Note on Prodi-Serrin-Ladyzhenskaya type regularity criteria for the Navier-Stokes equations

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In this article we prove new regularity criteria of the Prodi-Serrin-Ladyzhenskaya type for the Cauchy problem of the three-dimensional incompressible Navier-Stokes equations. Our results improve the classical $L^r(0,T;L^s)$ regularity criteria for both velocity and pressure by factors of certain negative powers of the scaling invariant norms $\|u\|_{L^3}$ and $\|u\|_{\dot{H}^{1/2}}$.

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I. INTRODUCTION

The Cauchy problem of the three-dimensional incompressible Navier-Stokes equations play an important role in not only mathematical fluid mechanics but also the development of the theory of general evolutionary equations. The system reads

$$u_t + u \cdot \nabla u = -\nabla p + \nu \triangle u, \qquad (x,t) \in \mathbb{R}^3 \times (0,\infty), \tag{1}$$

$$\mathrm{div} u = 0, \qquad (x,t) \in \mathbb{R}^3 \times (0,\infty), \tag{2}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^3, \tag{3}$$

where $u : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is the velocity field, $p : \mathbb{R}^3 \mapsto \mathbb{R}$ is the pressure, and $\nu > 0$ is the (dimensionless) viscosity.

Systematic study of this problem began in 1934 with the classical paper [22] by Jean Leray, where it is shown that for arbitrary $T \in (0,\infty]$ there is at least one function $u(x,t)$ satisfying the following.

i. $u \in L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d))$;

ii. u satisfies (1) and (2) in the sense of distributions;

iii. u takes the initial value in the L^2 sense: $\lim_{t\searrow 0} \|u(\cdot,t) - u_0(\cdot)\|_{L^2} = 0$;

iv. u satisfies the energy inequality

$$\|u(\cdot,t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\cdot,\tau)\|_{L^2}^2 \mathrm{d}\tau \leqslant \|u_0\|_{L^2}^2 \tag{4}$$

for all $0 \leqslant t \leqslant T$.

Such a function $u(x,t)$ is called a Leray-Hopf weak solution for (1)–(3) in $\mathbb{R}^d \times [0,T)$.

It is easy to show that if a Leray-Hopf weak solution is smooth, then it is a classical solution and is furthermore unique (in the class of Leray-Hopf weak solutions). However the smoothness of Leray-Hopf weak solutions is still a completely open problem. On the other hand, it has been long known that various additional assumptions guarantee such smoothness. One important class of such assumptions is the following so-called Prodi-Serrin-Ladyzhenskaya criteria, developed over three decades in [11], [21], [24], [23], [25], [26]. If a Leray-Hopf solution $u(x,t)$ further satisfies

$$u \in L^r(0,T;L^s(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} \leqslant 1, \qquad 3 < s \leqslant \infty, \tag{5}$$

then $u(x,t)$ is smooth. Here the norm of the mixed Lebesgue space $L^r(0,T:L^s(\mathbb{R}^3))$ is defined as

$$\|u\|_{L^r(0,T;L^s)} := \begin{cases} \left(\int_0^T \|u(\cdot,t)\|_{L^s}^r\right)^{1/r} & 1 \leqslant r < \infty \\ \mathrm{esssup}_{t\in(0,T)} \|u(\cdot,t)\|_{L^s} & r = \infty \end{cases}. \tag{6}$$

The proof of the criterion (5) is quite straightforward through standard energy estimate, though it should be mentioned that it is the much more non-trivial "localized" version of (5) that was proved in the references above. The borderline case $u \in L^\infty(0,T;L^3(\mathbb{R}^3))$ that is missing in (5) turned out to be much more complicated due to the criticality of $\|u\|_{L^3}$ under the rescaling transformation $u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)$ that keeps (1)–(3) invariant. This case could not be dealt with using the method that established (5), and was only recently settled by Escauriaza, Seregin, and Šverák in

[10] via a novel approach based on deep understanding of backward uniqueness of heat equations. Many generalizations and refinements of (5) have been proved, see e.g. [3], [5], [8], [12], [31], [34].

Mathematically the pressure p serves as the Lagrange multiplier of the incompressibility constraint $\mathrm{div} u = 0$. As a consequence there is no explicit equation governing the evolution of p in (1)–(3). The lack of such an equation is partially compensated through the following relation between u and p obtained via taking divergence of (1),

$$-\triangle p = \sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j). \tag{7}$$

For the convenience of analyzing (1)–(3) in the framework of functional analysis, (7) is usually written as

$$p = \sum_{i,j=1}^{3} R_i R_j (u_i u_j), \tag{8}$$

where $R_i, i = 1, 2, 3$ are the Riesz transforms. As Riesz transforms are zeroth order pseudo-differential operators, there holds

$$\|p\|_{L^s} \leqslant C \|u\|_{L^{2s}}^2 \quad \text{for all} \quad s \in (1, \infty). \tag{9}$$

From (9) it is natural to conjecture that

$$p \in L^r(0, T; L^s) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} \leqslant 2, \qquad \frac{3}{2} < s \leqslant \infty \tag{10}$$

may guarantee the smoothness of u. Note that thanks to (9), (10) is a weaker assumption than (5) as it is implied by the latter. The affirmative answer to this conjecture was established in [2], [7] and later refined in many follow-up papers, including [1], [3], [9], [13], [16], [17], [19], [27], [28], [30].

Roughly speaking, most of the aforementioned improvements of (5) or (10) can be categorized into two types. The first type replaces the Lebesgue norm L^s and/or L^r by weaker norms with the same scaling property. For example in [13] $\|p\|_{L^s}$ is replaced by the homogeneous Besov norm $\|p\|_{\dot{B}^0_{s,\sigma}}$ for some appropriate σ, and in [3] $\|u\|_{L^s}$ is replaced by $\|u\|_{L^{s,\infty}}$ where $L^{s,\infty}$ are the weak Lebesgue spaces. The second type of improvement weakens the conditions by a logarithmic factor. For example in [31] it is shown that u is smooth as long as

$$\int_0^T \frac{\|u\|_{L^s}^r}{\log(e + \|u\|_{L^\infty})} \mathrm{d}t < \infty, \qquad \frac{2}{r} + \frac{3}{s} \leqslant 1, \quad 3 < s \leqslant \infty. \tag{11}$$

Possibility of combining these two types of improvement has been well explored, see e.g. [12], [16]. We must also mention that criteria similar to (5) or (10) have also been proved for other physically meaningful quantities such as $\nabla u, \omega := \mathrm{curl} u$, and ∇p, see e.g. [4], [6], [14], [33], [35].

In this article we present and prove a new type of improvement of (5) and (10), of the form

$$\int_0^T \frac{\|u\|_{L^s}^r}{(1 + \|u\|_X)^\kappa} \mathrm{d}t < \infty, \qquad \frac{2}{r} + \frac{3}{s} \leqslant 1, \quad 3 < s < \infty, \tag{12}$$

or

$$\int_0^T \frac{\|p\|_{L^s}^r}{(1 + \|u\|_X)^\kappa} \mathrm{d}t < \infty, \qquad \frac{2}{r} + \frac{3}{s} \leqslant 2, \qquad \frac{3}{2} < s \leqslant \infty, \tag{13}$$

where $\|\cdot\|_X$ is a scaling-invariant norm for (1)–(3) and $\kappa > 0$. More specifically, we will prove the following theorems.

Theorem 1. *Let* $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ *for some* $q > 3$, *and satisfy* $\mathrm{div} u_0 = 0$. *Let* $u(t,x)$ *be a Leray-Hopf solution of NSE in* $[0,T)$. *If for some* $s \in (3,\infty)$ *and* $\frac{2}{r} + \frac{3}{s} = 1$ *there holds*

$$\int_0^T \frac{\|u\|_{L^s}^r}{(1+\|u\|_{\dot{H}^{1/2}})^\kappa} \mathrm{d}t < \infty, \tag{14}$$

where $\kappa = \begin{cases} 2 & 3 < s \leqslant 5 \\ \frac{4}{s-3} & 5 < s < \infty \end{cases}$, *then* $u(t,x)$ *is smooth up to* T *and could be extended beyond* T.

Theorem 2. *Let* $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ *for some* $q > 3$, *and satisfy* $\mathrm{div} u_0 = 0$. *Let* $u(t,x)$ *be a Leray-Hopf solution of NSE in* $[0,T)$. *If for some* $s \in (3,\infty)$ *and* $\frac{2}{r} + \frac{3}{s} = 1$ *there holds*

$$\int_0^T \frac{\|u\|_{L^s}^r}{(1+\|u\|_{L^3})^\kappa} \mathrm{d}t < \infty, \tag{15}$$

where $\kappa = \begin{cases} 3 & 3 < s \leqslant 5 \\ \frac{6}{s-3} & 5 < s < \infty \end{cases}$, *then* $u(t,x)$ *is smooth up to* T *and could be extended beyond* T.

Theorem 3. *Let* $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ *for some* $q > 3$, *and satisfy* $\mathrm{div} u_0 = 0$. *Let* $u(t,x)$ *be a Leray-Hopf solution of NSE in* $[0,T)$. *If for some* $s \in \left(\frac{3}{2},\infty\right)$ *and* $\frac{2}{r} + \frac{3}{s} = 2$ *there holds*

$$\int_0^T \frac{\|p\|_{L^s}^r}{(1+\|u\|_{L^3})^\kappa} \mathrm{d}t < \infty, \tag{16}$$

where $\kappa = \begin{cases} \frac{6}{2s-3} & s \geqslant 3 \\ \frac{2s}{2s-3} & \frac{9}{4} \leqslant s \leqslant 3 \\ 3 & \frac{3}{2} < s \leqslant \frac{9}{4} \end{cases}$, *then* $u(t,x)$ *is smooth up to* T *and could be extended beyond* T.

A few remarks are in order.

Remark 1. *It is straightforward to cover the sub-critical cases. For example, if* u *satisfies (14) for some* r, s *satisfying* $s \in (3,\infty)$ *and* $\frac{2}{r} + \frac{3}{s} < 1$, *then there is* $r' < r$ *such that* $\frac{2}{r'} + \frac{3}{s} = 1$ *and (14) holds for* r', s *thanks to Hölder's inequality.*

Remark 2. *We emphasize that unlike (5) and (10), where one is weaker than the other, Theorems 1, 2, and 3 are independent of each other. For example, although* $\|u\|_{L^3} \leqslant C\|u\|_{\dot{H}^{1/2}}$ *by Sobolev embedding, Theorem 2 does not follow from Theorems 1, as the values of* κ *are different.*

Remark 3. *When* $s = 6$, *criterion (15) can be seen to be comparable but not equivalent to the following criterion*

$$\int_0^T \frac{\|u\|_{L^6}^6}{\|u\|_{L^4}^4 + 2\|u\|_{\dot{H}^1}^2} < \infty, \tag{17}$$

which has been reported recently [29]. The two criteria would be equivalent if we could establish that $\|u\|_{L^4}^4 + 2\|u\|_{\dot{H}^1}^2 \approx (\|u\|_{L^3}^2 + 2)\|u\|_{L^6}^2$, *which is consistent in scaling yet does not hold for general divergence free vector field* u. *For all other* $s > 3$, *(15) may be regarded as an extention of (17). It is also of interest to explore the possibility of weakening (16) through replacing the pressure* p *by the "effective pressure"* $p + \mathcal{P}$ *in [29].*

Remark 4. *It is not clear whether the splitting into different cases in Theorems 1, 2, and 3 is purely technical, due to the limitations of the energy method, or reflects deeper properties of the Navier-Stokes dynamics.*

Remark 5. *From the proofs we will see that a logarithmic factor could easily be added "for free". For example, (15) could be replaced by*

$$\int_0^T \frac{\|u\|_{L^s}^r}{(1+\|u\|_{L^3})^\kappa \log(e+\|u\|_{L^3})} \mathrm{d}t < \infty. \tag{18}$$

However it does not seem likely that the $\|u\|_{L^3}$ inside the logarithm could be replaced by $\|u\|_{L^\infty}$. Thus our results are not stronger than, though still independent of, the previous logarithmic improvement results such as (11).

Remark 6. *Criteria in a sense similar to Theorems 1, 2, and 3 have been proved in [18] and [32]. There it is shown that smoothness of the solution u is guaranteed if one of the following holds.*

- *[32].*

$$\frac{p}{1+|u|^\delta} \in L^r(0,T;L^s), \qquad \frac{2}{r}+\frac{3}{s}=\frac{5-3\delta}{2}, \quad \frac{6}{5-3\delta} < s \leqslant \infty, \quad 1 \leqslant \delta \leqslant \frac{5}{3}; \tag{19}$$

- *[18].*

$$\frac{p}{1+|u|^\delta} \in L^r(0,T;L^s), \quad \textit{with } \frac{2}{r}+\frac{3}{s}=\frac{4-3\delta}{2}, \quad \frac{18}{8-9\delta} \leqslant s \leqslant \frac{6}{2-3\delta}, \quad 0 \leqslant \delta < \frac{2}{3}. \tag{20}$$

- *[18].*

$$\frac{p}{1+|u|^\delta} \in L^r(0,T;L^s), \quad \textit{with } \frac{2}{r}+\frac{3}{s}=\frac{4-3\delta}{2}, \quad \frac{18}{8-9\delta} \leqslant s \leqslant \frac{6}{2-3\delta}, \quad \frac{2}{3} \leqslant \delta \leqslant \frac{8}{9}. \tag{21}$$

We briefly discuss their relations to Theorems 1, 2, and 3.

1. *(19)–(21) are sub-critical from a scaling point of view and therefore does not improve (10) except for the case $\delta = 1$ in (19) (and $\delta = 0$ in (20) which reduces (20) to (10)). To see this we recall (8) which dictates that p scales as $|u|^2$. This makes $\frac{p}{1+|u|^\delta} \in L^r(0,T;L^s)$ roughly equivalent to, from the scaling point of view, $|u|^{2-\delta} \in L^r(0,T;L^s)$ that is $u \in L^{(2-\delta)r}(0,T;L^{(2-\delta)s})$. Thus for example (20) corresponds to*

$$\frac{2}{(2-\delta)r}+\frac{3}{(2-\delta)s}=\frac{4-3\delta}{4-2\delta}<1 \tag{22}$$

for all values of δ except $\delta = 0$. Similarly, in (19) we have

$$\frac{2}{(2-\delta)r}+\frac{3}{(2-\delta)s}=1+\frac{1-\delta}{4-2\delta}<1 \tag{23}$$

unless $\delta = 1$.

2. *On the other hand, in (19)–(21) the factor* $(1+|u|^{\delta})^{-1}$ *is inside the whole space-time integral, while in our theorems* $(1+\|u\|_{L^3})^{-\kappa}$ *or* $(1+\|u\|_{\dot{H}^{1/2}})^{-\kappa}$ *is only inside the time integral. Thus the conditions (19)–(21) are in a sense more "localized".*

Remark 7. *It is quite straightforward to generalize Theorems 1, 2, and 3 to d-dimensional Navier-Stokes equations. For simplicity of presentation we will focus on the physical case* $d = 3$ *in this article.*

In the next section we prove Theorems 1, 2, and 3.

II. PROOF OF THEOREMS

A. Preliminaries

Without loss of generality, we take $\nu = 1$ in (1) to simplify the presentation. We apply the following result, summarized from [15], [20], to guarantee short-time smoothness of the solution and thus relieving us from worrying about the legitimacy of the various integral and differential manipulations below.

Theorem 4. *Let* $u_0 \in L^s(\mathbb{R}^3)$, $s \geqslant 3$. *Then there exists* $T > 0$ *and a unique classical solution* $u \in \mathrm{BC}(0, T; L^s(\mathbb{R}^3))$. *Moreover, let* $(0, T_*)$ *be the maximal interval such that the solution* u *stays in* $C(0, T_*; L^s(\mathbb{R}^3))$, $s > 3$. *Then for any* $t \in (0, T_*)$,

$$\|u(\cdot,t)\|_{L^s} \geqslant \frac{C}{(T_* - t)^{\frac{s-3}{2s}}} \tag{24}$$

where the constant C *is independent of* T_* *and* s.

We will also need the following simple lemma.

Lemma 1. *Let* $X(t) \in C^1(0,T) \cap C([0,T))$ *be non-negative and solve* $\dot{X}(t) \leqslant \frac{A(t)}{X(t)^{\kappa}} X(t) + \frac{C}{X(t)^{\kappa}}$ *for some* $A(t) \geqslant 0$, $k > 0$. *Assume*

$$\int_0^T \frac{A(t)}{(1 + X(t))^{\kappa}} \mathrm{d}t < \infty. \tag{25}$$

Then $\limsup_{t \longrightarrow T-} X(t) < \infty$.

Proof. Denote $B(t) := \frac{A(t)}{\max\{1, X(t)\}^{\kappa}}$. It is clear that (25) is equivalent to $\int_0^T B(t)\mathrm{d}t < \infty$. Let $Y(t) := \max\{1, X(t)\}$. Then $Y(t) = X(t)$ on the union of at most countably many open intervals (t_{iL}, t_{iR}) with $Y(t_{iL}) = 1$. Now on (t_{iL}, t_{iR}) we have $Y(t) > 1$ and therefore

$$\begin{aligned} \dot{Y}(t) &\leqslant \frac{A(t)}{X(t)^k} X(t) + \frac{C}{X(t)^k} \\ &= \frac{A(t)}{Y(t)^k} Y(t) + \frac{C}{Y(t)^k} \\ &\leqslant B(t) Y(t) + C. \end{aligned} \tag{26}$$

The conclusion immediately follows. ∎

Finally we need the following result which is a special case of Theorem 1.3 in [10].

Theorem 5. *Suppose that u is a weak Leray-Hopf solution of the Cauchy problem (1)–(3). If furthermore $u \in L^\infty(0,T;L^3(\mathbb{R}^3))$, then u is smooth up to T and can be extended beyond T.*

Notation. *In the following we will use $A \lesssim B$ to denote $A \leqslant cB$ when the constant c is inconsequential to the validity of the proof.*

B. Proof of Theorem 1

Assume the contrary. By Theorem 4 there is $T^* \in (0,T)$ such that $u(x,t)$ is smooth for $t \in (0,T^*)$ but cease to be so at $t = T^*$. Thanks to Theorem 5 and the Sobolev embedding $\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, this implies $\limsup_{t\nearrow T^*} \|u\|_{\dot{H}^{1/2}} = \infty$. Therefore to obtain contradiction it suffices to prove that $\|u\|_{\dot{H}^{1/2}} \leqslant C$, $\forall t \in (0,T^*)$, for some constant $C > 0$.

Let $\Lambda := (-\triangle)^{1/2}$. Multiplying the equation by Λu and then integrate over $\mathbb{R}^3$, we reach

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{H}^{1/2}}^2 &= -\int_{\mathbb{R}^3} u\cdot\nabla u\cdot\Lambda u \mathrm{d}x - \int_{\mathbb{R}^3}\Lambda u\cdot\Lambda^2 u \mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^3}|u||\nabla u||\Lambda u|\mathrm{d}x - \|u\|_{\dot{H}^{3/2}}^2. \end{aligned} \tag{27}$$

In what follows we discuss the two regimes of s stated in the theorem separately.

- $3 < s \leqslant 5$.

  Let s' be the conjugate to s, that is $\frac{1}{s} + \frac{1}{s'} = 1$. We start by estimating using Hölder's inequality

  $$I := \int_{\mathbb{R}^3}|u||\nabla u||\Lambda u|\mathrm{d}x \leqslant \|u\|_{L^s}\|\nabla u\|_{L^{2s'}}\|\Lambda u\|_{L^{2s'}}. \tag{28}$$

  Next we notice that as $3 < s \leqslant 5 \implies 2s' \in \left[\frac{5}{2}, 3\right) \subset (1,\infty)$, the boundedness of Riesz transforms on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$ now yields $\|\Lambda u\|_{L^{2s'}} \lesssim \|\nabla u\|_{L^{2s'}}$ since $\Lambda u = -\Lambda^{-1}\nabla\cdot(\nabla u) = -\sum_{i=1}^3 R_i(\partial_i u)$. Consequently

  $$I \lesssim \|u\|_{L^s}\|\nabla u\|_{L^{2s'}}^2. \tag{29}$$

  Now thanks to the the interpolation inequality

  $$\|\nabla u\|_{L^{2s'}} \lesssim \|u\|_{L^s}^{\frac{s-3}{6}}\|u\|_{\dot{H}^{3/2}}^{\frac{9-s}{6}}, \tag{30}$$

  we further obtain

  $$I \lesssim \|u\|_{L^s}\|u\|_{L^s}^{(s-3)/3}\|u\|_{\dot{H}^{3/2}}^{(9-s)/3}. \tag{31}$$

  Finally by Young's inequality we conclude that

  $$I \leqslant C\|u\|_{L^s}^r + \frac{1}{2}\|u\|_{\dot{H}^{3/2}}^2 \tag{32}$$

with $r = \frac{2s}{s-3}$.

Substituting (32) into (27), we have

$$\|u\|_{\dot{H}^{1/2}} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{\dot{H}^{1/2}} \lesssim \|u\|_{L^s}^r \tag{33}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{\dot{H}^{1/2}} \lesssim \frac{\|u\|_{L^s}^r}{\|u\|_{\dot{H}^{1/2}}^2} \|u\|_{\dot{H}^{1/2}} \tag{34}$$

and the conclusion follows from Lemma 1.

- $5 < s < \infty$.

We first notice that

$$\int_{\mathbb{R}^3} |u||\nabla u||\Lambda u| \mathrm{d}x \leqslant \int_{\mathbb{R}^3} |u||\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |u||\Lambda u|^2 \mathrm{d}x =: I + II. \tag{35}$$

We estimate I first. By Hölder inequality we have

$$\begin{aligned} I &= \int_{\mathbb{R}^3} |u||\nabla u|^{\frac{2(s-5)}{s-2}} |\nabla u|^{\frac{6}{s-2}} \mathrm{d}x \\ &\leqslant \|u\|_{L^s} \||\nabla u|^{\frac{2(s-5)}{s-2}}\|_{L^{\frac{s-2}{s-5}}} \||\nabla u|^{\frac{6}{s-2}}\|_{L^{\frac{s(s-2)}{2(s+1)}}} \\ &= \|u\|_{L^s} \|\|\nabla u\|_{L^2}^{\frac{2(s-5)}{s-2}} \|\nabla u\|_{L^{\frac{3s}{s+1}}}^{\frac{6}{s-2}}. \end{aligned} \tag{36}$$

Note that as $s > 5$, there holds $1 < \frac{s-2}{s-5} < \infty$ and $1 < \frac{s}{4} < \frac{s(s-2)}{2(s+1)} < \infty$. Therefore the application of Hölder inequality is justified.

Next we apply the following Gagliardo-Nirenberg inequalities,

$$\|\nabla u\|_{L^2} \lesssim \|u\|_{\dot{H}^{1/2}}^{1/2} \|u\|_{\dot{H}^{3/2}}^{1/2}, \qquad \|\nabla u\|_{L^{3s/(s+1)}} \lesssim \|u\|_{L^s}^{1/3} \|u\|_{\dot{H}^{3/2}}^{2/3}, \tag{37}$$

to obtain

$$I \lesssim \|u\|_{L^s}^{s/(s-2)} \|u\|_{\dot{H}^{1/2}}^{(s-5)/(s-2)} \|u\|_{\dot{H}^{3/2}}^{(s-1)/(s-2)}. \tag{38}$$

Young's inequality now yields

$$I \leqslant C\|u\|_{L^s}^{2s/(s-3)} \|u\|_{\dot{H}^{1/2}}^{2(s-5)/(s-3)} + \frac{1}{2}\|u\|_{\dot{H}^{3/2}}^2. \tag{39}$$

Through almost identical arguments, the same estimate could be obtained for II.

$$II \leqslant C\|u\|_{L^s}^{2s/(s-3)} \|u\|_{\dot{H}^{1/2}}^{2(s-5)/(s-3)} + \frac{1}{2}\|u\|_{\dot{H}^{3/2}}^2. \tag{40}$$

Substituting these into (27) and dividing both sides by $\|u\|_{\dot{H}^{1/2}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{\dot{H}^{1/2}} \lesssim \frac{\|u\|_{L^s}^r}{\|u\|_{\dot{H}^{1/2}}^{4/(s-3)}} \|u\|_{\dot{H}^{1/2}} \tag{41}$$

and the conclusion now follows from Lemma 1.

C. Proof of Theorem 2

Similar to the proof of Theorem 1, we assume that u blows up at $T^* \in (0,T)$, and it suffices to show that $\|u\|_{L^3} \leqslant C$ on $(0,T^*)$ for some $C>0$ independent of t.

We multiply (1) by $|u|u\cdot$ and integrate in $\mathbb{R}^3$ to obtain

$$\begin{aligned}\|u\|_{L^3}^2 \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} &= -\int_{\mathbb{R}^3} |u|u\cdot\nabla p\mathrm{d}x + \int_{\mathbb{R}^3} |u|u\cdot\triangle u\mathrm{d}x \\ &= \int_{\mathbb{R}^3} pu\cdot\nabla(|u|)\mathrm{d}x + \int_{\mathbb{R}^3} |u|u\cdot\triangle u\mathrm{d}x \\ &= \int_{\mathbb{R}^3} p|u|(\hat{u}\cdot\nabla|u|)\mathrm{d}x + \int_{\mathbb{R}^3} |u|u\cdot\triangle u\mathrm{d}x. \end{aligned} \tag{42}$$

where $\hat{u} := \frac{u}{|u|}$ (if $u=0$, just defined $\hat{u}=0$ too).

Recalling the identity

$$u\cdot\triangle u = \nabla\cdot(|u|\nabla|u|) - |\nabla u|^2, \tag{43}$$

we easily derive

$$\int_{\mathbb{R}^3} |u|u\cdot\triangle u\mathrm{d}x = -\frac{4}{9}\|\nabla|u|^{3/2}\|_{L^2}^2 - \||\nabla u||u|^{1/2}\|_{L^2}^2, \tag{44}$$

and reach the following estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3}^3 + \||\nabla u||u|^{1/2}\|_{L^2}^2 + \|\nabla|u|^{3/2}\|_{L^2}^2 \lesssim \left|\int_{\mathbb{R}^3} p|u|(\hat{u}\cdot\nabla|u|)\mathrm{d}x\right|. \tag{45}$$

which gives

$$\|u\|_{L^3}^2\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} \lesssim C\int_{\mathbb{R}^3} |p||u|^{\frac{1}{2}}|\nabla|u|^{3/2}|\mathrm{d}x - \||u|^{3/2}\|_{\dot{H}^1}^2. \tag{46}$$

Application of Young's inequality and then Hölder and Sobolev inequalities to (46) gives

$$\begin{aligned}\|u\|_{L^3}^2\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} &\lesssim C\int_{\mathbb{R}^3} |p|^2|u|\mathrm{d}x - \||u|^{3/2}\|_{\dot{H}^1}^2 \\ &\lesssim C\|p^2\|_{L^{5/4}}\|u\|_{L^5} - \|u\|_{L^9}^3 \\ &= C\|p\|_{L^{5/2}}^2\|u\|_{L^5} - \|u\|_{L^9}^3 \\ &\lesssim C\|u\|_{L^5}^5 - \|u\|_{L^9}^3. \end{aligned} \tag{47}$$

Note that in the last inequality we have used (9).

In what follows we discuss the two regimes of s stated in the theorem separately.

- $3 < s \leqslant 5$. In this case we apply the interpolation inequality

$$\|u\|_{L^5} \leqslant \|u\|_{L^s}^{\theta}\|u\|_{L^9}^{1-\theta} \tag{48}$$

where $\theta = \frac{4}{5}\frac{s}{9-s}$. This gives

$$\begin{aligned}\|u\|_{L^3}^2\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} &\lesssim C\|u\|_{L^s}^{4s/(9-s)}\|u\|_{L^9}^{9(5-s)/(9-s)} - \|u\|_{L^9}^3 \\ &\lesssim C\|u\|_{L^s}^{2s/(s-3)} \end{aligned} \tag{49}$$

after application of Young's inequality. Dividing both sides by $\|u\|_{L^3}^2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} \lesssim \frac{\|u\|_{L^s}^{2s/(s-3)}}{\|u\|_{L^3}^3}\|u\|_{L^3}. \tag{50}$$

The conclusion now follows from Lemma 1.

- $5 < s < \infty$. In this case we apply the interpolation inequality

$$\|u\|_{L^5} \leqslant \|u\|_{L^3}^{\theta}\|u\|_{L^s}^{1-\theta} \tag{51}$$

where $\theta = \frac{3}{5}\frac{s-5}{s-3}$. This gives

$$\|u\|_{L^3}^2\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} \lesssim \|u\|_{L^3}^{3(s-5)/(s-3)}\|u\|_{L^s}^{2s/(s-3)}. \tag{52}$$

Dividing both sides by $\|u\|_{L^3}^2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} \lesssim \frac{\|u\|_{L^s}^{2s/(s-3)}}{\|u\|_{L^3}^{6/(s-3)}}\|u\|_{L^3}, \tag{53}$$

and the conclusion immediately follows from Lemma 1.

D. Proof of Theorem 3

Again we assume that u blows up at $T^* \in (0,T)$, and try to to show that $\|u\|_{L^3} \leqslant C$ on $(0,T^*)$ for some $C > 0$ independent of t.

Following (42)–(45) we have

$$\begin{aligned}
\|u\|_{L^3}^2\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} &\leqslant \int_{\mathbb{R}^3}|p||u||\nabla|u||\mathrm{d}x - \|\nabla|u|^{3/2}\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3}|p|^{\alpha}|p|^{1-\alpha}|u|^{1/2}|\nabla|u|^{3/2}|\mathrm{d}x - \|\nabla|u|^{3/2}\|_{L^2}^2 \\
&\leqslant \||p|^{\alpha}\|_{L^a}\||p|^{1-\alpha}\|_{L^b}\||u|^{1/2}\|_{L^c}\|\nabla|u|^{3/2}\|_{L^2} - \|\nabla|u|^{3/2}\|_{L^2}^2 \\
&= \|p\|_{L^{a\alpha}}^{\alpha}\|p\|_{L^{b(1-\alpha)}}^{1-\alpha}\|u\|_{L^{c/2}}^{1/2}\|\nabla|u|^{3/2}\|_{L^2} - \|\nabla|u|^{3/2}\|_{L^2}^2 \\
&\leqslant C\|p\|_{L^s}^{\alpha}\|u\|_{L^{2b(1-\alpha)}}^{2(1-\alpha)}\|u\|_{L^{c/2}}^{1/2}\|\nabla|u|^{3/2}\|_{L^2} - \|\nabla|u|^{3/2}\|_{L^2}^2 \\
&=: A - \|\nabla|u|^{3/2}\|_{L^2}^2.
\end{aligned} \tag{54}$$

for appropriate α, a, b, c with $a\alpha = s$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{2}$. We deal with the three cases stated in the theorem one by one.

- $3 < s < \infty$. We take $\alpha = \frac{s}{2s-3}$, $a = 2s-3$, $b = \frac{3(2s-3)}{2s-6}$, $c = 6$. This gives

$$A = C\|p\|_{L^s}^{\frac{s}{2s-3}}\|u\|_{L^3}^{\frac{6s-15}{4s-6}}\|\nabla|u|^{3/2}\|_{L^2} \leqslant C\|p\|_{L^s}^{\frac{2s}{2s-3}}\|u\|_{L^3}^{\frac{6s-15}{2s-3}} + \|\nabla|u|^{3/2}\|_{L^2}^2. \tag{55}$$

Here we have applied Young's inequality. Substituting (55) into (54) and dividing both sides by $\|u\|_{L^3}^2$, we reach

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} \leqslant \frac{\|p\|_{L^s}^{\frac{2s}{2s-3}}}{\|u\|_{L^3}^{\frac{6}{2s-3}}}\|u\|_{L^3}. \tag{56}$$

The conclusion now follows from Lemma 1.

- $\frac{9}{4} < s < 3$. We take $\alpha = 1$, $a = s$, $c = \frac{2s}{s-2}$. Note that in this case the term involving b is not present. Application of the following interpolation inequality and Sobolev inequality

$$\|u\|_{L^{\frac{s}{s-2}}} \leqslant \|u\|_{L^3}^{\frac{4s-9}{s}}\|u\|_{L^9}^{\frac{9-3s}{s}}, \qquad \|u\|_{L^9}^3 \lesssim \|\nabla|u|^{3/2}\|_{L^2}^2 \tag{57}$$

gives

$$A = C\|p\|_{L^s}\|u\|_{L^{s/(s-2)}}^{1/2}\|\nabla|u|^{3/2}\|_{L^2} \lesssim \|p\|_{L^s}\|u\|_{L^3}^{\frac{4s-9}{2s}}\|\nabla|u|^{3/2}\|_{L^2}^{\frac{3}{s}}. \tag{58}$$

Application of Young's inequality now gives

$$A \leqslant C\|p\|_{L^s}^{\frac{2s}{2s-3}}\|u\|_{L^3}^{\frac{4s-9}{2s-3}} + \|\nabla|u|^{3/2}\|_{L^2}^2. \tag{59}$$

Substituting this into (54) and dividing both sides by $\|u\|_{L^3}^2$, we reach

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} \leqslant \frac{\|p\|_{L^s}^{\frac{2s}{2s-3}}}{\|u\|_{L^3}^{\frac{2s}{2s-3}}}\|u\|_{L^3}. \tag{60}$$

The conclusion now follows from Lemma 1.

- $\frac{3}{2} < s \leqslant \frac{9}{4}$. We take $\alpha = \frac{2s}{9-2s}$, $a = \frac{9-2s}{2}$, $b = \frac{9}{2}\frac{9-2s}{9-4s}$, $c = 18$. Now application of Sobolev inequality $\|u\|_{L^9}^3 \lesssim \|\nabla|u|^{3/2}\|_{L^2}^2$ gives

$$A = C\|p\|_{L^s}^{\frac{2s}{9-2s}}\|u\|_{L^9}^{\frac{45-18s}{18-4s}}\|\nabla|u|^{3/2}\|_{L^2} \lesssim \|p\|_{L^s}^{\frac{2s}{9-2s}}\|\nabla|u|^{3/2}\|_{L^2}^{\frac{24-8s}{9-2s}}. \tag{61}$$

Note that since $\frac{3}{2} < s \leqslant \frac{9}{4}$, there holds $\frac{24-8s}{9-2s} < 2$. Thus we can apply Young's inequality and obtain

$$A \leqslant C\|p\|_{L^s}^{\frac{2s}{2s-3}} + \|\nabla|u|^{3/2}\|_{L^2}^2. \tag{62}$$

Substituting this into (54) and dividing both sides by $\|u\|_{L^3}^2$, we reach

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^3} \leqslant \frac{\|p\|_{L^s}^{\frac{2s}{2s-3}}}{\|u\|_{L^3}^3}\|u\|_{L^3}. \tag{63}$$

The conclusion now follows from Lemma 1.

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[1] Samir Benbernou. A note on the regularity criterion in terms of pressure for the Navier-Stokes equations. *Applied Mathematics Letters*, 22(9):1438–1443, September 2009.

[2] Luigi C. Berselli and Giovanni P. Galdi. Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations. *Proc. Amer. Math. Soc.*, 130:3585–3595, 2002.

[3] Stefano Bosia, Vittorino Pata and James C. Robinson. A weak-L^p Prodi-Serrin type regularity criterion for the Navier-Stokes equations. *J. Math. Fluid Mech.*, 16(4): 721–725, 2014.

[4] Chongsheng Cao, Junlin Qin, and Edriss S. Titi. Regularity criterion for solutions of three-dimensional turbulent channel flows. *Commun. PDE*, 33:419–428, 2008.

[5] Chongsheng Cao and Edriss S. Titi. Regularity criteria for the three-dimensional Navier-Stokes equations. *Indiana University Mathematics Journal*, 57 (6): 2643–2661, 2008.

[6] Chongsheng Cao and Edriss S. Titi. Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor. *Arch. Rational Mech. Anal.*, 202: 919–932, 2011.

[7] Dongho Chae and Jihoon Lee. Regularity criterion in terms of pressure for the Navier-Stokes equations. *Nonlinear Analysis: Theory, Methods & Applications*, 46(5):727–735, November 2001.

[8] Chi Hin Chan and Alexis Vasseur. Log improvement of the Prodi-Serrin criteria for Navier-Stokes equations. *Methods and Applications of Analysis*, 14(2):197–212, 2007.

[9] Qionglei Chen and Zhifei Zhang. Regularity criterion via the pressure on weak solutions to the 3D Navier-Stokes equations. *Proc. Amer. Math. Soc.*, 135:1829–1837, 2007.

[10] L. Escauriaza, G. Seregin and V. Šverák. $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk*, 58(2):3–44, 2003. Russian Math. Surveys 58(2): 211–250.

[11] E. B. Fabes, B. F. Jones and N. M. Riviere. The initial value problem for the Navier-Stokes equations with data in L^p. *Arch. Rational Mech. Anal.*, 45:222–248, 1972.

[12] Jishan Fan, Song Jiang, Gen Nakamura and Yong Zhou. Logarithmically improved regularity criteria for the Navier-Stokes and MHD equations. *Journal of Mathematical Fluid Mechanics*, 13(4):557–571, December 2011.

[13] Jishan Fan, Song Jiang and Guoxi Ni. On regularity criteria for the n-dimensional Navier-Stokes equations in terms of the pressure. *Journal of Differential Equations*, 244(11):2963–2979, June 2008.

[14] Sadek Gala. Remarks on regularity criterion for weak solutions to the Navier-Stokes equations in terms of the gradient of the pressure. *Applicable Analysis: An International Journal*, 92(1):96–103, 2013.

[15] Yoshikazu Giga. Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *Journal of Differential Equations*, 62(2):186–212, April 1986.

[16] Zhengguang Guo and Sadek Gala. Remarks on logarithmical regularity criteria for the Navier-Stokes equations. *Journal of Mathematical Physics*, 52(6):63503, 2011.

[17] Zhengguang Guo and Sadek Gala. A note on the regularity criteria for the Navier-Stokes equations. *Applied Mathematics Letters*, 25(3):305–309, March 2012.

[18] Zhengguang Guo, Peter Wittwer and Weiming Wang. Regularity issue of the Navier-Stokes equations involving the combination of pressure and velocity field. *Acta Applicandae Mathematicae*, 123(1):99–112, February 2013.

[19] Xiaowei He and Sadek Gala. Regularity criterion for weak solutions to the Navier-Stokes equations in terms of the pressure in the class $L^2(0,T;\dot{B}^{-1}_{\infty,\infty}(R^3))\}$. *Nonlinear Analysis: Real World Applications*, 12(6):3602–3607, December 2011.

[20] Tosio Kato. Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solutions. *Math. Z.*, 187:471–480, 1984.

[21] O. A. Ladyzhenskaya. On uniqueness and smoothness of generalized solutions to the Navier-Stokes equations. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 5:169–185, 1967. English transl., Sem. Math. V. A. Steklov Math. Inst. Leningrad 5 (1969), 60–66.

[22] Jean Leray. On the motion of a viscous liquid filling space. *Acta mathematica*, 63:193–248, 1934.

[23] T. Ohyama. Interior regularity of weak solutions to the Navier-Stokes equation. *Proc. Japan Acad.*, 36:273–277, 1960.

[24] G. Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl.*, 4(48):173–182, 1959.

[25] Jame Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 9:187–191, 1962.

[26] Michael Struwe. On partial regularity results for the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(4):437–458, 1988.

[27] Michael Struwe. On a Serrin-type regularity criterion for the Navier-Stokes equations in terms of the pressure. *Journal of Mathematical Fluid Mechanics*, 9(2):235–242, June 2007.

[28] Tomoyuki Suzuki. A remark on the regularity of weak solutions to the Navier-Stokes equations in terms of the pressure in Lorentz spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 75(9):3849–3853, June 2012.

[29] C.V. Tran and X. Yu. Pressure moderation and effective pressure in Navier–Stokes flows. *Nonlinearity*, 29: 2990–3005, 2016.

[30] Xingwei Zhang, Yan Jia and Bo-Qing Dong. On the pressure regularity criterion of the 3D Navier-Stokes equations. *Journal of Mathematical Analysis and Applications*, 393(2):413–420, September 2012.

[31] Yi Zhou and Zhen Lei. Logarithmically improved criteria for Euler and Navier-Stokes equations. *Communications on Pure and Applied Analysis*, 12(6):2715–2719, November 2013.

[32] Yong Zhou. Regularity criteria in terms of pressure for the 3-D Navier-Stokes equations in a generic domain. *Mathematische Annalen*, 328(1):173–192, January 2004.

[33] Yong Zhou. On regularity criteria in terms of pressure for the Navier-Stokes equations in R^3. *Proc. Amer. Math. Soc.*, 134:149–156, 2006.

[34] Yong Zhou and Sadek Gala. Logarithmically improved regularity criteria for the Navier-Stokes equations in multiplier spaces. *Journal of Mathematical Analysis and Applications*, 356(2):498–501, August 2009.

[35] Xiang'ou Zhu. A regularity criterion for the Navier-Stokes equations in the multiplier spaces. *Abstract and Applied Analysis*, 2012, 2012.