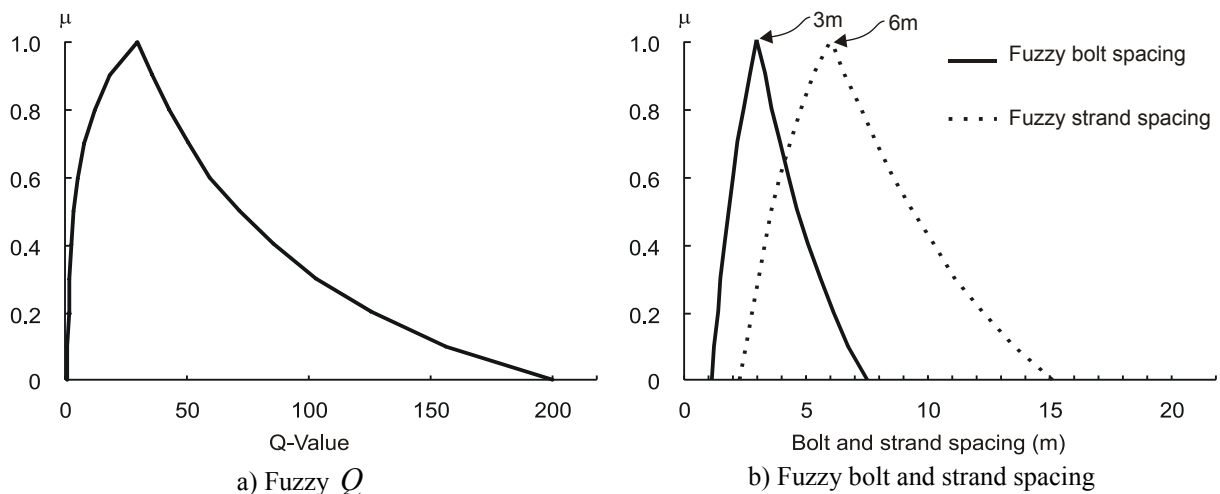
Figure 65: Resulting fuzzy numbers for Q and rock reinforcement spacing.

Unlike the Bayesian analysis presented above (section 6.1.2), the fuzzy analysis continues to bound the extreme values of Q calculated using the interval analysis, but importantly, now allows the following statements to be made: 'The value of Q will range between 0.6 and 200, but these values are least likely, with the most likely value being 30'. A similar statement can be made with respect to the rock reinforcement spacing: 'It is possible for the bolt spacing to range between 1.1 and 7.7m, but these values are least likely, with the most likely value being 3m. The equivalent strand spacing is twice these values'. The next section now investigates how the information from the interval and fuzzy analyses can be used to assist in making pragmatic decisions regarding the feasibility of the design.

6.1.4 Decision making: Assessment of feasibility

As the name implies, one of the main purposes of a feasibility study is to provide an assessment of the viability of a project. For a large scale project such as the Gjøvik cavern, a key consideration may be cost feasibility, a large part of which may be attributed to rock support and reinforcement. For example, Tzamos & Sofianos (2006) have presented a correlation between estimated support weight and Q-value for cost feasibility assessment. In our example, let us assume that a bolt spacing of 2.0m (and hence a strand spacing of 4.0m) is the minimum feasible in terms of support costs.

One means of assessing the likelihood of exceeding this threshold is through the possibility and necessity measures discussed earlier in section 3.2 (see Figure 37). For this example, Figure 66 illustrates the possibility and the necessity measures: $\Pi(s_b \leq 2.0m) \approx 0.6$ and $N(s_b \leq 2.0m) = 0$, respectively. The necessity measure of 0 indicates a greater possibility of the bolt spacing being greater than 2.0m. However, there is a possibility, at a level $\beta = 0.6$ in Figure 66, that the bolt spacing may be less than the 2.0m threshold. As an additional measure, we could de-fuzzify the fuzzy bolt spacing at the 2.0m threshold using the 'Agreement Index' presented earlier in Equation (5.2), as follows:

$$\text{AI}(S_b) = \frac{\int^{2.0} \mu_{Sb}(x)dx}{\int_x \mu_{Sb}(x)dx} = 0.094 \tag{6.5}$$

As shown by Equation (6.5), we obtain an agreement index of approximately 0.1 that the bolt spacing will be less than 2.0m.
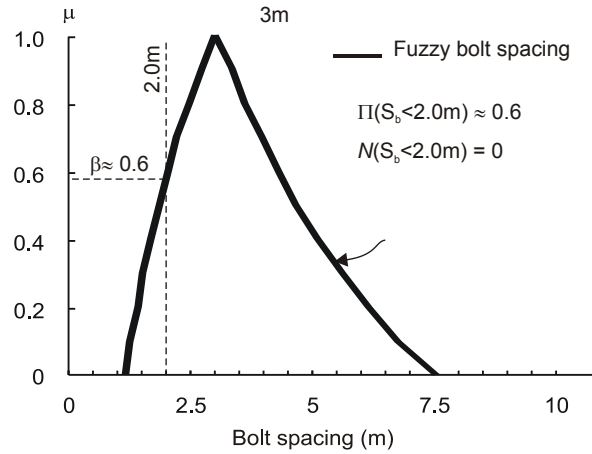
Figure 66: Possibility measure of fuzzy bolt spacing for feasibility assessment.

At present, there is no published literature on the engineering significance of the calculated values of these 'fuzzy confidence measures' (this will be discussed later in section 8.3 on further work). However, both the possibility measure and agreement index indicate that the limited data available suggests a higher possibility of a bolt spacing greater than the 2.0m threshold. With respect to the new taxonomy (Figure 17), given the low quality and quantity of information at this stage, the fuzzy analysis has been able to provide a faithful and robust indication of the full range of unpredictability in estimating the Q-value and bolt spacing. On this basis, one may decide to continue with further investigation and design for the Gjøvik cavern.

6.1.5 *Further investigation: Refining possibility*

With specific reference to the Gjøvik Cavern, a second phase of investigation was commissioned, which involved refining the assessment of the Q-value based on diamond cored holes drilled within the footprint of the cavern (Barton et al., 1994). The additional data obtained is presented in the histograms in Figure 67a.

Using this additional data, Barton et al. (1994) suggested refining the estimates of three of the parameters, as follows; $J_n = [4,9]$; $J_r = [1.5,3]$, and; $J_a = [1,3]$. Applying these refined bounds to the interval solution of Q as defined by Equation (6.4), above, we obtain an updated interval of Q:

$$[Q^L, Q^U] = \left[\frac{30}{9} \times \frac{1.5}{3} \times \frac{0.66}{1}, \frac{100}{4} \times \frac{3}{1} \times \frac{1}{1} \right] = [1.1, 75] \qquad (6.6)$$

By comparing this refined interval with that calculated at the project conception stage (i.e. in section 6.1.1, above), and specifically with the vertex method calculation (Figure 62), one can

see that this additional information has eliminated the two vertices that resulted in calculations of Q greater than 75. Now, using the fuzzy number calculated for Q in Figure 65a, this refined interval of the Q-value determined using Equation (6.6) approximates bounds represented by the α-cut at a possibility level (μ_Q) of approximately 0.44, as illustrated in Figure 65b. The corresponding interval of bolt spacing at this possibility level is then $[1.7, 4.9]$ m.
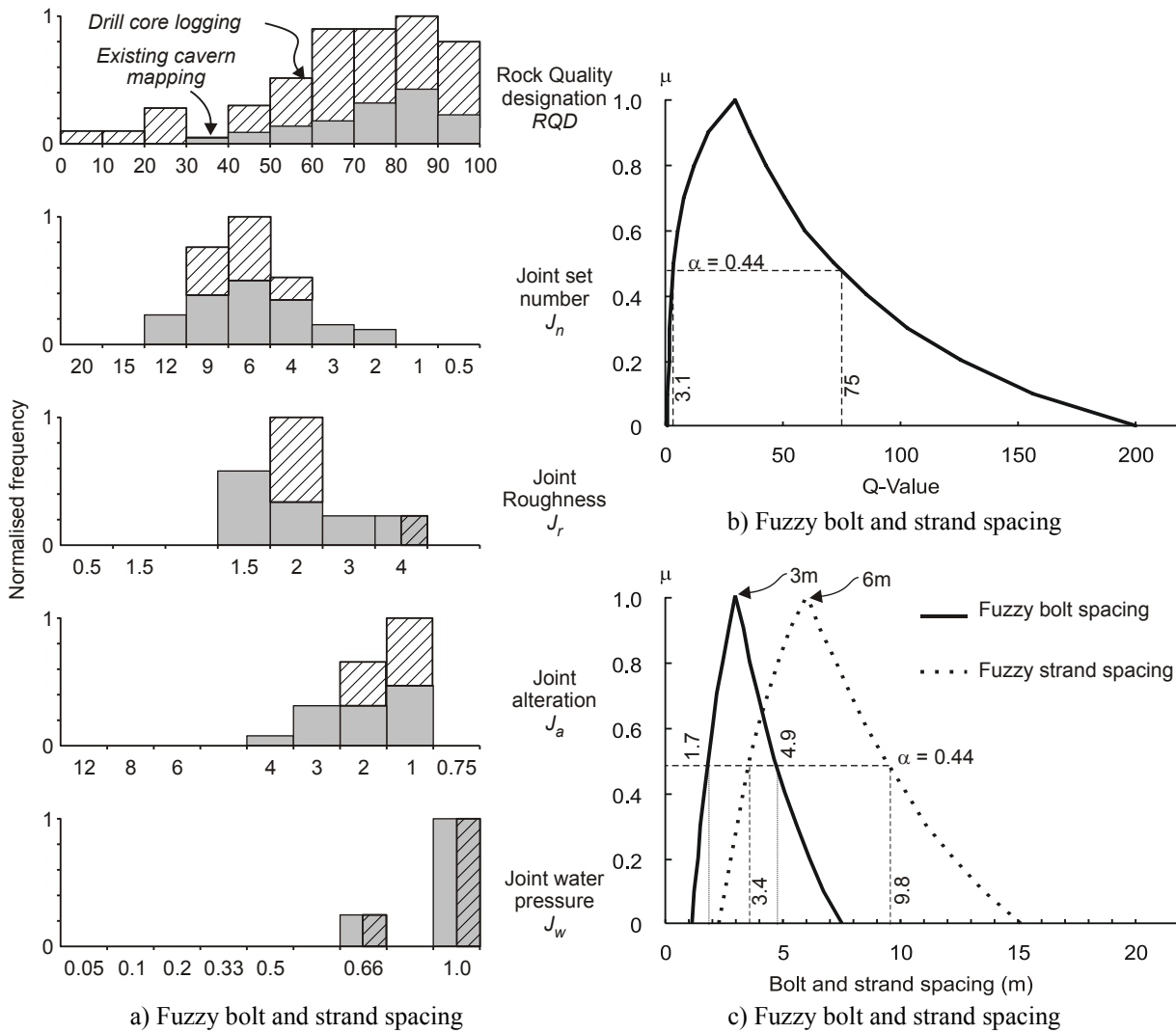
Figure 67: Normalised histograms of Q-mapping results from existing caverns and additional drill core data at proposed Gjøvik site and resulting fuzzy numbers for Q and rock reinforcement spacing.

In the context of the new taxonomy and framework, the intrinsically epistemic nature of the Q-system will not allow one to move from an epistemic to an aleatory model with additional information. This is once again exemplified by the data characterisation strategy sub-chart (Figure 49) which directs the analysis towards fuzzy numbers or intervals when the data is assessed subjectively. However, at the decision making stage in the proposed

framework (Figure 48), if the answer to the question *'Are the bounds small enough to generate a useable output'* is 'no', the framework directs further investigation. As this example has demonstrated, given the unpredictability is intrinsically epistemic, additional information only allows one to target further data collection at those parameters that will allow a reduction in the level of epistemic uncertainty. With this additional information, one can refine the original intervals of the intrinsically epistemic parameters, or alternatively move up the possibility level in the fuzzy number.

6.1.6 Comparison with design implemented at Gjøvik

The permanent rock reinforcement in the Gjøvik cavern consisted of systematic bolting and cable bolting in alternating 2.5 and 5. 0 m, centre-to-centre, patterns. The rock reinforcement was based on assessment of the Q-value during construction. With respect to the analysis presented in this Chapter, a bolt spacing of 2.5m represents a value close to the 'most possible' predicted by the fuzzy analysis. Given the intrinsically epistemic uncertainty inherent in rock mass classification systems, this demonstrates that with additional information one may refine the possibility measures further, however some imprecision is likely to remain in the final result.

6.2 Synopsis

The discussion in 2.8.1 concluded that empirical rock mass classification systems are inherently imprecise and thus must be recognised as being intrinsically epistemic. By using the new taxonomy and framework, the challenge problem presented in this Chapter showed that the unpredictability must therefore be characterised using non-stochastic methods. In this instance, intervals and fuzzy numbers.

In the case of the Gjøvik cavern, where the scale and complexity of the project was unprecedented (Barton et al., 1994), given the lack of information at the feasibility stage, interval-oriented methods provide a means of capturing approximation and imprecision in Q. That is, the interval analysis at the project conception stage demonstrated that knowledge was insufficient to make a potentially critical decision; 'to go, or not to go ahead' with the project. However, at the same level of information, a decision based on a Bayesian approach may have resulted in misinformed decisions (Figure 63). On the contrary, at the early stage when only interval data were available, given the wide intervals of Q and calculated bolt spacing, the only decision that could be reached was to gather further information. Undertaking some further mapping of the adjacent caverns allowed us to move to a fuzzy analysis.

This example showed how the fuzzy analysis can be used to estimate the range of bolt spacing and in turn, utilise this information to estimate cost feasibility of the project with using prior experience and a limited number of subjective measurements. A purely subjective interval may have resulted in the project costs being unfeasible. However, a small amount of additional data – in this case, limited mapping of adjacent caverns – results in a substantial reduction in epistemic uncertainty and thus assists in further decision-making whilst still presenting a robust assessment of the 'best and worst case' to the decision makers. As this example has demonstrated, this is not so if a Bayesian approach is used. Furthermore, this example demonstrates the usefulness of interval-oriented approaches in presenting a faithful representation of the available information.

As the parameters in this challenge problem were intrinsically epistemic (as discussed earlier in section 2.8.1), the framework of Chapter 4 does not allow anything more than an imprecise analysis. However, the next Chapter shows how the framework allows one to move to a higher unpredictability modelling method when the data are extrinsically epistemic and it is possible to collect further data collection is possible to reduce epistemic uncertainty.

Chapter 7

CHALLENGE PROBLEM 3 – EMPIRICAL STRENGTH CRITERIA

Section 2.8.3 discussed the application of the Hoek-Brown failure criterion (Hoek & Brown, 1980a; b) in modelling the peak strength of intact rock and rock masses. This earlier discussion demonstrated that as the parameters required to define intact rock strength can be obtained from objective laboratory measurements; they can be characterised as aleatory provided a sufficient number of triaxial test data are available with which to objectively fit probability distributions to define them. However, application of the Hoek-Brown criterion to estimate the strength of rock masses requires a subjective estimation of GSI. With respect to the new taxonomy (Figure 17), and as further discussed in section 2.8.1, the subjective assessment of GSI means the unpredictability resulting from its use must be characterised as intrinsically epistemic.

This section first presents an example that demonstrates the applicability of the new framework (Chapter 4) in characterising unpredictability in parameters used to estimate the strength of a jointed rock mass. In this example, the Hoek-Brown strength criterion requires GSI as an input to define the rock mass properties, and therefore the unpredictability must be characterised as intrinsically epistemic. This is followed by a second example in which the peak strength of the intact rock is estimated from parameters obtained through laboratory testing on intact rock specimens; the parameters are intrinsically aleatory. Using the concepts presented in this thesis, the applicability of the framework is demonstrated as data become progressively available. This second example shows that a quantitative lack of data requires the parameters to be characterised as extrinsically epistemic. The example presented in this section shows how in such an case, the proposed framework directs the user to a non-stochastic approach but with further data collection one can to move to an aleatory model. To facilitate the discussion, actual laboratory test results for Milbank granite obtained from the

literature (Ruffolo & Shakoor, 2009; Bauer et al., 2012) are used, with the exemplar rock mass also based on the Milbank granite data.

7.1 Strength of rock masses – intrinsically epistemic

Section 2.8.3 introduced the Hoek-Brown failure criterion for estimating the strength of jointed rock masses, as defined earlier by Equation (2.1). In this expression, the rock mass parameters, m_b and s, are derived through approximate correlations with GSI (Equations (2.1) to (2.3)). The Hoek-Brown equations used in the following example are reproduced below.

$$\sigma_1 = \sigma_3 + \sqrt{m_b \sigma_{ci} \sigma_3 + s\sigma_{ci}^2} \tag{2.1}$$

$$m_b = m_i \exp\left(\frac{GSI - 100}{28}\right) \tag{2.2}$$

$$\text{and } s = exp\left(\frac{GSI - 100}{9}\right) \tag{2.3}$$

The two laboratory properties required for the application of the Hoek-Brown criterion are the uniaxial compressive strength (UCS) of the intact rock (σ_{ci}) and the intact rock material constant m_i. Ideally these two parameters should be determined by triaxial tests on carefully prepared specimens as described by Hoek and Brown (1980b). In our example, we assume the minor principal stress σ_3 is deterministically known, ten UCS test results have been provided and there is no objective test data available with which to determine m_i.

For this example, the UCS data have been randomly drawn from a set of fifty tests undertaken on Milbank granite by Ruffolo & Shakoor (2009). A statistical analysis undertaken on these data concludes that σ_{ci} can be characterised by a normal distribution with a mean $\mu = 158 MPa$ and standard deviation $\sigma = 28 MPa$, as shown in Figure 68a. Hypothesis testing using the Kolmogorov-Smirnoff goodness-of-fit test concluded that the hypothesis that the data are drawn from a normal distribution cannot be rejected at the 95% confidence level. A Quantile-Quantile plot to visually confirm this is shown in Figure 68b. In this plot, data lying close to or on the diagonal indicate a good fit with a normal distribution. Using this information, following the data characterisation strategy, σ_{ci} is characterised as aleatory and an objective PDF fitted using statistical procedures.

a) Normal distribution fit to uniaxial compressive strength data

b) Quantile-Quantile plot of data against normal distribution
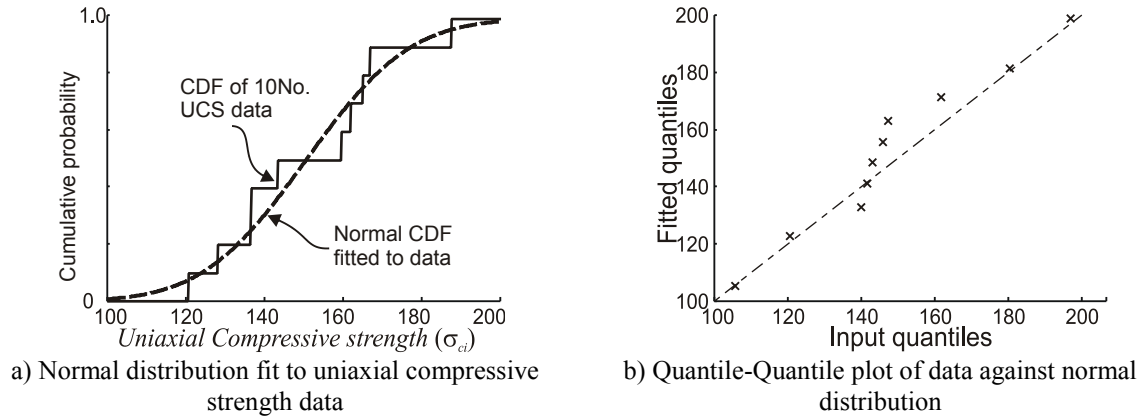
Figure 68: Statistical analysis on ten uniaxial compressive strength test data.

Given the absence of objective data for m_i, it must be determined subjectively through expert consultation or prior knowledge. Hoek (2007) provides an empirically derived table containing a range of values for m_i by rock group, which for granites is recommended as 32±3. It should be noted that no preference is given to any specific value within this range, nor is it considered that the values in this range are equi-probable. Consequently, following the data characterisation strategy, m_i is characterised by the box-interval $[29,35]$. It should be noted, that while it is known that m_i can be obtained from precise measurement, i.e. it is extrinsically epistemic (see section 2.8.3). Given the available information, this statement is by definition true because m_i has – at this stage, at least – been determined entirely subjectively.

Section 2.9.1 concluded that rock mass classification systems such as GSI require subjective estimation and incorporate nominal and ordinal scales of measurement, all of which mean they must be characterised as intrinsically epistemic and thus it is inappropriate to represent the unpredictability by stochastic models. On this basis, we now apply the proposed framework to an example of estimating the peak strength of a jointed rock mass. In this example, we assume that an expert geologist has provided the following classification of GSI: 'The rock mass structure is 'blocky' and the surface quality is good. The bounds of GSI are between 55 and 80, with a most likely value of 70'. Using this description and following the data characterisation strategy, GSI is represented by the fuzzy number

$G\hat{S}I = [55,70,70,80]$, and is shown in Figure 69a. The fuzzy number of GSI and the interval of m_i is then used to compute the fuzzy rock mass constants $\hat{m}_b$ and $\hat{s}$, using Equations (2.2) and (2.3), which are illustrated in Figure 69b and c.
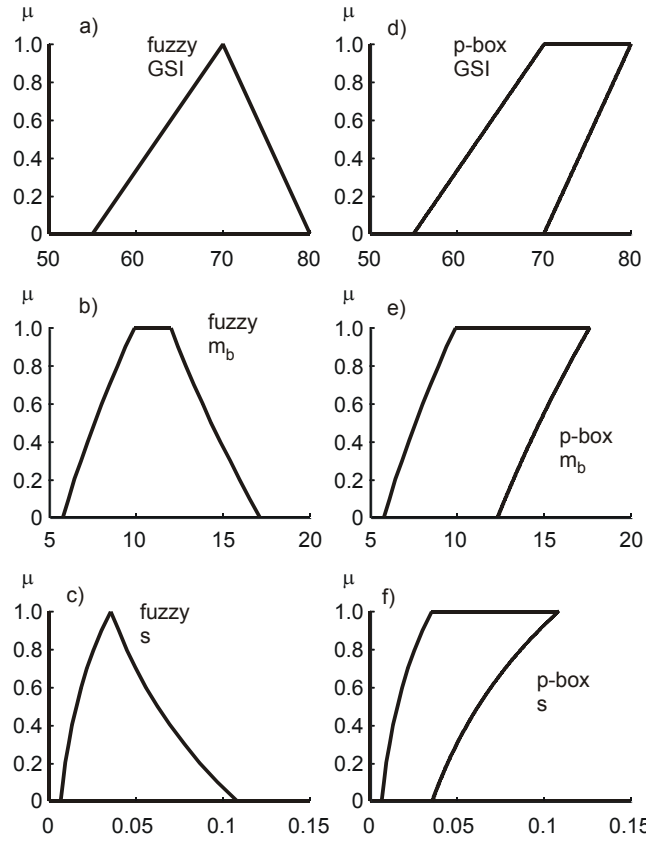
Figure 69: Fuzzy numbers and equivalent p-boxes for input parameters.

Equation (2.1) now consists of a mix of fuzzy numbers with an aleatory variable (σ_{ci}), which requires a hybrid analysis. For this analysis, the fuzzy numbers are considered within a possibilistic framework and the possibility and necessity measures used to construct equivalent p-boxes, as shown in Figure 69d,e and f. The output is in the form of a subjective bounded output; this p-box is shown in Figure 70. The calculations for undertaking this hybrid analysis are given in Appendix C.
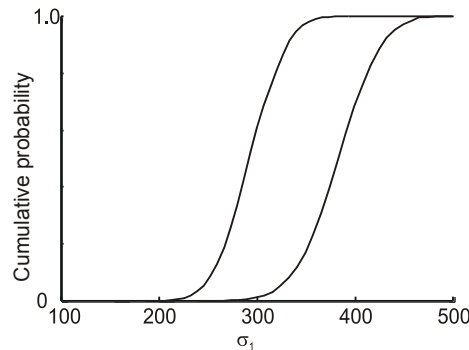
Figure 70: P-box from hybrid analysis to compute rock mass strength.

The paths followed in the main framework as well as the data characterisation and model selection strategy sub-charts for each of the parameters are presented in Table 10.

Table 10: Paths followed in framework for hybrid analysis.

Box #	Question	Answer
1	Preliminary investigation/data acquisition	10 UCS tests and no other objective data.
2	Select analytical model	Hoek-Brown failure criterion for rock mass strength using GSI
3	Identify parameters required for model	Uniaxial compressive strength σ_{ci}, material constant m_i and GSI are non-deterministic.
4	Further investigation/data acquisition	Not available
5	Uncertainty parameters characterisation	Move to 'Data characterisation strategy' sub-chart (i.e. Figure **49**)
Start Data characterisation strategy - σ_{ci}		
5.1	Can the data be objectively measured?	Yes – UCS tests performed in laboratory
5.2	A sufficient number of precise measurements are available?	Yes – 10 data available.
5.3	Statistical tests can be used to fit a unique PDF?	Yes. Hypothesis (by K-S test) that the data are drawn from a normal distribution cannot be rejected at the 95% level. See Q-Q plot for visual confirmation. Fit **Objective PDF**
Start Data characterisation strategy - m_i		
5.1	Can the data be objectively measured?	Yes – m_i can be measured through triaxial tests
5.2	A sufficient number of precise measurements are available?	No measurements available
5.3	Prior information on which to formulate a precise PDF is known?	No.
5.4	The type of distribution is known and intervals for its parameters can be provided?	There is no evidence to support any particular shape of distribution
5.5	A sufficient number of imprecise measurements are available?	Refer 5.2
5.6	An interval that bounds the parameters is known?	Yes. Prior published data and expert opinion can be used to provide bounds for each parameter (see Hoek, 2007)
5.7	An estimate of the most plausible values can be provided?	No. Empirical data (Hoek, 2007) only specifies a range with no preferred value.
5.8	An interval of more plausible values can be provided?	Not at this stage. Therefore characterise m_i using an **Interval**
Start Data characterisation strategy - GSI		
5.1	Can the data be objectively measured?	No – GSI must be determined subjectively by visual comparison against exemplar profile
5.2	A sufficient number of imprecise measurements are available?	No – An expert geologist has been requested to provide guidance.
5.3	An interval that bounds the parameters is known?	Yes. See above.
5.4	An interval of more plausible values can be provided?	Expert geologist advice based on nearby outcrop mapping: 'The rock mass is structure is 'blocky' and the surface quality is good. The bounds of GSI are between 55 and 80, with a most likely value of 70'. Therefore GSI is characterised using a triangular fuzzy number
6	Select appropriate uncertainty model	Move to 'Uncertainty model strategy' sub-chart (see Figure 50)

Table 10: Paths followed in framework for hybrid analysis (continued).

	Start Model Selection Strategy Sub-chart	
6.1	Parameter characterisation	σ_{ci} **is characterised as aleatory and** m_i **as intrinsically** epistemic as there is no justification to support it as stochastic and GSI as intrinsically epistemic as it is determined subjectively (qualitative lack of information)
6.2	Select uncertainty model	Combination of aleatory and epistemic parameters requires hybrid analysis
6.3	Analytical output	Subjective bounded output. We now return back to the main flowchart (Figure 48)
	Return to Main Framework	
7	Analysis	We undertake a hybrid analysis using p-boxes
8	Model propagation	The analysis results in a subjective bounded output
8a	Are the bounds small enough to generate a useable output	See following discussion

At the end of Table 10 we once again find ourselves at decision making stage, faced with the question: *'Are the bounds small enough to generate a useable output?'*. If the answer to this is 'no', the framework directs us back to the further data collection stage (Box 4 in Figure 48). However, as GSI can only be estimated subjectively, and is thus intrinsically epistemic, further data collection may reduce the uncertainty in the estimate of GSI (e.g. reducing the bounds) however, the refined estimate of GSI will still be imprecise. The intrinsic epistemicity of such parameters is captured by the first question posed in the data characterisation strategy (Figure 49): *'Are objective measurements available?'*. The consequence of this is that when extrinsically epistemic parameters are included in any analysis, the output will always be imprecise. With respect to this example, the data characterisation strategy sub-chart (Figure 49) illustrates that a number of imprecise measurements of GSI could be used to define a non-parametric p-box. An example of non-parametric p-boxes to characterise parameters such as GSI has been presented by Tonon et al. (2000).

The example that follows shows that if the parameters are extrinsically epistemic, the framework allows re-characterisation of the unpredictability from epistemic to aleatory as the level of information increases.

7.2 Intact rock strength - extrinsically epistemic

The Hoek-Brown failure criterion for estimating the strength of intact rock is given by:

$$\sigma_1 = \sigma_3 + \sqrt{m_i \sigma_{ci} \sigma_3 + s \sigma_{ci}^2} \qquad (7.1)$$

Using the aleatory and interval parameters for σ_{ci} and m_i, respectively, defined in the previous section, the model characterisation strategy (Figure 50) illustrates that a hybrid analysis is required, the result of which is a subjective bounded output. To undertake the hybrid analysis, the interval of m_i is modelled as an equivalent p-box shown in Figure 71 and combined with the aleatory model of σ_{ci}, using Equation (7.1). The resulting p-box, shown in Figure 72, accounts for both the imprecision in m_i and variability of σ_{ci}. The area between the upper and lower cumulative density functions of the output p-box in Figure 72 represents the region of unpredictability within which the value σ_1 must lie.
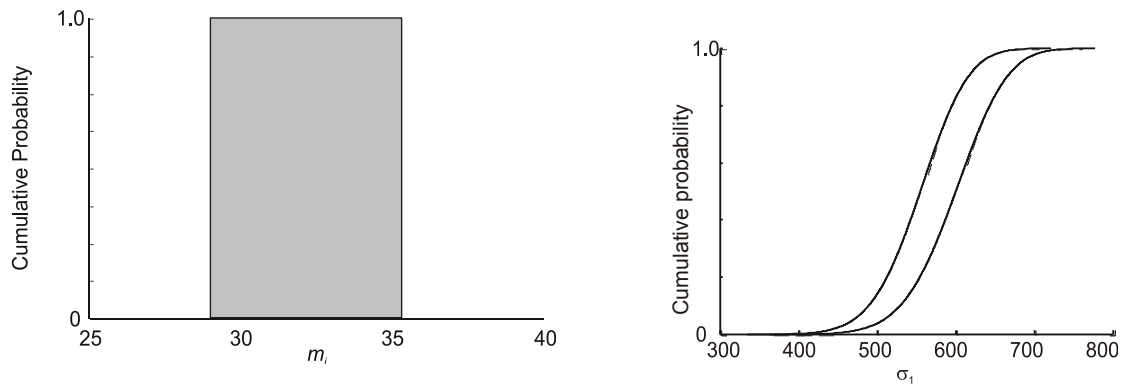
Figure 71: P-box representation of the interval of m_i.

Figure 72: P-box of intact rock strength calculated using the Hoek-Brown failure criterion.

Having obtained the bounded subjective output, the framework directs the user towards the decision making stage by asking the question: 'Are the bounds small enough to generate a useable output?' For demonstration purposes, we will assume the answer to this is 'no'. The framework then asks 'can more data be obtained?'. For this example, we assume the answer is 'yes', and thus return to 'Further investigation/data acquisition' stage.

At this stage, we could assume that there is a limited budget available for further testing. Thus, a decision would need to be made on whether to spend this budget on additional UCS tests or triaxial tests to obtain some objective data with which to fit m_i, or a combination of the two. In section 2.4, it was stated that once an aleatory model was fitted to a sufficient number of objective data, further testing would not reduce unpredictability but only serve to refine the precision of the parameters that define variability. On the contrary, it was stated that unpredictability could be reduced through further knowledge that would decrease imprecision, and hence uncertainty. The following two sections substantiate this claim through continuation of this example.

7.2.1 Refining the precision of the aleatory model

Continuing the example above, we now assume that further UCS tests were undertaken to refine the variability in UCS. Using the data of Ruffolo & Shakoor (2009), Figure 73a presents a comparison of the PDFs fitted to the original set of ten UCS tests, with a second dataset containing fifty samples with a mean $\mu = 159 MPa$ and standard deviation $\sigma = 25 MPa$.

Following the framework through a second time, Figure 73b presents a comparison of the p-boxes calculated using the PDFs of UCS obtained from a dataset containing ten samples to that with fifty samples. This figure demonstrates that only a small change in unpredictability is observed with the collection of an additional 40 UCS samples.
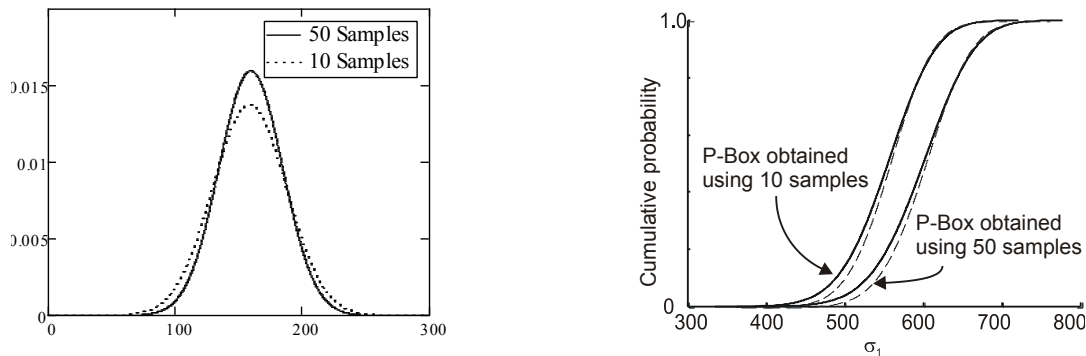
a) PDFs fitted to UCS data for a subset containing 10 samples and another containing 50samples

b) Resulting p-boxes for UCS defined with 10 and 50 samples.

Figure 73: Comparison of aleatory model and p-box obtained by UCS fit to 10 and 50 samples, respectively.

With respect to the earlier discussion in section Chapter 2, and especially the example of limiting precision in UCS presented in section 2.8.3, this example confirms the assertion that as aleatory variability is inherent in a system it cannot be reduced by additional investigation, though one may increase the precision of the parameters that describe it.

7.2.2 Reducing epistemic uncertainty

We now examine the effect of obtaining a better estimate of the parameter m_i through a series of triaxial tests and subsequent curve fitting to these, as described in Hoek & Brown (1980b). In our example, we have only been able to obtain six triaxial test results (Bauer et al., 2012), of which only two are at a confining stress greater than zero. The fitting procedure produces an estimate of m_i equal to 34.2, with an r^2 value of 0.86. Whist the triaxial test data are precise, given the low sample number, the quantity of information is not considered adequate

to warrant an aleatory model to characterise m_i. However, it may be used to reduce the interval to 34±1, as shown in Figure 74b.

a) m_i fit using (objective) triaxial test data

b) refined interval of m_i using results of triaxial test data
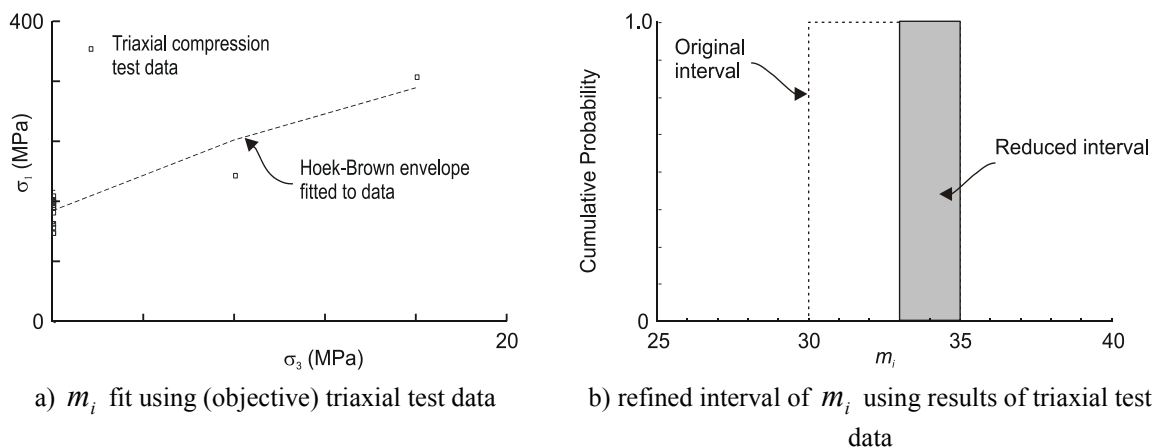
Figure 74: m_i fit to triaxial test data and p-box of reduced interval of m_i.

The p-box resulting from a hybrid analysis using the reduced interval of m_i compared to the original analysis in the preceding section, is shown in Figure 75. This figure shows a marked reduction in the distance between the lower and upper probability bounds, which in turn validates the earlier assertion that obtaining additional information can serve to reduce imprecision in epistemic parameters and hence reduce unpredictability.
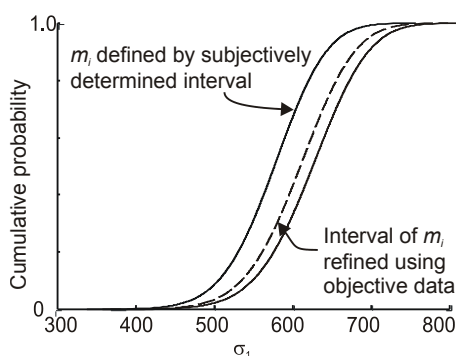
Figure 75: Comparison of p-boxes for m_i defined subjectively and as an interval refined based on a limited number of triaxial test data.

Hoek & Brown (1980b) show that for various rock types, very high values of r^2 can be obtained for m_i fit to precise triaxial data. Thus, if sufficient test data were available, following the framework further would allow m_i to be re-characterised as aleatory and modelled using a precise PDF. From this, one can conclude that while m_i may in fact be aleatory, if it is estimated entirely subjectively it must be classified as intrinsically epistemic.

On the other hand, if objective data is available, it may be characterised as extrinsically epistemic until sufficient data become available with which to fit an aleatory model.

7.3 Rock spalling around underground openings

A common problem that requires estimation of the strength of rock is that of rock spalling around underground openings. Rock spalling is usually defined as a function of the induced tangential stresses around the opening (σ_θ) as well as the spalling strength of the rock (σ_{spall}), and has traditionally been computed using a factor of safety approach given by (Harrison & Hudson, 2010):

$$FoS = \frac{\sigma_{spall}}{\sigma_\theta} \qquad (7.2)$$

Harrison & Hudson (2010), present a simple solution to compute the FoS against spalling as a function of the in-situ stress ratio k and overburden stress, σ_v, as follows:

$$FoS = \frac{\sigma_{spall}/\sigma_v}{3k-1} \text{ if } k \geq 1 \qquad (7.3)$$

$$FoS = \frac{\sigma_{spall}/\sigma_v}{3-k} \text{ if } k < 1 \qquad (7.4)$$

With respect to the examples presented in the preceding sections, 7.1 and 7.2, unpredictability in peak strength may be calculated using the Hoek-Brown strength criterion and characterised as intrinsically epistemic, extrinsically epistemic or aleatory depending on whether failure is considered in a fractured rock mass or intact rock. Furthermore, in the absence of a sufficient number of precise measurements of the in-situ stresses, empirical correlations are often used to estimate k from a calculated value of overburden stress, σ_v. One such correlation was presented previously in Figure 6 of section 2.2. With respect to the new taxonomy of Figure 17, the imprecision in such a correlation requires the parameter k to be characterised as epistemic.

In recognition of the epistemic nature of uncertainty that may result in both in rock spalling and the in-situ stress ratio, Harrison & Hudson (2010) present a fuzzy approach to calculating a fuzzy factor of safety ($F\hat{o}S$) by characterising unpredictability in rock strength (σ_{spall}) and in-situ stress ratio (k) using fuzzy numbers. In their assessment, Harrison & Hudson (2010) adopt the simplifying assumption that the rock mass under examination is

CHILE. Due to this simplifying assumption, as we have seen previously in section 2.8.4 and in the above example of section 7.2, given a sufficient number of precise data, the intact rock strength could have been characterised as aleatory, and with respect to the data characterisation strategy flowchart in the proposed framework (Figure 49 in Chapter 4) modelled using an alternative modelling method.

In contrast, Martin et al. (2003) and Martin & Christiansson (2009) present a probabilistic assessment of rock mass spalling around circular opening constructed in a fractured rock mass at the Aspo Hard Rock Laboratory. In their example, both the rock mass strength (σ_{spall}) and the in-situ stress ratio (k) are characterised by precise, triangular PDFs. Martin & Christiansson (2009) provide the following justification for the choice of distribution: *"The triangular distribution is typically used as subjective description when there is only limited sample data and the user wishes to provide the most likely value. Other distributions can be used if sufficient data are available"*.

With reference to the new taxonomy (i.e. Figure 17) presented in this thesis; characterisation of unpredictability by a PDF requires a sufficient quantity of objective data to justify the use of an aleatory. However, the subjective means of assigning the distribution as proposed by Martin & Christiansson (2009), will lead to a purely subjective output, which as we have seen previously in the rock slope stability challenge problem presented in Chapter 5, can lead to erroneous results and dissonance.

In the two sub-sections that follow, we apply the new taxonomy and framework to the assessment of spalling FoS around a circular opening using the parameters for the exemplar jointed rock mass and intact rock used in the previous examples of section 7.1 and 7.2, respectively.

7.3.1 Spalling around circular opening in jointed rock mass

Using the Hoek-Brown criterion (i.e. Equation (2.1)) to estimate the spalling strength of the jointed rock mass, the minor principal stress (σ_3) is set equal to zero. Simplifying this expression, we obtain an estimate of the rock mass spalling strength by:

$$\sigma_{spall} = \sqrt{\exp\left(\frac{GSI-100}{9}\right)\sigma_{ci}^2} \qquad (7.5)$$

This expression requires a subjective estimate of GSI, and as concluded earlier in section 7.1, must therefore be characterised as intrinsically epistemic. In line with the data characterisation strategy sub-chart (Figure 49), and as detailed previously in section 7.1, GSI is thus

characterised using a fuzzy number. The intact rock strength (σ_{ci}) is characterised using an aleatory model, with a normal distribution as defined previously in section 7.1.

In this example, we assume the depth of the opening is 500m below ground level and adopt a deterministic value for the overburden stress, $\sigma_v = 13.5 \text{MPa}$. The parameter defining the in-situ stress ratio, k, is estimated using the correlation of Figure 6 (see section 2.2) and assigned an interval [0.3,3].

By applying a hybrid analysis (see Appendix D), to Equations (7.2) to (7.4), we obtain a p-box of the resulting FoS against spalling for this rock mass, as shown in Figure 76.
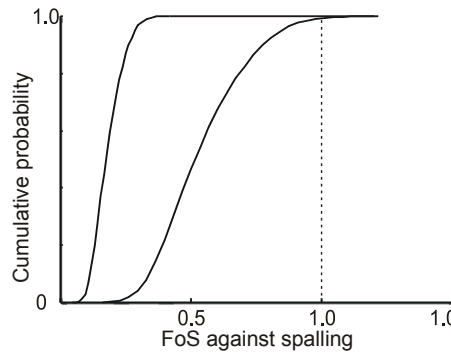
Figure 76: FoS against spalling in jointed rock mass using Hoek-Brown strength criterion.

Figure 76 shows that there is a low FoS against spalling in the rock mass, and thus a high potential for spalling at this depth. This conclusion is drawn accounting for both the imprecision in the rock mass spalling strength as well as the in-situ stress ratio. What this example demonstrates is; irrespective of the wide interval of k there is still a high potential for spalling in this rock mass. Therefore, a higher FoS could only be attained by increasing confidence in the spalling strength parameters of the rock mass. It is on those parameters where further data investigation should be focussed.

7.3.2 Spalling around circular opening intact rock mass

Similar to the example above, by setting the minor principal stress (σ_3) equal to zero in Equation (2.1), the spalling strength of intact rock is derived, as follows:

$$\sigma_{spall} = \sigma_{ci} \tag{7.6}$$

That is, the spalling strength is directly proportional to the uniaxial compressive strength of the rock. Similarly, and as previously defined by section 7.2, the intact rock strength (σ_{ci}) is characterised using an aleatory model, with a normal distribution.

Similar to the previous section, by applying a hybrid analysis (see Appendix D), to Equations (7.2) to (7.4), we obtain a p-box of the resulting FoS against spalling of the intact rock, as shown in Figure 77.
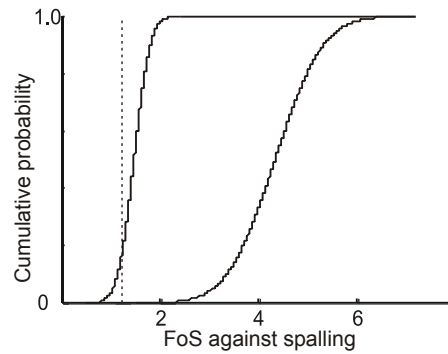
Figure 77: FoS against spalling in jointed rock mass using Hoek-Brown strength criterion.

Figure 77 shows that there is generally a high FoS against spalling of the intact rock, and thus a low potential for spalling at this depth. However, given that the intact rock strength σ_{ci} was modelled using a precise distribution, it is the imprecision in the in-situ stress ratio, k, that has resulted in an imprecise output. However, given that the upper bound CDF (the Belief function) is entirely to the right of a FoS of 1, and the lower bound CDF (Plausibility function) intersects a FoS of 1 at a low probability level, this may give sufficient confidence in assessing the low potential for spalling through the intact rock and thus eliminating any further need for investigation.

7.4 Synopsis

In this example, the varying levels of information for the various parameters means that unpredictability must be propagated using a hybrid analysis. The corollary of this is that hybrid methods do not yield a unique estimate of the probability. Although the very aim of these joint propagation methods is to promote consistency with available information (maintain robustness and faithfulness) and avoid the assumptions of Bayesian methods, the use of imprecise probabilities may become an impediment at the decision-making stage, since decision-makers may not feel comfortable with the notion of an imprecise probability of exceeding a threshold (Ferson & Ginzburg, 1996; Dubois & Guyonnet, 2011). Thus, if a

decision cannot be made based on a subjectively bounded output, the decision-maker has the following options; a) adopt the conservative bound of the p-box and form a decision; b) change the model which is used in the analysis by moving to one which contains only intrinsically aleatory parameters, or; c) develop a means to 'de-fuzzify' or 'de-box' the imprecise output in a way that allows a subjective decision to be made.

In this example, due to the lack of prior information – and for demonstration purposes – regarding the nature of unpredictability in characterising m_i, this parameter has been assigned an interval. However, it was also noted that in published literature there is evidence to support the aleatory nature of m_i. Therefore, it may be argued at m_i could have initially been represented by a uniform distribution – applying the principle of indifference – and updated once the regression data became available. This approach is perfectly valid with respect to the framework and supported with the arguments presented in this thesis. However, in this example we have chosen to demonstrate the reduction in epistemic uncertainty with increasing information. In this way, we support our earlier statement that epistemic uncertainty is reducible and can be re-characterised as aleatory variability if the parameter in question is intrinsically epistemic and further data becomes available to objectively fit a stochastic model.

The three challenge problems presented so far have served to demonstrate the effectiveness of the taxonomy in characterising unpredictability in parameters used in rock engineering, which may be obtained subjectively or objectively. These challenge problems have shown that irrespective of whether the unpredictability is due to epistemic uncertainty or aleatory variability, the novel framework proposed in this thesis provides a means of objectively characterising and propagating the unpredictability faithfully and robustly through analytical models. This concludes the demonstration of the applicability of the concepts presented in this thesis. The following Chapter provides a summary, conclusions drawn from and contributions made as a result of this work, as well as recommendations for further work.

Chapter 8

SUMMARY, CONCLUSIONS & FURTHER WORK

This final Chapter gives a summary of the concepts presented in this thesis. This is followed by a list of the conclusions drawn and the contributions made as part of this work. Finally, this work concludes by presenting areas for further work and development on the new concepts and contributions introduced in this thesis.

8.1 Summary

The discussion presented in this thesis, has shown that unpredictability in a parameter or system is due to the combination of epistemic uncertainty and aleatory variability. In the context of geotechnical engineering, unpredictability can be regarded as an accumulation of errors in sampling, observation, measurement, and the mathematical evaluation of data, together with concept and model uncertainty and inherent natural variability. In order to simply characterise unpredictability in rock mechanics and rock engineering, this thesis presented formal definitions for epistemic uncertainty and aleatory variability. Through a review of the wider literature, this thesis identified that aleatory variability – also known as stochastic uncertainty, objective uncertainty or irreducible uncertainty – describes the inherent variability in a physical system or environment; it can be modelled using stochastic models and handled using probabilistic methods. Epistemic uncertainty, on the other hand, is due to lack of knowledge; it is both subjective in nature and influenced by preconceptions of what is considered realistic for the system in question. It has also been called ignorance, imprecision or reducible uncertainty and can be reduced or eliminated through additional information or knowledge. Based on these concepts, this thesis presented justification for the notion that epistemic uncertainty cannot be modelled stochastically.

Through a critical review of the literature, this thesis has identifies that in geotechnical engineering, and rock mechanics and rock engineering in particular, the fundamental and intrinsic difference between epistemic uncertainty and aleatory variability appears to have not been correctly understood. Consequently, it appears that there is a lack of understanding regarding the need for characterising and propagating uncertainty and variability separately. Using examples specific to rock mechanics and rock engineering, this thesis showed that uncertainty and variability possess very different characteristics. Epistemic uncertainty is due to a qualitative or quantitative lack of knowledge; it is subjective in nature and can be reduced by improving the level of information. On the contrary, aleatory variability is objective and requires precise information to define a stochastic model with which to characterise it. Furthermore, as it is due to randomness, it is inherent in the system and thus irreducible.

Using these concepts, this thesis proposed a new taxonomy that, firstly, will allow geotechnical engineers to easily recognise these critical differences between epistemic uncertainty and aleatory variability and secondly, provide an objective means of characterising unpredictability through an assessment of the available information. The new taxonomy summarises the link between quantity and quality of information with respect to uncertainty characterisation. It demonstrates that aleatory variability can only be invoked once a sufficient quantity of precise information is available. The taxonomy is necessary to objectively fit a probability distribution to the data. It also confirms that a transition from epistemic uncertainty to aleatory variability can be achieved by gathering either more or better information. However, whether this is possible or desirable depends on the nature of the parameters or system under consideration.

Using the new taxonomy, this thesis put forward the notion of intrinsically aleatory parameters and suggested that such parameters may be characterised using statistics and propagated by the frequentist approach to probability. That is, one can assume that the variable under assessment (in our case, a parameter defining the ground property in question) is the result of a random process and can be characterised by a particular probability distribution; further knowledge would only refine the precision of the variability. This thesis suggested that one philosophical problem with this approach is that, in geotechnical engineering – rock engineering in particular – it is not always practical to obtain a sufficiently large data set, based on test results etc., from which to fit a representative probability distribution. In such cases, the parameters must be classified as extrinsically epistemic and characterised using alternate, non-stochastic means until sufficient data becomes available to formulate an aleatory model. However, many parameters used to characterise material, or

other, properties in rock mechanics are defined qualitatively or quantified entirely subjectively through expert judgement. For such parameters, while the underlying property or process may be the result of a random process, the subjective method used to characterise the variable means, irrespective of the amount of additional information or expert consultation, the intrinsic stochasticity, if present, will not be disclosed. In this thesis, such parameters are characterised as 'intrinsically epistemic'. This thesis concluded that such parameters are not amenable to characterisation using stochastic models – or propagation using the associated probabilistic analyses – which are suitable only for parameters exhibiting aleatory variability.

Despite the general recognition by geotechnical engineers that most rock masses are inherently heterogeneous and that there is also imprecision in the measurement or estimation of the engineering parameters used to describe their properties, there still appears to be much confusion regarding the nature of uncertainty. Consequently, various authors have suggested that total unpredictability, i.e. the combination of both epistemic uncertainty and aleatory variability, can be handled using the Bayesian approach and associated subjective probabilistic methods. However, this thesis has shown that the use of subjectively assigned probability distribution functions to characterise epistemic uncertainty can lead to erroneous results. Specifically, the Bayesian approach of assigning subjective 'priors' introduces information that is not actually available; thus this approach is identified as neither faithful nor robust.

This work presented arguments to support the thesis that the epistemic uncertainty and aleatory variability should be propagated, analytically, using different unpredictability modelling methods. Basically, interval-oriented approaches should be used to propagate epistemic uncertainty, and probability theory should be used to propagate variability. This thesis expand on the 'Level of Information' concept originally conjectured by Wenner & Harrison (1996) and propose a new framework for selecting an appropriate unpredictability model through a faithful assessment of the available information. This framework uses the concepts presented in the taxonomy and directs the user through a data characterisation strategy in order to determine whether the unpredictability is either epistemic or aleatory. The framework then leads the user to a model selection strategy in order to select an unpredictability model that faithfully propagates the available information through the analytical process. The development of this framework follows on from the taxonomy to provide an objective means of characterising unpredictability. Using this framework, once the unpredictability has been characterised as either epistemic or aleatory, an unpredictability model is selected that faithfully propagates the available information through the analytical

process. This supports the fundamental thesis that for any given amount of knowledge — and thus degree of uncertainty — there is an optimal model that should be applied

This thesis applied the new taxonomy and framework to three simple problems involving intrinsically and extrinsically epistemic parameters, as well as aleatory parameters. These examples served to demonstrate the fundamental errors that may result if a Bayesian approach, using subjective probabilities, is applied to intrinsically epistemic parameters. These examples showed that by arbitrarily assuming a prior probability density function, we are implying a greater level of information than is actually available: the greater level of information is in the definition of the shape of the PDF. The assumed prior PDF coupled with Monte-Carlo simulation has the effect of erroneously producing distribution tails that the information does not support. This may have detrimental consequences for engineering design, as it is often the extreme values represented by the tails of the distributions that govern design decisions. Through these examples, this thesis showed that the use of a more appropriate non-stochastic approach commensurate with the given level of information, selected using the framework, allows one to use all the available information and propagate the uncertainty faithfully through the analysis of an intrinsically epistemic system. Importantly, such methods will always contain the extreme events, however unlikely their occurrence may be. At the end of the modelling and risk analysis process the designer may then make a completely informed decision with regard to these unlikely events.

Using these, non-traditional method, as stated by (Dubois and Guyonnet, 2011), the advantage is that assessment of reliability takes place at the end of the risk analysis process, *"when no further collection of evidence is possible that might reduce the ambiguity due to epistemic uncertainty. This feature stands in contrast with the Bayesian methodology, where epistemic uncertainties on input parameters are modelled by single subjective probabilities at the beginning of the risk analysis process"*. This approach allows the epistemic uncertainty to be retained throughout the data collection and analysis phases with the expert opinion, or subjectivity, introduced at the final decision- making stage. This approach is advantageous in that it does not 'mask' epistemic uncertainty, as would occur if a Bayesian approach was applied from the beginning. The advantage of the approach proposed in this thesis is that it will, for the first time, allow an objective approach to faithfully characterise and propagate uncertainty and variability in rock mechanics and rock engineering. It will also beneficially reduce the dissonance between experts when faced with characterising epistemic uncertainty. Additionally, it allows the identification of areas where data acquisition will best serve to reduce unpredictability. We see that the methods proposed in this thesis can thus serve to

provide greater safety in engineering design as well as optimise data collection and investigation schemes.

8.2 Conclusions and contributions

The conclusions drawn and contributions made as a result of the work presented in this thesis are summarised below. These conclusions and contributions are divided in to two sub-groups: principal and supporting conclusions and contributions, and these are listed with respect to the Chapter of this thesis in which they were first introduced

8.2.1 Principal conclusions and contributions

1. Through an extensive review of the wider literature, Chapter 2 presents a discussion on the fundamental nature of unpredictability and, thus, provides formal definitions of epistemic uncertainty and aleatory variability as the two components that contribute to unpredictability. These definitions have been drawn from other fields of science and technology. Using these definitions, this Chapter demonstrates the importance of recognising the difference between uncertainty and variability and the means by which unpredictability associated with the parameters that characterise fractured rock masses are propagated through the modelling and design process. As a result, Chapter 2 contributes towards a novel understanding of the fundamental issues in characterising and propagating unpredictability in rock engineering design.

2. Using the new definitions proposed in section 2.2.3 and the level of information concept (Section 2.5), a new taxonomy is proposed in section 2.7 that will allow engineers preparing rock engineering designs to correctly and objectively identify the true nature of unpredictability. A further contribution of this new taxonomy is that it allows a means of identifying an appropriate, non-stochastic or stochastic, unpredictability model to propagate the unpredictability through the modelling and design process. The key contribution of this taxonomy is that it provides practitioners with one reference (Figure 17), with key terms identified by this work arranged in a simple manner, that can be used to objectively characterise the nature of unpredictability through an assessment of the available information. This table is supplemented by a key figure (Figure 17) that arranges these key terms with respect to the quantity and quality of information such that engineers can visualise the level of precision in the available information and thus gauge an appropriate means of modelling unpredictability. This table and figure that make up the proposed taxonomy succinctly summarise all the concepts presented in this Chapter.

3. The proposed taxonomy concludes that in order to characterise unpredictability as aleatory, a set of specific criteria need to be met; only when all of these criteria are fulfilled can the unpredictability be characterised as aleatory variability and modelled using probabilistic methods. The corollary of this is that failure to meet any criterion that defines aleatory variability means the unpredictability must be treated as epistemic uncertainty and thus handled using appropriate, non-stochastic models. The proposed taxonomy thus contributes to develop an understanding of unpredictability, which can be applied in rock engineering.

4. One of the major contributions of this work is the novel framework presented in Chapter 4. The framework has been developed by integrating the concepts presented in the new taxonomy (i.e. Figure 17) and the level of information concept (Figure 8 in section 2.5) with the unpredictability models introduced in Chapter 3. This new framework provides three flow-charts that, through a series of simple questions, directs the user to simply and objectively characterise the nature of unpredictability in a parameter or system before propagating it through the analysis and design process using the appropriate (mathematical) tools.

5. One contribution of this framework is to provides a tool for directing investigation (which can be costly) appropriately to reduce unpredictability. Secondly, it provides a protocol for objectively selecting an appropriate unpredictability analysis based on the available information. The practical contribution of this framework is that its application in practice will harmonise designs by reducing arbitrary choices in characterising and propagating unpredictability in rock mechanics and rock engineering. This will mean that designers and policy makers will for the first time have a framework against which rock mechanics designs can be assessed and scrutinised. As such, this would mean that safety of rock mechanics designs will be greatly improved as the unpredictability concepts, currently not properly understood, will be better incorporated in to designs.

6. Chapter 5 presents a challenge problem, that of planar slope stability, to demonstrate the applicability of the new taxonomy and framework. Through a critical review of existing analyses presented in the literature, this Chapter shows that in a situation where no objective data are available and expert assessment of slope stability is required, use of stochastic methods with subjectively assigned PDFs can lead to dissonance between experts in reaching conclusions on critical decisions such as the safety of a slope. This is due to the arbitrary choices required when characterising uncertainty in this manner. This Chapter shows how use of the framework provides an objective means of characterising and propagaing

unpredictability, which means that even with limited information, experts should converge to the same conclusions.

7. The second challenge problem investigates application of the framework when using empirical rock mass classification systems. This example illustrates the philosophical awkwardness in assigning a prior probability when presented with either limited or no objective information, or when the information is inherently imprecise. This challenge problem shows how for an intrinsically epistemic system such as Q, the framework directs the user to undertake a fuzzy analysis, which can be used to assist in making informed decisions during the feasibility stage of a major project. This example also concludes that the assignment of a subjectively determined probability distribution, given little or not evidence to support it, (i.e. applying the Bayesian approach without updating) may lead to either misinformed decisions or over-confidence in the accuracy of the resulting conclusions drawn from such analyses.

8. This thesis concludes with a final challenge problem involving estimation of the peak strength of jointed rock masses and intact rock. Through application of the framework, this challenge problem demonstrates how the new framework does not allow parameters that are inherently imprecise to be characterised using a probabilistic approach. Therefore, it is concluded that such parameters must always be handled using non-probabilistic methods. The final section in this last challenge problem re-applies the framework after additional data becomes available to show how one may re-characterise epistemic uncertainty as aleatory variability if the additional information meets the requirements of the latter presented in the taxonomy. This challenge problem is the first application of hybrid analysis to a problem in rock mechanics. A series of verified hybrid calculation algorithms have been developed and presented in the Appendices of this thesis using the program MathCAD. Whilst probabilistic approaches are widely applied to rock mechanics problems, fuzzy solutions are less common and this research has not uncovered any examples of hybrid analyses. This may be due to the perception that these latter methods are computationally challenging, or the lack of commercial software available to implement them. Thus, the hybrid algorithms developed for this challenge problem demonstrate the ease in which they may be applied. This may open up an avenue for application of fuzzy and hybrid analysis in routine geotechnical design.

8.2.2 Supporting conclusions and contributions

9. Section 2.2 identifies the necessity to characterise unpredictability through a review of the quality and quantity, as well as the type, of information available to the analyst. Using these concepts, the new definitions intrinsically epistemic, extrinsically epistemic, and intrinsically aleatory are presented. These new definitions allow identification of the underlying nature of unpredictability within a parameter or system.

10. Using these new definitions, section 2.3 concludes that as epistemic uncertainty is reducible, separating uncertainty and variability in an analysis allows one to understand what steps can be taken to reduce the unpredictability within a model. An important conclusion drawn from this discussion is that unpredictability is most significantly reduced by targeting data collection to reduce epistemic uncertainty, and in particular at re-categorisation of extrinsically epistemic parameters to aleatory. In this way, site investigation and data collection can be focussed at those aspects of the model which will benefit most from it. The practical implication of which is that site investigation can be performed more efficiently, thereby reducing both cost as well as reducing unpredictability in the final design.

11. Section 2.5 significantly develops the level of information concept (i.e. Figure 8) first conjectured by Wenner & Harrison (1996), by proposing a hierarchy of non-stochastic and stochastic approaches appropriate for propagating unpredictability. The conclusion drawn is that for any specified level of information an optimal model should be applied. Through simple examples, this discussion is able to confirm that non-stochastic methods commensurate with the given level of information allow one to use all the available information and propagate the uncertainty faithfully through the analysis of an intrinsically epistemic system. This confirmed level of information concept can thus be used as a basis for simply identifying the unpredictability modelling methods that can be applied to rock mechanics and rock engineering problems.

12. Section 2.8 applies the new taxonomy to examples specific to rock mechanics and rock engineering to show that many parameters – such as those used in rock mass classification systems – are intrinsically epistemic and that no matter the quantity of data, the inherent imprecision in such parameters means they can only ever be characterised as epistemic. On the contrary, parameters that can be objectively measured may be intrinsically aleatory, however if there is an insufficient quantity of data they must be characterised as extrinsically epistemic. Through application of the taxonomy, this review is able to conclude that the unpredictability in parameters used to characterise DIANE rock masses that are determined subjectively, must be modelled using non-stochastic methods. The conclusion

drawn is: use of stochastic analysis methods for such parameters is inappropriate and may in fact introduce a false sense of confidence in the output of designs on which they are based. Hence this discussion contributes to realising that many stochastic analyses presented in the literature where the unpredictability was epistemic may be potentially erroneous.

13. Another contribution of the discussion presented in section 2.8 is that it shows that whilst many of the parameters used to characterise DIANE rock masses are determined subjectively and so epistemic, others that can be determined objectively are in fact intrinsically aleatory. Through a review of the literature this section shows how parameters such as UCS are aleatory. This review can then form a basis for developing testing recommendations to assist in identifying those parameters which are aleatory and appropriate tests methods to characterise them.

14. The discussion in Chapter 3 details the mathematical basis of the various unpredictability models presented by the level of information concept (Figure 8 in section 2.5). Using examples specific to rock mechanics and rock engineering, this Chapter shows how these methods can be applied to rock mechanics problems. Of most importance, this section concludes that hybrid methods can be applied to rock engineering models where many parameters, each with a differing level of available information, need to be combined.

15. By using the methods introduced in the discussion presented in Chapter 3, algorithms for interval, fuzzy and hybrid analysis using MathCAD are developed. The basic algorithms are presented in the Appendices and can be used to develop further analytical models, e.g. tetrahedral wedge failure, if required.

16. A review of the literature revealed that Low (2008) presented a stochastic solution (using FORM) to planar slope stability using a simple spreadsheet program. However, to use the method of Low (2008) requires the unpredictability to be aleatory. In contrast, the case study presented in Chapter 5, characterises uncertainty using fuzzy numbers. As part of this work, a robust algorithm for calculating the fuzzy factor of safety for planar slope stability has been developed and presented in the Appendices of this thesis. These algorithms are implemented in MathCAD though they may be readily adapted to any similar software.

17. This thesis identifies that the imprecise output produced by non-stochastic methods can lead to difficulties in decision making. Thus, the challenge problem of Chapter 5 presents a new concept of 'the Agreement index', which uses a de-fuzzification procedure that may assist in decision making.

8.3 Further work

The fundamental nature of this work has naturally raised many questions with regard to further application to rock mechanics and rock engineering problems. These questions can be categorised in three major areas for future research: a) Significance of the new concepts of intrinsically epistemic, extrinsically epistemic and intrinsically aleatory with regards to characterising parameters commonly used in rock mechanics and rock engineering. More specifically, how these concepts will influence future testing directive and design methodologies; b) Further development and applicability of the non-stochastic and hybrid methods analysis methods with specific reference to rock engineering design, and; c) Decision making based on the imprecise outputs of the interval-oriented approaches. The following areas are each discusses herein.

8.3.1 Significance of the new concepts of intrinsically epistemic, extrinsically epistemic and intrinsically aleatory

1. The term intrinsically epistemic was introduced to define rock mass parameters which are inherently imprecise and for which, no matter the quantity of information, the quality of data could not be improved to reach a precise state; for these it is inappropriate to assign a precise probability distribution. It is apparent that this statement has significant repercussions; most notably, it implies that all parameters that are derived subjectively, through imprecise correlations and approximations can only ever be modelled using non-stochastic methods. This in turn implies that all probabilistic analyses undertaken to date, using such parameters are, strictly, in error. Therefore, there is a need to validate the appropriateness of applying subjectivist probabilistic methods in the context of rock engineering design. Specifically, there is a need to investigate whether geotechnical design codes should restrict the widespread use of such intrinsically epistemic parameters in detailed design calculations that are based on probabilistic methods or assumptions. Or, at least provide informative guidance on the need to recognise the imprecision inherent in these parameters.

2. Following on from this, it is apparent that there is a need to undertake research in to those parameters that may be intrinsically aleatory, though there is not enough evidence in the literature to support this. This thesis has identified a few properties, such as UCS and joint spacing that exhibit aleatory variability. However, it was also noted that it is not at all clear whether many objectively determined parameters are aleatory, and if so, why the

objective measurements produce such imprecise correlations. The most important of these parameters is the in-situ stress ratio (k). Figure 78a shows the imprecision in the correlation of the in-situ stress ratio with depth, which suggests that k is epistemic. However, the looking at the data between 400m and 600m depth, Figure 78b suggests that the in-situ stress ratio in this region may be characterised by a Weibull distribution. Therefore, it is not clear whether k is 'globally intrinsically epistemic' and perhaps 'locally intrinsically aleatory'. Furthermore, it is unclear as to why the imprecision appears to be greater at surface than at depth. The distribution of in-situ stress near a fault (Figure 79) also presents similar questions as to the nature of the unpredictability in k. At this stage, it is not clear how one would characterise k, and thus great deal of research is needed in to the nature of unpredictability in this parameter.

(a)

a) Ratio of major horizontal to vertical principal stresses b) Ratio of minor horizontal to vertical principal stresses with depth
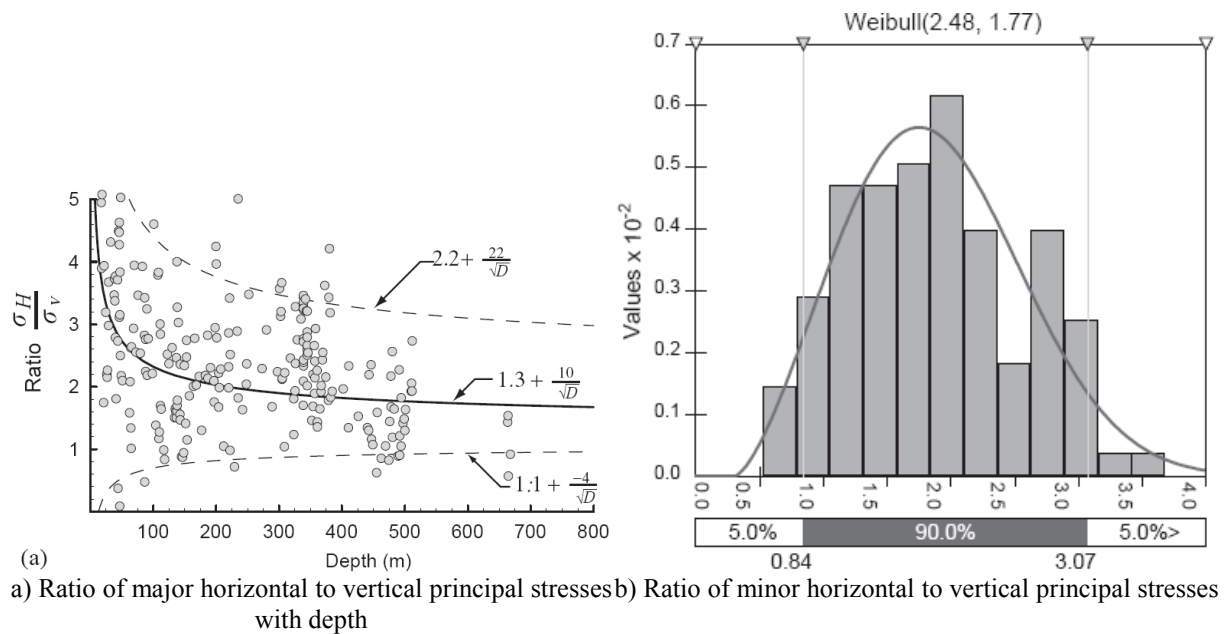
Figure 78: In-situ stress ratios determined from the Scandinavian database (from Martin et al., 2003).
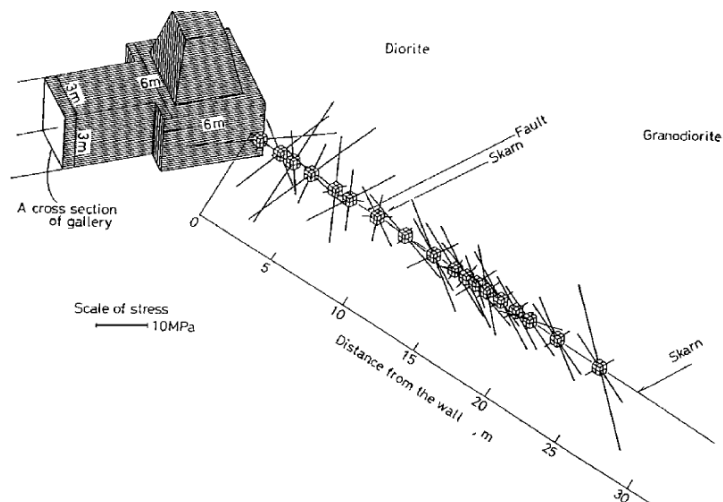
Figure 79: Rock stress distribution near a fault (from Obara & Sugawara, 2003).

8.3.2 Development and applicability of non-stochastic methods for rock engineering

3.   The challenge problems presented in this thesis were selected based on their simplicity to demonstrate the applicability of the framework to handle problems involving parameters at varying levels of information. However, there is great potential to expand the complexity of the challenge problems to account for various phenomena such as plasticity. For instance, Schweiger & Peschl (2005) have presented a preliminary investigation in to 'a random set finite element method' (i.e. using non-parametric p-boxes) for a retaining wall. Similarly, Peschl & Shweiger (2003) present a fuzzy finite element study of a footing on soil. There is also no shortage of literature on the stochastic finite element approach. However, in each of these analyses, the unpredictability has been characterised at the whim of the analyst. Therefore, an investigation in to application of the framework and a hybrid FEA approach is one avenue worth further investigation.

4.   Limit state design (LSD) codes (e.g. Eurocode 7 in Europe) have become legislative design standards for geotechnical engineering in many countries. These codes recognise the need for rock engineering designs to comply with the LSD paradigm. This paradigm requires that both the effect of actions (i.e. loads) and resistance in a structure be aleatory in nature (see Figure 80 for LSD model). However, as this thesis has shown, many parameters used to characterise DIANE rock masses are epistemic; some intrinsically epistemic. Therefore, it is evident that such parameters cannot be handled by LSD codes in their current form. Bedi & Harrison (2012) provide a detailed discussion on this matter. However, unlike LSD, the non-stochastic methods presented in this thesis can be used when the level of knowledge is inappropriate to characterise the unpredictability using an aleatory model. The examples presented in this thesis show that for many rock engineering structures, such as rock slopes, the load and resistance functions can be defined using fuzzy numbers or imprecise probability distributions. This opens up a research area aimed at investigating the applicability of the proposed framework and the concept of 'imprecise Limit State Design' principles. At this early stage of the development of the framework, on face value, it appears that there may be a place for this new framework to provide a means of directing geotechnical designs in the face of epistemic uncertainty whilst the link between LSD and this work is established.
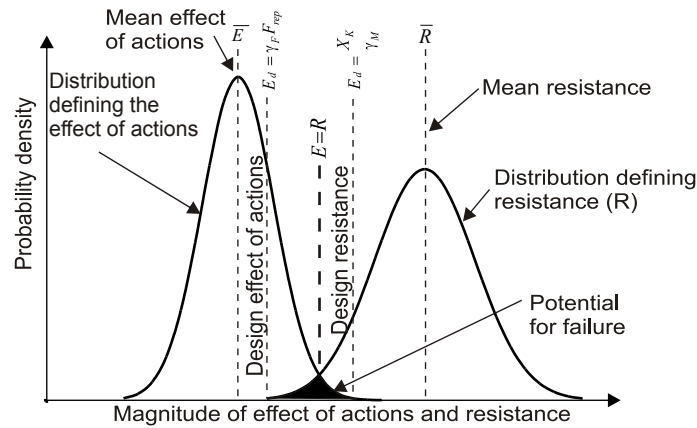
Figure 80: Both effects of actions and material resistance are considered as random variables in geotechnical LSD (from Bedi & Harrison, 2012).

The examples presented in this thesis show that for many rock engineering structures, such as rock slopes, the load and resistance functions can be defined using fuzzy numbers or imprecise probability distributions. Thus, there appears to be a space for investigation in to 'imprecise Limit State Design' principles.

8.3.3 Decision making based on imprecise outputs

5. In civil engineering, risk and reliability analyses using probabilistic methods have a long history. As such, levels for accepting a probability of failure or reliability index are generally well established. In terms of LSD, as Figure 80 also shows, the LSD concept uses partial factors to provide the required level of safety for structures designed in accordance with its principles. In this thesis, we presented the concept of an 'agreement index' by de-fuzzifying the outputs of a fuzzy analysis. Whilst the literature has presented means of undertaking 'fuzzy reliability analysis' (e.g. Yubin et al., 1997; Nunes & Sousa, 2009; Carvalho et al., 2011; Park et al., 2012), there does not appear to be any studies that present acceptable levels of fuzzy reliability measures. For example, for various structures in engineering acceptable probabilities of failure have been determined (e.g. Figure 81). Thus, investigation in to acceptable imprecise or 'de-fuzzified' reliability indices appears necessary.

6. This thesis expanded on the concept of 'Agreement index' (Kaufmann & Gupta, 1991) to propose an agreement index to assist in decision making. This index is only valid when the output is a fuzzy number. However, a similar index does not appear to have been published in the literature for p-boxes or the output from a hybrid analysis. This research has only revealed one such index presented by Dubois & Guyonnet (2011), who suggest the concept of a confidence index; this approach, however, appears highly subjective. On the
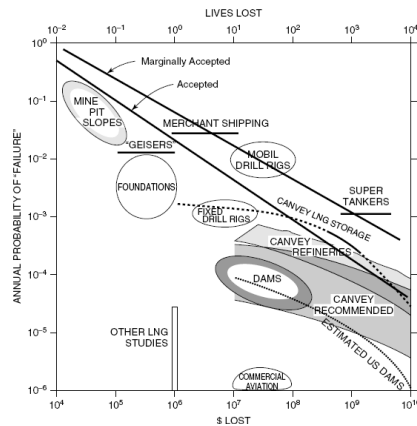
Figure 81: Acceptable limits on probabilities of failure for various structures (from Baecher & Christian, 2003).

contrary to the confidence index suggested by Dubois & Guyonnet (2011), the agreement index uses the information contained in the shape of a fuzzy number as a means of de-fuzzification. Thus, there appears to be a need to develop a similar 'de-boxing' method that uses the information in the p-box.

The work presented in this thesis has demonstrated a clear need to better understand uncertainty and variability in rock mechanics and rock engineering. The new taxonomy and framework developed and presented in this thesis aim to provide convenient tools in this endeavour. These new tools and further contributions made as part of their development can be applied immediately by practising engineers and rock mechanics. However, this section on further work illustrates the potential to build on the work presented in this thesis and apply the tools developed here to actual site-specific problems. Further development of the concepts and tools developed in this thesis will serve to improve both safety and efficiency in rock engineering designs.

REFERENCES

Ang, A. H-S. & Tang, W. H. (1984) *Probability concepts in engineering planning and design, Vol. 2, Decision, risk and reliability.* New York; Chichester, Wiley.

Ang, A. H-S. & Tang, W. H. (2007) *Probability concepts in engineering: emphasis on applications in civil & environmental engineering, 2nd ed.* Hoboken, N.J., Wiley.

Aughenbaugh, J. M. & Paredis, C. J. J. (2006) The value of using imprecise probabilities in engineering design. *J Mech Design,* **128** (4), 969–979.

Aven, T. (2010) On how to define, understand and describe risk. *Reliab Eng Syst Safe,* **95** (6), 623–631.

Aven, T. & Steen, R. (2010) The concept of ignorance in a risk assessment and risk management context. *Reliab Eng Syst Safe,* **95** (11), 1117–1122.

Aydin, A. (2004) Fuzzy set approaches to classification of rock masses. *Eng Geol,* **74**, 227–245.

Baecher, G. B. & Christian, J. T. (2003) *Reliability and statistics in geotechnical engineering,* Chichester, Wiley.

Bárdossy, G. & Fodor, J. (2004) *Evaluation of uncertainties and risks in geology: New mathematical approaches for their handling.* Berlin; London, Springer.

Barton, N. (1973) Review of a new shear strength criterion for rock joints. *Eng Geol,* **7**, 287–332.

Barton, N. (2002) Some new Q-value correlations to assist in site characterisation and tunnel design. *Int J Rock Mech Min,* **39** (2), 185–216.

Barton, N. & Choubey, V. (1977) The shear strength of rock and rock joints. *Rock Mech,* **13** (10), 1–24.

Barton, N., Lien, R. & Lunde, J. (1974) Engineering classification of rock masses for the design of tunnel support. *Rock Mech,* **6** (4), 89–239.

References

Barton, N., By, T. L., Chryssanthakis, P., Tunbridge, L., Kristiansen, J., Løset, F., Bhasin, R. K., Westerdahl, H. & Vik, G. (1994) Predicted and Measured Performance of the 62m Span Norwegian Olympic Ice Hockey Cavern at Gjøvik. *Int J Rock Mech Min,* **31** (6), 617–641.

Basarir, H. & Saiang, H. (2012) Assessment of slope stability using fuzzy sets and systems. *Int J Min Reclam Env.* [Online]. 1–17. Available from: DOI:10.1080/17480930.2012.702000 [Accessed 16 December 2012].

Baudrit, C., Couso, I. & Dubois, D. (2007) Joint propagation of probability and possibility in risk analysis: Towards a formal framework. *Int J Approx Reason,* **45** (1), 82–105.

Baudrit, C. & Dubois, D. (2005) Comparing methods for joint objective and subjective uncertainty propagation with an example in a risk assessment. In: *4th Int Symp on Imprecise Probabilities and their Applications, ISIPTA 2005, Pittsburg, USA.* pp. 31–40.

Baudrit, C. & Dubois, D. (2006) Practical representations of incomplete probabilistic knowledge. *Comput Stat Data An,* **51** (1), 86–108.

Baudrit, C., Dubois, D. & Guyonnet, D. (2006) Joint propagation and exploitation of probabilistic and possibilistic information in risk assessment. *IEEE T Fuzzy Syst,* **14** (5), 593–608.

Baudrit, C., Guyonnet, D. & Dubois, D. (2005) Postprocessing the hybrid method for addressing uncertainty in risk assessments. *J Environ Eng-ASCE,* **131** (12), 1750–1754.

Bauer, S.J., Gaither, K.N., Webb, S.W. & Nelson, C. (2012) *Compressed Air Energy Storage in Hard Rock Feasibility Study.* Sandia National Laboratories. Report number: SAND2012-0540. Available at: http://www.carebs.org/HardrockCRADAfinal-SAND2012-0540.pdf.

Becker, D. E. (1996) Eighteenth Canadian geotechnical colloquium: Limit states design for foundations . Part 2. Development for the National Building Code of Canada. *Can Geotech J,* **33** (6), 984–1007.

Becker, D. E. & Moore, I. D. (eds.) (2007) *Canadian foundation engineering manual,* Richmond, B.C., Canadian Geotechnical Society.

Bedi, A. & Harrison, J.P. (2012) Dealing with epistemic uncertainty in fractured rock masses. In: *Proc. International Symposium of the ISRM, Eurock 2012, 29–30 May, Stockholm, Sweden.*

Bedi, A. & Harrison, J.P. (2013a) Characterisation and propagation of epistemic uncertainty in rock engineering: a slope stability example. In: *Proc. International Symposium of the ISRM, Eurock 2013, 21–26 September, Wroclaw, Poland.* [In press].

Bedi, A. & Harrison, J.P. (2013b) A comparison of Bayesian techniques and non-probabilistic models in rock engineering design. In: *Proc. 47th U.S. Rock Mechanics/ Geomechanics Symposium, ARMA 2013, 23–26 June , California, USA*. [In press].

Beer, A. J., Stead, D. & Coggan, J. S. (2002) Estimation of the Joint Roughness Coefficient (JRC) by visual comparison. *Rock Mech Rock Eng,* **35** (1), 65–74.

Beer, M., Ferson, M. & Kreinovich, V. (2013) Imprecise probabilities in engineering analyses. *Mech Syst Signal Pr,* **37**, 4–29.

Beer, M., Zhang, Y., Quek, S. T. & Phoon, K. K. (2012) Reliability analysis with scarce information: Comparing alternative approaches in a geotechnical engineering context. *Struct saf,* **41** (2), 1–10.

Berger, J. O. (1985) *Statistical decision theory and Bayesian analysis.* New York; London, Springer.

Bieniawski, Z. T. (1989) *Engineering rock mass classifications.* New York, Wiley.

Booker, J. M. & McNamara, L. A. (2005) Expert knowledge in reliability characterisation: a rigorous approach to eliciting, documenting and analyzing expert knowledge. In: Nikolaidis E., Ghiocel D. & Singhal S. (eds.) *Engineering Design Reliability Handbook*. Boca Raton, Florida: CRC Press. pp. 13-1 – 13-31.

Brady, B. H. G. & Brown, E. T. (2004) *Rock mechanics for underground mining,* Dordrecht; London, Kluwer Academic Publishers.

Brown, E. T. (ed.). (1980) *Rock Characterization Testing and Monitoring. ISRM Suggested Methods*. Pergamon.

Cai, M. (2011) Rock mass characterization and rock property variability considerations for tunnel and cavern design. *Rock Mech Rock Eng,* **44** (4), 379–399.

Cai, M., Kaiser, P. K., Uno, H. & Tasaka, Y. (2000) Comparative study of rock support system design practice for large-scale underground excavations. In: *Proc. 4th North American Rock Mech. Symposium, Seattle, USA*. Rotterdam, Balkema. pp. 1027–1034.

Carter, T. G. & Miller, R. I. (1995) Crown-pillar risk assessment planning aid for cost-effective mine closurerRemediation. *T I Min Metall A,* **104**, A41–A57.

Carvalho, M., Nunes, E. & Telhada, J. (2011) Outlining fuzzy decision making in maintenance and planning. In: *Proc. X Congreso Galego de Estatística e Investigación de Operacións*, 3–5 November, *Pontevedra, Spain.*

Christian, J. T. (2004) Geotechnical engineering reliability: How well do we know what we are doing? *J Geotech Geoenviron,* **130** (10), 985–1003.

References

Christian, J. T., Ladd, C. C. & Baecher, G. B. (1994) Reliability applied to slope stability analysis. *J Geotech Eng-ASCE,* **120** (12), 2180–2207.

Chutia, R., Mahanta, S. and Baruha, H.K. (2011). Construction of a fuzzy number with fuzziness defnied around an interval. *Int J Mat Sci App*, **1** (1). 1–8.

Colyvan, M. (2008) Is probability the only coherent approach to uncertainty? *Risk Anal,* **28** (3), 645–652.

Cooper, J. A. & Ferson, S. (1999) New mathematical derivations applicable to safety and reliability analysis. In: *Proc. of the European Safety and Reliability Conference, ESREL 1999, 13–17 September, Munich, Germany.* pp. 975–980.

Cooper, J. A., Ferson, S. & Ginzburg, L. (1996) Hybrid processing of stochastic and subjective uncertainty data. *Risk Anal,* **16** (6), 785–791.

Crawford, J., Neretnieks, I. & Moreno, L. (2006) Epistemic and aleatory uncertainty in recommended, generic rock K-d values used in performance assessment studies. *Mater Res Soc Symp P,* **932**, 251–258.

Daftaribesheli, A., Ataei, M. & Sereshki, F. (2011) Assessment of rock slope stability using the Fuzzy Slope Mass Rating (FSMR) system. *Appl Soft Comput,* **11** (8), 4465–4473.

Davis, J. C. (2002) *Statistics and data analysis in geology, 3rd ed.* New York; Chichester, Wiley.

Deere, D.U. (1989) *Rock quality designation (RQD) after 20 years*. U.S. Army Corps Engrs Contract Report GL-89-1. Vicksburg, MS: Waterways Experimental Station.

Dong, W. M. & Shah, H. C. (1987) Vertex method for computing functions of fuzzy variables. *Fuzzy Set Syst,* **24** (1), 65–78.

Dubois, D. (2004) Forward. In*:* Bárdossy, G. & Fodor, J. (eds.) *Evaluation of uncertainties and risks in geology: new mathematical approaches for their handling.* Berlin; London, Springer. p. 5.

Dubois, D. (2006) Possibility theory and statistical reasoning. *Comput Stat Data An,* **51** (1), 47–69.

Dubois, D. (2010) The role of epistemic uncertainty in risk analysis. *Lect Notes Artif Int,* **6379**, 11–15.

Dubois, D. & Guyonnet, D. (2011) Risk-informed decision-making in the presence of epistemic uncertainty. *Int J Gen Syst,* **40** (2), 145–167.

Dubois, D. & Prade, H.M. (1988) *Possibility theory: an approach to computerized processing of uncertainty,* New York, Plenum Press.

Dubois, D. & Prade, H.M. (1989) Fuzzy sets, probability and measurement. *Eur J Oper Res,* **40** (2), 135–154.

Dubois, D. & Prade, H.M. (2009) Formal representations of uncertainty. In: Bouyssou, D., Dubois, D., Pirlot, M. & Prade, H.M. (eds.) *Decision-Making Process. Concepts and Methods.* London, ISTE & Wiley. pp. 85–156.

Duncan, J. M. (2001) Factor of safety and reliability in geotechnical engineering-Closure. *J Geotech Geoenviron,* **127** (8), 717–721.

Einstein, H. H. & Baecher, G. B. (1983) Probabilistic and statistical methods in engineering geology, Part one: Exploration. *Rock Mech Rock Eng,* **16** (1), 39–72.

Fellin, W., Lessmann, H., Oberguggenberger, M. & Vieider, R. (eds.) (2005) *Analyzing uncertainty in civil engineering.* Berlin, Springer.

Feng, P. & Lajtai, E. Z. (1998) Probabilistic treatment of the sliding wedge with EzSlide. *Eng Geol,* **50** (1–2), 153–163.

Feng, X. T. & Hudson, J. A. (2010) Specifying the information required for rock mechanics modelling and rock engineering design. *Int J Rock Mech Min,* **47** (2), 179–194.

Ferson, S. (1992) Using fuzzy arithmetic in Monte-Carlo simulation of fishery populations. *Low Wake Fi,* **10**, 595–608.

Ferson, S. (1996) What Monte Carlo methods cannot do. *Hum Ecol Risk Assess,* **2** (4), 990–1007.

Ferson, S. (2002) *RAMAS Risk Calc 4.0: Risk assessment with uncertain numbers,* Boca Raton, Florida: Lewis Press.

Ferson, S. & Ginzburg, L. R. (1996) Different methods are needed to propagate ignorance and variability. *Reliab Eng Syst Safe,* **54** (2-3), 133–144.

Ferson, S., Ginzburg, L., Kreinovich, V., Nguyen, H. T. & Starks, S. A. (2002) Uncertainty in risk analysis: towards a general second-order approach combining interval, probabilistic, and fuzzy techniques. In: *Proc. IEEE Int. Conference on Fuzzy Systems, FUZZ-IEEE 2002, 12–17May, Honolulu, HI.* pp. 1342–1347.

Ferson, S. & Hajagos, J. G. (2004) Arithmetic with uncertain numbers: rigorous and (often) best possible answers. *Reliab Eng Syst Safe,* **85** (1-3), 135–152.

References

Ferson, S., Joslyn, C. A., Helton, J. C., Oberkampf, W. L. & Sentz, K. (2004) Summary from the epistemic uncertainty workshop: consensus amid diversity. *Reliab Eng Syst Safe,* **85** (1-3), 355–369.

Ferson, S., Kreinovich, V., Ginzburg, L., Sentz, K. and Myers, D.S. (2003). *Experimental Uncertainty Estimation and Statistics for Data Having Interval Uncertainty*. Sandia National Laboratories. Report number: SAND2002-4015. Available at http://www.sandia.gov/epistemic/Reports/SAND2002-4015.pdf

Ferson, S., Kreinovich, V., Hajagos, J., Oberkampf, W. and Ginzburg, L. (2007). *Constructing probability boxes and Dempster-Shafer structures*. Sandia National Laboratories. Report number: SAND2007-0937. Available at http://www.ramas.com/intstats.pdf

Fookes, P. G. (1991) Geomaterials. *Q J Eng Geol,* **24** (1), 3–15.

Gill, D. E., Corthesy, R. & Leite, M. H. (2005) Determining the minimal number of specimens for laboratory testing of rock properties. *Eng Geol,* **78** (1–2), 29–51.

Gokceoglu, C., Sonmez, H. & Kayabasi, A. (2003) Predicting the deformation moduli of rock masses. *Int J Rock Mech Min,* **40** (5), 701–710.

Gokceoglu, C., Yesilnacar, E., Sonmez, H. & Kayabasi, A. (2004) A neuro-fuzzy model for modulus of deformation of jointed rock masses. *Comput Geotech,* **31** (5), 375–383.

Grimstad, E. and Barton, N. (1993) Updating the Q-System for NMT. In: *Proc. int. symp. On sprayed concrete - modern use of wet mix sprayed concrete for underground support, Fagernes.* 46-66. Oslo: Norwegian Concrete Assn.

Guo, J. & Du, X. P. (2007) Sensitivity analysis with mixture of epistemic and aleatory uncertainties. *AIAA J,* **45** (9), 2337–2349.

Guyonnet, D., Come, B., Perrochet, P. & Parriaux, A. (1999) Comparing two methods for addressing uncertainty in risk assessments. *J Environ Eng-ASCE,* **125** (7), 660–666.

Habibagahi, G. & Katebi, S. (1996) Rock mass classification using fuzzy sets. *Iran J Sci Technol,* **20** (3), 273–284.

Hanss, M. (2002) The transformation method for the simulation and analysis of systems with uncertain parameters. *Fuzzy Set Syst,* **130** (3), 277–289.

Hanss, M. (2005) *Applied fuzzy arithmetic: an introduction with engineering applications,* Berlin, Springer-Verlag.

Harrison, J.P. (2010) Rock mechanics. [Lectures] Imperial College London.

Harrison, J. P. & Hudson, J. A. (2010) Incorporating parameter variability in rock mechanics analyses: fuzzy mathematics applied to underground rock spalling. *Rock Mech Rock Eng,* **43** (2), 219–224.

Helton, J. C. & Oberkampf, W. L. (2004) Alternative representations of epistemic uncertainty. *Reliab Eng Syst Safe,* **85** (1-3), 1–10.

Helton, J. C., Johnson, J. D., Oberkampf, W. L. & Sallaberry, C. J. (2010) Representation of analysis results involving aleatory and epistemic uncertainty. *Int J Gen Syst,* **39** (6), 605–646.

Hoek, E. (1991) When Is a Design in Rock Engineering Acceptable. Müller lecture. In: *Proc. ISRM 7th International Congress on Rock Mechanics*, *September 16-20, Aachen.* Rotterdam, A.A. Balkema. pp. 1485–1497.

Hoek, E. (1994) Strength of rock and rock masses. *ISRM News J,* **2** (2), 4–16.

Hoek, E. (2007) *Practical rock engineering* [Online].Available from: http://www.rocscience.com/hoek/corner/Practical_Rock_Engineering.pdf [Accessed 18th June 2010].

Hoek, E. & Bray, J. (1974) *Rock slope engineering.* London, The Institution of Mining and Metallurgy.

Hoek, E. & Brown, E. T. (1980a) Empirical Strength Criterion for Rock Masses. *J Geotech Eng-ASCE,* **106** (9), 1013–1035.

Hoek, E. & Brown, E. T. (1980b) *Underground excavations in rock.* London, The Institution of Mining and Metallurgy.

Hoek, E. & Brown, E.T. (1988) The Hoek-Brown failure criterion - a 1988 update. In: *Proc. 15th Canadian rock mechanics symposium: Rock engineering for underground excavations, CARMA 1988, October 3–4, Toronoto, Canada*. pp. 31–38.

Hoek, E., Kaiser, P. K. & Bawden, W. F. (1995) *Support of underground excavations in hard rock.* Rotterdam, Balkema.

Hoek, E. & Londe, P. (1974) The design of rock slopes and foundations. General Report on Theme III. Proc. *Third Congress Intnl. Soc. Rock Mech*., Denver. **1**(A), 613–752.

Hoek, E., Wood, D. & Shah, S. (1992) A modified Hoek-Brown failure criterion for jointed rock masses. In: *International Symposium of the ISRM, Eurock 1992, Chester, UK, 14-17 September, Chester, UK*. pp. 209–214.

References

Hoffman, F. O. & Hammonds, J. S. (1994) Propagation of uncertainty in risk assessments - the need to distinguish between uncertainty due to lack of knowledge and uncertainty due to variability. *Risk Anal,* **14** (5), 707–712.

Howson, C. (2002) Bayesianism in statistics. In: Swinburne, R. (ed.) *Bayes's Theorem.* Proc. Brit. Acad., **113**. Oxford, Oxford University Press.

Hudson, J. A. & Harrison, J. P. (1997) *Engineering rock mechanics: an introduction to the principles.* Oxford; New York, Elsevier.

Insua, D. R. & Ruggeri, F. (2000) *Robust Bayesian analysis,* New York; London, Springer.

Jaynes, E. T. & Bretthorst, G. L. (2003) *Probability theory: the logic of science,* Cambridge; New York, Cambridge University Press.

Jeffreys, H. (1961) *Theory of probability.* Oxford, Clarendon.

Jimenez-Rodriguez, R., Sitar, N. & Chacon, J. (2006) System reliability approach to rock slope stability. *Int J Rock Mech Min,* **43** (6), 847–859.

Johari, A., Fazeli, A. & Javadi, A. A. (2013) An investigation into application of jointly distributed random variables method in reliability assessment of rock slope stability. *Comput Geotech,* **47**, 42–47.

Juang, C. H., Jhi, Y. Y. & Lee, D. H. (1998) Stability analysis of existing slopes considering uncertainty. *Eng Geol*, **49** (2), 111-122.

Juang, C. H.& Lee, D. H. (1990) Rock mass classification using fuzzy sets. In: *Proc. Tenth Southeast Asian Geotechnical Conference. Chinese Institute of Civil and Hydraulic Engineering,* Taipei, 309–314.

Kaufmann, A. & Gupta, M. M. (1991) *Introduction to fuzzy arithmetic: theory and applications.* New York, Van Nostrand Reinhold Co.

Kiureghian, A. D. & Ditlevsen, O. (2009) Aleatory or epistemic? Does it matter? *Struct Saf,* **31** (2), 105–112.

Klir, G. J. (1989) Is there more to uncertainty than some probability theorists might have us believe? *Int J Gen Syst,* **15** (4), 347–378.

Klir, G. J. & Yuan, B. (1995) *Fuzzy sets and fuzzy logic: theory and applications.* Upper Saddle River, N.J., Prentice-Hall.

Li, A. J., Cassidy, M. J., Wang, Y., Merifield, R. S. & Lyamin, A. V. (2012) Parametric Monte Carlo studies of rock slopes based on the Hoek-Brown failure criterion. *Comput Geotech,* **45**, 11–18.

Li, D. Q., Chen, Y. F., Lu, W. B. & Zhou, C. B. (2011) Stochastic response surface method for reliability analysis of rock slopes involving correlated non-normal variables. *Comput Geotech,* **38** (1), 58–68.

Lindley, D. V. (2000) The philosophy of statistics. *J Roy Stat Soc D-Sta,* **49**, 293–319.

Low, B. K. (2008) Efficient probabilistic algorithm illustrated for a rock slope. *Rock Mech Rock Eng,* **41** (5), 715–734.

Martin, C. D. & Christiansson, R. (2009) Estimating the potential for spalling around a deep nuclear waste repository in crystalline rock. *Int J Rock Mech Min*, **46**, 219–228.

Martin, C. D., Kaiser, P. K. & Christiansson, R. (2003) Stress, instability and design of underground excavations. *Int J Rock Mech Min,* **40** (7-8), 1027–1047.

Mayo, D. G. (1996) *Error and the growth of experimental knowledge,* Chicago, Ill; London, University of Chicago Press.

Miranda, T., Correia, A. G. & Sousa, L. R. E. (2009) Bayesian methodology for updating geomechanical parameters and uncertainty quantification. *Int J Rock Mech Min,* **46** (7), 1144–1153.

Merriam-Webster Inc. (2005) *The Merriam-Webster dictionary.* Springfield, Mass.; London, Merriam-Webster.

Moller, B. & Beer, M. (2008) Engineering computation under uncertainty - Capabilities of non-traditional models. *Comput Struct,* **86** (10), 1024–1041.

Moore, R. E. (1966) *Interval analysis.* Englewood Cliffs, N.J., Prentice-Hall.

Moore, R. E. & Bierbaum, F. (1979) *Methods and applications of interval analysis.* Philadelphia, SIAM.

Nadim, F. (2007) Tools and strategies for dealing with uncertainty in geotechnics. In: Griffiths, D. V. & Fenton, G. A. (eds.) *Probabilistic Methods in Geotechnical Engineering.* New York, Springer. pp. 71–95.

Naghibi, M. (2010) *Geotechnical resistance factors for ultimate limit state design of deep foundations under axial compression loading.* PhD Thesis, Dalhousie University, Nova Scotia, Canada.

Nguyen, V. U. (1985) Some fuzzy set applications in mining geomechanics. *Int J Rock Mech Min,* **22** (6), 369–379.

Nilsen, B. (2000) New trends in rock slope stability analysis. *Bull Eng Geol Env*, **58**, 173–178.

References

Nunes, E. & Sousa, S. (2009) Fuzzy performance measures in a high uncertainty context. In: *Proc. IX Congreso Galego de Estatística e Investigación de Operacións. 12–14 November, Ourense, Spain.*

Obara, Y. & Sugawara, K. (2003) Updating the use of the CCBO cell in Japan: overcoring case studies. *Int J Rock Mech Min,* **40** (7–8), 1189–1203.

Oberguggenberger, M. & Fellin, W. (2008) Reliability bounds through random sets: Non-parametric methods and geotechnical applications. *Comput Struct,* **86** (10), 1093–1101.

Oberkampf, W. L., Helton, J. C., Joslyn, C. A., Wojtkiewicz, S. F. & Ferson, S. (2004) Challenge problems: uncertainty in system response given uncertain parameters. *Reliab Eng Syst Safe,* **85** (1-3), 11–19.

Palmström, A. & Broch, E. (2006) Use and misuse of rock mass classification systems with particular reference to the Q-system. *Tunn Undergr Sp Tech,* **21** (6), 575–593.

Palmström, A. & Stille, H. (2007) Ground behaviour and rock engineering tools for underground excavations. *Tunn Undergr Sp Tech,* **22** (4), 363–376.

Park, H. J., Um, J. G., Woo, I. & Kim, J. W. (2012) Application of fuzzy set theory to evaluate the probability of failure in rock slopes. *Eng Geol,* **125**, 92–101.

Park, H. & West, T. R. (2001) Development of a probabilistic approach for rock wedge failure. *Eng Geol,* **59** (3–4), 233–251.

Park, H. J., West, T. R. & Woo, I. (2005) Probabilistic analysis of rock slope stability and random properties of discontinuity parameters, Interstate Highway 40, Western North Carolina, USA. *Eng Geol,* **79** (3–4), 230–250.

Peck, R. B. (1969). Ninth Rankine lecture: Advantages and limitations of the observational method in applied soil mechanics. *Geotechnique*, **19**(2), 171–187.

Peschl, G.M. & Schweiger, H.F. (2003). Reliability analysis in geotechnics with finite elements - comparison of probabilistic, stochastic and fuzzy set methods. In: *Proc. 3rd international symposium on imprecise probabilities and their applications (ISIPTA 03).* Canada: Carleton Scientific. 437–51.

Priest, S. D. (1985) *Hemspherical projections methods in rock mechanics.* George Allen & Unwin, London.

Priest, S. D. & Brown, E. T. (1983) Probabilistic stability analysis of variable rock slopes. *T I Min Metall A,* **92** (Jan), A1–A12.

Priest, S. D. & Hudson, J. A. (1976) Discontinuity spacings in rock. *Int J Rock Mech Min,* **13** (5), 135–148.

Raab, T. & Brosch, F. J. (1996) Uncertainty, subjectivity, experience: a comparative pilot study. *Eng Geol,* **44** (1-4), 129–145.

Rinderknecht, S. L., Borsuk, M. E. & Reichert, P. (2012) Bridging uncertain and ambiguous knowledge with imprecise probabilities. *Environ Modell Softw,* **36**, 122–130.

Ruffolo, R. M. & Shakoor, A. (2009) Variability of unconfined compressive strength in relation to number of test samples. *Eng Geol,* **108** (1–2), 16–23.

Sakurai S. & Shimuzu, N. (1987) Assessment of rock slope stability by fuzzy set theory. In: *Proc. 6th Int. Cong. Rock Mechs*. Montreal, 503–506.

Saltelli, A. (2004) *Sensitivity analysis in practice: a guide to assessing scientific models.* Chichester, John Wiley.

Schubert, W. 2012. Some remarks on current engineering design practices. In *Proc. XIV Cicolo di conferenze di meccanica e ingegneria delle rocce (eds. Barla G., Barla, M., Ferrero, A.M., Rotonda, T.). Torino, Italy, 21 November 2012*. Torino, Italy: Celid.

Schweiger, H. F. & Peschl, G. M. (2005) Reliability analysis in geotechnics with the random set finite element method. *Comput Geotech,* **32** (6), 422–435.

Sober, E. (2002) Bayesianism – its scope and limits. In: Swinburne, R. (ed.) *Bayes's Theorem.* Proc. Brit. Acad., **113**. Oxford, Oxford University Press.

Sonmez, H., Gokceoglu, C. & Ulusay, R. (2003) An application of fuzzy sets to the Geological Strength Index (GSI) system used in rock engineering. *Eng Appl Artif Intel,* **16** (3), 251–269.

Soundappan, P., Nikolaidis, E., Haftka, R. T., Grandhi, R. & Canfield, R. (2004) Comparison of evidence theory and Bayesian theory for uncertainty modeling. *Reliab Eng Syst Safe,* **85** (1–3), 295–311.

Stevens, S. S. (1946) On the theory of scales of measurement. *Science,* **103** (2684), 677–680.

Stille, H. & Palmström, A. (2003) Classification as a tool in rock engineering. *Tunn Undergr Sp Tech,* **18** (4), 331–345.

Swinburne, R. (ed.) (2002) *Bayes's Theorem*. Proceedings of the British Academy, **113**. Oxford, Oxford University Press.

Tamimi, S., Amadei, B. & Frangopol, D. M. (1989) Monte-Carlo simulation of rock slope reliability. *Comput Struct,* **33** (6), 1495–1505.

References

Tonon, F., Bernardini, A. & Mammino, A. (2000) Determination of parameters range in rock engineering by means of Random Set Theory. *Reliab Eng Syst Safe,* **70** (3), 241–261.

Tucker, W. T. & Ferson, S. (2003) *Probability bounds in environmental risk assessments.* New York, Applied Biomathematics.

Tzamos S. & Sofianos A.I. (2006) Extending the Q system's prediction of support in tunnels employing fuzzy logic and extra parameters. *Int. J. Rock Mech. Min. Sci.* **43**(6), 938–949.

Uzielli, M. (2008) Statistical analysis of geotechnical data. *Proc Monogr Eng Wate*, 173–193.

Verbraak, H. L. F. (1990) *The logic of objective Bayesianism: a theory of dynamic probability, or, against orthodox statistics.* Amsterdam, Verbraak.

Vose, D. (2000) *Risk analysis: a quantitative guide,* Chichester; New York, Wiley.

Walley, P. (1991) *Statistical reasoning with imprecise probabilities.* London, Chapman & Hall.

Walley, P. (1996) Inferences from multinomal data: learning about a bag of marbles (with discussion). *J Roy Stat Soc B Met,* **58** (1), 3–57.

Wang, L., Hwang, J. H., Juang, C. H. & Atamturktur, S. (2013) Reliability-based design of rock slopes - A new perspective on design robustness. *Eng Geol,* **154**, 56–63.

Wenner, D. & Harrison, J. P. (1996) Techniques to cope with uncertain parameters in geomechanics on different levels of information. In: *Proc. International Symposium of the ISRM*, Eurock 1996, 2–5 September, Torino, Italy. pp. 327–334.

Whitman, R. V. (2000) Organizing and evaluating uncertainty in geotechnical engineering. *J Geotech Geoenviron,* **126** (7), 583–593.

Williamson, R. C. & Downs, T. (1990) Probabilistic arithmetic I: numerical methods for calculating convolutions and dependency bounds. *Int J Approx Reason* **4**, 89–158.

Yamaguchi, U (1970) Number of Test-Pieces Required to Determine Strength of Rock. *Int J Rock Mech Min,* **7** (2), 209–227.

Yubin, L., Zhong, Q. & Guangyuan, W. (1997) Fuzzy random reliability of structures based on fuzzy random variables. *Fuzzy Set Syst*, **86**, 345–355.

Zadeh, L. A. (1965) Fuzzy sets. *Inform Control*, **8**, 338–353.

Zadeh, L.A. (1975) The concept of a linguistic variable and its application to approximate reasoning, *Inform Sci,* **8**, 199–249.

Zadeh, L. A. (1980) Fuzzy-sets versus probability. *P IEEE,* **68** (3), 421–421.

Zadeh, L. A. (1982) Is possibility different from probability? *Hum Syst Manage,* **3** (4), 253–254.

Zadeh, L. A. (1995) Probability-theory and fuzzy-logic are complementary rather than competitive - discussion. *Technometrics,* **37** (3), 271–276.

Zhang, L.M., Tang, W.H., Zhang, L.L. & Zheng, J.G. (2004) Reducing uncertainty of prediction from empirical correlations. *J Geotech Geoenviron,* **130** (5), 526–534.

Zhang, L.Y. & Einstein, H.H. (1998) Estimating the mean trace length of rock discontinuities. *Rock Mech Rock Eng,* **31** (4), 217–235.

Zhang, L.Y. & Einstein, H.H. (2004) Using RQD to estimate the deformation modulus of rock masses. *Int J Rock Mech Min,* **41** (2), 337–341.

Zhao, Y. G. & Ono, T. (2001) Moment methods for structural reliability. *Struct Saf,* **23** (1), 47–75.

Zimmermann, H. J. (2000) An application-oriented view of modeling uncertainty. *Eur J Oper Res,* **122** (2), 190–198.

Appendix A – Verification of generic MathCAD algorithm for hybrid analysis

A.1 Verification of hybrid calculation routines: Numerical example provided by Dubois & Guyonnet (2011)

Dubois & Guyonnet (2011) provide a numerical example of a hybrid calculation where the inputs are a mixture of fuzzy numbers and precise probability distributions. This example is used here to verify the performance of the numerical routines implemented in this thesis.

The function used in the hybrid calculation of Dubois & Guyonnet (2011) is given by the following expressions:

$$\text{IER} = \text{D} \cdot \text{UER} \tag{A.1}$$

$$\text{D} = \frac{\text{I} \cdot \text{C} \cdot \text{EF} \cdot \text{ED}}{\text{BW} \cdot \text{AT}} \tag{A.2}$$

The following table summarises the unpredictability model chosen to represent each parameter and the minimum, mode or core and upper bound values.

Table A.1: Parameters used for hybrid calculation (from Dubois & Guyonnet, 2011)

Parameter	Mode of representation	Lower bound	Mode or core	Upper bound
C	Precise PDF (triangular)	5	10	20
I	Fuzzy number (triangular)	1	1.5	2.5
EF	Fuzzy number (triangular)	200	250	350
ED	Precise PDF (triangular)	10	30	50
UER	Fuzzy number (triangular)	2×10^{-2}	5.7×10^{-2}	10^{-1}

The figure below presents the output obtained by Dubois & Guyonnet (2011). In their example, Dubois & Guyonnet (2011) undertake the simulation by 100 iterations of a hybrid Monte-Carlo technique.

In this figure, the Plausibility curve represents the upper bound CDF of the p-box, with the Belief curve representing the upper bound distribution of the p-box. The curve labelled 'Mote-Carlo' is the result of a solution using precise PDFs (triangular distributions) for all the parameters.
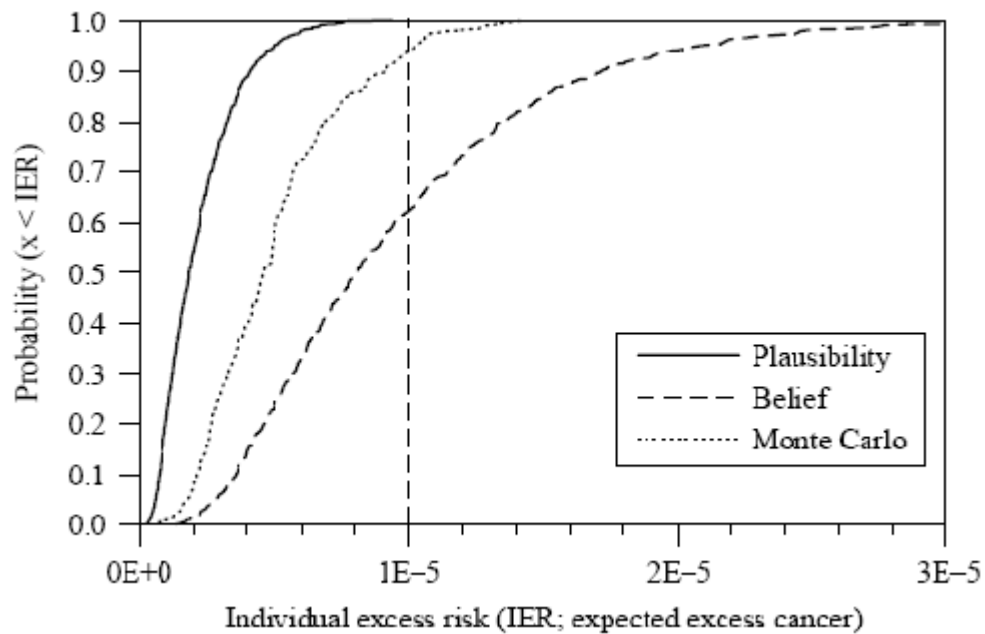
Figure A.1: Parameters used for hybrid calculation (from Dubois & Guyonnet, 2011)

In the verification calculation that follows, a similar hybrid Monte-Carlo routine with 5000 iterations is implemented.

A.1.1 Generic MathCAD routines

<u>Set up discretisation vectors</u>

$n := 1000$ Number of discretisations of each p-box

Create vector for ordinate of cumulative density function (CDF) with 'n' discretisations

$$v\_P := \begin{vmatrix} v \leftarrow 0 \\ j \leftarrow 0 \\ i \leftarrow 0 \\ \text{for } j \in 1..999 \\ \quad \begin{vmatrix} i \leftarrow \dfrac{j}{1000} \\ v_0 \leftarrow 0 \\ v_n \leftarrow 1 \\ v_j \leftarrow i \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

$k := 5000$ Input number of iterations for Monte-Carlo simulation

$bins := \text{round}\left(\sqrt{n}\right)$ Set bins

<u>Create user-defined triangular distribution functions</u>

Create vector of probability density

$$dtri(a,b,c) := \begin{vmatrix} v \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 0..n-1 \\ \quad v_j \leftarrow \begin{vmatrix} \dfrac{2 \cdot (j-a)}{(b-a)(c-a)} & \text{if } a \leq j \leq b \\ \dfrac{2 \cdot (c-j)}{(c-a)(c-b)} & \text{if } b < j \leq c \\ 0 & \text{otherwise} \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

$$v\_pd := \begin{vmatrix} v \leftarrow 0 \\ j \leftarrow 0 \\ i \leftarrow 0 \\ \text{for } j \in 0..n-1 \\ \quad v_j \leftarrow j \\ \text{return } v \end{vmatrix}$$

Create vector of cumulative density

$$ptri(a,b,c) := \begin{vmatrix} v \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 0..n-1 \\ \quad v_j \leftarrow \begin{vmatrix} 0 & \text{if } j < a \\ \dfrac{(j-a)^2}{(b-a)(c-a)} & \text{if } a \leq j \leq b \\ 1 - \dfrac{(c-j)^2}{(c-a)(c-b)} & \text{if } b < j \leq c \\ 1 & \text{otherwise} \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

Create vector of inverse cumulative density

$$\text{qtri}(a,b,c) := \begin{vmatrix} v \leftarrow 0 \\ i \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 1..n-1 \\ \quad \begin{vmatrix} i \leftarrow \dfrac{j}{n} \\ v_0 \leftarrow a \\ v_n \leftarrow c \\ v_j \leftarrow \begin{vmatrix} \sqrt{i \cdot (b-a) \cdot (c-a)} + a & \text{if } 0 < i < \dfrac{b-a}{c-a} \\ \left[c - \sqrt{(1-i) \cdot (c-a) \cdot (c-b)}\right] & \text{if } \dfrac{b-a}{c-a} \le i \le 1 \end{vmatrix} \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

Create vector of 'k' random numbers from triangular distribution

$$\text{rtri}(a,b,c) := \begin{vmatrix} v \leftarrow 0 \\ u \leftarrow 0 \\ j \leftarrow 0 \\ u \leftarrow \text{runif}(k,0,1) \\ \text{for } j \in 0..k-1 \\ \quad v_j \leftarrow \begin{vmatrix} \sqrt{u_j \cdot (b-a) \cdot (c-a)} + a & \text{if } 0 < u_j < \dfrac{b-a}{c-a} \\ \left[c - \sqrt{(1-u_j) \cdot (c-a) \cdot (c-b)}\right] & \text{if } \dfrac{b-a}{c-a} \le u_j \le 1 \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

Check output - test values for user defined triangular distribution

$$a := 5 \qquad b := 10 \qquad c := 20$$

$$\text{v\_dtri} := \text{dtri}(a,b,c) \qquad \text{v\_ptri} := \text{dtri}(a,b,c) \qquad \text{v\_qtri} := \text{qtri}(a,b,c)$$
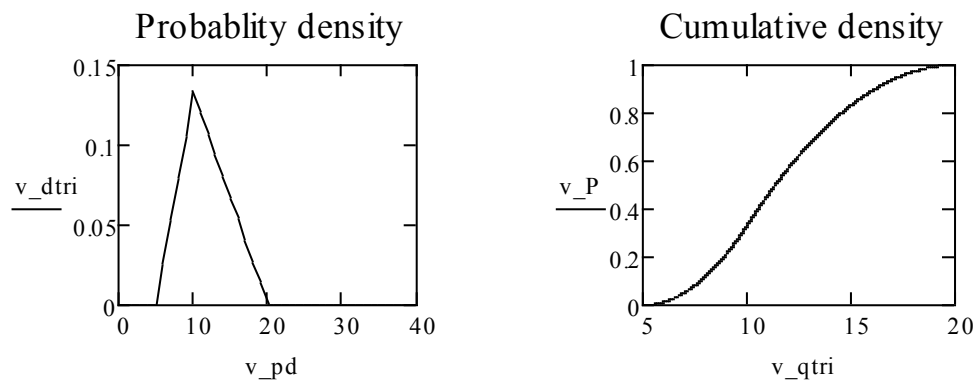
Figure A.2: MathCAD plots to check user-defined triangular distribution functions

Define Monte-Carlo simulation functions

$\text{v\_rtri} := \text{rtri}(a, b, c)$

Create vector or random numbers generated
from user-defined triangular PDF

$\text{v\_htri} := \text{sort}(\text{v\_rtri})$

Sort vector and create histrogram of random numb

$\text{h\_tri} := \text{histogram}(\text{bins}, \text{v\_htri})$
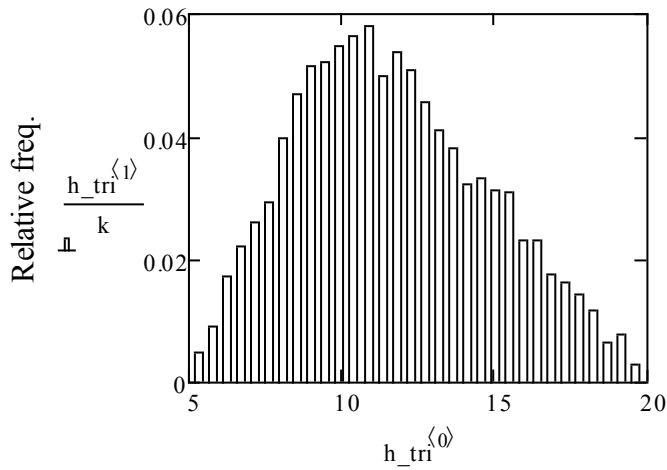
Figure A.3: Histogram of random numbers generated for triangular PDF .

Define functions to create p-box from fuzzy numbers

The L-R fuzzy numbers are defined as detailed in Chapter 3 of this thesis, i.e. fuz = [a,

$$\text{v\_fuz}_L(a, b, c, d) := \begin{vmatrix} v \leftarrow 0 \\ i \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 1..n-1 \\ \quad \begin{vmatrix} i \leftarrow \dfrac{j}{n} \\ v_0 \leftarrow a \\ v_n \leftarrow b \\ v_j \leftarrow \text{qunif}(i, a, b) \end{vmatrix} \\ \text{return } v \end{vmatrix} \qquad \text{v\_fuz}_R(a, b, c, d) := \begin{vmatrix} v \leftarrow 0 \\ i \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 1..n-1 \\ \quad \begin{vmatrix} i \leftarrow \dfrac{j}{n} \\ v_0 \leftarrow c \\ v_n \leftarrow d \\ v_j \leftarrow \text{qunif}(i, c, d) \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

The numerical example of Dubois & Guyonnet (2011) is now commenced on the next page using these defined functions

A.1.2 Verification of generic MathCAD routines

Apply generic routines to problem by Dubois & Guyonnet (2011).

Input parameters

    Deterministic input parameters

$BW := 70$ $AT := 70$

    Probabilistic input parameters

$C_i := qtri(5, 10, 20)$ $ED := qtri(10, 30, 50)$ Define triangular PDFs
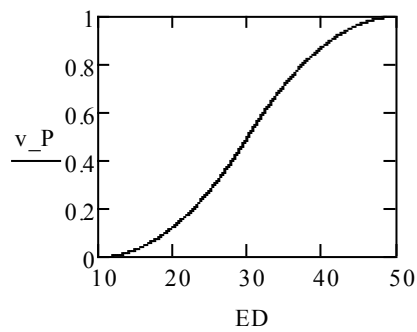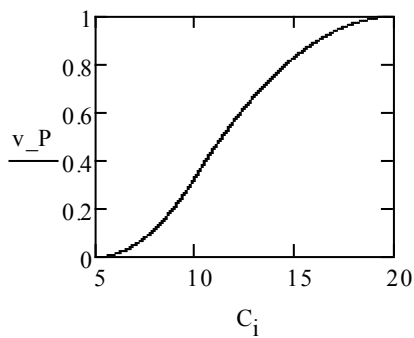
    Fuzzy input parameters

$I := (1 \quad 1.5 \quad 2.5)$

$a := I_{0,0}$ $b := I_{0,1}$ $c := I_{0,1}$ $d := I_{0,2}$

    Create p-box vector of fuzzy_I

$v\_I_L := v\_fuz_L(a, b, c, d)$

$v\_I_R := v\_fuz_R(a, b, c, d)$

$EF := (200 \quad 250 \quad 350)$

$a := EF_{0,0}$ $b := EF_{0,1}$ $c := EF_{0,1}$ $d := EF_{0,2}$

    Create p-box vector of fuzzy_IEF

$v\_EF_L := v\_fuz_L(a, b, c, d)$

$v\_EF_R := v\_fuz_R(a, b, c, d)$

$UER := \left(2 \cdot 10^{-2} \quad 5.7 \cdot 10^{-2} \quad 10^{-1}\right)$

$a := UER_{0,0}$ $b := UER_{0,1}$ $c := UER_{0,1}$ $d := UER_{0,2}$

    Create p-box vector of fuzzy_UER

$v\_UER_L := v\_fuz_L(a, b, c, d)$

$v\_UER_R := v\_fuz_R(a, b, c, d)$
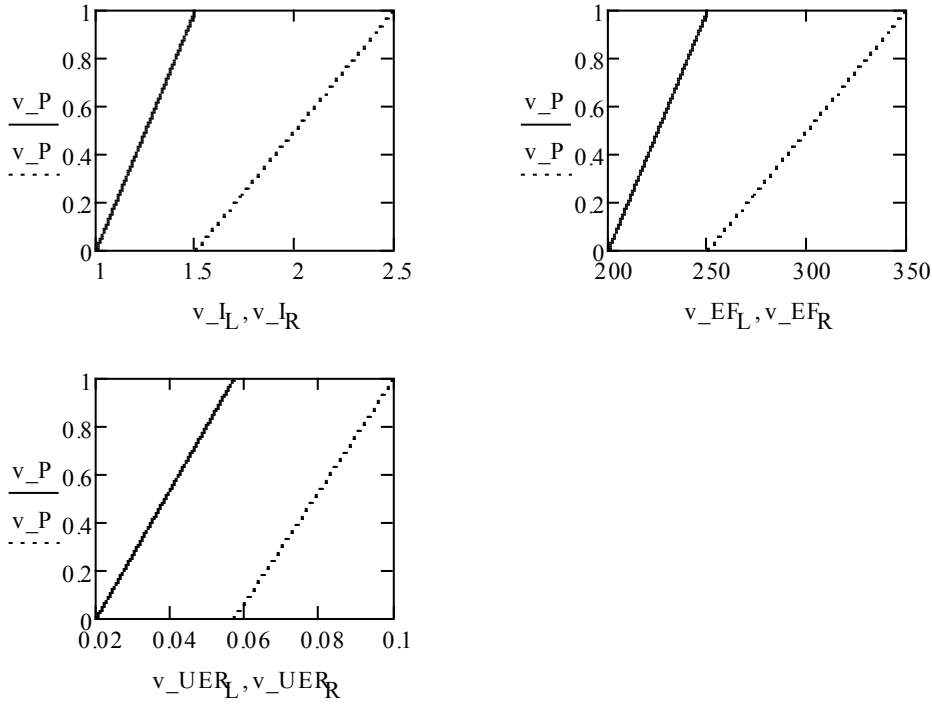
Figure A.4: P-boxe generated from fuzzy numbers.

Create vectors of random numbers for hybrid Monte-Carlo simulation

$$C_i := \text{rtri}(5, 10, 20) \qquad\qquad ED := \text{rtri}(10, 30, 50)$$

$$v\_I_L := \text{runif}\left(k, I_{0,0}, I_{0,1}\right) \qquad\qquad v\_I_R := \text{runif}\left(k, I_{0,1}, I_{0,2}\right)$$

$$v\_EF_L := \text{runif}\left(k, EF_{0,0}, EF_{0,1}\right) \qquad\qquad v\_EF_R := \text{runif}\left(k, EF_{0,1}, EF_{0,2}\right)$$

$$v\_UER_L := \text{runif}\left(k, UER_{0,0}, UER_{0,1}\right) \qquad\qquad v\_UER_R := \text{runif}\left(k, UER_{0,1}, UER_{0,2}\right)$$

Calculate lower and upperbound value of IER from random vectors

$$IER_L := \left[\left(\overrightarrow{\frac{v\_I_L \cdot C_i \cdot v\_EF_L \cdot ED}{BW \cdot AT} \cdot v\_UER_L}\right)\right] \cdot \left(2.74 \times 10^{-6}\right)$$

Note: a factor of 2.74x10^-6 needs to be applied to convert input units to be consistent with the ouput

$$IER_R := \left(\overrightarrow{\frac{v\_I_R \cdot C_i \cdot v\_EF_R \cdot ED}{BW \cdot AT} \cdot v\_UER_R}\right) \cdot \left(2.74 \times 10^{-6}\right)$$

***Sort upper and lower bound results from Monte-Carlo results
and historgrams for plot***

$$v\_hIER_L := sort\left(IER_L\right)$$ $\qquad\qquad$ $$v\_hIER_R := sort\left(IER_R\right)$$

$$h\_IER_L := histogram\left(bins, IER_L\right)$$ $\qquad$ $$h\_IER_R := histogram\left(bins, IER_R\right)$$

*Set up numerical integration to create CDF from PDFs
produced through Monte-Carlo simulation*

$$\_cdf(in\_hist) := \begin{vmatrix} v \leftarrow 0 \\ j \leftarrow 0 \\ i \leftarrow 0 \\ h \leftarrow 0 \\ \text{for } j \in 0..(bins - 1) \\ \quad \begin{vmatrix} h \leftarrow histogram(bins, in\_hist) \\ v_j \leftarrow \sum_{i=0}^{j} \dfrac{h_{i,1}}{k} \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

$$cdf\_IER_L := \_cdf\left(IER_L\right)$$
$$cdf\_IER_R := \_cdf\left(IER_R\right)$$

<u>**Plot histograms and lower and upper bounds of p-box**</u>
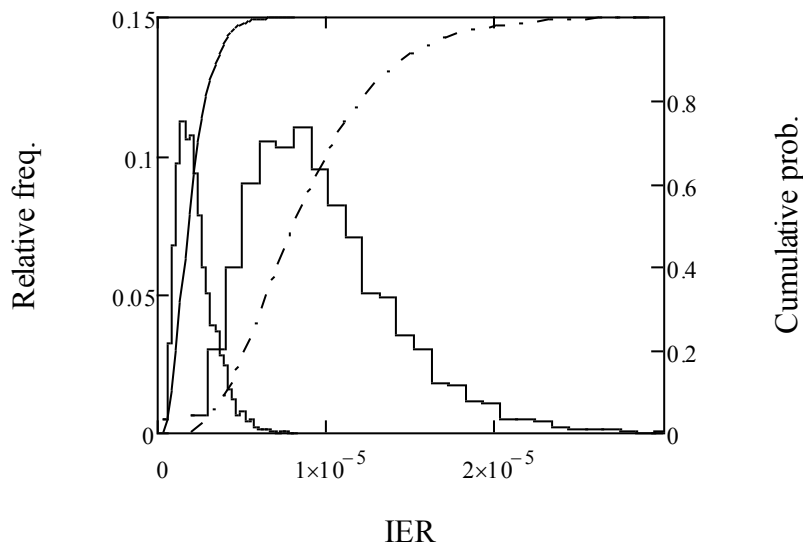
Figure A.5: MathCAD output of verification computation

Conclusion: The results produced by the hybrid Monte-Carlo simulation functions set up here re-produce the output calculated by Dubois & Guyonnet (2011). Minor differences in the output graphed in the figure above and that of Dubois & Guyonnet (2011) are due to Monte-Carlo sampling meaning two analyses will not produce identical results. However, the deviation between two calculations is minimal.

Appendix B – Algorithms for fuzzy planar slope stability analysis

Parameters:

H – Height of slope
z – Depth of tension crack
z_w – Height of water in tension crack
ψ_p – Angle of failure surface
ψ_f – Angle of slope face
c – Cohesion of failure surface
ϕ – Angle of friction of failure surface
T – Tension in bolt
θ – Angle of bolt installation
W – Weight of rock
U – Water pressure on sliding plane
V – Water pressure in tension crack
α – acceleration co-efficient
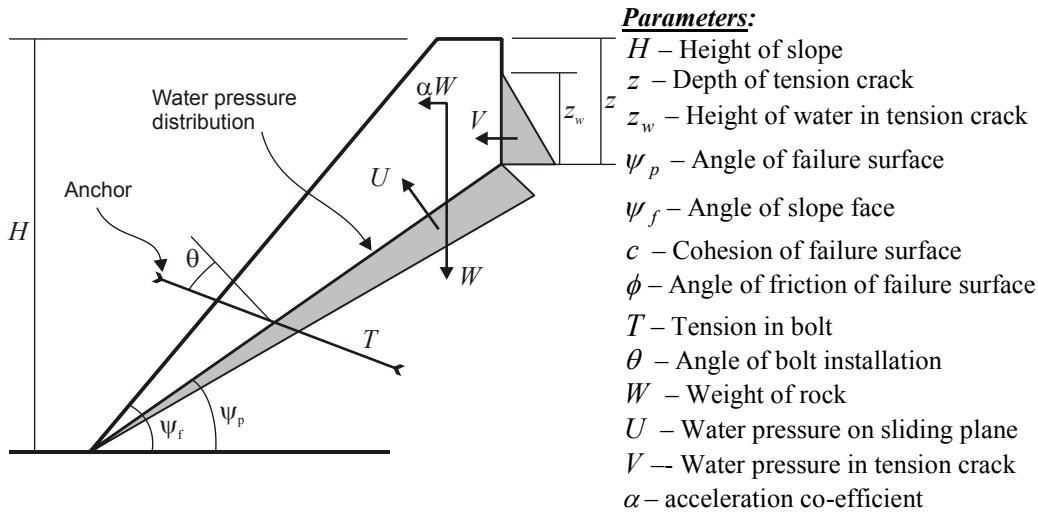
Figure B.1: MathCAD output of verification computation

Table B.1: Functions for definition of driving and resisting forces

Description of function	Variable used	Equation of function
A Area of block/m	$A = f(H, z, \psi_p)$	$A = \dfrac{H - z}{\sin \psi_p}$
W Weight of block/m	$W = f(H, z, \psi_p, \psi_f, \gamma)$	$W = \dfrac{1}{2} \gamma H^2 \left\{ \left[1 - \left(\dfrac{z}{H} \right)^2 \right] \cot \psi_p - \cot \psi_f \right\}$
U Water pressure normal to sliding plane	$U = f(H, z, \psi_p, \gamma_w, z_w)$	$U = \dfrac{1}{2} A \gamma_w z_w = \dfrac{1}{2} \gamma_w z_w \dfrac{H - z}{\sin \psi_p}$
V Horizontal component of water pressure	$V = f(\gamma_w, z_w)$	$V = \dfrac{1}{2} \gamma_w z_w^2$

Table B.2: Functions for definition of driving and resisting forces

Force components	U	V	W	
$\parallel$ to plane	0	$V_t = V \cos \psi_p$	$W_t = W \sin \psi_p$	$\sum$ = forces causing sliding
$\perp$ to plane	$-U$	$V_n = -V \sin \psi_p$	$W_n = W \cos \psi_p$	$\sum$ = effective normal force

B.1 MathCAD computation of fuzzy slope stability

Using the geometry in Figure B.1 and the functions defined in Table B1 and B2, the governing equation for planar slope stability is given by:

$$\text{FoS} = \frac{\sum \text{Resisting forces(R)}}{\sum \text{Driving forces(L)}} = \frac{cA + \left(W\left(\cos\psi_p - \alpha\sin\psi_p\right) - U - V\sin\psi_p + T\cos\theta\right)\tan\phi}{V\cos\psi_p + W\left(\sin\psi_p + \alpha\cos\psi_p\right) - T\sin\theta} \tag{B.1}$$

Deterministic inputs

Geometry inputs

$H := 60$

$\psi_f := 50\text{deg}$

$\psi_p := 35\text{deg}$

$z_{max} := H \cdot \left(1 - \dfrac{\tan\left(\psi_p\right)}{\tan\left(\psi_f\right)} \right) = 24.747$

$z_{wmax} := z_{max} = 24.747$

$b_{max} := H \cdot \left(\cot\left(\psi_p\right) - \cot\left(\psi_f\right) \right) = 35.343$

Weight density inputs

$\gamma := 2.6$

$\gamma_w := 1$

Bolt inclination

$\theta := 35\text{deg}$

Limit state

$\text{FOS} := 1$

Fuzzy inputs

$$\begin{pmatrix} \phi \\ c \\ z \\ b \\ z_w \\ a \end{pmatrix} \qquad a := \begin{pmatrix} 15\text{deg} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad b := \begin{pmatrix} 35\text{deg} \\ 10 \\ 0.5z_{max} \\ 0.5b_{max} \\ 0 \\ 0 \end{pmatrix} \qquad c := \begin{pmatrix} 35\text{deg} \\ 10 \\ 0.5z_{max} \\ 0.5b_{max} \\ 0 \\ 0 \end{pmatrix} \qquad d := \begin{pmatrix} 70\text{deg} \\ 25 \\ z_{max} \\ b_{max} \\ z_{wmax} \\ 0.16 \end{pmatrix}$$

Functions to define a-cuts of fuzzy numbers

Fuzzy membership function L-R

$$\mu_Y(x) = \begin{vmatrix} \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b \le x \le c \\ \dfrac{d-x}{d-c} & \text{if } c \le x \le d \end{vmatrix}$$

a-cuts of triangular fuzzy numbers

$n := 10$ Number of alpha cuts , range variable of cuts $:= 0..n$

$\alpha := 0, \dfrac{1}{n} .. 1$ membership values of a-cuts

$inc := \dfrac{1}{n} = 0.1$ Increment of a-cuts

$\alpha_{\phi min}(\alpha) := (b_0 - a_0)\alpha + a_0$ $\alpha_{\phi max}(\alpha) := d_0 - (d_0 - c_0)\cdot\alpha$

$\alpha_{cmin}(\alpha) := (b_1 - a_1)\alpha + a_1$ $\alpha_{cmax}(\alpha) := d_1 - (d_1 - c_1)\cdot\alpha$

$\alpha_{zmin}(\alpha) := (b_2 - a_2)\alpha + a_2$ $\alpha_{zmax}(\alpha) := d_2 - (d_2 - c_2)\cdot\alpha$

$\alpha_{bmin}(\alpha) := (b_3 - a_3)\alpha + a_3$ $\alpha_{bmax}(\alpha) := d_3 - (d_3 - c_3)\cdot\alpha$

$\alpha_{zwmin}(\alpha) := (b_4 - a_4)\alpha + a_4$ $\alpha_{zwmax}(\alpha) := d_4 - (d_4 - c_4)\cdot\alpha$

$\alpha_{amin}(\alpha) := (b_5 - a_5)\alpha + a_5$ $\alpha_{amax}(\alpha) := d_5 - (d_5 - c_5)\cdot\alpha$

Fuzzy variables as a-cut range variables

$f\_\phi_{min}(\alpha) := \alpha_{\phi min}(\alpha)$

$f\_\phi_{max}(\alpha) := \alpha_{\phi max}(\alpha)$ } Fuzzy ?

$f\_c_{min}(\alpha) := \alpha_{cmin}(\alpha)$

$f\_c_{max}(\alpha) := \alpha_{cmax}(\alpha)$ } Fuzzy c

$f\_z_{min}(\alpha) := \alpha_{zmin}(\alpha)$

$f\_z_{max}(\alpha) := \alpha_{zmax}(\alpha)$ } Fuzzy z

$f\_b_{min}(\alpha) := \alpha_{bmin}(\alpha)$

$f\_b_{max}(\alpha) := \alpha_{bmax}(\alpha)$ } Fuzzy b

$f\_zw_{min}(\alpha) := \alpha_{zwmin}(\alpha)$

$f\_zw_{max}(\alpha) := \alpha_{zwmax}(\alpha)$ } Fuzzy zw

$f\_a_{min}(\alpha) := \alpha_{amin}(\alpha)$

$f\_a_{max}(\alpha) := \alpha_{amax}(\alpha)$ } Fuzzy a
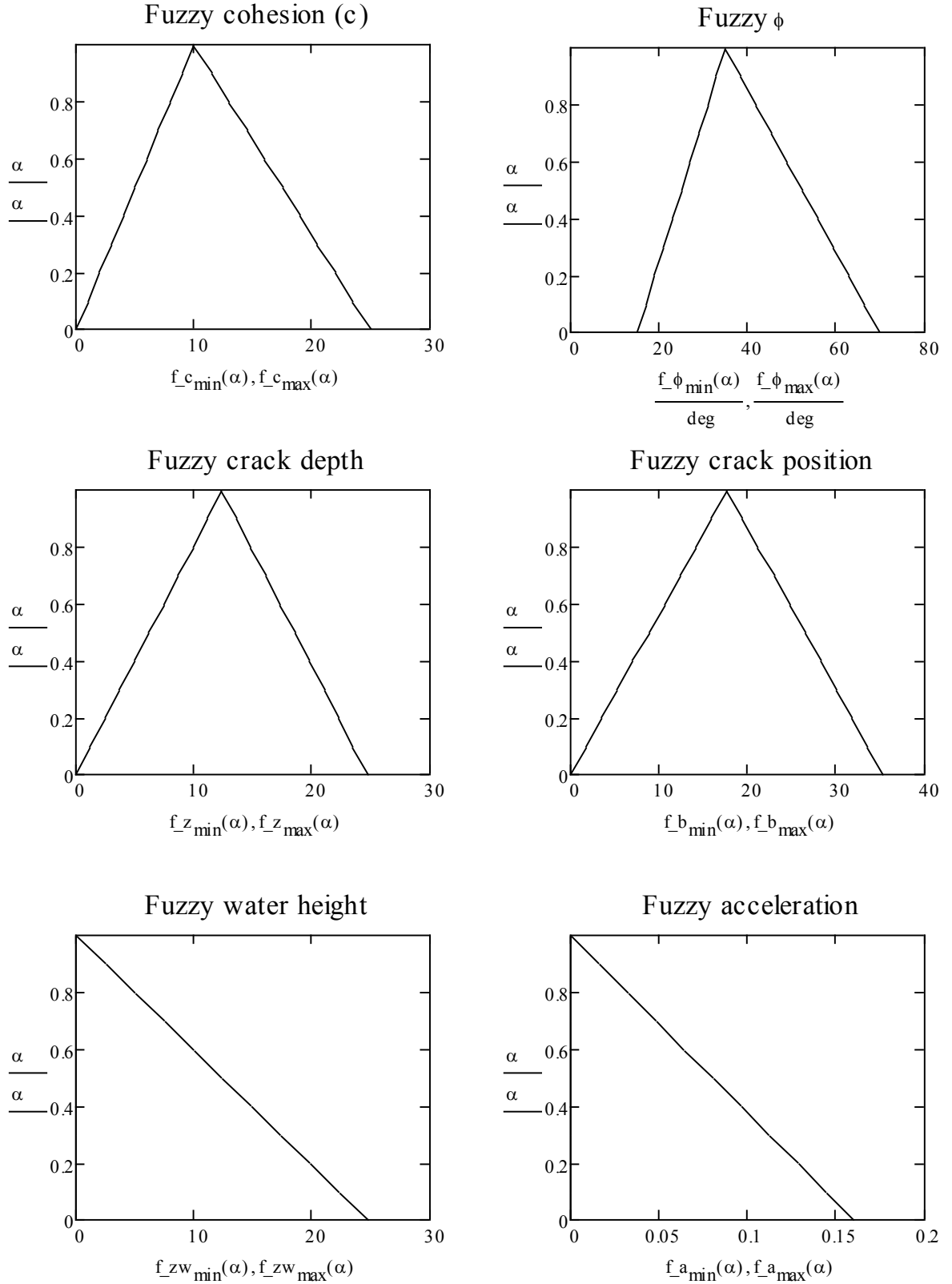
Figure B.2: Fuzzy numbers of input parameters produced from stacked array of α -cuts

Fuzzy geometry functions

$$f\_A_{min}(\alpha) := \frac{H - f\_z_{max}(\alpha)}{\sin(\psi_p)}$$

$$f\_A_{max}(\alpha) := \frac{H - f\_z_{min}(\alpha)}{\sin(\psi_p)}$$

$\left.\begin{array}{c}\end{array}\right\}$ Fuzzy Area of sliding plane

$$f\_W_{min}(\alpha) := \frac{1}{2}\gamma \cdot H^2 \left[\left[1 - \left(\frac{f\_z_{max}(\alpha)}{H}\right)^2\right]\cot(\psi_p) - \cot(\psi_f)\right]$$

$$f\_W_{max}(\alpha) := \frac{1}{2}\gamma \cdot H^2 \left[\left[1 - \left(\frac{f\_z_{min}(\alpha)}{H}\right)^2\right]\cot(\psi_p) - \cot(\psi_f)\right]$$

$\left.\begin{array}{c}\end{array}\right\}$ Fuzzy Weight of block

Fuzzy water pressures

$$f\_U_{min}(\alpha) := \frac{1}{2}\,f\_A_{min}(\alpha)\cdot\gamma_w\cdot f\_zw_{max}(\alpha)$$

$$f\_U_{max}(\alpha) := \frac{1}{2}\,f\_A_{max}(\alpha)\cdot\gamma_w\cdot f\_zw_{min}(\alpha)$$

$\left.\begin{array}{c}\end{array}\right\}$ Fuzzy water pressure on sliding plane

$$f\_V_{min}(\alpha) := \frac{1}{2}\gamma_w\cdot\left(f\_zw_{max}(\alpha)\right)^2$$

$$f\_V_{max}(\alpha) := \frac{1}{2}\gamma_w\cdot\left(f\_zw_{min}(\alpha)\right)^2$$

$\left.\begin{array}{c}\end{array}\right\}$ Fuzzy water pressure in crack

Fuzzy driving and resisting forces

$$f\_R_{min}(\alpha) := f\_c_{min}(\alpha)\cdot f\_A_{min}(\alpha) + \left[f\_W_{min}(\alpha)\cdot\left(\cos(\psi_p) - f\_a_{max}(\alpha)\cdot\sin(\psi_p)\right) - f\_U_{min}(\alpha) - f\_V_{min}(\alpha)\cdot\sin(\psi_p)\right]\cdot\tan\left(f\_\phi_{min}(\alpha)\right)$$

$$f\_R_{max}(\alpha) := f\_c_{max}(\alpha)\cdot f\_A_{max}(\alpha) + \left[f\_W_{max}(\alpha)\cdot\left(\cos(\psi_p) - f\_a_{min}(\alpha)\cdot\sin(\psi_p)\right) - f\_U_{max}(\alpha) - f\_V_{max}(\alpha)\cdot\sin(\psi_p)\right]\cdot\tan\left(f\_\phi_{max}(\alpha)\right)$$

$$f\_Q_{min}(\alpha) := f\_W_{min}(\alpha)\cdot\left(\sin(\psi_p) + f\_a_{max}(\alpha)\cdot\cos(\psi_p)\right) + f\_V_{min}(\alpha)\cdot\cos(\psi_p)$$

$$f\_Q_{max}(\alpha) := f\_W_{max}(\alpha)\cdot\left(\sin(\psi_p) + f\_a_{min}(\alpha)\cdot\cos(\psi_p)\right) + f\_V_{max}(\alpha)\cdot\cos(\psi_p)$$

Fuzzy Factor of Safety

$$f\_FOS_{min}(\alpha) := \frac{f\_R_{min}(\alpha)}{f\_Q_{min}(\alpha)}$$

$$f\_FOS_{max}(\alpha) := \frac{f\_R_{max}(\alpha)}{f\_Q_{max}(\alpha)}$$

$\left.\begin{array}{c}\end{array}\right\}$ Fuzzy Factor of safety

Fuzzy bolt tension to ensure FOS > 1

$$f\_T_{max}(\alpha) := \frac{FOS\cdot f\_Q_{min}(\alpha) - f\_R_{min}(\alpha)}{\cos(\theta)\cdot\tan\left(f\_\phi_{min}(\alpha)\right) + \sin(\theta)}$$

$$f\_T_{min}(\alpha) := \frac{FOS\cdot f\_Q_{max}(\alpha) - f\_R_{max}(\alpha)}{\cos(\theta)\cdot\tan\left(f\_\phi_{max}(\alpha)\right) + \sin(\theta)}$$

$\left.\begin{array}{c}\end{array}\right\}$ Fuzzy bolt tension

Figure B.3: Fuzzy numbers of functions produced from stacked array of α -cuts

Fuzzy FOS - Slope stability

Fuzzy bolt tension

$$\frac{\alpha}{\alpha}$$
$$\frac{\alpha}{\alpha}$$

f_FOS$_{min}(\alpha)$, f_FOS$_{max}(\alpha)$, FOS

f_T$_{min}(\alpha)$, f_T$_{max}(\alpha)$, 0
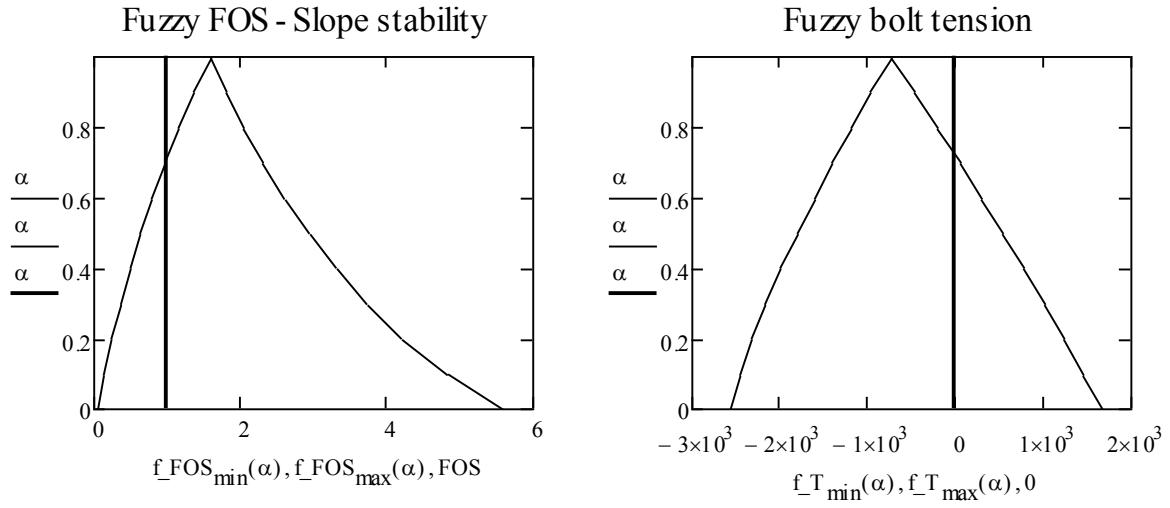
Fuzzy FOS function triple (lower, mode, upper):

$$\text{FOS}_{tri} := \left[\left(\text{f\_FOS}_{min}(0) \right) \quad \left(\text{f\_FOS}_{min}(1) \right) \quad \left(\text{f\_FOS}_{max}(0) \right) \right] = (0.047 \quad 1.586 \quad 5.578)$$

Figure B.4: Computed fuzzy factor of safety and bolt tension to ensure FoS > 1

B.2 MathCAD routines to compute Agreement index

De-fuzzification functions to calculate Agreement index from FoS

$FOS := 0.55$ Fos value at which AI is calculated

Function to convert fuzzy range variables to vectors f_FOSvec(f_FOS,n):
   - f_FOS = fuzzy FOS function as a range variable
   - n = number of alpha cuts

$$f\_FOSvec := \begin{vmatrix} N \leftarrow n + 1 \\ \text{for } j \in 0..(2N - 1) \\ \quad \begin{vmatrix} v_j \leftarrow \begin{vmatrix} f\_FOS_{min}\left(\dfrac{j}{n}\right) & \text{if } j < N \\ f\_FOS_{max}(1) & \text{if } j = N \\ f\_FOS_{max}\left[\dfrac{(2N-1)-j}{n}\right] & \text{otherwise} \end{vmatrix} \\ u_j \leftarrow \begin{vmatrix} \dfrac{j}{n} & \text{if } j < N \\ 1 & \text{if } j = N \\ \left[\dfrac{(2N-1)-j}{n}\right] & \text{otherwise} \end{vmatrix} \end{vmatrix} \\ \text{return } \begin{pmatrix} v \\ u \end{pmatrix} \end{vmatrix}$$

$$f\_FOS_x := f\_FOSvec_0$$

$$f\_FOS_y := f\_FOSvec_1$$
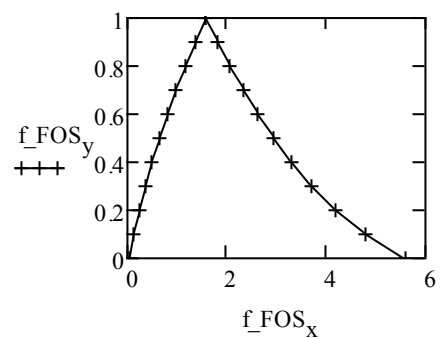
Function to trim fuzzy number - trim(f_V,c):
- f_V = fuzzy function to trim
- c = trimming condition

Bounds of integration for desired FOS value

$u_b := -\infty$ Lower integration limit $l_u := \infty$ Upper integration limit

$u_u := FOS$ Upper integration limit $l_b := FOS$ Lower integration limit

Create sub-array for linear interpolation of end co-ordinates

$$f\_FOSx_{interp} := stack\left(submatrix\left(f\_FOS_x, 0, n, 0, 0\right), submatrix\left(f\_FOS_x, n+2, rows\left(f\_FOS_x\right)-1, 0, 0\right)\right)$$

$$f\_FOSy_{interp} := stack\left(submatrix\left(f\_FOS_y, 0, n, 0, 0\right), submatrix\left(f\_FOS_y, n+2, rows\left(f\_FOS_y\right)-1, 0, 0\right)\right)$$

Define trimming boundary conditions

$$lim\_(f) := u_b \le f \le u_u$$

Trim to the left of bounds

$$trim(f\_V, c) := \begin{array}{|l}
i \leftarrow 0 \\
U \leftarrow 0 \\
\alpha_{ub} \leftarrow \begin{array}{|l} linterp\left(f\_FOSx_{interp}, f\_FOSy_{interp}, u_u\right) \text{ if } f\_FOS_{min}(1) = f\_FOS_{max}(1) \vee FOS < f\_FOS_{min}(1) \\
linterp\left(f\_FOSx_{interp}, f\_FOSy_{interp}, l_b\right) \text{ if } FOS > f\_FOS_{max}(1) \\
1 \text{ otherwise} \end{array} \\
x_{ub} \leftarrow \begin{array}{|l} linterp\left(f\_FOSx_{interp}, f\_FOSx_{interp}, u_u\right) \text{ if } f\_FOS_{min}(1) = f\_FOS_{max}(1) \vee FOS < f\_FOS_{min}(1) \\
linterp\left(f\_FOSx_{interp}, f\_FOSx_{interp}, l_b\right) \text{ if } FOS > f\_FOS_{max}(1) \\
FOS \text{ otherwise} \end{array} \\
\text{for } j \in 0..rows(f\_V) - 1 \\
\quad \text{if } lim\_\left(f\_V_j\right) = 1 \\
\qquad \begin{array}{|l} U_i \leftarrow f\_V_j \\
index \leftarrow lookup\left(U_i, f\_FOS_x, f\_FOS_y\right) \\
T_i \leftarrow index_0 \\
i \leftarrow i + 1 \end{array} \\
return \begin{pmatrix} stack\left(U, x_{ub}\right) \\ stack\left(T, \alpha_{ub}\right) \end{pmatrix}
\end{array}$$

$f\_FOS_{xtl} := trim\left(f\_FOS_x, lim\_\right)_0$

$f\_FOS_{ytl} := trim\left(f\_FOS_x, lim\_\right)_1$
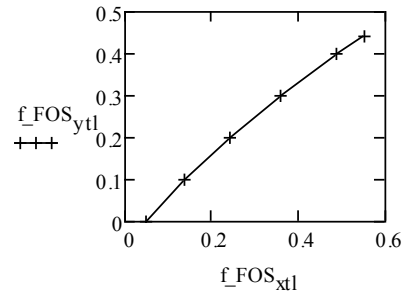
$$\lim\_f(f) := l_b \le f \le l_u$$

Trim to the right of bounds

$$\text{trim}_r(f\_V, c) := \begin{vmatrix} i \leftarrow 0 \\ U \leftarrow 0 \\ \alpha_{lb} \leftarrow \begin{vmatrix} \text{linterp}\left(f\_FOSx_{interp}, f\_FOSy_{interp}, l_b\right) & \text{if } f\_FOS_{min}(1) = f\_FOS_{max}(1) \vee FOS > f\_FOS_{max}(1) \\ \text{linterp}\left(f\_FOSx_{interp}, f\_FOSy_{interp}, u_u\right) & \text{if } FOS < f\_FOS_{min}(1) \\ 1 & \text{otherwise} \end{vmatrix} \\ x_{lb} \leftarrow \begin{vmatrix} \text{linterp}\left(f\_FOSx_{interp}, f\_FOSx_{interp}, l_b\right) & \text{if } f\_FOS_{min}(1) = f\_FOS_{max}(1) \vee FOS > f\_FOS_{max}(1) \\ \text{linterp}\left(f\_FOSx_{interp}, f\_FOSx_{interp}, u_u\right) & \text{if } FOS < f\_FOS_{min}(1) \\ FOS & \text{otherwise} \end{vmatrix} \\ \text{for } j \in 0..\text{rows}(f\_V) - 1 \\ \quad \text{if } \lim\_f\left(f\_V_j\right) = 1 \\ \qquad \begin{vmatrix} U_i \leftarrow f\_V_j \\ \text{index} \leftarrow \text{lookup}\left(U_i, f\_FOS_x, f\_FOS_y\right) \\ T_i \leftarrow \text{index}_0 \\ i \leftarrow i + 1 \end{vmatrix} \\ \text{return } \begin{pmatrix} \text{stack}\left(x_{lb}, U\right) \\ \text{stack}\left(\alpha_{lb}, T\right) \end{pmatrix} \end{vmatrix}$$

$$f\_FOS_{xtr} := \text{trim}_r\left(f\_FOS_x, \lim\_r\right)_0$$

$$f\_FOS_{ytr} := \text{trim}_r\left(f\_FOS_x, \lim\_r\right)_1$$
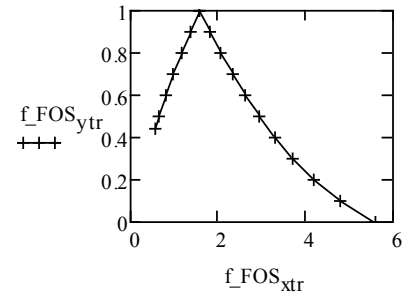
Create numerical integration sub-routines for fuzzy FOS

$$\text{int}_{FOS\_1} := \begin{vmatrix} j \leftarrow 0 \\ i \leftarrow 0 \\ \alpha \leftarrow 0 \\ \beta \leftarrow 0 \\ \alpha_{ub} \leftarrow \text{linterp}\left(f\_FOSx_{interp}, f\_FOSy_{interp}, u_u\right) \\ \text{for } j \in 0..n \qquad\qquad \text{if } \text{rows}\left(f\_FOS_{ytl}\right) > n + 1 \\ \quad \begin{vmatrix} T_j \leftarrow \sum_{i=j}^{\text{rows}(f\_FOS_{xtl})-2} \left[\frac{1}{2} \cdot \left[\left(f\_FOS_{ytl}\right)_{i+1} + \left(f\_FOS_{ytl}\right)_i \right] \cdot \left[\left(f\_FOS_{xtl}\right)_{i+1} - \left(f\_FOS_{xtl}\right)_i \right] \right] \\ \alpha \leftarrow \alpha + \text{inc} \end{vmatrix} \\ \text{for } j \in 0..n \qquad\qquad \text{otherwise} \\ \quad \begin{vmatrix} T_j \leftarrow \begin{vmatrix} \sum_{i=j}^{\text{rows}(f\_FOS_{xtl})-2} \left[\frac{1}{2} \cdot \left[\left(f\_FOS_{ytl}\right)_{i+1} + \left(f\_FOS_{ytl}\right)_i \right] \cdot \left[\left(f\_FOS_{xtl}\right)_{i+1} - \left(f\_FOS_{xtl}\right)_i \right] \right] & \text{if } \alpha < \alpha_{ub} \\ 0 & \text{otherwise} \end{vmatrix} \\ \alpha \leftarrow \alpha + \text{inc} \end{vmatrix} \\ T \end{vmatrix}$$

$$\text{int}_{FOS\_r} := \begin{vmatrix} j \leftarrow 0 \\ i \leftarrow 0 \\ \alpha \leftarrow 0 \\ \beta \leftarrow 0 \\ \alpha_{ub} \leftarrow \text{linterp}\left(f\_FOSx_{interp}, f\_FOSy_{interp}, u_u\right) \\ \text{for } j \in 0..n \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \text{rows}\left(f\_FOS_{ytr}\right) > n+1 \\ \quad \begin{vmatrix} T_j \leftarrow \displaystyle\sum_{i=0}^{\text{rows}\left(f\_FOS_{ytr}\right)-2-j} \left[\frac{1}{2}\cdot\left[\left(f\_FOS_{ytr}\right)_{i+1} + \left(f\_FOS_{ytr}\right)_i\right]\cdot\left[\left(f\_FOS_{xtr}\right)_{i+1} - \left(f\_FOS_{xtr}\right)_i\right]\right] \\ \alpha \leftarrow \alpha + \text{inc} \end{vmatrix} \\ \text{for } j \in 0..n \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \\ \quad \begin{vmatrix} T_j \leftarrow \begin{vmatrix} \displaystyle\sum_{i=0}^{\text{rows}\left(f\_FOS_{ytr}\right)-2-j} \left[\frac{1}{2}\cdot\left[\left(f\_FOS_{ytr}\right)_{i+1} + \left(f\_FOS_{ytr}\right)_i\right]\cdot\left[\left(f\_FOS_{xtr}\right)_{i+1} - \left(f\_FOS_{xtr}\right)_i\right]\right] & \text{if } \alpha < \alpha_{ub} \\ 0 & \text{otherwise} \end{vmatrix} \\ \alpha \leftarrow \alpha + \text{inc} \end{vmatrix} \\ T \end{vmatrix}$$

Check trimmed fuzzy integration sub-routines:

$$\text{Intf\_FOS} := \begin{vmatrix} n \leftarrow \text{rows}\left(f\_FOS_x\right) \\ \displaystyle\sum_{i=0}^{n-2} \left[\frac{1}{2}\cdot\left(f\_FOS_{y_{i+1}} + f\_FOS_{y_i}\right)\cdot\left(f\_FOS_{x_{i+1}} - f\_FOS_{x_i}\right)\right] \end{vmatrix} \qquad = 2.436$$

Integration of full functio

$$\text{intf\_FOS}_{tr} := \begin{vmatrix} n \leftarrow \text{rows}\left(f\_FOS_{xtr}\right) \\ \displaystyle\sum_{i=0}^{n-2} \left[\frac{1}{2}\cdot\left(f\_FOS_{ytr_{i+1}} + f\_FOS_{ytr_i}\right)\cdot\left(f\_FOS_{xtr_{i+1}} - f\_FOS_{xtr_i}\right)\right] \end{vmatrix} \qquad = 2.315$$

Integration to the righ

$$\text{intf\_FOS}_{tl} := \begin{vmatrix} n \leftarrow \text{rows}\left(f\_FOS_{xtl}\right) \\ \displaystyle\sum_{i=0}^{n-2} \left[\frac{1}{2}\cdot\left(f\_FOS_{ytl_{i+1}} + f\_FOS_{ytl_i}\right)\cdot\left(f\_FOS_{xtl_{i+1}} - f\_FOS_{xtl_i}\right)\right] \end{vmatrix} \qquad = 0.121$$

Integration to the le

$$\boxed{\text{sum\_FOS} := \text{intf\_FOS}_{tr} + \text{intf\_FOS}_{tl} = 2.436}$$ *Numerical integration is accurate*

$$AI_\alpha := \frac{\text{intf\_FOS}_{tl}}{\text{sum\_FOS}} = 0.05 \qquad \textbf{Agreement index}$$

Appendix C – MathCAD routine for calculation of fuzzy-Q

This appendix presents the MathCAD routines to calculate the fuzzy Q-value, bolt and strand spacing used in Challenge problem 2, presented in Chapter 6.

By applying interval analysis (as described in section 3.1), and specifically Equations (3.5) and (3.6), the minimum and maximum intervals of Q are obtained by Equation C.1. Similarly, the corresponding interval of required roof support pressure is defined by equation C.2. Both of these intervals are confirmed by numerical computations using the vertex method (described in section 3.2.2).

$$\left[Q^L, Q^U\right] = \left[\frac{RQD^L}{J_n^U} \frac{J_r^L}{J_a^U} \frac{J_w^L}{SRF^U}, \frac{RQD^U}{J_n^L} \frac{J_r^U}{J_a^L} \frac{J_w^{mU}}{SRF^L}\right] \tag{C.1}$$

$$\left[P_{roof}^L, P_{roof}^U\right] = \frac{2\sqrt{J_n^L}Q^{R^{1/3}}}{3J_r^R}, \frac{2\sqrt{J_n^R}Q^{L^{1/3}}}{3J_r^L} \tag{C.2}$$

The required bolt spacing is derived by assuming the strand spacing (S_s) is equal to twice the bolt spacing (S_b). The support pressure provided by each element is equal to the yield load of each element divided by the area over which it acts. Assuming a square pattern, the support pressure is provided by:

$$P_{roof} = \frac{T_b}{S_b^2} + \frac{T_s}{S_s^2} \tag{C.3}$$

Now, setting the strand spacing $S_s = 2S_b$, substituting this in to Equation C.3 and solving this for S_b, the required bolt spacing is obtained by:

$$S_b = \sqrt{\frac{T_b + 0.25T_s}{P_{roof}}} \tag{C.4}$$

Using the interval of required roof support pressure, i.e. Equation C.2, in Equation C.4, the required bolt spacing can be estimated from the Q-value.

C.1 MathCAD routines to compute fuzzy-*Q*

a-cuts of triangular fuzzy numbers

$n := 10$ Number of alpha cuts, range variable of cuts $cuts := 0..n$

$\alpha := 0, \dfrac{1}{n}..1$ membership values of a-cuts

$inc := \dfrac{1}{n} = 0.1$ Increment of a-cuts

$$\mu_Y(x) := \begin{vmatrix} \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b \le x \le c \\ \dfrac{d-x}{d-c} & \text{if } c \le x \le d \end{vmatrix}$$

$\alpha_{RQDmin}(\alpha) := (b_0 - a_0)\alpha + a_0$ $\alpha_{RQDmax}(\alpha) := d_0 - (d_0 - c_0)\cdot\alpha$

$\alpha_{Jnmin}(\alpha) := (b_1 - a_1)\alpha + a_1$ $\alpha_{Jnmax}(\alpha) := d_1 - (d_1 - c_1)\cdot\alpha$

$\alpha_{Jrmin}(\alpha) := (b_2 - a_2)\alpha + a_2$ $\alpha_{Jrmax}(\alpha) := d_2 - (d_2 - c_2)\cdot\alpha$

$\alpha_{Jamin}(\alpha) := (b_3 - a_3)\alpha + a_3$ $\alpha_{Jamax}(\alpha) := d_3 - (d_3 - c_3)\cdot\alpha$

$\alpha_{Jwmin}(\alpha) := (b_4 - a_4)\alpha + a_4$ $\alpha_{Jwmax}(\alpha) := d_4 - (d_4 - c_4)\cdot\alpha$

$\alpha_{SRFmin}(\alpha) := (b_5 - a_5)\alpha + a_5$ $\alpha_{SRFmax}(\alpha) := d_5 - (d_5 - c_5)\cdot\alpha$

Fuzzy variables as a-cut range variables

$f\_RQD_{min}(\alpha) := \alpha_{RQDmin}(\alpha)$

$f\_RQD_{max}(\alpha) := \alpha_{RQDmax}(\alpha)$ $\left.\right\}$ Fuzzy RQD

$f\_Jn_{min}(\alpha) := \alpha_{Jnmin}(\alpha)$

$f\_Jn_{max}(\alpha) := \alpha_{Jnmax}(\alpha)$ $\left.\right\}$ Fuzzy Jn

$f\_Jr_{min}(\alpha) := \alpha_{Jrmin}(\alpha)$

$f\_Jr_{max}(\alpha) := \alpha_{Jrmax}(\alpha)$ $\left.\right\}$ Fuzzy Jr

$f\_Ja_{min}(\alpha) := \alpha_{Jamin}(\alpha)$

$f\_Ja_{max}(\alpha) := \alpha_{Jamax}(\alpha)$ $\left.\right\}$ Fuzzy Ja

$f\_Jw_{min}(\alpha) := \alpha_{Jwmin}(\alpha)$

$f\_Jw_{max}(\alpha) := \alpha_{Jwmax}(\alpha)$ $\left.\right\}$ Fuzzy Jw

$f\_SRF_{min}(\alpha) := \alpha_{SRFmin}(\alpha)$

$f\_SRF_{max}(\alpha) := \alpha_{SRFmax}(\alpha)$ $\left.\right\}$ Fuzzy SRF

Fuzzy functions

$$f\_Q_{min}(\alpha) := min\left(\frac{f\_RQD_{min}(\alpha)}{f\_Jn_{min}(\alpha)} \cdot \frac{f\_Jr_{min}(\alpha)}{f\_Ja_{min}(\alpha)} \cdot \frac{f\_Jw_{min}(\alpha)}{f\_SRF_{min}(\alpha)}, \frac{f\_RQD_{min}(\alpha)}{f\_Jn_{max}(\alpha)} \cdot \frac{f\_Jr_{min}(\alpha)}{f\_Ja_{max}(\alpha)} \cdot \frac{f\_Jw_{min}(\alpha)}{f\_SRF_{max}(\alpha)}\right)$$

$$f\_Q_{max}(\alpha) := max\left(\frac{f\_RQD_{max}(\alpha)}{f\_Jn_{min}(\alpha)} \cdot \frac{f\_Jr_{max}(\alpha)}{f\_Ja_{min}(\alpha)} \cdot \frac{f\_Jw_{max}(\alpha)}{f\_SRF_{min}(\alpha)}, \frac{f\_RQD_{max}(\alpha)}{f\_Jn_{max}(\alpha)} \cdot \frac{f\_Jr_{max}(\alpha)}{f\_Ja_{max}(\alpha)} \cdot \frac{f\_Jw_{max}(\alpha)}{f\_SRF_{max}(\alpha)}\right)$$

$\left.\right\}$ Fuzzy Q

Figure C.1: Fuzzy inputs and calculated fuzzy-*Q*

Fuzzy functions for bolt spacing

Note: Factor of 11.023 applied to convert original units of kg/m^3 used by Grimstad & Barton (2003) to tons/m^2 used by Barton et al. (1994) for the Gjøvik cavern design.

$$\text{f\_Proof}_{min}(\alpha) := 11.023 \frac{2\sqrt{\text{f\_Jn}_{min}(\alpha) \cdot \text{f\_Q}_{max}(\alpha)}^{\left(\frac{-1}{3}\right)}}{3\,\text{f\_Jr}_{max}(\alpha)}$$

$$\text{f\_Proof}_{max}(\alpha) := 11.023 \frac{2\sqrt{\text{f\_Jn}_{max}(\alpha) \cdot \text{f\_Q}_{min}(\alpha)}^{\left(\frac{-1}{3}\right)}}{3\,\text{f\_Jr}_{min}(\alpha)}$$

} Fuzzy roof support pressure requried

Fuzzy P.roof

$\frac{\alpha}{\alpha}$

$\text{f\_Proof}_{min}(\alpha), \text{f\_Proof}_{max}(\alpha)$
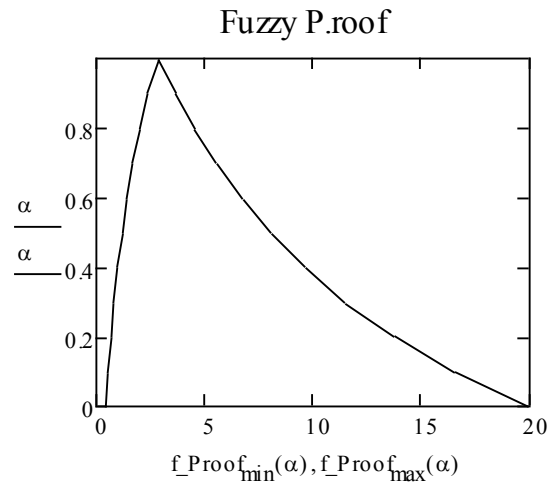
Fuzzy P.roof function triple (lower, mode, upper):

$$\text{Proof}_{fuz} := \left[\left(\text{f\_Proof}_{min}(0)\right) \quad \left(\text{f\_Proof}_{min}(1)\right) \quad \left(\text{f\_Proof}_{max}(1)\right) \quad \left(\text{f\_Proof}_{max}(0)\right) \right] = (0.444 \; 2.897 \; 2.897 \; 19.916)$$

Figure C.2: Fuzzy roof support pressure calculated from *Q*

Calculate fuzzy support requirements

$T_{bf} := 22$ Bolt yield strength

$T_{sf} := 16.7$ Strand yeild strength

$$f\_sb_{min}(\alpha) := min\left(\sqrt{\frac{T_{bf} + 0.25T_{sf}}{f\_Proof_{min}(\alpha)}}, \sqrt{\frac{T_{bf} + 0.25T_{sf}}{f\_Proof_{max}(\alpha)}}\right)$$

$$f\_sb_{max}(\alpha) := max\left(\sqrt{\frac{T_{bf} + 0.25T_{sf}}{f\_Proof_{min}(\alpha)}}, \sqrt{\frac{T_{bf} + 0.25T_{sf}}{f\_Proof_{max}(\alpha)}}\right)$$

} Fuzzy bolt spacing

$$f\_ss_{min}(\alpha) := 2min\left(\sqrt{\frac{T_{bf} + 0.25T_{sf}}{f\_Proof_{min}(\alpha)}}, \sqrt{\frac{T_{bf} + 0.25T_{sf}}{f\_Proof_{max}(\alpha)}}\right)$$

$$f\_ss_{max}(\alpha) := 2max\left(\sqrt{\frac{T_{bf} + 0.25T_{sf}}{f\_Proof_{min}(\alpha) - p_{sc}}}, \sqrt{\frac{T_{bf} + 0.25T_{sf}}{f\_Proof_{max}(\alpha) - p_{sc}}}\right)$$

} Fuzzy strand spacing

Fuzzy bolt & strand spacing
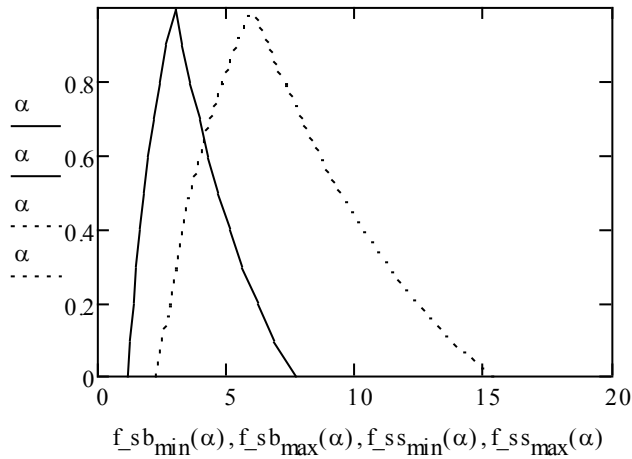
$$s_{b.fuz} := \left[\left(f\_sb_{min}(0)\right) \ \left(f\_sb_{min}(1)\right) \ \left(f\_sb_{max}(1)\right) \ \left(f\_sb_{max}(0)\right)\right] = (1.146 \ 3.006 \ 3.006 \ 7.676)$$

$$s_{s.fuz} := \left[\left(f\_ss_{min}(0)\right) \ \left(f\_ss_{min}(1)\right) \ \left(f\_ss_{max}(1)\right) \ \left(f\_ss_{max}(0)\right)\right] = (2.293 \ 6.012 \ 6.012 \ 15.351)$$

Figure C.3: Fuzzy bolt and strand spacing calculated from *Q*

Appendix D – MathCAD functions for hybrid analysis of peak strength using Hoek-Brown failure criterion

D.1 Strength of rock mass – extrinsically epistemic

Set up discretisation vectors

$n := 1000$ Number of discretisations of each p-box

Create vector for ordinate of cumulative density function (CDF) with 'n' discretisations   *Create vector of s.ci*

$$v\_P := \begin{vmatrix} v \leftarrow 0 \\ j \leftarrow 0 \\ i \leftarrow 0 \\ \text{for } j \in 1..999 \\ \quad \begin{vmatrix} i \leftarrow \dfrac{j}{1000} \\ v_0 \leftarrow 0 \\ v_n \leftarrow 1 \\ v_j \leftarrow i \end{vmatrix} \\ \text{return } v \end{vmatrix} \qquad v\_\sigma_{ci}(\mu, \sigma) := \begin{vmatrix} v \leftarrow 0 \\ j \leftarrow 0 \\ i \leftarrow 0 \\ \text{for } j \in 1..n-1 \\ \quad \begin{vmatrix} i \leftarrow \dfrac{j}{n} \\ v_0 \leftarrow \text{qnorm}(0.0001, \mu, \sigma) \\ v_n \leftarrow \text{qnorm}(0.9999, \mu, \sigma) \\ v_j \leftarrow \text{qnorm}(i, \mu, \sigma) \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

Define functions to create p-box from intervals

Create p-box vector of interval_mi

$$v\_int_L(a, b) := \begin{vmatrix} v \leftarrow 0 \\ i \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 1..n-1 \\ \quad \begin{vmatrix} i \leftarrow \dfrac{j}{n} \\ v_0 \leftarrow a \\ v_n \leftarrow a + 0.00001 \\ v_j \leftarrow \text{qunif}(i, a, a + 0.00001) \end{vmatrix} \\ \text{return } v \end{vmatrix} \qquad v\_int_R(a, b) := \begin{vmatrix} v \leftarrow 0 \\ i \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 1..n-1 \\ \quad \begin{vmatrix} i \leftarrow \dfrac{j}{n} \\ v_0 \leftarrow b \\ v_n \leftarrow b + 0.00001 \\ v_j \leftarrow \text{qunif}(i, b, b + 0.00001) \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

Define functions to create p-box from fuzzy numbers

The L-R fuzzy numbers are defined as detailed in Chapter 3 of this thesis, i.e. fuz = [a,

$$v\_fuz_L(a,b,c,d) := \begin{vmatrix} v \leftarrow 0 \\ i \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 1..n-1 \\ \begin{vmatrix} i \leftarrow \dfrac{j}{n} \\ v_0 \leftarrow a \\ v_n \leftarrow b \\ v_j \leftarrow qunif(i,a,b) \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

$$v\_fuz_R(a,b,c,d) := \begin{vmatrix} v \leftarrow 0 \\ i \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 1..n-1 \\ \begin{vmatrix} i \leftarrow \dfrac{j}{n} \\ v_0 \leftarrow c \\ v_n \leftarrow d \\ v_j \leftarrow qunif(i,c,d) \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

Input fuzzy GSI

$f\_GSI := (55\ \ 70\ \ 80)$ fuzzy min, mode & max

Create p-boxes of fuzzy GSI and calculate m & s

$a := f\_GSI_{0,0}$ $b := f\_GSI_{0,1}$ $c := f\_GSI_{0,1}$ $d := f\_GSI_{0,2}$

$vf\_GSI_L := v\_fuz_L(a,b,c,d)$ $vf\_GSI_R := v\_fuz_R(a,b,c,d)$

Create p-box vectors of m.b and s

$$v\_m_{bL} := \overrightarrow{\left(vi\_mi_L \cdot \exp\left(\frac{vf\_GSI_L - 100}{28} \right) \right)}$$

$$v\_m_{bR} := \overrightarrow{\left(vi\_mi_R \cdot \exp\left(\frac{vf\_GSI_R - 100}{28} \right) \right)}$$

$$v\_s_{GSI.L} := \overrightarrow{\exp\left(\frac{vf\_GSI_L - 100}{9} \right)}$$

$$v\_s_{GSI.R} := \overrightarrow{\exp\left(\frac{vf\_GSI_R - 100}{9} \right)}$$
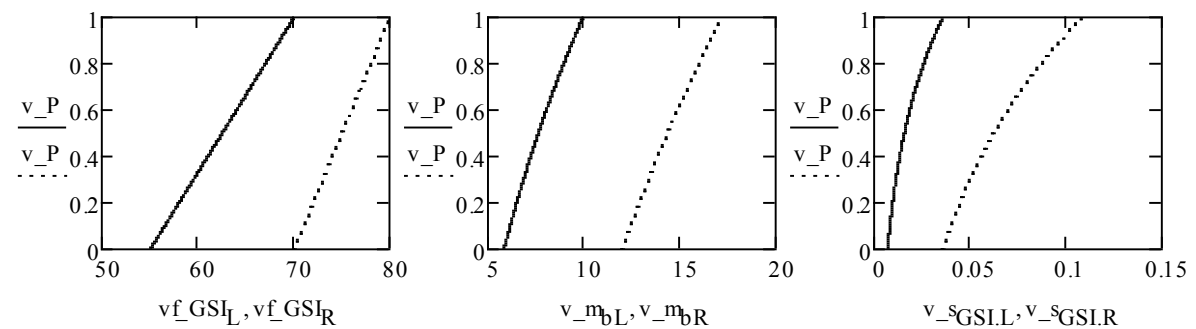
Figure D.1: P-boxes generated from possibility distributions of fuzzy numbers.

Create vectors of random numbers for hybrid Monte-Carlo simulation

$k := 5000$ Input number of iterations for Monte-Carlo simulation

$bins := round(\sqrt{n})$ Set bins

Function to generate vector of random variables from p-boxed calculated from fuzzy distribu

$$
v\_rand(v\_in, k, n) :=
\begin{vmatrix}
v \leftarrow 0 \\
i \leftarrow 0 \\
j \leftarrow 0 \\
\text{for } j \in 0..k-1 \\
\quad
\begin{vmatrix}
i \leftarrow rnd(n-1) \\
i \leftarrow round(i) \\
v_0 \leftarrow v\_in_{0,0} \\
v_{k-1} \leftarrow v\_in_{n,0} \\
v_j \leftarrow v\_in_{i,0}
\end{vmatrix} \\
\text{return } v
\end{vmatrix}
$$

Generate random variables from p-boxes derived from fuzzy distributions

$m_{bL} := v\_rand(v\_m_{bL}, k, n)$ $m_{bR} := v\_rand(v\_m_{bR}, k, n)$

$GSI_L := v\_rand(vf\_GSI_L, k, n)$ $GSI_R := v\_rand(vf\_GSI_L, k, n)$

$s_{GSI.L} := v\_rand(v\_s_{GSI.L}, k, n)$ $s_{GSI.R} := v\_rand(v\_s_{GSI.L}, k, n)$

$\sigma_{ci} := rnorm(k, \mu, \sigma)$ Vector containing uniform random variables for UCS

                Calculate Pl(s.1)   *Calculate Bel(s.1)*

$$
\sigma_{1\_GSI.L} := \overrightarrow{\left[\sigma_3 + \sigma_{ci} \cdot \left(m_{bL} \cdot \frac{\sigma_3}{\sigma_{ci}} + s_{GSI.L} \right)^{0.5} \right]}
\qquad
\sigma_{1\_GSI.R} := \overrightarrow{\left[\sigma_3 + \sigma_{ci} \cdot \left(m_{bR} \cdot \frac{\sigma_3}{\sigma_{ci}} + s_{GSI.R} \right)^{0.5} \right]}
$$

Sort upper and lower bound results from Monte-Carlo results and define historgrams for plot

$$\text{v\_h}\sigma_{1\_GSI.L} := \text{sort}\left(\sigma_{1\_GSI.L}\right) \qquad\qquad \text{v\_h}\sigma_{1\_GSI.R} := \text{sort}\left(\sigma_{1\_GSI.R}\right)$$

$$\text{h\_}\sigma_{1\_GSI.L} := \text{histogram}\left(\text{bins}, \sigma_{1\_GSI.L}\right) \qquad\qquad \text{h\_}\sigma_{1\_GSI.R} := \text{histogram}\left(\text{bins}, \sigma_{1\_GSI.R}\right)$$

Set up numerical integration to create CDF from PDFs produced through Monte-Carlo simulation

$$\_cdf(\text{in\_hist}) := \left|
\begin{array}{l}
v \leftarrow 0 \\
j \leftarrow 0 \\
i \leftarrow 0 \\
h \leftarrow 0 \\
\text{for } j \in 0..(\text{bins}-1) \\
\quad \left|
\begin{array}{l}
h \leftarrow \text{histogram}(\text{bins}, \text{in\_hist}) \\
v_j \leftarrow \displaystyle\sum_{i=0}^{j} \frac{h_{i,1}}{k}
\end{array}
\right. \\
\text{return } v
\end{array}
\right.$$

$$\text{cdf\_}\sigma_{1\_GSI.L} := \_cdf\left(\sigma_{1\_GSI.L}\right)$$

$$\text{cdf\_}\sigma_{1\_GSI.R} := \_cdf\left(\sigma_{1\_GSI.R}\right)$$
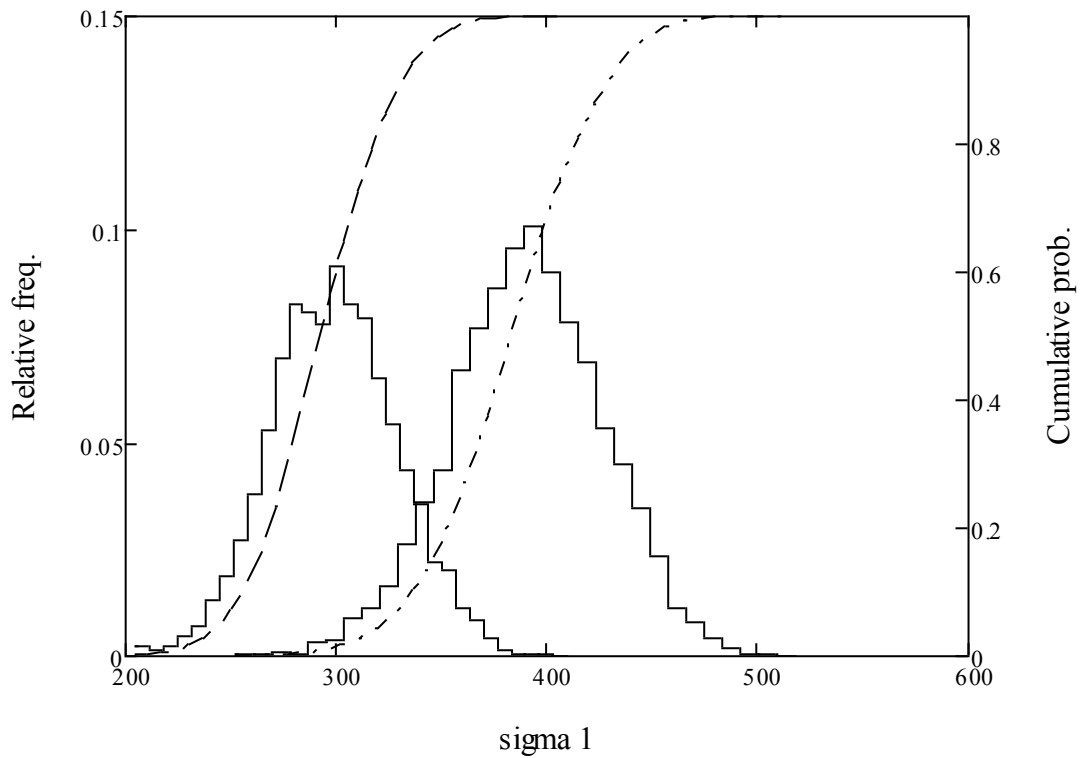
Figure D.2: P-boxes of rock mass strength and histograms from hybrid Monte-Carlo simulation.

D.2 Strength of Intact rock – extrinsically epistemic

Inputs

s.ci is aleatory and defined by a normal distribution.

$\mu := 158$ $\sigma := 28$ Moments to define UCS - units of MPa

$\sigma_3 := 50$ Assume s3 is deterministic

$s := 1$ s = 1 for intact rock

mi is defined by an interval

$m_i := (29 \ 35)$ Interval of mi

Define functions for PDF and CDF of normal distribution

$x := 0, 1 .. 300$

$$f_{\sigma ci}(x) := \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left[\frac{-1}{2\sigma^2} \cdot (x - \mu)^2\right]$$

$$F_{\sigma ci}(x) := \int_{-\infty}^{x} f_{\sigma ci}(x)\, dx$$
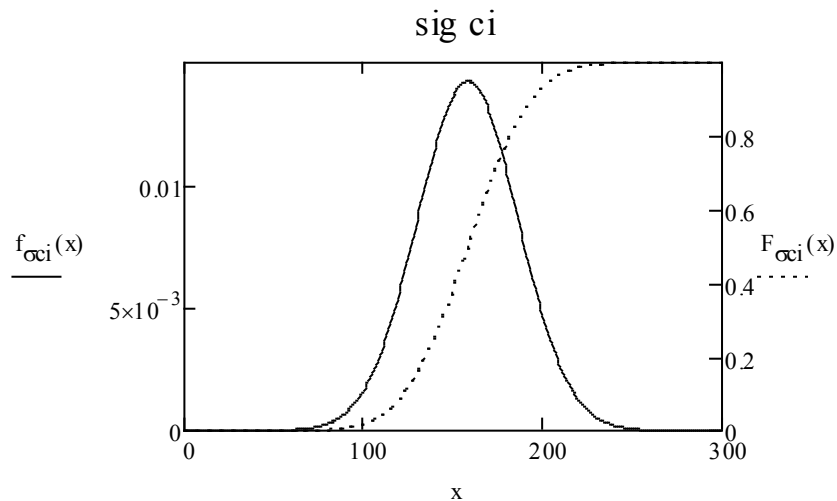
Figure D.3: Uniaxial compressive strength defined by normal distribution

Define functions to calculate p-box of s.1

Create p-box vector of interval_mi

Note: We can apply interval analysis at every probablity level to simply construct the p-box in t
analysis. The lower and upper fractile values for s.1 are then given by the following limiting fu

$\sigma_{ci} := v\_\sigma_{ci}(\mu, \sigma)$

Vector containing CDF of s.ci

$vi\_mi_L := v\_int_L\left(m_{i_{0,0}}, m_{i_{0,1}}\right)$

Vector containing lower bound of m.i

$vi\_mi_R := v\_int_R\left(m_{i_{0,0}}, m_{i_{0,1}}\right)$

Vector containing upper bound of m.i

$$\sigma_{1L} := \overrightarrow{\left[\sigma_3 + \sigma_{ci} \cdot \left(vi\_mi_L \cdot \frac{\sigma_3}{\sigma_{ci}} + s\right)^{0.5}\right]}$$

Lower bound of p-box

$$\sigma_{1U} := \overrightarrow{\left[\left[\sigma_3 + \sigma_{ci} \cdot \left(vi\_mi_R \cdot \frac{\sigma_3}{\sigma_{ci}} + s\right)^{0.5}\right]\right]}$$

Upper bound of p-box
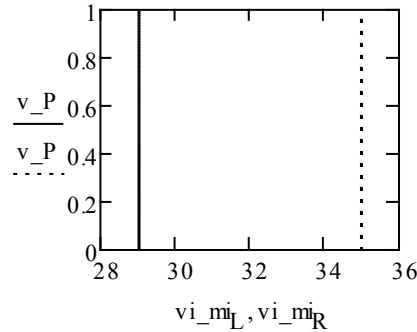
Figure D.4: P-box of interval of m_i
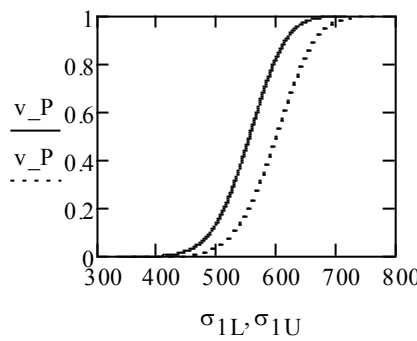
Figure D.5: P-box of intact rock strength using interval of m_i

D.2.1 *Refining the precision of the aleatory model*

$\mu\_new := 159$ $\sigma\_new := 25$ Moments to define UCS - units of MPa

Define functions to calculate p-box of s.1

Create p-box vector of interval_mi

Note: We can apply interval analysis at every probablity level to simply construct the p-box in t
analysis. The lower and upper fractile values for s.1 are then given by the following limiting fur

$\sigma_{ci\_new} := v\_\sigma_{ci}(\mu\_new, \sigma\_new)$ Vector containing updated CDF of s.ci

$$\sigma_{1L\_new} := \overrightarrow{\left[\sigma_3 + \sigma_{ci\_new} \cdot \left(vi\_m_L \cdot \frac{\sigma_3}{\sigma_{ci\_new}} + s \right)^{0.5} \right]}$$ Lower bound of p-box

$$\sigma_{1U\_new} := \overrightarrow{\left[\overrightarrow{\left[\sigma_3 + \sigma_{ci\_new} \cdot \left(vi\_m_R \cdot \frac{\sigma_3}{\sigma_{ci\_new}} + s \right)^{0.5} \right]} \right]}$$ Upper bound of p-box
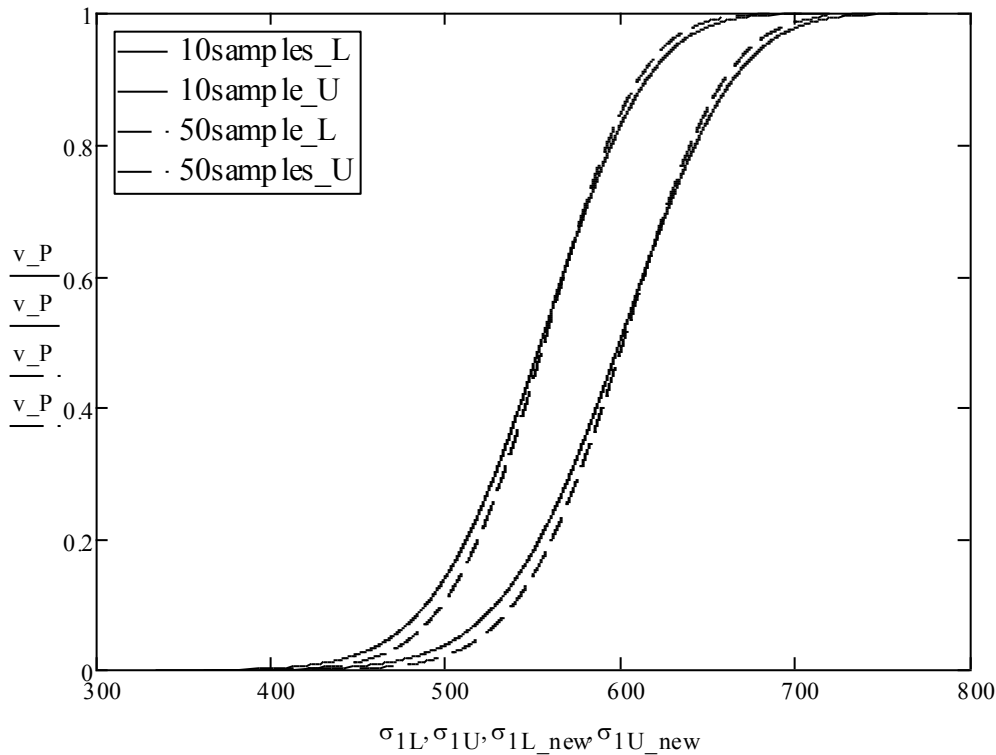
Figure D.6: P-box of intact rock strength using updated aleatory model of UCS.

D.2.2 Reducing epistemic uncertainty

$$m_{i\_new} := (33 \quad 35)$$ Updated interval of mi

Define functions to calculate p-box of s.1

Create p-box vector of interval_mi

Note: We can apply interval analysis at every probablity level to simply construct the p-box in t analysis. The lower and upper fractile values for s.1 are then given by the following limiting fu

$$vi\_mi\_new_L := v\_int_L\left(m_{i\_new_{0,0}}, m_{i\_new_{0,1}}\right)$$

$$vi\_mi\_new_R := v\_int_R\left(m_{i\_new_{0,0}}, m_{i\_new_{0,1}}\right)$$

$$\sigma_{1L\_mi} := \overrightarrow{\left[\sigma_3 + \sigma_{ci} \cdot \left(vi\_mi\_new_L \cdot \frac{\sigma_3}{\sigma_{ci}} + s\right)^{0.5}\right]}$$ Lower bound of p-box

$$\sigma_{1U\_mi} := \overrightarrow{\left[\left[\sigma_3 + \sigma_{ci} \cdot \left(vi\_mi\_new_R \cdot \frac{\sigma_3}{\sigma_{ci}} + s\right)^{0.5}\right]\right]}$$ Upper bound of p-box
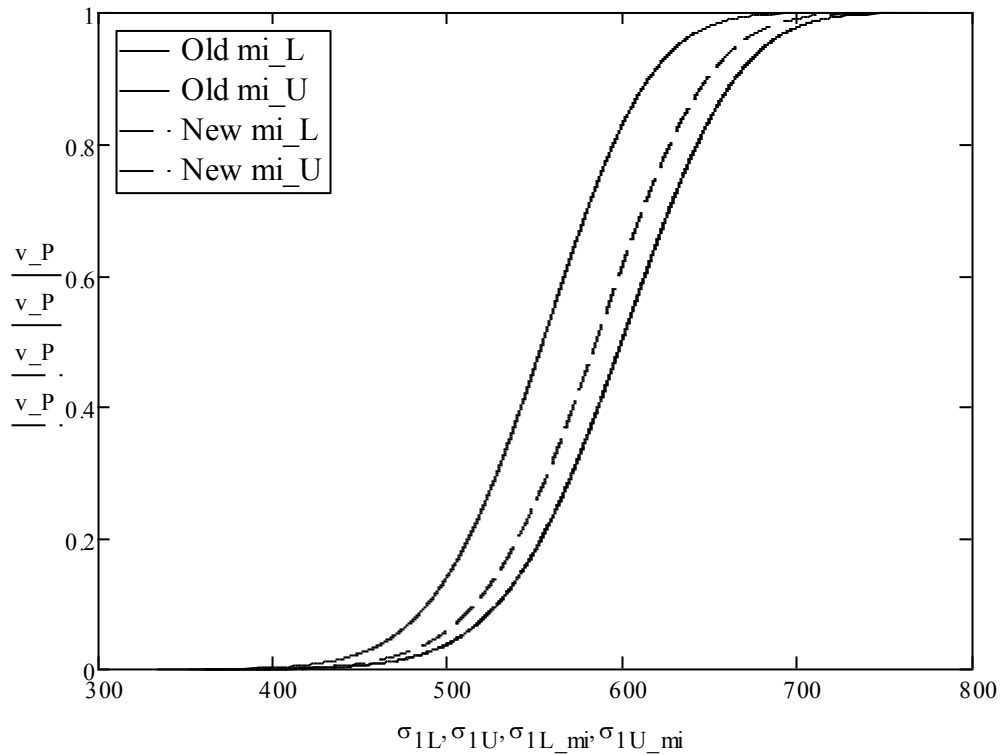
Figure D.7: P-box of intact rock strength using reducing interval of m_i.

D.3 Rock spalling around circular opening in intact rock

Inputs

s.ci is aleatory and defined by a normal distribution.

$\mu := 158$ $\sigma := 28$ Moments to define UCS - units of MPa

$\sigma_3 := 0$ Assume s3 is deterministic

$s := 1$ s = 1 for intact rock

mi is defined by an interval

$k_{stress} := (0.3 \quad 2)$ Interval of k

$\sigma_v := 500 \cdot 0.027 = 13.5$

Set up discretisation vectors

$n := 1000$ Number of discretisations of each p-box

Create vector for ordinate of cumulative density   *Create vector of s.ci*
function (CDF) with 'n' discretisations

$v_P := $
$$
\begin{aligned}
&v \leftarrow 0 \\
&j \leftarrow 0 \\
&i \leftarrow 0 \\
&\text{for } j \in 1..999 \\
&\quad \left| \begin{aligned}
&i \leftarrow \frac{j}{1000} \\
&v_0 \leftarrow 0 \\
&v_n \leftarrow 1 \\
&v_j \leftarrow i
\end{aligned} \right. \\
&\text{return } v
\end{aligned}
$$

$v_\sigma_{ci}(\mu, \sigma) := $
$$
\begin{aligned}
&v \leftarrow 0 \\
&j \leftarrow 0 \\
&i \leftarrow 0 \\
&\text{for } j \in 1..n-1 \\
&\quad \left| \begin{aligned}
&i \leftarrow \frac{j}{n} \\
&v_0 \leftarrow \text{qnorm}(0.0001, \mu, \sigma) \\
&v_n \leftarrow \text{qnorm}(0.9999, \mu, \sigma) \\
&v_j \leftarrow \text{qnorm}(i, \mu, \sigma)
\end{aligned} \right. \\
&\text{return } v
\end{aligned}
$$

Define functions to create p-box from intervals

Create p-box vector of interval_mi

$v_int_L(a, b) := $
$$
\begin{aligned}
&v \leftarrow 0 \\
&i \leftarrow 0 \\
&j \leftarrow 0 \\
&\text{for } j \in 1..n-1 \\
&\quad \left| \begin{aligned}
&i \leftarrow \frac{j}{n} \\
&v_0 \leftarrow a \\
&v_n \leftarrow a + 0.00001 \\
&v_j \leftarrow \text{qunif}(i, a, a + 0.00001)
\end{aligned} \right. \\
&\text{return } v
\end{aligned}
$$

$v_int_R(a, b) := $
$$
\begin{aligned}
&v \leftarrow 0 \\
&i \leftarrow 0 \\
&j \leftarrow 0 \\
&\text{for } j \in 1..n-1 \\
&\quad \left| \begin{aligned}
&i \leftarrow \frac{j}{n} \\
&v_0 \leftarrow b \\
&v_n \leftarrow b + 0.00001 \\
&v_j \leftarrow \text{qunif}(i, b, b + 0.00001)
\end{aligned} \right. \\
&\text{return } v
\end{aligned}
$$

Define functions to calculate p-box of s.1

Create p-box vector of interval_mi

Note: We can apply interval analysis at every probablity level to simply construct the p-box in t
analysis. The lower and upper fractile values for s.1 are then given by the following limiting fur

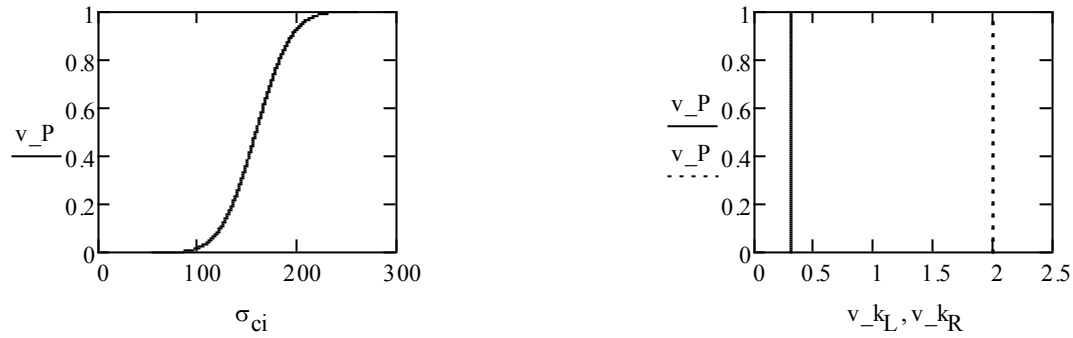$$\sigma_{ci} := v\_\sigma_{ci}(\mu, \sigma)$$

Vector containing CDF of s.ci

$$v\_k_L := v\_int_L\left(k_{stress_{0,0}}, k_{stress_{0,1}}\right)$$

Vector containing lower bound of k

$$v\_k_R := v\_int_R\left(k_{stress_{0,0}}, k_{stress_{0,1}}\right)$$
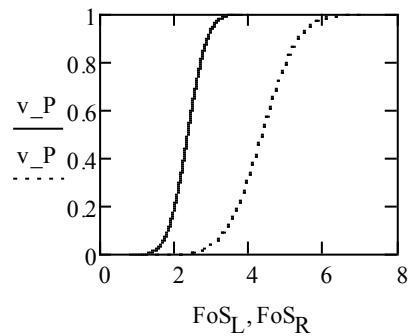
Vector containing upper bound of k

Figure D.8: Degenerate P-box of UCS and interval P-box of *k*.

$$FoS = \frac{\sigma_{spall}}{\sigma_v \cdot (3 - k)} \qquad \text{if } k < 1$$

$$FoS = \frac{\sigma_{spall}}{\sigma_v \cdot (3k - 1)} \qquad \text{if } k > 1$$

$$FoS_L := \overrightarrow{\left[\frac{\sigma_{ci}}{\sigma_v \cdot \left(3 \cdot v\_k_R - 1\right)}\right]} \qquad \text{Lower bound of p-box}$$

$$FoS_R := \overrightarrow{\left[\frac{\sigma_{ci}}{\sigma_v \cdot \left(3 - v\_k_L\right)}\right]} \qquad \text{Upper bound of p-box}$$

Figure D.8: P-box FoS for spalling in intact rock.

D.4   Rock spalling around circular opening in fractured rock mass

Define functions to create p-box from fuzzy numbers

The L-R fuzzy numbers are defined as detailed in Chapter 3 of this thesis, i.e. fuz = [a,

$$v\_fuz_L(a,b,c,d) :=
\begin{vmatrix}
v \leftarrow 0 \\
i \leftarrow 0 \\
j \leftarrow 0 \\
\text{for } j \in 1..n-1 \\
\quad
\begin{vmatrix}
i \leftarrow \dfrac{j}{n} \\
v_0 \leftarrow a \\
v_n \leftarrow b \\
v_j \leftarrow qunif(i,a,b)
\end{vmatrix} \\
\text{return } v
\end{vmatrix}
\qquad
v\_fuz_R(a,b,c,d) :=
\begin{vmatrix}
v \leftarrow 0 \\
i \leftarrow 0 \\
j \leftarrow 0 \\
\text{for } j \in 1..n-1 \\
\quad
\begin{vmatrix}
i \leftarrow \dfrac{j}{n} \\
v_0 \leftarrow c \\
v_n \leftarrow d \\
v_j \leftarrow qunif(i,c,d)
\end{vmatrix} \\
\text{return } v
\end{vmatrix}$$

Input fuzzy GSI

$$f\_GSI := (55 \quad 70 \quad 80) \qquad \text{fuzzy min, mode \& max}$$

Create p-boxes of fuzzy GSI and calculate s

$$a := f\_GSI_{0,0} \qquad b := f\_GSI_{0,1} \qquad c := f\_GSI_{0,1} \qquad d := f\_GSI_{0,2}$$

$$vf\_GSI_L := v\_fuz_L(a,b,c,d) \qquad vf\_GSI_R := v\_fuz_R(a,b,c,d)$$

Create p-box vectors of s

$$v\_s_{GSI.L} := \overrightarrow{\exp\left(\frac{vf\_GSI_L - 100}{9}\right)} \qquad v\_s_{GSI.R} := \overrightarrow{\exp\left(\frac{vf\_GSI_R - 100}{9}\right)}$$
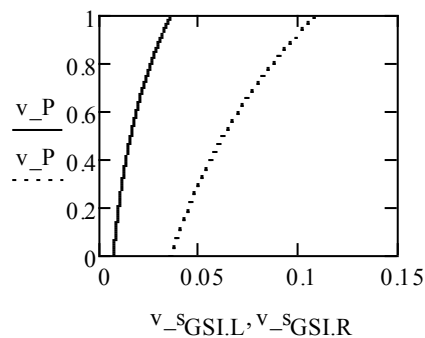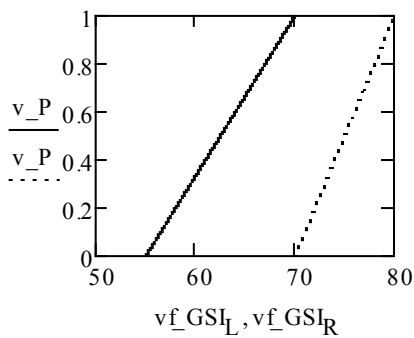

Figure D.8: P-boxes of GSI and s for rock mass.

Create vectors of random numbers for hybrid Monte-Carlo simulation

$k := 5000$ Input number of iterations for Monte-Carlo simulation

$bins := round(\sqrt{n})$ Set bins

Function to generate vector of random variables from p-boxed calculated from fuzzy distribu

$$v\_rand(v\_in, k, n) := \begin{vmatrix} v \leftarrow 0 \\ i \leftarrow 0 \\ j \leftarrow 0 \\ \text{for } j \in 0..k-1 \\ \quad \begin{vmatrix} i \leftarrow rnd(n-1) \\ i \leftarrow round(i) \\ v_0 \leftarrow v\_in_{0,0} \\ v_{k-1} \leftarrow v\_in_{n,0} \\ v_j \leftarrow v\_in_{i,0} \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

Generate random variables from p-boxes derived from fuzzy distributions

$GSI_L := v\_rand(vf\_GSI_L, k, n)$ $GSI_R := v\_rand(vf\_GSI_L, k, n)$

$s_{GSI.L} := v\_rand(v\_s_{GSI.L}, k, n)$ $s_{GSI.R} := v\_rand(v\_s_{GSI.L}, k, n)$

$\sigma_{ci} := rnorm(k, \mu, \sigma)$ Vector containing uniform random variables for UCS

            Calculate Pl(s.1)   *Calculate Bel(s.1)*

$$\sigma_{ci\_GSI.L} := \overrightarrow{\left(\sqrt{\sigma_{ci}^2 \cdot s_{GSI.L}}\right)} \qquad\qquad \sigma_{ci\_GSI.R} := \overrightarrow{\left(\sqrt{\sigma_{ci}^2 \cdot s_{GSI.R}}\right)}$$

$$FoS_L := \overrightarrow{\left[\overrightarrow{\left[\frac{\sigma_{ci\_GSI.L}}{\sigma_v \cdot \left(3 \cdot k_{stress_{0,1}} - 1\right)}\right]}\right]}$$ Lower bound of p-box

$$FoS_R := \overrightarrow{\left[\frac{\sigma_{ci\_GSI.R}}{\sigma_v \cdot \left(3 - k_{stress_{0,0}}\right)}\right]}$$ Upper bound of p-box

Sort upper and lower bound results from Monte-Carlo results
and define historgrams for plot

$$\text{v\_h}\sigma_{1\_GSI.L} := \text{sort}(FoS_L) \qquad\qquad \text{v\_h}\sigma_{1\_GSI.R} := \text{sort}(FoS_R)$$

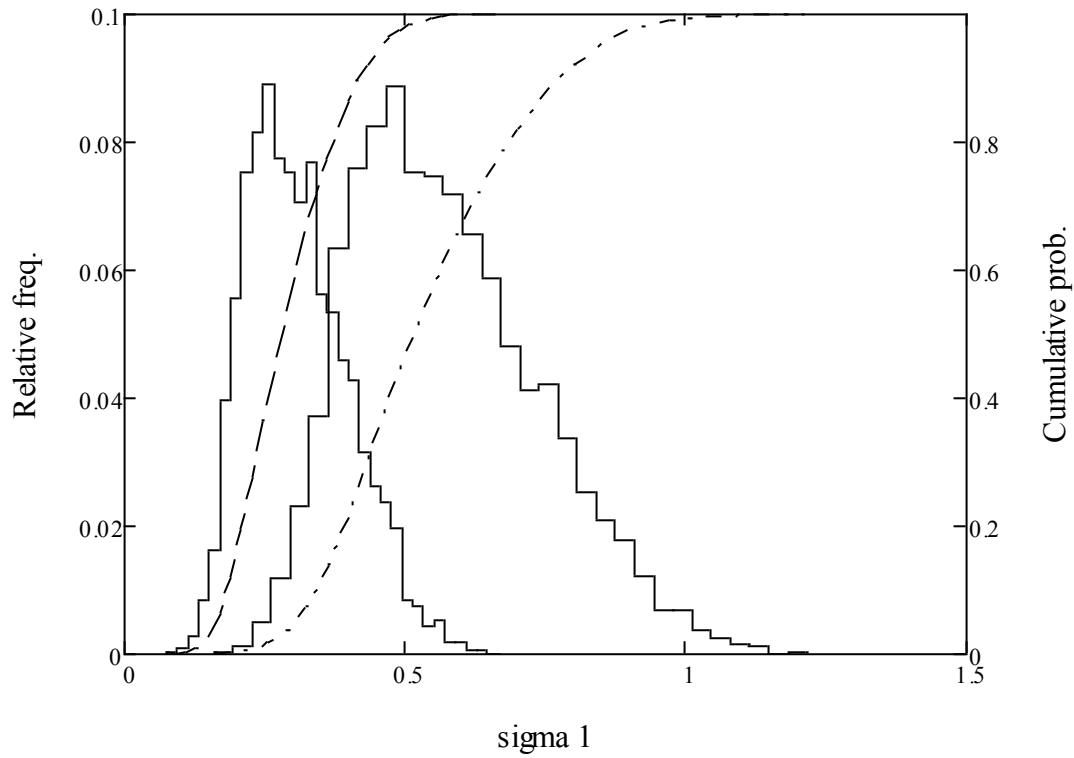$$\text{h\_}\sigma_{1\_GSI.L} := \text{histogram}(bins, FoS_L) \qquad\qquad \text{h\_}\sigma_{1\_GSI.R} := \text{histogram}(bins, FoS_R)$$

Set up numerical integration to create CDF from PDFs
produced through Monte-Carlo simulation

$$\_cdf(in\_hist) := \begin{vmatrix} v \leftarrow 0 \\ j \leftarrow 0 \\ i \leftarrow 0 \\ h \leftarrow 0 \\ \text{for } j \in 0..(bins - 1) \\ \quad \begin{vmatrix} h \leftarrow \text{histogram}(bins, in\_hist) \\ v_j \leftarrow \sum_{i=0}^{j} \dfrac{h_{i,1}}{k} \end{vmatrix} \\ \text{return } v \end{vmatrix}$$

$$\text{cdf\_}\sigma_{1\_GSI.L} := \_cdf(FoS_L)$$

$$\text{cdf\_}\sigma_{1\_GSI.R} := \_cdf(FoS_R)$$

Figure D.9: P-boxes of FoS for spalling in fractured rock mass from hybrid Monte-Carlo simulation.

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