Spectral Bounds for Infinite Dimensional Polydiagonal Symmetric Matrix Operators on Discrete Spaces

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Arman Šahović

Department of Mathematics

Imperial College

180 Queen's Gate, London SW7 2BZ

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Table of contents

Abstract

In this thesis, we prove a variety of discrete Agmon–Kolmogorov inequalities and apply them to prove Lieb–Thirring inequalities for discrete Schrödinger operators on $\ell^2(\mathbb{Z})$. We generalise these results in two ways: Firstly, to higher order difference operators, leading to spectral bounds for Tri-, Penta- and Polydiagonal Jacobi-type matrix operators. Secondly, to ℓ^2-spaces on higher dimensional domains, specifically on $\ell^2(\mathbb{Z}^2)$, $\ell^2(\mathbb{Z}^3)$ and finally $\ell^2(\mathbb{Z}^d)$.

In the Introduction we discuss previous work on Landau–Kolmogorov inequalities on a variety of Banach Spaces, Lieb–Thirring inequalities in $L^2(\mathbb{R}^d)$, and the use of Jacobi Matrices in relation to the discrete Schrödinger Operator. We additionally give our main results with some introduction to the notation at hand.

Chapters 2, 3 and 4 follow a similar structure. We first introduce the relevant difference operators and examine their properties. We then move on to prove the Agmon–Kolmogorov and Generalised Sobolev inequalities over $\mathbb{Z}$ of order 1, 2 and σ respectively. Furthermore, we prove the Lieb–Thirring inequality for the respective discrete Schrödinger-type operators, which we subsequently lift to arbitrary moments. Finally we apply this inequality to obtain spectral bounds for tri-, penta- and polydiagonal matrices.

In Chapter 5, we prove a variety of Agmon–Kolmogorov inequalities on $\ell^2(\mathbb{Z}^2)$ and $\ell^2(\mathbb{Z}^3)$. We use these intuitive ideas to obtain 2^{d-1} Agmon–Kolmogorov inequalities on $\ell^2(\mathbb{Z}^d)$. We continue from here in the same manner as before and prove the discrete Generalised Sobolev and Lieb–Thirring inequalities for a variety of exponent combinations on $\ell^2(\mathbb{Z}^d)$.

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Dedication

To my mother and father, for giving me life;
and to my mother again, because otherwise
she would take it away.

List of Symbols

$\mathbb{N}$ the set of natural numbers

$\overline{\mathbb{N}}$ the set $\mathbb{N} \cup \{+\infty\}$

$\mathbb{Z}$ the set of integers

$\mathbb{R}$ the set of real numbers

$\mathbb{C}$ the set of complex numbers

ℓ^p the space of sequences whose p^{th} power is summable

ℓ^∞ the space of all bounded sequences

L^p the space of functions whose p^{th} power is integrable

L^∞ the space of essentially bounded functions

δ_{ij} the Kronecker delta symbol

$\|.\|_B$ the norm over a Banach space B

${}^aC_b$ the binomial coefficient

D the difference operator

D^* the adjoint of the difference operator

H_D the discrete Schrödinger operator

H_D^d the multidimensional discrete Schrödinger operator

Δ_D the discrete Laplacian operator

∇_D the discrete divergence operator on $\ell^2(\mathbb{Z}^d)$

$\sigma_{ess}(A)$ the essential spectrum of an operator A

$(.)_\pm$ the positive/negative part of a sequence or function

Chapter 1

Introduction and Main Results

In this chapter, we give an account of previous work on three topics and introduce our main results. These three objects of study will, in the order presented, be examined throughout the majority of this thesis. However, we will study them on discrete rather than continuous spaces, answering the canonical question of whether the methods can be extended to such spaces, and what the differences might be. First we discuss functional inequalities on a variety of Banach spaces, broadly categorised as Landau–Kolmogorov inequalities. We then continue to give an account of previous advances on Lieb–Thirring inequalities for the eigenvalues of the Schrödinger operator. Thirdly, we observe the representation of the discrete Schrödinger operator as a Jacobi matrix operator, after which we finally present our main results.

1.1 Landau–Kolmogorov Inequalities

In 1912, G. H. Hardy, J. E. Littlewood and G. Pólya (see [HLP52]) proved the following inequalities for a function $f \in L^2(\mathbb{R})$:

$$\|f'\|_{L^2(-\infty,\infty)} \leq \|f\|^{1/2}_{L^2(-\infty,\infty)} \|f''\|^{1/2}_{L^2(-\infty,\infty)}, \tag{1.1}$$

$$\|f'\|_{L^2(0,\infty)} \leq \sqrt{2} \|f\|^{1/2}_{L^2(0,\infty)} \|f''\|^{1/2}_{L^2(0,\infty)}, \tag{1.2}$$

with the constants 1 and $\sqrt{2}$ being sharp. These results sparked interest in inequalities involving functions, their derivatives and integrals for a century to come. Specifically, in 1913, E. Landau (see [Lan13]) proved the following inequality: For $\Omega \subseteq \mathbb{R}$, and $f \in L^\infty(\Omega)$:

$$\|f'\|_{L^\infty(\Omega)} \leq \sqrt{2}\|f''\|_{L^\infty(\Omega)}^{1/2}\|f\|_{L^\infty(\Omega)}^{1/2},$$

with the constant $\sqrt{2}$ being sharp. This result in turn was motivation for A. Kolmogorov (see [Kol39]), where in 1939 he found sharp constants for the more general case, using a simple, but very effective inductive argument to extend the case to higher order derivatives:

$$\|f^{(k)}\|_{L^\infty(\Omega)} \leq C(k,n)\|f^{(n)}\|_{L^\infty(\Omega)}^{k/n}\|f\|_{L^\infty(\Omega)}^{1-k/n},$$

where, for $k, n \in \mathbb{N}$ with $1 \leq k < n$, he determined the best constants $C(k,n) \in \mathbb{R}$ for $\Omega = \mathbb{R}$. Namely, $C(k,n) = a_{n-k}a_n^{-1+k/n}$ where a_n are the Akhiezer–Krein–Favard constants:

$$a_n := \frac{4}{\pi}\sum_{k=0}^{\infty}\left[\frac{(-1)^k}{2k+1}\right]^{n+1}.$$

Since then, there has been a great deal of work on what are nowadays known as the Landau-Kolmogorov inequalities, which are in their most general form: .

$$\|f^{(k)}\|_{L^p} \leq K(k,n,p,q,r)\ \|f^{(n)}\|_{L^q}^{\alpha}\|f\|_{L^r}^{\beta},$$

with the minimal constant $K = K(k,n,p,q,r)$. The real numbers $1 \leq p,q,r \leq \infty$; $k,n \in \mathbb{N}$ with $1 \leq k < n$ and $\alpha, \beta \in \mathbb{R}$ take on values for which the constant K is finite (see [Gab67]). The simple inductive argument by Kolmogorov left many future papers on Landau–Kolmogorov inequalities to be solely concerned with the case $k = 1$, $n = 2$.

Besides A. Kolmogorov's result, and that of G. H. Hardy, J. E. Littlewood and G. Pólya, there is only one more case for which the sharp constant is known, namely, $p = q = r = 1$, as shown by E. M. Stein (see [Ste57]).

However, literature on discrete equivalents of those inequalities remained very limited for a long time. In 1979, E. T. Copson (see [Cop79]) was one of the first to find equivalent results for sequences, series and difference operators. Indeed, he found the discrete equivalent to (1.1) and (1.2). For a square summable sequence, $\{a(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and a difference operator $(Da)(n) := a(n+1) - a(n)$, we have:

$$\|Da\|_{\ell^2(-\infty,\infty)} \leq \|a\|^{1/2}_{\ell^2(-\infty,\infty)} \|D^2 a\|^{1/2}_{\ell^2(-\infty,\infty)}, \tag{1.3}$$

$$\|Da\|_{\ell^2(0,\infty)} \leq \sqrt{2}\|a\|^{1/2}_{\ell^2(0,\infty)} \|D^2 a\|^{1/2}_{\ell^2(0,\infty)}, \tag{1.4}$$

with the constants 1 and $\sqrt{2}$ yet again being sharp. For the case of $\ell^2(\mathbb{Z})$, these constants were reconfirmed to be sharp by H. A. Gindler and J. A. Goldstein (see [GG81]), by using Banach space techniques. They viewed the difference operator as a combination of the shift operator A, and the identity operator, I, and then used known results from dissipative operator theory. Z. Ditzian (see [Dit83]) then extended those results to establish best constants for a variety of Banach spaces, adding equivalent results for continuous shift operators $f(x+h) - f(x);\ x \in \mathbb{R}, f \in \text{Ł}^2(\mathbb{R})$.

Comparing inequalities such as (1.1) and (1.2), with (1.3) and (1.4) respectively, it was suspected that sharp constants were identical for equivalent discrete and continuous Landau–Kolmogorov inequalities for $1 \leq p = q = r \leq \infty$. Indeed, in the cases $p = 1, 2, \infty$, this was true for the whole and semi-axis. However, the general case has since been shown to be false, as for example demonstrated in [KKZ88] by M. K. Kwong and A. Zettl, where they prove that for many values of p, the discrete constants are strictly greater than the continuous ones.

Another important special case of the Landau-Kolmogorov inequalities is the Agmon inequality, proven by S. Agmon (see [Agm10]). Viewed as an interpolation inequality between $L^\infty(\mathbb{R})$ and $L^2(\mathbb{R})$, he states the following:

$$\|f\|_{L^\infty(\mathbb{R})} \leq \|f\|^{1/2}_{L^2(\mathbb{R})} \|f'\|^{1/2}_{L^2(\mathbb{R})}.$$

The Agmon inequality is widely known and specifically applied in Spectral Theory, as we shall see later in the case for Lieb–Thirring inequalities on $\mathbb{R}$. Thus, throughout this thesis we shall call, for a domain Ω, a function $f \in L^2(\Omega)$, a sequence $\varphi \in \ell^2(\Omega)$, α, β being $\mathbb{Q}$-valued functions of the integers k, n with $k \leq n$ and constants $C(\Omega, k, n)$, $D(\Omega, k, n) \in \mathbb{R}$:

$$\|f^{(k)}\|_{L^\infty(\Omega)} \leq C(\Omega, k, n)\ \|f\|_{L^2(\Omega)}^{\alpha(k,n)}\ \|f^{(n)}\|_{L^2(\Omega)}^{\beta(k,n)}, \tag{1.5}$$

$$\|D^k\varphi\|_{\ell^\infty(\Omega)} \leq D(\Omega, k, n)\, \|\varphi\|_{\ell^2(\Omega)}^{\alpha(k,n)}\ \|D^n\varphi\|_{\ell^2(\Omega)}^{\beta(k,n)}, \tag{1.6}$$

Agmon–Kolmogorov inequalities, where (1.6), for $\Omega := \mathbb{Z}^d$ will play a major role in this thesis, first with $d = 1$ and then with arbitrary $d \in \mathbb{N}$.

In continuous space, there has been much progress regarding the sharp constants of Agmon–Kolmogorov inequalities. In fact, Taikov found the best constant for (1.5) in [Tai68], with $\Omega = \mathbb{R}$, $\alpha(n,k) = \frac{n-k-1/2}{n}$, $\beta(n,k) = \frac{k+1/2}{n}$. Indeed, he found:

$$C(\mathbb{R}, k, n) = \left(\frac{1}{(2k+1)^{(2k+1)/2n}(2n-2k-1)^{(2n-2k-1)/2n}} \frac{1}{\sin \pi \frac{2k+1}{2n}} \right)^{1/2}.$$

A. Ilyin in turn found sharp constants for the same inequality (see [Ily98]) for periodic functions f with zero mean value.

Our approach specifically deals with Agmon–Kolmogorov inequalities of the type in (1.6), using a combination of the arguments by S. Agmon in [Agm10] and H. A. Gindler and J. A. Goldstein in [GG81], in addition to the induction argument by A. Kolmogorov in [Kol39]. We thus use these inequalities to obtain Lieb–Thirring inequalities using the widely-applied method by A. Eden and C. Foias in [EF91].

1.2 Lieb–Thirring Inequalities in $L^2(\mathbb{R}^d)$

Our second topic of interest will be the so called Lieb–Thirring inequalities. We will prove a variety of discrete equivalents and apply them to obtain spectral bounds for Jacobi-type matrix operators. As a motivation for the study of discrete problems, we recall known results for continuous multi-dimensional Schrödinger operators. Let H be the Schrödinger operator acting in $L^2(\mathbb{R}^d)$, let $V \in L^{\gamma+d/2}(\mathbb{R}^d)$, $\gamma \geq 0$, be the potential and let $\{e_j\}_{j=1}^N$ and $\{\psi_j(x)\}_{j=1}^N$ with $N \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ be the associated negative eigenvalues and eigenfunctions:

$$H\psi_j(x) := -\Delta\psi_j(x) + V\psi_j(x) = e_j\psi_j(x).$$

In 1975, E. Lieb and W. Thirring proved the so called Lieb–Thirring inequalities (see [LT75]), which relate γ-moments of these eigenvalues $\{e_j\}_{j=1}^N$ with the potential $V \in L^{\gamma+d/2}(\mathbb{R}^d)$ via the following estimate:

$$\sum_{j=1}^N |e_j|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{\gamma+d/2} dx, \tag{1.7}$$

where $V_- = (|V| - V)/2$ is the negative part of V and $L_{\gamma,d}$ is described below.

It is known that the constants $L_{\gamma,d}$ are finite if $\gamma \geq 1/2$ ($d = 1$), $\gamma > 0$ ($d = 2$), and $\gamma \geq 0$ ($d \geq 3$). If $\gamma = 0$ ($d \geq 3$) the inequality (1.7) is simply an upper bound for the number of eigenvalues, and is referred to as the CLR-inequality (Cwikel–Lieb–Rozenblum) (see [Cwi77], [Lie76] and [Roz72]). The case $\gamma = 1/2$ ($d = 1$) was proved by T. Weidl in [Wei96].

In all these cases we have the following Weyl-type asymptotic formula for the eigenvalues of the operator $H(\alpha) = -\Delta + \alpha V$:

$$\sum_{j=1}^N |e_j|^\gamma = \alpha^{\gamma+d/2} (2\pi)^{-d} \int\int (|\xi|^2 + V(x))_-^\gamma dx\, d\xi + o(\alpha^{\gamma+d/2})$$

$$= \alpha^{\gamma+d/2} L^{cl}_{\gamma,d} \int V_-(x)^{\gamma+d/2} dx + o(\alpha^{\gamma+d/2}), \quad \text{as} \quad \alpha \to \infty, \tag{1.8}$$

where

$$L^{cl}_{\gamma,d} = (2\pi)^{-d} \int (|\xi|^2 - 1)^{\gamma}_{-} d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma + d/2 + 1)}, \quad \gamma \geq 0.$$

Therefore the sharpness of the constants $L_{\gamma,d}$ appearing in (1.7) could be compared with the values of $L^{cl}_{\gamma,d}$. Then (1.8) implies that $L_{\gamma,d} \geq L^{cl}_{\gamma,d}$.

In some cases the values of sharp constants $L_{\gamma,d}$ are known. However, they do not always coincide with $L^{cl}_{\gamma,d}$. It has been proven in [LT76] that $L_{3/2,1} = 3/16$. In [AL78], Aizenman and Lieb were able to obtain sharp constants $L_{\gamma,1}$ for all $\gamma \geq 3/2$. They found a simple way to "lift" the moments of any result of the Lieb–Thirring inequality for a fixed γ to $\gamma + n$, $n \in \mathbb{N}$. This argument is widely used, and we shall employ this method in every chapter of this thesis.

Later A. Laptev and T. Weidl obtained sharp constants for $L_{\gamma,d}$ for all $\gamma \geq 3/2$ in any dimension (see [LW00]) , by using an operator version of the Buslaev–Faddeev–Zakharov trace formulae and then applying induction with respect to dimension. If $\gamma = 1/2$ and $d = 1$ then $L_{1/2,1} = 1/2$ was found by D. Hundertmark, E.B. Lieb and L. Thomas in [HLT98].

Several attempts have been made to improve estimates for the constants $L_{\gamma,d}$. For $1/2 \leq \gamma < 3/2$, Hundertmark, Laptev and Weidl (see [HLW00]) found the constant to be not greater than $2\,L^{cl}_{\gamma,d}$. Recently this has been improved for $1 \leq \gamma < 3/2$ by J. Dolbeault, A. Laptev and M. Loss (see [DLL08]) to $c\,L^{cl}_{\gamma,d}$, $c = 1.8\ldots$. They used methods derived from A. Eden and C. Foias (see [EF91]), who improved the constant in one dimension for $\gamma \geq 1$, using a simple and elegant proof. Having applied the Agmon inequality to suborthogonal sequences of orthonormal functions, they arrived at the Generalised Sobolev inequality, which is the dual to the Lieb Thirring inequality.

Although Lieb–Thirring inequalities for Schrödinger operators acting on $L^2(\mathbb{R}^d)$ attracted attention of many specialists during the last two decades, the literature on the study of their equivalent inequalities for discrete operators is limited. One of the main aims of this thesis is to emulate and combine the approaches in [EF91] and [DLL08] to obtain Lieb–Thirring inequalities for discrete Schrödinger-type operators of arbitrary order and arbitrary dimension.

1.3 The Discrete Schrödinger Operator as a Jacobi Matrix

The third topic to be discussed in this thesis is the representation of the discrete Schrödinger operator as a Jacobi-type matrix operator. A self-adjoint Jacobi operator is a symmetric linear operator acting on the Hilbert Space of square summable sequences $\ell^2(\mathbb{Z})$.
It is given by

$$W = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & b_{-1} & a_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & 0 & \dots \\ \dots & 0 & a_0 & b_1 & a_1 & \dots \\ \dots & 0 & 0 & a_1 & b_2 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

viewed as an operator acting on $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ via:

$$(W\varphi)(n) = a_{n-1}\varphi(n-1) + b_n\varphi(n) + a_n\varphi(n+1), \quad \text{for } n \in \mathbb{Z},$$

where $a_n > 0$ and $b_n \in \mathbb{R}$. The operator will then be bounded and can hence be viewed as the one-dimensional discrete Schrödinger operator if $a_n = 1$. We will use the Lieb–Thirring-type inequalities to obtain spectral bounds for tri-, penta- and polydiagonal Jacobi-type matrices; the latter two simply being generalisations of the former. By pentadiagonal we mean two diagonals above and below the main diagonal, and by polydiagonal arbitrarily many. This was inspired by the work of D. Hundertmark and B. Simon in [HS02], where they were able to find spectral bounds for these operators. We thus state their result:

If a_n, $b_n \in \mathbb{R}$ and $a_n \to 1$, $b_n \to 0$ rapidly enough, as $n \to \pm\infty$, the essential spectrum $\sigma_{ess}(W)$ of W is absolutely continuous and coincides with the interval $[-2, 2]$ (see for example [BG99]). Besides, W may have simple eigenvalues $\{E_j^{\pm}\}_{j=1}^{N_{\pm}}$ where $N_{\pm} \in \overline{\mathbb{N}}$, and

$$E_1^+ > E_2^+ > \ldots > 2 > -2 > \ldots > E_2^- > E_1^-.$$

Theorem 1.1 (Hundertmark-Simon [HS02]). *Let* $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n - 1\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. *Then*

$$\sum_{j=1}^{N_+} \left((E_j^+)^2 - 4\right)^{1/2} + \sum_{j=1}^{N_-} \left((E_j^-)^2 - 4\right)^{1/2} \leq \sum_{n=-\infty}^{\infty} |b_n| + 4 \sum_{n=-\infty}^{\infty} |a_n - 1|. \tag{1.9}$$

Moreover, if $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n - 1\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z})$, $\gamma \geq 1/2$, *then*

$$\sum_{j=1}^{N_+} |E_j^+ - 2|^\gamma + \sum_{j=1}^{N_-} |E_j^- + 2|^\gamma \leq k_\gamma \left[\sum_{n=-\infty}^{\infty} |b_n|^{\gamma+1/2} + 4 \sum_{n=-\infty}^{\infty} |a_n - 1|^{\gamma+1/2} \right], \tag{1.10}$$

where

$$k_\gamma = 2(3^{\gamma-1/2}) L_{\gamma,1}^{cl}, \qquad \textit{and} \qquad L_{\gamma,1}^{cl} = \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\,\Gamma(\gamma+3/2)}.$$

We note here that the inequality (1.9) is sharp and this is the only case when sharp constants in Lieb–Thirring inequalities are known for Jacobi matrices. By their method of generalisation, however, they lost optimality for $\gamma \geq 1/2$. We will specifically improve those constants in Chapter 2, which is based on the author's work in [Sah10]. Additionally, in Chapters 3 and 4, we generalise these ideas to higher order operators in the form of penta- and polydiagonal matrices.

Main Results

We present our main results, regarding the 2^{nd}, 4^{th} and σ^{th} order Schrödinger-type difference operators and their relation to tri-, penta- and polydiagonal Jacobi-type matrix operators. Additionally, we provide the Agmon–Kolmogorov and Lieb–Thirring inequalities on $\ell^2(\mathbb{Z}^d)$. For ease of reading, we introduce the following notation for a specific type of inequality: $Name\,of\ inequality - (operator,\ \mathbb{Z}^k)$, where we have two parameters: the operator under consideration and the ℓ^2-domain on which the underlying sequences act.

1.4 Spectral Bounds for Tridiagonal Jacobi Operators

For a sequence $\{\varphi(n)\}_{n\in\mathbb{Z}}$, let D and D^* be the difference operator and its adjoint (with respect to the standard inner product) respectively, denoted by $D\varphi(n) = \varphi(n+1) - \varphi(n)$, and $D^*\varphi(n) = \varphi(n-1) - \varphi(n)$.

We let $\{\psi_j\}_{j=1}^N$, $N \in \overline{\mathbb{N}}$, be the orthonormal system of eigensequences in $\ell^2(\mathbb{Z})$ corresponding to the negative eigenvalues $\{e_j\}_{j=1}^N$ of the discrete Schrödinger operator H_D:

$$(H_D\psi_j)(n) := D^*D\psi_j(n) - b_n\psi_j(n) = e_j\psi_j(n),$$

where $j \in \{1,\dots,N\}$ and we assume that $b_n \geq 0$ for all $n \in \mathbb{Z}$. The discrete Lieb–Thirring inequality is concerned with estimating those negative eigenvalues. We thus have:

Theorem 1.2 (Lieb–Thirring Inequality – $(H_D,\mathbb{Z})$)**.** *Let* $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z})$, $\gamma \geq 1$. *Then the negative eigenvalues* $\{e_j\}_{j=1}^N$ *of the operator* H_D *satisfy the inequality*

$$\sum_{j=1}^N |e_j|^\gamma \leq \eta_1^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2},$$

where

$$\eta_1^\gamma := \frac{\pi}{\sqrt{3}} L_{\gamma,1}^{cl} \quad \text{and} \quad L_{\gamma,1}^{cl} := \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\,\Gamma(\gamma+3/2)}.$$

Remark. We will use a reflection argument to prove that the Schrödinger-type operators $-H_D = -D^*D + b$ and $H_D' := (D^*D - 4) + b$ have identical spectra. Therefore the negative eigenvalues of H_D coincide with the positive eigenvalues of H_D'. We thus have a similar result for the positive eigenvalues of the operator H_D':

Corollary 1.3. *Let* $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z})$, $\gamma \geq 1$. *Then the positive eigenvalues* $\{e_j\}_{j=1}^N$ *of the operator* H_D' *satisfy the inequality*

$$\sum_{j=1}^N e_j^\gamma \leq \eta_1^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2}.$$

with η_1^γ *given above.*

We then use the above estimate and proceed with a spectral estimate for our tridiagonal Jacobi matrix operator, W_1, denoted by:

$$W_1 := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & b_{-1} & a_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & 0 & \dots \\ \dots & 0 & a_0 & b_1 & a_1 & \dots \\ \dots & 0 & 0 & a_1 & b_2 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

viewed as an operator acting on $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, via:

$$(W_1\varphi)(n) = a_{n-1}\varphi(n-1) + b_n\varphi(n) + a_n\varphi(n+1), \quad \text{for } n \in \mathbb{Z}.$$

As before, we assume that a_n, $b_n \in \mathbb{R}$ and $a_n \to 1$, $b_n \to 0$ rapidly enough as $n \to \pm\infty$. We have $\sigma_{ess}(W_1) = [-2, 2]$ and W_1 may have simple eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$ where $N_\pm \in \overline{\mathbb{N}}$. We then improve on the result in [HS02], by obtaining better constants:

Theorem 1.4. *Let $\gamma \geq 1$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n - 1\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z})$. Then the following inequality holds true for the eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$ of the operator W_1:*

$$\sum_{j=1}^{N_-} |E_j^- + 2|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - 2|^\gamma \leq \nu_1^\gamma \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/2} + 4 \sum_{n\in\mathbb{Z}} |a_n - 1|^{\gamma+1/2} \right).$$

where

$$\nu_1^\gamma := 3^{\gamma-1}\pi\, L_{\gamma,1}^{cl} \quad \textit{and} \quad L_{\gamma,1}^{cl} = \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\,\Gamma(\gamma+3/2)}.$$

Remark. We see that if we divide the constant obtained by Hundertmark and Simon in (1.10), namely $k_\gamma = 2\,(3^{\gamma-1/2})\,L_{\gamma,1}^{cl}$, by our factor above, we obtain $\frac{2}{\pi/\sqrt{3}} \approx 1.1$, a smaller constant, as claimed previously.

1.5 Spectral Bounds for Pentadiagonal Jacobi-type Operators

Here, we lift all previous arguments to 2^{nd} and 4^{th} order operators $\Delta_D := D^* D$ and H_D^2, and finally estimate the simple eigenvalues of pentadiagonal Jacobi-type operators.

Let $\{\psi_j\}_{j=1}^N$, $N \in \overline{\mathbb{N}}$ be the orthonormal system of eigensequences in $\ell^2(\mathbb{Z})$ corresponding to the negative eigenvalues $\{e_j\}_{j=1}^N$ of the 4^{th} order discrete Schrödinger-type operator:

$$(H_D^2 \psi_j)(n) := (\Delta_D^2 \psi_j)(n) - b_n \psi_j(n) = e_j \psi_j(n),$$

where $j \in \{1, \ldots, N\}$ and we assume that $b_n \geq 0$ for all $n \in \mathbb{Z}$. We then have:

Theorem 1.5 (Lieb–Thirring Inequality – $(H_D^2, \mathbb{Z})$)**.** *Let $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/4}(\mathbb{Z})$, $\gamma \geq 1$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D^2 satisfy the inequality*

$$\sum_{j=1}^N |e_j|^\gamma \leq \eta_2^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/4} \qquad \textit{where} \quad \eta_2^\gamma = \frac{4}{5^{5/4}} \frac{\Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

Remark. As the discrete spectrum of H_D^2 lies in $[-\infty, 0]$ and $[16, \infty]$, we shift our operator to the left by 16 and by analogy have an estimate for the positive eigenvalues of that operator. We hence immediately obtain Corollary 1.6.

Corollary 1.6. *Let $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/4}(\mathbb{Z})$, $\gamma \geq 1$. Then the positive eigenvalues $\{e_j\}_{j=1}^N$ of the operator $\Delta_D^2 - 16 + b$ satisfy the inequality*

$$\sum_{j=1}^N e_j^\gamma \quad \leq \quad \eta_2^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/4}.$$

with η_2^γ given above.

We proceed as before and we let W_2 be a 'pentadiagonal' self-adjoint Jacobi-type operator:

$$W_2 := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & b_{-1} & a_{-1} & c_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & c_0 & 0 & \dots \\ \dots & c_{-1} & a_0 & b_1 & a_1 & c_1 & \dots \\ \dots & 0 & c_0 & a_1 & b_2 & a_2 & \dots \\ \dots & 0 & 0 & c_1 & a_2 & b_3 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

viewed as an operator acting on $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, via:

$$(W_2\varphi)(n) = c_{n-2}\varphi(n-2) + a_{n-1}\varphi(n-1) + b_n\varphi(n) + a_n\varphi(n+1) + c_n\varphi(n+2), \quad \text{for } n \in \mathbb{Z}.$$

In what follows we assume that $a_n, b_n, c_n \in \mathbb{R}$ and $a_n \to -4$, $b_n \to 0$, $c_n \to 1$ rapidly enough as $n \to \pm\infty$. We have $\sigma_{ess}(W_2) = [-6, 10]$ and W_2 may have simple eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$ where $N_\pm \in \overline{\mathbb{N}}$, and

$$E_1^+ > E_2^+ > ... > 10 > -6 > ... > E_2^- > E_1^-.$$

We thus have the following bound for these eigenvalues:

Theorem 1.7. *Let $\gamma \geq 1$, and $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n + 4\}_{n\in\mathbb{Z}}$, $\{c_n - 1\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/4}(\mathbb{Z})$. Then for the eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$ of the operator W_2 we have:*

$$\sum_{j=1}^{N_-} |E_j^- + 6|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - 10|^\gamma \;\leq\; \nu_2^\gamma \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/4} + 4 \sum_{n\in\mathbb{Z}} |a_n + 4|^{\gamma+1/4} + 4 \sum_{n\in\mathbb{Z}} |c_n - 1|^{\gamma+1/4} \right),$$

where

$$\nu_2^\gamma := 5^{\gamma-2} \frac{4\,\Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

1.6 Spectral Bounds for Polydiagonal Jacobi-type Operators

We now complete our generalisation. For $\sigma \in \mathbb{N}$, we define the $2\sigma^{th}$ order difference operator $\Delta_D^\sigma := \Delta_D(\Delta_D^{\sigma-1})$, and we shall show that it will take the following explicit form:

$$(\Delta_D^\sigma \varphi)(n) = \sum_{k=0}^{2\sigma} {}^{2\sigma}C_k(-1)^{k+\sigma}\varphi(n-\sigma+k).$$

We then let $\{\psi_j\}_{j=1}^N$, $N \in \overline{\mathbb{N}}$ be the orthonormal system of eigensequences in $\ell^2(\mathbb{Z})$ corresponding to the negative eigenvalues $\{e_j\}_{j=1}^N$ of the $(2\sigma)^{th}$ order Schrödinger-type operator:

$$(H_D^\sigma \psi_j)(n) := (\Delta_D^\sigma \psi_j)(n) - b_n\psi_j(n) = e_j\psi_j(n),$$

where $j \in \{1,\ldots,N\}$ and we assume that $b_n \geq 0$ for all $n \in \mathbb{Z}$. Then H_D^σ may have discrete eigenvalues. Thus our estimate is concerned with exactly those eigenvalues:

Theorem 1.8 (Lieb–Thirring Inequality – $(H_D^\sigma, \mathbb{Z})$)**.** *Let* $b_n \geq 0$*,* $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z})$*,* $\gamma \geq 1$*. Then the negative eigenvalues* $\{e_j\}_{j=1}^N$ *of the operator* H_D^σ *satisfy the inequality*

$$\sum_{j=1}^N |e_j|^\gamma \leq \eta_\sigma^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2\sigma},$$

where

$$\eta_\sigma^\gamma := \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})}.$$

Remark. As the discrete spectrum of H_D^σ lies in $[-\infty, 0]$ and $[4^\sigma, \infty]$, we shift our operator to the left by 4^σ and by analogy have an estimate for the positive eigenvalues of that operator: We hence immediately obtain Corollary 1.9:

Corollary 1.9. *Let* $b_n \geq 0$*,* $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z})$*,* $\gamma \geq 1$*. Then the positive eigenvalues* $\{e_j\}_{j=1}^N$ *of the operator* $\Delta_D^\sigma - 4^\sigma + b$ *satisfy the inequality*

$$\sum_{j=1}^{N} e_j^\gamma \leq \eta_\sigma^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2\sigma}, \qquad \textit{with } \eta_\sigma^\gamma \textit{ given above.}$$

We continue as before, and let W_σ to be polydiagonal self-adjoint Jacobi-type operator:

$$W_\sigma := \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \iddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & a_{-2}^\sigma & 0 & 0 & \ddots \\ \ddots & \ddots & b_{-1} & a_{-1}^1 & \ddots & \ddots & a_{-1}^\sigma & 0 & \ddots \\ \ddots & \ddots & a_{-1}^1 & b_0 & a_0^1 & \ddots & \ddots & a_0^\sigma & \ddots \\ \ddots & \ddots & \ddots & a_0^1 & b_1 & a_1^1 & \ddots & \ddots & \ddots \\ \ddots & a_{-2}^\sigma & \ddots & \ddots & a_1^1 & b_2 & a_2^1 & \ddots & \ddots \\ \ddots & 0 & a_{-1}^\sigma & \ddots & \ddots & a_2^1 & b_3 & \ddots & \ddots \\ \ddots & 0 & 0 & a_0^\sigma & \ddots & \ddots & \ddots & \ddots & \ddots \\ \iddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

acting on $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, via: $(W_\sigma\varphi)(n) = \sum_{i=1}^\sigma a_{n-i}^i \varphi(n-i) + b_n\varphi(n) + \sum_{i=1}^\sigma a_n^i \varphi(n+i)$.
In what follows we assume that a_n^i, $b_n \in \mathbb{R}$ for all $i \in \{1, ..., \sigma\}$ and for $\omega_i = {}^{2\sigma}C_{\sigma+i}(-1)^i$, $a_n^i \to \omega_i$, $b_n \to 0$, rapidly enough as $n \to \pm\infty$. Then the essential spectrum $\varsigma_{ess}(W_\sigma)$ coincides with the interval $[-{}^{2\sigma}C_\sigma, 4^\sigma - {}^{2\sigma}C_\sigma]$ and W_σ may have simple eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$, where $N_\pm \in \overline{\mathbb{N}}$, and

$$E_1^+ > E_2^+ > ... > 4^\sigma - {}^{2\sigma}C_\sigma > -{}^{2\sigma}C_\sigma > ... > E_2^- > E_1^-.$$

Theorem 1.10. *Let* $\gamma \geq 1$*,* $\{b_n\}_{n\in\mathbb{Z}}$*,* $\{a_n^i - \omega_i\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z})$*, for all* $i \in \{1, \ldots, \sigma\}$*. Then for the eigenvalues* $\{E_j^\pm\}_{j=1}^{N_\pm}$ *of the operator* W_σ *we have:*

$$\sum_{j=1}^{N_-} |E_j^- + {}^{2\sigma}C_\sigma|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - (4^\sigma - {}^{2\sigma}C_\sigma)|^\gamma \leq \nu_\sigma^\gamma \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/2\sigma} + 4 \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\sigma} |a_n^k - \omega_k|^{\gamma+1/2\sigma} \right),$$

where

$$\nu_\sigma^\gamma = 2\sigma\ (2\sigma+1)^{\gamma-2} \frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})}.$$

1.7 Inequalities on $\ell^2(\mathbb{Z}^d)$

In Chapter 5, we generalise the Agmon–Kolmogorov, Generalised Sobolev, and Lieb–Thirring inequalities to higher dimensional domains. We note that the Agmon–Kolmogorov inequality is the most interesting case with no continuous equivalent. We thus consider $\ell^2(\mathbb{Z}^d)$ and define $\nabla_D\varphi(\zeta_1,\zeta_2,\dots,\zeta_d) = \big(D_1\varphi(\zeta), D_2\varphi(\zeta),\dots,D_d\varphi(\zeta)\big)$, with $\zeta \in \mathbb{Z}^d$.

We then have the following following family of inequalities:

Theorem 1.11 (Agmon–Kolmogorov Inequalities – $(\nabla_D, \ell^2(\mathbb{Z}^d))$)**.** *For a sequence $\{\varphi(\zeta)\}_{\zeta\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ and $p \in \{1,\dots,2^{d-1}\}$:*

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \le \mu_{p,d}\|\nabla_D\varphi\|^{p/2^d}_{\ell^2(\mathbb{Z}^d)}\,\|\varphi\|^{1-p/2^d}_{\ell^2(\mathbb{Z}^d)},$$

where

$$\mu_{p,d} := \left(\frac{\kappa_{p,d}}{d^{p/2}}\right)^{1/2^d}, \qquad \kappa_{p,d} = 2^{2^{d-1}d-p}.$$

Let $\{\psi_j\}_{j=1}^N$, $N \in \overline{\mathbb{N}}$ be the orthonormal system of eigensequences in $\ell^2(\mathbb{Z}^d)$ corresponding to the negative eigenvalues $\{e_j\}_{j=1}^N$ of the discrete Schrödinger operator of dimension d:

$$(H_D^d\psi_j)(\zeta) := (-\nabla_D^2\psi_j)(\zeta) - b_\zeta\psi_j(\zeta) = e_j\psi_j(\zeta),$$

where $j \in \{1,\dots,N\}$, and we assume that $b_\zeta \ge 0$ for all $\zeta \in \mathbb{Z}^d$. The essential spectrum $\sigma_{ess}(H_D^d)$ lies in the interval $[0,4^d]$, and thus for the negative eigenvalues of H_D^d, we have the following bound:

Theorem 1.12 (Lieb–Thirring Inequality – $(H_D, \mathbb{Z}^d)$)**.** *Let $b_\zeta \ge 0$, $\{b_\zeta\}_{\zeta\in\mathbb{Z}^d} \in \ell^{\gamma+\alpha}(\mathbb{Z}^d)$, for $\gamma \ge 1$ and $\alpha := p/2^d$, with $p \in \{1,\dots,2^{d-1}\}$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D^d are discrete and satisfy the inequality:*

$$\sum_{j=1}^N |e_j|^\gamma \quad\le\quad \eta_{p,d}\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{\gamma+\alpha},$$

where

$$\eta_{p,d} := \frac{\Gamma(2+\alpha)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}\frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}}\mu_{p,d}^2.$$

for $\mu_{p,d}$ given above.

Remark. Again, by analogy, we have the same bound for the positive eigenvalues of the operator $H_D^d - 4^d + b$, which we shall not repeat here.

Chapter 2

Spectral Bounds for Tridiagonal Jacobi Operators

We follow now with the proofs of our main results for discrete Schrödinger operators. The methods employed in this chapter will to some extent be reproduced similarly in subsequent chapters and thus we introduce all ideas and Lemmata in full, detailed form. We will then be able to call on these results later, and hence focus on the more complex operators studied.

In Section 2.1, we introduce our operator and examine some of its properties. In Section 2.2, we prove the relevant Agmon–Kolmogorov inequality which leads to the Generalised Sobolev inequality in Section 2.3. In Section 2.4 we prove the Lieb–Thirring inequality for $\gamma = 1$ eigenvalue moments and we then lift this to general $\gamma \geq 1$ in Section 2.5, by using the Aizenman–Lieb procedure. Finally we apply these spectral bounds to tridiagonal Jacobi matrix operators in Section 2.6.

2.1 The Operators D, D^*D and H_D

Definition 2.1. For $\{\varphi(n)\}_{n\in\mathbb{Z}}$, a sequence of real (or complex) numbers, we define the ℓ^2-space as the real (complex) inner product space such that $\sum_{n\in\mathbb{Z}} |\varphi(n)|^2$ is finite and hence endow it with the norm

$$\|\varphi(n)\|_{\ell^2(\mathbb{Z})} = \left(\sum_{n\in\mathbb{Z}} |\varphi(n)|^2\right)^{1/2}.$$

We then let $\langle .,. \rangle$ be the inner product on $\ell^2(\mathbb{Z})$:

$$\langle \varphi, \phi \rangle := \sum_{n=-\infty}^{\infty} \varphi(n)\overline{\phi(n)}.$$

Throughout this thesis we will be interested in operators derived from the following, basic difference operators:

Definition 2.2 (Difference operator)**.** For an infinite sequence $\{\varphi(n)\}_{n\in\mathbb{Z}}$, let D and D^* be the difference operator and its adjoint denoted by:

$$D\varphi(n) = \varphi(n+1) - \varphi(n), \qquad D^*\varphi(n) = \varphi(n-1) - \varphi(n).$$

Remark. We note here that this difference operator, as opposed to the differential operator in $L^2(\mathbb{R})$, in our framework within the space $\ell^2(\mathbb{Z})$, does not give rise to a 'discrete Sobolev Space' The reason for this is that for $\varphi \in \ell^2(\mathbb{Z})$, $D\varphi$ is already in $\ell^2(\mathbb{Z})$ by:

$$\begin{aligned} \left(\sum_{n\in\mathbb{Z}} |\varphi(n)|^2 + |D\varphi(n)|^2 \right)^{1/2} &= \left(\sum_{n\in\mathbb{Z}} |\varphi(n)|^2 + |\varphi(n+1) - \varphi(n)|^2 \right)^{1/2} \\ &\leq \left(\sum_{n\in\mathbb{Z}} |\varphi(n)|^2 + 2\big(|\varphi(n+1)|^2 + |\varphi(n)|^2\big) \right)^{1/2} \\ &\leq \left(5 \sum_{n\in\mathbb{Z}} |\varphi(n)|^2 \right)^{1/2} \leq \infty. \end{aligned}$$

Definition 2.3 (Discrete Laplacian)**.** Thus for a sequence $\{\varphi(n)\}_{n\in\mathbb{Z}}$, our 2^{nd} order, self-adjoint discrete Laplacian operator will be denoted by:

$$\Delta_D\varphi(n) := D^*D\,\varphi(n) = -\varphi(n+1) - \varphi(n-1) + 2\varphi(n).$$

Remark. Δ_D is self-adjoint, as $(\Delta_D)^* = (D^*D)^* = D^*D^{**} = D^*D = \Delta_D$ and the operator Δ_D^k for $k \in \mathbb{N}$ will hence also be self adjoint. The spectrum of Δ_D is absolutely continuous and its essential spectrum, denoted by σ_{ess}, can easily shown to be $\sigma_{\text{ess}}(D^*D) = [0,4]$. We will in fact prove a more general formula for finding essential spectra for any given difference operator, using the discrete Fourier Transform.

Definition 2.4 (Discrete Fourier Transform)**.** For $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, let $\mathcal{F}$ be the discrete Fourier Transform denoted by

$$(\mathcal{F}\varphi)(\theta) := \hat{\varphi}(\theta) = \sum_{n\in\mathbb{Z}} \varphi(n)e^{in\theta}, \qquad \theta \in (0, 2\pi).$$

The spectral inequalities of this thesis are all concerned with the eigenvalues of operators derived from the discrete version of the well-known Schrödinger operator:

Definition 2.5. For a sequence $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we let $b = \{b_n\}_{n\in\mathbb{Z}}$ be the multiplication operator defined on $\ell^2(\mathbb{Z})$ via $(b\,\varphi)(n) := b_n\varphi(n)$. If $b_n \geq 0$, then the discrete Schrödinger operator will be denoted by H_D, where:

$$(H_D\,\varphi)(n) := (D^*D - b)\varphi(n) = (\Delta_D - b)\varphi(n).$$

Remark. We note here that H_D may in fact have negative, simple eigenvalues. Our discrete Lieb–Thirring inequality $-\ (D, \mathbb{Z})$ estimates the sum of exactly those eigenvalues.

We give the following Lemma, providing us with a way of later symmetrising the Lieb–Thirring inequality $-\ (D, \mathbb{Z})$.

Lemma 2.6. *Let b be the multiplication operator acting on $\ell^2(\mathbb{Z})$. Then the discrete Schrödinger-type operators $-H_D = -D^*D + b$ and $H'_D := (D^*D - 4) + b$ have identical spectra, and therefore the negative eigenvalues of H_D coincide with the positive eigenvalues of H'_D.*

Proof.

$$\begin{aligned}
(\mathcal{F}H'_D\mathcal{F}^{-1}\hat{\varphi})(\theta) &= \mathcal{F}((D^*D-4+b_n)(\mathcal{F}^{-1}\hat{\varphi}))(\theta) \\
&= \sum_{n\in\mathbb{Z}} e^{in\theta}(D^*D-4)\varphi(n) + \mathcal{F}(b_n\varphi)(\theta) \\
&= \sum_{n\in\mathbb{Z}}\Big(-e^{i(n+1)\theta} - e^{i(n-1)\theta} + 2e^{in\theta} - 4e^{in\theta}\Big)\varphi(n) + \mathcal{F}(b_n\varphi)(\theta) \\
&= (-2\cos\theta-2)\Big(\sum_{n\in\mathbb{Z}} e^{in\theta}\varphi(n)\Big) + \mathcal{F}(b_n\varphi)(\theta) \\
&= (-2\cos\theta-2)\hat{\varphi}(\theta) + \frac{1}{2\pi}\int_0^{2\pi}\hat{b}(\theta-\tau)\hat{\varphi}(\tau)\,d\tau,
\end{aligned}$$

by the Convolution Theorem. We now take the inner product w.r.t. $\hat{\varphi}(\theta)$:

$$\int_0^{2\pi}(\mathcal{F}H'_D\mathcal{F}^{-1}\hat{\varphi})(\theta)\overline{\hat{\varphi}(\theta)}\,d\theta = \int_0^{2\pi}(-2\cos\theta-2)|\hat{\varphi}(\theta)|^2d\theta + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\hat{b}(\theta-\tau)\hat{\varphi}(\tau)\overline{\hat{\varphi}(\theta)}\,d\tau d\theta.$$

Therefore using the periodicity of $\hat{b}$ we now change the variable to alter the signature of $\cos(\theta)$ without affecting the other terms. We denote $\hat{\psi}(\theta) = \hat{\varphi}(\theta+\pi)$ and we find that:

$$RHS = \int_0^{2\pi}(2\cos\theta-2)|\hat{\psi}(\theta)|^2d\theta + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\hat{b}(\theta-\tau)\hat{\psi}(\tau)\overline{\hat{\psi}(\theta)}\,d\tau d\theta. \tag{2.1}$$

If we now follow the same argument with $\mathcal{F}(-H_D)\mathcal{F}^{-1}$, we obtain:

$$\begin{aligned}
(\mathcal{F}(-H_D)\mathcal{F}^{-1}\hat{\varphi})(\theta) &= \mathcal{F}((-D^*D+b_n)\mathcal{F}^{-1}\hat{\varphi})(\theta) \\
&= \Big(\sum_{n\in\mathbb{Z}}(e^{i(n+1)\theta} + e^{i(n-1)\theta} - 2e^{in\theta})\varphi(n)\Big) + \mathcal{F}(b_n\varphi)(\theta) \\
&= (2\cos\theta-2)\hat{\varphi}(\theta) + \frac{1}{2\pi}\int_0^{2\pi}\hat{b}(\theta-\tau)\hat{\varphi}(\tau)\,d\tau,
\end{aligned}$$

which implies:

$$\int_0^{2\pi}(\mathcal{F}(-H_D)\mathcal{F}^{-1}\hat{\varphi})(\theta)\overline{\hat{\varphi}(\theta)}\,d\theta = \int_0^{2\pi}(2\cos\theta-2)|\hat{\varphi}(\theta)|^2d\theta + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\hat{b}(\theta-\tau)\hat{\varphi}(\tau)\overline{\hat{\varphi}(\theta)}\,d\tau d\theta,$$

thus coinciding with (2.1).

□

2.2 Agmon–Kolmogorov Inequality – $(D, \mathbb{Z})$

The Agmon inequality is the starting point of the work of A. Eden and C. Foias in [EF91], where they use it to give a simple proof of the one-dimensional Generalised Sobolev inequality. This immediately extends to the Lieb–Thirring inequality. The basic idea by S. Agmon in [Agm10] was to estimate the L^∞-norm of a function by the L^2-norm of the function and its derivative. Let $f, f' \in L^2(\mathbb{R})$. Then:

$$\|f\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}^{1/2} \|f'\|_{L^2(\mathbb{R})}^{1/2}.$$

We now give the discrete version of this inequality:

Proposition 2.7 (Discrete Agmon Inequality). *Let $\varphi \in \ell^2(\mathbb{Z})$. Then we have the following inequality:*

$$\|\varphi\|_{\ell^\infty(\mathbb{Z})} \leq \|\varphi\|_{\ell^2(\mathbb{Z})}^{1/2} \|D\varphi\|_{\ell^2(\mathbb{Z})}^{1/2}.$$

Proof. We recognise the ability to express a single point of a sequence in terms of two sums that continually cancel each other out except for a central term. We thus have:

$$\begin{aligned}
|\varphi(n)|^2 &= 1/2 \left| \sum_{k=-\infty}^{n-1} D\varphi^2(k) - \sum_{k=n}^{\infty} D\varphi^2(k) \right| \\
&\leq 1/2 \Big(\sum_{k=-\infty}^{n-1} \left| D\varphi^2(k) \right| + \sum_{k=n}^{\infty} \left| D\varphi^2(k) \right| \Big) \\
&= 1/2 \sum_{k=-\infty}^{\infty} \left| \varphi^2(k+1) - \varphi^2(k) \right| \\
&= 1/2 \sum_{k=-\infty}^{\infty} |D\varphi(k)| \Big(|\varphi(k+1)| + |\varphi(k)| \Big).
\end{aligned}$$

We apply the Cauchy–Schwarz inequality and obtain:

$$\begin{aligned}
|\varphi(n)|^2 &\leq 1/2 \left(\sum_{k=-\infty}^{\infty} |D\varphi(k)|^2 \right)^{1/2} \left[\left(\sum_{k=-\infty}^{\infty} |\varphi(k+1)|^2 \right)^{1/2} + \left(\sum_{k=-\infty}^{\infty} |\varphi(k)|^2 \right)^{1/2} \right] \\
&= \Big(\sum_{k=-\infty}^{\infty} |D\varphi(k)|^2 \Big)^{1/2} \Big(\sum_{k=-\infty}^{\infty} |\varphi(k)|^2 \Big)^{1/2}.
\end{aligned}$$

□

Remark. As previously discussed, we will call our generalisations with respect to order and dimension of the above discrete inequality Agmon–Kolmogorov inequalities. This emphasises this particular class of inequalities being a special case of the Landau–Kolmogorov inequalities.

2.3 Generalised Sobolev Inequality – $(D,\mathbb{Z})$

The proof of the next inequality is the discrete equivalent to the method used by A. Eden and C. Foias in [EF91]. It will be essential in the proof for the discrete Lieb–Thirring inequality – $(D,\mathbb{Z})$.

Proposition 2.8 (Generalised Sobolev Inequality – $(D,\mathbb{Z})$). *Let $\{\psi_j\}_{j=1}^N$ be an orthonormal system of sequences in $\ell^2(\mathbb{Z})$, i.e. $\langle\psi_j,\psi_k\rangle = \delta_{jk}$, and let $\rho(n) := \sum_{j=1}^N |\psi_j(n)|^2$. We then have:*

$$\sum_{n\in\mathbb{Z}} \rho^3(n) \le \sum_{j=1}^N \sum_{n\in\mathbb{Z}} |D\psi_j(n)|^2 .$$

Proof. Let $\xi = (\xi_1,\xi_2,\dots,\xi_N) \in \mathbb{C}^N$. Then, by Proposition 2.7, for all $n\in\mathbb{Z}$, we have:

$$\begin{aligned}
\Big|\sum_{j=1}^N \xi_j\psi_j(n)\Big|^2 &\le \Big\|\sum_{j=1}^N \xi_j\psi_j\Big\|_{\ell^2(\mathbb{Z})} \Big\|D\sum_{j=1}^N \xi_j\psi_j\Big\|_{\ell^2(\mathbb{Z})}\\
&= \left(\sum_{n\in\mathbb{Z}}\sum_{j=1}^N \xi_j\psi_j(n)\,\overline{\sum_{k=1}^N \xi_k\psi_k(n)}\right)^{1/2}\left(\sum_{n\in\mathbb{Z}} D\sum_{j=1}^N \xi_j\psi_j(n)\,\overline{D\sum_{k=1}^N \xi_k\psi_k(n)}\right)^{1/2}\\
&= \Big(\sum_{j,k=1}^N \xi_j\bar{\xi}_k \sum_{n\in\mathbb{Z}} \psi_j(n)\,\overline{\psi_k(n)}\Big)^{1/2}\Big(\sum_{j,k=1}^N \xi_j\bar{\xi}_k \sum_{n\in\mathbb{Z}} D\psi_j(n)\,\overline{D\psi_k(n)}\Big)^{1/2}\\
&= \Big(\sum_{j,k=1}^N \xi_j\bar{\xi}_k\langle\psi_j,\psi_k\rangle\Big)^{1/2}\Big(\sum_{j,k=1}^N \xi_j\bar{\xi}_k\langle D\psi_j,D\psi_k\rangle\Big)^{1/2}.
\end{aligned}$$

Using the orthonormality of $\{\psi_j\}_{j=1}^N$, we obtain

$$\Big|\sum_{j=1}^N \xi_j\psi_j(n)\Big|^2 \le \Big(\sum_{j=1}^N |\xi_j|^2\Big)^{1/2}\Big(\sum_{j,k=1}^N \xi_j\bar{\xi}_k\langle D\psi_j,D\psi_k\rangle\Big)^{1/2}.$$

If we set $\xi_j := \overline{\psi_j(n)}$ and we have $\rho(n) = \sum_{j=1}^N |\psi_j(n)|^2$ then the latter inequality becomes

$$\begin{aligned}
\rho^2(n) &\le \rho^{1/2}(n)\Big(\sum_{j,k=1}^N \psi_j(n)\overline{\psi_k(n)}\langle D\psi_j,D\psi_k\rangle\Big)^{1/2}\\
\Rightarrow \rho^3(n) &\le \sum_{j,k=1}^N \psi_j(n)\overline{\psi_k(n)}\langle D\psi_j,D\psi_k\rangle.
\end{aligned}$$

If we sum both sides, and again use orthonormality, we arrive at

$$\sum_{n\in\mathbb{Z}} \rho^3(n) = \sum_{n\in\mathbb{Z}}\Big(\sum_{j=1}^N |\psi_j(n)|^2\Big)^3 \le \sum_{j=1}^N\Big(\sum_{n\in\mathbb{Z}} |D\psi_j(n)|^2\Big).$$

□

2.4 Lieb–Thirring Inequality – $(H_D, \mathbb{Z}, \gamma = 1)$

In this section, we start proving our first main result, the discrete version of the Lieb–Thirring inequality. We divide the proof into two steps, in the first of which we employ the discrete Generalised Sobolev Inequality and a maximising argument to obtain our desired inequality for $\gamma = 1$. In the following section, we then lift this result to higher moments using the Aizenman–Lieb procedure.

We let $\{\psi_j\}_{j=1}^N$, $N \in \overline{\mathbb{N}}$, be the orthonormal system of eigensequences in $\ell^2(\mathbb{Z})$ corresponding to the negative eigenvalues $\{e_j\}_{j=1}^N$ of the discrete Schrödinger operator H_D:

$$(H_D\psi_j)(n) := D^*D\psi_j(n) - b_n\psi_j(n) = e_j\psi_j(n), \tag{2.2}$$

where $j \in \{1, \dots, N\}$ and we assume that $b_n \geq 0$, for all $n \in \mathbb{Z}$. Our next result is concerned with estimating those negative eigenvalues:

Theorem 2.9 (Lieb–Thirring Inequality – $(H_D, \mathbb{Z}, \gamma = 1)$)**.** *Let $b_n \geq 0, \{b_n\}_{n\in\mathbb{Z}} \in \ell^{3/2}(\mathbb{Z})$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D are discrete and they satisfy the inequality*

$$\sum_{j=1}^N |e_j| \leq \frac{\pi}{\sqrt{3}}\, L_{1,1}^{cl} \sum_{n\in\mathbb{Z}} b_n^{3/2},$$

where

$$L_{1,1}^{cl} = \frac{2}{3\pi}.$$

Proof. We take the inner product w.r.t. $\psi_j(n)$ on (2.2) and sum both sides of the equation with respect to j:

$$\sum_{j=1}^N e_j = \sum_{j=1}^N \Big(\sum_{n\in\mathbb{Z}} |D\psi_j(n)|^2\Big) - \sum_{j=1}^N \Big(\sum_{n\in\mathbb{Z}} b_n|\psi_j(n)|^2\Big). \tag{2.3}$$

We now use the Generalised Sobolev inequality – $(D, \mathbb{Z})$, i.e. Proposition 2.8:

$$\sum_{n\in\mathbb{Z}} \Big(\sum_{j=1}^N |\psi_j(n)|^2\Big)^3 \leq \sum_{j=1}^N \sum_{n\in\mathbb{Z}} |D\psi_j(n)|^2,$$

on (2.3) and then in the following step the discrete Hölder inequality to obtain:

$$\begin{aligned}\sum_{j=1}^N e_j &\geq \sum_{n\in\mathbb{Z}}\Big(\sum_{j=1}^N |\psi_j(n)|^2\Big)^3 - \sum_{n\in\mathbb{Z}}\Big(\sum_{j=1}^N b_n|\psi_j(n)|^2\Big)\\ &\geq \sum_{n\in\mathbb{Z}}\Big(\sum_{j=1}^N |\psi_j(n)|^2\Big)^3 - \Big(\sum_{n\in\mathbb{Z}} b_n^{3/2}\Big)^{2/3}\Big(\sum_{n\in\mathbb{Z}}\Big(\sum_{j=1}^N |\psi_j(n)|^2\Big)^3\Big)^{1/3}. \end{aligned}\tag{2.4}$$

We now define:

$$\chi := \Big(\sum_{n\in\mathbb{Z}}\Big(\sum_{j=1}^N |\psi_j(n)|^2\Big)^3\Big)^{1/3}, \quad \varsigma := \Big(\sum_{n\in\mathbb{Z}} b_n^{3/2}\Big)^{2/3}.$$

Then (2.4) can be written as

$$\chi^3 - \varsigma\chi \leq \sum_{j=1}^N e_j.$$

The LHS is maximal when

$$\chi = \sqrt{\varsigma/3} = \frac{1}{\sqrt{3}}\Big(\sum_{n\in\mathbb{Z}} b_n^{3/2}\Big)^{1/3}.$$

Substituting this into (2.4), we obtain

$$\begin{aligned}\sum_{j=1}^N e_j &\geq \frac{1}{3\sqrt{3}}\sum_{n\in\mathbb{Z}} b_n^{3/2} - \frac{1}{\sqrt{3}}\sum_{n\in\mathbb{Z}} b_n^{3/2}\\ &= -\frac{2}{3\sqrt{3}}\sum_{n\in\mathbb{Z}} b_n^{3/2}.\end{aligned}$$

Thus:

$$\sum_{j=1}^N |e_j| \leq \frac{2}{3\sqrt{3}} \sum_{n\in\mathbb{Z}} b_n^{3/2}.$$

□

2.5 Aizenman–Lieb Procedure

In [AL78], M. Aizenman, and E. Lieb, were able to lift results about Lieb–Thirring inequalities for $\gamma = 1$ to arbitrary γ by using a standard result of the Beta function in conjunction with the Variational Principle:

Theorem 2.10 (Variational Principle)**.** *Let A be a self-adjoint operator in a Hilbert space H that is bounded from below, i.e., $A \geq cI$ for some c. Then let D be the domain of A and let:*

$$E_n^- = \sup_{\varphi_1,\dots,\varphi_{n-1}\in D} \quad \inf_{\substack{\|\psi\|=1;\psi\in D\\ \psi\in[\varphi_1,\dots,\varphi_{n-1}]^\perp}} \langle\psi, A\psi\rangle.$$

Similarly, if A is bounded from above:

$$E_n^+ = \inf_{\varphi_1,\dots,\varphi_{n-1}\in D} \sup_{\substack{\|\psi\|=1;\psi\in D\\ \psi\in[\varphi_1,\dots,\varphi_{n-1}]^\perp}} \langle\psi, A\psi\rangle.$$

From this definition, we have the following important property, for self-adjoint operators A and B:

$$A \le B \qquad \Rightarrow \qquad E_n^\pm(A) \le E_n^\pm(B),$$

and hence:

$$E_1^- \le E_2^+ \le \dots \le E_2^+ \le E_1^+.$$

Remark. This principle (Theorem XIII.1 in Reed-Simon [RS79]) asserts that:

(i) $E_\infty^\pm := \lim E_n^\pm$ has $E_\infty^+(A) = \sup \sigma_{ess}(A)$, and $E_\infty^-(A) = \inf \sigma_{ess}(A)$

(ii) If A has $N^\pm$ eigenvalues counting multiplicity in the interval (E_∞^+, ∞) and $(-\infty, E_\infty^-,)$, these eigenvalues are precisely $E_1^\pm, E_2^\pm, \dots, E_{N_\pm}^\pm$ and $E_j^\pm = E_\infty^\pm$ if $j > N_\pm$.

To use the Aizenman–Lieb "lifting" argument, we need to express any positive a^x, $a > 0$, $x \in \mathbb{R}$, in terms of the Beta function wherein a^x reoccurs as some function of a:

Lemma 2.11. *Let $\mathcal{B}$ be the Beta-function:*

$$\mathcal{B}(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1}\, d\tau,$$

with the property

$$\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Also let, for a, $b \in \mathbb{R}$:

$$(a-b)_+ = \begin{cases} a-b & \text{if } b < a, \\ 0 & \text{if } b \ge a. \end{cases}$$

Then for any $\gamma, \mu \in \mathbb{R}$ with $\gamma > 1$ we have:

$$\mu_+^\gamma = \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2}(\mu-\tau)_+ d\tau.$$

Proof.

$$\begin{aligned}
\int_0^\infty \tau^{\gamma-2}(\mu-\tau)_+ d\tau &= \mu_+ \int_0^\infty \tau^{\gamma-2}(1-\frac{\tau}{\mu})_+ \, d\tau \\
&= \mu_+ \int_0^1 (t\mu_+)^{\gamma-2}(1-t)_+\mu_+ \, dt \\
&= \mu_+^\gamma \int_0^1 t^{\gamma-2}(1-t)_+ \, dt \\
&= \mu_+^\gamma \mathcal{B}(\gamma-1,2),
\end{aligned}$$

where our integral delimiters changed as our argument is identically 0 outside of $[0,1]$. □

Thus we are finally equipped to prove our first main result, namely Theorem 1.2. We note here that we will use this method in the subsequent chapters to lift our various Lieb–Thirring-type inequalities to arbitrary moments. Due to the similarity, we will be careful in the discussion here, but give only very brief accounts in later chapters.

Theorem 1.2 Let $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z})$, $\gamma \geq 1$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D satisfy the inequality

$$\sum_{j=1}^N |e_j|^\gamma \leq \eta_1^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2},$$

where

$$\eta_1^\gamma = \frac{\pi}{\sqrt{3}} L^{cl}_{\gamma,1}. \qquad \text{and} \qquad L^{cl}_{\gamma,1} = \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\,\Gamma(\gamma+3/2)}.$$

Proof of Theorem 1.2. Let $\{e_j(\tau)\}_{j=1}^N$ be the negative eigenvalues of the operator $D^*D-(b_n-\tau)_+$, for $\tau > 0$. By the variational principle for the negative eigenvalues $\{-(|e_j|-\tau)_+\}_{j=1}^N$ of the operator $D^*D-(b_n-\tau)$ we have

$$D^*D-(b_n-\tau)_+ \leq D^*D-(b_n-\tau)$$

$$\Rightarrow e_j(\tau) \leq -(|e_j|-\tau)_+,$$

$$\Rightarrow (|e_j|-\tau)_+ \leq |e_j(\tau)|. \tag{2.5}$$

For any $\gamma, \mu \in \mathbb{R}$ with $\gamma > 1$ we have, using Lemma 2.11:

$$\begin{aligned}\sum_{j=1}^{N} |e_j|^\gamma &= \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2} (\sum_{j=1}^{N} |e_j| - \tau)_+ \, d\tau \\ &\leq \frac{1}{\mathcal{B}(\gamma-1,2)}, \int_0^\infty \tau^{\gamma-2} \sum_{j=1}^{N} e_j(\tau)_+ \, d\tau,\end{aligned}$$

by (2.5). Then we can apply our Lieb–Thirring inequality – $(H_D, \mathbb{Z}, \gamma = 1)$:

$$\begin{aligned}\sum_{j=1}^{N} |e_j|^\gamma &\leq \frac{2}{3\sqrt{3}} \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2} \sum_{n\in\mathbb{Z}} (b_n - \tau)_+^{3/2} d\tau \\ &= \frac{2}{3\sqrt{3}} \frac{1}{\mathcal{B}(\gamma-1,2)} \sum_{n\in\mathbb{Z}} b_n^{3/2} \int_0^\infty \tau^{\gamma-2} (1 - \frac{\tau}{b_n})_+^{3/2} d\tau \\ &= \frac{2}{3\sqrt{3}} \frac{1}{\mathcal{B}(\gamma-1,2)} \sum_{n\in\mathbb{Z}} b_n^{3/2} \int_0^1 (s\, b_n)^{\gamma-2} (1-s)_+^{3/2} \, b_n \, ds \\ &= \frac{2}{3\sqrt{3}} \frac{1}{\mathcal{B}(\gamma-1,2)} \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2} \int_0^1 s^{\gamma-2} (1-s)_+^{3/2} \, ds \\ &= \frac{2}{3\sqrt{3}} \frac{1}{\mathcal{B}(\gamma-1,2)} \mathcal{B}(\gamma-1,5/2) \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2}.\end{aligned}$$

Given that

$$\frac{\mathcal{B}(\gamma-1,5/2)}{\mathcal{B}(\gamma-1,2)} = \frac{\Gamma(5/2)\Gamma(\gamma+1)}{\Gamma(2)\Gamma(\gamma+3/2)} = \frac{3\sqrt{\pi}\,\Gamma(\gamma+1)}{4\,\Gamma(\gamma+3/2)},$$

we have:

$$\sum_{j=1}^{N} |e_j|^\gamma \leq \frac{2}{3\sqrt{3}} \frac{3\sqrt{\pi}\,\Gamma(\gamma+1)}{4\,\Gamma(\gamma+3/2)} \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2}.$$

We now recall:

$$L_{\gamma,1}^{cl} = \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\;\Gamma(\gamma+3/2)}.$$

Thus:

$$\sum_{j=1}^{N} |e_j|^\gamma \leq \eta_1^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2}, \qquad \text{where} \quad \eta_2^\gamma = \frac{\pi}{\sqrt{3}} L_{\gamma,1}^{cl}.$$

□

2.6 Tridiagonal Jacobi Matrix Operators

We recall that we let W_1 be a tridiagonal self-adjoint Jacobi matrix

$$W_1 := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & b_{-1} & a_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & 0 & \dots \\ \dots & 0 & a_0 & b_1 & a_1 & \dots \\ \dots & 0 & 0 & a_1 & b_2 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

viewed as a whole-line operator acting on $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, via:

$$(W_1\varphi)(n) = a_{n-1}\varphi(n-1) + b_n\varphi(n) + a_n\varphi(n+1), \quad \text{for } n \in \mathbb{Z}.$$

where $a_n, b_n \in \mathbb{R}$. Alternatively, we will sometimes denote our variables explicitly, i.e. $W_1(\{a_n\},\{b_n\}) := W_1$, where we understand $\{a_n\}$ to mean $\{a_n\}_{n\in\mathbb{Z}}$. We are then interested in perturbations of the special case:

$$(W_1^0\,\varphi)(n) := (W_1(\{a_n \equiv 1\}, \{b_n \equiv 0\})\ \varphi)(n) = \varphi(n-1) + \varphi(n+1)$$

called the free Jacobi matrix. In particular we examine the case where $W_1 - W_1^0$ is compact. Indeed, we can write:

$$W_1 = W_1^0 + (W_1 - W_1^0)$$

and as $W_1 - W_1^0$ is compact, the essential spectra of W_1 and W_0 coincide by Weyl's theorem (see [Wey10]). Bakic and Guljas (see [BG99]) gave a simple description of the compactness relying on the tri-diagonal entries tending to zero rapidly enough, as $n \to \pm\infty$. Thus, in what follows we assume that $a_n \to 1$, $b_n \to 0$ rapidly enough as $n \to \pm\infty$. Then the essential spectrum $\sigma_{ess}(W_1) = \sigma_{ess}(W_1^0) = [-2,2]$ and W_1 may have simple eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$ where $N_\pm \in \overline{\mathbb{N}}$, and

$$E_1^+ > E_2^+ > \dots > 2 > -2 > \dots > E_2^- > E_1^-.$$

Thus our second main result estimates the γ-moments of those eigenvalues:

Theorem 1.4 Let $\gamma \geq 1$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n - 1\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z})$. Then the following inequality holds true for the eigenvalues $\{E_j^{\pm}\}_{j=1}^{N_\pm}$ of the operator W_1:

$$\sum_{j=1}^{N_-} |E_j^- + 2|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - 2|^\gamma \leq \nu_1^\gamma \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/2} + 4 \sum_{n\in\mathbb{Z}} |a_n - 1|^{\gamma+1/2} \right),$$

where

$$\nu_1^\gamma := 3^{\gamma-1} \pi \, L_{\gamma,1}^{cl} \quad \text{where} \quad L_{\gamma,1}^{cl} = \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\,\Gamma(\gamma+3/2)}.$$

Before proceeding with the proof, we require two short Lemmata. Both will be used in subsequent chapters and therefore we shall give the most general cases here, which are to be referred to later. First, we are looking to estimate our perturbed matrix operator W_1 by an operator with the perturbation terms concentrated across the main diagonal, which will then be viewed as the new potential term. This will thus be more closely related to our Lieb–Thirring inequality. We define that for two square matrices A and B, $A \leq B$ means $x^T A x \leq x^T B x$. We need the following matrix inequalities:

$$\begin{pmatrix} -|a_n - 1| & 1 \\ 1 & -|a_n - 1| \end{pmatrix} \leq \begin{pmatrix} 0 & a_n \\ a_n & 0 \end{pmatrix} \leq \begin{pmatrix} |a_n - 1| & 1 \\ 1 & |a_n - 1| \end{pmatrix}, \tag{2.6}$$

to estimate our W_1 from above and below. This is in fact a special case of the following Lemma:

Lemma 2.12. *The following inequalities hold true for square, $m \times m$ matrices. Let $a_n^m, \omega_m \in \mathbb{R}$, then, for all $m \in \mathbb{N}$:*

$$\begin{pmatrix} -|a_n^m - \omega_m| & 0 & \dots & 0 & \omega_m \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \omega_m & 0 & \dots & 0 & -|a_n^m - \omega_m| \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & \dots & 0 & a_n^m \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ a_n^m & 0 & \dots & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} |a_n^m - \omega_m| & 0 & \dots & 0 & \omega_m \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \omega_m & 0 & \dots & 0 & |a_n^m - \omega_m| \end{pmatrix}$$

Proof. For the proof of the second inequality, we have

$$
\begin{pmatrix} x_1 & x_2 & \dots & x_m \end{pmatrix}
\begin{pmatrix}
|a_n^m - \omega_m| & 0 & \dots & 0 & (\omega_m - a_n^m) \\
0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 \\
(\omega_m - a_n^m) & 0 & \dots & 0 & |a_n^m - \omega_m|
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}
$$

$$
\begin{aligned}
&= |a_n^m - \omega_m| x_1^2 + (\omega_m - a_n^m) x_1 x_m + (\omega_m - a_n^m) x_1 x_m + |a_n^m - \omega_m| x_m^2 \\
&\geq \left(\sqrt{|a_n^m - \omega_m|}\, x_1 + \frac{(\omega_m - a_n^m)}{\sqrt{|a_n^m - \omega_m|}} x_m \right)^2 \ \geq \ 0.
\end{aligned}
$$

The proof of the first inequality works similarly. □

This immediately gives us (2.6). We additionally need the following application of Jensen's inequality. For $\alpha, \beta, \gamma, q \in \mathbb{R}, \ q \geq 1$:

$$
(\alpha + \beta + \gamma)^q \leq 3^{q-1}(\alpha^q + \beta^q + \gamma^q). \tag{2.7}
$$

We see this as a special case of:

Lemma 2.13. *Let* $\alpha_i, \ q \in \mathbb{R},$ *for all* $i \in \{1, 2, ..., N\}$ *and* $q \geq 1$. *Then we have:*

$$
(\sum_{i=1}^{N} \alpha_i)^q \leq N^{q-1} (\sum_{i=1}^{N} \alpha_i^q).
$$

Proof. For a convex function $f(x)$, $\lambda_i \in \mathbb{R}^+$, $N \in \mathbb{N}$ and $i \in \{1, 2, ..., N\}$ such that $\sum_{i=1}^{N} \lambda_i = 1$, Jensen's inequality yields

$$
f(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_N x_N) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + ... + \lambda_N f(x_N).
$$

for any $x_1, x_2, ..., x_N \in \mathbb{R}$. Then our result follows by choosing $f(x) = x^q$, $i = N$, $\lambda_i = 1/N$ and setting $x_i = \alpha_i$, for all $i \in \{1, 2, .., N\}$:

$$
(\sum_{i=1}^{N} \alpha_i)^q = N^q (\sum_{i=1}^{N} \frac{\alpha_i}{N})^q \leq N^{q-1} (\sum_{i=1}^{N} \alpha_i^q).
$$

□

We are finally in a position to prove our spectral inequality:

Proof of Theorem 1.4. We apply 2.6, i.e.:

$$\begin{pmatrix} -|a_n-1| & 1 \\ 1 & -|a_n-1| \end{pmatrix} \le \begin{pmatrix} 0 & a_n \\ a_n & 0 \end{pmatrix} \le \begin{pmatrix} |a_n-1| & 1 \\ 1 & |a_n-1| \end{pmatrix},$$

repeatedly at each point of indices:

$$W_1(\{a_n \equiv 1\}, \{b_n^{(-)}\}) \le W_1(\{a_n\}, \{b_n\}) \le W_1(\{a_n \equiv 1\}, \{b_n^{(+)}\}),$$

i.e.

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b_{-1}^{(-)} & 1 & 0 & 0 & \dots \\ \dots & 1 & b_0^{(-)} & 1 & 0 & \dots \\ \dots & 0 & 1 & b_1^{(-)} & 1 & \dots \\ \dots & 0 & 0 & 1 & b_2^{(-)} & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \le \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b_{-1} & a_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & 0 & \dots \\ \dots & 0 & a_0 & b_1 & a_1 & \dots \\ \dots & 0 & 0 & a_1 & b_2 & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \le \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b_{-1}^{(+)} & 1 & 0 & 0 & \dots \\ \dots & 1 & b_0^{(+)} & 1 & 0 & \dots \\ \dots & 0 & 1 & b_1^{(+)} & 1 & \dots \\ \dots & 0 & 0 & 1 & b_2^{(+)} & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $b_n^{(\pm)} = b_n \pm \big(|a_{n-1}-1| + |a_n-1|\big)$. Now we have:

$$-D^*D + b_n \;=\; \varphi(n+1) + \varphi(n-1) + (b_n-2)\varphi(n) \;=\; W(\{a_n \equiv 1\}, \{b_n-2\}),$$

$$-D^*D + 4 + b_n \;=\; \varphi(n+1) + \varphi(n-1) + (b_n+2)\varphi(n) \;=\; W(\{a_n \equiv 1\}, \{b_n+2\}).$$

Remark. These operators have opposite sign to the ones we used in Sections 4 and 5. The reason for this is to keep the previous Theorems in line with standard literature on Lieb–Thirring inequalities, and the following in line with the result by D. Hundertmark and B. Simon in [HS02].

Now $(E_j^+ - 2)$ are positive eigenvalues of $W_1(\{a_n\}, \{b_n-2\})$. Thus by using the above inequalities, and the Variational Principle (Theorem 2.10), we have:

$$W_1(\{a_n\}, \{b_n-2\}) \le W_1(\{a_n \equiv 1\}, \{b_n^{(+)}-2\})$$

$$\Rightarrow \quad |E_j^+ - 2| \le e_j^+, \tag{2.8}$$

where e_j^+ are the positive eigenvalues of $W(\{a_n \equiv 1\}, \{b_n^{(+)} - 2\}) = -D^*D + b_n^{(+)}$.

Let us now define $(b_n)_+ := \max(b_n, 0)$, $(b_n)_- := -\min(b_n, 0)$. By the Lieb–Thirring inequality – $(D, \mathbb{Z})$ for positive eigenvalues, i.e. Corollary 1.3, we have:

$$\sum_{j=1}^{N_+} (e_j^+)^\gamma \le \frac{\pi}{\sqrt{3}} L_{\gamma,1}^{cl} \sum_{n\in\mathbb{Z}} (b_n^{(+)})_+^{\gamma+1/2}.$$

Thus applying (2.8):

$$\sum_{j=1}^{N_+} |E_j^+ - 2|^\gamma \le \frac{\pi}{\sqrt{3}} L_{\gamma,1}^{cl} \sum_{n\in\mathbb{Z}} \Big((b_n)_+ + |a_{n-1} - 1| + |a_n - 1|\Big)^{\gamma+1/2}. \tag{2.9}$$

We follow the same method for our negative eigenvalues and obtain:

$$\sum_{j=1}^{N_-} |E_j^- + 2|^\gamma \le \frac{\pi}{\sqrt{3}} L_{\gamma,1}^{cl} \sum_{n\in\mathbb{Z}} \Big((b_n)_- + |a_{n-1} - 1| + |a_n - 1|\Big)^{\gamma+1/2}. \tag{2.10}$$

We apply (2.7), i.e.:

$$(\alpha + \beta + \gamma)^q \le 3^{q-1}(\alpha^q + \beta^q + \gamma^q),$$

to each of (2.9) and (2.10) as follows:

$$\Big((b_n)_\pm + |a_{n-1} - 1| + |a_n - 1|\Big)^{\gamma+1/2} \le 3^{\gamma-1/2}\Big((b_n)_\pm^{\gamma+1/2} + |a_{n-1} - 1|^{\gamma+1/2} + |a_n - 1|^{\gamma+1/2}\Big),$$

Then we combine our inequalities to finally arrive at

$$\begin{aligned}
\sum_{j=1}^{N_-} |E_j^- + 2|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - 2|^\gamma \quad &\le \quad 3^{\gamma-1/2} \frac{\pi}{\sqrt{3}} L_{\gamma,1}^{cl} \sum_{n\in\mathbb{Z}} \Big((b_n)_-^{\gamma+1/2} + |a_{n-1} - 1|^{\gamma+1/2} + |a_n - 1|^{\gamma+1/2} \\
&\qquad + (b_n)_+^{\gamma+1/2} + |a_{n-1} - 1|^{\gamma+1/2} + |a_n - 1|^{\gamma+1/2}\Big) \\
&\le \quad \nu_1^\gamma \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/2} + 4 \sum_{n\in\mathbb{Z}} |a_n - 1|^{\gamma+1/2} \right),
\end{aligned}$$

where

$$\nu_1^\gamma := 3^{\gamma-1} \pi L_{\gamma,1}^{cl} \quad \text{and} \quad L_{\gamma,1}^{cl} := \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\,\Gamma(\gamma+3/2)}, \quad \gamma \ge 1.$$

□

Chapter 3

Spectral Bounds for Pentadiagonal Jacobi-type Operators

This chapter serves as a bridge between the previous chapter and the more involved proofs in the subsequent chapter. We will thus reproduce the previous methods in somewhat shorter form, leaving only the key steps. We take most ideas and results in this chapter as an intuitive starting point for the generalisation to higher order operators in the next chapter.

In Section 3.1, we again identify our operators and their properties. We then employ a rather simple argument to obtain Agmon–Kolmogorov inequalities for second order difference operators in Section 3.2. We continue similarly as in the previous chapter, where in Sections 3.3 and 3.4 we prove the relevant Generalised Sobolev and Lieb–Thirring inequalities. Finally, we apply our results to the so called pentadiagonal Jacobi-type matrix operators in Section 3.5.

3.1 The Operators Δ_D, Δ_D^2 and H_D^2

We thus turn our attention to functional inequalities and spectral bounds related to the second order difference operator $\Delta_D := D^*D$ and the following fourth order difference operator: For a sequence $\varphi \in \ell^2(\mathbb{Z})$, we define $(\Delta_D^2\varphi)(n) := (\Delta_D(\Delta_D\varphi))(n)$, or explicitly:

$$(\Delta_D^2\varphi)(n) := \varphi(n+2) - 4\,\varphi(n+1) + 6\,\varphi(n) - 4\,\varphi(n-1) + \varphi(n-2).$$

We identify its essential spectrum, by applying our discrete Fourier Transform, and thus identifying the symbol:

$$\begin{aligned}
\mathcal{F}(\Delta_D^2\psi)(x) &= \sum_{n\in\mathbb{Z}} e^{inx}\Big(\psi_j(n+2) - 4\psi_j(n+1) + 6\psi_j(n) - 4\psi_j(n-1) + \psi_j(n-2)\Big) \\
&= \sum_{n\in\mathbb{Z}} e^{i(n-2)x}\psi_j(n) - \sum_{n\in\mathbb{Z}} 4e^{i(n-1)x}\psi_j(n) + \sum_{n\in\mathbb{Z}} 6e^{inx}\psi_j(n) \\
&\quad - \sum_{n\in\mathbb{Z}} 4e^{i(n+1)x}\psi_j(n) + \sum_{n\in\mathbb{Z}} e^{i(n+2)x}\psi_j(n) \\
&= \sum_{n\in\mathbb{Z}} e^{inx}\Big(e^{-2ix} - 4e^{-ix} + 6 - 4e^{ix} + e^{2ix}\Big)\psi_j(n) \\
&= \Big(6 - 8\cos(x) + 2\cos(2x)\Big)(\mathcal{F}\psi)(x).
\end{aligned}$$

Therefore, the essential spectrum coincides with the image of $f(x) := 6 - 8\cos(x) + 2\cos(2x)$, i.e. the interval $[0, 16]$.

We will be interested in spectral inequalities for discrete spectra of fourth order operators perturbed by a potential term. In essence, we extend the discrete version of the well-known Schrödinger operator:

Definition 3.1. For a sequence $\{\varphi(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we let $b = \{b_n\}_{n\in\mathbb{Z}}$ be the multiplication operator defined on $\ell^2(\mathbb{Z})$ via $(b\,\varphi)(n) := b_n\varphi(n)$. If $b_n \geq 0$ then the fourth order discrete Schrödinger-type operator will be denoted by H_D^2, where:

$$(H_D^2\,\varphi)(n) := (\Delta_D^2 - b)\varphi(n).$$

Remark. We note here that H_D^2 may in fact have negative, simple eigenvalues. We thus are interested in obtaining a bound for the sum of these eigenvalues.

3.2 Agmon–Kolmogorov Inequality – $(\Delta_D, \mathbb{Z})$

We remind ourselves of the work by E. T. Copson in [Cop79] where the author finds a sharp inequality of Agmon-type for second order difference operators. Indeed he found, for a square-summable sequence $\varphi \in \ell^2(\mathbb{Z})$:

$$\|D\varphi\|_{\ell^2(\mathbb{Z})} \leq \|\varphi\|^{1/2}_{\ell^2(\mathbb{Z})} \|D^2\varphi\|^{1/2}_{\ell^2(\mathbb{Z})} .$$

H. A. Gindler and J. A. Goldstein in [GG81], gave an alternative proof using a simple, more general fact. We extract their method to our discrete operator, giving us the following inequality:

Proposition 3.2 (Agmon–Kolmogorov Inequality – $(\Delta_D, \mathbb{Z})$)**.** *Let $\varphi \in \ell^2(\mathbb{Z})$. We then have the following inequality:*

$$\|\varphi\|_{\ell^\infty(\mathbb{Z})} \leq \|\varphi\|^{3/4}_{\ell^2(\mathbb{Z})} \|\Delta_D\varphi\|^{1/4}_{\ell^2(\mathbb{Z})}.$$

Proof. By our discrete Agmon inequality (Proposition 2.7), we have:

$$|\varphi(n)|^2 \leq \|\varphi\|_{\ell^2(\mathbb{Z})} \|D\varphi\|_{\ell^2(\mathbb{Z})}.$$

Now we obtain a bound for $\|D\varphi\|_{\ell^2(\mathbb{Z})}$:

$$\|D\varphi\|^2_{\ell^2(\mathbb{Z})} = \langle D\varphi, D\varphi\rangle = \langle D^* D\varphi, \varphi\rangle \leq \|D^* D\varphi\|_{\ell^2(\mathbb{Z})} \|\varphi\|_{\ell^2(\mathbb{Z})},$$

by Cauchy's inequality. Therefore we have:

$$\begin{aligned} |\varphi(n)|^2 &\leq \|\varphi\|_{\ell^2(\mathbb{Z})} \|D\varphi\|_{\ell^2(\mathbb{Z})} \\ &\leq \|\varphi\|_{\ell^2(\mathbb{Z})} \|\varphi\|^{1/2}_{\ell^2(\mathbb{Z})} \|D^* D\varphi\|^{1/2}_{\ell^2(\mathbb{Z})} \\ &= \|\varphi\|^{3/2}_{\ell^2(\mathbb{Z})} \|\Delta_D\varphi\|^{1/2}_{\ell^2(\mathbb{Z})}. \end{aligned}$$

□

3.3 Generalised Sobolev Inequality – $(\Delta_D, \mathbb{Z})$

We continue as in the previous chapter, by obtaining the following inequality:

Proposition 3.3 (Generalised Sobolev Inequality – $(\Delta_D, \mathbb{Z})$)**.** *Let $\{\psi_j\}_{j=1}^N$ be an orthonormal system of sequences in $\ell^2(\mathbb{Z})$, i.e. $\langle \psi_j, \psi_k \rangle = \delta_{jk}$, and let $\rho(n) := \sum_{j=1}^N |\psi_j(n)|^2$. Then*

$$\sum_{n\in\mathbb{Z}} \rho^5(n) \leq \sum_{j=1}^N \sum_{n\in\mathbb{Z}} |\Delta_D \psi_j(n)|^2 .$$

Proof. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{C}^N$. We follow exactly the same method as before, and hence move through the argument swiftly. By our Agmon–Kolmogorov inequality – $(\Delta_D, \mathbb{Z})$, i.e. Proposition (3.2), we have:

$$\begin{aligned}
\Big|\sum_{j=1}^N \xi_j \psi_j(n)\Big|^2 &\leq \Big\|\sum_{j=1}^N \xi_j \psi_j\Big\|_{\ell^2(\mathbb{Z})}^{3/2} \Big\|\Delta_D \sum_{j=1}^N \xi_j \psi_j\Big\|_{\ell^2(\mathbb{Z})}^{1/2} \\
&= \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \sum_{n\in\mathbb{Z}} \psi_j(n)\,\overline{\psi_k(n)}\Big)^{3/4} \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \sum_{n\in\mathbb{Z}} \Delta_D \psi_j(n)\,\overline{\Delta_D \psi_k(n)}\Big)^{1/4} \\
&\leq \Big(\sum_{j=1}^N |\xi_j|^2\Big)^{3/4} \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \langle \Delta_D \psi_j, \Delta_D \psi_k \rangle\Big)^{1/4}.
\end{aligned}$$

We set $\xi_j := \overline{\psi_j(n)}$, and as $\rho(n) = \sum_{j=1}^N |\psi_j(n)|^2$:

$$\begin{aligned}
\rho^2(n) &\leq \rho^{3/4}(n)\Big(\sum_{j,k=1}^N \psi_j(n)\overline{\psi_k(n)} \langle \Delta_D \psi_j, \Delta_D \psi_k \rangle\Big)^{1/4} \\
\Rightarrow \rho^5(n) &\leq \sum_{j,k=1}^N \psi_j(n)\overline{\psi_k(n)} \langle \Delta_D \psi_j, \Delta_D \psi_k \rangle \\
\Rightarrow \sum_{n\in\mathbb{Z}} \rho^5(n) &\leq \sum_{j=1}^N \Big(\sum_{n\in\mathbb{Z}} |\Delta_D \psi_j(n)|^2\Big).
\end{aligned}$$

$\square$

3.4 Lieb–Thirring Inequality – $(H_D^2, \mathbb{Z})$

As before, we apply the discrete second order Agmon and Generalised Sobolev inequality to the 4th order discrete Schrödinger type operator to obtain bounds for its negative eigenvalues.

Let $\{\psi_j\}_{j=1}^N$, $N \in \overline{\mathbb{N}}$ be the orthonormal system of eigensequences in $\ell^2(\mathbb{Z})$ corresponding to the negative eigenvalues $\{e_j\}_{j=1}^N$ of the fourth order discrete Schrödinger-type operator:

$$(H_D^2 \psi_j)(n) := (\Delta_D^2 \psi_j)(n) - b_n \psi_j(n) = e_j \psi_j(n), \tag{3.1}$$

where $j \in \{1, \dots, N\}$ and we assume that $b_n \geq 0$, for all $n \in \mathbb{Z}$. Our next result is concerned with estimating those negative eigenvalues:

Theorem 3.4 (Lieb–Thirring Inequality – $(H_D^2, \mathbb{Z}, \gamma = 1)$)**.** *Let $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{5/4}(\mathbb{Z})$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D^2 are discrete and they satisfy the inequality*

$$\sum_{j=1}^N |e_j| \leq \frac{4}{5^{5/4}} \sum_{n\in\mathbb{Z}} b_n^{5/4}.$$

Proof. By taking the inner product with $\psi_j(n)$ on (3.1) and summing both sides of the equation with respect to j, we have:

$$\sum_{j=1}^N e_j = \sum_{j=1}^N \Big(\sum_{n\in\mathbb{Z}} |\Delta_D \psi_j(n)|^2 \Big) - \sum_{j=1}^N \Big(\sum_{n\in\mathbb{Z}} b_n |\psi_j(n)|^2 \Big).$$

We now use the Generalised Sobolev inequality – $(\Delta_D, \mathbb{Z})$, (Proposition 3.3), and Hölder's inequality, to obtain:

$$\sum_{j=1}^N e_j \geq \sum_{n\in\mathbb{Z}} \Big(\sum_{j=1}^N |\psi_j(n)|^2 \Big)^5 - \Big(\sum_{n\in\mathbb{Z}} b_n^{5/4} \Big)^{4/5} \Big(\sum_{n\in\mathbb{Z}} \Big(\sum_{j=1}^N |\psi_j(n)|^2 \Big)^5 \Big)^{1/5}. \tag{3.2}$$

Define

$$\chi := \Big(\sum_{n\in\mathbb{Z}} \Big(\sum_{j=1}^N |\psi_j(n)|^2 \Big)^5 \Big)^{1/5}, \quad \varsigma := \Big(\sum_{n\in\mathbb{Z}} b_n^{5/4} \Big)^{4/5}.$$

The latter inequality can be written as

$$\chi^5 - \varsigma\chi \leq \sum_{j=1}^N e_j.$$

The LHS is maximal when

$$\chi = \frac{1}{5^{1/4}} \Big(\sum_{n \in \mathbb{Z}} b_n^{5/4} \Big)^{1/5}.$$

Substituting this into (3.2), we obtain:

$$\begin{aligned} \sum_{j=1}^{N} e_j \quad &\geq \quad \frac{1}{5^{5/4}} \sum_{n\in\mathbb{Z}} b_n^{5/4} - \frac{1}{5^{1/4}} \sum_{n\in\mathbb{Z}} b_n^{5/4} \\ &= \quad -\frac{4}{5^{5/4}} \sum_{n\in\mathbb{Z}} b_n^{5/4}. \end{aligned}$$

Therefore:

$$\sum_{j=1}^{N} |e_j| \leq \frac{4}{5^{5/4}} \sum_{n\in\mathbb{Z}} b_n^{5/4}.$$

□

We lift this result again to higher moments using the Aizenman–Lieb Procedure, to obtain:

Theorem 1.5 Let $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/4}(\mathbb{Z})$, $\gamma \geq 1$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D^2 satisfy the inequality

$$\sum_{j=1}^{N} |e_j|^{\gamma} \quad \leq \quad \eta_2^{\gamma} \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/4},$$

where

$$\eta_2^{\gamma} = \frac{4}{5^{5/4}} \frac{\Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

Proof of Theorem 1.5. Let $\{e_j(\tau)\}_{j=1}^N$ be the negative eigenvalues of the operator $\Delta_D^2 - (b_n - \tau)_+$. By the variational principle for the negative eigenvalues $\{-(|e_j| - \tau)_+\}_1^N$ of the operator $\Delta_D^2 - (b_n - \tau)$ we have

$$(|e_j| - \tau)_+ \leq |e_j(\tau)|.$$

Therefore, for any $\gamma > 1$, we apply our Aizenman–Lieb procedure, as before:

$$\begin{aligned}
\sum_{j=1}^{N} |e_j|^\gamma &\le \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2} \sum_{j=1}^{N} e_j(\tau)_+ \, d\tau \\
&\le \frac{4}{5^{5/4}} \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2} \sum_{n\in\mathbb{Z}} (b_n-\tau)_+^{5/4} d\tau \\
&= \frac{4}{5^{5/4}} \frac{1}{\mathcal{B}(\gamma-1,2)} \sum_{n\in\mathbb{Z}} b_n^{5/4} \int_0^\infty \tau^{\gamma-2} (1-\frac{\tau}{b_n})_+^{5/4} d\tau \\
&= \frac{4}{5^{5/4}} \frac{1}{\mathcal{B}(\gamma-1,2)} \sum_{n\in\mathbb{Z}} b_n^{5/4} \int_0^1 (s\, b_n)^{\gamma-2} (1-s)_+^{5/4} \, b_n \, ds \\
&= \frac{4}{5^{5/4}} \frac{1}{\mathcal{B}(\gamma-1,2)} \mathcal{B}(\gamma-1,9/4) \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/4} \\
&\le \eta_2^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/4},
\end{aligned}$$

where

$$\eta_2^\gamma := \frac{4}{5^{5/4}} \frac{\mathcal{B}(\gamma-1,9/4)}{\mathcal{B}(\gamma-1,2)} = \frac{4}{5^{5/4}} \frac{\Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

□

3.5 Pentadiagonal Jacobi-type Operators

We let W_2 be a pentadiagonal self-adjoint Jacobi-type matrix operator:

$$W_2 := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & b_{-1} & a_{-1} & c_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & c_0 & 0 & \dots \\ \dots & c_{-1} & a_0 & b_1 & a_1 & c_1 & \dots \\ \dots & 0 & c_0 & a_1 & b_2 & a_2 & \dots \\ \dots & 0 & 0 & c_1 & a_2 & b_3 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

viewed as a whole-line operator acting on $\ell^2(\mathbb{Z})$, via:

$$(W_2\varphi)(n) = c_{n-2}\varphi(n-2) + a_{n-1}\varphi(n-1) + b_n\varphi(n) + a_n\varphi(n+1) + c_n\varphi(n+2), \quad \text{for } n \in \mathbb{Z},$$

where a_n, b_n, $c_n \in \mathbb{R}$. Again we denote $(W_2(\{a_n\},\{b_n\},\{c_n\})\varphi)(n) := (W_2\varphi)(n)$. We are then interested in perturbations of the special case:

$$(W_2^0\varphi)(n) \;:=\; (W_2(\{a_n \equiv -4\},\{b_n \equiv 0\},\{c_n \equiv 1\})\varphi)(n) \;=\; \varphi(n-2) - 4\varphi(n-1) - 4\varphi(n+1) + \varphi(n+2).$$

which we shall call the free pentadiagonal Jacobi-type matrix. As in the previous chapter, we examine the case where $W_2 - W_2^0$ is compact. Thus in what follows we assume that $a_n \to -4$, $b_n \to 0$, $c_n \to 1$ rapidly enough as $n \to \pm\infty$. Then the essential spectrum σ_{ess} is given by $\sigma_{ess}(W_2) = \sigma_{ess}(W_2^0) = [-6, 10]$ and W_2 may have simple eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$ where $N_\pm \in \overline{\mathbb{N}}$, and

$$E_1^+ > E_2^+ > \dots > 10 > -6 > \dots > E_2^- > E_1^-.$$

We thus reiterate Theorem 1.7, a bound for exactly these eigenvalues:

Theorem 1.7 Let $\gamma \geq 1$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n+4\}_{n\in\mathbb{Z}}$, $\{c_n-1\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/4}(\mathbb{Z})$. Then for the eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$ of the operator W_2 we have:

$$\sum_{j=1}^{N_-} |E_j^- + 6|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - 10|^\gamma \leq \nu_2^\gamma \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/4} + 4\sum_{n\in\mathbb{Z}} |a_n+4|^{\gamma+1/4} + 4\sum_{n\in\mathbb{Z}} |c_n-1|^{\gamma+1/4} \right),$$

where

$$\nu_2^\gamma := 5^{\gamma-2}\,\frac{4\,\Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

Proof of Theorem 1.7. In the pentadiagonal matrix case, we have to combine two matrix inequalities, both of which are special cases of Lemma 2.12. We illustrate this method with simple block matrices. We have, for all $n \in \mathbb{Z}$:

$$\begin{pmatrix} -|c_n-1| & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -|c_n-1| \end{pmatrix} \le \begin{pmatrix} 0 & 0 & c_n \\ 0 & 0 & 0 \\ c_n & 0 & 0 \end{pmatrix} \le \begin{pmatrix} |c_n-1| & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & |c_n-1| \end{pmatrix},$$

and

$$\begin{pmatrix} -|a_n+4| & -4 \\ -4 & -|a_n+4| \end{pmatrix} \le \begin{pmatrix} 0 & a_n \\ a_n & 0 \end{pmatrix} \le \begin{pmatrix} |a_n+4| & -4 \\ -4 & |a_n+4| \end{pmatrix}.$$

We now extend these two inequalities to infinite-dimensional matrices and combine them by addition. Then these imply, by repeated use at each point of indices:

$$W_2(\{a_n \equiv -4\},\{b_n^{(-)}\},\{c_n \equiv 1\}) \le W_2(\{a_n\},\{b_n\},\{c_n\}) \le W_2(\{a_n \equiv -4\},\{b_n^{(+)}\},\{c_n \equiv 1\}), \tag{3.3}$$

i.e.

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b_{-1} & a_{-1} & c_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & c_0 & 0 & \dots \\ \dots & c_{-1} & a_0 & b_1 & a_1 & c_1 & \dots \\ \dots & 0 & c_0 & a_1 & b_2 & a_2 & \dots \\ \dots & 0 & 0 & c_1 & a_2 & b_2 & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ge \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b_{-1}^{(-)} & -4 & 1 & 0 & 0 & \dots \\ \dots & -4 & b_0^{(-)} & -4 & 1 & 0 & \dots \\ \dots & 1 & -4 & b_1^{(-)} & -4 & 1 & \dots \\ \dots & 0 & 1 & -4 & b_2^{(-)} & -4 & \dots \\ \dots & 0 & 0 & 1 & -4 & b_3^{(-)} & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & b_{-1} & a_{-1} & c_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & c_0 & 0 & \dots \\ \dots & c_{-1} & a_0 & b_1 & a_1 & c_1 & \dots \\ \dots & 0 & c_0 & a_1 & b_2 & a_2 & \dots \\ \dots & 0 & 0 & c_1 & a_2 & b_3 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \le \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & b_{-1}^{(+)} & -4 & 1 & 0 & 0 & \dots \\ \dots & -4 & b_0^{(+)} & -4 & 1 & 0 & \dots \\ \dots & 1 & -4 & b_1^{(+)} & -4 & 1 & \dots \\ \dots & 0 & 1 & -4 & b_2^{(+)} & -4 & \dots \\ \dots & 0 & 0 & 1 & -4 & b_3^{(+)} & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $b_n^{(\pm)} := b_n \pm (|a_{n-1}+4| + |a_n+4|) \pm (|c_{n-2}-1| + |c_n - 1|)$.

We also have:

$$\begin{aligned} \Delta_D^2 + b_n &= \varphi(n-2) - 4\varphi(n-1) - 4\varphi(n+1) + \varphi(n+2) + (b_n+6)\varphi(n) \\ &= W_2(\{a_n \equiv -4\}, \{b_n+6\}, \{c_n \equiv 1\}), \end{aligned} \tag{3.4}$$

and:

$$\begin{aligned} \Delta_D^2 - 16 + b_n &= \varphi(n-2) - 4\varphi(n-1) - 4\varphi(n+1) + \varphi(n+2) + (b_n-10)\varphi(n) \\ &= W_2(\{a_n \equiv -4\}, \{b_n-10\}, \{c_n \equiv 1\}). \end{aligned} \tag{3.5}$$

Now $(E_j^+ - 10)$ are positive eigenvalues of $W(\{a_n\}, \{b_n - 10\}, \{c_n\})$. Thus by using (3.3) and the Variational Principle, Theorem 2.10, we have:

$$W_2(\{a_n\}, \{b_n-10\}, \{c_n\}) \le W_2(\{a_n \equiv -4\}, \{b_n^{(+)} - 10\}, \{c_n \equiv 1\})$$

$$\Rightarrow |E_j^+ - 10| \le e_j^+, \tag{3.6}$$

where e_j^+ are the positive eigenvalues of $W(\{a_n \equiv -4\}, \{b_n^{(+)} - 10\}, \{c_n \equiv 1\}) = \Delta_D^2 - 16 + b_n^{(+)}$. Let us define as before, $(b_n)_+ := \max(b_n, 0)$, $(b_n)_- := -\min(b_n, 0)$. Then by the Lieb–Thirring inequality – $(\Delta_D, \mathbb{Z})$, for positive eigenvalues, i.e. Corollary 1.6, we have:

$$\sum_{j=1}^{N_+} (e_j^+)^\gamma \le \eta_2^\gamma \sum_{n\in\mathbb{Z}} (b_n^{(+)})_+^{\gamma+1/4}.$$

where

$$\eta_2^\gamma := \frac{4}{5^{5/4}} \frac{\Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

Thus applying (3.6), we obtain:

$$\sum_{j=1}^{N_+} |E_j^+ - 10|^\gamma \leq \eta_2^\gamma \sum_{n\in\mathbb{Z}} \Big((b_n)_+ + |a_{n-1}+4| + |a_n+4| + |c_{n-2}-1| + |c_n-1| \Big)^{\gamma+1/4}, \tag{3.7}$$

Following a similar procedure for (3.4) and using Theorem 1.5 we find:

$$\sum_{j=1}^{N_-} |E_j^- + 6|^\gamma \leq \eta_2^\gamma \sum_{n\in\mathbb{Z}} \Big((b_n)_- + |a_{n-1}+4| + |a_n+4| + |c_{n-2}-1| + |c_n-1| \Big)^{\gamma+1/4}. \tag{3.8}$$

Now we apply the 5-element case of Lemma 2.13, i.e.

$$(\alpha+\beta+\gamma+\delta+\epsilon)^q \leq 5^{q-1}(\alpha^q+\beta^q+\gamma^q+\delta^q+\epsilon^q).$$

to each of (3.7) and (3.8), via:

$$\begin{aligned}&\Big((b_n)_\pm + (|a_{n-1}+4| + |a_n+4|) + (|c_{n-2}-1| + |c_n-1|) \Big)^{\gamma+1/4} \\ &\qquad \leq 5^{\gamma-3/4}\Big((b_n)_\pm^{\gamma+1/4} + |a_{n-1}+4|^{\gamma+1/4} + |a_n+4|^{\gamma+1/4} + |c_{n-2}-1|^{\gamma+1/4} + |c_n-1|^{\gamma+1/4} \Big),\end{aligned}$$

and combine our inequalities to finally arrive at:

$$\begin{aligned}\sum_{j=1}^{N_-} |E_j^- + 6|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - 10|^\gamma \quad &\leq \quad 5^{\gamma-3/4}\, \eta_2^\gamma \sum_{n\in\mathbb{Z}} \Big((b_n)_-^{\gamma+1/4} + |a_{n-1}+4|^{\gamma+1/4} + |a_n+4|^{\gamma+1/4} \\ &\qquad + |c_{n-2}-1|^{\gamma+1/4} + |c_n-1|^{\gamma+1/4} + (b_n)_+^{\gamma+1/4} + |a_{n-1}+4|^{\gamma+1/4} \\ &\qquad + |a_n+4|^{\gamma+1/4} + |c_{n-2}-1|^{\gamma+1/4} + |c_n-1|^{\gamma+1/4} \Big) \\ &\leq \quad \nu_2^\gamma \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/4} + 4\sum_{n\in\mathbb{Z}} |a_n+4|^{\gamma+1/4} + 4\sum_{n\in\mathbb{Z}} |c_n-1|^{\gamma+1/4} \right),\end{aligned}$$

where

$$\nu_2^\gamma := 5^{\gamma-2} \frac{4\, \Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

□

Chapter 4

Spectral Bounds for Polydiagonal Jacobi-type Operators

We finally generalise the methods and ideas in the previous two chapters, and apply them to discrete Laplacians of arbitrary order and then polydiagonal Jacobi-type matrix operators. In contrast, we focus our attention on the combinatorial techniques required to construct and study our operators, and thus move over similarly seen methods quickly.

In Section 4.1, we introduce our operator and identify its spectrum. In Section 4.2, we prove the Discrete Agmon–Kolmogorov inequality for the operator D^σ, leading again to the Generalised Sobolev Inequality for the same operator in Section 4.3. In Section 4.4, we obtain the Lieb–Thirring inequality for an arbitrary order Schrödinger-type operator. We finally use this inequality to prove spectral bounds for polyidiagonal Jacobi-type matrix operators in Section 4.5.

4.1 The Operators Δ_D^σ and H_D^σ

We are now interested in arbitrary order difference operators, and specifically in our discrete Laplacian of aribtrary order. For $\sigma \in \mathbb{N}$, $n \in \mathbb{Z}$ and a sequence $\varphi \in \ell^2(\mathbb{Z})$, Δ_D^σ will be defined by:

$$(\Delta_D^\sigma \varphi)(n) \coloneqq (\Delta_D(\Delta_D^{\sigma-1}\varphi))(n).$$

Remark. We note here that we are in fact are talking about a difference operator of order 2σ. As seen before, Δ_D being self-adjoint immediately implies that Δ_D^σ is also self-adjoint.

We will obtain an explicit formula for Δ_D^σ, requiring a few combinatorial techniques. We first observe the following pattern:

$$\begin{aligned}
(\Delta_D\varphi)(n) &= -\varphi(n+1) - \varphi(n-1) + 2\varphi(n) \\
(\Delta_D^2\varphi)(n) &= \varphi(n+2) - 4\,\varphi(n+1) + 6\,\varphi(n) - 4\,\varphi(n-1) + \varphi(n-2), \\
(\Delta_D^3\varphi)(n) &= -\varphi(n+3) + 6\,\varphi(n+2) - 15\,\varphi(n+1) + 20\varphi(n) - \\
&\quad 15\,\varphi(n-1) + 6\varphi(n-2) - \varphi(n-3).
\end{aligned}$$

The coefficients for Δ_D, Δ_D^2 and Δ_D^3 coincide with the second, fourth and sixth row (denoted r) of Pascal's triangle respectively, considering a superimposed pattern of alternating signature:

$$\begin{array}{rccccccccccccc}
r=0: & & & & & & & 1 & & & & & & \\
r=1: & & & & & & 1 & & 1 & & & & & \\
\Delta_D \to r=2: & & & & & -1 & & +2 & & -1 & & & & \\
r=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
\Delta_D^2 \to r=4: & & & 1 & & -4 & & 6 & & -4 & & 1 & & \\
r=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
\Delta_D^3 \to r=6: & -1 & & 6 & & -15 & & 20 & & -15 & & 6 & & -1
\end{array}$$

In order to use this observation to obtain the formula for Δ_D^σ, we need the following standard identities for binomial coefficients:

Lemma 4.1. *Let* ${}^aC_b := \binom{a}{b} := a!/((a-b)!b!)$*, for* $a, b \in \mathbb{Z}$*. Then we have:*

$$\begin{array}{lll}
(i) & {}^aC_b + {}^aC_{b+1} & = {}^{a+1}C_{b+1}, \\
(ii) & 2\,{}^aC_0 + {}^aC_1 & = {}^{a+2}C_1, \\
(iii) & 2\,{}^aC_a + {}^aC_{a-1} & = {}^{a+2}C_{a+1}.
\end{array}$$

Proof.

(i) $$LHS = \frac{a!}{(a-b)!b!} + \frac{a!}{(a-b-1)!(b+1)!} = \frac{a!(b+1)}{(a-b)!(b+1)!} + \frac{a!(a-b)}{(a-b)!(b+1)!} = \frac{(a+1)!}{(a-b)!(b+1)!}$$

Now applying (i), and realising ${}^aC_0 = {}^{a+k}C_0 = {}^aC_a = {}^{a+k}C_{a+k} = 1$:

$$\text{(ii)} \qquad LHS = {}^aC_0 + {}^aC_0 + {}^aC_1 \; = \; {}^aC_0 + {}^{a+1}C_1 \; = \; {}^{a+1}C_0 + {}^{a+1}C_1 \; = \; {}^{a+2}C_1$$

$$\text{(iii)} \qquad LHS = {}^aC_a + {}^aC_a + {}^aC_{a-1} \; = \; {}^aC_a + {}^{a+1}C_a \; = \; {}^{a+1}C_{a+1} + {}^{a+1}C_a \; = \; {}^{a+2}C_{a+1}$$

□

Remark. We note here, that these identities define Pascal's Triangle, where a would symbolise the row number, and b the position in that row. So for example, (i) shows that two adjacent numbers (positions b and $b+1$) add up to the number in the row below $(a+1)$, with the same position of the second number $(b+1)$.

Having established the rules for dealing with those coefficients, we now give our formula for the σ^{th} order discrete Laplacian operator:

Proposition 4.2. *Let $\{\varphi(n)\}_{n\in\mathbb{Z}}$ be a sequence in $\ell^2(\mathbb{Z})$, then our σ^{th} order discrete Laplacian Δ_D^σ takes the following explicit form:*

$$(\Delta_D^\sigma \varphi)(n) = \sum_{k=0}^{2\sigma} {}^{2\sigma}C_k (-1)^{k+\sigma} \varphi(n-\sigma+k).$$

Proof. We proceed by induction, for $\sigma = 1$:

$$\begin{aligned} (\Delta_D^1 \varphi)(n) &= \sum_{k=0}^{2} {}^2C_k (-1)^{k+1} \varphi(n-1+k) \\ &= -{}^2C_0\, \varphi(n-1) + {}^2C_1\, \varphi(n) - {}^2C_2\, \varphi(n+1) \\ &= -\varphi(n+1) + 2\varphi(n) - \varphi(n-1), \end{aligned}$$

which coincides with our direct computation.

Now for the inductive step, we assume the formula holds for $\sigma = \nu$ and apply Δ_D:

$$\begin{aligned} (\Delta_D(\Delta_D^\nu \varphi))(n) &= \Delta_D \sum_{k=0}^{2\nu} {}^{2\nu}C_k (-1)^{k+\nu} \varphi(n-\nu+k) \\ &= -\sum_{k=0}^{2\nu} {}^{2\nu}C_k (-1)^{k+\nu} \varphi(n+1-\nu+k) + 2 \sum_{k=0}^{2\nu} {}^{2\nu}C_k (-1)^{k+\nu} \varphi(n-\nu+k) \\ &\quad - \sum_{k=0}^{2\nu} {}^{2\nu}C_k (-1)^{k+\nu} \varphi(n-1-\nu+k). \end{aligned}$$

For ease of reading we now let $a_k := (-1)^{\nu+k}\varphi(n-\nu+k)$ and adjust the signs accordingly:

$$\begin{aligned}(\Delta_D^{\nu+1}\varphi))(n) &= \sum_{k=0}^{2\nu} {}^{2\nu}C_k a_{k+1} + 2\sum_{k=0}^{2\nu} {}^{2\nu}C_k a_k + \sum_{k=0}^{2\nu} {}^{2\nu}C_k a_{k-1} \\ &= {}^{2\nu}C_{2\nu}a_{2\nu+1} + \sum_{k=0}^{2\nu-1} {}^{2\nu}C_k a_{k+1} + 2\sum_{k=0}^{2\nu} {}^{2\nu}C_k a_k + \sum_{k=1}^{2\nu} {}^{2\nu}C_k a_{k-1} + {}^{2\nu}C_0 a_{-1},\end{aligned}$$

where we separated the terms of highest and lowest argument for the sequence $\varphi(.)$. These will later remain the same.

We again separate the two outermost terms of the sums respectively, which correspond to the path along Pascal's triangle that walks along the outer diagonal of "1"s:

$$\begin{aligned}(\Delta_D^{\nu+1}\varphi)(n) &= {}^{2\nu}C_{2\nu}a_{2\nu+1} + {}^{2\nu}C_{2\nu-1}a_{2\nu} + 2\,{}^{2\nu}C_{2\nu}\,a_{2\nu} + \sum_{k=0}^{2\nu-2} {}^{2\nu}C_k a_{k+1} + 2\sum_{k=1}^{2\nu-1} {}^{2\nu}C_k a_k \\ &\quad + \sum_{k=2}^{2\nu} {}^{2\nu}C_k a_{k-1} + 2\,{}^{2\nu}C_0 a_0 + {}^{2\nu}C_1 a_0 + {}^{2\nu}C_0 a_{-1}.\end{aligned}$$

Then we apply Lemma 4.1, (ii) and (iii).

$$\begin{aligned}(\Delta_D^{\nu+1}\varphi))(n) &= {}^{2\nu}C_{2\nu}a_{2\nu+1} + {}^{2\nu+2}C_{2\nu+1}a_{2\nu} + \sum_{k=0}^{2\nu-2} {}^{2\nu}C_k a_{k+1} + 2\sum_{k=1}^{2\nu-1} {}^{2\nu}C_k a_k \\ &\quad + \sum_{k=2}^{2\nu} {}^{2\nu}C_k a_{k-1} + {}^{2\nu+2}C_1 a_0 + {}^{2\nu}C_0 a_{-1}.\end{aligned}$$

Our central three sums represent all coefficients not involved with the outer diagonal of Pascal's triangle. We will collect them all via Lemma 4.1 (i). We thus rearrange our sums to represent the respective paths down Pascal's Triangle. For ease of reading, we let $S_1 := {}^{2\nu}C_{2\nu}a_{2\nu+1} + {}^{2\nu+2}C_{2\nu+1}a_{2\nu}$ and $S_2 := {}^{2\nu+2}C_1 a_0 + {}^{2\nu}C_0 a_{-1}$:

$$\begin{aligned}(\Delta_D^{\nu+1}\varphi))(n) &= S_1 + \Big(\sum_{k=0}^{2\nu-2} {}^{2\nu}C_k a_{k+1} + \sum_{k=1}^{2\nu-1} {}^{2\nu}C_k a_k\Big) + \Big(\sum_{k=1}^{2\nu-1} {}^{2\nu}C_k a_k + \sum_{k=2}^{2\nu} {}^{2\nu}C_k a_{k-1}\Big) + S_2 \\ &= S_1 + \Big(\sum_{k=1}^{2\nu-1} \big({}^{2\nu}C_{k-1} + {}^{2\nu}C_k\big)\,a_k\Big) + \Big(\sum_{k=1}^{2\nu-1} \big({}^{2\nu}C_k + {}^{2\nu}C_{k+1}\big)\,a_k\Big) + S_2 \\ &= S_1 + \Big(\sum_{k=1}^{2\nu-1} \big({}^{2\nu+1}C_k + {}^{2\nu+1}C_{k+1}\big)\,a_k\Big) + S_2 \\ &= S_1 + \Big(\sum_{k=1}^{2\nu-1} {}^{2\nu+2}C_{k+1}a_k\Big) + S_2.\end{aligned}$$

We now reinsert the expressions for S_1 and S_2, and as ${}^{2\nu}C_{2\nu} = {}^{2\nu+2}C_{2\nu+2} = 1$ and ${}^{2\nu}C_0 = {}^{2\nu+2}C_0 = 1$, and thus we have:

$$\begin{aligned}
(\Delta_D^{\nu+1}\varphi)(n) &= {}^{2\nu+2}C_{2\nu+2}a_{2\nu+1} + {}^{2\nu+2}C_{2\nu+1}a_{2\nu} + \sum_{k=1}^{2\nu-1} {}^{2\nu+2}C_{k+1}a_k + {}^{2\nu+2}C_1 a_0 + {}^{2\nu+2}C_0 a_{-1} \\
&= \sum_{k=-1}^{2\nu+1} {}^{2\nu+2}C_{k+1}a_k \\
&= \sum_{k=-1}^{2\nu+1} {}^{2\nu+2}C_{k+1}(-1)^{k+\nu}\varphi(n-\nu+k) \\
&= \sum_{k=0}^{2\nu+2} {}^{2\nu+2}C_k(-1)^{k-1+\nu}\varphi(n-\nu+k-1) \\
&= \sum_{k=0}^{2(\nu+1)} {}^{2(\nu+1)}C_k(-1)^{k+(\nu+1)}\varphi(n-(\nu+1)+k),
\end{aligned}$$

where we note that $(-1)^{k+(\nu-1)} = (-1)^{k+(\nu+1)}$. This is then our desired form for the operator $(\Delta_D^{\nu+1}\varphi)(n)$, completing the inductive step. □

4.2 Spectral Analysis of Δ_D^σ

Having identified the explicit form of our operator, we now proceed to identify its essential spectrum. We again observe the following pattern for our essential spectra, which in this chapter will be denoted by ς_{ess}:

$$\begin{aligned}
\mathcal{F}(\Delta_D\varphi)(x) &= (2-2\cos(x))(\mathcal{F}\varphi(x)) \\
\mathcal{F}(\Delta_D^2\varphi)(x) &= \Big(6-8\cos(x)+2\cos(2x)\Big)(\mathcal{F}\varphi(x)) \\
\mathcal{F}(\Delta_D^3\varphi)(x) &= \Big(20-30\cos(x)+12\cos(2x)-2\cos(3x)\Big)(\mathcal{F}\varphi(x)).
\end{aligned}$$

These identities thus give us:

$$\varsigma_{\text{ess}}(\Delta_D) = [0,4], \qquad \varsigma_{\text{ess}}(\Delta_D^2) = [0,16], \qquad \varsigma_{\text{ess}}(\Delta_D^3) = [0,64].$$

We thus start by proving:

Lemma 4.3. *In general, we have, for $\sigma \in \mathbb{N}$, $\varphi \in \ell^2(\mathbb{Z})$ and for $x \in \mathbb{R}$:*

$$\mathcal{F}(\Delta_D^\sigma\varphi)(x) = \Big[{}^{2\sigma}C_\sigma + 2\sum_{k=0}^{\sigma-1} {}^{2\sigma}C_k(-1)^{k+\sigma}\cos((\sigma-k)x)\Big](\mathcal{F}\varphi)(x).$$

Proof.

$$
\begin{aligned}
\mathcal{F}(\Delta_D^\sigma \varphi)(x) &= \sum_{n\in\mathbb{Z}} e^{inx}\Big(\sum_{k=0}^{2\sigma} {}^{2\sigma}C_k(-1)^{k+\sigma}\varphi(n-\sigma+k)\Big)\\
&= \sum_{n\in\mathbb{Z}} e^{inx}\ {}^{2\sigma}C_0(-1)^{\sigma}\varphi(n-\sigma) + \ldots + \sum_{n\in\mathbb{Z}} e^{inx}\ {}^{2\sigma}C_{2\sigma}(-1)^{3\sigma}\varphi(n+\sigma)\\
&= \sum_{n\in\mathbb{Z}} e^{i(n+\sigma)x}\ {}^{2\sigma}C_0(-1)^{\sigma}\varphi(n) + \ldots + \sum_{n\in\mathbb{Z}} e^{i(n-\sigma)x}\ {}^{2\sigma}C_{2\sigma}(-1)^{3\sigma}\varphi(n).
\end{aligned}
$$

We now exploit the symmetry of the binomial coefficients, i.e. ${}^aC_b = {}^aC_{a-b}$, and match up all terms with equal coefficients:

$$
\begin{aligned}
\mathcal{F}(\Delta_D^\sigma \varphi)(x) &= \sum_{n\in\mathbb{Z}} e^{inx}\Big(\sum_{k=0}^{2\sigma} {}^{2\sigma}C_k(-1)^{k+\sigma}e^{i(\sigma-k)x}\Big)\varphi(n)\\
&= \sum_{n\in\mathbb{Z}} e^{inx}\Big({}^{2\sigma}C_\sigma + \sum_{k=0}^{\sigma-1} {}^{2\sigma}C_k(-1)^{k+\sigma}\big[e^{-i(\sigma-k)x} + e^{i(\sigma-k)x}\big]\Big)\varphi(n)\\
&= \Big[{}^{2\sigma}C_\sigma + 2\sum_{k=0}^{\sigma-1} {}^{2\sigma}C_k(-1)^{k+\sigma}\cos((\sigma-k)x)\Big](\mathcal{F}\varphi)(x).
\end{aligned}
$$

□

We are left to identify the range of the above symbol:

Proposition 4.4. *The essential spectrum of the operator Δ_D^σ, namely $\varsigma_{ess}(\Delta_D^\sigma)$, coincides with the interval $[0, 4^\sigma]$.*

Proof. Following the above formula for the operator, we investigate the range of:

$$f(x) = \Big[{}^{2\sigma}C_\sigma + 2\sum_{k=0}^{\sigma-1} {}^{2\sigma}C_k(-1)^{k+\sigma}\cos((\sigma-k)x)\Big].$$

We identify the extrema:

$$f'(x) = 2\sum_{k=0}^{\sigma-1} {}^{2\sigma}C_k(-1)^{k+\sigma+1}(\sigma-k)\sin((\sigma-k)x).$$

The principle solutions for $f'(x) = 0$ we are interested in will be $x_1 = 0$, $x_2 = \pi$. Indeed, using $x_2 = \pi$:

$$f(\pi) = {}^{2\sigma}C_\sigma + 2\sum_{k=0}^{\sigma-1} {}^{2\sigma}C_k(-1)^{k+\sigma}(-1)^{\sigma-k} = {}^{2\sigma}C_\sigma + 2\sum_{k=0}^{\sigma-1} {}^{2\sigma}C_k = \sum_{k=0}^{2\sigma} {}^{2\sigma}C_k,$$

where we used the symmetry of the binomial coefficients again. We then recognise the equivalence of the above equation to summing each even-numbered row of Pascal's triangle. We hence use the binomial formula to obtain:

$$\sum_{k=0}^{2\sigma} {}^{2\sigma}C_k = (1+1)^{2\sigma} = 4^\sigma.$$

Now we turn to $x_1 = 0$:

$$f(0) = {}^{2\sigma}C_\sigma + 2\sum_{k=0}^{\sigma-1} {}^{2\sigma}C_k(-1)^{k+\sigma} = \sum_{k=0}^{2\sigma} {}^{2\sigma}C_k(-1)^{k+\sigma} = (1-1)^{2\sigma} = 0.$$

Clearly we attain the maximum at x_2 and the minimum at x_1 and by the periodic nature of $f(x)$, these extrema remain the same disregarding of which solution of $f' = 0$ is employed, and hence our Proposition is proved. □

4.3 Agmon–Kolmogorov Inequality – $(D^\sigma, \mathbb{Z})$

Having established the explicit formulae for our operators and their spectra, we now turn to applying the same method as before to obtain spectral bounds of a more general Schrödinger-type operator. We thus commence again with the relevant Agmon–Kolmogorov inequality, in this case, for the operator D^σ. We first need the following discrete Kolmogorov-type inequality:

Lemma 4.5 (Discrete Kolmogorov-type inequality)**.** *For a sequence $\varphi \in \ell^2(\mathbb{Z})$, and for $n, k \in \mathbb{N}$, $n > k \geq 1$, we have the following inequality:*

$$\|D^k\varphi\|_{\ell^2(\mathbb{Z})} \leq \|\varphi\|_{\ell^2(\mathbb{Z})}^{1-k/n}\|D^n\varphi\|_{\ell^2(\mathbb{Z})}^{k/n}.$$

Proof. We see that the case $k = 1$, $n = 2$ is in fact our Proposition 3.2. Hence we proceed by strong induction and assume we have the required inequality for $n \leq m$. Then:

$$\begin{aligned}
\|D^m\varphi\|_{\ell^2(\mathbb{Z})}^2 &= \langle D^m\varphi, D^m\varphi\rangle \\
&= \langle D^*D^m\varphi, D^{m-1}\varphi\rangle \\
&\leq \|D^*D^m\varphi\|_{\ell^2(\mathbb{Z})}\|D^{m-1}\varphi\|_{\ell^2(\mathbb{Z})} \\
&= \|D^{m+1}\varphi\|_{\ell^2(\mathbb{Z})}\|D^{m-1}\varphi\|_{\ell^2(\mathbb{Z})},
\end{aligned}$$

where we used Cauchy's inequality and the equivalence in norm of D^* and D due to normality.

We thus apply our induction hypothesis, and set $k = m - 1$ and $n = m$:

$$\begin{aligned}
& \|D^m\varphi\|^2_{\ell^2(\mathbb{Z})} && \le \|D^{m+1}\varphi\|_{\ell^2(\mathbb{Z})}\|D^m\varphi\|^{(m-1)/m}_{\ell^2(\mathbb{Z})}\|\varphi\|^{1/m}_{\ell^2(\mathbb{Z})} \\
\Rightarrow \quad & \|D^m\varphi\|^{2-(m-1)/m}_{\ell^2(\mathbb{Z})} && \le \|D^{m+1}\varphi\|_{\ell^2(\mathbb{Z})}\|\varphi\|^{1/m}_{\ell^2(\mathbb{Z})} \\
\Rightarrow \quad & \|D^m\varphi\|_{\ell^2(\mathbb{Z})} && \le \|D^{m+1}\varphi\|^{m/(m+1)}_{\ell^2(\mathbb{Z})}\|\varphi\|^{1/(m+1)}_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

We now return to the induction hypothesis:

$$\begin{aligned}
\|D^k\varphi\|_{\ell^2(\mathbb{Z})} &\le \|D^m\varphi\|^{k/m}_{\ell^2(\mathbb{Z})}\|\varphi\|^{(m-k)/m}_{\ell^2(\mathbb{Z})} \\
&\le \|D^{m+1}\varphi\|^{k/(m+1)}_{\ell^2(\mathbb{Z})}\|\varphi\|^{k/m(m+1)}_{\ell^2(\mathbb{Z})}\|\varphi\|^{(m-k)/m}_{\ell^2(\mathbb{Z})} \\
&= \|D^{m+1}\varphi\|^{k/(m+1)}_{\ell^2(\mathbb{Z})}\|\varphi\|^{1-k/(m+1)}_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

Hence our statement holds for $n = m + 1$ and our inductive step is complete. □

We are now equipped to prove our relevant Agmon–Kolmogorov inequality:

Proposition 4.6 (Agmon–Kolmogorov Inequality – $(D^\sigma, \mathbb{Z})$)**.** *For a sequence $\varphi \in \ell^2(\mathbb{Z})$, we have for any $\sigma \in \mathbb{N}$, $\sigma \ge 2$:*

$$\|\varphi\|_{\ell^\infty(\mathbb{Z})} \le \|\varphi\|^{1-1/2\sigma}_{\ell^2(\mathbb{Z})}\|D^\sigma\varphi\|^{1/2\sigma}_{\ell^2(\mathbb{Z})}.$$

Proof. First we use Lemma 4.5 with $k = 1$, $n = \sigma$:

$$\|D\varphi\|_{\ell^2(\mathbb{Z})} \le \|\varphi\|^{1-\frac{1}{\sigma}}_{\ell^2(\mathbb{Z})}\|D^\sigma\varphi\|^{\frac{1}{\sigma}}_{\ell^2(\mathbb{Z})},$$

and we apply this estimate to our discrete Agmon inequality, Proposition 2.7:

$$\begin{aligned}
|\,\varphi(n)\,|^2 &\le \|\varphi\|_{\ell^2(\mathbb{Z})}\|D\varphi\|_{\ell^2(\mathbb{Z})} \\
&\le \|\varphi\|_{\ell^2(\mathbb{Z})}\|\varphi\|^{1-\frac{1}{\sigma}}_{\ell^2(\mathbb{Z})}\|D^\sigma\varphi\|^{\frac{1}{\sigma}}_{\ell^2(\mathbb{Z})} \\
&= \|\varphi\|^{2-\frac{1}{\sigma}}_{\ell^2(\mathbb{Z})}\|D^\sigma\varphi\|^{\frac{1}{\sigma}}_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

□

4.4 Generalised Sobolev Inequality – $(D^\sigma, \mathbb{Z})$

Proposition 4.7 (Generalised–Sobolev inequality – $(\mathrm{D}^\sigma, \mathbb{Z})$)**.** *Let $\{\psi_j\}_{j=1}^N$ be an orthonormal system of sequences in $\ell^2(\mathbb{Z})$, i.e. $\langle \psi_j, \psi_k \rangle = \delta_{jk}$, and let $\rho(n) := \sum_{j=1}^N |\psi_j(n)|^2$. Then*

$$\sum_{n\in\mathbb{Z}} \rho^{2\sigma+1}(n) \le \sum_{j=1}^N \sum_{n\in\mathbb{Z}} |D^\sigma \psi_j(n)|^2 .$$

Proof. We follow exactly the same method as before, and hence move through the argument swiftly. Let $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{C}^N$. By our Agmon–Kolmogorov inequality – $(D^\sigma, \mathbb{Z})$, i.e. Proposition 4.6, we have:

$$\begin{aligned} \Big|\sum_{j=1}^N \xi_j \psi_j(n)\Big|^2 &\le \Big\|\sum_{j=1}^N \xi_j \psi_j\Big\|_{\ell^2(\mathbb{Z})}^{(2\sigma-1)/\sigma} \Big\|D^\sigma \sum_{j=1}^N \xi_j \psi_j\Big\|_{\ell^2(\mathbb{Z})}^{1/\sigma} \\ &= \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \langle \psi_j, \psi_k \rangle\Big)^{(2\sigma-1)/2\sigma} \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \langle D^\sigma \psi_j, D^\sigma \psi_k \rangle\Big)^{1/2\sigma} \\ &\le \Big(\sum_{j=1}^N |\xi_j|^2\Big)^{(2\sigma-1)/2\sigma} \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \langle D^\sigma \psi_j, D^\sigma \psi_k \rangle\Big)^{1/2\sigma} . \end{aligned}$$

Let $\xi_j := \overline{\psi_j(n)}$ and as $\rho(n) = \sum_{j=1}^N |\psi_j(n)|^2$:

$$\begin{aligned} \rho^2(n) &\le \rho^{(2\sigma-1)/2\sigma}(n) \Big(\sum_{j,k=1}^N \psi_j(n) \overline{\psi_k(n)} \langle D^\sigma \psi_j, D^\sigma \psi_k \rangle\Big)^{1/2\sigma} \\ \Rightarrow \rho^{2\sigma+1}(n) &\le \sum_{j,k=1}^N \psi_j(n) \overline{\psi_k(n)} \langle D^\sigma \psi_j, D^\sigma \psi_k \rangle \\ \Rightarrow \sum_{n\in\mathbb{Z}} \rho^{2\sigma+1}(n) &\le \sum_{j=1}^N \Big(\sum_{n\in\mathbb{Z}} |D^\sigma \psi_j(n)|^2\Big). \end{aligned}$$

$\square$

4.5 Lieb–Thirring Inequality – $(H_D^\sigma, \mathbb{Z})$

We will now be interested in spectral inequalities for discrete spectra of σ^{th} order Laplacian operators perturbed by a potential term. We thus fully generalise the previous chapter's idea of the discrete Schrödinger-type operator.

We let $\{\psi_j\}_{j=1}^N$, $N \in \overline{\mathbb{N}}$ be the orthonormal system of eigensequences in $\ell^2(\mathbb{Z})$ corresponding to the negative eigenvalues $\{e_j\}_{j=1}^N$ of the $(2\sigma)^{th}$ order Schrödinger-type operator:

$$(H_D^\sigma \psi_j)(n) := (\Delta_D^\sigma \psi_j)(n) - b_n \psi_j(n) = e_j \psi_j(n), \tag{4.1}$$

where $j \in \{1, \ldots, N\}$ and we assume that $b_n \geq 0$ for all $n \in \mathbb{Z}$. Our next result is concerned with estimating those negative eigenvalues:

Theorem 4.8 (Lieb–Thirring inequality – $(H_D^\sigma, \mathbb{Z}, \gamma = 1)$)**.** *Let $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{(2\sigma+1)/2\sigma}(\mathbb{Z})$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D^σ are discrete and they satisfy the inequality:*

$$\sum_{j=1}^N |e_j| \quad \leq \quad \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma}.$$

Proof. We take the inner product with $\psi_j(n)$ on 4.1 and sum both sides of the equation with respect to j. We obtain:

$$\sum_{j=1}^N e_j = \sum_{j=1}^N \Big(\sum_{n\in\mathbb{Z}} |D^\sigma \psi_j(n)|^2 \Big) \; - \; \sum_{j=1}^N \Big(\sum_{n\in\mathbb{Z}} b_n |\psi_j(n)|^2 \Big).$$

We now use Proposition 4.7 and apply the appropriate Hölder's inequality, i.e:

$$\sum_{j=1}^K e_j \geq \sum_{n\in\mathbb{Z}} \Big(\sum_{j=1}^N |\psi_j(n)|^2 \Big)^{2\sigma+1} - \Big(\sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma} \Big)^{2\sigma/(2\sigma+1)} \Big(\sum_{n\in\mathbb{Z}} \Big(\sum_{j=1}^N |\psi_j(n)|^2 \Big)^{2\sigma+1} \Big)^{1/(2\sigma+1)}. \tag{4.2}$$

We define

$$\chi := \Big(\sum_{n\in\mathbb{Z}} \Big(\sum_{j=1}^N |\psi_j(n)|^2 \Big)^{2\sigma+1} \Big)^{1/(2\sigma+1)}, \qquad \varsigma := \Big(\sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma} \Big)^{2\sigma/(2\sigma+1)}.$$

The latter inequality can be written as

$$\chi^{2\sigma+1} - \varsigma\chi \leq \sum_{j=1}^N e_j.$$

The LHS is maximal when

$$\chi = \frac{1}{(2\sigma+1)^{1/2\sigma}} \Big(\sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma} \Big)^{1/(2\sigma+1)}.$$

Substituting this into (4.2), we obtain:

$$\begin{aligned} \sum_{j=1}^{N} e_j \quad &\geq \quad \Big(\frac{1}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma} \Big) - \frac{1}{(2\sigma+1)^{1/2\sigma}} \sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma}. \\ &= \quad \frac{-2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma}. \end{aligned}$$

Therefore:

$$\sum_{j=1}^{N} |e_j| \leq \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma}. \tag{4.3}$$

□

We can thus finally prove Theorem 1.8:

Theorem 1.8 Let $b_n \geq 0$, $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z})$, $\gamma \geq 1$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D^σ satisfy the inequality

$$\sum_{j=1}^{N} |e_j|^\gamma \quad \leq \quad \eta_\sigma^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2\sigma},$$

where

$$\eta_\sigma^\gamma \ := \ \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})}.$$

Proof. Let $\{e_j(\tau)\}_{j=1}^N$ be the negative eigenvalues of the operator $\Delta_D^\sigma - (b_n - \tau)_+$. By the variational principle for the negative eigenvalues $\{-(|e_j| - \tau)_+\}_{j=1}^N$ of the operator $\Delta_D^\sigma - (b_n - \tau)$ we have as before:

$$(|e_j| - \tau)_+ \leq |e_j(\tau)|.$$

By this estimate, and our Lemma 2.11, we find that

$$\begin{aligned}
\sum_{j=1}^{N} |e_j|^\gamma &= \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2} (\sum_{j=1}^{N} |e_j| - \tau)_+ \, d\tau \\
&\leq \frac{1}{\mathcal{B}(\gamma-1,2)}, \int_0^\infty \tau^{\gamma-2} \sum_{j=1}^{N} e_j(\tau)_+ \, d\tau \\
&\leq \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2} \sum_{n\in\mathbb{Z}} (b_n - \tau)_+^{(2\sigma+1)/2\sigma} d\tau,
\end{aligned}$$

by (4.3) above. Then, exactly as before:

$$\begin{aligned}
\sum_{j=1}^{N} |e_j|^\gamma &\leq \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{1}{\mathcal{B}(\gamma-1,2)} \sum_{n\in\mathbb{Z}} b_n^{(2\sigma+1)/2\sigma} \int_0^1 (s\, b_n)^{\gamma-2} (1-s)_+^{(2\sigma+1)/2\sigma} \, b_n \, ds \\
&= \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{1}{\mathcal{B}(\gamma-1,2)} \mathcal{B}(\gamma-1, \frac{4\sigma+1}{2\sigma}) \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2\sigma} \\
&= \eta_\sigma^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2\sigma},
\end{aligned}$$

where

$$\eta_\sigma^\gamma := \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{\mathcal{B}(\gamma-1,\frac{4\sigma+1}{2\sigma})}{\mathcal{B}(\gamma-1,2)} = \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})}.$$

□

4.6 Polydiagonal Jacobi-type Operators

We let W_σ be a polydiagonal self-adjoint Jacobi-type matrix operator:

$$W_\sigma := \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \iddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & a^\sigma_{-2} & 0 & 0 & \ddots \\ \ddots & \ddots & b_{-1} & a^1_{-1} & \ddots & \ddots & a^\sigma_{-1} & 0 & \ddots \\ \ddots & \ddots & a^1_{-1} & b_0 & a^1_0 & \ddots & \ddots & a^\sigma_0 & \ddots \\ \ddots & \ddots & \ddots & a^1_0 & b_1 & a^1_1 & \ddots & \ddots & \ddots \\ \ddots & a^\sigma_{-2} & \ddots & \ddots & a^1_1 & b_2 & a^1_2 & \ddots & \ddots \\ \ddots & 0 & a^\sigma_{-1} & \ddots & \ddots & a^1_2 & b_3 & \ddots & \ddots \\ \ddots & 0 & 0 & a^\sigma_0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \iddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

viewed as an operator acting on $\ell^2(\mathbb{Z})$ as follows. For $n \in \mathbb{Z}, i \in \{1, \ldots, \sigma\}$:

$$\begin{aligned} (W_\sigma\varphi)(n) &= \sum_{i=1}^{\sigma} a^i_{n-i}\varphi(n-i) + b_n\varphi(n) + \sum_{i=1}^{\sigma} a^i_n\varphi(n+i) \\ &= a^\sigma_{n-\sigma}\varphi(n-\sigma) + \ldots + a^1_{n-1}\varphi(n-1) + b_n\varphi(n) + a^1_n\varphi(n+1) + \ldots + a^\sigma_n\varphi(n+\sigma), \end{aligned}$$

where a^i_n, $b_n \in \mathbb{R}$, for all $i \in \{1, \ldots, \sigma\}$. Again we denote $(W_\sigma(\{a^1_n\}, \ldots, \{a^\sigma_n\}, \{b_n\})\varphi)(n) := (W_\sigma\varphi)(n)$ where we understand $\{.\}$ to mean $\{.\}_{n\in\mathbb{Z}}$. We are then interested in perturbations of the following special case:

$$(W^0_\sigma\varphi)(n) := (W_\sigma(\{a^1_n \equiv \omega_1\}, \ldots, \{a^\sigma_n \equiv \omega_\sigma\}, \{b_n \equiv 0\})\varphi)(n),$$

where $\omega_i := {}^{2\sigma}C_{\sigma+i}(-1)^i$, and explicitly:

$$(W^0_\sigma\varphi)(n) = ((\Delta^\sigma_D - {}^{2\sigma}C_\sigma)\varphi)(n) = \sum_{k=0,\ k\neq\sigma}^{2\sigma} {}^{2\sigma}C_k(-1)^{k+\sigma}\varphi(n-\sigma+k),$$

called the free Jacobi-type matrix of order σ. In particular, we examine the case where $W_\sigma - W^0_\sigma$ is compact. Thus in what follows we assume that our sequences tend to the operator coefficients rapidly enough, i.e. $a^i_n \to \omega_i$, $b_n \to 0$, rapidly enough as $n \to \pm\infty$. Then the essential spectrum ς_{ess} is given by $\varsigma_{ess}(W_\sigma) = \varsigma_{ess}(W^0_\sigma) = [-{}^{2\sigma}C_\sigma, 4^\sigma - {}^{2\sigma}C_\sigma]$ and W_σ may have simple eigenvalues $\{E^\pm_j\}_{j=1}^{N_\pm}$

where $N_\pm \in \overline{\mathbb{N}}$, and

$$E_1^+ > E_2^+ > \ldots > 4^\sigma - {}^{2\sigma}C_\sigma > -{}^{2\sigma}C_\sigma > \ldots > E_2^- > E_1^-.$$

We hence restate Theorem 1.10, the bound for these eigenvalues:

Theorem 1.10 Let $\gamma \geq 1$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n^i - \omega_i\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z})$ for all $i \in \{1,\ldots,\sigma\}$. Then for the eigenvalues $\{E_j^\pm\}_{j=1}^{N_\pm}$ of the operator W_σ we have:

$$\sum_{j=1}^{N_-}|E_j^- + {}^{2\sigma}C_\sigma|^\gamma + \sum_{j=1}^{N_+}|E_j^+ - (4^\sigma - {}^{2\sigma}C_\sigma)|^\gamma \leq \nu_\sigma^\gamma\left(\sum_{n\in\mathbb{Z}}|b_n|^{\gamma+1/2\sigma} + 4\sum_{n\in\mathbb{Z}}\sum_{k=1}^{\sigma}|a_n^k - \omega_k|^{\gamma+1/2\sigma}\right),$$

where

$$\nu_\sigma^\gamma = 2\sigma\ (2\sigma+1)^{\gamma-2}\ \frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})}.$$

Proof. We remind ourselves for our general matrix bounds for square, $m\times m$ matrices, i.e. Lemma 2.12. For $a_n^m, \omega_m \in \mathbb{R}$, we have:

$$\begin{pmatrix} -|a_n^m-\omega_m| & 0 & \ldots & 0 & \omega_m \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ \omega_m & 0 & \ldots & 0 & -|a_n^m-\omega_m| \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & \ldots & 0 & a_n^m \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ a_n^m & 0 & \ldots & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} |a_n^m-\omega_m| & 0 & \ldots & 0 & \omega_m \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ \omega_m & 0 & \ldots & 0 & |a_n^m-\omega_m| \end{pmatrix}.$$

We thus extrapolate the method used in the pentadiagonal case and use this on each block of indices:

$$\begin{aligned} W_\sigma(\{a_n^1 \equiv \omega_1\},\ldots,\{a_n^\sigma \equiv \omega_\sigma\},\{b_n^{(-)}\}) &\leq W_\sigma(\{a_n^1\},\ldots,\{a_n^\sigma\},\{b_n\}) \\ &\leq W_\sigma(\{a_n^1 \equiv \omega_1\},\ldots,\{a_n^\sigma \equiv \omega_\sigma\},\{b_n^{(+)}\}). \end{aligned} \tag{4.4}$$

where $b_n^{(\pm)}$ is given by

$$b_n^{(\pm)} = b_n \pm \Big((|a_{n-1}^1 - \omega_1| + |a_n^1 - \omega_1|) + \ldots + (|a_{n-\sigma}^\sigma - \omega_\sigma| + |a_n^\sigma - \omega_\sigma|)\Big),$$

i.e.

$$b_n^{(\pm)} = b_n \pm \Big(\sum_{k=1}^{\sigma}|a_{n-k}^k - \omega_k| + |a_n^k - \omega_k|\Big).$$

We give the following representation and note here that it is slightly misleading, as the coefficients do not lign up in such a simple way, however the more compact visualisation proves useful:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b_{-1} & a^1_{-1} & \dots & a^\sigma_{-1} & \dots \\ \dots & a^1_{-1} & b_0 & a^1_0 & \vdots & \dots \\ \dots & \vdots & a^1_0 & b_1 & a^1_1 & \dots \\ \dots & a^\sigma_{-1} & \dots & a^1_1 & b_2 & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b^{(-)}_{-1} & \omega_1 & \dots & \omega_\sigma & \dots \\ \dots & \omega_1 & b^{(-)}_0 & \omega_1 & \vdots & \dots \\ \dots & \vdots & \omega_1 & b^{(-)}_1 & \omega_1 & \dots \\ \dots & \omega_\sigma & \dots & \omega_1 & b^{(-)}_2 & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b_{-1} & a^1_{-1} & \dots & a^\sigma_{-1} & \dots \\ \dots & a^1_{-1} & b_0 & a^1_0 & \vdots & \dots \\ \dots & \vdots & a^1_0 & b_1 & a^1_1 & \dots \\ \dots & a^\sigma_{-1} & \dots & a^1_1 & b_2 & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \leq \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \dots & b^{(+)}_{-1} & \omega_1 & \dots & \omega_\sigma & \dots \\ \dots & \omega_1 & b^{(+)}_0 & \omega_1 & \vdots & \dots \\ \dots & \vdots & \omega_1 & b^{(+)}_1 & \omega_1 & \dots \\ \dots & \omega_\sigma & \dots & \omega_1 & b^{(+)}_2 & \dots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Now we relate these to our Schrödinger-type operators:

$$\Delta^\sigma_D - 4^\sigma + b_n = W^0_\sigma - (4^\sigma - {}^{2\sigma}C_\sigma) + b_n = W_\sigma\left(\{a^1_n \equiv \omega_1\}, \dots, \{a^\sigma_n \equiv \omega_\sigma\}, \{b_n - (4^\sigma - {}^{2\sigma}C_\sigma)\}\right), \tag{4.5}$$

and

$$\Delta^\sigma_D + b_n = W^0_\sigma + {}^{2\sigma}C_\sigma + b_n = W_\sigma\left(\{a^1_n \equiv \omega_1\}, \dots, \{a^\sigma_n \equiv \omega_\sigma\}, \{b_n + {}^{2\sigma}C_\sigma\}\right). \tag{4.6}$$

Now $(E^+_j - (4^\sigma - {}^{2\sigma}C_\sigma))$ are positive eigenvalues of $W_\sigma(\{a^1_n\}, \dots, \{a^\sigma_n\}, \{b_n - (4^\sigma - {}^{2\sigma}C_\sigma)\})$. Thus by using (4.4), and the Variational Principle, Theorem 2.10, we have

$$W_\sigma(\{a^1_n\}, \dots, \{a^\sigma_n\}, \{b_n - (4^\sigma - {}^{2\sigma}C_\sigma)\}) \leq W_\sigma\left(\{a^1_n \equiv \omega_1\}, \dots, \{a^\sigma_n \equiv \omega_\sigma\}, \{b^{(+)}_n - (4^\sigma - {}^{2\sigma}C_\sigma)\}\right),$$

$$\Rightarrow |E^+_j - (4^\sigma - {}^{2\sigma}C_\sigma)| \leq e^+_j, \tag{4.7}$$

where e^+_j are the positive eigenvalues of

$$W_\sigma\left(\{a^1_n \equiv \omega_1\}, \dots, \{a^\sigma_n \equiv \omega_\sigma\}, \{b^{(+)}_n - (4^\sigma - {}^{2\sigma}C_\sigma)\}\right) = \Delta^\sigma_D - 4^\sigma + b^{(+)}_n.$$

Let us now again define $(b_n)_+ := \max(b_n, 0)$, $(b_n)_- := -\min(b_n, 0)$. Then, by the Lieb–Thirring inequality – $(H^{\sigma}_{D}, \mathbb{Z})$ for the positive eigenvalues of our operator, i.e. Corollary 1.9, we have:

$$\sum_{j=1}^{N_+} (e_j^+)^{\gamma} \le \eta_\sigma^\gamma \sum_{n\in\mathbb{Z}} (b_n^{(+)})_+^{\gamma+1/2\sigma}.$$

Thus, applying (4.7):

$$\sum_{j=1}^{N_+} |E_j^+ - (4^\sigma - {}^{2\sigma}C_\sigma)|^\gamma \le \eta_\sigma^\gamma \sum_{n\in\mathbb{Z}} \left((b_n)_+ + \sum_{k=1}^{\sigma} \left(|a_{n-k}^k - \omega_k| + |a_n^k - \omega_k| \right) \right)^{\gamma+1/2\sigma}, \tag{4.8}$$

where

$$\eta_\sigma^\gamma := \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})}.$$

Similarly, following the same method and using the Lieb–Thirring inequality – $(H^{\sigma}_{D}, \mathbb{Z})$ for negative eigenvalues, i.e Theorem 1.8 on (4.6):

$$\sum_{j=1}^{N_-} |E_j^- + {}^{2\sigma}C_\sigma|^\gamma \le \eta_\sigma^\gamma \sum_{n\in\mathbb{Z}} \left((b_n)_- + \sum_{k=1}^{\sigma} \left(|a_{n-k}^k - \omega_k| + |a_n^k - \omega_k| \right) \right)^{\gamma+1/2\sigma}. \tag{4.9}$$

Applying Lemma 2.13 for $2\sigma+1$ elements, i.e. for $i \in \{1, \dots, 2\sigma+1\}$, let α_i, $q \in \mathbb{R}$, with $q \ge 1$,

$$\left(\sum_{i=1}^{2\sigma+1} \alpha_i \right)^q \le (2\sigma+1)^{q-1} \left(\sum_{i=1}^{2\sigma+1} \alpha_i^q \right),$$

to each of (4.8) and (4.9) we have:

$$\begin{aligned} &\left((b_n)_\pm + \sum_{k=1}^{\sigma} \left(|a_{n-k}^k - \omega_k| + |a_n^k - \omega_k| \right) \right)^{\gamma+1/2\sigma} \\ &\qquad \le (2\sigma+1)^{\gamma-(2\sigma-1)/2\sigma} \left((b_n)_\pm^{\gamma+1/2\sigma} + \sum_{k=1}^{\sigma} \left(|a_{n-k}^k - \omega_k|^{\gamma+1/2\sigma} + |a_n^k - \omega_k|^{\gamma+1/2\sigma} \right) \right). \end{aligned}$$

Adding them up we arrive at

$$\begin{aligned}
&\sum_{j=1}^{N_-} |E_j^- + {}^{2\sigma}C_\sigma|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - (4^\sigma - {}^{2\sigma}C_\sigma)|^\gamma \\
&\le (2\sigma+1)^{\gamma-(2\sigma-1)/2\sigma}\,\eta_\sigma^\gamma \left[\sum_{n\in\mathbb{Z}} (b_n)_-^{\gamma+1/2\sigma} + \sum_{k=1}^{\sigma}\left(\sum_{n\in\mathbb{Z}} |a_{n-k}^k - \omega_k|^{\gamma+1/2\sigma} + \sum_{n\in\mathbb{Z}} |a_n^k - \omega_k|^{\gamma+1/2\sigma}\right)\right] \\
&+ (2\sigma+1)^{\gamma-(2\sigma-1)/2\sigma}\,\eta_\sigma^\gamma \left[\sum_{n\in\mathbb{Z}} (b_n)_+^{\gamma+1/2\sigma} + \sum_{k=1}^{\sigma}\left(\sum_{n\in\mathbb{Z}} |a_{n-k}^k - \omega_k|^{\gamma+1/2\sigma} + \sum_{n\in\mathbb{Z}} |a_n^k - \omega_k|^{\gamma+1/2\sigma}\right)\right] \quad .
\end{aligned}$$

Therefore:

$$\sum_{j=1}^{N_-} |E_j^- + {}^{2\sigma}C_\sigma|^\gamma + \sum_{j=1}^{N_+} |E_j^+ - (4^\sigma - {}^{2\sigma}C_\sigma)|^\gamma \le \nu_\sigma^\gamma \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/2\sigma} + 4\sum_{n\in\mathbb{Z}}\sum_{k=1}^{\sigma} |a_n^k - \omega_k|^{\gamma+1/2\sigma}\right),$$

where

$$\nu_\sigma^\gamma = 2\sigma\ (2\sigma+1)^{\gamma-2}\frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})},$$

and the proof of Theorem 1.10 is complete.

□

Chapter 5

Inequalities on $\ell^2(\mathbb{Z}^d)$

In this chapter, we generalise our inequalities with regards to dimension of our domain, the focus of which will be obtaining an entire family of Agmon–Kolmogorov inequalities. We then employ our previous approach to generate a large class of Lieb–Thirring inequalities of arbitrary dimension.

In Section 5.1, we introduce our higher dimensional difference operators and give the relevant notation. In Section 5.2 we introduce the Agmon–Cauchy inequality, a simple operator bound and then use those to prove three Agmon–Kolmogorov inequalities on $\ell^2(\mathbb{Z}^2)$. In Section 5.3, we follow the same method to obtain four Agmon–Kolmogorov inequalities on $\ell^2(\mathbb{Z}^3)$, all with different exponents and constants. Finally, in Section 5.4, we generalise this to Agmon–Kolmogorov inequalities for arbitrary dimension, and in fact generate 2^{d-1} inequalities for each dimension, and find the general formula for each constant in Section 5.5. We thus give examples of specific formulae for these inequalities for two- and three-dimensional domains in Section 5.6. In Sections 5.7 and 5.8, we in turn use these to prove the discrete Generalised Sobolev and Lieb–Thirring inequalities for arbitrary dimension, and finally identify in Section 5.9 which choices of Agmon–Kolmogorov inequalities are in fact redundant when considering their application to spectral inequalities.

5.1 Discrete Operators of Arbitrary Dimension

We introduce our notation for the d-dimensional inner product space of square summable sequences:

Definition 5.1. For a vector of integers $\zeta := (\zeta_1, \dots, \zeta_d) \in \mathbb{Z}^d$, we say $\{\varphi(\zeta)\}_{\zeta \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, if and only if the following norm is finite:

$$\|\varphi\|_{\ell^2(\mathbb{Z}^d)} := \left(\sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta)|^p \right)^{1/2}.$$

Then, for φ, $\phi \in \ell^2(\mathbb{Z}^d)$, we let $< .,. >_d$ be the inner product on $\ell^2(\mathbb{Z}^d)$:

$$\langle \varphi, \phi \rangle_d := \sum_{\zeta \in \mathbb{Z}^d} \varphi(\zeta) \overline{\phi(\zeta)}.$$

We thus generalise our difference operators to higher dimension:

Definition 5.2. For a sequence $\varphi \in \ell^2(\mathbb{Z}^d)$, we let $D_1, \dots, D_d$ be the partial difference operators defined by:

$$(D_i \varphi)(\zeta) := \varphi(\zeta_1, \dots, \zeta_i + 1, \dots, \zeta_d) - \varphi(\zeta_1, \dots, \zeta_d),$$

The discrete gradiant ∇_D shall thus be defined by:

$$\nabla_D \varphi(\zeta_1, \zeta_2, \dots, \zeta_d) = \big(D_1 \varphi(\zeta), D_2 \varphi(\zeta), \dots, D_d \varphi(\zeta) \big).$$

Thus, combining this definition with that of our norm above, we obtain:

$$\|\nabla_D \varphi\|^2_{\ell^2(\mathbb{Z}^d)} = \|D_1 \varphi\|^2_{\ell^2(\mathbb{Z}^d)} + \dots + \|D_d \varphi\|^2_{\ell^2(\mathbb{Z}^d)}.$$

5.2 Agmon–Kolmogorov Inequalities – $(\nabla_D, \mathbb{Z}^2)$

In the continuous case, the two-dimensional (and in fact the higher dimensional) version of the Agmon inequality does not hold true, namely for any $f \in H^1(\mathbb{R}^2)$:

$$\sup_{x \in \mathbb{R}^2} |f(x)|^2 \nleq C \Big(\int_{\mathbb{R}^2} |f(x)|^2 \, dx \Big)^{1/2} \Big(\int_{\mathbb{R}^2} |\nabla f(x)|^2 \, dx \Big)^{1/2}, \tag{5.1}$$

for any constant $C \in \mathbb{R}$. The following counter-example illustrates this:

We consider a radially symmetric function $f : \mathbb{R}^2 \to \mathbb{R}$. For $x \in \mathbb{R}$, we let $r := |x|$; then for a real number α such that $0 < \alpha < 1/2$, we choose:

$$f(r) = \begin{cases} (-\ln r)^{\alpha} & 0 < r < 1/2, \\ 0 & r > 1/2. \end{cases}$$

This function then does not satisfy the two-dimensional Agmon inequality. Indeed:

$$\Big(\int_0^{\infty} |f(r)|^2 \, r \, dr \Big)^{1/2} = \Big(\int_0^{1/2} |-\ln r|^{2\alpha} \, r \, dr \Big)^{1/2} < \infty.$$

and

$$\begin{aligned} \Big(\int_0^{\infty} |f(r)'|^2 \, r \, dr \Big)^{1/2} &= \Big(\int_0^{\infty} |-\ln r|^{2(\alpha-1)} \, \frac{1}{r} \, dr \Big)^{1/2} \\ &= \Big(\int_0^{\infty} |-\ln r|^{2(\alpha-1)} \, d(\ln r) \Big)^{1/2} \\ &= \Big[\frac{(-\ln r)^{2\alpha-1}}{2\alpha-1} \Big]_0^{1/2} < \infty \qquad \text{for} \quad \alpha < 1/2. \end{aligned}$$

Thus, the RHS of (5.1) will thus be finite, but $\sup_{r\in\mathbb{R}+} |f(r)|^2$ is clearly unbounded near 0, thus giving us our counter-example.

However, we will show that the discrete Agmon inequality does generalise to higher dimension. Throughout this section, we consider the two-dimensional case only. We let $\varphi := \{\varphi(\zeta_1, \zeta_2)\}_{\zeta_1,\zeta_2\in\mathbb{Z}}$ be a sequence in $\ell^2(\mathbb{Z}^2)$, and thus we introduce the following norms:

$$\|\varphi\|^2_{(\ell^\infty(\mathbb{Z}),\ell^2(\mathbb{Z}))} := \sup_{\zeta_1\in\mathbb{Z}} \sum_{\zeta_2\in\mathbb{Z}} |\varphi(\zeta_1, \zeta_2)|^2,$$

and

$$\|\varphi\|^2_{(\ell^2(\mathbb{Z}),\ell^\infty(\mathbb{Z}))} := \sup_{\zeta_2\in\mathbb{Z}} \sum_{\zeta_1\in\mathbb{Z}} |\varphi(\zeta_1, \zeta_2)|^2.$$

We will prove three simple lemmata, which we then apply to obtain three different inequalities. The first allows us to estimate the $\ell^\infty(\mathbb{Z}, \ell^2(\mathbb{Z}))$-norm of a sequence by two $\ell^2(\mathbb{Z}^2)$-norms.

Lemma 5.3 (Agmon–Cauchy Inequality on $\ell^2(\mathbb{Z}^2)$)**.** *For a sequence $\varphi \in \ell^2(\mathbb{Z}^2)$, the following inequality holds true:*

$$\|\varphi\|^2_{(\ell^\infty(\mathbb{Z}),\ell^2(\mathbb{Z}))} \le \|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)} \, \|\varphi\|_{\ell^2(\mathbb{Z}^2)},$$

and by symmetry we have the same result for D_2, i.e.:

$$\|\varphi\|^2_{(\ell^2(\mathbb{Z}),\ell^\infty(\mathbb{Z}))} \leq \|D_2\varphi\|_{\ell^2(\mathbb{Z}^2)} \|\varphi\|_{\ell^2(\mathbb{Z}^2)}.$$

Proof. Using the one-dimensional Agmon–Kolmogorov inequality – $(D, \mathbb{Z})$ i.e. Proposition 2.7 on our second variable, we find:

$$|\varphi(\zeta_1,\zeta_2)|^2 \leq \Big(\sum_{l\in\mathbb{Z}}|D_2\varphi(\zeta_1,l)|^2\Big)^{1/2}\Big(\sum_{l\in\mathbb{Z}}|\varphi(\zeta_1,l)|^2\Big)^{1/2}.$$

Now we sum with respect to ζ_1:

$$\sum_{\zeta_1\in\mathbb{Z}}|\varphi(\zeta_1,\zeta_2)|^2 \leq \sum_{\zeta_1\in\mathbb{Z}}\bigg[\Big(\sum_{l\in\mathbb{Z}}|D_2\varphi(\zeta_1,l)|^2\Big)^{1/2}\Big(\sum_{l\in\mathbb{Z}}|\varphi(\zeta_1,l)|^2\Big)^{1/2}\bigg],$$

and use the Cauchy–Schwartz inequality:

$$\sum_{\zeta_1\in\mathbb{Z}}|\varphi(\zeta_1,\zeta_2)|^2 \leq \Big(\sum_{\zeta_1,l\in\mathbb{Z}}|D_2\varphi(\zeta_1,l)|^2\Big)^{1/2}\Big(\sum_{\zeta_1,l\in\mathbb{Z}}|\varphi(\zeta_1,l)|^2\Big)^{1/2}.$$

□

We estimate the $\ell^2(\mathbb{Z}^2)$-norm of a partial difference operator with the $\ell^2(\mathbb{Z}^2)$-norm of the sequence itself, showing it is bounded.

Lemma 5.4. *For a sequence $\varphi \in \ell^2(\mathbb{Z}^2)$ and $i \in \{1,2\}$, the following inequality holds true:*

$$\|D_i\varphi\|_{\ell^2(\mathbb{Z}^2)} \leq 2\|\varphi\|_{\ell^2(\mathbb{Z}^2)}.$$

Proof. We prove the inequality for $i = 1$ and due to symmetry the other case follows immediately.

$$\begin{aligned}
\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^2)} &= \sum_{\zeta_1,\zeta_2\in\mathbb{Z}}|\varphi(\zeta_1+1,\zeta_2)-\varphi(\zeta_1,\zeta_2)|^2\\
&\leq 2\Big(\sum_{\zeta_1,\zeta_2\in\mathbb{Z}}|\varphi(\zeta_1+1,\zeta_2)|^2+\sum_{\zeta_1,\zeta_2\in\mathbb{Z}}|\varphi(\zeta_1,\zeta_2)|^2\Big)\\
&= 4\sum_{\zeta_1,\zeta_2\in\mathbb{Z}}|\varphi(\zeta_1,\zeta_2)|^2\\
&= 4\|\varphi\|^2_{\ell^2(\mathbb{Z}^2)}.
\end{aligned}$$

□

The two Lemmata above will generalise to higher dimensions, and form the core of our method. We give one final Lemma:

Lemma 5.5. *Let φ be a sequence in $\ell^2(\mathbb{Z}^2)$, then we have:*

$$2\|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)}\,\|D_2\varphi\|_{\ell^2(\mathbb{Z}^2)} \quad\leq\quad \|\nabla_D\varphi\|^2_{\ell^2(\mathbb{Z}^2)}.$$

Proof.

$$\begin{aligned} 2\|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)}\,\|D_2\varphi\|_{\ell^2(\mathbb{Z}^2)} &= 2\Big(\sum_{\zeta_1,\zeta_2\in\mathbb{Z}}|D_1\varphi(\zeta_1,\zeta_2)|^2\Big)^{1/2}\Big(\sum_{\zeta_1,\zeta_2\in\mathbb{Z}}|D_2\varphi(\zeta_1,\zeta_2)|^2\Big)^{1/2} \\ &\leq \sum_{\zeta_1,\zeta_2\in\mathbb{Z}}\big(|D_1\varphi(\zeta_1,\zeta_2)|^2+|D_2\varphi(\zeta_1,\zeta_2)|^2\big) \\ &= \|\nabla_D\varphi\|^2_{\ell^2(\mathbb{Z}^2)}. \end{aligned}$$

□

Remark. We note here that this result does not generalise to higher dimension, as the inequality $2ab \leq a^2 + b^2$ does not hold true for more than 2 variables. This single lemma will give a special case of the Agmon–Kolmogorov inequality that, as we will see, is independent from the others.

We finally give three Agmon–Kolmogorov inequalities over $\mathbb{Z}^2$, showing the possibility of manipulating the exponents of the $\ell^2(\mathbb{Z}^2)$-norm values:

Proposition 5.6 (Agmon–Kolmogorov Inequalities over $\mathbb{Z}^2$)**.** *Let φ be a sequence in $\ell^2(\mathbb{Z}^2)$. Then the following inequalities hold true:*

(i)

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^2)} \leq 2^{5/8}\|\nabla_D\varphi\|^{1/4}_{\ell^2(\mathbb{Z}^2)}\|\varphi\|^{3/4}_{\ell^2(\mathbb{Z}^2)},$$

(ii)

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^2)} \leq 2^{1/4}\|\nabla_D\varphi\|^{1/2}_{\ell^2(\mathbb{Z}^2)}\|\varphi\|^{1/2}_{\ell^2(\mathbb{Z}^2)},$$

(iii)

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^2)} \leq 2^{-1/8}\|\nabla_D\varphi\|^{3/4}_{\ell^2(\mathbb{Z}^2)}\|\varphi\|^{1/4}_{\ell^2(\mathbb{Z}^2)}.$$

The proofs all share the same first step, namely "lifting" the one-dimensional Agmon–Kolmogorov inequality to an estimate in two-dimensional space, involving 4 separate terms. Then we can choose how to combine the different norms.

Proof.

Step 1: By applying the one-dimensional Agmon inequality 2.7 on the first variable, we have

$$|\varphi(\zeta_1,\zeta_2)|^4 \le \sum_{l\in\mathbb{Z}} |D_1\varphi(l,\zeta_2)|^2 \sum_{l\in\mathbb{Z}} |\varphi(l,\zeta_2)|^2.$$

Then by our Agmon–Cauchy Inequality, Lemma 5.3, considering $D_1\varphi(l,\zeta_2)$ as our sequence for the first term and $\varphi(l,\zeta_2)$ for the second, we have:

$$|\varphi(\zeta_1,\zeta_2)|^4 \le \|D_2D_1\varphi\|_{\ell^2(\mathbb{Z}^2)} \|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)} \|D_2\varphi\|_{\ell^2(\mathbb{Z}^2)} \|\varphi\|_{\ell^2(\mathbb{Z}^2)}. \tag{5.2}$$

Now we apply Lemma 5.4 to our mixed difference term, yielding:

$$|\varphi(\zeta_1,\zeta_2)|^4 \le 2\|D_2\varphi\|_{\ell^2(\mathbb{Z}^2)} \|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)} \|D_2\varphi\|_{\ell^2(\mathbb{Z}^2)} \|\varphi\|_{\ell^2(\mathbb{Z}^2)}. \tag{5.3}$$

Here we remember that $D_iD_j = D_jD_i$. Hence we could have estimated the mixed difference by $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)}$, which generally leads to the same inequalities, due to the symmetry of our argument. However, in (iii) this will prove useful, due to the different method employed.

Step 2:

(i) Now we use Lemma 5.4 on the first and third term on the RHS of (5.3):

$$\begin{aligned} |\varphi(\zeta_1,\zeta_2)|^4 &\le 2\cdot 2\|\varphi\|_{\ell^2(\mathbb{Z}^2)} \|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)} 2\|\varphi\|_{\ell^2(\mathbb{Z}^2)} \|\varphi\|_{\ell^2(\mathbb{Z}^2)} \\ &= 8\|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)} \|\varphi\|^3_{\ell^2(\mathbb{Z}^2)}. \end{aligned}$$

We square this inequality, repeat the argument symmetrically ending with a D_2-norm, and then add the resulting inequalities to one another (a process which we shall henceforth call symmetrising). We thus obtain:

$$\begin{aligned} 2|\varphi(\zeta_1,\zeta_2)|^8 &\le 64\left(\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^2)} + \|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^2)}\right) \|\varphi\|^6_{\ell^2(\mathbb{Z}^2)} \\ \Rightarrow \quad |\varphi(\zeta_1,\zeta_2)| &\le 2^{5/8}\|\nabla_D\varphi\|^{1/4}_{\ell^2(\mathbb{Z}^2)} \|\varphi\|^{3/4}_{\ell^2(\mathbb{Z}^2)}. \end{aligned}$$

(ii) We use Lemma 5.4 again, retaining two D_2-norms in this case:

$$\begin{aligned} |\varphi(\zeta_1,\zeta_2)|^4 &\leq 2\|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^2)}\,2\|\varphi\|_{\ell^2(\mathbb{Z}^2)}\|\varphi\|_{\ell^2(\mathbb{Z}^2)} \\ &= 4\|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^2)}\,\|\varphi\|^2_{\ell^2(\mathbb{Z}^2)}. \end{aligned}$$

We symmetrise:

$$\begin{aligned} 2|\varphi(\zeta_1,\zeta_2)|^4 &\leq 4\left(\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^2)} + \|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^2)}\right)\|\varphi\|^2_{\ell^2(\mathbb{Z}^2)}. \\ \Rightarrow \quad |\varphi(\zeta_1,\zeta_2)| &\leq 2^{1/4}\|\nabla_D\varphi\|^{1/2}_{\ell^2(\mathbb{Z}^2)}\|\varphi\|^{1/2}_{\ell^2(\mathbb{Z}^2)}. \end{aligned}$$

(iii) Using Lemma 5.5 on the first two terms on the RHS of (5.3), i.e.:

$$|\varphi(\zeta_1,\zeta_2)|^4 \leq \|\nabla_D\varphi\|^2_{\ell^2(\mathbb{Z}^2)}\|D_2\varphi\|_{\ell^2(\mathbb{Z}^2)}\|\varphi\|_{\ell^2(\mathbb{Z}^2)}. \tag{5.4}$$

We repeat the argument starting from (5.2), firstly estimating the mixed difference term by $\|D_1u\|_{\ell^2(\mathbb{Z}^2)}$:

$$|\varphi(\zeta_1,\zeta_2)|^4 \leq 2\|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)}\,\|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)}\|D_2\varphi\|_{\ell^2(\mathbb{Z}^2)}\|\varphi\|_{\ell^2(\mathbb{Z}^2)},$$

and then use Lemma 5.5 again to obtain:

$$|\varphi(\zeta_1,\zeta_2)|^4 \leq \|\nabla_D\varphi\|^2_{\ell^2(\mathbb{Z}^2)}\|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)}\|\varphi\|_{\ell^2(\mathbb{Z}^2)}. \tag{5.5}$$

We square both (5.4) and (5.5), and add them up to obtain:

$$2|\varphi(\zeta_1,\zeta_2)|^8 \leq \|\nabla_D\varphi\|^4_{\ell^2(\mathbb{Z}^2)}\left(\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^2)} + \|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^2)}\right)\|\varphi\|^2_{\ell^2(\mathbb{Z}^2)}$$

$$\Rightarrow \quad |\varphi(\zeta_1,\zeta_2)| \leq 2^{-1/8}\|\nabla_D\varphi\|^{3/4}_{\ell^2(\mathbb{Z}^2)}\|\varphi\|^{1/4}_{\ell^2(\mathbb{Z}^2)}.$$

$\square$

Remark. The last inequality, as we shall later see, is unique in the sense that the exponent of the $\ell^2(\mathbb{Z}^2)$-norm of the divergence is greater than that of the $\ell^2(\mathbb{Z}^2)$-norm of the sequence itself. This is due to the fact that we were able to generate it using Lemma 5.5, which only holds true on $\ell^2(\mathbb{Z}^2)$.

5.3 Agmon–Kolmogorov Inequalities – $(\nabla_D, \mathbb{Z}^3)$

We continue similarly, giving the Agmon–Cauchy Inequality and the operator bound on $\ell^2(\mathbb{Z}^3)$. Here we note that we have to "lift" our Agmon inequality twice. We introduce the following norms:

$$\|\varphi\|^2_{(\ell^\infty(\mathbb{Z}),\ell^2(\mathbb{Z}^2))} := \sup_{\zeta_1\in\mathbb{Z}} \sum_{\zeta_2,\zeta_3\in\mathbb{Z}} |\varphi(\zeta_1,\zeta_2,\zeta_3)|^2 \qquad \text{where} \quad i,j \in \{1,2,3\} \ \&\ i \neq j,$$

and:

$$\|\varphi\|^2_{(\ell^\infty(\mathbb{Z}^2),\ell^2(\mathbb{Z}))} := \sup_{\zeta_1,\zeta_2\in\mathbb{Z}} \sum_{\zeta_3\in\mathbb{Z}} |\varphi(\zeta_1,\zeta_2,\zeta_3)|^2 \qquad \text{where} \quad i,j \in \{1,2,3\} \ \&\ i \neq j.$$

Naturally, any combination of the three variables will be defined in the same way.

Lemma 5.7 (Agmon–Cauchy Inequality on $\ell^2(\mathbb{Z}^3)$)**.** *For a sequence $\varphi \in \ell^2(\mathbb{Z}^3)$, we have:*

$$\text{(i)} \qquad \sup_{\zeta_2\in\mathbb{Z}} \sum_{\zeta_1\in\mathbb{Z}} |\varphi(\zeta_1,\zeta_2,\zeta_3)|^2 \leq \Big(\sum_{\zeta_1,l\in\mathbb{Z}} |D_2\varphi(\zeta_1,l,\zeta_3)|^2 \Big)^{1/2} \Big(\sum_{\zeta_1,l\in\mathbb{Z}} |\varphi(\zeta_1,l,\zeta_3)|^2 \Big)^{1/2},$$

$$\text{(ii)} \qquad \|\varphi\|^2_{(\ell^2(\mathbb{Z}^2),\ell^\infty(\mathbb{Z}))} \leq \|D_3\varphi\|_{\ell^2(\mathbb{Z}^3)} \, \|\varphi\|_{\ell^2(\mathbb{Z}^3)}.$$

We note that, for all $i \in \{1,2,3\}$, we could have chosen any combination of D_i-terms by symmetry.

Proof. Using the one-dimensional Agmon–Kolmogorov inequality – $(D, \mathbb{Z})$, i.e. Proposition 2.7, on ζ_2 we find:

$$|\varphi(\zeta_1,\zeta_2,\zeta_3)|^2 \leq \Bigg[\Big(\sum_{l\in\mathbb{Z}} |D_2\varphi(\zeta_1,l,\zeta_3)|^2 \Big)^{1/2} \Big(\sum_{l\in\mathbb{Z}} |\varphi(\zeta_1,l,\zeta_3)|^2 \Big)^{1/2} \Bigg].$$

Now we sum with respect to ζ_1:

$$\sum_{\zeta_1\in\mathbb{Z}} |\varphi(\zeta_1,\zeta_2,\zeta_3)|^2 \leq \sum_{\zeta_1\in\mathbb{Z}} \Bigg[\Big(\sum_{l\in\mathbb{Z}} |D_2\varphi(\zeta_1,l,\zeta_3)|^2 \Big)^{1/2} \Big(\sum_{l\in\mathbb{Z}} |\varphi(\zeta_1,l,\zeta_3)|^2 \Big)^{1/2} \Bigg],$$

and use the Cauchy–Schwartz inequality:

$$\sum_{\zeta_1\in\mathbb{Z}} |\varphi(\zeta_1,\zeta_2,\zeta_3)|^2 \leq \Big(\sum_{\zeta_1,l\in\mathbb{Z}} |D_2\varphi(\zeta_1,l,\zeta_3)|^2 \Big)^{1/2} \Big(\sum_{\zeta_1,l\in\mathbb{Z}} |\varphi(\zeta_1,l,\zeta_3)|^2 \Big)^{1/2},$$

which completes the proof for (i).

To prove (ii), we apply Proposition 2.7 to ζ_3 and sum across both ζ_1 and ζ_2:

$$\sum_{\zeta_1,\in\mathbb{Z}}\sum_{\zeta_2\in\mathbb{Z}}|\varphi(\zeta_1,\zeta_2,\zeta_3)|^2 \le \sum_{\zeta_1,\in\mathbb{Z}}\sum_{\zeta_2\in\mathbb{Z}}\left[\Big(\sum_{m\in\mathbb{Z}}|D_3\varphi(\zeta_1,\zeta_2,m)|^2\Big)^{1/2}\Big(\sum_{m\in\mathbb{Z}}|\varphi(\zeta_1,\zeta_2,m)|^2\Big)^{1/2}\right],$$

and then we apply the Cauchy–Schwartz inequality twice:

$$\begin{aligned}\sum_{\zeta_1,\zeta_2\in\mathbb{Z}}|\varphi(\zeta_1,\zeta_2,\zeta_3)|^2 &\le \Big(\sum_{\zeta_1,\zeta_2,m\in\mathbb{Z}}|D_3\varphi(\zeta_1,\zeta_2,m)|^2\Big)^{1/2}\Big(\sum_{\zeta_1,\zeta_2,m\in\mathbb{Z}}|\varphi(\zeta_1,\zeta_2,m)|^2\Big)^{1/2}\\ &= \|D_3\varphi\|_{\ell^2(\mathbb{Z}^3)}\,\|\varphi\|_{\ell^2(\mathbb{Z}^3)}.\end{aligned}$$

□

We identify again an estimate of the $\ell^2(\mathbb{Z}^3)$-norm of a partial difference operator with the $\ell^2(\mathbb{Z}^3)$-norm of the sequence itself:.

Lemma 5.8. *For $\varphi \in \ell^2(\mathbb{Z}^3)$ and $i \in \{1,2,3\}$, we have:*

$$\|D_i\varphi\|_{\ell^2(\mathbb{Z}^3)} \le 2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}.$$

The proof is analogous to its two-dimensional equivalent, i.e. Lemma 5.4 and needs no repetition here. We thus apply these two Lemmata to generate four different Agmon–Kolmogorov inequalities on three-dimensional domains:

Proposition 5.9 (Agmon–Kolmogorov inequalities on $\ell^2(\mathbb{Z}^3)$)**.** *For a sequence $\varphi \in \ell^2(\mathbb{Z}^3)$, we have the following inequalities:*

(i)
$$\sup_{\zeta_1,\zeta_2,\zeta_3}|\varphi(\zeta_1,\zeta_2,\zeta_3)| \le \frac{2^{11/8}}{3^{1/16}}\|\nabla_D\varphi\|_{\ell^2(\mathbb{Z}^3)}^{1/8}\|\varphi\|_{\ell^2(\mathbb{Z}^3)}^{7/8},$$

(ii)
$$\sup_{\zeta_1,\zeta_2,\zeta_3}|\varphi(\zeta_1,\zeta_2,\zeta_3)| \le \frac{2^{5/4}}{3^{1/8}}\|\nabla_D\varphi\|_{\ell^2(\mathbb{Z}^3)}^{1/4}\|\varphi\|_{\ell^2(\mathbb{Z}^3)}^{3/4},$$

(iii)
$$\sup_{\zeta_1,\zeta_2,\zeta_3}|\varphi(\zeta_1,\zeta_2,\zeta_3)| \le \frac{2^{9/8}}{3^{3/16}}\|\nabla_D\varphi\|_{\ell^2(\mathbb{Z}^3)}^{3/8}\|\varphi\|_{\ell^2(\mathbb{Z}^3)}^{5/8},$$

(iv)
$$\sup_{\zeta_1,\zeta_2,\zeta_3}|\varphi(\zeta_1,\zeta_2,\zeta_3)| \le \frac{2}{3^{1/4}}\|\nabla_D\varphi\|_{\ell^2(\mathbb{Z}^3)}^{1/2}\|\varphi\|_{\ell^2(\mathbb{Z}^3)}^{1/2}.$$

Proof. By using the one-dimensional Agmon–Kolmogorov inequality – $(D, \mathbb{Z})$, Proposition 2.7, with regards to the first variable ζ_1, we have

$$|\varphi(\zeta_1,\zeta_2,\zeta_3)|^8 \leq \left(\sum_{l\in\mathbb{Z}} |D_1\varphi(l,\zeta_2,\zeta_3)|^2\right)^2 \left(\sum_{l\in\mathbb{Z}} |\varphi(l,\zeta_2,\zeta_3)|^2\right)^2.$$

Applying Lemma 5.7 (i) on each term:

$$|\varphi(\zeta_1,\zeta_2,\zeta_3)|^8 \leq \sum_{l,m\in\mathbb{Z}} |D_2D_1\varphi(l,m,\zeta_3)|^2 \sum_{l,m\in\mathbb{Z}} |D_1\varphi(l,m,\zeta_3)|^2 \sum_{l,m\in\mathbb{Z}} |D_2\varphi(l,m,\zeta_3)|^2 \sum_{l,m\in\mathbb{Z}} |\varphi(l,m,\zeta_3)|^2.$$

Now we employ Lemma 5.7 (ii), effectively 'lifting' the estimate to $\ell^2(\mathbb{Z}^3)$-norm values, yielding:

$$\begin{aligned} |\varphi(\zeta_1,\zeta_2,\zeta_3)|^8 \leq\; & \|D_3D_2D_1\varphi\|_{\ell^2(\mathbb{Z}^3)} \|D_2D_1\varphi\|_{\ell^2(\mathbb{Z}^3)} \|D_3D_1\varphi\|_{\ell^2(\mathbb{Z}^3)} \|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)} \cdot \\ & \|D_3D_2\varphi\|_{\ell^2(\mathbb{Z}^3)} \|D_2\varphi\|_{\ell^2(\mathbb{Z}^3)} \|D_3\varphi\|_{\ell^2(\mathbb{Z}^3)} \|\varphi\|_{\ell^2(\mathbb{Z}^3)}. \end{aligned}$$

We take this inequality as our starting point, and choose four different combinations of $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}$- and $\|\varphi\|_{\ell^2(\mathbb{Z}^3)}$-terms by applying Lemma 5.8 repeatedly.

(i) We now use Lemma 5.8, on all but one term above, leaving a single $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}$-term, and estimate the remaining terms by $\|\varphi\|_{\ell^2(\mathbb{Z}^3)}$ only:

$$\begin{aligned} |\varphi(\zeta_1,\zeta_2,\zeta_3)|^8 \leq\; & 4\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)} 4\|\varphi\|_{\ell^2(\mathbb{Z}^3)} 4\|\varphi\|_{\ell^2(\mathbb{Z}^3)} 2\|\varphi\|_{\ell^2(\mathbb{Z}^3)} 4\|\varphi\|_{\ell^2(\mathbb{Z}^3)} \cdot \\ & 2\|\varphi\|_{\ell^2(\mathbb{Z}^3)} 2\|\varphi\|_{\ell^2(\mathbb{Z}^3)} \|\varphi\|_{\ell^2(\mathbb{Z}^3)} \\ =\; & 2^{11} \|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)} \|\varphi\|^7_{\ell^2(\mathbb{Z}^3)}. \end{aligned}$$

We square and symmetrise:

$$3|\varphi(\zeta_1,\zeta_2,\zeta_3)|^{16} \leq 2^{22} (\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^3)} + \|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^3)} + \|D_3\varphi\|^2_{\ell^2(\mathbb{Z}^3)}) \|\varphi\|^{14}_{\ell^2(\mathbb{Z}^3)}$$

$$\Rightarrow \quad |\varphi(\zeta_1,\zeta_2,\zeta_3)| \leq \frac{2^{11/8}}{3^{1/16}} \|\nabla_D\varphi\|^{1/8}_{\ell^2(\mathbb{Z}^3)} \|\varphi\|^{7/8}_{\ell^2(\mathbb{Z}^3)}.$$

(ii) We again use 5.8 repeatedly, leaving two $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}$-terms, and estimating the remaining terms by $\|\varphi\|_{\ell^2(\mathbb{Z}^3)}$ only:

$$\begin{aligned}|\varphi(\zeta_1,\zeta_2,\zeta_3)|^8 &\le 4\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}\,2\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}4\|\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}\cdot\\ &\quad 4\|\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}\|\varphi\|_{\ell^2(\mathbb{Z}^3)}\\ &= 2^{10}\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^3)}\,\|\varphi\|^6_{\ell^2(\mathbb{Z}^3)}.\end{aligned}$$

We symmetrise:

$$3|\varphi(\zeta_1,\zeta_2,\zeta_3)|^8 \le 2^{10}(\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^3)}+\|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^3)}+\|D_3\varphi\|^2_{\ell^2(\mathbb{Z}^3)})\,\|\varphi\|^6_{\ell^2(\mathbb{Z}^3)}.$$

$$\Rightarrow\quad |\varphi(\zeta_1,\zeta_2,\zeta_3)| \le \frac{2^{5/4}}{3^{1/8}}\|\nabla_D\varphi\|^{1/4}_{\ell^2(\mathbb{Z}^3)}\|\varphi\|^{3/4}_{\ell^2(\mathbb{Z}^3)}.$$

(iii) Now we leave three $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}$-terms:

$$\begin{aligned}|\varphi(\zeta_1,\zeta_2,\zeta_3)|^8 &\le 4\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}\,2\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}\cdot\\ &\quad 4\|\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}\|\varphi\|_{\ell^2(\mathbb{Z}^3)}\\ &= 2^{9}\|D_1\varphi\|^3_{\ell^2(\mathbb{Z}^3)}\,\|\varphi\|^5_{\ell^2(\mathbb{Z}^3)}.\end{aligned}$$

We take it to the power 2/3 and symmetrise:

$$3|\varphi(\zeta_1,\zeta_2,\zeta_3)|^{16/3} \le 2^{18/3}(\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^3)}+\|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^3)}+\|D_3\varphi\|^2_{\ell^2(\mathbb{Z}^3)})\,\|\varphi\|^{10/3}_{\ell^2(\mathbb{Z}^3)}.$$

$$\Rightarrow\quad |\varphi(\zeta_1,\zeta_2,\zeta_3)| \le \frac{2^{9/8}}{3^{3/16}}\|\nabla_D\varphi\|^{3/8}_{\ell^2(\mathbb{Z}^3)}\|\varphi\|^{5/8}_{\ell^2(\mathbb{Z}^3)}.$$

(iv) Finally, we leave four $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}$-terms:

$$\begin{aligned}|\varphi(\zeta_1,\zeta_2,\zeta_3)|^8 &\le 4\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}\,2\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}\|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}\cdot\\ &\quad 4\|\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}2\|\varphi\|_{\ell^2(\mathbb{Z}^3)}\|\varphi\|_{\ell^2(\mathbb{Z}^3)}\\ &= 2^{8}\|D_1\varphi\|^4_{\ell^2(\mathbb{Z}^3)}\,\|\varphi\|^4_{\ell^2(\mathbb{Z}^3)}.\end{aligned}$$

We square root and symmetrise:

$$3|\varphi(\zeta_1,\zeta_2,\zeta_3)|^4 \le 2^{4}(\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^3)}+\|D_2\varphi\|^2_{\ell^2(\mathbb{Z}^3)}+\|D_3\varphi\|^2_{\ell^2(\mathbb{Z}^3)})\,\|\varphi\|^2_{\ell^2(\mathbb{Z}^3)}.$$

$$\Rightarrow \quad |\varphi(\zeta_1, \zeta_2, \zeta_3)| \leq \frac{2}{3^{1/4}} \|\nabla_D \varphi\|^{1/2}_{\ell^2(\mathbb{Z}^3)} \|\varphi\|^{1/2}_{\ell^2(\mathbb{Z}^3)}.$$

□

Remark. We note here that if we were to start with estimates by $\|D_2\varphi\|_{\ell^2(\mathbb{Z}^3)}$ or $\|D_3\varphi\|_{\ell^2(\mathbb{Z}^3)}$, we would obtain the same inequalities, due to the symmetrising argument.

5.4 Agmon–Kolmogorov Inequalities – $(\nabla_D, \mathbb{Z}^d)$

We finally generalise the previous method to arbitrary dimension, in addition to which we obtain a formula for the constant depending on both dimension and exponent of the $\ell^2(\mathbb{Z}^d)$-norm values chosen. We introduce the following notation:

Definition 5.10. For a sequence $\varphi(\zeta) \in \ell^2(\mathbb{Z}^d)$ with $\zeta := (\zeta_1, ..., \zeta_d) \in \mathbb{Z}^d$, for $0 \leq k \leq d$ we define:

$$[\varphi]_k := \left(\sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta)|^2 \right)^{1/2}.$$

Remark. We identify that $[\varphi]_0 := |\varphi(\zeta)|$ and if we apply this operator for $k = d$, i.e. sum across all coordinates, we obtain the $\ell^2(\mathbb{Z}^d)$-norm:

$$[\varphi]_d = \|\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

The following lifts the Agmon–Kolmogorov-type inequality by one dimension:

Lemma 5.11 (Agmon–Cauchy inequality – $(\nabla_D, \mathbb{Z}^d)$**.** *For the operator D_{k+1}, acting on a sequence $\varphi(\zeta) \in \ell^2(\mathbb{Z}^d)$, we have:*

$$\sup_{\zeta_{k+1} \in \mathbb{Z}} [\varphi]_k \leq [D_{k+1}\varphi]^{1/2}_{k+1} \, [\varphi]^{1/2}_{k+1}.$$

Proof. Using the discrete Agmon inequality on the $(k+1)^{th}$ coordinate, we find:

$$|\varphi(\zeta_1, \ldots, \zeta_d)|^2 \leq \Big(\sum_{l \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2} \Big(\sum_{l \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2}.$$

Now we sum with respect to the other coordinates:

$$\sum_{\zeta_1\in\mathbb{Z}}\cdots\sum_{\zeta_k\in\mathbb{Z}}|\varphi(\zeta_1,\ldots,\zeta_d)|^2 \leq$$

$$\sum_{\zeta_1\in\mathbb{Z}}\cdots\sum_{\zeta_k\in\mathbb{Z}}\Bigg[\Big(\sum_{l\in\mathbb{Z}}|D_{k+1}\varphi(\zeta_1,\ldots,\zeta_k,l,\zeta_{k+2},\ldots,\zeta_d)|^2\Big)^{1/2}\Big(\sum_{l\in\mathbb{Z}}|\varphi(\zeta_1,\ldots,\zeta_k,l,\zeta_{k+2},\ldots,\zeta_d)|^2\Big)^{1/2}\Bigg],$$

and use the Cauchy–Schwartz inequality on the k^{th} coordinate:

$$\begin{aligned}\sum_{\zeta_1\in\mathbb{Z}}\cdots\sum_{\zeta_k\in\mathbb{Z}}|\varphi(\zeta_1,\ldots,\zeta_d)|^2 \quad\leq\quad & \sum_{\zeta_1\in\mathbb{Z}}\cdots\sum_{\zeta_{k-1}\in\mathbb{Z}}\Big[\Big(\sum_{\zeta_k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}|D_{k+1}\varphi(\zeta_1,\ldots,\zeta_k,l,\zeta_{k+2},\ldots,\zeta_d,)|^2\Big)^{1/2}\cdot\\ & \Big(\sum_{\zeta_k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}|\varphi(\zeta_1,\ldots,\zeta_k,l,\zeta_{k+2},\ldots,\zeta_d)|^2\Big)^{1/2}\Big].\end{aligned}$$

We repeat this process to finally obtain:

$$\begin{aligned}\sum_{\zeta_1\in\mathbb{Z}}\cdots\sum_{\zeta_k\in\mathbb{Z}}|\varphi(\zeta_1,\ldots,\zeta_d)|^2 \quad\leq\quad & \Big(\sum_{\zeta_1\in\mathbb{Z}}\cdots\sum_{\zeta_k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}|D_{k+1}\varphi(\zeta_1,\ldots,\zeta_k,l,\zeta_{k+2},\ldots,\zeta_d)|^2\Big)^{1/2}\cdot\\ & \Big(\sum_{\zeta_1\in\mathbb{Z}}\cdots\sum_{\zeta_k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}|\varphi(\zeta_1,\ldots,\zeta_k,l,\zeta_{k+2},\ldots,\zeta_d)|^2\Big)^{1/2}.\end{aligned}$$

□

We again estimate the $\ell^2(\mathbb{Z}^d)$-norm of a partial difference operator with the $\ell^2(\mathbb{Z}^d)$-norm of the sequence itself.

Lemma 5.12 (Operator Bound on $\ell^2(\mathbb{Z}^d)$)**.** *For a sequence $\varphi\in\ell^2(\mathbb{Z}^d)$ and for $i\in\{1,\ldots,d\}$, we have:*

$$\|D_i\varphi\|_{\ell^2(\mathbb{Z}^d)}\leq 2\|\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

The proof is again analogous to its two-dimensional equivalent, i.e. Lemma 5.4 and needs no repetition here. This implies that we can obtain an estimate for any mixed difference operator as follows:

$$\|D_1\ldots D_k\varphi\|_{\ell^2(\mathbb{Z}^d)}\leq 2\|D_1\ldots D_{l-1}D_{l+1}\ldots D_k\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

As the operators all commute with each other, we can choose to estimate a k^{th} order mixed difference operator with one of order $(k-1)$, whilst generating a factor of 2. Therefore, by eliminating l difference operators, our inequality will contain the constant 2^l.

We arrive at our main Theorem for this chapter, the Agmon–Kolmogorov inequality on $\ell^2(\mathbb{Z}^d)$. Again for simplicity, we split the Theorem in two. The first proves the inequality, and the second finds a formula for the constant. The second proves much more challenging.

Theorem 5.13 (Agmon–Kolmogorov Inequality – $(\nabla_D, \mathbb{Z}^d)$)**.** *For a sequence $\varphi \in \ell^2(\mathbb{Z}^d)$, and $p \in \{1, \ldots, 2^{d-1}\}$, we have:*

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \mu_{p,d} \|\nabla_D \varphi\|^{p/2^d}_{\ell^2(\mathbb{Z}^d)} \|\varphi\|^{1-p/2^d}_{\ell^2(\mathbb{Z}^d)},$$

where

$$\mu_{p,d} := \left(\frac{\kappa_{p,d}}{d^{p/2}}\right)^{1/2^d},$$

and $\kappa_{p,d}$ is a constant to be determined later.

Proof.

For clarity, we divide the proof into 3 steps. The first 'lifts' our Agmon–Kolmogorov inequality to arbitrary dimension. The second analyses the structure of the estimate. The third step completes the proof of the inequality.

Step 1:

We initiate the process as in the case of $\mathbb{Z}^2$ and $\mathbb{Z}^3$ by using the one-dimensional Agmon–Kolmogorov, and then the Agmon–Cauchy inequality, Lemma 5.11, repeatedly:

$$\begin{aligned}
\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} &\leq [D_1\varphi]_1^{1/2}\,[\varphi]_1^{1/2} \\
&\leq [D_2D_1\varphi]_2^{1/4}\,[D_1\varphi]_2^{1/4}\,[D_2\varphi]_2^{1/4}\,[\varphi]_2^{1/4} \\
&\leq [D_3D_2D_1\varphi]_3^{1/8}\,[D_2D_1\varphi]_3^{1/8}\,[D_3D_1\varphi]_3^{1/8}\,\cdot \\
&\quad\; [D_1\varphi]_3^{1/8}[D_3D_2\varphi]_3^{1/8}[D_2\varphi]_3^{1/8}[D_3\varphi]_3^{1/8}[\varphi]_3^{1/8} \\
&\;\vdots \\
&\leq [D_d\ldots D_1\varphi]_d^{1/2^d}\;\ldots\;\ldots\;[\varphi]_d^{1/2^d} \\
&= \|D_d\ldots D_1\varphi\|^{1/2^d}_{\ell^2(\mathbb{Z}^d)}\;\ldots\;\ldots\;\|\varphi\|^{1/2^d}_{\ell^2(\mathbb{Z}^d)} \\
\Rightarrow \|\varphi\|^{2^d}_{\ell^\infty(\mathbb{Z}^d)} &\leq \|D_d\ldots D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}\;\ldots\;\ldots\;\|\varphi\|_{\ell^2(\mathbb{Z}^d)}.
\end{aligned}$$

Step 2:

We have generated an estimate by 2^d norms, with exactly 2^{d-1} norms originating from the term

$[D_1\varphi]_1^{1/2}$. All those will thus involve the operator D_1, or more formally: $|\Xi_1| = 2^{d-1}$, where we let

$$\Xi_1 := \left\{ \|D_{a_1}\dots D_{a_k} D_1\varphi\|_{\ell^2(\mathbb{Z}^d)} \,\middle|\, a_i \neq a_j \;\forall\, i \neq j\,;\, \{a_1,\dots,a_k\} \subset \{2,\dots,d\} \right\}.$$

We note that we could also employ estimates by $\|D_i\varphi\|_{\ell^2(\mathbb{Z}^d)}$ for any $i \in \{1,\dots,2^d\}$, but our inequality will not change due to our symmetrising argument. Similarly, we have 2^{d-1} norms originating from the term $[\varphi]_1^{1/2}$, whose estimates will not involve the operator D_1. Hence $|\Xi_2| = 2^{d-1}$, where we let

$$\Xi_2 := \left\{ \|D_{a_1}\dots D_{a_k}\varphi\|_{\ell^2(\mathbb{Z}^d)} \,\middle|\, a_i \neq a_j \;\forall\, i \neq j\,;\, \{a_1,\dots,a_k\} \subset \{2,\dots,d\} \right\}.$$

Step 3:

We will now apply Lemma 5.12 repeatedly, to reduce the order of the operator inside the norms to either 0 or 1. For this argument to proceed, as before in the two- and three-dimensional case, we recognise that we have to estimate all ${}^1\xi \in \Xi_1$ by ${}^1\xi_1 := \|D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}$ or alternatively by $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$. Hence, we choose a $p \in \{0,\dots,2^{d-1}\}$ to estimate p elements in Ξ_1 by $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}$, leaving $2^{d-1} - p$ elements in Ξ_1 to be estimated by $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$. However, for all 2^{d-1} elements ${}^2\xi \in \Xi_2$, we have to provide an estimate by ${}^2\xi_1 := \|\varphi\|_{\ell^2(\mathbb{Z}^d)}$ only. This means we have $2^d - p$ elements in $\Xi := \Xi_1 \bigcup \Xi_2$ to be estimated by $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$:

$$\|\varphi\|^{2^d}_{\ell^\infty(\mathbb{Z}^d)} \leq \kappa_{p,d}\|D_1\varphi\|^p_{\ell^2(\mathbb{Z}^d)}\,\|\varphi\|^{2^d-p}_{\ell^2(\mathbb{Z}^d)}.$$

where $\kappa_{p,d}$ remains a constant of the form 2^z with $z \in \mathbb{Q}$, which we leave to be identified in the next section. We continue as in the sections before:

$$\|\varphi\|^{2^{d+1}/p}_{\ell^\infty(\mathbb{Z}^d)} \leq \kappa_{p,d}^{2/p}\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^d)}\,\|\varphi\|^{(2^{d+1}-2p)/p}_{\ell^2(\mathbb{Z}^d)}.$$

We now symmetrise:

$$\begin{aligned} d\,\|\varphi\|^{2^{d+1}/p}_{\ell^\infty(\mathbb{Z}^d)} &\leq \kappa_{p,d}^{2/p}\left(\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^d)} + \ldots + \|D_d\varphi\|^2_{\ell^2(\mathbb{Z}^d)}\right)\|\varphi\|^{(2^{d+1}-2p)/p}_{\ell^2(\mathbb{Z}^d)} \\ &= \kappa_{p,d}^{2/p}\|\nabla_D\varphi\|^2_{\ell^2(\mathbb{Z}^d)}\,\|\varphi\|^{(2^{d+1}-2p)/p}_{\ell^2(\mathbb{Z}^d)}, \end{aligned}$$

and finally rearrange:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \left(\frac{\kappa_{p,d}}{d^{p/2}}\right)^{1/2^d}\|\nabla_D\varphi\|^{p/2^d}_{\ell^2(\mathbb{Z}^d)}\,\|\varphi\|^{1-p/2^d}_{\ell^2(\mathbb{Z}^d)}.$$

□

5.5 The Constant $\kappa_{p,d}$

It remains to identify the constant $\kappa_{p,d}$, where we will have to analyse its dependence on dimension and our choice of p, as we have up to $2^d - 1$ $\ell^2(\mathbb{Z}^d)$-norm values to estimate. Note that in this section we will use the notation $\binom{a}{b}$ instead of ${}^aC_b$, for ease of reading.

Theorem 5.14. *We have, for arbitrary dimension d and $p \in \{1, \ldots, 2^{d-1}\}$:*

$$\kappa_{p,d} = 2^{\,d \cdot 2^{d-1} - p}.$$

We will break the proof down into two Lemmata and a final step, due to the amount of combinatorics involved. The method for finding $\kappa_{p,d}$ will rely largely on the following observation:

Let $\tau(\xi)$ be the order of the operator contained in any given $\xi \in \Xi$. Then we let $\Omega_i := \{\xi \,|\, \tau(\xi) = i\}$, be the set of all terms in the estimate whose operator has a given order i. In Ξ_1 we have $1 \le i \le d$, and in Ξ_2, $0 \le i \le d-1$. Then we have the following structures for the operators occuring in ${}^1\xi \in \Xi_1$ and ${}^2\xi \in \Xi_2$ respectively:

Ξ_1	${}^1\xi_{2^{d-1}}$	$\ldots$	$\ldots$	$\ldots$	${}^1\xi_1$	$\lvert\Omega_1\rvert$	$\lvert\Omega_2\rvert$	$\lvert\Omega_3\rvert$	$\ldots$	$\lvert\Omega_d\rvert$
$\mathbb{Z}$:					D_1	1				
$\mathbb{Z}^2$:				D_2D_1	D_1	1	1			
$\mathbb{Z}^3$:		$D_3D_2D_1$	D_2D_1	D_3D_1	D_1	1	2	1		
$\mathbb{Z}^4$:		$\iddots$	$\iddots$	$\iddots$	D_1	1	3	3	1	
$\vdots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$
$\mathbb{Z}^d$:	$D_d \ldots D_1$	$\ldots$	$\ldots$	$\ldots$	D_1	$\binom{d-1}{0}$	$\binom{d-1}{1}$	$\binom{d-1}{2}$	$\ldots$	$\binom{d-1}{d-1}$

$$
\begin{array}{cccccccccccc}
\Xi_2 & {}^2\xi_{2^{d-1}} & \dots & \dots & \dots & \dots & {}^2\xi_1 & |\Omega_0| & |\Omega_1| & |\Omega_2| & \dots & |\Omega_{d-1}| \\
\mathbb{Z}: & & & & & & 1 & 1 & & & & \\
\mathbb{Z}^2: & & & & & D_2 & 1 & 1 & 1 & & & \\
\mathbb{Z}^3: & & & D_3D_2 & D_2 & D_3 & 1 & 1 & 2 & 1 & & \\
\mathbb{Z}^4: & & \iddots & \iddots & \iddots & D_4 & 1 & 1 & 3 & 3 & 1 & \\
\vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\mathbb{Z}^d: & D_d \dots D_2 & \dots & \dots & \dots & D_d & 1 & \binom{d-1}{0} & \binom{d-1}{1} & \binom{d-1}{2} & \dots & \binom{d-1}{d-1}
\end{array}
$$

Our first Lemma identifies $|\Omega_i|$, and thus justifies these diagrams:

Lemma 5.15. *For the size of Ω_i, we have for $d \geq 2$:*
For Ξ_1:

$$|\Omega_i| = \binom{d-1}{i-1}, \qquad 1 \leq i \leq d,$$

and Ξ_2:

$$|\Omega_i| = \binom{d-1}{i}, \qquad 0 \leq i \leq d-1.$$

Proof. We follow by induction and prove the case of Ξ_2, noting that the argument for Ξ_1 is symmetrically identical. We have already seen that the formula is correct for $d = 2$ (by the diagram above or in Section 5.2), and now we assume it is true for $d = l$, i.e. for $0 \leq i \leq l-1$:

$$|\Omega_i| = \binom{l-1}{i},$$

and thus we have the following list, equivalent to the last line of the diagram above:

$$
\begin{array}{cccccccccccc}
\Xi_2 & {}^2\xi_{2^{d-1}} & \dots & \dots & {}^2\xi_2 & {}^2\xi_1 & |\Omega_0| & |\Omega_1| & |\Omega_2| & \dots & |\Omega_{l-1}| \\
\mathbb{Z}^l: & D_l \dots D_2 & \dots & \dots & D_l & 1 & \binom{l-1}{0} & \binom{l-1}{1} & \binom{l-1}{2} & \dots & \binom{l-1}{l-1}
\end{array}
$$

Now each term of a given order τ will, by the Agmon–Cauchy inequality (Lemma 5.11), generate a term of order τ and one of order $\tau + 1$. Thus we have:

$$
\begin{array}{ccccccccccc}
\Xi_2 & {}^2\xi_{2^d} & \dots & \dots & {}^2\xi_2 & {}^2\xi_1 & |\Omega_0| & |\Omega_1| & |\Omega_2| & \dots & |\Omega_l| \\
\mathbb{Z}^{l+1}: & D_{l+1} \dots D_2 & \dots & \dots & D_{l+1} & 1 & \binom{l-1}{0} & \binom{l-1}{0} + \binom{l-1}{1} & \binom{l-1}{1} + \binom{l-1}{2} & \dots & \binom{l-1}{l-1}
\end{array}
$$

Now we apply Lemma 4.1 (i), i.e. ${}^{a}C_b + {}^{a}C_{b+1} = {}^{a+1}C_{b+1}$ and consider ${}^{a}C_0 = {}^{a}C_a = 1$, which immediately implies:

$$\begin{array}{ccccccccccc} \Xi_2 & {}^2\xi_{2^d} & \dots & \dots & {}^2\xi_2 & {}^2\xi_1 & |\Omega_0| & |\Omega_1| & |\Omega_2| & \dots & |\Omega_{l+1}| \\ \mathbb{Z}^{l+1}: & D_{l+1}\dots D_2 & \dots & \dots & D_2 & 1 & \binom{l}{0} & \binom{l}{1} & \binom{l}{2} & \dots & \binom{l}{l} \end{array}$$

and hence for $d = l+1$, we have:

$$|\Omega_i| = \binom{l}{i},$$

completing our inductive step.

□

As discussed previously, if we consider to estimate a given $\xi \in \Xi$ using Lemma 5.12, we will, for example, obtain:

$$\|D_1 \dots D_k \varphi\|_{\ell^2(\mathbb{Z}^d)} \leq 2\|D_1 \dots D_{l-1} D_{l+1} \dots D_k \varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

We can see that we generate a factor of 2 for every partial difference operator we eliminate, and thus have, for ${}^1\xi \in \Xi_1$ and ${}^2\xi \in \Xi_2$ with order $\tau({}^1\xi)$ and $\tau({}^2\xi)$ respectively:

$${}^1\xi \leq\ 2^{\tau({}^1\xi)-1}\,\|D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}, \qquad \text{and} \qquad {}^2\xi \leq\ 2^{\tau({}^2\xi)}\,\|\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

We note here that $\kappa_{p,d}$ will not depend on which $\ell^2(\mathbb{Z}^d)$-norms in Ξ_1 are chosen to be estimated by ${}^2\xi_1 := \|\varphi\|_{\ell^2(\mathbb{Z}^d)}$. The reason for this is transparent when considering that the sum of all the orders $\sum_{i=1}^{2^{d-1}} \tau({}^1\xi_i)$ is a constant and needs to be reduced to the constant $p \cdot \tau({}^1\xi_1) = p$, generating a unique $\kappa_{p,d}$.

Lemma 5.16. *The* $\min_p \kappa_{p,d}$ *will be attained at* $p = 2^{d-1}$ *and takes on the following explicit form:*

$$\kappa_{2^{d-1},d} = \prod_{i=0}^{d-1} 2^{2i\binom{d-1}{i}}.$$

Proof. Our minimum constant for Ξ_1 in fact occurs if we choose all ${}^1\xi_1 \in \Xi_1$ to be estimated by $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}$, i.e. choose $p = 2^{d-1}$, the maximum p possible. Our minimum constant, denoted by ρ_d^1, for all terms in Ξ_1 will thus be:

$$\rho_d^1 = \prod_{k=1}^{2^{d-1}} 2^{\tau({}^1\xi_k)-1}.$$

Instead of examining each individual element ${}^1\xi$, we consider that all ${}^1\xi$ of equal order i generate the same constant, namely 2^{i-1}. Thus we collect all ${}^1\xi$ of the same order, and obtain:

$$\rho_d^1 = \prod_{i=1}^{d} 2^{(i-1)|\Omega_i|} = \prod_{i=1}^{d} 2^{(i-1)\binom{d-1}{i-1}}.$$

Then we need to estimate all ${}^2\xi \in \Xi_2$, and we proceed as for Ξ_1. All ${}^2\xi$ need to be estimated by $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$, each generating the constant 2^i, forming the equivalent pattern as that of Ξ_1. We thus obtain, for the minimal constant ρ_d^2:

$$\rho_d^2 = \prod_{i=0}^{d-1} 2^{i|\Omega_i|} = \prod_{i=0}^{d-1} 2^{i\binom{d-1}{i}}.$$

We now see that $\rho_d^2 = \rho_d^1$, and:

$$\kappa_{2^{d-1},d} = \rho_d^2 \rho_d^1 = \prod_{i=0}^{d-1} 2^{2i\binom{d-1}{i}}.$$

□

We are now finally in a position to prove Theorem 5.14:

Proof of Theorem 5.14. We are left to analyse the constant's dependence on our choice of p. First we note that in addition to the constant generated above we will have chosen $2^{d-1} - p$ terms to be further reduced to $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$, each generating a power of 2. Hence we additionally need to multiply $\kappa_{2^{d-1},d}$ by $2^{2^{d-1}-p}$. Thus our final constant will be:

$$\kappa_{p,d} = 2^{2^{d-1}-p} \cdot \prod_{i=0}^{d-1} 2^{2i\binom{d-1}{i}} = 2^{2^{d-1}-p+2\sum_{i=0}^{d-1} i\binom{d-1}{i}},$$

Then we can simplify this further by considering the binomial formula $(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$. We differentiate with respect to X and set $X = 1$:

$$n \cdot 2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}.$$

Thus

$$2\sum_{i=0}^{d-1} i\binom{d-1}{i} + 2^{d-1} - p \quad = 2 \cdot (d-1)2^{d-2} + 2^{d-1} - p \quad = d \cdot 2^{d-1} - p,$$

and we finally have

$$\kappa_{p,d} = 2^{d \cdot 2^{d-1} - p},$$

as required.

□

5.6 Explicit Formulae for $\mu_{p,2}$ and $\mu_{p,3}$

We consolidate our inequalities for two- and three-dimensional domains into simple formulae, using the multidimensional version of the Agmon–Kolmogorov inequality, Theorem 5.13:

($\mathbb{Z}^2$) If we set $d = 2$, we have:

$$\mu_{p,2} := \left(\frac{\kappa_{p,2}}{2^{p/2}}\right)^{1/4}.$$

We obtain $\mu_{p,2}$, via:

$$\kappa_{p,2} = 2^{4-p} \quad \Rightarrow \quad \mu_{p,2} := \frac{2^{\frac{4-p}{4}}}{2^{p/8}} = 2^{1-3p/8}.$$

Hence our Agmon–Kolmogorov inequalities for $d = 2$ read:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^2)} \leq 2^{1-3p/8} \|\nabla_D \varphi\|_{\ell^2(\mathbb{Z}^2)}^{p/4} \|\varphi\|_{\ell^2(\mathbb{Z}^2)}^{1-p/4}, \qquad \text{for } p \in \{1,2,3\}.$$

where we note again that $p = 3$ is a special case, and occurs in two dimensions only.

($\mathbb{Z}^3$) Alternatively for $d = 3$:

$$\mu_{p,3} := \left(\frac{\kappa_{p,3}}{3^{p/2}}\right)^{1/8}.$$

We obtain $\mu_{p,3}$, via:

$$\kappa_{p,3} = 2^{3 \cdot 2^2 - p} = 2^{12-p} \quad \Rightarrow \quad \mu_{p,3} := \frac{2^{\frac{12-p}{8}}}{3^{p/16}}.$$

Hence our Agmon–Kolmogorov inequalities for $d = 3$ read:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^3)} \leq \frac{2^{\frac{12-p}{8}}}{3^{p/16}} \|\nabla_D \varphi\|_{\ell^2(\mathbb{Z}^3)}^{p/8} \|\varphi\|_{\ell^2(\mathbb{Z}^3)}^{1-p/8}, \qquad \text{for } p \in \{1,2,3,4\}.$$

5.7 Generalised Sobolev Inequality – $(\nabla_D, \mathbb{Z}^d)$

As before, we use our Agmon–Kolmogorov inequality – $(\nabla_D, \mathbb{Z}^d)$, Theorem 5.13, to find the Generalised Sobolev inequality – $(\nabla_D, \mathbb{Z}^d)$, and subsequently the Lieb–Thirring inequality – $(\nabla_D, \mathbb{Z}^d)$. As the exponents and constants can vary, we give the most general form.

Theorem 5.17. *Let $\{\psi_j\}_{j=1}^N$ be an orthonormal system of sequences in $\ell^2(\mathbb{Z}^d)$, i.e. $\langle \psi_j, \psi_k \rangle_d = \delta_{jk}$, $\zeta \in \mathbb{Z}^d$ and let $\rho(\zeta) := \sum_{j=1}^N |\psi_j(\zeta)|^2$, $\alpha := p/2^d$ with $p \in \{1, \dots, 2^{d-1}\}$. Then:*

$$\sum_{\zeta \in \mathbb{Z}^d} \rho^{\frac{1+\alpha}{\alpha}}(\zeta) \le \mu_{p,d}^{2/\alpha} \sum_{j=1}^N \Big(\sum_{\zeta \in \mathbb{Z}^d} |\nabla_D \psi_j(\zeta)|^2 \Big),$$

where

$$\mu_{p,d}^{2/\alpha} = \left(\frac{\kappa_{p,d}}{d^{p/2}} \right)^{2/p} = \frac{2^{d/p \cdot 2^d - 2}}{d}.$$

Proof. Let $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{C}^N$. Then, by the Agmon–Kolmogorov inequality – $(\nabla_D, \mathbb{Z}^d)$ for all $\zeta \in \mathbb{Z}^d$, letting $\alpha := p/2^d$, we have:

$$\begin{aligned}
\Big| \sum_{j=1}^N \xi_j \psi_j(\zeta) \Big|^2 &\le \mu_{p,d}^2 \Big\| \sum_{j=1}^N \xi_j \psi_j \Big\|_{\ell^2(\mathbb{Z}^d)}^{2(1-\alpha)} \Big\| \nabla_D \sum_{j=1}^N \xi_j \psi_j \Big\|_{\ell^2(\mathbb{Z}^d)}^{2\alpha} \\
&= \mu_{p,d}^2 \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \langle \psi_j, \psi_k \rangle_d \Big)^{1-\alpha} \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \langle \nabla_D \psi_j, \nabla_D \psi_k \rangle_d \Big)^{\alpha} \\
&= \mu_{p,d}^2 \Big(\sum_{j=1}^N |\xi_j|^2 \Big)^{1-\alpha} \Big(\sum_{j,k=1}^N \xi_j \bar{\xi}_k \langle \nabla_D \psi_j, \nabla_D \psi_k \rangle_d \Big)^{\alpha}.
\end{aligned}$$

If we set $\xi_j := \overline{\psi_j(\zeta)}$ and we have $\rho(\zeta) = \sum_{j=1}^N |\psi_j(\zeta)|^2$, then the latter inequality becomes

$$\begin{aligned}
\rho^2(\zeta) &\le \mu_{p,d}^2 \rho^{1-\alpha}(\zeta) \Big(\sum_{j,k=1}^N \psi_j(\zeta) \overline{\psi_k(\zeta)} \langle \nabla_D \psi_j, \nabla_D \psi_k \rangle_d \Big)^{\alpha} \\
\Rightarrow \rho^{\frac{2-(1-\alpha)}{\alpha}}(\zeta) &\le \mu_{p,d}^{2/\alpha} \sum_{j,k=1}^N \psi_j(\zeta) \overline{\psi_k(\zeta)} \langle \nabla_D \psi_j, \nabla_D \psi_k \rangle_d \\
\Rightarrow \sum_{\zeta \in \mathbb{Z}^d} \rho^{\frac{1+\alpha}{\alpha}}(\zeta) &\le \mu_{p,d}^{2/\alpha} \sum_{j=1}^N \Big(\sum_{\zeta \in \mathbb{Z}^d} |\nabla_D \psi_j(\zeta)|^2 \Big).
\end{aligned}$$

□

5.8 Lieb–Thirring Inequality – $(H_D, \mathbb{Z}^d)$

As before, we apply the Generalised Sobolev inequality to the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ to obtain bounds for its negative eigenvalues.

Let $\{\psi_j\}_{j=1}^N$, $N \in \overline{\mathbb{N}}$ be the orthonormal system of eigensequences in $\ell^2(\mathbb{Z}^d)$ corresponding to the negative eigenvalues $\{e_j\}_{j=1}^N$ of the discrete Schrödinger operator of dimension d:

$$(H_D^d \psi_j)(\zeta) \coloneqq (-\nabla_D^2 \psi_j)(\zeta) - b_\zeta \psi_j(\zeta) = e_j \psi_j(\zeta), \tag{5.6}$$

where $j \in \{1, \dots, N\}$ and we assume that $b_\zeta \geq 0$ for all $\zeta \in \mathbb{Z}^d$. Our next result is concerned with estimating those negative eigenvalues:

Theorem 5.18. *Let $b_\zeta \geq 0$, $\{b_\zeta\}_{\zeta \in \mathbb{Z}^d} \in \ell^{1+\alpha}(\mathbb{Z}^d)$, for $\alpha \coloneqq p/2^d$, and $p \in \{1, \dots, 2^{d-1}\}$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D^d are discrete and they satisfy the inequality:*

$$\sum_{j=1}^N |e_j| \leq \vartheta_{p,d} \sum_{\zeta \in \mathbb{Z}^d} b_\zeta^{1+\alpha},$$

where

$$\vartheta_{p,d} \coloneqq \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}} \frac{2^{d-2\alpha}}{d^\alpha} = \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}} \mu_{p,d}^2.$$

Proof. We take the inner product with $\psi_j(\zeta)$ on (5.6), and sum both sides of the equation with respect to j.

$$\sum_{j=1}^N e_j = \sum_{j=1}^N \Big(\sum_{\zeta \in \mathbb{Z}^d} |\nabla_D \psi_j(\zeta)|^2 \Big) - \sum_{j=1}^N \Big(\sum_{\zeta \in \mathbb{Z}^d} b_\zeta |\psi_j(\zeta)|^2 \Big).$$

We now use the Generalised Sobolev inequality – $(\nabla_D, \mathbb{Z}^d)$, i.e. Theorem 5.17, and the discrete Hölder inequality to obtain:

$$\sum_{j=1}^N e_j \geq \mu_{p,d}^{-2/\alpha} \sum_{\zeta \in \mathbb{Z}^d} \Big(\sum_{j=1}^N |\psi_j(\zeta)|^2 \Big)^{\frac{1+\alpha}{\alpha}} - \Big(\sum_{\zeta \in \mathbb{Z}^d} b_\zeta^{1+\alpha} \Big)^{\frac{1}{1+\alpha}} \Big(\sum_{\zeta \in \mathbb{Z}^d} \Big(\sum_{j=1}^N |\psi_j(\zeta)|^2 \Big)^{\frac{1+\alpha}{\alpha}} \Big)^{\frac{\alpha}{1+\alpha}}. \tag{5.7}$$

We define

$$\chi \coloneqq \Big(\sum_{\zeta \in \mathbb{Z}^d} \Big(\sum_{j=1}^N |\psi_j(\zeta)|^2 \Big)^{\frac{1+\alpha}{\alpha}} \Big)^{\frac{\alpha}{1+\alpha}}, \quad \varrho \coloneqq \Big(\sum_{\zeta \in \mathbb{Z}^d} b_\zeta^{1+\alpha} \Big)^{\frac{1}{1+\alpha}}.$$

Then (5.7) can be written as

$$\mu_{p,d}^{-2/\alpha} \chi^{\frac{1+\alpha}{\alpha}} - \varrho \chi \leq \sum_{j=1}^N e_j.$$

The LHS is maximal when

$$0 = \frac{1+\alpha}{\alpha}\mu_{p,d}^{-2/\alpha}\chi^{\frac{1+\alpha}{\alpha}-1} - \varrho$$

$$\Rightarrow \quad \chi = \left(\frac{\alpha\varrho}{\mu_{p,d}^{-2/\alpha}(1+\alpha)}\right)^{\alpha} = \mu_{p,d}^2\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\Big(\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha}\Big)^{\frac{\alpha}{1+\alpha}}.$$

Substituting this into (5.7), we obtain:

$$\begin{aligned}
\sum_{j=1}^{N} e_j \quad &\geq \quad \mu_{p,d}^{-2/\alpha}\Big(\mu_{p,d}^2\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\Big)^{\frac{1+\alpha}{\alpha}}\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha} - \Big(\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha}\Big)^{\frac{1}{1+\alpha}}\mu_{p,d}^2\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\Big(\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha}\Big)^{\frac{\alpha}{1+\alpha}} \\
&= \quad \mu_{p,d}^2\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha} - \mu_{p,d}^2\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha} \\
&= \quad \left(\frac{-1}{1+\alpha}\right)\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\mu_{p,d}^2\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha},
\end{aligned}$$

Thus:

$$\sum_{j=1}^{N} |e_j| \leq \vartheta_{p,d}\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha}, \tag{5.8}$$

where

$$\vartheta_{p,d} := \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}}\mu_{p,d}^2.$$

□

We thus prove our final theorem, namely:

Theorem 1.12 Let $b_\zeta \geq 0$, $\{b_\zeta\}_{\zeta\in\mathbb{Z}^d} \in \ell^{\gamma+\alpha}(\mathbb{Z}^d)$, for $\gamma \geq 1$ and $\alpha := p/2^d$, with $p \in \{1,\ldots,2^{d-1}\}$. Then the negative eigenvalues $\{e_j\}_{j=1}^N$ of the operator H_D^d are discrete and they satisfy the inequality

$$\sum_{j=1}^{N} |e_j|^\gamma \quad \leq \quad \eta_{p,d}\sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{\gamma+\alpha},$$

where

$$\eta_{p,d} \;:=\; \frac{\Gamma(2+\alpha)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}\frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}}\frac{2^{d-2\alpha}}{d^\alpha} \;=\; \frac{\Gamma(2+\alpha)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}\vartheta_{p,d}.$$

Proof of Theorem 1.12. Let $\{e_j(\tau)\}_{j=1}^N$ be the negative eigenvalues of the operator $-\nabla_D^2 - (b_\zeta - \tau)_+$. By the variational principle, we have for the negative eigenvalues $\{-(|e_j| - \tau)_+\}_1^N$ of the operator $-\nabla_D^2 - (b_\zeta - \tau)$:

$$-\nabla_D^2 - (b_\zeta - \tau)_+ \leq -\nabla_D^2 - (b_\zeta - \tau)$$

$$\Rightarrow e_j(\tau) \leq -(|e_j| - \tau)_+, \quad \Rightarrow (|e_j| - \tau)_+ \leq |e_j(\tau)|.$$

Therefore, for any $\gamma > 1$, we apply our Aizenman–Lieb procedure, as before:

$$\begin{aligned}\sum_{j=1}^{N} |e_j|^\gamma &= \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2} (\sum_{j=1}^{N} |e_j| - \tau)_+ \, d\tau \\ &\leq \frac{1}{\mathcal{B}(\gamma-1,2)}, \int_0^\infty \tau^{\gamma-2} \sum_{j=1}^{N} e_j(\tau)_+ \, d\tau.\end{aligned}$$

We now let $\vartheta_{p,d} := \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}} \mu_{p,d}^2$, and apply the Lieb–Thirring – $(H_D^d, \mathbb{Z}^d, \gamma = 1)$ inequality, i.e. Theorem 5.18:

$$\begin{aligned} LHS &\leq \vartheta_{p,d} \frac{1}{\mathcal{B}(\gamma-1,2)} \int_0^\infty \tau^{\gamma-2} \sum_{\zeta\in\mathbb{Z}^d} (b_\zeta - \tau)_+^{1+\alpha} d\tau \\ &= \vartheta_{p,d} \frac{1}{\mathcal{B}(\gamma-1,2)} \sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{1+\alpha} \int_0^1 (s\, b_\zeta)^{\gamma-2} (1-s)_+^{1+\alpha}\, b_\zeta \, ds \\ &= \vartheta_{p,d} \frac{1}{\mathcal{B}(\gamma-1,2)} \mathcal{B}(\gamma-1, 2+\alpha) \sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{\gamma+\alpha}.\end{aligned}$$

We thus have:

$$\sum_{j=1}^{N} |e_j|^\gamma \leq \eta_{p,d} \sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{\gamma+\alpha}, \tag{5.9}$$

$$\eta_{p,d} := \frac{\mathcal{B}(\gamma-1, 2+\alpha)}{\mathcal{B}(\gamma-1,2)} \vartheta_{p,d} = \frac{\Gamma(2+\alpha)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} \vartheta_{p,d}.$$

□

5.9 Obsolete Choices of p

We give an account of which choices of p would be obsolete with regards to the resulting spectral bounds for the discrete Schrödinger operator.

First, we consider that as $\ell^a(\mathbb{Z}^d) \subset \ell^b(\mathbb{Z}^d)$ for $a < b$, we will have the largest class of potentials for $\max_p \alpha$ which takes the value $\alpha = 1/2$ (as $\ell^{\gamma+\alpha}(\mathbb{Z}^d) \subset \ell^{\gamma+1/2}(\mathbb{Z}^d)$ for $\alpha \le 1/2$). There is only one exception in the case of $d = 2$ where $\max_p \alpha = 3/4$, which we discuss below. However, the constant could potentially be better for certain classes (i.e. other choices of p), so we are left to analyse which choices of p generate obsolete cases of our Lieb–Thirring inequality – $(\nabla_D, \mathbb{Z}^d)$, via creating larger constants as well as smaller classes of potentials. We initially consider the case of $\gamma = 1$ (i.e. Theorem 5.18), $d = 2$, and analyse the behaviour with changing p.
For $d = 2$, we have:

$$\mu_{p,2} = 2^{1-3p/8}.$$

Thus, for our Lieb–Thirring inequalities for $\gamma = 1$, we have the following constants:

$$\vartheta_{1,2} := \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}} 2^{1-3p/8}\Big|_{p=1} = \frac{4}{5^{5/4}} \cdot 2^{5/8} \approx 0.825\ldots,$$

$$\vartheta_{2,2} := \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}} 2^{1-3p/8}\Big|_{p=2} = \frac{2}{3^{3/2}} \cdot 2^{1/4} \approx 0.457\ldots,$$

$$\vartheta_{3,2} := \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}} 2^{1-3p/8}\Big|_{p=3} = \frac{4 \cdot 3^{3/4}}{7^{7/4}} \cdot 2^{-1/8} \approx 0.277\ldots\ .$$

We see that the case $p = 3$ creates estimates involving the largest class of potentials, namely $b_n \in \ell^{7/4}(\mathbb{Z}^d)$, and additionally has the best constant, rendering the cases $p = 1$ and $p = 2$ obsolete. Hence for $d = 2$, $\gamma = 1$, we can restate our inequality 5.8 as

$$\sum_{j=1}^{N} |e_j| \le \frac{2^{15/8} \cdot 3^{3/4}}{7^{7/4}} \sum_{\zeta \in \mathbb{Z}^2} b_\zeta^{7/4}.$$

Remark. This case of $\alpha > 1/2$ is unique across all dimensions. Generally, $\alpha = 1/2$ is the maximum exponent, and thus generates the largest class of potentials, namely $b_\zeta \in \ell^{3/2}(\mathbb{Z}^d)$. This is also true for arbitrary moments $b_\zeta \in \ell^{\gamma+1/2}(\mathbb{Z}^d)$.

We now fix $d \ge 2$ and $\gamma \ge 1$ and consider that the constant $\kappa_{p,d} = 2^{2^{d-1}d-p}$ being a monotonically decreasing function in p implies that so is $\mu_{p,d} := \left(\frac{\kappa_{p,d}}{d^{p/2}}\right)^{1/2^d}$. Additionally, as $f(\alpha) := \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}}$ and

$g(\alpha) := \frac{\Gamma(2+\alpha)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}$ are also monotically decreasing in α (and therefore in p as $\alpha := p/2^d$), then so is our constant $\eta_{p,d}$, as:

$$\eta_{p,d} := \frac{\Gamma(2+\alpha)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}} \mu_{p,d}^2.$$

Therefore $\eta_{p,d}$ is minimal when p is maximal, i.e. $p = 2^{d-1}$, which sets $\alpha = 1/2$. As mentioned above, this case includes the largest class of potentials across all p. Furthermore, we have just shown that this case also achieves the best constants. Therefore all other choices of p become obsolete. Hence our spectral inequality of arbitrary dimension, 5.9, simplifies to:

$$\begin{aligned} \sum_{j=1}^N |e_j|^\gamma &\leq \eta_{p,d}\Big|_{p=2^{d-1}} \sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{\gamma+1/2} \\ &= 2^{d-2}\sqrt{\frac{\pi}{3\,d}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)} \sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{\gamma+1/2}. \end{aligned}$$

where the computation was greatly facilitated by recognising that $\alpha = 1/2$ generates the same constant as the case of the Lieb–Thirring inequality – $(H_D, \mathbb{Z})$, with the additional coefficient $\mu_{2^{d-1},d}^2$.

Appendix A

List of Inequalities

In this appendix, we give a list of all inequalities proven in this thesis. We give the minimum notation and conditions required.

A.1 Auxiliary Inequalities

Discrete Kolmogorov Inequality

For $\varphi \in \ell^2(\mathbb{Z})$ and for $n, k \in \mathbb{N},\ n > k \geq 1$:

$$\|D^k \varphi\|_{\ell^2(\mathbb{Z})} \leq \|\varphi\|_{\ell^2(\mathbb{Z})}^{1-k/n} \|D^n \varphi\|_{\ell^2(\mathbb{Z})}^{k/n}.$$

Agmon–Cauchy Inequalities

The following inequalities hold true, and are symmetric across all coordinates:

($\mathbb{Z}^2$) For $\varphi \in \ell^2(\mathbb{Z}^2)$, we have:

$$\|\varphi\|^2_{(\ell^\infty(\mathbb{Z}),\ell^2(\mathbb{Z}))} \leq \|D_1\varphi\|_{\ell^2(\mathbb{Z}^2)}\, \|\varphi\|_{\ell^2(\mathbb{Z}^2)}.$$

($\mathbb{Z}^3$) For $\varphi \in \ell^2(\mathbb{Z}^3)$, we have:

$$\text{(i)} \qquad \sup_{\zeta_2 \in \mathbb{Z}} \sum_{\zeta_1 \in \mathbb{Z}} |\varphi(\zeta_1, \zeta_2, \zeta_3)|^2 \leq \Big(\sum_{\zeta_1, l \in \mathbb{Z}} |D_2 \varphi(\zeta_1, l, \zeta_3)|^2 \Big)^{1/2} \Big(\sum_{\zeta_1, l \in \mathbb{Z}} |\varphi(\zeta_1, l, \zeta_3)|^2 \Big)^{1/2}.$$

$$\text{(ii)} \qquad \|\varphi\|^2_{(\ell^\infty(\mathbb{Z}),\ell^2(\mathbb{Z}^2))} \leq \|D_1\varphi\|_{\ell^2(\mathbb{Z}^3)}\, \|\varphi\|_{\ell^2(\mathbb{Z}^3)}.$$

($\mathbb{Z}^d$) For $\varphi \in \ell^2(\mathbb{Z}^d)$, we have:

$$\sup_{\zeta_{k+1}} [\varphi]_k \leq [D_{k+1}\varphi]_{k+1}^{1/2} \, [\varphi]_{k+1}^{1/2} .$$

Bound on Partial Difference Operator on $\ell^2(\mathbb{Z}^d)$

For $\varphi \in \ell^2(\mathbb{Z}^d)$, and for $i \in \{1, \ldots, d\}$, we have:

$$\|D_i \varphi\|_{\ell^2(\mathbb{Z}^d)} \leq 2\|\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

A.2 Agmon–Kolmogorov Inequalities

For a sequence $\varphi \in \ell^2(\mathbb{Z})$, the following inequalities hold:

($D, \mathbb{Z}$) (Agmon)

$$\|\varphi\|_{\ell^\infty(\mathbb{Z})} \leq \|\varphi\|_{\ell^2(\mathbb{Z})}^{1/2} \|D\varphi\|_{\ell^2(\mathbb{Z})}^{1/2}.$$

($\Delta_D, \mathbb{Z}$)

$$\|\varphi\|_{\ell^\infty(\mathbb{Z})} \leq \|\varphi\|_{\ell^2(\mathbb{Z})}^{3/4} \|\Delta_D \varphi\|_{\ell^2(\mathbb{Z})}^{1/4}.$$

($D^\sigma, \mathbb{Z}$)

$$\|\varphi\|_{\ell^\infty(\mathbb{Z})} \leq \|\varphi\|_{\ell^2(\mathbb{Z})}^{1-1/2\sigma} \|D^\sigma \varphi\|_{\ell^2(\mathbb{Z})}^{1/2\sigma}.$$

($\nabla_D, \mathbb{Z}^2$) For $p \in \{1, 2, 3\}$, $\varphi \in \ell^2(\mathbb{Z}^2)$:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^2)} \leq 2^{1-3p/8} \|\nabla_D \varphi\|_{\ell^2(\mathbb{Z}^2)}^{p/4} \, \|\varphi\|_{\ell^2(\mathbb{Z}^2)}^{1-p/4}.$$

($\nabla_D, \mathbb{Z}^3$) For $p \in \{1, 2, 3, 4\}$, $\varphi \in \ell^2(\mathbb{Z}^3)$:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^3)} \leq \frac{2^{\frac{12-p}{8}}}{3^{p/16}} \|\nabla_D \varphi\|_{\ell^2(\mathbb{Z}^3)}^{p/8} \, \|\varphi\|_{\ell^2(\mathbb{Z}^3)}^{1-p/8}.$$

($\nabla_D, \mathbb{Z}^d$) For $p \in \{1, \ldots, 2^{d-1}\}$, $\varphi \in \ell^2(\mathbb{Z}^d)$:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \mu_{p,d} \|\nabla_D \varphi\|_{\ell^2(\mathbb{Z}^d)}^{p/2^d} \, \|\varphi\|_{\ell^2(\mathbb{Z}^d)}^{1-p/2^d},$$

where

$$\mu_{p,d} := \left(\frac{\kappa_{p,d}}{d^{p/2}}\right)^{1/2^d}, \qquad \kappa_{p,d} = 2^{2^{d-1}d-p}.$$

A.3 Generalised Sobolev Inequalities

Let $\{\psi_j\}_{j=1}^N$ be an orthonormal system of sequences in $\ell^2(\mathbb{Z})$ and let $\rho(n) \coloneqq \sum_{j=1}^N |\psi_j(n)|^2$:

$(D, \mathbb{Z})$

$$\sum_{n\in\mathbb{Z}} \rho^3(n) \le \sum_{j=1}^N \sum_{n\in\mathbb{Z}} |D\psi_j(n)|^2 .$$

$(\Delta_D, \mathbb{Z})$

$$\sum_{n\in\mathbb{Z}} \rho^5(n) \le \sum_{j=1}^N \sum_{n\in\mathbb{Z}} |\Delta_D\psi_j(n)|^2 .$$

$(D^\sigma, \mathbb{Z})$

$$\sum_{n\in\mathbb{Z}} \rho^{2\sigma+1}(n) \le \sum_{j=1}^N \sum_{n\in\mathbb{Z}} |D^\sigma\psi_j(n)|^2 .$$

$(D, \mathbb{Z}^d)$

Let $\{\psi_j\}_{j=1}^N$ be an orthonormal system of sequences in $\ell^2(\mathbb{Z}^d)$, $\zeta \in \mathbb{Z}^d$ and let $\rho(\zeta) \coloneqq \sum_{j=1}^N |\psi_j(\zeta)|^2$. Then

$$\sum_{\zeta\in\mathbb{Z}^d} \rho^{\frac{2^d+p}{p}}(\zeta) \le \mu_{p,d}^{\frac{2^{d+1}}{p}} \sum_{j=1}^N \sum_{\zeta\in\mathbb{Z}^d} |\nabla_D\psi_j(\zeta)|^2 .$$

A.4 Lieb–Thirring Inequalities

Let $b_n \geq 0$, $\gamma \geq 1$. Then we have the following bound for the negative eigenvalues $\{e_j\}_{j=1}^N$ of:

$(H_D, \mathbb{Z})$ Let $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z})$:

$$\sum_{j=1}^{N} |e_j|^\gamma \leq \eta_1^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2}, \qquad \eta_1^\gamma := \frac{\sqrt{\pi}}{2\sqrt{3}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)}.$$

$(H_D^2, \mathbb{Z})$ Let $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/4}(\mathbb{Z})$:

$$\sum_{j=1}^{N} |e_j|^\gamma \leq \eta_2^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/4}, \qquad \eta_2^\gamma := \frac{4}{5^{5/4}} \frac{\Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

$(H_D^\sigma, \mathbb{Z})$ Let $\{b_n\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z})$:

$$\sum_{j=1}^{N} |e_j|^\gamma \leq \eta_\sigma^\gamma \sum_{n\in\mathbb{Z}} b_n^{\gamma+1/2\sigma}, \qquad \eta_\sigma^\gamma := \frac{2\sigma}{(2\sigma+1)^{(2\sigma+1)/2\sigma}} \frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})}.$$

$(H_D^d, \mathbb{Z}^d)$ Let $b_\zeta \geq 0$, $\{b_\zeta\}_{\zeta\in\mathbb{Z}^d} \in \ell^{\gamma+\alpha}(\mathbb{Z}^d)$, $\alpha := p/2^d$, $p \in \{1, \dots, 2^{d-1}\}$:

$$\sum_{j=1}^{N} |e_j|^\gamma \leq \eta_{p,d} \sum_{\zeta\in\mathbb{Z}^d} b_\zeta^{\gamma+\alpha},$$

where

$$\eta_{p,d} := \frac{\Gamma(2+\alpha)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}} \frac{2^{d-2\alpha}}{d^\alpha}.$$

A.5 Spectral Bounds for Jacobi-type Matrix Operators

Let $\gamma \geq 1$, then we have the following bounds for the eigenvalues $\{E_j^{\pm}\}_{j=1}^{N_{\pm}}$ of the following operators:

Tridiagonal Case (W_1)

Let $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n - 1\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2}(\mathbb{Z})$:

$$\sum_{j=1}^{N_-} |E_j^- + 2|^{\gamma} + \sum_{j=1}^{N_+} |E_j^+ - 2|^{\gamma} \leq \nu_1^{\gamma} \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/2} + 4 \sum_{n\in\mathbb{Z}} |a_n - 1|^{\gamma+1/2} \right),$$

where

$$\nu_1^{\gamma} := 3^{\gamma-1} \pi \, L_{\gamma,1}^{cl} \; = \; 3^{\gamma-1} \frac{\sqrt{\pi}\,\Gamma(\gamma+1)}{2\,\Gamma(\gamma+3/2)}.$$

Pentadiagonal Case (W_2)

Let $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n + 4\}_{n\in\mathbb{Z}}$, $\{c_n - 1\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/4}(\mathbb{Z})$:

$$\sum_{j=1}^{N_-} |E_j^- \; + \; 6|^{\gamma} \; + \; \sum_{j=1}^{N_+} |E_j^+ \; - \; 10|^{\gamma} \quad \leq \quad \nu_2^{\gamma} \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/4} + 4 \sum_{n\in\mathbb{Z}} |a_n + 4|^{\gamma+1/4} + 4 \sum_{n\in\mathbb{Z}} |c_n - 1|^{\gamma+1/4} \right),$$

where

$$\nu_2^{\gamma} := 5^{\gamma-2} \frac{4\,\Gamma(9/4)\Gamma(\gamma+1)}{\Gamma(\gamma+5/4)}.$$

Polydiagonal Case (W_{σ})

Let $\{b_n\}_{n\in\mathbb{Z}}$, $\{a_n^i - \omega_i\}_{n\in\mathbb{Z}} \in \ell^{\gamma+1/2\sigma}(\mathbb{Z})$ for all $i \in \{1, \ldots, \sigma\}$:

$$\sum_{j=1}^{N_-} |E_j^- \; + \; {}^{2\sigma}C_{\sigma}|^{\gamma} \; + \; \sum_{j=1}^{N_+} |E_j^+ \; - \; (4^{\sigma} \; - \; {}^{2\sigma}C_{\sigma})|^{\gamma} \quad \leq \quad \nu_{\sigma}^{\gamma} \left(\sum_{n\in\mathbb{Z}} |b_n|^{\gamma+1/2\sigma} + 4 \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\sigma} |a_n^k - \omega_k|^{\gamma+1/2\sigma} \right),$$

where

$$\nu_{\sigma}^{\gamma} = 2\sigma \; (2\sigma+1)^{\gamma-2} \frac{\Gamma(\frac{4\sigma+1}{2\sigma})\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{2\sigma+1}{2\sigma})}.$$

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